Derived Invariance of the Tamarkin-Tsygan Calculus of an Associative Algebra

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Derived Invariance of the Tamarkin-Tsygan Calculus of an Associative Algebra

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Contents

0.1 Summary

0.1.1 English

In this thesis we prove that the Tamarkin-Tsygan calculus of a finite dimensional associative algebra over a field k is a derived invariant. In other words, the main result of this work goes as follows: a derived equivalence between two finite dimensional associative algebras over a field k induces an isomorphism between Hochschild homology and Hochschild cohomology that respects simultaneously the cup product, the cap product, the Gerstenhaber bracket and the Connes differential, see Theorem 5.2.15 in page 94.

We use homological constructions developed by Rickard [50, 51, 52] and Keller [34, 38] to extend results on derived invariance.

First, we use Morita theory for derived categories of Noetherian algebras over a commutative ring k that are finitely generated and projective as k-modules to obtain derived invariance of the cup product with coefficients in bimodules whose image under a derived equivalence is concentrated in degree zero, see Theorem 4.2.6 in page 70. We do this by extending the construction of the cup product in terms of morphisms in the derived category of the enveloping algebra given by Rickard in [50], see Proposition 4.2.4 in page 68.

We show that the cap product with coefficients in an algebra is a derived invariant of algebras over a field that have finite dimensional Hochschild homology in each degree, see Theorem 4.3.1 in page 71. The cap product is expressed in terms of Hochschild cohomology by using a canonical isomorphism. We then express the cap product in terms of morphisms in the derived category and with that interpretation the derived invariance of the cap product with coefficients in the algebra is obtained. In case k is a commutative ring, we have obtained with Keller [2] a generalization allowing bimodule coefficients whose image under a derived equivalence is concentrated in degree zero for k-projective Noetherian algebras that are finitely generated as k-modules, see Theorem 4.4.5 in page 79.

For the derived invariance of the Connes differential we assume k to be a field and consider two approaches. The first one gives an interpretation of the Connes differential in terms of morphisms in the derived category for algebras with finite dimensional Hochschild homology in each degree, see Proposition 5.1.4 in page 83 and the remark following it. However, it does not appear to be successful so far. It does not consider the relation of the Connes differential with cyclic homology.

The second approach was published in another joint work with Keller [3]. We prove derived invariance of the Connes periodicity long exact sequence for alge-

bras over a field using the cyclic functor defined by Keller in [34], see Corollary 5.2.13 in page 93. As a consequence we obtain the derived invariance of the Connes differential, see Corollary 5.2.14 in page 94.

Nevertheless, the isomorphism between homologies for the derived invariance of the Connes differential is not, a priori, the same than the one used for derived invariance of the cap product. Since the Tamarkin-Tsygan calculus of an algebra is the information given by the graded k-vector spaces of Hochschild homology and Hochschild cohomology, the cup product, the cap product, the Gerstenhaber bracket and the Connes differential, it is necessary that both isomorphisms coincide. We prove that this follows from the uniqueness of the cyclic functor, see Theorem 5.2.12 in page 89.

The results of this thesis can be summarized as follows. Let k be a field and let $\mathbf{ALG}_k$ be the category of differential graded algebras over k whose morphisms are complexes of bimodules $X : A \to B$ such that X is isomorphic to a bounded complex with finitely generated B-projective components. The composition is induced by the derived tensor product. Let $\mathbf{A}_k$ be the full subcategory of $\mathbf{ALG}_k$ of finite dimensional algebras.

Let $A \in \mathbf{A}_k$ and define $\mathbb{H}(A)$ to be the Tamarkin-Tsygan calculus of A given by its Hochschild theory. Let $\hat{\mathbb{H}}(A)$ be the Tamarkin-Tsygan calculus given by the interpretations of the Hochschild theory of A in terms of Hochschild cohomology. Define also $\widetilde{\mathbb{H}}(A)$ as the Tamarkin-Tsygan calculus given by the interpretations of the Hochschild theory of A as morphisms in the derived category of the enveloping algebra of A. See pages 96-98 for the constructions. Let $\mathbf{TT}-\mathbf{calc}$ be the category of Tamarkin-Tsygan calculi. We conclude with Theorem 5.3.2 in page 98.

Theorem. *Let k be a field. The assignments*

$$A \mapsto \mathbb{H}(A), \quad A \mapsto \hat{\mathbb{H}}(A) \quad \textit{and} \quad A \mapsto \widetilde{\mathbb{H}}(A)$$

define functors

$$\mathbb{H}, \hat{\mathbb{H}}, \widetilde{\mathbb{H}} : \mathbf{A}_k \to \textbf{\textit{TT-calc}}$$

that are constant on each class of derived equivalent algebras.

We finish this thesis by giving an example which shows that the Tamarkin-Tsygan calculus of an algebra is not a complete derived invariant.

0.1.2 French

Dans cette thèse nous démontrons que le calcul de Tamarkin-Tsygan d'une algèbre associative de dimension finie sur un corps k est un invariant dérivé. En d'autres mots, le résultat principal de ce travail est le suivant: une équivalence dérivée entre deux algèbres de dimension finie sur un corps k induit un isomorphisme entre l'homologie de Hochschild et la cohomologie de Hochschild qui respecte simultanément le cup produit, le cap produit, le crochet de Gerstenhaber et la différentielle de Connes, voir le Théorème 5.2.15 à la page 94.

Nous utilisons des constructions développées par Rickard [50, 51, 52] et Keller [34, 38] pour étendre des résultats sur l'invariance dérivée.

D'abord nous utilisons la théorie de Morita pour les catégories dérivées d'algèbres noethériennes sur un anneau commutatif k qui sont de génération finie et projectives comme k-modules pour obtenir l'invariance dérivée du cup produit avec des coefficients dans des bimodules dont l'image par une équivalence dérivée sont concentrés en degré zéro, voir le Théorème 4.2.6 à la page 70. Nous le faisons en étendant la construction du cup produit en termes de morphismes de la catégorie dérivée de l'algèbre enveloppante donnés par Rickard dans [50], voir le Proposition 4.2.4 à la page 68.

Nous montrons que le cap produit à coefficients dans une algèbre est un invariant dérivé pour des algè bres sur un corps dont l'homologie de Hochschild est de dimension finie en chaque degré, voir Théorème 4.3.1 à la page 71. Le cap produit est exprimé en termes de la cohomologie de Hochschild en utilisant un isomorphisme canonique. Nous exprimons alors le cap produit via les morphismes dans la catégorie dérivée et avec cette interprétation l'invariance dérivée du cap produit avec des coefficients dans l'algèbre est obtenue. Lorsque k est un anneau commutatif, nous avons obtenu avec Keller [2] une généralisation permettant des bimodules de coefficients dont l'image par une équivalence dérivée est concentrée en degré zéro pour des algèbres noethériennes k-projectives et de génération finie comme k-modules, voir Théorème 4.4.5 à la page 79.

Pour l'invariance dérivée de la différentielle de Connes, nous supposons que k est un corps et nous considérons deux approches. La première fournit une interprétation de la différentielle de Connes en termes de morphismes de la catégorie dérivée pour des algèbres dont l'homologie de Hochschild est finie en chaque degré, voir Proposition 5.1.4 à la page 83 et la remarque à la suite. Toutefois cette approche ne semble pas réussir, du moins jusqu'à maintenant. Elle ne considère pas la relation de la différentielle de Connes avec l'homologie cyclique.

La seconde approche a été publiée dans un autre travail en collaboration avec

Keller [3]. Nous prouvons l'invariance dérivée de la suite exacte longue de périodicité de Connes pour des algèbres sur un corps, en utilisant le foncteur cyclique défini par Keller dans [34], voir le Corollaire 5.2.13 à la page 93. En conséquence nous obtenons l'invariance dérivée de la différentielle de Connes, voir Corollaire 5.2.14 à la page 94.

Néanmoins l'isomorphisme entre les homologies pour l'invariance dérivée de la différentielle de Connes n'est pas, a priori, le même que celui utilisé pour l'invariance dérivée du cap produit. Puisque le calcul de Tamarkin-Tsygan d'une algèbre est l'information donnée par l'espace vectoriel gradué de l'homologie de Hochschild et celui de la cohomologie de Hochschild, le cup produit, le cap produit, le crochet de Gerstenhaber et la différentielle de Connes, il est nécessaire de montrer que les deux isomorphismes coïncident. Nous montrons que cela suit de l'unicité du foncteur cyclique, voir le Théorème 5.2.12 à la page 89.

Les résultats de cette thèse peuvent être résumés comme suit. Soit k un corps et soit $\mathbf{ALG}_k$ la catégorie des algèbres différentielles gradués sur k dont les morphismes sont les complexes de bimodules $X : A \to B$ tels que X est isomorphe à un complexe borné avec des composants B-projectifs de génération finie. La composition est induite par le produit tensoriel. Soit $\mathbf{A}_k$ la sous-catégorie pleine de $\mathbf{ALG}_k$ des algèbres de dimension finie.

Soit $A \in \mathbf{A}_k$ et définissons $\mathbb{H}(A)$ comme étant la calcul de Tamarkyn-Tsygan de A donné par sa théorie de Hochschild. Soit $\hat{\mathbb{H}}(A)$ le calcul de Tamarkyn-Tsygan donné par les interprétations de la théorie de Hochschild de A en termes de la cohomologie de Hochschild. Définissons aussi $\widetilde{\mathbb{H}}(A)$ comme le calcul de Tamarkyn-Tsygan donné par les interprétations de la théorie de Hochschild de A en tant que morphismes de la catégorie dérivée de l'algèbre enveloppante de A. Voir les pages 96-98 pour les constructions. Soit $\mathbf{TT}-\mathbf{calc}$ la catégorie des calculs de Tamarkyn-Tsygan . Nous concluons avec le Théorème 5.3.2 à la page 98.

Théorème. *Soit k un corps. Les associations*

$$A \mapsto \mathbb{H}(A), \quad A \mapsto \hat{\mathbb{H}}(A) \quad \textit{et} \quad A \mapsto \widetilde{\mathbb{H}}(A)$$

définissent des foncteurs

$$\mathbb{H}, \hat{\mathbb{H}}, \widetilde{\mathbb{H}} : \mathbf{A}_k \to \boldsymbol{TT\text{-}calc}$$

qui sont constants sur chaque classe d'équivalence dérivée d'algèbres.

Nous finissons cette thèse en donnant un exemple qui montre que le calcul de Tamarkin-Tsygan dune algèbre n'est pas un invariant dérivé complet.

Chapter 1

Introduction

In this thesis we prove that the Tamarkin-Tsygan calculus of a finite dimensional associative algebra over a field k is a derived invariant. In other words, the main result of this work goes as follows: a derived equivalence between two finite dimensional associative algebras over a field k induces an isomorphism between Hochschild homology and Hochschild cohomology that respects simultaneously the cup product, the cap product, the Gerstenhaber bracket and the Connes differential, see Theorem 5.2.15 in page 94.

Derived categories where introduced by Grothendieck and Verdier [18, 63], who used them as a formalism for hyperhomology. For algebras they have been studied by Happel [19], by Rickard [51, 50, 52] and Keller [34, 39, 37, 38]. Hochschild theory was introduced in [23] and [24], then Gerstenhaber [17] proved that Hochschild cohomology of an algebra has the structure of a *Gerstenhaber algebra*, which is a graded commutative algebra with a degree -1 Lie bracket such that the cup product is a derivation for the bracket. Then in [61] and [15] it was proven that Hochschild cohomology with these operations together with Hochschild homology, the cap product and Connes differential has the structure of a *Tamarkin-Tsygan calculus*. Since it has also been proven that the cup product and the Gerstenhaber bracket are derived invariants, the question arises to analyze the possible derived invariance of the cap product, of the Connes differential as well as of the entire Tamarkin-Tsygan calculus.

Next, we provide a summary of each Chapter.

In Chapter 2, we fix notation and give a quick review on the construction of (relative) derived categories of (differential graded) algebras [33, 35]. We also introduce derived functors of the tensor product and Hom functors [35, 63, 66, 70] on the three types of derived categories that we will be working with, namely derived categories of (ordinary) algebras, of differential graded algebras and relative derived categories of differential graded algebras. For example, we use the relative derived tensor product functor for differential graded algebras to define the R-relative derived Picard group introduced by Keller in [38], that he used to prove derived invariance of the Gerstenhaber bracket. We note that the non-relative version of the derived Picard group was introduced in [68] and in [54]. In the last section we recall Rickard's Morita theory for derived categories as well as Keller's work on the subject [32, 33, 35, 50, 51, 52]. We focus on the case of derived equivalences of standard type for Noetherian algebras that are finitely generated and projective over a commutative ring k, since we will strongly use them in Chapter 4. This Chapter ends with Proposition 2.3.16 in page 38, that fixes explicit natural transformations between equivalences of standard type, which are used in the next Chapters.

Chapter 3 is an introduction to Hochschild [7, 23, 24, 66, 67] and cyclic [14, 25, 26, 44] theories and their relation to derived categories [25, 34, 38, 52]. We present known results of derived invariance of Hochschild (co)homology, of the cup product and of the Gerstenhaber bracket. We recall the definition of the cap product and the Connes differential. The latter has (at least) three versions that we make explicit. We recall an expression of cyclic homology given by Kassel [25] in terms of derived functors of differential graded algebras and mixed complexes, used by Keller [34] to prove derived invariance of cyclic homology. At the end of this Chapter we give the definition of the Tamarkin-Tsygan calculus of an algebra provided by its Hochschild theory and we recall the Tamarkin-Tsygan calculus of Weyl algebras.

Chapters 4 and 5 are the main ones of this thesis. Chapter 4 begins by giving explicit morphisms between the spaces of morphisms in the derived categories in relation with Hochschild theory for Noetherian algebras that are finitely generated and projective as modules over a commutative ring k. We then make a detailed analysis of each of the operations in the theory. For the cup product we extend the interpretation in terms of morphisms in the derived category given by Rickard [52] to allow coefficients in arbitrary bimodules, by using the natural isomorphism

$$\gamma_N : H^n(A,N) \xrightarrow{\sim} Hom_{D^b(A^e)}(A,N[n]),$$

for an algebra A projective over a commutative ring k and an A-bimodule N, where A^e denotes $A \otimes A^{op}$ and A^{op} being the algebra with the same k-module structure, but opposite multiplication to that of A. Namely, for A-bimodules N and M we define an operation

$$Hom_{D^b(A^e)}(A,N[n]) \otimes Hom_{D^b(A^e)}(A,M[m]) \xrightarrow{\widetilde{\cup}} Hom_{D^b(A^e)}(A,(N\otimes_A M)[n+m])$$

and in Proposition 4.2.4 in page 68 we show that this is an extension of Rickard's interpretation of the cup product for k-projective algebras, by providing a commutative diagram

$$\begin{tikzcd} H^n(A,N)\otimes H^m(A,M) \arrow[r, "\cup"] \arrow[d, "\gamma_N\otimes\gamma_M"'] & H^{n+m}(A,N\otimes_A M) \arrow[d, "\gamma_{(N\otimes_A M)}"] \\ Hom_{D^b(A^e)}(A,N[n]) \otimes Hom_{D^b(A^e)}(A,M[m]) \arrow[r, "\widetilde{\cup}"] & Hom_{D^b(A^e)}(A,(N\otimes_A M)[n+m]). \end{tikzcd}$$

For Noetherian algebras which are projective and finitely generated as k-modules we prove derived invariance of the $\widetilde{\cup}$-product with coefficients in a bimodule M that is concentrated in degree zero under a derived equivalence, see Theorem 4.2.6 in page 70. As a consequence we get derived invariance of the cup product with coefficients in a bimodule M that is concentrated in degree zero under a derived equivalence for Noetherian algebras which are projective and finitely generated as k-modules.

The case of the cap product is more intricate. We consider k to be a field and we use the canonical morphism

$$\varphi_N : H_\bullet(A,N) \to H^\bullet(A,N^*)^*$$

which is defined via a morphism first considered in [7] page 181, where N^* is the k-dual of N, to provide a link of the cap product with Hochschild cohomology. The morphism φ_N is a monomorphism if k is a field, and is an isomorphism if each $H_n(A,N)$ is finite dimensional. In particular, for every k-algebra A that has finite dimensional Hochschild homology in each degree with coefficients in A there is an isomorphism

$$\varphi_A : HH_\bullet(A) \xrightarrow{\sim} H^\bullet(A,A^*)^*.$$

This also follows if the algebra is finite dimensional. We define an operation

$$\hat{\cap} : H^n(A,N^*)^* \otimes H^m(A,M) \to H^{n-m}(A,(N\otimes_A M)^*)^*,$$

that fits into a commutative diagram

$$\begin{tikzcd}
H_n(A,N)\otimes H^m(A,M) \arrow[r, "\cap"] \arrow[d, "\varphi_N\otimes 1"'] & H_{n-m}(A,(N\otimes_A M)) \arrow[d, "\varphi_{N\otimes_A M}"] \\
H^n(A,N^*)^*\otimes H^m(A,M) \arrow[r, "\hat{\cap}"] & H^{n-m}(A,(N\otimes_A M)^*)^*,
\end{tikzcd}$$

for A an algebra over field k and A-bimodules N and M. Observe that, for example, the vertical arrows are isomorphisms if A is a finite dimensional algebra and N and M are finite dimensional A-bimodules, see Proposition 4.3.7 in page 73. Moreover, we show that in the case of algebras that have finite dimensional Hochschild homology in each degree with both coefficients equal to A the $\hat{\cap}$-product is closely related to the cup product with coefficients in A^*. We extend the $\hat{\cap}$-product to morphisms in the derived category via the isomorphism γ, by first defining an operation

$$Hom_{D^b(A^e)}(A,N^*[n])^*\otimes Hom_{D^b(A^e)}(A,M[m]) \xrightarrow{\widetilde{\cap}} Hom_{D^b(A^e)}(A,(N\otimes_A M)^*[n-m])^*$$

for which we prove that the following diagram is commutative

$$\begin{tikzcd}
H^n(A,N^*)^*\otimes H^m(A,M) \arrow[r, "\hat{\cap}"] \arrow[d, "(\gamma^*_{N^*})^{-1}\otimes\gamma_M"'] & H^{n-m}(A,(N\otimes_A M)^*)^* \arrow[d, "(\gamma^*_{(N\otimes_A M)^*})^{-1}"'] \\
Hom_{D^b(A^e)}(A,N^*[n])^*\otimes Hom_{D^b(A^e)}(A,M[m]) \arrow[r, "\widetilde{\cap}"'] & Hom_{D^b(A^e)}(A,(N\otimes_A M)^*[n-m])^*.
\end{tikzcd}$$

The definition of the $\hat{\cap}$-product does not depend on the morphism φ, so the last diagram is commutative for an algebra A projective over a commutative ring k. Nevertheless, the $\hat{\cap}$-product is related to the cap product via isomorphisms if k is a field and Hochschild homology is finite dimensional in each degree. For example if A is finite dimensional and N and M are finite dimensional A-bimodules. We obtain the derived invariance of the usual cap product with both coefficients equal to A by proving the derived invariance of the $\widetilde{\cap}$-product, for algebras over a field with finite dimensional Hochschild homology in each degree.

There is another interpretation of the cap product, published in a joint work with Keller [2], for algebras that are projective over a commutative ring k. This interpretation of the cap product is given by the bottom morphism of the following commutative diagram

$$\begin{tikzcd} H_n(A,M) \arrow[r, "-\cap[f]"] \arrow[d, "\cong"'] & H_{n-m}(A,M) \arrow[d, "\cong"'] \\ H_0\left(M \otimes^{\mathbb{L}}_{A^e} A[-n]\right) \arrow[r, "H_0(1\otimes[f][-n])"'] & H_0\left(M \otimes^{\mathbb{L}}_{A^e} A[m-n]\right), \end{tikzcd}$$

for $[f] \in HH^m(A)$ and M an A-bimodule. This enables to give another proof for derived invariance of the cap product that allows bimodule coefficients in Hochschild homology for Noetherian algebras that are finitely generated and projective over a commutative ring k in the following way, see page 79. Let M be an A-bimodule such that under a derived equivalence $F : D(A^e) \xrightarrow{\sim} D(B^e)$ the complex $N := FM$ is concentrated in degree zero. In page 79 we prove the following.

Theorem. *4.4.5 Let A and B be derived equivalent Noetherian algebras that are projective and finitely generated as k-modules. For each*

$$[f] \in HH^m(A) = Hom_{D^b(A^e)}(A, A[m])$$

there is a commutative diagram

$$\begin{tikzcd} H_n(A,M) \arrow[r, "\cong"] \arrow[d, "-\cap[f]"'] & H_n(B,N) \arrow[d, "-\cap\widetilde{F}([f])"] \\ H_{n-m}(A,M) \arrow[r, "\cong"'] & H_{n-m}(B,N). \end{tikzcd}$$

This can also be proved for k-projective algebras by the use of model category theory, see [32] and [2]. However, in this work we make the constructions for Noetherian algebras that are finitely generated and projective as k-modules. Derived invariance of the cap product with this kind of coefficients in Hochschild homology for Noetherian algebras that are projective and finitely generated as k-modules follows from the last theorem. This proof extends the one given for algebras over a field with finite dimensional Hochschild homology in each degree.

In Chapter 5, we pursue by proving the derived invariance of the Connes differential for algebras over a field k. For this we give two approaches. The first one does not consider the relation of the Connes differential with cyclic homology. We use the canonical isomorphism φ to interpret the Connes differential in terms of Hochschild cohomology with coefficients in A^* for algebras with finite dimensional Hochschild homology in each degree with coefficients in A. Then we express the Connes differential in terms of morphisms in the derived category via the isomorphism γ. Namely, we define morphisms

$$\hat{B}_A : H^{n+1}(A,A^*) \to H^n(A,A^*)$$

and

$$\widetilde{B}_A : Hom_{D(A^e)}(A,A^*[n+1]) \to Hom_{D(A^e)}(A,A^*[n])$$

such that the diagrams

$$\begin{array}{ccc} HH_n(A) & \xrightarrow{B_A} & H_{n+1}(A) \\ \Big\downarrow \varphi_A & & \Big\downarrow \varphi_A \\ H^n(A,A^*)^* & \xrightarrow{\hat{B}_A^*} & H^{n+1}(A,A^*)^* \end{array}$$

and

$$\begin{array}{ccc} HH^{n+1}(A,A^*) & \xrightarrow{\hat{B}_A} & H^n(A,A^*) \\ \Big\downarrow \gamma_A & & \Big\downarrow \gamma_A \\ Hom_{D^b(A^e)}(A,A^*[n+1]) & \xrightarrow{\widetilde{B}_A} & Hom_{D^b(A^e)}(A,A^*[n]). \end{array}$$

are commutative. The last two diagrams are commutative for any k-projective algebra A over a commutative ring k, but the Connes differential is related to the $\widetilde{B}_A$ via isomorphisms only if k is a field and A has finite dimensional Hochschild homology in each degree with coefficients in A. We make explicit the reasons why the morphism $\widetilde{B}_A$ does not appear to be useful to prove derived invariance of the Connes differential.

The second approach is a joint work with Keller [3], in which we obtain derived invariance of the Connes periodicity long exact sequence by the use of the cyclic functor defined by Keller in [34]. As a consequence we get the derived invariance of the Connes differential between Hochschild homologies in the Tamarkin-Tsygan setting for algebras over a field k.

Note however that the isomorphism between homologies for the derived invariance above is not, a priori, the same than the one used for derived invariance of the cap product. In order to prove the derived invariance of the Tamarkin-Tsygan calculus of a finite dimensional algebra we use the uniqueness of the cyclic functor [34] to get that both isomorphisms are the same. The Connes differential in terms of Hochschild homology can be expressed as the composition of two maps in Connes periodicity long exact sequence, namely the periodicity map and the connecting morphism. These two maps involve cyclic homology, which is known to be a derived invariant by the work of Keller [34].

All of the above implies that the Tamarkin-Tsygan calculus of finite dimensional algebras is a derived invariant. Indeed, while the derived invariance of the cap product only requires that the algebras are Noetherian and finitely generated projective as modules over a commutative ring k, the derived invariance of the Connes differential is obtained when k is a field. In the meantime, the cup product and the Gerstenhaber bracket are derived invariants also with the finite dimensional hypothesis.

Finally, we summarize our results in Theorem 5.3.2 in page 98 for a field k. Let $\mathbf{Alg}_k$, as in [34], be the category whose objects are the associative DG k-algebras A such that the functor $Hom(A,-)$ sends quasi-isomorphisms to isomorphisms, and whose morphisms are morphisms of DG k-algebras which do not necessarily preserve the unit. Define $\mathbf{ALG}_k$ to be the category whose objects are those of $\mathbf{Alg}_k$ and morphisms from A to B are the isomorphism classes of $A-B$ bimodule complexes that are perfect over B, that is, the covariant Hom functor commutes with coproducts. The composition of morphisms in $\mathbf{ALG}_k$ is given by the total derived tensor product. The identity of $A \in \mathbf{ALG}_k$ is the isomorphism class of the bimodule ${}_AA_A$. Let $\mathbf{A}_k$ be the full subcategory of $\mathbf{ALG}_k$ formed by the finite dimensional k-algebras and let **TT-calc** be the category of Tamarkin-Tsygan calculi.

Let $A \in \mathbf{A}_k$ and define $\mathbb{H}(A)$ to be the Tamarkin-Tsygan calculus of A given by its Hochschild theory. Let $\hat{\mathbb{H}}(A)$ be the Tamarkin-Tsygan calculus given by the interpretations of the Hochschild theory of A in terms of Hochschild cohomology. Define also $\widetilde{\mathbb{H}}(A)$ as the Tamarkin-Tsygan calculus given by the interpretations of the Hochschild theory of A as morphisms in the derived category of the enveloping algebra of A. See pages 96-98 for the constructions. Let $\mathbf{TT}-\mathbf{calc}$ be the category of Tamarkin-Tsygan calculi. We conclude with Theorem 5.3.2 in page 98.

Theorem. *Let k be a field. The assignments*

$$A \mapsto \mathbb{H}(A), \quad A \mapsto \hat{\mathbb{H}}(A) \quad \textit{and} \quad A \mapsto \widetilde{\mathbb{H}}(A)$$

define functors

$$\mathbb{H}, \hat{\mathbb{H}}, \widetilde{\mathbb{H}} : \mathbf{A}_k \to \boldsymbol{\mathit{TT\text{-}calc}}$$

that are constant on each class of derived equivalent algebras.

This complements the results of Rickard [50, 51, 52], Keller [28, 34, 38] and Zimmermann [69] on derived invariance of Hochschild (co)homology, the cup product with coefficients in the algebra, the Gerstenhaber bracket and cyclic homology.

We finish this thesis by giving an example which shows that the Tamarkin-Tsygan calculus of an algebra is not a complete derived invariant.

Chapter 2

Derived homological algebra

In this Chapter we introduce derived categories of algebras and of differential graded algebras, as well as relative derived categories of differential graded algebras. We recall from [27], that *the machinery needed to define a derived category in full generality tends to obscure the simplicity of the phenomena*. We focus on the applications and properties of derived categories. Along this work we will denote by 1 the identity morphisms and by *id* the identity functors. We denote by " $\cong$ " the isomorphisms of objects in categories and isomorphisms of functors. Equivalences between categories will be denoted with the symbol " $\simeq$ ".

In general, one can construct the derived category of an abelian category, although this thesis focuses on the derived categories of the abelian categories of (differential graded) modules over (differential graded) algebras.

We assume the reader is familiar with the usual homological algebra of abelian and module categories, tensor product, Hom functors and their usual derived functors Tor and Ext. Let k be a commutative ring and let A be an associative k-algebra. For the tensor product over k we write $\otimes$ instead of $\otimes_k$, and we also write $Hom(X,Y)$ instead of $Hom_k(X,Y)$. Let $Mod(A)$ (resp. $A-Mod$) be the category of right (resp. left) A-modules and $Proj(A)$ (resp. $A-Proj$) the full subcategory of projective right (resp. left) A-modules. We will denote by $mod(A)$ (resp. $A-mod$) the full subcategory of the category $Mod(A)$ (resp. $A-Mod$) formed by the finitely generated right (resp. left) A-modules and by $proj(A)$ (resp. $A-proj$) the full subcategory of $mod(A)$, (resp. $A-mod$) formed by the finitely generated projective right (resp. left) A-modules. If A is Noetherian, then the categories $mod(A)$ and $proj(A)$ (resp. $A-mod$ and $A-proj$) are abelian. Moreover, the objects of $proj(A)$ (resp. $A-proj$) are the projective objects of $mod(A)$ (resp. $A-mod$).

