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Xingyu Li

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**THÈSE DE DOCTORAT**  
**DE L'UNIVERSITÉ PSL**

Préparée à Université Paris Dauphine

**Equations cinétiques et de diffusion: comportement asymptotique dans le temps long et hypocoercivité**

Soutenue par

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## Remerciements

Eight years ago, I was just a newbie of University of Science and Technology of China (USTC) and had no idea about which field of mathematics I would work on in the future. After two years, the analysis caught my most interest; this way of using basic calculations to prove complicated conclusions was really amazing in my eyes. Today, I'm about to finish my PhD studies soon and start a new journey in my life. I really wish to thank several people in particular.

First, I must thank my family members like my aunts, uncles, cousins, grandfather and grandmother. They always supported me, no matter financially, mentally or emotionally. They are my strongest and most powerful backing, especially my mother. She has given a lot of herself to support me for my studies and living since I was a kid. She always respected and supported my big decisions, even when she didn't agree. When I could not stop preparing mathematical competitions in high school, she thought I may not be able to make my way studying mathematics and that I should rather apply for National College Entrance Examination; nevertheless, she tried her best to provide for me with hundreds of books and extra classes, just because this was what I truly wanted. For all that she has done for me, I don't even know where to begin to thank her. Maybe the best to do would rather be saying nothing at all and warmly hug her instead.

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and benefit from his precious knowledge and way with maths.

To conclude, I would like to introduce you with a Chinese proverb, which can be translated by: "All is beyond the words".

一切盡在不言中。

## Preamble

As the title of this thesis, it is about analyzing the hypocoercivity and asymptotic behaviour of evolution partial differential equations. The original models of these equations come from different fields, such as physics, biology, engineering. In this thesis, we use the methods that both from the existing papers and the new created ones to study the behaviour of these models.

The first chapter is introduction, which provides the background, main results and the mathematical tools of the following chapters. The main structure of the introduction can be divided by two parts: (Part 1: Hypocoercivity, Chapter 3, Chapter 7, part 2: Asymptotic behaviour: Chapter 4, Chapter 6) and Chapter 5 provides the theoretical support and useful tool for both two parts. The following chapters are all come from articles, and the list of the work goes as follows:

- Chapter 3: article [79], in collaboration with Jean DOLBEAULT<sup>1</sup>, published in *Mathematical Models and Methods in Applied Sciences.*, 13(2018), 2637-2666.
- Chapter 4: article [124], submitted for publication.
- Chapter 5: article [80], in collaboration with Jean DOLBEAULT, accepted by *International Mathematics Research Notices*.
- Chapter 6: article [123], preprint in HAL and arXiv.
- Chapter 7: article [3], in collaboration with Lanoir ADDALA<sup>2</sup>, Jean DOLBEAULT, Lazhar TAYEB<sup>3</sup>, submitted for publication.

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## **Chapter 1**

# **Résumé de la thèse en français**

Cette thèse porte sur deux aspects importants des équations aux dérivées partielles d'évolution : le comportement des solutions en temps grand et les estimations hypocoercives. L'hypocoercivité est un ensemble de techniques qui permet d'étudier le taux de convergence vers l'équilibre de solutions d'équations cinétiques. Un cas standard d'équation cinétique en mécanique classique consiste à décrire un gaz de particules par une fonction de distribution qui dépend de la position, de la vitesse et du temps, c'est-à-dire, à donner une description probabiliste du gaz sur l'espace de phase. Pour mesurer la distance à l'équilibre, il est usuel d'utiliser une fonctionnelle d'*entropie* ou d'*énergie libre*. La dérivée en temps est souvent appelée la *production d'entropie*, et lorsqu'elle contrôle linéairement l'entropie, on obtient alors une décroissance exponentielle de l'entropie. Dans ce cas, on dira qu'il y a coercivité. Toutefois, dans beaucoup de modèles physiques, la relaxation n'est produite que sur les variables de vitesse, et la production d'entropie ne contrôle pas linéairement l'entropie. Pour autant, il est parfois possible de construire une fonctionnelle d'entropie modifiée qui, elle, décroît exponentiellement : les méthodes d'hypocoercivité consistent à construire de telles fonctionnelles. C'est en particulier l'objet du chapitre qui est consacré au système de Vlasov-Poisson-Fokker-Planck linéarisé. On considère aussi des modèles diffusifs pour lesquels on établit une relation entre l'entropie linéarisée, et la production d'entropie linéarisée. Non seulement cela permet de démontrer une convergence exponentielle des solutions en temps grand, mais dans certains cas, cette approche permet aussi de caractériser les taux optimaux de retour exponentiel à l'équilibre. Deux modèles sont étudiés en détail : le modèle de Cucker-Smale homogène (dynamique collective du vol des oiseaux en biologie mathématique) et les équations de Nernst-Planck (évolution de systèmes de particules chargées).

## 1.1 Partie 1: hypocoercivité

Les *méthodes hypocoercives* utilisées en théorie des équations cinétiques peuvent être rangées en deux classes principales: les méthodes dites  $H^1$  reposent sur des calculs de la dérivée en temps d'une information de Fisher. Par rapport à l'information de Fisher physique qui porte typiquement seulement sur des dérivées en vitesse, l'information de Fisher qui permet d'obtenir de la coercivité contient des termes supplémentaires avec des dérivées par rapport à la variable de position, et aussi des termes qui mélangent variable de vitesse et variable de position: on parle d'ailleurs d'information de Fisher *twistée*.

### 1.1.1 Équation de Fokker-Planck cinétique

Le premier résultat porte sur une équation tout à fait classique et qui a été abondamment étudiée du point de vue de l'hypo-ellipticité: l'équation de Fokker-

Planck cinétique avec un potentiel harmonique,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - x \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v f). \quad (1.1)$$

Parce que (1.1) est linéaire, nous supposons que  $\|f\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$ . Il est bien connu que la solution converge en temps grand vers une fonction gaussienne en position et en vitesse, notée  $f_\star$ , qui a la forme

$$f_\star(x, v) = (2\pi)^{-d} e^{-\frac{1}{2}(|x|^2 + |v|^2)} \quad \forall (x, v) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Nous définissons  $d\mu := f_\star dx dv$  pour la mesure invariante sur l'espace des phases  $\mathbb{R}^d \times \mathbb{R}^d$ . La fonction  $g := f/f_\star$  résout

$$\frac{\partial g}{\partial t} + \mathbb{T}g = \mathbb{L}g \quad (1.2)$$

où l'opérateur de transport  $\mathbb{T}$  et l'opérateur de collision  $\mathbb{L}$  sont

$$\mathbb{T}g := v \cdot \nabla_x g - x \cdot \nabla_v g \quad \text{and} \quad \mathbb{L}g := \Delta_v g - v \cdot \nabla_v g.$$

De plus,  $\mathbb{T}$  et  $\mathbb{L}$  sont respectivement anti-auto-adjoint et auto-adjoint sur l'espace  $L^2(\mathbb{R}^d, d\mu)$ . Pour étudier la convergence vers la solution stationnaire  $f_\star$ , nous considérons l'entropie

$$\mathcal{E}[g] := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(g) d\mu \quad (1.3)$$

où

$$\varphi(w) := \frac{1}{p-1} (w^p - 1 - p(w-1)), \quad p \in (1, 2] \quad (1.4)$$

en particulier, quand  $p = 2$ ,

$$\varphi_2(w) = (w-1)^2$$

et pour  $p \rightarrow 1_+$ ,

$$\varphi_1(w) := w \log w - (w-1)$$

et les entropies correspondantes constituent une famille qui interpole entre une norme  $L^2$  à poids gaussien (cas  $p = 2$ ) et une entropie de Gibbs correspondant à la limite  $p \rightarrow 1$ . Nous savons de [116] que l'entropie (1.3) contrôle la convergence vers l'état stationnaire en utilisant l'inégalité de Csiszár-Kullback.

Le taux de décroissance optimal de  $\mathcal{E}[g]$  a été établi par A. Arnold et J. Erb dans [7], qui est le résultat suivant.

**Proposition 1.1.** *Pour tout la solution nonnegative  $g \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$  de (1.2) avec une donnée initiale  $g_0$ , tel que  $\mathcal{E}[g_0] < \infty$ , nous pouvons une constante  $\mathcal{C} > 0$  qui satisfait*

$$\mathcal{E}[g(t, \cdot, \cdot)] \leq \mathcal{C} e^{-t} \quad \forall t \geq 0. \quad (1.5)$$

De plus, le taux  $e^{-t}$  est optimal pour  $t \rightarrow +\infty$ .



Dans cette thèse, notre objectif est de donner une estimation plus forte. La fonction  $h := g^{p/2}$  résout

$$\frac{\partial h}{\partial t} + \mathbb{T}h = \mathbb{L}h + \frac{2-p}{p} \frac{|\nabla_v h|^2}{h}. \quad (1.6)$$

Le principal résultat consiste à donner le taux de convergence exponentiel de l'entropie généralisée

$$\frac{1}{p-1} \left[ \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^2 f_\star dx dv - \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^{2/p} f_\star dx dv \right)^{p/2} \right]$$

notre principal outil est l'information Fisher

$$\mathcal{J}_\lambda[h] = (1-\lambda) \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu + (1-\lambda) \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu + \lambda \int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2 d\mu. \quad (1.7)$$

Et notre résultat principal est le suivant.

**Theorem 1.1.** *Laissons  $h$  pour la solution de (1.6) avec donnée initiale  $h_0 \in L^1 \cap L^p(\mathbb{R}^d, d\gamma)$ ,  $h_0 \neq 1$ . Alors il existe une fonction  $\lambda : \mathbb{R}^+ \rightarrow [1/2, 1)$  tel que  $\lambda(0) = \lim_{t \rightarrow +\infty} \lambda(t) = 1/2$  et une fonction continue  $\rho$  sur  $\mathbb{R}^+$  tel que  $\rho > 1/2$  a.e., et nous avons*

$$\frac{d}{dt} \mathcal{J}_{\lambda(t)}[h(t, \cdot)] \leq -2\rho(t) \mathcal{J}_{\lambda(t)}[h(t, \cdot)] \quad \forall t \geq 0.$$

En conséquence, pour tout  $t \geq 0$ , nous avons l'estimation globale

$$\mathcal{J}_{\lambda(t)}[h(t, \cdot)] \leq \mathcal{J}_{1/2}[h_0] \exp\left(-2 \int_0^t \rho(s) ds\right).$$

Nous donnons plus d'explications sur ce théorème. Le point remarquable de cette approche est que, dans le calcul de la dérivée en temps des informations de Fisher – qui ressemble beaucoup à des calculs de *carré du champ* dans une approche à la Bakry-Emery – il est possible lorsque  $p$  est strictement compris entre 1 et 2 d'obtenir non seulement le taux optimal de convergence exponentielle, mais aussi d'exploiter l'un des termes de reste et d'obtenir une amélioration, marginale, des taux de convergence. Cette amélioration met en évidence un phénomène intéressant d'oscillation dans l'espace des phases, qui n'est pas présent dans les méthodes d'entropie appliquées, par exemple, à des diffusions non-dégénérées.

La structure principale de notre preuve provient de [161] de C.Villani. Nous calculons

$$\frac{d}{dt} \mathcal{J}_\lambda[h] = \frac{d}{dt} \left( \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu + \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu + 2\lambda \int_{\mathbb{R}^d} \nabla_x h \cdot \nabla_v h d\mu \right).$$

nous définissons les notations suivantes

$$\begin{aligned} H_{vv} &= \left( \frac{\partial^2 h}{\partial v_i \partial v_j} \right)_{1 \leq i, j \leq d}, & H_{xv} &= \left( \frac{\partial^2 h}{\partial x_i \partial v_j} \right)_{1 \leq i, j \leq d}, \\ M_{vv} &= \left( \frac{\partial \sqrt{h}}{\partial v_i} \frac{\partial \sqrt{h}}{\partial v_j} \right)_{1 \leq i, j \leq d}, & M_{xv} &= \left( \frac{\partial \sqrt{h}}{\partial x_i} \frac{\partial \sqrt{h}}{\partial v_j} \right)_{1 \leq i, j \leq d}. \end{aligned}$$

et à partir du calcul direct,

$$-\frac{1}{2} \frac{d}{dt} \mathcal{J}_{\lambda(t)}[h(t, \cdot)] = X^\perp \cdot \mathfrak{M}_1 X - \frac{1}{2} \lambda'(t) X^\perp \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X + Y^\perp \cdot \mathfrak{M}_2 Y$$

où

$$\mathfrak{M}_0 = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d}, \quad \mathfrak{M}_1 = \begin{pmatrix} 1 - \lambda & \frac{\lambda}{2} \\ \frac{\lambda}{2} & \lambda \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d}$$

et

$$\mathfrak{M}_2 = \begin{pmatrix} 1 & \lambda & -\frac{\kappa}{2} & -\frac{\kappa\lambda}{2} \\ \lambda & 1 & -\frac{\kappa\lambda}{2} & -\frac{\kappa}{2} \\ -\frac{\kappa}{2} & -\frac{\kappa\lambda}{2} & 2\kappa & 2\kappa\lambda \\ -\frac{\kappa\lambda}{2} & -\frac{\kappa}{2} & 2\kappa\lambda & 2\kappa \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d \times \mathbb{R}^d}$$

sont des fonctions valorisées par matrice de blocs  $(\lambda, \nu)$ , ici  $\kappa = 8(2-p)/p$ . En plus,

$$X = (\nabla_\nu h, \nabla_x h), \quad Y = (H_{\nu\nu}, H_{x\nu}, M_{\nu\nu}, M_{x\nu})$$

de la théorie de l'algèbre linéaire, nous savons que

$$Y^\perp \cdot \mathfrak{M}_2 Y \geq \lambda_1(p, \lambda) |Y|^2$$

pour quelque  $\lambda_1(p, \lambda)$  tel que

$$\lambda_1(p, 1/2) = \frac{1}{4} \left( 2\kappa + 1 - \sqrt{5\kappa^2 - 4\kappa + 1} \right) > 0$$

En particulier, nous choisissons  $\lambda = 1$  et nous pouvons prouver le résultat de la décroissance exponentielle avec le taux 1.

Pour prouver le meilleur résultat, nous remarquons que pour tout  $p \in (1, 2)$ , par continuité nous savons que  $\lambda_1(p, \lambda) > 0$  si  $\lambda$  est assez proche de  $\frac{1}{2}$ . Nous obtenons que

$$-\frac{1}{2} \frac{d}{dt} \mathcal{J}_{\lambda(t)}[h(t, \cdot)] \geq X^\perp \cdot \mathfrak{M}_1 X + \frac{1}{2} \lambda'(t) X^\perp \cdot \mathfrak{M}_0 X + \varepsilon X^\perp \cdot \mathfrak{M}_3 X$$

avec

$$\mathfrak{M}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d}$$

et

$$\varepsilon = \frac{\lambda_1(p, \lambda) \int_{\mathbb{R}^d} |\nabla_\nu h|^2 d\mu}{1 + (p-1) \mathcal{E}[h_0^{2/p}]}$$

Cela découle de l'estimation  $|Y|^2 \geq \|M_{\nu\nu}\|^2$  et de l'inégalité de Cauchy-Schwarz. En suite, nous pouvons prouver que si pour quelque  $t_0 > 0$ ,

$$\int_{\mathbb{R}^d} |\nabla_\nu h|^2 d\mu = 0, \quad \int_{\mathbb{R}^d} |\nabla_\nu h|^2 + \nabla_\nu h \cdot \nabla_x h + |\nabla_x h|^2 d\mu \neq 0,$$

alors

$$\int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu > 0 \quad \text{pour } t \rightarrow t_{0+}$$

cela signifie que  $e^t \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu$  est positif, sauf pour les valeurs isolées de  $t > 0$ . Notre objectif est de trouver  $\lambda(t)$  et  $\rho(t) > 1/2$ , telles que

$$X^\perp \cdot \mathfrak{M}_1 X - \frac{1}{2} \lambda'(t) X^\perp \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X + \varepsilon X^\perp \cdot \mathfrak{M}_3 X \geq \rho(t) X^\perp \cdot \mathfrak{M}_0 X$$

pour  $X \in \mathbb{R}^{2d}$ . Le détail de la preuve peut être vu dans la section 3.3 de chapitre 3.

**Remarque 1.1.** *Nous considérons*

$$f_0(x, v) = f_\star(x - x_0, v - v_0) \quad \forall (x, v) \in \mathbb{R}^d \times \mathbb{R}^d$$

pour quelque  $(x_0, v_0) \neq (0, 0)$ . Des calculs directs, nous obtenons que

$$f(t, x, v) = f_\star(x - x_\star(t), v - v_\star(t))$$

$$\text{avec } \begin{cases} x_\star(t) = \left( \cos\left(\frac{\sqrt{3}}{2} t\right) x_0 + \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t\right) \left(v_0 + \frac{x_0}{2}\right) \right) e^{-\frac{t}{2}}, \\ v_\star(t) = \left( -\frac{\sqrt{3}}{2} \sin\left(\frac{\sqrt{3}}{2} t\right) \left(x_0 + \frac{v_0}{2}\right) + \cos\left(\frac{\sqrt{3}}{2} t\right) v_0 \right) e^{-\frac{t}{2}}, \end{cases} \quad (1.8)$$

résout (1.1) et nous pouvons vérifier que le taux  $e^{-t}$  est optimale.

### 1.1.2 Système de Vlasov-Poisson-Fokker-Planck linéarisé

Le deuxième cas d'application des méthodes hypocoercives porte sur le système de Vlasov-Poisson-Fokker-Planck linéarisé. L'équation non linéaire de Vlasov-Poisson-Fokker-Planck avec un potentiel externe  $V$  est

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f - (\nabla_x V + \nabla_x \phi) \cdot \nabla_v f &= \Delta_v f + \nabla_v \cdot (v f), \\ -\Delta_x \phi &= \rho_f = \int_{\mathbb{R}^d} f dv. \end{aligned} \quad (1.9)$$

Il décrit la dynamique d'un plasma de particules de Coulomb dans un réservoir thermique, qui dégénère en une équation de Vlasov-Poisson à basse température (ici la température est prise égale à 1 ainsi qu'à d'autres constantes physiques). La solution stationnaire

$$f_\star(x, v) = \rho_\star(x) (2\pi)^{-d/2} e^{-|v|^2/2}$$

où la densité spatiale  $\rho_\star$  est obtenue en résolvant l'équation de Poisson-Boltzmann

$$-\Delta_x \phi_\star = \rho_\star = M \frac{e^{-V - \phi_\star}}{\int_{\mathbb{R}^d} e^{-V - \phi_\star} dx}.$$

Pour linéariser cette équation, nous définissons  $f = f_*(1 + \varepsilon h)$ , laissons  $\varepsilon \rightarrow 0$ , et jetons  $O(\varepsilon^2)$  terme. Et (1.9) devient

$$\begin{aligned} \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_*) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h &= 0, \\ -\Delta_x \psi_h &= \int_{\mathbb{R}^d} h f_* dv. \end{aligned} \quad (1.10)$$

L'espace des phases est  $\mathbb{R}^d \times \mathbb{R}^d \ni (x, v)$  et le potentiel  $V$  est un potentiel de confinement qui permet de prévenir le phénomène de *runaway* et garantit l'existence d'une solution de masse finie  $f_*$ . La méthode repose sur une méthode d'hypocoercivité  $L^2$  qui consiste à construire la fonctionnelle de Lyapunov

$$H_\delta[h] := \frac{1}{2} \|h\|^2 + \delta \langle Ah, h \rangle$$

où le produit scalaire est défini par

$$\langle h_1, h_2 \rangle := \iint_{\mathbb{R}^d \times \mathbb{R}^d} h_1 h_2 d\mu + \int_{\mathbb{R}^d} \rho_{h_1} (-\Delta)^{-1} \rho_{h_2} dx \quad (1.11)$$

avec  $\rho_h = \int_{\mathbb{R}^d} h f_* dv$ , où  $d\mu = f_* dv dx$ . Les opérateurs de transport et de diffusion sont donnés respectivement par

$$\mathbb{T}h := v \cdot \nabla_x h - \nabla_x (V + \phi_*) \cdot \nabla_v h + v \cdot \nabla_x \psi_h \quad \text{et} \quad \mathbb{L}h := \Delta_v h - v \cdot \nabla_v h, \quad (1.12)$$

$\Pi$  est la projection orthogonale sur le noyau de  $\mathbb{L}$ , et l'opérateur  $A$  est défini par

$$A := (\text{Id} + (\mathbb{T}\Pi)^* \mathbb{T}\Pi)^{-1} (\mathbb{T}\Pi)^*.$$

**Théorème 1.1.** *Supposons que  $d \geq 1$ ,  $V(x) = |x|^\alpha$  pour un paramètre  $\alpha > 1$  et  $M > 0$ . Il existe deux constantes positives  $\mathcal{C}$  et  $\lambda$ , telles que toute solution  $h$  du système de Vlasov-Poisson-Fokker-Planck linéarisé, de moyenne nulle avec  $\|h_0\|^2 < \infty$ , vérifie*

$$\|h(t, \cdot, \cdot)\|^2 \leq \mathcal{C} \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0. \quad (1.13)$$

Un autre point est d'obtenir des estimations uniformes dans la *limite de diffusion*. Pour tout  $\varepsilon > 0$ , nous considérons la solution de l'équation Vlasov-Poisson-Fokker-Planck dans l'échelle parabolique donnée par

$$\varepsilon \partial_t f + v \cdot \nabla_x f - (\nabla_x V + \nabla_x \phi) \cdot \nabla_v f = \frac{1}{\varepsilon} (\Delta_v f + \nabla_v \cdot (v f)), \quad -\Delta_x \phi = \rho_f = \int_{\mathbb{R}^d} f dv. \quad (1.14)$$

Il a été prouvé que lorsque  $d = 2$  ou  $3$ , (1.14) a une solution faible  $(f^\varepsilon, \phi^\varepsilon)$  qui converge comme  $\varepsilon \rightarrow 0_+$  vers  $f^0$ , où la densité de charge  $\rho = \int_{\mathbb{R}^d} f^0 dv$  est une solution faible du système *drift-diffusion-Poisson*

$$\frac{\partial \rho}{\partial t} = \nabla_x \cdot (\nabla_x \rho + \rho \nabla_x (V + \phi)), \quad -\Delta_x \phi = \rho. \quad (1.15)$$

Une source d'inspiration pour la méthode est le comportement asymptotique des solutions de (1.15) pendant une longue période. Pour  $t \rightarrow +\infty$ , il est bien connu que  $(\rho, \phi)$  converge vers un état stable  $(\rho_\star, \phi_\star)$  donné par la équation de Poisson-Boltzmann

$$-\Delta_x \phi_\star = \rho_\star = e^{-V - \phi_\star} \quad (1.16)$$

à un taux exponentiel. Maintenant, nous linéarisons (1.14) pour obtenir l'équation

$$\begin{aligned} \varepsilon \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \frac{1}{\varepsilon} (\Delta_v h - v \cdot \nabla_v h) &= 0, \\ -\Delta_x \psi_h = \int_{\mathbb{R}^d} h f_\star dv, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star dx dv &= 0. \end{aligned} \quad (1.17)$$

On obtient une estimation de la décroissance qui est uniforme par rapport à  $\varepsilon \rightarrow 0_+$ .

**Théorème 1.2.** *Nous supposons que  $d \geq 1$ ,  $V(x) = |x|^\alpha$  pour quelque  $\alpha > 1$  et  $M > 0$ . Pour tout  $\varepsilon > 0$  assez petit, il existe deux constantes  $\mathcal{C}$  et  $\lambda$ , indépendantes de  $\varepsilon$ , tels que pour tout la solution  $h$  de (1.17) avec une donnée initiale  $h_0$  tel que  $\|h_0\|^2 < \infty$  satisfait (1.13).*

La preuve du Théorème s'appuie sur un certain nombre d'observations préliminaires:

- En raison du potentiel de confinement, l'intégration par parties peut être effectuée.
- Il existe une solution unique de l'équation de Poisson-Boltzmann.
- La mesure de probabilité construite au-dessus de la solution de l'équation de Poisson-Boltzmann satisfait une inégalité de Poincaré.
- Le produit scalaire est bien défini.

La méthode, qui a été introduite dans [82, 83], repose sur la limite de diffusion et un autre résultat montre que les estimations de décroissance sont en effet uniformes par rapport à un paramètre de changement d'échelle correspondant à la limite parabolique. Le point clef de la preuve consiste à introduire la norme  $\|h\|^2 = \langle h, h \rangle$  qui fait intervenir le terme non-local du à l'équation de Poisson. Nous considérerons un espace de fonctions de distribution tel que

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star dx dv = 0$$

et utiliser le produit scalaire (1.11) qui est adapté au couplage de Poisson. Nous pouvons vérifier que cet espace  $\mathcal{H}$  est Hilbert.

Avec ces préliminaires, il est possible de s'appuyer sur une méthode abstraite d'hypocoercivité qui va comme suit. Pour l'équation

$$\partial_t f + \mathbb{T}f = \mathbb{L}f \quad (1.18)$$

supposons qu'il existe des constantes  $\lambda_m$ ,  $\lambda_M$  et  $C_M > 0$ , telles que pour tout  $F \in \mathcal{H}$ , les propriétés suivantes sont valables:

▷ *coercivité microscopique*

$$-\langle LF, F \rangle \geq \lambda_m \|(\text{Id} - \Pi)F\|^2,$$

▷ *coercivité macroscopique*

$$\|\text{T}\Pi F\|^2 \geq \lambda_M \|\Pi F\|^2,$$

▷ *dynamique macroscopique parabolique*

$$\Pi\text{T}\Pi F = 0,$$

▷ *opérateurs auxiliaires bornés*

$$\|\text{A}\text{T}(\text{Id} - \Pi)F\| + \|\text{A}LF\| \leq C_M \|(\text{Id} - \Pi)F\|.$$

Avec ces propriétés, nous pouvons prouver la proposition suivante.

**Proposition 1.2.** *Il existe une constante  $\lambda > 0$  dépendant de  $\lambda_m$ ,  $\lambda_M$  et  $C_M$ , tels que  $\lambda H_\delta[F] \leq D_\delta[F]$  pour tout  $F \in \mathcal{H}$ . Par conséquent, pour une solution de (1.18), nous avons*

$$H_\delta[F(t, \cdot)] \leq H_\delta[F_0] e^{-\lambda t}.$$

Revenons au système Vlasov-Poisson-Fokker-Planck linéarisé (1.10). nous rappelons que les opérateurs de transport et de diffusion sont définis par (1.12). Ensuite, nous devons vérifier que les hypothèses ci-dessus sont satisfaites. Les trois premiers sont faciles à vérifier. La dernière hypothèse est plus compliquée à prouver. (les détails sont donnés dans chapitre 7:

- Faites la reformulation de l'inégalité comme une estimation de régularité elliptique.
- Obtenons une estimation du type  $H^1$ .
- Prouver les inégalités de Poincaré pondérées et les estimations de type  $H^1$ .
- Obtenons des estimations pour la deuxième période de commande.

**Remarque 1.2.** *L'hypocoercivité de l'équation (1.9) est toujours un problème ouvert. Pour l'équation*

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f - (\nabla_x V + \varepsilon \nabla_x \phi) \cdot \nabla_v f &= \Delta_v f + \nabla_v \cdot (v f), \\ -\Delta_x \phi &= \rho_f = \int_{\mathbb{R}^d} f dv. \end{aligned} \tag{1.19}$$

où  $\varepsilon$  est la charge totale du système. EHérau et L.Thomann ont prouvé en [109] que si  $\varepsilon > 0$  est assez petit, alors nous avons le résultat d'hypocoercivité similaire. En fait, en utilisant aussi la même méthode que l'équation cinétique de Fokker-Planck. Mais pour le cas plus général  $\varepsilon = 1$ , les difficultés proviennent de l'estimation du terme de Poisson.

## 1.2 Partie 2: comportement asymptotique de grand temps

### 1.2.1 Modèle de Cucker-Smale homogène

La deuxième partie de la thèse porte sur des modèles diffusifs avec termes de champ moyen. Les comportements collectifs émergents et l'auto-organisation dans les interactions multi-agents sont des sujets intéressants dans de nombreux domaines. Dans les systèmes biologiques, peu importe les cellules, les insectes ou les mammifères, tous les individus peuvent s'auto-organiser et se déplacer de manière cohérente. Bien sûr, des conditions spéciales doivent être remplies, sinon la dynamique aléatoire prédomine. Il est donc important de créer un modèle mathématique qui décrit les commutations entre les systèmes désorganisés et les systèmes présentant une phase ordonnée. Le *modèle de Cucker-Smale homogène*, dit encore modèle de *McKean-Vlasov*, est utilisé pour décrire la distribution des vitesses d'un groupe d'oiseaux. Ce modèle s'écrit

$$\frac{\partial f}{\partial t} = D \Delta_v f + \nabla_v \cdot (\nabla_v \phi(v) f + (v - \mathbf{u}_f) f) \quad (1.20)$$

où la vitesse moyenne vers laquelle tendent les oiseaux est donnée par

$$\mathbf{u}_f(t) := \frac{\int_{\mathbb{R}^d} v f(t, v) dv}{\int_{\mathbb{R}^d} f(t, v) dv}.$$

Le potentiel  $\phi(v) = \frac{\alpha}{4} |v|^4 - \frac{\alpha}{2} |v|^2$  quant à lui modélise le fait que les vitesses très grandes ou les vitesses nulles sont peu favorables, mais qu'aucune direction n'est *a priori* privilégiée. La tendance des oiseaux à aligner leurs vitesses est contrecarrée par les erreurs qu'ils commettent, ce qui est modélisé par un bruit dont l'intensité est donnée en fonction de  $D$ . Ainsi,  $D$  petit correspond à un régime de petit bruit dans lequel l'alignement des vitesses est possible, alors que pour  $D$  grand, tout alignement est dominé par le bruit et la seule solution stationnaire est isotrope. Dans [13] en particulier, les auteurs ont étudié le modèle dans les régimes asymptotiques  $D \rightarrow 0$  et  $D \rightarrow +\infty$ . Le but de l'étude est de compléter cette description en toute dimension et de démontrer l'existence d'une transition de phase entre un régime ordonné avec une vitesse moyenne non nulle pour  $D < D_*$  et un régime homogène pour  $D \geq D_*$ . Un premier résultat de classification s'énonce comme suit.

**Théorème 1.3.** *Supposons que  $d \geq 1$  et  $\alpha > 0$ . Alors il existe une valeur critique  $D_* > 0$  telle que*

- (i)  $D > D_*$ : *il existe une seule solution stationnaire, stable, avec  $\mathbf{u}_f = \mathbf{0}$ .*
- (ii)  $D < D_*$ : *il existe une solution stationnaire instable isotrope avec  $\mathbf{u}_f = \mathbf{0}$  et un ensemble continu de solutions stables positives ou nulles, non-symétriques, polarisées (c'est-à-dire de vitesse moyenne non-nulle). De plus, les solutions polarisées sont toutes identiques à une rotation près.*

Toute solution stationnaire a la forme

$$f_{\mathbf{u}}(v) = \frac{e^{-\frac{1}{D}\left(\frac{1}{2}|v-\mathbf{u}|^2 + \frac{\alpha}{4}|v|^4 - \frac{\alpha}{2}|v|^2\right)}}{\int_{\mathbb{R}^d} e^{-\frac{1}{D}\left(\frac{1}{2}|v-\mathbf{u}|^2 + \frac{\alpha}{4}|v|^4 - \frac{\alpha}{2}|v|^2\right)} dv}$$

où  $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{R}^d$  résout  $\int_{\mathbb{R}^d} (\mathbf{u} - v) f_{\mathbf{u}}(v) dv = 0$ . Après un rotation,  $\mathbf{u} = (u, 0, \dots, 0) = u e_1$  est donnée par  $\mathcal{H}(u) = 0$ , où

$$\mathcal{H}(u) := \int_{\mathbb{R}^d} (v_1 - u) e^{-\frac{1}{D}(\phi_\alpha(v) - u v_1)} dv \quad \text{et} \quad \phi_\alpha(v) := \frac{\alpha}{4}|v|^4 + \frac{1-\alpha}{2}|v|^2$$

donc pour montre que de théorème, on devrais étudié le zéros de fonction  $\mathcal{H}(u)$ . Parce que  $\mathcal{H}(0) = 0$  et si  $\mathcal{H}(u) = 0$ , alors  $\mathcal{H}(-u) = 0$ , donc nous considérons toujours le cas  $u \geq 0$ . D'abord, nous pouvons montrer que il existe un zéro unique  $D_*$  de  $\mathcal{H}'(0)$ , et  $\mathcal{H}'(0) > 0$  quand  $D < D_*$ ,  $\mathcal{H}'(0) < 0$  quand  $D > D_*$ .

Pour le cas  $d = 1$ , pour tout  $u > 0$ ,  $\mathcal{H}''(u) < 0$  si  $\mathcal{H}(u) \leq 0$ . Par conséquent,  $\mathcal{H}$  change de signe au plus une fois sur  $(0, +\infty)$ . Le cas  $d = 2$  est plus difficile. Nous établissons les propriétés suivantes:

- quand  $D \geq D_*$ , alors  $\mathcal{H}'(u) \leq 0$  pour tout  $u \geq 0$ , donc 0 est le solution unique de  $\mathcal{H}(u) = 0$ .
- quand  $D < D_*$ , nous savons que  $\mathcal{H}'(0) > 0$ , donc il existe un zéro  $u_1$  of  $\mathcal{H}(u)$ . De plus, nous prouverons que  $\mathcal{H}(u)$  est décroissant strictement sur  $(u_1, \infty)$ , donc  $u_1$  est le zéro unique de  $\mathcal{H}(u)$ .

Cette transition de phase correspond à un exemple remarquable de brisure de symétrie, au sens où *l'énergie libre*

$$\mathcal{F}[f] := D \int_{\mathbb{R}^d} f \log f dv + \int_{\mathbb{R}^d} f \phi dv - \frac{1}{2} |\mathbf{u}_f|^2$$

atteint son minimum pour les solutions polarisées si  $D < D_*$ . L'étude ne se limite pas à l'analyse des solutions stationnaires et de leur stabilité. En effet, pour une solution du problème d'évolution, il est remarquable que l'énergie libre est en fait une fonction de Lyapunov qui décroît suivant la relation

$$\frac{d}{dt} \mathcal{F}[f(t, \cdot)] = -\mathcal{I}[f(t, \cdot)] \quad \text{avec} \quad \mathcal{I}[f] := \int_{\mathbb{R}^d} \left| D \frac{\nabla_v f}{f} + \nabla_v \phi - \mathbf{u}_f \right|^2 f dv.$$

On montre par exemple le résultat suivant dans le régime de bruit élevé.

**Théorème 1.4.** *Pour tout  $d \geq 1$  et pour tout  $\alpha > 0$ , si  $D > D_*$ , alors toute solution  $f$  avec donnée initiale positive ou nulle  $f_{\text{in}}$  de masse 1 telle que  $\mathcal{F}[f_{\text{in}}] < \infty$  vérifie, pour une certaine constante  $C > 0$ , l'estimation*

$$0 \leq \mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_0] \leq C e^{-\mathcal{C}_D t} \quad \forall t \geq 0$$



Pour obtenir des taux de convergence, tout l'enjeu revient à comparer l'information de Fisher  $\mathcal{I}$  avec  $\mathcal{F}$ . Dans l'espace des fonctions

$$\mathcal{A} : \left\{ g \in L^2(f_0 dv), \int_{\mathbb{R}^d} g f_0 dv = 0 \right\}$$

nous introduisons les formes quadratiques

$$Q_{1,\mathbf{u}}[g] := \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^2} \mathcal{F}[f_{\mathbf{u}}(1 + \varepsilon g)] = D \int_{\mathbb{R}^d} g^2 f_{\mathbf{u}} dv - D^2 |\mathbf{v}_g|^2,$$

où  $\mathbf{v}_g := \frac{1}{D} \int_{\mathbb{R}^d} v g f_{\mathbf{u}} dv$ , et

$$Q_{2,\mathbf{u}}[g] := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \mathcal{I}[f_{\mathbf{u}}(1 + \varepsilon g)] = D^2 \int_{\mathbb{R}^d} |\nabla g - \mathbf{v}_g|^2 f_{\mathbf{u}} dv.$$

Nous montrerons le résultat de stabilité de la solution stationnaire.

- $Q_{1,\mathbf{0}} \geq 0 \iff D \geq D_*$ . Si  $D > D_*$ , alors

$$Q_{1,\mathbf{0}}[g] \geq \eta(D) \int_{\mathbb{R}^d} g^2 f_0 dv \quad (1.21)$$

pour quelque  $\eta(D) > 0$ .

- Pour  $D < D_*$ ,  $|\mathbf{u}| = u(D) \neq 0$ ,  $Q_{1,\mathbf{u}}[g] \geq 0$ .

Ensuite, pour coercivité, on se rappelle l'inégalité de Poincaré: il existe une constante optimale  $\Lambda_D > 0$ , tel que pour tout  $h \in H^1(\mathbb{R}^d, f_{\mathbf{u}} dv)$  qui satisfait  $\int_{\mathbb{R}^d} h f_{\mathbf{u}} dv = 0$ , nous avons

$$\int_{\mathbb{R}^d} |\nabla h|^2 f_{\mathbf{u}} dv \geq \Lambda_D \int_{\mathbb{R}^d} |h|^2 f_{\mathbf{u}} dv. \quad (1.22)$$

ici  $\mathbf{u}$  est une vitesse admissible, tel que  $\mathbf{u} = \mathbf{0}$  si  $D \geq D_*$ , ou  $|\mathbf{u}| = u(D)$  si  $D < D_*$ .

$$\mathbf{u}[f] = \mathbf{0} \text{ if } D \geq D_* \text{ or } \mathbf{u}_f = \mathbf{0} \text{ and } D < D_*,$$

$$\mathbf{u}[f] = \frac{u(D)}{|\mathbf{u}_f|} \mathbf{u}_f \text{ if } D < D_* \text{ and } \mathbf{u}_f \neq \mathbf{0}.$$

**Proposition 1.3.** *Nous supposons que  $d \geq 1$ ,  $\alpha > 0$ ,  $D > 0$  et  $\mathcal{C}_D = D \Lambda_D$ . Considérons une fonction de distribution non négative  $f \in L^1(\mathbb{R}^d)$  avec  $\int_{\mathbb{R}^d} f dv = 1$ , et  $\mathbf{u} \in \mathbb{R}^d$  satisfait soit  $\mathbf{u} = \mathbf{0}$  soit  $|\mathbf{u}| = u(D)$  si  $D < D_*$  et considérons  $g = (f - f_{\mathbf{u}})/f_{\mathbf{u}}$ . Nous supposons que  $g \in H^1(\mathbb{R}^d, f_{\mathbf{u}} dv)$ . Si  $\mathbf{u} = \mathbf{0}$ , alors*

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D Q_{1,\mathbf{u}}[g].$$

*Autrement, si  $\mathbf{u} \neq \mathbf{0}$  pour quelque  $D \in (0, D_*)$ , alors il existe  $0 < \kappa(D) < 1$ , tel que*

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D (1 - \kappa(D)) \frac{(\mathbf{v}_g \cdot \mathbf{u})^2}{|\mathbf{v}_g|^2 |\mathbf{u}|^2} Q_{1,\mathbf{u}}[g]$$

*avec  $\mathbf{v}_g := \frac{1}{D} \int_{\mathbb{R}^d} (v - \mathbf{u}) g f_{\mathbf{u}} dv$ .*

Nous allons maintenant étudier le comportement asymptotique à long temps de la solution de (1.20) pour le cas  $D > D_*$ . On fait d'abord la linéarisation de (1.20). On écrit

$$f = f_0(1 + g), \quad \mathbf{v}_g = \frac{1}{D} \int_{\mathbb{R}^d} v g f_0 dv$$

alors (1.20) devient

$$\frac{\partial g}{\partial t} = \mathcal{L} g - \mathbf{v}_g \cdot \left( D \nabla g - (v + \nabla \phi_\alpha) g \right), \quad (1.23)$$

ici

$$\mathcal{L} g := D \Delta g - (v + \nabla \phi_\alpha) \cdot (\nabla g - \mathbf{v}_g)$$

est l'opérateur linéarisé. En suite, sur l'espace

$$\mathcal{X} := \left\{ g \in L^2(f_0 dv) : \int_{\mathbb{R}^d} g f_0 dv = 0 \right\}$$

Nous définissons naturellement le produit scalaire

$$\langle g_1, g_2 \rangle := D \int_{\mathbb{R}^d} g_1 g_2 f_0 dv - D^2 \mathbf{v}_{g_1} \cdot \mathbf{v}_{g_2}$$

et il est équivalent à la norme standard  $L^2(f_0 dv)$ . De plus,

$$Q_{1,0}[g] = \langle g, g \rangle, \quad Q_{2,0}[g] = -\langle g, \mathcal{L} g \rangle.$$

Pour prouver le théorème de comportement asymptotique à long temps, nous rappelons que  $\mathcal{C}_D$  est la constante optimale dans l'inégalité

$$Q_{2,0}[g] \geq \mathcal{C}_D Q_{1,0}[g].$$

Pour l'équation linéarisée

$$\frac{\partial g}{\partial t} = \mathcal{L} g \quad (1.24)$$

avec donnée initiale  $g_0 \in \mathcal{X}$ ,

$$\frac{1}{2} \frac{d}{dt} Q_{1,0}[g] = \frac{1}{2} \frac{d}{dt} \langle g, g \rangle = \langle g, \mathcal{L} g \rangle = -Q_{2,0}[g]$$

et il a une décroissance exponentielle. Donc

$$\langle g(t, \cdot), g(t, \cdot) \rangle \leq \langle g_0, g_0 \rangle e^{-2\mathcal{C}_D t} \quad \forall t \geq 0.$$

Nous réécrivons (1.23) comme

$$f_0 \frac{\partial g}{\partial t} = D \nabla \cdot \left( (\nabla g - \mathbf{v}_g) f_0 \right) - D \mathbf{v}_g \cdot \nabla (g f_0)$$

nous trouvons que

$$\frac{1}{2} \frac{d}{dt} Q_{1,0}[g] + Q_{2,0}[g] = D^2 \mathbf{v}_g \cdot \int_{\mathbb{R}^d} g (\nabla g - \mathbf{v}_g) f_0 dv.$$

nous utilisons  $\mathbf{u}_f = D\mathbf{v}_g$ , inégalité de Cauchy-Schwartz et (1.21), et nous obtenons

$$\left( \int_{\mathbb{R}^d} g (\nabla g - \mathbf{v}_g) f_0 \, dv \right)^2 \leq \int_{\mathbb{R}^d} g^2 f_0 \, dv \int_{\mathbb{R}^d} |\nabla g - \mathbf{v}_g|^2 f_0 \, dv \leq \frac{Q_{1,0}[g]}{\eta(D)} \frac{Q_{2,0}[g]}{D^2}.$$

dans Proposition 1.3, nous avons

$$\frac{d}{dt} Q_{1,0}[g] \leq -2 \left( 1 - |\mathbf{u}_f(t)| \sqrt{\frac{\mathcal{C}_D}{\eta(D)}} \right) Q_{1,0}[g].$$

nous pouvons que  $\lim_{t \rightarrow +\infty} |\mathbf{u}_f(t)| = 0$ , donc

$$\limsup_{t \rightarrow +\infty} e^{2\mathcal{C}_D t} Q_{1,0}[g(t, \cdot)] < +\infty \quad (1.25)$$

de inégalité de Gronwall. De plus, on observe

$$f \log(f/f_0) - (f - f_0) \leq \frac{1}{2} (f - f_0)^2 / f_0$$

donc finalement le théorème est prouvé.

