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Jaime Bustillo. Rigidité symplectique et EDPs hamiltoniennes. Mathématiques générales [math.GM]. Université Paris sciences et lettres, 2018. Français. NNT : 2018PSLEE050 . tel-02393553

HAL Id: tel-02393553

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THÈSE DE DOCTORAT

de l'Université de recherche Paris Sciences et Lettres
PSL Research University

Préparée à l'École Normale Supérieure de Paris

Symplectic rigidity and Hamiltonian PDEs

Rigidité symplectique et EDPs hamiltoniennes

École doctorale n°574

ÉCOLE DOCTORALE DE MATHÉMATIQUES HADAMARD

Spécialité MATHÉMATIQUES FONDAMENTALES



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le 2 juillet 2018

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*A Sandra,
los teoremas son para ti.*

Remerciements

Je remercie chaleureusement mon directeur de thèse, Claude Viterbo, de m'avoir permis de réaliser mon doctorat dans le laboratoire qu'il dirigeait ainsi que pour son encadrement dans ce projet de recherche. Je lui suis très reconnaissant de sa volonté de partager avec moi ses connaissances mathématiques et de la confiance et de la liberté qu'il m'a accordée pour mener à terme ce travail.

Sergei Kuksin et Emmanuel Opshtein m'ont fait l'honneur d'être rapporteurs de ma thèse, ils ont pris le temps de m'écouter et de discuter avec moi. Leurs remarques m'ont permis d'envisager mon travail sous un autre angle et d'améliorer la version finale du manuscrit. Pour tout cela je les remercie.

Je tiens à remercier à Felix Schlenk, Sobhan Seyfaddini et Patrick Bernard pour avoir accepté de participer à mon jury de thèse et pour leur participation scientifique ainsi que le temps qu'ils ont consacré à ma recherche.

Ce travail n'aurait pas été possible sans le soutien de l'École Doctorale de Mathématiques Hadamard et de PSL, qui m'ont permis, grâce à une allocation de recherche, de me consacrer sereinement à l'élaboration de ma thèse. Je veux remercier aussi la Fundación Mutua Madrileña pour la bourse accordée en 2013 grâce à laquelle j'ai pu retourner à Paris pour suivre ma passion mathématique.

J'ai eu la chance de faire cette thèse à Paris, où il y a une grande communauté symplectique. Le séminaire Symplectix m'a toujours motivé tout le long de ma thèse et je veux remercier tous les participants pour l'ambiance stimulante qu'ils ont créée. Je remercie particulièrement Vincent Humilière pour l'intérêt porté à ma recherche et à mon parcours. C'est grâce à lui et à Laurent Charles que je suis allé parler avec Claude Viterbo pour lui demander de m'encadrer. Un grand merci aux deux.

Au cours de ces années j'ai fait partie du DMA où j'ai eu l'occasion de rencontrer et d'interagir avec de nombreux mathématiciens. Les discussions que j'ai pu avoir m'ont beaucoup apporté même si elles n'ont pas contribué directement au contenu de ces lignes. Je tiens à remercier particulièrement Rodolfo pour toutes nos discussions, conseils et support qui ont été précieux dans les étapes plus dures de cette thèse.

Pendant ces trois ans j'ai partagé le bureau avec Aymeric, Charles, David, Ephrème, Irène, Paolo et Valentine. Merci à tous pour les bons moments et la très bonne ambiance! Merci aussi à Jessica, Matias, Maxence, Michel, Joseph, Stefan, Thibault, Wei, Yusuke... pour votre sympathie et pour tous les repas ensemble.

Un grand merci aux doctorants du laboratoire PNCA, Chloé, Erwan, Florence, Gabrielle, Joanna, Olivier, Romain et Simon pour les très bons moments passés ensemble, au soleil, à la salle d'escalade, chez les uns et les autres pour les repas thésards... Merci aussi pour votre solidarité précieuse.

Cuando vives lejos de casa, tus amigos son como tu familia. Quiero aprovechar para darle las gracias a Charlotte, François, Lorena y Pedro que me han acompañado todos estos años en París. Hemos pasado muy buenos momentos juntos y he aprendido mucho con vosotros. ¡Gracias de corazón!

Gracias a mi familia de Madrid y Barcelona que siempre está ahí cuando lo necesito. Papá, Mamá, Bea, Yayi, ¡os quiero! Gracias por todo lo que me habéis enseñado y por todo el apoyo estos años. Aunque a veces estuviéramos viviendo cada uno en un país diferente siempre habéis estado cerca. Este trabajo también es vuestro.

Y por último gracias a Sandra, compañera de aventuras. Vinimos hace cinco años a París buscando hacer una tesis y ya casi lo hemos conseguido.

Résumé

On étudie les propriétés de rigidité symplectique des difféomorphismes hamiltoniens en dimension finie et en dimension infinie. En dimension finie, les outils principaux qu'on utilise sont les fonctions génératrices et les capacités symplectiques. En dimension infinie on regarde les flots des équations en dérivées partielles (EDPs) hamiltoniennes et, en particulier, les flots qui peuvent être approchés uniformément par des flots hamiltoniens de dimension finie.

Dans la première partie de la thèse on étudie les sélecteurs d'action définies à partir des fonctions génératrices et on construit des invariants hamiltoniens pour les sous-ensembles de $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. Cela nous permet de démontrer un théorème non-squeezing coisotrope pour les difféomorphismes hamiltoniens à support compact de \mathbb{R}^{2n} . On montre à continuation que cette propriété apparaisse dans certains cas non compacts. Finalement, on explique comment ce résultat donne aussi l'information sur le problème de rigidité symplectique en dimension intermédiaire. Encore en dimension finie, on démontre qu'on peut utiliser le théorème du chameau symplectique pour produire des sous-ensembles invariants compacts dans des surfaces d'énergie.

Dans la deuxième partie on étudie les propriétés de rigidité symplectique des flots des EDPs hamiltoniennes. On se place dans le contexte introduit par Kuksin et on étudie une classe particulière de EDPs semi-linéaires qui peuvent être approchées par flots hamiltoniens de dimension finie. D'abord on donne une nouvelle construction de capacité symplectique en dimension infinie à partir des capacités de Viterbo. Puis on démontre l'analogie de la rigidité intermédiaire pour certaines EDPs hamiltoniennes. Cette classe inclue l'équation d'ondes en dimension 1 avec une non-linéarité bornée, comme par exemple l'équation de Sine-Gordon. Dans la dernière partie de la thèse on s'intéresse à un analogue de la conjecture d'Arnold pour l'équation de Schrödinger périodique avec une non linéarité de convolution.

Abstract

We study symplectic rigidity properties in both finite and infinite dimension. In finite dimension, the main tools that we use are generating functions and symplectic capacities. In infinite dimension we study flows of Hamiltonian partial differential equations (PDEs) and, in particular, flows which can be uniformly approximated by finite dimensional Hamiltonian diffeomorphisms.

In the first part of this thesis we study the action selectors defined from generating functions and we build Hamiltonian invariants for subsets of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. This allows us to prove a coisotropic non-squeezing theorem for compactly supported Hamiltonian diffeomorphisms of \mathbb{R}^{2n} . We then extend this result to some non-compact settings. Finally we explain how this result can give information about the middle dimensional symplectic rigidity problem. Still in finite dimensions, we show that it is possible to use the symplectic camel theorem to create energy surfaces with compact invariant subsets.

In the second part of the thesis we study symplectic rigidity properties of flows of Hamiltonian PDEs. We work in the context introduced by Kuksin and study a particular class of semi-linear Hamiltonian PDEs that can be approximated by finite dimensional Hamiltonian diffeomorphisms. We first give a new construction of an infinite dimensional capacity using Viterbo's capacities. The main result of this part is the proof of the analogue of the middle dimensional rigidity for certain types of Hamiltonian PDEs. These include nonlinear string equations with bounded nonlinearity such as the Sine-Gordon equation. In the final part of this thesis we study an analogue of Arnold's conjecture for the periodic Schrödinger equations with a convolution nonlinearity.

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Introduction

La géométrie symplectique trouve ses origines dans la formulation mathématique de la mécanique classique. Au XIX^{ème} siècle, Hamilton donne une interprétation variationnelle des équations du mouvement. Considérons l'espace de phases \mathbb{R}^{2n} avec coordonnées (q, p) , où q représente la position d'une particule et p représente le moment. L'évolution $\gamma : [0, 1] \rightarrow \mathbb{R}^{2n}$ d'une particule dans l'espace de phases sous l'influence d'une fonction d'énergie $H : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ est un point stationnaire de l'action

$$\mathcal{A}_H(\gamma) = \int_0^1 p(t) \cdot \dot{q}(t) dt - \int_0^1 H(t, \gamma(t)) dt$$

Sous cette formulation, les points stationnaires sont décrits par les équations de Hamilton:

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases}$$

Par exemple, la fonction d'énergie d'une particule sous l'influence d'un potentiel $V(q)$ a une fonction d'énergie donnée par $H(q, p) = \frac{1}{2m}|p|^2 + V(q)$. Les équations de Hamilton donnent alors l'équation de Newton classique

$$m\ddot{q} = -\nabla V(q).$$

En général, les équations de Hamilton donnent lieu à une transformation de l'espace de phases. Lorsque l'espace de phases est de dimension deux, cette transformation préserve l'aire des sous-ensembles de \mathbb{R}^2 . Dans les dimensions supérieures, l'énoncé équivalent est le suivant: le flot des équations de Hamilton préserve la forme différentielle $\omega_0 = \sum_{i=1}^n dq_i \wedge dp_i$. Cette forme différentielle est ce que nous appelons la forme symplectique standard de \mathbb{R}^{2n} . Afin de comprendre la dynamique de la mécanique classique, on est amené à étudier les structures symplectiques et, en particulier, les transformations de l'espace qui préservent la forme symplectique. Le concept de forme symplectique se généralise à des variétés lisses M : une structure symplectique ω sur M est une deux-forme fermée non dégénérée. Le fait que ω soit non-dégénérée permet de généraliser les équations de Hamilton: pour toute fonction d'énergie H , le champ vectoriel hamiltonien X_H est défini comme l'unique champ de vecteurs qui satisfait $i_{X_H}\omega = dH$. En utilisant le fait que ω est fermée, nous obtenons que le flot généré par X_H préserve la structure symplectique. Le principal exemple de variété symplectique avec laquelle nous allons travailler sont les espaces cotangents T^*N . Ils représentent les espaces de phase de la mécanique classique où le mouvement est limité à une variété N . Chaque espace cotangent est équipé d'une structure symplectique canonique qui est donnée localement par la structure standard dans $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$. Plus généralement, la structure locale des variétés symplectiques est décrite par le théorème suivante:

Théorème (Darboux). *Toute variété symplectique de dimension $2n$ est localement symplectomorphe à $(\mathbb{R}^{2n}, \omega_0)$.*

Ce théorème implique que les variétés symplectiques n'ont pas d'invariants locaux. Les propriétés symplectiques de M sont déterminées par la géométrie de $(\mathbb{R}^{2n}, \omega_0)$ et par la structure globale des objets dans M . Afin d'étudier la géométrie de l'espace, on est alors conduit à l'étude des difféomorphismes globaux qui préservent la structure symplectique. Nous les appellerons symplectomorphismes. La première propriété vérifiée par symplectomorphismes est le théorème de Liouville: chaque difféomorphisme symplectique préserve le volume des sous-ensembles. L'un des premiers résultats qui a montré la différence entre les difféomorphismes conservant le volume et les symplectomorphismes, est le théorème non-squeezing de Gromov:

Théorème (Gromov 1985, [Gro85]). *Soit $(\mathbb{R}^{2n}, \omega_0)$ l'espace symplectique standard. Notons B_r^{2n} une boule de rayon r dans \mathbb{R}^{2n} et soit $B_R^2 \times \mathbb{R}^{2n-2}$ le cylindre symplectique standard de rayon R . Si $\varphi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ est un difféomorphisme symplectique alors*

$$\varphi(B_r^{2n}) \subseteq B_R^2 \times \mathbb{R}^{2n-2} \quad \text{implique} \quad r \leq R$$

Capacités symplectiques

La preuve originale du théorème non-squeezing de Gromov reposait sur la technique des courbes pseudoholomorphes. Peu après, plusieurs auteurs [EH90, HZ90, Vit92] ont donné des preuves indépendantes du théorème de Gromov en utilisant le concept de capacités symplectiques. Une capacité symplectique est une fonction $c : \mathcal{P}(\mathbb{R}^{2n}) \rightarrow [0, +\infty]$ qui vérifie les propriétés suivantes:

1. (croissance) Si $U \subseteq V$ alors $c(U) \leq c(V)$.
2. (conformalité) $c(\lambda U) = \lambda^2 c(U)$ pour tout $\lambda \in \mathbb{R}$.
3. (invariance symplectique) Si $\phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ est un symplectomorphisme, alors $c(\phi(U)) = c(U)$.
4. (non-trivialité+normalisation) $c(B_1^{2n}) = \pi = c(B_1^2 \times \mathbb{R}^{2n-2})$.

L'existence d'une fonction avec ces propriétés implique le théorème de Gromov. La construction de certaines capacités (cf. [EH90, HZ90, Vit92]) provient de l'étude des orbites périodiques des systèmes hamiltoniens. C'est l'approche qui va nous intéresser par la suite.

Le spectre d'action en tant qu'invariant symplectique

Rappelons que le principe de Hamilton affirme que les orbites périodiques d'un système hamiltonien associé à H sont les points extrêmes de l'action:

$$A_H(\gamma) = \int_0^1 p(t) \cdot \dot{q}(t) dt - \int_0^1 H(t, \gamma(t)) dt.$$

En particulier, les points critiques de A_H contiennent des informations dynamiques très importantes. Cette information peut être utilisée pour construire des capacités symplectiques. Considérons un difféomorphisme hamiltonien à support compact H dans \mathbb{R}^{2n} . Dans ce cas, nous pouvons intégrer les équations de Hamilton pour obtenir un flot global ψ_t^H de \mathbb{R}^{2n} . Une orbite périodique du système peut être vue comme un point fixe z_0 de ψ_1^H , c'est-à-dire un point vérifiant $\psi_1^H(z_0) = z_0$. Notons $A(z_0, H) = A_H(\psi_t^H(z_0))$ la valeur de l'action associée au point fixe z_0 . A première vue, cette quantité dépend de la fonction H et de tout le chemin ψ_t^H , mais en fait ce n'est pas le cas. On peut montrer que cette valeur d'action ne dépend pas du chemin hamiltonien

utilisé pour engendrer ψ_1^H . Plus précisément, si K une autre fonction hamiltonienne à support compact tel que $\psi_1^K = \psi_1^H$, alors

$$A(z_0, H) = A(z_0, K).$$

Les valeurs d'action définissent donc un sous-ensemble de \mathbb{R} qui dépend uniquement du temps un $\psi = \psi_1^H$. On peut alors associer à tout ψ son spectre d'action $\sigma(\psi) \subseteq \mathbb{R}$ défini par

$$\sigma(\psi) = \{A(z, H) \mid z \in \text{Fix}(\psi = \psi_1^H)\}.$$

Il s'avère que le spectre d'action est invariant par conjugaisons symplectiques: pour tout symplectomorphisme φ de \mathbb{R}^{2n} on a

$$\sigma(\varphi\psi\varphi^{-1}) = \sigma(\psi).$$

En particulier, le spectre d'action des difféomorphismes hamiltoniens supportés en U est exactement le même que celui qu'on trouve pour les difféomorphismes hamiltoniens supportés dans $\varphi(U)$. Cette remarque importante est le point de départ de la construction des capacités symplectiques que Viterbo a donné dans [Vit92].

Capacités de Viterbo

Pour chaque ensemble ouvert, U on note $\text{Ham}^c(U)$ l'ensemble de temps un des flots hamiltoniens avec un support compact contenu dans U . Pour un ensemble ouvert borné U , la capacité de Viterbo $c(U)$ est définie en utilisant l'information dynamique des éléments dans $\text{Ham}^c(U)$. Plus précisément, cette information va provenir de la valeur d'action associée à certaines orbites périodiques dynamiquement importantes. Ces valeurs d'action sont choisies par deux sélecteurs, désignés par $c(\mu, \cdot)$ et $c(1, \cdot)$, qui prennent un élément ψ^H de $\text{Ham}^c(\mathbb{R}^{2n})$ et donnent deux valeurs

$$c(\mu, \psi) = \mathcal{A}_H(\sigma(t)) \quad \text{et} \quad c(1, \psi) = \mathcal{A}_H(\beta(t))$$

où $\sigma(t) = \psi_t^H(x)$ et $\beta(t) = \psi_t^H(y)$ avec $x, y \in \text{Fix}(\psi_1^H)$. Ces valeurs vérifient $c(\mu, \psi) \geq 0$ et $c(1, \psi) \leq 0$. On peut prouver que la valeur $\gamma(\psi) = c(\mu, \psi) - c(1, \psi)$ est nulle si et seulement si ψ est l'identité. Intuitivement, cela signifie que ces valeurs d'action peuvent être interprétées comme un maximum et minimum dynamique de ψ . De plus, on peut prouver que γ définit une distance sur $\text{Ham}^c(\mathbb{R}^{2n})$. En utilisant l'invariance par conjugaison symplectique du spectre d'action, on peut voir que les sélecteurs sont invariants sous la conjugaison symplectique, c'est-à-dire,

$$c(\mu, \psi) = c(\mu, \varphi\psi\varphi^{-1}),$$

donc ces valeurs peuvent être utilisées pour définir deux invariants symplectiques sur des ensembles ouverts bornés :

$$c(U) = \sup\{c(\mu, \psi) \mid \psi \in \text{Ham}^c(U)\}$$

$$\gamma(U) = \inf\{\gamma(\psi) \mid \psi \in \text{Ham}^c(\mathbb{R}^{2n}) \text{ et } \psi(U) \cap U = \emptyset\}$$

Sur les ensembles ouverts non bornés V , la capacité $c(V)$ et $\gamma(V)$ est définie comme le sup sur tous les ensembles bornés ouverts contenus dans V . Enfin, sur des sous-ensembles arbitraires $X \subseteq \mathbb{R}^{2n}$ les deux capacités sont définies comme l'infimum sur tous les ensembles ouverts V qui contiennent X . La valeur $c(U)$ mesure la taille de ce maximum dynamique si le support est contenu dans U . Il a de plus la propriété que, sur un corps lisse convexe K , la valeur $c(K)$ coïncide avec la quantité géométrique de l'aire minimale d'une caractéristique fermée sur la frontière ∂K . D'autre part, γ est défini en utilisant la distance de Viterbo sur $\text{Ham}^c(\mathbb{R}^{2n})$: si nous définissons l'énergie d'un difféomorphisme comme la distance à l'identité, alors $\gamma(U)$ mesure l'énergie minimale dont on a besoin pour déplacer U . Les deux capacités sont toujours liées par l'inégalité $c(X) \leq \gamma(X)$. En dimension deux, si ψ est un difféomorphisme hamiltonien à support compact qui déplace un disque d'aire πr^2 , alors nous avons $\pi r^2 \leq \gamma(\psi)$.

Construction des sélecteurs d'action

Afin d'extraire l'information des points fixes de $\psi_1^H = \psi$ nous regardons le graphe de ψ , noté $\Gamma_\psi = \text{Id} \times \psi$, dans $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ et les points d'intersection de Γ_ψ avec Γ_{Id} . Si nous munissons $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ avec la structure symplectique $\omega_0 \oplus (-\omega_0)$ alors Γ_ψ devient une sous-variété lagrangienne. De plus, Γ_ψ est isotope à la diagonale et coïncide avec elle en dehors d'un compact. Travailler sur la variété linéaire de \mathbb{R}^{2n} a l'avantage qu'il y a un symplectomorphisme explicite

$$\mathcal{I} : (\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \omega_0 \oplus (-\omega_0)) \longrightarrow (T^*\mathbb{R}^{2n}, -d\lambda)$$

tel que Γ_{Id} est envoyé à la section nulle. En particulier, puisque l'application est symplectique, l'image de Γ_ψ est une sous-variété lagrangienne qui est isotope de façon hamiltonienne à section nulle et coïncide à l'infini avec la section nulle. Cela nous permet de considérer la compactification en un point de \mathbb{R}^{2n} à S^{2n} et de voir $T^*\mathbb{R}^{2n}$ comme un sous-ensemble de T^*S^{2n} . Chaque Γ_ψ est alors contenu dans une sphère lagrangienne $L_\psi \subseteq T^*S^{2n}$ qui est une isotopie hamiltonienne de la section nulle. Nous voulons extraire des informations des points d'intersection de L_ψ avec la section nulle.

Il y a une manière naturelle de décrire les Lagrangiens dans l'espace cotangent: toute fonction f définit une sous-variété lagrangienne via le graphe de son différentiel df . Pour cette famille de Lagrangiens, les points d'intersection avec la section nulle sont déterminés par les points critiques de f . Nous allons décrire L_ψ en utilisant une généralisation de cette description qui sont les fonctions génératrices. On introduit une variable auxiliaire et on considère les fonctions $S : S^{2n} \times \mathbb{R}^k \rightarrow \mathbb{R}$. Le graphe de la différentielle est une sous-variété lagrangienne dans $T^*(S^{2n} \times \mathbb{R}^k)$. Nous allons pousser en avant cette sous-variété lagrangienne de manière symplectique pour obtenir une lagrangienne dans T^*S^{2n} . Pour le faire, nous devons introduire le concept de réduction symplectique.

Réduction symplectique. Soit V un espace vectoriel symplectique. On dit que $W \subseteq V$ est un sous-espace coïso trope si W contient son orthogonal symplectique W^ω , ou en autres termes, un sous-espace W tel quel si $u \in V$ vérifie $\omega(u, w) = 0$ pour tout $w \in W$, alors $u \in W$. Puisque $W^\omega \subseteq W$, on peut considérer le quotient W/W^ω qui est par construction un espace vectoriel symplectique avec la forme symplectique induite par ω . Notons π la projection naturelle $\pi : W \rightarrow W/W^\omega$ et soit Z un sous-ensemble de V . La réduction symplectique de Z par W est

$$\text{Red}_W(Z) := \pi(Z \cap W),$$

qui est un sous-ensemble de W/W^ω . Une des propriétés importantes de cette opération est que si $L \subset V$ est un sous-espace vectoriel lagrangien transverse à W , alors $\text{Red}_W(L)$ est lagrangien dans le quotient. Cette définition s'applique au cas non linéaire, la seule difficulté qui peut survenir est que W/W^ω ne soit pas une variété.

Fonctions génératrices quadratiques à l'infini. Soit N une variété compacte et considérons une fonction $S : N \times \mathbb{R}^k \rightarrow \mathbb{R}$. Le graphe de la différentielle est une sous-variété lagrangienne dans $T^*(N \times \mathbb{R}^k)$. Considérons le sous-espace coïso trope $W := T^*N \times 0_{\mathbb{R}^k} \subseteq T^*N \times T^*\mathbb{R}^k \simeq T^*(N \times \mathbb{R}^k)$. On peut voir que $W/W^\omega \simeq T^*N$. Si dS est transverse à W alors

$$L_S = \text{Red}_W(dS)$$

est une sous-variété lagrangienne immergée dans T^*N . Dans ce cas, nous disons que S est une fonction génératrice ou *fg* de L_S . La condition de transversalité implique que $\Sigma_S := dS^{-1}(W)$ est une sous-variété de dimension $\dim N$. On note $i_S : \Sigma_S \rightarrow L_S \subseteq T^*N$ l'application $i_S(q, \xi) =$

$\text{Red}_W(dS(q, \xi))$, c'est une immersion. Si de plus $S(x, \xi) = Q(\xi)$ en dehors d'un compact avec Q une forme quadratique non dégénérée, alors nous appellerons S une fonction génératrice quadratique à l'infini ou *fgqi*. Il s'avère que cette description des lagrangiennes est suffisante pour décrire tous nos L_ψ dans T^*S^{2n} .

Théorème (Laudenbach et Sikorav [LS85, Sik86, Sik87]). *Soit T^*N l'espace cotangent d'une variété compacte et considérons un difféomorphisme hamiltonien $\Phi : T^*N \rightarrow T^*N$. Alors $\Phi(0_N)$ a une *fgqi* S telle que i_S est un difféomorphisme.*

Le fait que i_S soit bijective implique qu'il y a une correspondance bijective entre les points critiques de S et les points d'intersection de L avec la section nulle. Ce théorème d'existence a déjà de fortes implications pour la topologie de ces sous-variétés. Il implique que ce type de sous-variétés intersectent suffisamment la section nulle. Afin de définir les capacités de Viterbo, nous devons pouvoir comparer les fonctions génératrices d'une même lagrangienne. Considérons les opérations suivantes:

1. *Addition d'une constante.* Si $c \in \mathbb{R}$, on pose $S' = S + c : N \times \mathbb{R}^k \rightarrow \mathbb{R}$.
2. *Composition par un difféomorphisme.* Si $\phi : N \times \mathbb{R}^k \rightarrow N \times \mathbb{R}^k$ est un difféomorphisme qui satisfait $\phi(x, \xi) = (x, \varphi(x, \xi))$ alors on pose $S' = S \circ \phi$.
3. *Stabilisation.* Si $Q' : \mathbb{R}^{k'} \rightarrow \mathbb{R}$ est une autre forme quadratique non dégénérée, alors nous définissons $S' = S + Q' : N \times \mathbb{R}^k \times \mathbb{R}^{k'} \rightarrow \mathbb{R}$.

On dit que une *fgqi* S' est équivalente à une autre *fgqi* S si S' peut être obtenue à partir de S par une succession d'opérations ci-dessus. On peut vérifier facilement que pour chacune des opérations précédentes, nous avons

$$\text{Red}_W(dS') = \text{Red}_W(dS),$$

$$\text{CritVal}(S') = \text{CritVal}(S) + c.$$

Le théorème suivant assure que ces trois opérations sont suffisantes pour aller d'une *fgqi* à une autre *fgqi* à condition qu'elles décrivent une isotopie lagrangienne à la section nulle.

Théorème (Viterbo [Vit92] et Théret [Thé99a]). *Soit T^*N l'espace cotangent d'une variété compacte et considérons un difféomorphisme hamiltonien $\Phi : T^*N \rightarrow T^*N$. Alors toutes les *fgqi* S de $\Phi(0_N)$ telles que i_S est un difféomorphisme sont équivalentes.*

Il convient de remarquer que dans l'article de Théret l'hypothèse de que i_S soit un difféomorphisme n'est pas incluse dans la définition de fonction génératrice [Thé99a, Définition 2.1] mais elle est incluse dans la terminologie " S engendre L " [Thé99a, Définition 2.2].

Définition de $c(\mu, \cdot)$ et $c(1, \cdot)$. Puisque les *fgqi* sont de degré un, les points d'intersection de L_ψ avec la section nulle sont en correspondance bijective avec les points critiques de S . Rappelons que ces points d'intersection sont en bijection avec les points fixes de ψ . Après avoir normalisé la *fgqi* pour que la valeur critique à l'infini soit 0, on peut prouver que les valeurs critiques de S coïncident avec le spectre d'action de ψ

$$\text{CritVal}(S) = \sigma(\psi).$$

Pour sélectionner une valeur d'action, il suffit alors de sélectionner une valeur critique de S . Viterbo le fait dans [Vit92] via la théorie de Liusternik-Schilermann. Dans cette théorie, les

valeurs critiques sont considérées comme témoins des changements topologiques des ensembles de niveaux de S . Soit S une *fgqi* qui coïncide avec la forme quadratique Q à l'infini. Notons S^λ l'ensemble des points $(x, \xi) \in N \times \mathbb{R}^K$ tels que $S(x, \xi) \leq \lambda$. Le fait que S coïncide avec Q à l'infini implique que pour $|\lambda|$ suffisamment grand les ensembles de niveau ne dépendent que de Q , en d'autres termes: $S^\lambda = Q^\lambda$. Pour un λ_0 fixe assez grand qui vérifie cette propriété, on définit $S^\infty = S^{\lambda_0}$ et $S^{-\infty} = S^{-\lambda_0}$. Alors la topologie de la paire $(S^\infty, S^{-\infty})$ ne dépend que de l'indice de la forme quadratique et nous avons

$$(S^\infty, S^{-\infty}) \simeq (S^{2n} \times D^-, S^{2n} \times \partial D^-)$$

où D^- est le disque de dimension l'indice de Q . En particulier nous obtenons les isomorphismes en cohomologie sur un corps

$$H^*(S^\infty, S^{-\infty}) \simeq H^*(N) \otimes H(D^-, \partial D^-),$$

et nous pouvons définir une application $T : H^*(N) \rightarrow H^*(S^\infty, S^{-\infty})$ qui est appelé l'isomorphisme de Thom. Afin de trouver les changements dans la topologie des ensembles de niveau, nous déformons la paire $(S^\infty, S^{-\infty})$. Considérons l'inclusion naturelle $i_\lambda : S^\lambda \rightarrow S^\infty$. Nous obtenons une application en cohomologie

$$i_\lambda : H^*(S^\infty, S^{-\infty}) \rightarrow H^*(S^\lambda, S^{-\infty}).$$

Enfin, si on note μ le générateur de $H^{2n}(S^{2n})$ et par 1 le générateur de $H^0(S^{2n})$ nous peut définir deux valeurs:

$$c(\mu, S) = \inf\{\lambda \mid i_\lambda^* T(\mu) \neq 0\} \quad \text{and} \quad c(1, S) = \inf\{\lambda \mid i_\lambda^* T(1) \neq 0\}$$

Ils détectent un ensemble de niveau qui subit un changement topologique, donc ils sont des valeurs critiques de S . On peut alors définir $c(\mu, \psi) = c(\mu, S)$ où S est un *fgqi* normalisé. Cette valeur est bien définie grâce au théorème d'unicité des *fgqi*.

Rigidité coïsothrope

Pour le premier résultat original de cette thèse, nous prouvons un nouveau résultat de rigidité pour une famille large de symplectomorphismes. Au lieu d'étudier l'évolution des boules, comme dans le non-squeezing de Gromov, nous allons étudier l'évolution des cylindres coïsootropes. Pour rappel, (cf. section précédente), tout sous-espace coïsothrope $W \subseteq \mathbb{R}^{2n}$ induit un espace symplectique W/W^ω . On note $\pi : W \rightarrow W/W^\omega$ la projection. La réduction de $Z \subseteq \mathbb{R}^{2n}$ est définie comme

$$\text{Red}_W(Z) = \pi(Z \cap W).$$

Dans le cas particulier de $W = \mathbb{C}^m \times i\mathbb{R}^{n-m} \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$, $\pi = \pi_m$ est la projection sur les m premières coordonnées complexes. Un cylindre coïsothrope est un sous-ensemble de la forme $X \times \mathbb{R}^{n-m} \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$. On peut ainsi écrire $X \times \mathbb{R}^{n-m} = X \times 0_{\mathbb{R}^{n-m}}$ et $W = T^*\mathbb{R}^m \times T_0^*\mathbb{R}^{n-m}$.

Théorème (Non-squeezing coïsothrope). *Soit $X \subseteq \mathbb{R}^{2m}$ un ensemble compact, on considère $X \times \mathbb{R}^{n-m} \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$ et on note $W = \mathbb{C}^m \times i\mathbb{R}^{n-m}$. Pour chaque difféomorphisme hamiltonien à support compact ψ nous avons*

$$c(X) \leq \gamma(\text{Red}_W[\psi(X \times \mathbb{R}^{n-m})]).$$

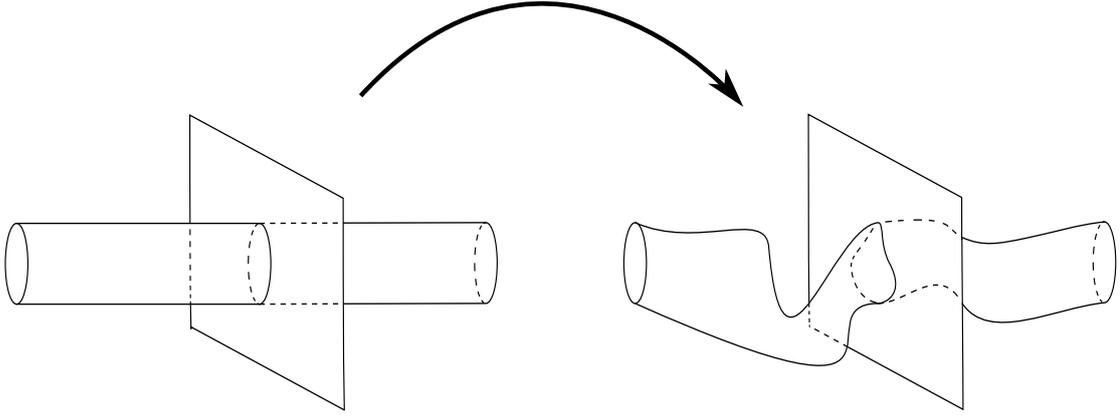


Figure 1: Cette figure représente l'image du cylindre coïsothrope par un difféomorphisme hamiltonien à support compact ψ . Le plan transverse représente le sous-espace coïsothrope complémentaire W . Le théorème précédent donne des informations sur la capacité de la projection de l'intersection avec W .

La démonstration de ce théorème est contenu dans l'article [Bus17]. Explicitement, la réduction de W est $\pi_m(\psi(X \times \mathbb{R}^{n-m}) \cap (\mathbb{C}^m \times i\mathbb{R}^{n-m}))$, qui est la projection d'un ensemble borné. En particulier nous étudions des trajectoires où la composante de $T^*\mathbb{R}^{n-m}$ commence avec un moment nul et arrive à un point précis $T_0^*\mathbb{R}^{n-m}$. Nous soulignons que les capacités de Viterbo ne donnent pas à elles seules des informations de rigidité pour l'image des cylindres coïsootropes puisque

$$c(X \times \mathbb{R}^k) = 0 = \gamma(X \times \mathbb{R}^k).$$

Le cas $X = (S^1)^m$, et où $\text{Red}_W[\psi(X \times \mathbb{R}^{n-m})]$ est contenu dans un cylindre a été prouvé par Buhovski et Opshtein en utilisant la théorie de courbes pseudo-holomorphes. La preuve de notre théorème est obtenue par une série d'inégalités entre les capacités de Viterbo d'ensembles et la réduction symplectique de ces ensembles. L'avantage d'utiliser les capacités de Viterbo est qu'elles sont construites en utilisant des fonctions génératrices, et la réduction symplectique peut être vue comme une opération explicite sur les fonctions génératrices. Cette opération peut être ensuite étudiée en détail. La preuve vient de l'extension des sélecteurs d'action à $\mathbb{R}^{2m} \times T^*\mathbb{T}^{n-m}$.

Sélecteurs d'action sur $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. La construction des sélecteurs d'action de Viterbo s'appuie fortement sur la structure linéaire de l'espace. Nous ne pouvons pas utiliser la même construction pour généraliser ces invariants à chaque variété symplectique, néanmoins nous sommes encore capables de le faire si nous ajoutons un espace cotangent d'un tore.

Notons π la projection naturelle de $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^{2k}$ sur $T^*\mathbb{R}^{2m} \times T^*\mathbb{T}^{2k}$. C'est la projection induite par le quotient de $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^{2k}$ par translation de vecteurs dans \mathbb{Z}^k . Prenons une isotopie hamiltonienne à support compact ψ de $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ et considérons le relevé $\tilde{\psi}$ sur $\mathbb{R}^{2m} \times T^*\mathbb{R}^k$. L'application $\tilde{\psi}$ commute avec l'action de \mathbb{Z}^k par translation, donc les points fixes de $\tilde{\psi}$ sont regroupés dans \mathbb{Z}^k orbites. De plus, on peut voir que ces orbites sont en bijection avec des points fixes contractiles de ψ . On peut définir le spectre d'action contractile de ψ comme

$$\sigma_c(\psi) = \sigma(\tilde{\psi}).$$

Pour chaque difféomorphisme hamiltonien de support compact φ de $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ on peut montrer que

$$\sigma_c(\varphi\psi\varphi^{-1}) = \sigma_c(\psi).$$

Comme dans le cas de \mathbb{R}^{2n} on considère le graphe de $\tilde{\psi}$ et on peut utiliser le symplectomorphisme \mathcal{I} pour le voir comme une sous-variété lagrangienne de $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^{2k}$. Dans ce cas, les points d'intersection avec la section nulle viennent dans des \mathbb{Z}^k orbites où chaque point de l'orbite a la même valeur d'action. Afin de se débarrasser de cette redondance, nous utilisons les propriétés périodiques du flot pour prendre le quotient par cette action et trouver une sous-variété lagrangienne dans $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{T}^k$. Les points d'intersection de cette lagrangienne avec la section nulle sont en bijection avec les points fixes contractiles de ψ . Nous pouvons alors compactifier la base pour obtenir une lagrangienne

$$L_\psi \subseteq T^*S^{2m} \times T^*S^k \times T^*\mathbb{T}^k.$$

Comme expliqué précédemment, cette sous-variété est isotope de façon hamiltonienne à la section nulle donc elle a une *fgqi*. On peut alors utiliser les classes cohomologiques $\alpha \otimes \beta \otimes \gamma \in H^*(S^{2m}) \otimes H^*(S^k) \otimes H^*(\mathbb{T}^k)$ pour sélectionner les valeurs critiques de S . Après normalisation, nous avons à nouveau la propriété très importante

$$\text{CritVal}(S) = \sigma_c(\psi).$$

De la même façon que dans le cadre classique, nous pouvons alors définir une notion de capacité pour les sous-ensembles de $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. Pour les ensembles ouverts bornés U dans $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ on définit

$$c(\alpha \otimes \beta \otimes \gamma, U) = \sup\{c(\alpha \otimes \beta \otimes \gamma, \psi) \mid \psi \in \text{Ham}^c(U)\}.$$

Cette fois ces quantités ne sont pas invariantes par les difféomorphismes symplectiques généraux de $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ mais seulement par les isotopies hamiltoniennes.

Sélecteurs de valeurs critiques et réduction symplectique. L'étape suivante consiste à relier ces nouveaux invariants aux capacités classiques de Viterbo. Ceci est fait via une séquence d'inégalités qui relient les fonctions génératrices à l'opération de réduction symplectique. La remarque clé est que si $S : N \times B \times \mathbb{R}^l \rightarrow \mathbb{R}$ est une fonction génératrice pour $L_S \subseteq T^*N \times T^*B$ et si L_S est transverse au sous-espace coïso trope $W = T^*N \times T_b^*B$ alors $S_b = S(\cdot, b, \cdot)$ engendre $\text{Red}_W(L_S)$. En plus, les sélecteurs de valeurs critiques se comportent bien par rapport à cette opération. Plus précisément pour $\alpha \in H^*(N)$ et $\mu \in H^*(B)$ la classe d'orientation dans B nous avons la chaîne d'inégalités

$$c(\alpha \otimes 1, S) \leq c(\alpha, S_b) \leq c(\alpha \otimes \mu, S).$$

Ce résultat nous permet d'obtenir la relation désirée avec les capacités de Viterbo. Nous obtenons d'abord la non trivialité de l'invariant via l'inégalité suivante:

Proposition. *Si $X \subseteq \mathbb{R}^{2m}$ est compacte alors $c(X) \leq c(\mu \otimes \mu \otimes 1, X \times \{0\} \times \mathbb{T}^k)$.*

Dans cette proposition, il est crucial d'avoir un ensemble avec \mathbb{T}^k tout entier dans la variable de la classe cohomologique 1 afin d'obtenir quelque chose de non trivial. La borne supérieure provient d'une relation avec l'énergie de déplacement de Viterbo:

Proposition. *Soit $Z \subseteq \mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k$ un ensemble compact. Pour $w \in \mathbb{T}^k$ on considère le sous-espace coïso trope $W = \mathbb{R}^{2m} \times \mathbb{R}^k \times \{w\}$. On a*

$$c(\mu \otimes \mu \otimes 1, Z) \leq \gamma(\text{Red}_W(Z)).$$

Afin de prouver le théorème de non-squeezing, nous étendons l'ensemble $X \times \mathbb{R}^k$ et le voyons comme $X \times \{0\} \times \mathbb{T}^k \subset \mathbb{R}^{2m} \times T^*\mathbb{T}^k$. Nous étendons ensuite le difféomorphisme hamiltonien à support compact à tout l'espace $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ et appliquons les propositions précédentes.

Cas non compact

L'étape suivante consiste à étendre le théorème de non-squeezing coïsothrope à des cas où le difféomorphisme hamiltonien n'a pas un support compact. Puisque nous regardons l'évolution d'un ensemble non borné $X \times \mathbb{R}^k \subseteq \mathbb{C}^m \times \mathbb{C}^k$ nous devons avoir un flot globalement défini. Pour cette raison nous restreignons l'étude à la classe des difféomorphismes hamiltoniens qui satisfont

$$|\nabla H(z)| \leq A + B|z| \quad \text{for every } (t, z) \in [0, T] \times \mathbb{R}^{2n}$$

pour deux constantes positives A and B . Ces difféomorphismes hamiltoniens définissent des applications globales ψ_t^H pour tout $t \in [0, T]$. La première remarque est qu'il existe un difféomorphisme hamiltonien de ce type qui ne satisfait pas l'énoncé du théorème: considérons le difféomorphisme

$$\phi(z_1, \dots, z_n) = (z_{m+1}, \dots, z_n, z_1, \dots, z_m)$$

avec $z_j = q_j + ip_j$. C'est un difféomorphisme hamiltonien qui est engendré par une fonction hamiltonienne quadratique. On peut voir facilement que

$$\phi(X \times \mathbb{R}^k) = \mathbb{R}^k \times X \subseteq \mathbb{R}^k \times \mathbb{C}^m$$

de façon que l'image de π_m est également contenue dans un sous-espace linéaire propre. Puisque l'énergie de déplacement associée à un sous-espace linéaire est nulle nous obtenons

$$\gamma(\text{Red}_W[\phi(X \times \mathbb{R}^k)]) = 0.$$

Si l'on considère un cylindre coïsothrope de la forme $X \times \mathbb{R}^k$ avec $c(X) > 0$ alors l'énoncé du non-squeezing coïsothrope n'est pas vrai pour cette application ϕ . Nous prouvons qu'il existe une classe d'applications qui vérifient l'énoncé. Plus précisément, nous prouvons:

Théorème. *Considérons une fonction hamiltonienne de la forme*

$$H_t(z) = \frac{1}{2} \langle Az, z \rangle + U_t(z), \quad \text{avec } |\nabla U(z)| \leq C \quad \text{pour tout } (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}$$

avec un C positive. Supposons que $A = A_1 \oplus \dots \oplus A_n$ où A_k est une application linéaire sur $\text{Vect}\{\frac{\partial}{\partial q_k}, \frac{\partial}{\partial p_k}\}$ pour tout $1 \leq k \leq n$. Soit $X \subseteq \mathbb{R}^{2m}$ un ensemble compact et considérons le sous-espace coïsothrope $W := \mathbb{C}^m \times i\mathbb{R}^{n-m}$. Alors pour tout $t \in \mathbb{R}$ tel que

$$\mathbb{R}^{n-m} + e^{-tJA}i\mathbb{R}^{n-m} = \mathbb{C}^{n-m},$$

nous avons

$$c(X) \leq \gamma(\text{Red}_W[\psi_t(X \times \mathbb{R}^{n-m})]).$$

Un exemple de difféomorphisme hamiltonien qui vérifie les hypothèses est la classe des flots définis à partir des fonctions hamiltoniennes mécaniques de la forme

$$H_t(q, p) = \frac{1}{2}|p|^2 + U_t(q), \quad \text{avec } |\nabla U_t(z)| \leq C.$$

En particulier, le théorème est vrai si U est un potentiel périodique en q .

Rigidité coïsothrope sur $T^*\mathbb{T}^n$

En utilisant cette extension non-compacte, nous pouvons regarder le véritable cas périodique $T^*\mathbb{T}^n$. Dans ce cas, nous prouvons que nous avons toujours ce comportement dans le cas des difféomorphismes hamiltoniens à support compact.

Théorème. *Soit ψ un difféomorphisme hamiltonien à support compact de $T^*\mathbb{T}^n = T^*\mathbb{T}^m \times T^*\mathbb{T}^{n-m}$. Pour tout $w \in \mathbb{T}^{n-m}$ on note $W_f = T^*\mathbb{T}^m \times T_w^*\mathbb{T}^{n-m}$ et $W_0 = T^*\mathbb{T}^m \times 0_{\mathbb{T}^{n-m}}$. Soit B_r une boule de rayon r dans $T^*\mathbb{T}^m$. Alors l'inclusion*

$$\text{Red}_{W_f}[\psi(B_r \times 0_{\mathbb{T}^{n-m}})] \subseteq B_R \quad \text{ou} \quad \text{Red}_{W_0}[\psi(B_r \times T_w^*\mathbb{T}^{n-m})] \subseteq B_R$$

entraîne $r \leq R$.

Ce théorème est également vrai pour les difféomorphismes hamiltoniens mécaniques. Il indique également que ce type de rigidité coïsothrope peut apparaître dans le cadre des produits d'espaces cotangent de bases compactes. Par boule symplectique de rayon r dans une variété symplectique (M, ω) on entend une boule de rayon r dans \mathbb{R}^{2n} qui est plongée de façon symplectique dans M . En particulier, nous pouvons conjecturer le comportement suivant:

Conjecture. Soit N et M deux variétés compactes et ψ une isotopie hamiltonienne compacte de $T^*N \times T^*M$. Pour tout $m \in M$ on note $W_f = T^*N \times T_m^*M$ et $W_0 = T^*N \times 0_M$. Soit B_r une boule symplectique de rayon r dans T^*N . Alors l'inclusion

$$\text{Red}_{W_f}[\psi(B_r \times 0_M)] \subseteq B_R \quad \text{ou} \quad \text{Red}_{W_0}[\psi(B_r \times T_m^*M)] \subseteq B_R$$

implique $r \leq R$.

Rigidité symplectique de dimension intermédiaire

L'une des conséquences du théorème non-squeezing coïsothrope est son interprétation dans le contexte du problème de rigidité de dimension intermédiaire. Une des premières questions concernant ce problème est apparue dans [Hof90b] où Hofer s'est interrogé sur la généralisation des capacités aux dimensions intermédiaires. Les capacités symplectiques sont une sorte de mesure bidimensionnelle des ensembles car, du point de vue des capacités symplectiques, tous les cylindres d'un même rayon mais avec une base de dimensions différentes, sont exactement identiques. Afin d'essayer de voir si ces cylindres sont différents pour la géométrie symplectique, Hofer a demandé s'il existe une capacité symplectique k -intermédiaire c^k satisfaisant la monotonie, la k -conformité, l'invariance symplectique et telle que

$$c^k(B_1^{2k} \times \mathbb{R}^{2n-2k}) < +\infty \quad \text{mais} \quad c^k(B_1^{2k-2} \times \mathbb{R}^2 \times \mathbb{R}^{2n-2k}) = +\infty?$$

La première inégalité est déjà vérifiée par les capacités standards et c'est la deuxième qui diffère. L'un des premiers résultats indiquant que les capacités intermédiaires n'existaient pas est apparu dans une publication de Guth [Gut08]. Il s'est intéressé à la question des plongements des polydisques $P = B_{R_1}^2 \times \cdots \times B_{R_n}^2$ avec $R_1 \leq \cdots \leq R_n$ dans un autre $P' = B_{R'_1}^2 \times \cdots \times B_{R'_n}^2$ avec $R'_1 \leq \cdots \leq R'_n$ en utilisant une application symplectique. Il y a deux obstacles évidents à ce problème. Le premier provient du non-squeezing de Gromov qui implique $R_1 \leq R'_1$. Le deuxième est le volume qui implique que $R_1 \cdots R_n \leq R'_1 \cdots R'_n$. En utilisant les techniques de l'article de Traynor [Tra95] on peut voir que l'on pourrait plonger P dans P' dès que $R_1 \cdots R_k \lesssim R'_1 \cdots R'_k$ pour chaque k entre 1 et n . Guth a montré que, modulo une constante dimensionnelle, $k = 1$ et $k = n$ sont les deux seules obstructions. Plus précisément:

Théorème (Guth 2008 [Gut08]). *Il existe une constante dimensionnelle $C(n)$ telle que si $C(n)R_1 \leq R'_1$ et $C(n)R_1 \cdots R_n \leq R'_1 \cdots R'_n$ alors P se plonge symplectiquement dans P' .*

Ce théorème a donné une réponse partielle à la question de Hofer. Cela implique que si $1 < k < n$ alors k -capacités qui vérifient l'hypothèse de continuité suivante :

$$\lim_{R \rightarrow +\infty} c(B_1^{2k} \times B_R^{2n-2k}) < +\infty \quad \text{et} \quad \lim_{R \rightarrow +\infty} c(B_1^{2k-2} \times B_R^{2n-2k+2}) = +\infty$$

n'existe pas. Son théorème était quasiment une réponse définitive, mais la question des capacités moins régulières restait ouverte. Cela a été récemment résolu par la négative de Pelayo et Vũ Ngọc dans [PVuN15]. Ils ont appliqué un argument limite à la construction de Guth et Hind-Kerman [HK14] afin de prouver la version non-bornée du théorème de Guth.

Théorème (Pelayo-Vũ Ngọc 2015 [PVuN15]). *Si $n \geq 2$ alors le cylindre $B_1^2 \times \mathbb{R}^{2n-2}$ peut être plongé de façon symplectiquement dans le produit $B_R^{2n-2} \times \mathbb{R}^2$ pour tout $R \geq \sqrt{2^{n-1} + 2^{n-2} - 2}$.*

Avec ce théorème, la question des capacités symplectiques intermédiaires était définitivement résolue. Ils ont montré que les difféomorphismes symplectiques généraux sont trop flexibles pour capturer ce type de rigidité cylindrique. Un autre point de vue sur le problème de la dimension intermédiaire vient d'une reformulation du théorème non-squeezing de Gromov. En dimension 2 les symplectomorphismes sont les mêmes que les applications préservant les aires donc Eliashberg et Gromov [EG91] ont souligné que (en utilisant un théorème de Moser sur l'existence de difféomorphismes conservant l'aire) le théorème de Gromov est équivalent à

$$\text{area}(\Pi_1 \phi(B_r^{2n})) \geq \pi r^2 \quad \text{pour tout symplectomorphisme } \phi.$$

On note Π_k la projection sur les $2k$ premières coordonnées. Une généralisation possible de cet énoncé aux dimensions supérieures est

$$\text{Vol}(\Pi_k \phi(B_r^{2n})) \geq \text{Vol}(\Pi_k B_r^{2n}) = \text{Vol}(B_r^{2k}) \quad \text{pour chaque symplectomorphisme } \phi.$$

Ce problème a été étudié par Abbondandolo et Matveyev dans [AM13]. Dans leur article, ils ont d'abord prouvé que l'inégalité est vraie dans le cas linéaire:

Théorème (Abbondandolo-Matveyev 2013 [AM13]). *Soit Φ automorphisme symplectique linéaire de \mathbb{R}^{2n} , et soit $P : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ soit la projection orthogonale sur un sous-espace linéaire complexe $V \subseteq \mathbb{R}^{2n}$ de dimension $2k$, $1 \leq k \leq n$. Alors*

$$\text{Vol}(P\Phi(B_r^{2n})) \geq \text{Vol}(B_r^{2k})$$

avec égalité si et seulement si le sous-espace linéaire $\Phi^T V$ est complexe.

Contrairement au résultat linéaire, ils prouvent que les difféomorphismes symplectiques sont à nouveau trop flexibles pour avoir cette sorte de rigidité symplectique de dimension intermédiaire. Plus précisément, cela montre qu'on peut écraser arbitrairement la projection symplectique de l'image de la boule par des symplectomorphismes.

Théorème (Abbondandolo-Matveyev 2013 [AM13]). *Soit $P : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ la projection orthogonale sur un sous-espace linéaire complexe $V \subseteq \mathbb{R}^{2n}$ de dimension $2k$, $1 < k < n$. Pour chaque $\epsilon > 0$ il existe une application symplectique lisse $\phi : B_1^{2n} \rightarrow \mathbb{R}^{2n}$ telle que*

$$\text{Vol}(P\phi(B_r^{2n})) < \epsilon.$$

Malgré tous ces résultats négatifs, il y a eu un résultat récent de Rigolli dans [Rig15] qui prouve qu'il y a aussi une rigidité intermédiaire locale si l'on restreint la classe des symplectomorphismes aux analytiques.

Théorème (Rigolli 2015 [Rig15]). *Soit $\phi_t : \overline{B}_1^{2n} \rightarrow \mathbb{R}^{2n}$ un chemin analytique des plongements symplectiques, avec $t \in [0, 1]$, tel que ϕ_0 est linéaire. Alors l'inégalité de non-squeezing de dimension intermédiaire*

$$\text{Vol}(P\phi_t(B_1^{2n})) \geq \text{Vol}(B_r^{2k})$$

est vérifiée pour tout t assez petit.

Ce résultat indique que cette rigidité intermédiaire pourrait être possible en restreignant la classe des difféomorphismes symplectiques que nous considérons. De plus, ce résultat a une autre implication locale:

Théorème (Rigolli 2015 [Rig15]). *Pour chaque inclusion analytique symplectique d'un domaine D il existe une fonction $r_0 : D \rightarrow (0, +\infty)$ telle que l'inégalité à dimension intermédiaire*

$$\text{Vol}(P\phi(B_r^{2n}(x))) \geq \text{Vol}(B_r^{2k})$$

est valable pour tout $x \in D$ et pour tout $r < r(x)$. De plus, r_0 est borné inférieurement par une constante positive sur des sous-ensembles compacts $K \subseteq D$.

Nous aimerions souligner une autre généralisation possible du problème de dimensions intermédiaires. Dans la dimension 2, la valeur des capacités symplectique des disques topologiques coïncide avec l'aire, donc on peut aussi réécrire le théorème de Gromov comme

$$c(\Pi_1\phi(B_r^{2n})) \geq \pi r^2 \quad \text{pour tout symplectomorphisme } \phi,$$

où c est une capacité symplectique. On peut alors se demander si cette inégalité est vraie avec Π_1 remplacé par Π_k , et plus généralement, s'intéresser aux sous-ensembles Z différents de B_r^{2n} et remplacer πr^2 par la capacité de $\Pi_k Z$. Les résultats précédents montrent qu'il y a peu d'espoir qu'une telle inégalité soit toujours vraie, il faut donc restreindre le type de symplectomorphismes considérés. Notre théorème de non-squeezing coïsothrope implique une rigidité de dimension intermédiaire pour les déformations des cylindres coïsootropes $Z = X \times \mathbb{R}^{n-m} \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$ à condition de restreindre la classe de symplectomorphismes aux isotopies hamiltoniennes à vitesse bornée.

Théorème. *Soit $X \subset \mathbb{R}^{2m}$ un ensemble compact. On considère $X \times \mathbb{R}^{n-m} \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$ et soit ψ un difféomorphisme hamiltonien engendré par une fonction avec un gradient borné. Alors*

$$c(X) \leq \gamma(\Pi_m\psi(X \times \mathbb{R}^{n-m})).$$

On remarque que, contrairement à ce qui se passe dans le problème de rigidité intermédiaire du volume, cette affirmation n'est pas vraie pour tout symplectomorphisme linéaire. Comme il est habituel en géométrie symplectique, ce résultat de rigidité peut être utilisé pour définir un invariant. L'exemple classique de ce fait est la définition de la largeur de Gromov après avoir prouvé le théorème non-squeezing. Dans notre cas, nous devrions considérer la quantité suivante:

$$\gamma_G^m(U) = \inf\{\gamma(\Pi_m\phi(U)) \mid \phi \in G\}$$

où G est un sous-groupe du groupe des difféomorphismes symplectiques. Pour $G = \text{Sympl}(\mathbb{R}^{2n})$ on sait que γ_G^m est nul dans les cylindres coïsootropes de dimension m . En particulier, il pourrait

être trivial pour chaque sous-ensemble de \mathbb{R}^{2n} . D'autre part, si les éléments de G sont des difféomorphismes hamiltoniens engendrés par des fonctions à gradient borné, alors le théorème précédent implique que γ_G^m est borné inférieurement sur les cylindres coisotropes de dimension m . A titre d'exemple de G on peut prendre le sous-groupe des difféomorphismes hamiltoniens φ_t^H où H , φ_t^H , et $(\varphi_t^H)^{-1}$ sont Lipschitz sur la variable d'espace sur des intervalles de temps compacts. Pour ce sous-groupe, le théorème précédent donne

$$c(X) \leq \gamma_G^m(X \times \mathbb{R}^{n-m}) \leq \gamma(X).$$

Plus précisément γ_G^m possède les propriétés suivantes:

1. (monotonie) Si $U \subseteq V$ donc $\gamma_G^m(U) \leq \gamma_G^m(V)$.
2. (conformalité) $\gamma_G^m(\lambda U) = \lambda^2 \gamma_G^m(U)$ pour tout $\lambda \in \mathbb{R}$.
3. (G invariance) Si $\phi \in G$ on a $\gamma_G^m(\phi(U)) = \gamma_G^m(U)$.
4. (non-trivialité+normalisation) $\gamma_G^m(B_r^{2m} \times \mathbb{R}^{n-m}) = \pi r^2$.

Enfin, on veut souligner que cet invariant satisfait une propriété 2-conformalité au lieu de la k -conformalité demandée par Hofer pour les capacités symplectiques de dimension intermédiaire

Rigidité sur les surfaces d'énergie

Dans la partie suivante de cette thèse, nous étudions les propriétés de rigidité du flot caractéristique des surfaces d'énergie. Soit (M, ω) une variété symplectique et soit $H : M \rightarrow \mathbb{R}$ une fonction hamiltonienne lisse. Chaque valeur régulière λ définit une hypersurface lisse Σ de M . Les propriétés du flot ψ^H sur Σ ont toujours été un sujet de recherche important (voir par exemple [HZ94, Chapitre 4]). A titre d'exemple, nous rappelons l'étude des orbites périodiques dans les surfaces d'énergie. Nous nous intéressons aux sous-ensembles compacts de Σ qui sont invariants par le flot. Ceux-ci incluent les orbites périodiques et aussi d'autres types de sous-ensembles de dimension supérieure. Nous utilisons le théorème du chameau symplectique pour créer des sous-ensembles invariants compacts dans les déformations de Σ . Rappelons que l'énoncé du théorème du chameau symplectique est:

Théorème ([Vit92] et aussi [MT93] pour $\dim = 4$). *On considère l'espace $E = \mathbb{R}^{2n-1} \times \{0\} \setminus B^{2n}(0, r)$ et on suppose qu'il existe une isotopie symplectique φ^t de \mathbb{R}^{2n} à support dans $\mathbb{R}^{2n} \setminus E$ tel que*

$$B_R^{2n} \subseteq \{p_n < 0\} \quad \text{et} \quad \varphi^1(B_R^{2n}) \subseteq \{p_n > 0\}.$$

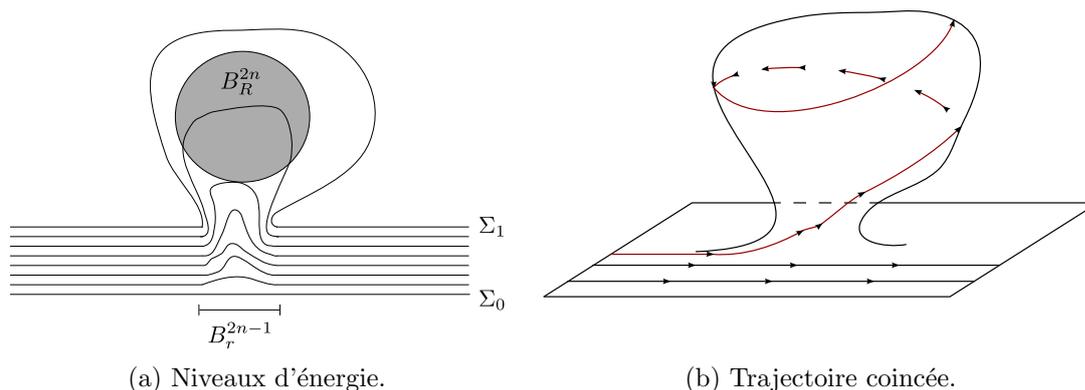
Alors $r \geq R$.

Il y a aussi une démonstration non publiée de ce résultat par Gromov et Eliashberg.

Localement nous pouvons trouver un ouvert assez petit $U \subseteq M$, et un petit voisinage ouvert de l'origine dans \mathbb{R}^{2n} tel que nous avons un symplectomorphisme

$$U \cap \Sigma \simeq V \cap \mathbb{R}^{2n-1} \times \{0\}.$$

C'est pour cela qu'on va regarder des déformations de $\mathbb{R}^{2n-1} \times \{0\}$ contenus dans V avec les propriétés désirés. On définit d'abord le concept de fonction (r, R) -simple (voir Figure 2 (a)) pour laquelle les niveaux d'énergie sont des déformations compactes de $\mathbb{R}^{2n-1} \times \{\lambda\}$. Puis on définit les perturbations simples: perturbations C^0 dans la classe des fonctions simples.



(a) Niveaux d'énergie.

(b) Trajectoire coincée.

Figure 2: (a) Les niveaux d'énergie entre 0 et 1 pour une fonction (r, R) -simple avec $r < R$. (b) Une représentation d'une trajectoire coincée dans un sous-ensemble borné. Cette trajectoire produit un sous-ensemble invariant compact.