The derived category of an algebra is constructed in three steps. First, one forms the category of differential complexes $Ch(A)$ of A-modules (we chose right modules unless specified otherwise), whose objects are pairs (X,d) where

$$X = \bigoplus_{n\in\mathbb{Z}} X_n$$

in which each X_n is an A-module and $d : X_\bullet \to X_{\bullet+1}$ is a *graded map* of degree 1, which means that the n-th component of d is a map $d_n : X_n \to X_{n+1}$, called the *differential* of X such that $d^2 = 0$. The morphisms $f : X \to Y$ in $Ch(A)$ are families of morphisms of A-modules $(f_n : X_n \to Y_n)_{n\in\mathbb{Z}}$ that commute with the differentials.

Second, one constructs the homotopy category in the following way. A *zero-homotopic* morphism of complexes $f : X \to Y$ is a morphism that admits a morphism $h : X \to Y$ of *degree* -1, i.e. h is equal to a family $(h_n : X_n \to Y_{n-1})_{n\in\mathbb{Z}}$, such that $f = hd + dh$. Two morphisms of complexes $f, g : X \to Y$ are called *homotopic* if $f - g$ is zero-homotopic. The homotopy category $K(A)$ has the same objects as $Ch(A)$ and the morphisms are homotopy equivalence classes of morphisms of complexes.

Finally, the derived category is described in terms of homology.

2.1 The derived category

The n-th *homology* of a complex X is defined as $H_n(X) = ker(d_n)/Im(d_{n-1})$ and a morphism of complexes $f : X \to Y$ inducing an isomorphism $H_n(f) : H_n(X) \to H_n(Y)$ for all n is called a *quasi-isomorphism*.

Lemma 2.1.1. *([66] page 17) A zero-homotopic morphism of complexes induces the zero map in homology. Therefore, two homotopic morphisms of complexes induce the same map in homology.*

Definition 2.1.2. *([66] page 17) We say that a morphism of complexes $h : X \to Y$ is a homotopy equivalence if there is a morphism of complexes $t : Y \to X$ such that $t \circ h$ is homotopic to the identity morphism of X and $h \circ t$ is homotopic to the identity morphism of Y.*

The derived category identifies all complexes for which there is a specified quasi-isomorphism between them. Of course, this last result proves that in particular homotopic complexes are quasi-isomorphic. We will follow [70] chapter

3 for the construction of the derived category. Other equivalent construction concerning localization of categories can be found in [18, 19, 63, 66].

Consider diagrams of the form $X \xleftarrow{v} Z \xrightarrow{\alpha} Y$ in the category $Ch(A)$, where v is a quasi-isomorphism. We say that a diagram $X \xleftarrow{v} Z \xrightarrow{\alpha} Y$ *covers* $X \xleftarrow{v'} Z' \xrightarrow{\alpha'} Y$ if there is a morphism of complexes $\gamma : Z \to Z'$ such that the following diagram is commutative

$$\begin{tikzcd} & Z' \arrow[dl, "v'"'] \arrow[dr, "\alpha'"] \arrow[dd, "\gamma"] & \\ X & & Y. \\ & Z \arrow[ul, "v"] \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

Since $v' = v\gamma$, then $H(\gamma) = H(v')H(v)^{-1}$ and therefore γ is necessarily a quasi-isomorphism. Two diagrams $X \xleftarrow{v} Z \xrightarrow{\alpha} Y$ and $X \xleftarrow{v'} Z' \xrightarrow{\alpha'} Y$ are equivalent if there is a third diagram $X \xleftarrow{v''} Z'' \xrightarrow{\alpha''} Y$ covering both of them, or equivalently one of them covers the other. The *derived category* $D(A)$ of A is the category whose objects are complexes of A-modules and the morphisms $Hom_{D(A)}(X,Y)$ are the equivalence classes by the covering relation of diagrams $\left[X \xleftarrow{v} Z \xrightarrow{\alpha} Y\right]$, which are called *roofs* or *fractions*. The identity morphism of a complex X is the roof $\left[X \xleftarrow{1} X \xrightarrow{1} X\right]$. To define composition we need the following result. Recall that the morphisms from X to Y in the homotopy category $K(A)$ of A are homotopy equivalence classes of morphisms of complexes from X to Y.

Lemma 2.1.3. *([66] page 381) Given objects $Z, Y, Z' \in K(A)$, a morphism $\alpha : Z \to Y$ and a quasi-isomorphism $\beta : Z' \to Y$ there exists $Z'' \in K(A)$, a quasi-isomorphism $\gamma : Z'' \to Z$ and a morphism $\delta : Z'' \to Z'$ such that the following diagram is commutative in the category $K(A)$*

$$\begin{tikzcd} Z'' \arrow[r, "\delta"] \arrow[d, "\gamma"'] & Z'. \arrow[d, "\beta"] \\ Z \arrow[r, "\alpha"'] & Y. \end{tikzcd}$$

Given two roofs $X \xleftarrow{v} Z \xrightarrow{\alpha} Y$ and $Y \xleftarrow{v'} Z' \xrightarrow{\alpha'} W$ we use the last lemma to construct a diagram

$$\begin{tikzcd}[column sep=small]
 & & Z'' \arrow[dl, "v''"'] \arrow[dr, "\alpha''"] & & \\
 & Z \arrow[dl, "v"'] \arrow[dr, "\alpha"] & & Z' \arrow[dl, "v'"'] \arrow[dr, "\alpha'"] & \\
X & & Y & & W.
\end{tikzcd}$$

In which v'' is a quasi-isomorphism. Define composition of roofs as

$$\left[Y \xleftarrow{v} Z' \xrightarrow{\alpha} W\right] \circ \left[X \xleftarrow{v} Z \xrightarrow{\alpha} Y\right] = \left[X \xleftarrow{vv''} Z \xrightarrow{\alpha'\alpha''} W\right].$$

This composition does not depend on the representative of the roof and it is associative with units given by roofs of the form $\left[X \xleftarrow{1} X \xrightarrow{1} X\right]$, see [70] section 3.4. The categories $Ch(A), K(A)$ and $D(A)$ are equipped with an auto-equivalence called the *shift-functor*. The *shift* $X[1]$ of a complex X is defined as $X[1]_n = X_{n-1}$ with differential $d_n[1] = -d_{n-1}$, and we inductively define $X[n] = X[n-1][1]$ and $d_m[n] = d_m[n-1][1]$, analogously for morphisms. The category $Ch(A)$ is an abelian category with component-wise direct sum (kernel, cokernel) as biproduct (kernel, cokernel). The shift functor endows the homotopy category $K(A)$ and the derived category $D(A)$ with triangulated category structures. The categories $K(A)$ and $D(A)$ are not abelian in general, but they have another similar and rich structure, that of a triangulated category. Triangulated categories are constructed to mimic the structure of abelian categories and short exact sequences.

Let $\mathbb{T}$ be an additive category with an autoequivalence $T : \mathbb{T} \to \mathbb{T}$ and let X, Y and Z be objects of $\mathbb{T}$. A *triangle* on (X,Y,Z) is a triple (u,v,w) of morphisms in $\mathbb{T}$ such that

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX\ .$$

Given exact triangles (u,v,w) on (X,Y,Z) and (u',v',w') on (X',Y',Z'), a *morphism of triangles* is a triple (f,g,h) of morphisms of $\mathbb{T}$ such that $f : X \to X'$ together with $g : Y \to Y'$ and $h : Z \to Z'$ make commutative the following diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & TX \arrow[d, "Tf"'] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX'.
\end{tikzcd}$$

Two triangles are called *isomorphic* if there is a morphism of triangles (f,g,h) such that f,g and h are isomorphisms.

Definition 2.1.4. *[63] An additive category $\mathbb{T}$ is called a triangulated category if it is equipped with an autoequivalence $T:\mathbb{T}\to\mathbb{T}$, called the translation functor, and with a distinguished family of triangles called exact triangles, which are subject to the following four axioms:*

- *Every morphism $u:X\to Y$ can be embedded in an exact traingle (u,v,w). If $X=Y$ and $Z=0$, then the triangle $(1,0,0)$ is exact. If (u,v,w) is a triangle on (X,Y,Z), isomorphic to an exact triangle (u',v',w') on (X',Y',Z'), then (u,v,w) is also exact.*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\cong"'] & Y \arrow[r, "v"] \arrow[d, "\cong"'] & Z \arrow[r, "w"] \arrow[d, "\cong"'] & TX \arrow[d, "\cong"'] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX'. \end{tikzcd}$$

- *If (u,v,w) is an exact triangle on (X,Y,Z), then both its rotates $(v,w,-Tw)$ and $(-T^{-1}w,u,v)$ are exact triangles on (Y,Z,TX) and $(T^{-1}Z,X,Y)$, respectively.*

- *Given exact triangles (u,v,w) on (X,Y,Z) and (u',v',w') on (X',Y',Z') with morphisms $f:X\to X'$ and $g:Y\to Y'$ such that $gu=u'f$, there exists a morphism $h:Z\to Z'$ such that (f,g,h) is a morphism of triangles*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, dashed, "\exists\, h"'] & TX \arrow[d, "Tf"'] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX' \end{tikzcd}$$

- *Given objects X,Y,Z,X',Y',Z' in $\mathbb{T}$ suppose there are exact triangles (u,j,∂) on (X,Y,Z') and (v,x,i) on (Y,Z,X') as well as (vu,y,δ) on (X,Z,Y'). Then there is an exact triangle $(f,g,(Tj)i)$ on (Z',Y',X') such that the following equations are satisfied:*

$$x=gy,\quad \partial=(Tj)if,\quad (Tj)i=(Tx)i,\quad ig=(Tu)\delta,\quad fj=yv.$$

Remark 2.1.5. *The last axiom of triangulated category is commonly known as the octahedral axiom. It is called this way because it can be represented in a three-dimensional diagram with the form of an octahedral given by the involved exact triangules. More details about the octahedral axiom can be found in [66, 70].*

Proposition 2.1.6. *([66] pages 376 and 382) The categories $K(A)$ and $D(A)$ are triangulated categories.*

Proof. Define the translation functor as the shift functor. Let $u : X \to Y$ be a morphism of complexes. Define the *cone* of u as the complex

$$Cone(u) = \bigoplus_{n\in\mathbb{Z}} (X_{n-1} \oplus Y_n)$$

with differential

$$\begin{pmatrix} -d^X & 0 \\ u & d^Y \end{pmatrix}.$$

The embedding into the first component gives a morphism of complexes $v : Y \to Cone(u)$ and the projection onto the second component gives a morphism of complexes $w : Cone(u) \to X[1]$. The triangles isomorphic to triangles of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Cone(u) \xrightarrow{w} X[1]$$

are the exact triangles. The rest of the proof consists on the verifications that this exact triangles triangulate the categories $K(A)$ and $D(A)$. For details we refer to [66, 70]. □

We will denote by $Ch^+(A)$ the full subcategory of $Ch(A)$ formed by the *bounded from below* complexes, i.e. complexes X for which there is an n such that $X_i = 0$ for all $i < n$. Let $Ch^-(A)$ be the full subcategory of $Ch(A)$ formed by the *bounded from above* complexes, i.e. complexes X for which there is an n such that $X_i = 0$ for all $i > n$. We will also denote $Ch^b(A)$ the full subcategory of $Ch(A)$ formed by the *bounded complexes*, that is, complexes that are in $Ch^+(A)$ and in $Ch^-(A)$ at the same time. Likewise for $K^+(A)$, $K^-(A)$, $K^b(A)$, $D^+(A)$, $D^-(A)$ and $D^b(A)$. Let $Ch^{-,b}(A)$ (resp. $Ch^{+,b}(A)$) be the full subcategory of $Ch(A)$ formed by the bounded from above (resp. below) complexes with bounded homology, i.e. bounded from above (resp. below) complexes for which there is an integer $m \geq 0$ such that for all $i > 0$ the homology in degree i (resp. degree $-i$) is zero. Similarly for $K^{+,b}(A)$ and $K^{-,b}(A)$.

Corollary 2.1.7. *([66] page 382 and 386) The categories $K^+(A)$, $K^-(A)$, $K^b(A)$, $K^{+,b}(A)$, $K^{-,b}(A)$, $D^+(A)$, $D^-(A)$ and $D^b(A)$ are triangulated categories.*

Definition 2.1.8. *([66] page 377) A morphism of triangulated categories is an additive functor $F : \mathbb{T} \to \mathbb{T}'$ that commutes with the translation functors and sends exact triangles to exact triangles.*

There is a canonical morphism of triangulated categories $Q : K(A) \to D(A)$ called the *localization functor*, that is equal to the identity on objects and it sends a morphism $f : X \to Y$ to the roof $\left[X \xleftarrow{1} X \xrightarrow{f} Y\right]$. Observe that if f is a quasi-isomorphism then $Q(f)$ is an invertible morphism in $D(A)$. This property characterizes the derived category as the following well known result shows.

Theorem 2.1.9. *([66] page 386) (Universal property) Let $\mathbb{T}$ be a triangulated category and let $F : K(A) \to \mathbb{T}$ be an additive functor that sends quasi-isomorphisms to invertible morphisms. Then there exists a unique functor $G : D(A) \to \mathbb{T}$ such that $F = G \circ Q$, i.e. the following diagram is commutative*

$$\begin{tikzcd} K(A) \arrow[r, "Q"] \arrow[d, "F"'] & D(A). \arrow[dl, dashed, "\exists!\, G"] \\ \mathbb{T} & \end{tikzcd}$$

The functor Q induces equivalences of triangulated categories when we restrict it to certain subcategories, depending on the algebra.

Proposition 2.1.10. *([70] page 332) Let A be an algebra over a commutative ring k. Then the functor Q induces triangulated equivalences*

- $D^-(A) \simeq K^-(A-Proj)$
- $D^b(A) \simeq K^{-,b}(A-Proj)$

and if A is Noetherian, then Q also induces triangulated equivalences

- $D^-(A-mod) \simeq K^-(A-proj)$
- $D^b(A-mod) \simeq K^{-,b}(A-proj)$.

Proof. The proof proceeds by constructing a quasi-inverse of the functor Q. For each object X in $D^-(A)$ we construct an object P_X in $K^-(A-Proj)$ such that $X \cong P_X$ in $D^-(A)$. We call P_X a *projective resolution* of X. Without loss of generality, we can assume that X is isomorphic to a complex whose negative degree components are 0, and $H_0(X) \neq 0$. That is,

$$X : \cdots \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \longrightarrow 0 \longrightarrow \cdots$$

where d_1 is not surjective. Let $P_0 \to X_0$ be a projective cover and form the pullback with d_1:

$$\begin{tikzcd}
 & & Q_1 \arrow[r, "\delta_1'"] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & X_2 \arrow[r, "d_2"] & X_1 \arrow[r, "d_1"] & X_0 \arrow[r] & 0.
\end{tikzcd}$$

In general, Q_1 will not be projective, so we take a projective cover $\rho_1 : P_1 \to Q_1$ and define $\delta_1 = \delta_1' \rho_1 : P_1 \to P_0$. Then form the following pullback with d_2:

$$\begin{tikzcd}
 & Q_2 \arrow[r] \arrow[dd] & ker\delta_1 \arrow[d] & & \\
 & & P_1 \arrow[r, "\delta_1"] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & X_2 \arrow[r, "d_2"] & X_1 \arrow[r, "d_1"] & X_0 \arrow[r] & 0.
\end{tikzcd}$$

This gives a commutative diagram whose rows are complexes

$$\begin{tikzcd}
 & Q_2 \arrow[r, "\delta_2'"] \arrow[d] & P_1 \arrow[r, "\delta_2"] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r, "d_3"] & X_2 \arrow[r, "d_2"] & X_1 \arrow[r, "d_1"] & X_0 \arrow[r] & 0.
\end{tikzcd}$$

Inductively we obtain a complex (P, δ) of projective modules and a morphism of complexes $\phi : P \to X$

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_3"] & P_2 \arrow[r, "\delta_2"] \arrow[d, "\phi_2"'] & P_1 \arrow[r, "\delta_1"] \arrow[d, "\phi_1"'] & P_0 \arrow[r] \arrow[d, "\phi_0"'] & 0 \\
\cdots \arrow[r, "d_3"] & X_2 \arrow[r, "d_2"] & X_1 \arrow[r, "d_1"] & X_0 \arrow[r] & 0.
\end{tikzcd}$$

The rest of the proof are straightforward verifications. For a detailed proof we refer to [70] page 333. □

Our next aim is to extend the constructions of derived categories from algebras to differential graded algebras. Once we do this, we will introduce relative derived categories of differential graded algebras. Let k be a commutative ring.

Definition 2.1.11. *[31] A differential graded k-algebra A (DG-algebra) is a $\mathbb{Z}$-graded associative k-algebra*

$$A = \bigoplus_{n\in\mathbb{Z}} A_n$$

endowed with a k-linear differential $d : A \to A$ which is homogeneous of degree 1, that is $d(A_n) \subset A_{n+1}$ for each $n \geq 0$. The differential satisfies the graded Leibniz rule

$$d(ab) = (da)b + (-1)^n a(db)\ \forall a \in A_n,\ \forall b \in A.$$

For example, if B is an algebra, it gives rise to a DG-algebra A defined by

$$A_n = \begin{cases} B,\ n = 0 \\ 0,\ n \neq 0, \end{cases}$$

with differential zero and the product of A in degree zero is the product of B and zero otherwise. Conversely, any DG-algebra A which is concentrated in degree 0 is obtained in this way.

Definition 2.1.12. *[31] Let A be a DG-algebra. A differential graded A-module (DG-module) is a $\mathbb{Z}$-graded module over A,*

$$M = \bigoplus_{n\in\mathbb{Z}} M_n$$

endowed with a k-linear differential $d : M \to M$ which is homogeneous of degree 1 and satisfies the graded Liebniz rule

$$d(ma) = (dm)a + (-1)^n m(da),\ \forall m \in M_n,\ \forall a \in A.$$

A *morphism of DG-modules* is a morphism of the underlying graded k-modules $f : M \to N$ which is homogeneous of degree 0 and commutes with the differentials. In the situation that the DG-algebra A comes from an associative algebra B concentrated in degree 0, the category of DG A-modules identifies with the category of complexes of B-modules.

Let A be a DG k-algebra, we will define its *derived category* $D(A)$ as the category with objects the DG A-modules and morphisms given by roofs $[gf^{-1}]$ of DG A-modules whose representatives are diagrams

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

where f is a quasi-isomorphism. For example, if A is an algebra and we consider it as a DG k-algebra concentrated in degree 0, then $D(A)$ is the usual derived category of A.

Let R be a *commutative* DG-algebra over k, that is $rs = (-1)^{|r||s|}sr$ for homogeneous elements $r, s \in R$.

Definition 2.1.13. *[31] Let E be a DG-algebra over R. A morphism of DG E-modules is an R-relative quasi-isomorphism if it is a homotopy equivalence as morphism of DG R-modules.*

Definition 2.1.14. *[38] Let E be a DG-algebra over R. The R-relative derived category $D_R(E)$ of E has as objects all the DG E-modules, and the morphisms are roofs $[gf^{-1}]$ where f is an R-relative quasi-isomorphism.*

For example, $D_k(E)$ is the usual derived category $D(E)$ of the DG-algebra E. Of course, all the properties we have proved for derived categories also hold for derived categories of DG-algebras and relative derived categories of DG-algebras. Namely, they are triangulated categories and satisfy an analogous universal property, as well as the analogous to Proposition 2.1.10 in page 24.

2.2 Derived functors

We follow [66] chapter 10 and [70] chapter 3 for the construction of derived functors. The fact that module categories have enough projectives and injectives allows us to lift the tensor product functor and the Hom functor to the derived category. Let A and B be k-algebras. Suppose given an additive functor $F : Mod(A) \to Mod(B)$, since F preserves chain homotopy equivalences it extends to additive functors $Ch(A) \to Ch(B)$ and $K(A) \to K(B)$. We would like to extend F to a functor $D(A) \to D(B)$. If $F : Mod(A) \to Mod(B)$ is an exact functor, then it sends quasi-isomorphisms to quasi-isomorphisms and therefore the functor F is well-defined in the derived category

$$F : D(A) \to D(B).$$

If F is not exact we proceed as follows. Let K be any of the subcategories $K^+(A)$, $K^-(A)$ or $K^b(A)$ of $K(A)$ and let D be the image of K under the localization functor Q, considered as a full subcategory of $D(A)$.

Definition 2.2.1. *([66] page 391) Let $F : K \to K(B)$ be a morphism of triangulated categories. A right derived functor of F on K is a morphism $\mathbb{R}F : D \to D(B)$ of triangulated categories together with a natural transformation $\eta : QF \to (\mathbb{R}F)Q$ which is universal in the sense that if $G : D \to D(B)$ is another morphism equipped with a natural transformation $\varepsilon : QF \to GQ$ then there exists a unique natural transformation $\sigma : \mathbb{R}F \to G$ such that $\varepsilon_X = \sigma_{Q(X)} \circ \eta_X$ for all $X \in D$.*

$$\begin{tikzcd} K \arrow[r, "F"] \arrow[d, "Q"'] & K(B) \arrow[d, "Q"'] \\ D \arrow[r, "\mathbb{R}F"'] & D(B) \end{tikzcd} \qquad \eta : QF \to (\mathbb{R}F)Q.$$

The left derived functor of a functor $F : K \to K(B)$ is a morphism of triangulated categories $\mathbb{L}F : D \to D(B)$ together with a natural transformation $\eta : (\mathbb{L}F)Q \to QF$ satisfying the dual univeral property. That is, for any morphism of triangulated categories $G : D \to D(B)$ and any natural transformation $\varepsilon : GQ \to QF$ there exists a unique natural transformation $\sigma : G \to \mathbb{L}F$ such that $\varepsilon_X = \eta_X \circ \sigma_{Q(X)}$ for all $X \in K$.

$$\begin{tikzcd} K \arrow[r, "F"] \arrow[d, "Q"'] & K(B) \arrow[d, "Q"'] \\ D \arrow[r, "\mathbb{L}F"'] & D(B) \end{tikzcd} \qquad \eta : (\mathbb{L}F)Q \to QF.$$

Theorem 2.2.2. *([66] page 392) Let $F : K^+(A) \to K^+(B)$ be a morphism of triangulated categories, then the right derived functor $\mathbb{R}F$ exists on $D^+(A)$, i.e. the following diagram is commutative*

$$\begin{tikzcd} K^+(A) \arrow[r, "F"] \arrow[d, "Q"'] & K^+(B) \arrow[d, "Q"'] \\ D^+(A) \arrow[r, "\mathbb{R}F"'] & D^+(B) \end{tikzcd} ,$$

and if I is a bounded from below complex of injectives then

$$\mathbb{R}F(I) = QF(I).$$

Dually, let $G : K^-(A) \to K^-(B)$ be a morphism of triangulated categories, then the *left derived functor $\mathbb{L}G$ exists on $D^-(A)$, i.e. the following diagram is commutative*

$$\begin{tikzcd} K^-(A) \arrow[r, "G"] \arrow[d, "Q"'] & K^-(B) \arrow[d, "Q"'] \\ D^-(A) \arrow[r, "\mathbb{L}G"'] & D^-(B) \end{tikzcd},$$

and if P is a bounded from above complex of projectives then

$$\mathbb{L}G(P) = QG(P).$$

Remark 2.2.3. *The proof relies on the fact that module categories have both enough injectives and enough projectives.*

The tensor product of modules can be extended to complexes in the following way. Let X be a complex of right A-modules and Y be a complex of left A-modules. Define a complex $X \otimes_A Y$ which degree n component is

$$(X \otimes_A Y)_n = \bigoplus_{i+j=n} X_i \otimes_A Y_j$$

and differential

$$d_n = \left((-1)^j d_i^X \otimes 1 + 1 \otimes d_j^Y\right)_{i+j=n}.$$

If X is a complex in $Ch(A \otimes B^{op})$ (complexes of $A-B$ bimodules), then the functor $-\otimes_A X$ takes values in $Ch(B)$. In fact, the tensor product of complexes can be extended to an additive bifunctor

$$-\otimes_B - : Ch(A \otimes B^{op}) \times Ch(B \otimes C^{op}) \to Ch(A \otimes C^{op}).$$

Since the tensor product is an additive functor, if we take X a bounded from above complex of $A-B$ bimodules, then the tensor product with X over A defines an additive functor

$$-\otimes_A X : K^-(A) \to K^-(B).$$

By the existence theorem of derived functors we can define the following.

Definition 2.2.4. *([70] page 354) Let X be a bounded from above complex of right A-modules and Y be a bounded from above complex of left A-modules. The total tensor product of X and Y is defined as the left derived functor of the tensor product $-\otimes_A Y : K^-(A) \to K^-(B)$, that is*

$$X \otimes_A^{\mathbb{L}} Y := \mathbb{L}(-\otimes_A Y)(X).$$

Categories of modules have enough projective objects and therefore every object X has a projective resolution P_X as in Proposition 2.1.10 in page 24, then the functor $X \otimes^{\mathbb{L}}_A Y$ can be calculated as $P_X \otimes_A Y$. One can also consider the complex $X \otimes_A P_Y$. The next result shows that both procedures give the same value and therefore $X \otimes^{\mathbb{L}}_A Y$ can be calculated on either way. We say that the functor $- \otimes^{\mathbb{L}}_A -$ is *balanced*.

Proposition 2.2.5. *([66] page 395) Let X be a complex in $K^-(A)$ of right A-modules and Y a complex in $K^-(A)$ of left A-modules. Then there is an isomorphism*

$$\mathbb{L}(X \otimes_A -)(Y) \cong \mathbb{L}(- \otimes_A Y)(X).$$

We define $Hom_A(X,Y)$ for X and Y complexes of right A-modules, to be the complex with degree n component

$$Hom_A(X,Y)_n = \prod_{i-j=n} Hom_A(X_i, Y_j)$$

and differential

$$d_n = \prod_{i-j=n} (d^Y_j)_* + (-1)^j (d^X_{i+1})^*,$$

see for example [70] page 354. Where we have denoted $(d^Y_j)_* = Hom_A(X, d^Y_j)$ and $(d^X_{i+1})^* = Hom_A(d_{i+1}, Y)$. This defines an additive bifunctor

$$Hom_A(-,-) : Ch(A) \times Ch(A) \to Ch(k).$$

If X is a bounded from below complex of $B-A$ bimodules, then we have an additive functor

$$Hom_A(X,-) : Ch^+(A) \to Ch^+(B).$$

The *Hom*-functor is an additive functor and therefore it extends to an additive functor

$$Hom_A(X,-) : K^+(A) \to K^+(B),$$

in case X is a bounded from below complex of $B-A$ bimodules.

Definition 2.2.6. *([70] page 355) Let X and Y be complexes of right A-modules. The right derived functor of the Hom functor is defined as the right derived functor of the functor $Hom_A(X,-) : K^+(A) \to K^+(B)$, that is*

$$\mathbb{R}Hom_A(X,Y) = \mathbb{R}Hom_A(X,-)(Y).$$

This construction can be extended to a bifunctor

$$\mathbb{R}Hom_A(-,-): Ch(A \otimes B^{op}) \times Ch(A \otimes C^{op}) \to Ch(B \otimes C^{op}).$$

Categories of modules have both enough projectives and enough injectives. Construct I_Y an injective resolution of the complex Y dual to the projective resolution defined in Proposition 2.1.10 in page 24. We can compute $\mathbb{R}Hom_A(X,Y)$ as $Hom_A(X,I_Y)$. Of course, this derived functor is also balanced, and can be computed as $Hom_A(P_X,Y)$.

Proposition 2.2.7. *([66] page 400) Let X and Y be complexes in $K^+(A)$. Then there is an isomorphism*

$$\mathbb{R}Hom_A(X,-)(Y) \simeq \mathbb{R}Hom_A(-,Y)(X).$$

It is well-known that the tensor product of modules and the Hom functor form an adjoint pair of functors. Under certain circumstances this is the case for their derived functors.