Le cas  $D \leq D_*$  est plus compliqué mais il est néanmoins possible d'obtenir des résultats analogues en temps grand pour la décroissance de l'entropie et, dans des cas simples ( $d = 1$ , ou  $d \geq 2$  et données initiales possédant une symétrie) avec conditions *ad hoc* sur l'entropie, d'établir la convergence vers une solution stationnaire polarisée. Dans le cas général, on monte aussi la décroissance exponentielle de  $\mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_{\mathbf{u}}]$  si  $\mathcal{F}[f_{\text{in}}] < \mathcal{F}[f_0]$ , ce qui démontre la convergence vers l'ensemble des solutions stationnaires polarisées.

**Proposition 1.4.** *Nous supposons que  $d \geq 2$ ,  $\alpha > 0$  et  $D \in (0, D_*)$ . Nous considérons une solution  $f$  de (1.20) avec donnée initiale non négative  $f_{\text{in}}$  de masse 1, tel que  $\mathcal{F}[f_{\text{in}}] < \mathcal{F}[f_0]$  et supposons que  $\mathbf{u} = \lim_{t \rightarrow +\infty} \mathbf{u}_f(t)$  est uniquement défini. Si*

$$|(\mathbf{u}_f - \mathbf{u}) \cdot \mathbf{u}| \geq \varepsilon u(D) |\mathbf{u}_f - \mathbf{u}|$$

pour quelques  $\varepsilon > 0$  et  $t > 0$  assez grand, alors il existe deux constantes  $C$ ,  $\lambda$  et quelque  $\mathbf{u} \in \mathbb{R}^d$ , tel que

$$0 \leq \mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_{\mathbf{u}}] \leq C e^{-\lambda t} \quad \forall t \geq 0.$$

Il y a encore des problèmes ouverts à ce sujet. Le premier problème est, avons-nous plus d'informations sur la valeur seuil  $D_*$  et  $u(D)$ ? Nous avons déjà prouvé que

$$\lim_{\alpha \rightarrow 0} D_*(\alpha, d) = \frac{1}{d+2}, \quad \lim_{\alpha \rightarrow \infty} D_*(\alpha, d) = \frac{1}{d}.$$

quand  $\alpha \rightarrow \infty$ , existe-t-il une constante  $\eta > 0$ , telle que

$$\lim_{\alpha \rightarrow \infty} \alpha^\eta \left( D_* - \frac{1}{d} \right)$$

existe? En plus, nous savons dans le section 4.2.6 et [13] que

$$\lim_{D \rightarrow 0} u(D) = 1, \quad \lim_{D \rightarrow D_*} \frac{u(D)^2}{D_* - D} = \alpha \frac{(1 - \alpha)(1 - dD_*) - 2D_*}{1 - (d + 2)D_*}.$$

Pouvons-nous donner une description plus délicate du comportement de  $u(D)$  sur  $(0, D_*)$ ?

Le deuxième problème concerne le cas polarisé  $0 < D < D_*$ . Nous ne pouvons pas dire exactement que quand la convergence vers la solution stationnaire isotrope ou la solution stationnaire polarisée apparaît. Si la solution converge vers la solution stationnaire isotrope  $f_0$ , nous pouvons déduire de l'inégalité de Log-Sobolev que

$$\|f(t, \cdot) - f_0\|_{L^1(\mathbb{R}^d)} \rightarrow 0 \quad \text{quand } t \rightarrow \infty$$

Mais maintenant  $f_0$  n'est pas linéairement stable, nous ne pouvons donc pas utiliser la méthode ci-dessus pour étudier le comportement asymptotique à long temps. Nous supposons qu'en utilisant la méthode  $\phi$ -entropie pour prouver le résultat suivant: pour chaque  $0 < D < D_*$ , il existe  $1 < p < 2$ ,  $\mathcal{C}_*$ ,  $\lambda > 0$ , tel que

$$\int_{\mathbb{R}^d} \varphi_p \left( \frac{f}{f_0} \right) f_0 dv \leq \mathcal{C}_* e^{-\lambda t}.$$

## 1.2.2 Le système de Nernst-Planck

Le système de Nernst-Planck est d'abord étudié par W.Nernst et M.Planck en électromagnétisme. Et ils ont utilisé ce modèle pour décrire la diffusion de particules chargées dans un soluté sous l'influence d'un potentiel, en présence de forces électrostatiques. En dimension  $d = 2$ , le modèle d'origine est un système sans confinement avec couplage Poisson a la forme

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla v) \\ v = -\frac{1}{2\pi} \log|x| * u \\ u(0, x) = n_0 \geq 0 \end{cases} \quad x \in \mathbb{R}^2, t > 0. \quad (1.26)$$

Et dans cette thèse, nous considérons le système de Poisson-Nernst-Planck avec confinement s'écrit

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n + \nabla \cdot (n \nabla c) + \nabla \cdot (n \nabla \phi) \\ -\Delta c = n \\ n(0, x) = n_0 \geq 0, \quad \int_{\mathbb{R}^d} n(0, x) dx = M > 0 \end{cases} \quad x \in \mathbb{R}^d, t > 0. \quad (1.27)$$

Il décrit un modèle de particules chargées soumises à la fois à une diffusion et à une force de dérive donnée au travers d'une équation de Poisson par un terme non-linéaire de champ moyen. Un tel système, dans lequel les particules se repoussent par interaction électrostatique, a tendance à tendre vers zéro

ponctuellement, et c'est pourquoi on introduit un potentiel extérieur de confinement  $\phi$ . De plus, pour le cas particulier  $d = 2, \phi = \frac{\mu}{2}|x|^2$ , nous pouvons relier ces deux systèmes en changeant les variables

$$u(t, x) = R^{-d} n(\tau, \xi), v(t, x) = c(\tau, \xi) \text{ où } \xi = \frac{x}{R}, \tau = \log R, R = (1 + 2\mu t)^{\frac{1}{2}}$$

les solutions stationnaires de (1.27) sont les formes

$$-\Delta c_\infty = n_\infty = M \frac{e^{-c_\infty - \phi}}{\int_{\mathbb{R}^d} e^{-c_\infty - \phi} dx}.$$

Comme dans le cas du modèle de Cucker-Smale homogène, nous définissons l'énergie libre et les informations de Fisher

$$\mathcal{F}[n] := \int_{\mathbb{R}^d} n \log n dx + \int_{\mathbb{R}^d} n \phi dx + \frac{1}{2} \int_{\mathbb{R}^d} n c dx$$

$$\mathcal{I}[n] := -\frac{d}{dt} \mathcal{F}[n] = \int_{\mathbb{R}^d} n |\nabla(\log n + c + \phi)|^2 dx.$$

Notre but est de prouver le résultat de convergence de la solution de (1.27) vers les solutions stationnaires  $(n_\infty, c_\infty)$ . Nous élaborons d'abord notre théorème principal, qui donne le taux optimal du poids exponentiel en termes de norme  $L^2$  pondérée.

**Théorème 1.5.** *Nous supposons que  $d = 2$  et le potentiel  $\phi = \frac{\mu}{2}|x|^2$ . nous supposons aussi que  $n$  résout (1.27) avec donnée initiale  $n(0, x) = n_0 \geq 0, \int_{\mathbb{R}^d} n_0 dx = M$ , tel que  $\mathcal{F}[n_0] < \infty$ . Alors il existe une constante positive  $C$ , tel que pour tout les temps  $t > 0$ ,*

$$\int_{\mathbb{R}^d} \frac{|n(t, \cdot) - n_\infty|^2}{n_\infty} dx \leq C e^{-2\mu t}.$$

Nous prouverons également que le taux exponentiel  $2\mu$  est optimal. En fait, nous pouvons prouver le résultat similaire pour le potentiel général  $\phi$  aussi. Comme dans le cas du modèle de Cucker-Smale homogène, c'est une linéarisation appropriée autour de la solution stationnaire qui décide du taux de convergence. Un point particulièrement intéressant vient du fait que, avec le produit scalaire correspondant à l'analogue de la forme quadratique  $Q_{1,0}$ , l'opérateur d'évolution linéarisé est la forme polaire de la forme quadratique  $Q_{2,0}$ . Le problème est en fait plus simple que pour le modèle de flocking, car il est possible de caractériser la solution stationnaire comme minimum de l'énergie libre, qui est une fonctionnelle strictement convexe (et bornée inférieurement si  $\phi$  est confinant). On montre alors que, sous des hypothèses de croissance sur  $\phi$ , il y a convergence avec un taux exponentiel, mais nous ne pouvons pas garantir que le taux est optimal. Pour des résultats précis, nous renvoyons au chapitre dédié. .

D'abord, nous étudions l'énergie libre  $\mathcal{F}[n]$ . Nous pouvons prouver sur le ensemble

$$\mathcal{X} := \left\{ f \in L^1_+(\mathbb{R}^d) : \int_{\mathbb{R}^d} f(x) dx = M, f \log f \in L^1(\mathbb{R}^d), f \phi \in L^1(\mathbb{R}^d) \right\},$$

$\mathcal{F}$  est semi-borné par le bas et il a un minimiseur unique  $n_\infty$  sur  $\mathcal{X}$ . En suite, nous affirmons que si  $n$  résout (1.27), alors pour tout  $p \in [1, \infty)$ ,  $q \in [2, \infty)$ ,

$$\lim_{t \rightarrow \infty} \|n(t, \cdot) - n_\infty\|_{L^p(\mathbb{R}^d)} = 0, \quad \lim_{t \rightarrow \infty} \|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^q(\mathbb{R}^d)} = 0.$$

Pour prouver cette affirmation, nous devons d'abord montrer que  $\|n(t, \cdot)\|_{L^p(\mathbb{R}^d)}$ ,  $\|\nabla c(t, \cdot)\|_{L^q(\mathbb{R}^d)}$  sont uniformément borné à partir de calculs directs. Pour prouver la convergence vers la solution stationnaire, le cas  $d = 3$  est simple, et le cas  $d = 2$  est plus difficile. Nous utiliserons le lemme de Auber-Lions.

L'étape suivante, comme dans le cas du modèle de Cucker-Smale homogène, nous étudions les perturbations autour des solutions stationnaires. Nous définissons les formes quadratiques, qui sont respectivement la linéarisation de l'énergie libre et l'information de Fisher:

$$Q_1[f] := \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^2} \mathcal{F}[n_\infty(1 + \varepsilon f)] = \int_{\mathbb{R}^d} f^2 n_\infty dx + \int_{\mathbb{R}^d} |\nabla(g c_\infty)|^2 dx$$

$$Q_2[f] := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \mathcal{I}[n_\infty(1 + \varepsilon f)] = \int_{\mathbb{R}^d} |\nabla(f + g c_\infty)|^2 n_\infty dx$$

D'abord, si  $n_\infty$  satisfait l'inégalité de Poincaré: il existe  $C_\star > 0$ , tel que

$$\text{pour tout } u \in H^1(\mathbb{R}^d), \int_{\mathbb{R}^d} u n_\infty dx = 0, \int_{\mathbb{R}^d} |\nabla_x u|^2 n_\infty dx \geq C_\star \int_{\mathbb{R}^d} u^2 n_\infty dx \quad (1.28)$$

En particulier, pour  $d = 2$ ,  $\phi = \frac{\mu}{2}|x|^2$  ( $\mu > 0$ ), la constante de coercivité optimale entre  $Q_1[f]$  et  $Q_2[f]$  est juste  $\mu$ . La preuve de cette conclusion peut être divisée en trois parties:

- Étape 1: pour les fonctions radiales, le problème est devenu un problème EDO. Nous parvenons à trouver une valeur propre non triviale  $\lambda = 2\mu > 0$  et les fonctions propres correspondantes.
- Étape 2: utiliser la théorie de Sturm-Liouville pour montrer que la valeur propre à l'étape 1 est la meilleure.
- Étape 3: pour des fonctions non radiales plus générales, nous faisons la décomposition des harmoniques sphériques, et la constante de coercivité optimale est atteinte aux fonctions dans l'une des composantes non radiales  $k = 1$  avec la valeur propre  $\mu$ .

Maintenant, nous revenons à l'équation (1.27). Pour prouver le théorème sur le comportement asymptotique à grande temps, nous devons linéariser l'équation. Nous écrivons  $n = n_\infty(1 + f)$ ,  $c = c_\infty(1 + g)$ , et (1.27) devient

$$\frac{\partial f}{\partial t} - \mathcal{L}f = \frac{1}{n_\infty} \nabla(f n_\infty \nabla(g c_\infty)), \quad -\Delta(g c_\infty) = f n_\infty$$

ici

$$\mathcal{L}f := \frac{1}{n_\infty} \nabla [n_\infty \nabla (f + gc_\infty)]$$

est l'opérateur linéarisé. Nous définissons  $G_d$  comme la fonction de Green du Laplacien dans  $\mathbb{R}^d$ . Sur l'ensemble

$$\mathcal{A} := \left\{ f \in L^2(\mathbb{R}^d, n_\infty dx) : \int_{\mathbb{R}^d} f n_\infty dx = 0 \right\}$$

nous pouvons définir le produit scalaire

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^d} f_1 f_2 n_\infty dx + \int_{\mathbb{R}^d} n_\infty f_1 (G_d * (f_2 n_\infty)) dx$$

de plus, nous avons

$$Q_1[f] = \langle f, f \rangle, \quad Q_2[f] = -\langle f, \mathcal{L}f \rangle.$$

$$\frac{1}{2} \frac{d}{dt} Q_1[f] = \left\langle \frac{\partial f}{\partial t}, f \right\rangle = -Q_2[f] - \int_{\mathbb{R}^d} \nabla(f + gc_\infty) f n_\infty \nabla(gc_\infty) dx$$

Nous supposons que  $C_\star$  est la constante de coercivité optimale entre  $Q_1$  et  $Q_2$  définie dans (1.28). Nous rappelons que  $C_\star = \mu$  quand  $d = 2, \phi = \frac{\mu}{2}|x|^2 (\mu > 0)$ . Donc

$$\frac{1}{2} \frac{d}{dt} Q_1[f] \leq -C_\star \left( 1 - \frac{\|\nabla(gc_\infty)\|_{L^\infty(\mathbb{R}^d)}}{\sqrt{\mathcal{E}_\star}} \right) Q_1[f].$$

En faisant l'estimation de  $\|\nabla(gc_\infty)\|_{L^\infty(\mathbb{R}^d)}$ , le théorème est finalement prouvé par lemme de Gronwall. En fait, Le problème est en fait plus simple que pour le modèle de flocking, car il est possible de caractériser la solution stationnaire comme minimum de l'énergie libre, qui est une fonctionnelle strictement convexe.

Nous avons plus de résultats pour le cas  $d = 2, \phi = \frac{\mu}{2}|x|^2 (\mu > 0)$ . D'abord, nous pouvons prouver que  $\|n(t, \cdot) - n_\infty\|_{L^\infty(\mathbb{R}^d)} \rightarrow 0$  pour  $t \rightarrow \infty$ . La preuve est basée sur le noyau associé à l'équation de Fokker-Planck et à la formule de Duhamel. De plus, nous combinons avec le théorème du comportement asymptotique à grand temps, nous avons un meilleur résultat, qui est

$$\|n(t, \cdot) - n_\infty\|_{L^\infty(\mathbb{R}^d)} = O(e^{-\lambda t}) \quad \text{pour tout } 0 < \lambda < \mu.$$

Nous revenons au système de Nernst-Planck sans confinement (1.26). Du résultat ci-dessus, nous obtenons un résultat sur l'asymptotique intermédiaire pour les solutions de (1.26) en l'absence de tout potentiel externe de confinement.

**Remarque 1.3.** *Un problème plus général concerne le comportement asymptotique à long temps des solutions aux systèmes de dérive-diffusion*

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla(\nabla u + u\nabla\phi + u\nabla\phi_0), & u(0, \cdot) = u_0(x) \geq 0 \\ \frac{\partial v}{\partial t} = \nabla(\nabla v - v\nabla\phi + v\nabla\phi_0), & v(0, \cdot) = v_0(x) \geq 0 \\ -\Delta\phi = u - v \end{cases} \quad x \in \mathbb{R}^d, t > 0, \quad (1.29)$$

ici la dimension  $d$  est encore 2 ou 3. De plus,

$$\int_{\mathbb{R}^d} u_0 dx = M_u, \int_{\mathbb{R}^d} v_0 dx = M_v$$

nous pouvons toujours prouver le résultat similaire en utilisant la méthode ci-dessus. Mais attention, maintenant le résultat dépend vraiment des masses  $M_u$  et  $M_v$ . D'autres travaux sur ce sujet sont en cours.

### 1.3 Partie 3: inégalité de Hardy-Littlewood-Sobolev logarithmique généralisée

En fait, la véritable difficulté pour le système de Poisson-Nernst-Planck consiste à déterminer les conditions sous lesquelles l'énergie libre est bornée inférieurement. C'est d'ailleurs aussi l'une des difficultés dans l'étude du système de Vlasov-Poisson-Fokker-Planck linéarisé. Si cela ne pose pas de réel problème particulier en dimension  $d \geq 3$ , en revanche le problème n'est pas complètement simple en dimension  $d = 2$ . Lorsque l'on résout une équation de Poisson de la forme  $-\Delta c = n$ , le potentiel  $c$  est donné à une constante additive près au moyen d'une convolution qui prend la forme  $c(x) = (-\Delta)^{-1} n = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| n(y) dy$  et il n'est pas évident de décider si l'énergie libre

$$\mathcal{F}[n] = \int_{\mathbb{R}^2} n \log n dx + \int_{\mathbb{R}^2} n \phi dx + \frac{1}{2} \int_{\mathbb{R}^2} n (-\Delta)^{-1} n dx$$

est bornée inférieurement, étant donné que le noyau de convolution n'a pas de signe défini. C'est l'objet du dernier chapitre de cette thèse de donner une réponse précise sur la croissance nécessaire du potentiel extérieur pour garantir une borne inférieure. L'estimation principale consiste à établir l'*inégalité de Hardy-Littlewood-Sobolev logarithmique généralisée suivante*.

**Théorème 1.6.** *Considérons le potentiel extérieur  $V(x) = 2 \log(1 + |x|^2) + \log \pi$ . Pour tout  $\alpha \geq 0$ , on a*

$$\begin{aligned} \int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f dx + M(1-\alpha)(1 + \log \pi) \\ \geq \frac{2}{M} (\alpha - 1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log|x-y| dx dy \end{aligned}$$

pour toute fonction  $f \in L^1_+(\mathbb{R}^2)$  de masse  $M = \int_{\mathbb{R}^2} f dx > 0$ . De plus, le cas d'égalité est atteint par  $f_\star = M e^{-V}$  et  $f_\star$  est l'unique fonction optimale pour tout  $\alpha > 0$ .

Dans le cas limite  $\alpha = 0$ , l'inégalité n'est autre que l'inégalité de Hardy-Littlewood-Sobolev logarithmique établie par Carlen et Loss. Pour  $\alpha \in (0, 1]$ , le résultat se déduit assez simplement du cas  $\alpha = 0$ , en utilisant une simple inégalité de Jensen. Par contre, pour  $\alpha > 1$ , le coefficient du terme de convolution change



de signe est c'est donc une inégalité d'une autre nature qui apparait. Il se trouve que Carlen, Carrillo et Loss ont donné dans [51] une preuve du cas  $\alpha = 0$  basée sur l'utilisation d'un flot non-linéaire. Cette approche peut être généralisée dans le cas d'un potentiel extérieur et en particulier dans le cas de  $V$  avec  $\alpha \neq 0$ , au prix d'un flot de diffusion non-linéaire avec un terme de dérive. En pratique cela revient à considérer la différence des deux termes de l'inégalité et à montrer qu'elle est monotone sous l'action de l'équation d'évolution

$$\frac{\partial f}{\partial t} = \Delta \sqrt{f} + 2\sqrt{\pi} \nabla \cdot (x f).$$

Plus clairement, nous considérons l'énergie gratuite

$$\begin{aligned} \mathcal{F}[f] := & \int_{\mathbb{R}^2} f \log f \, dx + \alpha \int_{\mathbb{R}^2} V f \, dx + (1 - \alpha) (1 + \log \pi) \\ & + 2(1 - \alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x - y| \, dx \, dy \end{aligned}$$

À partir des calculs directs et de l'inégalité de Gagliardo-Nirenberg,  $t \mapsto \mathcal{F}[f(t, \cdot)]$  est monotone noncroissant pour tout  $t \geq 0$ . Le résultat de Carlen, Carrillo et Loss fait principalement appel à l'homogénéité de l'équation et de l'inégalité, ainsi qu'à la forme des profils asymptotiques. Ici, c'est une méthode d'entropie nouvelle, dans un cas inhomogène, basée sur un flot non-linéaire, qui permet d'établir l'inégalité fonctionnelle du théorème et de caractériser le cas d'égalité.

Ensuite, nous nous concentrons sur l'application du théorème. Considérons le modèle de dérive-diffusion-Poisson

$$\frac{\partial f}{\partial t} = \Delta f + \beta \nabla \cdot (f \nabla V) + \nabla \cdot (f \nabla \phi) \quad \text{où} \quad -\varepsilon \Delta \phi = f$$

Quand  $\varepsilon = -1$ , c'est le cas attractant. Et quand  $\varepsilon = +1$ , c'est le cas répulsif (modèle de Keller-Segel). Compte tenu de *méthodes d'entropie*, considérons la *énergie libre fonctionnelle*

$$\mathcal{F}_\beta[f] := \int_{\mathbb{R}^2} f \log f \, dx + \beta \int_{\mathbb{R}^2} V f \, dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi f \, dx.$$

Nous avons le corollaire suivant.

**Corollaire 1.1.** *Supposons que  $M > 0$ . Le fonctionnel  $\mathcal{F}_\beta$  est borné par le bas et admet un minimiseur sur l'ensemble des fonctions*

$$\mathcal{B} := \left\{ f \in L^1_+(\mathbb{R}^2), \quad \text{tel que} \quad \int_{\mathbb{R}^2} f \, dx = M \right\}$$

*si soit  $\varepsilon = +1$  et  $\beta \geq 1 + \frac{M}{8\pi}$ , soit  $\varepsilon = -1$ ,  $\beta \geq 1 - \frac{M}{8\pi}$  et  $M \leq 8\pi$ . Si  $\varepsilon = +1$ , le minimiseur est unique.*





## **Chapter 2**

### **Introduction**

Partial differential equations (PDEs) can describe the models from many fields such as physics, biology, ecology and economic studies. In this thesis, the models we consider are all depending on the time  $t$ , which let the mathematical models become *evolution PDEs*. We mainly study two aspects of the behaviour of these models: the *hypocoercivity* and the *large time asymptotic behaviour*.

We first give the basic explanations about the meanings of *hypocoercivity* and *asymptotic behaviour*.

- What is hypocoercivity? It is a typical issue of *kinetic models*. The word "kinetic" comes from physics and refers in the language of classical mechanics to a model that depends on the *position variable*  $x$  and the *velocity variable*  $v$ . One of the important topics concerning these models is about the convergence to a stationary state when the time is large enough and the rate of convergence. In the language of mathematical PDE theory, it is to study the qualitative estimate of the rate of the solutions towards a *global equilibrium*, or so-called *stationary solution*. A useful tool is the *entropy*, or the so-called *free energy*, which is a nonnegative functional which is decaying with respect to the time  $t$ . Our goal is to study the exponential rate of the convergence of the free energy of the solutions towards the free energy of the stationary solution. If the *entropy dissipation* is coercive with respect to the entropy functional, then we can directly have the coercivity result from Grönwall's inequality. However, in *kinetic transport models*, this method cannot be directly implemented, because the decay of the entropy functional only controls the convergence towards a subspace. The key tool to solve this problem is to consider the *modified entropy functional*, which is equivalent to the entropy functional, such that the decay of the modified entropy functional is exponential. This strategy is called *hypocoercivity*, see [161, 106, 133, 83] for more details. In this thesis, Chapter 3 and Chapter 7 focus on this topic.

- What is large time asymptotic behaviour? In this thesis, what we care about is the behaviour of the solutions predicted by the original model after large time ( $t \rightarrow \infty$ ). Is there an exponential rate of convergence of the solutions towards stationary solutions? If yes, can we find the *optimal* rate? Can we compute the exact value of the optimal rate in some special cases? These questions will be answered in Chapter 4 and Chapter 6 for the homogeneous Cucker-Smale model of flocking and for the Nernst-Planck equations for charged particles. These models rely on non-linear equations in both cases. Our strategy of proof is to do the linearization and show that the exponential rate of convergence to stationary states is given by the spectral gap of the linearized operator around the stationary solutions. The main tools are the *free energy* and the *relative Fisher information*. We consider the corresponding quadratic forms around the stationary solutions and prove a coercivity result between the quadratic forms. Moreover, we will show in Chapter 4 that for the flocking model, there exists a threshold

value of the noise parameter which drives a phase transition and we will classify all stationary solutions.

The mathematical method for studying the models depends on the dimension  $d$  of the space. Usually we can divide into three cases:  $d = 1$ ,  $d = 2$  and  $d \geq 3$ . This is because the corresponding tools and equations, such as the Poisson equation or the Sobolev inequalities, differ according to these three cases. The 1-dimensional case is not the main target of this thesis, and we will discuss it only in Chapter 4 and Chapter 7. The case  $d = 2$  is often more complicated, because the Poisson kernel involves a logarithm, and the Onofri or the logarithmic Hardy-Littlewood-Sobolev (log HLS) inequalities apply instead of normal Sobolev or Hardy-Littlewood-Sobolev (HLS) inequalities. Chapter 5 is the theoretical preparation for  $d = 2$ , which will be used to prove the results of Chapter 6 and Chapter 7.

This introduction goes as follows. Part 2.1 is devoted to hypocoercivity. We introduce the kinetic Fokker-Planck equation and the linearized Vlasov-Poisson-Fokker-Planck equation, give the main results and main ideas of the proofs. Part 2.2 is concerned with the large time asymptotic behaviour of the Cucker-Smale model of flocking and of the Nernst-Planck equation. Moreover, we describe a phase transition in the flocking model. In part 2.3, we introduce a generalized logarithmic Hardy-Littlewood-Sobolev inequality and give an important application: the free energy is bounded from below, and admits a unique minimizer under rather general conditions.

## 2.1 Part 1: hypocoercivity

In this thesis, we mainly focus on the *kinetic transport models*. The ordinary mathematical model to describe them can be written as

$$\partial_t f + \mathbb{T}f = \mathbb{L}f \quad (2.1)$$

where the distribution function  $f(t, x, v)$  is defined on the phase space  $x, v \in \mathbb{R}^d$ ,  $t \in [0, \infty)$  is the time, and  $f$  belongs to a Hilbert space  $\mathcal{H}$  with the corresponding scalar product and associated norm. The operator  $\mathbb{T}$  is called *transport operator* and the operator  $\mathbb{L}$  is called *collision operator* and they are respectively antisymmetric and symmetric under the scalar product. Here we focus on two equations: the *kinetic Fokker-Planck equation* in an external harmonic potential,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - x \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v f), \quad (2.2)$$

that we use as a benchmark for testing optimal estimate, and the *linearized Vlasov-Poisson-Fokker-Planck equation*

$$\begin{aligned} \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h &= 0, \\ -\Delta_x \psi_h &= \int_{\mathbb{R}^d} h f_\star d\nu, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star dx d\nu = 0. \end{aligned} \quad (2.3)$$

### 2.1.1 Kinetic Fokker-Planck equation

The  $\varphi$ -entropy of a nonnegative function  $w \in L^1(\mathbb{R}^d, d\gamma)$  is usually defined as

$$\mathcal{E}[w] := \int_{\mathbb{R}^d} \varphi(w) d\gamma,$$

where  $d\gamma$  denotes a nonnegative, usually bounded measure and  $\varphi$  is a nonnegative convex continuous function on  $\mathbb{R}^+$  such that

$$\varphi(1) = 0, \quad \varphi'' \geq 0, \quad \varphi \geq \varphi(1) = 0 \quad \text{and} \quad (1/\varphi'')'' \leq 0. \quad (2.4)$$

A classic example is

$$\varphi_p(w) := \frac{1}{p-1} (w^p - 1 - p(w-1)) \quad p \in (1, 2] \quad (2.5)$$

when  $p = 2$ ,  $\varphi_2(w) = (w-1)^2$  and the limit as  $p \rightarrow 1_+$  is the Gibbs entropy

$$\varphi_1(w) := w \log w - (w-1).$$

Now we consider the extension of the notion of  $\varphi$ -entropy to the kinetic Fokker-Planck equation.

Because (2.2) is linear, we can suppose that  $\|f\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$ . The stationary solution is

$$f_\star(x, v) = (2\pi)^{-d} e^{-\frac{1}{2}(|x|^2 + |v|^2)} \quad \forall (x, v) \in \mathbb{R}^d \times \mathbb{R}^d.$$

The function  $g := f/f_\star$  solves

$$\frac{\partial g}{\partial t} + \mathbb{T}g = \mathbb{L}g \quad (2.6)$$

where the transport operator  $\mathbb{T}$  and the collision operator  $\mathbb{L}$  are

$$\mathbb{T}g := v \cdot \nabla_x g - x \cdot \nabla_v g \quad \text{and} \quad \mathbb{L}g := \Delta_v g - v \cdot \nabla_v g.$$

Let  $d\mu := f_\star dx d\nu$  be the invariant measure on the phase space  $\mathbb{R}^d \times \mathbb{R}^d$ , so that  $\mathbb{T}$  and  $\mathbb{L}$  are respectively anti-self-adjoint and self-adjoint on the space  $L^2(\mathbb{R}^d, d\mu)$ .

For the equation (2.6), we study the convergence of  $g(t, \cdot)$  towards the stationary state 1. We need a lemma, and its proof can be found in [116].

**Lemma 2.1.** *Suppose that  $f, g \in L^1(\mathbb{R}^d)$  satisfy  $f \geq 0, g > 0$ , and  $\int_{\mathbb{R}^d} f(x) dx = \int_{\mathbb{R}^d} g(x) dx = 1$ . Then for  $\varphi_p$  defined in (2.5), we have*

$$\left( \int_{\mathbb{R}^d} |f - g| dx \right)^2 \leq 2 \int_{\mathbb{R}^d} \varphi_p \left( \frac{f}{g} \right) g dx$$

From the lemma above, if we can prove that the  $\varphi$ -potential

$$\mathcal{E}[g] := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(g) d\mu$$

converges to 0 for large time  $t$ , then the  $L^1$ -convergence of  $g$  towards 1 for large time  $t$  is also concluded. Moreover, for (2.2), the convergence of  $f$  towards the stationary state  $f_\star$  can be naturally deduced.

First, the function  $h := g^{p/2}$  solves

$$\frac{\partial h}{\partial t} + \mathbb{T}h = \mathbb{L}h + \frac{2-p}{p} \frac{|\nabla_v h|^2}{h}. \quad (2.7)$$

With this notation,  $\mathcal{E}[g] = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(f/f_\star) d\mu$  so that, with  $f = g f_\star = h^{2/p} f_\star$  we have

$$\mathcal{E}[g] = \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^2 \log \left( \frac{h^2}{\iint_{\mathbb{R}^d \times \mathbb{R}^d} h^2 d\mu} \right) d\mu \quad \text{if } \varphi = \varphi_1,$$

$$\mathcal{E}[g] = \mathcal{E}[h^{2/p}] = \frac{1}{p-1} \left[ \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^2 d\mu - \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^{2/p} d\mu \right)^{p/2} \right] \\ \text{if } \varphi = \varphi_p, p \in (1, 2].$$

The optimal rate of decay of  $\mathcal{E}[g]$  has been established by A. Arnold and J. Erb in [7], which is the following result.

**Proposition 2.1.** *Take  $\varphi = \varphi_p$  for some  $p \in [1, 2]$ . To any nonnegative solution  $g \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$  of (2.6) with an initial datum  $g_0$  such that  $\mathcal{E}[g_0] < \infty$ , we can associate a constant  $\mathcal{C} > 0$  for which*

$$\mathcal{E}[g(t, \cdot, \cdot)] \leq \mathcal{C} e^{-t} \quad \forall t \geq 0. \quad (2.8)$$

Moreover the rate  $e^{-t}$  is sharp as  $t \rightarrow +\infty$ .

In this thesis, our main tool is the *Fisher information functional*

$$\mathcal{I}_\lambda[h] = (1-\lambda) \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu + (1-\lambda) \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu + \lambda \int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2 d\mu. \quad (2.9)$$

We can prove that even if this result of Proposition 2.1 is optimal, it is possible to obtain a slightly stronger result.



**Theorem 2.1.** *Let  $p \in (1, 2)$  and  $h$  be a solution of (2.7) with initial datum  $h_0 \in L^1 \cap L^p(\mathbb{R}^d, d\gamma)$ ,  $h_0 \neq 1$ , and  $d\gamma$  be the Gaussian probability measure corresponding to the harmonic potential  $\psi(x) = |x|^2/2$ . Then there exists a function  $\lambda : \mathbb{R}^+ \rightarrow [1/2, 1)$  such that  $\lambda(0) = \lim_{t \rightarrow +\infty} \lambda(t) = 1/2$  and a continuous function  $\rho$  on  $\mathbb{R}^+$  such that  $\rho > 1/2$  a.e., for which we have*

$$\frac{d}{dt} \mathcal{I}_{\lambda(t)}[h(t, \cdot)] \leq -2\rho(t) \mathcal{I}_{\lambda(t)}[h(t, \cdot)] \quad \forall t \geq 0.$$

As a consequence, for any  $t \geq 0$  we have the global estimate

$$\mathcal{I}_{\lambda(t)}[h(t, \cdot)] \leq \mathcal{I}_{1/2}[h_0] \exp\left(-2 \int_0^t \rho(s) ds\right).$$

The strategy is based on a *carré du champ* computation as in Villani's approach of the  $H^1$ -hypo-coercivity, where the improvement comes from the remainder terms when  $p \in (1, 2)$ . We prove that for some  $\lambda \in (0, 1)$  that depends on time  $t$ , the rate of decay is faster than  $e^{-t}$  up to a zero-measure set in  $t$ . Define the notations

$$\begin{aligned} H_{vv} &= \left( \frac{\partial^2 h}{\partial v_i \partial v_j} \right)_{1 \leq i, j \leq d}, & H_{xv} &= \left( \frac{\partial^2 h}{\partial x_i \partial v_j} \right)_{1 \leq i, j \leq d}, \\ M_{vv} &= \left( \frac{\partial \sqrt{h}}{\partial v_i} \frac{\partial \sqrt{h}}{\partial v_j} \right)_{1 \leq i, j \leq d}, & M_{xv} &= \left( \frac{\partial \sqrt{h}}{\partial x_i} \frac{\partial \sqrt{h}}{\partial v_j} \right)_{1 \leq i, j \leq d}. \end{aligned}$$

From direct computation,

$$-\frac{1}{2} \frac{d}{dt} \mathcal{I}_{\lambda(t)}[h(t, \cdot)] = X^\perp \cdot \mathfrak{M}_1 X - \frac{1}{2} \lambda'(t) X^\perp \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X + Y^\perp \cdot \mathfrak{M}_2 Y$$

where

$$\mathfrak{M}_0 = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d}, \quad \mathfrak{M}_1 = \begin{pmatrix} 1 - \lambda & \frac{\lambda}{2} \\ \frac{\lambda}{2} & \lambda \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d}$$

and

$$\mathfrak{M}_2 = \begin{pmatrix} 1 & \lambda & -\frac{\kappa}{2} & -\frac{\kappa\lambda}{2} \\ \lambda & 1 & -\frac{\kappa\lambda}{2} & -\frac{\kappa}{2} \\ -\frac{\kappa}{2} & -\frac{\kappa\lambda}{2} & 2\kappa & 2\kappa\lambda \\ -\frac{\kappa\lambda}{2} & -\frac{\kappa}{2} & 2\kappa\lambda & 2\kappa \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d \times \mathbb{R}^d}$$

are bloc-matrix valued functions of  $(\lambda, \nu)$ , here  $\kappa = 8(2-p)/p$ . And

$$X = (\nabla_v h, \nabla_x h), \quad Y = (H_{vv}, H_{xv}, M_{vv}, M_{xv})$$

from linear algebra theory, we learn that

$$Y^\perp \cdot \mathfrak{M}_2 Y \geq \lambda_1(p, \lambda) |Y|^2$$

for some  $\lambda_1(p, \lambda)$  such that

$$\lambda_1(p, 1/2) = \frac{1}{4} \left( 2\kappa + 1 - \sqrt{5\kappa^2 - 4\kappa + 1} \right) > 0$$

if  $p \in (1, 2)$ , For any  $p \in (1, 2)$ , by continuity we know that  $\lambda_1(p, \lambda) > 0$  if  $\lambda$  is close enough to  $\frac{1}{2}$ . We obtain that

$$-\frac{1}{2} \frac{d}{dt} \mathcal{J}_{\lambda(t)}[h(t, \cdot)] \geq X^\perp \cdot \mathfrak{M}_1 X + \frac{1}{2} \lambda'(t) X^\perp \cdot \mathfrak{M}_0 X + \varepsilon X^\perp \cdot \mathfrak{M}_3 X$$

with

$$\mathfrak{M}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d}$$

and

$$\varepsilon = \frac{\lambda_1(p, \lambda) \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu}{1 + (p-1) \mathcal{E}[h_0^{2/p}]}$$

. This follows from the estimate  $|Y|^2 \geq \|M_{\nu\nu}\|^2$  and from the Cauchy-Schwarz inequality. Notice that  $e^t \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu$  is positive except for isolated values of  $t > 0$ . Our goal is to find  $\lambda(t)$  and  $\rho(t) > 1/2$  such that

$$X^\perp \cdot \mathfrak{M}_1 X - \frac{1}{2} \lambda'(t) X^\perp \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X + \varepsilon X^\perp \cdot \mathfrak{M}_3 X \geq \rho(t) X^\perp \cdot \mathfrak{M}_0 X$$

for any  $X \in \mathbb{R}^{2d}$ . The detail of the proof can be seen in section 3.3 of Chapter 3.

**Remark 2.1.** We are able to give an example of a function that reaches the optimal rate. Let

$$f_0(x, v) = f_\star(x - x_0, v - v_0) \quad \forall (x, v) \in \mathbb{R}^d \times \mathbb{R}^d$$

for some  $(x_0, v_0) \neq (0, 0)$ . From direct computations, we obtain that

$$f(t, x, v) = f_\star(x - x_\star(t), v - v_\star(t))$$

$$\text{with } \begin{cases} x_\star(t) = \left( \cos\left(\frac{\sqrt{3}}{2} t\right) x_0 + \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t\right) \left(v_0 + \frac{x_0}{2}\right) \right) e^{-\frac{t}{2}}, \\ v_\star(t) = \left( -\frac{\sqrt{3}}{2} \sin\left(\frac{\sqrt{3}}{2} t\right) \left(x_0 + \frac{v_0}{2}\right) + \cos\left(\frac{\sqrt{3}}{2} t\right) v_0 \right) e^{-\frac{t}{2}}, \end{cases} \quad (2.10)$$

solves (2.2). For the entropy with  $g = f/f_\star$  and  $\varphi = \varphi_p$ , we obtain as  $t \rightarrow +\infty$  that

$$\begin{aligned} \mathcal{E}[g(t, \cdot, \cdot)] &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_p(g) d\mu = \frac{p}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |g - 1|^2 d\mu (1 + o(1)) \\ &= \frac{p}{2} (|x_\star(t)|^2 + |v_\star(t)|^2) (1 + o(1)) = O(e^{-t}). \end{aligned}$$

This proves that the rate  $e^{-t}$  is optimal.

### 2.1.2 Linearized Vlasov-Poisson-Fokker-Planck equation

The nonlinear Vlasov-Poisson-Fokker-Planck equation with an external potential  $V$  is

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f - (\nabla_x V + \nabla_x \phi) \cdot \nabla_v f &= \Delta_v f + \nabla_v \cdot (v f), \\ -\Delta_x \phi &= \rho_f = \int_{\mathbb{R}^d} f dv. \end{aligned} \quad (2.11)$$

It describes the dynamics of a plasma of Coulomb particles in a thermal reservoir, which degenerates into a Vlasov-Poisson equation at low temperature (here the temperature is taken to be 1 as well as other physical constants).

Under appropriate conditions on  $V$ , the system (2.11) has a unique nonnegative integrable stationary solution  $f_*$  with associated potential  $\phi_*$ . Set

$$M = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_* dx dv > 0$$

as the mass. Now we linearize (2.11) around  $f_*$ . We define the function  $h$  such that

$$f = f_*(1 + \eta h).$$

The mass constraint  $\iint_{\mathbb{R}^d \times \mathbb{R}^d} f dx dv = M$  can then be rewritten as

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_* dx dv = 0,$$

and (2.11) becomes

$$\begin{aligned} \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_*) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h &= \eta \nabla_x \psi_h \cdot \nabla_v h, \\ -\Delta_x \psi_h &= \int_{\mathbb{R}^d} h f_* dv. \end{aligned}$$

The linearized equation (2.3) is obtained by dropping  $\mathcal{O}(\eta)$  term. Define the norm

$$\|h\|^2 := \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^2 f_* dx dv + \int_{\mathbb{R}^d} |\nabla_x \psi_h|^2 dx.$$

Our main result is devoted to the large time behaviour of a solution of the linearized system (2.3) on  $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \ni (t, x, v)$  with given initial datum  $h_0$  at  $t = 0$ .

**Theorem 2.2.** *Let us assume that  $d \geq 1$ ,  $V(x) = |x|^\alpha$  for some  $\alpha > 1$  and  $M > 0$ . Then there exist two constants  $\mathcal{C}$  and  $\lambda$  such that any solution  $h$  of (2.3) with an initial datum  $h_0$  of zero average with  $\|h_0\|^2 < \infty$  is such that*

$$\|h(t, \cdot, \cdot)\|^2 \leq \mathcal{C} \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0. \quad (2.12)$$

This theorem can be extended to a general external potential  $V(x)$  under technical conditions. The estimates are robust and compatible with the *diffusion limit*, which is discussed more precisely in Chapter 7 (also see below). The

strategy for proving Theorem 2.2 relies on two ideas:

- Use the norm obtained by a Taylor expansion of the entropy around the asymptotic state, that can measure the asymptotic rate of convergence. In presence of a Poisson coupling, this norm has to involve a non-local term, which is the main source of technical difficulties. See [50, 123, 124] for similar situations.
- On the basis of the previous norm, build a *twisted* norm, or Lyapunov functional, which has a coercivity property and reflects the spectral gap that determines the rate of convergence. This idea is inspired from [83, 106].