Dans cette section on montre d'abord qu'on peut déformer facilement Σ pour produire des sous-ensembles invariants compacts et puis on démontre en utilisant le théorème du chameau symplectique que ces sous-ensembles persistent sous des perturbations C^0 de la fonction H . Plus précisément:

Théorème. *Soit $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ une fonction (r, R) -simple avec $r < R$. Alors il existe un $\lambda \in [0, 1]$ tel que Σ_λ a un sous-ensemble invariant compact. En conséquence, pour toute perturbation ϵ -simple H_ϵ of H avec $\epsilon \in [0, \frac{R-r}{2}[$ il existe un $\lambda_\epsilon \in [0, 1]$ tel que $\Sigma_\lambda^\epsilon = H_\epsilon^{-1}(\lambda_\epsilon)$ a un sous-ensemble invariant compact.*

La preuve est par l'absurde. S'il n'y a pas de sous-ensemble invariant compact, alors chaque trajectoire caractéristique va d'un côté de l'espace à l'autre (voir Figure 2). Dans ce cas, nous pouvons construire une isotopie symplectique qui vérifie l'hypothèse du théorème du chameau symplectique mais qui contredit la conclusion. Puisque ce n'est pas possible, nous concluons qu'il doit y avoir une trajectoire qui est coincée dans un ensemble borné. Cette trajectoire doit alors être arbitrairement proche d'un sous-ensemble invariant compact.

EDPs hamiltoniennes

Dans la deuxième partie de ce manuscrit, nous étudions les propriétés de rigidité symplectique dans les espaces de Hilbert de dimension infinie. De nombreuses équations en dérivées partielles (EDPs) apparaissant en physique peuvent être vues comme des systèmes hamiltoniens de dimension infinie

$$\dot{u} = J\nabla H_t(u).$$

Ici, la fonction hamiltonienne H est définie sur un espace de Hilbert de dimension infinie. J est un opérateur anti-symétrique non dégénéré qui définit une structure symplectique via le produit scalaire. Les principaux exemples d'EDP hamiltoniennes sont:

1. Equation d'onde non-linéaire

$$u_{tt} - \Delta u + f(t, x, u) = 0,$$

2. Equation de Schrödinger non-linéaire

$$iu_t - \Delta u + f(t, x, |u|^2)u = 0,$$

3. Equation des membranes

$$u_{tt} + \Delta^2 u + f(t, x, u) = 0,$$

4. Equation de Korteweg-de Vries (KdV)

$$u_t - uu_x + u_{xxx} = 0.$$

Contrairement à ce qui se passe dans les dimensions finies, la fonction hamiltonienne H_t utilisée pour définir le champ vectoriel n'est pas définie sur tout l'espace de phase symplectique E , mais seulement sur un sous-espace dense de E . Cela implique également que le champ de vecteurs ne sera défini que sur un sous-espace dense. En particulier, avant d'examiner les propriétés symplectiques du flot, il faut d'abord se demander quelle est la définition précise du flot, et en particulier quelle est la définition d'une solution. Une façon de résoudre ce problème est de regarder des équations semi-linéaires, c'est-à-dire, des équations dont la fonction hamiltonienne peut être écrite comme $H_t(u) = \frac{1}{2}\langle Au, u \rangle + h_t(u)$. Ici A est un opérateur linéaire possiblement non borné et h_t est une fonction lisse définie sur tout l'espace E . Par exemple, dans le cas de l'équation de Schrödinger, l'espace de Hilbert est $E = L^2(\mathbb{T}^d, \mathbb{C})$, l'opérateur est $A = \Delta$ et h est défini par une intégrale en utilisant f . Pour ce type d'équations, on peut utiliser la formule de Duhamel pour définir des solutions. Ceci donne la notion de solution généralisée, et (dans des conditions de régularité sur la non-linéarité) les résultats d'existence et d'unicité peuvent être dérivés en utilisant des arguments de points fixes.

Rigidité Symplectique

Dans le cas de dimension infinie, on connaît peu les propriétés de rigidité des symplectomorphismes généraux. L'étude des propriétés de rigidité symplectique pour les EDPs a commencé avec Kuksin en 1995 [Kuk95]. Il s'est intéressé à un type d'équations qui sont une perturbation compacte d'une application linéaire symplectique. La propriété clé de ces applications symplectiques est qu'ils peuvent être approximés sur des ensembles bornés par des flots hamiltoniens de dimension finie. Cette approximation lui a permis de construire un analogue infini de la capacité symplectique définie par Hofer-Zehnder. Cette capacité est construite comme une sorte de limite de la dimension finie, et hérite ainsi en grande partie les propriétés des capacités symplectiques. En particulier, elle vérifie la propriété de normalisation, de sorte qu'on peut l'utiliser pour dériver un analogue du théorème de non-squeezing. Cependant, contrairement à la capacité en dimension finie, la capacité de Kuksin n'est pas invariante par des symplectomorphismes généraux de l'espace symplectique de Hilbert, mais seulement par ceux qui sont une perturbation compacte d'une application linéaire symplectique, et où l'application linéaire est bien adaptée à la base qui est utilisé pour construire la capacité.

Peu après ce travail, plusieurs auteurs se sont intéressés au comportement non-squeezing dans des flots plus généraux. Bourgain a été le premier à prouver le non-squeezing pour un flot qui n'est pas une perturbation compacte d'une application linéaire [Bou94]. Il a étudié l'équation de Schrödinger non linéaire cubique

$$i\partial_t u = -\Delta u + |u|^2 u,$$

sur le tore. Encore une fois, la technique consistait à approximer d'une certaine façon le flot de l'équation par des flots hamiltoniens en dimension finie afin d'appliquer le théorème de Gromov. Ensuite, Colliander et ses collaborateurs [CKS⁺05] ont prouvé, en utilisant à nouveau une approximation en dimension finie, que l'équation KdV sur $H^{-\frac{1}{2}}(\mathbb{T})$ a aussi une propriété non-squeezing. Encore une fois, le principal défi de la preuve est de prouver l'approximation. D'autres équations où le non-squeezing a été prouvée sont : l'équation de Klein-Gordon périodique, par

Mendelson [Men17] et l'équation de BBM périodique [Rou10]. Il est important de citer l'article de Killip-Visan-Zhang [KVZ16] où ils ont donné la première preuve du non-squeezing dans un volume infini. Ils considèrent de nouveau le NLS cubique sur \mathbb{R}^2 et montrent qu'il a une bonne approximation par des flots de dimension finie.

Toutes les démonstrations précédentes s'appuient sur la structure particulière des équations. Un point de vue différent est venu dans l'article [AM15] d'Abbondandolo et Majer. Ils ont étudié la propriété de non-squeezing pour des difféomorphismes symplectiques arbitraires d'un espace de Hilbert séparable en utilisant une approche géométrique. Ils ont prouvé que l'écrasement ne peut pas se produire au moins pour les images convexes de la boule. Ils l'ont fait en construisant une capacité pour les ensembles convexes en utilisant le principe d'action dual. Leur argument est le premier à ne pas être basé sur l'approximation par des flots de dimension finie. Nous citons également aussi les articles de Sukhov et Tumanov [ST16a, ST16b] où ils donnent une nouvelle preuve du non-squeezing de Gromov et l'utilisent ensuite pour explorer les limites de cette preuve dans le contexte des espaces de Hilbert.

Capacités Symplectiques

Le premier résultat de cette partie de la thèse est une construction de capacités en dimensions infinies, différente de celle donnée par Kuksin dans [Kuk95] mais aussi basée sur l'approximation en dimension finie. Nous nous limitons aux EDPs semi-linéaires du type décrit dans [Kuk95]. La structure symplectique sur E est définie en utilisant le produit scalaire et une structure presque complexe J . Les fonctions hamiltoniennes sont de la forme

$$H_t(u) = \frac{1}{2} \langle Au, u \rangle + h_t(u),$$

où A est un opérateur linéaire auto-adjoint (peut être non-borné) et h_t est une fonction lisse. Le champ de vecteurs hamiltonien est

$$X_H(u) = JAu + J\nabla h_t(u)$$

Si e^{tJA} est borné, les solutions peuvent être définies à partir de la formule de Duhamel. On note $\{\varphi_j^\pm \mid j \geq 1\}$ une base orthonormée de E et on suppose que $J\varphi_j^\pm = \pm\varphi_j^\mp$ et que A est diagonal pour cette base. Soit E_n le sous-espace de dimensions finie engendré par $\{\varphi_j^\pm \mid 1 \leq j \leq n\}$ et Π_n la projection naturelle sur E_n . Sous des hypothèses de compacité sur la non-linéarité, les flots sont des symplectomorphismes qui peuvent être approchés sur des ensembles bornés par les flots hamiltoniens de dimension finie engendrés par la projection du champ de vecteurs sur E_n . Des exemples spécifiques de ce type d'équations sont (voir [Kuk95]): équation de cordes non linéaire dans \mathbb{T} avec une non linéarité qui a au plus une croissance polynomiale à l'infini, équation d'onde non linéaire quadratique dans \mathbb{T}^2 , équation des membranes non-linéaire sur \mathbb{T}^2 avec une non-linéarité qui a au plus une croissance polynomiale à l'infini et l'équation de Schrödinger avec une non-linéarité de convolution dans \mathbb{T}^n .

Definition d'une capacité Comme dans le cas de dimension finie, pour calculer la capacité d'un ensemble, on examine l'action des orbites périodiques associées aux fonctions avec support dans cet ensemble. Dans les dimensions infinies, on se doit d'être plus prudents car une fonction avec un support borné peut ne pas induire un flot hamiltonien globalement défini. On considère un sous-ensemble ouvert borné $\mathcal{U} \subseteq E$. On définit l'ensemble de la façon suivante

$$\mathcal{F}(\mathcal{U}) = \{ \text{fonctions lisses } f : E \rightarrow \mathbb{R} \text{ telles que } \text{Supp } f \subseteq \mathcal{U} \text{ et } d(\text{Supp } f, \partial\mathcal{U}) > 0 \}.$$

On considère le sélecteur d'action $c(\mu, \cdot)$ défini au début de l'introduction et on écrit $c(\mu, H) = c(\mu, \psi_1^H)$. On note $f_n = f|_{E_n}$ et on définit

$$c_o(\mu, f) = \liminf_n c(\mu, f_n) \quad \text{et} \quad c^o(\mu, f) = \limsup_n c(\mu, f_n)$$

On remarque que le support de f_n est contenu dans $\text{Supp } f \cap E_n$ qui est borné, donc f_n a un support compact. En particulier $c(\mu, f_n)$ est bien défini. De plus, on remarque que pour chaque n il existe une orbite périodique $z_n : [0, 1] \rightarrow E_n$ tel que $c(\mu, f_n)$ est l'action de z_n , mais ces orbites peuvent ne pas converger. On utilise $c(\mu, f)$ pour définir la capacité des ensembles ouverts bornés \mathcal{U} comme

$$c_o(\mathcal{U}) = \sup\{c_o(\mu, f) \text{ pour } f \in \mathcal{F}(\mathcal{U})\} \quad \text{et} \quad c^o(\mathcal{U}) = \sup\{c^o(\mu, f) \text{ pour } f \in \mathcal{F}(\mathcal{U})\},$$

Il s'avère que cette définition est invariante par le flot des équations précédentes et possède les propriétés des capacités symplectiques. En particulier, il profite de la normalisation et on peut donc l'utiliser pour dériver le théorème de non-squeezing:

Théorème (Kuksin [Kuk95]). *Si $\Phi : E \rightarrow \mathbb{R}$ est le flot d'une EDP hamiltonienne satisfaisant la condition de compacité, alors $\Phi(B_r) \subseteq Z_R$ implique $r \leq R$.*

La construction originale d'une capacité symplectique de dimension infinie par Kuksin était une généralisation de la capacité de Hofer-Zehnder. Il utilise la notion d'orbites périodiques rapides des systèmes hamiltoniens: orbites de période $T \leq 1$ qui ne sont pas des points fixes. La capacité de \mathcal{U} est alors (grossièrement) définie comme la variation maximale d'un difféomorphisme hamiltonien qui ne produit pas de trajectoires rapides. Comme dans notre cas, la définition implique une sorte de limite de la capacité à mesure que la dimension croit.

Rigidité en dimensions intermédiaires

Le deuxième résultat de ce chapitre concerne l'étude de la rigidité en dimension intermédiaire pour les équations de Kuksin. La première étape consiste à définir le cylindre coïsothrope dans ce contexte. Nous expliquons le résultat pour l'exemple concret de l'équation des cordes non linéaire.

$$\ddot{u} = u_{xx} - f(t, x, u), \quad u = u(t, x),$$

où $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Si on note par B l'opérateur $B = (-\partial^2/\partial x^2 + 1)^{1/2}$ alors on peut écrire l'équation sous la forme

$$\begin{aligned} \dot{u} &= -Bv, \\ \dot{v} &= (B - B^{-1})u + B^{-1}f(t, x, u). \end{aligned}$$

On définit $E = H^{\frac{1}{2}}(\mathbb{T}) \times H^{\frac{1}{2}}(\mathbb{T})$ comme le produit des espaces de Hilbert. Maintenant, on écrit $A = (B - B^{-1}) \times B$ et on définit $J : E \rightarrow E$ par $J(u, v) = (-v, u)$ on peut écrire l'équation de cordes non-linéaire comme une EDPs semi-linéaire dans E comme

$$(\dot{u}, \dot{v}) = JA(u, v) + J\nabla h_t(u, v).$$

On suppose que f est borné et que toutes ses dérivées sont à croissance polynomiale. On note $E = E_+ \times E_- = H^{\frac{1}{2}}(\mathbb{T}) \times H^{\frac{1}{2}}(\mathbb{T})$. On écrit un élément de $E = E^+ \times E^-$ dans la base de Fourier et on note par E_k le sous-espace de dimension finie de E dont les éléments (u, v) sont une combinaison linéaire de termes de fréquences allant jusqu'à k . On note par $\Pi_k : E \rightarrow E_k$ la projection naturelle et soit $X \times E_+^k$ le sous-espace de E formé d'éléments (u, v) qui se projettent sur X sous Π_k et

tels que $(0, v) \in E_k$. Ceci est la définition naturelle d'un cylindre coïsothrope en dimension infinie. Comme dans le cas de dimension finie, les capacités précédemment construites ne donnent pas d'information de rigidité sur ces cylindres coïsootropes. On note que le sous-ensemble $X \times E_+^k$ n'est pas borné, donc nous ne pouvons pas déduire directement la rigidité à partir de l'approximation en dimension finie sur les ensembles bornés. Nous avons besoin d'une étape intermédiaire et cela vient du théorème de non-squeezing coïsothrope. Soit $\Phi : E \rightarrow E$ le flot de l'équation de cordes non linéaire, on le décompose en

$$\Phi^t(u) = e^{tJA}V^t(u).$$

L'application V^t peut être approchée sur des ensembles bornés par une application dimension finie $V_n^t(u_n, u^n) = (\varphi_n^t(u_n), u^n)$ (par les hypothèse sur f) où φ_n^t vérifie la propriété du non-squeezing coïsothrope (car f est bornée). Le fait que V_n^t soit l'identité sur les grands modes de Fourier permet de prouver que si ∇h est borné sur l'ensemble de l'espace, alors l'ensemble

$$V^t(X \times E_+^k) \cap \{\Pi_+^k = 0\}$$

n'est pas seulement borné, mais est aussi compact. Cela nous permet d'avoir un contrôle précis sur cet sous-ensemble. En particulier, nous prouvons que cette intersection est symplectiquement rigide:

Proposition. *On note $\Phi^t : E \rightarrow E$ le flot de l'équation de cordes non linéaire avec f bornée et ses dérivées avec croissance polynomiale. Pour chaque $k \in \mathbb{N}$, pour chaque ensemble compact X de E_k , et pour tout $t \in \mathbb{R}$ on a*

$$c(X) \leq \gamma(\Pi_k(V^t(X \times E_+^k) \cap \{\Pi_+^k = 0\})).$$

Une fois que nous avons ceci, nous pouvons utiliser l'invariance symplectique des capacités et le fait que e^{tJA} est une application symplectique qui commute avec Π_k pour obtenir le résultat suivant :

Théorème. *On note $\Phi^t : E \rightarrow E$ le flot de l'équation de cordes non linéaire avec f bornée et ses dérivées avec croissance polynomiale. Pour chaque $k \in \mathbb{N}$, pour chaque ensemble compact X de E_k , et pour tout $t \in \mathbb{R}$ on a*

$$c(X) \leq \gamma(\Pi_k \Phi^t(X \times E_+^k)).$$

La démonstration de ce théorème est contenu dans l'article [Bus17]. Un exemple particulier d'équation qui vérifie ces l'hypothèses est l'équation de Sine-Gordon:

$$u_{tt} = u_{xx} - \sin u, \quad u = u(t, x),$$

où $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Pour $k = 0$ l'ensemble $X \times E_+^k$ est constitué d'éléments (u, v) où la paire (u_0, v_0) des valeurs moyennes est contenu dans $X \subset \mathbb{R}^2$ et telles que $(0, v) \in E_0$, c'est-à-dire que, v est une constante égale à v_0 . Nous voyons que ce théorème donne des informations sur le comportement global des solutions à vitesse initiale constante et telle que la paire des valeurs moyennes est en X . D'un autre côté, si nous échangeons les rôles de E^+ et E^- nous obtenons des informations sur le comportement des solutions où la position initiale de la corde est constante.

Conjecture d'Arnold pour EDPs hamiltoniennes

Toujours dans la recherche de la rigidité symplectique dans les EDPs hamiltoniennes, nous attirons notre attention sur la conjecture d'Arnold. Cette conjecture affirme que les difféomorphismes hamiltoniens ont toujours des points fixes. Plus précisément:

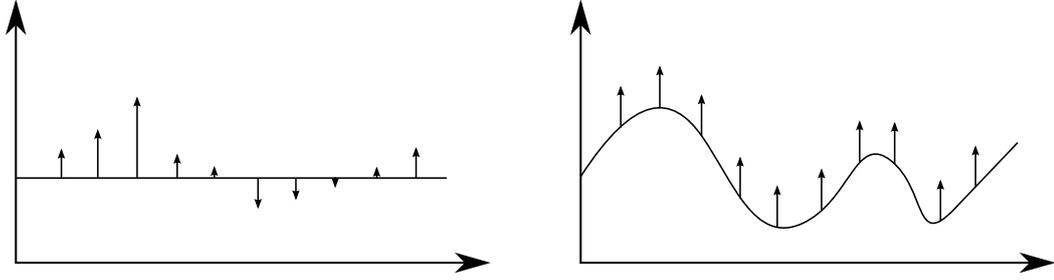


Figure 3: Le graphe de la condition initiale $u(0, x)$ et à chaque point le vecteur de vitesse initiale $v(0, x)$. A gauche, $u(0, x)$ est constant. A droite, $v(0, x)$ est constant.

Conjecture. Tout difféomorphisme hamiltonien sur une variété symplectique compacte (M, ω) a au moins autant de points fixes qu'une fonction sur M a des points critiques.

La version non-dégénérée de cette conjecture a été établie sur chaque variété symplectique. La preuve a donné lieu à l'homologie de Floer, une théorie homologique construite à partir de points critiques de la fonctionnelle d'action. Un exemple particulier où cette conjecture est vraie, est l'espace projectif complexe $\mathbb{C}P^n$ où le nombre de points fixes est $n + 1$. La preuve originale de ce cas [For85] n'utilise pas les techniques de Floer mais profite de la symétrie de $\mathbb{C}P^n$ pour étudier la fonctionnelle d'action pour des difféomorphismes hamiltoniens dans \mathbb{R}^{2n} .

Afin de relier ce contexte compact avec le flot des EDPs hamiltoniennes sur les espaces de Hilbert, nous devons considérer un contexte spécifique introduit par Fabert dans [Fab18]. On considère l'équation de Schrödinger non-linéaire sur \mathbb{T} avec non-linéarité de convolution,

$$iu_t = u_{xx} + [f(|u * \xi|^2, x, t)u * \xi] * \xi,$$

où $u : \mathbb{T} \rightarrow \mathbb{C}$. C'est l'un des exemples donnés par Kuksin dans [Kuk95] des équations qui peuvent être approchées par des difféomorphismes hamiltoniens de dimension finie. De plus, cette équation définit un flot Φ^t sur tout l'espace $L^2(\mathbb{T}, \mathbb{C})$. Après restriction à la sphère unité \mathbb{S} de L^2 le difféomorphisme Φ^t descend en une application sur l'espace projectif PL^2 . Cette application est approchée par des applications de dimensions finies $\Psi_n^t : \mathbb{C}P^n \rightarrow \mathbb{C}P^n$ et chacune de ces applications a au moins $n + 1$ points fixes.

Conjecture. Pour tout $t \in \mathbb{R}$, l'application $\Psi_t : PL^2 \rightarrow PL^2$ a un nombre infini de points fixes.

Nous prouvons le cas indépendant du temps. Dans ce cas, il suffit de chercher les points critiques de H restreints à la sphère unité. Cela équivaut à chercher des ondes stationnaires, c'est-à-dire des solutions de l'équation de la forme

$$u(t, x) = e^{i\mu t}v(x), \quad \text{avec } v \in L^2.$$

Théorème. *On suppose que f est indépendant de t . Alors, pour tout $t \in \mathbb{R}$ le flot Ψ^t a un nombre infini de points fixes. Plus précisément, il y a une séquence de points $u_n \in H^1$ et $\mu_n \in \mathbb{R}$ tels que*

$$\Phi^t(u_n) = u_n(t, x) = e^{i\mu_n t}u_n(x) \quad \text{et} \quad H(u_n) \rightarrow +\infty \quad \text{si } n \rightarrow +\infty,$$

pour tout $(t, x) \in \mathbb{R} \times \mathbb{T}$.

La première étape de la démonstration consiste à établir une sorte de condition de Palais-Smale pour les restrictions dimensionnelles H_n . En autres termes, nous montrons qu'une suite de points critiques u_n de H_n avec une énergie bornée converge vers un point critique de H . Ensuite, il faut appliquer des méthodes variationnelles pour trouver ces séquences à énergie bornée indépendantes de n .

Chapter 1

Introduction

Symplectic geometry has its origins in the mathematical formulation of classical mechanics. In the 19th century Hamilton gave a variational interpretation of the equations of motion. Consider the phase space \mathbb{R}^{2n} with coordinates (q, p) , where q represents the position of a particle and p represents the momentum. Let $H : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be an energy function. Then the evolution $\gamma : [0, 1] \rightarrow \mathbb{R}^{2n}$ of a particle in the phase space is a stationary point of the action

$$\mathcal{A}_H(\gamma) = \int_0^1 p(t) \cdot \dot{q}(t) dt - \int_0^1 H(t, \gamma(t)) dt$$

Under this formulation the extremal points are described by Hamilton's equations:

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases}$$

For example the movement of a particle under the influence of a potential $V(q)$ has an energy function given by $H(q, p) = \frac{1}{2m}|p|^2 + V(q)$. Hamilton's equations for the motion associated to this Hamiltonian give the classical Newtonian equation

$$m\ddot{q} = -\nabla V(q).$$

In general, Hamilton's differential equations give rise to a transformation of the phase space. When the phase space has dimension two this transformation will preserve the area of subsets of \mathbb{R}^2 . In higher dimensions, the equivalent statement is the following: the flow of Hamilton's equations preserves the differential form $\omega_0 = \sum_{i=1}^n dq_i \wedge dp_i$. This differential form is what we call the standard symplectic structure of \mathbb{R}^{2n} . In order to understand the dynamics of classical mechanics one is led to the study of symplectic structures and, in particular, the transformations of the space that preserve the symplectic form.

The concept of symplectic form generalizes to arbitrary smooth manifolds M : a symplectic structure ω on M is a non-degenerate closed two form. The fact that ω is non-degenerate allows us to generalize Hamilton's equations: for any energy function H , the Hamiltonian vector field X_H is defined as the unique vector field that satisfies $i_{X_H}\omega = dH$. Using the fact that ω is closed we then get that the flow generated by X_H preserves the symplectic structure.

The main example of symplectic manifold that we are going to work with are cotangent bundles T^*N . They represent the phase spaces of classical mechanics where the movement is confined to a manifold N . Every cotangent bundle is equipped with a canonical symplectic structure which is locally given by the standard structure in $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$. This local behavior is shared by every symplectic manifold M :

Theorem (Darboux). *Every symplectic manifold of dimension $2n$ is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.*

This theorem implies that symplectic manifolds do not have local invariants. The symplectic properties of M are determined by the geometry of $(\mathbb{R}^{2n}, \omega_0)$ and by the global structure of the objects in M . In order to study the geometry of the space one is then led to the study of the global diffeomorphisms that preserve the symplectic structure. We will refer to them as symplectomorphisms. The first property verified by symplectomorphisms is Liouville's theorem: every symplectic map preserves the volume of subsets. One of the first results that pointed out the difference between volume preserving maps and symplectomorphisms was Gromov's influential non-squeezing theorem:

Theorem (Gromov 1985, [Gro85]). *Let $(\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic space. Denote by B_r^{2n} a ball of radius r in \mathbb{R}^{2n} and let $B_R^2 \times \mathbb{R}^{2n-2}$ be the standard symplectic cylinder of radius R . If $\varphi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a symplectic diffeomorphism then*

$$\varphi(B_r^{2n}) \subseteq B_R^2 \times \mathbb{R}^{2n-2} \quad \text{implies} \quad r \leq R$$

1.1 Symplectic capacities

The original proof of Gromov's nonsqueezing theorem relied on the technique of pseudoholomorphic curves. Shortly after, several authors [EH90, HZ90, Vit92] gave independent proofs of Gromov's theorem using the concept of symplectic capacities. A symplectic capacity is a function $c : \mathcal{P}(\mathbb{R}^{2n}) \rightarrow [0, +\infty]$ that verifies the following properties:

1. (monotonicity) If $U \subseteq V$ then $c(U) \leq c(V)$.
2. (conformality) $c(\lambda U) = \lambda^2 c(U)$ for all $\lambda \in \mathbb{R}$.
3. (symplectic invariance) If $\phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a symplectomorphism then $c(\phi(U)) = c(U)$.
4. (non-triviality+normalization) $c(B_1^{2n}) = \pi = c(B_1^2 \times \mathbb{R}^{2n-2})$.

The existence of a function with these properties implies Gromov's theorem. The construction of some of the capacities (cf. [EH90, HZ90, Vit92]) comes from the study of periodic orbits of Hamiltonian systems. This is the approach that we are interested in.

The action spectrum as a symplectic invariant

Recall that Hamilton's principle states that the periodic orbits of a Hamiltonian system associated to H are the extremal points of the action:

$$A_H(\gamma) = \int_0^1 p(t) \cdot \dot{q}(t) dt - \int_0^1 H(t, \gamma(t)) dt.$$

In particular, the critical points of A_H contain very important dynamical information. This information can be used to build symplectic capacities.

Consider a compactly supported Hamiltonian H in \mathbb{R}^{2n} . In this case we can integrate Hamilton's equations to get a global flow ψ_t^H of \mathbb{R}^{2n} . A one periodic orbit of the system can be seen as a fixed point z_0 of ψ_1^H , that is, a point verifying $\psi_1^H(z_0) = z_0$. Denote by $A(z_0, H) = A_H(\psi_t^H(z_0))$ the action value associated to the fixed point z_0 . At first sight, this quantity depends on the function H and on the whole path ψ_t^H , but this is surprisingly not true. One can prove that this

action value does not depend on the particular Hamiltonian path used to generate ψ_1^H . More precisely, let K be another compactly supported Hamiltonian such that $\psi_1^K = \psi_1^H$, then

$$A(z_0, H) = A(z_0, K).$$

The action values define a subset of \mathbb{R} that depends only on the time one map $\psi = \psi_1^H$. We can then associate to every ψ its action spectrum $\sigma(\psi) \subseteq \mathbb{R}$ defined by

$$\sigma(\psi) = \{A(z, H) \mid z \in \text{Fix}(\psi = \psi_1^H)\}.$$

It turns out that the action spectrum is invariant by symplectic conjugations: for every symplectomorphisms φ of \mathbb{R}^{2n} we have

$$\sigma(\varphi\psi\varphi^{-1}) = \sigma(\psi).$$

In particular, the action spectrum of Hamiltonians supported in U is exactly the same as the one found for Hamiltonians supported in $\varphi(U)$. This important remark is the starting point of the construction of symplectic capacities that Viterbo gave in [Vit92].

Viterbo's capacities

For every open set U we denote by $\text{Ham}^c(U)$ the set of time one maps of Hamiltonian flows with compact support contained in U . For an open bounded set U Viterbo's capacity $c(U)$ is defined using the dynamical information of the elements in $\text{Ham}^c(U)$. More precisely, this information is going to come from the action value associated to certain important periodic orbits. These action values are chosen by two selectors, denoted by $c(\mu, \cdot)$ and $c(1, \cdot)$, that take as an input an element ψ^H of $\text{Ham}^c(\mathbb{R}^{2n})$ and give two values

$$c(\mu, \psi) = \mathcal{A}_H(\sigma(t)) \quad \text{and} \quad c(1, \psi) = \mathcal{A}_H(\beta(t))$$

where $\sigma(t) = \psi_t^H(x)$ and $\beta(t) = \psi_t^H(y)$ with $x, y \in \text{Fix}(\psi_1^H)$. These values verify $c(\mu, \psi) \geq 0$ and $c(1, \psi) \leq 0$ for every ψ . One can prove that the value $\gamma(\psi) = c(\mu, \psi) - c(1, \psi)$ is zero if and only if ψ is the identity. Intuitively, this means that these action values can be interpreted as a dynamical maximum and minimum of ψ . Moreover, one can prove that γ has the property of defining a distance on $\text{Ham}^c(\mathbb{R}^{2n})$. Using the invariance by symplectic conjugation of the action spectrum one can see that the selectors are invariant under symplectic conjugation, i.e.

$$c(\mu, \psi) = c(\mu, \varphi\psi\varphi^{-1}),$$

so these values can be used to define two symplectic invariants on open bounded sets:

$$c(U) = \sup\{c(\mu, \psi) \mid \psi \in \text{Ham}^c(U)\}$$

$$\gamma(U) = \inf\{\gamma(\psi) \mid \psi \in \text{Ham}^c(\mathbb{R}^{2n}) \text{ and } \psi(U) \cap U = \emptyset\}$$

On open unbounded sets V , the capacity $c(V)$ and $\gamma(V)$ is defined as the sup over every open bounded sets contained in V . Finally, on arbitrary subsets $X \subseteq \mathbb{R}^{2n}$ both capacities are defined as the infimum over all open sets V that contain X .

The value $c(U)$ measures how big this dynamical maximum can be provided that the support is contained in U . It moreover has the property that, on a convex smooth body K , the value $c(K)$ coincides with the geometric quantity of the minimal area of a closed characteristic on the boundary ∂K . On the other hand γ is defined using Viterbo's distance on $\text{Ham}^c(\mathbb{R}^{2n})$: if we define the energy of a diffeomorphism as the distance to the identity, then $\gamma(U)$ measures the minimal energy that one needs to displace U from itself. Both capacities are always related by the inequality $c(X) \leq \gamma(X)$. In dimension two, if ψ is a compactly supported Hamiltonian diffeomorphism which displaces a disc of area πr^2 , then we have $\pi r^2 \leq \gamma(\psi)$.

Construction of the action selectors

In order to extract information from the fixed points of $\psi_1^H = \psi$ we look at the graph of ψ , denoted by $\Gamma_\psi = \text{Id} \times \psi$ in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ and look at the intersection points of Γ_ψ with Γ_{Id} . If we endow $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with the symplectic structure $\omega_0 \oplus (-\omega_0)$ then Γ_ψ becomes a Lagrangian sub-manifold. In addition Γ_ψ is isotopic to the diagonal and coincides with it outside a compact set. Working on the linear manifold of \mathbb{R}^{2n} has the advantage that there is an explicit symplectomorphism

$$\mathcal{I} : (\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \omega_0 \oplus (-\omega_0)) \longrightarrow (T^*\mathbb{R}^{2n}, -d\lambda)$$

such that Γ_{Id} is sent to the zero section. In particular, since the map is symplectic, the image of every Γ_ψ is a Lagrangian submanifold which is Hamiltonian isotopic to the zero section and coincides at infinity with the zero section. This allows us to consider the one point compactification of \mathbb{R}^{2n} to S^{2n} and see $T^*\mathbb{R}^{2n}$ as a subset of T^*S^{2n} . Every Γ_ψ is contained in a Lagrangian sphere $L_\psi \subseteq T^*S^{2n}$ which is Hamiltonian isotopic to the zero section. We want to extract information from the intersection points of L_ψ with the zero section.

There is a natural way of describing Lagrangians in the cotangent space: every function f defines a Lagrangian submanifold via the graph of its differential df . For this family of Lagrangians, the intersection points with the zero section are determined by the critical points of f . We are going to describe L_ψ using a generalization of this description called generating functions. Introduce an auxiliary variable and consider functions $S : S^{2n} \times \mathbb{R}^k \rightarrow \mathbb{R}$. The graph of the differential is a Lagrangian submanifold in $T^*(S^{2n} \times \mathbb{R}^k)$. We are going to push forward this Lagrangian submanifold in a symplectic way to get a Lagrangian in T^*S^{2n} . To do this we need to introduce the concept of symplectic reduction.

Symplectic reduction. Let V be a symplectic vector space. We say that $W \subseteq V$ is a coisotropic subspace if W contains its symplectic orthogonal W^ω , or in other words, a subspace W such that if $u \in V$ verifies $\omega(u, w) = 0$ for every $w \in W$, then $u \in W$. Since $W^\omega \subseteq W$ we can consider the quotient W/W^ω which is by construction a symplectic vector space with symplectic form induced by ω . Denote by π the natural projection $\pi : W \rightarrow W/W^\omega$ and let Z be a subset of V . The symplectic reduction of Z by W is

$$\text{Red}_W(Z) := \pi(Z \cap W),$$

which is a subset of W/W^ω . One of the important properties of this operation is that if $L \subset V$ is a Lagrangian vector subspace transverse to W , then $\text{Red}_W(L)$ is Lagrangian in the quotient. This definition carries over to the nonlinear case at the level of tangent spaces. The only difficulty that may arise is that W/W^ω may not be a manifold.

Generating functions quadratic at infinity. Let N be a compact manifold and consider a function $S : N \times \mathbb{R}^k \rightarrow \mathbb{R}$. The graph of the differential is a Lagrangian submanifold in $T^*(N \times \mathbb{R}^k)$. Consider the coisotropic subspace $W := T^*N \times 0_{\mathbb{R}^k} \subseteq T^*N \times T^*\mathbb{R}^k \simeq T^*(N \times \mathbb{R}^k)$. One may see that $W/W^\omega \simeq T^*N$. If dS is transverse to W then

$$L_S = \text{Red}_W(dS)$$

is an immersed Lagrangian submanifold in T^*N . In this case we say that S is a generating function or *gf* of L_S . The transversality condition implies that $\Sigma_S := dS^{-1}(W)$ is a sub-manifold of dimension $\dim N$. We denote by $i_S : \Sigma_S \rightarrow L_S \subseteq T^*N$ the induced map $i_S(q, \xi) = \text{Red}_W(dS(q, \xi))$, it is an immersion. If moreover $S(x, \xi) = Q(\xi)$ outside a compact set with Q a nondegenerate quadratic form, then we will call S a generating function quadratic at infinity or *gfqi*. It turns out that this description of Lagrangians is enough to describe all of our L_ψ in T^*S^{2n} .

Theorem (Laudenbach and Sikorav [LS85, Sik86, Sik87]). *Let T^*N be the cotangent space of a compact manifold and consider a Hamiltonian diffeomorphism $\Phi : T^*N \rightarrow T^*N$. Then $\Phi(0_N)$ has a *gfqi* S such that i_S is a diffeomorphism.*

The fact that i_S is bijective implies that there is a bijective correspondence between the critical points of S and the intersection points of L with the zero section. This existence theorem already has strong implications for the intersection points of these Lagrangians with the zero section. It implies that these type of submanifolds intersect the zero section enough times. In order to define Viterbo's capacities, we need to be able to compare generating functions of the same Lagrangian. Consider the following operations:

1. *Addition of a constant.* If $c \in \mathbb{R}$, we set $S' = S + c : N \times \mathbb{R}^k \rightarrow \mathbb{R}$.
2. *Diffeomorphism operation.* If $\phi : N \times \mathbb{R}^K \rightarrow N \times \mathbb{R}^k$ is a diffeomorphism that satisfies $\phi(x, \xi) = (x, \varphi(x, \xi))$ then we set $S' = S \circ \phi$.
3. *Stabilization.* If $Q' : \mathbb{R}^{k'} \rightarrow \mathbb{R}$ is another nondegenerate quadratic form, then we set $S' = S + Q' : N \times \mathbb{R}^k \times \mathbb{R}^{k'} \rightarrow \mathbb{R}$.

We will say that a *gfqi* S' is equivalent to another *gfqi* S if S' can be obtained from S after a succession of the above basic operations. It is an easy exercise to verify that for every operation in the above definition we have

$$\text{Red}_W(dS') = \text{Red}_W(dS),$$

$$\text{CritVal}(S') = \text{CritVal}(S) + c.$$

The following theorem assures that these three operations are enough to go from one *gfqi* to another *gfqi* provided that they describe a Lagrangian isotopic to the zero section.

Theorem (Viterbo [Vit92] and Théret [Thé99a]). *Let T^*N be the cotangent bundle of a compact manifold and consider a Hamiltonian diffeomorphism $\Phi : T^*N \rightarrow T^*N$. Then all *gfqi* S of $\Phi(0_N)$ such that i_S is a diffeomorphism are equivalent.*

Note that in Théret's article the hypothesis of i_S diffeomorphism is not included in the definition of *gf* [Thé99a, Definition 2.1] but it is included in the terminology " S generates L " [Thé99a, Definition 2.2].

Definition of $c(\mu, \cdot)$ and $c(1, \cdot)$. By the existence theorem, every Lagrangian L_ψ has a *gfqi* with i_S bijective. Since i_S is bijective, intersection points of L_ψ with the zero section are in one to one correspondence with critical points of S . Recall that these intersection points are in bijective correspondence with the fixed points of ψ . After normalizing the *gfqi* so that the critical value at infinity is 0, one can prove that the critical values of S coincide with the action spectrum of ψ

$$\text{CritVal}(S) = \sigma(\psi).$$

In order to select an action value it is then enough to select a critical value of S . Viterbo does this in [Vit92] via Liusternik-Schilermann theory. In this theory, the critical values are seen as witnesses of the topological changes of the level sets of S . Let S be a *gfqi* which coincides with the quadratic form Q at infinity. Define S^λ to be the level set of points $(x, \xi) \in N \times \mathbb{R}^K$ such that $S(x, \xi) \leq \lambda$. The fact that S coincided with Q at infinity implies that for $|\lambda|$ large enough the level sets only depend on Q , in other words: $S^\lambda = Q^\lambda$. For a fixed λ_0 big enough that verifies

this property we define $S^\infty = S^{\lambda_0}$ and $S^{-\infty} = S^{-\lambda_0}$. Then the topology of the pair $(S^\infty, S^{-\infty})$ only depends on the index of the quadratic form and we have

$$(S^\infty, S^{-\infty}) \simeq (S^{2n} \times D^-, S^{2n} \times \partial D^-)$$

where D^- is the disc of dimension the index of Q . In particular we get the isomorphisms in cohomology over a field

$$H^*(S^\infty, S^{-\infty}) \simeq H^*(N) \otimes H(D^-, \partial D^-),$$

and we can define a map $T : H^*(N) \rightarrow H^*(S^\infty, S^{-\infty})$ which is called the Thom isomorphism. In order to find the change in the topology of the sublevel sets we deform the pair $(S^\infty, S^{-\infty})$. Consider the natural inclusion $i_\lambda : S^\lambda \rightarrow S^\infty$. We get a map in cohomology

$$i_\lambda : H^*(S^\infty, S^{-\infty}) \rightarrow H^*(S^\lambda, S^{-\infty}).$$

Finally, if we denote by μ the generator of $H^{2n}(S^{2n})$ and by 1 the generator of $H^0(S^{2n})$ we may define two values:

$$c(\mu, S) = \inf\{\lambda \mid i_\lambda^* T(\mu) \neq 0\} \quad \text{and} \quad c(1, S) = \inf\{\lambda \mid i_\lambda^* T(1) \neq 0\}$$

They find a level set that undergoes a topological change so they are critical values of S . We can then define $c(\mu, \psi) = c(\mu, S)$ where S is a normalized *gfqi*. This value is well defined thanks to the uniqueness theorem for *gfqi*.

1.2 Coisotropic rigidity

For the first original result of this thesis we use a generalization of Viterbo's spectral invariants in order to prove a new rigidity result for a wide family of symplectomorphisms. Instead of studying the evolution of balls, as in Gromov's nonsqueezing, we are going to study the evolution of coisotropic cylinders. Recall from the previous section that every coisotropic subspace $W \subseteq \mathbb{R}^{2n}$ induces a symplectic space W/W^ω . Denote by $\pi : W \rightarrow W/W^\omega$ the projection. The reduction of $Z \subseteq \mathbb{R}^{2n}$ is defined as

$$\text{Red}_W(Z) = \pi(Z \cap W).$$

In the particular case of $W = \mathbb{C}^m \times i\mathbb{R}^{n-m} \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$, $\pi = \pi_m$ is the projection onto the first m complex coordinates. A coisotropic cylinder is a subset of the form $X \times \mathbb{R}^{n-m} \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$. One may also write $X \times \mathbb{R}^{n-m} = X \times 0_{\mathbb{R}^{n-m}}$ and $W = T^*\mathbb{R}^m \times T_0^*\mathbb{R}^{n-m}$.

Theorem (Coisotropic non-squeezing). *Let $X \subseteq \mathbb{R}^{2m}$ be a compact set, consider $X \times \mathbb{R}^{n-m} \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$ and denote $W = \mathbb{C}^m \times i\mathbb{R}^{n-m}$. For every compactly supported Hamiltonian diffeomorphism ψ we have*

$$c(X) \leq \gamma(\text{Red}_W[\psi(X \times \mathbb{R}^{n-m})]).$$

The proof of this theorem is contained in the article [Bus17]. Explicitly, the reduction by W is $\pi_m(\psi(X \times \mathbb{R}^{n-m}) \cap (\mathbb{C}^m \times i\mathbb{R}^{n-m}))$, which is the projection of a bounded set. In particular we are studying trajectories where the component of $T^*\mathbb{R}^{n-m}$ starts with zero momentum and arrives at a precise point $T_0^*\mathbb{R}^{n-m}$. We point out that Viterbo's capacities all alone do not give rigidity information for the image of coisotropic cylinders since

$$c(X \times \mathbb{R}^k) = 0 = \gamma(X \times \mathbb{R}^k).$$

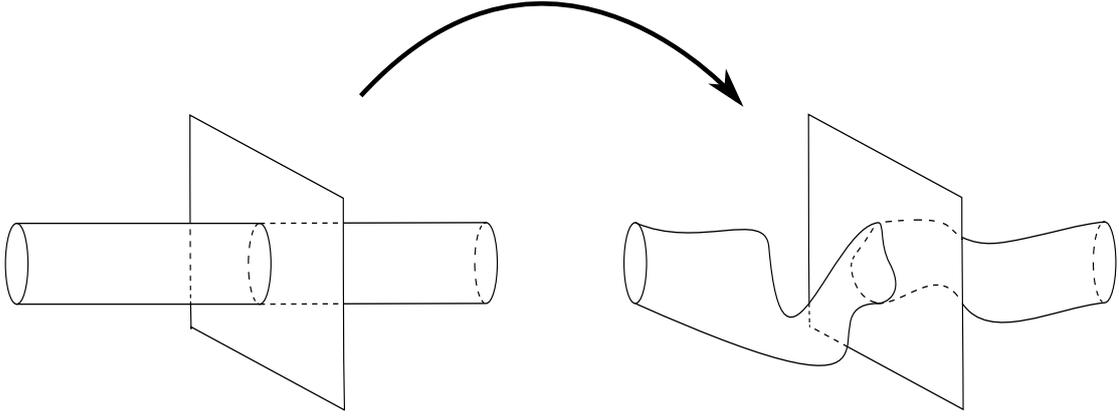


Figure 1.1: This figure represents the image of the coisotropic cylinder by a compactly supported Hamiltonian diffeomorphism ψ . The transverse plane represents the complementary coisotropic subspace W . The previous theorem gives information about the capacity of the projection of the intersection with W .

The case $X = (S^1)^m$ and $\text{Red}_W[\psi(X \times \mathbb{R}^{n-m})]$ contained in a cylinder was proved by Buhovski and Opshtein using the theory of pseudo-holomorphic curves. The proof of our theorem is achieved by a series of inequalities between Viterbo's capacities of subsets and the symplectic reduction of these subsets. The advantage of using Viterbo's capacities is that they are constructed using generating functions, and symplectic reduction can be seen as an explicit operation on generating functions. This operation can be then studied in detail. The proof comes from the extension of the action selectors to $\mathbb{R}^{2m} \times T^*\mathbb{T}^{n-m}$.

Action selectors on $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. The construction Viterbo's action selectors relies heavily on the linear structure of the space. We cannot use the same construction to generalize these invariants to every symplectic manifold, but nevertheless we are still able to do it if we add a contangent space of a torus.

Denote by π the natural projection of $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^{2k}$ onto $T^*\mathbb{R}^{2m} \times T^*\mathbb{T}^{2k}$. It is the projection induced by the quotient of $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^{2k}$ by translation of vectors in \mathbb{Z}^k . Take a compactly supported Hamiltonian isotopy ψ of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ and consider the lift $\tilde{\psi}$ on $\mathbb{R}^{2m} \times T^*\mathbb{R}^k$. The map $\tilde{\psi}$ commutes with the action of \mathbb{Z}^k by translation so the fixed points of $\tilde{\psi}$ come in \mathbb{Z}^k orbits. Moreover one may see that these orbits are in bijective correspondence with contractible fixed points of ψ . Define the contractible action spectrum of ψ as

$$\sigma_c(\psi) = \sigma(\tilde{\psi}).$$

For every compactly supported Hamiltonian φ of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ one can prove that

$$\sigma_c(\varphi\psi\varphi^{-1}) = \sigma_c(\psi).$$

As in the case of \mathbb{R}^{2n} we consider the graph of $\tilde{\psi}$ and we can use the symplectomorphism \mathcal{I} to see it as a Lagrangian submanifold of $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^{2k}$. In this case intersection points with the zero section come in \mathbb{Z}^k orbits where every point in the orbit has the same action value. In order to get rid of this redundancy, we use the periodic properties of the flow to quotient out by this action and find a Lagrangian submanifold in $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{T}^k$ whose intersection points with the zero section are in bijective correspondence with contractible fixed points of ψ . We may then compactify the basis to get a Lagrangian

$$L_\psi \subseteq T^*S^{2m} \times T^*S^k \times T^*\mathbb{T}^k.$$

As before, this submanifold is Hamiltonian isotopic to the zero section so it has a *gfgi*. We can then use the cohomological classes $\alpha \otimes \beta \otimes \gamma \in H^*(S^{2m}) \otimes H^*(S^k) \otimes H^*(\mathbb{T}^k)$ to select critical values of S . After normalization, we again have the very important property

$$\text{CritVal}(S) = \sigma_c(\psi).$$

We can then define a notion of capacity for subsets of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ in the same way as in the classical setting. For open bounded sets U in $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ we define

$$c(\alpha \otimes \beta \otimes \gamma, U) = \sup\{c(\alpha \otimes \beta \otimes \gamma, \psi) \mid \psi \in \text{Ham}^c(U)\}.$$

This time these quantities are not invariant by general symplectic diffeomorphisms of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ but only by Hamiltonian isotopies.

Critical values selectors and symplectic reduction. The next step is to relate these new invariants to Viterbo's classical capacities. This is done via a sequence of inequalities that relate generating functions with the operation of symplectic reduction. The key remark is that if $S : N \times B \times \mathbb{R}^l \rightarrow \mathbb{R}$ is a generating function for $L_S \subseteq T^*N \times T^*B$ and if L_S is transverse to the coisotropic subspace $W = T^*N \times T_b^*B$ then $S_b = S(\cdot, b, \cdot)$ generates $\text{Red}_W(L_S)$. In addition, the critical value selectors are also well behaved by this operation. More precisely for $\alpha \in H^*(N)$ and $\mu \in H^*(B)$ the orientation class in B we have the chain of inequalities

$$c(\alpha \otimes 1, S) \leq c(\alpha, S_b) \leq c(\alpha \otimes \mu, S).$$

This result allows us to get the desired relation with Viterbo's capacities if we look at the invariants with respect to the cohomological class $\mu \otimes \mu \otimes 1$. We first get the non triviality of the invariant via the following inequality:

Proposition. *If $X \subseteq \mathbb{R}^{2m}$ is compact then $c(X) \leq c(\mu \otimes \mu \otimes 1, X \times \{0\} \times \mathbb{T}^k)$.*

In this proposition it is crucial that we have a set with the whole \mathbb{T}^k in the variable of the cohomological class 1 in order to get something non-trivial. The upper bound comes from a relation with Viterbo's displacement energy:

Proposition. *Let $Z \subseteq \mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k$ be a compact set. For $w \in \mathbb{T}^k$ consider the coisotropic subspace $W = \mathbb{R}^{2m} \times \mathbb{R}^k \times \{w\}$. We have*

$$c(\mu \otimes \mu \otimes 1, Z) \leq \gamma(\text{Red}_W(Z)).$$

In order to prove the coisotropic non-squeezing theorem we extend the set $X \times \mathbb{R}^k$ and see it as $X \times \{0\} \times \mathbb{T}^k \subset \mathbb{R}^{2m} \times T^*\mathbb{T}^k$. We then extend the compactly supported Hamiltonian to the whole space $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ and apply the previous propositions.

Non-compact setting

The next step is to extend the coisotropic non-squeezing theorem to settings where the Hamiltonian does not have compact support. Since we look at the evolution of an unbounded set $X \times \mathbb{R}^k \subseteq \mathbb{C}^m \times \mathbb{C}^k$ we need to have a globally defined flow. For this reason we restrict the study to the class of Hamiltonians that satisfy

$$|\nabla H(z)| \leq A + B|z| \quad \text{for every } (t, z) \in [0, T] \times \mathbb{R}^{2n}$$

for two positive constants A and B . These Hamiltonians define global maps ψ_t^H for every $t \in [0, T]$. The first remark is that there exist a Hamiltonian diffeomorphism of this type that does not satisfy the statement of the theorem: consider the map

$$\phi(z_1, \dots, z_n) = (z_{m+1}, \dots, z_n, z_1, \dots, z_m)$$

with $z_j = q_j + ip_j$. It is a Hamiltonian diffeomorphisms which is generated by a quadratic Hamiltonian. One can easily see that

$$\phi(X \times \mathbb{R}^k) = \mathbb{R}^k \times X \subseteq \mathbb{R}^k \times \mathbb{C}^m$$

so the image by π_m is also contained in a proper linear subspace. Since the displacement energy associated to a linear subspace is zero we get

$$\gamma(\text{Red}_W[\phi(X \times \mathbb{R}^k)]) = 0.$$

If we consider a coisotropic cylinder of the form $X \times \mathbb{R}^k$ with $c(X) > 0$ then the statement of the coisotropic non-squeezing is not true for this map ϕ . We prove that there is a class of maps in-between that verify the coisotropic non-squeezing. More precisely we prove:

Theorem. *Consider a Hamiltonian function of the form*

$$H_t(z) = \frac{1}{2} \langle Az, z \rangle + U_t(z), \quad \text{with } |\nabla U(z)| \leq C \quad \text{for every } (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}$$

with a positive C . Suppose that $A = A_1 \oplus \dots \oplus A_n$ where A_k is a linear map on $\text{Vect}\{\frac{\partial}{\partial q_k}, \frac{\partial}{\partial p_k}\}$ for every $1 \leq k \leq n$. Let $X \subseteq \mathbb{R}^{2m}$ be a compact set and consider the two coisotropic subspace $W := \mathbb{C}^m \times i\mathbb{R}^{n-m}$. Then for every $t \in \mathbb{R}$ such that

$$\mathbb{R}^{n-m} + e^{-tJA}i\mathbb{R}^{n-m} = \mathbb{C}^{n-m}$$

we have

$$c(X) \leq \gamma(\text{Red}_W[\psi_t(X \times \mathbb{R}^{n-m})]).$$

As an example of one such Hamiltonian, we have the class of mechanical Hamiltonians of the form

$$H_t(q, p) = \frac{1}{2}|p|^2 + U_t(q), \quad \text{with } |\nabla U_t(z)| \leq C.$$

In particular, the theorem is true if U is a periodic potential in q .

Coisotropic rigidity on $T^*\mathbb{T}^n$

Using this non-compact extension we are able to look at the true periodic case $T^*\mathbb{T}^n$. In this case we prove that we still have this behavior for compactly supported Hamiltonians. It can be stated as the following nonsqueezing statement.

Theorem. *Let ψ be a compactly supported Hamiltonian isotopy of $T^*\mathbb{T}^n = T^*\mathbb{T}^m \times T^*\mathbb{T}^{n-m}$. For any $w \in \mathbb{T}^{n-m}$ denote $W_f = T^*\mathbb{T}^m \times T_w^*\mathbb{T}^{n-m}$ and $W_0 = T^*\mathbb{T}^m \times 0_{\mathbb{T}^{n-m}}$. Let B_r denote a ball of radius r in $T^*\mathbb{T}^m$. Then the inclusion*

$$\text{Red}_{W_f}[\psi(B_r \times 0_{\mathbb{T}^{n-m}})] \subseteq B_R \quad \text{or} \quad \text{Red}_{W_0}[\psi(B_r \times T_w^*\mathbb{T}^{n-m})] \subseteq B_R$$

imply $r \leq R$.

This theorem is also true for mechanical Hamiltonians. It also indicates that this type of coisotropic rigidity may also appear in the setting of product cotangent spaces of compact manifolds. By symplectic ball of radius r in a symplectic manifold (M, ω) we understand a ball of radius r of \mathbb{R}^{2n} which is symplectically embedded in M . We may conjecture the following behavior:

Conjecture. Let N and M be two compact manifolds and consider ψ a compactly supported Hamiltonian isotopy of $T^*N \times T^*M$. For any $m \in M$ denote $W_f = T^*N \times T_m^*M$ and $W_0 = T^*N \times 0_M$. Let B_r denote a symplectic ball of radius r in T^*N . Then the inclusion

$$\text{Red}_{W_f}[\psi(B_r \times 0_M)] \subseteq B_R \quad \text{or} \quad \text{Red}_{W_0}[\psi(B_r \times T_m^*M)] \subseteq B_R$$

imply $r \leq R$.

Middle dimensional symplectic rigidity

One of the consequences of the coisotropic non-squeezing theorem is its interpretation in the context of the middle dimensional rigidity problem. One of the first questions regarding this problem appeared in [Hof90b] where Hofer asked about the generalization of capacities to middle dimensions. Symplectic capacities are a sort of two dimensional way of measuring sets since from the point of view of symplectic capacities all cylinders of the same radius but with a base of different dimensions look exactly the same. In order to try and see if those cylinders are different for symplectic geometry Hofer asked if there exists a k -intermediate symplectic capacity c^k satisfying monotonicity, k -conformality, symplectic invariance and such that

$$c^k(B_1^{2k} \times \mathbb{R}^{2n-2k}) < +\infty \quad \text{but} \quad c^k(B_1^{2k-2} \times \mathbb{R}^2 \times \mathbb{R}^{2n-2k}) = +\infty?$$

The first inequality is already verified by the standard capacities and it is the second one which differs. One of the first results indicating that intermediate capacities do not exist appeared in an article by Guth [Gut08]. He was interested in the question of embeddings of polydiscs $P = B_{R_1}^2 \times \cdots \times B_{R_n}^2$ with $R_1 \leq \cdots \leq R_n$ into another one $P' = B_{R'_1}^2 \times \cdots \times B_{R'_n}^2$ with $R'_1 \leq \cdots \leq R'_n$ using a symplectic map. There are two obvious obstructions to this problem. The first one comes from Gromov's nonsqueezing which implies $R_1 \leq R'_1$. The second one is the volume which implies that $R_1 \cdots R_n \leq R'_1 \cdots R'_n$. Using the techniques in the article by Traynor [Tra95] one can see that one could embed P into P' roughly if and only if $R_1 \cdots R_k \lesssim R'_1 \cdots R'_k$ for every k between 1 and n . Guth proved that, modulo a dimensional constant, $k = 1$ and $k = n$ are the only two obstructions. More precisely:

Theorem (Guth 2008 [Gut08]). *There is a dimensional constant $C(n)$ such that if $C(n)R_1 \leq R'_1$ and $C(n)R_1 \cdots R_n \leq R'_1 \cdots R'_n$ then P symplectically embeds into P' .*

This theorem gave a partial answer to Hofer's question. It implies that if $1 < k < n$ then k -capacities that verify the following continuity hypothesis:

$$\lim_{R \rightarrow +\infty} c(B_1^{2k} \times B_R^{2n-2k}) < +\infty \quad \text{and} \quad \lim_{R \rightarrow +\infty} c(B_1^{2k-2} \times B_R^{2n-2k+2}) = +\infty$$

do not exist. His theorem was a close definitive answer but the question of less regular capacities remained open. This was recently solved in the negative by Pelayo and Vũ Ngọc in [PVuN15]. They applied a limit argument to the construction of Guth and Hind-Kerman [HK14] in order to prove the unbounded version of Guth's theorem:

Theorem (Pelayo-Vũ Ngọc 2015 [PVuN15]). *If $n \geq 2$ then the cylinder $B_1^2 \times \mathbb{R}^{2n-2}$ can be symplectically embedded into the product $B_R^{2n-2} \times \mathbb{R}^2$ for all $R \geq \sqrt{2^{n-1} + 2^{n-2} - 2}$.*

With this theorem the question of intermediate symplectic capacities was definitely settled. It showed that general symplectic diffeomorphisms are too flexible to capture this type of cylindrical rigidity.

Another point of view for the middle dimensional problem comes from a reformulation of Gromov's non-squeezing theorem. In dimension 2 symplectomorphisms are the same as area preserving maps so in [EG91] Eliashberg and Gromov pointed out that (using a theorem of Moser about the existence of area preserving diffeomorphisms) Gromov's theorem is equivalent to

$$\text{area}(\Pi_1\phi(B_r^{2n})) \geq \pi r^2 \quad \text{for every symplectomorphism } \phi.$$

Denote by Π_k the projection on the first $2k$ coordinates. A possible generalization of this statement to higher dimensions is

$$\text{Vol}(\Pi_k\phi(B_r^{2n})) \geq \text{Vol}(\Pi_k B_r^{2n}) = \text{Vol}(B_r^{2k}) \quad \text{for every symplectomorphism } \phi.$$

This problem was studied by Abbondandolo and Matveyev in [AM13]. In their article they first proved that the inequality is true in the linear case:

Theorem (Abbondandolo-Matveyev 2013 [AM13]). *Let Φ be a linear symplectic automorphism of \mathbb{R}^{2n} , and let $P : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be the orthogonal projector onto a complex linear subspace $V \subseteq \mathbb{R}^{2n}$ of dimension $2k$, $1 \leq k \leq n$. Then*

$$\text{Vol}(P\Phi(B_r^{2n})) \geq \text{Vol}(B_r^{2k})$$

with equality if and only if the linear subspace $\Phi^T V$ is complex.

In contrast with the linear result, they prove that symplectic diffeomorphisms are again too flexible to have this kind of middle dimensional symplectic rigidity. More precisely, this shows that one can squeeze arbitrarily the symplectic projection of image of the ball by symplectomorphisms.

Theorem (Abbondandolo-Matveyev 2013 [AM13]). *Let $P : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be the orthogonal projector onto a complex linear subspace $V \subseteq \mathbb{R}^{2n}$ of dimension $2k$, $1 < k < n$. For every $\epsilon > 0$ there exists a smooth symplectic embedding $\phi : B_1^{2n} \rightarrow \mathbb{R}^{2n}$ such that*

$$\text{Vol}(P\phi(B_1^{2n})) < \epsilon.$$

In spite of all this negative results, there was a recent result by Rigolli in [Rig15] where he proved that there is also local middle dimensional volume rigidity if one restricts the class of symplectomorphisms to analytic ones.

Theorem (Rigolli 2015 [Rig15]). *Let $\phi_t : \overline{B}_1^{2n} \rightarrow \mathbb{R}^{2n}$ be an analytic path of symplectic embeddings, with $t \in [0, 1]$, such that ϕ_0 is linear. Then the middle dimensional non-squeezing inequality*

$$\text{Vol}(P\phi_t(B_1^{2n})) \geq \text{Vol}(B_r^{2k})$$

holds for t small enough.

This result points out in the direction that this middle dimensional rigidity might be possible upon restriction of the class of symplectic diffeomorphisms that we consider. Moreover, this result has another local implication:

Theorem (Rigolli 2015 [Rig15]). *For every analytic symplectic embedding of a domain D there exists a function $r_0 : D \rightarrow (0, +\infty)$ such that the middle dimensional inequality*

$$\text{Vol}(P\phi(B_r^{2n}(x))) \geq \text{Vol}(B_r^{2k})$$

holds for every $x \in D$ and for every $r < r_0(x)$. Moreover r_0 is bounded away from zero on compact subsets $K \subseteq D$.

We would like to point out another possible middle dimensional generalization of the squeezing problem. In dimension 2 the value of any normalized symplectic capacity on topological discs coincides with the area, so one may also rewrite Gromov's theorem as

$$c(\Pi_1\phi(B_r^{2n})) \geq \pi r^2 \quad \text{for every symplectomorphism } \phi,$$

where c is a symplectic capacity. One can then ask if this inequality is true with Π_1 replaced by Π_k , and more generally look at subsets Z different from B_r^{2n} and replace πr^2 with the capacity of $\Pi_k Z$. The previous results show that there is little hope that one such inequality is always true so one has to restrict the type of symplectomorphisms considered. Our coisotropic non-squeezing theorem implies middle dimensional rigidity for deformations of coisotropic cylinders $Z = X \times \mathbb{R}^{n-m} \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$ provided that we restrict the class of symplectomorphisms to Hamiltonian isotopies with bounded speed.