Proposition 2.2.8. *([70] page 358) Let A and B be k-algebras and let X be a bounded complex of $A-B$ bimodules. Suppose A is k-projective. Then*

$$-\otimes_A^{\mathbb{L}} X : D^-(A) \to D^-(B)$$

has a right adjoint

$$\mathbb{R}Hom_B(X,-): D^-(B) \to D^-(A).$$

Remark 2.2.9. *These definitions of derived functors generalize the definitions of Tor and Ext in the sense that there are natural isomorphisms*

$$H_n(X \otimes_A^{\mathbb{L}} Y) \cong Tor_n^A(X,Y)$$

and

$$H^n(\mathbb{R}Hom_A(X,Y)) \cong Ext_A^n(X,Y),$$

for each $n \geq 0$. It is also true that there is a natural isomorphism

$$H_n(\mathbb{R}Hom_A(X,Y)) \cong Hom_{D(A)}(X,Y[n]),$$

for each $n \geq 0$. Intuitively, one can think that the derived category and derived functors comprise the homological algebra of the functors Tor and Ext

Remark 2.2.10. *Let k be a field, then the tensor product of complexes $\otimes_k$ is its own left derived functor $\otimes_k^{\mathbb{L}}$. Then it is also symmetric, i.e. $X \otimes_k^{\mathbb{L}} Y \cong Y \otimes_k^{\mathbb{L}} X$.*

Consider DG-algebras A and B, the tensor product and Hom functors can be defined in an analogous way in the derived categories of the DG-algebras A and B. Let X be a DG $A \otimes B^{op}$-module bounded as a complex, we denote these derived functors as

$$- \otimes_A^{\mathbb{L}} X : D^-(A) \to D^-(B)$$

and

$$\mathbb{R}Hom_A(X, -) : D^+(A) \to D^+(B).$$

Let R be a commutative DG-algebra over k. Consider DG-algebras A and B over R and take X a DG $A \otimes B^{op}$-module. The tensor product functor also allows a derived functor on the relative derived category, namely

$$- \otimes_A^{\mathbb{L},rel} X : D_R(A) \to D_R(B).$$

Definition 2.2.11. *[38] Let A be an algebra. A complex $U \in D_R(R \otimes A^e)$ is R-semifree if its underlying graded R-module is free.*

We will now define the *derived Picard group* of an (ordinary) algebra A. An object of the R-relative derived category $U \in D_r(R \otimes A^{op} \otimes A)$ is *invertible* if it is R-semifree and there exists an R-semifree complex $V \in D_R(R \otimes A \otimes A^{op})$ such that there are isomorphisms

$$U \otimes_{A\otimes R}^{\mathbb{L},rel} V \cong R \otimes A \quad \textit{and} \quad V \otimes_{A\otimes R}^{\mathbb{L},rel} U \cong R \otimes A.$$

This group was defined by Keller in [38] to prove derived invariance of the Gerstenhaber bracket in Hochschild cohomology.

2.3 Morita theory for derived categories

We follow [70] for the introduction to the work of Keller and Rickard in [29, 30, 36, 33, 35, 32, 51, 52]. Let k be a commutative ring and let A and B be k-algebras. Let $F : B - Mod \to A - Mod$ be a functor, and let $M := F(B)$ which is an $A - B$ bimodule. If $End_A(M) \cong B$, Morita theory tells that

$$M \otimes_B - : B - Mod \xrightarrow{\sim} A - Mod$$

is an equivalence. Furthermore, whenever there is a bimodule M which is projective as an A-module, projective as a B-module and such that $End_A(M) \cong B$ and $End_B(M) \cong A$, then

$$M \otimes_B - : B - Mod \tilde{\to} A - Mod$$

is an equivalence. There is an embedding of $A - Mod$ as full subcategory of $D^-(A)$, that takes a module M to the complex that has M concentrated in degree 0. This is called the *stalk complex* of M. We will denote as M the image of M under this embedding. Let $F : D^-(A) \to D^-(B)$ be an additive functor that commutes with the shift functor, then there is a morphism of algebras

$$A^{op} \cong End_A(A) \cong End_{D^-(A)}(A) \to End_{D^-(B)}(F(A))$$

and if F is an equivalence, then $End_{D^-(B)}(F(A)) \cong A^{op}$ as algebras. Moreover, since $Ext^i_A(A,A) = 0$ for all $i \neq 0$, then

$$\begin{aligned} Hom_{D^-(B)}(F(A),F(A)[i]) &\cong Hom_{D^-(B)}(F(A),F(A[i])) \\ &\cong Hom_{D^-(A)}(A,A[i]) \\ &\cong Ext^i_A(A,A) \\ &= 0 \end{aligned}$$

for all $i \neq 0$. Also, the smallest triangulated subcategory of $D^-(A)$ containing all direct summands of finite direct sums of A is $K^b(A - proj)$. The following lemma characterices $K^b(A - Proj)$ as a subcategory of $K^-(A - Proj)$.

Lemma 2.3.1. *([70] page 559) $K^b(A - Proj)$ is the subcategory of $K^-(A - Proj)$ consisting of those objects X such that for all object Y of $K^-(A - Proj)$ there is an integer $i(Y)$ such that*

$$Hom_{K^-(A-Proj)}(Y,X[i]) = 0$$

for all $i < i(Y)$.

The notion of compact objects, described below, is linked to the definition of a tilting complex, which is crucial for derived equivalences.

Definition 2.3.2. *([70] page 283) Let C be an additive category. An object $X \in C$ is compact if the covariant Hom-functor commutes with arbitrary coproducts, i.e. there is an isomorphism*

$$Hom_C(X, \bigoplus_{\lambda \in \Lambda} Y_\lambda) \cong \bigoplus_{\lambda \in \Lambda} Hom_C(X, Y_\lambda),$$

for every family $\{Y_\lambda\}_{\lambda \in \Lambda}$ of objects of C.

It is straightforward, but tedious, to prove the following characterization about the compact objects of $D^{-}(A)$.

Proposition 2.3.3. *([70] page 560) The compact objects in $D^{-}(A)$ are the objects isomorphic to objects in $K^{b}(A-proj)$.*

A tilting complex is a special kind of compact object, that generalizes the ideas behind tilting modules in tilting theory [19, 30, 50], see for example Chapter VI of [5] in page 184 for tilting theory.

Definition 2.3.4. *[51] Let A be an algebra over a commutative ring k. A tilting complex T over A is a compact object of $D^{-}(A)$ such that*

- *$Hom_{D^{-}(A)}(T, T[i]) = 0$ for all $i \neq 0$.*
- *The smallest triangulated full subcategory of $D^{-}(A)$ containing all direct factors of finite direct sums of T is $K^{b}(A-proj)$.*

The following proposition gives a direct construction of a tilting complex given a derived equivalence between algebras.

Proposition 2.3.5. *[51] Let $F : D^{-}(A) \to D^{-}(B)$ be a triangulated equivalence. Then $T := F(A)$ is a tilting complex over B with endomorphism ring A^{op}.*

As observed in [70] section 6.1, the disadvantage of Rickard's theorem is that it just gives a necessary and sufficient criterion for the existence of an equivalence between derived categories of algebras, but it does not give an explicit equivalence. However, Keller [36] gave a constructive proof of an explicit X once T is known.

Theorem 2.3.6. *[36] Let A and B be k-algebras and suppose that A is k-projective. Let T be a bounded complex of projective B-modules and suppose given a morphism of k-algebras $\alpha : A \to End_{K^{-}(B-Proj)}(T)$. Suppose moreover that*

$$Hom_{K^{-}(B-Proj)}(T, T[n]) = 0$$

for each $n > 0$. Then there is a complex X in $K^{-}(B-Proj)$, a quasi-isomorphism $\varphi : T \to X$ in $K^{-}(B-Proj)$, and a morphism of complexes $\beta : A \to End_{C^{-}(B-Proj)}(X)$ such that for all $a \in A$ the diagram

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[d, "\alpha(a)"'] & X \arrow[d, "\beta(a)"] \\ T \arrow[r, "\varphi"] & X \end{tikzcd}$$

is commutative in $K^{-}(B-Proj)$.

This theorem is proved using strong homotopy actions and is due to Keller. It has the following very useful consequence.

Corollary 2.3.7. *[36] Let k be a commutative ring and let A and B be k-algebras and suppose that A is k-projective. Let X be a right bounded complex of $B-A$ bimodules. Then there is a complex X' of $B-A$ bimodules such that $X' \cong X$ in $D^-(B\otimes A^{op})$ and such that the image of X' in $D^-(B)$ is a right bounded complex of projective B-modules. If $B\otimes A^{op}$ is Noetherian, and if all homogeneous components of X are finitely generated, then all homogeneous components of X' can be chosen to be finitely generated.*

Let T be a bounded complex of projective B-modules and suppose that $X \cong T$ in $D^-(B)$. Then there is a bounded complex X'' isomorphic to X in $D^-(B\otimes A^{op})$ such that each homogeneous component of X'' is projective as a $B\otimes A^{op}$-module except the component in the highest degree m, where X''_m is B-projective. Moreover, if $T_i = 0$, then $X''_i = 0$.

If B is also k-projective, and if there is a bounded complex T' of projective A-modules which is isomorphic to X in $D^-(A)$, then X'' can be chosen so that all its homogeneous components are A-projective and B-projective.

We will always mean equivalence given by a morphism of triangulated categories when we refer to equivalences of derived categories. The original result [51] of Rickard is as follows, see [70] on pages 589-593 for a detailed proof.

Theorem 2.3.8. *[51] Let k be a commutative ring and let A and B be k-algebras such that A is k-projective. Then*

- *$D^-(A)$ and $D^-(B)$ are equivalent if and only if there is a tilting complex T over B with endomorphism algebra A^{op}. If there is an equivalence $D^-(A)\simeq D^-(B)$, then T is the image of A under this equivalence.*

- *Moreover,*

$$\begin{aligned} D^-(A)\simeq D^-(B) &\iff D^b(A)\simeq D^b(B)\\ &\iff K^b(A-proj)\simeq K^b(B-proj)\\ &\iff K^b(A-Proj)\simeq K^b(B-Proj)\\ &\iff K^-(A-Proj)\simeq K^-(A-Proj). \end{aligned}$$

- *If A and B are Noetherian and finitely generated projective as k-modules, then*

$$D^-(A)\simeq D^-(B) \iff D^b(A-mod)\simeq D^b(B-mod).$$

Remark 2.3.9. *Let k be a field and let A and B be finite dimensional k-algebras. If $D^b(A) \simeq D^b(B)$, then there is a complex X of $A-B$ bimodules such that*

$$-\otimes_A^{\mathbb{L}} X : D^b(A) \to D^b(B)$$

is an equivalence. It is not known if this implies that every equivalence of derived categories is of this form, see [70] on page 593. No equivalence is known which is not given by a tensor product with a complex of bimodules.

Remark 2.3.10. *Since we are interested in finite dimensional algebras, we can restrict our study to equivalences of the form*

$$D^b(A-mod) \xrightarrow{\sim} D^b(B-mod).$$

We will denote $D^b(A) := D^b(A-mod)$ when we assume that A is a Noetherian algebra which is k-projective and finitely generated as k-module.

Definition 2.3.11. *([70] page 594) Let A and B be algebras projective over a commutative ring k. A triangulated equivalence $F : D^b(A) \to D^b(B)$ is called of standard type if there is a complex X of $A-B$ bimodules such that*

$$F \cong -\otimes_A^{\mathbb{L}} X.$$

Although not all equivalences of derived categories have to be of standard type, every equivalence of derived categories gives an equivalence of standard type.

Proposition 2.3.12. *([70] page 594) Let A and B be algebras over a commutative ring k. Assume B is a projective k-module. If there is an equivalence $F : D^b(B) \to D^b(A)$ then there is an equivalence of standard type $-\otimes_A^{\mathbb{L}} X : D^b(A) \to D^b(B)$. Moreover, we can choose $X = F(B)$.*

It turns out that once one has an equivalence of standard type, a quasi-inverse, also of standard type, can be described.

Proposition 2.3.13. *([70] page 594) Let k be a commutative ring and let A and B be Noetherian k-projective algebras, and suppose that they are finitely generated k-modules. Let*

$$-\otimes_A^{\mathbb{L}} X : D^b(A) \to D^b(B)$$

be an equivalence of standard type. Then

$$\mathbb{R}Hom_B(X, -) : D^b(B) \to D^b(A)$$

is a quasi-inverse of $-\otimes^{\mathbb{L}}_A X$. *Moreover, there is an isomorphism of functors*

$$\mathbb{R}Hom_B(X,-) \cong -\otimes^{\mathbb{L}}_B \mathbb{R}Hom_B(X,B).$$

Corollary 2.3.14. *([70] page 595) Let k be a commutative ring and let A and B be Noetherian k-projective algebras that are finitely generated as k-modules. Let*

$$-\otimes^{\mathbb{L}}_A X : D^b(A) \to D^b(B)$$

be an equivalence of standard type. Then there is a complex $X^\vee$ *of* $B-A$ *bimodules, such that it has components which are projective as A-modules and as B-modules, and*

$$X^\vee \otimes^{\mathbb{L}}_A X \cong B \;\; in \;\; D^b(B^e) \;\; and \;\; X \otimes^{\mathbb{L}}_B X^\vee \cong A \;\; in \;\; D^b(A^e).$$

Of course, we put $X^\vee := \mathbb{R}Hom_B(X,B)$. Let $A^e = A \otimes_k A^{op}$ be the *enveloping algebra* of A with product

$$(a\otimes b)(a' \otimes b') = aa' \otimes b'b,$$

for $a,a' \in A$ and $b,b' \in A^{op}$. The category of A-bimodules is equivalent to the category of left A^e-modules and to the category of right A^e-modules. Indeed, an A-bimodule M is turned into a left A^e-module by $(a\otimes b)m = amb$ and a right A^e-module by $m(a\otimes b) = bma$.

Proposition 2.3.15. *([70] page 612) Let k be a commutative ring and let A and B be Noetherian k-algebras that are finitely generated and projective as k-modules. Let*

$$-\otimes^{\mathbb{L}}_A X : D^b(A) \to D^b(B)$$

be a derived equivalence with quasi-inverse

$$-\otimes^{\mathbb{L}}_A X^\vee : D^b(B) \to D^b(A).$$

Then

$$F := \left(X^\vee \otimes^{\mathbb{L}}_A -\right) \otimes^{\mathbb{L}}_A X : D^b(A^e) \to D^b(B^e)$$

is an equivalence of triangulated categories of standard type. Moreover, there is an isomorphism of functors

$$F \cong X^\vee \otimes^{\mathbb{L}}_A (-\otimes^{\mathbb{L}}_A X).$$

Proof. Since both A and B are Noetherian and finitely generated projective k-modules, we may choose X and Y so that all their homogeneous components are projective except for the last one which is projective as left and as right module, by Keller's theorem [36]. We then can replace the derived tensor product by the tensor product of complexes. This argument also shows that the tensor product is associative. The fact that A is k-projective implies that $A \otimes B^{op}$ is a finitely generated free right B-module, and likewise for A. The quasi-inverse of F is given by

$$G := X \otimes_B^{\mathbb{L}} - \otimes_B^{\mathbb{L}} X^\vee : D^b(B^e) \to D^b(A^e).$$

Finally,

$$X^\vee \otimes_A - \otimes_A X \cong - \otimes_{A^e} (X \otimes X^\vee),$$

therefore

$$X^\vee \otimes_A^{\mathbb{L}} - \otimes_A^{\mathbb{L}} X \cong - \otimes_{A^e}^{\mathbb{L}} (X \otimes X^\vee).$$

□

The following result will be extensively used in Chapters 4 and 5.

Proposition 2.3.16. *[52] Let k be a commutative ring and let A and B be derived equivalent Noetherian k-algebras that are finitely generated and projective as k-modules. There are isomorphisms*

$$u : A \xrightarrow{\sim} X \otimes_B^{\mathbb{L}} X^\vee \quad \text{and} \quad v : X^\vee \otimes_A^{\mathbb{L}} X \xrightarrow{\sim} B$$

in $D(A^e)$ and $D(B^e)$ respectively, that induce a functor

$$F = - \otimes_{A^e}^{\mathbb{L}} (X \otimes_k^{\mathbb{L}} X^\vee) \cong X^\vee \otimes_A^{\mathbb{L}} - \otimes_A^{\mathbb{L}} X : D^b(A^e) \xrightarrow{\sim} D(B^e)$$

and a quasi-inverse for it

$$G = - \otimes_{B^e}^{\mathbb{L}} (X^\vee \otimes_k^{\mathbb{L}} X) \cong X \otimes_B^{\mathbb{L}} - \otimes_B^{\mathbb{L}} X^\vee : D(B^e) \xrightarrow{\sim} D^b(A^e).$$

Proof. As in the proof of the previous result, we can replace the derived tensor product by the tensor product of complexes. Consider an equivalence of standard type

$$- \otimes_A^{\mathbb{L}} X : D^b(A) \xrightarrow{\sim} D^b(B)$$

it has as quasi-inverse the functor

$$- \otimes_B^{\mathbb{L}} X^\vee : D^b(B) \xrightarrow{\sim} D^b(A),$$

and therefore there is a natural transformation

$$u : id_{D^b(A)} \xrightarrow{\sim} - \otimes^{\mathbb{L}}_A X \otimes^{\mathbb{L}}_B X^\vee,$$

which is an isomorphism of functors. This natural transformation extends to

$$u : id_{D^b(A^e)} \xrightarrow{\sim} - \otimes^{\mathbb{L}}_A X \otimes^{\mathbb{L}}_B X^\vee,$$

by functoriality. Evaluating on the A^e-module A we get an isomorphism in $D^b(A^e)$ that we also denote as

$$u : A \xrightarrow{\sim} X \otimes^{\mathbb{L}}_B X^\vee.$$

We choose $v : X^\vee \otimes^{\mathbb{L}}_A X \to B$ as the preimage of the identity morphism 1_X of X under the following composition of isomorphisms in $D(k)$

$$\begin{aligned} Hom_{D(k)}(X^\vee \otimes^{\mathbb{L}}_B X, B) &\xrightarrow{\sim} Hom_{D(k)}(X \otimes^{\mathbb{L}}_B X^\vee \otimes^{\mathbb{L}}_A X, X \otimes^{\mathbb{L}}_B B) \\ &\xrightarrow{\sim} Hom_{D(k)}(A \otimes^{\mathbb{L}}_A X, X \otimes^{\mathbb{L}}_B B) \\ &\xrightarrow{\sim} Hom_{D(k)}(X, X). \end{aligned}$$

Where the first one is induced by $1_X \otimes^{\mathbb{L}}_B -$, the second one by $u \otimes^{\mathbb{L}}_A 1_X$ and the last one is induced by the canonical identifications $A \otimes^{\mathbb{L}}_A X = X$ and $X \otimes^{\mathbb{L}}_B B = X$. That is,

$$1_X = (1_X \otimes^{\mathbb{L}}_B v) \circ (u \otimes^{\mathbb{L}}_A 1_X),$$

therefore $1_X \otimes^{\mathbb{L}}_B v^{-1} = (u \otimes^{\mathbb{L}}_A 1_X)$ and so the diagram

$$\begin{tikzcd} X \arrow[r, "u \otimes^{\mathbb{L}}_A 1"] \arrow[dr, "1"'] & X \otimes^{\mathbb{L}}_B X^\vee \otimes^{\mathbb{L}}_A X \arrow[d, "1 \otimes^{\mathbb{L}}_B v"] \\ & X \end{tikzcd}$$

is commutative. We can extend v to an isomorphism of functors

$$v : - \otimes^{\mathbb{L}}_B X^\vee \otimes^{\mathbb{L}}_A X \to id_{D^b(B^e)}$$

and as a consequence the following diagram is commutative

$$\begin{tikzcd} X^\vee \otimes^{\mathbb{L}}_A X \otimes^{\mathbb{L}}_B X^\vee \otimes^{\mathbb{L}}_A X \arrow[r, "1 \otimes^{\mathbb{L}}_A 1 \otimes^{\mathbb{L}}_B v"] \arrow[d, "v \otimes^{\mathbb{L}}_B 1 \otimes^{\mathbb{L}}_A 1"'] & X^\vee \otimes^{\mathbb{L}}_A X \arrow[d, "v"] \\ X^\vee \otimes^{\mathbb{L}}_A X \arrow[r, "v"'] & B. \end{tikzcd}$$

The way we have chosen v implies also that the diagram

$$\begin{tikzcd}
X^\vee \otimes^{\mathbb{L}}_A X \arrow[r, "1\otimes^{\mathbb{L}}_A u \otimes^{\mathbb{L}}_B 1"] \arrow[dr, "1"'] & X^\vee \otimes^{\mathbb{L}}_A X \otimes^{\mathbb{L}}_B X^\vee \otimes^{\mathbb{L}}_A X \arrow[d, "1\otimes^{\mathbb{L}}_A 1 \otimes^{\mathbb{L}}_B v"] \\
& X^\vee \otimes^{\mathbb{L}}_A X
\end{tikzcd}$$

is commutative. From this we get that the following diagram commutes

$$\begin{tikzcd}
X^\vee \otimes^{\mathbb{L}}_A X \otimes^{\mathbb{L}}_B X^\vee \arrow[r, "1\otimes^{\mathbb{L}}_A u"] \arrow[dr, "v\otimes^{\mathbb{L}}_B 1"'] & X^\vee \arrow[d, "1"'] \\
& X^\vee .
\end{tikzcd}$$

Since the isomorphism $id \to FG$ is induced by u and the isomorphism $GF \to id$ is induced by v, we get that F and G are quasi-inverse functors. $\square$

Chapter 3

Hochschild and cyclic theories

3.1 Hochschild (co)homology

Hochschild (co)homology of an algebra is the algebraic analogue of the topological (co)homology theory of a topological space, see [7] Chapter IX, section 4 and [21] Chapter 2. It was introduced by Hochschild [23, 24] then Cartan-Eilenberg [7], Gerstenhaber [17], Happel [20], have studied its structure given by operations and more generally by Keller [37, 38] in the context of differential graded algebras and B_∞-algebras. More recently, computations of the theory with some of its operations has been developed by Cibils, Lanzilotta, Marcos, Redondo and Solotar [10, 11, 12, 13, 48, 49, 55, 56], as well as Suárez-Alvarez [58, 60], Lambre [41, 42], Kordon and Suárez-Alvarez [40], Negron and Witherspoon [47], Volkov [65, 64], Witherspoon [67] and several other authors, see for example [1, 4, 8, 9, 22, 46].

Let k be a commutative ring and let A be a k-algebra. Denote by $\mu : A \otimes A \to A$ the product of the algebra A. Denote the n-folded tensor product of A as

$$A^{\otimes n} = \underbrace{A \otimes \cdots \otimes A}_{n \text{ copies}}.$$

The *Bar resolution* of A is denoted by $\mu : Bar_\bullet(A) \to A$ where we consider μ as a morphism concentrated in degree 0. It is given by $Bar_n(A) = A^{\otimes(n+2)}$ for $n \geq 0$ and $Bar_i(A) = 0$ for $i < 0$, that is

$$Bar_\bullet(A): \ \cdots \to A^{\otimes n} \to A^{\otimes(n-1)} \to \cdots \to A \otimes A \to 0,$$

with $\delta_n : A^{\otimes(n+2)} \to A^{\otimes(n+1)}$ as differential, where

$$\delta_n(a_1 \otimes \cdots \otimes a_{n+2}) = \sum_{i=1}^{n+1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+2}.$$

Let N be an A^e-module, where $A^e = A \otimes A^{op}$. The complexes $N \otimes_{A^e} Bar_\bullet(A)$ and $Hom_{A^e}(Bar_\bullet(A), N)$ are called *Hochschild complexes*.

Definition 3.1.1. *[23] The Hochschild homology k-modules of A with coefficients in the A^e-module N are defined as*

$$H_n(A,N) := H_n\big(N \otimes_{A^e} Bar_\bullet(A)\big),$$

for every $n \geq 0$. The Hochschild cohomology k-modules of A with coefficients in the A^e-module N are defined as

$$H^n(A,N) := H^n\big(Hom_{A^e}(Bar_\bullet(A), N)\big),$$

for every $n \geq 0$. In case $N = A$, we write

$$HH_n(A) := H_n(A,A) \quad \textit{and} \quad HH^n(A) := H^n(A,A).$$

If the algebra is projective as a module over k, Hochschild (co)homology is given by the classical derived functors of the tensor product and Hom functors.

Lemma 3.1.2. *([70] page 341) If A is projective as a k-module, then the complex $Bar_\bullet(A)$ is exact and each $Bar_n(A)$ is an A^e-projective module for every $n \geq 0$. Therefore, $Bar_\bullet(A)$ is a projective resolution of A as an A^e-module.*

Corollary 3.1.3. *([70] page 342) If A is a projective k-module then there are natural isomorphisms*

$$H_n(A,N) \cong Tor_n^{A^e}(N,A)$$

and

$$H^n(A,N) \cong Ext^n_{A^e}(A,N)$$

for every $n \geq 0$.

We will construct the Hochschild complexes and define the operations we are interested in, given in terms of them. There are natural isomorphisms

$$\begin{array}{rcl} N \otimes_{A^e} A^{\otimes(n+2)} & \to & N \otimes A^{\otimes n} \\ x \otimes a_0 \otimes \cdots \otimes a_{n+1} & \mapsto & (a_{n+1} x a_0) \otimes a_1 \otimes \cdots \otimes a_n \end{array}$$

for every $n \geq 0$, that induce an isomorphism of complexes, in which the complex $N \otimes A^{\otimes \bullet}$ has differential

$$\begin{array}{rcl} d: \quad N \otimes A^{\otimes n} & \to & N \otimes A^{\otimes (n-1)} \\ x \otimes a_1 \otimes \cdots \otimes a_n & \mapsto & xa_1 \otimes \cdots \otimes a_{n-1} \\ & & + \sum_{i=1}^{n-1} (-1)^i x \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & & + (-1)^n a_n x \otimes a_1 \otimes \cdots \otimes a_{n-1}, \end{array}$$

for $n > 1$, and $d(x \otimes a) = xa - ax$ when $n = 1$. Dually, there are isomorphisms

$$\begin{array}{rcl} Hom_{A^e}(A^{\otimes (n+2)}, N) & \to & Hom(A^{\otimes n}, N) \\ f & \mapsto & f' \end{array}$$

for every $n \geq 0$, where $f'(a_1 \otimes \cdots \otimes a_n) = f(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$, and such that they extend to an isomorphism of complexes where the complex $Hom(A^{\otimes \bullet}, M)$ has the differential

$$\partial: \quad Hom(A^{\otimes n}, N) \quad \to \quad Hom(A^{\otimes (n+1)}, N)$$

defined as

$$\begin{array}{rcl} \partial f(a_1 \otimes \cdots \otimes a_{n+1}) & = & a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ & & + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ & & + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}, \end{array}$$

for all $f \in Hom(A^{\otimes n}, N)$. From now on we will be working with the Hochschild complexes $(N \otimes A^{\otimes \bullet}, d)$ and $(Hom(A^{\otimes \bullet}, N), \partial)$, unless specified otherwise.

Definition 3.1.4. *([7] pages 216-219) Let N and M be A-bimodules. The cup product in the Hochschild complex $Hom(A^{\otimes \bullet}, M)$ is given by morphisms*

$$\cup : Hom(A^{\otimes n}, N) \otimes Hom(A^{\otimes m}, M) \to Hom(A^{\otimes (n+m)}, N \otimes_A M)$$

for every pair of integers n and m, defined as

$$\alpha \cup \beta(a_1 \otimes \cdots \otimes a_{n+m}) := \alpha(a_1 \otimes \cdots \otimes a_n) \otimes_A \beta(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

for $\alpha \in Hom(A^{\otimes n}, N)$ and $\beta \in Hom(A^{\otimes m}, M)$. In here we establish the convention that $A^{\otimes 0} = k$. If $n = 0$, we define

$$\alpha \cup \beta(a_1 \otimes \cdots \otimes a_m) := \alpha(1) \otimes_A \beta(a_1 \otimes \cdots \otimes a_m).$$

If $m = 0$, we define

$$\alpha \cup \beta(a_1 \otimes \cdots \otimes a_n) := \alpha(a_1 \otimes \cdots \otimes a_n) \otimes_A \beta(1).$$

This operation passes to cohomology,

$$\cup : H^n(A,N) \otimes H^m(A,M) \to H^{n+m}(A, N \otimes_A M)$$

by defining $[\alpha] \cup [\beta] := [\alpha \cup \beta]$. When $N = M = A$ this product can be interpreted in terms of the Yoneda product of exact sequences, and also as the shift functor followed by the composition in the derived category of A^e, see [50].