Another point is to obtain estimates which are uniform in the *diffusion limit*. For any  $\varepsilon > 0$ , we consider the solution of the *Vlasov-Poisson-Fokker-Planck equation in the parabolic scaling* given by

$$\varepsilon \partial_t f + v \cdot \nabla_x f - (\nabla_x V + \nabla_x \phi) \cdot \nabla_v f = \frac{1}{\varepsilon} (\Delta_v f + \nabla_v \cdot (v f)), \quad -\Delta_x \phi = \rho_f = \int_{\mathbb{R}^d} f dv. \quad (2.13)$$

It has been proved that when  $d = 2$  or  $3$ , (2.13) has a weak solution  $(f^\varepsilon, \phi^\varepsilon)$  which converges as  $\varepsilon \rightarrow 0_+$  to  $f^0$ , where the charge density  $\rho = \int_{\mathbb{R}^d} f^0 dv$  is a weak solution of the *drift-diffusion-Poisson* system

$$\frac{\partial \rho}{\partial t} = \nabla_x \cdot (\nabla_x \rho + \rho \nabla_x (V + \phi)), \quad -\Delta_x \phi = \rho. \quad (2.14)$$

A source of inspiration for the method is the asymptotic behavior of the solutions of (2.14) for large time. As  $t \rightarrow +\infty$ , it is well known that  $(\rho, \phi)$  converges a steady state  $(\rho_\star, \phi_\star)$  given by the Poisson-Boltzmann equation

$$-\Delta_x \phi_\star = \rho_\star = e^{-V - \phi_\star} \quad (2.15)$$

at an exponential rate. Now we linearize (2.13) to get the equation

$$\begin{aligned} \varepsilon \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \frac{1}{\varepsilon} (\Delta_v h - v \cdot \nabla_v h) &= 0, \\ -\Delta_x \psi_h = \int_{\mathbb{R}^d} h f_\star dv, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star dx dv &= 0. \end{aligned} \quad (2.16)$$

We obtain a decay estimate which is uniform with respect to  $\varepsilon \rightarrow 0_+$ .

**Theorem 2.3.** *Let us assume that  $d \geq 1$ ,  $V(x) = |x|^\alpha$  for some  $\alpha > 1$  and  $M > 0$ . For any  $\varepsilon > 0$  small enough, there exist two constants  $\mathcal{C}$  and  $\lambda$ , which do not depend on  $\varepsilon$ , such that any solution  $h$  of (2.16) with an initial datum such that  $\|h_0\|^2 < \infty$  satisfies (2.12).*

Let us explain the strategy of the method. We denote by  $\mathbb{T}$  and  $\mathbb{L}$  the transport operator and the diffusion operator. Let us define  $\text{Id}$  as the identity,  $\Pi$  is the orthogonal projection onto the null space of  $\mathbb{L}$ , and assume that  $*$  denotes the adjoint with respect to  $\langle \cdot, \cdot \rangle$ . As in [82, 83], we define the operator  $\mathbb{A}$  by

$$\mathbb{A} := (\text{Id} + (\mathbb{T}\Pi)^* \mathbb{T}\Pi)^{-1} (\mathbb{T}\Pi)^*$$

and consider the Lyapunov functional

$$H_\delta[h] := \frac{1}{2} \|h\|^2 + \delta \langle Ah, h \rangle$$

for some  $\delta > 0$ . If  $h$  solves (2.1), then

$$-\frac{d}{dt} H_\delta[h] = D_\delta[h],$$

where

$$D_\delta[h] := -\langle Lh, h \rangle + \delta \langle AT\Pi h, h \rangle - \delta \langle TA h, h \rangle + \delta \langle AT(\text{Id} - \Pi)h, h \rangle - \delta \langle ALh, h \rangle.$$

We shall consider a space of distribution functions such that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star dx dv = 0$$

and use the scalar product

$$\langle h_1, h_2 \rangle := \iint_{\mathbb{R}^d \times \mathbb{R}^d} h_1 h_2 d\mu + \int_{\mathbb{R}^d} \rho_{h_1} (-\Delta)^{-1} \rho_{h_2} dx$$

which is adapted to the Poisson coupling. With these preliminaries, it is possible to rely on an abstract method for hypocoercivity on a Hilbert space  $\mathcal{H}$  that goes as follows. Suppose that there exist constants  $\lambda_m, \lambda_M$  and  $C_M > 0$ , such that for any  $F \in \mathcal{H}$ , the following properties hold:

▷ *microscopic coercivity*

$$-\langle LF, F \rangle \geq \lambda_m \|(\text{Id} - \Pi)F\|^2, \quad (\text{H1})$$

▷ *macroscopic coercivity*

$$\|\text{T}\Pi F\|^2 \geq \lambda_M \|\Pi F\|^2, \quad (\text{H2})$$

▷ *parabolic macroscopic dynamics*

$$\Pi\text{T}\Pi F = 0, \quad (\text{H3})$$

▷ *bounded auxiliary operators*

$$\|\text{AT}(\text{Id} - \Pi)F\| + \|\text{AL}F\| \leq C_M \|(\text{Id} - \Pi)F\|. \quad (\text{H4})$$

The first observation is the following abstract result, taken from [82, 83].

**Proposition 2.2.** *There exists a constant  $\lambda > 0$  depending on  $\lambda_m, \lambda_M$  and  $C_M$ , such that  $\lambda H_\delta[F] \leq D_\delta[F]$  for any  $F \in \mathcal{H}$ . As a consequence, for a solution of (2.1), we have  $H_\delta[F(t, \cdot)] \leq H_\delta[F_0] e^{-\lambda t}$ .*

Let us come back to the linearized VFPF system (2.3). The transport and the diffusion operators are defined by

$$\mathbb{T}h := v \cdot \nabla_x h - \nabla_x W_\star \cdot \nabla_v h + v \cdot \nabla_x \psi_h, \quad \mathbb{L}h := \Delta_v h - v \cdot \nabla_v h. \quad (2.17)$$

The proof of Theorem 2.2 relies on a number of preliminary observations:

- Due to the confinement potential, integration by parts can be performed.
- There exists a unique solution of the Poisson-Boltzmann equation.
- The probability measure build on top of the solution of the Poisson-Boltzmann equation satisfies a Poincaré inequality.
- The scalar product is well defined.

Next we need to check (H1)-(H4) are satisfied. (H1)-(H3) are not difficult to prove. (H4) is more complicated (the details are given in section 7.4 of Chapter 7):

- Do the reformulation of the inequality as an elliptic regularity estimate.
- Get an estimate of  $H^1$ -type.
- Prove weighted Poincaré inequalities and weighted  $H^1$ -type estimates.
- Get estimates for the second order term.

As a consequence, we can identify the best possible rates of convergence for the nonlinear Vlasov-Poisson-Fokker-Planck system. We establish this rate in the  $d = 1$  case, but in higher dimensions, some global regularity estimates are still missing.

**Remark 2.2.** *We suppose in the theorem that  $V(x) = |x|^\alpha$  with  $\alpha > 1$ . It also apply for more general  $V$ , which should satisfy a number of technical assumptions, see section 7.3. It is easy to check that  $V(x) = |x|^\alpha$  satisfies all assumptions.*

The method also applies to the linearized VFPF system (2.16) in the parabolic scaling, with estimates which are independent of the scaling parameter  $\varepsilon > 0$ . See section 7.5 of Chapter 7 for the details.

## 2.2 Part 2: Large time asymptotic behaviour

In this part, we introduce the Cucker-Smale for flocking model in the homogeneous version, which is a Vlasov-McKean type model, and the Nernst-Planck system. In both cases, we state a result on the exponential rate of convergence to the set of stationary solution, and for the flocking model, we also give a result on a phase transition which corresponds to a symmetry breaking phenomenon.

### 2.2.1 flocking model

Emerging collective behaviours and self-organization in multi-agents interactions are interesting topics in many fields. In biological systems, no matter cells, insects or mammals, all the individuals can be self-organized and moving coherently. Of course, special conditions have to be fulfilled, otherwise random

dynamics predominate. So it is important to create a mathematical model that describes the switches between disorganized systems and systems exhibiting an ordered phase. Additionally, the order parameter which decides whether the system is ordered or not and measure the stability or the unstability of some special stationary solutions can also be used in the dynamical regime to get an estimate of the rates of convergence.

In this chapter, we consider the *flocking* behaviour. The original homogeneous Cucker-Smale model, also called McKean-Vlasov model, describes the velocities of a group of birds. In [154, 13], the authors have analyzed the equation

$$\frac{\partial f}{\partial t} = D \Delta_v f + \nabla_v \cdot (\nabla_v \phi_\alpha(v) f - \mathbf{u}_f f) \quad (2.18)$$

where

$$\mathbf{u}_f(t) := \frac{\int_{\mathbb{R}^d} v f(t, v) dv}{\int_{\mathbb{R}^d} f(t, v) dv}$$

is the average velocity,  $D > 0$  is the parameter of noise,  $f$  is a probability measure, and the potential  $\phi$  is given by

$$\phi_\alpha(v) = \frac{\alpha}{4} |v|^4 + \frac{1-\alpha}{2} |v|^2.$$

In particular, in [13], the emphasis is put on the analysis of the asymptotic regimes as  $D \rightarrow 0_+$  (small noise regime) and  $D \rightarrow +\infty$  (large noise regime). Our goal is to give a more complete picture and analyse stability and dynamical issues. Fig. 2.1 shows that there exists *phase transition* between disordered and collective state.

In order to study the dynamics, it is convenient to introduce the *free energy*

$$\mathcal{F}[f] := D \int_{\mathbb{R}^d} f \log f dv + \int_{\mathbb{R}^d} f \phi_\alpha dv - \frac{1}{2} |\mathbf{u}_f|^2$$

which decays according to

$$\frac{d}{dt} \mathcal{F}[f(t, \cdot)] = - \int_{\mathbb{R}^d} \left| D \frac{\nabla_v f}{f} + \nabla_v \phi_\alpha - \mathbf{u}_f \right|^2 f dv.$$

Let us assume first that  $d = 1$ . For any  $D \in (0, D_*)$ , for some  $D_* > 0$ , there are three stationary solutions:  $f_\star^{(0)}$ , and  $f_\star^{(\pm)}$ , with  $\mathcal{F}[f_\star^{(\pm)}] < \mathcal{F}[f_\star^{(0)}]$ . We prove the following result: if  $\mathcal{F}[f(t=0, \cdot)] < \mathcal{F}[f_\star^{(0)}]$  and  $D < D_*$ , then

$$\mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_\star^{(\pm)}] \leq C e^{-\lambda t}$$

for some positive constant  $C$ , where  $\lambda$  is the eigenvalue of the linearized problem at  $f_\star^{(\pm)}$  in the weighted space  $L^2\left((f_\star^{(\pm)})^{-1}\right)$  with scalar product

$$\langle f, g \rangle_\pm := D \int_{\mathbb{R}} f g (f_\star^{(\pm)})^{-1} dv - \mathbf{u}_f \mathbf{u}_g.$$

The question that we address next is to describe the set of the stationary solutions in higher dimensions, establish the (local) stability properties of these solutions and relate the spectral gap of the linearized problem with the rate of convergence of the solutions of the evolution problem.

In Chapter 4, we consider (2.18) in dimension  $d \geq 1$ . Our first result is

**Theorem 2.4.** *Let  $d \geq 1$  and  $\alpha > 0$ . There exists a critical  $D_* > 0$  such that*

- (i)  $D > D_*$ : *there exists only one stable stationary distribution with  $\mathbf{u}_f = \mathbf{0}$ .*
- (ii)  $D < D_*$ : *there exists one instable isotropic stationary distribution with  $\mathbf{u}_f = \mathbf{0}$  and a continuum of stable non-negative non-symmetric polarized stationary distributions (unique up to a rotation).*

Any stationary solution of (2.18) can be written as

$$f_{\mathbf{u}}(v) = \frac{e^{-\frac{1}{D} \left( \frac{1}{2} |v - \mathbf{u}|^2 + \frac{\alpha}{4} |v|^4 - \frac{\alpha}{2} |v|^2 \right)}}{\int_{\mathbb{R}^d} e^{-\frac{1}{D} \left( \frac{1}{2} |v - \mathbf{u}|^2 + \frac{\alpha}{4} |v|^4 - \frac{\alpha}{2} |v|^2 \right)} dv}$$

where  $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{R}^d$  solves  $\int_{\mathbb{R}^d} (\mathbf{u} - v) f_{\mathbf{u}}(v) dv = 0$ . Up to a rotation,  $\mathbf{u} = (u, 0, \dots, 0) = u e_1$  is given by

$$\mathcal{H}(u) = 0$$

where

$$\mathcal{H}(u) := \int_{\mathbb{R}^d} (v_1 - u) e^{-\frac{1}{D} (\phi_\alpha(v) - u v_1)} dv \quad \text{and} \quad \phi_\alpha(v) := \frac{\alpha}{4} |v|^4 + \frac{1-\alpha}{2} |v|^2.$$

The key steps of the proof of Theorem 2.4 are as follows:

- First from direct calculations, we obtain the expression

$$\mathcal{H}(u) = \alpha \int_{\mathbb{R}^d} (1 - |v|^2) v_1 e^{-\frac{1}{D} (\phi_\alpha(v) - u v_1)} dv$$

and we show that there exists a unique zero  $D_*$  of  $\mathcal{H}'(0)$ , and  $\mathcal{H}'(0) > 0$  when  $D < D_*$ ,  $\mathcal{H}'(0) < 0$  when  $D > D_*$ .

- If  $d = 1$ , for any  $u > 0$ ,  $\mathcal{H}''(u) < 0$  if  $\mathcal{H}(u) \leq 0$ . As a consequence,  $\mathcal{H}$  changes sign at most once on  $(0, +\infty)$ .

- For  $d \geq 2$ , we establish the following properties:

(1) When  $D \geq D_*$ , then  $\mathcal{H}'(u) \leq 0$  for any  $u \geq 0$ , then 0 is the only solution of  $\mathcal{H}(u) = 0$ .

(2) When  $D < D_*$ , we know that  $\mathcal{H}'(0) > 0$ , so there exists a positive zero  $u_1$  of  $\mathcal{H}(u)$ . We will prove that  $\mathcal{H}(u)$  is strictly decreasing on  $(u_1, \infty)$ , so  $u_1$  is the only zero of  $\mathcal{H}(u)$ .

The details of the proof can be seen in Chapter 4.



The threshold value  $D_*$  is a function of  $\alpha$  and  $d$ . To prove the existence of  $D_*$ , we prove that

$$\frac{1}{d+2} < D_* < \frac{1}{d} \quad \text{for any } \alpha > 0, \quad d \in \mathbb{N}^*.$$

In the last section of Chapter 4, we prove that

$$\lim_{\alpha \rightarrow 0} D_*(\alpha, d) = \frac{1}{d+2}, \quad \lim_{\alpha \rightarrow \infty} D_*(\alpha, d) = \frac{1}{d}.$$

The proof of Theorem 2.4 requires some additional technical work that we shall omit here.

After discovering the role of the threshold value  $D_*$  in phase transition, we consider the dynamical aspects of the problem. For this purpose, we need the following list of objects:

- *Free energy*

$$\mathcal{F}[f] := D \int_{\mathbb{R}^d} f \log f \, dv + \int_{\mathbb{R}^d} f \phi_\alpha \, dv - \frac{1}{2} |\mathbf{u}_f|^2,$$

- *Relative entropy* with respect to a stationary solution  $f_{\mathbf{u}}$

$$\mathcal{F}[f] - \mathcal{F}[f_{\mathbf{u}}] = D \int_{\mathbb{R}^d} f \log \left( \frac{f}{f_{\mathbf{u}}} \right) \, dv - \frac{1}{2} |\mathbf{u}_f - \mathbf{u}|^2,$$

- *Relative Fisher information*

$$\mathcal{I}[f] := \int_{\mathbb{R}^d} \left| D \frac{\nabla f}{f} + \alpha v |v|^2 + (1 - \alpha) v - \mathbf{u}_f \right|^2 f \, dv,$$

- *Non-equilibrium Gibbs state*

$$G_f(v) := \frac{e^{-\frac{1}{D} \left( \frac{1}{2} |v - \mathbf{u}_f|^2 + \frac{\alpha}{4} |v|^4 - \frac{\alpha}{2} |v|^2 \right)}}{\int_{\mathbb{R}^d} e^{-\frac{1}{D} \left( \frac{1}{2} |v - \mathbf{u}_f|^2 + \frac{\alpha}{4} |v|^4 - \frac{\alpha}{2} |v|^2 \right)} \, dv}.$$

A direct computation shows that  $\mathcal{F}[f]$  is a Lyapunov function in the sense that

$$\frac{d}{dt} \mathcal{F}[f(t, \cdot)] = -\mathcal{I}[f(t, \cdot)]$$

while  $\mathcal{F}[f] - \mathcal{F}[f_{\mathbf{u}}]$  can be considered as a relative entropy to an arbitrary stationary state  $f_{\mathbf{u}}$  because

$$\mathcal{F}[f] - \mathcal{F}[f_{\mathbf{u}}] = D \int_{\mathbb{R}^d} f \log \left( \frac{f}{f_{\mathbf{u}}} \right) \, dv - \frac{1}{2} |\mathbf{u}_f - \mathbf{u}|^2.$$

The Fisher information can be considered as a relative functional as well, as we can write it as

$$\mathcal{I}[f] = D^2 \int_{\mathbb{R}^d} \left| \nabla \log \left( \frac{f}{G_f} \right) \right|^2 f \, dv,$$

but the point is that it is relative to a non-equilibrium Gibbs state which is not a stationary state. To be precise, we have that  $\frac{d}{dt} \mathcal{F}[f(t, \cdot)] = 0$  if and only if  $f = G_f$  is a stationary solution.

Next, we study the quadratic forms obtained by expanding the relative entropy and the Fisher information around a stationary solution. Here we consider the space of the functions  $g \in L^2(f_{\mathbf{u}} d\nu)$  such that

$$\int_{\mathbb{R}^d} g f_{\mathbf{u}} d\nu = 0$$

and define

$$Q_{1,\mathbf{u}}[g] := \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^2} \mathcal{F}[f_{\mathbf{u}}(1 + \varepsilon g)] = D \int_{\mathbb{R}^d} g^2 f_{\mathbf{u}} d\nu - D^2 |\mathbf{v}_g|^2$$

$$\text{where } \mathbf{v}_g := \frac{1}{D} \int_{\mathbb{R}^d} \nu g f_{\mathbf{u}} d\nu,$$

$$Q_{2,\mathbf{u}}[g] := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \mathcal{I}[f_{\mathbf{u}}(1 + \varepsilon g)] = D^2 \int_{\mathbb{R}^d} |\nabla g - \mathbf{v}_g|^2 f_{\mathbf{u}} d\nu.$$

We have two questions:

- (1) Is  $Q_{1,\mathbf{u}}$  nonnegative?
- (2) If yes, is  $Q_{2,\mathbf{u}} \geq \lambda Q_{1,\mathbf{u}}$  for some  $\lambda > 0$ ?

For the first question, the answer is given by the following result.

**Lemma 2.2.**  $Q_{1,\mathbf{0}}$  is a nonnegative quadratic form if and only if  $D \geq D_*$ . If  $D < D_*$ ,  $|\mathbf{u}| = u(D) \neq 0$ , then  $Q_{1,\mathbf{u}}[g] \geq 0$ .

For the second question, we rely on the *Poincaré inequality*:

$$\int_{\mathbb{R}^d} |\nabla h|^2 f_{\mathbf{u}} d\nu \geq \Lambda_D \int_{\mathbb{R}^d} |h|^2 f_{\mathbf{u}} d\nu \quad \text{for any } h \text{ such that } \int_{\mathbb{R}^d} h f_{\mathbf{u}} d\nu = 0.$$

Let  $f \in L^1(\mathbb{R}^d)$  with  $\int_{\mathbb{R}^d} f d\nu = 1$ ,  $g = (f - f_{\mathbf{u}})/f_{\mathbf{u}}$  and let  $\mathbf{u}[f] = \frac{u(D)}{|\mathbf{u}_f|} \mathbf{u}_f$  if  $D < D_*$  and  $\mathbf{u}_f \neq \mathbf{0}$ . Otherwise take  $\mathbf{u}[f] = \mathbf{0}$ . So we can prove the proposition below.

**Proposition 2.3.** Let  $d \geq 1$ ,  $\alpha > 0$ ,  $D > 0$ . If  $\mathbf{u} = \mathbf{0}$ , then

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D Q_{1,\mathbf{u}}[g].$$

Otherwise, if  $|\mathbf{u}| = u(D) \neq 0$  for some  $D \in (0, D_*)$ , then

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D (1 - \kappa(D)) \frac{(\mathbf{v}_g \cdot \mathbf{u})^2}{|\mathbf{v}_g|^2 |\mathbf{u}|^2} Q_{1,\mathbf{u}}[g].$$

In both cases,  $\mathcal{C}_D$  is a positive constant and  $\kappa(D) \in (0, 1)$ .

We shall assume that  $\mathcal{C}_D$  denotes the optimal constant. Next we can prove a theorem on the convergence to the isotropic solution in the high noise case.

**Theorem 2.5.** *For any  $d \geq 1$  and any  $\alpha > 0$ , if  $D > D_*$ , then for any solution  $f$  with nonnegative initial datum  $f_{\text{in}}$  of mass 1 such that  $\mathcal{F}[f_{\text{in}}] < \infty$ , there is a positive constant  $C$  such that, for any time  $t > 0$ ,*

$$0 \leq \mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_0] \leq C e^{-\mathcal{C}_D t}$$

and  $\mathcal{C}_D$  is the optimal rate of convergence.

We first notice that from *Logarithmic Sobolev inequality*, we have

$$\int_{\mathbb{R}^d} \left| \nabla \log \left( \frac{f}{f_0} \right) \right|^2 f \, d\nu \geq \mathcal{K}_0 \int_{\mathbb{R}^d} f \log \left( \frac{f}{f_0} \right) \, d\nu = \mathcal{F}[f] - \mathcal{F}[f_0] \quad (2.19)$$

so we can prove that

**Proposition 2.4.** *A solution  $f \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^d))$  of with radially symmetric initial datum  $f_{\text{in}} \in L^1_+(\mathbb{R}^d)$  such that  $\mathcal{F}[f_{\text{in}}] < \infty$ . Then*

$$0 \leq \mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_0] \leq C e^{-\lambda t}$$

for some  $\lambda > 0$ .

The general case relies on a more complicated method. The point in Proposition 2.3 is to find the *optimal* exponential rate. In terms of  $f = f_0(1 + g)$ , (2.18) becomes

$$\frac{\partial g}{\partial t} = \mathcal{L}g - \mathbf{v}_g \cdot (D \nabla g - (v + \nabla \phi_\alpha)g)$$

where

$$\mathcal{L}g := D \Delta g - (v + \nabla \phi_\alpha) \cdot (\nabla g - \mathbf{v}_g)$$

is the linearized operator. Next, we define

$$\langle g_1, g_2 \rangle := D \int_{\mathbb{R}^d} g_1 g_2 f_0 \, d\nu - D^2 \mathbf{v}_{g_1} \cdot \mathbf{v}_{g_2}$$

as the scalar product on the space  $\mathcal{X} := \{g \in L^2(f_0 \, d\nu) : \int_{\mathbb{R}^d} g f_0 \, d\nu = 0\}$ . We use this scalar product and the Grönwall inequality to prove the result.

The large time behaviour for  $0 < D < D_*$  is more delicate. We get a similar result about convergence to a polarized stationary solution in a special case.

**Proposition 2.5.** *Assume that  $d \geq 2$ ,  $\alpha > 0$  and  $D \in (0, D_*)$ . Let us consider a solution  $f$  of (2.18) with nonnegative initial datum  $f_{\text{in}}$  of mass 1 such that  $\mathcal{F}[f_{\text{in}}] < \mathcal{F}[f_0]$  and assume that  $\mathbf{u} = \lim_{t \rightarrow +\infty} \mathbf{u}_f(t)$  is uniquely defined. If  $|(\mathbf{u}_f - \mathbf{u}) \cdot \mathbf{u}| \geq \varepsilon u(D) |\mathbf{u}_f - \mathbf{u}|$  for some  $\varepsilon > 0$  and  $t > 0$  large enough, then there are two positive constants  $C, \lambda$  and some  $\mathbf{u} \in \mathbb{R}^d$  such that*

$$0 \leq \mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_{\mathbf{u}}] \leq C e^{-\lambda t} \quad \forall t \geq 0.$$

**Remark 2.3.** *There are still open problems about the flocking model, which are mainly at the polarized case  $0 < D < D_*$ . We know that the solution converges to either the isotropic stationary solution  $f_0$  or the set of the polarized stationary solution  $f_u$ .*

### 2.2.2 Nernst-Planck model

The Nernst-Planck model was introduced in electromagnetism in order to describe the diffusion of charged particles in a solute under the influence of a potential, in presence of electrostatic forces. Nowadays, it is applied to many other areas. For example, in biology, it is used to describe the movements of ions crossing the cell membrane.

In dimension  $d = 2$ , the original model is a non-confined system with Poisson coupling given by

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla v) \\ v = -\frac{1}{2\pi} \log|x| * u \\ u(0, x) = n_0 \geq 0 \end{cases} \quad x \in \mathbb{R}^2, t > 0. \quad (2.20)$$

and in this thesis, we consider the Nernst-Planck equation with confining potential in  $\mathbb{R}^d$  with  $d = 2$  or  $3$ . The distribution function is obtained as a solution of

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n + \nabla \cdot (n \nabla c) + \nabla \cdot (n \nabla \phi) \\ -\Delta c = n \\ n(0, x) = n_0 \geq 0, \quad \int_{\mathbb{R}^d} n(0, x) dx = M > 0 \end{cases} \quad x \in \mathbb{R}^d, t > 0. \quad (2.21)$$

When  $d = 2$  and, for some  $\mu > 0$ ,  $\phi(x) = \frac{\mu}{2}|x|^2$  is the harmonic potential, we can relate (2.21) and (2.20) by changing variables according to

$$\begin{aligned} u(t, x) &= R^{-d} n(\tau, \xi), \quad v(t, x) = c(\tau, \xi), \\ \xi &= \frac{x}{R}, \quad \tau = \log R, \quad R = R(t) := (1 + 2\mu t)^{\frac{1}{2}}. \end{aligned} \quad (2.22)$$

If it exists, the stationary solution  $(n_\infty, c_\infty)$  of (2.21) is given by

$$-\Delta c_\infty = n_\infty = M \frac{e^{-c_\infty - \phi}}{\int_{\mathbb{R}^d} e^{-c_\infty - \phi} dx}. \quad (2.23)$$

Our main goal is to determine the asymptotic behaviour of the solution of (2.21).

**Theorem 2.6.** *Let  $d = 2$  or  $3$  and assume that  $\phi(x) = \frac{\mu}{2}|x|^2$ , for some  $\mu > 0$ . Assume that  $n$  solves (2.21) with initial datum  $n(0, x) = n_0 \in L^2_+(n_\infty^{-1} dx)$  such that  $\mathcal{F}[n_0] < \infty$  with mass  $\int_{\mathbb{R}^d} n_0 dx = M$ . Then there exist two positive constants  $C$  and  $\Lambda$  such that, for any time  $t > 0$ ,*

$$\int_{\mathbb{R}^d} \frac{|n(t, \cdot) - n_\infty|^2}{n_\infty} dx \leq C e^{-\Lambda t}.$$

*In particular,  $\Lambda = 2\mu$  when  $d = 2$ .*

For proving Theorem 2.6, it is interesting to introduce the *free energy* and the *Fisher information* respectively defined by

$$\mathcal{F}[n] := \int_{\mathbb{R}^d} n \log n \, dx + \int_{\mathbb{R}^d} n \phi \, dx + \frac{1}{2} \int_{\mathbb{R}^d} n c \, dx \quad (2.24)$$

with  $c = (-\Delta)^{-1} n$ , and

$$\mathcal{I}[n] := \int_{\mathbb{R}^d} n |\nabla(\log n + c + \phi)|^2 \, dx.$$

Under appropriate conditions on  $\phi$ , we can show that  $(n_\infty, c_\infty)$  is the minimizer of  $\mathcal{F}$ . The case  $d = 2$  deserves some care and will be addressed in the next section. By direct computation, we observe that

$$\frac{d}{dt} \mathcal{F}[n(t, \cdot)] = -\mathcal{I}[n(t, \cdot)]. \quad (2.25)$$

**Proposition 2.6.** *For any  $p \in [1, \infty)$  and  $q \in [2, \infty)$ , we have*

$$\lim_{t \rightarrow \infty} \|n(t, \cdot) - n_\infty\|_{L^p(\mathbb{R}^d)} = 0, \quad \lim_{t \rightarrow \infty} \|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^q(\mathbb{R}^d)} = 0$$

The proof of Proposition 2.6 when  $d = 3$  is simple and relies, among various estimates, on the Hardy-Littlewood-Sobolev inequality. The case  $d = 2$  is more complicated, and the proof relies on the Aubin-Lions lemma.

In order to understand the asymptotic behaviour of the solutions, we study the quadratic forms derived from the free energy  $\mathcal{F}$  and the Fisher information  $\mathcal{I}$  by perturbing  $(n_\infty, c_\infty)$ . Define

$$Q_1[f] := \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^2} \mathcal{F}[n_\infty(1 + \varepsilon f)] = \int_{\mathbb{R}^d} f^2 n_\infty \, dx + \int_{\mathbb{R}^d} |\nabla(g c_\infty)|^2 \, dx,$$

$$Q_2[f] := \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^2} \mathcal{I}[n_\infty(1 + \varepsilon f)] = \int_{\mathbb{R}^d} |\nabla(f + g c_\infty)|^2 n_\infty \, dx.$$

Here  $g c_\infty := (-\Delta)^{-1}(f n_\infty)$ . Our goal is to show the following coercivity result.

**Proposition 2.7.** *Assume that  $\phi$  is such that the measure  $n_\infty \, dx$  admits a Poincaré inequality. For any  $f \in H^1(\mathbb{R}^d)$  such that  $\int_{\mathbb{R}^d} f n_\infty \, dx = 0$ , we have*

$$Q_2[f] \geq C_\star Q_1[f]$$

for some  $C_\star > 0$ . In particular, when  $d = 2$ ,  $\phi = \frac{\mu}{2} |x|^2$ , we have  $C_\star = \mu$ .

Sufficient conditions such that  $n_\infty \, dx$  satisfies a Poincaré inequality are given in Section 7.3 of Chapter 7. The main steps of the proof in the case of the harmonic potential go as follows:

- First consider the radial case: the problem becomes an eigenvalue of an ODE. We find that  $2\mu$  is an eigenvalue by direct computation.

- Still in the radial function case, we prove that  $2\mu$  is the smallest positive eigenvalue by using the Sturm-Liouville theory.
- For the general, non-radial case, we use a spherical harmonics decomposition to prove the result.

Now let us come back to the equation (2.21). Set

$$n(t, x) = n_\infty(x) (1 + f(t, x)), \quad c(t, x) = c_\infty (1 + g(t, x)).$$

The linearized equation becomes

$$\begin{cases} \frac{\partial f}{\partial t} - \mathcal{L}f = \frac{1}{n_\infty} \nabla \cdot [f n_\infty \nabla (g c_\infty)] \\ -\Delta (g c_\infty) = f n_\infty \end{cases} \quad x \in \mathbb{R}^d, t > 0 \quad (2.26)$$

for any  $x \in \mathbb{R}^d$ ,  $t \geq 0$ , with the linear operator  $\mathcal{L}$  defined by

$$\mathcal{L}f := \frac{1}{n_\infty} \nabla [n_\infty \nabla (f + g c_\infty)].$$

On the admissible set  $\mathcal{A} := \{f \in L^2(\mathbb{R}^d, n_\infty dx) : \int_{\mathbb{R}^d} f n_\infty dx = 0\}$ , define the scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^d} f_1 f_2 n_\infty dx + \int_{\mathbb{R}^d} n_\infty f_1 (G_d * (f_2 n_\infty)) dx. \quad (2.27)$$

Then  $\mathcal{L}$  is self-adjoint on  $\mathcal{A}$ , and for any  $f \in \mathcal{A}$ , we have

$$Q_1[f] = \langle f, f \rangle, \quad Q_2[f] = -\langle f, \mathcal{L}f \rangle.$$

A solution of the linearized equation

$$\frac{\partial g}{\partial t} = \mathcal{L}g$$

with initial datum  $g_0 \in \mathcal{A}$  satisfies

$$\frac{1}{2} \frac{d}{dt} Q_1[g] = -Q_2[g]$$

and we obtain from Grönwall's inequality that, for any  $t \geq 0$ ,

$$Q_1[g(t, \cdot)] \leq Q_1[g_0] e^{-2\mathcal{C}_* t}.$$

The solution of the nonlinear equation (2.21) satisfies

$$\begin{aligned} \frac{d}{dt} Q_1[f] &= -2Q_2[f] - 2 \int_{\mathbb{R}^d} \nabla(f + g c_\infty) f n_\infty \nabla(g c_\infty) dx \\ &\leq -2Q_2[f] + \sqrt{Q_2[f] Q_1[f]} \cdot \|\nabla(g c_\infty)\|_{L^\infty(\mathbb{R}^d)} \end{aligned}$$

where the inequality arises from the Cauchy-Schwarz inequality. This completes the proof of Theorem 2.6.

### 2.3 Part 3: Generalized log-HLS inequality

In this section, we consider functions defined on  $\mathbb{R}^2$ . Define

$$\mu(x) := \frac{1}{\pi(1+|x|^2)^2}$$

as the reference *probability measure* and

$$V(x) := -\log \mu(x) = 2 \log(1+|x|^2) + \log \pi$$

as the *external potential*. Define the space  $L^1_+(\mathbb{R}^2)$  as the set of a.e. nonnegative functions in  $L^1(\mathbb{R}^2)$ . For any function  $f \in L^1_+(\mathbb{R}^2)$  with  $M = \int_{\mathbb{R}^2} f dx > 0$ , we have the two inequalities :

- *logarithmic Hardy-Littlewood-Sobolev inequality*

$$\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log|x-y| dx dy + M(1 + \log \pi) \geq 0, \quad (2.28)$$

- *Jensen's inequality*

$$\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \int_{\mathbb{R}^2} V f dx = \int_{\mathbb{R}^2} f \log \left( \frac{f}{f_\star} \right) dx \geq 0. \quad (2.29)$$

It is possible to interpolate between these two inequalities, which corresponds to some parameter  $\alpha \in (0, 1)$ , but also to consider the case  $\alpha > 1$  in the following sense.

**Theorem 2.7.** *For any  $\alpha \geq 0$ , we have*

$$\begin{aligned} \int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f dx + M(1-\alpha)(1 + \log \pi) \\ \geq \frac{2}{M}(\alpha-1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log|x-y| dx dy \end{aligned} \quad (2.30)$$

for any function  $f \in L^1_+(\mathbb{R}^2)$  with  $M = \int_{\mathbb{R}^2} f dx > 0$ . Moreover, the equality case is achieved by  $f_\star = M\mu$  and  $f_\star$  is the unique optimal function for any  $\alpha > 0$ .

When  $0 \leq \alpha \leq 1$ , the theorem can be proved by multiplying (2.28) by  $1-\alpha$  and multiplying (2.29) by  $\alpha$ . The difficult case is of course  $\alpha > 1$ . The strategy of the proof is inspired by E. Carlen, J. Carrillo and M. Loss in [51]. Assume that  $M = 1$  without losing generality. Consider the evolution equation

$$\frac{\partial f}{\partial t} = \Delta \sqrt{f} + 2\sqrt{\pi} \nabla \cdot (x f)$$

and the free energy

$$\begin{aligned} \mathcal{F}[f] := \int_{\mathbb{R}^2} f \log f dx + \alpha \int_{\mathbb{R}^2} V f dx + (1-\alpha)(1 + \log \pi) \\ + 2(1-\alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log|x-y| dx dy. \end{aligned}$$

After elementary calculations, we obtain that

$$\begin{aligned} \frac{d}{dt} \mathcal{F}[f(t, \cdot)] &= -8 \left( \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 dx - \pi \int_{\mathbb{R}^2} f^{3/2} dx \right) \\ &\quad - 8\pi\alpha \int_{\mathbb{R}^2} \left( f^{3/2} - \mu\sqrt{f} - \sqrt{\mu}f + \mu^{3/2} \right) dx. \end{aligned}$$

On the one hand, we notice that

$$\int_{\mathbb{R}^2} \left( f^{3/2} - \mu\sqrt{f} - \sqrt{\mu}f + \mu^{3/2} \right) dx = \int_{\mathbb{R}^2} \varphi\left(\frac{f}{\mu}\right) \mu^{3/2} dx$$

where

$$\varphi(t) := t^{3/2} - t - \sqrt{t} + 1$$

is a strictly convex function on  $\mathbb{R}^+$  such that  $\varphi(1) = \varphi'(1) = 0$ , so that  $\varphi$  is non-negative.

On the other hand, according to [67], the Gagliardo-Nirenberg inequality

$$\|\nabla g\|_2^2 \|g\|_4^4 \geq \pi \|g\|_6^6 \quad (2.31)$$

applied to  $g = f^{1/4}$  means that

$$\int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 dx - \pi \int_{\mathbb{R}^2} f^{3/2} dx \geq 0,$$

see also [51]. Hence we have proved that  $t \mapsto \mathcal{F}[f(t, \cdot)]$  is monotone nonincreasing and for any  $t \geq 0$  we have that

$$\mathcal{F}[f_0] \geq \mathcal{F}[f(t, \cdot)] \geq \lim_{t \rightarrow +\infty} \mathcal{F}[f(t, \cdot)] = \mathcal{F}[f_\star] = 0.$$

The proof of (2.30) is completed.

Let us draw some consequences of Theorem 2.7. We consider the *drift-diffusion-Poisson* model

$$\frac{\partial f}{\partial t} = \Delta f + \beta \nabla \cdot (f \nabla V) + \nabla \cdot (f \nabla \phi) \quad \text{where} \quad -\varepsilon \Delta \phi = f \quad (2.32)$$

when  $V = -\log \mu$ . For  $\varepsilon = +1$ , the solution  $f$  converges as  $t \rightarrow \infty$  to the stationary solution  $M \frac{e^{-\beta V - \phi}}{\int_{\mathbb{R}^2} e^{-\beta V - \phi} dx}$ , which is obtained by solving the equation

$$-\Delta \psi = M \left( \frac{e^{-\gamma V - \psi}}{\int_{\mathbb{R}^2} e^{-\gamma V - \psi} dx} - \mu \right), \quad \psi = (\beta - \gamma)V + \phi, \quad \gamma = \beta - \frac{M}{8\pi}.$$

It is possible to consider the solution as a critical point of the strictly convex functional

$$\psi \mapsto \mathcal{J}_{M, \gamma}[\psi] := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 dx + M \int_{\mathbb{R}^2} \psi \mu dx + M \log \left( \int_{\mathbb{R}^2} e^{-\gamma V - \psi} dx \right).$$



For  $\varepsilon = -1$ , the proof goes as in the case  $\beta = 0$ , see [33] for more details.

When  $\varepsilon = +1$ , (2.32) corresponds to the mean field model with repulsive electrostatic forces. It can be used in electrolytes, plasmas and charged particles models, and we refer to Chapter 6 for further details. When  $\varepsilon = -1$ , the model corresponds to attractive forces, as in the Keller-Segel model. See [33] for more details.

In view of *entropy methods*, we consider the *free energy functional*

$$\mathcal{F}_\beta[f] := \int_{\mathbb{R}^2} f \log f \, dx + \beta \int_{\mathbb{R}^2} V f \, dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi f \, dx. \quad (2.33)$$

**Corollary 2.1.** *Let  $M > 0$ . The functional  $\mathcal{F}_\beta$  is bounded from below and admits a minimizer on the set of the functions*

$$\mathcal{B} := \left\{ f \in L_+^1(\mathbb{R}^2), \quad \text{tel que} \quad \int_{\mathbb{R}^2} f \, dx = M \right\}$$

*if either  $\varepsilon = +1$  and  $\beta \geq 1 + \frac{M}{8\pi}$ , or  $\varepsilon = -1$ ,  $\beta \geq 1 - \frac{M}{8\pi}$  and  $M \leq 8\pi$ . If  $\varepsilon = +1$ , the minimizer is unique.*

We will see in Chapter 6 and Chapter 7 that the scalar products associated with the Nernst-Planck equation and the linearized Vlasov-Poisson-Fokker-Planck equation are well defined in dimension 2 because of Theorem 2.7.

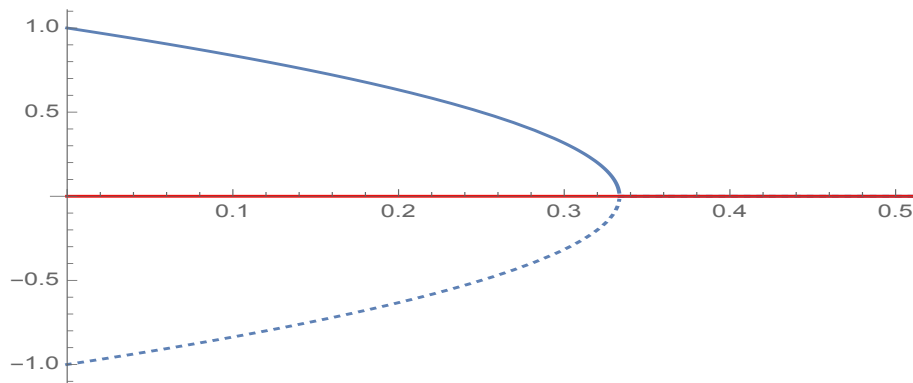


Figure 2.1: This figure describes the phase transition. Solutions of  $D \mapsto u(D)$  are represented. Here  $d = 1$ . There exists a bifurcation point  $D = D_*$  such that the only stationary solution corresponds to  $\mathbf{u}_f = 0$  if  $D > D_*$  and there are three solutions corresponding to  $\mathbf{u}_f = 0, \pm u(D)$  if  $D < D_*$  and  $\mathbf{u}_f = 0$  is linearly unstable if  $D < D_*$ .