Theorem. *Let $X \subset \mathbb{R}^{2m}$ be a compact set. Consider $X \times \mathbb{R}^{n-m} \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$ and let ψ be a Hamiltonian diffeomorphism generated by a function with bounded gradient. Then*

$$c(X) \leq \gamma(\Pi_m\psi(X \times \mathbb{R}^{n-m})).$$

Remark that, in contrast to what happens in the middle volume rigidity problem, this statement is not true for every linear symplectomorphism. As it is usual in symplectic geometry, this rigidity result can be used to define an invariant. The classical example of this fact is the definition of Gromov's width after he proved the nonsqueezing theorem. In our case we should consider the following quantity:

$$\gamma_G^m(U) = \inf\{\gamma(\Pi_m\phi(U)) \mid \phi \in G\}$$

where G is a subgroup of the group of symplectic diffeomorphisms. For $G = \text{Symp}(\mathbb{R}^{2n})$ we know that γ_G^m is zero on coisotropic cylinders of dimension m . In particular, it might be trivial for every subset of \mathbb{R}^{2n} . On the other hand, if the elements of G are Hamiltonian diffeomorphisms generated by functions with bounded gradient then the previous theorem implies that γ_G^m is bounded from below on coisotropic cylinders of dimension m . As an example of G one can take the subgroup of Hamiltonian diffeomorphisms φ_t^H where H , φ_t^H , and $(\varphi_t^H)^{-1}$ are Lipschitz on the space variable over compact time intervals. For this subgroup the previous theorem gives

$$c(X) \leq \gamma_G^m(X \times \mathbb{R}^{n-m}) \leq \gamma(X).$$

More precisely γ_G^m has the following properties:

1. (monotonicity) If $U \subseteq V$ then $\gamma_G^m(U) \leq \gamma_G^m(V)$.
2. (conformality) $\gamma_G^m(\lambda U) = \lambda^2 \gamma_G^m(U)$ for all $\lambda \in \mathbb{R}$.
3. (G invariance) If $\phi \in G$ we have $\gamma_G^m(\phi(U)) = \gamma_G^m(U)$.
4. (non-triviality+normalization) $\gamma_G^m(B_r^{2m} \times \mathbb{R}^{n-m}) = \pi r^2$.

Finally we want to point out that this invariant satisfies a 2-conformality property instead of the k -conformality asked by Hofer for the middle dimensional symplectic capacities.

1.3 Rigidity on energy surfaces

In the next part of this thesis we study rigidity properties of the characteristic flow of energy surfaces. Let (M, ω) be a symplectic manifold and let $H : M \rightarrow \mathbb{R}$ be a smooth Hamiltonian function. Every regular value λ defines a smooth hypersurface Σ of M . The properties of the flow ψ^H on Σ have historically been an important subject of research (see for example [HZ94, Chapter 4]). As an example we recall the study of periodic orbits in energy surfaces. We are interested in the compact subsets of Σ which are invariant by the flow. These include the periodic orbits and also other types subsets of higher dimension. We use the symplectic camel theorem to create compact invariant subsets in deformations of Σ . Recall that the statement of the symplectic camel theorem is:

Theorem ([Vit92] and [MT93] in dimension 4). *Consider the space $E = \mathbb{R}^{2n-1} \times \{0\} \setminus B^{2n}(0, r)$ and suppose that there is a symplectic isotopy φ^t of \mathbb{R}^{2n} with support in $\mathbb{R}^{2n} \setminus E$ such that*

$$B_R^{2n} \subseteq \{p_n < 0\} \quad \text{and} \quad \varphi^1(B_R^{2n}) \subseteq \{p_n > 0\}.$$

Then $r \geq R$.

There is also an unpublished proof of this result by Gromov and Eliashberg.

Recall that locally we may find a small enough open set $U \subseteq M$, and a small enough open neighbourhood of the origin in \mathbb{R}^{2n} such that we have a symplectomorphism

$$U \cap \Sigma \simeq V \cap \mathbb{R}^{2n-1} \times \{0\}.$$

Taking this into account we look at deformations of $\mathbb{R}^{2n-1} \times \{0\}$ contained in V with the desired properties. We first define the concept of (r, R) -simple function (see Figure 1.2 (a)) whose energy levels are compact deformations of $\mathbb{R}^{2n-1} \times \{0\}$. We then define simple perturbations: these are C^0 perturbations inside our class of simple functions.

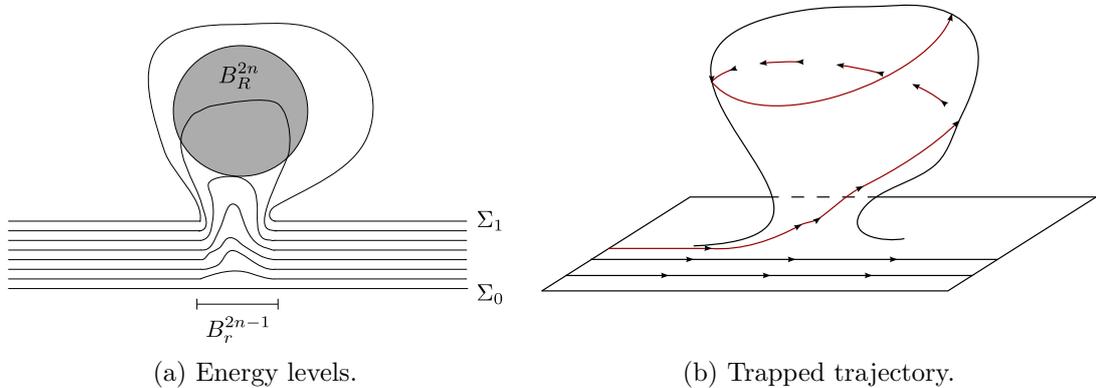


Figure 1.2: (a) The energy levels between 0 and 1 of a (r, R) -simple function with $r < R$. (b) A representation of a trajectory that is trapped in a bounded set. The closure of the trajectory creates a compact invariant subset.

In this section we first show that one can easily deform Σ to produce compact invariant subsets and then we prove using the symplectic camel theorem that these subsets are going to persist under C^0 perturbations of the energy function H . More precisely:

Theorem. *Let $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a (r, R) -simple function with $r < R$. Then there exists a $\lambda \in [0, 1]$ such that Σ_λ has a compact invariant subset. As a consequence, for every ϵ -simple perturbation H_ϵ of H with $\epsilon \in [0, \frac{R-r}{2}[$ there exists a $\lambda_\epsilon \in [0, 1]$ such that $\Sigma_\lambda^\epsilon = H_\epsilon^{-1}(\lambda_\epsilon)$ has a compact invariant subset.*

The proof is by contradiction. If there is no compact invariant subset, then every characteristic trajectory goes from one side of the space to the other (see Figure 1.2). If this is the case, we can build a symplectic isotopy that verifies the hypothesis of the symplectic camel theorem but which contradicts the conclusion. Since this is not possible, we conclude that there must be a trajectory that gets trapped in a bounded set. This trajectory then has to be arbitrarily close to a compact invariant subset.

1.4 Hamiltonian PDEs

In the second part of this manuscript we study symplectic rigidity properties in infinite dimensional Hilbert spaces. Many partial differential equations (PDEs) arising in physics can be seen as infinite dimensional Hamiltonian systems

$$\dot{u} = J\nabla H_t(u).$$

Here the Hamiltonian function H is defined on some infinite dimensional Hilbert space and J is a nondegenerate anti-symmetric operator that defines a symplectic structure via the scalar product. The main examples of Hamiltonian PDEs are:

1. the nonlinear wave equation

$$u_{tt} - \Delta u + f(t, x, u) = 0,$$

2. the nonlinear Schrödinger equation

$$iu_t - \Delta u + f(t, x, |u|^2)u = 0,$$

3. the membrane equation

$$u_{tt} + \Delta^2 u + f(t, x, u) = 0,$$

4. The Korteweg-de Vries (KdV) equation

$$u_t - uu_x + u_{xxx} = 0.$$

In contrast to what happens in finite dimensions, the Hamiltonian function H_t that is used to define the vector field is not defined on the whole symplectic phase space E , but only on a dense subspace of E . This also implies that the vector field is going to be defined only on a dense subspace. In particular, before looking at the symplectic properties of the flow, one has to first ask what is the precise definition of the flow, and in particular what is the definition of a solution.

One way of solving this issue is to look at semilinear equations, that is, equations whose Hamiltonian can be written as $H_t(u) = \frac{1}{2}\langle Au, u \rangle + h_t(u)$. Here A is a possibly unbounded linear operator and h_t is a smooth function defined on the whole space E . For example, in the case of the Schrödinger equation, the Hilbert space is $E = L^2(\mathbb{T}^d, \mathbb{C})$, the operator is $A = \Delta$ and h is defined via an integral using f . For these type of equations one can use Duhamel's formula to define solutions. This gives the notion of generalized solution, and (under smoothness conditions on the nonlinearity) the existence and uniqueness results can be derived using fixed point arguments.

Symplectic rigidity: prior work

In the infinite dimensional case little is known about the rigidity properties of general symplectomorphisms. The investigation of symplectic rigidity properties for PDEs started with Kuksin in 1995 [Kuk95]. He studied a type of equations which are a compact perturbation of a symplectic linear flow. The key property of these symplectic mappings is that they can be approximated

on bounded sets by finite dimensional Hamiltonian flows. This approximation allowed him to construct an infinite dimensional analogue of the symplectic capacity defined by Hofer-Zehnder. This capacity is built as a sort of limit of the finite dimensional one, and as such inherits much of the properties of symplectic capacities. In particular it verifies the normalization property so one can use it to derive an analogue of the nonsqueezing theorem. However, in contrast to the finite dimensional capacity, Kuksin's capacity is not invariant by general symplectomorphism of the symplectic Hilbert space, but only by those which are a compact perturbation of a symplectic linear map, and where the linear map is well adapted to the basis which is used to construct the capacity.

Shortly after Kuksin's work, several authors took the task of investigating if one could expect nonsqueezing behavior in more general flows. Bourgain was the first to prove nonsqueezing for a flow which is not a compact perturbation of a linear map [Bou94]. He studied the cubic nonlinear Schrödinger equation

$$i\partial_t u = -\Delta u + |u|^2 u,$$

on the torus. Again, the technique was to approximate in a certain way the flow of the equation by finite dimensional Hamiltonian flows in order to apply Gromov's theorem. The fact of looking at periodic solutions allows to apply a Fourier cutoff to the nonlinearity. Next, Colliander et al. [CKS⁺05] proved, using again a finite dimensional approximation that the KdV equation on $H^{-\frac{1}{2}}(\mathbb{T})$ also has the nonsqueezing property. Again, the main challenge in the proof is to prove the approximation. Other equations where nonsqueezing has been proven are the periodic Klein-Gordon, by Mendelson [Men17] or the periodic BBM [Rou10]. It is worth mentioning the article of Killip-Visan-Zhang [KVZ16] where they gave the first proof of non-squeezing in infinite volume. They consider again the cubic NLS on \mathbb{R}^2 and show again that it has a good approximation by finite dimensional flows.

All the previous proofs rely on the particular structure of the equations. A different point of view came in the article [AM15] of Abbondandolo and Majer. They studied the non-squeezing property for arbitrary symplectic diffeomorphisms of a separable Hilbert space using a geometric approach. They proved that squeezing cannot happen at least for convex images of the ball. They did this by constructing a capacity for convex sets profiting from the dual action principle. Their argument is the first one which is not based on the approximation by finite dimensional flows. We also mention the articles by Sukhov and Tumanov [ST16a, ST16b] where they give a new proof of Gromov's non-squeezing and then use it to explore the limits of this proof in the context of Hilbert spaces.

Symplectic capacities

The first result of this part of the thesis is a construction of infinite dimensional capacities different from the one that Kuksin gave in [Kuk95] but also based in the finite dimensional approximation. We restrict ourselves to semilinear PDEs of the type described in [Kuk95]. The symplectic structure on E is defined using the scalar product and an almost complex structure J . The Hamiltonian functions are of the form

$$H_t(u) = \frac{1}{2} \langle Au, u \rangle + h_t(u),$$

where A is a (possibly unbounded) self-adjoint linear operator and h_t is a smooth function. The Hamiltonian vector field is

$$X_H(u) = JAu + J\nabla h_t(u)$$

If e^{tJA} is bounded, then solutions can be defined via Duhamel's formula. Denote by $\{\varphi_j^\pm \mid j \geq 1\}$ a Hilbert orthonormal basis of E and suppose that $J\varphi_j^\pm = \pm\varphi_j^\mp$ and that A is diagonal for this

basis. Let E_n be the finite dimensional subspace generated by $\{\varphi_j^\pm \mid 1 \leq j \leq n\}$ and Π_n the natural projection on E_n . Under compactness assumptions on the nonlinearity, flow-maps are symplectomorphisms which can be approximated on bounded sets by finite dimensional Hamiltonian flows by projection of the vector field on E_n . Specific examples of this type of equations are (see [Kuk95]): nonlinear string equation in \mathbb{T} with a nonlinearity which has at most polynomial growth at infinity, quadratic nonlinear wave equation in \mathbb{T}^2 , nonlinear membrane equation on \mathbb{T}^2 with a nonlinearity which has at most polynomial growth at infinity and Schrödinger equation with a convolution nonlinearity in \mathbb{T}^n .

Definition of the capacity As in the finite dimensional case, in order to calculate the capacity of a set we are going to look at the action of periodic orbits associated to functions with support in that set. In infinite dimensions we have to be more careful since a function with bounded support may not induce a globally defined Hamiltonian flow. Consider a bounded open subset $\mathcal{U} \subseteq E$. We define the set

$$\mathcal{F}(\mathcal{U}) = \{\text{smooth functions } f : E \rightarrow \mathbb{R} \text{ such that } \text{Supp } f \subseteq \mathcal{U} \text{ And } d(\text{Supp } f, \partial\mathcal{U}) > 0\}.$$

Consider the action selector $c(\mu, \cdot)$ defined at the beginning of the introduction and write $c(\mu, H) = c(\mu, \psi_1^H)$. Denote $f_n = f|_{E_n}$ and define

$$c_o(\mu, f) = \liminf_n c(\mu, f_n) \quad \text{and} \quad c^o(\mu, f) = \limsup_n c(\mu, f_n)$$

Remark that the support of f_n is contained in $\text{Supp } f \cap E_n$ which is bounded, so f_n has compact support. In particular $c(\mu, f_n)$ is well defined. Remark moreover that for every n there is a periodic orbit $z_n : [0, 1] \rightarrow E_n$ such that $c(\mu, f_n)$ is the action of z_n , but these orbits may not converge in any way. We now use $c(\mu, f)$ to define the capacity of open bounded sets \mathcal{U} as

$$c_o(\mathcal{U}) = \sup\{c_o(\mu, f) \text{ for } f \in \mathcal{F}(\mathcal{U})\} \quad \text{and} \quad c^o(\mathcal{U}) = \sup\{c^o(\mu, f) \text{ for } f \in \mathcal{F}(\mathcal{U})\},$$

It turns out that this definition has the properties of symplectic capacities and in particular it benefits from normalization property. Moreover under compactness assumptions on the flow Φ we have $c_o(\mathcal{U}) \leq c_o(\Phi(\mathcal{U}))$ and $c^o(\mathcal{U}) \leq c^o(\Phi(\mathcal{U}))$ so one may use it to derive the nonsqueezing theorem:

Theorem (Kuksin [Kuk95]). *If $\Phi : E \rightarrow \mathbb{R}$ is the flow of a Hamiltonian PDEs satisfying the compactness condition, then $\Phi(B_r) \subseteq Z_R$ implies $r \leq R$.*

The original construction of an infinite dimensional symplectic capacity by Kuksin was a generalization of Hofer-Zehnder's capacity. It uses the notion of fast periodic orbits of Hamiltonian systems: orbits with period $T \leq 1$ which are not fixed points. The capacity of \mathcal{U} is then (roughly) defined as the maximal variation of a Hamiltonian which does not produce a fast trajectory. As in our case, the definition involves some sort of limit of the capacity as the dimension grows.

Middle dimensional rigidity

The second result of this chapter concerns the study of middle dimensional rigidity for Kuksin's equations. The first step is to define the coisotropic cylinder in this context. We explain the result for the concrete example of the nonlinear string equation.

$$\ddot{u} = u_{xx} - f(t, x, u), \quad u = u(t, x),$$

where $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Recall that if we denote by B the operator $B = (-\partial^2/\partial x^2 + 1)^{1/2}$ then we may write the equation in the form

$$\begin{aligned}\dot{u} &= -Bv, \\ \dot{v} &= (B - B^{-1})u + B^{-1}f(t, x, u).\end{aligned}$$

Suppose that f is bounded and that all its derivatives have polynomial growth. Define $E = H^{\frac{1}{2}}(\mathbb{T}) \times H^{\frac{1}{2}}(\mathbb{T})$ as the product of Hilbert spaces. Now putting $A = (B - B^{-1}) \times B$ and defining $J : E \rightarrow E$ by $J(u, v) = (-v, u)$ we can write the nonlinear string equation as the semilinear PDE on E as

$$(\dot{u}, \dot{v}) = JA(u, v) + J\nabla h_t(u, v).$$

Denote $E = E_+ \times E_- = H^{\frac{1}{2}}(\mathbb{T}) \times H^{\frac{1}{2}}(\mathbb{T})$. Write an element of $E = E^+ \times E^-$ in Fourier basis and denote by E_k the finite dimensional subspace of E whose elements (u, v) are a linear combination of terms of frequencies up to k . Denote by $\Pi_k : E \rightarrow E_k$ be the natural projection and let $X \times E_+^k$ be the subset of E that consists of elements (u, v) that project onto X under Π_k and such that $(0, v) \in E_k$. This is the natural definition of an infinite dimensional coisotropic cylinder.

As in the finite dimensional case, the capacities previously constructed do not give rigidity information about these coisotropic cylinders. Note that the subset $X \times E_+^k$ is not bounded so we cannot deduce directly middle dimensional rigidity for these Hamiltonian PDEs from the finite dimensional approximation on bounded sets. We need an intermediate step and this will come from the coisotropic non-squeezing theorem. Let $\Phi : E \rightarrow E$ be the flow of the nonlinear string equation and decompose it as

$$\Phi^t(u) = e^{tJA}V^t(u).$$

The map V^t can be approximated on bounded sets by a finite dimensional map $V_n^t(u_n, u^n) = (\varphi_n^t(u_n), u^n)$ (by the hypothesis on f) where φ_n^t verifies the coisotropic non-squeezing (since f is bounded). The fact that V_n^t acts as the identity on big Fourier modes allows us to prove that if ∇h is bounded on the whole space, then the set

$$V^t(X \times E_+^k) \cap \{\Pi_+^k = 0\}$$

is not only bounded, but compact. This allows us to have a precise control of the subset. In particular we prove that this intersection is symplectically rigid:

Proposition. *Denote by $\Phi^t : E \rightarrow E$ the flow of the nonlinear string equation with f bounded and such that all its derivatives have polynomial growth. For every $k \in \mathbb{N}$, every compact subset X of E_k and every $t \in \mathbb{R}$ we have*

$$c(X) \leq \gamma(\Pi_k(V^t(X \times E_+^k) \cap \{\Pi_+^k = 0\})).$$

Once we have this, we can use the symplectic invariance of the capacities together with the fact that e^{tJA} is a symplectic map which commutes with Π_k to get the middle dimensional rigidity for the actual flow of the equation:

Theorem. *Denote by $\Phi^t : E \rightarrow E$ the flow of the nonlinear string equation with f bounded and such that all its derivatives have polynomial growth. For every $k \in \mathbb{N}$, every compact subset X of E_k and every $t \in \mathbb{R}$ we have*

$$c(X) \leq \gamma(\Pi_k\Phi^t(X \times E_+^k)).$$

The proof of this theorem is contained in the article [Bus17]. A particular example of equation which verifies all the hypothesis is the Sine-Gordon equation:

$$u_{tt} = u_{xx} - \sin u, \quad u = u(t, x),$$

where $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For $k = 0$ the set $X \times E_+^k$ consists of elements (u, v) where the pair (u_0, v_0) of the mean values is in $X \subset \mathbb{R}^2$ and such that $(0, v) \in E_0$, that is, v is constant equal to v_0 . We see that these theorem gives information on the global behavior of solutions with constant initial velocity such that the pair of the mean values is in X . On the other hand, if we interchange the roles of E^+ and E^- we get information on the behavior of solutions where the initial position of the string is constant.

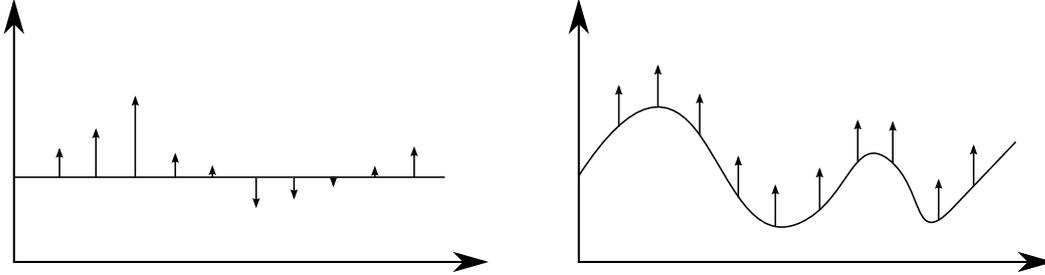


Figure 1.3: The representation the graph of the initial condition $u(0, x)$ and at each point the vector of the initial speed $v(0, x)$. On the left, $u(0, x)$ is constant. On the right, $v(0, x)$ is constant.

Arnold conjecture for Hamiltonian PDEs

Still in the search of symplectic rigidity in Hamiltonian PDEs, we point our attention to Arnold's conjecture. This conjecture asserts that Hamiltonian diffeomorphisms always have fixed points. More precisely:

Conjecture. On every compact symplectic manifold (M, ω) a Hamiltonian diffeomorphism has at least as many fixed points as a function on M has critical points.

The non-degenerate version of this conjecture has been established on every symplectic manifold. The proof gave rise to Floer's homology, a homological theory built from critical points of the action functional. As a particular example where this conjecture is true we have the complex projective space $\mathbb{C}P^n$ where the number of fixed points is $n + 1$. The original proof of this case [For85] did not use Floer's techniques but profited from the symmetry of $\mathbb{C}P^n$ to study the action functional for Hamiltonians in \mathbb{R}^{2n} .

In order to relate this compact setting with the flow of Hamiltonian PDEs on Hilbert spaces we need to consider a specific setting introduced by Fabert in [Fab18]. Consider the nonlinear Schrödinger equation on \mathbb{T} with convolution nonlinearity,

$$iu_t = u_{xx} + [f(|u * \xi|^2, x, t)u * \xi] * \xi,$$

where $u : \mathbb{T} \rightarrow \mathbb{C}$. This is one of the examples given by Kuksin in [Kuk95] of equations that can be approximated by finite dimensional Hamiltonian diffeomorphisms. Moreover this equation defines a flow Φ^t on the whole space $L^2(\mathbb{T}, \mathbb{C})$. After restriction to the unit sphere \mathbb{S} of L^2 the diffeomorphism Φ^t descends to a map on the projective space PL^2 . This map is approximated by finite dimensional maps $\Psi_n^t : \mathbb{C}P^n \rightarrow \mathbb{C}P^n$ and each of this maps has at least $n + 1$ fixed points.

Conjecture. For every $t \in \mathbb{R}$, the map $\Psi_t : PL^2 \rightarrow PL^2$ has infinitely many fixed points.

We prove the time-independent case. In this case it is enough to look for critical points of H restricted to the unit sphere. This is equivalent to looking for standing waves, that is, solutions of the equation of the form

$$u(t, x) = e^{i\mu t}v(x), \quad \text{with } v \in L^2.$$

Theorem. *Suppose that f is independent of t . Then for every $t \in \mathbb{R}$ the flow Ψ^t has infinitely many fixed points. More precisely, there is a sequence of points $u_n \in H^1$ and $\mu_n \in \mathbb{R}$ such that*

$$\Phi^t(u_n) = u_n(t, x) = e^{it\mu_n} u_n(x) \quad \text{and} \quad H(u_n) \rightarrow +\infty \quad \text{as} \quad n \rightarrow +\infty,$$

for every $(t, x) \in \mathbb{R} \times \mathbb{T}$.

The first step in the proof is establishing a kind of Palais-Smale condition for the finite dimensional reductions H_n of the Hamiltonian. In other words, we show that a sequence of critical points u_n of H_n with bounded energy converges to a critical point of H . Then one has to apply variational methods to find those sequences with bounded energy independent of n .

Chapter 2

The action spectrum as a symplectic invariant

In the variational approach of symplectic geometry Hamiltonian trajectories can be identified with critical values of the action functional. In this setting, looking for periodic orbits of Hamiltonian systems amounts to finding critical values of a functional on an infinite dimensional space. In this chapter we recall the basic properties of the set of critical values of this functional and explain how it can be used to define symplectic invariants. A more detailed exposition of these results can be found in Hofer and Zehnder's book [HZ94].

2.1 The action functional.

Consider \mathbb{R}^{2n} with coordinates $z = (q_1, p_1, \dots, q_n, p_n)$. We endow \mathbb{R}^{2n} with the standard symplectic structure

$$\omega_0 = \sum_{i=1}^n dq_i \wedge dp_i.$$

It is a non-degenerate, closed two form. For any function $H : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ there exists a unique time-dependent vector field X_H on \mathbb{R}^{2n} such that

$$\omega_0(X_H, \cdot) = dH_t(\cdot).$$

We call X_H the Hamiltonian vector field generated by H . Using the isomorphism $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ we have

$$X_H(z) = -i\nabla H_t(z) = J\nabla H_t(z),$$

where J is the multiplication by $-i$. Denote by \mathcal{H} the set of smooth functions $H : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ such that $\text{Supp } H$ is contained in a compact set. Here, by support of H we mean

$$\text{Supp } H = \bigcup_{t \in [0, 1]} \text{Supp } H_t \quad \text{with} \quad \text{Supp } H_t = \overline{\{z \in \mathbb{R}^{2n} \mid H_t(z) \neq 0\}}.$$

Every element of \mathcal{H} generates a compactly supported vector field, which in turn, generates a globally defined flow $\psi_t^H = \psi_t$ on \mathbb{R}^{2n} . We denote by $\text{Ham}^c(\mathbb{R}^{2n})$ the set of time one maps $\psi_1^H = \psi^H$ of the flows generated by functions on \mathcal{H} . By Cartan's formula these flows preserve the symplectic structure:

$$\frac{d}{dt} \psi_t^* \omega_0 = \mathcal{L}_{X_H} \omega_0 = di_{X_H} \omega_0 + i_{X_H} d\omega_0 = ddH_t = 0,$$

We are interested in the dynamical information coming from periodic orbits, that is, fixed points of ψ_1^H . More precisely, this information is going to come from the action value of the orbits.

2.1.1 Periodic orbits as critical points.

Let $z : [0, 1] \rightarrow \mathbb{R}^{2n}$ be a closed loop, i.e. a smooth curve such that $z(0) = z(1)$. For every smooth function $H : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ we consider the value

$$A_H(z(t)) = \int_0^1 p(t) \cdot \dot{q}(t) dt - \int_0^1 H_t(z(t)) dt.$$

Using integration by parts and the fact that $z(t)$ is periodic we get the equivalent expression

$$A_H(z(t)) = - \int_0^1 \frac{1}{2} J \dot{z}(t) \cdot z(t) dt - \int_0^1 H_t(z(t)) dt,$$

where the dot represents the real scalar product. Suppose that $z \in L^2(S^1, \mathbb{R}^{2n})$, then we may decompose it as a sum of Fourier modes

$$z(t) = \sum_{k \in \mathbb{Z}} e^{2i\pi kt} z_k,$$

with $z_k \in \mathbb{R}^{2n}$. With this decomposition the first part of the action becomes

$$\int_0^1 \frac{1}{2} J \dot{z}(t) \cdot z(t) dt = \pi \sum_{k \in \mathbb{Z}} k |z_k|^2,$$

which defines a continuous quadratic form on the dense subspace $H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n}) \subseteq L^2(S^1, \mathbb{R}^{2n})$. We want to see A_H as a function defined on $H^{\frac{1}{2}}$. The functions in $H^{\frac{1}{2}}$ may not be continuous so one has to impose some condition on H in order to, for example, have the second term bounded. In what follows we suppose that H is compactly supported although one may make the weaker assumption that H and all its derivatives have polynomial growth (see [HZ94] appendix A.3).

Lemma 2.1. *If H is compactly supported the functional $A_H : H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n}) \rightarrow \mathbb{R}$ is smooth and its critical points are exactly the smooth closed loops that solve*

$$\dot{z}(t) = X_H(z(t)).$$

Proof. The smoothness follows from the fact that H and all its derivatives have polynomial growth. For a proof see [HZ94] appendix A.3. For the second claim, a straightforward calculation yields

$$dA_H(z)(w) = \int_0^1 [-J \dot{z}(t) - \nabla H_t(z(t))] w(t) dt.$$

A point z verifies $dA_H(z)(w) = 0$ for every $w \in H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n})$ if and only if

$$\dot{z}(t) = J \nabla H_t(z(t)) = X_H(z(t)).$$

It remains to show that $z(t)$ is actually in $C^\infty([0, 1], \mathbb{R}^{2n})$. First, $dA_H(z)(1) = 0$ implies that $\nabla H_t(z(t))$ has vanishing mean. Moreover, since H is compactly supported we have $\nabla H_t(z(t)) \in L^2(S^1, \mathbb{R}^{2n})$. These two facts imply that there exists a $\xi \in H^1(S^1, \mathbb{R}^{2n})$ such that

$$\dot{\xi}(t) = J \nabla H_t(z(t)),$$

so $z = \xi \in H^1(S^1, \mathbb{R}^{2n})$. In particular z is a continuous weak solution of the Hamiltonian equation. This implies that

$$z(t) = z(0) + \int_0^t J \nabla H_t(z(t))$$

must be C^1 . The right hand side is then C^2 so z is C^2 . By recursion we get the z is in fact smooth. \square

2.1.2 The action spectrum.

For $H \in \mathcal{H}$ we denote the set of fixed points of the Hamiltonian map $\psi^H = \psi_1^H$ by

$$\text{Fix}(\psi^H) = \{z \in \mathbb{R}^{2n} \mid \psi^H(z) = z\}.$$

If $z_0 \in \text{Fix}(\psi^H)$ then the curve $z = (q, p) : [0, 1] \rightarrow \mathbb{R}^{2n}$ defined by $t \mapsto \psi_t^H(z_0)$ satisfies $z(0) = z(1)$ and hence is a loop in \mathbb{R}^{2n} . Its action is the real number

$$A(z_0, H) = A_H(z(t)) = \int_0^1 p(t) \cdot \dot{q}(t) dt - \int_0^1 H_t(z(t)) dt.$$

Note that if $t \mapsto z(t)$ is a constant curve, then the first term is zero. If in addition the Hamiltonian function H is time-independent then the action is just

$$A(z_0, H) = A_H(z(t)) = -H(z_0).$$

It turns out that the $A(z_0, H)$ depends only on the fixed point z_0 and on the map ψ^H and not on the particular H that generates ψ^H . This observation is contained in the following lemma:

Lemma 2.2 (see [HZ94] p. 151). *If H and K in \mathcal{H} generate the same time one map $\psi_1^H = \psi_1^K$, then*

$$A(z_0, H) = A(z_0, K)$$

for every $z_0 \in \text{Fix}(\psi^H) = \text{Fix}(\psi^K)$.

We can then associate to every fixed point z_0 of a map $\psi \in \text{Ham}^c(\mathbb{R}^{2n})$ the action

$$A(z_0, \psi) = A(z_0, H) \quad \text{if } \psi = \psi_1^H,$$

and we may consider the set of all such actions.

Definition 2.3. The action spectrum of $\psi \in \text{Ham}^c(\mathbb{R}^{2n})$ is the set $\sigma(\psi) \subseteq \mathbb{R}$ defined by

$$\sigma(\psi) = \{A(z, \psi) \mid z \in \text{Fix}(\psi)\}.$$

Note that $0 \in \sigma(\psi)$ for every $\psi \in \text{Ham}^c(\mathbb{R}^{2n})$ since the Hamiltonians functions have compact support. The following lemma allows us to use the action spectrum to extract symplectic information. Denote by \mathcal{G} the group of conformally symplectic diffeomorphisms, that is, diffeomorphisms ϕ which satisfy

$$\phi^* \omega_0 = \alpha \omega_0$$

for some $\alpha = \alpha(\phi) \in]0, \infty[$.

Lemma 2.4 (see [HZ94] p.152). *If $\psi \in \text{Ham}^c(\mathbb{R}^{2n})$ and $\phi \in \mathcal{G}$ with $\phi^* \omega_0 = \alpha \omega_0$ then*

$$A(\phi(z), \phi\psi\phi^{-1}) = \alpha A(z, \psi).$$

In particular

$$\sigma(\phi\psi\phi^{-1}) = \alpha\sigma(\psi).$$

The last property that we need from the action spectrum is the following lemma. A possible proof can be given using generating functions as in Chapter 3. It is the key property that allows us to prove the symplectic invariance for continuous action selectors.

Lemma 2.5 (see [HZ94] p.152). *The action spectrum $\sigma(\psi)$ is compact and nowhere dense.*

2.2 Action value selectors

In view of Lemma 2.4 we may use the action spectrum to define symplectic invariants. For example, since the spectrum is compact by Lemma 2.5 we can define the quantity

$$s(\psi) = \sup \sigma(\psi) = \sup \{A(z, \psi) \mid z \in \text{Fix}(\psi)\}.$$

Then, for every open set $U \subseteq \mathbb{R}^{2n}$ we may consider

$$s(U) = \sup \{c(\psi) \mid \psi \in \text{Ham}^c(\mathbb{R}^{2n})\}.$$

If ϕ is a symplectic diffeomorphism then by symplectic invariance of the spectrum $s(\psi) = s(\phi\psi\phi^{-1})$ so as a consequence $s(\varphi(U)) = s(U)$. Unfortunately this invariant is trivial. Indeed, for every open set one may construct an autonomous Hamiltonian with a minimum as small as we want, and this minimum generates a fixed point with action value equal to the value of $-H$ at this point. Since this minimum is arbitrary, we conclude $c(U) = +\infty$ for every open set. In order to construct a nontrivial capacity from an action selector we have to make a less obvious choice. In particular this choice has to be of a dynamical nature, so that it has useful information.

Over the group $\text{Ham}^c(\mathbb{R}^{2n})$ we can define the action spectrum "bundle"

$$\mathcal{A} = \bigcup_{\psi \in \text{Ham}^c(\mathbb{R}^{2n})} \{\psi\} \times \sigma(\psi).$$

Every fiber is a compact nowhere dense set. Since $0 \in \sigma(\psi)$ for every ψ there is a trivial section of \mathcal{A} . In the following chapter we give constructions of nontrivial sections

$$c : \text{Ham}^c(\mathbb{R}^{2n}) \rightarrow \mathcal{A}$$

i.e. maps that verify $c(\psi) \in \sigma(\psi)$. We use these sections to define a symplectic capacity. The following lemma assures that this can be done under some continuity hypothesis on c .

Lemma 2.6. *Let $c : \text{Ham}^c(\mathbb{R}^{2n}) \rightarrow \mathcal{A}$ be a section of the action bundle such that for any symplectic isotopy ψ_t the map*

$$t \mapsto c(\psi_t)$$

is continuous. Then for every conformally symplectic isotopy ϕ_t starting at the identity and every $\psi \in \text{Ham}^c(\mathbb{R}^{2n})$ we have

$$c(\phi_t\psi(\phi_t)^{-1}) = \alpha(t)c(\psi) \quad \text{for every } t \in [0, 1].$$

Proof. Every $\psi \in \text{Ham}^c(\mathbb{R}^{2n})$ and every conformally symplectic map defines a path of compactly supported symplectic maps (which is in fact a path in $\text{Ham}^c(\mathbb{R}^{2n})$) given by $t \mapsto \phi_t\psi(\phi_t)^{-1}$ for $t \in [0, 1]$. For $\alpha(t) = \alpha(\phi_t) > 0$ the map

$$t \mapsto \frac{1}{\alpha(t)}c(\phi_t\psi(\phi_t)^{-1})$$

is continuous and by Lemma 2.4 takes values in $\sigma(\psi)$. By Lemma 2.5 the subset $\sigma(\psi)$ is nowhere dense so this map must be constant equal to $c(\phi_0\psi\phi_0^{-1}) = c(\psi)$. \square

In particular, we see that a section satisfying this continuity hypothesis is invariant by symplectic conjugation. We can use it to define a symplectic invariant for open bounded sets of \mathbb{R}^{2n} by

$$\bar{c}(U) = \sup \{c(\psi) \mid \psi \in \text{Ham}^c(U)\},$$

where the notation $\text{Ham}^c(U)$ stands for the set of compactly supported Hamiltonian diffeomorphisms with support contained in U . Recall that the definition of a symplectic capacity is:

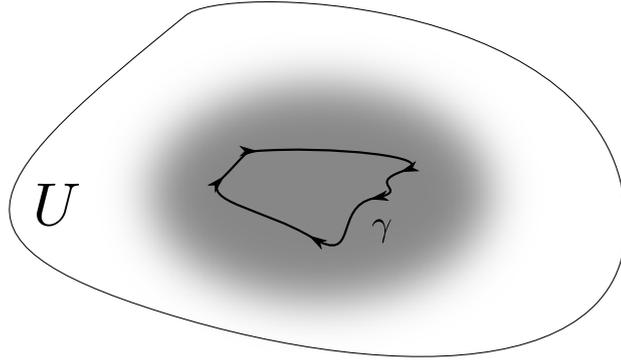


Figure 2.1: The shaded region represents the support of a Hamiltonian diffeomorphism ψ . The curve represents the periodic orbit γ such that $c(\psi) = A_H(\gamma)$.

Definition 2.7. Denote by $\mathcal{P}(\mathbb{R}^{2n})$ the power set of \mathbb{R}^{2n} . A symplectic capacity on \mathbb{R}^{2n} is a function $c : \mathcal{P}(\mathbb{R}^{2n}) \rightarrow [0, +\infty]$ that verifies the following properties:

1. (monotonicity) If $U \subseteq V$ then $c(U) \leq c(V)$.
2. (conformality) $c(\lambda U) = \lambda^2 c(U)$ for all $\lambda \in \mathbb{R}$.
3. (symplectic invariance) If $\phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a symplectomorphism then $c(\phi(U)) = c(U)$.
4. (non-triviality+normalization) $c(B_1^{2n}) = \pi = c(B_1^2 \times \mathbb{R}^{2n-2})$.

The invariant \bar{c} defined using a section satisfying the hypothesis of Lemma 2.6 verifies the three first properties of symplectic capacities. Let us explain a classical way of obtaining the normalization property:

For every small ϵ find a smooth compactly supported decreasing function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $h(t) = 1 - \epsilon$ for $t \leq 0$, $h(t) = 0$ for $t \geq 1 + \epsilon$, and $0 \leq |h'(t)| \leq (1 - \epsilon)$. We can use it to defined an autonomous Hamiltonian $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ by $H(z) = -\pi h(\|z\|^2)$. The gradient for this function is $\nabla H(z) = -2\pi h'(\|z\|^2)z$ so the Hamiltonian equation is

$$\dot{z} = J\nabla H(z) = 2\pi i h'(\|z\|^2)z.$$

In order to understand its flow, note that if $z : \mathbb{R} \rightarrow \mathbb{R}^{2n}$ is a Hamiltonian trajectory then

$$\frac{d}{dt}\|z(t)\|^2 = \frac{d}{dt}\langle z(t), z(t) \rangle = 2\langle \dot{z}(t), z(t) \rangle = 2\pi h'(\|z(t)\|^2)\langle iz(t), z(t) \rangle = 0,$$

so the norm is an invariant of the flow. In particular we can now write the explicit form of the flow as

$$\psi_t(z) = e^{2\pi i t h'(\|z\|^2)} z.$$

We conclude using the bound for h' that the only 1-periodic orbits are the ones with $h' = 0$. The orbits are the trivial ones with $h(z) = 0$ and the ones with $h(z) = -1 + \epsilon$. In particular

$$\sigma(\psi_H) = \{0, \pi - \epsilon\pi\}$$

If the selector c is nontrivial on this map we have $c(\psi) = \pi - \epsilon\pi$. Since ϵ can be chosen arbitrarily close to zero, we may prove the following lemma.

Lemma 2.8. *Suppose that c is an action selector verifying the assumptions of Lemma 2.6 and non-zero on functions h as above. Then the invariant defined by c verifies $\bar{c}(B_1^{2n}) \geq \pi$.*

2.2.1 Energy-capacity inequality

The classical way of obtaining the reverse inequality for $\bar{c}(B_1^{2n})$, of more precisely, of obtaining $\bar{c}(B_1^2 \times \mathbb{R}^{2n-2}) \leq \pi$ comes from the concept of displacement energy: for an open bounded subset $U \subseteq \mathbb{R}^{2n}$ we say that a Hamiltonian diffeomorphism ψ^H displaces U if

$$\psi^H(U) \cap U = \emptyset.$$

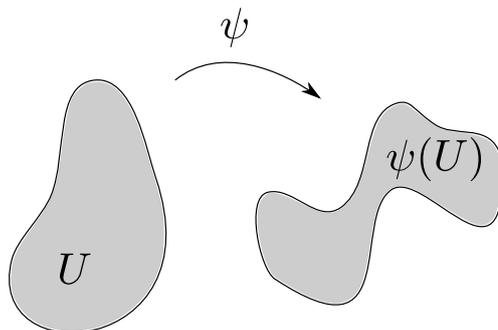


Figure 2.2: An open set U displaced by ψ .

One way of measuring the energy of a diffeomorphism ψ^H is via the following quantity:

Definition 2.9 (see [HZ94] p.146 or [Hof90a]). For every $H \in \mathcal{H}$ define

$$\|H\| = \int_0^1 \sup_z H(t, z) dt - \int_0^1 \inf_z H(t, z) dt,$$

and for every $\psi \in \text{Ham}^c(\mathbb{R}^{2n})$ define

$$E(\psi) = \inf\{\|H\| \mid \psi = \psi_1^H\}.$$

This measures the minimal variation of H that we need in order to generate the Hamiltonian diffeomorphism ψ . A priori it is not clear if $E(\psi) > 0$ for a non-identity element. It is possible to prove (for example using the tools of the following chapter) the following important result:

Theorem 2.10 (see [HZ94] p.146 or [Hof90a]). *Let ψ be an element of $\text{Ham}^c(\mathbb{R}^{2n})$. Then $E(\psi) = 0$ if and only if $\psi = \text{Id}$.*

For every compactly supported diffeomorphisms $\psi, \varphi \in \text{Ham}^c(\mathbb{R}^{2n})$ Hofer defines

$$d(\psi, \varphi) = E(\psi\varphi^{-1}).$$

The previous result implies that $d(\psi, \varphi) = 0$ is and only if $\psi = \varphi$. Using the definition for E one can moreover prove that d is bi-invariant by symplectic diffeomorphisms and that it moreover verifies the triangle inequality. In particular, it is a bi-invariant distance on the group of compactly supported Hamiltonian diffeomorphisms called the Hofer distance on $\text{Ham}^c(\mathbb{R}^{2n})$ (see [HZ94] for more details). We will not explicitly use this distance on this manuscript, but we present it to give an intuition for the closely related Viterbo displacement energy. Note that with this vocabulary, the energy $E(\psi)$ is the distance of ψ to the identity. Hofer uses E to define a new symplectic capacity:

Definition 2.11 (see [HZ94]). For every bounded open set $U \subseteq \mathbb{R}^{2n}$ we define the displacement energy of U as

$$E(U) = \inf\{E(\psi) \mid \psi(U) \cap U = \emptyset\}.$$

For every open subset $V \subseteq \mathbb{R}^{2n}$ we define $E(V)$ as the supremum of $E(U)$ for every open bounded set $U \subseteq V$. For every set $X \subseteq \mathbb{R}^{2n}$ we define $E(X)$ as the infimum of $E(V)$ for every open set V that contains X .

This time, the inequality that one may easily prove goes in the opposite direction.

Theorem 2.12 ([HZ94] p. 171). *Let $Z_r^{2n} = \{z \in \mathbb{R}^{2n} \mid q_1^2 + p_1^2 \leq r^2\}$ be a standard cylinder. Then*

$$E(Z_r^{2n}) \leq \pi r^2.$$

Proof. Fix $r = 1$. Every bounded open subset of Z_1^{2n} is contained in $B_1^2 \times B_R^{2n-2}$ for some $R > 0$, so it is enough to prove that

$$E(B_1^2 \times B_R^{2n-2}) \leq \pi$$

for every $R > 0$. We start by constructing a Hamiltonian in \mathbb{R}^2 that displaces B_1^2 . For this we first use an area- and orientation- preserving diffeomorphism φ of \mathbb{R}^2 to send the ball inside C , the square of sides $\sqrt{\pi} + \epsilon$ on the positive quadrant with a vertex on $(0, 0)$. Remark that the Hamiltonian $H(x, y) = (\sqrt{\pi} + 2\epsilon)x$ displaces C vertically for $t = 1$. In order to make it compactly supported we may multiply H by a function ξ which is equal to one on a neighborhood of the trajectory of C . In particular we can choose $\chi = 0$ outside $-\epsilon \leq x \leq \sqrt{\pi} + 2\epsilon$. Then $K = \chi H$ has norm $\|K\| = \pi + o(\epsilon)$. Moreover $\varphi^{-1} \circ \psi_t^K \circ \varphi$ displaces the ball from itself and it is generated by $K \circ \varphi$ (see appendix A). In order to get the diffeomorphism on \mathbb{R}^{2n} we can just multiply $K \circ \varphi$ by another compactly supported function of \mathbb{R}^{2n} with values on $[0, 1]$ and equal to one on the trajectory of $B_1^2 \times B_R^{2n-2}$. \square

In order to prove that both \bar{c} and E are normalized one usually proves an energy-capacity inequality: suppose that for every ψ Hamiltonian diffeomorphisms supported on a bounded open set U , and every $\varphi \in \text{Ham}^c(\mathbb{R}^{2n})$ such that it displaces U from itself, i.e. $\varphi(U) \cap U = \emptyset$, we have

$$c(\psi) \leq E(\varphi).$$

Then the definition of the invariants imply that

$$\bar{c}(U) \leq E(U).$$

Taking into account Lemma 2.8 and Theorem 2.12 we conclude that both c and E are normalized. The energy-capacity inequality is always a consequence of the following very important property of the action spectrum (see Definition 2.3).

Lemma 2.13 (see [HZ94] p.166). *Let ψ_t be a Hamiltonian flow supported on a bounded open set U , and let $\varphi \in \text{Ham}^c(\mathbb{R}^{2n})$ be such that it displaces U from itself, i.e. $\varphi(U) \cap U = \emptyset$. Then for every t we have*

$$\sigma(\psi_t \circ \varphi) = \sigma(\varphi) = \sigma(\varphi \circ \psi_t).$$

2.3 Other symplectic capacities

We end this chapter with a short description of another type of symplectic capacity which is not defined via a section of the action spectrum. This capacity also verifies an energy-capacity inequality. We introduce it since it was used by Kuksin in [Kuk95] to define an infinite dimensional capacity for Hamiltonian PDEs so it will appear again in Chapter 6.

We say that an autonomous Hamiltonian (i.e. a time independent Hamiltonian function) is slow if it does not have any periodic orbit of period smaller 1 other than its critical points. In [HZ90] they define the following capacity:

Definition 2.14. For every open set U of \mathbb{R}^{2n} we define

$$c_{HZ}(U) = \sup\{\max H \mid H \text{ is slow and compactly supported in } U\}.$$

Using a function as in Lemma 2.8 one may prove that $c_{HZ}(B_1^{2n}) \geq \pi$. The nontrivial inequality is again the opposite one, which can be recovered with an energy-capacity inequality.

Theorem 2.15 (see [HZ94] p.166). *For every open bounded set U we have*

$$c_{HZ}(U) \leq E(U).$$

This was one of the first capacities that was discovered.

Chapter 3

Generating functions and spectral invariants

In the previous section we saw that the action spectrum depends only on the time one map of the Hamiltonian diffeomorphism, and not on the precise Hamiltonian function that generates it. In particular all the symplectic information of the action is contained on the fixed points of $\psi = \psi_1^H$. A classical way of studying the fixed points of diffeomorphisms is to consider its graph

$$\Gamma_\psi = \{(z, \psi(z)) \mid z \in \mathbb{R}^{2n}\}$$

and look at the intersection points with the diagonal, i.e., the graph of the identity. Remark that ψ is compactly supported so both graphs coincide outside a compact set. One may use the symplectic structure of \mathbb{R}^{2n} to see Γ_ψ as a Lagrangian submanifold of the symplectic manifold $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \omega_0 \oplus (-\omega_0)) = \mathbb{R}^{2n} \times \overline{\mathbb{R}^{2n}}$. Working with linear manifolds has the advantage that there exists an explicit global symplectomorphism $\mathcal{I} : \mathbb{R}^{2n} \times \overline{\mathbb{R}^{2n}} \rightarrow T^*\mathbb{R}^{2n}$ such that $\mathcal{I}(\Gamma_{\text{Id}}) = 0_{\mathbb{R}^{2n}}$. This symplectomorphism is given by

$$\mathcal{I}(q, p, Q, P) = \left(\frac{q+Q}{2}, \frac{p+P}{2}, p-P, Q-q \right).$$

Since \mathcal{I} is a symplectomorphism, $\mathcal{I}(\Gamma_\psi)$ continues to be a Lagrangian submanifold which is Hamiltonian isotopic to the zero section. We will see that after compactification of the base manifold of $T^*\mathbb{R}^{2n}$ to T^*S^{2n} , these Lagrangian submanifolds can be described by the symplectic reduction of differential of functions $S : S^{2n} \times \mathbb{R}^m \rightarrow \mathbb{R}$ called generating functions. The critical points of S are in one to one correspondence with the intersection points of $\mathcal{I}(\Gamma_\psi)$ with the zero section $\mathcal{I}(\Gamma_{\text{Id}})$. We show that after normalization of S the set of critical values of S is exactly the action spectrum of ψ . We then review how to use minmax selectors of critical values to build a continuous sections of the action spectrum bundle verifying Lemma 2.8. These invariants are used to define a symplectic capacity and a displacement energy. Most of the results of this section can be found in Viterbo's original article [Vit92].

3.1 Definition

Let N be a compact smooth manifold and let L be a Lagrangian submanifold of T^*N which is Hamiltonian isotopic to the zero section. In light of the previous discussion we want to give a description of this submanifold which allows us to extract information from the intersection points with the zero section. Recall that the standard symplectic structure of T^*N is given by $\omega = -d\lambda$ where λ is the unique one form that verifies $\alpha^*\lambda = \alpha$ for every section of the cotangent space.

This characterization already yields a description of a big family of Lagrangian submanifolds: the graph of every closed one form of T^*N . In particular, the differential of a function gives a description of Lagrangian submanifolds but this description is not enough for Hamiltonian deformations of the zero section. These type of deformations are too flexible and they need not project one-to-one onto the zero section (see Figure 3.1), so we need a more general description.

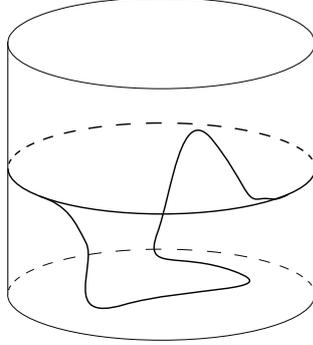


Figure 3.1: A possible Hamiltonian deformation of the zero section in $T^*\mathbb{T}$.

Introduce an auxiliary variable and consider functions $S : N \times \mathbb{R}^m \rightarrow \mathbb{R}$. The graph of the differential dS is a Lagrangian submanifold (still denoted dS) of the cotangent space $T^*(N \times \mathbb{R}^m) = T^*N \times T^*\mathbb{R}^m$. We will push forward the Lagrangian in a symplectic way by the projection onto the first coordinate to get a Lagrangian in T^*N . This operation is called symplectic reduction and is based on the following lemma:

Lemma 3.1. *Let V be a symplectic vector space, let $W \subseteq V$ a coisotropic subspace and let $L \subseteq V$ be a Lagrangian subspace. Denote by $\pi : W \rightarrow W/W^\omega$ the natural projection. If L is transverse to W then $\pi(L \cap W) \subseteq W/W^\omega$ is a Lagrangian subspace.*

Proof. First note that L is isotropic so $L \cap W$ is also isotropic and this implies that $\pi(L \cap W)$ is also isotropic. In order to prove that $\pi(L \cap W)$ is Lagrangian it remains to prove that it has half of the dimension of W/W^ω . Now remark that $L + W = V$ implies $L^\omega \cap W^\omega = V^\omega$ and using that L is Lagrangian we get $L \cap W^\omega = \{0\}$. This means that π is injective on $L \cap W$ so $\pi(L \cap W)$ has the same dimension as $L \cap W$. Using $\dim W + \dim W^\omega = 2n$ we get

$$\dim W/W^\omega = 2n - 2 \dim W^\omega.$$

On the other hand we have

$$\dim L \cap W = \dim L + \dim W - 2n = \dim W - n = (2n - \dim W^\omega) - n = n - \dim W^\omega,$$

and using the previous two equalities we get

$$\dim L \cap W = \frac{1}{2} \dim W/W^\omega,$$

which finishes the proof. \square

We are going to apply this lemma to $L = dS$ and $W = T^*N \times 0_{\mathbb{R}^m}$. We are no longer in the linear case so we have to make sure that the result carries over. A coisotropic submanifold is a submanifold with coisotropic tangent spaces (see Appendix A). The symplectic orthogonal will define an integrable distribution and a foliation called characteristic foliation (see Lemma A.7). In order to consider subsets on the quotient, one first needs to be sure that the quotient by the characteristic foliation is a well defined manifold. This is not always be the case.

Example 3.2. Consider the cotangent space $T^*N \times T^*M$ for two smooth manifolds N and M . Then

$$T^*N \times 0_M \quad \text{and} \quad T^*N \times T_m^*M \quad \text{with} \quad m \in M$$

are coisotropic submanifolds. The symplectic reduction W/W^ω is in both cases naturally isomorphic to T^*N .

Definition 3.3 (Symplectic reduction). Let M be a symplectic manifold and let $W \subseteq M$ be a coisotropic submanifold. Consider a subset $Z \subseteq M$, then we will denote

$$\text{Red}_W(Z) := \pi(Z \cap W) \subseteq W/W^\omega.$$

Together Lemma 3.1 and Example 3.2 imply that if dS is transverse to $T^*N \times 0_{\mathbb{R}^m}$ then the projection π restricted to dS gives a Lagrangian immersion into T^*N . This is the definition of a generating function. We will moreover ask for a condition at infinity. This condition will allow us to have a better control of the function and in particular guarantees that S has enough critical points.

Definition 3.4. We say that a function $S : N \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a generating function or *gf* for an immersed Lagrangian submanifold $L \subseteq T^*N$ if

- dS is transverse to $W = T^*N \times 0_{\mathbb{R}^m}$.
- $L = \text{Red}_W(dS)$.

Moreover we say that S is quadratic at infinity or *gfqi* if S coincides with a nondegenerate quadratic form Q outside a compact set.

Explicitly, the first condition amounts to asking that the map

$$(q, \xi) \mapsto \frac{\partial S}{\partial \xi}(q, \xi)$$

has zero as a regular value. The transversality condition implies that

$$\Sigma_S := dS^{-1}(dS \cap W) = \frac{\partial S^{-1}}{\partial \xi}(0)$$

is a submanifold contained in $N \times \mathbb{R}^m$ of the same dimension of N . We will denote by $i_S : \Sigma_S \mapsto T^*N$ the immersion

$$(q, \xi) \mapsto \left(q, \frac{\partial S}{\partial q}(q, \xi) \right).$$

By definition of S the image of this map is L . The condition of being exactly quadratic at infinity may seem a little strong. An important remark is that it can be relaxed to a condition on the fiber derivative. This is the content of the following proposition which is taken from Théret's thesis.

Proposition 3.5 (see [Thé95]). *Let N be a compact manifold and let $S : N \times \mathbb{R}^k \rightarrow \mathbb{R}$ be a gf for an immersed Lagrangian L . Suppose that there exists a nondegenerate quadratic form Q of \mathbb{R}^l for which $\partial_\xi(S - Q)$ is bounded. Then there exists a fiber preserving diffeomorphism $\varphi : N \times \mathbb{R}^k \rightarrow N \times \mathbb{R}^k$ such that $S' = S \circ \varphi$ is a gfqi for L with quadratic form Q .*

Proof. Denote $R = S - Q$ and $S_t = Q + tR = (1 - t)Q + tS$. We look for a fiber isotopy $\varphi_t(q, \xi) = (q, \sigma_t(q, \xi))$ such that $S_t \circ \varphi_t = Q$ outside a compact set. Differentiating with respect to t , and denoting by $(0, X_t)$ the infinitesimal generator of φ_t we want X_t to verify

$$\partial_\xi S_t(X_t) + \partial_t S_t = 0 \quad \text{outside a compact set.}$$

We may write the condition as

$$(\nabla Q + t\partial_\xi(S - Q))(X_t) = -R.$$

Note that Q is a non-degenerate quadratic form so $|\nabla Q(\xi)|$ goes to infinity with ξ . Note moreover that by hypothesis $t\partial_\xi(S - Q)$ is bounded. Using these two facts we see that outside a big enough compact set, $\partial_\xi S_t$ is non zero so we may define

$$X_t = \frac{-R}{|\partial_\xi S_t|^2} \partial_\xi S_t$$

which verifies the desired equality outside a compact set. Now make this vector field global with the help of a step function which is zero on a big enough compact set, and equal to one at infinity. In order to finish the proof we just need to check that this vector field is globally integrable. We have

$$|X_t| \leq \frac{|R|}{|\partial_\xi S_t|} \leq a|R(q, \xi)| \leq a|R(q, 0) + \int_0^1 \partial_\xi R(q, t\xi) \cdot \xi dt| \leq a' + b'|\xi|$$

where we used that N is compact and that $\partial_\xi R$ is bounded. We may now use Gronwall's Lemma to prove that trajectories do not explode in finite time, so the flow is globally defined. \square

In the case where L is not only immersed but also embedded, one can characterize the map i_S . This is important since we want to have a bijective correspondence between critical points of S and intersection points with the zero section.

Lemma 3.6. *Let N be a connected compact manifold and let S be a *gfqi* for a Hamiltonian isotopy of the zero section in T^*N . Then Σ_S is a covering space of L and i_S is a covering map of finite degree.*

Proof. By definition Σ_S is a closed set. Moreover, the quadratic at infinity condition implies that Σ_S is contained in a compact set so we conclude that it is a compact submanifold. The map i_S is an immersion and both Σ_S and L have the same dimension so i_S is in fact a local diffeomorphism. Note that Σ_S is compact and L connected so a standard argument shows that the fibers are finite with the same cardinal, which gives the result. \square

Example 3.7. Let N be a compact manifold, we are going to build a *gfqi* for the zero section where i_S is not a diffeomorphism. We look for *gf* $S : N \times \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$S(q, \xi) = F(\xi),$$

with $F : \mathbb{R} \rightarrow \mathbb{R}$. Once the transversality condition is verified, it is clear that S will generate the zero section. Denote by f the derivative of F . In this case the transversality condition of Definition 3.4 amounts to asking that the map

$$(q, \xi) \mapsto \frac{\partial S}{\partial \xi}(q, \xi) = f(\xi)$$

has zero as a regular value. In other words, we want that for every $\xi \in f^{-1}(0)$ we have $f'(\xi) \neq 0$. In order to address the quadratic at infinity condition we first recall that quadratic forms of \mathbb{R}

are of the form $Q(\xi) = a\xi^2$. If we want an S quadratic at infinity, using Proposition 3.5 we see that it is enough to find a function f and a constant C such that $|f(\xi) - 2a\xi| \leq C$ for every $\xi \in \mathbb{R}$. Fix $2a = 1$ and consider the function

$$f(\xi) = \begin{cases} \xi - 1 & \text{if } 1 \leq \xi \\ -\frac{\sin \pi \xi}{\pi} & \text{if } -1 \leq \xi \leq 1 \\ \xi + 1 & \text{if } \xi \leq -1 \end{cases}$$

We have $f(\xi) = 0$ if and only if $\xi \in \{-1, 0, 1\}$. Moreover $f'(-1) = 1$, $f'(0) = -1$ and $f'(1) = 1$

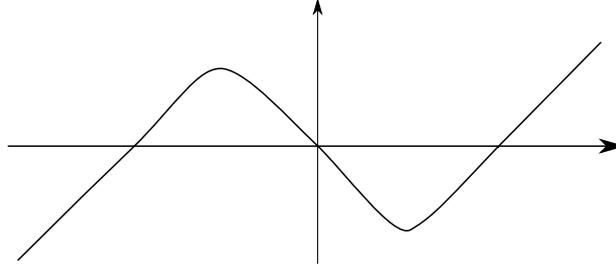


Figure 3.2: A representation of the graph of f .

so the transversality condition is verified. We may now use Proposition 3.5 in order to find a fiber preserving diffeomorphism $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that $S \circ \varphi = F \circ \varphi$ is exactly quadratic at infinity. Moreover we may choose φ to be the identity on a neighbourhood of $[-1, 1]$. In particular we have

$$\Sigma_S = N \times \{-1\} \cup N \times \{0\} \cup N \times \{1\}$$

and $i_S : \Sigma_S \rightarrow 0_N \subseteq T^*N$ is a 3-to-1 covering map. We see that in this case the critical points of S are not in one to one correspondence with the intersection points with the zero section.