Theorem 3.1.5. *[17] The k-module*

$$HH^\bullet(A) := \bigoplus_{n \geq 0} HH^n(A)$$

equipped with the cup product is a graded commutative algebra.

We will consider $Ext^n_{A^e}(A,A)$ as the k-module of equivalence classes of exact sequences with length n of A^e-modules starting and ending in A, see for example section 2.6 in page 38 of [6]. The *Yoneda product* of two exact sequences starting and ending in A is defined as their concatenation, i.e. given two exact sequences of A^e-modules

$$X: \qquad 0 \longrightarrow A \xrightarrow{\varepsilon_0} X_1 \xrightarrow{\varepsilon_1} \cdots \xrightarrow{\varepsilon_{n-1}} X_n \xrightarrow{\varepsilon_n} A \longrightarrow 0$$

and

$$Y: \qquad 0 \longrightarrow A \xrightarrow{\eta_0} Y_1 \xrightarrow{\eta_1} \cdots \xrightarrow{\eta_{m-1}} Y_m \xrightarrow{\eta_m} A \longrightarrow 0$$

their *Yoneda product* $X \Diamond Y$ is defined as the exact sequence

$$0 \longrightarrow A \xrightarrow{\varepsilon_0} X_1 \xrightarrow{\varepsilon_1} \cdots \xrightarrow{\varepsilon_{n-1}} X_n \xrightarrow{\eta_0 \varepsilon_n} Y_1 \xrightarrow{\eta_1} \cdots \xrightarrow{\eta_{m-1}} Y_m \xrightarrow{\eta_m} A \longrightarrow 0.$$

Theorem 3.1.6. *([45] page 85) The k-module*

$$Ext^\bullet_{A^e}(A,A) := \bigoplus_{n \geq 0} Ext^n_{A^e}(A,A)$$

with the Yoneda product is a graded commutative algebra.

Let $f \in Hom_{D^b(A^e)}(A,A[n])$ and $g \in Hom_{D^b(A^e)}(A,A[m])$, define $f \star g$ as the morphism $g[n] \circ f$ from A to $A[n+m]$ in $D^b(A^e)$, so that we have a morphism

$$\star : Hom_{D^b(A^e)}(A,A[n]) \otimes Hom_{D^b(A^e)}(A,A[m]) \to Hom_{D^b(A^e)}(A,A[n+m]).$$

Theorem 3.1.7. *([70] page 613) The k-module*

$$Hom_{D^b(A^e)}(A,A[\bullet]) := \bigoplus_{n \geq 0} Hom_{D^b(A^e)}(A,A[n])$$

with the $\star$-product is a graded commutative algebra.

The last result can be proved also with the methods of [59]. There is a well known result relating these three algebras. We will use the Hochschild complex $Hom_{A^e}(Bar(A),A)$ in the following proof.

Theorem 3.1.8. *([67] pages 23-26) Let A be a k-projective algebra. The graded commutative algebras $HH^\bullet(A)$ and $Ext^\bullet_{A^e}(A,A)$ as well as $Hom_{D^b(A^e)}(A,A[\bullet])$ are isomorphic.*

Proof. We will describe the isomorphisms

$$Ext^\bullet_{A^e}(A,A) \tilde{\to} HH^\bullet(A)$$

and

$$Ext^\bullet_{A^e}(A,A) \tilde{\to} Hom_{D^b(A^e)}(A,A[\bullet]),$$

as well as the isomorphism

$$HH^\bullet(A) \tilde{\to} Hom_{D^b(A^e)}(A,A[\bullet])$$

to which we also give its inverse. The last one will be extensively used in the next Chapters. Let

$$X = (\ 0 \longrightarrow A \xrightarrow{\varepsilon_0} X_1 \xrightarrow{\varepsilon_1} \cdots \xrightarrow{\varepsilon_{n-1}} X_n \xrightarrow{\varepsilon_n} A \longrightarrow 0\ )$$

be an exact sequence of A^e-modules representing an element of $Ext^n_{A^e}(A,A)$. Since $Bar(A)$ is a projective resolution of A as an A^e-module, by the comparison theorem, see [53] pages 340 and 341, there exists a morphism of complexes $\phi_\bullet$:

$Bar_\bullet(A) \to X$ unique up to homotopy, lifting the identity map of A, that is, there is a commutative diagram with exact rows

$$\begin{tikzcd} A^{\otimes(n+3)} \arrow[r] \arrow[d] & A^{\otimes(n+2)} \arrow[r, "\delta"] \arrow[d, "\phi_n"'] & A^{\otimes(n+1)} \arrow[r, "\delta"] \arrow[d, "\phi_{n-1}"'] & \cdots \arrow[r, "\delta"] & A \otimes A \arrow[r, "\delta"] \arrow[d, "\phi_0"'] & A \arrow[r] \arrow[d, "1"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "\varepsilon_0"] & X_1 \arrow[r, "\varepsilon_1"] & \cdots \arrow[r, "\varepsilon_{n-1}"] & X_n \arrow[r, "\varepsilon_n"] & A \arrow[r] & 0. \end{tikzcd}$$

The isomorphism $\eta : Ext^n_{A^e}(A,A) \xrightarrow{\sim} HH^n(A)$ maps X to $[\phi_n]$. It can be shown by direct computations that

$$\eta(X \Diamond Y) = \eta(X) \cup \eta(Y).$$

For the isomorphism $\lambda : Ext^n_{A^e}(A,A) \xrightarrow{\sim} Hom_{D^b(A^e)}(A,A[n])$ consider X as before and define

$$X' = (\ 0 \longrightarrow A \xrightarrow{\varepsilon_0} X_1 \xrightarrow{\varepsilon_1} \cdots \xrightarrow{\varepsilon_{n-1}} X_n \longrightarrow 0\ ).$$

Denote by ε_n the morphism of complexes $X' \to A$ given by ε_n concentrated in degree 0. Observe that ε_n is a quasi-isomorphism. Using the morphism of complexes $1'_A : X' \to A[n]$ defined as the identity morphism of A concentrated in degree n, we form the roof

$$\lambda(X) := [1'_A \varepsilon_n^{-1}] = \left[\ A \xleftarrow{\varepsilon_n} X' \xrightarrow{1'_A} A[n]\ \right]$$

which represents a morphism from A to $A[n]$ in $D^b(A^e)$. This defines the isomorphism $\lambda : Ext^n_{A^e}(A,A) \xrightarrow{\sim} Hom_{D^b(A^e)}(A,A[n])$, which satisfies that

$$\lambda(X \Diamond Y) = \lambda(X) \star \lambda(Y).$$

We will now describe the isomorphism

$$HH^n(A) \xrightarrow{\sim} Hom_{D^b(A^e)}(A,A[n])$$

and its inverse. Let $[f] \in HH^n(A)$ be represented by a map $f : A^{\otimes(n+2)} \to A$ of A-bimodules. The image of $[f]$ in $Hom_{D^b(A^e)}(A,A[n])$ is given by the roof represented by the diagram

$$A \xleftarrow{\mu} Bar_\bullet(A) \xrightarrow{f} A[n],$$

where f is considered as a morphism concentrated in degree n. That is, we define

$$\gamma : HH^n(A) \to Hom_{D^b(A^e)}(A, A[n])$$

as the roof $\gamma[f] := [f\mu^{-1}]$. This isomorphism sends the cup product to the $\star$-product,

$$\gamma([f] \cup [g]) = \gamma[f] \star \gamma[g].$$

Let $f = [f_2 f_1^{-1}]$ be a morphism from A to $A[n]$ in $D^b(A^e)$, i.e. f is represented by the diagram

$$A \xleftarrow{f_1} Z \xrightarrow{f_2} A[n].$$

Since the bar resolution is a projective resolution of A as an A^e-module, by the comparison theorem, see for instance [53] pages 340 and 341, there is a morphism of complexes $\omega : Bar_\bullet(A) \to Z$ such that $f_1\omega = \mu$, which means that the following diagram is commutative

$$\begin{tikzcd} & Bar_\bullet(A) \arrow[dl, "\mu"'] \arrow[d, "\omega"] & \\ A & Z \arrow[l, "f_1"'] \arrow[r, "f_2"] & A[n]. \end{tikzcd}$$

Compose f_2 with the n-th component of ω, namely

$$\omega_n : A^{\otimes(n+2)} \to Z_n,$$

to obtain a morphism $f_2 \circ \omega_n : A^{\otimes(n+2)} \to A$ of A^e-modules whose homology is an element $\gamma_{-1}[f_2 f_1^{-1}] := [f_2\omega_n]$ in $HH^n(A)$. The rest of the proof are straightforward verifications. $\square$

The following result was first proven by Happel in [19] Chapter 3, in the tilting approach. The proof presented here is due to Rickard [52]. It uses the algebra identification

$$HH^\bullet(A) = Hom_{D^b(A^e)}(A, A[\bullet])$$

to obtain a direct proof of derived invariance of the cup product.

Theorem 3.1.9. *[52] Let A and B be derived equivalent algebras that are projective and finitely generated as k-modules with equivalence $- \otimes_A^{\mathbb{L}} X : D^b(A) \xrightarrow{\sim} D^b(B)$, then X induces an isomorphism of graded (commutative) k-algebras*

$$HH^\bullet(A) \xrightarrow{\sim} HH^\bullet(B).$$

Proof. Since the equivalence $F := -\otimes_A^{\mathbb{L}} X$ respects composition and the shift we get that

$$F(f \cup g) = F(g[n] \circ f) = F(g[n]) \circ F(f) = F(g)[n] \circ F(f) = F(f) \cup F(g).$$

□

The Hochschild cohomology of an algebra has a deeper structure that is also preserved under derived equivalence [38], namely the structure of a Gerstenhaber algebra [17].

Definition 3.1.10. *[17] Let $H^\bullet$ be a graded k-module. We say that $(H^\bullet, \cup, [-,-])$ is a Gerstenhaber-algebra if*

1. *For every pair of integers n and m, there exists a morphism*

$$\cup : H^n \otimes H^m \to H^{n+m}$$

such that it induces an associative graded commutative product on $H^\bullet$.

2. *For every pair of integers n and m, there exists a morphism*

$$[-,-] : H^n \otimes H^m \to H^{n+m-1}$$

such that

$$[\alpha \cup \beta, \gamma] = [\alpha, \gamma] \cup \beta + (-1)^{|\alpha|(|\gamma|-1)} \alpha \cup [\beta, \gamma]$$

for homogeneous elements $\alpha, \beta, \gamma \in H^\bullet$.

3. *The morphism $[-,-]$ also satisfies the graded Jacobi identity*

$$[\alpha, [\beta, \gamma]] = [[\alpha, \beta], \gamma] + (-1)^{(|\alpha|-1)(|\beta|-1)} [\beta, [\alpha, \beta]]$$

for homogeneous elements $\alpha, \beta, \gamma \in H^\bullet$.

We will now construct an operation $[-,-]$ that will endow $(HH^\bullet(A), \cup)$ with the structure of a Gerstenhaber algebra [17]. Let $\alpha \in Hom(A^{\otimes n}, A)$ and $\beta \in Hom(A^{\otimes m}, A)$ and let i be an integer such that $1 \leq i \leq n-1$. Define $\alpha \bullet_i \beta \in Hom(A^{\otimes (n+m-1)}, A)$ as the morphism that maps $a_1 \otimes \cdots \otimes a_{n+m-1}$ to

$$\alpha(a_1 \otimes \cdots \otimes a_{i-1} \otimes \beta(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1}).$$

The *Gerstenhaber* product is by definition

$$\alpha \bullet \beta = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} \alpha \bullet_i \beta.$$

Definition 3.1.11. *[17] The Gerstenhaber bracket is defined as*

$$[\alpha,\beta] = \alpha \bullet \beta - (-1)^{(|\alpha|-1)(|\beta|-1)}\beta \bullet \alpha,$$

for homogeneous elements $\alpha,\beta \in HH^\bullet(A)$.

Theorem 3.1.12. *[17] Hochschild cohomology of an algebra endowed with the Gerstenhaber bracket is a Gerstenhaber algebra.*

Theorem 3.1.13. *[38] Let A and B be derived equivalent k-projective algebras. There is a canonical isomorphism of Gerstenhaber algebras*

$$HH^\bullet(A) \xrightarrow{\sim} HH^\bullet(B).$$

Keller proves this by showing that the Gerstenhaber bracket is the induced bracket of the Lie algebra of a group valued functor defined on a category of commutative algebras. That group is the *R-relative derived Picard group*, which is immediately a derived invariant. We make the required constructions and sketch the proof of the following theorem. Next we recall the definition of the R-relative derived Picard group.

Definition 3.1.14. *[38] Let R be a commutative DG-algebra over k and let A be a k-algebra. The R-relative derived Picard group of R, denoted $DPic_A(R)$ is the set of isomorphism classes of invertible objects of the relative derived category $D_R(R\otimes A^{op}\otimes A)$. The product in $DPic_A(R)$ is given by the relative derived tensor product.*

The relative derived Picard group of R can be considered as a group valued functor defined on the category CDG_k of commutative differential graded k-algebras. Denote Gps the category of groups, then we have a functor

$$DPic_A : CDG_k \to Gps.$$

For every group valued functor there exists a Lie algebra associated to it. Let $G : CDG_k \to Gps$ be a functor, define

$$ev_0 : k[\varepsilon]/(\varepsilon^2) \to k$$

the morphism that evaluates at $\varepsilon = 0$. Then the Lie algebra of G is the kernel of the map $G(ev_0)$, i.e. the space of tangent vectors at the origin. The bracket is induced from the commutator of $G(k[\varepsilon]/(\varepsilon^2))$.

Let i be an integer and define $R_i = k[\varepsilon]/(\varepsilon^2)$ where ε is of degree $-i$. For the relative derived Picard group, we get a graded Lie algebra whose i-th degree $LieDPic^i_A$ is the set of isomorphism classes of objects $U \in D_{R_i}(R_i \otimes A^{op} \otimes A)$ that are R_i-semifree and $U \otimes_{R_i} k \cong A$ in $D(A^e)$. Denote

$$LieDPic^{\bullet}_A := \bigoplus_{i\in\mathbb{Z}} LieDPic^i_A.$$

Let A and B be derived equivalent Noetherian k-projective finitely generated algebras, then there is an isomorphism

$$\begin{array}{ccc} DPic_A(R) & \xrightarrow{\sim} & DPic_B(R) \\ U & \to & X^{\vee} \otimes^{\mathbb{L}}_A U \otimes^{\mathbb{L}}_A X. \end{array}$$

Therefore, the product of $DPic_A(R)$ is a derived invariant for A. Keller proves in [38] that the Lie bracket of the Lie algebra of the relative derived Picard group coincides with the Gerstenhaber bracket. As a consequence the Gerstenhaber algebra structure on Hochschild cohomology is derived invariant.

Theorem 3.1.15. *[38] There is a canonical isomorphism of graded Lie algebras*

$$HH^{\bullet+1}(A)^{op} \xrightarrow{\sim} LieDPic^{\bullet}_A$$

which is functorial with respect to invertible bimodule complexes $X \in D(A^{op} \otimes B)$.

Note that $HH^{\bullet+1}(A)^{op}$ is the graded algebra $HH^{\bullet+1}(A)$ with opposite Lie bracket. The Hochschild homology of an algebra $HH_{\bullet}(A) := \bigoplus_{n\geq 0} HH_n(A)$ is also a derived invariant. The first appearance of the isomorphism between Hochschild homologies of two derived equivalent algebras that are projective over a commutative ring k was in [28]. It is also proved in [69] which we follow now for the proof. Let A and B be derived equivalent Noetherian algebras that are finitely generated and projective as k-modules and take X and $X^{\vee}$ as in Proposition 2.3.16 in page 38, then

$$\begin{array}{rcl} A \otimes^{\mathbb{L}}_{A^e} A & \xrightarrow{\sim} & A \otimes^{\mathbb{L}}_{A^e} (X \otimes^{\mathbb{L}}_B X^{\vee}) \\ & \xrightarrow{\sim} & A \otimes^{\mathbb{L}}_{A^e} (X \otimes^{\mathbb{L}}_k X^{\vee}) \otimes^{\mathbb{L}}_{B^e} B \\ & \xrightarrow{\sim} & B \otimes^{\mathbb{L}}_{B^e} (X^{\vee} \otimes^{\mathbb{L}}_k X) \otimes^{\mathbb{L}}_{A^e} A \\ & \xrightarrow{\sim} & B \otimes^{\mathbb{L}}_{B^e} (X^{\vee} \otimes^{\mathbb{L}}_A X) \\ & \xrightarrow{\sim} & B \otimes^{\mathbb{L}}_{B^e} B, \end{array}$$

and by taking homology we get the following.

Theorem 3.1.16. *[69] Let A and B be derived equivalent Noetherian k-algebras that are finitely generated and projective as k-modules. There is a canonical isomorphism*

$$HH_\bullet(A) \xrightarrow{\sim} HH_\bullet(B).$$

Hochschild homology and cohomology interact via a pairing, called the cap product. This is an algebraic version of the topological cap product which gives rise to Poincaré duality on homology and cohomology of topological manifolds, see [21] Chapter 3 section 3.

Definition 3.1.17. *([7] page 217) The cap product in the Hochschild complexes*

$$\cap : \left(N \otimes A^{\otimes n}\right) \otimes Hom(A^{\otimes m}, M) \to N \otimes_A M \otimes A^{\otimes (n-m)},$$

is defined as

$$z \cap \beta := (-1)^{nm} x \otimes_A \beta(a_1 \otimes \cdots \otimes a_m) \otimes a_{m+1} \otimes \cdots \otimes a_n,$$

for $\beta \in Hom(A^{\otimes m}, M)$ and every $z = x \otimes a_1 \otimes \cdots \otimes a_n \in N \otimes A^{\otimes n}$.

The cap product also provides a well-defined cap product in (co)homology, see [7] pages 216-219.

$$\cap : H_n(A,N) \otimes H^m(A,M) \to H_{n-m}(A, N \otimes_A M).$$

3.2 Cyclic homology

We consider now cyclic homology, as used by Connes [14]. This theory can be defined for non-unital algebras, but in this work we develop it for unital algebras. We follow [44] to introduce cyclic homology. Let k be a commutative ring. Recall that a *bicomplex* is a $\mathbb{Z}^2$-graded k-module $M_{\bullet,\bullet}$ with differentials Δ^v of bidegree $(1,0)$ and Δ^h of bidegree $(0,1)$ such that

$$\Delta^v \Delta^h + \Delta^h \Delta^v = 0.$$

Every bicomplex M gives rise to a complex, called its *total complex* $Tot(M)$, where the n-th component of the total complex of M is by definition

$$Tot(M)_n = \bigoplus_{i+j=n} M_{i,j}$$

with differential

$$\Delta : Tot(M)_n \to Tot(M)_{n+1}$$

given by

$$\Delta = \Delta^h + \Delta^v.$$

The *homology of the bicomplex* $M_{\bullet,\bullet}$ is defined to be the homology of its total complex $(Tot(M), \Delta)$, that is

$$H_n(M) := H_n(Tot(M)),$$

for all integers n. We apply this to specific operations in the tensor powers of a k-algebra A involving an action of the cyclic group. The multiplicative cyclic group of order $n+1$ with generator $t = t_n$ acts on $A^{\otimes(n+1)}$ by

$$t_n(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

Let $N = 1 + t + t^2 + \cdots + t^n$ denote the *norm operator* on $A^{\otimes(n+1)}$. Define the operations $d_0(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := a_0 a_1 \otimes \cdots \otimes a_n$ and $d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := a_n a_0 \otimes \cdots \otimes a_{n-1}$, as well as

$$d_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

for all $1 \leq i \leq n-1$ and all generators $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ of $A^{\otimes(n+1)}$. Define

$$b := \sum_{i=0}^{n} (-1)^i d_i : A \otimes A^{\otimes n} \to A \otimes A^{\otimes(n-1)}$$

and

$$b' = \sum_{i=0}^{n-1} (-1)^i d_i : A \otimes A^{\otimes n} \to A \otimes A^{\otimes(n-1)}.$$

The *cyclic bicomplex* $C(A) = C_{\bullet,\bullet}(A)$ of an algebra A is defined as the following first quadrant bicomplex

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"] & \cdots \arrow[l] \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] & \cdots \arrow[l] \\
A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & A \arrow[l, "1-t"'] & \cdots . \arrow[l]
\end{tikzcd}$$

By direct computations one checks that b and b' are differentials and that $Nb + b'N = 0$ and also that $b(1-t) = (1-t)b'$.

Definition 3.2.1. *([44] page 53) The n-th cyclic homology of the algebra A is*

$$HC_n(A) := H_n(C_{\bullet,\bullet}(A)),$$

for all $n \geq 0$.

Next we will construct other (bi)complexes whose homology will also be cyclic homology. First we consider the multiplication by $1-t$ as an endomorphism of $A^{\otimes(n+1)}$. Let $C_n^\lambda(A) := Coker(1-t)$ be the cokernel of $1-t : A^{\otimes(n+1)} \to A^{\otimes(n+1)}$. The *Connes complex* [14] is defined as

$$C_\bullet^\lambda(A) : \cdots \xrightarrow{b} C_n^\lambda(A) \xrightarrow{b} C_{n-1}^\lambda(A) \xrightarrow{b} \cdots \xrightarrow{b} C_0^\lambda(A).$$

We can consider $C_\bullet^\lambda(A)$ as a bicomplex in which the 0-column is $C_\bullet^\lambda(A)$, and zero elsewhere. Define a morphism of bicomplexes $p : C_\bullet(A) \to C_\bullet^\lambda(A)$, i.e. it commutes with both differentials, defined as the quotient by the action of $1-t$

$$A^{\otimes(n+1)} \to C_n^\lambda(A)$$

on the 0-column, and zero on the others.

Theorem 3.2.2. *([44] page 54) For any algebra A over a ring k which contains $\mathbb{Q}$ the natural map*

$$p_\bullet : HC_\bullet(A) \tilde{\to} H_\bullet(C^\lambda(A))$$

is an isomorphism.

We now construct a bicomplex $\bar{\mathscr{B}}(A) = \bar{\mathscr{B}}_{\bullet,\bullet}(A)$ whose (p,q)-term is

$$\bar{\mathscr{B}}_{p,q}(A) = A^{\otimes(q-p+1)}.$$

The vertical differential is the map b and the horizontal differential of $\bar{\mathscr{B}}(A)$ is by definition

$$\bar{B}_A = (1-t)sN,$$

where $s : A^{\otimes n} \to A^{\otimes(n+1)}$ is called *extra-degeneracy* and is given by

$$s(x) = 1 \otimes x.$$

That is, $\bar{\mathscr{B}}_{\bullet,\bullet}(A)$ is the bicomplex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
A^{\otimes 4} & \xleftarrow[\bar{B}_A]{} & A^{\otimes 3} & \xleftarrow[\bar{B}_A]{} & A^{\otimes 2} & \xleftarrow[\bar{B}_A]{} & A \\
\downarrow b & & \downarrow b & & \downarrow b & & \\
A^{\otimes 3} & \xleftarrow[\bar{B}_A]{} & A^{\otimes 2} & \xleftarrow[\bar{B}_A]{} & A & & \\
\downarrow b & & \downarrow b & & & & \\
A^{\otimes 2} & \xleftarrow[\bar{B}_A]{} & A & & & & \\
\downarrow b & & & & & & \\
A & & & & & &
\end{array}$$

Theorem 3.2.3. *([44] page 57) For any k-algebra A there are isomorphisms*

$$HC_n(A) \cong H_n(\bar{\mathscr{B}}(A))$$

for every $n \geq 0$.

The bicomplex $\bar{\mathscr{B}}(A)$ can be reduced further to a normalized complex $\mathscr{B}(A)$. Consider $\bar{A} = A/k$, the quotient by the module generated by the identity of A. Let $\mathscr{B}(A)_{p,q}$ be equal to $A \otimes \bar{A}^{\otimes(q-p)}$, let b be its vertical differential and let the horizontal differential be $B_A := sN$, that is

$$B_A(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{in}(1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}),$$

for every $a_0 \otimes \cdots \otimes a_n \in A \otimes \bar{A}^{\otimes n}$. That is, $\mathscr{B}_{\bullet,\bullet}(A)$ is the bicomplex

$$\begin{tikzcd}
\vdots \arrow[d, "b"'] & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "b"'] \\
A \otimes \bar{A}^{\otimes 3} \arrow[d, "b"'] & A \otimes \bar{A}^{\otimes 2} \arrow[l, "B_A"] \arrow[d, "b"'] & A \otimes \bar{A} \arrow[l, "B_A"] \arrow[d, "b"'] & A \arrow[l, "B_A"] \\
A \otimes \bar{A}^{\otimes 2} \arrow[d, "b"'] & A \otimes \bar{A} \arrow[l, "B_A"] \arrow[d, "b"'] & A \arrow[l, "B_A"] & \\
A \otimes \bar{A} \arrow[d, "b"'] & A \arrow[l, "B_A"] & & \\
A & & &
\end{tikzcd}$$

Theorem 3.2.4. *([44] page 58) For any k-algebra A there are canonical isomorphisms*

$$HC_n(A) \cong H_n(\mathscr{B}(A)),$$

for every $n \geq 0$.

Another interpretation of cyclic homology has been given by Kassel, as a Tor functor over the DG-algebra of dual numbers.

Definition 3.2.5. *([44] page 75) The triple (M, b, B) is called a mixed complex if M is a $\mathbb{Z}$-graded k-module, b and B are graded endomorphisms of M of degrees 1 and -1, respectively, that satisfy the equations $b^2 = 0$ and $B^2 = 0$ as well as $bB + Bb = 0$.*

Definition 3.2.6. *([44] page 76) The cyclic homology of the mixed complex (M, b, B)*

is the homology of the bicomplex

$$\begin{tikzcd}
\vdots \arrow[d, "b"'] & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "b"'] \\
M_3 \arrow[d, "b"'] & M_2 \arrow[l, "B_A"] \arrow[d, "b"'] & M_1 \arrow[l, "B_A"] \arrow[d, "b"'] & M_0 \arrow[l, "B_A"] \\
M_2 \arrow[d, "b"'] & M_1 \arrow[l, "B_A"] \arrow[d, "b"'] & M_0 \arrow[l, "B_A"] & \\
M_1 \arrow[d, "b"'] & M_0 \arrow[l, "B_A"] & & \\
M_0. & & &
\end{tikzcd}$$

For example, the triple $(\mathscr{B}(A), b, B_A)$ is a mixed complex that we presented as a bicomplex and cyclic homology of this bicomplex is cyclic homology of the algebra A. We will now interpret mixed complexes as DG-modules over the DG-algebra of dual numbers.

Let Λ be the DG-algebra of dual numbers, i.e. $\Lambda := k[\varepsilon]/(\varepsilon^2)$ where the degree of ε is -1 and the differential vanishes, so that its underlying complex is

$$0 \longrightarrow k \xrightarrow{0} k\varepsilon \longrightarrow 0.$$

As in [25, 34], the category of DG Λ-modules and the category of mixed complexes are identified. The identifications is defined as follows. Given a mixed complex (M,b,B), we regard (M,b) as a DG k-module and the action of ε is given by $\varepsilon m := B(m)$ for every $m \in M$. If (M,b) is a DG Λ-module we define a mixed complex (M,b,B) by $B(m) := \varepsilon m$ for every $m \in M$.

Theorem 3.2.7. *[25] Let (M,b,B) be a mixed complex, or equivalently a DG-module over Λ. There is an isomorphism*

$$HC_n(M) \cong Tor_n^{\Lambda}(k,M),$$

for every $n \geq 0$, where k is the Λ-module given by the augmentation.