## Chapter 3

# $\phi$ -entropy of kinetic Fokker-Planck equation

**Article  $\phi$ -entropies for Fokker-Planck and kinetic Fokker-Planck equations, in collaboration with Jean DOLBEAULT, published in *Mathematical Models and Methods in Applied Sciences*, 28 (13): 2637-2666, 2018.**

This chapter is devoted to  $\phi$ -entropies applied to Fokker-Planck and kinetic Fokker-Planck equations in the whole space, with confinement. The so-called  $\phi$ -entropies are Lyapunov functionals which typically interpolate between Gibbs entropies and  $L^2$  estimates. We review some of their properties in the case of diffusion equations of Fokker-Planck type, give new and simplified proofs, and then adapt these methods to a kinetic Fokker-Planck equation acting on a phase space with positions and velocities. At kinetic level, since the diffusion only acts on the velocity variable, the transport operator plays an essential role in the relaxation process. Here we adopt the  $H^1$  point of view and establish a sharp decay rate. Rather than giving general but quantitatively vague estimates, our goal here is to consider simple cases, benchmark available methods and obtain sharp estimates on a key example. Some  $\phi$ -entropies give rise to improved entropy – entropy production inequalities and, as a consequence, to faster decay rates for entropy estimates of solutions to non-degenerate diffusion equations. We prove that faster entropy decay also holds at kinetic level away from equilibrium and that optimal decay rates are achieved only in asymptotic regimes.

### 3.1 Introduction

By definition, the  $\phi$ -entropy of a nonnegative function  $w \in L^1(\mathbb{R}^d, d\gamma)$  is the functional

$$\mathcal{E}[w] := \int_{\mathbb{R}^d} \phi(w) d\gamma,$$

where  $\phi$  is a nonnegative convex continuous function on  $\mathbb{R}^+$  such that  $\phi(1) = 0$  and  $1/\phi''$  is concave on  $(0, +\infty)$ , i.e.,

$$\phi'' \geq 0, \quad \phi \geq \phi(1) = 0 \quad \text{and} \quad (1/\phi'')'' \leq 0. \quad (3.1)$$

Notice that the last condition means  $2(\phi''')^2 \leq \phi'' \phi^{(iv)}$  a.e. A classical example of such a function  $\phi$  is given by

$$\phi_p(w) := \frac{1}{p-1} (w^p - 1 - p(w-1)) \quad p \in (1, 2]$$

where, in the case  $p = 2$ ,  $\phi_2(w) = (w-1)^2$  and the limit case as  $p \rightarrow 1_+$  is given by the standard Gibbs entropy

$$\phi_1(w) := w \log w - (w-1).$$

Many results corresponding to the case  $p = 2$  can be obtained, e.g., by spectral methods. The case  $p = 1$  is important in probability theory and statistical physics. Our goal is to emphasize that they share properties which can be put in a common framework. Throughout this paper we shall assume that  $d\gamma$  is a nonnegative bounded measure, which is absolutely continuous with respect to Lebesgue's measure and write

$$d\gamma = e^{-\psi} dx$$

where  $\psi$  is a *potential* such that  $e^{-\psi}$  is in  $L^1(\mathbb{R}^d, dx)$ . Up to the addition of a constant to  $\psi$ , we can assume without loss of generality that  $d\gamma$  is a probability measure. A review of the *main properties of  $\phi$ -entropies*, new and simplified proofs and key references are given in Section 3.2.

Without entering the technical details, let us illustrate the use of the  $\phi$ -entropy in the case of diffusion equations. A typical application of the  $\phi$ -entropy is the control of the rate of relaxation of the solution to the *Ornstein-Uhlenbeck equation*

$$\frac{\partial w}{\partial t} = \mathbb{L} w := \Delta w - \nabla \psi \cdot \nabla w, \quad (3.2)$$

which is also known as the *backward Kolmogorov equation*. If we solve the equation with a nonnegative initial datum  $w_0$  such that  $\int_{\mathbb{R}^d} w_0 d\gamma = 1$ , then the solution satisfies  $\int_{\mathbb{R}^d} w(t, \cdot) d\gamma = 1$  for any  $t > 0$  and  $\lim_{t \rightarrow +\infty} w(t, \cdot) = 1$ . The Ornstein-Uhlenbeck operator  $\mathbb{L}$  defined on  $L^2(\mathbb{R}^d, d\gamma)$  is indeed self-adjoint and such that

$$-\int_{\mathbb{R}^d} (\mathbb{L} w_1) w_2 d\gamma = \int_{\mathbb{R}^d} \nabla w_1 \cdot \nabla w_2 d\gamma \quad \forall w_1, w_2 \in H^1(\mathbb{R}^d, d\gamma).$$

As a consequence, it is also straightforward to observe that for any solution  $w$  with initial datum  $w_0$  such that  $\mathcal{E}[w_0]$  is finite, then

$$\frac{d}{dt} \mathcal{E}[w] = - \int_{\mathbb{R}^d} \varphi''(w) |\nabla_x w|^2 d\gamma =: -\mathcal{I}[w],$$

where  $\mathcal{I}[w]$  denotes the  $\varphi$ -Fisher information functional. If for some  $\Lambda > 0$  we can establish the *entropy – entropy production inequality*

$$\mathcal{I}[w] \geq \Lambda \mathcal{E}[w] \quad \forall w \in H^1(\mathbb{R}^d, d\gamma), \quad (3.3)$$

then we deduce that

$$\mathcal{E}[w(t, \cdot)] \leq \mathcal{E}[w_0] e^{-\Lambda t} \quad \forall t \geq 0,$$

which controls the convergence of  $w$  to 1 as  $t \rightarrow +\infty$ , for instance in  $L^p(\mathbb{R}^d, d\gamma)$  by a generalized *Csiszár-Kullback inequality* if  $\varphi = \varphi_p$ ,  $1 \leq p \leq 2$ . The entropy – entropy production inequality is the Poincaré inequality associated with  $d\gamma$  if  $\varphi = \varphi_2$ , and the logarithmic Sobolev inequality if  $\varphi = \varphi_1$ .

We recall that the study of (3.2) is equivalent to the study of the *Fokker-Planck equation*

$$\frac{\partial u}{\partial t} = \Delta u + \nabla_x \cdot (u \nabla_x \psi). \quad (3.4)$$

A nonnegative solution with initial datum  $u_0 \in L^1(\mathbb{R}^d, dx)$  and  $\int_{\mathbb{R}^d} u_0 dx = M > 0$  has constant mass  $M = \int_{\mathbb{R}^d} u(t, \cdot) dx$  for any  $t > 0$ , and converges towards the unique stationary solution

$$u_\star = M \frac{e^{-\psi}}{\int_{\mathbb{R}^d} e^{-\psi} dx}.$$

Without loss of generality, we shall assume that  $M = 1$ . Then one observes that  $w = u/u_\star$  solves (3.2), which allows to control the rate of convergence of  $u$  to  $u_\star$ . A list of results concerning the solutions of (3.2) and (3.4) is also collected in Section 3.2.

The third section of this paper is devoted to the extension of  $\varphi$ -entropy methods to kinetic equations. Section 3.3 of this paper deals with the *kinetic Fokker-Planck equation*, or *Vlasov-Fokker-Planck equation*, that can be written as

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v f). \quad (3.5)$$

Our basic example corresponds to the case of the harmonic potential  $\psi(x) = |x|^2/2$ . Unless it is explicitly specified, we will only consider this case. Notice that this problem has an explicit Green function whose expression can be found in [53].

Since (3.5) is linear, we can assume at no cost that  $\|f\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$  and consider the stationary solution

$$f_\star(x, v) = (2\pi)^{-d} e^{-\psi(x)} e^{-\frac{1}{2}|v|^2} = (2\pi)^{-d} e^{-\frac{1}{2}(|x|^2 + |v|^2)} \quad \forall (x, v) \in \mathbb{R}^d \times \mathbb{R}^d.$$

The function

$$g := \frac{f}{f_\star}$$

solves

$$\frac{\partial g}{\partial t} + \mathbb{T}g = \mathbb{L}g \quad (3.6)$$

where the transport operator  $\mathbb{T}$  and the Ornstein-Uhlenbeck operator  $\mathbb{L}$  are defined respectively by

$$\mathbb{T}g := v \cdot \nabla_x g - x \cdot \nabla_v g \quad \text{and} \quad \mathbb{L}g := \Delta_v g - v \cdot \nabla_v g.$$

Let  $d\mu := f_\star dx dv$  be the invariant measure on the phase space  $\mathbb{R}^d \times \mathbb{R}^d$ , so that  $\mathbb{T}$  and  $\mathbb{L}$  are respectively anti-self-adjoint and self-adjoint. The function

$$h := g^{p/2}$$

solves

$$\frac{\partial h}{\partial t} + \mathbb{T}h = \mathbb{L}h + \frac{2-p}{p} \frac{|\nabla_v h|^2}{h}. \quad (3.7)$$

At the kinetic level, we consider the  $\phi$ -entropy given by

$$\mathcal{E}[g] := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \phi(g) d\mu.$$

With this notation,  $\mathcal{E}[g] = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \phi(f/f_\star) d\mu$  so that, with  $f = g f_\star = h^{2/p} f_\star$  we have

$$\mathcal{E}[g] = \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^2 \log\left(\frac{h^2}{\iint_{\mathbb{R}^d \times \mathbb{R}^d} h^2 d\mu}\right) d\mu \quad \text{if } \phi = \phi_1,$$

$$\mathcal{E}[g] = \mathcal{E}[h^{2/p}] = \frac{1}{p-1} \left[ \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^2 d\mu - \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^{2/p} d\mu \right)^{p/2} \right] \quad \text{if } \phi = \phi_p, p \in (1, 2].$$

The optimal rate of decay of  $\mathcal{E}[g]$  has been established by A. Arnold and J. Erb in [7]. In the special case of a harmonic potential, their result goes as follows.

**Proposition 3.1.** *Assume that  $\psi(x) = |x|^2/2$  for any  $x \in \mathbb{R}^d$ . Take  $\phi = \phi_p$  for some  $p \in [1, 2]$ . To any nonnegative solution  $g \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$  of (3.6) with initial datum  $g$  such that  $\mathcal{E}[g_0] < \infty$ , we can associate a constant  $\mathcal{C} > 0$  for which*

$$\mathcal{E}[g(t, \cdot, \cdot)] \leq \mathcal{C} e^{-t} \quad \forall t \geq 0. \quad (3.8)$$

Moreover the rate  $e^{-t}$  is sharp as  $t \rightarrow +\infty$ .

The striking point of this *hypocoercivity* result is to identify the sharp rate of decay. The rate is specific of the harmonic potential  $\psi(x) = |x|^2/2$ , but it turns out to be useful for the comparison with rates obtained by other methods. Although probably not optimal, a precise estimate of  $\mathcal{C}$  will be given in Section 3.3, with a simplified proof of Proposition 3.1.

The method is based on the use of a *Fisher information* type functional

$$\mathcal{J}[h] = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2 d\mu \quad (3.9)$$

which involves derivatives in  $x$  and  $v$ . If  $h$  solves (3.6), then the key estimate is to prove that

$$\frac{d}{dt} \mathcal{J}[h(t, \cdot)] \leq -\mathcal{J}[h(t, \cdot)].$$

The result of Proposition 3.1 follows from the entropy – entropy production inequality (3.16) that will be established in Proposition 3.4: since

$$\Lambda \mathcal{E}[g(t, \cdot, \cdot)] = \Lambda \mathcal{E}[h^{2/p}] \leq \mathcal{J}[h],$$

then  $\mathcal{E}[g(t, \cdot, \cdot)]$  has an exponential decay. However, we underline the fact that

$$\frac{d}{dt} \mathcal{E}[g(t, \cdot)] = - \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu \neq -\mathcal{J}[h(t, \cdot)].$$

At the level of non-degenerate diffusions, a distinctive property of the  $\varphi$ -entropy with  $\varphi = \varphi_p$  and  $p \in (1, 2)$  is that the entropy – entropy production inequality  $\mathcal{J} \geq \Lambda \mathcal{E}$  with an optimal constant  $\Lambda > 0$  can be improved in the sense that there exists a strictly convex function  $F$  on  $\mathbb{R}^+$  with  $F(0) = 0$  and  $F'(0) = 1$  such that  $\mathcal{J} \geq \Lambda F(\mathcal{E})$ . This has been established in [6] and details will be given in Section 3.2.5. The key issue is to prove that for some function  $\rho$  on  $\mathbb{R}^+$ , which depends on the solution  $w$ , such that  $\rho > \Lambda$  a.e., we have  $\frac{d}{dt} \mathcal{J}[w(t, \cdot)] \leq -\rho(t) \mathcal{J}[w(t, \cdot)]$ . One may wonder if a similar result also holds in the hypocoercive kinetic Fokker-Planck equation. So far, no global improved inequality has been established. What we shall prove is that, if we consider the more general *Fisher information* functional

$$\mathcal{J}_\lambda[h] = (1 - \lambda) \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu + (1 - \lambda) \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu + \lambda \int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2 d\mu, \quad (3.10)$$

then for an appropriate choice of  $\lambda$  (which turns out to be  $t$ -dependent), the rate of decay is faster than  $e^{-t}$  up to a zero-measure set in  $t$ . The precise statement, which is our main result, goes as follows.

**Theorem 3.1.** *Let  $p \in (1, 2)$  and  $h$  be a solution of (3.7) with initial datum  $h_0 \in L^1 \cap L^p(\mathbb{R}^d, d\gamma)$ ,  $h_0 \not\equiv 1$ , and  $d\gamma$  be the Gaussian probability measure corresponding to the harmonic potential  $\psi(x) = |x|^2/2$ . Then there exists a function*



$\lambda : \mathbb{R}^+ \rightarrow [1/2, 1)$  such that  $\lambda(0) = \lim_{t \rightarrow +\infty} \lambda(t) = 1/2$  and a continuous function  $\rho$  on  $\mathbb{R}^+$  such that  $\rho > 1/2$  a.e., for which we have

$$\frac{d}{dt} \mathcal{I}_{\lambda(t)}[h(t, \cdot)] \leq -2\rho(t) \mathcal{I}_{\lambda(t)}[h(t, \cdot)] \quad \forall t \geq 0.$$

As a consequence, for any  $t \geq 0$  we have the global estimate

$$\mathcal{I}_{\lambda(t)}[h(t, \cdot)] \leq \mathcal{I}_{1/2}[h_0] \exp\left(-2 \int_0^t \rho(s) ds\right).$$

This result is weaker than the result for non-degenerate diffusions. The qualitative issues are easy to understand and to some extent classical in the hypocoercivity theory, but no quantitative estimate of  $\rho$  in terms of  $h$  is known so far. If  $\phi_p$ -entropies were initially thought as interesting objects which interpolate between the Gibbs entropy and standard  $L^2$  estimates, improved entropy – entropy production inequalities and the result of Theorem 3.1 capture an important feature when  $p \in (1, 2)$ : faster rates of decay for finite values of  $t$ . As  $t \rightarrow +\infty$ , we cannot expect a faster decay rate, but we gain a pre-factor which is less than 1. See Section 3.3.4 for more details.

Let us conclude this introduction with a brief review of the literature. Fokker-Planck equations like (3.4) are ubiquitous in various areas of physics ranging from the description of the motion of particles in a gas or a solute to semiconductor physics, models of stars in astrophysics or models of populations in biology and social sciences, as microscopic dynamics involving Brownian motion are represented at macroscopic scales by diffusion equations. Second order dynamics (in which forces produce acceleration) in random environments obey in many cases to the Langevin equation and at macroscopic scale the corresponding distribution function solves (3.5). A typical example is given by particles having random encounters with some background obstacles, a situation that can be encountered in many areas of physical modeling. It has to be emphasized that (3.4) appears in the diffusion limit of the solutions of (3.5), that is, in the *overdamped regime* in which friction and other forces equilibrate very fast, so that the velocity instantaneously adapts to the forces, which results in first order dynamics. For some general properties of (3.4) and (3.5), a review of stochastic and PDE methods and some entries to applied cases, we refer for instance to [145, 139], among many other books on this topic.

The word "hypocoercivity" is apparently due to T. Gallay and was made popular by C. Villani in [159]. Our computations are based on Villani's ideas in Section 3 of [159] (also see [161]), but the use of *twisted gradients* involving simultaneously derivatives in  $x$  and  $v$  can be also found in [107] and in earlier works like [108]. It is actually a consequence of Hörmander's hypoelliptic theory, which covers simultaneously regularization properties and large time behaviour. One can refer for instance to [92, 108, 105] and, much earlier, to [113]. The seed

for such an approach can actually be traced back to Kolmogorov's computation of Green's kernel for the kinetic Fokker-Planck equation in [119], which has been reconsidered by [?] from a more modern point of view and successfully applied, for instance, to the study of the Vlasov-Poisson-Fokker-Planck system in [158, 41, 42]. In case of the kinetic Fokker-Planck equation, we can refer to [105, 107] in the case of a general potential of confinement, and more specifically to [7]. In this last paper, the authors deal with the issue of accurate rates: "while the main theorem in [161] covers a wide class of problems, the price paid is in the estimate for the decay rate, which is off by orders of magnitude." The result of Proposition 3.1 addresses the issue of the optimal rate in a very simple case. For completion, one also has to mention [122] and [114] for further theoretical and numerical results.

A twin problem of the kinetic Fokker-Planck equation is the linear BGK model, which has no regularizing properties but shares many common features with the kinetic Fokker-Planck equation as soon as we are concerned with rates of convergence. We refer to [106, 133] for early contributions, to [82, 83, 45, 130, 1] for more recent ones, and especially to [95]. In this last paper, J. Evans studies the linear BGK model and a kinetic Fokker-Planck equation on the torus using the  $\varphi$ -entropies.

In [159], only the cases  $p = 1$  and  $p = 2$  were considered, but it is well known since the founding work [12] of Bakry and Emery that intermediate values of  $p$  can then be considered. In the case of  $\varphi$ -entropies associated with non-degenerate diffusions, this idea was invoked on many occasions, for instance in [21, 121, 58, 9, 37, 35] in relation with spectral estimates or the *carré du champ* methods. For *carré du champ* techniques in kinetic equations, we can refer to [20], also [131, 129], and finally Remark 6.7 in [11] for an early contribution on  $\varphi$ -entropies. Although  $\varphi$ -entropies are natural in the context of the kinetic Fokker-Planck equation, precise connections were made only quite recently. In [7], A. Arnold and J. Erb discuss  $\varphi$ -entropies in the context of the kinetic Fokker-Planck equation and prove, among more general results, Proposition 3.1. We can also refer to [1, 2, 131] for various related results. As far as we know, no result such as Theorem 3.1 has been established yet.

## 3.2 A review of results on $\varphi$ -entropies

In this section we consider a  $\varphi$ -entropy defined by  $\mathcal{E}[w] := \int_{\mathbb{R}^d} \varphi(w) d\gamma$  where  $d\gamma = e^{-\Psi} dx$  is a probability measure and  $\varphi$  satisfies (3.1). Most of the results presented here are known, but they are scattered in the literature. Our purpose here is to collect some essential statements and present simple proofs.

### 3.2.1 Generalized Csiszár-Kullback-Pinsker inequality

By assumption (3.1), we know that  $\mathcal{E}$  is nonnegative and achieves its minimum at  $w \equiv 1$ . It results from the strict convexity of  $\varphi$  that  $\mathcal{E}[w]$  controls a norm of  $(w - 1)$  under a generic assumption compatible with the expression of  $\varphi_p$ . The classical result of [141, 60, 120] has been extended in [117, 155, 47, 61]. Here is a statement, with a short proof taken from Section 1.4 of [19], for completeness.

**Proposition 3.2.** *Let  $p \in [1, 2]$ ,  $w \in L^1 \cap L^p(\mathbb{R}^d, d\gamma)$  be a nonnegative function, and assume that  $\varphi \in C^2(0, +\infty)$  is a nonnegative strictly convex function such that  $\varphi(1) = \varphi'(1) = 0$ . If  $A := \inf_{s \in (0, \infty)} s^{2-p} \varphi''(s) > 0$ , then*

$$\mathcal{E}[w] \geq 2^{-\frac{2}{p}} A \min \left\{ 1, \|w\|_{L^p(\mathbb{R}^d, d\gamma)}^{p-2} \right\} \|w - 1\|_{L^p(\mathbb{R}^d, d\gamma)}^2.$$

When  $\varphi = \varphi_p$ , we find that  $A = p$ . This inequality has many variants and extensions: it is not limited to  $\mathbb{R}^d$  but also holds on bounded domains or manifolds and the relative  $\varphi$ -entropy  $\int_{\mathbb{R}^d} (\varphi(w_1) - \varphi(w_2) - \varphi'(w_1)(w_2 - w_1)) d\gamma$  can also be used to measure  $\|w_2 - w_1\|_{L^p(\mathbb{R}^d, d\gamma)}^2$ .

*Proof.* Up to the addition of a small constant, we can assume that  $w > 0$  and argue by density. A Taylor expansion at order two shows that

$$\mathcal{E}[w] = \frac{1}{2} \int_{\mathbb{R}^d} \varphi''(\xi) |w - 1|^2 d\gamma \geq \frac{A}{2} \int_{\mathbb{R}^d} \xi^{p-2} |w - 1|^2 d\gamma$$

where  $\xi$  lies between 1 and  $w$ . With  $\alpha = p(2-p)/2$  and  $h > 0$ , for any measurable set  $\mathcal{A} \subset \mathbb{R}^d$ , we get

$$\int_{\mathcal{A}} |w - 1|^p h^{-\alpha} h^\alpha d\gamma \leq \left( \int_{\mathcal{A}} |w - 1|^2 h^{p-2} d\gamma \right)^{p/2} \left( \int_{\mathcal{A}} h^p d\gamma \right)^{(2-p)/2}$$

by Hölder's inequality. We apply this formula to two different sets.

On  $\mathcal{A} = \{x \in \mathbb{R}^d : w(x) > 1\}$ , we use  $\xi^{p-2} \geq w^{p-2}$  and take  $h = w$ :

$$\int_{\{w>1\}} |w - 1|^2 \xi^{p-2} d\gamma \geq \left( \int_{\{w>1\}} |w - 1|^p d\gamma \right)^{2/p} \|w\|_{L^p(\mathbb{R}^d, d\gamma)}^{p-2}.$$

On  $\mathcal{A} = \{x \in \mathbb{R}^d : w(x) \leq 1\}$ , we use  $\xi^{p-2} \geq 1$  and take  $h = 1$ :

$$\int_{\{w \leq 1\}} |w - 1|^2 \xi^{p-2} d\gamma \geq \left( \int_{\{w \leq 1\}} |w - 1|^p d\gamma \right)^{2/p}.$$

By adding these two estimates and using with  $r = 2/p \geq 1$  the elementary inequality  $(a + b)^r \leq 2^{r-1}(a^r + b^r)$  for any  $a, b \geq 0$  allows us to conclude the proof.  $\square$

### 3.2.2 Convexity, tensorization and sub-additivity

Let us turn our attention to (3.3). To start with, we observe that the functional  $w \mapsto \mathcal{I}[w] = \int_{\mathbb{R}^d} \varphi''(w) |\nabla w|^2 d\gamma$  is convex if and only if  $1/\varphi''$  is concave. Now let us consider two probability measures  $d\gamma_1$  and  $d\gamma_2$  defined respectively on  $\mathbb{R}^{d_1}$  and  $\mathbb{R}^{d_2}$ , such that Inequality (3.3) holds with  $\gamma = \gamma_i$ , and  $i = 1, 2$ :

$$\int_{\mathbb{R}^{d_i}} \varphi''(w) |\nabla w|^2 d\gamma_i =: \mathcal{I}_{\gamma_i}[w] \geq \Lambda_i \mathcal{E}_{\gamma_i}[w] \quad \forall w \in H^1(\mathbb{R}^{d_i}, d\gamma_i), \quad (3.11)$$

Here we denote by  $\mathcal{E}_\gamma$  the  $\varphi$ -entropy for functions which are not normalized, that is,

$$\mathcal{E}_\gamma[w] := \int_{\mathbb{R}^d} \varphi(w) d\gamma - \varphi\left(\int_{\mathbb{R}^d} w d\gamma\right).$$

Assuming that  $d\gamma$  is a probability measure, by Jensen's inequality we know that  $w \mapsto \mathcal{E}_\gamma[w]$  is nonnegative because  $\varphi$  is convex. As we shall see below,  $w \mapsto \mathcal{E}_\gamma[w]$  is also convex, which is the key ingredient for *tensorization*. The question at stake is to know if Inequality (3.3) holds on  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$  for the measure  $d\gamma = d\gamma_1 \otimes d\gamma_2$ . Most of the results of Section 3.2.2 have been stated in [58] or are considered as classical. Our contribution here is to give simplified proofs.

**Theorem 3.2.** *Assume that  $\varphi$  satisfies (3.1). If  $d\gamma_1$  and  $d\gamma_2$  are two probability measures on  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$  satisfying (3.11) with positive constants  $\Lambda_1$  and  $\Lambda_2$ , then  $d\gamma_1 \otimes d\gamma_2$  is such that the following inequality holds:*

$$\begin{aligned} \mathcal{I}_{\gamma_1 \otimes \gamma_2}[w] &= \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \varphi''(w) |\nabla w|^2 d\gamma_1 d\gamma_2 \\ &\geq \min\{\Lambda_1, \Lambda_2\} \mathcal{E}_{\gamma_1 \otimes \gamma_2}[w] \quad \forall w \in H^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, d\gamma). \end{aligned}$$

It is straightforward to notice that the Fisher information is additive

$$\mathcal{I}_{\gamma_1 \otimes \gamma_2}[w] = \int_{\mathbb{R}^{d_2}} \mathcal{I}_{\gamma_1}[w] d\gamma_2 + \int_{\mathbb{R}^{d_1}} \mathcal{I}_{\gamma_2}[w] d\gamma_1,$$

so that the proof of Theorem 3.2 can be reduced to the proof of a *sub-additivity* property of the  $\varphi$ -entropies that goes as follows.

**Proposition 3.3.** *Assume that  $\varphi$  satisfies (3.1) and consider two probability measures  $d\gamma_1$  and  $d\gamma_2$  on  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ . Then for any  $w \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, d\gamma_1 \otimes d\gamma_2)$ , we have*

$$\mathcal{E}_{\gamma_1 \otimes \gamma_2}[w] \leq \int_{\mathbb{R}^{d_2}} \mathcal{E}_{\gamma_1}[w] d\gamma_2 + \int_{\mathbb{R}^{d_1}} \mathcal{E}_{\gamma_2}[w] d\gamma_1 \quad \forall w \in L^1(d\gamma_1 \otimes d\gamma_2).$$

This last result relies on convexity properties that we are now going to study. As a preliminary step, we establish an inequality of Jensen type.

**Lemma 3.1.** *Let  $w \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, d\gamma_1 \otimes d\gamma_2)$  be a function of two variables  $(x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ . If  $\mathcal{F}_{\gamma_1}$  is a convex functional on  $L^1(d\gamma_1)$  such that*

$$\frac{d}{dt} \int_{\mathbb{R}^{d_2}} \mathcal{F}_{\gamma_1} \left[ t w + (1-t) \int_{\mathbb{R}^{d_2}} w d\gamma_2 \right] d\gamma_2 \Big|_{t=0} = 0, \quad (3.12)$$

then the following inequality holds:

$$\int_{\mathbb{R}^{d_2}} \mathcal{F}_{\gamma_1}[w] d\gamma_2 \geq \mathcal{F}_{\gamma_1} \left[ \int_{\mathbb{R}^{d_2}} w d\gamma_2 \right].$$

*Proof.* Let  $w_t = t w + (1-t) \int_{\mathbb{R}^{d_2}} w d\gamma_2$ . By convexity of  $\mathcal{F}_{\gamma_1}$ ,

$$\mathcal{F}_{\gamma_1}[w_t] \leq t \mathcal{F}_{\gamma_1}[w] + (1-t) \mathcal{F}_{\gamma_1} \left[ \int_{\mathbb{R}^{d_2}} w d\gamma_2 \right].$$

Hence it follows that

$$\mathcal{F}_{\gamma_1}[w_t] - \mathcal{F}_{\gamma_1} \left[ \int_{\mathbb{R}^{d_2}} w d\gamma_2 \right] \leq t \left( \mathcal{F}_{\gamma_1}[w] - \mathcal{F}_{\gamma_1} \left[ \int_{\mathbb{R}^{d_2}} w d\gamma_2 \right] \right),$$

from which we deduce that

$$0 = \frac{d}{dt} \mathcal{F}_{\gamma_1}[w_t] |_{t=0} \leq \mathcal{F}_{\gamma_1}[w] - \mathcal{F}_{\gamma_1} \left[ \int_{\mathbb{R}^{d_2}} w d\gamma_2 \right].$$

Conclusion holds after integrating with respect to  $\gamma_2$ .  $\square$

The second observation is the proof of the convexity of  $w \mapsto \mathcal{E}_\gamma[w]$ . The following result is taken from [121].

**Lemma 3.2.** *If  $\phi$  satisfies (3.1), then  $\mathcal{E}_\gamma$  is convex.*

*Proof.* We give a two steps proof of this result, for completeness.

• Define  $x_t = t y + (1-t)x$ ,  $t \in (0, 1)$ . Since  $1/\phi''$  is concave,

$$\frac{1}{\phi''(x_t)} \geq \frac{t}{\phi''(y)} + \frac{1-t}{\phi''(x)}. \quad (3.13)$$

The function  $\phi$  is convex, hence  $\phi''(x) > 0$  and  $\phi''(y) > 0$  and so

$$\frac{1}{\phi''(x_t)} \geq \frac{t}{\phi''(y)} \quad \text{and} \quad \frac{1}{\phi''(x_t)} \geq \frac{1-t}{\phi''(x)}.$$

This means

$$\phi''(y) \geq t \phi''(x_t) \quad \text{and} \quad \phi''(x) \geq (1-t) \phi''(x_t).$$

We can also rewrite (3.13) as

$$\phi''(x) \phi''(y) \geq (t \phi''(x) + (1-t) \phi''(y)) \phi''(x_t).$$

Consider the function

$$F_t(x, y) := t \phi(y) + (1-t) \phi(x) - \phi(x_t)$$

and observe that

$$\text{Hess}(F_t) = \begin{pmatrix} (1-t) \phi''(x) - (1-t)^2 \phi''(x_t) & -t(1-t) \phi''(x_t) \\ -t(1-t) \phi''(x_t) & t \phi''(y) - t^2 \phi''(x_t) \end{pmatrix}$$

is nonnegative since both diagonal terms are nonnegative and the determinant is nonnegative. The matrix  $\text{Hess}(F_t)$  is therefore nonnegative and  $F_t$  is convex.

• We observe that

$$\begin{aligned} & t \mathcal{E}_\gamma[w_1] + (1-t) \mathcal{E}_\gamma[w_0] - \mathcal{E}_\gamma[t w_1 + (1-t) w_0] \\ &= \int_{\mathbb{R}^d} F_t(w_1, w_0) d\gamma - F_t\left(\int_{\mathbb{R}^d} w_1 d\gamma, \int_{\mathbb{R}^d} w_0 d\gamma\right) \end{aligned}$$

is nonnegative by Jensen's inequality, which proves the result.  $\square$

**Proof of Proposition 3.3.** We claim that  $\mathcal{F}_{\gamma_1} = \mathcal{E}_{\gamma_1}$  satisfies (3.12). Indeed, let us consider  $w_t = t w + (1-t) w_0$  with  $w_0 := \int_{\mathbb{R}^{d_2}} w d\gamma_2$ . A simple computation shows that

$$\frac{d}{dt} \mathcal{F}_{\gamma_1}[w_t] = \int_{\mathbb{R}^{d_1}} \varphi'(w_t) (w - w_0) d\gamma_1 - \varphi'\left(\int_{\mathbb{R}^{d_1}} w_t d\gamma_1\right) \int_{\mathbb{R}^{d_1}} (w - w_0) d\gamma_1,$$

and, as a consequence at  $t = 0$ ,

$$\frac{d}{dt} \mathcal{F}_{\gamma_1}[w_t]|_{t=0} = \int_{\mathbb{R}^{d_1}} \varphi'(w_0) (w - w_0) d\gamma_1 - \varphi'\left(\int_{\mathbb{R}^{d_1}} w_0 d\gamma_1\right) \int_{\mathbb{R}^{d_1}} (w - w_0) d\gamma_1.$$

Since  $w_0$  does not depend on  $x_2$ , an integration with respect to  $\gamma_2$  concludes the proof of (3.12). From Lemma 3.1, we get

$$\int_{\mathbb{R}^{d_2}} \mathcal{E}_{\gamma_1}[w] d\gamma_2 \geq \mathcal{E}_{\gamma_1}\left[\int_{\mathbb{R}^{d_2}} w d\gamma_2\right].$$

By definition of  $\mathcal{E}_{\gamma_1}$ , this means

$$\begin{aligned} & \int_{\mathbb{R}^{d_2}} \left[ \int_{\mathbb{R}^{d_1}} \varphi(w) d\gamma_1 - \varphi\left(\int_{\mathbb{R}^{d_1}} w d\gamma_1\right) \right] d\gamma_2 \\ & \geq \int_{\mathbb{R}^{d_1}} \varphi\left(\int_{\mathbb{R}^{d_2}} w d\gamma_2\right) d\gamma_1 - \varphi\left(\iint_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} w d\gamma_1 \otimes \gamma_2\right), \end{aligned}$$

from which we deduce

$$\begin{aligned} & \int_{\mathbb{R}^{d_2}} \left[ \int_{\mathbb{R}^{d_1}} \varphi(w) d\gamma_1 - \varphi\left(\int_{\mathbb{R}^{d_1}} w d\gamma_1\right) \right] d\gamma_2 \\ & \quad + \int_{\mathbb{R}^{d_1}} \left[ \int_{\mathbb{R}^{d_2}} \varphi(w) d\gamma_2 - \varphi\left(\int_{\mathbb{R}^{d_2}} w d\gamma_2\right) \right] d\gamma_1 \\ & \geq \iint_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \varphi(w) d\gamma_1 \otimes \gamma_2 - \varphi\left(\iint_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} w d\gamma_1 \otimes \gamma_2\right). \end{aligned}$$

This ends the proof of Proposition 3.3.  $\square$

**Proof of Theorem 3.2.** The proof is an easy consequence of Proposition 3.3 and of the observation that

$$\begin{aligned} & \min\{\Lambda_1, \Lambda_2\} \mathcal{E}_{\gamma_1 \otimes \gamma_2}[w] \\ & \leq \Lambda_1 \int_{\mathbb{R}^{d_2}} \mathcal{E}_{\gamma_1}[w] d\gamma_2 + \Lambda_2 \int_{\mathbb{R}^{d_1}} \mathcal{E}_{\gamma_2}[w] d\gamma_1 \\ & \leq \iint_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \varphi''(w) [|\nabla_{x_1} w|^2 + |\nabla_{x_2} w|^2] d\gamma_1 \otimes \gamma_2 \\ & \leq \iint_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \varphi''(w) |\nabla w|^2 d\gamma_1 \otimes \gamma_2 = \mathcal{I}_{\gamma_1 \otimes \gamma_2}[w]. \end{aligned}$$

□

As a concluding remark, we observe that tensorization is not limited to probability measures on  $\mathbb{R}^d$ . The main interest of such an approach when dealing with  $\mathbb{R}^d$  is that it is enough to establish the inequality when  $d = 1$ . In the case  $d = 1$ , sharp criteria can be found in [17] (also see [16]). There are many related issues that can be traced back to the work of Muckenhoupt, e.g., [134] and Hardy (see [104]).

### 3.2.3 Entropy – entropy production inequalities: perturbation results

Perturbing the measure in the case of a Poincaré inequality is essentially trivial. In the case of the logarithmic Sobolev inequality, this has been done by Holley and Stroock in [112]. More general entropy functionals have been considered in [155], which cover all  $\varphi$ -entropies. Also see [4, 58].

Assume that for some probability measure  $d\gamma$  and for some  $\Lambda > 0$ , Inequality (3.3) holds, that is,

$$\Lambda \left[ \int_{\mathbb{R}^d} \varphi(w) d\gamma - \varphi(\bar{w}) \right] \leq \int_{\mathbb{R}^d} \varphi''(w) |\nabla w|^2 d\gamma \quad \forall w \in H^1(d\gamma). \quad (3.14)$$

Here we denote by  $\bar{w}$  the average of  $w$  with respect to  $d\gamma$ :  $\bar{w} := \int_{\mathbb{R}^d} w d\gamma$ . Assume that  $d\mu$  is a measure which is absolutely continuous with respect to  $d\gamma$  and such that

$$e^{-b} d\gamma \leq d\mu \leq e^{-a} d\gamma$$

for some constants  $a, b \in \mathbb{R}$ . The statement below generalizes the one of Lemma 5.2 of [32].

**Lemma 3.3.** *Under the above assumption, if  $\varphi$  is a  $C^2$  function such that  $\varphi'' > 0$ , then*

$$e^{a-b} \Lambda \int_{\mathbb{R}^d} [\varphi(w) - \varphi(\tilde{w}) - \varphi'(\tilde{w})(w - \tilde{w})] d\mu \leq \int_{\mathbb{R}^d} \varphi''(w) |\nabla w|^2 d\mu \quad \forall w \in H^1(d\mu),$$

where  $\tilde{w} := \int_{\mathbb{R}^d} w d\mu / \int_{\mathbb{R}^d} d\mu$ .

*Proof.* We start by observing that

$$\begin{aligned} e^b \int_{\mathbb{R}^d} \varphi''(w) |\nabla w|^2 d\mu &\geq \int_{\mathbb{R}^d} \varphi''(w) |\nabla w|^2 d\gamma = \mathcal{I}_\gamma[w] \\ &\geq \Lambda \mathcal{E}_\gamma[w] = \Lambda \left[ \int_{\mathbb{R}^d} \varphi(w) d\gamma - \varphi(\bar{w}) \right] \\ &= \Lambda \int_{\mathbb{R}^d} (\varphi(w) - \varphi(\bar{w}) - \varphi'(\bar{w})(w - \bar{w})) d\gamma. \end{aligned}$$

By convexity of  $\varphi$ , we know that  $\varphi(w) - \varphi(\bar{w}) - \varphi'(\bar{w})(w - \bar{w}) \geq 0$ , so that

$$\begin{aligned} \Lambda \mathcal{E}_\gamma[w] &\geq \Lambda e^a \int_{\mathbb{R}^d} (\varphi(w) - \varphi(\bar{w}) - \varphi'(\bar{w})(w - \bar{w})) d\mu \\ &= \Lambda e^a \int_{\mathbb{R}^d} (\varphi(w) - \varphi(\bar{w}) - \varphi'(\bar{w})(\tilde{w} - \bar{w})) d\mu. \end{aligned}$$

By convexity of  $\varphi$  again,  $\varphi(\bar{w}) + \varphi'(\bar{w})(\tilde{w} - \bar{w}) \leq \varphi(\tilde{w})$ , which shows that

$$\Lambda \mathcal{E}_\gamma[w] \geq \Lambda e^a \int_{\mathbb{R}^d} (\varphi(w) - \varphi(\tilde{w})) d\mu = e^a \Lambda \int_{\mathbb{R}^d} [\varphi(w) - \varphi(\tilde{w}) - \varphi'(\tilde{w})(w - \tilde{w})] d\mu$$

and completes the proof.  $\square$

### 3.2.4 Entropy – entropy production inequalities and linear flows

Let us consider the counterpart of the *Ornstein-Uhlenbeck equation* (3.2) on a smooth convex bounded domain  $\Omega$

$$\frac{\partial w}{\partial t} = \mathbb{L} w := \Delta w - \nabla \psi \cdot \nabla w, \quad (3.15)$$

supplemented with homogenous Neumann boundary conditions

$$\nabla w \cdot \nu = 0 \quad \text{on} \quad \partial\Omega,$$

where  $\nu$  denotes a unit outward pointing normal vector orthogonal to  $\partial\Omega$ . Let us consider the measure  $d\gamma = (\int_{\Omega} e^{-\psi} dx)^{-1} e^{-\psi} dx$ . If  $w$  solves (3.15) with a nonnegative initial datum  $w_0$  such that  $\int_{\Omega} w_0 d\gamma = 1$ , then mass is conserved so that  $\int_{\Omega} w(t, \cdot) d\gamma = 1$  for any  $t \geq 0$  and converges to 1 as  $t \rightarrow +\infty$ . The next question is how to measure the rate of convergence using the  $\varphi$ -entropy. For simplicity, let us assume that  $\varphi = \varphi_p$  for some  $p \in [1, 2]$ . An answer is given by the formal computation of Section 7.1, adapted to the bounded domain  $\Omega$ . Because of the boundary condition, it is straightforward to check that

$$\frac{d}{dt} \int_{\Omega} \frac{w^p - 1}{p-1} d\gamma = -\frac{4}{p} \int_{\Omega} |\nabla w^{p/2}|^2 d\gamma$$

if  $p > 1$  and a similar results holds when  $p = 1$ . Hence, if for some  $\Lambda > 0$  we can prove that

$$\int_{\Omega} \frac{w^p - 1}{p-1} d\gamma \leq \frac{4}{p\Lambda} \int_{\Omega} |\nabla w^{p/2}|^2 d\gamma \quad \text{for any } w \text{ such that } \int_{\Omega} w d\gamma = 1, \quad (3.16)$$



then we can conclude that  $\int_{\Omega} \frac{w^p - 1}{p-1} d\gamma$  decays like  $e^{-\Lambda t}$ . The main idea of the Bakry-Emery method, or *carré du champ* method, as it is exposed in [12] is that (3.16) can be established using the flow itself, by computing  $\frac{d}{dt} \int_{\Omega} |\nabla z|^2 d\gamma$  with  $z := w^{p/2}$ . Let us sketch the main steps of the proof.