3.2 Existence

In this section we prove the existence of generating functions quadratic at infinity S where the map i_S is a diffeomorphism for compactly supported Hamiltonian deformations of the zero section in $T^*\mathbb{R}^n$. The proof that we present is based on the one that appears in [Vit06]. The proof for general compact manifolds N follows using the argument in Brunella's article [Bru91]. Let $\psi = \psi_t^H$ be a compactly supported Hamiltonian diffeomorphisms in $T^*\mathbb{R}^n$. The key observation is that for small t , the generating function of $\psi_t(L)$ can be explicitly described using the concept of generating functions for symplectic diffeomorphisms and the gf for L .

Consider the graph of a symplectomorphism φ as a Lagrangian submanifold in $T^*\mathbb{R}^n \times \overline{T^*\mathbb{R}^n}$. Then using the involution $(q, p) \mapsto (q, -p)$ for the second coordinate we may send it in a symplectic way into $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ to get another Lagrangian submanifold. We want to identify the diffeomorphisms φ whose image in $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ can be described by the differential of a function h .

Definition 3.8. Let φ be a symplectic diffeomorphisms of $T^*\mathbb{R}^n$. We say that a function $h : \mathbb{R}_x^n \times \mathbb{R}_y^n \rightarrow \mathbb{R}$ is a generating function for φ if

$$\varphi(q_1, p_1) = (q_2, p_2) \quad \text{if and only if} \quad p_1 = \frac{\partial h}{\partial x}(q_1, q_2) \quad \text{and} \quad p_2 = -\frac{\partial h}{\partial y}(q_1, q_2).$$

In that case we write $\varphi = \varphi_h$.

Example 3.9. Consider the function $h_0(x, y) = \frac{1}{2}|x - y|^2$. In this case we want

$$p_1 = \frac{\partial h}{\partial x}(q_1, q_2) = q_1 - q_2 \quad \text{and} \quad p_2 = -\frac{\partial h}{\partial y}(q_1, q_2) = q_1 - q_2,$$

so h_0 generates the diffeomorphism

$$\varphi(q_1, p_1) = (q_1 - p_1, p_1).$$

Moreover, the same easy calculation tells us that φ^{-1} is generated by $-h_0$.

Lemma 3.10. *Every compactly supported Hamiltonian diffeomorphism $\psi = \psi_1^H$ can be written as a composition*

$$\psi = \varphi_{h_m} \circ \cdots \circ \varphi_{h_1},$$

where h_i is equal to h_0 or $-h_0$ outside a compact set.

Proof. Write $\psi_t = \psi_t \circ \varphi_{h_0} \circ \varphi_{-h_0}$. The map $\psi_t \circ \varphi_{h_0}$ is symplectic and coincides with φ_{h_0} outside a compact set. Its image in $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ is Lagrangian and coincides with dh_0 outside a compact set. For t small enough it is also the graph of a closed one form, which is exact since we are in $\mathbb{R}^n \times \mathbb{R}^n$. It is then the graph of dh and coincides with dh_0 outside a compact set. We conclude that $\psi_t \circ \varphi_{h_0} = \varphi_h$. For bigger t we just need to decompose ψ_t in small pieces. The map ψ_t is compactly supported so for every ϵ we may find a decomposition such that each piece is at least ϵ close to the identity. \square

We now give the stability of the property of having gf by symplectic maps of the form φ_h .

Proposition 3.11 (Chekanov's composition formula). *Let $L \subseteq T^*\mathbb{R}^n$ be a Lagrangian submanifold generated by a generating function $S : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$. Let φ be a symplectomorphism of $T^*\mathbb{R}^n$ that has a generating function h . If h is sufficiently C^2 close to h_0 then $\varphi(L)$ has a generating function given by*

$$\begin{aligned} K : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n &\rightarrow \mathbb{R} \\ (q, \xi, \eta) &\mapsto -h(\eta, q) + S(\eta, \xi) \end{aligned}$$

Proof. We have that $(q, \xi, \eta) \in \Sigma_K$ if and only if

$$\frac{\partial S}{\partial \xi}(\eta, \xi) = 0 \quad \text{and} \quad \frac{\partial h}{\partial x}(\eta, q) = \frac{\partial S}{\partial q}(\eta, \xi).$$

Moreover, using the fact that φ is generated by h (see Definition 3.8) we get

$$i_K(q, \xi, \eta) = (q, \frac{\partial K}{\partial q}(q, \xi, \eta)) = (q, -\frac{\partial h}{\partial y}(\eta, q)) = \varphi(\eta, \frac{\partial h}{\partial x}(\eta, q)) = \varphi(\eta, \frac{\partial S}{\partial q}(\eta, \xi))$$

which implies $L_K = \varphi(L)$. To end the proof we need to see that K verifies the transversality condition. Denote by F the function

$$F(q, \xi, \eta) = \left(\frac{\partial K}{\partial \xi}, \frac{\partial K}{\partial \eta} \right) = \left(\frac{\partial S}{\partial \xi}, \frac{\partial S}{\partial q} - \frac{\partial h}{\partial x} \right)$$

By hypothesis S satisfies the transversality condition. If we denote Π_m the projection on the first m coordinates, then for $x \in \Sigma_K$,

$$\Pi_m D_x F(\{0\} \times \mathbb{R}^{m+n}) = \mathbb{R}^m \times \{0\}.$$

To end the proof it is enough to show that

$$D_x F(\mathbb{R}^n \times \{0\}) = \{0\} \times \mathbb{R}^n.$$

This means that we have to show that $q \mapsto \frac{\partial h}{\partial x}(\eta, q)$ is a submersion. To conclude, recall that $x \mapsto \frac{\partial h_0}{\partial x}(x, y) = x - y$ is a submersion for every y , so the same property will be true for every h sufficiently C^2 close to h_0 . \square

Definition 3.12. We say that a generating function $S : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ satisfies property (P) if

- there exists a non-degenerate quadratic form $Q : \mathbb{R}^m \rightarrow \mathbb{R}$ such that for every $C > 0$

$$\text{Supp}(S - Q) \cap \{(q, \xi) \in \mathbb{R}^n \times \mathbb{R}^m \mid |\xi| \leq C\}$$

is compact.

- $S - Q$ has bounded C^k norm for every k .

We will say that a Lagrangian submanifold L satisfies property (P) if it is generated by a gf which satisfies property (P).

Remark 3.13. The importance of property (P) is that it guarantees that S has a smooth extension to $S^n \times \mathbb{R}^m$ by $S(N, \xi) = Q(\xi)$.

Lemma 3.14. *Let $L \subseteq T^*\mathbb{R}^n$ be a Lagrangian submanifold that satisfies property (P). Then if h is sufficiently close to h_0 (or $-h_0$) and coincides with it outside a compact set, the Lagrangian $\varphi_h(L) \subseteq T^*\mathbb{R}^n$ satisfies property (P).*

Proof. Let S be a gf satisfying property (P) and let K be the gf obtained using Proposition 3.11. Define $f(q, \xi, \eta) = (q, \xi, q - \eta)$. We claim that $K \circ f$ satisfies property (P). The map f is a fiber preserving diffeomorphism so $K \circ f$ generates the same Lagrangian. Note that $h_0(q - \eta, q) = \frac{1}{2}|\eta|^2$ is a non-degenerate quadratic form and denote it by $Q_0(\eta)$. By definition of K we have

$$K \circ f(q, \xi, \eta) = K(q, \xi, q - \eta) = -h(q - \eta, q) + S(q - \eta, \xi).$$

We know that $h = h_0$ outside a compact set and that S satisfies property (P) so for any $C > 0$, if $|(\xi, \eta)| \leq C$ then for q big enough we will have $K \circ f(q, \xi, \eta) = -Q_0(\eta) + Q(\xi)$. Finally, using again these same properties one can see that $K \circ f + Q_0 - Q$ has bounded C^k norm for every k . \square

Corollary 3.15. *Let $L \subseteq T^*\mathbb{R}^n$ be a Lagrangian submanifold that satisfies property (P) and let ψ_t be a compactly supported hamiltonian isotopy of $T^*\mathbb{R}^n$. Then for sufficiently small t_0 , there is a continuous path $t \mapsto S_t$ of gf satisfying property (P) for the same Q such that S_t generates $\psi_t(L)$ for $0 \leq t \leq t_0$.*

Proof. We just need to check at every step that if we introduce the parameter t nothing changes. First one checks that dh_t depends continuously on t , so h_t too. And then that we can choose $h_t = h_0$ outside a compact set independent of t , this is the case since the support of ψ_t is contained in a compact set independent of t . The result follows from the previous lemma since for small enough t we have $\psi_t = \psi_t \circ \varphi_{h_0} \circ \varphi_{-h_0} = \varphi_h \circ \varphi_{-h_0}$. \square

Corollary 3.16. *Let $L \subseteq T^*\mathbb{R}^n$ be a Lagrangian submanifold that satisfies property (P) and let ψ_t be a compactly supported hamiltonian isotopy of $T^*\mathbb{R}^n$. Then for $t \in [0, 1]$, there is a continuous path S_t of gf satisfying property (P) for the same Q such that S_t generates $\psi_t(L)$ for $0 \leq t \leq 1$.*

Proof. Since ψ_t is compactly supported, we can divide the time interval in a fixed number of pieces as we did in Lemma 3.10, such that on each piece we have a small enough diffeomorphisms which will verify Corollary 3.15. \square

Theorem 3.17 (Laudenbach and Sikorav [LS85, Sik86, Sik87]). *Let ψ_t be a compactly supported Hamiltonian isotopy of \mathbb{R}^{2n} and for every t consider the image of its graph $\mathcal{I}(\Gamma_{\psi_t})$ in $T^*\mathbb{R}^{2n}$ (see the beginning of the chapter). Let L_t be the one point compactification of the Lagrangians $\mathcal{I}(\Gamma_{\psi_t})$ in T^*S^{2n} . Then for $t \in [0, 1]$, there exists a continuous path $t \mapsto S_t$ of $gfqi$ for the same Q such that S_t generates L_t and i_{S_t} is a diffeomorphism.*

Proof. Denote by $\bar{\pi} : \mathbb{R}^{2n} \times \overline{\mathbb{R}^{2n}} \rightarrow \overline{\mathbb{R}^{2n}}$ the projection. The Hamiltonian isotopy $\psi_t = \psi_t^H$ induces a Hamiltonian isotopy in $\mathbb{R}^{2n} \times \overline{\mathbb{R}^{2n}}$ via the Hamiltonian function $H \circ \bar{\pi}$ which is not compactly supported. We are interested in the deformation of the zero section given by $\mathcal{I}(\Gamma_{\psi_t})$ in $T^*\mathbb{R}^{2n}$. This deformation is induced by the Hamiltonian $K = H \circ \bar{\pi} \circ \mathcal{I}^{-1}$. An explicit calculation yields that $\mathcal{I}^{-1} : T^*\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \times \overline{\mathbb{R}^{2n}}$ is given by

$$\mathcal{I}^{-1}(x, y, X, Y) = \left(\frac{2x - Y}{2}, \frac{2y + X}{2}, \frac{2x + Y}{2}, \frac{2y - X}{2} \right).$$

Note that ψ_t is compactly supported so the deformation of the diagonal in $\mathbb{R}^{2n} \times \overline{\mathbb{R}^{2n}}$ happens in a compact set. As a consequence for $t \in [0, 1]$ the deformation $\mathcal{I}(\Gamma_{\psi_t})$ will not leave the disc bundle $D_R \subseteq T^*\mathbb{R}^{2n}$ for some R where D_R denotes the subset of points $(x, y, X, Y) \in T^*\mathbb{R}^{2n}$ with $|(X, Y)| \leq R$. Consider now a smooth function ρ equal to one on D_R , and equal to zero on the complement of $D_{R'}$ for some $R' > R$. The Hamiltonian ρK is compactly supported (use the expression of \mathcal{I}^{-1} and the fact that H is compactly supported) and it generates a compactly supported Hamiltonian isotopy of $T^*\mathbb{R}^{2n}$ whose deformation of the zero section coincides with $\mathcal{I}(\Gamma_{\psi_t})$. Since the zero section has the zero function as a $gfqi$ that satisfies property (P), we may use Corollary 3.16 to get a continuous deformation of $gf S_t$ satisfying property (P). Property (P) now guarantees that we can extend the functions S_t smoothly to S^{2n} . In order to make them exactly quadratic at infinity we use Proposition 3.5.

To see that the map i_{S_t} for the $gfqi S_t$ is a diffeomorphism, note that by Lemma 3.6 we know that i_S is a covering map with finite fiber. By definition of the extension to S^{2n} of S we have $S(N, \xi) = Q(\xi)$ and this implies that $i_S^{-1}(N, 0) = (N, 0)$ is the only preimage. \square

3.3 Uniqueness

We now explain the uniqueness theorem for $gfqi$. There are three ways of obtaining a new gf from an existing one. These basic operations are the following:

Definition 3.18. Let $S : N \times \mathbb{R}^m \rightarrow \mathbb{R}$ be a generating function.

- (Addition of a constant) If $c \in \mathbb{R}$, we set $S' := S + c : N \times \mathbb{R}^m \rightarrow \mathbb{R}$.
- (Diffeomorphisms operation) Let Φ be a fiber preserving diffeomorphisms of $N \times \mathbb{R}^m$, i.e. $\Phi(q, \xi) = (q, \phi(q, \xi))$. We set $S' := S \circ \Phi : N \times \mathbb{R}^m \rightarrow \mathbb{R}$.
- (Stabilization) Let $Q' : \mathbb{R}^{m'} \rightarrow \mathbb{R}$ be a non-degenerate quadratic form. We set $S' := S + Q' : N \times \mathbb{R}^m \times \mathbb{R}^{m'} \rightarrow \mathbb{R}$.

Remark 3.19. If we start with a $gfqi$ then the addition of a constant or stabilization do not give a $gfqi$. Nevertheless, using Proposition 3.5 one may compose by a fiber preserving diffeomorphism which makes the gf exactly quadratic at infinity. The diffeomorphism operation does not always preserve the quadratic at infinity property, but it will be the case if the diffeomorphism is compactly supported.

Definition 3.20. Two generating functions are equivalent if they can be made equal by a succession of basic operations.

Remark 3.21. The set of critical values of two equivalent generating functions differ only by a constant.

It was Viterbo's idea to use the set of critical values of *gfqi* to define symplectic invariants. The following theorem is the starting point of the construction.

Theorem 3.22 (Viterbo [Vit92] and Théret [Thé99a]). *Let N be a compact manifold and let $\psi : T^*N \rightarrow T^*N$ be a Hamiltonian diffeomorphism. Then the *gfqi* S of $\psi(0_N)$ such that i_S is a diffeomorphism are all equivalent.*

Outline of the proof. The proof is carried out in two steps:

(1) *The set of closed Lagrangian submanifolds whose *gfqi* of degree one are all equivalent is stable under Hamiltonian isotopies.* To see this, let L_0 and $L_1 = \psi(L_0)$ be two Lagrangian submanifolds. Suppose that the *gfqi* of L_0 are all equivalent and let S and S' be two *gfqi* for L_1 . Using Sikorav's existence theorem (a generalization of Theorem 3.17) one may build two paths of *gfqi* of degree one S_t and S'_t that generate $\psi_t(L_0)$ for all $t \in [0, 1]$. Now since L_0 has the uniqueness property, we may build a loop of *gfqi* which starts and ends at L_1 . One now proves that since this loop is contractible at the level of Lagrangians it will also be contractible at the level of *gfqi* (more precisely that the map that send S to L_S is a Serre fibration). This reduces the case to proving that a smooth path of *gfqi* which generate the same L_1 are all equivalent. To prove this one may use Moser's trick and look for a fiber preserving isotopy Φ_t such that $S_t \circ \Phi_t = S_0$.

(2) *The *gfqi* S of the zero section such that i_S is a diffeomorphism are all equivalent.* This is a purely differential problem. Let $S : N \times \mathbb{R}^m \rightarrow \mathbb{R}$ be one such *gfqi*. We are going to consider the family $S_q = S(q, \cdot) : \mathbb{R}^m \rightarrow \mathbb{R}$. Since S generates the zero section and i_S is a diffeomorphism, each S_q has only one critical point which is non-degenerate and we may suppose that it is located at the origin. Then one proves that S is equivalent to

$$Q(q, \xi) = \frac{1}{2} d^2 S_q(0)(\xi, \xi).$$

S_q is a Morse function so locally we always have $S_q \simeq Q(q, \cdot)$. The difficulty arises when we try to find a global equality which depends smoothly on the variable q . Remark that Q is not constant on the base variable, but one may easily construct a fiber preserving diffeomorphism which makes it constant. \square

Remark 3.23. In Théret's article the hypothesis of i_S diffeomorphism is not included in the definition of *gf* [Thé99a, Definition 2.1] but it is included in the terminology " S generates L " [Thé99a, Definition 2.2].

3.4 Critical value selectors

In this section we will exploit the quadratic at infinity condition in order to find critical values for *gfqi*. We are going to use the minmax principle on the sublevel sets of a *gfqi*. Denote

$$S^c := \{(q, \xi) \in N \times \mathbb{R}^m \mid S(q, \xi) \leq c\}.$$

Let $[a, b]$ be an interval of regular values of S . The classical argument of deformation of sublevel sets in Morse theory tells us that S^a is diffeomorphic to S^b . Now remark that since S is quadratic at infinity, all the critical values of S are contained in a compact set. In particular for $c \in \mathbb{R}$ big

enough all the pairs of sublevel sets (S^c, S^{-c}) are diffeomorphic. We will fix one such c and denote $(S^{+\infty}, S^{-\infty}) = (S^{c_0}, S^{-c_0})$. We use the topology of the sublevel set to find critical values of the generating function using minmax arguments. More precisely, we look for the sub-levels where particular cohomological classes disappear. This indicates that there is a topological change of the sub-level, so there has to be a critical point.

It turns out that for *gfqi* there is a way of choosing the cohomological class α that depends only on N . To see this, note that for $c \in \mathbb{R}$ big enough

$$(S^{+\infty}, S^{-\infty}) = (N \times Q^{+\infty}, N \times Q^{-\infty}).$$

Now Q is a Morse function with only one critical point at the origin of index $\text{Ind}(Q) = k$. A standard argument in Morse theory now tells us that $Q^{+\infty}$ is a deformation retract of $Q^{-\infty}$ with one k cell attached. Going to cohomology over all field, one finds that

$$H^*(S^{+\infty}, S^{-\infty}) = H^*(N \times (Q^{+\infty}, Q^{-\infty})) \simeq H^*(N) \otimes H^*(Q^{+\infty}, Q^{-\infty}) \simeq H^*(N) \otimes H^*(D^k, \partial D^k).$$

where $H^n(D^k, \partial D^k)$ is non-zero if and only if $n = k$. In that case $H^k(D^k, \partial D^k)$ is a vector space of dimension one. Using the previous chain of isomorphisms, we get an isomorphism

$$T : H^*(N) \rightarrow H^{*+k}(S^{+\infty}, S^{-\infty})$$

that we will call the Thom isomorphism.

Definition 3.24. For every $\alpha \in H^*(N)$ we define

$$c(\alpha, S) = \inf\{c \mid i_c^* T \alpha \neq 0\}.$$

It is a critical value of S (see Appendix C or for example [Nic11, section 2.7]).

3.4.1 Properties

The advantage of this critical value selector is that one may exploit the algebraic properties of the cohomology groups. The following theorem tells us that these selectors are continuous with respect to the C^0 norm on *gfqi*, that we have a triangular inequality and a kind of nondegeneracy. The interested reader can look at the original reference [Vit92] for a proof using algebraic topology or the article [Mil99] for a proof using Morse theory.

Theorem 3.25 ([Vit92]). *Let N be a compact connected and oriented manifold of dimension n and consider a *gfqi* $S : N \times \mathbb{R}^m \rightarrow \mathbb{R}$ which generates a Lagrangian submanifold $L \subseteq T^*N$.*

- *For every $\alpha \in H^*(N)$, the map $c(\alpha, \cdot) : S \mapsto c(\alpha, S)$ is C^0 continuous. Moreover, it has the following behavior with respect to the basic operations on *gfqi* (cf. Definition 3.18):*

$$\begin{aligned} c(\alpha, S + c) &= c(\alpha, S) + c \\ c(\alpha, S \circ \Phi) &= c(\alpha, S) \\ c(\alpha, S \oplus Q) &= c(\alpha, S) \end{aligned}$$

- *If $\mu \in H^n(N)$ and $1 \in H^0(N)$ are the generators of $H^n(N)$ and $H^0(N)$, then*

$$c(1, -S) = -c(\mu, S).$$

Moreover, we have the inequality

$$c(1, S) \leq c(\mu, S)$$

with $c(\mu, S) = c(1, S)$ if and only if S generates the zero section.

- Let $S_1 : N \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $S_2 : N \times \mathbb{R}^{m'} \rightarrow \mathbb{R}$ be two generating functions, and consider

$$S_1 \# S_2(q, \xi, \eta) = S_1(q, \xi) + S_2(q, \eta).$$

It is a function which can be made quadratic at infinity via a fiber preserving diffeomorphism (Proposition 3.5) so we can use the minmax selector of critical values. For every $u, v \in H^*(N)$ we have

$$c(u \cup v, S_1 \# S_2) \geq c(u, S_1) + c(v, S_2).$$

The function $S_1 \# S_2$ is not always a *gfqi* since the transversality condition may not be verified. Nevertheless, in some cases we can prove that the critical value that we select for this function coincides with the critical value that we select for a true *gfqi*. Let $\psi, \varphi \in \text{Ham}^c(\mathbb{R}^{2n})$ and consider L_ψ and L_φ be the two lagrangian submanifolds in T^*S^{2n} induced by them. They coincide with the zero section on a neighborhood of the north pole. If we normalize the *gfqi* of degree one asking that $S(i_S^{-1}(N, 0)) = 0$ then the previous theorem implies that the critical value of S is independent of the *gfqi* used to calculate it. We denote by S_ψ a normalized *gfqi* for L_ψ .

Proposition 3.26 (see [Vit92] proposition 3.5). *Using the notation of the previous paragraph for every $u \in H^*(S^{2n})$ we have*

$$c(u, S_{\psi \circ \varphi}) = c(u, S_\varphi \# (-S_{\psi^{-1}})).$$

3.5 Viterbo's capacities

We now use the previous results to study the properties of compactly supported Hamiltonian diffeomorphisms of \mathbb{R}^{2n} . Recall (Lemma 2.2) that the action spectrum depends only on the time one map of the Hamiltonian diffeomorphism, and not on the precise Hamiltonian function that generates it. For this reason we will just study the fixed points of $\psi = \psi_1^H$ instead of the whole path. Consider the graph of ψ

$$\Gamma_\psi = \{(z, \psi(z)) \mid z \in \mathbb{R}^{2n}\}$$

and look at the intersection points with the diagonal, i.e., the graph of the identity. Γ_ψ is a Lagrangian submanifold of $\mathbb{R}^{2n} \times \overline{\mathbb{R}^{2n}}$ and we can see it as a Lagrangian inside $T^*\mathbb{R}^{2n}$ using the explicit global symplectomorphism $\mathcal{I} : \mathbb{R}^{2n} \times \overline{\mathbb{R}^{2n}} \rightarrow T^*\mathbb{R}^{2n}$ given by

$$\mathcal{I}(q, p, Q, P) = \left(\frac{q+Q}{2}, \frac{p+P}{2}, p-P, Q-q \right).$$

Since \mathcal{I} is a symplectomorphism, $\mathcal{I}(\Gamma_\psi)$ continues to be a Lagrangian submanifold which is Hamiltonian isotopic to the zero section. The fixed points of ψ are then sent to intersection points with the zero section. The manifold $\mathcal{I}(\Gamma_\psi)$ is still Hamiltonian isotopic to the zero section and coincides with it at infinity. Now note that the diffeomorphism $\mathbb{R}^{2n} \simeq S^{2n} \setminus \{N\}$ induces a diffeomorphism $T^*\mathbb{R}^{2n} \simeq T^*(S^{2n} \setminus \{N\})$ and that by definition of the symplectic structure on cotangent bundles, it is a symplectomorphism. Using this symplectomorphism we see that the one point compactification of $\mathcal{I}(\Gamma_\psi)$ in T^*S^{2n} , which we denote by L_ψ , is also Hamiltonian isotopic to the zero section. Theorem 3.17 tells us that this Lagrangian submanifold has a generating function quadratic at infinity S such that i_S is a diffeomorphism. Intersection points with the zero section are then in one to one correspondence with the critical points of S . We want to extract symplectic information from these intersection points. This information is going to come from the critical values. The key remark is that if S is suitably normalized, then its critical values are exactly the action spectrum associated to ψ . In the following we will suppose that generating functions are normalized by having the critical value associated to points close to infinity to be zero, i.e. $S(i_S^{-1}(N, 0)) = 0$. Remark that we also have $S(i_S^{-1}(q, 0)) = 0$ if $|q| > R$ for a certain radius R that verifies $\Phi(\Gamma_\psi) \cap \{|q| > R\} = \{(q, 0) \mid |q| > R\}$.

The action spectrum as the set of critical values of a generating function. Let $\tilde{\lambda} = pdq$ be the Liouville form of $T^*\mathbb{R}^{2n}$, an easy calculation yields $i_S^*\tilde{\lambda} = dS|_{\Sigma_S}$. A simple calculation shows that one can recover the critical values of S integrating paths: let $\tilde{\gamma}$ be a path on $L_S = \mathcal{I}(\Gamma_\psi)$ with endpoints on the zero section. Then $i_S^{-1}(\tilde{\gamma}(0))$ and $i_S^{-1}(\tilde{\gamma}(1))$ are critical values of S and, if $\tilde{\gamma}(0)$ is close to infinity, we have by normalization $S(i_S^{-1}(\tilde{\gamma}(0))) = 0$. Integration of $\tilde{\lambda}$ along $\tilde{\gamma}$ yields

$$\int_{\tilde{\gamma}} \tilde{\lambda} = \int_{i_S^{-1}(\tilde{\gamma})} i_S^*\tilde{\lambda} = \int_{i_S^{-1}(\tilde{\gamma})} dS|_{\Sigma_S} = S(i_S^{-1}(\tilde{\gamma}(1))).$$

Lemma 3.27. *Let $\psi = \psi_1^H$ be a compactly supported Hamiltonian of \mathbb{R}^{2n} . Consider the image of the graph $\mathcal{I}(\Gamma_\psi)$ in $T^*\mathbb{R}^{2n}$ and its one point compactification $L_\psi \subseteq T^*S^{2n}$. Let S be a normalized *gfqi* for L_ψ . Then*

$$\text{CritVal}(S) = \sigma(\psi).$$

Proof. Choose a critical value of S and find it as integration over a path $\tilde{\gamma}$ as before. Let $\tilde{\lambda}$ be the Liouville form on $T^*\mathbb{R}^{2n}$. We have $\tilde{\gamma} = \mathcal{I}(\gamma, \psi(\gamma))$. Using the fact that $d(\mathcal{I}^*\tilde{\lambda} - \lambda \oplus (-\lambda)) = 0$ and that $\tilde{\gamma}$ is a closed path we get

$$\int_{\tilde{\gamma}} \tilde{\lambda} = \int_{(\gamma, \psi(\gamma))} \mathcal{I}^*\tilde{\lambda} = \int_{(\gamma, \psi(\gamma))} \lambda \oplus (-\lambda).$$

so in particular

$$S(i_S^{-1}(\tilde{\gamma}(1))) = \int_{\gamma} \lambda - \int_{\psi(\gamma)} \lambda.$$

If $\Psi : [0, 1]^2 \rightarrow \mathbb{R}^{2n}$ is the mapping $\Psi(t, s) = \psi_t(\gamma(s))$ then one may see that on the extended phase space we have

$$\int_{\Psi[0,1]^2} d(\lambda - Hdt) = 0.$$

Now by Stokes theorem we have

$$\int_{\gamma} \lambda + \int_{\psi_t(\gamma(0))} (\lambda - Hdt) - \int_{\psi(\gamma)} \lambda - \int_{\psi_t(\gamma(1))} (\lambda - Hdt) = 0.$$

We now use the fact that $\gamma(0)$ is outside the support of H to conclude that

$$S(i_S^{-1}(\tilde{\gamma}(1))) = \int_{\psi_t(\gamma(1))} (\lambda - Hdt) = A(\gamma(1), H).$$

□

This Lemma tells us that the critical values of the generating function carry indeed symplectic information in the form of the action spectrum. The next step is to build a section of the action spectrum bundle using the minmax critical value selectors that we saw in Section 3.4.

Consider the one point smooth compactification $L_\psi \subseteq T^*S^{2n}$ of $\mathcal{I}(\Gamma_\psi)$. We know by Theorem 3.17 that this submanifold is generated by a *gfqi* S . By Section 3.4 we may use the two cohomological classes $\mu \in H^{2n}(S^{2n})$ and $1 \in H^0(S^{2n})$ so select two critical values $c(\mu, S)$ and $c(1, S)$. Then by Section 3.3 together with Theorem 3.25 we know that if we normalize the *gfqi* then the indeterminacy of the addition by a constant is removed. In particular, these values do not depend on the *gfqi* used to select it. In our case we are going to suppose that de *gfqi* are normalized at infinity, that is, we require that $S(i_S^{-1}(N, 0)) = 0$. With this normalization the critical value selection depends only on ψ . and not on the particular *gfqi* that we use. We will denote the

selectors by $c(\mu, \psi)$ and $c(1, \psi)$. By Lemma 3.27 we know that they take values in the spectrum of ψ : we have constructed two sections of the action spectrum bundle \mathcal{A} . We now give proofs of the properties that we saw in Chapter 2 that guarantee that we can build a symplectic capacity with these invariants.

Proposition 3.28. *The functions $c(\mu, \cdot) : \text{Ham}^c(\mathbb{R}^{2n}) \rightarrow \mathcal{A}$ are sections of the action bundle such that for any compactly supported symplectic isotopy ψ_t the map*

$$t \mapsto c(\mu, \psi_t)$$

is continuous. Moreover $c(\mu, \cdot)$ is nontrivial over the maps defined by h as in Lemma 2.8.

Proof. Using the existence theorem (Theorem 3.17) we see that for every compactly supported symplectic isotopy ψ_t there is a continuous path of normalized *gfqi* S_t generating L_{ψ_t} that coincide with the same quadratic form Q outside a compact set. We now use that the critical value selectors are continuous with respect to the C^0 norm (see Theorem 3.25) to conclude that

$$t \mapsto c(\mu, \psi_t) = c(\mu, S_t)$$

is continuous with respect to t .

For the second claim, recall that by Theorem 3.25 for every ψ different from the identity we have $c(1, \psi) < c(\mu, \psi)$. Since for the map ψ defined by h as in Lemma 2.8 we have $\sigma(\psi) = \{0, \pi - \epsilon\pi\}$ we conclude that $c(1, \psi) = 0$ and $c(\mu, \psi) = \pi - \epsilon\pi$. \square

Definition 3.29 (see [Vit92]). For every open bounded subset $U \in \mathbb{R}^{2n}$ the Viterbo capacity is defined as

$$c(U) = \sup\{c(\mu, \psi) \mid \psi \in \text{Ham}^c(U)\}.$$

For every general open subset $V \in \mathbb{R}^{2n}$ we define $c(V)$ as the supremum of $c(U)$ for every open bounded set $U \in V$. For every set $X \subseteq \mathbb{R}^{2n}$ we define $c(X)$ as the infimum of $c(V)$ for every open set V that contains X .

Remark 3.30. In the article [Vit92] Viterbo defines the capacities using $c_+(\psi) = -c(1, \psi) = c(\mu, \psi^{-1})$. He does this in order to have $c_+(\psi) > 0$ if ψ is generated by a non-negative Hamiltonian. An easy exercise shows that the capacity defined this way is the same.

Proposition 3.28 together with Lemma 2.8 tell us that $c(B_r^{2n}) \geq \pi r^2$. In the following subsection we are going to deduce the reverse inequality using an energy-capacity inequality, as we did in Chapter 2. Before that, we explain that there is another geometric way to prove the normalization of Viterbo's capacity.

Representation theorem. Let C be a smooth convex body with smooth boundary ∂C . It is a hypersurface so it is a coisotropic submanifold of \mathbb{R}^{2n} . Its symplectic orthogonal defines a foliation called the characteristic foliation, which in the case of hypersurfaces is a line bundle. Since it is one dimensional it is integrable so we may talk about the trajectory of a point, and in particular we may talk about closed trajectories. We define the symplectic action of a closed characteristic γ in ∂C as the integral along the Liouville form λ of \mathbb{R}^{2n} :

$$\mathcal{A}(\gamma) = \int_{\gamma} \lambda$$

The action is clearly a symplectic invariant.

Example 3.31. Consider the ball B_r^{2n} in \mathbb{R}^{2n} . The boundary is the energy surface of the function $H(z) = \|z\|^2$ at level r^2 . The characteristic foliation in this case is defined by the Hamiltonian vector field of H ,

$$X_H(z) = J\nabla H(z) = -2iz$$

which generates the Hamiltonian flow

$$\psi_t^H(z) = e^{-2it}z.$$

In this case every trajectory is periodic. Denoting $z(t) = \psi_t^H$ we get

$$\mathcal{A}(\gamma) = \int_\gamma \lambda = -\frac{1}{2} \int_0^\pi J\dot{z}(t) \cdot z(t) dt = \int_0^\pi r^2 dt = \pi r^2.$$

Theorem 3.32 (Representation theorem). *Let $C \subset \mathbb{R}^{2n}$ be a convex body with smooth boundary ∂C . Then the Viterbo capacity equals the action of some closed characteristic on ∂C .*

In this sense we say that the capacity of C can be represented by a closed characteristic on the boundary. A proof of this theorem for starshaped domains can be found in [San14]. Note that the action and the capacity are invariant by symplectomorphisms but the property of being convex is not. In fact, the previous theorem is also true if C verifies the symplectic property of having a boundary of restricted contact type. We can then use this theorem together with the previous example to get following corollary:

Corollary 3.33. *The Viterbo capacity of the ball B_r^{2n} is πr^2 .*

If we look at the ellipsoid

$$E^{2n}(r_1, \dots, r_n) = \{z \in \mathbb{R}^{2n} \mid \sum_{i=1}^n \frac{1}{r_i} (q_i^2 + p_i^2) < 1\}$$

with $r_1 \leq \dots \leq r_n$, then the set of actions of closed characteristics is $\{\pi r_1^2, \dots, \pi r_n^2\}$. In order to identify which of these actions define the capacity we have to understand what type of orbit is going to be selected by the action selector $c(\mu, \cdot)$. One may see that there is a relation between the index of the critical point associated to the orbit, and the index of the characteristic trajectory (see [San14] for details). This explicit relation allows us to get the following result

Proposition 3.34. *The Viterbo capacity of the ellipsoid $E^{2n}(r_1, \dots, r_n)$ with $r_1 \leq \dots \leq r_n$ is equal to πr_1^2 .*

For every open bounded subset $U \subseteq C_r^{2n}$ of the standard symplectic cylinder one may find an $R > 0$ and an ellipsoid $E(r, R, \dots, R)$ containing U . By monotonicity of the invariants and by Proposition 3.34 we see that $c(U) \leq c(E(r, R, \dots, R)) = \pi r^2$. As a corollary we get that Viterbo's invariant c is normalized, so it is indeed a symplectic capacity.

3.5.1 Energy-capacity inequality

The action selectors that Viterbo defined can be used to define a displacement energy for subsets of \mathbb{R}^{2n} . This time (compare with Definition 2.9), the maximum and minimum of a Hamiltonian function that displaces a bounded open set are going to be a "dynamical" maximum and minimum. More precisely, we consider the quantity

$$\gamma(\psi) = c(\mu, \psi) - c(1, \psi).$$

One may prove (for example using Proposition 4.8) that $c(1, \psi) \leq 0 \leq c(\mu, \psi)$ so $\gamma(\psi) \geq 0$. Moreover Theorem 3.25 implies that $\gamma(\psi) = 0$ if and only if $\psi = \text{Id}$. This is one of the reasons why we say that $c(\mu, \psi)$ is a "dynamical maximum" and $c(1, \psi)$ is a "dynamical minimum". The following proposition gives an interpretation of the triangular inequality for generating functions that we saw in Theorem 3.25 in terms of spectral invariants for compactly supported Hamiltonian diffeomorphisms.

Proposition 3.35 (see [Vit92]). *For every two compactly supported Hamiltonian diffeomorphisms ψ and φ we have*

$$\begin{aligned} c(\mu, \psi \circ \varphi) &\leq c(\mu, \psi) + c(\mu, \varphi) \\ c(1, \psi \circ \varphi) &\geq c(1, \psi) + c(1, \varphi) \\ \gamma(\psi \circ \varphi) &\leq \gamma(\psi) + \gamma(\varphi) \end{aligned}$$

These properties together with Lemma 2.13 are used to prove an energy-capacity inequality:

Proposition 3.36. *Let ψ be a Hamiltonian diffeomorphisms supported on a bounded open set U , and let $\varphi \in \text{Ham}^c(\mathbb{R}^{2n})$ be such that it displaces U from itself, i.e. $\varphi(U) \cap U = \emptyset$. Then*

$$c(\mu, \psi) \leq \gamma(\varphi).$$

Proof. By Proposition 3.28 the map $t \mapsto c(\mu, \psi_t \circ \varphi)$ is continuous. By Lemma 3.27 this function takes values on $\sigma(\psi_t \circ \varphi)$ and by Lemma 2.13 we know that $\sigma(\psi_t \circ \varphi) = \sigma(\varphi)$. The set $\sigma(\varphi)$ is totally discontinuous so we conclude that the function is constant and

$$c(\mu, \varphi) = c(\mu, \psi_0 \circ \varphi) = c(\mu, \psi_1 \circ \varphi) = c(\mu, \psi \circ \varphi).$$

To conclude use the previous proposition to get

$$c(\mu, \psi) \leq c(\mu, \psi \circ \varphi) + c(\mu, \varphi^{-1})$$

together with $c(\mu, \varphi^{-1}) = -c(1, \varphi)$. □

Definition 3.37 (see [Vit92]). For every bounded open set $U \subseteq \mathbb{R}^{2n}$ we define Viterbo's displacement energy of U as

$$\gamma(U) = \inf\{\gamma(\psi) \mid \psi(U) \cap U = \emptyset\}.$$

For every open subset $V \in \mathbb{R}^{2n}$ we define $\gamma(V)$ as the supremum of $\gamma(U)$ for every open bounded set $U \in V$. For every set $X \subseteq \mathbb{R}^{2n}$ we define $\gamma(X)$ as the infimum of $\gamma(V)$ for every open set V that contains X .

Proposition 3.36 gives the classical energy-capacity inequality:

Theorem 3.38. *For every subset $X \subseteq \mathbb{R}^{2n}$ we have*

$$c(X) \leq \gamma(X).$$

In order to prove that γ is also a symplectic capacity, we have to prove the normalization property. Taking into account the fact that $c(X) \leq \gamma(X)$, it is enough to show that the capacity of the standard cylinder is less or equal than πr^2 . We are going to use the same map as in Theorem 2.12. Remark that we have no control on the periodic orbits of this map. In order to solve this problem, we use that the selector γ is continuous with respect to the C^0 topology in \mathcal{H} . More precisely we have the following proposition:

Proposition 3.39 (see [Vit92]). *Let H_1, H_2 be two compactly supported Hamiltonians. Let ψ_1, ψ_2 be the associated time one flows. If $|H_1 - H_2| \leq \epsilon$ then we have $|\gamma(\psi_1) - \gamma(\psi_2)| \leq \epsilon$. In particular*

$$\gamma(\psi) \leq \|H\|_{C^0}.$$

This proposition allows us to reason as in Chapter 2 and use the same Hamiltonian diffeomorphisms as in Theorem 2.12 to get the following result.

Corollary 3.40. *Let $Z_r^{2n} \subseteq \mathbb{R}^{2n}$ be the standard symplectic cylinder. Then $\gamma(Z_r^{2n}) = \pi r^2$.*

This is an example of the utility of the energy capacity inequality. Another example which is going to be useful is the calculation of the displacement energy of a vector subspace of \mathbb{R}^{2n} .

Proposition 3.41. *Consider the coisotropic subspace $\mathbb{R}^{2k} \times \mathbb{R}^{n-k} \subseteq \mathbb{C}^k \times \mathbb{C}^{n-k}$ with $0 \leq k < n$. We have*

$$c(\mathbb{R}^{2k} \times \mathbb{R}^{n-k}) = 0 = \gamma(\mathbb{R}^{2k} \times \mathbb{R}^{n-k}).$$

Proof. For every $\lambda \neq 0$ we have $\lambda \cdot (\mathbb{R}^{2k} \times \mathbb{R}^{n-k}) = \mathbb{R}^{2k} \times \mathbb{R}^{n-k}$ so by homogeneity of symplectic capacities we deduce that the capacity is either 0 or $+\infty$. Since we have the inequality $c(\mathbb{R}^{2k} \times \mathbb{R}^{n-k}) \leq \gamma(\mathbb{R}^{2k} \times \mathbb{R}^{n-k})$ it is enough to show that $\gamma(\mathbb{R}^{2k} \times \mathbb{R}^{n-k}) < +\infty$. By definition

$$\gamma(\mathbb{R}^{2k} \times \mathbb{R}^{n-k}) = \inf\{\gamma(V) \mid V \text{ is open and } \mathbb{R}^{2k} \times \mathbb{R}^{n-k} \subseteq V\},$$

so we have to find an open set V containing $\mathbb{R}^{2k} \times \mathbb{R}^{n-k}$ with finite γ value. We will use Proposition 3.39. Find a smooth bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$ with values on $]0, 1[$ and $f'(s) > 0$ for every $s \in \mathbb{R}$ (every strictly positive integrable function f' works).

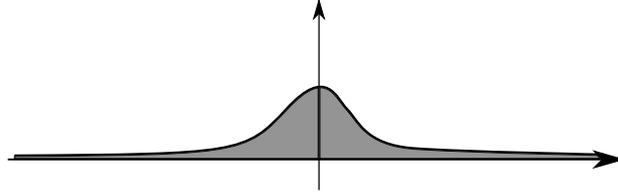


Figure 3.3: A possible graph of f' .

Define the open set

$$V = \{(q_1, p_1, \dots, q_n, p_n) \in \mathbb{R}^{2n} \text{ such that } |p_n| < f'(q_n)\}.$$

By hypothesis $k < n$ so $\mathbb{R}^{2k} \times \mathbb{R}^{n-k} \subseteq V$. We claim that $\gamma(V) < +\infty$. To see this consider the bounded Hamiltonian $H(q, p) = -2f(q_n)$ which generates the flow

$$\psi_t(q, p) = (q_1, p_1, \dots, q_n, p_n + t2f'(q_n)).$$

If $(q, p) \in V$ then

$$|p_n + 2f'(q_n)| \geq 2f'(q_n) - |p_n| > f'(q_n)$$

which implies that $\psi_1(V) \cap V = \emptyset$. Let U be an open bounded set contained in V , we have $\psi_1(U) \cap U = \emptyset$. Find a compactly supported smooth function $\chi : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ with values on $[0, 1]$ and constant equal to 1 on a neighbourhood of $\bigcup_{t \in [0, 1]} \psi_t(U)$. Then χH verifies $\|\chi H\|_{C^0} \leq \|H\|_{C^0}$ and by construction its flow still displaces the open set U . We conclude by Proposition 3.39 that $\gamma(U) \leq \|H\|_{C^0}$. Since the bound does not depend on U this implies that $\gamma(V) \leq \|H\|_{C^0}$ which finally gives

$$\gamma(\mathbb{R}^{2k} \times \mathbb{R}^{n-k}) \leq \|H\|_{C^0} < +\infty$$

concluding the proof. \square

Lagrangian non-triviality. The previous proof was possible because a vector subspace is invariant by multiplication by constants. One may think that the property of having zero capacity is shared by the subsets with empty interior. This is clearly not true for the displacement energy since the boundary of a connected open set has at least the displacement energy of the set that it bounds. This is not clear for the capacity c . The following theorem contained in the article of Théret [Thé99b] tells us that there are Lagrangian submanifolds of \mathbb{R}^{2n} with nontrivial c capacity.

Theorem 3.42 (see [Thé99b]). *Let $L = S(r_1) \times S(r_2) \times \cdots \times S(r_n)$ be a split torus in $\mathbb{C}^n = \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}$, each $S(r_k)$ being the euclidean circle of radius r_k . Then*

$$c(L) = \pi(\min r_k)^2.$$

Chapter 4

Generating functions and symplectic reduction

In this chapter we extend Viterbo's construction to build capacities of subsets of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. This type of extension was already pointed out in [Vit92, Section 5] in order to prove the symplectic camel theorem, but we will consider a slightly different setting. It is in this chapter that the true advantage of using generating functions is going to appear.

In the previous chapter we saw that generating functions are by definition a symplectic reduction of the graph of the differential of functions. The key observation of this chapter is that under some transversality condition the restriction of the domain gives a new *gfi* from a known one. Moreover, the spectral invariants associated to this two generating functions are related by an explicit inequality. This observation is used to relate Viterbo's capacities to the new capacities defined on $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$.

Using these relations we prove a coisotropic non-squeezing theorem: a new rigidity theorem for compactly supported Hamiltonian diffeomorphisms. We then extend this result to some non-compact settings. As a corollary we get a type of middle dimensional rigidity for Hamiltonian flows under some growth condition on the Hamiltonian function.

4.1 Restriction as a symplectic reduction

Recall that any function $S : N \times \mathbb{R}^k \rightarrow \mathbb{R}$ defines a Lagrangian submanifold of $T^*(N \times \mathbb{R}^k)$ via the graph of dS . We say that S is a generating function if dS is transverse to $T^*N \times 0_{\mathbb{R}^k}$. In this case we may apply symplectic reduction by the coisotropic submanifold $W = T^*N \times 0_{\mathbb{R}^k}$ to get a Lagrangian submanifold

$$L_S = \text{Red}_W(dS).$$

Consider a submanifold V of N and look at the restriction $S|_V : V \times \mathbb{R}^k \rightarrow \mathbb{R}$. The graph of $dS|_V$ defines a Lagrangian submanifold in $T^*(V \times \mathbb{R}^k)$. We want to understand the cases in which this function is going to be a generating function. The first remark is that $dS|_V$ is in fact a symplectic reduction.

Lemma 4.1. *Let N be a smooth manifold and let $V \subseteq N$ be a smooth submanifold. Denote by i the natural inclusion. Let $f : N \rightarrow \mathbb{R}$ be a smooth function and consider the restriction $f|_V = f \circ i$. Then $i^*T^*N = T^*N|_V \subseteq T^*N$ is a coisotropic submanifold and*

$$df|_V = \text{Red}_{T^*N|_V}(df).$$

Proof. We start by proving that $T^*N|_V$ is a coisotropic submanifold. Recall that for every $(q, p) \in T^*N$ we have

$$T_{(q,p)}T^*N = T_qN \oplus T_q^*N.$$

Restriction on the base variable gives

$$T_{(q,p)}T^*N|_V = T_qV \oplus T_q^*N = T_qV \oplus T_q^*V \oplus N_q^*V,$$

where N_q^*V is the conormal of T_q^*V in T_q^*N . Denote by ω the standard symplectic form in T^*N . Using normal coordinates for $V \subseteq N$ we get the identification

$$(T_{(q,p)}T^*N|_V)^\omega = N_q^*V,$$

so $W = T^*N|_V$ is a coisotropic submanifold and the coisotropic leaves are given by the conormal subspaces. In particular there is a canonical identification between W/W^ω and T^*V . It is now clear that $\text{Red}_{T^*N|_V}(df)$ are the differential of f on points of V , where we forget the conormal direction, and this is exactly $df|_V$. \square

Remark 4.2. The transversality condition between L and W is naturally verified in this case, so we get a Lagrangian submanifold in the reduction.

Example 4.3. Consider a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Then

$$df(x, y) = \frac{\partial f}{\partial x}(x, y)dx + \frac{\partial f}{\partial y}(x, y)dy.$$

Its intersection with $W = T^*\mathbb{R}^2_{|\mathbb{R} \times \{0\}}$ is $df(x, 0)$ and the reduction is just

$$\text{Red}_W(df) = \frac{\partial f}{\partial x}(x, 0)dx$$

which is precisely $d(f \circ i)$.

The following proposition gives a sufficient condition for S_V to be a generating function, and moreover describes the relation between the Lagrangian submanifolds generated by S and S_V .

Proposition 4.4. *Let V be a submanifold of N and let $L \subseteq T^*N$ be a Lagrangian submanifold generated by a gf $S : N \times \mathbb{R}^k \rightarrow \mathbb{R}$ such that i_S is a diffeomorphism. If L is transverse to $T^*N|_V$ then the restriction S_V is a generating function of*

$$L_V = \text{Red}_{T^*N|_V}(L)$$

Remark 4.5. Let $i : V \hookrightarrow N$ be the inclusion. One may also write the last equality as

$$\text{Red}_{T^*V \times 0_{\mathbb{R}^k}}(\text{Red}_{i^*T^*(N \times \mathbb{R}^k)}(dS)) = \text{Red}_{i^*T^*N \times 0_{\mathbb{R}^k}}(dS) = \text{Red}_{i^*T^*N}(\text{Red}_{T^*N \times 0_{\mathbb{R}^k}}(dS)),$$

which means the natural fact that the reduction by the zero section commutes with the reduction by restriction.

Proof of Proposition 4.4. Taking into account Lemma 4.1 we just need to verify the transversality condition. Denote by π_k the projection $\pi_k : T^*N \times T^*\mathbb{R}^k \rightarrow \mathbb{R}^k$ given by $\pi_k(q, p, \xi, \eta) = \eta$. Define $F = \pi_k \circ dS : N \times \mathbb{R}^k \rightarrow \mathbb{R}^k$. By definition of gf zero is a regular value of F . Remark that the restriction of F to V , denoted by $F|_V$ is equal to $\pi_k \circ d(S|_V)$. In particular we have

$$\Sigma_{S|_V} = F|_V^{-1}(0) = F^{-1}(0) \cap (V \times \mathbb{R}^k) = \Sigma_S \cap (V \times \mathbb{R}^k)$$

4.1. Restriction as a symplectic reduction

We need to show that 0 is a regular value of $F|_V$, or in other words, that for $(q, \xi) \in \Sigma_{S|_V}$ we have the following equality:

$$d_{(q,\xi)}F(T_{(q,\xi)}(V \times \mathbb{R}^k)) = \mathbb{R}^k.$$

The transversality condition for S tells us that for $(q, \xi) \in \Sigma_S$ we have

$$d_{(q,\xi)}F(T_{(q,\xi)}(N \times \mathbb{R}^k)) = \mathbb{R}^k.$$

If we use that $T_{(q,\xi)}\Sigma_S = \ker d_{(q,\xi)}F$ we see that it is enough to show that

$$T_{(q,\xi)}\Sigma_S + T_{(q,\xi)}(V \times \mathbb{R}^k) = T_{(q,\xi)}(N \times \mathbb{R}^k).$$

Note that we already have $\{0\} \times \mathbb{R}^k \subseteq T_{(q,\xi)}(V \times \mathbb{R}^k)$, in order to prove the previous equality it is enough to prove that the first projection is onto T_qN . More precisely, for $\pi_N : N \times \mathbb{R}^k \rightarrow N$, we need to show that

$$d_{(q,\xi)}\pi_N(T_{(q,\xi)}\Sigma_S + T_{(q,\xi)}(V \times \mathbb{R}^k)) = T_qN.$$

Denote by π the bundle projection associated to T^*N . Since i_S is a diffeomorphism we have

$$T_{i_S(q,\xi)}L = d_{(q,\xi)}i_S(T_{(q,\xi)}\Sigma_S).$$

The transversality condition on L implies

$$T_qN = d_{i_S(q,\xi)}\pi(T_{i_S(q,\xi)}L) + T_qV$$

We have $\pi_N = \pi \circ i_S$ so using the two previous equalities we get

$$T_qN = d_{i_S(q,\xi)}\pi(T_{i_S(q,\xi)}L) + T_qV = d_{(q,\xi)}\pi_N(T_{(q,\xi)}\Sigma_S + T_{(q,\xi)}(V \times \mathbb{R}^k)),$$

which gives the desired result. □

The example that we are going to use is the case of the cotangent bundle of a product manifold. This is contained in the following corollary.

Corollary 4.6. *Let N and B be two smooth manifolds, and consider a $gfqi$ $S : N \times B \times \mathbb{R}^k \rightarrow \mathbb{R}$ for an embedded submanifold $L \subseteq T^*(N \times B) = T^*N \times T^*B$ such that i_S is a diffeomorphism. For any $b \in B$, if L is transverse to $W = T^*N \times T_b^*B$ then $S_b = S(\cdot, b, \cdot) : N \times \mathbb{R}^k \rightarrow \mathbb{R}$ is a $gfqi$ for the immersed Lagrangian submanifold $L_b = \text{Red}_W(L) \subseteq T^*N$.*

Remark 4.7. If $L = \Psi(0_{N \times B})$ is Hamiltonian isotopic to the zero section in $T^*(N \times B)$, then there is a smooth path S_t of $gfqi$ that generate $L_t = \Psi_t(0)$ for every $t \in [0, 1]$. If $L = L_1$ verifies the transversality condition of the previous corollary, then $(S_1)_b = S_1(\cdot, b, \cdot)$ is a $gfqi$, but this may not be the case for every $t \in [0, 1]$. Nevertheless one can still find the critical values using minmax selectors $c((S_t)_b, \alpha)$. In particular this quantity is continuous with respect to t . Finally, the path of Lagrangians $L_t = \Psi_t(0)$ may not descend to a path of immersed Lagrangian submanifolds, and L_b may not be Hamiltonian isotopic to the zero section. This means in particular that L_b may not verify the uniqueness property. All these remarks will be useful in the following sections.

4.1.1 Relation with the critical value selectors

In this subsection we present the behavior of the minmax selectors under restriction of the domain. Critical values are selected using cohomology classes of the base manifold, so we first need a relation between the cohomology of N and the cohomology of the submanifold V . Consider the cohomology with coefficients on a field, then Kunnet's formula gives

$$H^*(N \times B) \simeq H^*(N) \otimes H^*(B).$$

The first part of the following proposition comes from proposition 5.1 in [Vit92]. We also give a proof for a sufficient condition that guarantees the equality of the minmax critical values. This property will be very useful in the sequel.

Proposition 4.8. *Let N and B be two connected compact oriented manifolds, S a gfi for a Lagrangian submanifold in $T^*(N \times B)$, b a point in B and $S_b := S(\cdot, b, \cdot)$. Let $\alpha \in H^*(N)$ and $\mu_B \in H^{\dim B}(B)$ the orientation class of B , and $1 \in H^0(B)$. Then,*

$$c(\alpha \otimes 1, S) \leq c(\alpha, S_b) \leq c(\alpha \otimes \mu_B, S).$$

Moreover, if $\tilde{K}(x, b, \xi) = K(x, \xi)$ for all $(x, b, \xi) \in N \times B \times \mathbb{R}^N$ then

$$c(\alpha \otimes 1, \tilde{K}) = c(\alpha, K) = c(\alpha \otimes \mu_B, \tilde{K}).$$

Proof. Let as before $E^\lambda := \{S \leq \lambda\}$, and $E_b^\lambda := \{S_b \leq \lambda\}$. Consider the commutative diagram

$$\begin{array}{ccccc} H^*(N \times B) & \xrightarrow{T} & H^*(E^\infty, E^{-\infty}) & \xrightarrow{i_\lambda^*} & H^*(E^\lambda, E^{-\infty}) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(N) & \xrightarrow{T} & H^*(E_b^\infty, E_b^{-\infty}) & \xrightarrow{i_\lambda^*} & H^*(E_b^\lambda, E_b^{-\infty}) \end{array}$$

where the map $H^*(N \times B) \rightarrow H^*(N)$ is induced by the injection $N \rightarrow N \times \{b\} \rightarrow N \times B$, and coincides with the composition of the projection on $H^*(N) \otimes H^0(B)$ and the obvious identification $H^*(N) \otimes H^0(B) \rightarrow H^*(N)$. Since the diagram is commutative, $i_\lambda^* T(\alpha) \neq 0$ implies $i_\lambda^* T(\alpha \otimes 1) \neq 0$, so $c(\alpha \otimes 1, S) \leq c(\alpha, S_b)$. To get the second inequality, we need to introduce spectral invariants defined *via* homology. The Thom isomorphism is now $T : H_*(S^{2n}) \xrightarrow{\sim} H_*(E^{+\infty}, E^{-\infty})$, and

$$c(A, S) = \inf\{\lambda \mid TA \in \text{Im}(i_{\lambda*})\}.$$

The homological and cohomological invariants are related by the following equality $c(\alpha, S) = -c(\text{PD}(\alpha), -S)$ [Vit92, Proposition 2.7]. In the homology setting, the commutative diagram becomes

$$\begin{array}{ccccc} H_*(N \times B) & \xrightarrow{T} & H_*(E^\infty, E^{-\infty}) & \xleftarrow{i_{\lambda*}} & H_*(E^\lambda, E^{-\infty}) \\ \uparrow & & \uparrow & & \uparrow \\ H_*(N) & \xrightarrow{T} & H_*(E_b^\infty, E_b^{-\infty}) & \xleftarrow{i_{\lambda*}} & H_*(E_b^\lambda, E_b^{-\infty}) \end{array}$$

As before, if $A \in H_*(N)$ verifies $T(A) \in \text{Im}(i_{\lambda*})$, then $T(A \otimes [b]) \in \text{Im}(i_{\lambda*})$, so $c(A \otimes [b], S) \leq c(A, S_b)$ for all $A \in H_*(N)$ (and all S). Thus,

$$c(\alpha, S_b) = -c(\text{PD}(\alpha), -S_b) \leq -c(\text{PD}(\alpha) \otimes [b], -S) = -c(\text{PD}(\alpha \otimes \mu_B), -S)$$

and $-c(\text{PD}(\alpha \otimes \mu_B), -S) = c(\alpha \otimes \mu_B, S)$ so we get $c(\alpha, S_b) \leq c(\alpha \otimes \mu_B, S)$. Finally, if $\tilde{K}(x, b, \xi) = K(x, \xi)$ for all $(x, b, \xi) \in N \times B \times \mathbb{R}^N$ then $E^\lambda = E_b^\lambda \times B$ so $i_\lambda^*(\alpha \otimes \beta) = (i_\lambda^* \alpha) \otimes \beta$. This gives $c(\alpha \otimes 1, \tilde{K}) = c(\alpha \otimes \mu_B, \tilde{K}) = c(\alpha, K)$. \square

To understand how this proposition may be useful and the subtleties involved, consider the following setting: suppose that we are interested in Lagrangian submanifolds of $T^*(N \times B) = T^*N \times T^*B$ which are isotopic to the zero section. Suppose moreover that we are only looking at Lagrangian submanifolds which intersect the zero section at a particular point $(q, b) \in N \times B$. We may then suppose that all *gfi* of degree one are normalized such that the critical value associated to this point is 0. Then for every $\Psi(0) \subseteq T^*(N \times B)$ and $\alpha \otimes 1 \in H^*(N) \otimes H^0(B)$ we may consider the value

$$c(\alpha \otimes 1, \Psi(0)) = c(\alpha \otimes 1, S),$$

which does not depend on the normalized generating function quadratic at infinity S . Of course, we may do the same for Lagrangian submanifolds $\psi(0) \subseteq T^*N$ and consider

$$c(\alpha, \psi(0)) = c(\alpha, K),$$

for a *gfi* K normalized at $q \in N$. Suppose moreover that $\Psi(0)$ intersects transversely $T^*N \times T_b^*B$. Then Proposition 4.8 gives

$$c(\alpha \otimes 1, \Psi(0)) \leq c(\alpha, S_b).$$

Note that this last quantity is not always equal to $c(\alpha, \Psi(0)_b)$ since this is only meaningful if $\Psi(0)_b$ is Hamiltonian isotopic to the zero section (so it has the uniqueness property). Even if this is the case, S_b , which is a *gfi*, may not verify the fact that i_{S_b} is a diffeomorphism, so the uniqueness theorem may not apply, and the equality

$$c(\alpha, \Psi(0)_b) = c(\alpha, S_b)$$

may not be true. To sum up, in order for the previous proposition to be useful, one has to verify that

- $\Psi(0)$ is transverse to $T^*N \times T_b^*B$.
- $\Psi(0)_b$ is Hamiltonian isotopic to the zero section.
- i_{S_b} is a diffeomorphism.

4.2 Spectral invariants on $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$

We now move on to extend the definition of Viterbo's spectral invariants to the periodic case $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. Recall that the general idea of this invariants is to use generating functions to build a section of the action spectrum bundle associated to a Hamiltonian diffeomorphism. The facts that made this possible are that the spectrum does not depend on the Hamiltonian function, that it is invariant by symplectic conjugation, and finally, that the action spectrum coincides with the critical values of a normalized *gfi*. The following subsection contains the justification of these facts in the periodic case.