Proof. The DG Λ-module k has the following free Λ resolution

$$\cdots \to \Lambda[2] \to \Lambda[1] \to \Lambda \to k.$$

Denote by L the deleted resolution of k. Define X_M to be the complex with degree n component given by

$$M_n \oplus M_{n-2} \oplus M_{n-4} \oplus \cdots$$

with differential

$$d(m_n, m_{n-2}, m_{n-4}, \cdots) = (bm_n + Bm_{n-2}, bm_{n-2} + Bm_{n-4}, bm_{n-4} + Bm_{n-6}, \cdots).$$

Note that $HC_n(M) = H_n(X_M)$. Since the total complex of the bicomplex $L \otimes_\Lambda M$ coincides with X_M, we get the desired isomorphism. $\square$

Consider the cyclic bicomplex $C(A)$, denote by $C(A)^{\{2\}}$ the bicomplex equal to $C(A)$ on the first two columns and zero elsewhere. Let $C(A)[2,0]$ be the bicomplex shifted as $C(A)[2,0]_{p,q} = C_{p-2,q}(A)$, then there is an exact sequence of bicomplexes, where kernels and images are taken bidegreewise,

$$0 \to C(A)^{\{2\}} \to C(A) \to C(A)[2,0] \to 0.$$

In which the first map is the inclusion of the first two columns. The last map is induced by moding out by the first two columns of $C(A)$. The homology of this exact sequence gives the well-known *Connes periodicity long exact sequence*. Let I_A and S_A be the homology of the inclusion and the homology of the quotient maps of the last exact sequence, respectively. Let B'_A to be the connecting morphism.

Theorem 3.2.8. *[14] For any k-algebra A there is a natural long exact sequence*

$$\cdots \longrightarrow HH_n(A) \xrightarrow{I_A} HC_n(A) \xrightarrow{S_A} HC_{n-2}(A) \xrightarrow{B'_A} HH_{n-1}(A) \xrightarrow{I_A} \cdots.$$

We have identified $HH_n(A)$ with $H_n(C(A)^{\{2\}})$ via the inclusion of the first column

$$A^{\otimes(\bullet+1)} \to Tot_n(C(A)^{\{2\}}),$$

which is a quasi-isomorphisms. The maps I_A and B'_A can be composed to give a morphism between Hochschild homologies defined as

$$B'_A I_A : HH_n(A) \to HH_{n+1}(A).$$

Let A be an associative unital algebra over a commutative ring k. Then it is straightforward to prove that $B_A = B'_A I_A$.

Definition 3.2.9. *([44] pages 55-66) The maps $\bar{B}_A$, B'_A and B_A are called Connes differential's.*

The map B_A is given in terms of Hochschild homology, which gives an extra structure to the Hochschild theory of an algebra, first studied by Tamarkin and Tsygan in [61] and by Dalentski, Gelfand and Tsygan in [15]. It has been studied further, for example in [42] and [62].

3.3 Tamarkin-Tsygan calculus of an algebra

Definition 3.3.1. *[61] A Tamarkin-Tsygan calculus (or differential calculus) is the datum $(H_\bullet, H^\bullet, \cup, [-,-], \cap, B)$, where*

1. $(H^\bullet, \cup, [-,-])$ *is a Gerstenhaber algebra.*
2. *For every pair of integers n, m there is a map*
$$\cap : H_n \otimes H^m \to H_{n-m}$$
that provides $H_\bullet$ with the structure of a graded $(H^\bullet, \cup)$-module.
3. *For each $j \geq 0$ define $i_\alpha : H_j \to H_{j-n}$ by*
$$i_\alpha(z) := (-1)^{jn} z \cap \alpha,$$
for $\alpha \in H^n$. Let $\alpha \in H^n$ and $\beta \in H^m$, the map $B : H_\bullet \to H_{\bullet+1}$ is such that $B^2 = 0$ and
$$[[B, i_\alpha]_{gr}, i_\beta]_{gr} = i_{[\alpha,\beta]},$$
where the graded bracket $[-,-]_{gr}$ is defined as
$$[B, i_\alpha]_{gr} := Bi_\alpha - (-1)^n i_\alpha B.$$

Remark 3.3.2. *Let $(H_\bullet, H^\bullet, \cup, [-,-], \cap, B)$ be a Tamarkin-Tsygan calculus. The statement (2) from the previous definition is equivalent to the commutativity of the following diagram*

$$\begin{tikzcd}
H_n \otimes H^m \otimes H^l \arrow[r, "1\otimes\cup"] \arrow[d, "\cap\otimes 1"'] & H_n \otimes H^{m+l} \arrow[d, "\cap"'] \\
H_{n-m} \otimes H^l \arrow[r, "\cap"'] & H_{n-m-l}
\end{tikzcd}$$

for every triple of integers n,m and l. The equation (3) from the previous definition is given by the commutativity of the following diagram

$$\begin{tikzcd} H_i \otimes H^j \otimes H^l \arrow[r, "1\otimes[-,-]"] \arrow[dr, "{[[B,i_\alpha]_{gr},i_\beta]_{gr}}"'] & H_i \otimes H^{j+l-1} \arrow[d, "\cap"] \\ & H_{i-j-l+1}, \end{tikzcd}$$

for every triple of integers n,m and l.

The main example of a Tamarkin-Tsygan calculus is the Hochschild theory of an algebra endowed with the cup product, the Gerstenhaber bracket, the cap product and the Connes differential.

Theorem 3.3.3. *[61] Let A be a k-algebra. The datum*

$$(HH_\bullet(A), HH^\bullet(A), \cup_A, [-,-]_A, \cap_A, B_A)$$

is a Tamarkin-Tsygan calculus.

We call $(HH_\bullet(A), HH^\bullet(A), \cup_A, [-,-]_A, \cap_A, B_A)$ the *Tamarkin-Tsygan calculus of A*.

3.4 Weyl algebras

In the following we will make explicit the Tamarkin-Tsygan calculus of Weyl algebras. Let $n \geq 1$ be an integer. The *Weyl algebra A_n* is the quotient of the free algebra on $2n$ variables

$$k < x_1, \cdots, x_n, y_1, \cdots, y_n >$$

modulo the relations

$$\begin{array}{ll} x_i x_j = x_j x_i & \text{for } i \neq j, \\ y_i y_j = y_j y_i & \text{for } i \neq j, \\ x_i y_j = y_j x_i & \text{for } i \neq j, \\ y_i x_i - x_i y_i = 1. & \end{array}$$

It was calculated in [57] that

$$HH_i(A_n) = \begin{cases} 0 & i \neq 2n \\ k & i = 2n. \end{cases}$$

As it is well-known, see for example [55], we have

$$HH^i(A_n) = \begin{cases} 0 & i > 0 \\ k & i = 0. \end{cases}$$

Since Hochschild homology of the Weyl algebra A_n is concentrated in degree $2n$, the Connes differential is the zero map. The cap product of A_n

$$\cap : HH_\bullet(A_n) \otimes HH^0(A_n) \to HH_\bullet(A_n).$$

has as only non-zero component

$$\cap : HH_{2n}(A_n) \otimes HH^0(A_n) \to HH_{2n}(A_n),$$

and is the multiplication action of $HH^0(A_n) \cong k$ on $HH_{2n}(A_n) \cong k$. These operations describe the Tamarkin-Tsygan calculus structure of the Weyl algebra A_n.

Observe now that for integers $n, m \geq 0$, the Weyl algebras A_n and A_m have isomorphic Gerstenhaber algebra structure on their Hochschild cohomology. Indeed, the Gerstenhaber bracket of A_1 is trivial because the Hochschild cohomology is concentrated in degree 0. The cup product of A_n is the product of its center

$$\cup : HH^0(A_n) \otimes HH^0(A_n) \to HH^0(A_n),$$

and likewise for A_m,

$$\cup : HH^0(A_m) \otimes HH^0(A_m) \to HH^0(A_m).$$

In other words, the Gerstenhaber algebra structure does not enables to distinguish possible different derived equivalence classes. On the other hand, Hochschild homology of A_n is concentrated in degree $2n$, so that the Weyl algebras A_n and A_m are derived equivalent if and only if $n = m$.

Chapter 4

Derived invariance of operations

This Chapter presents the main results of this thesis. We begin by giving constructions between morphisms in the derived categories of the enveloping algebras of derived equivalent algebras. Then we perform a careful analysis of the cup product and the cap product. We let k be a commutative ring unless stated otherwise.

4.1 Morphisms induced by derived equivalences

We will consider interpretations in the derived category of Hochschild homology and cohomology. In this section we construct morphisms between these interpretations induced by derived equivalent algebras. Let k be a commutative ring and let A and B be derived equivalent Noetherian k-projective algebras that are finitely generated as k-modules. We will strongly use Proposition 2.3.16 given in page 38. There are bimodule complexes X and $X^\vee$ that are projective on either side, so that the derived tensor product is associative, such that

$$-\otimes^{\mathbb{L}}_A X : D^b(A) \tilde{\to} D^b(B) \text{ and } -\otimes^{\mathbb{L}}_B X^\vee : D^b(B) \tilde{\to} D^b(A)$$

are equivalences. The equivalence $F := X^\vee \otimes^{\mathbb{L}}_A - \otimes^{\mathbb{L}}_A X : D^b(A^e) \tilde{\to} D(B^e)$ induces an isomorphism

$$\widetilde{F}_A : Hom_{D^b(A^e)}(A, A[n]) \tilde{\to} Hom_{D^b(B^e)}(B, B[n]),$$

see for instance [38], given by $\widetilde{F}_A(f) = v[n] \circ F(f) \circ v^{-1}$. Its inverse is induced by $G := X \otimes^{\mathbb{L}}_B - \otimes^{\mathbb{L}}_B X^\vee : D^b(B^e) \tilde{\to} D^b(A^e)$, namely

$$\widetilde{G}_A : Hom_{D^b(B^e)}(B, B[n]) \tilde{\to} Hom_{D^b(A^e)}(A, A[n])$$

which is given by $\widetilde{G}_A(g) = u^{-1}[n] \circ G(g) \circ u$.

Proposition 4.1.1. *Let A and B be Noetherian algebras that are finitely generated and projective as k-modules. For each $f \in Hom_{D^b(A^e)}(A, A[n])$ and each $g \in Hom_{D(B^e)}(B, B[n])$ the following equations hold*

$$\begin{aligned} f &= (u^{-1} \circ G(v))[n] \circ GF(f) \circ G(v^{-1}) \circ u \quad \textit{and} \\ g &= (v \circ F(u^{-1}))[n] \circ FG(g) \circ F(u) \circ v^{-1}. \end{aligned}$$

Proof. We prove the first equation, the second is dual. We choose u, v and the equivalences F and G as well as the complexes X and $X^\vee$ as in Proposition 2.3.16 in page 38 and recall that the natural isomorphism $id \xrightarrow{\sim} GF$ is induced by u, that is, the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u \otimes^{\mathbb{L}}_A u"'] & A[n] \arrow[d, "u \otimes^{\mathbb{L}}_A u[n]"] \\ GF(A) \arrow[r, "GF(f)"'] & GF(A)[n] \end{tikzcd}$$

is commutative for all f. We have that

$$G(v^{-1}) = 1_X \otimes^{\mathbb{L}}_B v^{-1} \otimes^{\mathbb{L}}_B 1_{X^\vee},$$

and since $1_X \otimes^{\mathbb{L}}_B v^{-1} = u \otimes^{\mathbb{L}}_A 1_X$ we get $G(v^{-1}) = u \otimes^{\mathbb{L}}_A 1_X \otimes^{\mathbb{L}}_B 1_{X^\vee}$. then

$$\begin{aligned} (u^{-1} \otimes^{\mathbb{L}}_A u^{-1}) \circ G(v^{-1}) &= (u^{-1} \otimes^{\mathbb{L}}_A u^{-1}) \circ (u \otimes^{\mathbb{L}}_A 1) \\ &= (u^{-1} \circ u) \otimes^{\mathbb{L}}_A (u^{-1} \circ 1) \\ &= 1_A \otimes^{\mathbb{L}}_A u^{-1} \\ &= u^{-1} \end{aligned}$$

that is, the diagram

$$\begin{tikzcd} G(B) \arrow[r, "u^{-1}"] \arrow[d, "G(v^{-1})"'] & A \\ G(X^\vee \otimes^{\mathbb{L}}_A X) \arrow[ur, "(u \otimes^{\mathbb{L}}_A u)^{-1}"'] & \end{tikzcd}$$

is commutative and therefore $G(v^{-1}) = (u \otimes^{\mathbb{L}}_A u) \circ u^{-1}$. Observe that

$$\begin{aligned} GF(f) \circ G(v^{-1}) \circ u &= (u \otimes^{\mathbb{L}}_A u)[n] \circ f \circ (u \otimes^{\mathbb{L}}_A u)^{-1} \circ (u \otimes^{\mathbb{L}}_A u) \circ u^{-1} \circ u \\ &= (u \otimes^{\mathbb{L}}_A u)[n] \circ f \end{aligned}$$

and recall from the bottom diagram of the previous page that

$$G(v) = u \circ (u \otimes_A^{\mathbb{L}} u)^{-1},$$

then

$$\begin{aligned}
&u^{-1}[n] \circ G(v)[n] \circ GF(f) \circ G(v^{-1}) \circ u \\
&= u^{-1}[n] \circ G(v)[n] \circ (u \otimes_A^{\mathbb{L}} u)[n] \circ f \\
&= u^{-1}[n] \circ u[n] \circ (u \otimes_A^{\mathbb{L}} u)[n]^{-1} \circ (u \otimes_A^{\mathbb{L}} u)[n] \circ f \\
&= f.
\end{aligned}$$

□

Recall from Theorem 3.1.8 in page 45, that we may consider Hochschild cohomology as morphisms in the derived category via the following isomorphism.

Definition 4.1.2. *For a k-projective algebra A and an A-bimodule N we define*

$$\gamma_N : H^n(A,N) \to Hom_{D^b(A^e)}(A, N[n])$$

as the morphism that sends a class $[f]$ *to the roof* $\left[A \xleftarrow{\mu} Bar_\bullet(A) \xrightarrow{f} N[n]\right]$ *where* f *is considered as a morphism concentrated in degree n.*

For an arbitrary A-bimodule N define a map

$$\widetilde{F}_N : Hom_{D^b(A^e)}(A, N[n]) \to Hom_{D(B^e)}(B, F(N)[n])$$

given by $\widetilde{F}_N(f) = F(f) \circ v^{-1}$. As in the last proposition, one checks that it has an inverse induced by G, which is given by

$$\begin{array}{rccc}
\widetilde{G}_N : & Hom_{D(B^e)}(B, F(N)[n]) & \to & Hom_{D^b(A^e)}(A, N[n]) \\
& g & \mapsto & (u \otimes_A^{\mathbb{L}} 1_N \otimes_A^{\mathbb{L}} u)^{-1}[n] \circ G(g) \circ u.
\end{array}$$

Remark 4.1.3. *In general,* $F(N)$ *is not concentrated in a single degree since* F *is a functor* $D(A^e) \to D(B^e)$*. If* $N = A$ *we have that* $F(A) \cong B$*, then there exists A-bimodules N such that* $F(N)$ *is concentrated in degree zero.*

If we assume that N is an A-bimodule such that FN is concentrated in degree zero, we can consider FN as a B-bimodule and define the isomorphism

$$H^n(A,N) \xrightarrow{\sim} H^n(B, FN)$$

as the composition $\gamma_B^{-1} \circ \widetilde{F}_N \circ \gamma_A$, that is,

$$\begin{array}{rcl} H^n(A,N) & \tilde{\to} & Hom_{D^b(A^e)}(A,N[n]) \\ & \tilde{\to} & Hom_{D(B^e)}(B,FN[n]) \\ & \tilde{\to} & H^n(B,FN). \end{array}$$

Let us analyze the particular case when N is equal to the k-dual of A. We denote $(-)^* = Hom(-,k)$. Moreover, we give the canonical A-bimodule structure to A^*

$$afb(x) = f(bxa) \ \ \forall a,b,x \in A, f \in A^*.$$

Proposition 4.1.4. *([70] page 617) Let A and B be derived equivalent Noetherian algebras that are projective and finitely generated as k-modules. Then there are isomorphisms which are given by taking homology*

$$Z : F(A^*) \tilde{\to} F(A)^* \quad \textit{and} \quad Z' : G(B^*) \tilde{\to} G(B)^*,$$

in $D^b(B^e)$ and $D^b(A^e)$, respectively.

Proof. By using several adjointness formulas we get the following sequence of isomorphisms,

$$\begin{array}{l} Hom_{D^b(B^e)}(B \otimes B^{op}, F(A^*)[n]) \\ \cong Hom_{D^b(A^e)}(X \otimes \mathbb{R}Hom_A(X,A), Hom(A,k)[n]) \\ \cong Hom_{D^b(A)}(X, Hom_{A^{op}}(\mathbb{R}Hom_A(X,A), Hom(A,k)[n])) \\ \cong Hom_{D^b(A)}(X[-n], Hom(\mathbb{R}Hom_A(X,A) \otimes_A A, k)) \\ \cong Hom_{D^b(A)}(X[-n], Hom(\mathbb{R}Hom_A(X,A), k)) \\ \cong Hom(\mathbb{R}Hom_A(X,A) \otimes^{\mathbb{L}} X[-n], k) \\ \cong Hom(Hom_{D^b(A)}(X, X[-n]), k) \end{array}$$

Since X is isomorphic to a tilting complex T, the homology of $F(A^*)$ is $Hom_{D^b(A)}(T,T[n])^*$, which is $F(A)^*[n] = B^*$ if $n = 0$ and zero elsewhere. In homology we get that $Z : F(A^*) \tilde{\to} F(A)^*$ is an isomorphism in $D^b(B^e)$. The isomorphism Z' is constructed dually. □

Let A and B be as above. We define a morphism of k-modules

$$\widetilde{F}_{A^*} : Hom_{D^b(A^e)}(A, A^*[n]) \to Hom_{D(B^e)}(B, B^*[n])$$

given by

$$\widetilde{F}_{A^*}(f) := (\nu^{-1})^*[n] \circ Z[n] \circ F(f) \circ \nu^{-1},$$

that is, $\widetilde{F}_{A^*}(f)$ is equal to the following composition

$$B \xrightarrow{v^{-1}} F(A) \xrightarrow{F(f)} F(A^*)[n] \xrightarrow{Z[n]} F(A)^*[n] \xrightarrow{(v^{-1})^*[n]} B^*[n].$$

Dually, we define

$$\widetilde{G}_{A^*} : Hom_{D(B^e)}(B, B^*[n]) \to Hom_{D^b(A^e)}(A, A^*[n]),$$

by

$$\widetilde{G}_{A^*}(g) := u^*[n] \circ Z'[n] \circ G(g) \circ u.$$

Remark 4.1.5. *Although $\widetilde{F}$ and $\widetilde{G}$ are canonical morphisms, the author do not see why they should be inverse to each other. In the following result we give sufficient conditions for this to hold.*

Proposition 4.1.6. *Let A and B be derived equivalent Noetherian algebras that are projective and finitely generated as k-modules. Assume that the diagrams*

$$\begin{tikzcd}
GF(A^*) \arrow[r, "G(Z)"] \arrow[dd, "u^{-1}\otimes^{\mathbb{L}}_A(1_{A^*})\otimes^{\mathbb{L}}_A u^{-1}"'] & G(F(A)^*) \arrow[dr, "G((v^{-1})^*)"] & \\
 & & G(B^*) \arrow[dl, "Z'"] \\
A^* \arrow[r, "(u^{-1})^*"'] & G(B)^* &
\end{tikzcd}$$

and

$$\begin{tikzcd}
FG(B^*) \arrow[r, "F(Z')"] \arrow[dd, "v\otimes^{\mathbb{L}}_B(1_{B^*})\otimes^{\mathbb{L}}_B v"'] & F(G(B)^*) \arrow[dr, "F(u^*)"] & \\
 & & F(A^*) \arrow[dl, "Z"] \\
B^* \arrow[r, "v^*"'] & F(A)^* &
\end{tikzcd}$$

are commutative. There are isomorphisms induced by $\widetilde{F}$ and $\widetilde{G}$

$$H^n(A, A^*) \cong H^n(B, B^*)$$

for every $n \geq 0$.

Proof. We will prove that for every morphism $f : A \to A^*[n]$ in $D(A^e)$ we have $\widetilde{G}\widetilde{F}(f) = f$. The natural isomorphism $GF \to id_{D^b(A^e)}$ implies that

$$GF(f) = \left(u \otimes_A^{\mathbb{L}} (1_{A^*}[n]) \otimes_A^{\mathbb{L}} u\right) \circ f \circ (u \otimes_A^{\mathbb{L}} u)^{-1}.$$

The way we have chosen u and v implies that

$$G(v) = (u \otimes_A^{\mathbb{L}} u)^{-1} \circ u.$$

Therefore

$$\begin{aligned}
&\widetilde{G}\widetilde{F}(f)\\
&= \widetilde{G}\left((v^{-1})^*[n] \circ Z[n] \circ F(f) \circ v^{-1}\right)\\
&= u^*[n] \circ Z'[n] \circ G\left((v^{-1})^*[n] \circ Z[n] \circ F(f) \circ (v^{-1})\right) \circ u\\
&= u^*[n] \circ Z'[n] \circ G\left((v^{-1})^*[n]\right) \circ G(Z)[n] \circ GF(f) \circ G(v^{-1}) \circ u.
\end{aligned}$$

We will compute this from right to left. Observe that $GF(f) \circ G(v^{-1}) \circ u$ is equal to

$$\left(u \otimes_A^{\mathbb{L}} (1_A^*[n]) \otimes_A^{\mathbb{L}} u\right)^{-1} \circ f.$$

The commutativity of the first diagram in the statement implies that

$$\begin{aligned}
&\widetilde{G}\widetilde{F}(f)\\
&= u^*[n] \circ Z'[n] \circ G\left((v^{-1})^*[n]\right) \circ G(Z)[n] \circ \left(u \otimes_A^{\mathbb{L}} (1_A^*[n]) \otimes_A^{\mathbb{L}} u\right)^{-1} \circ f\\
&= f.
\end{aligned}$$

Dually, the natural isomorphism $FG \to id_{D^b(B^e)}$ and the second diagram in the statement imply that $\widetilde{F}\widetilde{G}(g) = g$ for all morphism $g : B \to B^*[n]$ in $D(B^e)$. $\square$

4.2 Cup product

Let A be a k-projective algebra. Let N and M be A-bimodules. We extend the interpretation of the cup product in terms of the derived category of the enveloping algebra given by Rickard [52], to allow coefficients in arbitrary bimodules in the following way.

Lemma 4.2.1. *Let A be a k-projective algebra. For every roof represented by*

$$A \xleftarrow{g_1} Z \xrightarrow{g_2} M[m]$$

in $D^b(A^e)$, the diagram

$$N \xleftarrow{1_N \otimes_A g_1} N \otimes_A Z \xrightarrow{1_N \otimes_A g_2} N \otimes_A M[m]$$

represents a roof in $D^b(A^e)$, that we will denote by $1_N \otimes_A g$.

Proof. Apply the functor $N \otimes_A -$ to the diagram $A \xleftarrow{g_1} Z \xrightarrow{g_2} M[m]$ to obtain

$$N \xleftarrow{1_N \otimes_A g_1} N \otimes_A Z \xrightarrow{1_N \otimes_A g_2} N \otimes_A M[m] \ .$$

Since A is k-projective the universal coefficient theorem, see [53] page 448, asserts that the natural morphism

$$\begin{array}{rcl} N \otimes_A H_0(Z) & \to & H_0(N \otimes_A Z) \\ n \otimes [x] & \mapsto & [n \otimes x], \end{array}$$

has cokernel given by $Tor_0^A(H_{-1}(Z), N)$. Observe that A being k-projective implies that Z and $Bar(A)$ are quasi-isomorphic, then

$$H_{-1}(Z) \cong H_{-1}(Bar(A)) = 0.$$

Therefore $N \otimes_A H_0(Z) \cong H_0(N \otimes_A Z)$. We have a commutative diagram

$$\begin{tikzcd} N \otimes_A H(Z) \arrow[d, "\cong"] \arrow[dr, "1_N \otimes_A H(g_1)"] & \\ H(N \otimes_A Z) \arrow[r, "H(1_N \otimes_A g_1)"'] & N. \end{tikzcd}$$

Since g_1 is a quasi-isomorphism we get that $1_N \otimes_A g_1$ is a quasi-isomorphism and then

$$1_N \otimes_A g := \left[N \xleftarrow{1_N \otimes_A g_1} N \otimes_A Z \xrightarrow{1_N \otimes_A g_2} N \otimes_A M[m] \right]$$

is a morphism in $D^b(A^e)$. $\square$

Definition 4.2.2. *Let N and M be A-bimodules for A a k-projective algebra. For each pair of integers n and m let*

$$\widetilde{\cup} : Hom_{D^b(A^e)}(A, N[n]) \otimes Hom_{D^b(A^e)}(A, M[m]) \to Hom_{D^b(A^e)}(A, (N \otimes_A M)[n+m])$$

be the operation defined by

$$f \widetilde{\cup} g := (1_N \otimes_A g[n]) \circ f.$$

Remark 4.2.3. *Let A be a k-projective algebra. The tensor product of complexes over A commutes with the shift functor, and then we can define $1_N \otimes_A g[n] := 1_N \otimes_A (g[n])$, which is also equal to $(1_N \otimes_A g)[n]$. Therefore $f \widetilde{\cup} g$ is given by the following composition of morphisms in $D^b(A^e)$*

$$A \xrightarrow{\quad f \quad} N[n] \xrightarrow{1_N \otimes_A g[n]} N \otimes_A M[n+m].$$

Proposition 4.2.4. *Let A be a k-projective algebra. The following diagram is commutative*

$$\begin{tikzcd}
H^n(A,N) \otimes H^m(A,M) \arrow[r, "\cup"] \arrow[d, "\gamma_N \otimes \gamma_M"'] & H^{n+m}(A, N \otimes_A M) \arrow[d, "\gamma_{N \otimes_A M}"] \\
Hom_{D^b(A^e)}(A, N[n]) \otimes Hom_{D^b(A^e)}(A, M[m]) \arrow[r, "\widetilde{\cup}"] & Hom_{D^b(A^e)}(A, (N \otimes_A M)[n+m]).
\end{tikzcd}$$

Moreover, the following diagram is commutative in $D^b(A^e)$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & N[n] \arrow[d, "1_N \otimes_A g[n]"] \\
M[m] \arrow[r, "f \otimes_A 1_M[m]"'] & N \otimes_A M[n+m]
\end{tikzcd}$$

for all morphisms $f : A \to N[n]$ and $g : A \to M[m]$ in $D(A^e)$.