As a preliminary observation, we notice that  $L$  is self-adjoint in  $L^2(\Omega, d\gamma)$  in the sense that

$$\int_{\Omega} w_1 (L w_2) d\gamma = - \int_{\Omega} \nabla w_1 \cdot \nabla w_2 d\gamma = \int_{\Omega} (L w_1) w_2 d\gamma$$

and also that

$$[\nabla, L] = -\text{Hess } \psi.$$

Using  $w = z^{2/p}$  we deduce from (3.15) that

$$\frac{\partial z}{\partial t} = L z + \frac{2-p}{p} \frac{|\nabla z|^2}{z}. \quad (3.17)$$

We adopt the convention that  $a \cdot b = \sum_{i=1}^d a_i b_i$  if  $a = (a_i)_{1 \leq i \leq d}$  and  $b = (b_i)_{1 \leq i \leq d}$  are two vectors with values in  $\mathbb{R}^d$ . If  $m = (m_{i,j})_{1 \leq i,j \leq d}$  and  $n = (n_{i,j})_{1 \leq i,j \leq d}$  are two matrices, then  $m : n = \sum_{i,j=1}^d m_{i,j} n_{i,j}$ . Also  $a \otimes b$  denotes the matrix  $(a_i b_j)_{1 \leq i,j \leq d}$ . We shall use  $|a|^2 = a \cdot a$  and  $\|m\|^2 = m : m$  for vectors and matrices respectively. With these notations, let us use (3.17) to compute

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla z|^2 d\gamma = \int_{\Omega} \nabla z \cdot \nabla \left( L z + \frac{2-p}{p} \frac{|\nabla z|^2}{z} \right) d\gamma \\ &= \int_{\Omega} \nabla z \cdot (L \nabla z - \text{Hess } \psi \nabla z) d\gamma + \frac{2-p}{p} \int_{\Omega} \nabla z \cdot \left( 2 \text{Hess } z \frac{\nabla z}{z} - \frac{|\nabla z|^2}{z} \nabla z \right) d\gamma \\ &= - \int_{\Omega} \|\text{Hess } z\|^2 d\gamma - \int_{\Omega} \text{Hess } \psi : \nabla z \otimes \nabla z d\gamma + \int_{\partial\Omega} \text{Hess } z : \nabla z \otimes \nu e^{-\psi} d\sigma \\ &\quad + 2 \frac{2-p}{p} \int_{\Omega} \text{Hess } z : \frac{\nabla z \otimes \nabla z}{z} d\gamma - \frac{2-p}{p} \int_{\Omega} \left\| \frac{\nabla z \otimes \nabla z}{z} \right\|^2 d\gamma \\ &= - \frac{2}{p} (p-1) \int_{\Omega} \|\text{Hess } z\|^2 d\gamma - \int_{\Omega} \text{Hess } \psi : \nabla z \otimes \nabla z d\gamma \\ &\quad - \frac{2-p}{p} \int_{\Omega} \left\| \text{Hess } z - \frac{\nabla z \otimes \nabla z}{z} \right\|^2 d\gamma + \int_{\partial\Omega} \text{Hess } z : \nabla z \otimes \nu e^{-\psi} d\sigma. \end{aligned}$$

Here  $d\sigma$  denotes the surface measure induced by Lebesgue's measure on  $\partial\Omega$ . We learn from Grisvard's lemma, see for instance Lemma 5.1 in [?] or [?], that  $\int_{\partial\Omega} \text{Hess } z : \nabla z \otimes \nu e^{-\psi} d\sigma$  is nonpositive as soon as  $\Omega$  is convex and  $\nabla z \cdot \nu = 0$  on  $\partial\Omega$ . As soon as we know that either

$$\text{Hess } \psi \geq \Lambda \star \text{Id}$$

for some  $\Lambda_\star > 0$ , or the inequality

$$\frac{2}{p}(p-1) \int_{\Omega} |\nabla X|^2 d\gamma + \int_{\Omega} \text{Hess } \psi : X \otimes X d\gamma \geq \Lambda(p) \int_{\Omega} |X|^2 d\gamma \quad \forall X \in H^1(\Omega, d\gamma)^d$$

holds for some  $\Lambda(p) > 0$ , which is a weaker assumption for any  $p > 1$ , then we obtain that

$$\frac{d}{dt} \int_{\Omega} |\nabla z|^2 d\gamma \leq -2\Lambda(p) \int_{\Omega} |\nabla z|^2 d\gamma.$$

Of course we know that  $\Lambda(p) \geq \Lambda_\star$ . By convention, we take  $\Lambda(1) = \Lambda_\star$ .

**Proposition 3.4.** *Assume that  $p \in [1, 2]$ ,  $\varphi = \varphi_p$  and, with the above notations,  $\Lambda(p) > 0$ . If  $\Omega$  is a smooth convex bounded domain in  $\mathbb{R}^d$ , then (3.16) holds with  $\Lambda = 2\Lambda(p)$ .*

*Proof.* It is straightforward. In view of the above computations, we know that

$$\frac{d}{dt} \left( \frac{4}{p\Lambda} \int_{\Omega} |\nabla w^{p/2}|^2 d\gamma - \int_{\Omega} \frac{w^p - 1}{p-1} d\gamma \right) \leq 0$$

and  $\lim_{t \rightarrow +\infty} \int_{\Omega} \frac{w^p - 1}{p-1} d\gamma = \lim_{t \rightarrow +\infty} \int_{\Omega} |\nabla w^{p/2}|^2 d\gamma = 0$ . This is enough to conclude that, for any  $t \geq 0$ ,

$$\frac{4}{p\Lambda} \int_{\Omega} |\nabla w^{p/2}|^2 d\gamma - \int_{\Omega} \frac{w^p - 1}{p-1} d\gamma \geq 0.$$

□

We conclude this section with the unbounded case  $\Omega = \mathbb{R}^d$ . For any given  $p \in [1, 2]$ , let us assume that the inequality

$$\frac{2}{p}(p-1) \int_{\mathbb{R}^d} |\nabla X|^2 d\gamma + \int_{\mathbb{R}^d} \text{Hess } \psi : X \otimes X d\gamma \geq \Lambda(p) \int_{\mathbb{R}^d} |X|^2 d\gamma \quad \forall X \in H^1(\mathbb{R}^d, d\gamma)^d$$

holds for some  $\Lambda(p) > 0$ . For  $p > 1$ , this assumption is a spectral gap condition on a vector valued Schrödinger operator: see for instance [85] for further details. With this assumption in hand, we have the following functional inequality, which interpolates between the logarithmic Sobolev inequality and the Poincaré inequality.

**Corollary 3.1.** *Assume that  $q \in [1, 2]$  and let us consider the probability measure  $d\gamma = e^{-\psi} dx$  on  $\mathbb{R}^d$ . Then with  $\Lambda = \Lambda(2/q)$ , we have*

$$\frac{\|f\|_{L^2(\mathbb{R}^d, d\gamma)}^2 - \|f\|_{L^q(\mathbb{R}^d, d\gamma)}^2}{2-q} \leq \frac{1}{\Lambda} \int_{\mathbb{R}^d} |\nabla f|^2 d\gamma \quad \forall f \in H^1(\mathbb{R}^d, d\gamma). \quad (3.18)$$

*Proof.* By homogeneity, we know from Proposition 3.4 that

$$\int_{\Omega} \frac{w^p - \bar{w}^p}{p-1} d\gamma \leq \frac{2}{p\Lambda(p)} \int_{\Omega} |\nabla w^{p/2}|^2 d\gamma$$

for all  $w$  such that  $f = w^{p/2}$ . Here we take  $p = 2/q$ . The conclusion holds by approximating  $\mathbb{R}^d$  by a growing sequence of bounded convex domains. □

An equivalent form of (3.18) is

$$\mathcal{I}[w] \geq \Lambda \mathcal{E}[w] \quad \forall w \in H^1(\mathbb{R}^d, d\gamma) \text{ such that } \int_{\mathbb{R}^d} w d\gamma = 1 \quad (3.19)$$

with the notation of Section 7.1,  $\varphi = \varphi_p$  and  $p = 2/q \in [1, 2]$ .

**Remark 3.1.** *The optimality of the constant  $\Lambda = 1$  in (3.18) is easy to obtain when  $\psi(x) = \frac{1}{2}|x|^2$ . With  $q = 1$ , (3.18) is the Gaussian Poincaré inequality*

$$\|f - \bar{f}\|_{L^2(\mathbb{R}^d, d\gamma)}^2 \leq \int_{\mathbb{R}^d} |\nabla f|^2 d\gamma \quad \forall f \in H^1(\mathbb{R}^d, d\gamma) \quad \text{with } \bar{f} = \int_{\mathbb{R}^d} f d\gamma,$$

with equality if  $f = f_1$ ,  $f_1(x) = x_1$ . By taking the limit as  $q \rightarrow 2_-$  in (3.18), we recover Gross' logarithmic Sobolev inequality

$$\int_{\mathbb{R}^d} f^2 \log \left( \frac{f^2}{\|f\|_{L^2(\mathbb{R}^d, d\gamma)}^2} \right) d\gamma \leq 2 \int_{\mathbb{R}^d} |\nabla f|^2 d\gamma \quad \forall f \in H^1(\mathbb{R}^d, d\gamma).$$

For any  $q \in [1, 2)$ , the equality case in (3.18) with  $\Lambda = 1$  is achieved by considering  $1 + \varepsilon f_1$  as a test function in the limit as  $\varepsilon \rightarrow 0$ .

From the point of view of the evolution equation, it is easy to see that the equality in (3.16) is achieved asymptotically as  $t \rightarrow +\infty$  by taking  $w = u/u_\star$  where  $u$  is the solution of (3.4) given by

$$u(t, x) = u_\star (x - x_\star(t))$$

with  $x_\star(t) = x_0 e^{-t}$  for any fixed  $x_0 \in \mathbb{R}^d$ .

### 3.2.5 Improved entropy – entropy production inequalities

In the proof of Proposition 3.4, the term  $\int_{\mathbb{R}^d} \|\text{Hess } z - \nabla z \otimes \nabla z / z\|^2 d\gamma$  has been dropped. In some cases, one can recombine the other terms differently and obtain an improved inequality if  $q \in (1, 2)$ . See [6] (and also [5] for a spectral point of view or [74] in the case of the sphere). The boundary term  $\int_{\partial\Omega} \text{Hess } z : \nabla z \otimes \nu e^{-\psi} d\sigma$  may also be of importance, as it is suggested in nonlinear problems by [78].

Let us give an example of an improvement, based on [6], in the special case  $\psi(x) = |x|^2/2$ . Using  $\text{Hess } \psi = \text{Id}$ , after approximating  $\mathbb{R}^d$  by bounded domains, we obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla z|^2 d\gamma + \int_{\mathbb{R}^d} |\nabla z|^2 d\gamma \leq & - \int_{\mathbb{R}^d} \left\| \text{Hess } z - \frac{2-p}{p} \frac{\nabla z \otimes \nabla z}{z} \right\|^2 d\gamma \\ & - \frac{2}{p} \kappa_p \int_{\mathbb{R}^d} \frac{|\nabla z|^4}{z^2} d\gamma \end{aligned}$$

with  $\kappa_p = (p-1)(2-p)/p$ . A simple Cauchy-Schwarz inequality shows that

$$\left( \int_{\mathbb{R}^d} |\nabla z|^2 d\gamma \right)^2 \leq \int_{\mathbb{R}^d} \frac{|\nabla z|^4}{z^2} d\gamma \int_{\mathbb{R}^d} z^2 d\gamma.$$

With the notations of Section 7.1, we have  $\int_{\mathbb{R}^d} z^2 d\gamma = \int_{\mathbb{R}^d} w^p d\gamma = 1 + (p-1)\mathcal{E}[w]$  and  $\int_{\mathbb{R}^d} |\nabla z|^2 d\gamma = \frac{p}{4}\mathcal{I}[w]$  so that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla z|^2 d\gamma + \int_{\mathbb{R}^d} |\nabla z|^2 d\gamma \leq -\frac{2}{p} \kappa_p \frac{\left( \int_{\mathbb{R}^d} |\nabla z|^2 d\gamma \right)^2}{\int_{\mathbb{R}^d} |z|^2 d\gamma}$$

can be rewritten as

$$\frac{d}{dt} \mathcal{I}[w] + 2\mathcal{I}[w] \leq -\kappa_p \frac{\mathcal{I}[w]^2}{1 + (p-1)\mathcal{E}[w]}. \quad (3.20)$$

We recall that we consider here the case  $\varphi = \varphi_p$ ,  $p \in (1, 2)$ , so that  $\kappa_p$  is positive and we can take advantage of (3.20) to obtain an improved version of Corollary 3.1. The following result follows the scheme of Theorem 2 in [6].

**Proposition 3.5.** *Assume that  $q \in (1, 2)$  and let us consider the Gaussian probability measure  $d\gamma = (2\pi)^{-d/2} e^{-|x|^2/2} dx$ . Then there exists a strictly convex function  $F$  on  $\mathbb{R}^+$  such that  $F(0) = 0$  and  $F'(0) = 1$ , for which*

$$\frac{1}{q} F \left( q \frac{\|f\|_{L^2(\mathbb{R}^d, d\gamma)}^2 - 1}{2 - q} \right) \leq \|\nabla f\|_{L^2(\mathbb{R}^d, d\gamma)}^2$$

for any  $f \in H^1(\mathbb{R}^d, d\gamma)$  such that  $\|f\|_{L^q(\mathbb{R}^d, d\gamma)} = 1$ .

*Proof.* The proof follows the strategy of [6]. Let  $e(t) := \frac{1}{p-1} \left( \int_{\mathbb{R}^d} z^2 d\gamma - 1 \right)$  where  $z = w^{p/2}$  solves (3.17) with initial datum  $f$ . We deduce from (3.20) that

$$e'' + 2e' \geq \frac{\kappa_p |e'|^2}{1 + (p-1)e} \geq \frac{\kappa_p |e'|^2}{1 + e}.$$

The function  $F(s) := \frac{1}{1-\kappa_p} [1 + s - (1+s)^{\kappa_p}]$  solves  $F' = 1 + \kappa_p \frac{F}{1+s}$  and we can check that (3.20) is equivalent to

$$\frac{d}{dt} \left( \frac{e' + 2F(e)}{(1+e)^{\kappa_p}} \right) \geq 0.$$

Since  $\lim_{t \rightarrow +\infty} (e'(t) + 2F(e(t))) = 0$ , we have shown that  $e' + 2F(e) \leq 0$  for any  $t \geq 0$ . This is true in particular at  $t = 0$ , with  $z(t=0, \cdot) = f$ .  $\square$

From the point of view of entropy – production of entropy inequalities, we have obtained that

$$\mathcal{I}[w] \geq 2F(\mathcal{E}[w])$$

where  $F$  is a strictly convex function such that  $F(0) = 0$  and  $F'(0) = 1$ . Using the homogeneity and substituting  $f/\|f\|_{L^q(\mathbb{R}^d, d\gamma)}$  to  $f$ , similar estimates have been used in [6] to prove that

$$\frac{2}{(2-q)^2} \left[ \|f\|_{L^2(\mathbb{R}^d, d\gamma)}^2 - \|f\|_{L^q(\mathbb{R}^d, d\gamma)}^{2(2-q)} \|f\|_{L^2(\mathbb{R}^d, d\gamma)}^{2(q-1)} \right] \leq \|\nabla f\|_{L^2(\mathbb{R}^d, d\gamma)}^2 \quad \forall f \in H^1(\mathbb{R}^d, d\gamma).$$

### 3.2.6 Interpolation inequalities: comments and extensions

The inequality of Corollary 3.1 appears in many papers. It is proved for the first time by the *carré du champ* method and any  $q \in [1, 2]$  in [12] in the case of a compact manifold, but special cases were known long before. For instance the case  $q = 2$  corresponding to the logarithmic Sobolev inequality can be traced back to [103, 96] (also see [162, 90] for related issues) but was already known as the Blachmann-Stam inequality [149]: see [160, 151] for a more detailed historical account. The case  $q = 1$  when  $\psi(x) = \frac{1}{2}|x|^2$  is known as the Gaussian Poincaré inequality. It appears for instance in [135] but was probably known much earlier in the framework of the theory of Hermite functions. In the case  $q \in (1, 2)$  when  $\psi(x) = \frac{1}{2}|x|^2$ , we may refer to [21] for a proof based on spectral methods, which has been extended in [5] to more general potentials.

One of the technical limitations of the *carré du champ* method is the difficulty of controlling the boundary terms in the various integrations by parts. In the above proof, we used Grisvard's lemma for convex domains. Alternative methods, which will not be exposed here, rely on the properties of Green's functions, or use direct spectral estimates.

Let us list some possible extensions:

- In Corollary 3.1, for any given  $q \in [1, 2]$ , we need that  $\Lambda(p)$  is positive only for  $p = 2/q$ . The condition for  $p = 1$ , which is equivalent to  $\text{Hess}\psi \geq \Lambda(1)\text{Id}$  with  $\Lambda(1) > 0$ , is not required unless  $q = 2$ . For any  $q < 2$ , the positivity condition of  $\Lambda(2/q)$  is a nonlocal condition, which allows  $\psi$  to be a non-uniformly strictly convex potential: see [85] for details.

- The case of unbounded convex domains can be considered. Reciprocally, according to [9], the case of a bounded convex domain  $\Omega$  can be deduced from the Euclidean case, by approximating a function  $\psi$  which takes the value  $+\infty$  on  $\Omega^c$  by smooth locally bounded potentials.

- Spectral methods can be used to establish that the family of inequalities of Corollary 3.1 interpolates between the logarithmic Sobolev inequality and the Poincaré inequality: this approach has been made precise in [21, 121], with extensions in [18, 5].

- Exhibiting a whole family of Lyapunov functionals for the same evolution equation needs an explanation that has been given in [84, 86]: to each entropy, we associate a notion of distance such that the equation appears as the gradient flow of the entropy.

In the context of linear diffusions and Markov processes,  $\phi$ -entropies are very natural objects which put the Gibbs entropy and the quadratic form associ-

ated to the Poincaré inequality in a common framework. It is therefore evident to ask the same question in a kinetic framework involving a degenerate diffusion operator coupled to a transport operator. Much less has been done so far and the next section is a contribution to the issue of optimal rates of convergence measured by  $\varphi_p$ -entropies, with a special emphasis on  $p \neq 1, 2$ .

### 3.3 Sharp rates for the kinetic Fokker-Planck equation

In this section, our goal is to provide a computation of the sharp exponential rate in Proposition 3.1 and establish the improvement of Theorem 3.1 by generalizing the estimate of Proposition 3.5 to the kinetic setting. The method follows the strategy of Section 3 of [159] in case  $p = 2$ , which is sometimes referred to as the  $H^1$  hypocoercivity method of C. Villani. This method is also known to cover the case  $p = 1$ . We extend it to any  $p \in [1, 2]$  and compute the precise algebraic expressions, which allows us to identify the sharp rate. Similar computations have been done in [7, 2, 1, 95, 131]. According to [53] (see earlier references therein), the Green function associated with (3.5) is a Gaussian kernel, so that integrations by parts can be performed on  $\mathbb{R}^d \times \mathbb{R}^d$  without any special precaution.

#### 3.3.1 $H^1$ hypocoercive estimates

Using the notation of Section 7.1, our strategy is to consider the solution  $h = g^{p/2}$  of (3.7), where  $g = f/f_\star$ , define

$$\mathcal{J}[h] := \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu + 2\lambda \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_x h d\mu + \nu \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu$$

and adjust the parameters  $\lambda$  and  $\nu$  in order to maximize  $\lambda_\star = \lambda_\star(\lambda, \nu) > 0$  so that

$$\frac{d}{dt} \mathcal{J}[h(t, \cdot, \cdot)] \leq -\lambda_\star(\lambda, \nu) \mathcal{J}[h(t, \cdot, \cdot)].$$

Since (3.6) is linear and preserves positivity, we recall that we can assume that  $g$  is nonnegative and such that  $\|g\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d, d\mu)} = 1$ . Let us define the notations:

$$\begin{aligned} H_{\nu\nu} &= \left( \frac{\partial^2 h}{\partial v_i \partial v_j} \right)_{1 \leq i, j \leq d}, & H_{x\nu} &= \left( \frac{\partial^2 h}{\partial x_i \partial v_j} \right)_{1 \leq i, j \leq d}, \\ M_{\nu\nu} &= \left( \frac{\partial \sqrt{h}}{\partial v_i} \frac{\partial \sqrt{h}}{\partial v_j} \right)_{1 \leq i, j \leq d}, & M_{x\nu} &= \left( \frac{\partial \sqrt{h}}{\partial x_i} \frac{\partial \sqrt{h}}{\partial v_j} \right)_{1 \leq i, j \leq d}. \end{aligned}$$

We start by observing that, up to a few integrations by parts, we obtain the identities

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu \\
&= - \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_v (v \cdot \nabla_x h - x \cdot \nabla_v h) d\mu + \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_v (\Delta_v h - v \cdot \nabla_v h) d\mu \\
&\quad + \left(\frac{2}{p} - 1\right) \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_v \left(\frac{|\nabla_v h|^2}{h}\right) d\mu \\
&= - \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_x h d\mu - \left( \int_{\mathbb{R}^d} \|H_{vv}\|^2 d\mu + \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu \right) \\
&\quad + \kappa \int_{\mathbb{R}^d} (H_{vv} : M_{vv} - 2 \|M_{vv}\|^2) d\mu \quad (3.21)
\end{aligned}$$

with  $\kappa = 8(2-p)/p$ ,

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu \\
&= - \int_{\mathbb{R}^d} \nabla_x h \cdot \nabla_x (v \cdot \nabla_x h - x \cdot \nabla_v h) d\mu + \int_{\mathbb{R}^d} \nabla_x h \cdot \nabla_x (\Delta_v h - v \cdot \nabla_v h) d\mu \\
&\quad + \left(\frac{2}{p} - 1\right) \int_{\mathbb{R}^d} \nabla_x h \cdot \nabla_x \left(\frac{|\nabla_v h|^2}{h}\right) d\mu \\
&= \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_x h d\mu - \int_{\mathbb{R}^d} \|H_{xv}\|^2 d\mu + \kappa \int_{\mathbb{R}^d} (H_{xv} : M_{xv} - 2 \|M_{xv}\|^2) d\mu, \quad (3.22)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_x h d\mu &= \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu - \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu - \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_x h d\mu \\
&\quad - 2 \int_{\mathbb{R}^d} H_{vv} : H_{xv} d\mu \\
&\quad + \kappa \int_{\mathbb{R}^d} (H_{vv} : M_{xv} + H_{xv} : M_{vv} - 4 M_{vv} : M_{xv}) d\mu. \quad (3.23)
\end{aligned}$$

Collecting these estimates shows that

$$\begin{aligned}
& -\frac{1}{2} \frac{d}{dt} \mathcal{F}[h(t, \cdot, \cdot)] \\
&= -\frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu + 2\lambda \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_x h d\mu + \nu \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu \right) \\
&= (1-\lambda) \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu + (1+\lambda-\nu) \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_x h d\mu + \lambda \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu \\
&\quad + \int_{\mathbb{R}^d} \|H_{vv}\|^2 d\mu - \kappa \int_{\mathbb{R}^d} (H_{vv} : M_{vv} - 2 \|M_{vv}\|^2) d\mu \\
&+ 2\lambda \int_{\mathbb{R}^d} H_{vv} : H_{xv} d\mu - \kappa \lambda \int_{\mathbb{R}^d} (H_{vv} : M_{xv} + H_{xv} : M_{vv} - 4 M_{vv} : M_{xv}) d\mu \\
&\quad + \nu \int_{\mathbb{R}^d} \|H_{xv}\|^2 d\mu - \kappa \nu \int_{\mathbb{R}^d} (H_{xv} : M_{xv} - 2 \|M_{xv}\|^2) d\mu
\end{aligned}$$

where  $\kappa = 8(2 - p)/p$ . This can be rewritten as

$$-\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} X^\perp \cdot \mathfrak{M}_0 X d\mu = \int_{\mathbb{R}^d} X^\perp \cdot \mathfrak{M}_1 X d\mu + \int_{\mathbb{R}^d} Y^\perp \cdot \mathfrak{M}_2 Y d\mu$$

where

$$\mathfrak{M}_0 = \begin{pmatrix} 1 & \lambda \\ \lambda & \nu \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d}, \quad \mathfrak{M}_1 = \begin{pmatrix} 1 - \lambda & \frac{1 + \lambda - \nu}{2} \\ \frac{1 + \lambda - \nu}{2} & \lambda \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d}$$

and

$$\mathfrak{M}_2 = \begin{pmatrix} 1 & \lambda & -\frac{\kappa}{2} & -\frac{\kappa\lambda}{2} \\ \lambda & \nu & -\frac{\kappa\lambda}{2} & -\frac{\kappa\nu}{2} \\ -\frac{\kappa}{2} & -\frac{\kappa\lambda}{2} & 2\kappa & 2\kappa\lambda \\ -\frac{\kappa\lambda}{2} & -\frac{\kappa\nu}{2} & 2\kappa\lambda & 2\kappa\nu \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d \times \mathbb{R}^d}$$

are bloc-matrix valued functions of  $(\lambda, \nu)$ , and

$$X = (\nabla_\nu h, \nabla_x h), \quad Y = (H_{\nu\nu}, H_{x\nu}, M_{\nu\nu}, M_{x\nu}).$$

The problem is reduced to a problem of linear algebra, namely to maximize

$$\lambda_\star(\lambda, \nu) := \min_{X \in \mathbb{R}^{2d}} \frac{X^\perp \cdot \mathfrak{M}_1(\lambda, \nu) X}{X^\perp \cdot \mathfrak{M}_0(\lambda, \nu) X}$$

on the set of parameters  $(\lambda, \nu) \in \mathbb{R}^2$  such that

$$\min_{Y \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}} \frac{Y^\perp \cdot \mathfrak{M}_2 Y}{\|Y\|^2} \geq 0.$$

Here  $X$  and  $Y$  now arbitrary vectors and matrices respectively in  $\mathbb{R}^{2d}$  and  $\mathbb{R}^{2d} \times \mathbb{R}^{2d}$ . Elementary computations show that  $\lambda$  and  $\nu$  must satisfy the condition  $\lambda^2 \leq \nu$  and also that  $\lambda_\star(\lambda, \nu)$  achieves its maximum at  $(\lambda, \nu) = (\frac{1}{2}, 1)$ , so that  $\lambda_\star(\frac{1}{2}, 1) = \frac{1}{2}$ . For  $(\lambda, \nu) = (\frac{1}{2}, 1)$ ,  $\mathfrak{M}_1(\frac{1}{2}, 1) = \frac{1}{2} \mathfrak{M}_0(\frac{1}{2}, 1)$  and the eigenvalues of  $\mathfrak{M}_2(\frac{1}{2}, 1)$  are given as a function of  $\kappa = 8(2 - p)/p$  by

$$\lambda_1(\kappa) := \frac{1}{4} \left( 2\kappa + 1 - \sqrt{5\kappa^2 - 4\kappa + 1} \right), \quad \lambda_2(\kappa) := \frac{3}{4} \left( 2\kappa + 1 - \sqrt{5\kappa^2 - 4\kappa + 1} \right),$$

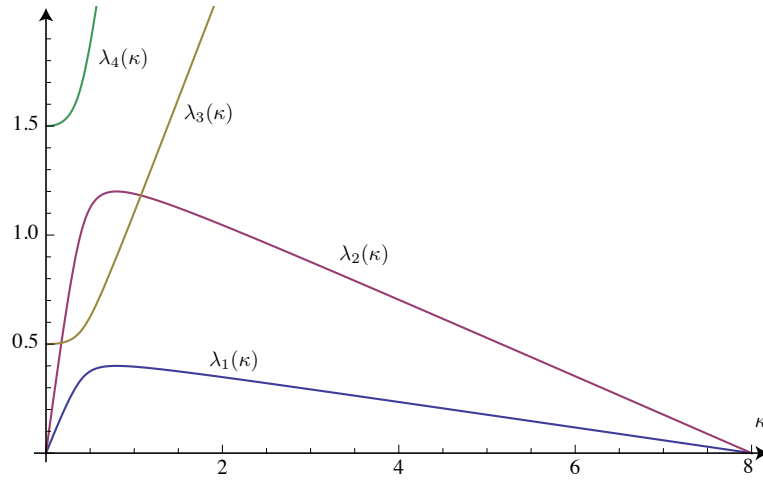
$$\lambda_3(\kappa) := \frac{1}{4} \left( 2\kappa + 1 + \sqrt{5\kappa^2 - 4\kappa + 1} \right), \quad \lambda_4(\kappa) := \frac{3}{4} \left( 2\kappa + 1 + \sqrt{5\kappa^2 - 4\kappa + 1} \right).$$

In the range  $p \in [1, 2]$ , which means  $\kappa \in [0, 8]$ , they are all nonnegative: see Fig. 3.1. Since  $\lambda_1(\kappa)$  is the lowest eigenvalue, we have proved the following result.

**Lemma 3.4.** *With the above notations and  $(\lambda, \nu) = (\frac{1}{2}, 1)$ , we have the estimate*

$$\int_{\mathbb{R}^d} X^\perp \cdot \mathfrak{M}_1 X d\mu + \int_{\mathbb{R}^d} Y^\perp \cdot \mathfrak{M}_2 Y d\mu \geq \frac{1}{2} \int_{\mathbb{R}^d} X^\perp \cdot \mathfrak{M}_0 X d\mu + \frac{1}{4} \left( 2\kappa + 1 - \sqrt{5\kappa^2 - 4\kappa + 1} \right) |Y|^2.$$



Figure 3.1: Plot of the eigenvalues of  $\mathfrak{M}_2(\frac{1}{2}, 1)$  as a function of  $\kappa$ .

### 3.3.2 Proof of Proposition 3.1

Assume that  $h$  solves (3.7). With  $(\lambda, \nu) = (\frac{1}{2}, 1)$ , we deduce from Lemma 3.4 that  $\mathcal{J}[h]$  is defined by (3.9). Then it satisfies the differential inequality

$$\frac{d}{dt} \mathcal{J}[h(t, \cdot, \cdot)] \leq -\mathcal{J}[h(t, \cdot, \cdot)],$$

from which we deduce that

$$\mathcal{J}[h(t, \cdot, \cdot)] \leq \mathcal{J}[h(0, \cdot, \cdot)] e^{-t} \quad \forall t \geq 0.$$

Using (3.3) with  $d\gamma = \mu dx dv$ ,  $\lambda = 1$  and  $\varphi = \varphi_p$  for any  $p \in [1, 2]$  (also see Remark 3.1), we obtain that

$$\mathcal{E}[h(t, \cdot, \cdot)] \leq \mathcal{J}[h_0] e^{-t} \quad \forall t \geq 0$$

if  $h$  is the solution of (3.7) with initial datum  $h_0$ .

The optimality of the rate is established by considering an initial datum which is a decentred stationary solution. With the notations of Section 7.1, let

$$f_0(x, v) = f_\star(x - x_0, v - v_0) \quad \forall (x, v) \in \mathbb{R}^d \times \mathbb{R}^d$$

for some  $(x_0, v_0) \neq (0, 0)$ . The reader is invited to check that

$$f(t, x, v) = f_\star(x - x_\star(t), v - v_\star(t))$$

with

$$\begin{cases} x_\star(t) = \left( \cos\left(\frac{\sqrt{3}}{2} t\right) x_0 + \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t\right) \left(v_0 + \frac{x_0}{2}\right) \right) e^{-\frac{t}{2}}, \\ v_\star(t) = \left( -\frac{\sqrt{3}}{2} \sin\left(\frac{\sqrt{3}}{2} t\right) \left(x_0 + \frac{v_0}{2}\right) + \cos\left(\frac{\sqrt{3}}{2} t\right) v_0 \right) e^{-\frac{t}{2}}, \end{cases} \quad (3.24)$$

solves (3.5). Now let us compute the entropy as  $t \rightarrow +\infty$ : with  $g = f/f_\star$  and  $\varphi = \varphi_p$ , we obtain that, as  $t \rightarrow +\infty$ ,

$$\begin{aligned} \mathcal{E}[g(t, \cdot, \cdot)] &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_p(g) d\mu = \frac{p}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |g-1|^2 d\mu (1+o(1)) \\ &= \frac{p}{2} (|x_\star(t)|^2 + |v_\star(t)|^2) (1+o(1)) = O(e^{-t}). \end{aligned}$$

This proves that the rate  $e^{-t}$  of Proposition 3.1 is optimal and completes the proof.  $\square$

Compared to the proof of Proposition 3.1, a refined estimate can be obtained by observing that, in the computation of  $\frac{d}{dt} \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu$  and  $\frac{d}{dt} \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu$ , we have

$$\begin{aligned} \|H_{vv}\|^2 - \kappa H_{vv} : M_{vv} + 2\kappa \|M_{vv}\|^2 &\geq 0, \\ \|H_{xv}\|^2 - \kappa H_{xv} : M_{xv} + 2\kappa \|M_{xv}\|^2 &\geq 0, \end{aligned}$$

with  $\kappa = 8(2-p)/p$ . Let us define

$$\begin{aligned} a := e^t \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu, \quad b := e^t \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_x h d\mu, \quad c := e^t \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu, \\ \text{and } j := a + b + c. \end{aligned}$$

We deduce from (3.21), (3.22) and (3.23) that

$$\frac{da}{dt} \leq a - 2(j-c), \quad \frac{dc}{dt} \leq 2(j-a) - c \quad \text{and} \quad \frac{dj}{dt} \leq 0$$

while we know by definition of  $a$ ,  $b$  and  $c$  and by the Cauchy-Schwarz estimate that

$$a \geq 0, \quad c \geq 0 \quad \text{and} \quad b^2 \leq ac.$$

In terms of  $a$  and  $c$ , the inequality  $b^2 = (a+c-j)^2 \leq ac$  means that the problem is constrained to the interior of an ellipse, and that  $a=0$  if and only if  $c=j$ : see Fig. 3.2. Finally, let us observe that we have the following property.

**Lemma 3.5.** *Assume that  $p \in [1, 2]$ ,  $\psi(x) = |x|^2/2$  and let  $h$  be a solution of (3.7) with initial datum  $h_0 \in L^1 \cap L^p(\mathbb{R}^d, d\gamma)$ . With the above notations, if for some  $t_0 > 0$ ,  $a(t_0) = 0$  and  $j(t_0) \neq 0$ , then for any  $t > t_0$  with  $t - t_0$  small enough, we have  $a(t) > 0$ .*

*Proof.* From the equivalence of (3.5) and (3.7), we know that  $h$  is smooth because of the expression of Green's function. By definition of  $b$  and  $j$ , we have that  $b(t_0) = 0$  and  $c(t_0) = j(t_0) > 0$ . Since  $a(t_0) = 0$  means that  $h$  does not depend on  $v$ , we know that  $\frac{dj}{dt}(t_0) = j(t_0) > 0$ , hence proving that  $a(t) > 0$  for  $t - t_0 > 0$ , small, because of the condition  $b^2 \leq ac$  and  $\frac{dc}{dt} \leq 0$ , which means that  $t \mapsto (a(t), c(t))$  is constrained to the interior of the ellipse of Fig. 3.2.  $\square$

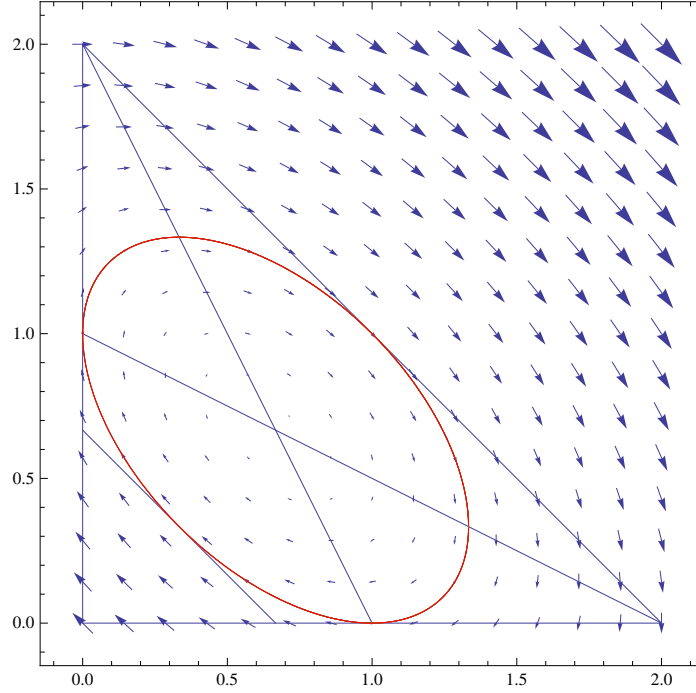


Figure 3.2: Plot of the vector field associated with the ODEs  $\frac{da}{dt} = a - 2(j - c)$  and  $\frac{dc}{dt} = 2(j - a) - c$ . The coordinates are  $a/j$  (horizontal axis) and  $c/j$  (vertical axis). The two straight lines intersecting at the center of the ellipse are defined by  $2(j - a) - c = 0$  and  $a - 2j + 2c = 0$ .

### 3.3.3 Proof of Theorem 3.1

Let us consider the *Fisher information* functional as defined in (3.10). A computation shows that

$$-\frac{1}{2} \frac{d}{dt} \mathcal{F}_{\lambda(t)}[h(t, \cdot)] = X^\perp \cdot \mathfrak{M}_1 X - \frac{1}{2} \lambda'(t) X^\perp \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X + Y^\perp \cdot \mathfrak{M}_2 Y$$

where  $\mathfrak{M}_0$ ,  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are defined as before, with  $v = 1$ , and  $X = (\nabla_v h, \nabla_x h)$ ,  $Y = (H_{vv}, H_{xv}, M_{vv}, M_{xv})$ . We take of course  $\lambda = \lambda(t)$ . We know that

$$Y^\perp \cdot \mathfrak{M}_2 Y \geq \lambda_1(p, \lambda) |Y|^2$$

for some  $\lambda_1(p, \lambda)$  such that  $\lambda_1(p, 1/2) = \frac{1}{4} (2\kappa + 1 - \sqrt{5\kappa^2 - 4\kappa + 1}) > 0$  if  $p \in (1, 2)$ , and  $\kappa = 8(2 - p)/p$ . For any  $p \in (1, 2)$ , by continuity we know that  $\lambda_1(p, \lambda) > 0$  if  $\lambda - 1/2 > 0$  is taken small enough. From  $|Y|^2 \geq \|M_{vv}\|^2$  and, by Cauchy-Schwarz,

$$\left( \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu \right)^2 \leq \int_{\mathbb{R}^d} h^2 d\mu \int_{\mathbb{R}^d} \|M_{vv}\|^2 d\mu \leq c_0 \int_{\mathbb{R}^d} \|M_{vv}\|^2 d\mu$$

where  $c_0 := 1 + (p-1)\mathcal{E}[h_0^{2/p}]$ , we obtain

$$-\frac{1}{2} \frac{d}{dt} \mathcal{J}_{\lambda(t)}[h(t, \cdot)] \geq X^\perp \cdot \mathfrak{M}_1 X + \frac{1}{2} \lambda'(t) X^\perp \cdot \mathfrak{M}_0 X + \varepsilon X^\perp \cdot \mathfrak{M}_3 X$$

with  $\varepsilon = \lambda_1(p, \lambda) c_0^{-1} \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu$  and  $\mathfrak{M}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \text{Id}_{\mathbb{R}^d}$ . We recall that  $\mathfrak{a}$  is defined by  $\mathfrak{a} = e^t \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu$  is positive except for isolated values of  $t > 0$ . Our goal is to find  $\lambda(t)$  and  $\rho(t) > 1/2$  such that

$$X^\perp \cdot \mathfrak{M}_1 X - \frac{1}{2} \lambda'(t) X^\perp \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X + \varepsilon X^\perp \cdot \mathfrak{M}_3 X \geq \rho(t) X^\perp \cdot \mathfrak{M}_0 X$$

for any  $X \in \mathbb{R}^{2d}$ .

To establish the existence of  $\rho > 1/2$  a.e., we proceed in several steps.

- If  $\mathfrak{a} \geq \mathfrak{a}_\star$  for some constant  $\mathfrak{a}_\star > 0$ , then we define  $\varepsilon(t) = \nu e^{-t}$  with  $\nu = \lambda_1(p, \lambda) c_0^{-1} \mathfrak{a}_\star$ ,  $\lambda(t) = (1 + \varepsilon(t))/2$  and  $\rho(t) = \frac{1}{2} (1 + \nu/(\nu + 3e^t))$ . The same estimate holds on any subinterval of  $\mathbb{R}^+$ .

- If  $\mathfrak{a}(t_0) = 0$  for some  $t_0 \geq 0$ , then in a neighborhood of  $(t_0)_+$ , we can solve

$$\frac{d\lambda}{dt} = \nu \varepsilon(t), \quad \lambda(t_0) = \frac{1}{2}.$$

An eigenvalue computation shows that

$$\mathfrak{M}_1 + \frac{1}{2} \nu \varepsilon \mathfrak{M}_0 + \varepsilon \mathfrak{M}_3 \geq \zeta(\varepsilon, \lambda, \nu) \mathfrak{M}_0$$

with

$$\zeta(0, \frac{1}{2}, \nu) = \frac{1}{2}, \quad \frac{\partial \zeta}{\partial \varepsilon}(0, \frac{1}{2}, \nu) = \frac{2 + \sqrt{3} - 2\nu}{3}, \quad \frac{\partial \zeta}{\partial \lambda}(0, \frac{1}{2}, \nu) = -\frac{2}{\sqrt{3}}.$$

We choose an arbitrary  $\nu \in (0, 1 + \sqrt{3}/2)$ . Since  $0 < \lambda(t) - 1/2 = o(\varepsilon(t))$  for  $t - t_0 > 0$ , small enough, this guarantees that  $\rho(t) = \zeta(\varepsilon(t), \lambda(t), \nu)$  satisfies  $\rho(t) > 1/2$  on a neighborhood of  $(t_0)_+$ .

- If  $\zeta(t_0) = 0$  for some  $t_0 > 0$ , then in a neighborhood of  $(t_0)_-$ , we proceed as above with some  $\nu < 0$ .

- If  $(t_n)_{n \in \mathbb{N}}$  is the increasing sequence of points such that  $\mathfrak{a}(t_n) = 0$  and if  $\mathfrak{a}(t) > 0$  for any  $t \in \mathbb{R}^+$  such that  $t \neq t_n$  for any  $n \in \mathbb{N}$ , we can choose a constant  $\mathfrak{a}_\star$ , small enough, on any interval  $(t_n, t_{n+1})$  and glue the above solutions to obtain a function  $\rho(t) > 1/2$  on  $(0, t_0)$  and  $\cup_{n \in \mathbb{N}} (t_n, t_{n+1})$ . It is an open question to decide

if there is an increasing sequence, finite or infinite, of times  $t_n$  such that  $a(t_n) = 0$ , or if  $a(t)$  is positive for any  $t > 0$ . We can of course impose that  $a(t_0) = 0$  at  $t_0 = 0$  by taking an initial datum  $h_0$  which does not depend on  $\nu$ . If such a sequence  $(t_n)_{n \in \mathbb{N}}$  exists, then we know that  $\lambda(t_n) = 1/2$  so that we have the remarkable decay estimate

$$\mathcal{J}_{\frac{1}{2}}[h(t_{n+1}, \cdot)] \leq \mathcal{J}_{\frac{1}{2}}[h(t_n, \cdot)] e^{-2 \int_{t_n}^{t_{n+1}} \rho(s) ds} < \mathcal{J}_{\frac{1}{2}}[h(t_n, \cdot)] e^{-(t_{n+1} - t_n)}$$

for any  $p \in (1, 2)$ . As far as  $a$  is concerned, we expect that it has some oscillatory behaviour as indicated by the vector field in Fig. 3.2, but since terms involving  $Y$  are neglected, this is so far formal. In any case, we can choose  $\lambda(t)$  such that  $\lim_{t \rightarrow +\infty} \lambda(t) = 1/2$ . This concludes the proof of Theorem 3.1.  $\square$

### 3.3.4 Concluding remarks

Even if the global rate cannot be improved because it is determined by the large time asymptotics, at any finite time the instantaneous rate of decay is strictly higher in the case of the diffusions studied in Sections 3.2.4-3.2.5, or at least higher at almost any time in the case of the kinetic equation, according to Theorem 3.1.