4.2.1 Preliminary results

Consider the space $T^*\mathbb{R}^m \times T^*\mathbb{R}^k$ with coordinates (z, \tilde{q}, p) . Then $\mathbb{R}^{2m} \times T^*\mathbb{T}^k = T^*\mathbb{R}^m \times T^*\mathbb{T}^k$ is defined as the quotient by the \mathbb{Z}^k -action given by $k \cdot (z, \tilde{q}, p) = (z, \tilde{q} + k, p)$. We will denote the coordinates of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ by (z, q, p) and the projection by

$$\pi : T^*\mathbb{R}^m \times T^*\mathbb{R}^k \rightarrow T^*\mathbb{R}^m \times T^*\mathbb{T}^k.$$

Remark that

$$\pi^* p = p,$$

and that, even if q is only defined up to a constant, dq is globally well defined and

$$\pi^* dq = d\tilde{q}.$$

In particular, the Liouville form $\tilde{\lambda}$ on $T^*\mathbb{R}^m \times T^*\mathbb{R}^k$ is \mathbb{Z}^k -invariant and induces in the quotient a form λ which coincides with the Liouville form in $T^*\mathbb{R}^m \times T^*\mathbb{T}^k$ and verifies

$$\pi^* \lambda = \tilde{\lambda}.$$

Let $H_t : \mathbb{R}^{2m} \times T^*\mathbb{T}^k \rightarrow \mathbb{R}$ be a compactly supported Hamiltonian and denote its flow by $\psi_t^H = \psi_t = (\psi_z, \psi_q, \psi_p)$. Consider moreover the new Hamiltonian function $\tilde{H}_t = H_t \circ \pi$. It will also define a new global Hamiltonian flow $\psi_t^{\tilde{H}} = \tilde{\psi}_t = (\tilde{\psi}_z, \tilde{\psi}_{\tilde{q}}, \tilde{\psi}_p)$. Note that $\tilde{H}_t(k \cdot (z, \tilde{q}, p)) = \tilde{H}_t(z, \tilde{q}, p)$, and since dk is the identity we get

$$X_H(k \cdot (z, \tilde{q}, p)) = X_H(z, \tilde{q}, p).$$

In particular this implies that $\tilde{\psi}_t(k \cdot (z, \tilde{q}, p)) = k \cdot \tilde{\psi}_t(z, \tilde{q}, p)$, or in coordinates

$$\begin{cases} \tilde{\psi}_z(z, \tilde{q} + k, p) = \tilde{\psi}_z(z, \tilde{q}, p) \\ \tilde{\psi}_{\tilde{q}}(z, \tilde{q} + k, p) = \tilde{\psi}_{\tilde{q}}(z, \tilde{q}, p) + k \\ \tilde{\psi}_p(z, \tilde{q} + k, p) = \tilde{\psi}_p(z, \tilde{q}, p) \end{cases}$$

The flow $\tilde{\psi}_t$ being equivariant, it induces a diffeomorphism in the quotient, which is precisely ψ_t . In other words, we have the following commutative diagram:

$$\begin{array}{ccc} T^*\mathbb{R}^m \times T^*\mathbb{R}^k & \xrightarrow{\tilde{\psi}_t} & T^*\mathbb{R}^m \times T^*\mathbb{R}^k \\ \downarrow \pi & & \downarrow \pi \\ T^*\mathbb{R}^m \times T^*\mathbb{T}^k & \xrightarrow{\psi_t} & T^*\mathbb{R}^m \times T^*\mathbb{T}^k \end{array}$$

We start with the following key remark:

Lemma 4.9. *The natural projection π sends periodic orbits of $\tilde{\psi}_t$ to contractible periodic orbits of ψ_t . Moreover the set of fixed points of $\tilde{\psi}_t$ is invariant by the \mathbb{Z}^k action.*

Proof. If $\tilde{\gamma}(t) := \tilde{\psi}_t(z_0)$ is a periodic orbit then the equality $\pi \circ \tilde{\psi}_t = \psi_t \circ \pi$ implies that $\gamma(t) := \pi \circ \tilde{\gamma}(t) = \psi_t \circ \pi(z_0)$ is also a periodic orbit. The second claim is a consequence of the \mathbb{Z}^k equivariance of $\tilde{\psi}_t$: if z_0 is a fixed point of $\tilde{\psi}_t$ then $\tilde{\psi}_t(k \cdot z_0) = k \cdot \tilde{\psi}_t(z_0) = k \cdot z_0$. \square

Remark 4.10. Choose a fixed point of $z_0 = (z, \tilde{q}, p)$ of $\tilde{\psi}$ and denote $z(t) = \tilde{\psi}_t(z_0)$. The action of this point is given by

$$A(z_0, \tilde{H}) = \int_0^1 \frac{1}{2} \langle -J\dot{z}(t), z(t) \rangle dt - \int_0^1 \tilde{H}_t(z(t)) dt.$$

For every $k \in \mathbb{Z}^k$ the path associated to $k \cdot z_0$ is given by $k \cdot z(t)$ which has derivative equal to $\dot{z}(t)$. Moreover $\tilde{H}_t(k \cdot z(t)) = \tilde{H}_t(z(t))$. In particular the action of $k \cdot z_0$ is equal to

$$\begin{aligned} A(k \cdot z_0, \tilde{H}) &= \int_0^1 \frac{1}{2} \langle -J\dot{z}(t), k \cdot z(t) \rangle dt - \int_0^1 \tilde{H}_t(z(t)) dt \\ &= A(z_0, \tilde{H}) + \int_0^1 \frac{1}{2} \langle -J\dot{z}(t), k \rangle dt \\ &= A(z_0, \tilde{H}) + \frac{1}{2} \langle -Jz(1), k \rangle - \frac{1}{2} \langle -Jz(0), k \rangle \\ &= A(z_0, \tilde{H}) \end{aligned}$$

We see that the whole \mathbb{Z}^k orbit of a fixed point of $\tilde{\psi}$ shares the same dynamical information.

Definition 4.11. The contractible action spectrum $\sigma_c(\psi)$ of $\psi = \psi_1$ is defined as the action spectrum $\sigma(\tilde{\psi})$ associated to $\tilde{\psi} = \tilde{\psi}_1$.

We start with the verification that the previous definition makes sense, and then we show that the contractible action spectrum is invariant by conjugation by compactly supported Hamiltonian diffeomorphisms. This time the contractible action spectrum is not invariant by general symplectic diffeomorphisms. This is a consequence of the fact that in $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ there are symplectic diffeomorphisms which are not Hamiltonian, in contrast to \mathbb{R}^{2n} .

Lemma 4.12. *The contractible action spectrum is well defined for compactly supported Hamiltonian diffeomorphisms.*

Proof. We have to show that the contractible action spectrum does not depend on the particular path chosen to generate the Hamiltonian map. Let ψ_t^H and ψ_t^K be two paths such that $\psi_1^H = \psi = \psi_1^K$. Then $\psi_1^{\tilde{H}}$ and $\psi_1^{\tilde{K}}$ are two maps that lift ψ which coincide outside a compact set, so they must be equal on the whole space. Lemma 2.2 now gives $\sigma(\psi_1^{\tilde{H}}) = \sigma(\psi_1^{\tilde{K}})$ and this finishes the proof. \square

Lemma 4.13. *For every compactly supported Hamiltonian isotopy φ of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ we have*

$$\sigma_c(\varphi\psi\varphi^{-1}) = \sigma_c(\psi).$$

Proof. By Lemma 2.4 it is enough to check that the lift of $\varphi\psi\varphi^{-1}$ is $\tilde{\varphi}\tilde{\psi}\tilde{\varphi}^{-1}$. Recall that if ψ is generated by H_t , then $\varphi\psi\varphi^{-1}$ and $\tilde{\varphi}\tilde{\psi}\tilde{\varphi}^{-1}$ are generated by

$$H_\varphi(t, z, q, p) = H(t, \varphi^{-1}(z, q, p)) \quad \text{and} \quad \tilde{H}_{\tilde{\varphi}}(t, z, q, p) = \tilde{H}(t, \tilde{\varphi}^{-1}(z, q, p)).$$

Now we use that $\tilde{H} = H \circ \pi$ and that $\varphi \circ \pi = \pi \circ \tilde{\varphi}$ to get

$$\tilde{H}_\varphi(t, z, \tilde{q}, p) = H(t, \varphi^{-1}(\pi(z, \tilde{q}, p))) = H(t, \pi(\tilde{\varphi}^{-1}(z, \tilde{q}, p))) = \tilde{H}(t, \tilde{\varphi}^{-1}(z, \tilde{q}, p)),$$

and this implies that the lift of $\varphi\psi\varphi^{-1}$ is $\tilde{\varphi}\tilde{\psi}\tilde{\varphi}^{-1}$, finishing the proof. \square

4.2.2 Definition of the invariants

As in the case of the classical Viterbo capacities, we want to associate to each ψ a Lagrangian submanifold Hamiltonian isotopic to the zero section whose intersection points with the zero section capture the information about the contractible action spectrum. As before, we work on $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ which we will reorder as $\mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k$ endowed with coordinates (z, p, q) . Let $\psi \in \text{Ham}^c(\mathbb{R}^{2m} \times T^*\mathbb{T}^k)$ be a compactly supported Hamiltonian diffeomorphism with coordinates (ψ_z, ψ_p, ψ_q) generated by H_t . The Hamiltonian $\tilde{H}_t = H_t \circ \pi$ generates a lift $\tilde{\psi} \in \text{Ham}(\mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{R}^k)$ with coordinates $(\tilde{\psi}_z, \tilde{\psi}_p, \tilde{\psi}_{\tilde{q}})$ such that

$$\begin{cases} \tilde{\psi}_z(z, p, \tilde{q} + 1) = \tilde{\psi}_z(z, p, \tilde{q}) = \psi_z(z, p, q) & (\text{with } (z, p, q) = \pi(z, p, \tilde{q})) \\ \tilde{\psi}_p(z, p, \tilde{q} + 1) = \tilde{\psi}_p(z, p, \tilde{q}) = \psi_p(z, p, q) \\ \tilde{\psi}_{\tilde{q}}(z, p, \tilde{q} + 1) = \tilde{\psi}_{\tilde{q}}(z, p, \tilde{q}) + 1 \end{cases}$$

Again, the graph of $\tilde{\psi}$ is a Lagrangian submanifold $\Gamma(\tilde{\psi}) \subset \mathbb{R}^{2m} \times \mathbb{R}^{2k} \times \overline{\mathbb{R}^{2m} \times \mathbb{R}^{2k}}$. Recall that for every n there is a symplectomorphism $\mathcal{I} : \mathbb{R}^{2n} \times \overline{\mathbb{R}^{2n}} \rightarrow T^*\mathbb{R}^{2n}$ such that $\mathcal{I}(\Gamma_{\text{Id}}) = 0_{\mathbb{R}^{2n}}$. This symplectomorphism is given by

$$\mathcal{I}(q, p, Q, P) = \left(\frac{q+Q}{2}, \frac{p+P}{2}, p-P, Q-q \right).$$

Under \mathcal{I} , the graph $\Gamma(\tilde{\psi})$ becomes a Lagrangian submanifold of $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^{2k}$ whose points are denoted by $\Gamma_{\tilde{\psi}}(z, p, \tilde{q})$ equal to

$$(\mathcal{I}(z, \psi_z(z, p, q)), \frac{p + \psi_p(z, p, q)}{2}, \frac{\tilde{q} + \tilde{\psi}_{\tilde{q}}(z, p, \tilde{q})}{2}, \tilde{q} - \tilde{\psi}_{\tilde{q}}(z, p, \tilde{q}), \psi_p(z, p, q) - p).$$

As usual, the intersection points of this Lagrangian submanifold are in bijective correspondence with the fixed points of $\tilde{\psi}$. Moreover, it is Hamiltonian isotopic to the zero section. Remark that this Lagrangian submanifold does not coincide at infinity with the zero section since $\tilde{\psi}$ is periodic in \tilde{q} . In order to get this property we have to take the quotient.

Taking the quotient. As we saw in Remark 4.10, the whole \mathbb{Z}^k orbit of a fixed point of $\tilde{\psi}$ shares the same action value. In particular, the repetition of these intersection points with the zero section will not give any new dynamical information. We are now going to quotient out by this action in order to get a Lagrangian which coincides outside a compact set with the zero section and such that it only has one intersection point with the zero section for each \mathbb{Z}^k orbit. First remark that the fact that $\tilde{\psi}_{\tilde{q}}(z, p, \tilde{q} + 1) = \tilde{\psi}_{\tilde{q}}(z, p, \tilde{q}) + 1$ implies that $\Gamma_{\tilde{\psi}}$ descends to an embedding $\tilde{\Gamma}_{\tilde{\psi}} : \mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k \rightarrow T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{T}^k$, given by

$$\tilde{\Gamma}_{\tilde{\psi}}(z, p, q) = (\mathcal{I}(z, \psi_z), \frac{p + \psi_p}{2}, q - \psi_q, \frac{q + \psi_q}{2}, \psi_p - p).$$

Lemma 4.14. *The image of $\tilde{\Gamma}_{\tilde{\psi}}$ is a Lagrangian submanifold Hamiltonian isotopic to the zero section, which coincides with the zero section outside a compact set. Moreover, the intersection points of $\tilde{\Gamma}_{\tilde{\psi}}$ with the zero section are in one to one correspondence with \mathbb{Z}^k orbits of fixed points of $\tilde{\psi}$.*

Proof. Everything is a consequence of the natural construction of the Lagrangian. The manifold $\Gamma_{\tilde{\psi}}$ is the quotient of $\Gamma(\tilde{\psi})$ in $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^{2k}$ which is Hamiltonian to the zero section via $\tilde{K} = \tilde{H} \circ \tilde{\pi} \circ \mathcal{I}^{-1}$ where $\tilde{\pi} : \mathbb{R}^{2n} \times \overline{\mathbb{R}^{2n}} \rightarrow \overline{\mathbb{R}^{2n}}$ is the projection (here $n = m + k$). An explicit calculation yields that $\mathcal{I}^{-1} : T^*\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \times \overline{\mathbb{R}^{2n}}$ is given by

$$\mathcal{I}^{-1}(x, y, X, Y) = \left(\frac{2x - Y}{2}, \frac{2y + X}{2}, \frac{2x + Y}{2}, \frac{2y - X}{2} \right).$$

In particular,

$$\mathcal{I}^{-1}(x + k, y, X, Y) = \mathcal{I}^{-1}(x, y, X, Y) + (k, 0, k, 0).$$

Remark that $\tilde{H} \circ \tilde{\pi}$ does not take into account the first pair of variables, so if \tilde{H} is periodic on the base variable, it will also be the case for \tilde{K} , so it induces a Hamiltonian K on the quotient such that $\tilde{K} = K \circ \pi$. Denoting by $\tilde{\Psi}_t$ and Ψ_t the Hamiltonian diffeomorphisms associated to \tilde{K} and K , the same discussion as the one in Subsection 4.2.1 shows that we have a commutative diagram

$$\begin{array}{ccc} T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{R}^k & \xrightarrow{\tilde{\Psi}_t} & T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{R}^k \\ \downarrow \pi & & \downarrow \pi \\ T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{T}^k & \xrightarrow{\Psi_t} & T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{T}^k \end{array}$$

By construction $\tilde{\Psi}(0) = \Gamma(\tilde{\psi})$ and $\Psi(0) = \tilde{\Gamma}_{\tilde{\psi}}$, which proves the first claim. In order to see that it coincides with the zero section at infinity, first recall that H is compactly supported. This

means that if (z, p) is outside a fixed bounded set, then $\tilde{H}(z, \tilde{q}, p) = 0$ and $\tilde{\psi}(z, \tilde{q}, p) = (z, \tilde{q}, p)$. Moreover, we may see that for every (z, \tilde{q}, p) in the support of \tilde{H} , the base points

$$\left(\frac{z + \psi_z}{2}, \frac{p + \psi_p}{2}\right) \in \mathbb{R}^{2m} \times \mathbb{R}^k$$

are contained in a bounded set. In particular, outside this bounded set, we only find points coming from outside the support, so they are in the zero section.

Finally, to justify the last claim, remark that every point of $\Psi(0) \cap 0$ is a projection of points in $\tilde{\Psi}(0) \cap 0$, and that these points are divided into \mathbb{Z}^k orbits in $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{R}^k$. It is then enough to prove that the \mathbb{Z}^k orbits of points in $\tilde{\Psi}(0) \cap 0$ are in one to one correspondence with \mathbb{Z}^k orbits of fixed points of $\tilde{\psi}$. But this is trivial by construction. \square

Remark 4.15. We point out again that the intersection points of $\tilde{\Gamma}_\psi$ with the zero section are not in one to one correspondence with the fixed points of ψ . Indeed if $\psi(z, q, p) = (z, q, p)$ and $\gamma(t) = \psi_t(z, p, q)$ is not contractible, it lifts to a point that verifies $\tilde{\psi}(z, \tilde{q}, p) = k \cdot (z, \tilde{q}, p)$ for some $k \in \mathbb{Z}^k \setminus \{0\}$, and then

$$\tilde{q} - \tilde{\psi}_{\tilde{q}} = k \neq 0.$$

As in the classical setting, using Lemma 4.14, the Lagrangian image of $\tilde{\Gamma}_\psi$ can be compactified to a Lagrangian submanifold

$$L_\psi \subset T^*(S^{2m} \times S^k \times \mathbb{T}^k),$$

which is Hamiltonian isotopic to the zero-section, and coincides with the zero-section on a neighbourhood of $\{N\} \times S^k \times \mathbb{T}^k$ and of $S^{2m} \times \{N\} \times \mathbb{T}^k$. By Sikorav's Theorem for arbitrary compact base manifolds [LS85, Sik86, Sik87] (or Subsection 4.2.3 for this particular case), L_ψ is generated by a *gfqi* of degree one, which is moreover unique up to the three classical operations. After normalization at infinity by $S(i_S^{-1}(N, N, 0, 0)) = 0$, we may select critical values of S by minmax over cohomological classes. These values do not depend on the *gfqi* used for L_ψ so we get spectral values $c(\alpha \otimes \beta \otimes \gamma, \psi)$ for $\alpha \in H^*(S^{2m})$, $\beta \in H^*(S^k)$ and $\gamma \in H^*(\mathbb{T}^k)$.

Definition 4.16. For every open bounded subset $U \in \mathbb{R}^{2m} \times T^*\mathbb{T}^k$ and every $\alpha \in H^*(S^{2m})$, $\beta \in H^*(S^k)$ and $\gamma \in H^*(\mathbb{T}^k)$ we define

$$c(\alpha \otimes \beta \otimes \gamma, U) = \sup\{c(\alpha \otimes \beta \otimes \gamma, \psi) \mid \psi \in \text{Ham}^c(U)\}.$$

For every general open subset $V \in \mathbb{R}^{2m} \times T^*\mathbb{T}^k$ we define $c(\alpha \otimes \beta \otimes \gamma, V)$ as the supremum of $c(\alpha \otimes \beta \otimes \gamma, U)$ for every open bounded set $U \in V$. For every set $Z \subseteq \mathbb{R}^{2m} \times T^*\mathbb{T}^k$ we define $c(\alpha \otimes \beta \otimes \gamma, Z)$ as the infimum of $c(\alpha \otimes \beta \otimes \gamma, V)$ for every open set V that contains Z .

Remark 4.17. Remark that we have chosen the compactification by $S^{2m} \times S^k \times \mathbb{T}^k$ instead of $S^{2m+k} \times \mathbb{T}^k$ as in [Vit92]. The reason for this is that the first one allows us to prove Proposition 4.25 which relates this new invariants with Viterbo's capacity.

As in the classical case, the important property of these values is that they are contained in the action spectrum. This is the content of the following proposition:

Proposition 4.18. *Let ψ be a compactly supported Hamiltonian of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. Consider the image of the graph $\tilde{\Gamma}_\psi$ in $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{T}^k$. It is a Lagrangian submanifold Hamiltonian isotopic to the zero section. Let S be a normalized *gfqi* of degree one for the compactification L_ψ . Then*

$$\text{CritVal}(S) = \sigma_c(\psi).$$

Proof. As in Lemma 3.27 we know that any critical value of S can be written as

$$S(i_S^{-1}(\gamma(1))) = \int_{\gamma} \lambda$$

where γ is a path on L_{ψ} between an intersection point at infinity and an intersection point associated to the desired critical value. Here λ is the Liouville form of $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{T}^k$. On the other hand, the same argument as in Lemma 3.27 tells us that we may find the action of a point $z_0 \in \mathbb{R}^{2m} \times T^*\mathbb{T}^k$ associated to \tilde{H} by

$$A(z_0, \tilde{H}) = \int_{\tilde{\gamma}} \tilde{\lambda}$$

where $\tilde{\gamma}$ is a path on $\Gamma(\tilde{\psi}) \subseteq T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{R}^k$ between an intersection point at infinity and an intersection point associated to the desired fixed point. Now remark that for every critical point one may find a γ of the form $\gamma = \pi \circ \tilde{\gamma}$, and viceversa, every $\tilde{\gamma}$ gives rise to one such γ . In conclusion, it is enough to prove that for $\gamma = \pi \circ \tilde{\gamma}$ we have the equality

$$\left(\int_{\tilde{\gamma}} \pi^* \lambda \right) = \int_{\tilde{\gamma}} \tilde{\lambda} = \int_{\gamma} \lambda \quad \left(= \int_{\pi \circ \tilde{\gamma}} \lambda \right),$$

and this can be easily verified. \square

Corollary 4.19. *For every compactly supported Hamiltonian diffeomorphism φ of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ and every subset $Z \in \mathbb{R}^{2m} \times T^*\mathbb{T}^k$ we have*

$$c(\alpha \otimes \beta \otimes \gamma, \varphi(Z)) = c(\alpha \otimes \beta \otimes \gamma, Z).$$

Proof. By definition of the invariants it is enough to prove that for every compactly supported Hamiltonian isotopy φ of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ we have

$$c(\alpha \otimes \beta \otimes \gamma, \varphi\psi\varphi^{-1}) = c(\alpha \otimes \beta \otimes \gamma, \psi).$$

By Proposition 4.18 we know that $c(\alpha \otimes \beta \otimes \gamma, \varphi_t\psi\varphi_t^{-1})$ takes values on the contractible action spectrum of $\varphi_t\psi\varphi_t^{-1}$. By Lemma 4.13 we know that the spectrum is independent of t . Let S_t be a path of $gfqi$ of degree one for $L_{\varphi_t\psi\varphi_t^{-1}}$. Then

$$t \mapsto c(\alpha \otimes \beta \otimes \gamma, \varphi_t\psi\varphi_t^{-1}) = c(\alpha \otimes \beta \otimes \gamma, S_t)$$

is continuous with respect to t , so using Lemma 2.5 we conclude that it must be constant. \square

4.2.3 The diffeomorphism property of $gfqi$ in the reduction

We now give a proof of the existence of $gfqi$ for Lagrangian submanifolds L of $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{T}^k$ which are isotopic to the zero section by a compactly supported Hamiltonian diffeomorphism. The main reason for giving the explicit construction is to be able to build concrete examples where the reduction S_b of $gfqi$ seen in Section 4.1.1 still verifies the fact that i_{S_b} is a diffeomorphism.

Consider a compactly supported Hamiltonian H defined on $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{T}^k$. Let $j : \mathbb{T}^k \hookrightarrow \mathbb{C}^k$ be the standard embedding and consider the standard projection given by $r \cdot (e^{i\theta_1}, \dots, e^{i\theta_k}) \mapsto (e^{i\theta_1}, \dots, e^{i\theta_k})$ with $r = (r_1, \dots, r_k)$, which is defined on an open set W of \mathbb{C}^k that contains \mathbb{T}^k . This projection induces a map $\pi : T^*W \subseteq T^*\mathbb{C}^k \rightarrow T^*\mathbb{T}^k$, which consists of forgetting r and dr . Using π we can extend H to the whole $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{R}^{2k}$ as a function \tilde{H} which verifies

$$\tilde{H} = H \circ \pi \quad \text{if} \quad |r - 1| \leq \frac{1}{3}$$

and which is zero if $|r - 1| \geq \frac{1}{2}$. Remark that \tilde{H} is not compactly supported but the support of the image of the zero section is contained in a compact set, so we may turn \tilde{H} into a compactly supported Hamiltonian without changing the image of the zero section. Denote by $\tilde{\psi}_t$ the flow associated to \tilde{H} and by ψ_t the flow associated to H .

Lemma 4.20. *The Lagrangian submanifold $\tilde{\psi}_t(0)$ is transverse to the restriction of $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{R}^{2k}$ to $\mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k$.*

The proof is left as an exercise (or see [Bru91] or [Vit06]). This transversality condition allows us to reduce the Lagrangian submanifold $\tilde{\psi}_t(0)$, to get a Lagrangian in $T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{T}^k$. The reduced Lagrangian is precisely $\psi_t(0)$. Moreover, using Proposition 4.4 we see that any gf of $\tilde{\psi}_t(0)$ descends to a gf of $\psi_t(0)$ by restriction of the domain.

Proposition 4.21. *The Lagrangian $\psi_t(0)$ seen as a submanifold of $T^*S^{2m} \times T^*S^k \times T^*\mathbb{T}^k$ has a generating function quadratic at infinity $S : S^{2m} \times S^k \times \mathbb{T}^k \times \mathbb{R}^l \rightarrow \mathbb{R}$ such that $S(N, p, q, \xi) = Q(\xi)$ and $S(z, N, q, \xi) = Q(\xi)$. In particular for any x in S^{2m} , S^k or \mathbb{T}^k , if S_x is a $gfqi$ for an embedded Lagrangian then i_{S_x} is a diffeomorphism.*

Proof. By Theorem 3.17, we know that $\tilde{\psi}_t(0)$ admits a gf which extends smoothly at infinity by Q , and this property is preserved by restriction. Since $S(N, p, q, \xi) = Q(\xi)$ and $S(z, N, q, \xi) = Q(\xi)$ we see that there is just one critical point at infinity, so by Lemma 3.6 i_S is a diffeomorphism. Moreover, since we have two infinities, the same will be true for S_x . \square

4.2.4 First calculations

In this subsection we use the inequalities in Section 4.1.1 to get our first understanding of the new invariants that we have built. We start with the following basic properties:

Proposition 4.22. *Let ψ be a compactly supported Hamiltonian diffeomorphism of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. Denote by μ and 1 the top and bottom cohomology classes of S^{2m} and S^k . Then for every $\alpha \in H^*(S^{2m})$, $\beta \in H^*(S^k)$ and $\gamma \in H^*(\mathbb{T}^k)$,*

- $c(\mu \otimes \beta \otimes \gamma, \psi) \geq 0$ and $c(1 \otimes \beta \otimes \gamma, \psi) \leq 0$.
- $c(\alpha \otimes \mu \otimes \gamma, \psi) \geq 0$ and $c(\alpha \otimes 1 \otimes \gamma, \psi) \leq 0$.

As a consequence we have $c(\mu \otimes 1 \otimes \gamma, \psi) = 0$ and $c(1 \otimes \mu \otimes \gamma, \psi) = 0$.

Proof. By Proposition 4.8 we know that for every $gfqi$ S associated to ψ and every $z \in S^{2m}$ we have

$$c(1 \otimes \beta \otimes \gamma, S) \leq c(\beta \otimes \gamma, S_z) \leq c(\mu \otimes \beta \otimes \gamma, S).$$

Moreover we know by Proposition 4.21 that there exists a $gfqi$ S for ψ such that $S(N, p, q, \xi) = Q(\xi)$ so we get

$$c(\beta \otimes \gamma, S_N) = c(\beta \otimes \gamma, Q) = 0,$$

which gives the desired result. \square

Remark 4.23. At first sight, the previous proposition may indicate that we do not gain any new information by compactifying to $S^{2m} \times S^k \times \mathbb{T}^k$ instead of $S^{2m+k} \times \mathbb{T}^k$ since every mixed cohomology class ($\mu \otimes 1 \otimes \gamma$ and $1 \otimes \mu \otimes \gamma$) finds trivial critical values. This question will be settled in the next proposition. In contrast to this, if in the classical setting of $T^*\mathbb{R}^{2n}$ we compactify to $T^*S^{2k} \times T^*S^{2n-2k}$ the same arguments give that we still only get two non-trivial action selectors, so in that case there is no gain of information.

The following lemma shows that this trivial critical value selectors may be used to normalize a continuous path of $gfqi$. We will not use it, but it may be useful in some situations.

Lemma 4.24. *Let ψ be a compactly supported Hamiltonian diffeomorphism of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$ and let $\Psi_t(0)$ be the associated Lagrangian path in $T^*S^{2m} \times T^*S^k \times T^*\mathbb{T}^k$. Suppose that S_t is a continuous path of $gfqi$ of degree one that generate $\Psi_t(0)$. Then there exists a continuous family of normalized generating functions for $\Psi_t(0)$.*

Proof. To start with, we know that both functions $S_N^t(p, q, \xi) = S^t(N, p, q, \xi)$ and $S_N^t(z, q, \xi) = S^t(z, N, q, \xi)$ generate the zero sections so they have just one critical value. Moreover we see that $S^t(i_{S^t}^{-1}(N, N, q, 0))$ is a common critical value so they are both the same. Using Proposition 4.8 we get $c(1 \otimes \gamma, S_N^t) \leq c(\mu \otimes 1 \otimes \gamma, S^t) \leq c(\mu \otimes \gamma, S_N^t)$ so $c(\mu \otimes 1 \otimes \gamma, S^t) = S^t(i_{S^t}^{-1}(N, N, q, 0))$ determines continuously the critical value at infinity. Now define $c_t := c(\mu \otimes 1 \otimes \gamma, S^t)$ and consider $S_t - c_t$. It is a normalized continuous path of gf which can be modified into a path of $gfqi$ using Proposition 3.5. \square

We continue with the first nontrivial calculation associated to our new generalized capacities in $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. The following proposition relates the classical Viterbo capacity c with our new invariant for the particular open bounded subsets containing compact subsets of the form $X \times \{0\} \times \mathbb{T}^k$.

Proposition 4.25. *Take γ equal to $\mu \in H^k(\mathbb{T}^k)$ or $1 \in H^0(\mathbb{T}^k)$. For any open bounded subset $U \times V \times \mathbb{T}^k \subseteq \mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k$ we have*

$$c(U) \leq c(\mu \otimes \mu \otimes \gamma, U \times V \times \mathbb{T}^k).$$

Proof. Suppose for simplicity that $0 \in V$. Let $\phi \in \text{Ham}^c(U)$. It is enough to find a $\Psi \in \text{Ham}^c(U \times V \times \mathbb{T}^k)$ such that $c(\mu, \phi) \leq c(\mu \otimes \mu \otimes 1, \Psi)$. Let $H : \mathbb{R} \times \mathbb{R}^{2m} \rightarrow \mathbb{R}$ be a generator of ϕ , and $\chi \in C_c^\infty(V)$ with $\chi(p) = 1$ on a neighborhood of 0 (so $\frac{\partial \chi}{\partial p}(p) = 0$ on a neighborhood of 0). The Hamiltonian χH of $\mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k$ generates a compactly supported Hamiltonian diffeomorphism that we will denote $\Psi = (\psi_z, \psi_p, \psi_q)$. It is easy to see that

$$\Psi(z, p, q) = (\psi_z(z, p), p, q + C(z, p))$$

with $C(z, p) = \int_0^1 \frac{\partial \chi}{\partial p}(p) H(t, z) dt$, and that $C(z, 0) = 0$ and $\psi_z(z, 0) = \phi(z)$. The embedding $\Gamma_\Psi : \mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k \rightarrow T^*\mathbb{R}^{2m} \times T^*\mathbb{R}^k \times T^*\mathbb{T}^k$ it thus given by

$$\tilde{\Gamma}_\Psi(z, p, q) = (\mathcal{I}(z, \psi_z(z, p)), p, -C(z, p), q + \frac{1}{2}C(z, p), 0).$$

By definition, when we compactify $\text{Im } \tilde{\Gamma}_\Psi$ we get L_Ψ which, by the previous expression, is easily seen to be transverse to $T^*S^{2m} \times T_0S^k \times T^*\mathbb{T}^k$. Now

$$\tilde{\Gamma}_\Psi(z, 0, q) = (\mathcal{I}(z, \phi(z)), 0, 0, q, 0).$$

so $L_\Psi \cap T^*S^{2m} \times T_0S^k \times T^*\mathbb{T}^k = L_\phi \times \{(0, 0)\} \times 0_{\mathbb{T}^k}$ and the reduction is $L_\phi \times 0_{\mathbb{T}^k}$ which is also Hamiltonian isotopic to the zero-section. Therefore, by Proposition 4.21, there exists a $gfqi$ S for L_Ψ with i_S diffeomorphism such that S_0 is a $gfqi$ for $L_\phi \times 0_{\mathbb{T}^k}$ with i_{S_0} diffeomorphism. On the other hand, if K is a $gfqi$ for L_ϕ then $\tilde{K}(z, q, \xi) = K(z, \xi)$ is also a $gfqi$ for $L_\phi \times 0_{\mathbb{T}^k}$. Moreover, both S_0 and \tilde{K} have 0 as the critical value associated to $\{N\} \times \{q\}$, so by uniqueness of $gfqi$ $c(\mu \otimes \gamma, \tilde{K}) = c(\mu \otimes \gamma, S_0)$. By Proposition 4.8,

$$c(\mu, K) = c(\mu \otimes \gamma, \tilde{K}) = c(\mu \otimes \gamma, S_0) \leq c(\mu \otimes \mu \otimes \gamma, S),$$

which precisely means that $c(\mu, \phi) \leq c(\mu \otimes \mu \otimes \gamma, \Psi)$. \square

Remark 4.26. One may also prove that for the previous maps ϕ and Ψ we have

$$\gamma(\phi) \leq c(\mu \otimes \mu \otimes \gamma, \Psi) - c(1 \otimes 1 \otimes \gamma, \Psi).$$

4.2.5 Reduction energy-capacity inequality

The next proposition is a modified version of [Vit92, proposition 5.2] in order to adapt it to our setting. For a subset $U \subset \mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k$ and a point $w \in \mathbb{T}^k$ we denote by U_w the reduction of U by the coisotropic subspace $W = \mathbb{R}^{2m} \times \mathbb{R}^k \times \{w\} = \mathbb{R}^{2m} \times T_w^*\mathbb{T}^k$, that is, $U_w = \text{Red}_W(U)$ (cf. Definition 3.3). We will also explicitly write $U_w := (U \cap \{q = w\})/\mathbb{R}^k$ which indicates that we quotient out $U \cap \{q = w\}$ by the characteristic foliation \mathbb{R}^k .

Proposition 4.27. *Consider an open bounded subset $U \subset \mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k$, a point $w \in \mathbb{T}^k$ and the reduction $U_w := (U \cap \{q = w\})/\mathbb{R}^k$. Then*

$$c(\mu \otimes \mu \otimes 1, U) \leq \gamma(U_w).$$

Proof. Note that if U is open and bounded, then U_w is also open and bounded. Notice moreover that any Hamiltonian diffeomorphism of \mathbb{R}^{2m} that displaces U_w also displaces its filling $\tilde{U}_w := \mathbb{R}^{2m} \setminus F_w^\infty$, where F_w^∞ is the unbounded connected component of $\mathbb{R}^{2m} \setminus U_w$. Thus $\gamma(U_w) = \gamma(\tilde{U}_w)$. Replacing U by the bigger open subset

$$(\mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k \setminus \{w\}) \cup (\tilde{U}_w \times \mathbb{R}^k \times \mathbb{T}^k).$$

we may as well assume that $\mathbb{R}^{2m} \setminus U_w$ is connected and unbounded, which we do henceforth. Let $\psi \in \text{Ham}^c(U)$ and $\varphi \in \text{Ham}^c(\mathbb{C}^m)$ be such that $\varphi(U_w) \cap U_w = \emptyset$. By definition of the capacities on open bounded subsets we need to prove that

$$c(\mu \otimes \mu \otimes 1, \psi) \leq \gamma(\varphi).$$

We know that the Lagrangian submanifold L_φ in T^*S^{2m} is isotopic to the zero section by a Hamiltonian diffeomorphism Φ and has a *gfqi* $K : S^{2m} \times \mathbb{R}^l \rightarrow \mathbb{R}$. This diffeomorphism Φ induces a Hamiltonian diffeomorphism $\tilde{\Phi} := \Phi \times \text{Id}$ on $T^*S^{2m} \times T^*S^k$ that verifies $\tilde{\Phi}(0) = L_\varphi \times 0_{S^k}$ and $\tilde{K}(z, p, \xi) := K(p, \xi)$ (defined on $S^{2m} \times S^k \times \mathbb{R}^d$) is a *gfqi* for this submanifold. Now for a normalized *gfqi* S of L_ψ we have

$$c(\mu \otimes \mu \otimes 1, \psi) = c(\mu \otimes \mu \otimes 1, S) \leq c(\mu \otimes \mu, S_w) \leq c(\mu \otimes \mu, S_w \#(-\tilde{K})) - c(1 \otimes 1, -\tilde{K}).$$

The first inequality above follows from Proposition 4.4, while the second one is the triangle inequality for spectral invariants in Theorem 3.25 (because $(\mu \otimes \mu) \cup (1 \otimes 1) = \mu \otimes \mu$). The following Lemma 4.28 ensures that $c(\mu \otimes \mu, S_w \#(-\tilde{K})) = c(\mu \otimes \mu, -\tilde{K})$, so applying Proposition 4.4 and Theorem 3.25:

$$c(\mu \otimes \mu \otimes 1, \psi) \leq c(\mu \otimes \mu, -\tilde{K}) - c(1 \otimes 1, -\tilde{K}) = c(\mu, -K) - c(1, -K) = \gamma(\varphi),$$

where we used that for any *gfqi* we have $c(\mu, -S) = -c(1, S)$.

Consider a Hamiltonian path ψ^t from the identity to ψ in $\text{Ham}^c(U)$ and a Hamiltonian path Ψ^t of $T^*S^{2m} \times T^*S^k \times T^*\mathbb{T}^k$ such that $\Psi^t(0) = L_{\psi^t}$. This path gives rise to a family of *gfqi* S^t , continuous in t , that generate L_{ψ^t} for all t and that coincide with a fixed quadratic form Q outside a compact set independent of t (see Subsection 4.2.3). Moreover the path S^t is normalized (see Subsection 4.2.3 or Lemma 4.24).

Lemma 4.28. *Let S^t be a continuous family of normalized *gfqi* for the Lagrangian L_{ψ^t} . Then $c(\mu \otimes \mu, S_w^t \#(-\tilde{K}))$ is a critical value of $-\tilde{K}$ and as a consequence*

$$c(\mu \otimes \mu, S_w \#(-\tilde{K})) = c(\mu \otimes \mu, -\tilde{K}).$$

Proof. Recall that points in L_{ψ^t} are of the form

$$\tilde{\Gamma}_{\psi^t}(z, p, q) = \left(\mathcal{I}(z, \psi_z^t), \frac{p + \psi_p^t}{2}, q - \psi_q^t, \frac{q + \psi_q^t}{2}, \psi_p^t - p \right)$$

plus other points on the zero section that come from compactifying. Moreover, the functions S_w^t formally generate the sets of points

$$\left(\mathcal{I}(z, \psi_z^t), \frac{p + \psi_p^t}{2}, q - \psi_q^t \right) \quad \text{for points } (z, p, q) \text{ that verify } \frac{q + \psi_q^t}{2} = w,$$

plus other points in the zero section. This set is denoted henceforth L_w^t . Recall that the notation $S_w^t \#(-\tilde{K})$ stands for the function $(z, p, \xi, \eta) \mapsto S^t(z, p, w, \xi) - K(z, \eta)$. It is enough to prove that all critical points (z, p, ξ, η) of $S_w^t \#(-\tilde{K})$ are such that (z, η) is a critical point of $-\tilde{K}$, while (z, p, ξ) is a critical point of S_w^t with critical value 0. A critical point of $S_w^t \#(-\tilde{K})$ verifies

$$\frac{\partial S_w^t}{\partial z} = \frac{\partial \tilde{K}}{\partial z}, \quad \frac{\partial S_w^t}{\partial p} = \frac{\partial \tilde{K}}{\partial p} = 0 \quad \text{and} \quad \frac{\partial S_w^t}{\partial \xi} = \frac{\partial \tilde{K}}{\partial \eta} = 0,$$

so it is associated to an intersection point of L_w^t and $L_\varphi \times 0_{S^k}$ in the fiber of (z, p) . This intersection point therefore verifies:

$$\frac{\partial S_w^t}{\partial p} = q - \psi_q^t = 0 \quad \text{and} \quad \frac{q + \psi_q^t}{2} = w \quad \text{so } q = \psi_q^t = w,$$

or will be on the zero section coming from critical points of S^t at infinity.

We first claim that such a point of intersection must lie on $\mathcal{I}(U_w \times U_w)^c \times T^*S^k$: Indeed, if $\mathcal{I}(U_w \times U_w) \times T^*S^k \cap (L_\varphi \times 0_{S^k}) \neq \emptyset$, then $\Phi^{-1}(\mathcal{I}(U_w \times U_w)) \cap 0_{S^{2m}} \neq \emptyset$. But $\Phi^{-1}(\mathcal{I}(U_w \times U_w)) = \mathcal{I}(U_w \times \varphi^{-1}(U_w))$ does not intersect the zero section because φ displaces U_w . We now claim that this implies that the intersection point is on the zero section: if a point of L_w^t is in $\mathcal{I}(U_w \times U_w)^c \times T^*S^k$, $(z, \psi_z^t) \in (U_w \times U_w)^c$ so $z \notin U_w$ or $\psi_z^t \notin U_w$. In both cases, $\psi^t(z, p, w) = (z, p, w)$ because $q = \psi_q^t = w$, and ψ^t has support in U , whose intersection with $\{q = w\}$ is contained in $U_w \times \mathbb{R}^k \times \{w\}$. Thus, the point $\tilde{\Gamma}_{\psi^t}(z, p, w)$ is on the zero section, (z, p, w, ξ) is indeed a critical point of S_t and as a consequence (z, η) is a critical point of $-\tilde{K}$. In addition (z, p, w) is in U^c because $z \notin U_w$.

Now we prove that all the points in U^c have critical value 0. Since $\text{Supp } \psi_t \Subset U$ and U^c is connected, there is an open connected set W that contains U^c and that does not intersect $\text{Supp } \psi_t$ (for all t). Then $0_W \subset L_t$ so if $j : W \hookrightarrow L_t$ is the inclusion on the zero section, $f := i_{S^t}^{-1} \circ j : W \rightarrow \Sigma_{S^t}$ is an embedding into the set of critical points. The open set W is connected so $S^t \circ f$ is constant and all the points in W have the same critical value. The fact that S^t is normalized now implies that this value is zero.

Finally, Sard's theorem ensures that the set of critical values of $-\tilde{K}$ has measure zero, so it is totally disconnected. By continuity of the invariants, $c(\mu \otimes \mu, S_w^t \#(-\tilde{K}))$ is therefore constant, so

$$c(\mu \otimes \mu, -\tilde{K}) = c(\mu \otimes \mu, S_w^0 \#(-\tilde{K})) = c(\mu \otimes \mu, S_w^1 \#(-\tilde{K})).$$

□

□

Corollary 4.29. *We have the following inequalities for the generalized capacities on $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$.*

- For every open bounded subsets $U \subset \mathbb{R}^{2m}$ and $V \subset \mathbb{R}^k$ we have

$$c(U) \leq c(\mu \otimes \mu \otimes 1, U \times V \times \mathbb{T}^k) \leq \gamma(U),$$

and

$$c(\mu \otimes \mu \otimes 1, U \times V \times (\mathbb{T}^k \setminus \{w\})) = 0.$$

- Let $B_r^{2m} \subseteq \mathbb{R}^{2m}$ be the open ball of radius r , then

$$c(\mu \otimes \mu \otimes 1, B_r^{2m} \times \{0\} \times \mathbb{T}^k) = \pi r^2,$$

and

$$c(\mu \otimes \mu \otimes 1, B_r^{2m} \times T^*\mathbb{T}^k) = \pi r^2,$$

- For every compact subset $X \subseteq \mathbb{R}^{2m}$,

$$c(X) \leq c(\mu \otimes \mu \otimes 1, X \times \{0\} \times \mathbb{T}^k) \leq \gamma(X).$$

- Let $Z \subset \mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k$ be a compact set and consider a point $w \in \mathbb{T}^k$. Denote by $Z_w := (Z \cap \{q = w\})/\mathbb{R}^k$ the reduction. Then

$$c(\mu \otimes \mu \otimes 1, Z) \leq \gamma(Z_w).$$

Proof. The only one that requires some justification is the last one. Let $Z \subset \mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k$ and $Z_w := (Z \cap \{q = w\})/\mathbb{R}^k$. We need to show that $c(\mu \otimes \mu \otimes 1, Z) \leq \gamma(Z_w)$. Let $V \subset \mathbb{R}^{2m}$ be an arbitrary neighbourhood of Z_w , and

$$U := (\mathbb{R}^{2m} \times \mathbb{R}^k \times \mathbb{T}^k \setminus \{w\}) \cup (V \times \mathbb{R}^k \times \mathbb{T}^k).$$

Obviously, $Z \subset U$ and $U_w = V$, so by monotonicity of c , it is enough to prove that $c(\mu \otimes \mu \otimes 1, U) \leq \gamma(U_w)$. And this is justified in Proposition 4.27. \square

4.3 Coisotropic non-squeezing

Corollary 4.29 can be used to prove a new non-squeezing statement for compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n} = \mathbb{R}^{2m} \times \mathbb{R}^{2k}$. The first step is to place ourselves in the previous setting of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. Denote by N the north pole of $S^1 = \mathbb{T}$. The diffeomorphism $\mathbb{R} \simeq \mathbb{T} \setminus \{N\}$ induces a diffeomorphism $T^*\mathbb{R} \simeq T^*(\mathbb{T} \setminus \{N\})$ and by definition of the symplectic structure on cotangent bundles, it is a symplectomorphism. This symplectomorphism gives a symplectomorphism $\mathbb{R}^{2m} \times \mathbb{R}^{2k} \simeq \mathbb{R}^{2m} \times T^*(\mathbb{T} \setminus \{N\}) \times \cdots \times T^*(\mathbb{T} \setminus \{N\})$. Remark now that if $\psi_t \in \text{Ham}^c(\mathbb{R}^{2n})$, then it is the identity outside a compact set so we may extend it as the identity at infinity to get a diffeomorphism of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. Using the previous symplectomorphism we conclude that the extension is also Hamiltonian. Now recall that if $Z \subseteq M$ with M a symplectic manifold, and W is a coisotropic submanifold of M , then the symplectic reduction of Z is defined by $\text{Red}_W(Z) = \pi_W(Z \cap W)$ where $\pi_W : W \rightarrow W/W^\omega$ is the natural projection by the coisotropic foliation (see Lemma 3.1 and Definition 3.3).

Example 4.30. Consider a subset $Z \subseteq \mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{n-m}$ and the coisotropic subspace $W = \mathbb{C}^m \times i\mathbb{R}^{n-m}$. Then $\pi_W = \pi_m : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is the projection onto the first m coordinates, and

$$\text{Red}_W(Z) = \pi_m(Z \cap \mathbb{C}^m \times i\mathbb{R}^{n-m}).$$

Theorem 4.31. *Let $X \subseteq \mathbb{R}^{2m}$ be a compact set and consider $X \times \mathbb{R}^{n-m} \subset \mathbb{C}^m \times \mathbb{C}^{n-m}$ and denote by $W := \mathbb{C}^m \times i\mathbb{R}^{n-m}$. For every compactly supported Hamiltonian diffeomorphism ψ of \mathbb{C}^n we have*

$$c(X) \leq \gamma(\text{Red}_W(\psi(X \times \mathbb{R}^{n-m}))).$$

Proof. Since ψ has compact support, we can view it as a symplectomorphism of $\mathbb{C}^m \times T^*\mathbb{T}^{n-m} \simeq \mathbb{R}^{2m} \times \mathbb{R}^{n-m} \times \mathbb{T}^{n-m}$. In this setting $X \times \mathbb{R}^{n-m}$ is seen as $X \times \{0\} \times \mathbb{T}^{n-m}$ and $\psi(X \times \{0\} \times \mathbb{T}^k)_0$ coincides with $\text{Red}_W(\psi(X \times \mathbb{R}^{n-m}))$. Here $0 \in \mathbb{T}^k$ is the image of $0 \in \mathbb{R}^k$ by the symplectomorphism described in the beginning of Section 4.3. Now applying Corollary 4.29, invariance (Corollary 4.19) and again Corollary 4.29 we get the chain of inequalities:

$$\begin{aligned} c(X) &\leq c(\mu \otimes \mu \otimes 1, X \times \{0\} \times \mathbb{T}^k) = c(\mu \otimes \mu \otimes 1, \psi(X \times \{0\} \times \mathbb{T}^k)) \\ &\leq \gamma(\psi(X \times \{0\} \times \mathbb{T}^k)_0) = \gamma(\text{Red}_W(\psi(X \times \mathbb{R}^{n-m}))). \end{aligned}$$

□

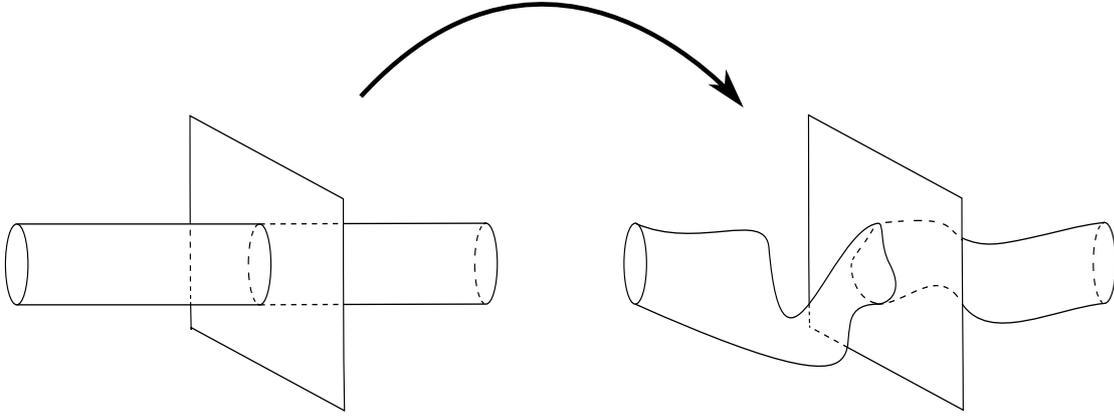


Figure 4.1: This figure represents the image of the coisotropic cylinder by a compactly supported Hamiltonian diffeomorphism ψ . The transverse plane represents the complementary coisotropic subspace. Theorem 1.2 gives information about the capacity of the projection of the intersection with W .

Remark 4.32. Recall that we saw in Proposition 3.41 that for the coisotropic subspace $\mathbb{R}^{2k} \times \mathbb{R}^{n-k} \subseteq \mathbb{C}^k \times \mathbb{C}^{n-k}$ with $0 \leq k < n$ we have

$$c(\mathbb{R}^{2k} \times \mathbb{R}^{n-k}) = 0 = \gamma(\mathbb{R}^{2k} \times \mathbb{R}^{n-k}).$$

In particular classical capacities do not give rigidity information on the image of coisotropic cylinders by arbitrary symplectic maps.

Remark 4.33. The subset $X \times \mathbb{R}^{n-m} \subset \mathbb{C}^m \times \mathbb{C}^{n-m}$ can also be seen as $X \times 0_{\mathbb{R}^{n-m}} \subset T^*\mathbb{R}^m \times T^*\mathbb{R}^{n-m}$ and $W := \mathbb{C}^m \times i\mathbb{R}^{n-m}$ can be seen as $T^*\mathbb{R}^m \times T_0^*\mathbb{R}^{n-m}$. With this notation we see that Theorem 4.31 talks about the trajectories that start with zero momentum in the last $n-m$ coordinates and get to a particular point $0 \in \mathbb{R}^{n-m}$.

As a particular example we get the following rigidity theorem for Lagrangian tubes. There is also an unpublished proof of the following corollary by Buhovsky and Opshtein using the theory of pseudo-holomorphic curves.

Corollary 4.34 (Buhovsky and Opshtein). *Let $L := S^1(r)^m \times \mathbb{R}^{n-m} \subset \mathbb{C}^m \times \mathbb{C}^{n-m}$ be a standard Lagrangian tube. Assume that there is a compactly supported Hamiltonian diffeomorphism ψ of \mathbb{C}^n such that $\psi(L) \cap (\mathbb{C}^m \times i\mathbb{R}^{n-m}) \subset Z(R) \times i\mathbb{R}^{n-m}$ where $Z(R)$ is a symplectic cylinder of capacity R . Then $r \leq R$.*

Proof. Theorem 4.31 gives $c(S^1(r)^m) \leq \gamma(Z(R)) = \pi R^2$ and $c(S^1(r)^m) = \pi r^2$ by Theorem 3.42. \square

It will be useful to note that Theorem 4.31 is also true for two complementary Lagrangian subspaces of \mathbb{C}^{n-m} different from \mathbb{R}^{n-m} and $i\mathbb{R}^{n-m}$. The proof follows from a simple argument of symplectic linear algebra.

Lemma 4.35. *Let \mathbb{R}^{2n} be the standard symplectic vector space and let L_0, L_1 be two Lagrangian subspaces such that $\mathbb{R}^{2n} = L_0 \oplus L_1$. Then there exists a linear symplectic isomorphism $A : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that $A(\mathbb{R}^n) = L_0$ and $A(i\mathbb{R}^n) = L_1$.*

Proof. Define the map $\rho : L_1 \rightarrow L_0^*$ by

$$\rho(v) = \omega(v, \cdot)|_{L_0}$$

It is an isomorphism since $\rho(v) = 0$ implies $v \in L_0^{\perp\omega} = L_0$ and thus $v = 0$ since $\mathbb{R}^{2n} = L_0 \oplus L_1$. Let $(b_i)_{i=1}^n$ be a basis of L_0 , then $(b_{i+n} := -\rho^{-1}(b_i^*))_{i=1}^n$ is a basis of L_1 . We claim that $(b_i)_{i=1}^{2n}$ is a symplectic basis of \mathbb{R}^{2n} . Indeed

$$\omega(b_i, b_{j+n}) = -\omega(b_{j+n}, b_i) = -\rho(b_{j+n})(b_i) = b_j^*(b_i) = \delta_{i,j}$$

Define $A : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ by $A(e_j) = b_j$ and $A(ie_j) = b_{j+n}$ which is symplectic. \square

Theorem 4.36. *Let \mathbb{C}^{n-m} be the standard symplectic vector space and let L_0, L_1 be two Lagrangian subspaces such that $\mathbb{C}^{n-m} = L_0 \oplus L_1$. Let X be a compact set of \mathbb{R}^m and consider $X \times L_0 \subset \mathbb{C}^m \times \mathbb{C}^{n-m}$ and denote by $W = \mathbb{C}^m \times L_1$. For every compactly supported Hamiltonian diffeomorphism ψ of \mathbb{C}^n we have*

$$c(X) \leq \gamma(\text{Red}_W(\psi(X \times L_0))).$$

Proof. Use Proposition 4.35 and consider the compactly supported Hamiltonian diffeomorphism $(I \times A^{-1}) \circ \psi \circ (I \times A)$. Then apply Theorem 4.31. \square

4.3.1 Non-compact setting

In this subsection we explore the limits of Theorem 4.31 as we look into the flow of Hamiltonians without compact support. The first remark is that there are Hamiltonian diffeomorphisms of \mathbb{R}^{2n} that do not verify the statement of Theorem 4.31. In particular the statement is not verified by the linear Hamiltonian diffeomorphism

$$\phi(z_1, \dots, z_n) = (z_{m+1}, \dots, z_n, z_1, \dots, z_m)$$

with $z_j = q_j + ip_j$. It is a rotation, and a Hamiltonian map which is generated by a quadratic function. For this map one can easily see that

$$\phi(\mathbb{R}^{2m} \times \mathbb{R}^{n-m}) = \mathbb{R}^{n-m} \times \mathbb{R}^{2m}.$$

If $m \neq n$ then the projection by π_m is contained in a coisotropic subspace of \mathbb{R}^{2m} so by Proposition 3.41 and by monotonicity we get for every subset $X \subseteq \mathbb{R}^{2m}$

$$\gamma(\text{Red}_W(\varphi(X \times \mathbb{R}^{n-m}))) = 0.$$

In particular if $c(X) > 0$ this gives an example of Hamiltonian diffeomorphism for which

$$c(X) > \gamma(\text{Red}_W(\varphi(X \times \mathbb{R}^{n-m}))).$$

In order to explain where the statement starts to fail, we need some vocabulary:

Definition 4.37. Let ψ_1 be a Hamiltonian diffeomorphism of $\mathbb{R}^{2n} = \mathbb{C}^n$. We will say that $\gamma : [0, 1] \rightarrow \mathbb{R}^{2n}$ given by $\gamma(s) = \psi_s(z)$ is a camel trajectory if

$$\gamma(0) = z \in X \times \mathbb{R}^{n-m} \quad \text{and} \quad \gamma(1) = \psi_1(z) \in \mathbb{C}^m \times i\mathbb{R}^{n-m}.$$

For a camel trajectory γ we call $\gamma(0) = z$ a camel point. The set of camel points of ψ_1 is then equal to

$$\text{Cpt}(\psi_1) = X \times \mathbb{R}^{n-m} \cap (\psi_1)^{-1}(\mathbb{C}^m \times i\mathbb{R}^{n-m}),$$

and the set of camel trajectories is equal to

$$\text{Ctr}(\psi_1) = \bigcup_{s \in [0,1]} \psi_s(\text{Cpt}(\psi_1)).$$

In order to extend Theorem 4.31 to the case without compact support we need to have some control over the set of Camel points. More precisely we will need that this set is bounded. Remark that an easy calculation yields that $\text{Cpt}(\phi)$ is unbounded where ϕ is the map of the beginning of the subsection $\phi(z_1, \dots, z_n) = (z_{m+1}, \dots, z_n, z_1, \dots, z_m)$. Moreover, since the statement of Theorem 4.31 deals with unbounded sets, we need to consider flows that are globally defined. The following results are based on the previous two remarks. We start with the following classical lemma:

Lemma 4.38. Let $H : \mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a Hamiltonian function such that

$$|\nabla H_t(z)| \leq A + B|z| \quad \text{for every } (t, z) \in [0, T] \times \mathbb{R}^{2n},$$

for two positive constants A and $B \neq 0$. Then for every $(t, z) \in [0, T] \times \mathbb{R}^{2n}$

$$|\psi_t(z)| \leq e^{Bt}|z| + \frac{A}{B}(e^{Bt} - 1) \quad \text{and} \quad |\psi_t(z) - z| \leq (|z| + \frac{A}{B})(e^{Bt} - 1).$$

In particular, solutions do not blow up and the flow is globally defined. If $B = 0$ then

$$|\psi_t(z)| \leq |z| + tA \quad \text{and} \quad |\psi_t(z) - z| \leq tA.$$

Proof. By the hypothesis on the gradient we have

$$|\psi_t(z)| \leq |z| + \int_0^t (A + B|\psi_s(z)|) ds = |z| + At + B \int_0^t |\psi_s(z)| ds =: R(t)$$

Then

$$R'(t) = A + B|\psi_t(z)| \leq A + BR(t) \quad \text{and} \quad R(0) = |z|.$$

We may now use Gronwall's lemma to get

$$R(t) \leq e^{Bt}|z| + A \int_0^t e^{Bs} ds = e^{Bt}|z| + \frac{A}{B}(e^{Bt} - 1).$$

For the second inequality write

$$|\psi_t(z) - z| \leq \int_0^t (A + B|\psi_s(z)|) ds = At + B \int_0^t |\psi_s(z)| ds = R(t) - |z|$$

and use the previous inequality. \square

Now consider $\mathbb{R}^{2n} = \mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{n-m}$. We denote by π_m the projection on the first m coordinates, by π^m the projection on the second $n - m$ coordinates, by π_+^m the projection on the real parts, and by π_-^m the projection on the imaginary parts. In particular, we may describe the set $X \times \mathbb{R}^{n-m} \in \mathbb{C}^m \times \mathbb{C}^{n-m}$ by

$$X \times \mathbb{R}^{n-m} = \{z \in \mathbb{C}^n \mid \pi_m(z) \in X \text{ and } \pi_-^m(z) = 0\}.$$

Theorem 4.39. *Let $H : \mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a Hamiltonian function such that*

$$|\nabla H(z)| \leq A + B|z| \text{ for every } (t, z) \in [0, T] \times \mathbb{R}^{2n}$$

for two positive constants A and B , and denote by ψ_t the Hamiltonian flow generated by H . Let $X \subseteq \mathbb{R}^{2m}$ be a compact set and consider the coisotropic subspace $W := \mathbb{C}^m \times i\mathbb{R}^{n-m}$. Then we have

$$c(X) \leq \gamma(\text{Red}_W[\psi_t(X \times \mathbb{R}^{n-m})])$$

for every $|t| < \min(\frac{\ln 2}{3B}, T)$ if $B \neq 0$ and for every $t \in [0, T]$ if $B = 0$.