Proof. Let $[f] \in H^n(A,N)$ and $[g] \in H^m(A,M)$, then

$$\gamma_{N \otimes_A M}([f] \cup [g]) \quad = \quad [(f \cup g)\mu^{-1}].$$

On the other hand

$$\begin{aligned}
\gamma_N([f]) \widetilde{\cup} \gamma_M([g]) &= [f\mu^{-1}] \widetilde{\cup} [g\mu^{-1}] \\
&= (1_N \otimes_A [g\mu^{-1}][n]) \circ [f\mu^{-1}].
\end{aligned}$$

The composition $(1_N \otimes_A [g\mu^{-1}][n]) \circ [f\mu^{-1}]$ is represented by the following diagram

$$\begin{tikzcd}
 & & Bar_\bullet(A) \arrow[dl, "1"'] \arrow[dr, "\widetilde{f}"] & & \\
 & Bar_\bullet(A) \arrow[dl, "\mu"'] \arrow[dr, "f"] & & N \otimes_A Bar_\bullet(A)[n] \arrow[dl, "1 \otimes_A \mu[n]"'] \arrow[dr, "1 \otimes_A g[n]"] & \\
A & & N[n] & & N \otimes_A M[n+m],
\end{tikzcd}$$

where $\widetilde{f}$ is chosen to be $\widetilde{f}(a_1 \otimes \cdots \otimes a_i) = f(a_1 \otimes \cdots \otimes a_n \otimes 1) \otimes 1 \otimes a_{n+1} \otimes \cdots \otimes a_i$ in case $i \geq n+1$ and zero otherwise. Then

$$\begin{aligned}
&(1 \otimes g[n]) \circ \widetilde{f}(a_1 \otimes \cdots \otimes a_i) \\
&= (1 \otimes g[n])\Big(f(a_1 \otimes \cdots \otimes a_n \otimes 1) \otimes 1 \otimes a_{n+1} \otimes \cdots \otimes a_i\Big) \\
&= f(a_1 \otimes \cdots \otimes a_n \otimes 1) \otimes g(1 \otimes a_{n+1} \otimes \cdots \otimes a_i)
\end{aligned}$$

and recall from [67] section 2.2 page 27 that in terms of the Hochschild complex $Hom_{A^e}(Bar_\bullet(A), N \otimes_A M)$ we have

$$f(a_1 \otimes \cdots \otimes a_n \otimes 1) \otimes g(1 \otimes a_{n+1} \otimes \cdots \otimes a_i) = f \cup g(a_1 \otimes \cdots \otimes a_i).$$

Therefore

$$\gamma_{N \otimes_A M}([f] \cup [g]) = \gamma_N([f]) \widetilde{\cup} \gamma_M([g]),$$

and then the first diagram in the statement commutes. To prove that

$$(f \otimes_A 1_M[m]) \circ g = (1_N \otimes_A g[n]) \circ f$$

we observe that there is a morphism $\widetilde{g} : Bar(A) \to Bar_\bullet(A) \otimes_A M[m]$ given by

$$\widetilde{g}(a_1 \otimes \cdots \otimes a_i) = a_1 \otimes \cdots \otimes a_{i-m} \otimes 1 \otimes g(1 \otimes a_{i-m+1} \otimes \cdots \otimes a_i),$$

for $i \geq m+1$ and zero otherwise. Therefore the composition $(f \otimes_A 1[m]) \circ g$ is represented by the diagram

$$\begin{tikzcd}
 & & Bar_\bullet(A) \arrow[dl, "1"'] \arrow[dr, "\widetilde{g}"] & & \\
 & Bar_\bullet(A) \arrow[dl, "\mu"'] \arrow[dr, "g"] & & Bar_\bullet(A) \otimes_A M[m] \arrow[dl, "1 \otimes_A \mu[n]"'] \arrow[dr, "f \otimes_A 1[m]"] & \\
A & & M[m] & & N \otimes_A M[n+m]
\end{tikzcd} .$$

The analogous argument as before gives $(f \otimes_A 1[m]) \circ g = f \cup g = (1 \otimes_A g[n]) \circ f$. $\square$

Remark 4.2.5. *Observe that if $N = A$, then $f \widetilde{\cup} g = g[n] \circ f$ and if $M = A$, then $f \widetilde{\cup} g = f[m] \circ g$.*

Theorem 4.2.6. *Let A and B be derived equivalent Noetherian k-algebras that are projective and finitely generated as k-modules. Let N and M be bounded complexes of A-bimodules. Then there are commutative diagrams*

$$\begin{tikzcd}
Hom_{D^b(A^e)}(A,N[n]) \otimes Hom_{D^b(A^e)}(A,A[m]) \arrow[r, "\widetilde{\cup}_A"] \arrow[d, "\widetilde{F}_N \otimes \widetilde{F}_A"'] & Hom_{D^b(A^e)}(A,N[n+m]) \arrow[d, "\widetilde{F}_N"] \\
Hom_{D(B^e)}(B,F(N)[n]) \otimes Hom_{D(B^e)}(B,B[m]) \arrow[r, "\widetilde{\cup}_B"] & Hom_{D(B^e)}(B,F(N)[n+m])
\end{tikzcd}$$

and

$$\begin{tikzcd}
Hom_{D^b(A^e)}(A,A[n]) \otimes Hom_{D^b(A^e)}(A,M[m]) \arrow[r, "\widetilde{\cup}_A"] \arrow[d, "\widetilde{F}_A \otimes \widetilde{F}_M"'] & Hom_{D^b(A^e)}(A,M[n+m]) \arrow[d, "\widetilde{F}_M"] \\
Hom_{D(B^e)}(B,B[n]) \otimes Hom_{D(B^e)}(B,F(M)[m]) \arrow[r, "\widetilde{\cup}_B"] & Hom_{D(B^e)}(B,F(M)[n+m]).
\end{tikzcd}$$

Proof. Let $f \in Hom_{D^b(A^e)}(A,N[n])$ and $g \in Hom_{D^b(A^e)}(A,A[m])$, then

$$\begin{aligned}
\widetilde{F}_N(f)\widetilde{\cup}_B\widetilde{F}_A(g) &= \big(F(f)\circ v^{-1}\big)\widetilde{\cup}_B\big(v[m]\circ F(g)\circ v^{-1}\big) \\
&= F(f)[m]\circ v^{-1}[m]\circ v[m]\circ F(g)\circ v^{-1} \\
&= F(f)[m]\circ F(g)\circ v^{-1} \\
&= F\Big(f[m]\circ g\Big)\circ v^{-1} \\
&= \widetilde{F}_N\Big(f[m]\circ g\Big) \\
&= \widetilde{F}_N(f\widetilde{\cup}_A g).
\end{aligned}$$

For the second diagram consider morphisms $f : A \to A[n]$ and $g : A \to M[m]$ in $D^b(A^e)$, then

$$\begin{aligned}
\widetilde{F}_A(f)\widetilde{\cup}_B\widetilde{F}_M(g) &= \big(v[n]\circ F(f)\circ v^{-1}\big)\widetilde{\cup}_B\big(F(g)\circ v^{-1}\big) \\
&= F(g)[n]\circ v^{-1}[n]\circ v[n]\circ F(f)\circ v^{-1} \\
&= F(g)[n]\circ F(f)\circ v^{-1} \\
&= F\Big(g[n]\circ f\Big)\circ v^{-1} \\
&= \widetilde{F}_M\Big(g[n]\circ f\Big) \\
&= \widetilde{F}_M(f\widetilde{\cup}_A g).
\end{aligned}$$

$\square$

Corollary 4.2.7. *Let A and B be derived equivalent k-projective Noetherian algebras that are finitely generated as k-modules. Let N and M be A-bimodules such that $F(N)$ and $F(M)$ are concentrated in degree zero. There are canonical isomorphisms*

$$\gamma_{FN}^{-1} \circ \widetilde{F}_N \circ \gamma_N : H^n(A,N) \xrightarrow{\sim} H^n(B,FN)$$

and

$$\gamma_{FM}^{-1} \circ \widetilde{F}_M \circ \gamma_M : H^m(A,M) \xrightarrow{\sim} H^m(B,FM)$$

such that the diagrams

$$\begin{tikzcd} HH^n(A) \otimes H^m(A,M) \arrow[r, "\cup_A"] \arrow[d, "\cong"'] & H^{n+m}(A,M) \arrow[d, "\cong"'] \\ HH^n(B) \otimes H^m(B,FM) \arrow[r, "\cup_B"'] & H^{n+m}(B,FM) \end{tikzcd}$$

and

$$\begin{tikzcd} H^n(A,N) \otimes HH^m(A) \arrow[r, "\cup_A"] \arrow[d, "\cong"'] & H^{n+m}(A,N) \arrow[d, "\cong"'] \\ H^n(B,FN) \otimes HH^m(B) \arrow[r, "\cup_B"'] & H^{n+m}(B,FN) \end{tikzcd}$$

are commutative.

4.3 Cap product with coefficients in an algebra over a field

In this section we let k be a field. Our next aim is to provide an interpretation of the cap product in terms of the derived category to prove the following.

Theorem 4.3.1. *The cap product is a derived invariant of algebras over a field with finite dimensional Hochschild homology in each degree with isomorphisms*

$$HH_\bullet(X) : HH_\bullet(A) \to HH_\bullet(B)$$

and

$$HH^\bullet(X) : HH^\bullet(A) \to HH^\bullet(B)$$

given by the following formulas:

$$HH_\bullet(X) = \varphi_B^{-1} \circ \gamma_{B^*}^* \circ \widetilde{G}_{A^*}^* \circ (\gamma_{A^*}^{-1})^* \circ \varphi_A$$

and

$$HH^\bullet(X) = \gamma_B^{-1} \circ \widetilde{F}_A \circ \gamma_A.$$

We consider Hochschild cohomology with coefficients in the dual of an algebra to give an interpretation of the cap product in terms of cohomology by using the following morphism.

Definition 4.3.2. *Let $\varphi_N : H_\bullet(A,N) \to H^\bullet(A,N^*)^*$ be the graded map defined by*

$$\varphi_N[x \otimes a_1 \otimes \cdots \otimes a_n][f] = f(a_1 \otimes \cdots \otimes a_n)(x),$$

for each $[x \otimes a_1 \otimes \cdots \otimes a_n] \in H_n(A,N)$ and $[f] \in H^\bullet(A,N^*)$.

Lemma 4.3.3. *Let A be a k-algebra and N an A-bimodule. The morphism φ_N is a monomorphism. Moreover, if $H_n(A,N)$ is finite dimensional for every $n \geq 0$, then φ_N is an isomorphism.*

Proof. The morphism φ_N is equal to the composition

$$H_n(A,N) \to H_n(A,N)^{**} \to H^n(A,N^*)^*$$

where the first map is the evaluation map, which is a monomorphism and an isomorphism when $H_n(A,N)$ is finite dimensional. The second is the k-dual of the map

$$\mathscr{P} : H^n(A,N^*) \to H_n(A,N)^*$$

defined by

$$\mathscr{P}[f][x \otimes a_1 \otimes \cdots \otimes a_n] = f(a_1 \otimes \cdots \otimes a_n)(x).$$

Since $\mathscr{P}$ is an isomorphism when k is a field, see [7] page 181, and the evaluation map is a monomorphism of vector spaces, we get that φ_N is a monomorphism. This implies that if each $H_n(A,N)$ is finite dimensional then φ_N is an isomorphism. □

Proposition 4.3.4. *If A is a finite dimensional algebra over a field k and N is a finitely generated A-bimodule, then φ_N is an isomorphism.*

Proof. Note first that a finitely generated A bimodule is finite dimensional. For finite dimensional algebras, the Hochschild complex is finite dimensional in each degree. Since Hochschild homology is a subquotient in each degree, the dimension of $H_n(A,N)$ is finite for all $n \geq 0$. The previous lemma implies that φ_N is an isomorphism. □

Remark 4.3.5. *The Weyl algebra A_1 is infinite dimensional but $HH_2(A_1) = k$ and $HH_n(A) = 0$ for $n \neq 2$, see [57].*

We consider the cap product in terms of cohomology by using φ. Define a morphism $\Omega_{N,M} : M \otimes_A (N \otimes_A M)^* \to N^*$ given by

$$\Omega_{N,M}(y \otimes \sigma)(x) = \sigma(x \otimes y).$$

Observe that $\Omega_{N,A}$ is the identity morphism of N^* after the identifications $N \otimes_A A = N$ and $A \otimes_A N^* = N^*$. The inverse of $\Omega_{N,A}$ is $\Omega_{N,A}^{-1} : N^* \to A \otimes_A (N \otimes_A A)^*$ which is given by

$$\Omega_{N,A}^{-1}(\sigma) = 1 \otimes \tilde{\sigma}$$

where $\tilde{\sigma}(x \otimes a) = \sigma(xa)$. Clearly $\Omega_{A,M} : M \otimes_A M^* \to A^*$ is not an isomorphism in general. We identify $\Omega_{A^*,A}$ with the identity morphism of A^{**}.

Definition 4.3.6. *Let*

$$\hat{\cap} : H^n(A, N^*)^* \otimes H^m(A, M) \to H^{n-m}(A, (N \otimes_A M)^*)^*$$

be given by

$$(\sigma \hat{\cap} [f])[g] := \sigma([\Omega_{N,M} \circ f \cup g]),$$

for $n \geq m \geq 0$, and is defined as zero otherwise.

If $[f] \in H^m(A, M)$ and $[g] \in H^{n-m}(A, (N \otimes_A M)^*)$, then

$$[f \cup g] \in H^n(A, M \otimes_A (N \otimes_A M)^*)$$

in which $f \cup g$ is a cocycle from $A^{\otimes(n+2)}$ to $M \otimes_A (N \otimes_A M)^*$ and can be composed with $\Omega_{N,M}$ to get a cocycle from $A^{\otimes(n+2)}$ to N^* to which we can apply σ. We will consider $\Omega_{N,M}$ as a morphism concentrated in degree zero in the derived category, i.e. as the roof represented by the diagram

$$\begin{tikzcd} & M \otimes_A (N \otimes_A M)^* \arrow[dl, "1"'] \arrow[dr, "\Omega_{N,M}"] & \\ M \otimes_A (N \otimes_A M)^* & & N^*. \end{tikzcd}$$

Proposition 4.3.7. *Let A be an algebra over a field k. For every pair of integers $n,m \geq 0$, the following diagram is commutative*

$$\begin{array}{ccc}
H_n(A,N)\otimes H^m(A,M) & \xrightarrow{\ \cap\ } & H_{n-m}(A,N\otimes_A M) \\
\Big\downarrow \varphi_N \otimes 1 & & \Big\downarrow \varphi_{N\otimes_A M} \\
H^n(A,N^*)^*\otimes H^m(A,M) & \xrightarrow{\ \hat{\cap}\ } & H^{n-m}(A,(N\otimes_A M)^*)^*.
\end{array}$$

Proof. If $n < m$, then the bottom and the top maps are zero. Assume $n \geq m \geq 0$ and let $[z] = [x\otimes a_1 \otimes \cdots \otimes a_n] \in H_n(A,N)$ and $[f] \in H^m(A,M)$ as well as $[g] \in H^{n-m}(A,(N\otimes_A M)^*)$, then

$$\begin{aligned}
\varphi_{N\otimes_A M}[z\cap f][g] &= \varphi_{N\otimes_A M}[x\otimes f(a_1\otimes\cdots\otimes a_m)\otimes a_{m+1}\otimes\cdots\otimes a_n][g] \\
&= g(a_{m+1}\otimes\cdots\otimes a_n)\big(x\otimes f(a_1\otimes\cdots\otimes a_m)\big) \\
&= [\Omega_{N,M}(f\cup g)](a_1\otimes\cdots\otimes a_n)(x) \\
&= \varphi_N[z][\Omega_{N,M}(f\cup g)] \\
&= \big(\varphi_N[z]\hat{\cap}[f]\big)[g].
\end{aligned}$$

□

Remark 4.3.8. *The diagram in Proposition 4.3.7 is commutative even if k is a commutative ring and A is an arbitrary k-algebra. The vertical maps are monomorphisms if k is a field, and they are isomorphisms if in addition $H_n(A,N)$ and $H_{n-m}(A,N\otimes_A M)$ are finite dimensional. For example, if k is a field, A is a finite dimensional k-algebra and the A-bimodules N and M are finite dimensional, then the vertical maps are isomorphisms.*

The previous result shows that via the morphism φ, the cap product is related to the cup product in cohomology. We use this to define a cap product in terms of the derived category by using the $\widetilde{\cup}$-product. Assume given morphisms $g : A \to M[m]$ and $h : A \to (N\otimes_A M)^*[n-m]$ in $D^b(A^e)$. Their $\widetilde{\cup}$-product is a morphism

$$g\widetilde{\cup}h : A \to M\otimes_A (N\otimes_A M)^*[n],$$

and therefore $g\widetilde{\cup}h$ can be composed with $\Omega_{N,M}[n]$ in $D^b(A^e)$ to get a morphism $A \to N^*[n]$.

Definition 4.3.9. *Let*

$$\widetilde{\cap} : Hom_{D^b(A^e)}(A,N^*[n])^*\otimes Hom_{D^b(A^e)}(A,M[m]) \to Hom_{D^b(A^e)}(A,(N\otimes_A M)^*[n-m])^*$$

be given by

$$\sigma \widetilde{\cap} g(h) := \sigma(\Omega_{N,M}[n] \circ g \widetilde{\cup} h),$$

for every $\sigma \in Hom_{D^b(A^e)}(A, N^*[n])^*$ *and every morphisms* $g \in Hom_{D^b(A^e)}(A, M[m])$ *and* $h \in Hom_{D^b(A^e)}(A, (N \otimes_A M)^*[n-m])$.

Proposition 4.3.10. *Let k be a commutative ring and A be a k-projective algebra. The following diagram is commutative*

$$\begin{array}{ccc} Hom_{D^b(A^e)}(A, N^*[n])^* \otimes Hom_{D^b(A^e)}(A, M[m]) & \xrightarrow{\widetilde{\cap}} & Hom_{D^b(A^e)}(A, (N \otimes_A M)^*[n-m])^* \\ {\scriptstyle (\gamma_{N^*})^* \otimes (\gamma_M)^{-1}} \downarrow & & \downarrow {\scriptstyle (\gamma_{(N \otimes_A M)^*})^*} \\ H^n(A, N^*)^* \otimes H^m(A, M) & \xrightarrow{\hat{\cap}} & H^{n-m}(A, (N \otimes_A M)^*)^*. \end{array}$$

Proof. Let $\sigma \in Hom_{D^b(A^e)}(A, N^*[n])^*$ and $[g\mu^{-1}] \in Hom_{D^b(A^e)}(A, M[m])$.
Let $[h] \in H^{n-m}(A, (N \otimes_A M)^*)$, then

$$\begin{aligned} \big(\gamma_{(N \otimes_A M)^*}\big)^*(\sigma \widetilde{\cap}[g\mu^{-1}])([h]) &= \big(\sigma \widetilde{\cap}[g\mu^{-1}]\big)[h\mu^{-1}] \\ &= \sigma\big(\Omega_{N,M}[n] \circ ([g\mu^{-1}] \widetilde{\cup} [h\mu^{-1}])\big). \end{aligned}$$

On the other hand

$$\begin{aligned} (\gamma_{N^*})^*(\sigma) \hat{\cap} (\gamma_M)^{-1}[g\mu^{-1}]([h]) &= (\sigma \circ \gamma_{N^*}) \hat{\cap} [g]([h]) \\ &= \sigma \circ \gamma_{N^*}\big(\Omega_{N,M}[n] \circ ([g] \cup [h])]\big) \\ &= \sigma\big(\Omega_{N,M}[n] \circ ([g] \cup [h])\mu^{-1}\big). \end{aligned}$$

Since $[g\mu^{-1}] \widetilde{\cup} [h\mu^{-1}] = \big[([g] \cup [h])\mu^{-1}\big]$ the proof is finished. □

Remark 4.3.11. *Observe that we do not require k to be a field for the last proposition. We only need the algebra A to be k-projective in order for* γ *to have an inverse.*

Theorem 4.3.12. *Let k be a commutative ring and let A and B be derived equivalent algebras which are finitely generated and projective as k-modules. There are commutative diagrams*

$$\begin{array}{ccc} Hom_{D^b(A^e)}(A, A^*[n])^* \otimes Hom_{D^b(A^e)}(A, A[m]) & \xrightarrow{\widetilde{\cap}_A} & Hom_{D^b(A^e)}(A, A^*[n-m])^*. \\ {\scriptstyle (\widetilde{G}_{A^*})^* \otimes \widetilde{F}_A} \downarrow & & \downarrow {\scriptstyle (\widetilde{G}_{A^*})^*} \\ Hom_{D^b(B^e)}(B, B^*[n])^* \otimes Hom_{D^b(B^e)}(B, B[m]) & \xrightarrow[\widetilde{\cap}_B]{} & Hom_{D^b(B^e)}(B, B^*[n-m])^* \end{array}$$

and

$$\begin{tikzcd}
Hom_{D^b(B^e)}(B,B^*[n])^* \otimes Hom_{D^b(B^e)}(B,B[m]) \arrow[r, "\widetilde{\frown}_B"] \arrow[d, "(\widetilde{F}_{A^*})^* \otimes \widetilde{G}_A"'] & Hom_{D^b(B^e)}(B,B^*[n-m])^* \arrow[d, "(\widetilde{F}_{A^*})^*"] \\
Hom_{D^b(A^e)}(A,A^*[n])^* \otimes Hom_{D^b(A^e)}(A,A[m]) \arrow[r, "\widetilde{\frown}_A"'] & Hom_{D^b(A^e)}(A,A^*[n-m])^*,
\end{tikzcd}$$

where F_A and G_A are equivalences of standard type chosen as in Proposition 2.3.16 in page 38.

Proof. We will prove the commutativity of the second diagram. The other one is dual. Recall that we identify $\Omega_{A^*,A}$ with the identity morphism of A^{**}. Let $\sigma \in Hom_{D^b(B^e)}(B,B^*[n])^*$ and $g \in Hom_{D^b(B^e)}(B,B[m])$. Let $Z : F(A^*) \to F(A)^*$ be the isomorphism from Proposition 4.1.4. Recall from Proposition 4.1.1 that we have an equality

$$v^{-1}[m] \circ g = F(u^{-1})[m] \circ FG(g) \circ F(u) \circ v^{-1}.$$

Using this in the sixth equality of the following sequence of equalities for $h \in Hom_{D^b(A^e)}(A,A^*[n-m])$, we get that

$$\begin{aligned}
&\left(\left((\widetilde{F}_{A^*})^*(\sigma)\right) \widetilde{\frown}_A \left((\widetilde{G}_A(g))\right)\right)(h) \\
&= \left(\sigma \circ \widetilde{F}_{A^*}\right) \widetilde{\frown}_A \left(u^{-1}[m] \circ G(g) \circ u\right)(h) \\
&= \left(\sigma \circ \widetilde{F}_{A^*}\right) \left(h \widetilde{\smile}_A (u^{-1}[m] \circ G(g) \circ u)\right) \\
&= \left(\sigma \circ \widetilde{F}_{A^*}\right) \left(h[m] \circ u^{-1}[m] \circ G(g) \circ u\right) \\
&= \sigma\left((v^{-1})^*[m] \circ Z[m] \circ F\left(h[m] \circ u^{-1}[m] \circ G(g) \circ u\right) \circ v^{-1}\right) \\
&= \sigma\left((v^{-1})^*[m] \circ Z[m] \circ F(h)[m] \circ F(u^{-1})[m] \circ FG(g) \circ F(u) \circ v^{-1}\right) \\
&= \sigma\left((v^{-1})^*[m] \circ Z[m] \circ F(h)[m] \circ v^{-1}[m] \circ g\right) \\
&= (\sigma \widetilde{\frown}_B g)\left((v^{-1})^*[n-m] \circ Z[n-m] \circ F(h)[n-m] \circ v^{-1}[n-m]\right) \\
&= (\sigma \widetilde{\frown}_B g) \circ \widetilde{F}_{A^*}(h) \\
&= \left(\widetilde{F}_{A^*}\right)^* (\sigma \widetilde{\frown}_B g)(h).
\end{aligned}$$

□

Proposition 4.3.13. *Let A and B be derived equivalent algebras over a field k. Assume that A has finite dimensional Hochschild homology $HH_n(A)$ for each $n \geq 0$. For every pair of integers n and m, the following diagram is commutative*

$$\begin{tikzcd} HH_n(A) \otimes HH^m(A) \arrow[r, "\cap_A"] \arrow[d, "\cong"'] & HH_{n-m}(A) \arrow[d, "\cong"'] \\ HH_n(B) \otimes HH^m(B) \arrow[r, "\cap_B"'] & HH_{n-m}(B). \end{tikzcd}$$

Proof. Since A has finite dimensional Hochschild homology in each degree, it is sufficient, due to Propositions 4.3.7 and 4.3.10, to prove derived invariance of the $\widetilde{\cap}$-product. This is given by Theorem 4.3.12. □

Remark 4.3.14. *We do not require Hochschild cohomology to be finite dimensional. Observe that the finite dimension hypothesis on Hochschild homology is used to express the cap product as the $\hat{\cap}$-product, see Proposition 4.3.7.*

4.4 Cap product with coefficients in a bimodule over an algebra over a commutative ring

Next we will provide a different proof of the derived invariance of the cap product through another interpretation of the cap product in the derived category. This approach was considered in a joint work with Keller [2] and is valid for k a commutative ring.

Lemma 4.4.1. *[2] Let A be a k-projective algebra and let $[f] \in HH^m(A)$. There is a commutative diagram*

$$\begin{tikzcd} H_n(A,M) \arrow[r, "-\cap[f]"] \arrow[d, "\cong"'] & H_{n-m}(A,M) \arrow[d, "\cong"'] \\ H_0\left(M \otimes^{\mathbb{L}}_{A^e} A[-n]\right) \arrow[r, "H_0(1\otimes[f][-n])"'] & H_0\left(M \otimes^{\mathbb{L}}_{A^e} A[m-n]\right). \end{tikzcd}$$

Proof. Since A is k-projective, then $Bar_\bullet(A)$ is a projective A^e-resolution of A and therefore

$$M \otimes^{\mathbb{L}}_{A^e} A = Tot\left(M \otimes_{A^e} Bar_\bullet(A)\right) = M \otimes_{A^e} Bar_\bullet(A).$$

We have that

$$H_0(1\otimes[f][-n])[x\otimes y] = [x\otimes[f](y)] = [x\otimes y]\cap[f],$$

for every $x\in M$ and $y\in Bar_\bullet(A)$. $\square$

Remark 4.4.2. *Observe that the hypothesis on the algebras being Noetherian and finitely generated projective as k-modules is not required since we are not using morphisms induced by a derived equivalence, see Proposition 2.3.16 in page 38.*

Assume that A and B are derived equivalent Noetherian algebras that are projective and finitely generated as k-modules. Let M be an A-bimodule and suppose that $N = FM$ is concentrated in degree zero. There are canonical isomorphisms

$$M\otimes^{\mathbb{L}}_{A^e} A \cong M\otimes^{\mathbb{L}}_{A^e}\left(X\otimes^{\mathbb{L}}_{B} X^\vee\right) \cong M\otimes^{\mathbb{L}}_{A^e}\left(X^\vee\otimes^{\mathbb{L}}_{k} X\right)\otimes^{\mathbb{L}}_{B^e} B \cong FM\otimes^{\mathbb{L}}_{B^e} B = N\otimes^{\mathbb{L}}_{B^e} B,$$

first given in [28] and more explicitly in [69]. In homology we obtain isomorphisms

$$H_n(A,M)\xrightarrow{\sim} H_n(B,N),$$

for every $n\geq 0$. We will use the following result that can be proved by straightforward computations.

Lemma 4.4.3. *[69] Let A, B and C be algebras over a field k. Let X be a complex of $A-B$ bimodules and Y a complex of $B-C$ bimodules. The complex of vector spaces $X\otimes Y$ is a complex of $(A\otimes C^{op})-B^e$ bimodules with action given by*

$$(a\otimes c)(x\otimes y)(b_1\otimes b_2) := (axb_1)\otimes(b_2yc).$$

Moreover, the isomorphism

$$\begin{array}{ccc} X\otimes^{\mathbb{L}}_{B} Y & \xrightarrow{\sim} & (X\otimes Y)\otimes^{\mathbb{L}}_{B^e} B \\ x\otimes y & \mapsto & x\otimes y\otimes 1 \end{array}$$

has inverse given by $x\otimes y\otimes b\mapsto xb\otimes y$.