As  $t \rightarrow +\infty$ , Theorem 3.1 provides us with an improved estimate of the leading order term. The exponential decay rate cannot be improved as shown by (3.24), but we prove that there is a constant less than 1 to be taken into account. This observation is reminiscent of what happens for nonlinear diffusions of porous medium or fast diffusion type, which goes as follows. When looking at the relative entropy with respect to the *best matching* (in the sense of relative entropy) profiles, it turns out that there is a delay  $\tau$  compared to the relative entropy with respect to a fixed Barenblatt profile. As a result, we obtain a multiplicative factor  $e^{-\tau}$  corresponding to an improved estimate in an asymptotic expansion as  $t \rightarrow +\infty$  [89]. We have a similar property when we study the large time behavior of the solutions of (3.6) using a  $\phi_p$ -entropy for any given  $p \in (1, 2)$ .

The key estimate of Theorem 3.1 asserts that

$$\frac{d}{dt} \mathcal{J}_{\lambda(t)}[h(t, \cdot)] \leq -2\rho(t) \mathcal{J}_{\lambda(t)}[h(t, \cdot)] \leq -\mathcal{J}_{\lambda(t)}[h(t, \cdot)]$$

where the last inequality is strict for almost any value of  $t \geq 0$  (unless  $h$  is a stationary solution). Now, let us consider the large time asymptotics and define

$$\tau := \lim_{t \rightarrow +\infty} \left( 2 \int_0^t \rho(s) ds - t \right).$$

We cannot expect that  $\tau = +\infty$  for any initial datum but at least show that  $\tau$  is positive (unless  $h$  is a stationary solution), so that for large values of  $t$  we have

$$\mathcal{J}_{1/2}[h(t, \cdot)] \lesssim e^{-\tau} \mathcal{J}_{1/2}[h_0] e^{-t}. \quad (3.25)$$

For instance, in case of (3.24), one can prove that  $\rho(t) - 1/2$  is of the order of  $e^{-t}$  and  $\tau$  is finite. With  $e^{-\tau} < 1$ , (3.25) is anyway a strict improvement of the usual estimate as  $t \rightarrow +\infty$ .

The improvement of Theorem 3.1 is obtained only for *almost any* time: according to Lemma 3.5, the optimal decay rate could eventually be realized at an increasing sequence of times  $t_n \nearrow +\infty$ , but the solution will then deviate and temporarily regain a faster decay rate. Qualitatively, this comes from the oscillations in the phase space corresponding to the ODE associated with the vector field shown in Fig. 3.2. Such a pattern is consistent with what is known of the rates measured by *hypocoercive methods* in kinetic equations.



## Chapter 4

# Flocking model

**Article *Flocking: phase transition and asymptotic behaviour*. Appeared in <https://hal.archives-ouvertes.fr/hal-02143985> and <https://arxiv.org/abs/1906.07517>**

This chapter is devoted to a continuous McKean-Vlasov model with noise, which has isotropic and polarized stationary solutions depending on the intensity of the noise. The first result establishes the threshold value of the noise parameter which drives the phase transition. This threshold value is used to classify all stationary solutions and their linear stability properties. Using an entropy, these stability properties are extended to the non-linear regime. The second result is concerned with the asymptotic behaviour of the solutions of the evolution problem. In several cases, we prove that stable solutions attract the other solutions with an optimal exponential rate of convergence determined by the spectral gap of the linearized problem around the stable solutions. The spectral gap has to be computed in a norm adapted to the non-local term.



## 4.1 Introduction

In many fields such as biology, ecology or economic studies, emerging collective behaviours and self-organization in multiagent interactions have attracted the attention of many researchers. In this paper we consider the McKean-Vlasov model in order to describe *flocking*. The original model of [63] is Cucker-Smale model, which describes a population of  $N$  birds moving in  $\mathbb{R}^3$  by the equations

$$v_i(t_n + \Delta t) - v_i(t_n) = \frac{\lambda \Delta t}{N} \sum_{j=1}^N a_{ij}(v_j(t_n) - v_i(t_n)), \quad i = 1, 2 \dots N$$

at discrete times  $t_n = n\Delta t$  with  $n \in \mathbb{N}$  and  $\Delta t > 0$ . Here  $v_i$  is the velocity of the  $i$ th bird, the model is homogeneous in the sense that there is no position variable, and the coefficients  $a_{ij}$  model the interaction between pairs of birds as a function of their relative velocities, while  $\lambda$  is an overall coupling parameter. The authors proved that under certain conditions on the parameters, the solution converges to a state in which all birds fly with the same velocity. Another model is the Vicsek model [157] which was derived earlier to study the evolution of a population in which individuals have a given speed but the direction of their velocity evolves according to a diffusion equation with a local alignment term. This model exhibits phase transitions. In [65, 66, 98, 154], phase transition has been shown in a continuous version of the model: with high noise, the system is disordered and the average velocity is zero, while for low noise a direction is selected.

Here we consider a model on  $\mathbb{R}^d$ ,  $d \geq 1$  with noise as in [36, 62]. The population is described by a distribution function  $f(v, t)$  in which the interaction occurs through a mean-field nonlinearity known as *local velocity consensus* and we also equip the individuals with a so-called *self-propulsion* mechanism which privileges a speed (without a privileged direction) but does not impose a single value to the speed as in the Vicsek model. The distribution function solves

$$\frac{\partial f}{\partial t} = D \Delta f + \nabla \cdot \left( (v - \mathbf{u}_f) f + \alpha v (|v|^2 - 1) f \right), \quad f(\cdot, 0) = f_{\text{in}} > 0 \quad (4.1)$$

where  $t \geq 0$  denotes the time variable and  $v \in \mathbb{R}^d$  is the velocity variable. Here  $\nabla$  and  $\Delta$  are the gradient and the Laplacian with respect to  $v$  respectively. The parameter  $D > 0$  measures the intensity of the noise,  $\alpha > 0$  is the parameter of self-propulsion which tends to force the distribution to be centered on velocities  $|v|$  of the order of 1 when  $\alpha$  becomes large, and

$$\mathbf{u}_f(t) = \frac{\int_{\mathbb{R}^d} v f(t, v) dv}{\int_{\mathbb{R}^d} f(t, v) dv}$$

is the *mean velocity*. We refer to [14] for more details. Notice that (4.1) is one-homogeneous: from now on, we will assume that the mass satisfies  $\int_{\mathbb{R}^d} f(t, v) dv =$

1 for any  $t \geq 0$ , without loss of generality. In (4.1), the velocity consensus term  $v - \mathbf{u}_f$  can be interpreted as a friction force which tends to align  $v$  and  $\mathbf{u}_f$ . Altogether, individuals are driven to a velocity corresponding to a speed of order 1 and a direction given by  $\mathbf{u}_f$ , but this mechanism is balanced by the noise which pushes the system towards an isotropic distribution with zero average velocity. The Vicsek model can be obtained as a limit case in which we let  $\alpha \rightarrow +\infty$ : see [40]. The competition between the two mechanisms, relaxation towards a non-zero average velocity and noise, is responsible for a phase transition between an ordered state for small values of  $D$ , with a distribution function  $f$  centered around  $\mathbf{u}$  with  $\mathbf{u} \neq \mathbf{0}$ , and a disordered, *symmetric* state with  $\mathbf{u} = \mathbf{0}$ . This phase transition can also be interpreted as a *symmetry breaking* mechanism from the *isotropic distribution* to an ordered, asymmetric or *polarized distribution*, with the remarkable feature that nothing but the initial datum determines the direction of  $\mathbf{u}_f$  for large values of  $t$  and any stationary solution generates a continuum of stationary solutions by rotation. We refer to [154] for more detailed comments and additional references on related models.

So far, a phase transition has been established in [154] when  $d = 1$  and it has been proved in [13] by A. Barbaro, J. Canizo, J. Carrillo and P. Degond that stationary solutions are isotropic for large values of  $D$  while symmetry breaking occurs as  $D \rightarrow 0$ . The bifurcation diagram showing the phase transition has also been studied numerically in [13] and the phase diagram can be found in [154, Theorem 2.1]. The first purpose of this paper is to classify all stable and unstable stationary solutions and establish a complete description of the phase transition.

**Theorem 4.1.** *Let  $d \geq 1$  and  $\alpha > 0$ . There exists a critical intensity of the noise  $D_* > 0$  such that*

- (i) *if  $D \geq D_*$  there exists one and only one non-negative stationary distribution which is isotropic, and stable if  $D > D_*$ ,*
- (ii) *if  $D < D_*$  there exist one and only one non-negative isotropic stationary distribution which is unstable, and a continuum of stable non-negative non-symmetric stationary distributions, but this non-symmetric stationary solution is unique up to a rotation.*

Under the assumption of mass normalization to 1, it is straightforward to observe that any stationary solution can be written as

$$f_{\mathbf{u}}(v) = \frac{e^{-\frac{1}{D} \left( \frac{1}{2} |v - \mathbf{u}|^2 + \frac{\alpha}{4} |v|^4 - \frac{\alpha}{2} |v|^2 \right)}}{\int_{\mathbb{R}^d} e^{-\frac{1}{D} \left( \frac{1}{2} |v - \mathbf{u}|^2 + \frac{\alpha}{4} |v|^4 - \frac{\alpha}{2} |v|^2 \right)} dv}$$

where  $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{R}^d$  solves  $\int_{\mathbb{R}^d} (\mathbf{u} - v) f_{\mathbf{u}}(v) dv = 0$ . Up to a rotation, we can assume that  $\mathbf{u} = (u, 0, \dots, 0) = u e_1$  and the question of finding stationary solutions

to (4.1) is reduced to solve  $u \in \mathbb{R}$  such that

$$\mathcal{H}(u) = 0 \quad (4.2)$$

where

$$\mathcal{H}(u) := \int_{\mathbb{R}^d} (v_1 - u) e^{-\frac{1}{D}(\phi_\alpha(v) - u v_1)} dv \quad \text{and} \quad \phi_\alpha(v) := \frac{\alpha}{4}|v|^4 + \frac{1-\alpha}{2}|v|^2.$$

Obviously  $u = 0$  is always a solution. Moreover, if  $u$  is a solution of (4.2), then  $-u$  is also a solution. As a consequence, from now on, we always suppose that  $u \geq 0$ . Theorem 4.1 is proved in Section 4.2 by analyzing (4.2). Taking  $\mathbf{u} = u e_1$  is a key but straightforward idea in case of stationary solution, which however does not adapt so easily to non-stationary solutions of the evolution problem.

The second purpose of this paper is to study the stability of the stationary states and the rates of convergence of the solutions of the evolution problem. A key tool is the *free energy*

$$\mathcal{F}[f] := D \int_{\mathbb{R}^d} f \log f dv + \int_{\mathbb{R}^d} f \phi_\alpha dv - \frac{1}{2} |\mathbf{u}_f|^2 \quad (4.3)$$

and we shall also consider the *relative entropy* with respect to  $f_{\mathbf{u}}$  defined as

$$\mathcal{F}[f] - \mathcal{F}[f_{\mathbf{u}}] = D \int_{\mathbb{R}^d} f \log \left( \frac{f}{f_{\mathbf{u}}} \right) dv - \frac{1}{2} |\mathbf{u}_f - \mathbf{u}|^2$$

where  $f_{\mathbf{u}}$  is a stationary solution to be determined. Notice that  $f_{\mathbf{u}}$  is a critical point of  $\mathcal{F}$  under the mass constraint. Since there is only one stationary solution  $f_{\mathbf{u}}$  corresponding to  $\mathbf{u} = \mathbf{0}$  if  $D > D_*$ , we know that  $f_{\mathbf{0}}$  is the unique minimizer of  $\mathcal{F}$ , it is non-linearly stable and in particular we have that  $\mathcal{F}[f] - \mathcal{F}[f_{\mathbf{u}}] \geq 0$ . See Section 4.4 for more details.

To a distribution function  $f$ , we associate the *non-equilibrium Gibbs state*

$$G_f(v) := \frac{e^{-\frac{1}{D} \left( \frac{1}{2} |v - \mathbf{u}_f|^2 + \frac{\alpha}{4} |v|^4 - \frac{\alpha}{2} |v|^2 \right)}}{\int_{\mathbb{R}^d} e^{-\frac{1}{D} \left( \frac{1}{2} |v - \mathbf{u}_f|^2 + \frac{\alpha}{4} |v|^4 - \frac{\alpha}{2} |v|^2 \right)} dv}. \quad (4.4)$$

Unless  $f$  is a stationary solution of (4.1), let us notice that  $G_f$  does not solve (4.1). A crucial observation is that

$$\mathcal{F}[f] = D \int_{\mathbb{R}^d} f \log f dv + \frac{1}{2} \int_{\mathbb{R}^d} |v - \mathbf{u}_f|^2 f dv + \int_{\mathbb{R}^d} \left( \frac{\alpha}{4} |v|^4 - \frac{\alpha}{2} |v|^2 \right) f dv$$

is a Lyapunov function in the sense that

$$\frac{d}{dt} \mathcal{F}[f(t, \cdot)] = -\mathcal{I}[f(t, \cdot)]$$

if  $f$  solves (4.1), where  $\mathcal{F}[f]$  is the *relative Fisher information* of  $f$  defined as

$$\mathcal{F}[f] := \int_{\mathbb{R}^d} \left| D \frac{\nabla f}{f} + \alpha v |v|^2 + (1 - \alpha) v - \mathbf{u}_f \right|^2 f dv = D^2 \int_{\mathbb{R}^d} \left| \nabla \log \left( \frac{f}{G_f} \right) \right|^2 f dv. \quad (4.5)$$

It is indeed clear that  $\mathcal{F}[f(t, \cdot)]$  is monotone non-increasing and  $\frac{d}{dt} \mathcal{F}[f(t, \cdot)] = 0$  if and only if  $f = G_f$  is a stationary solution of (4.1). This is consistent with our first stability result.

**Proposition 4.1.** *For any  $d \geq 1$  and any  $\alpha > 0$ ,  $f_0$  is a linearly stable critical point if and only if  $D > D_*$ .*

Actually, from the dynamical point of view, we have a better, global result.

**Theorem 4.2.** *For any  $d \geq 1$  and any  $\alpha > 0$ , if  $D > D_*$ , then for any solution  $f$  of (4.1) with nonnegative initial datum  $f_{\text{in}}$  of mass 1 such that  $\mathcal{F}[f_{\text{in}}] < \infty$ , there are two positive constants  $C$  and  $\lambda$  such that, for any time  $t > 0$ ,*

$$0 \leq \mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_0] \leq C e^{-\lambda t}. \quad (4.6)$$

We shall also prove that

$$\int_{\mathbb{R}^d} |f(t, \cdot) - f_0|^2 f_0^{-1} dv \leq C e^{-\lambda t}$$

with same  $\lambda > 0$  as in Theorem 4.2, but eventually for a different value of  $C$ , and characterize  $\lambda$  as the spectral gap of the linearized evolution operator in an appropriate norm. A characterization of the optimal rate  $\lambda$  is given in Theorem 4.3.

For  $D < D_*$ , the situation is more subtle. The solution of (4.1) can in principle converge either to the *isotropic stationary solution*  $f_0$  or to a *polarized, non-symmetric stationary solution*  $f_{\mathbf{u}}$  with  $\mathbf{u} \neq \mathbf{0}$ . We will prove that  $\mathcal{F}[f] - \mathcal{F}[f_{\mathbf{u}}]$  decays with an exponential rate which is also characterized by a spectral gap in Section 4.6. In non-symmetric case, the question of the rate of convergence to a solution with a uniquely defined limiting  $\mathbf{u}$  or a set of polarized solutions is still open.

This chapter is organized as follows. In Section 4.2, we classify all stationary solutions, prove Theorem 3.1 and deduce that a phase transition occurs at  $D = D_*$ . Section 4.3 is devoted to the linearization. The relative entropy and the relative Fisher information provide us with two quadratic forms which are related by the linearized evolution operator. The main result here is to prove a spectral gap property for this operator in the appropriate norm, which is inspired by a similar method used in [50] to study the sub-critical Keller-Segel model: see Proposition 4.6. It is crucial to take into account all terms in the linearization, including the term arising from the non-local mean velocity. The proof of Theorem 4.2 follows using a Grönwall type estimate, in Section 4.5 (isotropic case). In Section 4.6, we also give some results in the polarized case.

## 4.2 Stationary solutions and phase transition

The aim of this section is to classify all stationary solutions of (4.1) as a first step of the proof of the phase transition result of Theorem 4.1. Our proofs are based on elementary although somewhat painful computations. We refer to [154] for more references for the case  $d = 1$ .

### 4.2.1 A technical observation

Let us start by the simple observation that

$$-D \frac{\partial}{\partial v_1} \left( e^{-\frac{1}{D}(\phi_\alpha(v) - u v_1)} \right) = (v_1 - u + \alpha(|v|^2 - 1)v_1) e^{-\frac{1}{D}(\phi_\alpha(v) - u v_1)}$$

can be integrated on  $\mathbb{R}^d$  to rewrite  $\mathcal{H}$  as

$$\mathcal{H}(u) = \alpha \int_{\mathbb{R}^d} (1 - |v|^2) v_1 e^{-\frac{1}{D}(\phi_\alpha(v) - u v_1)} dv$$

and compute

$$\mathcal{H}'(u) = \frac{\alpha}{D} \int_{\mathbb{R}^d} (1 - |v|^2) v_1^2 e^{-\frac{1}{D}(\phi_\alpha(v) - u v_1)} dv.$$

We observe that  $\mathcal{H}'(0) = \frac{\alpha}{D} |\mathbb{S}^{d-1}| h_d(D)$  where

$$h_d(D) := \int_0^\infty (s^{d+1} - s^{d+3}) e^{-\frac{\varphi_\alpha(s)}{D}} ds \quad \text{and} \quad \varphi_\alpha(s) := \frac{\alpha}{4} s^4 + \frac{1-\alpha}{2} s^2.$$

With these notations, we are now in a position to state a key ingredient of the proof.

**Proposition 4.2.** *For any  $d \geq 1$  and any  $\alpha > 0$ ,  $h_d$  has a unique positive root  $D_*$ . Moreover  $h_d$  is positive on  $(0, D_*)$  and negative on  $(D_*, +\infty)$ .*

*Proof.* Our goal is to prove that  $h_d = j_{d+1} - j_{d+3}$  is positive on  $(0, D_*)$  and negative on  $(D_*, +\infty)$  for some  $D_* > 0$ , where

$$j_d(D) := \int_0^\infty s^d e^{-\frac{1}{D}\varphi_\alpha(s)} ds. \quad (4.7)$$

Let us start with two useful identities. A completion of the square shows that for any  $n \in \mathbb{N}$ ,

$$j_{n+5} - 2j_{n+3} + j_{n+1} = \int_0^\infty s^{n+1} (s^2 - 1)^2 e^{-\frac{\varphi_\alpha}{D}} ds > 0. \quad (4.8)$$

With an integration by parts, we obtain that

$$\alpha j_{n+5} + (1 - \alpha) j_{n+3} = \int_0^\infty s^{n+2} \varphi'_\alpha e^{-\frac{1}{D}\varphi_\alpha} ds = (n+2) D j_{n+1}. \quad (4.9)$$

Next, we split the proof in a series of claims.

• *The function  $h_d$  is positive on  $(0, 1/(d+2)]$  and negative on  $[1/d, +\infty)$ .* Let us prove this claim. With  $n = d$  and  $n = d-2$ , we deduce from (4.8) and (4.9) that

$$h_d > \frac{1 - (d+2)D}{1 + \alpha} j_{d+1} \quad \text{and} \quad h_d < \frac{1 - dD}{1 + \alpha} j_{d-1}.$$

As a consequence, if  $h_d(D) = 0$ , then  $D \in (1/(d+2), 1/d)$ .

• *If  $\alpha \leq 1$ , then  $h_d(D) = 0$  has a unique solution.* By a direct computation, we observe that

$$4D^2 h'_d = \alpha h_{d+4} + 2(1 - \alpha) h_{d+2}$$

using (4.9) with  $n = d+2$ . If  $\alpha \in (0, 1)$ , it follows that  $h'_d < 0$  on  $[1/(d+2), +\infty)$ , which proves the claim.

• *If  $\alpha > 1$  and  $h'_d(D_\circ) = 0$  for some  $D_\circ \in (1/(d+2), 1/d)$ , then  $h_d(D_\circ) > 0$ .* Indeed, using

$$4D^2 h'_d = -\alpha j_{d+7} + (3\alpha - 2) j_{d+5} + 2(1 - \alpha) j_{d+3} = 0,$$

combined with (4.9) for  $n = d+2$  and  $n = d$ , we find that, at  $D = D_\circ$ ,

$$h_d(D_\circ) = \frac{(d+2)D - 1 + \alpha(1 - dD)}{\alpha - 1 + (d+4)D\alpha} j_{d+1}.$$

Collecting our observations concludes the proof. See Fig. 4.1 for an illustration.  $\square$

## 4.2.2 The one-dimensional case

**Lemma 4.1.** *Let us consider a continuous positive function  $\psi$  on  $\mathbb{R}^+$  such that the function  $s \mapsto \psi(s) e^{s^2}$  is integrable and define*

$$H(u) := \int_0^{+\infty} (1 - s^2) \psi(s) \sinh(su) ds \quad \forall u \geq 0.$$

*For any  $u > 0$ ,  $H''(u) < 0$  if  $H(u) \leq 0$ . As a consequence,  $H$  changes sign at most once on  $(0, +\infty)$ .*

*Proof.* We first observe that

$$H''(u) - H(u) = \int_0^{+\infty} (1 - s^2) (s^2 - 1) \psi(s) \sinh(su) ds < 0 \quad \forall u > 0. \quad (4.10)$$

Let  $u_* > 0$  be such that  $H(u_*) = 0$ . If  $H'(u_*) < 0$ , there is a neighborhood of  $(u_*)_+$  such that both  $H$  and  $H'$  are negative. As a consequence, by continuation,  $H'(u) < H'(u_*) < 0$  for any  $u > u_*$ . We also get that  $H'(u) < 0$  for any  $u > u_*$  if  $H'(u_*) = 0$  because we know that  $H''(u_*) < 0$ . We conclude by observing that  $H'(u_*) > 0$  would imply  $H'(u) > H'(u_*)$  for any  $u \in (0, u_*)$ , a contradiction with  $H(0) = 0$ .  $\square$

**Proposition 4.3.** *Assume that  $d = 1$  and  $\alpha > 0$ . With the notations of Proposition 4.2, Equation (4.2), i.e.,  $\mathcal{H}(u) = 0$ , has as a solution  $u = u(D) > 0$  if and only if  $D < D_*$  and  $\lim_{D \rightarrow (D_*)_-} u(D) = 0$ .*

In other words, there exists a solution to (4.2) if and only if  $\mathcal{H}'(0) > 0$ .

*Proof.* Since  $\mathcal{H}(0) = 0$ , for any  $D \neq D_*$ ,  $h_d(D)$  and  $\mathcal{H}(u)$  have the same sign in a neighborhood of  $u = 0_+$ . Next we notice that

$$-\frac{1}{\alpha} \mathcal{H}(u) = \int_0^\infty (v^2 - 1) v e^{-\frac{\phi_\alpha(v)}{D}} e^{\frac{uv}{D}} dv - \int_0^\infty (v^2 - 1) v e^{-\frac{\phi_\alpha(v)}{D}} e^{-\frac{uv}{D}} dv.$$

The second term of the right-hand side converges to 0 as  $u \rightarrow \infty$  by the dominated convergence theorem. Concerning the first term, let us notice that  $|(v^2 - 1)v|e^{-\phi_\alpha(v)/D}$  is bounded on  $(0, 3)$ , so that

$$\begin{aligned} & \int_0^\infty (v^2 - 1) v e^{-\frac{\phi_\alpha(v)}{D}} e^{\frac{uv}{D}} dv \\ & \geq \int_0^1 (v^2 - 1) v e^{-\frac{\phi_\alpha(v)}{D}} e^{\frac{uv}{D}} dv + \int_2^3 (v^2 - 1) v e^{-\frac{\phi_\alpha(v)}{D}} e^{\frac{uv}{D}} dv \\ & \geq -C_1 e^{u/D} + C_2 e^{2u/D} \rightarrow +\infty \quad \text{as } u \rightarrow +\infty \end{aligned}$$

for some positive constants  $C_1$  and  $C_2$ . This proves that  $\lim_{u \rightarrow +\infty} \mathcal{H}(u) = -\infty$  and shows the existence of at least one positive solution of (4.2) if  $h_d(D) > 0$ .

The fact that (4.2) has at most one solution on  $(0, +\infty)$  follows from Lemma 4.1 applied with  $H(u) = \mathcal{H}(Du)$  and  $\psi(v) = 2\alpha v e^{-\frac{\phi_\alpha(v)}{D}}$ . Finally, as consequence of the regularity of  $H$  and of (4.10), the solution  $u = u(D)$  of (4.2) is such that  $\lim_{D \rightarrow (D_*)_-} u(D) = 0$ .

For  $D = D_*$ , notice that  $\mathcal{H}'(0) = \mathcal{H}''(0) = 0$ , and

$$\mathcal{H}'''(0) = \frac{\alpha}{D^2} \int_{\mathbb{R}} (1 - v^2) v^3 e^{-\frac{\phi_*}{D_*}} dv < 0$$

because

$$-\int_{\mathbb{R}} (1 - v^2) v^3 e^{-\frac{\phi_*}{D_*}} dv = \int_{\mathbb{R}} v(1 - v^2)^2 e^{-\frac{\phi_*}{D_*}} dv > 0$$

so we deduce by using the similar method above that  $\mathcal{H}(u)$  has no positive solutions.  $\square$

### 4.2.3 The case of a dimension $d \geq 2$

We extend the result of Proposition 4.3 to higher dimensions.

**Proposition 4.4.** *Assume that  $d \geq 2$  and  $\alpha > 0$ . With the notations of Proposition 4.2, Equation (4.2), i.e.,  $\mathcal{H}(u) = 0$ , has as a solution  $u = u(D) > 0$  if and only if  $D < D_*$  and  $\lim_{D \rightarrow (D_*)_-} u(D) = 0$ .*

Qualitatively, the result is the same as in dimension  $d = 1$ : there exists a solution to (4.2) if and only if  $\mathcal{H}'(0) > 0$ . See Fig. 4.2.

In radial coordinates such that  $s = |v|$  and  $v_1 = s \cos \theta$ , with  $\theta \in [0, \pi]$ ,

$$\mathcal{H}(u) = \alpha \left| \mathbb{S}^{d-2} \right| \int_0^\pi \int_0^{+\infty} (1-s^2) s^d e^{-\frac{\varphi_\alpha(s)}{D}} \cos \theta (\sin \theta)^{d-2} e^{\frac{us}{D} \cos \theta} ds d\theta$$

written with the convention that  $|\mathbb{S}^0| = 2$  can also be rewritten as

$$\mathcal{H}(u) = 2\alpha \left| \mathbb{S}^{d-2} \right| \int_0^{\pi/2} \int_0^{+\infty} (1-s^2) s^d e^{-\frac{\varphi_\alpha(s)}{D}} \cos \theta (\sin \theta)^{d-2} \sinh\left(\frac{us}{D} \cos \theta\right) ds d\theta.$$

Lemma 4.1 does not apply directly. Let us consider

$$h(s) := \int_0^{\pi/2} \cos \theta (\sin \theta)^{d-2} \sinh(s \cos \theta) d\theta. \quad (4.11)$$

**Lemma 4.2.** *Assume that  $d \geq 2$ . The function  $h$  defined by (4.11) is such that  $s \mapsto sh'(s)/h(s)$  is monotone increasing on  $(0, +\infty)$ .*

*Proof.* Let  $s_1$  and  $s_2$  be such that  $0 < s_1 < s_2$  and consider a series expansion. With

$$P_n := \int_0^\pi (\cos \theta)^{2n} (\sin \theta)^{d-2} d\theta,$$

we know that

$$\begin{aligned} s_2 h'(s_2) h(s_1) &= \sum_{m=0}^{\infty} \frac{s_2^{2m+1}}{(2m)!} P_{m+1} \sum_{n=0}^{\infty} \frac{s_1^{2n+1}}{(2n+1)!} P_{n+1}, \\ s_1 h'(s_1) h(s_2) &= \sum_{m=0}^{\infty} \frac{s_1^{2m+1}}{(2m)!} P_{m+1} \sum_{n=0}^{\infty} \frac{s_2^{2n+1}}{(2n+1)!} P_{n+1}. \end{aligned}$$

These series are absolutely converging and we can reindex the difference of the two terms using  $i = \min\{m, n\}$  to get

$$\begin{aligned} &s_2 h'(s_2) h(s_1) - s_1 h'(s_1) h(s_2) \\ &= \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{(s_1 s_2)^{2i+1}}{(2i+2j+1)!(2i+1)!} P_{i+1} P_{j+1} \frac{2i+2j+1}{2(i+j+1)} (s_2^j - s_1^j) > 0. \end{aligned}$$

□

*Proof of Proposition 4.4.* We prove that  $\lim_{u \rightarrow +\infty} \mathcal{H}(u) = -\infty$  as in the case  $d = 1$  by considering the domains defined in the coordinates  $(s, \theta)$  by  $0 \leq s \leq 1$  and  $\theta \in [0, \pi/2]$  on the one hand, and  $2 \leq s \leq 3$  and  $0 \leq \theta \leq \theta_*$  for some  $\theta_* \in (0, \pi/6)$  on the other hand.



If  $D \geq D_*$ , we obtain from  $\mathcal{H}'(0) \leq 0$  that

$$\int_0^1 (1-s^2) s^{d+1} e^{-\frac{\varphi_\alpha(s)}{D}} ds \leq \int_1^\infty (s^2-1) s^{d+1} e^{-\frac{\varphi_\alpha(s)}{D}} ds$$

obviously  $h'(s)$  is strictly increasing on  $(0, \infty)$ , which means that for any  $u > 0$ ,

$$\begin{aligned} \int_0^1 (1-s^2) s^{d+1} e^{-\frac{\varphi_\alpha(s)}{D}} h'\left(\frac{us}{D}\right) ds &< \int_0^1 (1-s^2) s^{d+1} e^{-\frac{\varphi_\alpha(s)}{D}} h'\left(\frac{u}{D}\right) ds \\ &= \int_1^\infty (s^2-1) s^{d+1} e^{-\frac{\varphi_\alpha(s)}{D}} h'\left(\frac{u}{D}\right) ds \\ &< \int_1^\infty (s^2-1) s^{d+1} e^{-\frac{\varphi_\alpha(s)}{D}} h'\left(\frac{us}{D}\right) ds \end{aligned}$$

so  $\mathcal{H}'(u) < 0$  for any  $u > 0$ , which proves that  $\mathcal{H}(u)$  has no positive solutions when  $D \geq D_*$ .

For  $D < D_*$ , the existence of at least one solution  $u > 0$  of  $\mathcal{H}(u) = 0$  follows from Proposition 4.3. If there exist  $0 < u_1 < u_2$  such that  $\mathcal{H}(u_1) = \mathcal{H}(u_2) = 0$ , then

$$\int_0^1 (1-s^2) s^d e^{-\frac{\varphi_\alpha(s)}{D}} h(\tilde{u}_1 s) ds = \int_1^\infty (s^2-1) s^d e^{-\frac{\varphi_\alpha(s)}{D}} h(\tilde{u}_1 s) ds$$

where  $\tilde{u}_1 := u_1/D < u_2/D =: \tilde{u}_2$ . We deduce from Lemma 4.2 that the function  $s \mapsto k(s) := h(\tilde{u}_2 s)/h(\tilde{u}_1 s)$  is a monotone increasing function on  $(0, +\infty)$ . Using  $\mathcal{H}(u_1) = 0$ , we obtain

$$\begin{aligned} \int_0^1 (1-s^2) s^d e^{-\frac{\varphi_\alpha(s)}{D}} h(\tilde{u}_2 s) ds &= \int_0^1 (s^2-1) s^d e^{-\frac{\varphi_\alpha(s)}{D}} h(\tilde{u}_1 s) k(s) ds \\ &< \int_0^1 (s^2-1) s^d e^{-\frac{\varphi_\alpha(s)}{D}} h(\tilde{u}_1 s) k(1) ds \\ &= \int_1^\infty (s^2-1) s^d e^{-\frac{\varphi_\alpha(s)}{D}} h(\tilde{u}_1 s) k(1) ds \\ &< \int_1^\infty (s^2-1) s^d e^{-\frac{\varphi_\alpha(s)}{D}} h(\tilde{u}_1 s) k(s) ds \\ &= \int_1^\infty (s^2-1) s^d e^{-\frac{\varphi_\alpha(s)}{D}} h(\tilde{u}_2 s) ds, \end{aligned}$$

a contradiction with  $\mathcal{H}(u_2) = 0$ . □

#### 4.2.4 Classification of the stationary solutions and phase transition

We learn from the expression of  $\mathcal{S}$  in (4.5) that any stationary solution of (4.1) is of the form  $f_{\mathbf{u}}$  with  $\mathbf{u} = u e_1$  for some  $u$  which solves (4.2) up to an rotation. Since  $\mathcal{H}(0) = 0$ ,  $u = 0$  is always a solution. According to Propositions 4.3 and 4.4, Equation (4.2) has a solution  $u = u(D)$  if and only if  $D > D_*$  where  $D_*$  is obtained as the unique positive root of  $h_d$  by Proposition 4.2.

**Corollary 4.1.** *Let  $d \geq 2$  and  $\alpha > 0$ . With the above notations and  $D_*$  defined as in Proposition 4.2, we know that*

- (i) *if  $D \geq D_*$  there exists one and only one non-negative stationary distribution  $f_{\mathbf{u}}$  given by  $\mathbf{u} = \mathbf{0}$ , which is isotropic,*
- (ii) *if  $d \geq 2$  and  $D < D_*$  there exists one and only one non-negative isotropic stationary distribution with  $\mathbf{u} = \mathbf{0}$ , and a continuum of stable non-negative non-symmetric stationary distributions  $f_{\mathbf{u}}$  with  $\mathbf{u} = u(D)\mathbf{e}$  for any  $e \in \mathbb{S}^{d-1}$ , with the convention that  $\mathbb{S}^0 = \{-1, 1\}$ .*

*There are no other stationary solutions.*

In other words, we have obtained the complete classification of the stationary solutions of (4.1), which shows that there are two phases of stationary solutions: the isotropic one with  $\mathbf{u} = \mathbf{0}$ , and the non-isotropic ones with  $\mathbf{u} \neq \mathbf{0}$  which are unique up to a rotation and exist only if  $D < D_*$ . To complete the proof of Theorem 4.1, we have to study the linear stability of these stationary solutions.

#### 4.2.5 An important estimate

The next result is a technical estimate which is going to play a key role in our analysis.

**Lemma 4.3.** *Assume that  $d \geq 1$ ,  $\alpha > 0$  and  $D > 0$ .*

- (i) *In the case  $\mathbf{u} = \mathbf{0}$ , we have that  $\int_{\mathbb{R}^d} |v|^2 f_{\mathbf{0}} dv > dD$  if and only if  $D < D_*$ .*
- (ii) *In the case  $D \in (0, D_*)$  and  $\mathbf{u} \neq \mathbf{0}$ , we have that*

$$\int_{\mathbb{R}^d} |(v - \mathbf{u}) \cdot \mathbf{u}|^2 f_{\mathbf{u}} dv < D|\mathbf{u}|^2.$$

- (iii) *In the case  $d \geq 2$  and  $D \in (0, D_*)$  and  $\mathbf{u} \neq \mathbf{0}$ , we have that*

$$\int_{\mathbb{R}^d} |(v - \mathbf{u}) \cdot \mathbf{w}|^2 f_{\mathbf{u}} dv = D|\mathbf{w}|^2 \quad \forall \mathbf{w} \in \mathbb{R}^d \quad \text{such that} \quad \mathbf{u} \cdot \mathbf{w} = 0.$$

*Proof.* Using Definition (4.7), we observe that  $\int_{\mathbb{R}^d} |v|^2 f_{\mathbf{0}} dv - dD$  has the sign of

$$j_{d+1} - dDj_{d-1} = \alpha(j_{d+1} - j_{d+3}) = \alpha h_d(D)$$

by (4.9) with  $n = d - 2$ . This proves (i) according to Proposition 4.2 and Corollary 4.1.

By integrating  $D\mathbf{u} \cdot \nabla((\mathbf{u} \cdot v) f_{\mathbf{u}})$ , we obtain that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} \left( D|\mathbf{u}|^2 - (\mathbf{u} \cdot v)^2 (\alpha |v|^2 + 1 - \alpha) + u(\mathbf{u} \cdot v) \right) f_{\mathbf{u}} dv \\ &= D|\mathbf{u}|^2 - \int_{\mathbb{R}^d} |(v - \mathbf{u}) \cdot \mathbf{u}|^2 f_{\mathbf{u}} dv + D|\mathbf{u}|^2 \mathcal{H}'(|\mathbf{u}|) \end{aligned}$$

Then (ii) follows from Propositions 4.3 and 4.4 because  $\mathcal{H}'(u) < 0$  if  $u = u(D) = |\mathbf{u}|$ .

With no loss of generality, we can assume that  $\mathbf{u} = (u, 0, \dots, 0) \neq \mathbf{0}$ . By integrating  $\frac{\partial}{\partial v_1} f_{\mathbf{u}}$  on  $\mathbb{R}^d$ , we know that  $\int_{\mathbb{R}^d} (|v|^2 - 1) v_1 f_{\mathbf{u}} dv = 0$ . Let us consider radial coordinates such that  $s = |v|$  and  $v_1 = s \cos \theta$ , with  $\theta \in [0, \pi]$ . From the integration by parts

$$(d-1)D \int_0^\pi \cos \theta (\sin \theta)^{d-2} e^{\frac{us}{D} \cos \theta} d\theta = us \int_0^\pi (\sin \theta)^d e^{\frac{us}{D} \cos \theta} d\theta,$$

we deduce that  $\int_{\mathbb{R}^d} (|v|^2 - 1) (1 - v_1^2) f_{\mathbf{u}} dv = 0$  because  $s^2 (\sin \theta)^2 = 1 - v_1^2$  and

$$\int_{\mathbb{R}^d} (|v|^2 - 1) v_i^2 f_{\mathbf{u}} dv = 0 \quad \forall i \geq 2$$

by symmetry among the variables  $v_2, v_3, \dots, v_d$ . We conclude by integrating  $\frac{\partial}{\partial v_i} f_{\mathbf{u}}$  on  $\mathbb{R}^d$  that

$$\int_{\mathbb{R}^d} |v_i|^2 f_{\mathbf{u}} dv = D \quad \forall i \geq 2,$$

which concludes the proof of (iii).  $\square$

**Corollary 4.2.** *Assume that  $d \geq 1$ ,  $\alpha > 0$  and  $\mathbf{e} \in \mathbb{S}^{d-1}$ . There exists a function  $D \mapsto \kappa(D)$  on  $(0, D_*)$  which is continuous with values in  $(0, 1)$  such that, with  $\mathbf{u} = u(D)\mathbf{e}$ ,*

$$\frac{1}{D} \int_{\mathbb{R}^d} |(v - \mathbf{u}) \cdot \mathbf{w}|^2 f_{\mathbf{u}} dv = \kappa(D) (\mathbf{w} \cdot \mathbf{e})^2 + |\mathbf{w}|^2 - (\mathbf{w} \cdot \mathbf{e})^2 \quad \forall \mathbf{w} \in \mathbb{R}^d.$$

With  $\kappa(D) := \frac{1}{u(D)^2} \int_{\mathbb{R}^d} |(v - \mathbf{u}) \cdot \mathbf{u}|^2 f_{\mathbf{u}} dv$  and  $\mathbf{u} = u(D)\mathbf{e}$  for an arbitrary  $\mathbf{e} \in \mathbb{S}^{d-1}$ , the proof is a straightforward consequence of Lemma 4.3.

#### 4.2.6 An additional result on $u(D)$

The main goal of this subsection is to show a qualitative result on the behaviour of  $u(D)$  as  $D \rightarrow (D_*)_-$ .

**Proposition 4.5.** *Let  $0 < D < D_*$ . If  $u(D)$  denotes the positive solution of  $\mathcal{H}(u) = 0$ , then*

$$\lim_{D \rightarrow D_*} \frac{u(D)^2}{D_* - D} = \alpha \frac{(1 - \alpha)(1 - dD_*) - 2D_*}{1 - (d+2)D_*}.$$

*Proof.* According to the implicit function theorem,  $u(D)$  is a differentiable function of  $D$  on  $(0, D_*)$  and

$$\frac{\partial \mathcal{H}}{\partial u} \frac{\partial u}{\partial D} = -\frac{\partial \mathcal{H}}{\partial D} = -\frac{1}{D^2} \int_{\mathbb{R}^d} (v_1 - u) (\phi_\alpha - u v_1) e^{-\frac{1}{D}(\phi_\alpha - u v_1)} dv.$$

Notice that at  $D = D_*$ ,

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \text{ and } \frac{\partial^2 \mathcal{H}}{\partial u^2} = D^{-2} \int_{\mathbb{R}^d} (1 - |v|^2) v_1^3 e^{-\phi_\alpha/D_*} dv = 0$$

so we obtain that

$$\frac{\partial \mathcal{H}}{\partial u} \sim \beta u^2$$

as  $D \rightarrow (D_*)_-$ , where

$$\beta = \frac{1}{2} \frac{\partial^3 \mathcal{H}}{\partial u^3}(0) = \frac{1}{2D_*^3} \int_{\mathbb{R}^d} (1 - |v|^2) v_1^4 e^{-\frac{\phi_\alpha}{D_*}} dv < 0.$$

On the other hand, using integrations by parts and the identity

$$\int_{\mathbb{R}^d} v_1^2 e^{-\phi_\alpha/D_*} dv = D_* \int_{\mathbb{R}^d} e^{-\phi_\alpha/D_*} dv$$

deduced from Lemma 4.3, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} (v_1 - u) (\phi_\alpha - u v_1) e^{-\frac{1}{D}(\phi_\alpha - u v_1)} dv \\ &= \frac{1}{4} (\alpha - 4) u \int_{\mathbb{R}^d} v_1^2 e^{-\frac{1}{D}(v_1^2 \phi_\alpha - u v_1)} dv \\ & \quad + \frac{1}{4} (2D + 1 - \alpha - dD + \alpha dD - \alpha D) u + 3u^3 \int_{\mathbb{R}^d} e^{-\frac{1}{D}(\phi_\alpha - u v_1)} dv, \end{aligned}$$

so that

$$\int_{\mathbb{R}^d} (v_1 - u) (\phi_\alpha - u v_1) e^{-\frac{1}{D}(\phi_\alpha - u v_1)} dv \sim \frac{1}{4} ((1 - \alpha)(1 - dD_*) - 2D_*) u \int_{\mathbb{R}^d} e^{-\frac{\phi_\alpha}{D_*}} dv$$

as  $D \rightarrow (D_*)_-$ . Notice that  $(1 - \alpha)(1 - dD_*) - 2D_* < 0$  because  $\frac{1}{d+2} < D_* < \frac{1}{d}$ . By using (4.9) and  $\int_{\mathbb{R}^d} (1 - |v|^2) v_1^2 e^{-\phi_\alpha/D_*} dv = 0$ , we obtain that

$$\frac{\int_{\mathbb{R}^d} e^{-\frac{\phi_\alpha}{D_*}} dv}{\int_{\mathbb{R}^d} (1 - |v|^2) v_1^4 e^{-\frac{\phi_\alpha}{D_*}} dv} = \frac{1}{D_*} \frac{\alpha}{1 - (d+2)D_*},$$

which concludes the proof using

$$\lim_{D \rightarrow D_*} \frac{(u(D))^2}{D_* - D} = -2 \lim_{D \rightarrow D_*} u \frac{\partial u}{\partial D}.$$

□

**Remark 4.1.** We already know from [13] that  $\lim_{D \rightarrow 0} u(D) = 1$ . Combined with this corollary,  $\frac{(u(D))^2}{D_* - D}$  is uniformly bounded on  $(0, D_*)$ . Moreover, we can give a more explicit lower bound of  $\frac{(u(D))^2}{D_* - D}$ . The proof is similar to the method from Section 4.2.3, and we leave it to the reader.