Proof. We start with the case $B \neq 0$. By considering the Hamiltonian $\frac{1}{B}H_{\frac{t}{B}}$ we may suppose $B = 1$. We start by bounding the camel points and camel trajectories for small times. Suppose that z is a camel point at time t , then since X is compact, there exists an r such that

$$|z| \leq |\pi_m(z)| + |\pi_+^m(z)| + |\pi_-^m(z)| \leq r + |\pi_+^m(z)|.$$

Moreover, since z is a camel point at time t we have

$$\pi_+^m(\psi_t(z)) = 0.$$

By Lemma 4.38 we have

$$|\psi_t(z) - z| \leq (e^t - 1)(|z| + A)$$

so we conclude that

$$|z| \leq r + |\pi_+^m(z)| = r + |\pi_+^m(\psi_t(z) - z)| \leq r + (e^t - 1)(|z| + A).$$

In particular we see that if $e^t < 2$, so if $t < \ln 2$, the camel points verify

$$|z| \leq \frac{r + A}{2 - e^t},$$

and using Lemma 4.38 again we see that the set of camel trajectories $\text{Ctr}(\psi_t)$ is also bounded. The idea now is to build from ψ_t a compactly supported Hamiltonian diffeomorphism φ_t such that

$$\text{Ctr}(\varphi_t) = \text{Ctr}(\psi_t),$$

for $|t| \leq t_0$ for some $t_0 > 0$. Then we can apply Theorem 4.31 to get

$$c(X) \leq \gamma(\text{Red}_W(\varphi_t(X \times \mathbb{R}^{n-m}))) = \gamma(\text{Red}_W(\psi_t(X \times \mathbb{R}^{n-m})))$$

which is the desired result. Let $\rho : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function with values on $[0, 1]$ that equals 1 over the interval $[0, R]$, vanishes over $[2R, +\infty[$ and such that $|\rho'| \leq 2/R$. Note that on the support of ρ' we have $|z| \leq 2R$ so

$$|\rho'(|z|)| \leq \frac{2}{R} \leq \frac{4}{|z|}.$$

Define $G_t(z) = \rho(|z|)H_t(z)$ (the value of R will be chosen later). It is a compactly supported function that generates a Hamiltonian diffeomorphism φ_t . It is easy to see that if $H(0) = 0$ then (since $B = 1$)

$$|H(z)| \leq A|z| + \frac{|z|^2}{2}.$$

We have

$$|\nabla G_s(z)| = |\rho'(|z|)| \frac{z}{|z|} H_s(z) + \rho(|z|) \nabla H_s(z) \leq 4A + 2|z| + A + |z| \leq 5A + 3|z|$$

(Remark that since G is compactly supported, the gradient is bounded by a constant. But the key of the bound that we have is that it does not depend on R .) In particular, if $|t| \leq \frac{\ln 2}{3}$ we know that the camel trajectories of φ_t are bounded by a constant independent of R . If we chose R big enough then $\text{Ctr}(\psi_t)$ and $\text{Ctr}(\varphi_t)$ are contained in a bounded set where both flows coincide, so a fortiori $\text{Ctr}(\psi_t) = \text{Ctr}(\varphi_t)$ which gives the desired result.

We now prove the case $B = 0$. The only change with respect to the previous reasoning is that this time we have

$$|\pi_+^m(\psi_t(z) - z)| = |\pi_+^m \int_0^t X_H(\psi_s(z)) ds| \leq |t|A.$$

so if $z \in \text{Cpt}(\psi_t)$ then $|z| \leq r + |t|A$ and the camel points are bounded for every $t \in \mathbb{R}$ (see Figure 4.2). Moreover, the camel trajectories are also bounded by a constant that depends on t and A . This time $|H(z)| \leq A|z|$ so if we consider the same G as in the previous paragraph we have $|\nabla G_s(z)| \leq 5A$. In particular the camel trajectories of φ_t are bounded for all t by a constant that depends only on t and A and not on R . For R big enough we will have $\text{Ctr}(\psi_t) = \text{Ctr}(\varphi_t)$ which gives the desired result. \square

Remark 4.40. The time bound for $B \neq 0$ in the previous theorem is not optimal and one may get a better one modifying the bound for $|\rho'|$. This bound cannot be extended much more since the statement fails for bigger t as we saw with ϕ at the beginning of this subsection.

Remark 4.41. The general version with two Lagrangian subspaces $L_0 \oplus L_1 = \mathbb{C}^{n-m}$ as in Theorem 4.36 is also true. Indeed if H verifies the bound on the gradient, then the Hamiltonian associated to $(I \times A^{-1}) \circ \psi \circ (I \times A)$ will also verify a similar bound. Note that the bound on the time for the case $B \neq 0$ might change.

Corollary 4.42. Consider a Hamiltonian function of the form

$$H_t(z) = \frac{1}{2} \langle Az, z \rangle + U_t(z),$$

Where U has bounded gradient. Suppose that $A = A_1 \oplus \dots \oplus A_n$ where A_k is a linear map on $\text{Vect}\{\frac{\partial}{\partial q_k}, \frac{\partial}{\partial p_k}\}$ for every $1 \leq k \leq n$. Let $X \subseteq \mathbb{R}^{2m}$ be a compact set and consider the coisotropic subspace $W := \mathbb{C}^m \times i\mathbb{R}^{n-m}$. Then for every $t \in \mathbb{R}$ such that

$$\mathbb{R}^{n-m} + e^{-tJA}i\mathbb{R}^{n-m} = \mathbb{C}^{n-m}$$

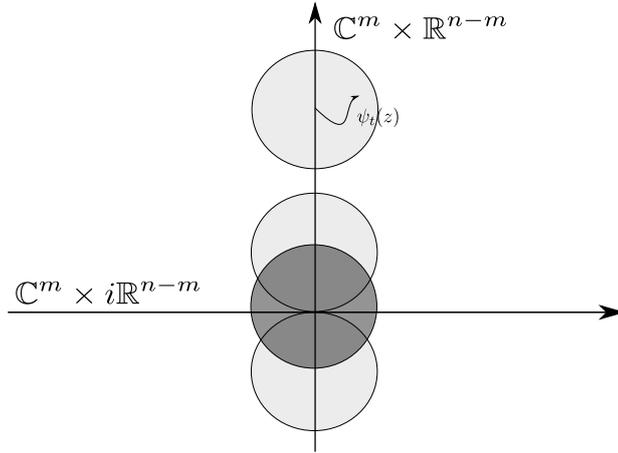


Figure 4.2: The light shaded balls represents the maximum length of a trajectory in $[0, t]$. The dark shaded region represents the bounded sets that contains $\text{Cpt}(\psi_t)$.

we have

$$c(X) \leq \gamma(\text{Red}_W[\psi_t(X \times \mathbb{R}^{n-m})]).$$

Proof. Write $H_t(z) = Q(z) + U_t(z)$. Then using Lemma A.9 in the Appendix we see that

$$\psi_t^H = \psi_t^Q \circ \psi_t^G \quad \text{with} \quad G_t(z) = (-Q + H)(t, \psi_t^Q(z)) = U_t(\psi_t^Q(z)).$$

We have the equality $\psi_t^Q(z) = e^{tJA}z$ so that

$$\nabla G_t(z) = e^{t(JA)^t} \nabla U_t(e^{tJA}z).$$

In particular we see that

$$|\nabla G_t(z)| \leq e^{t\|A\|} C$$

where $C > 0$ is a constant that bounds the gradient of U . This implies that φ_t^G verifies the hypothesis of the previous theorem. In particular (using Remark 4.41) we may take the two transverse coisotropic subspaces

$$\mathbb{C}^m \times \mathbb{R}^{n-m} \quad \text{and} \quad W_e = \mathbb{C}^m \times e^{-tJA}i\mathbb{R}^{n-m}$$

and we conclude that

$$c(X) \leq \gamma(\text{Red}_{W_e}[\psi_t^G(X \times \mathbb{R}^{n-m})]).$$

Now recall that

$$\text{Red}_{W_e}[\psi_t^G(X \times \mathbb{R}^{n-m})] = \pi_m(\psi_t^G(X \times \mathbb{R}^{n-m}) \cap W_e)$$

and that by hypothesis $e^{tJA}\pi_m = \pi_m e^{tJA}$ so

$$e^{tJA} \text{Red}_{W_e}[\psi_t^G(X \times \mathbb{R}^{n-m})] = \text{Red}_W[\psi_t(X \times \mathbb{R}^{n-m})].$$

Finally e^{tJA} restricts to a symplectomorphism of \mathbb{C}^m so we conclude using the invariance of γ . \square

Example 4.43. Mechanical systems in $T^*\mathbb{R}^n$. Consider a smooth Hamiltonian function $H : \mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ of the form

$$H_t(q, p) = \frac{1}{2}|p|^2 + U_t(q), \quad \text{with} \quad |\nabla U_t(z)| \leq C \quad \text{for} \quad (t, z) \in [-T, T] \times \mathbb{R}.$$

Then H is of the form of Corollary 4.42. Remark that in this case A is zero on $\mathbb{R}^n \subseteq \mathbb{C}^n$ and the identity on $i\mathbb{R}^n \subseteq \mathbb{C}^n$. In particular A clearly preserves every subspace of the form $\text{Vect}\{\frac{\partial}{\partial q_k}, \frac{\partial}{\partial p_k}\}$. Now, the system $\dot{z}(t) = JAz(t)$ is a decoupled system on every subspace $\text{Vect}\{\frac{\partial}{\partial q_k}, \frac{\partial}{\partial p_k}\}$. In those planes we have

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad iA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We see that JA is nilpotent so the exponential map is just

$$e^{tJA} = \text{Id} + tJA = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The transversality condition

$$\mathbb{R}^{n-m} + e^{-tJA}i\mathbb{R}^{n-m} = \mathbb{C}^{n-m}$$

is verified for every m and every $t \in \mathbb{R}$. We now give a possible interpretation of Corollary 4.42 in the context of mechanical systems. Consider N particles moving in \mathbb{R}^3 under the influence of a smooth potential $U : \mathbb{R}^{3N} \rightarrow \mathbb{R}$ which moreover has bounded gradient. Suppose that we have the following information at $t = 0$:

- The first particle has initial position and momentum on an ϵ neighborhood of $(x_1, y_1) \in \mathbb{R}^6$.
- The rest of the $N - 1$ particles start at fixed positions x_2, \dots, x_N .

In other words, if we denote by B_ϵ the ball of center (x_1, y_1) and radius ϵ in \mathbb{R}^6 , we know that the initial state of our system is contained in the coisotropic cylinder

$$B_\epsilon \times T_{x_2}^*\mathbb{R}^3 \times \dots \times T_{x_N}^*\mathbb{R}^3 \subseteq \mathbb{C}^3 \times \mathbb{C}^{3(N-1)}.$$

Suppose moreover that we are interested in the solutions such that at time t the $N - 1$ particles have stopped, so solutions that get to the subspace $W = \mathbb{C}^3 \times 0_{\mathbb{R}^{3(N-1)}}$. Then we will never know the state of the first particle with more precision than the one we had at $t = 0$, i.e. if the set of states of the first particle are contained in B_δ then

$$\pi\epsilon^2 \leq \gamma(\text{Red}_W[\psi_t^H(B_\epsilon \times T_{x_2}^*\mathbb{R}^3 \times \dots \times T_{x_N}^*\mathbb{R}^3)]) \leq \gamma(B_\delta) = \pi\delta^2,$$

so $\epsilon \leq \delta$.

4.3.2 Coisotropic rigidity in $T^*\mathbb{T}^n$.

We now apply the previous results in the periodic setting. Denote by $\pi : \mathbb{R}^{2n} \rightarrow T^*\mathbb{T}^n$ the standard projection. We start with a definition for a displacement energy in this context.

Definition 4.44. For every subset $Z \in T^*\mathbb{T}^n$ we set

$$c_{\mathbb{T}}(Z) = c(\pi^{-1}(Z)) \quad \text{and} \quad \gamma_{\mathbb{T}}(Z) = \gamma(\pi^{-1}(Z)).$$

Both quantities are invariant by Hamiltonian diffeomorphisms ψ of $T^*\mathbb{T}^n$. Indeed, any ψ lifts to a Hamiltonian symplectomorphism $\tilde{\psi}$ of \mathbb{R}^{2n} such that $\pi \circ \tilde{\psi} = \psi \circ \pi$. In particular

$$\pi^{-1}(\psi(Z)) = (\psi^{-1} \circ \pi)^{-1}(Z) = (\pi \circ \tilde{\psi}^{-1})^{-1}(Z) = \tilde{\psi}^{-1}(\pi^{-1}(Z)).$$

Using this chain of inequalities and the symplectic invariance of γ we find

$$\gamma_{\mathbb{T}}(\psi(Z)) = \gamma(\pi^{-1}(\psi(Z))) = \gamma(\tilde{\psi}^{-1}(\pi^{-1}(Z))) = \gamma(\pi^{-1}(Z)) = \gamma_{\mathbb{T}}(Z).$$

Remark $\gamma_{\mathbb{T}}$ is not a true displacement energy on $T^*\mathbb{T}^n$ since $\gamma_{\mathbb{T}}(0_{\mathbb{T}^n}) = 0$ even though one cannot displace the zero section from itself using a Hamiltonian isotopy in $T^*\mathbb{T}^n$. The following lemma gives a calculation of these invariants for standard balls.

Lemma 4.45. *Let $B_r \subseteq T^*\mathbb{T}^n$ be a standard ball such that*

$$\pi^{-1}(B_r) = \bigcup_{k \in \mathbb{Z}^n} k \cdot \tilde{B}_r$$

is a disjoint union. Then

$$c_{\mathbb{T}}(B_r) = \pi r^2 = \gamma_{\mathbb{T}}(B_r).$$

Proof. By monotonicity we clearly have $\pi r^2 \leq c_{\mathbb{T}}(B_r)$ so we have to prove that $\gamma_{\mathbb{T}}(B_r) \leq \pi r^2$. Since $\pi^{-1}(B_r)$ is unbounded, it is enough to prove the inequality for a finite union of disjoint balls. In particular it is enough to exhibit, for every $\epsilon > 0$ a compactly supported Hamiltonian diffeomorphism φ of \mathbb{R}^{2n} which displaces the finite union and that verifies $\gamma(\varphi) \leq \pi r^2 + \epsilon$. Any of the balls $k \cdot \tilde{B}_r$ in the disjoint union $\pi^{-1}(B_r)$ can be displaced vertically using a conjugation by k , $\varphi_k = k \circ \psi \circ k^{-1}$, of the map ψ in Theorem 2.12 which is generated by a Hamiltonian that verifies $\|H\|_{C^0} \leq \pi r^2 + \epsilon$. Since all these diffeomorphisms have disjoint support, the composition $\varphi_{k_1} \circ \dots \circ \varphi_{k_m}$ will displace the finite union and will be generated by Hamiltonian $H_m = H_{k_1} + \dots + H_{k_m}$ where each term has disjoint support from one another. In particular

$$\|H\|_{C^0} = \max_m \|H_{k_m}\|_{C^0} \leq \pi r^2 + \epsilon$$

now Proposition 3.39 implies that $\gamma(\varphi_{k_1} \circ \dots \circ \varphi_{k_m}) \leq \pi r^2 + \epsilon$ giving the desired result. \square

Theorem 4.46. *Let ψ be a compactly supported Hamiltonian diffeomorphism of $T^*\mathbb{T}^n = T^*\mathbb{T}^m \times T^*\mathbb{T}^{n-m}$. For any $w \in \mathbb{T}^{n-m}$ denote $W_f = T^*\mathbb{T}^m \times T_w^*\mathbb{T}^{n-m}$ and $W_0 = T^*\mathbb{T}^m \times 0_{\mathbb{T}^{n-m}}$. Let B_r denote a ball of radius r in $T^*\mathbb{T}^m$. Then*

$$\text{Red}_{W_f}[\psi(B_r \times 0_{\mathbb{T}^{n-m}})] \subseteq B_R \quad \text{or} \quad \text{Red}_{W_0}[\psi(B_r \times T_w^*\mathbb{T}^{n-m})] \subseteq B_R$$

implies $r \leq R$.

Proof. Every compactly supported Hamiltonian diffeomorphism ψ_t of $T^*\mathbb{T}^n$ lifts to a Hamiltonian diffeomorphism $\tilde{\psi}_t$ of \mathbb{R}^{2n} generated by a Hamiltonian with bounded gradient. In particular it verifies the hypothesis of Theorem 4.39. Remark also that the ball B_r lifts to an infinite disjoint union of balls of the same radius. We take for $X \subset \mathbb{R}^{2m}$ one of these balls \tilde{B}_r which project to $T^*\mathbb{T}^m$ as the B_r . The subset $\tilde{B}_r \times \mathbb{R}^{n-m}$ projects onto $B_r \times 0_{\mathbb{T}^{n-m}}$. We are interested in image of this set by $\tilde{\psi}_t$. Remark that we have

$$\pi(\tilde{\psi}_t(\tilde{B}_r \times \mathbb{R}^{n-m}) \cap \mathbb{C}^m \times i\mathbb{R}^{n-m}) \subseteq \psi_t(B_r \times 0_{\mathbb{T}^{n-m}}) \cap T^*\mathbb{T}^m \times T_0^*\mathbb{T}^{n-m}.$$

In particular, if we set $\tilde{W}_f = \mathbb{C}^m \times i\mathbb{R}^{n-m}$ and $W_f = T^*\mathbb{T}^m \times T_0^*\mathbb{T}^{n-m}$ we get

$$\pi(\text{Red}_{\tilde{W}_f}[\tilde{\psi}_t(\tilde{B}_r \times \mathbb{R}^{n-m})]) \subseteq \text{Red}_{W_f}[\psi_t(B_r \times 0_{\mathbb{T}^{n-m}})],$$

so we have

$$\text{Red}_{\tilde{W}_f}[\tilde{\psi}_t(\tilde{B}_r \times \mathbb{R}^{n-m})] \subseteq \pi^{-1}(\text{Red}_{W_f}[\psi_t(B_r \times 0_{\mathbb{T}^{n-m}})]) \subseteq \pi^{-1}(B_R).$$

By hypothesis $\pi^{-1}(B_R)$ is the disjoint union of copies of B_r so if we use Theorem 4.39 and Lemma 4.45 we get

$$\pi r^2 = c(\tilde{B}_r) \leq \gamma(\text{Red}_{\tilde{W}_f}[\tilde{\psi}_t(\tilde{B}_r \times \mathbb{R}^{n-m})]) \leq \gamma_{\mathbb{T}}(\text{Red}_{W_f}[\psi_t(B_r \times 0_{\mathbb{T}^{n-m}})]) = \gamma_{\mathbb{T}}(B_r) = \pi R^2$$

which gives the desired result. A similar reasoning gives the result for the second inclusion. \square

In the light of this result, a very interesting question is if this type of rigidity appears on the product of cotangent bundles of compact manifolds other than the torus. By symplectic ball of radius r in a symplectic manifold (M, ω) we understand a ball of radius r of \mathbb{R}^{2n} which is symplectically embedded in M . We may conjecture the following behavior:

Conjecture 4.47. Let N and M be two compact manifolds and consider ψ a compactly supported Hamiltonian isotopy of $T^*N \times T^*M$. For any $m \in M$ denote $W_f = T^*N \times T_m^*M$ and $W_0 = T^*N \times 0_M$. Let B_r be a symplectic ball of radius r in T^*N . Then the inclusion

$$\text{Red}_{W_f}[\psi(B_r \times 0_M)] \subseteq B_R \quad \text{or} \quad \text{Red}_{W_0}[\psi(B_r \times T_m^*M)] \subseteq B_R$$

imply $r \leq R$.

Example 4.48. Mechanical systems in $T^*\mathbb{T}^n$. One may see that Theorem 4.46 is still true if instead of compactly supported Hamiltonians we consider mechanical Hamiltonians $H : \mathbb{R} \times T^*\mathbb{T}^n \rightarrow \mathbb{R}$ of the form

$$H_t(q, p) = \frac{1}{2}|p|^2 + U_t(q),$$

as in Example 4.43.

4.3.3 Middle dimensional symplectic rigidity

An important consequence of Theorem 4.39 is its interpretation in the context of the middle dimensional rigidity problem. One of the first questions regarding this problem appeared in [Hof90b] where Hofer asked about the generalization of capacities to middle dimensions. Remark that all symplectic cylinders of the same radius but with a base of different dimensions have the same symplectic capacity. In order to try and see if those cylinders are different for symplectic geometry Hofer asked if there exists a k -intermediate symplectic capacity c^k satisfying monotonicity, k -conformality, symplectic invariance and

$$c^k(B_1^{2k} \times \mathbb{R}^{2n-2k}) < +\infty \quad \text{but} \quad c^k(B_1^{2k-2} \times \mathbb{R}^2 \times \mathbb{R}^{2n-2k}) = +\infty?$$

The first inequality is already verified by the standard capacities and it is the second one which differs. One of the first results indicating that intermediate capacities do not exist appeared in an article by Guth [Gut08]. He was interested in the question of when can we embed a polydisc $P = B_{R_1}^2 \times \cdots \times B_{R_n}^2$ with $R_1 \leq \cdots \leq R_n$ into another one $P' = B_{R'_1}^2 \times \cdots \times B_{R'_n}^2$ with $R'_1 \leq \cdots \leq R'_n$ using a symplectic map. There are two obvious obstructions to this problem, the first one comes from Gromov's nonsqueezing which implies $R_1 \leq R'_1$. The second one is the volume which implies that $R_1 \cdots R_n \leq R'_1 \cdots R'_n$. Using the techniques in the article by Traynor [Tra95] one can prove that one could embed P into P' if and only if $R_1 \cdots R_k \lesssim R'_1 \cdots R'_k$ for every k between 1 and n . Traynor's results could indicate the existence of middle dimensional conditions for symplectic embeddings of polydiscs. It was not known whether these middle dimensional conditions were necessary until an article of Guth where he proved that, modulo a dimensional constant, these are the only two invariants. More precisely:

Theorem 4.49 (Guth 2008 [Gut08]). *There is a dimensional constant $C(n)$ so that the following holds. If $C(n)R_1 \leq R'_1$ and $C(n)R_1 \cdots R_n \leq R'_1 \cdots R'_n$ then P symplectically embeds into P' .*

This theorem gave a partial answer to Hofer's question. It implies that if $1 < k < n$ then k -capacities that verify the following continuity hypothesis:

$$\lim_{R \rightarrow +\infty} c(B_1^{2k} \times B_R^{2n-2k}) < +\infty \quad \text{and} \quad \lim_{R \rightarrow +\infty} c(B_1^{2k-2} \times B_R^{2n-2k+2}) = +\infty$$

do not exist. His theorem was a close definitive answer but the question of less regular capacities remained open. This was recently solved in the negative by Pelayo and Vũ Ngọc in [PVuN15]. They applied a limit argument to the construction of Guth and Hind-Kerman [HK14] in order to prove the unbounded version of Guth's theorem:

Theorem 4.50 (Pelayo-Vũ Ngọc 2015 [PVuN15]). *If $n \geq 2$ then the cylinder $B_1^2 \times \mathbb{R}^{2n-2}$ can be symplectically embedded into the product $B_R^{2n-2} \times \mathbb{R}^2$ for all $R \geq \sqrt{2^{n-1} + 2^{n-2} - 2}$.*

With this theorem the question of intermediate symplectic capacities was definitely settled. It showed that general symplectic diffeomorphisms are too flexible to capture this type of cylindrical rigidity.

Another point of view for the middle dimensional problem comes from a reformulation of Gromov's non-squeezing theorem. In dimension 2 symplectomorphisms are the same as area preserving maps so in [EG91] Eliashberg and Gromov pointed out that (using a theorem of Moser about the existence of area preserving diffeomorphisms) Gromov's theorem is equivalent to

$$\text{area}(\Pi_1 \phi(B_r^{2n})) \geq \pi r^2 \quad \text{for every symplectomorphism } \phi.$$

Denote by Π_k the projection on the first $2k$ coordinates. A possible generalization of this statement to higher dimensions is

$$\text{Vol}(\Pi_k \phi(B_r^{2n})) \geq \text{Vol}(\Pi_k B_r^{2n}) = \text{Vol}(B_r^{2k}) \quad \text{for every symplectomorphism } \phi.$$

This problem was studied by Abbondandolo and Matveyev in [AM13]. In their article, Abbondandolo and Matveyev first proved that the inequality is true in the linear case:

Theorem 4.51 (Abbondandolo-Matveyev 2013 [AM13]). *Let Φ be a linear symplectic automorphism of \mathbb{R}^{2n} , and let $P : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be the orthogonal projector onto a complex linear subspace $V \subseteq \mathbb{R}^{2n}$ of dimension $2k$, $1 \leq k \leq n$. Then*

$$\text{Vol}(P\Phi(B_r^{2n})) \geq \text{Vol}(B_r^{2k})$$

with equality if and only if the linear subspace $\Phi^T V$ is complex.

As they point out in their article, unlike the projection of the ball, the intersection with the linear subspace has small volume, in other words one always has the reverse inequality

$$\text{Vol}(V \cap B_r^{2n}) \leq \text{Vol}(B_r^{2k}).$$

In contrast to the linear result, they moreover proved that symplectic diffeomorphisms are again too flexible to have this kind of middle dimensional symplectic rigidity. More precisely, they show that one can squeeze arbitrarily the symplectic projection of image of the ball by symplectomorphisms.

Theorem 4.52 (Abbondandolo-Matveyev 2013 [AM13]). *Let $P : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be the orthogonal projector onto a complex linear subspace $V \subseteq \mathbb{R}^{2n}$ of dimension $2k$, $1 < k < n$. For every $\epsilon > 0$ there exists a smooth symplectic embedding $\phi : B_1^{2n} \rightarrow \mathbb{R}^{2n}$ such that*

$$\text{Vol}(P\phi(B_1^{2n})) < \epsilon.$$

In spite of all this negative results for middle dimensional rigidity, there was a recent result by Rigolli in [Rig15] where he proved that there is also local middle dimensional volume rigidity if one restricts the class of symplectomorphisms to analytic ones.

Theorem 4.53 (Rigoli 2015 [Rig15]). *Let $\phi_t : \overline{B}_1^{2n} \rightarrow \mathbb{R}^{2n}$ be an analytic path of symplectic embeddings, with $t \in [0, 1]$, such that ϕ_0 is linear. Then the middle dimensional non-squeezing inequality*

$$\text{Vol}(P\phi_t(B_1^{2n})) \geq \text{Vol}(B_r^{2k})$$

holds for t small enough.

This result points out in the direction that this middle dimensional rigidity might be possible upon restriction of the class of symplectic diffeomorphisms that we consider. Moreover, this result has another local implication:

Theorem 4.54 (Rigoli 2015 [Rig15]). *For every analytic symplectic embedding of a domain D there exists a function $r_0 : D \rightarrow (0, +\infty)$ such that the middle dimensional inequality*

$$\text{Vol}(P\phi_t(B_r^{2n}(x))) \geq \text{Vol}(B_r^{2k})$$

holds for every $x \in D$ and for every $r < r_0(x)$. Moreover r_0 is bounded away from zero on compact subsets $K \subseteq D$.

We would like to point out another possible middle dimensional generalization of the squeezing problem. In dimension 2 the value of any normalized symplectic capacity on topological discs coincides with the area, so one may also rewrite Gromov's theorem as

$$c(\Pi_1\phi(B_r^{2n})) \geq \pi r^2 \quad \text{for every symplectomorphism } \phi,$$

where c is a symplectic capacity. One can then ask if this inequality is true with Π_1 replaced by Π_k , and more generally look at subsets Z different from B_r^{2n} and replace πr^2 with the capacity of $\Pi_k Z$. As we have seen in the previous results, there is little hope that one such inequality is always true so one has to restrict the type of symplectomorphisms considered. Theorem 4.39 sheds some light to this problem for deformations of coisotropic cylinders $Z = X \times \mathbb{R}^{n-m} \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$ provided that we restrict the class of symplectomorphisms to Hamiltonian isotopies with bounded speed, that is, diffeomorphisms generated by functions H verifying the hypothesis of Theorem 4.39. More precisely, using the monotonicity of γ we get the following corollary:

Theorem 4.55. *Let $X \subset \mathbb{R}^{2m}$ be a compact set. Consider $X \times \mathbb{R}^{n-m} \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$ and Let $H : \mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a Hamiltonian function such that*

$$|\nabla H(z)| \leq A \quad \text{for every } (t, z) \in [0, T] \times \mathbb{R}^{2n}$$

that generates the flow ψ_t . Then for every $t \in [0, T]$

$$c(X) \leq \gamma(\Pi_m\psi_t(X \times \mathbb{R}^{n-m})).$$

In contrast with the volume interpretation of the middle dimensional problem (see Theorem 4.51), This theorem is not true for every linear symplectic map, only for small times of linear Hamiltonian diffeomorphisms. Remark moreover that classical symplectic capacities do not give any rigidity information for coisotropic cylinders since by Proposition 3.41 we have

$$c(X \times \mathbb{R}^{n-m}) = 0 = \gamma(X \times \mathbb{R}^{n-m}).$$

As it is usual in symplectic geometry, this rigidity result can be used to define an invariant. The classical example of this fact is the definition of Gromov's width after he proved the nonsqueezing theorem. In our case Theorem 4.55 indicates that we should consider the following quantity:

$$\gamma_G^k(U) = \inf\{\gamma(\Pi_k\phi(U)) \mid \phi \in G\}$$

where G is a subgroup of the group of symplectic diffeomorphisms. For $G = \text{Sympl}(\mathbb{R}^{2n})$ the permutation of coordinates shows that γ_G^k is zero on coisotropic cylinders of dimension k . In particular, it might be trivial for every subset of \mathbb{R}^{2n} . On the other hand, if the elements of G are Hamiltonian diffeomorphisms generated by Lipschitz functions then Theorem 4.55 implies that γ_G^k is bounded from below on coisotropic cylinders of dimension k . As an example of G one can take the subgroup of Hamiltonian diffeomorphisms φ_t^H where H , φ_t^H , and $(\varphi_t^H)^{-1}$ are Lipschitz on the space variable over compact time intervals.

Proposition 4.56. *Denote $\text{Ham}^{dL}(\mathbb{R}^{2n})$ the set of Hamiltonian diffeomorphisms φ_t^H such that H_t , φ_t^H and $(\varphi_t^H)^{-1}$ are all Lipschitz in space over compact time intervals. Then $\text{Ham}^{dL}(\mathbb{R}^{2n})$ is a subgroup of $\text{Sympl}(\mathbb{R}^{2n})$. Moreover $\text{Ham}^{dL}(\mathbb{R}^{2n})$ is strictly bigger than the group of compactly supported Hamiltonian diffeomorphisms.*

Remark 4.57. The superscript dL on $\text{Ham}^{dL}(\mathbb{R}^{2n})$ stands for double Lipschitz condition.

Proof. First recall the following formulas:

$$\varphi_t^H \circ \varphi_t^K = \varphi_t^{H\#K} \quad \text{and} \quad (\varphi_t^H)^{-1} = \varphi_t^{\bar{H}},$$

where

$$\begin{aligned} H\#K(t, z) &= H(t, z) + K(t, (\varphi_t^H)^{-1}(z)), \\ \bar{H}(t, z) &= -H(t, \varphi_t^H(z)). \end{aligned}$$

The identity is clearly in $\text{Ham}^{dL}(\mathbb{R}^{2n})$ and it is an easy exercise to use these formulas to prove that $\text{Ham}^{dL}(\mathbb{R}^{2n})$ has a group structure. For the second statement, consider a Lipschitz autonomous Hamiltonian H with Lipschitz gradient and use Gronwall's lemma to prove that φ_t^H (and therefore $(\varphi_t^H)^{-1} = \varphi_{-t}^H$) is Lipschitz. \square

Definition 4.58. For every subset $Z \subseteq \mathbb{R}^{2n}$ we define

$$\gamma^k(Z) = \inf\{\gamma(\Pi_k \phi(Z)) \mid \phi \in \text{Ham}^{dL}(\mathbb{R}^{2n})\}$$

The properties of γ_k are summarized in the following proposition:

Proposition 4.59. *The map γ^k verifies the following properties:*

- (monotonicity) If $Z_1 \subseteq Z_2$ then $\gamma^k(Z_1) \leq \gamma^k(Z_2)$.
- (conformality) $\gamma^k(\lambda Z) = \lambda^2 \gamma^k(Z)$ for every $\lambda \in \mathbb{R}$.
- (Lipschitz invariance) If $\phi \in \text{Ham}^{dL}(\mathbb{R}^{2n})$ we have $\gamma^k(\phi(Z)) = \gamma^k(Z)$.
- (non-triviality+normalization) $\gamma^k(B_r^{2k} \times \mathbb{R}^{n-k}) = \pi r^2$.

Moreover for every compact $X \subseteq \mathbb{R}^{2m}$ we have

$$c(X) \leq \gamma^k(X \times \mathbb{R}^{n-k}) \leq \gamma(X).$$

Finally we want to point out that this invariant satisfies a 2-conformality property instead of the k -conformality asked by Hofer for the middle dimensional symplectic capacities.

Chapter 5

Rigidity on energy surfaces

Let (M, ω) be a symplectic manifold and consider a smooth function $H : M \rightarrow \mathbb{R}$. Every regular value λ defines a smooth hypersurface Σ of M . The Hamiltonian vector field is easily seen to be everywhere tangent to Σ , so the Hamiltonian flow preserves Σ . The properties of this flow have historically been an important subject of research. In this chapter we are going to study the influence that the famous symplectic camel theorem has on the dynamics on Σ . We explain how the symplectic camel theorem can be used to create compact invariant subsets. Moreover, we show that these invariant subsets persist under C^0 perturbation of the energy surfaces of H .

5.1 C^0 perturbations and compact invariant sets

Let (M, ω) be a symplectic manifold, consider a smooth function $H : M \rightarrow \mathbb{R}$ and let λ be a regular value of H . Then $\Sigma_\lambda = H^{-1}(\lambda)$ is a smooth hypersurface invariant by the flow of H . We are interested in the existence of compact invariant subsets of Σ_λ and its C^0 perturbations.

For historical context on the problem of compact invariant subsets the interested reader can look at the book [HZ94][Chapter 4] which has an excellent chapter on the existence of closed characteristics in Σ . The surfaces Σ that are considered in that chapter are compact and the invariant subsets are circles. In order to gain some intuition, we present an existence result contained in that chapter. Let (M, ω) be a symplectic manifold and denote by c_0 the capacity constructed by Hofer and Zehnder (which is a generalization of the one explained in Chapter 2, Section 2.3). Using the definition of this special capacity c_0 they are able to prove the following result:

Theorem 5.1 (see [HZ94]). *Let H be a Hamiltonian function on (M, ω) and let $\Sigma_{\lambda_0} = H^{-1}(\lambda_0)$ be a compact energy surface. Suppose that there is an open neighborhood U of Σ_{λ_0} such that U has bounded capacity: $c_0(U, \omega) < +\infty$. Then there exists an interval of regular values I containing λ_0 and a dense subset $D \subseteq I$ such that for every $\lambda \in D$ the energy surface Σ_λ has a closed characteristic.*

As a particular example of such an energy surface, we have every compact regular Σ_λ in $(\mathbb{R}^{2n}, \omega_0)$. Indeed, by monotonicity of the capacity, every bounded set U has capacity less than the capacity of a big ball B_R , so $c_0(U, \omega) < +\infty$. This result shows that, at least for compact energy surfaces, closed characteristics are a common phenomenon that appears under perturbation of the energy level.

In this section we explain how the symplectic camel theorem can be used to create compact invariant subsets in arbitrary energy surfaces Σ . Recall that the statement of the theorem is:

Theorem 5.2 ([Vit92, MT93]). *Consider the space $E = \mathbb{R}^{2n-1} \times \{0\} \setminus B^{2n}(0, r)$ and suppose that there is a symplectic isotopy φ^t of \mathbb{R}^{2n} with support in $\mathbb{R}^{2n} \setminus E$ such that*

$$B_R^{2n} \subseteq \{p_n < 0\} \quad \text{and} \quad \varphi^1(B_R^{2n}) \subseteq \{p_n > 0\}.$$

Then $r \geq R$.

There is also an unpublished proof of this result by Gromov and Eliashberg.

Every hypersurface Σ is locally symplectomorphic to $\mathbb{R}^{2n-1} \times \{0\} \subseteq \mathbb{R}^{2n}$ so every point $p \in \Sigma$ has a small open neighborhood U symplectomorphic to a small open neighborhood V of the origin in \mathbb{R}^{2n} such that

$$U \cap \Sigma \simeq V \cap \mathbb{R}^{2n-1} \times \{0\}.$$

Using this symplectomorphism we see that a particular class of deformations of Σ are the local deformations given by deformations of $\Sigma_0 := \mathbb{R}^{2n-1} \times \{0\}$. In light of this remark we restrict ourselves to compactly supported deformations of the standard hypersurface Σ_0 . Remark that Σ_0 is the energy surface associated to the coordinate function p_n . We are going to consider Hamiltonian functions given by $H = p_n \circ \varphi$ where $\varphi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a compactly supported diffeomorphism.

Definition 5.3. Denote (z, q_n, p_n) the coordinates of \mathbb{R}^{2n} . We say that $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is a (r, R) -simple function if the following properties are satisfied:

- $H = p_n \circ \varphi$ for a compactly supported diffeomorphism φ .
- The support of φ is contained in $\{p_n > 1\} \cup (B^{2n-1}(z_0, r) \times [0, 1])$ for a $z_0 \in \mathbb{R}^{2n-1}$.
- $B^{2n}(z_1, R) \subseteq \{H < 1\} \cap \{p_n > 1\}$ for a $z_1 \in \mathbb{R}^{2n}$.

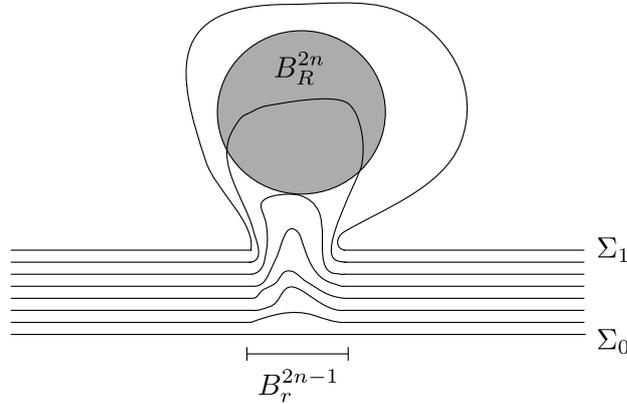


Figure 5.1: The energy surfaces of a (r, R) -simple function.

Remark 5.4. The choice of the interval $[0, 1]$ is arbitrary, and is chosen for simplicity in the notation of the proofs. The second condition implies that for every $\lambda \in [0, 1]$ we have

$$\{p_n = \lambda\} \setminus B^{2n}((z_0, \lambda), r) \subseteq H^{-1}(\lambda).$$

Compare with the hypothesis of Theorem 5.2.

We are moreover interested in perturbations of this type of functions. We consider perturbations inside the class of (r, R) -simple functions:

Definition 5.5. Let H be a (r, R) -simple function. We say that H_ϵ is a ϵ -simple perturbation of H if $H_\epsilon = p_n \circ \varphi_\epsilon$ with $d_{C^0}(\varphi, \varphi_\epsilon) \leq \epsilon$ and such that the support of φ_ϵ is contained in $\{p_n > 1\} \cup (B^{2n-1}(z_0, r + \epsilon) \times [0, 1])$.

If $H = p_n \circ \varphi$ is a (r, R) -simple function then

$$\Sigma_\lambda = H^{-1}(\lambda) = \varphi^{-1}(p_n^{-1}(\lambda)) = \varphi^{-1}(\mathbb{R}^{2n-1} \times \{\lambda\}).$$

If we denote $\Sigma_\lambda^\epsilon = H_\epsilon^{-1}(\lambda)$ then the C^0 distance between Σ_λ and Σ_λ^ϵ is bounded by the C^0 distance between φ and φ_ϵ . As a consequence one may see that an ϵ -simple perturbation of a (r, R) -simple function is a $(r + \epsilon, R - \epsilon)$ -simple function. The main result of this chapter is the following theorem:

Theorem 5.6. Let $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a (r, R) -simple function with $r < R$. Then there exists a $\lambda \in [0, 1]$ such that Σ_λ has a compact invariant subset. As a consequence, for every ϵ -simple perturbation H_ϵ of H with $\epsilon \in [0, \frac{R-r}{2}[$ there exists a $\lambda_\epsilon \in [0, 1]$ such that $\Sigma_{\lambda_\epsilon}^\epsilon = H_\epsilon^{-1}(\lambda_\epsilon)$ has a compact invariant subset.

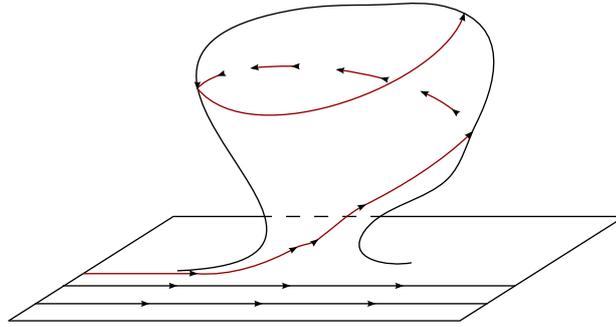


Figure 5.2: A representation of the energy surface Σ_λ where the invariant compact subset appears. In this case there is a trajectory trapped in a bounded set and this trajectory creates a compact invariant subset.

The idea of the proof goes as follows: if there is no compact invariant subset then every characteristic trajectory goes from one side of the space to the other (see Figure 5.2). If this is the case, we can build a symplectic isotopy that verifies the hypothesis of the Camel Theorem but which contradicts the conclusion. Since this is not possible, we conclude that there must be a trajectory trapped in a bounded set. This trajectory then creates a compact invariant subset. Note that the compact invariant sets persist for small simple perturbations, but we have no control on the energy level where it appears.

5.1.1 Trapping trajectories

The proof of Theorem 5.6 will be carried out in several steps, and is based on the following lemma. For simplicity of notation we will suppose that φ and thus H is supported inside $\{0 < q_n < 1\}$. We know that outside this set all the characteristic trajectories are just translations in the coordinate q_n .

Lemma 5.7. Let H be a Hamiltonian function of the form $H = p_n \circ \varphi$ where φ is a diffeomorphism supported in a compact set contained in $\{0 < q_n < 1\}$. Then the following statements are equivalent:

1. There exists characteristic trajectory in Σ_λ which goes through $\{q_n = 0\}$ but does not intersect $\{q_n = 1\}$. (see Figure 5.2)
2. Σ_λ has a compact invariant set.

Proof. (1 \implies 2). Suppose that 1 is true. First note that all trajectories enter $\{0 < q_n < 1\}$ and that if a trajectory intersects $\{q_n = 1\}$ then it goes to infinity. Thus, the hypothesis tells us that there is a point $z_0 \in \mathbb{R}^{2n}$ such that $\psi_t(z_0) = \psi_t^H(z_0)$ stays in a compact set for $t \in [0, +\infty[$. The ω -limit set associated to z_0 is then non-empty and generates the desired compact invariant set. More precisely consider the set

$$B = \{z \in \mathbb{R}^{2n} \mid \text{there exists a sequence } t_n \rightarrow +\infty \text{ such that } \lim_{n \rightarrow +\infty} \psi_{t_n}(z_0) = z\}$$

B is clearly nonempty, bounded and invariant by the flow. We now prove that it is closed. Take a sequence of points $\{z_n\}$ in B which converges to a point z . For every $m \in \mathbb{N}$ there exists a z_n such that $d(z_n, z) \leq \frac{1}{2m}$. Chose a sequence s_k adapted to z_n (cf. the definition of B) and define

$$t_m = \min\{s_k > t_{m-1} + 1 \mid d(\psi_{s_k}(z_0), z_n) \leq \frac{1}{2m}\}.$$

By definition of the points in B , we have $t_m < +\infty$ and we clearly have $t_m \rightarrow +\infty$. Finally

$$d(\psi_{t_m}(z_0), z) \leq d(\psi_{t_m}(z_0), z_n) + d(z_n, z) \leq \frac{1}{m},$$

so $z \in B$ and B is closed. This proves 1 \implies 2.

(-1 \implies -2). Now suppose that 1 is not true, so every characteristic trajectory in Σ_λ which goes through $\{q_n = 0\}$ intersects $\{q_n = 1\}$. Denote $C_\lambda = \mathbb{R}^{2n-2} \times]-\infty, 0[\times \{\lambda\}$, we have $C_\lambda \subseteq \Sigma_\lambda$. Since every trajectory in C_λ is unbounded, it is enough to show that

$$\bigcup_{t \in \mathbb{R}} \psi_t(C_\lambda) = \Sigma_\lambda.$$

Remark that $\Sigma_\lambda = \varphi^{-1}(\mathbb{R}^{2n-1} \times \{\lambda\})$ is connected, and that C_λ is open in Σ_λ and non-empty. Since ψ_t restricts to a diffeomorphism of Σ_λ , we see that the left hand side of the inequality is a open set. To prove the equality, it is then enough to prove that the lhs is also closed in Σ_λ . It is clear that all the trajectories of C_λ intersect $\mathbb{R}^{2n-2} \times \{0\} \times \{\lambda\}$. We claim that there exists a T such that

$$\psi_T(\mathbb{R}^{2n-2} \times \{0\} \times \{\lambda\}) \subseteq \mathbb{R}^{2n-2} \times [1, +\infty[\times \{\lambda\}.$$

Since φ has compact support, for a radius K big enough we have

$$\psi_1((B_K^{2n-2})^c \times \{0\} \times \{\lambda\}) = (B_K^{2n-2})^c \times \{1\} \times \{\lambda\}.$$

For the points in $B_K^{2n-2} \times \{0\} \times \{\lambda\}$, one may use the compactness of the closed ball and that by hypothesis every point arrives to the other side to find the desired T . We now use the existence of T to prove the closedness of the union of the images of C_λ . Take a point z in the closure, and a sequence of points $\{z_m\}$ which approaches it. Since $\{z_m\}$ converges we know that $q_n(z_m)$ are bounded. In particular, the existence of T implies that there is a t_0 such that $q_n(\psi_{-t_0}(z_m)) < -1$. By continuity of ψ_t we get that $q_n(\psi_{-t_0}(z)) \leq -1$. This implies that $z \in \psi_{t_0}(C_\lambda)$, and finishes the proof. \square

One may moreover prove the following result:

Lemma 5.8. *Let H be a Hamiltonian function which coincides with the function p_1 outside $\{0 < q_n < 1\}$. Then the set of unbounded trajectories that do not go through the wall $\{q_n = 1\}$ has measure 0.*

Proof. Let B be the set of $z \in \mathbb{R}^{2n}$ such that $\psi_t(z) \in \{q_n \leq 1\}$ for all $t \in \mathbb{R}$ and $\psi_t(z) \rightarrow \infty$ when $t \rightarrow -\infty$. Suppose that B has positive measure. Since

$$B = \bigcup_{n \geq 0} B \cap \{-n \leq q_n < -n + 1\}$$

then there is an $n_0 \geq 0$ such that $\mu(B \cap \{-n_0 \leq q_n < -n_0 + 1\}) > 0$. We prove that this is impossible studying the cases $n_0 > 0$ and $n_0 = 0$. If $n_0 > 0$ then the fact that for $t < 0$

$$\psi_t(\{-n_0 \leq q_n < -n_0 + 1\}) = \{-n_0 + t \leq q_n < -n_0 + 1 + t\}$$

implies that for each $C > 0$ we may find an $m_0 > 0$ such that

$$\mu\left(\bigcup_{n=n_0}^{m_0} B \cap \{-n \leq q_n < -n + 1\}\right) > C.$$

Now, ψ_t preserves the measure so this contradicts the fact that ψ_{m_0} sends the above union into $B \cap \{0 \leq q_n < +1\}$ which is bounded. Suppose now that $n_0 = 0$. Denote $B_0 = B \cap \{0 \leq q_n < 1\}$ and $B_n = \{z \in B_0 \mid \psi_{-n} \in \{-n \leq q_n < -n + 1\}\}$. We clearly have a disjoint union $B_0 = \bigcup_{n \geq 1} B_n$ so we may find a $k_0 \geq 1$ such that $\mu(B_{k_0}) > 0$. The same argument as for the case $n_0 > 0$ gives a contradiction which end the proof of the lemma. \square

5.1.2 Construction of the symplectic isotopy

Consider a smooth Hamiltonian of the form $H = p_n \circ \varphi$ with φ a diffeomorphism supported in $\{0 < q_n < 1\} \cap \{p_n > 0\}$. In particular $\Sigma_0 = \mathbb{R}^{2n-1} \times \{0\}$. Taking into account Lemma 5.7 we will make the following assumption:

Definition 5.9. We say that a (r, R) -simple function is directed if every characteristic trajectory in every Σ_λ which goes through $\{q_n = 0\}$ intersects $\{q_n = 1\}$.

In other words, we say that the trajectories go from one side of the space to the other.

Lemma 5.10. *Let H be a directed (r, R) -simple function at denote by ψ_t its flow. Then there is a smooth function $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ such that the map*

$$\Psi_t(z) := \psi_{f(z)t}(z) \quad \text{verifies} \quad q_n(\Psi_0(x, 0, \lambda)) = 0 \quad \text{and} \quad q_n(\Psi_1(x, 0, \lambda)) = 1$$

for every $(x, 0, \lambda) \in \times \mathbb{R}^{2n-2} \times \{0\} \times \mathbb{R}$.

Proof. We first define f on $\mathbb{R}^{2n-2} \times \{0\} \times \mathbb{R}$. Using the same arguments as in the second part of the proof of Lemma 5.7 we may find a $T > 0$ such that $q_n(\psi_T(\mathbb{R}^{2n-2} \times \{0\} \times \mathbb{R})) \subseteq]1, \infty[$. Define f to be

$$f(x, 0, \lambda) = T + 1 - q_n(\psi_T(x, 0, \lambda)).$$

Note that the definition is independent of the choice of T : take another T' , and suppose that $T < T'$. Then

$$\psi_{T'}(x, 0, \lambda) = \psi_{T'-T} \psi_T(x, 0, \lambda) = \psi_T(x, 0, \lambda) + (0, T - T', 0),$$

which gives

$$T' + 1 - q_n(\psi_{T'}(x, 0, \lambda)) = T + 1 - q_n(\psi_T(x, 0, \lambda)).$$

Extend f to the whole \mathbb{R}^{2n} by saying that it is constant on trajectories, that is, $f \circ \psi_t = f$. It is smooth since it is clearly smooth on $\{q_n \leq 0\}$, by invariance and by the existence of T . Using the fact that ψ_t is the flow of an autonomous vector field one sees that

$$\begin{aligned} \Psi_1(x, 0, \lambda) &= \psi_{f(x, 0, \lambda)}(x, 0, \lambda) \\ &= \psi_{-q_n(\psi_T(x, 0, \lambda))} \psi_1 \psi_T(x, 0, \lambda). \end{aligned}$$

Now remark that using the definition of T and that the support of H is contained in $\{0 < q_n < 1\}$ we have

$$q_n(\psi_1 \psi_T(x, 0, \lambda)) = q_n(\psi_T(x, 0, \lambda)) + 1.$$

Using again that ψ_t acts by translations outside of $\{0 < q_n < 1\}$ we get $q_n(\Psi_1(x, 0, \lambda)) = 1$. \square

Remark 5.11. The fact that $q_n(\Psi_1(x, 0, \lambda)) = 1$ implies that $f(z) \neq 0$ for every $z \in \mathbb{R}^{2n}$. One may also define $f(x, 0, \lambda)$ as the unique time s such that $q_n(\psi_s(x, 0, \lambda)) = 1$. Moreover since f is invariant along the trajectories we have

$$\{f, H\} = 0.$$

Note that f constant along the trajectories implies that Ψ is an autonomous flow:

$$\Psi_t(\Psi_s(z)) = \psi_{f(\psi_{f(z)s}(z))t}(\psi_{f(z)s}(z)) = \psi_{f(z)t}(\psi_{f(z)s}(z)) = \Psi_{t+s}(z)$$

The isotopy is generated by $Y = fX_H$ since:

$$\frac{d}{dt} \Psi_t(z) = f(z)X_H(\psi_{f(z)t}(z)) = f(\psi_{f(z)t}(z))X_H(\psi_{f(z)t}(z)) = Y(\Psi_t(z)).$$

Denote by $i_\lambda : \Sigma_\lambda \hookrightarrow \mathbb{R}^{2n}$ the natural inclusion, and by ω the standard symplectic form of \mathbb{R}^{2n} .

Lemma 5.12. Define $S_\lambda : \Sigma_0 \rightarrow \Sigma_\lambda$ by

$$S_\lambda(x, t, 0) = \Psi_t(x, 0, \lambda).$$

Then S_λ is a smooth isotopy that verifies $S_\lambda^* \omega = i_0^* \omega$.

Proof. It is clear that S is smooth in all the variables. To see that S_λ is onto (resp. one to one), remark that by (the proof of) Lemma 5.7 every trajectory of Σ_λ intersects once (resp. only once) the set $\mathbb{R}^{2n-2} \times \{0\} \times \{\lambda\}$. We now prove the differential equality. Remark that for $Y = fX_H$ we have $i_Y \omega = fdH$ so that

$$\frac{d}{dt} \Psi_t^* \omega = \Psi_t^* \mathcal{L}_Y \omega = \Psi_t^* di_Y \omega = \Psi_t^* (df \wedge dH) = df \wedge dH,$$

since both f and H are invariant for ψ_t . Denote by j_λ the map $j_\lambda(x) = (x, 0, \lambda)$ and by u_t the map $u_t(x) = (x, t, 0)$. Then clearly $j_\lambda^* dH = 0$ which implies that for all t

$$j_\lambda^* \Psi_t^* \omega = j_\lambda^* \Psi_0^* \omega = j_\lambda^* \omega = u_t^* i_0^* \omega.$$

In particular using $\Psi_t \circ j_\lambda = S_\lambda \circ u_t$ we get $u_t^* S_\lambda^* \omega = u_t^* i_0^* \omega$ which gives the result for vectors in $\mathbb{R}^{2n-2} \times \{0_{\mathbb{R}^2}\}$. It is now enough to prove that for $v \in \{0_{\mathbb{R}^{2n-2}}\} \times \mathbb{R} \times \{0\}$ we have $S_\lambda^* \omega(v, \cdot) = 0$. For this remark that

$$dS\left(\frac{\partial}{\partial q_n}\right) = \frac{d}{dt} \Psi_t = Y(\Psi_t).$$

5.1. C^0 perturbations and compact invariant sets

On the other hand, X^H defines the coisotropic foliation of Σ_λ , or in other words, $i_\lambda^* \omega(X^H, \cdot) = 0$. Joining these two facts, and using that $Y = fX$, we get

$$S_\lambda^* \omega\left(\frac{\partial}{\partial q_n}, \cdot\right) = 0$$

which is the desired result, and finishes the proof. \square

The isotopy S_λ is not compactly supported. Indeed one may see that outside a compact set we have

$$S_\lambda(x, t, 0) = (\varphi_\lambda(x), tf(x, 0, \lambda), \lambda)$$

where φ_λ is a compactly supported symplectic isotopy of \mathbb{R}^{2n-2} (one may see that it is symplectic using the proof of the previous lemma). Moreover, if x is big enough, we have $f(x, 0, \lambda) = 1$ and $\varphi_\lambda(x) = x$ so S_λ is the identity for x big enough. Consider $T_{-\lambda}(x, t, s) = (x, t, s - \lambda)$ and denote $\Sigma_\lambda^0 = T_{-\lambda}(\Sigma_\lambda)$ (see Figure 5.3). The two submanifolds Σ_0 and Σ_λ^0 coincide at infinity. Define the isotopy $S_\lambda^0 := T_{-\lambda} \circ S_\lambda : \Sigma_0 \rightarrow \Sigma_\lambda^0$. It is easily seen to be symplectic in the sense of the previous lemma. Outside a compact set we have

$$S_\lambda^0(x, t, 0) = (\varphi_\lambda(x), tf(x, 0, \lambda), 0).$$

We now extend S_λ^0 to a global isotopy.

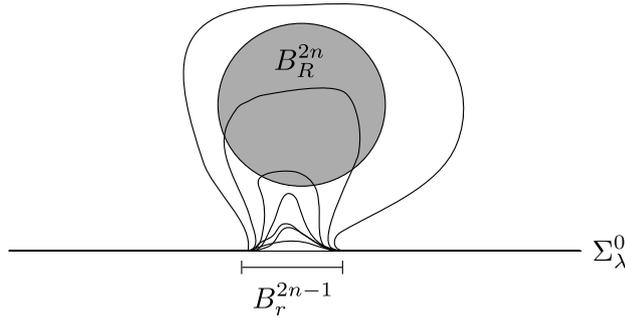


Figure 5.3: The representation of the Σ_λ^0 for the simple function of Figure 5.2.

Lemma 5.13. *There exists a global isotopy F_λ of \mathbb{R}^{2n} such that*

$$F_\lambda(\Sigma_0) = \Sigma_\lambda^0, \quad i_0^* F_\lambda^* \omega = i_0^* \omega \quad \text{and} \quad F_\lambda^* \omega = \omega \quad \text{outside a compact set.}$$

Proof. Let $K \subseteq \mathbb{R}^{2n}$ be a compact such that on its complement S_λ^0 has the simple expression

$$S_\lambda^0(x, t, 0) = (\varphi_\lambda(x), tf(x, 0, \lambda), 0) \quad \text{so} \quad (S_\lambda^0)^{-1}(x, t, 0) = \left(\varphi_\lambda^{-1}(x), \frac{t}{f(\varphi_\lambda^{-1}(x), 0, \lambda)}, 0\right).$$

Remark that outside K we have the extension $S_\lambda^0(x, t, s) = (\varphi_\lambda(x), tf(x, 0, \lambda), s)$ which is a diffeomorphism for every s . Inside of K extend the vector field

$$\frac{\partial S_\lambda^0}{\partial \lambda} \circ (S_\lambda^0)^{-1}$$

to a neighborhood U_λ^ϵ of Σ_λ . We do it the same way as we did outside K , that is, a constant extension of the vector field in the normal direction. One may chose the same ϵ for every $\lambda \in [0, 1]$ since outside K the extension is defined for every ϵ . We now multiply the vector field by a

function which is zero outside a small normal neighborhood and which is one in a smaller normal neighborhood. This gives a vector field on the whole \mathbb{R}^{2n} which outside K and on a neighborhood of Σ_0 is equal to

$$\tilde{X}_\lambda(x, t, s) = \left(\frac{\partial \varphi_\lambda}{\partial \lambda}, \frac{t}{f} \frac{\partial f}{\partial \lambda}, 0 \right),$$

evaluated at $(\varphi_\lambda^{-1}(x), 0, \lambda)$. This vector is less than linear since φ_λ is compactly supported and f and all its derivatives are bounded (because $f = 1$ at infinity). This means that the vector field that we have built is integrable and generates an isotopy \tilde{F}_λ such that $\tilde{F}_\lambda \circ i_0 = S_\lambda^0$. For the moment it does not verify $\tilde{F}_\lambda^* \omega = \omega$ outside a compact set so we need to make one last modification. Indeed outside a compact set, and for big s we know that \tilde{F}_λ is the identity, the same happens for big x since φ_λ is compactly supported, but not for big t . In order to solve this issue, take a compactly supported function $\rho(t)$ which is equal to one on a big interval containing the projection of K . Then modify \tilde{X}_λ for big t by considering the vector field

$$X_\lambda(x, t, s) = \left(\frac{\partial \varphi_\lambda}{\partial \lambda}, \rho(t) \frac{t}{f} \frac{\partial f}{\partial \lambda}, 0 \right),$$

Note that this defines a vector field tangent to Σ_0 (which coincides with Σ_λ outside K). This vector field is integrable and generates what we are going to call F_λ . Inside K we have

$$F_\lambda(\Sigma_0 \cap K) = S_\lambda^0(\Sigma_0 \cap K) = \Sigma_\lambda^0 \cap K.$$

Outside K the vector field preserves Σ_0 which coincides with Σ_λ^0 so we conclude that

$$F_\lambda(\Sigma_0) = \Sigma_\lambda^0.$$

Now, since inside K we have $\tilde{F}_\lambda = F_\lambda$, we get

$$i_0^* F_\lambda^* \omega = (F_\lambda \circ i_0)^* \omega = (\tilde{F}_\lambda \circ i_0)^* \omega = (S_\lambda^0)^* \omega = i_0^* \omega.$$

Outside K we have $F_\lambda(x, t, 0) = (\varphi_\lambda(x), g_\lambda(x, t, 0), 0)$ with φ_λ symplectic and $g = t$ outside a compact set. In particular a straightforward calculation gives that outside K we also have $(F_\lambda \circ i_0)^* \omega = i_0^* \omega$. Moreover, if x or s are big enough we have $F_\lambda = \text{Id}$ which is symplectic, and if only t is big we have $F_\lambda(x, t, s) = (\varphi_\lambda(x), t, s)$ which is also symplectic. We conclude that $F_\lambda^* \omega = \omega$ outside a compact set. \square

Proposition 5.14. *Let H be a directed (r, R) -simple function. Then there is a symplectic isotopy Φ_λ on the whole \mathbb{R}^{2n} such that $\Phi_\lambda(\Sigma_0) = \Sigma_\lambda^0$.*

The proof of this result is contained in the following subsection. It is based on the existence of F_λ . For the moment we use it to prove Theorem 5.6.

Proof of Theorem 5.6. Consider a directed (R, r) -simple function, then by Proposition 5.14 there is a symplectic isotopy $R_\lambda = \Phi_{1-t} \circ \Phi_1^{-1}$ such that $R_\lambda(\Sigma_1^0) = \Sigma_\lambda^0$. We claim that this implies

$$R_\lambda(T_{-1}B^{2n}(z_1, R)) \cap \Sigma_0 \setminus B(z_0, r) = \emptyset \quad \text{for every } \lambda \in [0, 1].$$

(see Definition 5.3) Indeed, since $\Sigma_0 \setminus B(z_0, r) \subseteq \Sigma_\lambda^0$ for every $\lambda \in [0, 1]$, the non-empty intersection would imply that there is a point in $T_{-1}B(z_1, R) \cap \Sigma_1^0 = T_{-1}(B(z_1, R) \cap \Sigma_1)$. But this is impossible since by definition of H we have $H(B(z_1, R)) < 1$. Finally we conclude by Theorem 5.2 that if $R > r$ then this isotopy does not exist. Our initial assumption the H is directed (Definition 5.9) must be wrong so there must be $\lambda \in [0, 1]$ such that Σ_λ has a characteristic trajectory which goes through $\{q_n = 0\}$ but does not intersect $\{q_n = 1\}$. By Lemma 5.7 we conclude that the energy surface Σ_λ has a compact invariant set. \square

5.1.3 Symplectic extension: proof of Proposition 5.14

In the previous subsection we saw that there exists a compactly supported isotopy F_λ of \mathbb{R}^{2n} such that

$$F_\lambda(\Sigma_0) = \Sigma_\lambda^0, \quad i_0^* F_\lambda^* \omega = i_0^* \omega \quad \text{and} \quad F_\lambda^* \omega = \omega \quad \text{outside a compact set.}$$

Definition 5.15. We say that a property (P) is verified if we go far enough in the direction of Σ_0 if there exists a compact set $K \subseteq \Sigma_0$ such that (P) is verified for every point in $(\Sigma_0 \setminus K) \times \mathbb{R} \subseteq \mathbb{R}^{2n}$.