Remark 4.4.4. *In case $C = A$, we have that the components of the complex $X\otimes Y$ consist of $A^e - B^e$ bimodules.*

Theorem 4.4.5. *[2] Let A and B be derived equivalent Noetherian algebras that are projective and finitely generated as k-modules. For each* $[f] \in HH^m(A) \cong Hom_{D^b(A^e)}(A, A[m])$ *there is a commutative diagram*

$$\begin{tikzcd} H_n(A,M) \arrow[r, "\cong"] \arrow[d, "-\cap[f]"'] & H_n(B,N) \arrow[d, "-\cap\widetilde{F}([f])"] \\ H_{n-m}(A,M) \arrow[r, "\cong"'] & H_{n-m}(B,N). \end{tikzcd}$$

Proof. Since the algebras are Noetherian and finitely generated projective as k-modules, we can choose morphisms u and v as in Proposition 2.3.16 in page 38. The following diagram is commutative

$$\begin{tikzcd} X \otimes^{\mathbb{L}}_B X^\vee \otimes^{\mathbb{L}}_A A \otimes^{\mathbb{L}}_A X \otimes^{\mathbb{L}}_B X^\vee \arrow[r, "1\otimes^{\mathbb{L}}_A v \otimes^{\mathbb{L}}_A 1", "\cong"'] \arrow[d, "GF(f)"'] & X \otimes^{\mathbb{L}}_B B \otimes^{\mathbb{L}}_B X^\vee \arrow[d, "G\left(\widetilde{F}_A(f)\right)"'] \\ X \otimes^{\mathbb{L}}_B X^\vee \otimes^{\mathbb{L}}_A A[m] \otimes^{\mathbb{L}}_A X \otimes^{\mathbb{L}}_B X^\vee \arrow[r, "\cong", "1\otimes^{\mathbb{L}}_A v \otimes^{\mathbb{L}}_A 1[m]"'] & X \otimes^{\mathbb{L}}_B B[m] \otimes^{\mathbb{L}}_B X^\vee. \end{tikzcd}$$

Let $\omega : X \otimes^{\mathbb{L}}_B X^\vee \xrightarrow{\sim} (X^\vee \otimes^{\mathbb{L}}_k X) \otimes^{\mathbb{L}}_{B^e} B$ be the isomorphism given by the last lemma. The following diagram is commutative

$$\begin{tikzcd} X \otimes^{\mathbb{L}}_B B \otimes^{\mathbb{L}}_B X^\vee \arrow[r, "\omega", "\cong"'] \arrow[d, "G\left(\widetilde{F}_A(f)\right)"'] & (X^\vee \otimes^{\mathbb{L}}_k X) \otimes^{\mathbb{L}}_{B^e} B \arrow[d, "1\otimes\widetilde{F}(f)"] \\ X \otimes^{\mathbb{L}}_B B[m] \otimes^{\mathbb{L}}_B X^\vee \arrow[r, "\cong", "\omega[m]"'] & (X^\vee \otimes^{\mathbb{L}}_k X) \otimes^{\mathbb{L}}_{B^e} B[m]. \end{tikzcd}$$

The naturality of the isomorphism of functors $id \xrightarrow{\sim} GF$ and the commutativity of the above diagrams imply that the square

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[d, "f"'] & (X^\vee \otimes^{\mathbb{L}}_k X) \otimes^{\mathbb{L}}_{B^e} B \arrow[d, "1\otimes\widetilde{F}(f)"] \\ A[m] \arrow[r, "\cong"] & (X^\vee \otimes^{\mathbb{L}}_k X) \otimes^{\mathbb{L}}_{B^e} B[m] \end{tikzcd}$$

is commutative. Recall that

$$F \cong - \otimes^{\mathbb{L}}_{A^e} (X^\vee \otimes X),$$

therefore, applying the functor $M \otimes^{\mathbb{L}}_{A^e} ?[-n]$ to the last commutative diagram gives a commutative diagram

$$\begin{tikzcd}
M \otimes^{\mathbb{L}}_{A^e} A[-n] \arrow[r, "\cong"] \arrow[d, "1\otimes f[-n]"'] & N \otimes^{\mathbb{L}}_{B^e} B[-n] \arrow[d, "1\otimes F(f)[-n]"] \\
M \otimes^{\mathbb{L}}_{A^e} A[m-n] \arrow[r, "\cong"] & N \otimes^{\mathbb{L}}_{B^e} B[m-n],
\end{tikzcd}$$

by taking homology and using last lemma we get the expected diagram. □

Theorem 4.4.6. *Let A and B be derived equivalent Noetherian algebras that are finitely generated and projective as k-modules. Let M be an A-bimodule such that $N := FM$ is concentrated in degree zero. There are canonical isomorphisms*

$$H_\bullet(A,M) \xrightarrow{\sim} H_\bullet(B,N) \ \ \text{and} \ \ HH^\bullet(A) \xrightarrow{\sim} HH^\bullet(B)$$

such that the following diagram is commutative

$$\begin{tikzcd}
H_n(A,M) \otimes HH^m(A) \arrow[r, "\cap_A"] \arrow[d, "\cong"'] & H_{n-m}(A,M) \arrow[d, "\cong"'] \\
H_n(B,N) \otimes HH^m(B) \arrow[r, "\cap_B"'] & H_{n-m}(B,N)
\end{tikzcd}$$

for all $n,m \geq 0$.

Proof. First we put the cap products of A and B in the form of Lemma 4.4.1. The result follows from the previous theorem since A and B are Noetherian algebras that are finitely generated and projective as k-modules. □

Remark 4.4.7. *The Theorems 4.4.5 and 4.4.6 hold for algebras projective over a commutative ring due to an argument of model category theory. In this thesis we make the explicit constructions for Noetherian algebras that are finitely generated and projective over a commutative ring.*

Chapter 5

Derived invariance of the Connes differential

Let A be an algebra over a field k. Recall that the multiplicative cyclic group of order $n+1$ acts on the k-vector space $A^{\otimes(n+1)}$ via its generator t by permutation of the factors. The morphism $N = 1+t+\cdots+t^n$ is the norm operator and s is the extra-degeneracy operator. There are three versions of Connes differential, as provided in page 53. The first one is given as the horizontal differential of the bicomplex $\bar{\mathscr{B}}(A)$ in which

$$\bar{B}_A := (1-t)sN.$$

When passing to the normalized bicomplex $\mathscr{B}(A)$, Connes differential is expressed as

$$B_A := sN.$$

Connes differential also appears as the connecting morphism on the homology long exact sequence obtained by applying homology to the short exact sequence of bicomplexes

$$0 \to C(A)^{\{2\}} \to C(A) \to C(A)[2,0] \to 0$$

namely

$$\cdots \longrightarrow HH_n(A) \xrightarrow{I_A} HC_n(A) \xrightarrow{S_A} HC_{n-2}(A) \xrightarrow{B'_A} HH_{n-1}(A) \xrightarrow{I_A} \cdots .$$

It can be proved that $B_A = B'_A \circ I_A$, and therefore we can consider Connes differential between Hochschild homologies, that is

$$B_A : HH_n(A) \to HH_{n+1}(A).$$

We will consider the derived invariance of Connes differential in two different ways. In the first one, we will not consider that Connes differential has a relation with cyclic homology and we will use the canonical isomorphisms γ_A and φ_A to give an interpretation of

$$B_A : HH_n(A) \to HH_{n+1}(A)$$

in terms of the derived category in case k is a field and A has finite dimensional Hochschild homology in each degree. We will see that it is not clear how this could be useful.

The second approach to prove derived invariance of Connes differential relies on more fruitful properties. We prove that Connes periodicity long exact sequence is a derived invariant. Therefore the morphisms I_A, S_A and B'_A are derived invariants and as a consequence the Connes differential between Hochschild homologies B_A is also a derived invariant since $B_A = B'_A \circ I_A$.

However, the isomorphism induced by a derived equivalence between the Hochschild homologies appearing in Connes periodicity long exact sequence is not, a priori, the same than the one we used when we proved derived invariance of the cap product. This means that we have to prove that these isomorphisms are the same in order to prove derived invariance of the Tamarkin-Tsygan calculus structure on the Hochschild theory of an algebra. We use the uniqueness of the cyclic functor introduced in [34] to prove that the last mentioned isomorphisms are equal. This was proved in another joint work with Keller [3].

5.1 The Connes differential in the derived category

We proceed to use the morphisms φ and γ, given in Theorem 3.1.8 on page 45 and Definition 4.3.2 on page 72. Let k be a commutative ring and A a k-projective algebra. Recall that

$$B_A[a_0 \otimes \cdots \otimes a_n] = \sum_{i=0}^{n} (-1)^{ni} [1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}],$$

for $[a_0 \otimes \cdots \otimes a_n] \in HH_n(A)$.

Definition 5.1.1. *Let $\hat{B}_A : H^{n+1}(A, A^*) \to H^n(A, A^*)$ be the morphism obtained by sending a class $[f] \in H^{n+1}(A, A^*)$ to the homology class of the morphism*

$$\hat{B}_A[f] : A^{\otimes n} \to A^*$$

defined as

$$\hat{B}_A[f](a_1\otimes\cdots\otimes a_n)(a_0) := \sum_{i=0}^{n}(-1)^{ni} f(a_i\otimes\cdots\otimes a_n\otimes a_0\otimes\cdots\otimes a_{i-1})(1),$$

for $a_1\otimes\cdots\otimes a_n \in A^{\otimes n}$ *and* $a_0\in A$.

Definition 5.1.2. *Let A be a k-projective algebra. We define*

$$\begin{array}{rccc} \widetilde{B}_A: & Hom_{D^b(A^e)}(A,A^*[n+1]) & \to & Hom_{D^b(A^e)}(A,A^*[n]) \\ & [f\mu^{-1}] & \mapsto & [\hat{B}_A[f]\mu^{-1}]. \end{array}$$

Remark 5.1.3. *The map* $\widetilde{B}_A$ *is well-defined since for roofs* $[f_1\mu^{-1}]$ *and* $[f_2\mu^{-1}]$ *representing the same morphism, a roof covers the other via a quasi-isomorphism* $\eta : Bar(A)\to Bar(A)$*, i.e. there is a commutative diagram*

$$\begin{tikzcd} & Bar(A) \arrow[dl, "\mu"'] \arrow[d, "\eta"'] \arrow[dr, "f_1"] & \\ A & Bar(A) \arrow[l, "\mu"] \arrow[r, "f_2"'] & A^*[n+1]. \end{tikzcd}$$

Therefore $f_1 = f_2\circ\eta$ *and then* $[f_1]=[f_2]$ *as elements of* $H^{n+1}(A,A^*)$*, hence* $\hat{B}_A[f_1]=\hat{B}_A[f_2]$ *and as a consequence*

$$\widetilde{B}_A[f_1\mu^{-1}] = \widetilde{B}_A[f_2\mu^{-1}].$$

Proposition 5.1.4. *Let k be a commutative ring and let A be a k-algebra. The following diagram is commutative*

$$\begin{tikzcd} HH_n(A) \arrow[r, "B_A"] \arrow[d, "\varphi_A"] & HH_{n+1}(A) \arrow[d, "\varphi_A"] \\ H^n(A,A^*)^* \arrow[r, "\hat{B}_A^*"] & H^{n+1}(A,A^*)^*, \end{tikzcd}$$

for each $n\geq 0$*. Moreover, if A is k-projective the following diagram is commutative*

$$\begin{tikzcd} H^{n+1}(A,A^*) \arrow[r, "\hat{B}_A"] \arrow[d, "\gamma_A"] & H^n(A,A^*) \arrow[d, "\gamma_A"] \\ Hom_{D^b(A^e)}(A,A^*[n+1]) \arrow[r, "\widetilde{B}_A"] & Hom_{D^b(A^e)}(A,A^*[n]). \end{tikzcd}$$

Proof. Let $[a_0\otimes\cdots\otimes a_n]\in HH_n(A)$, and for $[g]\in H^{n+1}(A,A^*)$ we have that

$$\begin{aligned}
&\varphi_A(B_A[a_0\otimes\cdots\otimes a_n])[g]\\
&=\varphi_A\left(\left[\sum_{i=0}^{n}(-1)^{in}1\otimes a_i\otimes\cdots\otimes a_n\otimes a_0\otimes\cdots\otimes a_{i-1}\right]\right)[g]\\
&=\sum_{i=0}^{n}(-1)^{in}\varphi(1\otimes a_i\otimes\cdots\otimes a_n\otimes a_0\otimes\cdots\otimes a_{i-1})[g]\\
&=\sum_{i=0}^{n}(-1)^{in}g(a_i\otimes\cdots\otimes a_n\otimes a_0\otimes\cdots\otimes a_{i-1})(1)\\
&=\hat{B}_A[g](a_1\otimes\cdots\otimes a_n)(a_0)\\
&=\varphi_A[a_0\otimes\cdots\otimes a_n]\big(\hat{B}_A[g]\big)\\
&=\hat{B}_A^*\big(\varphi_A[a_0\otimes\cdots\otimes a_n]\big)[g],
\end{aligned}$$

and then the upper square is commutative. For the commutativity of the second diagram let $[g]\in H^{n+1}(A,A^*)$, then

$$\begin{aligned}
\gamma\circ\hat{B}_A[g] &= (\hat{B}_A[g])\mu^{-1}\\
&= \widetilde{B}_A[g\mu^{-1}]\\
&= \widetilde{B}_A\circ\gamma[g].
\end{aligned}$$

□

Remark 5.1.5. *Observe that if k is a field and $HH_n(A)$ is finite dimensional for all $n\geq 0$, then the map B_A is related by isomorphisms to the k-dual of $\widetilde{B}_A$.*

Remark 5.1.6. *This interpretation of Connes differential in terms of the derived category does not allows us to give a direct proof of derived invariance as for the cup and the cap products. Let k be a field and let A and B be derived equivalent algebras with finite dimensional Hochschild homology in each degree. Choose isomorphisms u and v as in Proposition 2.3.16 in page 38. We would like the following diagram to be commutative*

$$\begin{array}{ccc}
Hom_{D^b(A^e)}(A,A^*[n+1]) & \xrightarrow{\widetilde{B}_A} & Hom_{D^b(A^e)}(A,A^*[n])\\
\Big\downarrow{\scriptstyle \widetilde{F}_{A^*}} & & \Big\downarrow{\scriptstyle \widetilde{F}_{A^*}}\\
Hom_{D^b(B^e)}(B,B^*[n+1]) & \xrightarrow{\widetilde{B}_B} & Hom_{D^b(B^e)}(B,B^*[n]).
\end{array}$$

Let $[f\mu^{-1}]$ be a morphism from A to $A^*[n+1]$ in $D^b(A^e)$, then

$$\begin{aligned}
\widetilde{F}_{A^*}\circ\widetilde{B}_A\left([f\mu^{-1}]\right) &= \widetilde{F}_{A^*}\big((\hat{B}_A[f]\mu^{-1}\big)\\
&= (v^{-1})^*[n]\circ Z[n]\circ F\left(\hat{B}_A[f]\mu^{-1}\right)\circ v^{-1}.
\end{aligned}$$

On the other hand,

$$\widetilde{B}_B \circ \widetilde{F}_{A^*}[f\mu^{-1}] \;\; = \;\; \widetilde{B}_B\big((v^{-1})^*[n+1] \circ Z[n+1] \circ F[f\mu^{-1}] \circ v^{-1}\big),$$

for which we need a roof of the form $[h\mu^{-1}]$ *to represent the morphism*

$$(v^{-1})^*[n+1] \circ Z[n+1] \circ F[f\mu^{-1}] \circ v^{-1}$$

in order to apply $\widetilde{B}_B$*. There is, at least for the author, no way to pursue from this point.*

5.2 Derived invariance of the Connes periodicity long exact sequence over a field

In this section we let k be a field. All versions of Connes differential involve cyclic homology in a way or another, then it is natural to prove derived invariance of Connes differential using the ideas of [34], in which it is proved derived invariance of cyclic homology. This was done in another joint work with Keller in [3].

5.2.1 The cyclic functor

We now follow the work of Keller [34] for the construction of the cyclic functor, which uses ideas of Kassel [25, 26] about mixed complexes. After this, we will prove derived invariance of Connes differential as in [3]. We need the following definitions.

Definition 5.2.1. *[25] Let* $\mathbf{Alg}_k$ *be the category whose objects are the associative DG k-algebras A such that the functor* $Hom(A,-)$ *sends quasi-isomorphisms to isomorphisms, and whose morphisms are morphisms of DG k-algebras which do not necessarily preserve the unit.*

The idea is to consider a category of DG-algebras that allows to manage derived invariance. It turns out that the right choice of morphisms in such a category are as follows.

Definition 5.2.2. *[25] For* $A, B \in \mathbf{Alg}_k$ *define* $rep(A,B)$ *as the full subcategory of the derived category* $D(A^{op} \otimes B)$ *whose objects are the DG bimodules X such that the restriction* X_B *is perfect in* $D(B)$*, that is, the covariant Hom functor commutes with coproducts.*

Definition 5.2.3. *[34] Define $\mathbf{ALG}_k$ to be the category whose objects are those of $\mathbf{Alg}_k$ and morphisms from A to B are the isomorphism classes of objects of $rep(A,B)$. The composition of morphisms in $\mathbf{ALG}_k$ is given by the total derived tensor product. The identity morphism of an object $A \in \mathbf{ALG}_k$ is the isomorphism class of the bimodule ${}_AA_A$.*

Definition 5.2.4. *[34] The localization functor is the functor*

$$\mathscr{L} : \mathbf{Alg}_k \to \mathbf{ALG}_k$$

given as the identity on objects and that associates to a morphism $f : A \to B$ of DG-algebras the bimodule ${}_fB_B$ with underlying space $f(1)B := \{f(1)b | b \in B\}$ and $A - B$-action given by

$$a.f(1)b.b' := f(a)bb' = f(1)f(a)bb'$$

for all $a \in A$ and all $b, b' \in B$.

The cyclic functor is a functor from the category $\mathbf{Alg}_k$ that will extend to $\mathbf{ALG}_k$ via the localization functor. The values of the cyclic functor will be on a derived category associated to mixed complexes.

Definition 5.2.5. *[34] The derived mix category **DMix** is the derived category of the DG-algebra of dual numbers $\Lambda = k[\varepsilon]/(\varepsilon^2)$, where ε is considered to be of degree -1.*

Definition 5.2.6. *[34] The cyclic functor*

$$\mathscr{C} : \mathbf{Alg}_k \to \mathbf{DMix}$$

is defined as follows. Let A be an object of $\mathbf{Alg}_k$, then $\mathscr{C}(A)$ is the mixed complex whith underlying graded k-vector space the mapping cone over $1-t$ viewed as a morphism of complexes

$$1 - t : (A^{\otimes \bullet + 1}, b') \to (A^{\otimes \bullet + 1}, b).$$

The first and second differentials of the mixed complex $\mathscr{C}(A)$ are

$$\begin{bmatrix} b & 1-t \\ 0 & -b' \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix},$$

respectively.

Remark 5.2.7. *When there is no confusion, we will also denote by $\mathscr{C}(A)$ the complex obtained from the image of A under the cyclic functor by forgetting the second differential.*

Each morphism of DG-algebras $f : A \to B$, even if it does not preserves the unit, induces morphisms of complexes $(A^{\otimes \bullet +1}, b') \to (B^{\otimes \bullet +1}, b')$ and $(A^{\otimes \bullet +1}, b) \to (B^{\otimes \bullet +1}, b)$ and therefore f induces a *morphism of mixed complexes*

$$\mathscr{C}(f) : \mathscr{C}(A) \to \mathscr{C}(B),$$

i.e. a graded map of degree 0 that commutes with both differentials.

Remark 5.2.8. *Observe that the underlying complex of $\mathscr{C}(A)$ is equal to the total complex of the bicomplex $C(A)^{\{2\}}$ formed by the two first columns of the cyclic bicomplex $C(A)$ in page 52, namely*

$$\begin{tikzcd}
\vdots \arrow[d, "b"'] & \vdots \arrow[d, "-b'"'] \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] \\
A & A. \arrow[l, "1-t"']
\end{tikzcd}$$

Keller extends the cyclic functor $\mathscr{C}$ to $\mathbf{ALG}_k$ in the sense that there is a commutative diagram

$$\begin{tikzcd}
\mathbf{Alg}_k \arrow[r, "\mathscr{C}"] \arrow[d, "\mathscr{L}"'] & \mathbf{DMix} \\
\mathbf{ALG}_k, \arrow[ur, dashed, "\mathscr{C}"'] &
\end{tikzcd}$$

in the following way. Let $X : A \to B$ be a morphism in $\mathbf{ALG}_k$. Since X_B is perfect as a DG B-module, there is an isomorphism of functors

$$Hom_{D(B)}(X, -) \cong Hom_{K(B)}(X, -).$$

Define a morphism $\alpha_X : A \to End_B(B \oplus X)$ given by

$$\alpha_X(a)(b,x) := (0, ax)$$

where $End_B(B \oplus X)$ is the differential graded endomorphism algebra of the DG B-module $B \oplus X$. Let $\beta_X : B \to End_B(B \oplus X)$ be defined as

$$\beta_X(b')(b,x) := (b'b, 0).$$

Note that the morphisms α_X and β_X do not preserve the units.

Proposition 5.2.9. *[34] The morphism $\mathscr{C}(\beta_X) : \mathscr{C}(B) \to \mathscr{C}(End_B(B \oplus X))$ is invertible in **DMix**.*

Consider the diagram of DG-algebras

$$A \xrightarrow{\alpha_X} End_B(B \oplus X) \xleftarrow{\beta_X} B$$

and define a morphism in **DMix** by

$$\mathscr{C}(X) := \mathscr{C}(\beta_X)^{-1} \circ \mathscr{C}(\alpha_X) : \mathscr{C}(A) \to \mathscr{C}(B).$$

Theorem 5.2.10. *([34] page 231) The functor $\mathscr{C} : \boldsymbol{Alg}_k \to \boldsymbol{DMix}$ extends uniquely to a functor $\mathscr{C} : \boldsymbol{ALG}_k \to \boldsymbol{DMix}$.*

5.2.2 Derived invariance

Let $X : A \to B$ be a morphism in $\mathbf{ALG}_k$. We choose morphisms $u_X : A \to X \otimes_B^{\mathbb{L}} X^\vee$ and $v_X : X^\vee \otimes_B^{\mathbb{L}} X \to B$ as in Proposition 2.3.16 in page 38. Define functors

$$F := X^\vee \otimes_A^{\mathbb{L}} - \otimes_A^{\mathbb{L}} X \cong - \otimes_{A^e}^{\mathbb{L}} (X \otimes X^\vee) : D^b(A^e) \to D(B^e)$$

and

$$G := X \otimes_B^{\mathbb{L}} - \otimes_B^{\mathbb{L}} X^\vee \cong - \otimes_{B^e}^{\mathbb{L}} (X^\vee \otimes X) : D(B^e) \to D^b(A^e),$$

induced by u_X and v_X as in Proposition 2.3.16 in page 38. This ensures that (F, G) is an adjoint pair of functors. Indeed, the morphisms u_X and v_X are given by the adjointness of the functors

$$- \otimes_A^{\mathbb{L}} X : D(A) \to D(B) \quad \text{and} \quad - \otimes_B^{\mathbb{L}} X^\vee : D(B) \to D(A),$$

and therefore for each $Y \in D(A^e)$ the morphism

$$u_X \otimes_A^{\mathbb{L}} 1_Y \otimes_A^{\mathbb{L}} u_X : GF(Y) \to Y$$

defines the unit, and for each $Z \in D(B^e)$ the morphism

$$v_X \otimes_B^{\mathbb{L}} 1_Z \otimes_B^{\mathbb{L}} v_X : Z \to FG(Z)$$

defines the co-unit of the adjoint pair (F,G).

We identify $X \otimes_B^{\mathbb{L}} X^\vee$ with $(X \otimes X^\vee) \otimes_{B^e}^{\mathbb{L}} B$ and $X^\vee \otimes_A^{\mathbb{L}} X$ with $(X^\vee \otimes X) \otimes_{A^e}^{\mathbb{L}} A$ via the isomorphism described by Lemma 4.4.3 in page 78. For the sake of simplicity, we still denote by u_X and v_X the same morphisms when composed with this identifications.
We will now define a functor

$$\psi : \mathbf{Alg}_k \to D(k)$$

that induces the isomorphism we used to prove derived invariance of the cap product. Put

$$\psi(A) = A \otimes_{A^e}^{\mathbb{L}} A$$

and $\psi(f) = f \otimes f$ for a morphism $f : A \to B$ of DG-algebras. We define ψ on morphisms of $\mathbf{ALG}_k$ as follows. Let $X : A \to B$ be a morphism in $\mathbf{ALG}_k$. The morphism $\psi(X)$ is defined as the following composition

$$\begin{array}{rcl} A \otimes_{A^e}^{\mathbb{L}} A & \to & A \otimes_{A^e}^{\mathbb{L}} X \otimes X^\vee \otimes_{B^e}^{\mathbb{L}} B \\ & \tilde{\to} & B \otimes_{B^e}^{\mathbb{L}} X^\vee \otimes X \otimes_{A^e}^{\mathbb{L}} A \\ & \to & B \otimes_{B^e}^{\mathbb{L}} B. \end{array}$$

That is, we put $\psi(X) := (1 \otimes v_X) \circ \tau \circ (1 \otimes u_X)$.

Remark 5.2.11. *The twist isomorphism τ for the tensor product over k is given by $a \otimes x \otimes y \otimes b \mapsto b \otimes y \otimes x \otimes a$. We will also denote by τ the twist morphisms over other tensor products, though they are not in general isomorphisms.*

Theorem 5.2.12. *The assignments $A \mapsto \psi(A)$, $X \mapsto \psi(X)$ define a functor on $\mathbf{ALG}_k$ that extends the cyclic functor. Therefore $\mathscr{C} = \psi$.*

Proof. There is a canonical quasi-isomorphism $\psi(A) \to Tot(C(A)^{\{2\}})$ induced by the inclusion in the first column

$$\psi(A) = A \otimes_{A^e}^{\mathbb{L}} A = Bar(A) \otimes_{A^e} A \hookrightarrow Cone(1-t) = \mathscr{C}(A),$$

for any algebra A. Therefore, the functors $\mathscr{C}$ and ψ provide isomorphic values on objects. Let $f : A \to B$ be a morphism of $\mathbf{Alg}_k$. The associated morphism in $\mathbf{ALG}_k$

via the localization functor is $X = {}_fB_B$. Note that $X^\vee \cong \mathbb{R}Hom_B(X,B) \cong {}_BB_f$ in $D^b(B^e)$. The following diagram is commutative

$$\begin{tikzcd} A \otimes^{\mathbb{L}}_{A^e} ({}_fB \otimes^{\mathbb{L}}_B B_f) \arrow[d, "\simeq"'] \arrow[dr, "\simeq"] & \\ A \otimes^{\mathbb{L}}_{A^e} ({}_fB \otimes B_f) \otimes^{\mathbb{L}}_{B^e} B \arrow[r, "\simeq"'] & A \otimes^{\mathbb{L}}_{A^e} {}_fB_f. \end{tikzcd}$$

The isomorphism on the left is given by

$$a \otimes f(1)b_1 \otimes b_2 f(1) \mapsto a \otimes (f(1)b_1 \otimes b_2 f(1)) \otimes 1,$$

and the one in the bottom is given by

$$a \otimes f(1)b_1 \otimes b_2 f(1) \otimes b \mapsto a \otimes f(1)b_1 b b_2 f(1).$$

Consider Lemma 4.4.3 in the case where $C = A$, and the complexes $X = {}_fB$ and $Y = X^\vee = B_f$. The isomorphism that the lemma describes is

$${}_fB \otimes^{\mathbb{L}}_B B_f \xrightarrow{\sim} ({}_fB \otimes B_f) \otimes^{\mathbb{L}}_{B^e} B.$$

The isomorphism on the left of last diagram is equal to this isomorphism tensored on the left by A. The bottom isomorphism of the same diagram is the inverse of the isomorphism described by Lemma 4.4.3 tensored on the left by A. The following diagram is also commutative

$$\begin{tikzcd} A \otimes^{\mathbb{L}}_{A^e} ({}_fB \otimes B_f) \otimes^{\mathbb{L}}_{B^e} B \arrow[r, "\simeq"'] \arrow[d, "\tau"'] & A \otimes^{\mathbb{L}}_{A^e} {}_fB_f \arrow[d, "\tau"] \\ B \otimes^{\mathbb{L}}_{B^e} B_f \otimes {}_fB \otimes^{\mathbb{L}}_{A^e} A \arrow[r, "\simeq"] & {}_fB_f \otimes^{\mathbb{L}}_{A^e} A, \end{tikzcd}$$

where the bottom isomorphism is given by

$$b \otimes b_1 f(1) \otimes f(1) b_2 \otimes a \mapsto f(1) b_2 b b_1 f(1) \otimes a.$$

This isomorphism can be described in terms of the dual of Lemma 4.4.3. The diagram

$$\begin{tikzcd} {}_fB_f \otimes^{\mathbb{L}}_{A^e} A \arrow[r, "\tau"] \arrow[d, "1\otimes f"'] & A \otimes^{\mathbb{L}}_{A^e} {}_fB_f \arrow[d, "f\otimes 1"] \\ B \otimes^{\mathbb{L}}_{B^e} B \arrow[r, "\tau"] & B \otimes^{\mathbb{L}}_{B^e} B \end{tikzcd}$$

is commutative and the bottom morphism equals the identity since

$$\tau(b \otimes b') = b' \otimes b = 1 \otimes (bb') = b \otimes b'.$$

We get that $\psi({}_f B_B)$ is the morphism induced by f from $A \otimes^{\mathbb{L}}_{A^e} A$ to $B \otimes^{\mathbb{L}}_{B^e} B$. Therefore

$$\psi({}_f B_B) = \mathscr{C}({}_f B_B).$$

Let $X : A \to B$ and $Y : B \to C$ be morphisms in $\mathbf{ALG}_k$ and define $Z = X \otimes^{\mathbb{L}}_B Y$. We have canonical isomorphisms

$$\begin{aligned}\mathbb{R}Hom_C(Y,C) \otimes^{\mathbb{L}}_B \mathbb{R}Hom_B(X,B) &\cong \mathbb{R}Hom_B(X, \mathbb{R}Hom_C(Y,C)) \\ &\cong \mathbb{R}Hom_C(X \otimes^{\mathbb{L}}_B Y, C).\end{aligned}$$