### 4.3 The linearized problem: local properties of the stationary solutions

This section is devoted to the quadratic forms associated with the expansion of the free energy  $\mathcal{F}$  and the Fisher information  $\mathcal{I}$  around the stationary solution  $f_{\mathbf{u}}$  studied in Section 4.2. These quadratic forms are defined for a smooth perturbation  $g$  of  $f_{\mathbf{u}}$  such that  $\int_{\mathbb{R}^d} g f_{\mathbf{u}} dv = 0$  by

$$\begin{aligned} Q_{1,\mathbf{u}}[g] &:= \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^2} (\mathcal{F}[f_{\mathbf{u}}(1 + \varepsilon g)] - \mathcal{F}[f_{\mathbf{u}}]) \\ &= D \int_{\mathbb{R}^d} g^2 f_{\mathbf{u}} dv - D^2 |\mathbf{v}_g|^2 \quad \text{where } \mathbf{v}_g := \frac{1}{D} \int_{\mathbb{R}^d} v g f_{\mathbf{u}} dv, \\ Q_{2,\mathbf{u}}[g] &:= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \mathcal{I}[f_{\mathbf{u}}(1 + \varepsilon g)] = D^2 \int_{\mathbb{R}^d} |\nabla g - \mathbf{v}_g|^2 f_{\mathbf{u}} dv. \end{aligned}$$

#### 4.3.1 Stability of the isotropic stationary solution

The first result is concerned with the linear stability of  $\mathcal{F}$  around  $f_0$ .

**Lemma 4.4.** *On the space of the functions  $g \in L^2(f_0 dv)$  such that  $\int_{\mathbb{R}^d} g f_0 dv = 0$ ,  $Q_{1,0}$  is a nonnegative (resp. positive) quadratic form if and only if  $D \geq D_*$  (resp.  $D > D_*$ ). Moreover, for any  $D > D_*$ , let  $\eta(D) := \alpha \mathcal{C} h_d(D)$  for some explicit  $\mathcal{C} = \mathcal{C}(D) > 0$ . Then*

$$Q_{1,0}[g] \geq \eta(D) \int_{\mathbb{R}^d} g^2 f_0 dv \quad \forall g \in L^2(f_0 dv) \quad \text{such that} \quad \int_{\mathbb{R}^d} g f_0 dv = 0. \quad (4.12)$$

*Proof.* On one hand, if  $D < D_*$ , let  $\mathbf{e} \in \mathbb{S}^{d-1}$ . We consider  $g(v) = v \cdot \mathbf{e}$  and, using (4.9) with  $n = d - 2$ , compute

$$Q_{1,0}[g] = D \int_{\mathbb{R}^d} v_1^2 f_0 dv - \left( \int_{\mathbb{R}^d} v_1^2 f_0 dv \right)^2 = \mathcal{C} \int_0^\infty (d D s^{d-1} - s^{d+1}) e^{-\frac{\alpha s}{D}} ds$$

where the last equality determines the value of  $\mathcal{C}$ . This proves that  $Q_{1,0}[g] = -\alpha \mathcal{C} h_d(D) < 0$ . So the necessary condition for the linear stability of  $f_0$  is  $D \geq D_*$ .

On the other hand, let  $g$  be a function in  $L^2(\mathbb{R}^d, f_0 dv)$  such that  $\int_{\mathbb{R}^d} g^2 f_0 dv = \int_{\mathbb{R}^d} v_1^2 f_0 dv$ . We can indeed normalize  $g$  with no loss of generality. With  $v_1 = v \cdot \mathbf{e}$ ,  $\mathbf{e} \in \mathbb{S}^{d-1}$  such that  $\mathbf{u}_g f_0 = u \mathbf{e}$  for some  $u \in \mathbb{R}$ , we know by the Cauchy-Schwarz inequality that

$$\left( \int_{\mathbb{R}^d} v_1 g f_0 dv \right)^2 \leq \int_{\mathbb{R}^d} g^2 f_0 dv \int_{\mathbb{R}^d} v_1^2 f_0 dv = \left( \int_{\mathbb{R}^d} v_1^2 f_0 dv \right)^2 = \left( \frac{1}{d} \int_{\mathbb{R}^d} |v|^2 f_0 dv \right)^2,$$

hence

$$Q_{1,0}[g] \geq D \int_{\mathbb{R}^d} v_1^2 f_0 dv - \left( \int_{\mathbb{R}^d} v_1^2 f_0 dv \right)^2 = -\alpha \mathcal{C} h_d(D).$$

This proves the linear stability of  $f_0$  if  $D > D_*$ .  $\square$

The classification result of Theorem 4.1 is a consequence of Corollary 4.1 and Lemma 4.4.

### 4.3.2 A coercivity result

Let us start by recalling the *Poincaré inequality*

$$\int_{\mathbb{R}^d} |\nabla h|^2 f_{\mathbf{u}} dv \geq \Lambda_D \int_{\mathbb{R}^d} |h|^2 f_{\mathbf{u}} dv \quad \forall h \in H^1(\mathbb{R}^d, f_{\mathbf{u}} dv) \quad \text{such that} \quad \int_{\mathbb{R}^d} h f_{\mathbf{u}} dv = 0. \quad (4.13)$$

Here  $\mathbf{u}$  is an admissible velocity such that  $\mathbf{u} = \mathbf{0}$  if  $D \geq D_*$ , or  $|\mathbf{u}| = u(D)$  if  $D < D_*$ , and  $\Lambda_D$  denotes the corresponding optimal constant. Since  $\varphi_\alpha$  can be seen as a uniformly strictly convex potential perturbed by a bounded perturbation, it follows from the *carré du champ* method and the Holley-Stroock lemma that  $\Lambda_D$  is a positive constant. Let

$$\begin{aligned} \mathbf{u}[f] &= 0 \quad \text{if } D \geq D_* \quad \text{or} \quad \mathbf{u}_f = \mathbf{0} \quad \text{and} \quad D < D_*, \\ \mathbf{u}[f] &= \frac{u(D)}{|\mathbf{u}_f|} \mathbf{u}_f \quad \text{if } D < D_* \quad \text{and} \quad \mathbf{u}_f \neq \mathbf{0}. \end{aligned}$$

Based on (6.13), we have the following coercivity result.

**Proposition 4.6.** *Let  $d \geq 1$ ,  $\alpha > 0$ ,  $D > 0$  and  $\mathcal{C}_D = D \Lambda_D$  with  $\Lambda_D$  as in (6.13). Let us consider a nonnegative distribution function  $f \in L^1(\mathbb{R}^d)$  with  $\int_{\mathbb{R}^d} f dv = 1$ , let  $\mathbf{u} \in \mathbb{R}^d$  be such that either  $\mathbf{u} = \mathbf{0}$  or  $|\mathbf{u}| = u(D)$  if  $D < D_*$  and consider  $g = (f - f_{\mathbf{u}})/f_{\mathbf{u}}$ . We assume that  $g \in H^1(\mathbb{R}^d, f_{\mathbf{u}} dv)$ . If  $\mathbf{u} = \mathbf{0}$ , then*

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D Q_{1,\mathbf{u}}[g].$$

Otherwise, if  $\mathbf{u} \neq \mathbf{0}$  for some  $D \in (0, D_*)$  with  $D_*$  as in Corollary 4.1, then we have

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D (1 - \kappa(D)) \frac{(\mathbf{v}_g \cdot \mathbf{u})^2}{|\mathbf{v}_g|^2 |\mathbf{u}|^2} Q_{1,\mathbf{u}}[g]$$

with  $\mathbf{v}_g := \frac{1}{D} \int_{\mathbb{R}^d} (v - \mathbf{u}) g f_{\mathbf{u}} dv$  and  $\kappa(D) < 1$  defined as in Corollary 4.2. As a special case, if  $\mathbf{u} = \mathbf{u}[f]$ , then  $Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D (1 - \kappa(D)) Q_{1,\mathbf{u}}[g]$ .

By construction,  $\mathbf{v}_g$  is such that  $D \mathbf{v}_g = \int_{\mathbb{R}^d} (v - \mathbf{u}) g f_{\mathbf{u}} dv = \int_{\mathbb{R}^d} v g f_{\mathbf{u}} dv = \mathbf{u}_f - \mathbf{u}$  because  $\int_{\mathbb{R}^d} g f_{\mathbf{u}} dv = 0$ .

*Proof.* Let us apply (6.13) to  $h(v) = g(v) - (v - \mathbf{u}) \cdot \mathbf{v}_g$ . Using  $\mathbf{v}_g = \frac{1}{D} \int_{\mathbb{R}^d} (v - \mathbf{u}) g f_{\mathbf{u}} dv$  and  $\int_{\mathbb{R}^d} g f_{\mathbf{u}} dv = 0$ , we obtain

$$\begin{aligned} \frac{1}{D^2} Q_{2,\mathbf{u}}[g] &= \int_{\mathbb{R}^d} |\nabla g - \mathbf{v}_g|^2 f_{\mathbf{u}} dv \\ &\geq \Lambda_D \int_{\mathbb{R}^d} (g^2 + |\mathbf{v}_g \cdot (v - \mathbf{u})|^2 - 2 \mathbf{v}_g \cdot (v - \mathbf{u}) g) f_{\mathbf{u}} dv \\ &= \Lambda_D \left[ \int_{\mathbb{R}^d} |g|^2 f_{\mathbf{u}} dv + \int_{\mathbb{R}^d} |\mathbf{v}_g \cdot (v - \mathbf{u})|^2 f_{\mathbf{u}} dv - 2D |\mathbf{v}_g|^2 \right]. \end{aligned}$$

If  $\mathbf{u} = \mathbf{0}$ , either  $\mathbf{v}_g = \mathbf{0}$  and the result is proved, or we know that  $\frac{1}{d} \int_{\mathbb{R}^d} |v|^2 f_0 dv \geq D$  by Lemma 4.3 because  $D \geq D_*$  by assumption. In that case we can estimate the r.h.s. by

$$\int_{\mathbb{R}^d} |g|^2 f_0 dv + |\mathbf{v}_g|^2 \left( \frac{1}{d} \int_{\mathbb{R}^d} |v|^2 f_0 dv - 2D \right) \geq \int_{\mathbb{R}^d} |g|^2 f_0 dv - D |\mathbf{v}_g|^2 = \frac{1}{D} Q_{1,0}[g],$$

which again proves the result whenever  $\mathbf{u} = \mathbf{0}$ .

If  $\mathbf{u} \neq \mathbf{0}$ , let us apply Corollary 4.2 with  $\mathbf{w} = \mathbf{v}_g$  and  $\kappa = \kappa(D)$ :

$$\int_{\mathbb{R}^d} |\mathbf{v}_g \cdot (v - \mathbf{u})|^2 f_{\mathbf{u}} dv = \mathcal{K} D |\mathbf{v}_g|^2 \quad \text{with} \quad \mathcal{K} = 1 - (1 - \kappa) \frac{(\mathbf{v}_g \cdot \mathbf{u})^2}{|\mathbf{v}_g|^2 |\mathbf{u}|^2}.$$

We deduce from the Cauchy-Schwarz inequality

$$D^2 |\mathbf{v}_g|^4 = \left( \int_{\mathbb{R}^d} \mathbf{v}_g \cdot (v - \mathbf{u}) f_{\mathbf{u}} dv \right)^2 \leq \int_{\mathbb{R}^d} |g|^2 f_0 dv \int_{\mathbb{R}^d} |\mathbf{v}_g \cdot (v - \mathbf{u})|^2 f_{\mathbf{u}} dv$$

that  $D |\mathbf{v}_g|^2 \leq \mathcal{K} \int_{\mathbb{R}^d} |g|^2 f_0 dv$ . Hence, if  $\beta \in (0, 1)$ , we obtain

$$\frac{1}{D^2} Q_{2,\mathbf{u}}[g] - \frac{\beta}{D^2} Q_{2,\mathbf{u}}[g] \geq (1 - \beta - (2 - \mathcal{K} - \beta) \mathcal{K}) \int_{\mathbb{R}^d} |g|^2 f_0 dv.$$

With  $\beta = 1 - \mathcal{K}$ , we obtain  $1 - \beta - (2 - \mathcal{K} - \beta) \mathcal{K} = 0$ , which proves the result.  $\square$

## 4.4 Properties of the free energy and consequences

We consider the free energy  $\mathcal{F}$  and the Fisher information  $\mathcal{I}$  defined respectively by (4.3) and (4.5).

### 4.4.1 Basic properties of the free energy

**Proposition 4.7.** *Assume that  $f_{\text{in}}$  is a nonnegative function in  $L^1(\mathbb{R}^d)$  such that  $\mathcal{F}[f_{\text{in}}] < \infty$ . Then there exists a solution  $f \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^d))$  of (4.1) with initial datum  $f_{\text{in}}$  such that  $\mathcal{F}[f(t, \cdot)]$  is nonincreasing and a.e. differentiable on  $[0, \infty)$ . Furthermore*

$$\frac{d}{dt} \mathcal{F}[f(t, \cdot)] \leq -\mathcal{I}[f(t, \cdot)], \quad t > 0 \quad \text{a.e.}$$

This result is classical and we shall skip its proof: see for instance [?, Proposition 2.1] for further details. One of the difficulties in the study of  $\mathcal{F}$  is that in (4.3), the term  $|\mathbf{u}_f|^2$  has a negative coefficient, so that the functional  $\mathcal{F}$  is not convex. A smooth solution realizes the equality, and by approximations, we obtain the result.

**Proposition 4.8.**  $\mathcal{F}$  is bounded from below on the set

$$\left\{ f \in L^1_+(\mathbb{R}^d) : \int_{\mathbb{R}^d} f \, d\nu = 1 \text{ and } \int_{\mathbb{R}^d} |v|^4 f \, d\nu < \infty \right\}$$

and

$$\int_{\mathbb{R}^d} |v|^4 f \, d\nu \leq \frac{1}{\alpha^2} \left( D + \alpha + \sqrt{(D + \alpha)^2 + 4\alpha \left( \mathcal{F}[f] + \frac{d}{2} \log(2\pi) D \right)} \right)^2.$$

*Proof.* Let  $g = f/\mu$  where  $\mu(v) := (2\pi)^{-d/2} e^{-\frac{1}{2}|v|^2}$  and  $d\mu = \mu \, d\nu$ . Since  $g \log g \geq g - 1$  and  $\int_{\mathbb{R}^d} (g - 1) \, d\mu = 0$ , we have the classical estimate

$$\int_{\mathbb{R}^d} f \log f \, d\nu + \frac{1}{2} \int_{\mathbb{R}^d} |v|^2 f \, d\nu = \int_{\mathbb{R}^d} g \left( \log g - \frac{d}{2} \log(2\pi) \right) \, d\mu \geq -\frac{d}{2} \log(2\pi).$$

By the Cauchy-Schwarz inequality,

$$|\mathbf{u}|^2 \leq \int_{\mathbb{R}^d} |v|^2 f \, d\nu \text{ and } \int_{\mathbb{R}^d} |v|^2 f \, d\nu \leq \sqrt{\int_{\mathbb{R}^d} |v|^4 f \, d\nu}$$

and we deduce that

$$\mathcal{F}[f] \geq -\frac{d}{2} \log(2\pi) D + \frac{\alpha}{4} X^2 - \frac{D + \alpha}{2} X \quad \text{with} \quad X := \sqrt{\int_{\mathbb{R}^d} |v|^4 f \, d\nu}.$$

A minimization of the r.h.s. with respect to  $X > 0$  shows that  $\mathcal{F}[f] \geq -\frac{(D + \alpha)^2}{4\alpha} - \frac{d}{2} \log(2\pi) D$  while the inequality provides the bound on  $X$ .  $\square$

#### 4.4.2 The minimizers of the free energy

**Corollary 4.3.** Let  $d \geq 1$  and  $\alpha > 0$ . The free energy  $\mathcal{F}$  as defined by (4.3) has a unique nonnegative minimizer with unit mass,  $f_0$ , if  $D \geq D_*$ . Otherwise, if  $D < D_*$ , we have

$$\min \mathcal{F}[f] = \mathcal{F}[f_{\mathbf{u}}] < \mathcal{F}[f_0]$$

for any  $u \in \mathbb{R}^d$  such that  $|\mathbf{u}| = u(D)$ . The above minimum is taken on all nonnegative functions in  $L^1(\mathbb{R}^d, (1 + |v|^4) \, d\nu)$  such that  $\int_{\mathbb{R}^d} f \, d\nu = 1$ .

*Proof.* Any minimizing sequence convergence is relatively compact in  $L^1(\mathbb{R}^d, d\nu)$  by the Dunford-Pettis theorem,  $f \mapsto \mathbf{u}_f$  is relatively compact and the existence of a minimizer follows by lower semi-continuity.  $\square$

#### 4.4.3 Proof of Theorem 4.1

By Corollary 4.3,  $f_0$  is the unique minimizer if and only if  $D \geq D_*$ . It is moreover linearly stable by Lemma 4.4. Otherwise  $f_{\mathbf{u}}$  with  $|\mathbf{u}| = u(D)$  is a minimizer of  $\mathcal{F}$  and it is unique up to a rotation. Combined with the results of Corollary 4.1, this completes the proof of Theorem 4.1.  $\square$



#### 4.4.4 Stability of the polarized stationary solution

Another interesting consequence of Corollary 4.3 is the linear stability of  $\mathcal{F}$  around  $f_{\mathbf{u}}$  when  $D < D_*$ .

**Lemma 4.5.** *Let  $D \in (0, D_*)$  and  $\mathbf{u} \in \mathbb{R}^d$  such that  $|\mathbf{u}| = u(D)$ . On the space of the functions  $g \in L^2(f_{\mathbf{u}} dv)$  such that  $\int_{\mathbb{R}^d} g f_{\mathbf{u}} dv = 0$ ,  $Q_{1, \mathbf{u}}$  is a nonnegative quadratic form.*

The proof is straightforward as, in the range  $D < D_*$ ,  $f_0$  is not a minimizer of  $\mathcal{F}$  and the minimum of  $\mathcal{F}$  is achieved by any  $f_{\mathbf{u}}$  with  $|\mathbf{u}| = u(D)$ . Details are left to the reader.

#### 4.4.5 An exponential rate of convergence for radially symmetric solutions

**Proposition 4.9.** *Let  $\alpha > 0$ ,  $D > 0$  and consider a solution  $f \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^d))$  of (4.1) with radially symmetric initial datum  $f_{\text{in}} \in L^1_+(\mathbb{R}^d)$  such that  $\mathcal{F}[f_{\text{in}}] < \infty$ . Then (4.6) holds for some  $\lambda > 0$ .*

*Proof.* According to Proposition 4.7, we know that

$$\frac{d}{dt} (\mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_0]) \leq -\mathcal{I}[f(t, \cdot)]$$

where  $\mathcal{I}$  defined by (4.5) and  $\mathbf{u}_f = \mathbf{0}$  because the radial symmetry is preserved by the evolution. We have a *logarithmic Sobolev inequality*

$$\int_{\mathbb{R}^d} \left| \nabla \log \left( \frac{f}{f_0} \right) \right|^2 f dv \geq \mathcal{K}_0 \int_{\mathbb{R}^d} f \log \left( \frac{f}{f_0} \right) dv = \mathcal{F}[f] - \mathcal{F}[f_0] \quad (4.14)$$

for some constant  $\mathcal{K}_0 > 0$ . This inequality holds for the same reason as for the Poincaré inequality (6.13): since  $\varphi_\alpha$  can be seen as a uniformly strictly convex potential perturbed by a bounded perturbation, it follows from the *carré du champ* method and the Holley-Stroock lemma that  $\mathcal{K}_0$  is a positive constant. Hence

$$\frac{d}{dt} (\mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_0]) \leq -\frac{\mathcal{K}_0}{D} \int_{\mathbb{R}^d} f \log \left( \frac{f}{f_0} \right) dv = -\frac{\mathcal{K}_0}{D} (\mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_0])$$

and we conclude that

$$\mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_0] \leq (\mathcal{F}[f_{\text{in}}] - \mathcal{F}[f_0]) e^{-\lambda t}$$

with  $\lambda = \mathcal{K}_0/D$ . The fact that  $\mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_0] \geq 0$  is a consequence of Corollary 4.3.  $\square$

#### 4.4.6 Continuity and convergence of the norm of the velocity average

**Proposition 4.10.** *Let  $\alpha > 0$ ,  $D > 0$  and consider a solution  $f \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^d))$  of (4.1) with initial datum  $f_{\text{in}} \in L^1_+(\mathbb{R}^d)$  such that  $\mathcal{F}[f_{\text{in}}] < \infty$ . Then  $t \mapsto \mathbf{u}_f(t)$  is a Lipschitz continuous function on  $\mathbb{R}^+$  such that  $\lim_{t \rightarrow +\infty} \mathbf{u}_f(t) = \mathbf{0}$  if  $D \geq D_*$ . If  $D \in (0, D_*)$ , along any increasing sequence  $(n_k)_{k \in \mathbb{N}}$  of integers, one can extract a subsequence, that we still denote by  $(n_k)_{k \in \mathbb{N}}$ , such that, uniformly in  $t \in (0, 1)$ , we obtain that  $\lim_{k \rightarrow +\infty} \mathbf{u}_f(t + n_k) = \mathbf{u}$  with either  $\mathbf{u} = \mathbf{0}$  or  $|\mathbf{u}| = u(D)$  if  $D \in (0, D_*)$ .*

*Proof.* Using (4.1), a straightforward computation shows that

$$\frac{d\mathbf{u}_f}{dt} = -\alpha \int_{\mathbb{R}^d} v(|v|^2 - 1) f dv$$

where the right hand side is bounded by Hölder interpolations using Propositions 4.7 and 4.8. By Proposition 4.8 and Hölder's inequality, we also know that  $\mathbf{u}_f$  is bounded.

We have a *logarithmic Sobolev inequality* analogous to (4.14) if we consider the relative entropy with respect to the *non-equilibrium Gibbs state*  $G_f$  defined by (4.4) instead of the relative entropy with respect to  $f_0$ : for some constant  $\mathcal{K} > 0$ ,

$$\int_{\mathbb{R}^d} \left| \nabla \log \left( \frac{f}{G_f} \right) \right|^2 f dv \geq \mathcal{K} \int_{\mathbb{R}^d} f \log \left( \frac{f}{G_f} \right) dv = \mathcal{F}[f] - \mathcal{F}[G_f].$$

By the *Csiszár-Kullback inequality*

$$\int_{\mathbb{R}^d} f \log \left( \frac{f}{G_f} \right) dv \geq \frac{1}{4} \|f - G_f\|_{L^1(\mathbb{R}^d)}^2, \quad (4.15)$$

we end up with the fact that  $\lim_{t \rightarrow +\infty} \int_t^{+\infty} (\int_{\mathbb{R}^d} |f - G_f| dv)^2 ds = 0$ . Using Hölder's inequality

$$\left| \int_{\mathbb{R}^d} v(f - G_f) dv \right| \leq \left( \int_{\mathbb{R}^d} |f - G_f| dv \right)^{3/4} \left( \int_{\mathbb{R}^d} |v|^4 (f + G_f) dv \right)^{1/4}$$

the decay of  $\mathcal{F}[f(t, \cdot)]$  and Proposition 4.8, we learn that  $\lim_{t \rightarrow +\infty} \int_{\mathbb{R}^d} v(f - G_f) dv = 0$ . Let  $\mathcal{C}(u) := \int_{\mathbb{R}^d} e^{-\frac{1}{D}(\phi_\alpha(v) - u v_1)} dv$ . By definition of  $\mathcal{H}$ , we have that

$$\int_{\mathbb{R}^d} v(f - G_f) dv = \mathbf{u}_f - \int_{\mathbb{R}^d} v G_f dv = \int_{\mathbb{R}^d} (\mathbf{u}_f - v) G_f dv = -\frac{\mathcal{H}(u)}{\mathcal{C}(u)} \frac{\mathbf{u}_f}{|\mathbf{u}_f|} \quad \text{with } u = |\mathbf{u}_f|.$$

Since  $\mathbf{u}_f$  is bounded,  $\mathcal{C}(u)$  is uniformly bounded by some positive constant and we deduce that

$$\lim_{t \rightarrow +\infty} \mathcal{H}(|\mathbf{u}_f|) = 0.$$

□

## 4.5 Large time asymptotic behaviour in the isotropic case

In this section, our main goal is to prove Theorem 4.2. In this section, we shall assume that  $D > D_*$ .

### 4.5.1 A non-local scalar product for the linearized evolution operator

We adapt the strategy of [50] to (4.1). With  $\mathbf{v}_g = \frac{1}{D} \int_{\mathbb{R}^d} v g f_0 dv$  as in Section 4.3,

$$\langle g_1, g_2 \rangle := D \int_{\mathbb{R}^d} g_1 g_2 f_0 dv - D^2 \mathbf{v}_{g_1} \cdot \mathbf{v}_{g_2} \quad (4.16)$$

is a scalar product on the space  $\mathcal{X} := \{g \in L^2(f_0 dv) : \int_{\mathbb{R}^d} g f_0 dv = 0\}$  by Lemma 4.4 because  $\langle g, g \rangle = Q_{1,0}[g]$ . Let us recall that  $f_0$  depends on  $D$  and, as a consequence, also  $D\mathbf{v}_g$ . Equation (4.1) means

$$\frac{\partial f}{\partial t} = \nabla \cdot \left( D \nabla f + (v - \mathbf{u}_f + \nabla \phi_\alpha) f \right)$$

and  $D \nabla f_0 = -(v + \nabla \phi_\alpha) f_0$ . Hence (4.1) is rewritten in terms of  $f = f_0(1 + g)$  as

$$f_0 \frac{\partial g}{\partial t} = D \nabla \cdot \left( (\nabla g - \mathbf{v}_g) f_0 - \mathbf{v}_g g f_0 \right)$$

using  $\mathbf{u}_f = D\mathbf{v}_g$ , that is,

$$\frac{\partial g}{\partial t} = \mathcal{L} g - \mathbf{v}_g \cdot \left( D \nabla g - (v + \nabla \phi_\alpha) g \right) \quad \text{with} \quad \mathcal{L} g = D \Delta g - (v + \nabla \phi_\alpha) \cdot (\nabla g - \mathbf{v}_g) \quad (4.17)$$

and collect some basic properties of  $\mathcal{X}$  endowed with the scalar product  $\langle \cdot, \cdot \rangle$  and  $\mathcal{L}$  considered as an operator on  $\mathcal{X}$ .

**Lemma 4.6.** *Assume that  $D > D_*$  and  $\alpha > 0$ . Let us consider the scalar product defined by (4.16) on  $\mathcal{X}$ . The norm  $g \mapsto \sqrt{\langle g, g \rangle}$  is equivalent to the standard norm on  $L^2(f_0 dv)$  according to*

$$\eta(D) \int_{\mathbb{R}^d} g^2 f_0 dv \leq \langle g, g \rangle \leq D \int_{\mathbb{R}^d} g^2 f_0 dv \quad \forall g \in \mathcal{X}. \quad (4.18)$$

Here  $\eta$  is as in (4.12). The linearized operator  $\mathcal{L}$  is self-adjoint on  $\mathcal{X}$  with the scalar product defined by (4.16) in the sense that  $\langle g_1, \mathcal{L} g_2 \rangle = \langle \mathcal{L} g_1, g_2 \rangle$  for any  $g_1, g_2 \in \mathcal{X}$ , and such that

$$-\langle g, \mathcal{L} g \rangle = Q_{2,0}[g]. \quad (4.19)$$

*Proof.* Inequality (4.18) is a straightforward consequence of Definition (4.16) and (4.12). The self-adjointness of  $\mathcal{L}$  is a consequence of elementary computations. By starting with

$$\mathcal{L} g_1 = \left[ D \Delta g_1 - (v + \nabla \phi_\alpha) \cdot \nabla g_1 \right] + (v + \nabla \phi_\alpha) \cdot \mathbf{v}_{g_1},$$

we first observe that

$$\int_{\mathbb{R}^d} [D\Delta g_1 - (v + \nabla\phi_\alpha) \cdot \nabla g_1] g_2 f_0 dv = -D \int_{\mathbb{R}^d} \nabla g_1 \cdot \nabla g_2 f_0 dv$$

and, as a consequence (take  $g_2 = v_i$  for some  $i = 1, 2, \dots, d$ ),

$$\mathbf{v}_{\mathcal{L} g_1} = \mathbf{v}_{g_1} - \int_{\mathbb{R}^d} \nabla g_1 f_0 dv.$$

Hence

$$-\langle \mathcal{L} g_1, g_2 \rangle = D^2 \int_{\mathbb{R}^d} (\nabla g_1 - \mathbf{v}_{g_1}) \cdot (\nabla g_2 - \mathbf{v}_{g_2}) dv,$$

which proves the self-adjointness of  $\mathcal{L}$  and Identity (4.19).  $\square$

The scalar product  $\langle \cdot, \cdot \rangle$  is well adapted to the linearized evolution operator in the sense that a solution of the *linearized equation*

$$\frac{\partial g}{\partial t} = \mathcal{L} g \tag{4.20}$$

with initial datum  $g_0 \in \mathcal{X}$  is such that

$$\frac{1}{2} \frac{d}{dt} Q_{1,0}[g] = \frac{1}{2} \frac{d}{dt} \langle g, g \rangle = \langle g, \mathcal{L} g \rangle = -Q_{2,0}[g]$$

and has exponential decay. According to Proposition 4.6, we know that

$$\langle g(t, \cdot), g(t, \cdot) \rangle \leq \langle g_0, g_0 \rangle e^{-2\mathcal{C}_D t} \quad \forall t \geq 0.$$

#### 4.5.2 Proof of Theorem 4.2

Let us consider the nonlinear term and prove that a solution  $g$  of (4.17) has the same asymptotic decay rate as a solution of the linearized equation (4.20). By rewriting (4.17) as

$$f_0 \frac{\partial g}{\partial t} = D \nabla \cdot (\nabla g - \mathbf{v}_g) f_0 - D \mathbf{v}_g \cdot \nabla (g f_0)$$

with  $f = f_0(1 + g)$  and using  $\int_{\mathbb{R}^d} g f_0 dv = 0$ , we find that

$$\frac{1}{2} \frac{d}{dt} Q_{1,0}[g] + Q_{2,0}[g] = D^2 \mathbf{v}_g \cdot \int_{\mathbb{R}^d} g (\nabla g - \mathbf{v}_g) f_0 dv.$$

Using  $\mathbf{u}_f = D \mathbf{v}_g$ , by the Cauchy-Schwarz inequality and (4.12), we obtain

$$\left( \int_{\mathbb{R}^d} g (\nabla g - \mathbf{v}_g) f_0 dv \right)^2 \leq \int_{\mathbb{R}^d} g^2 f_0 dv \int_{\mathbb{R}^d} |\nabla g - \mathbf{v}_g|^2 f_0 dv \leq \frac{Q_{1,0}[g]}{\eta(D)} \frac{Q_{2,0}[g]}{D^2}.$$

After taking into account Proposition 4.6, we have

$$\frac{d}{dt} Q_{1,0}[g] \leq -2 \left( 1 - |\mathbf{u}_f(t)| \sqrt{\frac{\mathcal{C}_D}{\eta(D)}} \right) Q_{1,0}[g].$$

By Proposition 4.10, we know that  $\lim_{t \rightarrow +\infty} |\mathbf{u}_f(t)| = 0$ , which proves that

$$\limsup_{t \rightarrow +\infty} e^{2(\mathcal{C}_D - \varepsilon)t} Q_{1,0}[g(t, \cdot)] < +\infty \quad (4.21)$$

for any  $\varepsilon \in (0, \mathcal{C}_D)$ . After observing that  $f \log(f/f_0) - (f - f_0) \leq \frac{1}{2}(f - f_0)^2/f_0$ , this completes the proof of Theorem 4.2.  $\square$

### 4.5.3 A sharp rate of convergence

We know from Proposition 4.6 that  $Q_{2,0}[g] \geq \mathcal{C}_D Q_{1,0}[g]$  for any  $g \in H^1(\mathbb{R}^d, f_0 dv)$  such that  $\int_{\mathbb{R}^d} g f_0 dv = 0$ . At no cost, we can assume that  $\mathcal{C}_D$  is the optimal constant.

**Theorem 4.3.** *For any  $d \geq 1$  and any  $\alpha > 0$ , if  $D > D_*$ , then the result of Theorem 4.2 holds with optimal rate  $\lambda = 2\mathcal{C}_D$ .*

*Proof.* We have to prove that 4.21 holds with  $\varepsilon = 0$ . By definition of  $\mathbf{u}_f$ , we have that

$$|\mathbf{u}_f|^2 = \left( \int_{\mathbb{R}^d} v(f - f_0) dv \right)^2 \leq \int_{\mathbb{R}^d} g^2 f_0 dv \int_{\mathbb{R}^d} |v|^2 f_0 dv$$

where  $g := (f - f_0)/f_0$ . This guarantees that  $|\mathbf{u}_f(t)| \leq c \sqrt{\eta(D) \mathcal{C}_D} e^{-\lambda t/2}$ . Then the function  $y(t) := Q_{1,0}[g(t, \cdot)]$  obeys to the differential inequality

$$y' \leq -2\mathcal{C}_D \left(1 - c e^{-\lambda t/2}\right) y$$

and we deduce as in Section 4.5.2 that  $\limsup_{t \rightarrow +\infty} e^{2\mathcal{C}_D t} y(t)$  is finite by a Grönwall estimate. This rate is optimal as shown by using test functions based on perturbations of  $f_0$ .  $\square$

## 4.6 Large time asymptotic behaviour in the polarized case

In this section, we shall assume that  $0 < D < D_*$ . The situation is more delicate than in the isotropic case  $D > D_*$ , as several asymptotic behaviours can occur.

### 4.6.1 Symmetric and non-symmetric stationary states

By perturbation of  $f_0$ , we know that the set of the functions  $f$  such that  $\mathcal{F}[f] < \mathcal{F}[f_0]$  is non-empty. Notice that the minimum of  $\mathcal{F}$  on radial functions is achieved by  $f_0$ . It follows that any function  $f$  such that  $\mathcal{F}[f] < \mathcal{F}[f_0]$  is non-radial.

**Lemma 4.7.** *For any  $d \geq 1$  and any  $\alpha > 0$ , if  $D < D_*$ , then for any solution  $f \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^d))$  of (4.1) with initial datum  $f_{\text{in}} \geq 0$  of mass 1 such that  $\mathcal{F}[f_{\text{in}}] < \mathcal{F}[f_0]$ . Then  $\lim_{t \rightarrow +\infty} |\mathbf{u}_f(t)| = u(D)$  and  $\lim_{t \rightarrow +\infty} \mathcal{F}[f(t, \cdot)] = \mathcal{F}[f_{\mathbf{u}}]$  for any  $\mathbf{u} \in \mathbb{R}^d$  such that  $|\mathbf{u}| = u(D)$  and there exists a sequence  $\{t_n\}$ , such that*

$$f(t_n + \cdot, \cdot) \longrightarrow f_{\mathbf{u}} \quad \text{in } L^1(\mathbb{R}^+(0, 1) \times \mathbb{R}^d) \quad \text{as } n \rightarrow +\infty.$$

*Proof.* We reconsider the proof of Proposition 4.10. Since  $\mathbf{u} = \mathbf{0}$  is forbidden by Proposition 4.7 and  $t \mapsto \mathbf{u}_f(t)$  is a converging Lipschitz function, there exists a unique limit  $\mathbf{u}$  such that  $|\mathbf{u}| = u(D)$ . The convergence result follows from the *logarithmic Sobolev inequality* and the *Csiszár-Kullback inequality* (4.15).  $\square$

#### 4.6.2 An exponential rate of convergence for partially symmetric solutions

Let us start with a simple case, which is to some extent the analogous of the case of Proposition 4.9 in the polarized case.

**Proposition 4.11.** *Let  $\alpha > 0$ ,  $D > 0$  and consider a solution  $f \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^d))$  of (4.1) with initial datum  $f_{\text{in}} \in L^1_+(\mathbb{R}^d)$  such that  $\mathcal{F}[f_{\text{in}}] < \mathcal{F}[f_0]$  and  $\mathbf{u}_{f_{\text{in}}} = (u, 0 \dots 0)$  for some  $u \neq 0$ . We further assume that*

$$f_{\text{in}}(v_1, v_2, \dots, v_{i-1}, v_i, \dots) = f_{\text{in}}(v_1, v_2, \dots, v_{i-1}, -v_i, \dots)$$

for any  $i = 2, 3, \dots, d$ . Then (4.6) holds with  $\lambda = \mathcal{C}_D(1 - \kappa(D)) > 0$ , with the notations of Proposition 4.6.

Here we assume that  $f_{\text{in}}(v_1, v_2, \dots, v_{i-1}, v_i, \dots)$  is even with respect to all coordinate of index  $i \geq 2$ , so that  $\mathbf{u}[f] = \mathbf{0}$  or  $\mathbf{u}[f] = (\pm u(D), 0 \dots 0)$  at any time  $t \geq 0$ .

*Proof.* According to Proposition 4.10, we know that  $\mathbf{u}_f$  is continuous. On the other hand, if  $\mathbf{u}_f = \mathbf{0}$ , then

$$\mathcal{F}[f] - \mathcal{F}[f_0] = \int_{\mathbb{R}^d} f \log\left(\frac{f}{f_0}\right) dv = \int_{\mathbb{R}^d} \frac{f}{f_0} \log\left(\frac{f}{f_0}\right) f_0 dv \geq X \log X|_{X=f_{\mathbb{R}^d} f} dv = 0$$

by Jensen's inequality, a contradiction with the assumption that  $\mathcal{F}[f_{\text{in}}] < \mathcal{F}[f_0]$  and Proposition 4.7. Hence  $\mathbf{u} = \mathbf{u}[f]$  is constant and we can reproduce with  $Q_{1,\mathbf{u}}[n]$  the proof done for  $Q_{1,0}[n]$  in Section 4.5.  $\square$

#### 4.6.3 Convergence to a polarized stationary state

To study the rate of convergence towards the stationary solutions  $f_{\mathbf{u}}$  with  $\mathbf{u} \neq \mathbf{0}$  in the range  $D \in (0, D_*)$ , we face a severe difficulty if  $\mathbf{u}_f$  converges tangentially to the set  $u(D)\mathbb{S}^{d-1}$  of admissible velocities for stationary solutions. Otherwise we obtain an exponential rate of convergence as in Theorem 4.2.

**Proposition 4.12.** *Assume that  $d \geq 2$ ,  $\alpha > 0$  and  $D \in (0, D_*)$ . Let us consider a solution  $f$  of (4.1) with nonnegative initial datum  $f_{\text{in}}$  of mass 1 such that  $\mathcal{F}[f_{\text{in}}] < \mathcal{F}[f_0]$  and assume that  $\mathbf{u} = \lim_{t \rightarrow +\infty} \mathbf{u}_f(t)$  is uniquely defined. If  $|(\mathbf{u}_f - \mathbf{u}) \cdot \mathbf{u}| \geq \varepsilon u(D) |\mathbf{u}_f - \mathbf{u}|$  for some  $\varepsilon > 0$  and  $t > 0$  large enough, then there are two positive constants  $C, \lambda$  and some  $\mathbf{u} \in \mathbb{R}^d$  such that*

$$0 \leq \mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_{\mathbf{u}}] \leq C e^{-\lambda t} \quad \forall t \geq 0.$$

*Proof.* We adapt the setting of Section 4.5.2 to  $g = (f - f_{\mathbf{u}})/f_{\mathbf{u}}$  and get that

$$\frac{1}{2} \frac{d}{dt} Q_{1,\mathbf{u}}[g] + Q_{2,\mathbf{u}}[g] = D^2 \mathbf{v}_g \cdot \int_{\mathbb{R}^d} g (\nabla g - \mathbf{v}_g) f_{\mathbf{u}} d\nu.$$

With  $Z(t) := \mathcal{C}_D (1 - \kappa(D)) \frac{(\mathbf{v}_g \cdot \mathbf{u})^2}{|\mathbf{v}_g|^2 |\mathbf{u}|^2}$ , we can rewrite Proposition 4.6 and the estimate of the nonlinear term as

$$Q_{2,\mathbf{u}}[g] \geq Z(t) Q_{1,\mathbf{u}}[g] \quad \text{and} \quad D^2 \mathbf{v}_g \cdot \int_{\mathbb{R}^d} g (\nabla g - \mathbf{v}_g) f_{\mathbf{u}} d\nu \leq D |\mathbf{v}_g| \frac{\sqrt{Q_{1,\mathbf{u}}[g] Q_{2,\mathbf{u}}[g]}}{\sqrt{\eta(D)}}$$

By assumption,  $Z(t) \geq \mathcal{C}_D (1 - \kappa(D)) \varepsilon^2$ . The conclusion follows as in Section 4.5.2.  $\square$

## 4.7 Some additional properties of $D_*$

In this section, we collect some plots which illustrate Section 4.2 and state related qualitative properties of  $D_*$ .

**Proposition 4.13.** *For any  $\alpha > 0$  and  $d \geq 1$ , the critical value  $D_* = D_*(\alpha, d)$  is monotone decreasing as a function of  $d$ , such that*

$$\frac{1}{d+2} < D_*(\alpha, d) < \frac{1}{d}$$

with lower and upper bounds achieved respectively as  $\alpha \rightarrow 0_+$  and  $\alpha \rightarrow +\infty$ .