The proof of the extension is a combination of classical arguments in symplectic geometry adapted to our particular context.

Lemma 5.16. Denote by $\omega_\lambda = F_\lambda^* \omega$ which is a symplectic form of \mathbb{R}^{2n} such that $\omega_\lambda = \omega$ outside a compact set and $i_0^* \omega_1 = i_0^* \omega$. For ϵ small enough there exists a compactly supported isotopy $\varphi_\lambda : U_0^\epsilon \rightarrow U_\lambda$ where U_λ is a neighbourhood of Σ_0 such that $\varphi_\lambda|_{\Sigma_0} = \text{Id}$, $\varphi_\lambda = \text{Id}$ if go far enough in the direction of Σ_0 and

$$\varphi_\lambda^* \omega_1|_{\Sigma_0} = \omega|_{\Sigma_0} \in \Gamma(\Sigma_0, \Lambda^2(T^*\mathbb{R}^{2n})).$$

Proof. Recall that given a symplectic form and a metric on a vector space V one can recover an almost complex structure on V which is moreover a symplectomorphism (see for example [HZ94, p.14]). For the case $V = \mathbb{R}^{2n}$ with symplectic form ω_0 and the standard euclidean metric one recovers the standard complex structure J_0 . Since this process depends smoothly on the parameters we can use ω_λ and the standard euclidean metric to get an almost complex structure J^λ such that it equals the standard structure J_0 at infinity and which depends smoothly on λ . Define

$$\varphi_\lambda(x, t, s) = (x, t, 0) + sJ^\lambda\left(\frac{\partial}{\partial q_n}\right)(x, t, 0)$$

then clearly $\varphi_\lambda|_{\Sigma_0} = \text{Id}$, and for points in Σ_0 , that is, points with $s = 0$,

$$d_z \varphi_\lambda|_{T_z \Sigma_0} = \text{Id} \quad \text{and} \quad d_z \varphi_\lambda\left(\frac{\partial}{\partial p_n}\right) = J^\lambda \frac{\partial}{\partial q_n}(z).$$

In particular, by construction φ_λ is symplectic between ω_λ and ω_0 over Σ_0 , that is, $\varphi_\lambda^* \omega_\lambda|_{\Sigma_0} = \omega|_{\Sigma_0}$. We now have to prove that φ_λ is a diffeomorphism on U_0^ϵ for some ϵ . In order to see this one may use the fact that $\varphi_\lambda = \text{Id}$ if (x, t) is big enough, and that the domain of definition of λ is the compact set $[0, 1]$. \square

The previous lemma gives an isotopy φ_λ on a neighbourhood of Σ_0 such that

$$F_\lambda \circ \varphi_\lambda(\Sigma_0) = \Sigma_\lambda^0 \quad \text{and} \quad (F_\lambda \circ \varphi_\lambda)^* \omega|_{\Sigma_0} = \varphi_\lambda^* F_\lambda^* \omega|_{\Sigma_0} = \omega|_{\Sigma_0},$$

and $\varphi_\lambda^* F_\lambda^* \omega = \omega$ if we go far enough in the direction of Σ_0 . We now use the following classical Darboux lemma to construct a symplectic diffeomorphism on a neighborhood of Σ_0 .

Proposition 5.17 (Darboux lemma). *Let ω_0 and ω_λ be two symplectic forms on \mathbb{R}^{2n} such that*

$$\omega_0|_{\Sigma_0} = \omega_\lambda|_{\Sigma_0} \in \Gamma(\Sigma_0, \Lambda^2(T^*\mathbb{R}^{2n}))$$

and $\omega_0 = \omega_\lambda$ outside a compact set. Then there is a diffeomorphism $\phi_\lambda : U_0 \rightarrow U_\lambda$ between two neighborhoods of Σ_0 such that $\phi|_{\Sigma_0} = \text{Id}_{\Sigma_0}$, $\phi_\lambda^ \omega_\lambda = \omega_0$ and ϕ_λ is the identity if we go far enough in the direction of Σ_0 .*

Proof. Set $\omega_\lambda^t = (1-t)\omega_0 + t\omega_\lambda$. The fact that $\omega_0|_{\Sigma_0} = \omega_\lambda|_{\Sigma_0}$ together with $\omega_0 = \omega_\lambda$ outside a compact set implies that ω_λ^t is a symplectic form on some ϵ neighborhood of Σ_0 . By Moser's trick we need to find X_λ^t a smooth family of vector fields on a neighborhood of Σ_0 such that X_λ^t is vanishing on Σ_0 and

$$\mathcal{L}_{X_\lambda^t}\omega_\lambda^t + \omega_\lambda - \omega_0 = 0.$$

By Lemma 5.18 below we can find a $\beta_\lambda \in \Omega^1(\mathbb{R}^{2n})$ such that $d\beta_\lambda = \omega_\lambda - \omega_0$, $\beta_\lambda|_{\Sigma_0} = 0$ and β_λ is zero if we go far enough in the direction of Σ_0 . As ω_λ^t is non degenerate on an ϵ neighborhood of Σ_0 , for $t \in [0, 1]$ there exists a vector field X_λ^t on this neighborhood such that

$$i_{X_\lambda^t}\omega_\lambda^t + \beta_\lambda = 0.$$

By Cartan's formula X_λ^t will also verify the first equation. Now since β_λ is zero when we go far enough in the direction of Σ_0 , the same will happen to X_λ^t , so we are able to find a neighborhood of Σ_0 for which the time one flow of X_λ^t is defined. The existence of β_λ is guaranteed by the following lemma.

Lemma 5.18. *Let $\alpha \in \Omega_c^2(\mathbb{R}^{2n})$ such that $i_0^*\alpha = 0$ and $d\alpha = 0$. Then there is a 1-form $\beta \in \Omega^1(\mathbb{R}^{2n})$ such that $d\beta = \alpha$ and $\beta|_{\Sigma_0} = 0$. Moreover β is zero if we go far enough in the direction of Σ_0 .*

Proof. Consider the isotopy $\phi_r(x, t, s) = (x, t, rs)$ and denote Z its infinitesimal generator. Define

$$K(\alpha) = \int_0^1 \phi_r^* i_Z \alpha dr.$$

Since $\phi_r|_{\Sigma_0} = Id$ and $Z|_{\Sigma_0} = 0$ we get $K(\alpha)|_{\Sigma_0} = 0$. Moreover, $i^*\alpha = 0$ implies that $\phi_0^*\alpha = 0$ and Cartan's formula together with $d\alpha = 0$ imply

$$\alpha = \phi_1^*\alpha - \phi_0^*\alpha = \int_0^1 \frac{d}{dr} \phi_r^* \alpha dr = d \int_0^1 \phi_r^* i_Z \alpha dr = dK(\alpha).$$

Finally since α is compactly supported, if z is a point far enough in the direction of Σ then $\phi_r(z)$ will be outside the support of α for $r \in [0, 1]$. The integral definition of $K(\alpha)$ now gives $K(\alpha)(z) = 0$. \square

\square

Applying the previous result, we get an isotopy ϕ_λ defined on a neighborhood U_0^ϵ of Σ_0 such that

$$F_\lambda \circ \varphi_\lambda \circ \phi_\lambda(\Sigma_0) = \Sigma_\lambda^0 \quad \text{and} \quad (F_\lambda \circ \varphi_\lambda \circ \phi_\lambda)^*\omega = \phi_\lambda^* \varphi_\lambda^* F_\lambda^* \omega = \omega$$

and $(F_\lambda \circ \varphi_\lambda \circ \phi_\lambda)^*\omega = \omega$ if we go far enough in the direction of Σ_0 . We finish with the construction of the global symplectic extension. This is a version of Banyaga's theorem [Ban78] that can be found in [MS17][p.115].

Proposition 5.19. *The isotopy $F_\lambda \circ \varphi_\lambda \circ \phi_\lambda$ can be extended to a symplectic isotopy $\Phi_\lambda : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$. In particular it verifies $\Phi_\lambda(\Sigma_0) = \Sigma_\lambda^0$.*

Proof. Consider $\mathcal{N} = U_0^{\epsilon'}$ with $\epsilon' < \epsilon$ so that $\mathcal{N} \subseteq U_0^\epsilon$. Now choose any compactly supported extension ψ_λ of $\varphi_\lambda \circ \phi_\lambda$ and consider the isotopy $E_\lambda = F_\lambda \circ \psi_\lambda$. Define $\omega_\lambda = E_\lambda^* \omega$ and note that ω_λ agrees with ω on \mathcal{N} and outside a compact set. Hence the forms $\tau_\lambda = \frac{d}{d\lambda} \omega_\lambda$ vanish on \mathcal{N} . Consider $\sigma_\lambda := K(\tau_\lambda)$ where K is defined on Lemma 5.18 as

$$K(\tau_\lambda) = \int_0^1 \phi_r^* i_Z \tau_\lambda dr.$$

Since φ_r contracts \mathcal{N} for $0 < r < 1$ we see that $\sigma_\lambda = 0$ on \mathcal{N} . Now we use Moser's trick to find a compactly supported diffeomorphism χ_λ of \mathbb{R}^{2n} which restrict to the identity on \mathcal{N} and satisfy $\chi_\lambda^* \omega_\lambda = \omega$ for all λ . To finish the proof have to make sure that we will end up with a globally defined diffeomorphism. Recall that the vector field which defines χ_λ is given by the equation

$$i_{X_\lambda} \omega_\lambda = -\sigma_\lambda.$$

Since we are only interested in the behavior at infinity this equation becomes

$$i_{X_\lambda} \omega = -\sigma_\lambda.$$

By definition $\phi_r(x, t, s) = (x, t, rs)$ so that $Z = s \frac{\partial}{\partial s}$. Now τ_λ has compact support independent of λ so one may easily bound the norm and check that there is a constant C such that

$$\|X_\lambda\| \leq C|s|$$

which implies that the vector field is globally integrable. Thus $\Phi_\lambda := E_\lambda \circ \chi_\lambda$ is the desired extension. \square

Chapter 6

Symplectic rigidity in Hamiltonian PDEs

In this chapter we study symplectic rigidity properties of infinite dimensional Hamiltonian systems. We will replace the classical phase space \mathbb{R}^{2n} by an infinite dimensional separable Hilbert space E . The symplectic forms that we consider are continuous 2-forms $\omega : E \times E \rightarrow \mathbb{R}$ which are non-degenerate in the sense that the associated linear mapping

$$\Omega : E \rightarrow E^* \quad \text{defined by} \quad \xi \mapsto \omega(\xi, \cdot)$$

is an isomorphism. Remark that in infinite dimensions we also have the concept of weak symplectic form if Ω is only injective. Let $H : E \rightarrow \mathbb{R}$ be a smooth Hamiltonian function. In the same way as we do in the finite dimensional case, one can define the vector field $X_H(u) = \Omega^{-1}(dH(u))$ and we can consider the ordinary differential equation

$$\dot{u} = X_H(u).$$

If this equation defines a flow Φ_t , one can then investigate if the properties that we saw in \mathbb{R}^{2n} appear also in this context.

As we saw in the introduction, equations of this type arise naturally as PDEs when studying conservative physical systems, but the situation is not exactly the same as the one described. In most cases the Hamiltonian H is not defined on the whole space E but only on a dense Hilbert subspace $D_H(E) \subseteq E$. As a consequence, the Hamiltonian vector field is also densely defined, so the concept of solutions of the Hamiltonian equation on E is not entirely clear. There is no universal way of solving this problem, and most of the times one has to examine the properties of the particular example in order to find a "good" definition of the generalized solutions on E .

The approach that we choose in this manuscript is to look at Hamiltonian equations of the form

$$\dot{u} = X_H(u) = JAu + J\nabla h_t(u).$$

where JA is a linear operator only defined on a dense subspace $D_H(E) \subseteq E$ and h_t is a smooth function defined on the whole space E (see Section 6.1). If the exponential e^{tJA} is bounded on E for every $t \in \mathbb{R}$, one can then define these "generalized" solutions as continuous curves on E that verify an integral equation (Duhamel's formula) which only involves e^{tJA} and ∇h . We can then define the flow Φ_t , and study the symplectic properties that may appear.

This chapter starts by recalling the proof by Kuksin [Kuk95] that under some compactness assumptions on the nonlinearity, Φ_t is a symplectomorphism that can be approximated by finite dimensional Hamiltonian flows. We then study symplectic rigidity properties of Φ_t . We start with

a construction of an infinity dimensional symplectic capacity based on Viterbo's action selectors. This construction is similar to the one that Kuksin gave in [Kuk95] using Hofer-Zehnder's capacity, and can be thought of as a sort of limit of finite dimensional capacities. As in Kuksin's classical construction, our capacity is invariant by Φ_t and inherits the normalization property. In particular it can be used to prove an analogue of Gromov's non-squeezing in infinite dimension. We then continue with the extension of the middle dimensional rigidity theorem (Theorem 4.55) to the infinite dimensional case. A particular example where this theorem applies is the Sine-Gordon equation.

6.1 Semilinear Hamiltonian equations

Denote by $\langle \cdot, \cdot \rangle$ the scalar product of E . Consider an anti-self-adjoint isomorphism $\bar{J} : E \rightarrow E$ and supply E with the strong symplectic structure

$$\omega(\cdot, \cdot) = \langle \bar{J}\cdot, \cdot \rangle.$$

Denote $J = (\bar{J})^{-1}$ which is also an anti-self-adjoint isomorphism of E . Take a possibly unbounded linear operator A with dense domain such that JA generates a C^0 group of (symplectic) transformations

$$\{e^{tJA} \mid t \in \mathbb{R}\} \quad \text{with} \quad \|e^{tJA}\|_E \leq Me^{N|t|}$$

and consider the Hamiltonian function

$$H_t(u) = \frac{1}{2} \langle Au, u \rangle + h_t(u),$$

where $h : E \times \mathbb{R} \rightarrow \mathbb{R}$ is smooth. The corresponding Hamiltonian equation has the form

$$\dot{u} = X_H(u) = JAu + J\nabla h_t(u).$$

In this case the domain of definition of the Hamiltonian vector field is the same as the domain $D(A)$ of A which is a dense subspace of E . This implies that classical solutions can only be defined on $D(A)$. More precisely by a *classical solution* we mean a function $u : [0, T[\rightarrow E$ continuous on $[0, T[$, continuously differentiable on $]0, T[$, with $u(t) \in D(A)$ for $0 < t < T$ and such that the equation is satisfied on $]0, T[$. Nevertheless, the boundedness of the exponential allows us to define solutions in the whole space E via Duhamel's formula:

Definition 6.1. A continuous curve $u(t) \in \mathcal{C}([0, T]; E)$ is a (*mild*) *solution* of the Hamiltonian equation in E with initial condition $u(0) = u_0$ if for $0 \leq t \leq T$,

$$u(t) = e^{tJA}u_0 + \int_0^t e^{(t-s)JA} J\nabla h_s(u(s)) ds.$$

One can easily verify that if $u(t)$ is a classical solution, then it is also a mild solution. For semilinear equations we know (see for example [Paz83]) that if ∇h is locally Lipschitz continuous, then for each initial condition there exists a unique solution which is defined until blow-up time. If moreover ∇h is continuously differentiable then the mild solutions with $u_0 \in D(A)$ are classical solutions of the initial value problem. Locally we get a smooth flow map $\Phi_t : \mathcal{O} \subseteq E \rightarrow E$ defined on an open set \mathcal{O} . If every solution satisfies an a priori estimate

$$\|u(t)\|_E \leq g(t, u(0)) < \infty$$

where g is a continuous function on $\mathbb{R} \times E$, then all flow maps $\Phi_t : E \rightarrow E$ are well defined and smooth. This is the case for example if $\|\nabla h_t(u)\|_E \leq C$. Remark that the choice of the linear

map A is arbitrary. Indeed if JA generates a continuous group of transformations and B is a bounded linear operator then $J(A+B)$ is an infinitesimal generator of a group $e^{tJ(A+B)}$ on E satisfying $\|e^{tJ(A+B)}\|_E \leq Me^{N+M\|B\|\|t\|}$. One can then consider the linear part $J(A+B)$ and set $J\nabla h_t - JB$ as the nonlinear part. This indeterminacy is only apparent: classical solutions verify Duhamel's formula for JA and $J(A+B)$ so both flow maps coincide over the dense subspace $D(A)$ which by continuity implies that the two flows are equal.

6.1.1 Examples

There are many important partial differential equation in mathematical physics that can be written as a semilinear Hamiltonian equation. We just present two families of examples just to give the reader an intuition and justification of why we restrict our study to this type of equations. We present them with periodic boundary conditions but the same applies for more general settings.

Nonlinear wave equation. Consider the periodic nonlinear string equation

$$\ddot{u} = \Delta u - \tilde{f}(t, x, u), \quad u = u(t, x),$$

where $x \in \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ and \tilde{f} is a smooth function which has at most a polynomial growth in u , as well as its u - and t -derivatives. Set $f = \tilde{f} - u$ and denote by B the operator $B = \sqrt{1 - \Delta}$. We may write the previous equation in the form

$$\begin{aligned} \dot{u} &= -Bv, \\ \dot{v} &= Bu + B^{-1}f(t, x, u). \end{aligned}$$

Define $E = H^{\frac{1}{2}}(\mathbb{T}^d) \times H^{\frac{1}{2}}(\mathbb{T}^d)$ as the product of Hilbert spaces where the scalar product of $H^{\frac{1}{2}}(\mathbb{T}^d)$ is given by

$$\langle u_1, u_2 \rangle = \int_{\mathbb{T}^d} Bu_1(x)u_2(x)dx.$$

Now put $A = B \times B$ and $J : E \rightarrow E$ by $J(u, v) = (-v, u)$. The symplectic form that we are going to work with is simply $\omega(\cdot, \cdot) = \langle -J\cdot, \cdot \rangle$. It is clearly a strong symplectic form on E . Define

$$h_t(u, v) = \int_{\mathbb{T}^d} F(t, x, u(x))dx, \quad F = \int_0^u f du,$$

so that

$$\nabla h_t(u, v) = (B^{-1}f(t, x, u(x)), 0).$$

The Hamiltonian function

$$H(u, v) = \frac{1}{2} \langle A(u, v), (u, v) \rangle + h_t(u, v),$$

generates the Hamiltonian vector field

$$X^H(u, v) = JA(u, v) + J\nabla h_t(u, v),$$

which defines the system associated to the nonlinear wave equation.

Nonlinear Schrödinger equation. Consider the periodic nonlinear Schrödinger equation

$$i\dot{u} = \Delta u + \frac{\partial g}{\partial \bar{u}}(t, x, u, \bar{u}) \quad u = u(t, x),$$

where $x \in \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$. In this case the phase space is $E = L^2(\mathbb{T}^d, \mathbb{C})$. The real scalar product is

$$\langle u, v \rangle = \frac{1}{(2\pi)^d} \operatorname{Re} \int_{\mathbb{T}^d} u(x) \bar{v}(x) dx,$$

and the complex structure is given by $u \mapsto Ju = iu(x)$. This gives the strong symplectic on E

$$\omega(u, v) = -\frac{1}{(2\pi)^d} \operatorname{Im} \int_{\mathbb{T}^d} u(x) \bar{v}(x) dx.$$

Set $A = -\Delta$ and

$$h_t(u) = \frac{1}{2(2\pi)^d} \int_{\mathbb{T}^d} g(t, x, u(x), \bar{u}(x)) dx.$$

Then

$$\nabla h_t(u) = \frac{\partial g}{\partial \bar{u}}(t, x, u, \bar{u})$$

and the Hamiltonian vector field

$$X^H(u) = JA(u) + J\nabla h_t(u)$$

defines via the Hamiltonian equation the nonlinear Schrödinger equation.

6.1.2 Hilbert scales

In the previous examples the symplectic spaces that we worked with form part of one parameter families of Hilbert spaces $H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$ and $H^s(\mathbb{T}^d, \mathbb{C})$, which receive the name of Hilbert scales. In order to get the finite dimensional approximation of the flow, we need to ask for a compactness condition which is better stated in the context of Hilbert scales. We now present the definition and basic properties of these spaces.

Let $E = F_0$ be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and Hilbert basis $\{\varphi_k, k \in \tilde{\mathbb{Z}}\}$ where $\tilde{\mathbb{Z}}$ is an even subset of some \mathbb{Z}^d (so $-\tilde{\mathbb{Z}} = \tilde{\mathbb{Z}}$). Let $\{\nu_k \mid k \in \tilde{\mathbb{Z}}\}$ be a positive sequence such that $\nu_k = \nu_{-k}$ and $\nu_k \rightarrow +\infty$ as $|k| \rightarrow +\infty$. For any real number s we define F_s as the Hilbert space with Hilbert basis $\{\varphi_k \nu_k^{-s} \mid k \in \tilde{\mathbb{Z}}\}$. Denote by $\|\cdot\|_s$ and $\langle \cdot, \cdot \rangle_s$ the norm and the scalar product associated to F_s . Then

$$\|u\|_s^2 = \sum |u_k|^2 \nu_k^{2s} \quad \text{if } u = \sum u_k \varphi_k.$$

For $s = 0$ we omit the index so for example we denote $\|\cdot\|_0 = \|\cdot\|$. The totality of $\{F_s\}$ is called a Hilbert scale, and the basis $\{\varphi_k\}$ is called a basis of the scale. A Hilbert scale may be continuous or discrete depending on if the parameter s is real or integer. The following lemma summarizes some of the key properties of Hilbert scales.

Lemma 6.2 (see for example [Kuk00]). *Let $\{F_s\}$ be a Hilbert scale. Then*

- *If $r < s$ then F_s is compactly embedded into F_r .*
- *The spaces F_s and F_{-s} are conjugated with respect to the scalar product $\langle \cdot, \cdot \rangle$: for any $u \in F_s \cap F_0$ we have*

$$\|u\|_s = \sup\{|\langle u, v \rangle| \mid v \in F_{-s} \cap F_0, \|v\|_{-s} = 1\}.$$

In particular, for any space F_s we can identify the dual $(F_s)^$ with the space F_{-s} .*

- For $-\infty < a < b < +\infty$ and $0 \leq \theta \leq 1$ the space F_c with $c = (1 - \theta)a + \theta b$ interpolates the spaces F_a and F_b , that is, $F_c = [F_a, F_b]$. In particular, for any $u \in F_b$, the following interpolation inequality holds:

$$\|u\|_c \leq \|u\|_a^{1-\theta} \|u\|_b^\theta.$$

Example 6.3. Take for $E = F_0$ the space of square integrable 2π periodic real functions, $L^2(\mathbb{T}, \mathbb{R})$. It has a basis given by

$$\varphi_0 = \frac{1}{\sqrt{2\pi}}, \quad \varphi_k = \frac{1}{\sqrt{\pi}} \cos kx, \quad \varphi_{-k} = \frac{1}{\sqrt{\pi}} \sin kx,$$

for $k \in \mathbb{N} \setminus \{0\}$. Take $\nu_k = \max(1, |k|)$. Then the resulting spaces F_s can be identified with the Sobolev spaces $H^s(\mathbb{T}, \mathbb{R})$.

6.1.3 Finite dimensional approximation

In this subsection we reproduce the basic results of the finite dimensional approximation contained in Kuksin's article [Kuk95]. We include the proofs for completeness. What follows can be explained in slightly more generality, but we choose to restrict to the particular form of the examples that we have in mind. Let $\{F_s\}$ be a Hilbert scale with Hilbert basis given by $\{\varphi_j^\pm \mid j \in \mathbb{N}\}$ with $F_0 = E$. Denote E_n the vector space generated by $\{\varphi_j^\pm \mid j \leq n\}$. It is a real vector space isomorphic to \mathbb{R}^{2n+2} which is contained in every F_s . Let E^n be the Hilbert subspace of E with basis $\{\varphi_j^\pm \mid j > n\}$ so that $E = E_n \oplus E^n$, and write $u = (u_n, u^n)$ for an element $u \in E$. Denote by $\Pi_n : E \rightarrow E_n$ be the natural projection. The idea is to approximate the flow using the flow of the projection of the vector field on E_n . We are going to restrict the class of the linear operators A so that they commute with the projection on E_n .

Let $\{\lambda_j\}$ be a sequence of positive real numbers (with possibly $\lambda_j \rightarrow \infty$) and consider the linear operator A whose action on the basis is given by

$$A\varphi_j^\pm = \lambda_j \varphi_j^\pm.$$

It is a self-adjoint operator in E with the natural domain of definition. Now consider the complex structure J on E defined by the action on the basis

$$J\varphi_j^\pm = \pm \varphi_j^\mp.$$

It is easy to see that the operator JA is anti self adjoint and generates a one parameter group of isometries e^{tJA} such that the restriction to the plane $\mathbb{R}\varphi_j^+ \oplus \mathbb{R}\varphi_j^- = (\mathbb{R} \oplus \mathbb{R}J)\varphi_j^+ = \mathbb{C}\varphi_j^+$ is just the rotation $e^{it\lambda_j}$. The fact that J and A preserve E_n for all n will allow us to define the finite dimensional approximations just by projecting the vector field onto E_n . More precisely, for a smooth function h_t on E consider the Hamiltonian function

$$H_n(u) = \frac{1}{2} \langle Au, u \rangle + h_n(u) \quad \text{where} \quad h_n(u) := h_t(\Pi_n(u)).$$

The Hamiltonian equation now becomes

$$\dot{u} = X_{H_n}(u) = JAu + J\nabla h_n(u),$$

where $\nabla h_n(u) = \Pi_n(\nabla h_t(\Pi_n(u)))$. The vector field X_{H_n} generates a local flow Φ_n^t . This flow can be decomposed as

$$\Phi_n^t = e^{tJA} \circ V_n^t \quad \text{with} \quad V_n^t(u) = (\phi_n^t(u_n), u^n).$$

Here ϕ_n^t is a finite dimensional Hamiltonian flow on E_n generated by the time dependent function $h_n \circ e^{tJA}$. In order to make sure that the flow of X^H and X^{H_n} are going to be globally well defined, we need to make some assumptions on the nonlinearity h . The following conditions allow us to work as if we were on a locally compact space.

Definition 6.4. Let $\{F_s\}$ be a Hilbert scale with $F_0 = E$. We say that the Hamiltonian equation

$$\dot{u} = X_{H_n}(u) = JAu + J\nabla h_n(u),$$

with $A\varphi_j^\pm = \lambda_j\varphi_j^\pm$, complex structure $J\varphi_j^\pm = \pm\varphi_j^\mp$ and a smooth function $h : \mathbb{R} \times E \rightarrow \mathbb{R}$ satisfies the compactness condition 6.4 if

- h has a C^2 extension to $\mathbb{R} \times F_s \rightarrow \mathbb{R}$ for some $s < 0$.
- For every $T > 0$ there is a constant $C_T > 0$ such that

$$\|\nabla h_t(z)\| \leq C_T \quad \text{for every } (t, z) \in [-T, T] \times E.$$

Remark 6.5. For the results in this section the second condition can be weakened to just asking that $\nabla h : \mathbb{R} \times F_s \rightarrow E$ sends bounded sets to bounded sets (see [Kuk95]). We choose our second condition for simplicity since these are the hypothesis needed to extend the middle dimensional rigidity theorem (Theorem 4.55) to infinite dimensions.

The first consequence of satisfying the compactness assumption is that the Hamiltonian equation defines a global flow made up of symplectomorphisms. More precisely we have the following lemma:

Lemma 6.6. *Suppose that the Hamiltonian equation satisfies the compactness condition 6.4. Then the gradient of h is locally Lipschitz, i.e., for every $T > 0$ and $R > 0$ there is a constant $C_T(R)$ such that*

$$\|\nabla h_t(u) - \nabla h_t(v)\| \leq C_T(R)\|u - v\| \quad \text{for every } (t, z) \in [-T, T] \times B_R.$$

Moreover, the Hamiltonian equation defines a globally defined flow $\Phi^t : E \rightarrow E$ such that $(\Phi^t)^*\omega = \omega$.

Proof. For the first claim one has to use that h has a C^2 extension to some F_s $s < 0$ where E is compactly included. Then use that $[-T, T] \times B_R^E$ is precompact in $\mathbb{R} \times F_s$ to bound $d\nabla h$ on a suitable norm. The existence of the local flow Φ^t follows from classical arguments of fixed point theorems (see [Paz83] or [Kuk95]) applied to Duhamel's formula. The second property in the compactness assumption then guarantees that solutions do not blow up in finite time, so the flow is globally defined. For the fact the Φ^t is a symplectomorphism see [Kuk00]. \square

We continue with the finite dimensional approximation. Remark that since $\|\nabla h_t(z)\|$ is bounded, the same will be true for h_n , so the flow Φ_n^t is also globally defined. The key point of the approximation is the following lemma which is a slight modification of a lemma in [Kuk95, appendix 2]. Remark the Π_n is defined on all F_s and commutes with the inclusion $F_s \hookrightarrow F_r$.

Lemma 6.7. *Let K be a compact subset of F_s for some s . Let $g : \mathbb{R} \times F_s \rightarrow E$ be a continuous map and fix a $T > 0$. Then*

$$\sup_{(t,u) \in [-T, T] \times K} \|g_t(u) - g_t(\Pi_n u)\|$$

converges to zero as n goes to infinity. Moreover suppose that h satisfies the compactness condition 6.4. Then for every $R > 0$ there exists a decreasing function $\epsilon_R : \mathbb{N} \rightarrow \mathbb{R}$ such that $\epsilon_R(n) \rightarrow 0$ as $n \rightarrow \infty$ and

$$\|\nabla h_t(u) - \nabla h_n(u)\| \leq \epsilon_R(n) \quad \text{for every } (t, u) \in [-T, T] \times B_R.$$

Proof. By contradiction suppose that there is a sequence $\{(s_n, z_n)\} \subset [-T, T] \times K$ such that $\|g_{s_n}(z_n) - g_{s_n}(\Pi_n z_n)\|_E \geq \delta > 0$ for every $n \in \mathbb{N}$. By compactness we may suppose that there is a converging subsequence $(s_{n_k}, z_{n_k}) \rightarrow (s, z)$. This sequence will also verify $\Pi_{n_k} z_{n_k} \rightarrow z$. We have

$$\|g_{s_{n_k}}(z_{n_k}) - g_{s_{n_k}}(\Pi_{n_k} z_{n_k})\| \leq \|g_{s_{n_k}}(z_{n_k}) - g_s(z)\| + \|g_s(z) - g_{s_{n_k}}(\Pi_{n_k} z_{n_k})\|$$

and the quantity of the rhs converges to zero as n_k goes to infinity by continuity of g . In particular, for n_k big enough we get $\|g_{s_{n_k}}(z_{n_k}) - g_{s_{n_k}}(\Pi_{n_k} z_{n_k})\| < \delta$, a contradiction.

For the second claim remark that ∇h_t has an extension to F_s for some $s < 0$. Denote by $\nabla \tilde{h}_t$ the extension and let $i : E \rightarrow F_s$ be the compact inclusion so that $\nabla h_t(u) = \nabla \tilde{h}_t(i(u))$. Recall that $\nabla h_n(u) = \Pi_n \nabla h_t(\Pi_n(u))$. We have

$$\begin{aligned} \|\nabla h_t(u) - \nabla h_n(u)\| &\leq \|\nabla h_t(u) - \Pi_n \nabla h_t(u)\| + \|\Pi_n \nabla h_t(u) - \Pi_n \nabla h_t(\Pi_n u)\| \\ &\leq \|\nabla \tilde{h}_t(i(u)) - \Pi_n \nabla \tilde{h}_t(i(u))\| + \|\nabla \tilde{h}_t(i(u)) - \nabla \tilde{h}_t(\Pi_n i(u))\|. \end{aligned}$$

For every $R > 0$ the sets $\bigcup_{|t| \leq T} \nabla \tilde{h}_t(i(B_E(0, R)))$ and $i(B_E(0, R))$ are precompact in E and F_s respectively, so we may take the sup in $B_E(0, R)$ and $|t| \leq T$ and apply the first part of the lemma to conclude. \square

Now we have all the tools we need for the finite dimensional approximation.

Proposition 6.8 (see [Kuk95]). *Fix a $t \in \mathbb{R}$. For each $R > 0$ and $\epsilon > 0$ there exists an N such that if $n \geq N$ then*

$$\|V^t(u) - V_n^t(u)\| \leq \epsilon$$

for all $u \in B_R$.

Proof. Duhamel's formula and the fact that e^{tJA} is a bounded operator give

$$\begin{aligned} \|V^t(u) - V_n^t(u)\| &\leq C \int_0^t \|\nabla h_s(\Phi^s(u)) - \nabla h_n(\Phi_n^s(u))\| ds \leq \\ &\leq C \int_0^t \|\nabla h_s(\Phi^s(u)) - \nabla h_s(\Phi_n^s(u))\| ds + C \int_0^t \|\nabla h_s(\Phi_n^s(u)) - \nabla h_n(\Phi_n^s(u))\| ds. \end{aligned}$$

If $u \in B_R$ and $s \in [0, t]$ then $\|\nabla h\|$ bounded implies that for all $n \in \mathbb{N}$ the element $\Phi_n^s(u)$ does not leave a ball of radius $R'(R, t)$. We can now use Lemma 6.7, the fact that ∇h is locally Lipschitz (Lemma 6.6) and the decomposition $\Phi^t(u) = e^{tJA}V^t(u)$ to get

$$\|V^t(u) - V_n^t(u)\| \leq \tilde{C} \int_0^t \|V^s(u) - V_n^s(u)\| ds + Ct\epsilon(n).$$

By Gronwall's lemma we conclude that

$$\|V^t(u) - V_n^t(u)\| \leq \epsilon(n)C(t)$$

where $C(t)$ depends continuously on t . The function $\epsilon(n)$ is decreasing and converges to zero so there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $\epsilon(n)C(t) \leq \epsilon$ which gives the result. \square

6.1.4 Examples

Nonlinear string equation. In order to satisfy the compactness condition of Definition 6.4 we need to restrict the type of nonlinearity as well as the dimension of the equation. We will change the operator A considered in the Hamiltonian formulation so that our class of PDEs include the well known Sine-Gordon equation. Consider the one dimensional equation

$$\ddot{u} = u_{xx} - f(t, x, u), \quad u = u(t, x),$$

where $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Recall that if we denote by B the operator $B = (-\partial^2/\partial x^2 + 1)^{1/2}$ then we may write the equation in the form

$$\begin{aligned} \dot{u} &= -Bv, \\ \dot{v} &= (B - B^{-1})u + B^{-1}f(t, x, u). \end{aligned}$$

Define $E = H^{\frac{1}{2}}(\mathbb{T}) \times H^{\frac{1}{2}}(\mathbb{T})$ as the product of Hilbert spaces where the scalar product of $H^{\frac{1}{2}}(\mathbb{T})$ is given by

$$\langle u_1, u_2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} B u_1(x) u_2(x) dx.$$

If we define the function

$$h_t(u, v) = -\frac{1}{2\pi} \int_0^{2\pi} F(t, x, u(x)) dx, \quad F = \int_0^u f du.$$

we get

$$\nabla h_t(u, v) = (B^{-1}f(t, x, u(x)), 0).$$

Suppose that f is a smooth function which is bounded over compact time intervals and has at most a polynomial growth in u , as well as its u - and t -derivatives:

$$\left| \frac{\partial^a}{\partial u^a} \frac{\partial^b}{\partial t^b} f(t, x, u) \right| \leq C_k (1 + |u|)^{M_k}, \quad \text{for } a + b = k \text{ and all } k \geq 0,$$

with $M_0 = 0$, C_k are positive constants bounded for bounded t and nonnegative M_k 's are t independent. Then the gradient verifies $\|\nabla h_t\| \leq C_0$ over the compact time interval associated to C_0 . Moreover the polynomial growth condition on f guarantees that there exists a $0 < \theta < 1/2$ such that ∇h has a C^1 extension to $H^{\frac{1}{2}-\theta}(\mathbb{T}) \times H^{\frac{1}{2}-\theta}(\mathbb{T})$ (see [Kuk95]). A particular case where such properties can be verified is $f(t, x, u) = \sin u$ which corresponds to the Sine-Gordon equation. Now putting $A = (B - B^{-1}) \times B$ and defining $J : E \rightarrow E$ by $J(u, v) = (-v, u)$ we can write the nonlinear string equation as the semilinear PDE:

$$(\dot{u}, \dot{v}) = JA(u, v) + J\nabla h_t(u, v).$$

Consider the symplectic Hilbert basis $\{\varphi_j^\pm \mid j \in \mathbb{Z}\}$ where

$$\varphi_j^+ = \frac{1}{(j^2 + 1)^{\frac{1}{4}}}(\varphi_j(x), 0), \quad \varphi_j^- = \frac{1}{(j^2 + 1)^{\frac{1}{4}}}(0, -\varphi_j(x)),$$

with

$$\varphi_j(x) = \begin{cases} \sqrt{2} \sin jx, & j > 0, \\ \sqrt{2} \cos jx, & j \leq 0. \end{cases}$$

In this basis we have $(B \times B)\varphi_j^\pm = \sqrt{j^2 + 1}\varphi_j^\pm$ so if we denote $\lambda_j = \sqrt{j^2 + 1}$ we get that

$$A\varphi_j^+ = (\lambda_j - \frac{1}{\lambda_j})\varphi_j^+ \quad \text{and} \quad A\varphi_j^- = \lambda_j\varphi_j^-.$$

Now remark that JA has eigenvalues $\{\pm i\sqrt{\lambda_j^2 - 1} = \pm ij\}$. If we calculate e^{tJA} we get that its action on each symplectic plane $\varphi_j^+\mathbb{R} \oplus \varphi_j^-\mathbb{R}$ is given by the matrix

$$\begin{pmatrix} \cos tj & -\frac{\sqrt{j^2+1}}{j} \sin tj \\ \frac{j}{\sqrt{j^2+1}} \sin tj & \cos tj \end{pmatrix}$$

which gets closer and closer to a rotation as j goes to infinity. In particular we get a bounded group of symplectic linear maps. In this case the exponential does not define an isometry, but one may see that the results of the previous sections still apply. In particular we have the finite dimensional approximation.

For higher dimensions, it is known (see [Kuk95]) that the weak compactness assumption is verified for $d = 2$ if f is a polynomial of degree less than or equal to 4, and for $d = 3$ if f is a polynomial of at most degree 2.

Nonlinear Schrödinger equation. In this case it is unlikely that the equation satisfies the compactness assumption under conditions on the nonlinearity. In order to guarantee the compactness condition we smooth out the Hamiltonian and consider

$$h_t(u) = \frac{1}{2(2\pi)^d} \int_{\mathbb{T}^d} g(t, x, U(x), \bar{U}(x)) dx, \quad \text{with} \quad U = u * \xi,$$

and $\xi \in C^\infty(\mathbb{T}^d, \mathbb{R})$. The corresponding Hamiltonian equation is

$$i\dot{u} = \Delta u + \frac{\partial g}{\partial \bar{u}}(t, x, U, \bar{U}) * \xi \quad U = u * \xi.$$

In this case it is enough to ask that g is a smooth function such that $\frac{\partial g}{\partial \bar{u}}$ is bounded.

6.2 Infinite dimensional symplectic rigidity

In this section we will explore the rigidity properties of the flow of semilinear Hamiltonian PDEs where the nonlinearity satisfies the compactness assumption. As in the finite dimensional case, symplectic rigidity starts with Gromov's nonsqueezing. The first proof of the analogue of Gromov's theorem for infinite dimensional Hamiltonian systems appeared in Kuksin's article [Kuk95] in the setting that we are working with. We want to point out that other authors followed him giving proofs for the theorem in different contexts for equations which are not compact perturbations of linear flows. For more details we refer the reader to the Introduction of this manuscript.

6.2.1 Non-squeezing theorem

Once we have the finite dimensional approximation, the non-squeezing theorem comes as a direct corollary. This is a consequence of the fact that the hypothesis of the theorem are open in the C^0 topology. More precisely let φ be a diffeomorphism of \mathbb{R}^{2n} and suppose that for every $\epsilon > 0$ there exists a symplectomorphism ψ such that

$$d_{C^0}(\varphi, \psi) \leq \epsilon,$$

then one can prove that φ satisfies the statement of Gromov's non-squeezing. The proof goes along the same lines as the one we explain in Theorem 6.11. In fact, in finite dimensions we have the following stronger and surprising result:

Theorem 6.9 (Eliashberg [Eli87] and Gromov [Gro85]). *The group of symplectomorphism is C^0 closed in the group of diffeomorphisms.*

This fact came as a surprise in the field. In spite of the fact that being a symplectomorphism is a C^1 condition, the property is stable by C^0 approximations. This theorem is the starting point of a rapidly growing field called C^0 symplectic geometry. The interested reader may look at Humilière's HDR [Hum18] for an introduction to the main concepts and results.

We now continue with the proof of the theorem for our class of PDEs. As before, we are working on a Hilbert space E with Hilbert basis given by $\{\varphi_j^\pm \mid j \in \mathbb{N}\}$. Denote

$$u = \sum_j (u_j^+ \varphi_j^+ + u_j^- \varphi_j^-) = \sum_j (u_j^+ + u_j^- J) \varphi_j^+$$

and consider the ball B_r of radius r centered at the origin and and the cylinder

$$C_R^k = \{u \in E \mid |u_k^+|^2 + |u_k^-|^2 < R^2\}.$$

Denote Π_k^{k-1} the projection on the symplectic plane generated by $\{\varphi_k^+, \varphi_k^-\}$.

Remark 6.10. Throughout this section having k as an exponent means that we take the elements of the basis with $|j| > k$, and having k as an index means that we take the elements with $|j| \leq k$.

Theorem 6.11. (see [Kuk95]) *Let $\Phi = \Phi_t : E \rightarrow E$ be the flow of a Hamiltonian satisfying the compactness condition 6.4. Then $\Phi(B_r) \subseteq C_R^k$ implies $r \leq R$.*

Proof. For each $\epsilon > 0$ and for the radius r of the ball use Lemma 6.7 to find the $N \in \mathbb{N}$ such that V_N is ϵ close to V over B_r . Suppose that for every $u \in B_r$ we have

$$\|\Pi_k^{k-1}(\Phi(u))\| \leq R.$$

The map e^{tJA} is an isometry on every symplectic plane $\{\varphi_k^+, \varphi_k^-\}$ and commutes with Π_k^{k-1} so for every $u \in B_r$

$$\|\Pi_k^{k-1}(V_N(u))\| \leq \|\Pi_k^{k-1}(V(u) - V_N(u))\| + \|\Pi_k^{k-1}(V(u))\| \leq \epsilon + \|\Pi_k^{k-1}(\Phi(u))\| \leq \epsilon + R.$$

Since $B_{E_n}(0, r) \subset B_r$ we deduce that for every $u \in B_{E_n}(0, r)$

$$\|\Pi_k^{k-1}(\phi_N^t(u))\| = \|\Pi_k^{k-1}(V_N(u))\| \leq \epsilon + R$$

where ϕ_N^t is a finite dimensional Hamiltonian flow in \mathbb{R}^{2N+2} . By Gromov's non-squeezing we conclude that $r \leq R + \epsilon$. Since ϵ was arbitrary we get $r \leq R$. \square

What type of information does this theorem give about the behavior of solutions of the Hamiltonian equation? We now give an interpretation in terms of the problem of energy transition to high frequencies. This behavior can also be seen as a kind of instability in the Fourier coefficients (see Corollary 6.14 or [Tao06]). We start with the remark that the Hamiltonian equation of Definition 6.4 preserves the subset $D(A)$ (see [Paz83]), and since H is defined on this subspace one can prove that

$$H(\Phi_t(u)) = H(u) \quad \text{for every } u \in D(A).$$

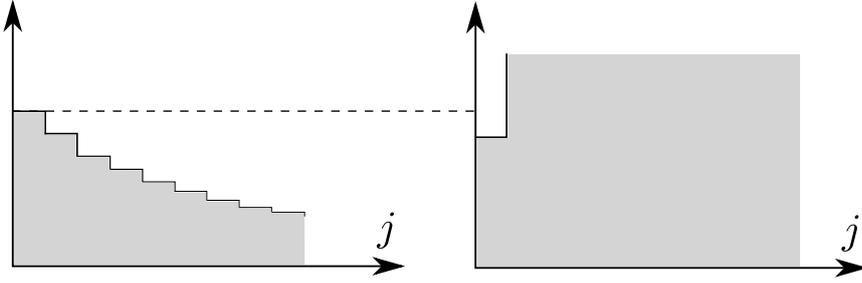


Figure 6.1: The representation of the projection on each E_k^{k-1} for a ball and a cylinder in symplectic space $E = H^{\frac{1}{2}} \times H^{\frac{1}{2}}$ with coefficients in the $L^2 \times L^2$ basis. To understand the figure recall that if φ_k^+ is an orthonormal element of the basis in $H^{\frac{1}{2}}$ (see Subsection 6.1.4) then the coefficient of this element φ_k^+ on the L^2 basis is $(1 + k^2)^{-\frac{1}{4}}$ which goes to zero when k goes to infinity.

Now recall that the Hamiltonian can be written as

$$H(u) = \frac{1}{2} \langle Au, u \rangle + h_t(u) = \sum_{j \in \mathbb{N}} \lambda_j (|u_j^+|^2 + |u_j^-|^2) + h_t(u).$$

In particular, by the conservation of energy, if one of the coefficients $|u_k^\pm(t)|^2$ with big k decreases with time then there has to be an growth of energy coming from another place. By the compactness assumption h_t is bounded over bounded sets and Φ_t sends bounded sets to bounded sets. This implies that h_t cannot compensate a big decrease of one coefficient, so the growth of energy has to come from the growth of another coefficient $|u_l^\pm|^2$. This phenomenon is called transfer of energy between the frequencies. In light of this behavior, the following question arises: Can this transfer of energy between the coefficients happen in a uniform way for a fixed set \mathcal{U} ? The nonsqueezing statement can be interpreted as:

Corollary 6.12. *The energy contained in a given frequency cannot be transferred to other frequencies in a locally uniform way (see Figure 6.1).*

The problem of the transfer of energy has been studied for several other Hamiltonian PDEs. We shortly explain the example of the Szegő equation where one can explicitly describe the transfer of energy for particular solutions, instead of uniform groups of solutions as in the previous corollary. For more details we refer to [GG16, GG17]. Consider on $L^2(\mathbb{T}, \mathbb{C})$ the projector

$$\Pi \left(\sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx} \right) = \sum_{k=0}^{+\infty} \hat{f}(k) e^{ikx}.$$

For initial conditions in $C^\infty(\mathbb{T}, \mathbb{C}) \cap \text{Ran}(\Pi)$ the cubic Szegő equation is defined as

$$i\dot{u} = \Pi(|u|^2 u).$$

One can prove that the L^2 norm of the solution is a constant of motion, so again a decrease of one of the coefficients has to be compensated with the growth of another one. The following theorem studies the transfer of energy to higher frequencies by studying the H^s norm for high values of s .

Theorem 6.13 (see [GG16]). *For any solution u to the Szegő equation, we have, for all $s > \frac{1}{2}$*

$$\|u(t)\|_{H^s} \leq C_s e^{C'_s t}.$$

where C_s and C'_s are positive constant which depend only on s and $\|u_0\|_{H^s}$. Moreover there exists a dense G_δ subset of $\Pi(C^\infty(\mathbb{T}, \mathbb{C}))$ denoted by G such that for every solution with $u_0 \in G$, for all $s > \frac{1}{2}$ and all $M \geq 1$ we have

$$\limsup_{t \rightarrow \infty} \frac{\|u(t)\|_{H^s}}{|t|^M} = +\infty$$

and on the other hand

$$\liminf_{t \rightarrow \infty} \|u(t)\|_{H^s} = \|u_0\|_{H^s}.$$

The fact that the L^2 norm is constant together with an H^s norm which goes to infinity implies that there is a transfer of energy to high frequencies. The result for the growth of the H^s norm implies that the coefficients for the higher frequencies become bigger, so the wave has more oscillation. This is why this phenomenon is sometimes called wave turbulence. For this equation this transfer of energy is less than exponential, but on a dense subspace it is faster than any polynomial growth. The last equality also indicates that the trajectories are always coming back to the H^s norm of $t = 0$, that is, the movement has a certain recurrent aspect. Remark that there is no locally uniform control on this transfer of energy.

The second interpretation of the non-squeezing behavior is the following corollary of Gromov's non-squeezing which exhibits the result as a instability for solutions (see [Tao06]). A solution u can loose the energy of a certain frequency k at some point, but this cannot happen to all the solutions with initial condition near u at the same time. If we look at the evolution of a given ball B_R centered at $u(0)$, for every $\epsilon > 0$ and every t there is at least one solution whose frequency is outside the disc of radius $R - \epsilon$ centered at $u(t)$.

Corollary 6.14. *Let u be a global classical solution of the Hamiltonian equation satisfying the compactness condition. Denote $u_k = u_k^+ + iu_k^- \in \mathbb{C}$. Then for any $R > r > 0$, any times $t_0, t_1 \in \mathbb{R}$ and any $k \in \mathbb{N}$ there exists another classical solution \tilde{u} such that*

$$\|u(t_0) - \tilde{u}(t_0)\|_E \leq R \quad \text{but} \quad |u_k(t_1) - \tilde{u}_k(t_1)| > r.$$

6.2.2 Symplectic capacities

In this subsection we define a type of infinite dimensional symplectic capacities. These capacities are constructed as a kind of limit of the classical ones, so they are going to inherit the basic properties. In particular, they verify the normalization property. Moreover, using the finite dimensional approximation we are able to prove that these capacities are invariant by our privileged class of Hamiltonian flows. The first construction of such a capacity appeared in [Kuk95] where Kuksin defined them via Hofer-Zehnder's capacity. In this manuscript we give a new construction using action selectors for compactly supported Hamiltonians. In particular one may use Viterbo's action selector.

Definition of the capacity. We start with the definition of admissible functions that will be used to select the limit action values. For any open subset $\mathcal{U} \subseteq E$ we define

$$\mathcal{F}(\mathcal{U}) = \{\text{smooth functions } f : E \rightarrow \mathbb{R} \text{ such that } \text{Supp } f \subseteq \mathcal{U} \text{ and } d(\text{Supp } f, \partial\mathcal{U}) > 0\}$$

Consider the action selector $c(\mu, \cdot)$ defined in Chapter 3. For any Hamiltonian H we define $c(\mu, H) = c(\mu, \psi_1^H)$. Denote $f_n = f|_{E_n}$ and define

$$c_o(\mu, f) = \liminf_n c(\mu, f_n) \quad \text{and} \quad c^o(\mu, f) = \limsup_n c(\mu, f_n)$$

Note that the support of f_n is contained in $\text{Supp } f \cap E_n$ which is bounded, so f_n has compact support. In particular $c(\mu, f_n)$ is well defined. Moreover, since \mathcal{U} is bounded, there exists an $R > 0$ such that $0 \leq c(\mu, f_n) \leq \pi R^2$ for every n and both limits are well defined finite numbers. Remark moreover that for every n there is a periodic orbit $z_n : [0, 1] \rightarrow E_n$ for the Hamiltonian vector field X^{f_n} such that

$$c(\mu, f_n) = A(z_n(0), f_n).$$

One could hope that the orbits z_n converge for example uniformly to a periodic orbit $z(t)$ associated to f . There is little hope for such a result: even though we control the action, there is a priori no C^0 control on the orbits z_n so they might be jumping from one place of the phase space to another. We now use $c(\mu, f)$ to define the capacity of open bounded sets \mathcal{U} as

$$c_o(\mathcal{U}) = \sup\{c_o(\mu, f) \text{ for } f \in \mathcal{F}(\mathcal{U})\} \quad \text{and} \quad c^o(\mathcal{U}) = \sup\{c^o(\mu, f) \text{ for } f \in \mathcal{F}(\mathcal{U})\}.$$

The definition for open unbounded sets \mathcal{V} is the sup over the open bounded sets \mathcal{U} contained in \mathcal{V} . For arbitrary sets \mathcal{X} , the capacity is defined as the inf over the open sets that contain \mathcal{X} . We clearly have $c_o(\mathcal{X}) \leq c^o(\mathcal{X})$.

Properties of the capacity. We now follow the structure of Kuksin's article [Kuk95] and prove the analogues of the results of his article for our new capacity. We denote by c Viterbo's capacity without specifying the dimension of the phase space. We start relating the new capacity to the classical one:

Proposition 6.15. *For any open subset $\mathcal{V} \subseteq E$ we have*

$$c_o(\mathcal{V}) \leq \liminf_n c(\mathcal{V} \cap E_n) \quad \text{and} \quad c^o(\mathcal{V}) \leq \limsup_n c(\mathcal{V} \cap E_n)$$

Proof. Let $\mathcal{U} \subset E$ be an open set. Then $\mathcal{U} \cap E_n$ is open (indeed, for $z_n \in E_n$, and an open ball $B_E(z_n, \epsilon) \subseteq \mathcal{U}$, we have the equality $B_{E_n}(z_n, \epsilon) = B_E(z_n, \epsilon) \cap E_n$). Now if $f \in \mathcal{F}(\mathcal{U})$, then f_n is compactly supported in $\mathcal{U} \cap E_n$ so by definition of the capacity c we get $c(\mu, f_n) \leq c(\mathcal{U} \cap E_n)$. If we now take the limits we get

$$c_o(f) \leq \liminf_n c(\mathcal{U} \cap E_n) \quad \text{and} \quad c^o(f) \leq \limsup_n c(\mathcal{U} \cap E_n).$$

Finally, taking the sup of $f \in \mathcal{U}$ we get the desired inequality. Now consider an unbounded open set \mathcal{V} . Every bounded open set \mathcal{U} contained in \mathcal{V} defines bounded open sets $\mathcal{U} \cap E_n \subseteq \mathcal{V} \cap E_n$. By monotonicity $c(\mathcal{U} \cap E_n) \leq c(\mathcal{V} \cap E_n)$ so for every such set \mathcal{U} we have

$$c_o(\mathcal{U}) \leq \liminf_n c(\mathcal{U} \cap E_n) \leq \liminf_n c(\mathcal{V} \cap E_n)$$

and the analogue for $c^o(\mathcal{U})$. Now just take the sup over \mathcal{U} to get the result. \square

The problem for arbitrary sets \mathcal{X} is that, for an open set \mathcal{V} that contains \mathcal{X} , we get by monotonicity $c(\mathcal{X} \cap E_n) \leq c(\mathcal{V} \cap E_n)$. Here the inequality goes in the same direction that the one we have for $c^o(\mathcal{V})$ and $c(\mathcal{V} \cap E_n)$. Nevertheless, it is possible to get some information for sets which are particularly stable by intersection with E_n (see Lemma 6.23).

Corollary 6.16. *For the standard cylinder we have*

$$c_o(C_r^k) \leq c^o(C_r^k) \leq \pi r^2.$$

Proof. Just remark that for $n \geq k$ we have $c(C_r^k \cap E_n) = \pi r^2$ and use the previous proposition. \square

Proposition 6.17. *For subsets $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X} \subseteq E$ and $\lambda \neq 0$ we have*

- *If $\mathcal{X}_1 \subseteq \mathcal{X}_2$ then $c_o(\mathcal{X}_1) \leq c_o(\mathcal{X}_2)$ and $c^o(\mathcal{X}_1) \leq c^o(\mathcal{X}_2)$.*
- *$c_o(\lambda\mathcal{X}) = \lambda^2 c_o(\mathcal{X})$ and $c^o(\lambda\mathcal{X}) = \lambda^2 c^o(\mathcal{X})$.*

Proof. To prove the first claim just observe that for open bounded sets $\mathcal{U}_1 \subseteq \mathcal{U}_2$ we have $\mathcal{F}(\mathcal{U}_1) \subseteq \mathcal{F}(\mathcal{U}_2)$, which implies the inequality for open bounded subsets. To prove the second claim observe that multiplication by λ denoted by $m_\lambda : E \rightarrow E$ is smooth, bounded and $d(m_\lambda(x), m_\lambda(y)) = \lambda d(x, y)$. This shows that $\lambda^* : \mathcal{F}(\tau\mathcal{O}) \rightarrow \mathcal{F}(\mathcal{O})$ given by $f \mapsto f \circ m_\lambda$ is a bijection. By the properties of the finite dimensional capacity we get $c(\mu, f_n \circ m_\lambda) = \lambda^2 c(\mu, f_n)$ and this implies the result. \square

Proposition 6.18. *For the ball B_r and the standard cylinder C_r^k we have*

$$c_o(C_r^k) = c_o(B_r) = \pi r^2 = c^o(B_r) = c^o(C_r^k).$$

Proof. By the previous proposition and Corollary 6.16 it is enough to show that $c_o(B_1) \geq \pi$. Consider our now classical (see Lemma 2.8) smooth function $h : [0, 1] \rightarrow \mathbb{R}$ such that $h(0) = 1 - \epsilon$, $h(t) = 0$ near 1 and $0 \leq |h'(t)| \leq 1 - \epsilon$. Consider the Hamiltonian $f : E \rightarrow \mathbb{R}$ defined by $f(z) = -\pi h(\|z\|^2)$, it is easy to see that $f \in \mathcal{F}(B_r)$. As in Remark 2.8 we see that for every n the spectrum associated to f_n is $\{0, \pi - \pi\epsilon\}$. We conclude that $c(\mu, f_n) = \pi - \pi\epsilon$, then $c(\mu, f) = \pi - \pi\epsilon$ and finally that $c_o(B_1) \geq \pi$. \square

We end this subsection with the proof of the fact that these capacities are indeed invariant by Hamiltonian flows satisfying compactness assumption 6.4.

Proposition 6.19. *Let $\Phi = \Phi_t$ be the flow of the Hamiltonian equation satisfying the compactness assumption 6.4. Then for every $\mathcal{X} \subseteq E$ we have*

$$c_o(\mathcal{X}) \leq c_o(\Phi(\mathcal{X})) \quad \text{and} \quad c^o(\mathcal{X}) \leq c^o(\Phi(\mathcal{X})).$$

If moreover Φ^{-1} is bounded, then we have equality.

Proof. Remark that the nonlinearity h has bounded gradient so V sends bounded open sets to bounded open sets and $V(\mathcal{U})$ is open and bounded. In particular $c(V(\mathcal{U}))$ is defined directly using $\mathcal{F}(\mathcal{U})$. It is enough to prove the result for open bounded sets. Recall that we have the decomposition $\Phi_t = e^{tJA}V^t$ and that e^{tJA} is a symplectic isometry which commutes with every projection Π_n . For every smooth function f , we have

$$(f \circ e^{tJA})_n = f_n \circ e^{tJA} \quad \text{and} \quad c(\mu, (f \circ e^{tJA})_n) = c(\mu, f_n).$$

This implies $c_o(\mu, f \circ e^{tJA}) = c_o(\mu, f)$ and $c^o(\mu, f \circ e^{tJA}) = c^o(\mu, f)$. In addition, the fact that e^{tJA} is an isometry implies that the map $(e^{tJA})^* : \mathcal{F}(e^{tJA}\mathcal{U}) \rightarrow \mathcal{F}(\mathcal{U})$ given by $f \mapsto f \circ e^{tJA}$ is a bijection. Using these two facts together we get

$$c_o(\mathcal{U}) = c_o(e^{tJA}\mathcal{U}) \quad \text{and} \quad c^o(\mathcal{U}) = c^o(e^{tJA}\mathcal{U}).$$

It is now enough to prove that the capacity is invariant by V^t . The subset \mathcal{U} is bounded so we may apply the approximation lemma, and for every $\epsilon > 0$ there exists an n such that for every $u \in \mathcal{U}$ we have $\|V_n(u) - V(u)\| \leq \epsilon$. We claim that for every $f \in \mathcal{F}(\mathcal{U})$ there exists an n such that $f \circ V_n^{-1} \in \mathcal{F}(V(\mathcal{U}))$. Once this is established, if we use that V_n is finite dimensional, we have $c_o(f) = c_o(f \circ V_n^{-1})$ and $c^o(f) = c^o(f \circ V_n^{-1})$. In particular we get

$$c_o(\mathcal{U}) \leq c_o(V(\mathcal{U})) = c_o(\Phi(\mathcal{U})) \quad \text{and} \quad c^o(\mathcal{U}) \leq c^o(V(\mathcal{U})) = c^o(\Phi(\mathcal{U})).$$

In order to prove that there exists an n such that $f \circ V_n^{-1} \in \mathcal{F}(V(\mathcal{U}))$ we need to show that

$$d(\text{Supp}(f \circ V_n^{-1}), \partial V(\mathcal{U})) > 0.$$

For any f we have the equality $\text{Supp}(f \circ V_n^{-1}) = V_n(\text{Supp } f)$, and since V_n is a diffeomorphism in finite dimensions we have $d(\text{Supp}(f \circ V_n^{-1}), \partial V_n(\mathcal{U})) = \delta > 0$. Now for every ϵ we may find an n such that $d(\partial V(\mathcal{U}), \partial V_n(\mathcal{U})) \leq \epsilon$. To conclude choose an n for an ϵ that verifies $\epsilon < \delta$ and use the previous inequalities.

In order to prove the reverse inequality one has to apply the same arguments to Φ^{-1} . To do this, one has to make sure that it is also approximated by finite dimensional flows. We leave this as an exercise or see [Kuk95, Lemma 6]. \square

Remark 6.20. As an example of Φ with Φ^{-1} satisfying compactness assumption 6.4 we have the autonomous Hamiltonian PDEs that satisfy the assumption 6.4.