Therefore we can identify

$$(X \otimes^{\mathbb{L}}_B Y)^{\vee} = Y^{\vee} \otimes^{\mathbb{L}}_B X^{\vee}.$$

There are canonical isomorphisms

$$\gamma : Y^{\vee} \otimes^{\mathbb{L}}_B X^{\vee} \otimes X \otimes^{\mathbb{L}}_B Y \xrightarrow{\sim} Y^{\vee} \otimes Y \otimes^{\mathbb{L}}_{B^e} X^{\vee} \otimes X$$

and

$$\delta : X \otimes X^{\vee} \otimes^{\mathbb{L}}_{B^e} Y \otimes Y^{\vee} \xrightarrow{\sim} X \otimes^{\mathbb{L}}_B Y \otimes Y^{\vee} \otimes^{\mathbb{L}}_B X^{\vee}$$

given by the obvious reordering of the factors, that make the diagrams

$$\begin{tikzcd} C \otimes^{\mathbb{L}}_{C^e} Y^{\vee} \otimes Y \otimes^{\mathbb{L}}_{B^e} X^{\vee} \otimes X \otimes^{\mathbb{L}}_{A^e} A \arrow[r, "1\otimes v_X"] \arrow[d, "1\otimes\gamma\otimes 1"'] & C \otimes^{\mathbb{L}}_{C^e} Y^{\vee} \otimes Y \otimes^{\mathbb{L}}_{B^e} B \arrow[d, "1\otimes v_Y"] \\ C \otimes^{\mathbb{L}}_{C^e} Y^{\vee} \otimes^{\mathbb{L}}_B X^{\vee} \otimes X \otimes^{\mathbb{L}}_B Y \otimes^{\mathbb{L}}_{A^e} A \arrow[r, "1\otimes v_Z"'] & C \otimes^{\mathbb{L}}_{C^e} C \end{tikzcd}$$

and

$$\begin{tikzcd} A \otimes^{\mathbb{L}}_{A^e} A \arrow[r, "1\otimes u_X"] \arrow[d, "1\otimes u_Z"'] & A \otimes^{\mathbb{L}}_{A^e} X \otimes X^{\vee} \otimes^{\mathbb{L}}_{B^e} B \arrow[d, "1\otimes u_Y"] \\ A \otimes^{\mathbb{L}}_{A^e} X \otimes^{\mathbb{L}}_B Y \otimes Y^{\vee} \otimes^{\mathbb{L}}_B X^{\vee} \otimes^{\mathbb{L}}_{C^e} C \arrow[r, "1\otimes\delta\otimes 1"'] & A \otimes^{\mathbb{L}}_{A^e} X \otimes X^{\vee} \otimes^{\mathbb{L}}_{B^e} Y \otimes Y^{\vee} \otimes^{\mathbb{L}}_{C^e} C \end{tikzcd}$$

commutative. By definition, $\psi(X \otimes^{\mathbb{L}}_B Y) = (1 \otimes u_Z)\tau(1 \otimes v_Z)$, and therefore it is equal to the following composition

$$
\begin{aligned}
A \otimes^{\mathbb{L}}_{A^e} A &\to A \otimes^{\mathbb{L}}_{A^e} \left(X \otimes X^\vee \otimes^{\mathbb{L}}_{B^e} B\right) \\
&\to A \otimes^{\mathbb{L}}_{A^e} X \otimes X^\vee \otimes^{\mathbb{L}}_{B^e} \left(Y \otimes Y^\vee \otimes^{\mathbb{L}}_{C^e} C\right) \\
&\xrightarrow{\sim} A \otimes^{\mathbb{L}}_{A^e} \left(X \otimes^{\mathbb{L}}_B Y \otimes Y^\vee \otimes^{\mathbb{L}}_B X^\vee\right) \otimes^{\mathbb{L}}_{C^e} C \\
&\xrightarrow{\sim} C \otimes^{\mathbb{L}}_{C^e} Y^\vee \otimes^{\mathbb{L}}_B X^\vee \otimes X \otimes^{\mathbb{L}}_B Y \otimes^{\mathbb{L}}_{A^e} A \\
&\xrightarrow{\sim} C \otimes^{\mathbb{L}}_{C^e} \left(Y^\vee \otimes Y \otimes^{\mathbb{L}}_{B^e} X^\vee \otimes X\right) \otimes^{\mathbb{L}}_{A^e} A \\
&\to C \otimes^{\mathbb{L}}_{C^e} Y^\vee \otimes Y \otimes^{\mathbb{L}}_{B^e} B \\
&\to C \otimes^{\mathbb{L}}_{C^e} C.
\end{aligned}
$$

There is also a canonical isomorphism

$$\eta : A \otimes^{\mathbb{L}}_{A^e} X \otimes X^\vee \otimes^{\mathbb{L}}_{B^e} Y \otimes Y^\vee \otimes^{\mathbb{L}}_{C^e} C \xrightarrow{\sim} C \otimes^{\mathbb{L}}_{C^e} Y^\vee \otimes Y \otimes^{\mathbb{L}}_{B^e} X^\vee \otimes X \otimes^{\mathbb{L}}_{A^e} A$$

that fits into a commutative diagram

$$
\begin{tikzcd}
A \otimes^{\mathbb{L}}_{A^e} X \otimes X^\vee \otimes^{\mathbb{L}}_{B^e} B \arrow[r, "1\otimes u_Y"] \arrow[d, "\tau"'] & A \otimes^{\mathbb{L}}_{A^e} X \otimes X^\vee \otimes^{\mathbb{L}}_{B^e} Y \otimes Y^\vee \otimes^{\mathbb{L}}_{B^e} B \arrow[d, "\eta"] \\
B \otimes^{\mathbb{L}}_{B^e} X^\vee \otimes X \otimes^{\mathbb{L}}_{A^e} A \arrow[r, "u_Y\otimes 1"'] & C \otimes^{\mathbb{L}}_{C^e} Y^\vee \otimes Y \otimes^{\mathbb{L}}_{B^e} X^\vee \otimes X \otimes^{\mathbb{L}}_{A^e} A.
\end{tikzcd}
$$

This diagram together with the following commutative diagram

$$
\begin{tikzcd}
B \otimes^{\mathbb{L}}_{B^e} X^\vee \otimes X \otimes^{\mathbb{L}}_{A^e} A \arrow[r, "u_y\otimes 1"] \arrow[d, "1\otimes v_X"'] & (Y \otimes Y^\vee \otimes^{\mathbb{L}}_{C^e} C) \otimes^{\mathbb{L}}_{B^e} X^\vee \otimes X \otimes^{\mathbb{L}}_{A^e} A \arrow[d, "\tau\otimes 1"] \\
B \otimes^{\mathbb{L}}_{B^e} B \arrow[d, "1\otimes u_X"'] & C \otimes^{\mathbb{L}}_{C^e} Y^\vee \otimes Y \otimes^{\mathbb{L}}_{B^e} X^\vee \otimes X \otimes^{\mathbb{L}}_{A^e} A \arrow[d, "1\otimes v_X"] \\
B \otimes^{\mathbb{L}}_{B^e} Y^\vee \otimes Y \otimes^{\mathbb{L}}_{C^e} C \arrow[r, "\tau"'] & C \otimes^{\mathbb{L}}_{C^e} Y^\vee \otimes Y \otimes^{\mathbb{L}}_{B^e} B,
\end{tikzcd}
$$

imply that $\psi(Y)\circ\psi(X)$ is equal to the following composition

$$\begin{aligned} A\otimes^{\mathbb{L}}_{A^e}A &\to A\otimes^{\mathbb{L}}_{A^e}\left(X\otimes X^{\vee}\otimes^{\mathbb{L}}_{B^e}B\right)\\ &\to A\otimes^{\mathbb{L}}_{A^e}X\otimes X^{\vee}\otimes^{\mathbb{L}}_{B^e}\left(Y\otimes Y^{\vee}\otimes^{\mathbb{L}}_{C^e}C\right)\\ &\xrightarrow{\sim} C\otimes^{\mathbb{L}}_{C^e}Y^{\vee}\otimes Y\otimes^{\mathbb{L}}_{B^e}X^{\vee}\otimes X\otimes^{\mathbb{L}}_{A^e}A\\ &\to C\otimes^{\mathbb{L}}_{C^e}Y^{\vee}\otimes Y\otimes^{\mathbb{L}}_{B^e}B\\ &\to C\otimes^{\mathbb{L}}_{C^e}C. \end{aligned}$$

Finally, since $\eta=(1\otimes\gamma\otimes 1)\tau(1\otimes\delta\otimes 1)$ we get that

$$\psi(X\otimes^{\mathbb{L}}_{B}Y)=\psi(Y)\circ\psi(X).$$

□

Let A and B be derived equivalent algebras over a field k and let $X:A\to B$ be an isomorphism in $\mathbf{ALG}_k$, so that the functor $-\otimes^{\mathbb{L}}_{A}X:D(A)\xrightarrow{\sim}D(B)$ is an equivalence. Then $\mathscr{C}(X)$ is an isomorphism of **DMix** and also an isomorphism of $D(k)$, and there is an isomorphism of functors

$$-\otimes^{\mathbb{L}}_{\Lambda}\mathscr{C}(A)\xrightarrow{\sim}-\otimes^{\mathbb{L}}_{\Lambda}\mathscr{C}(B),$$

where Λ is the DG-algebra of dual numbers $k[\varepsilon]/(\varepsilon^2)$.

Corollary 5.2.13. *[3] Let A and B be derived equivalent algebras over a field k. There is an isomorphism of exact sequences induced by a derived equivalence*

$$\begin{tikzcd} \cdots \arrow[r] & HC_{n-1}(A) \arrow[r, "B'_{n-1}"] \arrow[d, "\cong"'] & HH_n(A) \arrow[r, "I_n"] \arrow[d, "\cong"'] & HC_n(A) \arrow[r, "S_n"] \arrow[d, "\cong"'] & HC_{n-2}(A) \arrow[r] \arrow[d, "\cong"'] & \cdots \\ \cdots \arrow[r] & HC_{n-1}(B) \arrow[r, "B'_{n-1}"] & HH_n(B) \arrow[r, "I_n"] & HC_n(B) \arrow[r, "S_n"] & HC_{n-2}(B) \arrow[r] & \cdots . \end{tikzcd}$$

Proof. There is a canonical short exact sequence of DG Λ-modules

$$0\to k[1]\to\Lambda\to k\to 0$$

which gives rise to a triangle in the derived category $D(\Lambda)$

$$k[1]\xrightarrow{B'}\Lambda\xrightarrow{I}k\xrightarrow{S}k[2]\ .$$

We apply the isomorphism of functors $-\otimes^{\mathbb{L}}_{\Lambda}\mathscr{C}(A)\xrightarrow{\sim}-\otimes^{\mathbb{L}}_{\Lambda}\mathscr{C}(B)$ to this triangle to get an isomorphism of triangles in $D(k)$,

$$\begin{tikzcd}
k[1]\otimes^{\mathbb{L}}_{\Lambda}\mathscr{C}(A) \arrow[r, "B'_A"] \arrow[d, "\cong"'] & \mathscr{C}(A) \arrow[r, "I_A"] \arrow[d, "\cong"'] & k\otimes^{\mathbb{L}}_{\Lambda}\mathscr{C}(A) \arrow[r, "S_A"] \arrow[d, "\cong"'] & k[2]\otimes^{\mathbb{L}}_{\Lambda}\mathscr{C}(A) \arrow[d, "\cong"'] \\
k[1]\otimes^{\mathbb{L}}_{\Lambda}\mathscr{C}(B) \arrow[r, "B'_A"] & \mathscr{C}(B) \arrow[r, "I_A"] & k\otimes^{\mathbb{L}}_{\Lambda}\mathscr{C}(B) \arrow[r, "S_A"] & k[2]\otimes^{\mathbb{L}}_{\Lambda}\mathscr{C}(B).
\end{tikzcd}$$

We consider homology and recall from Theorem 3.2.7 on page 56 that for every $j\geq 0$ there are isomorphisms $H_j(k\otimes^{\mathbb{L}}_{\Lambda}\mathscr{C}(A))\cong HC_j(A)$. This gives the expected diagram. □

Corollary 5.2.14. *[3] Let A and B be derived equivalent algebras over a field k. Then for every $n\geq 0$ there is a commutative diagram*

$$\begin{tikzcd}
HH_n(A) \arrow[r, "B_A"] \arrow[d, "\cong"'] & HH_{n+1}(A) \arrow[d, "\cong"'] \\
HH_n(B) \arrow[r, "B_B"] & HH_{n+1}(B).
\end{tikzcd}$$

Proof. From Corollary 5.2.13 we get that there is a commutative diagram

$$\begin{tikzcd}
HH_n(A) \arrow[r, "I_A"] \arrow[d, "\cong"'] & HC_n(A) \arrow[r, "B'_A"] \arrow[d, "\cong"'] & HH_{n+1}(A) \arrow[d, "\cong"'] \\
HH_n(B) \arrow[r, "I_B"] & HC_n(B) \arrow[r, "B'_B"] & HH_{n+1}(B).
\end{tikzcd}$$

We conclude by noticing that $B_A=B'_A\circ I_A$. □

Theorem 5.2.15. *[3] Let A and B be derived equivalent algebras over a field k. The Tamarkin-Tsygan calculi of A and B are isomorphic.*

Proof. Let $-\otimes^{\mathbb{L}}_{A}X : D(A)\xrightarrow{\sim}D(B)$ be an equivalence of standard type. Let

$$HH_{\bullet}(X) : HH_{\bullet}(A)\to HH_{\bullet}(B)$$

and

$$HH^{\bullet}(X) : HH^{\bullet}(A)\to HH^{\bullet}(B)$$

be the isomorphisms

$$HH_\bullet(X) := H_\bullet\big((1\otimes v_X)\circ\tau\circ(1\otimes u_X)\big)$$

and

$$HH^\bullet(X) := \gamma_B^{-1}\circ\widetilde{F}_A\circ\gamma_A$$

respectively. It was proved by Rickard [50] that the isomorphism $HH^\bullet(X)$ respects the cup product. Keller [38] proved that this isomorphism also respects the Gerstenhaber bracket. The isomorphisms $HH_\bullet(X)$ and $HH^\bullet(X)$ preserve the cap product, see Theorem 4.4.6 on page 80 and Theorem 5.2.12 in page 89. The isomorphism induced by the cyclic functor

$$\mathscr{C}(X) : \mathscr{C}(A) \xrightarrow{\sim} \mathscr{C}(B)$$

preserves the Connes differential and it sends derived tensor products to composition, see the previous corollary. Therefore, it only remains to prove that $\mathscr{C}(X) = HH_\bullet(X)$. This is precisely Theorem 5.2.12 on page 89. $\square$

5.3 The category of Tamarkin-Tsygan calculi

We now summarize our results in terms of the category of Tamarkin-Tsygan calculi. In this section we let k be a field. Let $H = (\mathscr{H}_\bullet, \mathscr{H}^\bullet, \cup, [-,-], \cap, B)$ and $K = (\mathscr{K}_\bullet, \mathscr{K}^\bullet, \cup, [-,-], \cap, B)$ be Tamarkin-Tsygan calculi. A *morphism of Tamarkin-Tsygan calculi* is a pair of graded maps $(f,g) : H \to K$ of k-vector spaces such that $f : \mathscr{H}_\bullet \to \mathscr{K}_\bullet$ and $g : \mathscr{H}^\bullet \to \mathscr{K}^\bullet$ satisfy the following commutative diagrams

$$\begin{tikzcd}
\mathscr{H}^n\otimes\mathscr{H}^m \arrow[r, "\cup"] \arrow[d, "g_n\otimes g_m"'] & \mathscr{H}^{n+m} \arrow[d, "g_{n+m}"'] \\
\mathscr{K}^n\otimes\mathscr{K}^m \arrow[r, "\cup"'] & \mathscr{K}^{n+m},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{H}^n\otimes\mathscr{H}^m \arrow[r, "{[-,-]}"] \arrow[d, "g_n\otimes g_m"'] & \mathscr{H}^{n+m-1} \arrow[d, "g_{n+m-1}"'] \\
\mathscr{K}^n\otimes\mathscr{K}^m \arrow[r, "{[-,-]}"'] & \mathscr{K}^{n+m-1},
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{H}_n\otimes\mathscr{H}^m \arrow[r, "\cap"] \arrow[d, "f_n\otimes g_m"'] & \mathscr{H}_{n-m} \arrow[d, "f_{n-m}"'] \\
\mathscr{K}_n\otimes\mathscr{K}^m \arrow[r, "\cap"'] & \mathscr{K}_{n-m}
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathscr{H}_n \arrow[r, "B"] \arrow[d, "f_n"'] & \mathscr{H}_{n+1} \arrow[d, "f_{n+1}"'] \\
\mathscr{K}_n \arrow[r, "B"'] & \mathscr{K}_{n+1}.
\end{tikzcd}$$

Denote **TT-calc** the category of Tamarkin-Tsygan calculi, whose objects are Tamarkin-Tsygan calculi and morphisms are morphisms of Tamarkin-Tsygan calculi. The composition of morphisms in **TT-calc** is the obvious one

$$(f,g)\circ(h,l) := (f\circ h, g\circ l).$$

The identity morphism of an object $\mathscr{H}$ is $(1_{\mathscr{H}_\bullet}, 1_{\mathscr{H}^\bullet})$. Denote $\mathbf{A}_k$ the full subcategory of $\mathbf{ALG}_k$ formed by the finite dimensional k-algebras. For $A\in\mathbf{A}_k$ let $\mathbb{H}(A)$ be equal to the Tamarkin-Tsygan calculus

$$\big(HH_\bullet(A), HH^\bullet(A), \cup_A, [-,-]_A, \cap_A, B_A\big).$$

Recall from Proposition 4.3.7 in page 73 that there is a commutative diagram

$$\begin{tikzcd}
HH_n(A)\otimes HH^m(A) \arrow[r, "\cap"] \arrow[d, "\varphi_A\otimes 1"] & HH_{n-m}(A) \arrow[d, "\varphi_A"] \\
H^n(A,A^*)^*\otimes HH^m(A) \arrow[r, "\hat{\cap}"] & H^{n-m}(A,A^*)^*,
\end{tikzcd}$$

for all $n,m\geq 0$, where the vertical maps are isomorphisms. From Proposition 5.1.4 in page 83 there is also a commutative diagram

$$\begin{tikzcd}
HH_n(A) \arrow[r, "B_A"] \arrow[d, "\varphi_A"] & HH_{n+1}(A) \arrow[d, "\varphi_A"] \\
H^n(A,A^*)^* \arrow[r, "\hat{B}_A^*"] & H^{n+1}(A,A^*)^*,
\end{tikzcd}$$

where the vertical maps are isomorphisms. Therefore,

$$\hat{\mathbb{H}}(A) := \big(HH^\bullet(A,A^*)^*, HH^\bullet(A), \cup_A, [-,-]_A, \hat{\cap}_A, \hat{B}_A^*\big)$$

is a Tamarkin-Tsygan calculus by transport of structure. Let

$$\widetilde{\mathbb{H}}(A) := \big(Hom_{D^b(A^e)}(A,A^*[\bullet])^*, Hom_{D^b(A^e)}(A,A[\bullet]), \widetilde{\cup}_A, \widetilde{[-,-]}_A, \widetilde{\cap}_A, \widetilde{B}_A^*\big),$$

where the bracket $\widetilde{[-,-]}_A$ is defined as

$$\big[\widetilde{[f\mu^{-1}],[g\mu^{-1}]}\big]_A := \big[[f,g]\mu^{-1}\big].$$

The morphism $\mu : A \otimes A \to A$ is the product of A and we represent morphisms $A \to N$ of $D^b(A^e)$ as roofs $[f\mu^{-1}]$ since k is a field. It is easy to see that the bracket $\widetilde{[-,-]}$ fits into a commutative diagram

$$\begin{tikzcd} HH^n(A) \otimes HH^m(A) \arrow[r, "{[-,-]}"] \arrow[d, "\gamma_A \otimes \gamma_A"'] & HH^{n+m-1}(A) \arrow[d, "\gamma_A"] \\ Hom_{D^b(A^e)}(A, A[n]) \otimes Hom_{D^b(A^e)}(A, A[m]) \arrow[r, "\widetilde{[-,-]}"] & Hom_{D^b(A^e)}(A, A[n+m-1]). \end{tikzcd}$$

Therefore $\widetilde{\mathbb{H}}(A)$ is a Tamarkin-Tsygan calculus by transport of structure since there are also commutative diagrams whose vertical maps are isomorphisms

$$\begin{tikzcd} HH^n(A) \otimes HH^m(A) \arrow[r, "\cup"] \arrow[d, "\gamma_A \otimes \gamma_A"'] & HH^{n+m}(A) \arrow[d, "\gamma_A"] \\ Hom_{D^b(A^e)}(A, A[n]) \otimes Hom_{D^b(A^e)}(A, A[m]) \arrow[r, "\widetilde{\cup}"] & Hom_{D^b(A^e)}(A, A[n+m]) \end{tikzcd}$$

and

$$\begin{tikzcd} Hom_{D^b(A^e)}(A, A^*[n])^* \otimes Hom_{D^b(A^e)}(A, A[m]) \arrow[r, "\widetilde{\cap}"] \arrow[d, "(\gamma_{A^*}^{-1})^* \otimes \gamma_A^{-1}"'] & Hom_{D^b(A^e)}(A, A^*[n-m])^* \arrow[d, "(\gamma_{A^*})^*"] \\ H^n(A, A^*)^* \otimes HH^m(A) \arrow[r, "\hat{\cap}"] & H^{n-m}(A, A^*)^* \end{tikzcd}$$

as well as

$$\begin{tikzcd} H^{n+1}(A, A^*) \arrow[r, "\hat{B}_A"] \arrow[d, "\gamma_{A^*}"] & H^n(A, A^*) \arrow[d, "\gamma_{A^*}"] \\ Hom_{D^b(A^e)}(A, A^*[n+1]) \arrow[r, "\widetilde{B}_A"] & Hom_{D^b(A^e)}(A, A^*[n]), \end{tikzcd}$$

see Proposition 4.2.4 in page 68, Proposition 4.3.10 in page 75 and Proposition 5.1.4 in page 83. We have proved the following.

Theorem 5.3.1. *Let k be a field. For any algebra $A \in \boldsymbol{A}_k$ there are isomorphisms of Tamarkin-Tsygan calculi*

$$\mathbb{H}(A) \xrightarrow[(\varphi_A,1)]{\cong} \hat{\mathbb{H}}(A) \xrightarrow[((\gamma_{A^*}^{-1})^*,\gamma_A)]{\cong} \widetilde{\mathbb{H}}(A).$$

Recall from Theorem 4.3.12 in page 75 that there is a commutative diagram

$$\begin{tikzcd} Hom_{D^b(A^e)}(A,A^*[n])^* \otimes Hom_{D^b(A^e)}(A,A[m]) \arrow[r, "\widetilde{\cap}_A"] \arrow[d, "(\widetilde{G}_{A^*})^* \otimes \widetilde{F}_A"'] & Hom_{D^b(A^e)}(A,A^*[n-m])^* \arrow[d, "(\widetilde{G}_{A^*})^*"] \\ Hom_{D^b(B^e)}(B,B^*[n])^* \otimes Hom_{D^b(B^e)}(B,B[m]) \arrow[r, "\widetilde{\cap}_B"'] & Hom_{D^b(B^e)}(B,B^*[n-m])^* \end{tikzcd}$$

for any morphism $X : A \to B$ of $\mathbf{A}_k$. We define the morphism of Tamarkin-Tsygan calculi

$$\widetilde{\mathbb{H}}(X) : \widetilde{\mathbb{H}}(A) \to \widetilde{\mathbb{H}}(B)$$

by the tensor factors of the map on the left of the previous commutative diagram in each degree. Let $\hat{\mathbb{H}}(X)$ be equal to the following composition of morphisms of Tamarkin-Tsygan calculi

$$\hat{\mathbb{H}}(A) \cong \widetilde{\mathbb{H}}(A) \to \widetilde{\mathbb{H}}(B) \cong \hat{\mathbb{H}}(B).$$

Finally, we define the morphism

$$\mathbb{H}(X) : \mathbb{H}(A) \to \mathbb{H}(B)$$

as the following composition of morphisms of Tamarkin-Tsygan calculi

$$\mathbb{H}(A) \cong \hat{\mathbb{H}}(A) \cong \widetilde{\mathbb{H}}(A) \to \widetilde{\mathbb{H}}(B) \cong \hat{\mathbb{H}}(B) \cong \mathbb{H}(B).$$

Theorem 5.3.2. *Let k be a field. The assignments*

$$A \mapsto \mathbb{H}(A), \quad A \mapsto \hat{\mathbb{H}}(A) \quad \textit{and} \quad A \mapsto \widetilde{\mathbb{H}}(A)$$

define functors

$$\mathbb{H}, \hat{\mathbb{H}}, \widetilde{\mathbb{H}} : \mathbf{A}_k \to \textit{\textbf{TT-calc}}$$

that are constant on each class of derived equivalent algebras.

Proof. The identity morphism of an object $A \in \mathbf{A}_k$ is the A-bimodule A. Since the tensor product $-\otimes_A^{\mathbb{L}} A$ is isomorphic to the identity functor we can choose $u = v = 1_A$ and therefore the morphism induced in **TT-calc** from $\widetilde{\mathbb{H}}(A)$ to itself is equal to the identity morphism. This implies that $\hat{\mathbb{H}}$ and so does $\mathbb{H}$ map identities to identities. Let $X : A \to B$ and $Y : B \to C$ be morphisms in $\mathbf{A}_k$. We get

$$\widetilde{\mathbb{H}}(X \otimes_B^{\mathbb{L}} Y) = \widetilde{\mathbb{H}}(Y) \circ \widetilde{\mathbb{H}}(X),$$

since the morphisms u_X, u_Y, v_X and v_Y are natural and the cyclic functor induces the isomorphism we defined in homology, see Theorem 5.2.12 in page 89. Therefore, $\widetilde{\mathbb{H}}$ is a functor. Since the morphisms φ and γ are natural we conclude that $\hat{\mathbb{H}}$ and so does $\mathbb{H}$ are functors.

Finally, Theorem 5.2.15 in page 94 ensures that the Tamarkin-Tsygan calculus of an algebra is invariant under derived equivalences. This finishes the proof. $\square$

We finish this thesis by giving an example which shows that the Tamarkin-Tsygan calculus of an algebra is not a complete derived invariant.

Remark 5.3.3. *Let k be an algebraically closed field. Let A be the path algebra of the quiver*

$$\begin{tikzcd}
2 \arrow[dr] & & 3 \arrow[dl] \\
& 1 & \\
& 4 \arrow[u] &
\end{tikzcd}$$

and let B be the path algebra of the following quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4.$$

Observe that both quivers are trees and the algebras A and B are hereditary. Then their Hochschild homology, see [10], and Hochschild cohomology, see [11] Proposition 2.6, vanishes in degrees greater or equal than 1. From [10] we get that

$$HH_0(A) \cong k^4 \cong HH_0(B).$$

It is clear that the centers of A and B are of dimension one, that is

$$HH^0(A) \cong k \cong HH^0(B).$$

Therefore, the Gerstenhaber bracket and the Connes differential of both algebras A and B are zero in every degree. Their cup product is given by the product of k in degree $(0,0)$ *and is zero in every other degree. Their cap product is given by the k-vector space structure of* k^4 *in degree* $(0,0)$ *and it is zero in every other degree. Then the Tamarkin-Tsygan calculi of A and B are isomorphic.*

Recall from [16] Proposition 2.3 page 908, that the Coxeter polynomial is a derived invariant for algebras of finite global dimension, see also [43]. The Coxeter polynomial of A is

$$\phi_A(x) = (x+1)^2(x^2 - x + 1)$$

and the Coxeter polynomial of B is

$$\phi_B(x) = x^4 + x^3 + x^2 + x + 1.$$

Therefore A and B are not derived equivalent and then the Tamarkin-Tsygan calculus of an algebra is not a complete derived invariant.

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