Remark 6.21. In Kuksin's article [Kuk95] the capacities are defined using the Hofer-Zehnder capacity described in Subsection 2.3. An autonomous function $f \in \mathcal{F}(\mathcal{U})$ is called fast, if f_n is fast for $n \geq n_0$ for some n_0 . One then defines the capacity as the infimum of the maximal variation of fast solutions.

Corollary 6.22. *For any domain $\mathcal{O} \subseteq E$ and $\xi \in E$ we have*

$$c_o(\mathcal{X}) = c_o(\mathcal{X} + \xi) \quad \text{and} \quad c^o(\mathcal{X}) = c^o(\mathcal{X} + \xi).$$

Proof. It is clear that the addition $A_\xi : E \rightarrow E$ is a symplectomorphism. Moreover for each N we can write $\xi = \xi^N + \xi_N$ and A_{ξ^N} is a finite dimensional symplectomorphism such that $d(A_{\xi^N}(z), z) = \|\xi_N\| \rightarrow 0$. We may then apply the same arguments used in the previous proposition to prove one of the inequalities. The same argument also applies to $A_{-\xi}$ which gives the reverse inequality and the desired result. \square

6.2.3 Coisotropic Camel

We now move towards the extension of Theorem 4.55 to the context of Hamiltonian PDEs. Recall that to state the theorem we had to divide the symplectic phase space into two transverse coisotropic subspaces. In the infinite dimensional case we will denote $E = E_+ \oplus E_-$ where E_+ (resp. E_-) is generated by $\{\varphi_j^+ \mid j \in \mathbb{Z}\}$ (resp. $\{\varphi_j^- \mid j \in \mathbb{Z}\}$). Moreover denote E_k (resp. E_+^k and E_-^k) the Hilbert subspace generated by $\{\varphi_j^\pm \mid |j| \leq k\}$ (resp. $\{\varphi_j^+ \mid |j| \geq k+1\}$ and $\{\varphi_j^- \mid |j| \geq k+1\}$) and $\Pi_k : E \rightarrow E_k$ (resp. Π_+^k and Π_-^k) the corresponding projection. Fix a $k \geq 1$ and let X be a compact set contained in E_k . we are going to be interested in the image of sets of the form

$$C = \{u \in E \mid \Pi_k u \in X \text{ and } \Pi_-^k u = 0\}.$$

As in the finite-dimensional case, we first verify that the capacities constructed in the previous subsection do not give rigidity information for the image of these sets. Remark that we cannot use directly the inequality given in Proposition 6.15 since C is not an open set.

Lemma 6.23. *We have the following equalities*

$$c_o(E_k \times E_+^k) = 0 = c^o(E_k \times E_+^k).$$

Proof. By homogeneity the value of these capacities is either 0 or $+\infty$. By definition of the capacities it is then enough to exhibit an open set \mathcal{V} containing $E_k \times E_+^k$ with bounded capacity. By Proposition 6.15 it is enough to find an open set \mathcal{V} such that $c(\mathcal{V} \cap E_n)$ is bounded by a

constant independent of n for every n big enough. We use the same argument that we used in Proposition 3.41. Find a smooth bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$ with values on $]0, 1[$ and $f'(s) > 0$ for every $s \in \mathbb{R}$ (every strictly positive integrable function f' works). Define

$$\mathcal{V} = \{u \in E \text{ such that } |u_k^-| < f'(u_k^+)\}.$$

By hypothesis if $u \in E_k \times E_+^k$ then $u_k^- = 0$, so $E_k \times E_+^k \subseteq \mathcal{V}$. Now using the arguments in Proposition 3.41 we see that for $n > k$ the capacity $c(\mathcal{V} \cap E_n)$ is bounded by a constant that only depends on f . \square

To begin our study of the evolution of C we start with the study of the camel trajectories (see Definition 4.37). In the infinite dimensional context, the camel points associated to V^t are not only bounded, but also compact. This will allow us to have a precise control of the trajectories.

Lemma 6.24. *1. Fix a $k \geq 1$ and let X be a compact set contained in E_k . Then for every $t \in \mathbb{R}$ and $n > k$ we have $c(X) \leq \gamma(\Pi_k(V_n^t(C) \cap \{\Pi_+^k = 0\}))$.*

2. The set $\cup_n \{u \in C \mid \Pi_+^k V_n^t u = 0\} \subseteq E$ is bounded by a constant $R(t)$.

3. The set $\{u \in C \mid \Pi_+^k V^t u = 0\}$ is compact and so is $V^t(C) \cap \{\Pi_+^k = 0\}$.

Proof. Recall that $V_n^t u = (\phi_n^t(u_n), u^n)$ where ϕ_n^t is a finite dimensional flow generated by a Hamiltonian function with bounded gradient so it verifies Theorem 4.39. An easy computation shows that V_n^t verifies the statement if and only if ϕ_n^t verifies Theorem 4.39 on E_n .

For the second claim let $u \in E$ and decompose its norm as $\|u\| \leq \|\Pi_k u\| + \|\Pi_+^k u\| + \|\Pi_-^k u\|$. If $u \in C$ then by definition $\Pi_k u$ belongs to X which is compact contained in a ball of a certain radius r and $\Pi_-^k u = 0$ so $\|u\| \leq r + \|\Pi_+^k u\|$. We now show that $\Pi_+^k V_n^t u = 0$ implies $\|\Pi_+^k u\| \leq c(t)$. Duhamel's formula and the fact that $\sup_{(t,u) \in [0,t] \times E} \|\nabla h_n(u)\| \leq \sup_{(t,u) \in [0,t] \times E} \|\nabla h(u)\|$ is bounded imply that $\|V_n^t u - u\| \leq c(t)$ where $c(t)$ does not depend on n . We get that $\|\Pi_+^k u\| = \|\Pi_+^k V_n^t u - \Pi_+^k u\| \leq \|V_n^t u - u\| \leq c(t)$ and the result follows with $R(t) = r + c(t)$.

For the third claim we start by using the same argument as before to prove that $\{u \in C \mid \Pi_+^k V^t u = 0\}$ is bounded. Now let $\{z_n\} \subset E$ be a sequence such that

$$\Pi_k z_n \in X, \quad \Pi_-^k z_n = 0, \quad \text{and} \quad \Pi_+^k V^t z_n = 0 \quad \text{for all } n \in \mathbb{N}.$$

We claim that $\{z_n\}$ has a convergent subsequence. First remark that, by the decomposition of V_N in $E_N \oplus E^N$, for every $u \in E$ and $N \in \mathbb{N}$ we have $\Pi^N V_N^t u = \Pi^N u$. Moreover, by definition of z_n , if $N \geq k$ then $\Pi_-^N z_n = 0$ and $\Pi_+^N V^t z_n = 0$. For $N \geq k$ we have

$$\|\Pi^N z_n\| = \|\Pi_+^N z_n\| = \|\Pi_+^N V_N^t z_n\| = \|\Pi_+^N V_N^t z_n - \Pi_+^N V^t z_n\| \leq \|V_N^t z_n - V^t z_n\|.$$

Now $\{z_n\}_n$ is a bounded sequence so we can apply Proposition 6.8 and for every $\epsilon > 0$ there exists a $N_0(\epsilon) \in \mathbb{N}$ such that if $N \geq N_0$ then $\|V_N^t z_n - V^t z_n\| \leq \epsilon$. By the previous inequalities this implies that for $N \geq N_0$ we have $\|\Pi^N z_n\| \leq \epsilon$. On the other hand, $\{z_n\}_n$ bounded implies that it has a weakly converging subsequence (still denoted by $\{z_n\}$ for simplicity) that converges when projected onto any finite dimensional subspace E_N . We conclude that for any $\delta > 0$, with $\epsilon = \delta/3$ and $N \geq N_0(\epsilon)$, if $p, q \in \mathbb{N}$ are big enough we have

$$\|z_p - z_q\| \leq \|\Pi_N z_p - \Pi_N z_q\| + \|\Pi^N z_p\| + \|\Pi^N z_q\| < \delta$$

which implies that z_n is a Cauchy sequence. \square

We may now prove the infinite dimensional version of Theorem 4.39 for the map V^t . The proof of the following proposition relies on the finite dimensional result and it is the finite dimensional approximation of the flow that allows us to go from finite to infinite dimensions.

Proposition 6.25. *Fix a $k \geq 1$ and let X be a compact set contained in E_k . Define*

$$C = \{u \in E \mid \Pi_k u \in X \text{ and } \Pi_-^k u = 0\}.$$

Then for every $t \in \mathbb{R}$ we have

$$c(X) \leq \gamma(\Pi_k(V^t(C) \cap \{\Pi_+^k = 0\})).$$

Proof. Let \mathcal{V}_ϵ be the open ϵ neighbourhood of $\Pi_k(V^t(C) \cap \{\Pi_+^k = 0\})$. We will show that for each $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $\Pi_k(V_n^t(C) \cap \{\Pi_+^k = 0\}) \subseteq \mathcal{V}_\epsilon$. Once this is proven, Lemma 6.24 part 1 and monotonicity of the capacity γ imply that $c(X) \leq \gamma(\mathcal{V}_\epsilon)$ for every $\epsilon > 0$ so $c(X) \leq \lim_{\epsilon \rightarrow 0} \gamma(\mathcal{V}_\epsilon)$. We then use that $\Pi_k(V^t(C) \cap \{\Pi_+^k = 0\})$ is compact by Lemma 6.24 part 3 to conclude that $\lim_{\epsilon \rightarrow 0} \gamma(\mathcal{V}_\epsilon) = \gamma(\Pi_k(V^t(C) \cap \{\Pi_+^k = 0\}))$ which is the desired result.

The proof is by contradiction. Suppose that there exist an $\epsilon_0 > 0$ and a sequence $\{z_n\} \subset E$ such that for all $n \in \mathbb{N}$

$$\Pi_k z_n \in X, \quad \Pi_-^k z_n = 0, \quad \Pi_+^k V_n^t z_n = 0 \quad \text{and} \quad d(\Pi_k V_n^t z_n, \mathcal{V}_0) \geq \epsilon_0.$$

We claim that $\{z_n\}$ has a convergent subsequence. We use the same argument as in Lemma 6.24 part 3. For $N \geq k$ we have

$$\|\Pi^N z_n\| = \|\Pi_+^N z_n\| = \|\Pi_+^N V_N^t z_n\| = \|\Pi_+^N V_N^t z_n - \Pi_+^N V_n^t z_n\| \leq \|V_N^t z_n - V_n^t z_n\|.$$

By Lemma 6.24 part 2 we know that z_n is a bounded sequence so we can apply Proposition 6.8 and for every $\delta > 0$ there exists a $N_0(\delta) \in \mathbb{N}$ such that if $n, N \geq N_0$ then $\|V_N^t z_n - V_n^t z_n\| \leq \delta$. By the previous inequalities this implies that for $n, N \geq N_0$ we have $\|\Pi^N z_n\| \leq \delta$. On the other hand, $\{z_n\}$ bounded implies that it has a weakly converging subsequence (still denoted by $\{z_n\}$ for simplicity) that converges when projected onto any finite dimensional subspace E_N . We conclude that for any $\delta > 0$, with $\epsilon = \delta/3$ and $N \geq N_0(\epsilon)$, if $p, q \geq N_0$ are big enough we have

$$\|z_p - z_q\| \leq \|\Pi_N z_p - \Pi_N z_q\| + \|\Pi^N z_p\| + \|\Pi^N z_q\| < \delta$$

which implies that z_n is a Cauchy sequence. Denote z its limit in E . The set X is closed so $\Pi_k z \in X$ and Π_-^k is continuous so $\Pi_-^k z = 0$. This means that z is an element of C . Moreover remark that

$$\|V^t z - V_n^t z_n\| \leq \|V^t z - V^t z_n\| + \|V^t z_n - V_n^t z_n\|.$$

so by continuity of V^t and again by Proposition 6.8 we get that $V_n^t z_n$ converges to $V^t z$ in E . Using the hypothesis $\Pi_+^k V_n^t z_n = 0$ we find that $\Pi_+^k V^t z = 0$ which allows us to conclude that $\Pi_k V^t z$ belongs to \mathcal{V}_0 . This contradicts the fact that $d(\Pi_k V_n^t z_n, \mathcal{V}_0) \geq \epsilon_0 > 0$ for all $n \in \mathbb{N}$ achieving the proof of the theorem. \square

This is not a statement about the actual flow of our Hamiltonian equation. Nevertheless using the fact that e^{tJA} restricts to a symplectic isomorphism on each E_n we get the following middle dimensional nonsqueezing:

Theorem 6.26. *Denote by $\Phi^t : E \rightarrow E$ the flow of a Hamiltonian equation satisfying the compactness condition 6.4. For every $k \in \mathbb{N}$, every compact subset X of E_k and every $t \in \mathbb{R}$ we have*

$$c(X) \leq \gamma(\Pi_k \Phi^t(X \times E_+^k)).$$

Proof. We always have the inclusion $\Pi_k(V^t(C) \cap \{\Pi_+^k = 0\}) \subseteq \Pi_k V^t(C)$ so by Proposition 6.25 and monotonicity of the symplectic capacity γ we have

$$c(X) \leq \gamma(\Pi_k(V^t(C) \cap \{\Pi_+^k = 0\})) \leq \gamma(\Pi_k V^t(C)).$$

The linear operator e^{tJA} restricts to a symplectic isomorphism on each E_n which commutes with Π_k and the capacity γ is invariant under symplectic transformations so

$$\gamma(\Pi_k V^t(C)) = \gamma(e^{-tJA} \Pi_k e^{tJA} V^t(C)) = \gamma(\Pi_k \Phi^t(C)).$$

which gives the desired result. \square

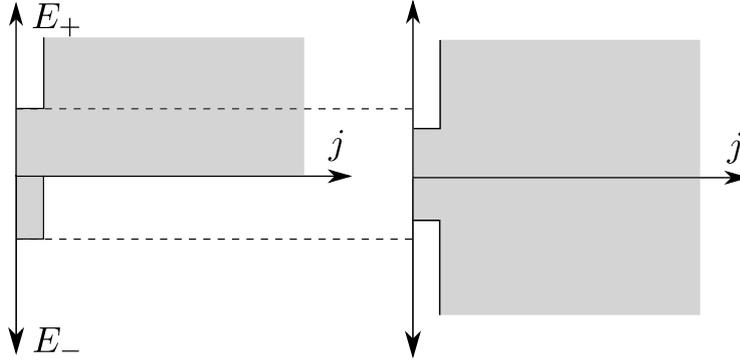


Figure 6.2: The representation of coisotropic cylinder and a standard cylinder in E .

Taking into account the discussion in Subsection 6.2.1, we can also interpret this result as a statement that talks about the energy transition between frequencies. This time, instead of having an negative result for the transfer of energy between frequencies in a locally uniform way, we have a negative result on unbounded sets. These sets are the coisotropic cylinders $X \times E_+^k$, which are better understood on particular examples. Consider the nonlinear string equation

$$\ddot{u} = u_{xx} - f(t, x, u), \quad u = u(t, x),$$

where $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Recall that in this case $E = H^{\frac{1}{2}}(\mathbb{T}) \times H^{\frac{1}{2}}(\mathbb{T})$ and a solution $U(t) = (u(t), v(t))$ verifies $\dot{u}(t) = -Bv(t)$ where B is the operator $B = (-\partial^2/\partial x^2 + 1)^{1/2}$. In this case the symplectic Hilbert basis $\{\varphi_j^\pm \mid j \in \mathbb{Z}\}$ is given by

$$\varphi_j^+ = \frac{1}{(j^2 + 1)^{\frac{1}{4}}}(\varphi_j(x), 0), \quad \varphi_j^- = \frac{1}{(j^2 + 1)^{\frac{1}{4}}}(0, -\varphi_j(x)),$$

with

$$\varphi_j(x) = \begin{cases} \sqrt{2} \sin jx, & j > 0, \\ \sqrt{2} \cos jx, & j \leq 0. \end{cases}$$

The graph of $u(t, \cdot)$ represents the position of a string on a plane, and the vector $(0, \dot{u}(t, x)) \in \mathbb{R}^2$ represents the speed of the point $u(t, x)$. For this equation, initial conditions in the space E_+ represents solutions with zero initial speed, and initial conditions on E_- represent solution where the graph of the string starts at $x = 0$. In light of this, coisotropic cylinders represent solutions where we allow initial conditions with one particular frequency (or k chosen frequencies) nonzero. As an example denote E_k the space generated by φ_j^\pm for $|j| \leq k$ and consider $E_0 \simeq \mathbb{R}^2$ which consists of constant functions.

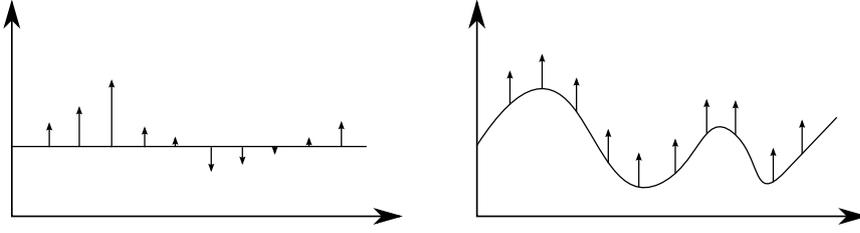


Figure 6.3: The representation the graph of the initial condition $u(0, x)$ and at each point the vector of the initial speed $v(0, x)$. On the left, $u(0, x)$ is constant. On the right, $v(0, x)$ is constant.

Corollary 6.27. *Let Φ_t be the flow of the Sine-Gordon equation. Let B_R be the ball of radius R in E_0 . Then if for a $t \in \mathbb{R}$ we have*

$$\Pi_0 \Phi_t(B_r \times E_+^0) \subseteq B_R \quad \text{or} \quad \Pi_0 \Phi_t(B_r \times E_-^0) \subseteq B_R.$$

then $r \leq R$. In particular for every $R > r > 0$ and every $t_0 \in \mathbb{R}$ there is a solution $(u(t), v(t)) = U(t) = \Phi_t(U_0)$ with constant initial position (or constant initial speed) such that

$$\left| \int_{\mathbb{T}} U(0, x) dx \right| \leq R \quad \text{but} \quad \left| \int_{\mathbb{T}} U(t_0, x) dx \right| > r.$$

If we compare this result with the analogue of Gromov's non-squeezing described in Subsection 6.2.1 we see two main differences. The first one is that instead of looking at the evolution of an open set (the ball), we look at the evolution of a subset which is contained in a vector subspace of infinite codimension ($E_k \times E_+^k \subseteq E$). In this sense the coisotropic cylinder can be thought of as a "thinner" subset. The second difference is that instead of looking at a bounded set we now look at the evolution of an unbounded set. In this sense the coisotropic cylinder can be thought of as a "bigger" subset. One way of understanding these differences is that if we want to have more control over one aspect of the set of initial conditions (for example, we only consider solutions where the initial velocity has non-zero Fourier coefficients only of frequency less than k) we need to lose control over some other property of the initial conditions (in this case the norm).

Chapter 7

Exploring the Arnold conjecture for Hamiltonian PDEs

Let (M, ω) be a closed symplectic manifold. Every smooth function $H : \mathbb{R} \times M \rightarrow \mathbb{R}$ defines via ω a vector field X^H on M . This vector field can be then integrated and the time t map of the flow generates a diffeomorphism $\psi_t : M \rightarrow M$ which is called Hamiltonian diffeomorphism.

Conjecture 7.1 (Arnold). Let M be a closed symplectic manifold and ψ_t a Hamiltonian diffeomorphism. Then the number of fixed points of ψ_t is at least the number of critical points of functions on M . Moreover, if all the fixed points are non-degenerate, i.e. 1 is not an eigenvalue of $d\psi_t$ at any of its fixed points, then the number is at least the number of critical points of a Morse function on M .

Remark that the number of critical points of functions can be bounded from below using topological invariants (cup-length, Betti numbers...). The Arnold conjecture has historically been one of the major forces driving the development of symplectic topology. We briefly recall some of the results. It was proven in [Wei78] and [Arn65] that every exact diffeomorphism which is C^1 -close to the identity has at least as many fixed points as a smooth function on M has critical points. In 1983, Conley and Zehnder [CZ83] proved the conjecture for the torus \mathbb{T}^{2n} . In 1985 Fortune [For85] proved the case $M = \mathbb{C}P^n$. The revolution came with the powerful theory of Floer Homology [Flo89a, Flo89b] which was developed as a tool to prove this conjecture. Further generalizations of his ideas followed shortly after and now the non-degenerate case has been established for every compact symplectic manifold. In this section we are interested in the case of $\mathbb{C}P^n$. In this case we have

Theorem 7.2 (Fortune 1985 [For85]). *Every Hamiltonian diffeomorphism of $\mathbb{C}P^n$ has at least $n + 1$ fixed points.*

We are interested in generalizations of this statement to infinite dimension. More precisely we look at the Schrödinger equation where the nonlinearity depends only on $|u|^2$, In this case the L^2 norm of functions is constant along solutions. If the flow is well defined, then it descends to a map on the projective Hilbert space. The aim of this section is to explore if the analogue of Theorem 7.2 is true in this context. We assume a compactness property for the nonlinearity (see Definition 6.4) and give a proof for the autonomous case.

7.1 Schrödinger equation with convolution nonlinearity

Consider the Hilbert space of periodic square integrable functions $L^2(\mathbb{T}, \mathbb{C})$ where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The operator $A = -\partial_x^2$ has a complete system of complex eigenfunctions given by $\{\exp(ikx) \mid k \in \mathbb{Z}\}$.

The real eigenfunctions are the real and imaginary part, which give a real Hilbert basis of $L^2(\mathbb{T}, \mathbb{C})$ for the real scalar product. More precisely define $\varphi_0 = 1$, $\varphi_{2j-1} = \sin jx$ and $\varphi_{2j} = \cos jx$ for $j \geq 1$. We define the real Hilbert basis by $\varphi_j^+ = \varphi_j$ and $\varphi_j^- = i\varphi_j$. Now let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $\xi : \mathbb{T} \rightarrow \mathbb{R}$ be two smooth functions. We are going to consider the Hamiltonian functions of the form explained in Subsection 6.1.4 with

$$h_t(u) = \frac{1}{2(2\pi)} \int_{\mathbb{T}} f(|u * \xi|^2, x, t) dx$$

The Hamiltonian equation in this case is

$$i\dot{u} = -\Delta u + \partial_1 f(|u * \xi|^2, x, t)(u * \xi) * \xi$$

where $\partial_1 f$ means derivative with respect to the first coordinate. We know (see Lemma 6.6 and Proposition 6.8) that the flow of this equation is globally well defined and is approximated by finite dimensional Hamiltonian diffeomorphisms. We denote the flow by $\Phi^t : L^2(\mathbb{T}, \mathbb{C}) \rightarrow L^2(\mathbb{T}, \mathbb{C})$. A straightforward calculation shows that the flow preserves the L^2 norm. In particular, this symmetry gives that Φ_t commutes with complex numbers of norm 1, i.e.

$$\Phi^t(e^{i\mu}u) = e^{i\mu}\Phi^t(u).$$

If we restrict the flow to the unit sphere \mathbb{S} in $L^2(\mathbb{T})$ we can deduce a map $\Psi^t : \mathbb{P}L^2 \rightarrow \mathbb{P}L^2$ where $\mathbb{P}L^2 := \mathbb{S}/S^1$. We are interested in fixed points of Ψ^t , that is, points $u \in \mathbb{S}$ such that there exists a $\mu \in \mathbb{R}$ with

$$\Phi^t(u) = e^{i\mu}u.$$

The analogue of the statement of Theorem 7.2 in this setting is:

Conjecture 7.3. The map Ψ^t has infinitely many fixed points.

Remark 7.4. One is tempted to state the conjecture for every Hamiltonian diffeomorphism which preserves the L^2 norm. The following example shows that this completely general generalization is false (see [Fab18]). Consider the function $G : L^2(\mathbb{C}, \mathbb{R}) \rightarrow \mathbb{R}$ given by

$$G(u) = \frac{1}{2} \int_{\mathbb{T}} V(x)|u(x)|^2$$

where V is a smooth periodic real function. The flow is given by $\Phi_G^t(u) = e^{itV(x)}u$ which clearly descends to $\mathbb{P}L^2$. In this case we see that for a $u \in \mathbb{S}$ we have $\Phi_G^1(u) = e^{i\mu}u$ if and only if $V(x) - \mu \in 2\pi\mathbb{Z}$ for every almost every x . In particular, a generic choice of V yields a Hamiltonian map with no fixed points.

In this section we explore the validity of Conjecture 7.3 and we prove that it is true in the autonomous case. Remark that in this case it is enough to study the critical points of the Hamiltonian, so the problem is purely topological in nature, and not symplectic. What makes this possible is the quadratic term in the Hamiltonian given by the Laplacian which guarantees enough critical points.

Theorem 7.5. *Suppose that f is independent of t . Then for every $t \in \mathbb{R}$ the map Ψ^t has infinitely many fixed points. More precisely, there is a sequence of points $u_n \in H^1$ and $\mu_n \in \mathbb{R}$ such that*

$$\Phi^t(u_n) = u_n(t, x) = e^{i\mu_n}u_n(x) \quad \text{for every } (t, x) \in \mathbb{R} \times \mathbb{T}$$

and

$$H(u_n) \rightarrow +\infty \quad \text{as } n \rightarrow +\infty,$$

This type of solutions receive the name of standing waves. We want to point out that standing waves are a natural type of solution which arise even if the nonlinearity is not of convolution type. For example, the nonlinear cubic Schrödinger equation

$$i\dot{u} = -u_{xx} + c|u|^2u$$

has infinitely many standing waves given by the plane waves

$$u_k(t, x) = e^{ikx} e^{-i\xi t}$$

with $\xi = k^2 + c$. Moreover in this case the energy of the waves also goes to infinity. In particular, for $c = 0$, the linear equation has infinitely many fixed points. Our case is a compact perturbation of this linear equation.

This theorem is to be compared with the one in [Fab18]. In this article Fabert gives a proof of the conjecture for non-autonomous Hamiltonians under some conditions on the nonlinearity. More precisely on the Hofer norm of the nonlinear term and on the function ξ . His proof uses symplectic topology for the existence of a particular type of Floer strips together with nonstandard analysis.

7.1.1 Converging sequences of fixed points

The first step is to use the fact that Φ^t can be approximated by finite dimensional Hamiltonian flows Φ_n^t in order to reduce the proof to a statement about converging sequences of standing waves for Φ_n^t .

Lemma 7.6. *Let $u_n \in \mathbb{S} \cap E_n$ and $\mu_n \in [0, 2\pi]$ be such that*

$$\Phi_n^t(u_n) = e^{i\mu_n t} u_n$$

for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Suppose that both u_n and μ_n converge in L^2 and $[0, 2\pi]$ respectively. Then the limits u and μ verify

$$\Phi^t(u) = e^{i\mu t} u$$

for every $t \in \mathbb{R}$.

Proof. Fix $t \in \mathbb{R}$. Use the triangular inequality and the fact that $\Phi_n^t(u_n) = e^{i\mu_n t} u_n$ to get

$$\|\Phi^t(u) - e^{i\mu t} u\| \leq \|\Phi^t(u) - \Phi^t(u_n)\| + \|\Phi^t(u_n) - \Phi_n^t(u_n)\| + \|e^{i\mu_n t} u_n - e^{i\mu t} u\| + \|e^{i\mu_n t} u - e^{i\mu t} u\|.$$

The right hand side converges to zero when n goes to infinity by (in order) continuity of Φ^t , by Proposition 6.8 and the fact that the u_n are bounded and by convergence of u_n and μ_n . \square

Now the problem reduces to the existence of these converging sequences. Recall that since the nonlinearity is of convolution type, the function h can be extended as a \mathcal{C}^2 function on the whole space $H^{-1}(\mathbb{T}, \mathbb{C})$. In particular, since the inclusion $L^2(\mathbb{T}, \mathbb{C}) \hookrightarrow H^{-1}(\mathbb{T}, \mathbb{C})$ is compact and that \mathbb{S} is bounded we can find a constant $c > 0$ such that

$$\sup_{u \in \mathbb{S}} |h_n(u)| \leq \sup_{u \in \mathbb{S}} |h(u)| \leq c \quad \text{and} \quad \sup_{u \in \mathbb{S}} \|\nabla h_n(u)\| \leq \sup_{u \in \mathbb{S}} \|\nabla h(u)\| \leq c.$$

The following lemma shows that H verifies a kind of Palais-Smale condition by approximation. More precisely:

Lemma 7.7. *Suppose that there is a constant $C > 0$, and two sequences $u_n \in \mathbb{S} \cap E_n$ and $\mu_n \in \mathbb{R}$ such that*

$$H_n(u_n) \leq C \quad \text{and} \quad \nabla H_n(u_n) + \mu_n u_n = 0$$

for every $n \in \mathbb{N}$. Then $\{u_n\}$ is compact in $H^2(\mathbb{T}, \mathbb{C})$.

Proof. We may write H as

$$H(u) = \frac{1}{2}(\|u\|_{H^1}^2 - \|u\|_{L^2}^2) + h(u)$$

so that

$$\|u_n\|_{H^1}^2 \leq \|u_n\|_{L^2}^2 + 2|h(u_n)| + 2|H(u_n)| \leq 1 + 2c + 2C$$

and u_n are bounded in $H^1(\mathbb{T}, \mathbb{C})$. The second hypothesis can be written as

$$Au_n + \nabla h_n(u_n) + \mu_n u_n = 0$$

where $\nabla h_n(u_n) = \Pi_n \nabla h(u_n)$. We first prove that μ_n is bounded. Taking the L^2 product with u_n the previous equality implies that

$$\langle Au_n, u_n \rangle + \langle \nabla h_n(u_n), u_n \rangle + \langle \mu_n u_n, u_n \rangle = 0.$$

This can be written as

$$\frac{1}{2}(\|u_n\|_{H^1}^2 - \|u_n\|_{L^2}^2) + \langle \nabla h_n(u_n), u_n \rangle + \mu_n \|u_n\|_{L^2}^2 = 0$$

We conclude using the fact that u_n has L^2 norm equal to one, that the sequence is also bounded in H^1 , and that the gradient of h is bounded in \mathbb{S} . Now we continue with the proof that $\{u_n\}$ has a converging subsequence in H^2 . Let $\Pi^0 : E \rightarrow E^0$ be the natural projection. It is easy to see that it commutes with A . The hypothesis of equality for the gradient gives

$$A\Pi^0 u_n = -\Pi^0(\Pi_n \nabla h(u_n) + \mu_n u_n),$$

and since A is invertible over E^0 we have

$$\Pi^0 u_n = -A^{-1}\Pi^0(\Pi_n \nabla h(u_n) + \mu_n u_n).$$

First, $\{\mu_n u_n\}$ is bounded in $H^1(\mathbb{T}, \mathbb{C})$ so it has a convergent subsequence in L^2 . Moreover, ∇h is continuous on L^2 and u_n is bounded in $H^1(\mathbb{T}, \mathbb{C})$. This implies that $\{\nabla h(u_n)\}$ is compact in L^2 so it also has a convergent subsequence. Now since $A^{-1} : \Pi^0 L^2(\mathbb{T}, \mathbb{C}) \rightarrow \Pi^0 H^2(\mathbb{T}, \mathbb{C})$ is continuous we conclude that $\Pi^0 u_n$ has a convergent subsequence in $H^2(\mathbb{T}, \mathbb{C})$. On the other hand, $\Pi_0 u_n$ is bounded contained in $E_0 \simeq \mathbb{R}^2$ so it also has a convergent subsequence in $H^2(\mathbb{T}, \mathbb{C})$. We conclude that $u_n = \Pi_0 u_n + \Pi^0 u_n$ has a convergent subsequence in $H^2(\mathbb{T}, \mathbb{C})$ which is the desired result. \square

Remark 7.8. The hypothesis on u_n of the theorem are equivalent to the existence of a critical point $[u_n]$ for the function that H_n induces on $\mathbb{C}P^N$ satisfying $H_n(u_n) \leq C$. In particular, taking into account Lemmas 7.6 and 7.7, the proof of Theorem 7.5 reduces to proving, for each $n \in \mathbb{N}$, the existence of critical points $[u_n]$ of the reduction of H_n to $\mathbb{C}P^N$ such that $H(u_n) \leq C$ for each n .

7.1.2 Finding critical points

Let M be a closed oriented manifold and $f \in \mathcal{C}^\infty(M)$. Denote by $f^c = \{x \in M \mid f(x) \leq c\}$ and denote by i_c the natural map $i_c : H^*(M) \rightarrow H^*(f^c)$. Let $\omega \in H^*(M) \setminus \{0\}$. We may select critical values (see Appendix C or for example [Nic11, section 2.7]) by the formula

$$c(\omega, f) = \inf\{c \mid i_c(\omega) \neq 0\}.$$

We recall that if two functions satisfy $f \leq g$ then

$$c(\omega, f) \leq c(\omega, g),$$

and that for any $a \in \mathbb{R}$ we have

$$c(\omega, f + a) = c(\omega, f) + a.$$

We will use the following classical result:

Theorem 7.9 (Lusternik-Schnirelmann, see for example [Vit97]). *If $\alpha, \omega \in H^*(M)$ such that $\alpha \cup \omega \neq 0$ then*

$$c(\alpha, f) \leq c(\alpha \cup \omega, f).$$

Moreover, suppose that we have an equality and denote K_c the set of critical points at level $c = c(\alpha, f) = c(\alpha \cup \omega, f)$. Then ω is nonzero on $H^(K_c)$. As a result $\dim K_c \geq \deg \omega$ and in particular, if $\deg \omega \neq 0$, K_c is uncountable.*

Consider $M = \mathbb{C}P^n$. In this case

$$H^{2k}(\mathbb{C}P^n) = \mathbb{Z} \quad \text{if } 0 \leq k \leq n$$

and the other homology groups are zero. Moreover, if we denote by α the generator of $H^2(\mathbb{C}P^n)$ we know that $H^{2k}(\mathbb{C}P^n)$ is generated by $\alpha^{\cup k}$. Consider the restriction of $\frac{1}{2}\langle Au, u \rangle$ to $E_n \simeq \mathbb{C}^{2n+1}$ which we denote by Q_n . On complex coordinates we have the explicit expression (see the definition of φ^\pm at the beginning of this section)

$$Q_n(z_0, z_1, \dots, z_{2n}) = \frac{1}{2} \sum_{k=1}^n k^2 (|z_{2k-1}|^2 + |z_{2k}|^2).$$

Denote by \underline{Q}_n the induced function on $\mathbb{C}P^{2n}$. The following lemma explains that we are able to find every critical value of \underline{Q}_n using the minmax selector.

Lemma 7.10. *The function $Q_n : \mathbb{C}P^{2n} \rightarrow \mathbb{R}$ has $n + 1$ critical values $\{k^2/2 \mid 0 \leq k \leq n\}$. The set of critical points of critical value $k^2/2$ is a point for $k = 0$ and a projective space $\mathbb{C}P^1$ in the coordinates z_{2k-1}, z_{2k} for $k \geq 1$. Moreover, if α is the generator of $H^2(\mathbb{C}P^{2n})$ then*

$$c(\alpha^{\cup 2k}, \underline{Q}_n) = \frac{k^2}{2}.$$

Proof. The first claim is a straightforward. For the second part remark that for every critical value c we have $\dim K_c \leq 2 < 4 = \deg \alpha \cup \alpha$. We may use Theorem 7.9 to get

$$c(1, \underline{Q}_n) < c(\alpha^{\cup 2}, \underline{Q}_n) < c(\alpha^{\cup 4}, \underline{Q}_n) < \dots < c(\alpha^{\cup 2n}, \underline{Q}_n)$$

so the selector spans the $n + 1$ critical values of \underline{Q}_n . □

7.1.3 Proof of Theorem 7.5

Denote by \underline{H}_n and \underline{h}_n the functions that H_n and h_n induce on $\mathbb{C}P^{2n}$. We have

$$\underline{H}_n = \underline{Q}_n + \underline{h}_n.$$

Denote again by c a positive constant such that

$$\sup_{u \in \mathbb{S}} |h_n(u)| \leq \sup_{u \in \mathbb{S}} |h(u)| \leq c.$$

By the properties of the minimax selector that we saw in the previous section, and by Lemma 7.10 we have that

$$|c(\alpha^{\cup 2k}, \underline{\mathbb{H}}_n) - c(\alpha^{\cup 2k}, \underline{\mathbb{Q}}_n)| = |c(\alpha^{\cup 2k}, \underline{\mathbb{H}}_n) - \frac{k^2}{2}| \leq c.$$

Let k_n be a sequence of natural numbers such that

$$[\frac{k_n^2}{2} - c, \frac{k_n^2}{2} + c] \cap [\frac{k_m^2}{2} - c, \frac{k_m^2}{2} + c] = \emptyset$$

for every $n, m \in \mathbb{N}$. Then

$$c(\alpha^{\cup 2k_m}, \underline{\mathbb{H}}_n)$$

defines a critical value of $\underline{\mathbb{H}}_n$ which belongs to $[\frac{k_m^2}{2} - c, \frac{k_m^2}{2} + c]$. This gives us a sequence u_n of points in \mathbb{S} that verify

$$H_n(u_n) \in [\frac{k_m^2}{2} - c, \frac{k_m^2}{2} + c] \quad \text{and} \quad \nabla H_n(u_n) + \mu_n u_n = 0$$

We now use Lemma 7.7 to conclude that $\{u_n\}$ converges in H^2 , and since H is continuous in H^1 we conclude that the limit u verifies

$$H(u) \in [-c + \frac{k_m^2}{2}, \frac{k_m^2}{2} + c].$$

Finally we use Lemma 7.6 to conclude that $u \in \mathbb{S}$ generates a standing wave for our equation. We may now do this for every k_m to find standing wave with different energy levels and whose energy tends to infinity.

Appendix A

Basic symplectic geometry

The aim of this appendix is to introduce the basic concepts and definitions of symplectic geometry that we use in this thesis.

A.1 Symplectic vector spaces

Definition A.1. A symplectic vector space is a pair (V, ω) where V is a real vector space and ω is a non-degenerate skew-symmetric bilinear form.

Let g be a scalar product on V . Then there exists a unique linear map $A : V \rightarrow V$ such that

$$\omega(u, v) = g(Au, v) \quad \text{for every } u, v \in V$$

The fact that ω is skew symmetric implies $A^t = -A$. This gives $\det A = \det A^t = (-1)^{\dim V} \det A$ so ω non-degenerate implies that the dimension of V is even.

Example A.2. For any a real vector space E , the direct sum $V = E \oplus E^*$ has a canonical symplectic structure determined by the formula

$$\omega_0(u \oplus \alpha, v \oplus \beta) = \beta(u) - \alpha(v).$$

If e_1, \dots, e_n is a basis of E and f_1, \dots, f_n is the dual basis, then $e_1 \oplus 0, \dots, e_n \oplus 0, 0 \oplus f_1, \dots, 0 \oplus f_n$ is a symplectic basis for V .

Given a linear subspace W of a symplectic vector space (V, ω) , its symplectic orthogonal is the subspace

$$W^\omega := \{v \in V \mid \omega(v, u) = 0 \quad \text{for every } u \in W\}.$$

By non-degeneracy we have

$$\dim W + \dim W^\omega = \dim V$$

and $(W^\omega)^\omega = W$. Moreover for subspaces W and Y we have $(W \cap Y)^\omega = W^\omega + Y^\omega$. There are special types of linear subspaces of a symplectic vector space (V, ω) .

Definition A.3. Let (V, ω) be a symplectic space and let $W \subset V$ be a linear subspace. We say that W is:

- Symplectic, if $W \cap W^\omega = \{0\}$.
- Isotropic, if $W \subseteq W^\omega$.
- Coisotropic, if $W^\omega \subseteq W$.
- Lagrangian, if $W = W^\omega$.

A Lagrangian subspace verifies $\dim W = \frac{1}{2} \dim V$.

A.2 Symplectic manifolds

Definition A.4. A symplectic manifold is a pair (M, ω) where M is a manifold and ω is a closed two form on M such that ω_p is symplectic for every $p \in M$.

Example A.5. Let N be a manifold and consider its cotangent space T^*N . Let (U, q_1, \dots, q_n) be a coordinate chart for N , with associated coordinates $(T^*U, q_1, \dots, q_n, p_1, \dots, p_n)$. Define a symplectic form on T^*U by

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i.$$

This ω is defined by considering the one form on T^*U

$$\lambda = \sum_{i=1}^n p_i dq_i$$

which satisfies $\omega = -d\lambda$. One can check that λ is coordinate independent: in terms of the natural projection $\pi : T^*N \rightarrow N$ given by $(q, p) \mapsto q$, the form λ may be equivalently defined pointwise without coordinates by

$$\lambda_{(q,p)} = (d_{(q,p)}\pi)^* p \in T_{(q,p)}(T^*N)$$

where $(d_{(q,p)}\pi)^* : T_q^*N \rightarrow T_{(q,p)}^*(T^*N)$ is the transpose of $d_{(q,p)}\pi$, that is,

$$\lambda_{(q,p)}(v) = p(d_{(q,p)}\pi(v)).$$

The form λ is uniquely characterized by the property that $\alpha^*\lambda = \alpha$ for every one form $\alpha : N \rightarrow T^*N$.

Symplectic manifolds are even dimensional. Moreover the n th exterior product ω^n is a volume form. When (M, ω) is compact, the cohomology class $[\omega^n]$ is non-zero by Stokes theorem. Therefore $[\omega]$ and all its powers $[\omega^k]$ are non-zero. In particular exact symplectic forms can only exist on non-compact manifolds.

Definition A.6. Let (M, ω) be a symplectic manifold and let $W \subset M$ be a submanifold. Then we say that W is symplectic (resp. isotropic, coisotropic or Lagrangian) if $T_w W$ is symplectic (resp. isotropic, coisotropic or Lagrangian) for every $w \in W$.

Lemma A.7. Let (M, ω) be a symplectic manifold and let $W \subseteq M$ be a coisotropic submanifold. Then TW^ω defines an integrable distribution.

Proof. (from [MS17]) Let X and Y be vector fields in W with values in TW^ω and fix a point $w \in W$. Given a tangent vector $v \in T_w W$ choose any vector field $Z : W \rightarrow TW$ such that $Z(w) = v$. Since ω is closed we have

$$\begin{aligned} 0 &= d\omega(X, Y, Z) \\ &= \mathcal{L}_X(\omega(Z, Y)) + \mathcal{L}_Y(\omega(X, Z)) + \mathcal{L}_Z(\omega(Y, X)) \\ &\quad + \omega([Y, Z], X) + \omega([Z, X], Y) + \omega([X, Y], Z) \\ &= \omega([X, Y], Z) \end{aligned}$$

In the last equality we have used that $\omega(X, Z) = \omega(Y, Z) = 0$ for every vector field Z in W . It follows that $[X, Y](w) \in T_w W^\omega$. \square

A.3 Hamiltonian diffeomorphisms

Let (M, ω) be a symplectic manifold. For every function $H : \mathbb{R} \times M \rightarrow \mathbb{R}$ we can define a vector field X^H given by the relation

$$\omega(X^H, \cdot) = dH_t(\cdot).$$

If the manifold is compact, or H is compactly supported, then X^H defines a flow ψ_t^H on M .

Definition A.8. The set of compactly supported Hamiltonian diffeomorphisms $\text{Ham}^c(M)$ is defined as the set which contains all time one maps ψ_1^H of Hamiltonian flows generated by compactly supported functions H .

Let H and K be two Hamiltonian functions on M . In order to study the properties of $\text{Ham}^c(M)$ we introduce the following definitions.

$$\begin{aligned}\bar{H}(t, z) &= -H(t, \psi_t^H(z)) \\ (H \# K)(t, z) &= H(t, z) + K(t, (\psi_t^H)^{-1}(z)) \\ (\bar{H} \# K)(t, z) &= (K - H)(t, \psi_t^H(z)) \\ H_\varphi(t, z) &= H(t, \varphi^{-1}(z)) \\ H^\rho(t, z) &= \rho'(t)H(\rho(t), z)\end{aligned}$$

Lemma A.9 (see for example [HZ94] p.144). *Let H and K be two Hamiltonian functions on M , let φ be a symplectomorphism of M and let ρ be a non-decreasing function. The following formulae hold true*

$$\begin{aligned}\psi_t^{\bar{H}} &= (\psi_t^H)^{-1} \\ \psi_t^{H \# K} &= \psi_t^H \circ \psi_t^K \\ \psi_t^{\bar{H} \# K} &= (\psi_t^H)^{-1} \circ \psi_t^K \\ \psi_t^{H_\varphi} &= \varphi \circ \psi_t^H \circ \varphi^{-1} \\ \psi_t^{H^\rho} &= \psi_{\rho(t)}^H\end{aligned}$$

Using these properties it is straightforward to prove that $\text{Ham}^c(M)$ is a normal subgroup of the group of symplectomorphisms which contains every time t map of the flow.

Appendix B

The action functional as a generating function

This section is based on [ABK⁺94] which is itself based on [Vit87, Vit90]. We give a variational intuition for the definition of generating functions. Consider a compactly supported Hamiltonian $H = H(t, q, p)$ which generates the flow ψ_t on $\mathbb{R}^{2n} = T^*\mathbb{R}^n$. Set $L_0 = \mathbb{R}^n \times \{0\} = 0_{\mathbb{R}^n}$ the zero section, and consider the Hamiltonian deformation $L_1 = \psi_t(L_0) \subseteq \mathbb{R}^{2n}$. Denote by $\gamma : [0, 1] \rightarrow \mathbb{R}^{2n}$ paths on \mathbb{R}^{2n} . We will work with paths in the Sobolev space $H^1 = H^1([0, 1], \mathbb{R}^{2n})$. Recall that the paths in this space can be represented by continuous functions. Consider the space of paths that start on the zero section:

$$\mathcal{P} = \{\gamma \in H^1 \mid p(0) = 0\}.$$

Note that one can recover γ as

$$\gamma = (q(1) - \int_t^1 \dot{q}(s) ds, \int_0^t \dot{p}(s) ds),$$

so we will use the notation $\gamma = (q(1), \dot{\gamma})$. This shows that $\mathcal{P} \simeq \mathbb{R}^n \times L^2([0, 1], \mathbb{R}^{2n})$. Remark that we have a fibration $\pi : \mathcal{P} \rightarrow \mathbb{R}^n$ defined by $\pi(\gamma) = q(1)$. The vector space structure on $\mathcal{P}_{q(1)}$ is the one induced by the derivative $\dot{\gamma}$. We denote by $\tilde{S} : \mathcal{P} \rightarrow \mathbb{R}$ the action functional on this space:

$$\tilde{S}(\gamma) = \int_0^1 -\frac{1}{2} J\dot{\gamma}(t) \cdot \gamma(t) dt - \int_0^1 H_t(\gamma(t)) dt.$$

A straightforward calculation gives that the differential of \tilde{S} on $\gamma = (q(1), \dot{\gamma})$ applied to $\delta\gamma = (\delta q(1), \delta\dot{\gamma})$ is

$$\begin{aligned} d\tilde{S}(\gamma)(\delta\gamma) &= \int_0^1 -\frac{1}{2} J\dot{\gamma} \cdot \delta\gamma dt - \int_0^1 -\frac{1}{2} J\delta\dot{\gamma} \cdot \gamma dt - \int_0^1 \nabla H_t(\gamma) \cdot \delta\gamma dt \\ &= -\frac{1}{2} [J\dot{\gamma} \cdot \delta\gamma]_0^1 - \int_0^1 [J\dot{\gamma} + \nabla H] \cdot \delta\gamma dt \\ &= p(1)\delta q(1) - \int_0^1 [J\dot{\gamma} + \nabla H] \cdot \delta\gamma dt \end{aligned}$$

Suppose that $\gamma = (q(1), \dot{\gamma})$ is a critical value of \tilde{S} . Then the previous calculation yields

$$\begin{aligned} p(1) &= 0, \\ J\dot{\gamma} + \nabla H &= 0. \end{aligned}$$

so γ is a Hamiltonian trajectory that verifies $p(0) = 0 = p(1)$. In particular, we see that critical points are in one to one correspondence with points in $L_1 \cap L_0$. In order to study these intersection points one may then study the critical values. Instead of doing it directly on \tilde{S} , we are going to find a finite dimensional subspace $E \subseteq L^2$ and a function $F : \mathbb{R}^n \times E \rightarrow \mathbb{R}^n \times L^2$ such that $S = \tilde{S} \circ F$ has the same critical values as \tilde{S} .

Finite dimensional reduction Every element $z(t) \in L^2([0, 1], \mathbb{R}^{2n})$ can be represented by their Fourier series

$$z(t) = \sum_{k \in \mathbb{Z}} e^{2\pi ktJ} z_k, \quad z_k \in \mathbb{R}^{2n}$$

Denote de projection operators $P = P_N$ and $Q = Q_N$ by

$$Pz = \sum_{|k| \leq N} e^{2\pi ktJ} z_k,$$

$$Qz = \sum_{|k| > N} e^{2\pi ktJ} z_k.$$

If we denote by $\nabla_Q \tilde{S}$ the L^2 -gradient of \tilde{S} in the direction of the Q component of the fiber, we get

$$\nabla_Q \tilde{S}(\gamma) = -QJ\dot{\gamma} - Q\nabla H(t, \gamma).$$

Using the implicit function theorem (see for example [ABK⁺94]) one can prove the existence of a C^1 map $f : \mathbb{R}^n \times PL^2 \rightarrow QL^2$ which assigns to every $(q(1), \xi) \in \mathbb{R}^n \times PL^2$ the unique value $f(q(1), \xi) \in QL^2$ such that

$$\nabla_Q \tilde{S}(q(1), \xi + f(q(1), \xi)) = 0.$$

Lemma B.1. *Denote by F the function $F(q(1), \xi) = (q(1), \xi + f(q(1), \xi))$. Vertical critical points of $S = \tilde{S} \circ F$ are in one to one correspondence with vertical critical points of \tilde{S} . In particular, both functions have the same critical points and critical values.*

Proof. We need to show that

$$\nabla_{L^2} \tilde{S}(\gamma) = \nabla_P \tilde{S} + \nabla_Q \tilde{S} = 0 \quad \text{if and only if} \quad \nabla_P S(q, \xi) = 0 \quad \text{with} \quad \gamma = F(q, \xi)$$

By definition of f we have $\nabla_Q \tilde{S}(\gamma) = 0$ if and only if $\gamma = F(q(1), \xi)$. We conclude using this, the chain rule and the explicit expression of $\nabla_P F$. \square

The function S generates $\psi_1(L_0) = \psi_1(0_{\mathbb{R}^n})$ via its differential. More precisely:

Lemma B.2. *Let $S = \tilde{S} \circ F$ be the function previously constructed. Then*

$$\psi_1(L_0) = \left\{ (q, \frac{\partial S}{\partial q}(q, \xi)) \mid (q, \xi) \in \mathbb{R}^n \times \mathbb{R}^m \quad \text{such that} \quad \frac{\partial S}{\partial \xi}(q, \xi) = 0 \right\}$$

Proof. By Lemma B.1, the points satisfying $\nabla_P S(q, \xi) = 0$ are in one to one correspondence with Hamiltonian trajectories $\gamma = F(q, \xi)$ that start on the zero section which satisfy $q(1) = q$. Moreover, using the formula for $d\tilde{S}$ one may see that $\frac{\partial S}{\partial q}(q, \xi) = p(1)$. This shows that (q, ξ) is a critical point if and only if γ starts and ends in the zero section, and this proves the claim. \square

In order to study the critical points of this modified action, one can look at the critical points of this finite dimensional reduction. This way of generating Hamiltonian isotopies of the zero section gives a variational interpretation of the definition of generating functions (see Definition 3.4).

Appendix C

Critical value selectors

We give the basic properties of the critical value selectors that appeared in Chapters 3 and 6. For simplicity we will consider compact manifolds M . The same results apply for generating functions quadratic at infinity S since in that case the gradient ∇S is linear at infinity, so it induces a global flow. Let $f : M \rightarrow \mathbb{R}$ be a smooth function and consider the level sets

$$f^\lambda := \{x \in M \mid f(x) \leq \lambda\}.$$

Denote by $i_\lambda : f^\lambda \rightarrow M$ the natural inclusion. It induces two maps

$$(i_c)_* : H_*(f^\lambda) \rightarrow H_*(M) \quad \text{and} \quad i_c^* : H^*(M) \rightarrow H^*(f^\lambda).$$

Definition C.1. For every non-zero class $\alpha \in H_*(M)$ define

$$c(\alpha, f) = \inf\{\lambda \mid \alpha \in \text{Im}((i_\lambda)_*)\}.$$

For every non-zero class $\beta \in H^*(M)$ define

$$c(\omega, f) = \inf\{\lambda \mid i_\lambda^*(\omega) \neq 0\}.$$

An easy verification yields $\min f \leq c(\cdot, f) \leq \max f$. In the case of generating functions quadratic at infinity one needs to consider (co)homology classes for the pair $(S^{-\infty}, S^{+\infty})$. In the homological case $c(\alpha, f)$ has the simple equivalent minmax definition

$$c(\alpha, f) = \min_{[\gamma]=\alpha} \max f(\gamma)$$

where $[\gamma] = \alpha$ means that the simplex γ represents the class α .

Lemma C.2. *The values $c(\alpha, f)$ and $c(\omega, f)$ are critical values of f .*

Proof. Suppose that $c = c(\alpha, f)$ is a regular value. Then there is an $\epsilon > 0$ such that $I_\epsilon = [c-\epsilon, c+\epsilon]$ is an interval of regular values. Chose a metric g on M and consider the gradient ∇f . Find a compactly supported function ρ in I_ϵ which is equal to one on I_δ for a $0 < \delta < \epsilon$. Consider the vector field

$$X(x) = -\rho(f(x)) \frac{\nabla f(x)}{|\nabla f(x)|^2}.$$

Using the fact that ρ is non-zero only on regular values we see that X is well defined. Moreover it has bounded speed so it induces a global flow φ_t . We claim that $\varphi_{2\delta}(f^{c+\delta}) = f^{c-\delta}$. To see this take an $x \in f^{c+\delta}$, then

$$\frac{d}{dt} f(\varphi_t(x)) = d_{\varphi_t(x)} f(X(\varphi_t(x))) = -\rho(f(\varphi_t(x))) \leq 0.$$

This implies that for every $t \geq 0$ we have $f(\varphi_t(x)) \leq f(x)$ so $\varphi_t(f^{c-\delta}) \subseteq f^{c-\delta}$. Now if $x \in f^{c+\delta} - f^{c-\delta}$ then one of the following two things happen: either $\varphi_t(x) \in f^{c-\delta}$ for some $t \in [0, 2\delta[$, and in that case $\varphi_{2\delta} = \varphi_{2\delta-t} \circ \varphi_t$ implies that $\varphi_{2\delta}(x) \subseteq f^{c-\delta}$. Or $\varphi_t(x)$ stays in $f^{c+\delta} - f^{c-\delta}$ for $t \in [0, 2\delta[$, and in that case for those t

$$\frac{d}{dt}f(\varphi_t(x)) = -\rho(f(\varphi_t(x))) = -1,$$

so $f(\varphi_{2\delta}(x)) = f(x) - 2\delta \leq c - \delta$. This proves $\varphi_{2\delta}(f^{c+\delta}) \subseteq f^{c-\delta}$. The same argument for $\varphi_{-2\delta}$ shows that $\varphi_{-2\delta}(f^{c-\delta}) \subseteq f^{c+\delta}$ which gives the reverse inclusion.

We have the following commutative diagram

$$\begin{array}{ccc} f^{c+\delta} & \xrightarrow{i_{c+\delta}} & M \\ \downarrow \varphi_{2\delta} & & \downarrow \varphi_{2\delta} \\ f^{c-\delta} & \xrightarrow{i_{c-\delta}} & M \end{array}$$

Since $c(\alpha, f) = c$ we have $\alpha = (i_{c+\delta})^* \varphi_{2\delta}^*(\gamma)$ for some $\gamma \in H_*(f^{c+\delta})$. Using the commutative diagram we get

$$\alpha = (i_{c+\delta})^* \varphi_{2\delta}^*(\gamma) = (i_{c-\delta})^* \varphi_{2\delta}^*(\gamma)$$

so $\alpha \in \text{Im}(i_{c-\delta})^*$ which implies $c(\alpha, f) \leq c - \delta$, a contradiction. So we conclude that $c(\alpha, f)$ is a critical value. A similar argument using the commutative diagram also gives that $c(\omega, f)$ is a critical value. \square

Lemma C.3. *Let $f, g : M \rightarrow \mathbb{R}$ be two smooth functions and consider a nonzero cohomological class $\omega \in H^*(M)$. If $f \leq g$ then*

$$c(\omega, f) \leq c(\omega, g).$$

Moreover, for every $c \in \mathbb{R}$ we have $c(\omega, f + a) = c(\omega, f) + a$.

Proof. The hypothesis $f \leq g$ implies that $g^c \subseteq f^c$ so we have a commutative diagram

$$\begin{array}{ccc} g^c & \xrightarrow{i_c^g} & M \\ \downarrow i_f^g & & \downarrow \text{Id} \\ f^c & \xrightarrow{i_c^f} & M \end{array}$$

If $c(\omega, g) \leq c$ then $(i_c^g)^*(\omega) \neq 0$ and using that $i_c^g = i_c^f \circ i_f^g$ we get $(i_c^f)^*(\omega) \neq 0$. This implies $c(\omega, f) \leq c$ so we conclude that $c(f, \omega) \leq c(g, \omega)$. If now $g = f + a$ we have $g^{c+a} = f^c$. This gives $(i_{c+a}^g)^*(\omega) \neq 0$ if and only if $(i_c^f)^*(\omega) \neq 0$ so we get $c(\omega, g) = c(\omega, f) + a$. \square

Corollary C.4. *Let $f, g : M \rightarrow \mathbb{R}$ be two smooth functions and consider a nonzero cohomological class $\omega \in H^*(M)$. Then*

$$|c(\omega, g) - c(\omega, f)| \leq \|g - f\|_{C^0}.$$

Proof. Just remark that

$$g - \|g - f\|_{C^0} \leq f \leq g + \|g - f\|_{C^0}$$

and apply the previous lemma. \square

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Résumé

On étudie les propriétés de rigidité symplectique des difféomorphismes hamiltoniens en dimension finie et en dimension infinie. En dimension finie, les outils principaux qu'on utilise sont les fonctions génératrices et les capacités symplectiques. En dimension infinie on regarde les flots des équations en dérivées partielles (EDPs) hamiltoniennes et, en particulier, les flots qui peuvent être approchés uniformément par des flots hamiltoniens de dimension finie.

Dans la première partie de la thèse on étudie les sélecteurs d'action définies à partir des fonctions génératrices et on construit des invariants hamiltoniens pour les sous-ensembles de $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. Cela nous permet de démontrer un théorème non-squeezing coisotrope pour les difféomorphismes hamiltoniens à support compact de \mathbb{R}^{2n} . On montre à continuation que cette propriété apparaisse dans certains cas non compacts. Finalement, on explique comment ce résultat donne aussi l'information sur le problème de rigidité symplectique en dimension intermédiaire. Encore en dimension finie, on démontre qu'on peut utiliser le théorème du chameau symplectique pour produire des sous-ensembles invariants compacts dans des surfaces d'énergie.

Dans la deuxième partie on étudie les propriétés de rigidité symplectique des flots des EDPs hamiltoniennes. On se place dans le contexte introduit par Kuksin et on étudie une classe particulière de EDPs semi-linéaires qui peuvent être approchées par flots hamiltoniens de dimension finie. D'abord on donne une nouvelle construction de capacité symplectique en dimension infinie à partir des capacités de Viterbo. Puis on démontre l'analogie de la rigidité intermédiaire pour certaines EDPs hamiltoniennes. Cette classe inclue l'équation d'ondes en dimension 1 avec une nonlinéarité bornée, comme par exemple l'équation de Sine-Gordon. Dans la dernière partie de la thèse on s'intéresse à un analogue de la conjecture d'Arnold pour l'équation de Schrödinger périodique avec une non linéarité de convolution.

Mots Clés

Géométrie symplectique, fonctions génératrices, capacités symplectiques, EDPs hamiltoniennes.

Abstract

We study symplectic rigidity properties in both finite and infinite dimension. In finite dimension, the main tools that we use are generating functions and symplectic capacities. In infinite dimension we study flows of Hamiltonian partial differential equations (PDEs) and, in particular, flows which can be uniformly approximated by finite dimensional Hamiltonian diffeomorphisms.

In the first part of this thesis we study the action selectors defined from generating functions and we build Hamiltonian invariants for subsets of $\mathbb{R}^{2m} \times T^*\mathbb{T}^k$. This allows us to prove a coisotropic non-squeezing theorem for compactly supported Hamiltonian diffeomorphisms of \mathbb{R}^{2n} . We then extend this result to some non-compact settings. Finally we explain how this result can give information about the middle dimensional symplectic rigidity problem. Still in finite dimensions, we show that it is possible to use the symplectic camel theorem to create energy surfaces with compact invariant subsets.

In the second part of the thesis we study symplectic rigidity properties of flows of Hamiltonian PDEs. We work in the context introduced by Kuksin and study a particular class of semi-linear Hamiltonian PDEs that can be approximated by finite dimensional Hamiltonian diffeomorphisms. We first give a new construction of an infinite dimensional capacity using Viterbo's capacities. The main result of this part is the proof of the analogue of the middle dimensional rigidity for certain types of Hamiltonian PDEs. These include nonlinear string equations with bounded nonlinearity such as the Sine-Gordon equation. In the final part of this thesis we study an analogue of Arnold's conjecture for the periodic Schrödinger equations with a convolution nonlinearity.

Keywords

Symplectic geometry, generating functions, symplectic capacities, Hamiltonian PDEs.