*Proof.* The monotonicity with respect to  $d$  can be read from

$$h_d(D) - h_{d+1}(D) = \int_0^\infty s^{d+1} (s^2 - 1)^2 e^{-\frac{\phi \alpha}{D}} ds > 0.$$

The lower bound is a consequence of

$$\int_0^\infty (s^{d+1} - s^{d+3}) e^{-\frac{1}{2}(d+2)s^2} ds = 0.$$

As for the upper bound, for any  $D > 0$ , by considering the derivatives with respect to  $\alpha$  of  $j_{d+1}$  and  $j_{d-1}$  as defined in (4.7), we notice that

$$\frac{j_{d+1}}{j_{d-1}} \sim \frac{2j_{d+3} - j_{d+5}}{2j_{d+1} - j_{d+3}} \sim \frac{\frac{\alpha+1}{\alpha} j_{d+3} - \frac{d+2}{\alpha} D j_{d+1}}{2j_{d+1} - j_{d+3}}$$

by L'Hôpital's rule as  $\alpha \rightarrow +\infty$ . We recall that  $j_{d+1}(D) = j_{d+3}(D)$  at  $D = D_*$ . By letting  $\alpha \rightarrow +\infty$  with  $D = D_*$ , we conclude that  $j_{d+1}/j_{d-1} \rightarrow 1$ . On the other hand, (4.9) with  $n = d - 2$  means that  $j_{d+1}(D_*) = d D_* j_{d-1}(D_*)$ , from which we conclude that  $\lim_{\alpha \rightarrow +\infty} D_*(\alpha, d) = 1/d$ .  $\square$

We conclude this chapter by computations of  $D_*$  for specific values of the parameters.

- If  $d = 1$ ,  $\alpha = 2$ ,  $D_*$  solves  $(1-4D)I_{-1/4}\left(\frac{1}{16D}\right) + (1+4D)I_{1/4}\left(\frac{1}{16D}\right) + I_{3/4}\left(\frac{1}{16D}\right) + I_{5/4}\left(\frac{1}{16D}\right) = 0$  where  $I_\gamma$  denotes the modified Bessel function of the first kind. Numerically, we find that  $D_* \approx 0.529$  matches [13, Fig. 1, p. 4].

- If  $d = 2$ ,  $\alpha = 2$ , we remind that  $D_* \approx 0.354$ : see Fig. 4.2.

- If  $d = 2$ ,  $\alpha = 4$ ,  $D_* \approx 0.398$  solves  $(16\Gamma\left(\frac{3}{2}, \frac{9}{16D}\right) - 16\sqrt{\pi})D - 8\Gamma\left(1, \frac{9}{16D}\right)\sqrt{D} + 6\sqrt{\pi} - 3\Gamma\left(\frac{1}{2}, \frac{9}{16D}\right) = 0$ .

For further numerical examples, we refer the reader to [154, 13].



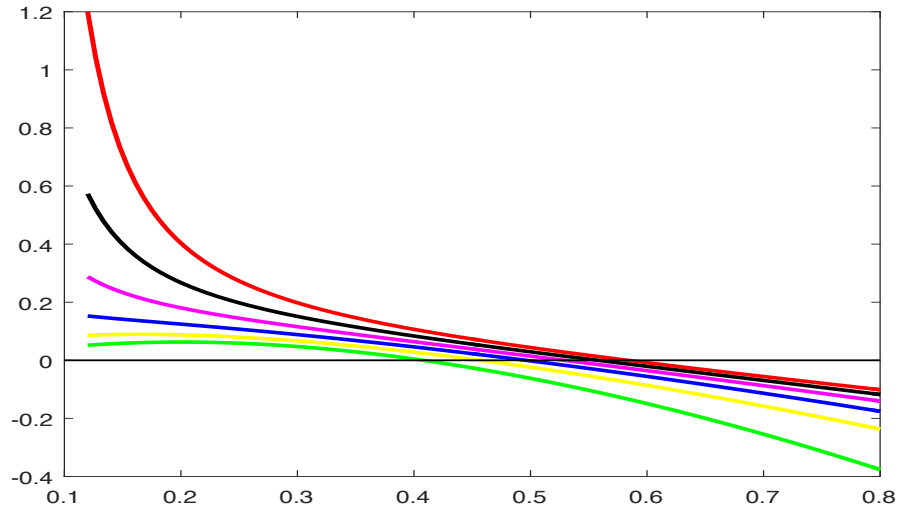


Figure 4.1: Plot of  $h_d(D)$  against  $D$  when  $d = 1$  with  $\alpha = 0.5, 1, \dots, 3.0$ .

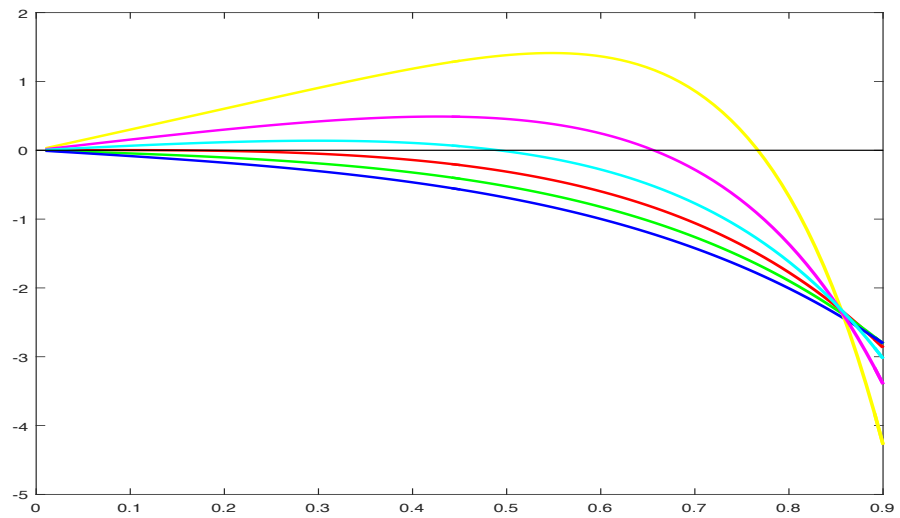


Figure 4.2: Plot of  $u \mapsto \mathcal{H}(u)$  when  $d = 2$ ,  $\alpha = 2$ , and  $D = 0.2, 0.25, \dots, 0.45$ . In this particular case,  $D_* \approx 0.354$  solves  $(8 \Gamma(\frac{3}{2}, \frac{1}{8D}) - 8\sqrt{\pi})D - \Gamma(\frac{1}{2}, \frac{1}{8D}) + 2\sqrt{\pi} = 0$ .





## Chapter 5

# Generalized log-HLS inequality

**Article *Generalized logarithmic Hardy-Littlewood-Sobolev inequality*, in collaboration with Jean DOLBEAULT, accepted by International Mathematics Research Notices, 2019.**

This chapter is devoted to logarithmic Hardy-Littlewood-Sobolev inequalities in the two-dimensional Euclidean space, in presence of an external potential with logarithmic growth. The coupling with the potential introduces a new parameter, with two regimes. The attractive regime reflects the standard logarithmic Hardy-Littlewood-Sobolev inequality. The second regime corresponds to a reverse inequality, with the opposite sign in the convolution term, that allows us to bound the free energy of a drift-diffusion-Poisson system from below. Our method is based on an extension of an entropy method proposed by E. Carlen, J. Carrillo and M. Loss, and on a nonlinear diffusion equation.

## 5.1 Main result and motivation

On  $\mathbb{R}^2$ , let us define the *density of probability*  $\mu = e^{-V}$  and the *external potential*  $V$  by

$$\mu(x) := \frac{1}{\pi(1+|x|^2)^2} \quad \text{and} \quad V(x) := -\log \mu(x) = 2 \log(1+|x|^2) + \log \pi \quad \forall x \in \mathbb{R}^2.$$

We shall denote by  $L_+^1(\mathbb{R}^2)$  the set of a.e. nonnegative functions in  $L^1(\mathbb{R}^2)$ . Our main result is the following *generalized logarithmic Hardy-Littlewood-Sobolev inequality*.

**Theorem 5.1.** *For any  $\alpha \geq 0$ , we have that*

$$\begin{aligned} \int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \alpha \int_{\mathbb{R}^2} V f dx + M(1-\alpha)(1+\log \pi) \\ \geq \frac{2}{M}(\alpha-1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)f(y) \log|x-y| dx dy \end{aligned} \quad (5.1)$$

for any function  $f \in L_+^1(\mathbb{R}^2)$  with  $M = \int_{\mathbb{R}^2} f dx > 0$ . Moreover, the equality case is achieved by  $f_\star = M\mu$  and  $f_\star$  is the unique optimal function for any  $\alpha > 0$ .

With  $\alpha = 0$ , the inequality is the classical *logarithmic Hardy-Littlewood-Sobolev inequality*

$$\int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)f(y) \log|x-y| dx dy + M(1+\log \pi) \geq 0. \quad (5.2)$$

In that case  $f_\star$  is an optimal function as well as all functions generated by a translation and a scaling of  $f_\star$ . As long as the parameter  $\alpha$  is in the range  $0 \leq \alpha < 1$ , the coefficient of the right-hand side of (5.1) is negative and the inequality is essentially of the same nature as the one with  $\alpha = 0$ . It can indeed be written as

$$\begin{aligned} \int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \alpha \int_{\mathbb{R}^2} V f dx + M(1-\alpha)(1+\log \pi) \\ + \frac{2}{M}(1-\alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)f(y) \log|x-y| dx dy \geq 0 \end{aligned}$$

For reasons that will be made clear below, we shall call this range the *attractive range*.

If  $\alpha = 1$ , the inequality is almost trivial since

$$\int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \int_{\mathbb{R}^2} V f dx = \int_{\mathbb{R}^2} f \log\left(\frac{f}{f_\star}\right) dx \geq 0 \quad (5.3)$$

is a straightforward consequence of Jensen's inequality. Now it is clear that by adding (5.2) multiplied by  $(1-\alpha)$  and (5.3) multiplied by  $\alpha$ , we recover (5.1)

for any  $\alpha \in [0, 1]$ . As a consequence (5.1) is a straightforward interpolation between (5.2) and (5.3) in the *attractive range*.

Now, let us consider the *repulsive range*  $\alpha > 1$ . It is clear that the inequality is no more the consequence of a simple interpolation. We can also observe that the coefficient  $(\alpha - 1)$  in the right-hand side of (5.1) is now positive. Since

$$G(x) = -\frac{1}{2\pi} \log|x|$$

is the Green function associated with  $-\Delta$  on  $\mathbb{R}^2$ , so that we can define

$$(-\Delta)^{-1}f(x) = (G * f)(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| f(y) dy,$$

it is interesting to write (5.1) as

$$\int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \alpha \int_{\mathbb{R}^2} V f dx + \frac{4\pi}{M} (\alpha - 1) \int_{\mathbb{R}^2} f (-\Delta)^{-1} f dx \geq M(\alpha - 1) (1 + \log \pi). \quad (5.4)$$

If  $f$  has a sufficient decay as  $|x| \rightarrow +\infty$ , for instance if  $f$  is compactly supported, we know that  $(-\Delta)^{-1}f(x) \sim -\frac{M}{2\pi} \log|x|$  for large values of  $|x|$  and as a consequence,

$$\alpha V + \frac{4\pi}{M} (\alpha - 1) (-\Delta)^{-1} f \sim 2(\alpha + 1) \log|x| \rightarrow +\infty \quad \text{as } |x| \rightarrow +\infty.$$

In a minimization scheme, this prevents the runaway of the left-hand side in (5.4). On the other hand,  $\int_{\mathbb{R}^2} f \log f dx$  prevents any concentration, and this is why it can be heuristically expected that the left-hand side of (5.4) indeed admits a minimizer.

Inequality (5.2) was proved in [52] by E. Carlen and M. Loss (also see [22]). An alternative method based on nonlinear flows was given by E. Carlen, J. Carrillo and M. Loss in [51]: see Section 7.4 for a sketch of their proof. Our proof of Theorem 5.1 relies on an extension of this approach which takes into account the presence of the external potential  $V$ . A remarkable feature of this approach is that it is insensitive to the sign of  $\alpha - 1$ .

One of the key motivations for studying (5.4) arises from entropy methods applied to *drift-diffusion-Poisson* models which, after scaling out all physical parameters, are given by

$$\frac{\partial f}{\partial t} = \Delta f + \beta \nabla \cdot (f \nabla V) + \nabla \cdot (f \nabla \phi) \quad (5.5)$$

with a nonlinear coupling given by the *Poisson* equation

$$-\varepsilon \Delta \phi = f. \quad (5.6)$$

Here  $V = -\log \mu$  is the external *confining potential* and we choose it as in the statement of Theorem 5.1, while  $\beta \geq 0$  is a coupling parameter with  $V$ , which measures the strength of the external potential. We shall consider more general potentials at the end of this paper. The coefficient  $\varepsilon$  in (5.6) is either  $\varepsilon = -1$ , which corresponds to the *attractive* case, or  $\varepsilon = +1$ , which corresponds to the *repulsive* case. In terms of applications, when  $\varepsilon = -1$ , (5.6) is the equation for the mean field potential obtained from Newton's law of attraction in gravitation, for applications in astrophysics, or for the Keller-Segel concentration of chemo-attractant in chemotaxis. The case  $\varepsilon = +1$  is used for repulsive electrostatic forces in semi-conductor physics, electrolytes, plasmas and charged particle models.

In view of *entropy methods* applied to PDEs (see for instance [116]), it is natural to consider the *free energy functional*

$$\mathcal{F}_\beta[f] := \int_{\mathbb{R}^2} f \log f \, dx + \beta \int_{\mathbb{R}^2} V f \, dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi f \, dx \quad (5.7)$$

because, if  $f > 0$  solves (5.5)-(5.6) and is smooth enough, with sufficient decay properties at infinity, then

$$\frac{d}{dt} \mathcal{F}_\beta[f(t, \cdot)] = - \int_{\mathbb{R}^2} f |\nabla \log f + \beta \nabla V + \nabla \phi|^2 \, dx \quad (5.8)$$

so that  $\mathcal{F}_\beta$  is a Lyapunov functional. Of course, a preliminary question is to establish under which conditions  $\mathcal{F}_\beta$  is bounded from below. The answer is given by the following result.

**Corollary 5.1.** *Let  $M > 0$ . The functional  $\mathcal{F}_\beta$  is bounded from below and admits a minimizer on the set of the functions  $f \in L^1_+(\mathbb{R}^2)$  such that  $\int_{\mathbb{R}^2} f \, dx = M$  if either  $\varepsilon = +1$  and  $\beta \geq 1 + \frac{M}{8\pi}$ , or  $\varepsilon = -1$ ,  $\beta \geq 0$  and  $M \leq 8\pi$ . If  $\varepsilon = +1$ , the minimizer is unique.*

As we shall see in Section 5.3.1, Corollary 5.1 is a simple consequence of Theorem 5.1. In the case of the parabolic-elliptic Keller-Segel model, that is, with  $\varepsilon = -1$  and  $\beta = 0$ , this has been used in [87, 33] to provide a sharp range of existence of the solutions to the evolution problem. In [50], the case  $\varepsilon = -1$  with a potential  $V$  with quadratic growth at infinity was also considered, in the study of intermediate asymptotics of the parabolic-elliptic Keller-Segel model.

Concerning the *drift-diffusion-Poisson* model (5.5)-(5.6) and considerations on the *free energy*, in the electrostatic case, we can quote, among many others, [100, 91] and subsequent papers. In the Euclidean space with confinig potentials, we shall refer to [71, 72, 28, 9]. However, as far as we know, these papers are primarily devoted to dimensions  $d \geq 3$  and the sharp growth condition on  $V$  when  $d = 2$  has not been studied so far. The goal of this paper is to fill this gap.

The specific choice of  $V$  has been made to obtain explicit constants and optimal inequalities, but the confining potential plays a role only at infinity if we are interested in the boundedness from below of the free energy. In Section 5.3.3, we shall give a result for general potentials on  $\mathbb{R}^2$ : see Theorem 5.2 for a statement.

## 5.2 Proof of the main result

As an introduction to the key method, we briefly sketch the proof of (5.2) given by E. Carlen, J. Carrillo and M. Loss in [51]. The main idea is to use the nonlinear diffusion equation

$$\frac{\partial f}{\partial t} = \Delta \sqrt{f}$$

with a nonnegative initial datum  $f_0$ . The equation preserves the mass  $M = \int_{\mathbb{R}^2} f dx$  and is such that

$$\begin{aligned} \frac{d}{dt} \left( \int_{\mathbb{R}^2} f \log f dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f ((-\Delta)^{-1} f) dx \right) \\ = -\frac{8}{M} \left( \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 dx \int_{\mathbb{R}^2} f dx - \pi \int_{\mathbb{R}^2} f^{3/2} dx \right). \end{aligned}$$

According to [67], the Gagliardo-Nirenberg inequality

$$\|\nabla g\|_2^2 \|g\|_4^4 \geq \pi \|g\|_6^6 \quad (5.9)$$

applied to  $g = f^{1/4}$  guarantees that the right-hand side is nonpositive. By the general theory of fast diffusion equations (we refer for instance to [156]), we know that the solution behaves for large values of  $t$  like a self-similar solution, the so-called Barenblatt solution, which is given by  $B(t, x) := t^{-2} f_\star(x/t)$ . As a consequence, we find that

$$\begin{aligned} \int_{\mathbb{R}^2} f_0 \log f_0 dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f_0 ((-\Delta)^{-1} f_0) dx \\ \geq \lim_{t \rightarrow +\infty} \int_{\mathbb{R}^2} B \log B dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} B ((-\Delta)^{-1} B) dx \\ = \int_{\mathbb{R}^2} f_\star \log f_\star dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f_\star ((-\Delta)^{-1} f_\star) dx \end{aligned}$$

After an elementary computation, we observe that the above inequality is exactly (5.2) written for  $f = f_0$ .

The point is now to adapt this strategy to the case with an external potential. This justifies why we have to introduce a nonlinear diffusion equation with a drift. As we shall see below, the method is insensitive to  $\alpha$  and applies when  $\alpha > 1$  exactly as in the case  $\alpha \in (0, 1)$ . A natural question is whether solutions are regular enough to perform the computations below and in particular if they have



a sufficient decay at infinity to allow all kinds of integrations by parts needed by the method. The answer is twofold. First, we can take an initial datum  $f_0$  which is as smooth and decaying as  $|x| \rightarrow +\infty$  as needed, prove the inequality and argue by density. Second, integrations by parts can be justified by an approximation scheme consisting in a truncation of the problem in larger and larger balls. We refer to [156] for regularity issues and to [116] for the truncation method. In the proof, we will therefore leave these issues apart, as they are purely technical.

**Proof of Theorem 5.1.** By homogeneity, we can assume that  $M = 1$  without loss of generality and consider the evolution equation

$$\frac{\partial f}{\partial t} = \Delta \sqrt{f} + 2\sqrt{\pi} \nabla \cdot (x f).$$

1) Using simple integrations by parts, we compute

$$\int_{\mathbb{R}^2} (1 + \log f) \Delta \sqrt{f} \, dx = -8 \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 \, dx$$

and

$$\int_{\mathbb{R}^2} (1 + \log f) \nabla \cdot (x f) \, dx = - \int_{\mathbb{R}^2} \frac{\nabla f}{f} \cdot (x f) \, dx = - \int_{\mathbb{R}^2} x \cdot \nabla f \, dx = 2 \int_{\mathbb{R}^2} f \, dx = 2.$$

As a consequence, we obtain that

$$\frac{d}{dt} \int_{\mathbb{R}^2} f \log f \, dx = -8 \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 \, dx + 8\pi \int_{\mathbb{R}^2} \mu^{3/2} \, dx \quad (5.10)$$

using

$$\int_{\mathbb{R}^2} \mu^{3/2} \, dx = \frac{1}{2\sqrt{\pi}}.$$

2) By elementary considerations again, we find that

$$4\pi \int_{\mathbb{R}^2} f (-\Delta)^{-1} (\Delta \sqrt{f}) \, dx = -4\pi \int_{\mathbb{R}^2} f^{3/2} \, dx$$

and

$$\begin{aligned} 4\pi \int_{\mathbb{R}^2} \nabla \cdot (x f) (-\Delta)^{-1} f \, dx &= -4\pi \int_{\mathbb{R}^2} x f \cdot \nabla (-\Delta)^{-1} f \, dx \\ &= 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) x \cdot \frac{x-y}{|x-y|^2} \, dx \, dy \\ &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) (x-y) \cdot \frac{x-y}{|x-y|^2} \, dx \, dy = 1 \end{aligned}$$

where, in the last line, we exchanged the variables  $x$  and  $y$  and took the half sum of the two expressions. This proves that

$$\frac{d}{dt} \left( 4\pi \int_{\mathbb{R}^2} f (-\Delta)^{-1} f \, dx \right) = -8\pi \int_{\mathbb{R}^2} (f^{3/2} - \mu^{3/2}) \, dx. \quad (5.11)$$

3) We observe that

$$\mu(x) = \frac{1}{\pi(1+|x|^2)^2} = e^{-V(x)}$$

solves

$$\Delta V = -\Delta \log \mu = 8\pi \mu \quad (5.12)$$

and, as a consequence,

$$\int_{\mathbb{R}^2} V \Delta \sqrt{f} dx = \int_{\mathbb{R}^2} \Delta V \sqrt{f} dx = 8\pi \int_{\mathbb{R}^2} \mu \sqrt{f} dx.$$

Since

$$\begin{aligned} 2\sqrt{\pi} \int_{\mathbb{R}^2} V \nabla \cdot (x f) dx &= -2\sqrt{\pi} \int_{\mathbb{R}^2} f x \cdot \nabla V dx = -8\sqrt{\pi} \int_{\mathbb{R}^2} \frac{|x|^2}{1+|x|^2} f dx \\ &= -8\sqrt{\pi} + 8\sqrt{\pi} \int_{\mathbb{R}^2} \frac{f}{1+|x|^2} dx = -8\sqrt{\pi} + 8\pi \int_{\mathbb{R}^2} \sqrt{\mu} f dx, \end{aligned}$$

we conclude that

$$\frac{d}{dt} \int_{\mathbb{R}^2} f V dx = 8\pi \int_{\mathbb{R}^2} (\mu \sqrt{f} + \sqrt{\mu} f - 2\mu^{3/2}) dx. \quad (5.13)$$

Let us define

$$\begin{aligned} \mathcal{F}[f] &:= \int_{\mathbb{R}^2} f \log f dx + \alpha \int_{\mathbb{R}^2} V f dx + (1-\alpha)(1+\log \pi) \\ &\quad + 2(1-\alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x-y| dx dy. \end{aligned}$$

Collecting (5.10), (5.11) and (5.13), we find that

$$\begin{aligned} \frac{d}{dt} \mathcal{F}[f(t, \cdot)] &= -8 \left( \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 dx - \pi \int_{\mathbb{R}^2} f^{3/2} dx \right) \\ &\quad - 8\pi \alpha \int_{\mathbb{R}^2} (f^{3/2} - \mu \sqrt{f} - \sqrt{\mu} f + \mu^{3/2}) dx. \end{aligned}$$

Notice that

$$\int_{\mathbb{R}^2} (f^{3/2} - \mu \sqrt{f} - \sqrt{\mu} f + \mu^{3/2}) dx = \int_{\mathbb{R}^2} \varphi\left(\frac{f}{\mu}\right) \mu^{3/2} dx \quad \text{with} \quad \varphi(t) := t^{3/2} - t - \sqrt{t} + 1$$

and that  $\varphi$  is a strictly convex function on  $\mathbb{R}^+$  such that  $\varphi(1) = \varphi'(1) = 0$ , so that  $\varphi$  is nonnegative. On the other hand, by (5.9), we know that

$$\int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 dx - \pi \int_{\mathbb{R}^2} f^{3/2} dx \geq 0$$

as in the proof of [51]. Altogether, this proves that  $t \mapsto \mathcal{F}[f(t, \cdot)]$  is monotone nonincreasing. Hence

$$\mathcal{F}[f_0] \geq \mathcal{F}[f(t, \cdot)] \geq \lim_{t \rightarrow +\infty} \mathcal{F}[f(t, \cdot)] = \mathcal{F}[f_\star] = 0.$$

This completes the proof of (5.1).  $\square$

### 5.3 Consequences

#### 5.3.1 Proof of Corollary 5.1

To prove the result of Corollary 5.1, we have to establish first that the *free energy* functional  $\mathcal{F}_\beta$  is bounded from below. Instead of using standard variational methods to prove that a minimizer is achieved, we can rely on the flow associated with (5.5)-(5.6).

• **Repulsive case.** Let us consider the *free energy* functional defined in (5.7) where  $\phi$  is given by (5.6) with  $\varepsilon = +1$ , i.e.,  $\phi = -\frac{1}{2\pi} \log|\cdot| * f$ .

**Lemma 5.1.** *Let  $M > 0$  and  $\varepsilon = +1$ . Then  $\mathcal{F}_\beta$  is bounded from below on the set of the functions  $f \in L^1_+(\mathbb{R}^2)$  such that  $\int_{\mathbb{R}^2} f dx = M$  if  $\beta \geq 1 + \frac{M}{8\pi}$ .*

*Proof.* With  $g = \frac{f}{M}$  and  $\alpha = 1 + \frac{M}{8\pi}$ , this means that

$$\begin{aligned} \frac{1}{M} \mathcal{F}_\beta[f] - \log M &= \int_{\mathbb{R}^2} g \log g dx + \beta \int_{\mathbb{R}^2} V g dx - \frac{M}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log|x-y| dx dy \\ &= (\beta - \alpha) \int_{\mathbb{R}^2} V g dx + \int_{\mathbb{R}^2} g \log g dx + \alpha \int_{\mathbb{R}^2} V g dx \\ &\quad - 2(\alpha - 1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log|x-y| dx dy \\ &\geq (\beta - \alpha) \int_{\mathbb{R}^2} V g dx - (1 - \alpha) (1 + \log \pi) \end{aligned}$$

according to Theorem 5.1: the condition  $\beta \geq \alpha$  is enough to prove that  $\mathcal{F}_\beta[f]$  is bounded from below.

□

**Proof of Corollary 5.1 with  $\varepsilon = +1$ .** Let us consider a smooth solution of (5.5)-(5.6). We refer to [123] for details and to [9] for similar arguments in dimension  $d \geq 3$ . According to (5.8),  $f$  converges as  $t \rightarrow +\infty$  to a solution of

$$\nabla \log f + \beta \nabla V + \nabla \phi = 0.$$

Notice that this already proves the existence of a stationary solution. The equation can be solved as

$$f = M \frac{e^{-\beta V - \phi}}{\int_{\mathbb{R}^2} e^{-\beta V - \phi} dx}$$

after taking into account the conservation of the mass. With (5.6), the problem is reduced to solving

$$-\Delta \psi = M \left( \frac{e^{-\gamma V - \psi}}{\int_{\mathbb{R}^2} e^{-\gamma V - \psi} dx} - \mu \right), \quad \psi = (\beta - \gamma) V + \phi, \quad \gamma = \beta - \frac{M}{8\pi}$$

using (5.12). It is a critical point of the functional  $\psi \mapsto \mathcal{J}_{M,\gamma}[\psi] := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 dx + M \int_{\mathbb{R}^2} \psi \mu dx + M \log \left( \int_{\mathbb{R}^2} e^{-\gamma V - \psi} dx \right)$ . Such a functional is strictly convex as, for instance, in [71, 72]. We conclude that  $\psi$  is unique up to an additional constant.  $\square$

• **Attractive case.** Let us consider the *free energy* functional (5.7)  $\mathcal{F}_\beta$  where  $\phi$  is given by (5.6) with  $\varepsilon = -1$ , i.e.,  $\phi = \frac{1}{2\pi} \log |\cdot| * f$ . Inspired by [87], we have the following estimate.

**Lemma 5.2.** *Let  $\beta \geq 0$  and  $\varepsilon = -1$ . Then  $\mathcal{F}_\beta$  is bounded from below on the set of the functions  $f \in L_+^1(\mathbb{R}^2)$  such that  $\int_{\mathbb{R}^2} f dx = M$  if  $M \leq 8\pi$ .*

*Proof.* With  $g = \frac{f}{M}$  and  $\alpha = 1 - \frac{M}{8\pi}$ , Theorem 5.1 applied to

$$\begin{aligned} \frac{1}{M} \mathcal{F}_\beta[f] - \log M &= (\beta - \alpha) \int_{\mathbb{R}^2} V g dx + \int_{\mathbb{R}^2} g \log g dx + \alpha \int_{\mathbb{R}^2} V g dx \\ &\quad + 2(1 - \alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log |x - y| dx dy \\ &\geq (\beta - \alpha) \int_{\mathbb{R}^2} V g dx - (1 - \alpha) (1 + \log \pi) \end{aligned}$$

proves that the free energy is bounded from below if  $M \leq 8\pi$ .  $\square$

**Proof of Corollary 5.1 with  $\varepsilon = -1$ .** The proof goes as in the case  $\beta = 0$ . We refer to [33] and leave details to the reader.  $\square$

**Remark 5.1.** *Let us notice that  $\mathcal{F}_\beta$  is unbounded from below if  $\beta < 0$ . This follows from the observation that  $\lim_{|y| \rightarrow \infty} \mathcal{F}_\beta[f_y] = -\infty$  where  $f_y(x) = f(x + y)$  for any admissible  $f$ .*

### 5.3.2 Duality

When  $\alpha > 1$ , we can write a first inequality by considering the *repulsive case* in the proof of Corollary 5.1 and observing that

$$\mathcal{J}_{M,\gamma}[\psi] \geq \min \mathcal{J}_{M,\gamma}$$

where  $\psi \in W_{\text{loc}}^{2,1}(\mathbb{R}^2)$  is such that  $\int_{\mathbb{R}^2} (\Delta \psi) dx = 0$  and the minimum is taken on the same set of functions.

When  $\alpha \in [0, 1)$ , it is possible to argue by duality as in [49, Section 2]. Since  $f_\star$  realizes the equality case in (5.1), we know that

$$\begin{aligned} \int_{\mathbb{R}^2} f_\star \log \left( \frac{f_\star}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f_\star dx + M(1 - \alpha) (1 + \log \pi) \\ = \frac{2}{M} (\alpha - 1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f_\star(x) f_\star(y) \log |x - y| dx dy \end{aligned}$$

and, using the fact that  $f_\star$  is a critical point of the difference of the two sides of (5.1), we also have that

$$\begin{aligned} \int_{\mathbb{R}^2} \log\left(\frac{f}{f_\star}\right) (f - f_\star) dx + \alpha \int_{\mathbb{R}^2} V (f - f_\star) dx \\ = \frac{4}{M} (\alpha - 1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (f(x) - f_\star(x)) f_\star(y) \log|x - y| dx dy. \end{aligned}$$

By subtracting the first identity to (5.1) and adding the second identity, we can rephrase (5.1) as

$$\mathcal{F}_{(1)}[f] := \int_{\mathbb{R}^2} f \log\left(\frac{f}{f_\star}\right) dx \geq \frac{4\pi}{M} (1 - \alpha) \int_{\mathbb{R}^2} (f - f_\star) (-\Delta)^{-1} (f - f_\star) dx := \mathcal{F}_{(2)}[f].$$

Let us consider the Legendre transform

$$\mathcal{F}_{(i)}^*[g] := \sup_f \left( \int_{\mathbb{R}^2} g f dx - \mathcal{F}_{(i)}[f] \right)$$

where the supremum is restricted to the set of the functions  $f \in L_+^1(\mathbb{R}^2)$  such that  $M = \int_{\mathbb{R}^2} f dx$ . After taking into account the Lagrange multipliers associated with the mass constraint, we obtain that

$$M \log\left(\int_{\mathbb{R}^2} e^{g-V} dx\right) = \mathcal{F}_{(1)}^*[g] \leq \frac{M}{16\pi(1-\alpha)} \int_{\mathbb{R}^2} |\nabla g|^2 dx + M \int_{\mathbb{R}^2} g e^{-V} dx = \mathcal{F}_{(2)}^*[g].$$

We can get rid of  $M$  by homogeneity and recover the standard Euclidean form of the Onofri inequality in the limit case as  $\alpha \rightarrow 0_+$ , which is clearly the sharpest one for all possible  $\alpha \in [0, 1)$ .

### 5.3.3 Extension to general confining potentials with critical asymptotic growth

As a concluding observation, let us consider a general potential  $W$  on  $\mathbb{R}^2$  such that

$$W \in C(\mathbb{R}^2) \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} \frac{W(x)}{V(x)} = \beta \quad (\mathcal{H}_W)$$

and the associated *free energy functional*

$$\mathcal{F}_{\beta,W}[f] := \int_{\mathbb{R}^2} f \log f dx + \beta \int_{\mathbb{R}^2} W f dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi f dx$$

where  $\phi$  is given in terms of  $f > 0$  by (5.6). With previous notations,  $\mathcal{F}_\beta = \mathcal{F}_{\beta,V}$ . Our last result is that the asymptotic behaviour obtained from  $(\mathcal{H}_W)$  is enough to decide whether  $\mathcal{F}_{\beta,W}$  is bounded from below or not. The precise result goes as follows.

**Theorem 5.2.** *Under Assumption  $(\mathcal{H}_W)$ ,  $\mathcal{F}_{\beta,W}$  defined as above is bounded from below if either  $\varepsilon = +1$  and  $\beta > 1 + \frac{M}{8\pi}$ , or  $\varepsilon = -1$ ,  $\beta > 1 - \frac{M}{8\pi}$  and  $M \leq 8\pi$ . The result is also true in the limit case if  $(W - \beta V) \in L^\infty(\mathbb{R}^2)$  and either  $\varepsilon = +1$  and  $\beta = 1 + \frac{M}{8\pi}$ , or  $\varepsilon = -1$ ,  $\beta > 1 - \frac{M}{8\pi}$  and  $M \leq 8\pi$ .*

*Proof.* If  $(W - \beta V) \in L^\infty(\mathbb{R}^2)$ , we can write that

$$\mathcal{F}_{\beta,W}[f] \geq \mathcal{F}_\beta[f] - M \|W - \beta V\|_{L^\infty(\mathbb{R}^2)}.$$

This completes the proof in the limit case. Otherwise, we redo the argument using  $\tilde{\beta} V - (\tilde{\beta} V - W)_+$  for some  $\tilde{\beta} \in (0, \beta)$  if  $\varepsilon = -1$ , and for some  $\tilde{\beta} \in (1 + \frac{M}{8\pi}, \beta)$  if  $\varepsilon = +1$ .  $\square$



## Chapter 6

# Nernst-Planck equation

**Article *Asymptotic behaviour of Nernst-Planck equation*, Appeared in <https://hal.archives-ouvertes.fr/hal-02143985> and <https://arxiv.org/abs/1906.07517>**

This paper is devoted to the Nernst-Planck system of equations with an external potential of confinement. The main result is concerned with the asymptotic behaviour of the solution of the Cauchy problem. We will prove that the optimal exponential rate of convergence of the solution to the unique stationary solution is determined by the spectral gap of the linearized problem around the minimizer of the free energy. The key issue is to consider an adapted notion of scalar product.



## 6.1 Introduction

At the end of nineteenth century, Nernst and Planck introduced a system of equations for representing the evolution of charged particles subject to electrostatic forces. The original model is exposed in [136, 142]: electrically charged particles diffuse under the action of a drift caused by an electrostatic potential. Nowadays we use this system in various frameworks like, for instance, phenomenological models for electrolytic behaviour in membranes. The original model is the non-confined Nernst-Planck system. If we take into account a mean-field Poisson coupling, in dimension  $d = 2$ , the system takes the form

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla v) \\ v = G_2 * u \\ u(0, x) = n_0 \geq 0 \end{cases} \quad x \in \mathbb{R}^2, t > 0, \quad (6.1)$$

where  $G_2(x) = -\frac{1}{2\pi} \log|x|$  denotes the Green function of the Laplacian in  $\mathbb{R}^2$ . We shall call this model the *Poisson-Nernst-Planck system*, which was also considered by Debye and Hückel in [64] and is sometimes called the *Debye-Hückel system* in the literature. Up to a sign change in the mean-field term, the model is similar to the Keller-Segel model, which is going to be a source of inspiration (see [33, 34, 50] for more details) for the study of the large time behaviour and this is a reason why we consider the two-dimensional case of the model.

Now let us introduce the notion of *confinement*. In the whole space, particles repel themselves and a well-known *runaway* phenomenon occurs: solutions locally vanish while the mass escapes at infinity. This can be prevented using a container (a bounded domain, with convenient boundary conditions) with walls, or a confinement potential. Actually, it is possible to obtain the bounded domain case as a limit of a whole space case with an external potential of *confinement* taking larger and larger values outside of the domain. Here we shall consider the Poisson-Nernst-Planck system with confinement in  $\mathbb{R}^d$ , where the dimension is  $d = 2$  or  $d = 3$ . The density function  $n$  solves

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n + \nabla \cdot (n \nabla c) + \nabla \cdot (n \nabla \phi) \\ c = G_d * n \\ n(0, x) = n_0 \geq 0, \quad \int_{\mathbb{R}^d} n(0, x) dx = M > 0 \end{cases} \quad x \in \mathbb{R}^d, t > 0. \quad (6.2)$$

The convolution kernel  $G_d$  is the Green function of the Laplacian in  $\mathbb{R}^d$ , namely

$$G_2(x) = -\frac{1}{2\pi} \log|x| \quad \text{for any } x \in \mathbb{R}^2 \quad \text{and} \quad G_3(x) = \frac{1}{4\pi|x|} \quad \text{for any } x \in \mathbb{R}^3.$$

In other words, we ask that  $c$  solves the Poisson equation

$$-\Delta c = n \quad x \in \mathbb{R}^d,$$

while  $\phi$  is a given external potential. In the special case of  $d = 2$  and  $\phi(x) = \frac{\mu}{2}|x|^2$  for some  $\mu > 0$ , if we use the change of variables

$$u(t, x) = R^{-d} n(\tau, \xi), \quad v(t, x) = c(\tau, \xi),$$

$$\xi = \frac{x}{R}, \quad \tau = \log R, \quad R = R(t) := \sqrt{1 + 2\mu t}, \quad (6.3)$$

then we observe that  $(n, c)$  solves (6.2) if and only if  $(u, v)$  solves (6.1). Studying the convergence rates of the solutions of (6.2) amounts to study the intermediate asymptotics of the solutions of (6.1) when runaway occurs. Obviously, the mass of a solution of (6.2) is conserved, and we shall write that  $\int_{\mathbb{R}^d} n(t, x) dx = M$  for any  $t \geq 0$ . The mass of a solution of (6.1) is also conserved, but one can prove that, for a solution of (6.1), the mass contained in any given compact set in  $\mathbb{R}^2$  decays to zero.

From here on, we shall assume that  $M > 0$  is fixed. Now let us turn our attention to the conditions on the confinement potential. From now on, we shall assume that  $\phi \in W_{\text{loc}}^{1, \infty}(\mathbb{R}^d)$  is such that  $\nabla \phi \in W^{1, \infty}(\mathbb{R}^d)$  and

$$\liminf_{|x| \rightarrow +\infty} \frac{\phi(x)}{\log|x|} > d, \quad (C1)$$

and also that the bounded measure  $e^{-\phi} dx$  admits a spectral gap (or Poincaré inequality, *i.e.*, that there exists a positive constant  $\Lambda_\phi$  such that

$$\int_{\mathbb{R}^d} |\nabla u|^2 e^{-\phi} dx \geq \Lambda_\phi \int_{\mathbb{R}^d} |u|^2 e^{-\phi} dx$$

$$\forall u \in H^1(\mathbb{R}^d; e^{-\phi} dx) \quad \text{such that} \quad \int_{\mathbb{R}^d} u e^{-\phi} dx = 0. \quad (C2)$$

Based on Persson's lemma, a sufficient condition is obtained by requesting that

$$\sigma_\phi := \lim_{r \rightarrow +\infty} \inf_{x \in B_r^c} \left( \frac{1}{4} |\nabla \phi|^2 - \frac{1}{2} \Delta_x \phi \right) > 0 \quad \text{and} \quad \lim_{r \rightarrow +\infty} \inf_{x \in B_r^c} |\nabla \phi| > 0. \quad (C3)$$

Let us refer to [3] for details and further references. We learn from [9, 28] that the stationary solutions  $(n_\infty, c_\infty)$  of (6.2) are obtained as solutions of the *Poisson-Boltzmann equation*

$$-\Delta c_\infty = n_\infty = M \frac{e^{-c_\infty - \phi}}{\int_{\mathbb{R}^d} e^{-c_\infty - \phi} dx}. \quad (6.4)$$

Under Assumption (C1) and the additional condition

$$\liminf_{|x| \rightarrow +\infty} \frac{\phi(x)}{\log|x|} > 4 + \frac{M}{2\pi} \quad \text{if} \quad d = 2, \quad (C4)$$

we know (see [3, Lemma 5] and earlier references therein) that the unique solution of (6.4) is obtained as a minimizer of the *free energy*  $\mathcal{F}$  defined by

$$\mathcal{F}[n] := \int_{\mathbb{R}^d} n \log n dx + \int_{\mathbb{R}^d} n \phi dx + \frac{1}{2} \int_{\mathbb{R}^d} n (-\Delta)^{-1} n dx. \quad (6.5)$$

Further details are given in Section 6.2. A simple consequence of the minimization procedure is that

$$\mathcal{F}[n] - \mathcal{F}[n_\infty] \geq 0 \quad \forall n \in L_+^1(\mathbb{R}^d)$$

with the convention that  $\mathcal{F}[n]$  can take the value  $+\infty$  if, for instance  $n \log n$  is not integrable. For sake of brevity, we shall say that  $\phi$  is a *confinement potential satisfying Assumption (C)* if (C1), (C3) and (C4) hold.

Our goal is to study the asymptotic behaviour of a solution of (6.2) with initial datum  $n_0$  at  $t = 0$  such that  $\mathcal{F}[n_0]$  is finite. It is a standard observation that the free energy  $\mathcal{F}[n(t, \cdot)]$  of a solution of (6.2) is monotone non-increasing along the flows and obeys to

$$\frac{d}{dt} \mathcal{F}[n(t, \cdot)] = -\mathcal{I}[n(t, \cdot)] \quad (6.6)$$

where the *Fisher information*  $\mathcal{I}$  is defined by

$$\mathcal{I}[n] := \int_{\mathbb{R}^d} n |\nabla(\log n + c + \phi)|^2 dx.$$

Our main result is that, as  $t \rightarrow +\infty$ ,  $\mathcal{F}[n(t, \cdot)]$  is bounded by  $\mathcal{I}[n(t, \cdot)]$  up to a multiplicative constant which shows that  $n(t, \cdot)$  converges to  $n_\infty$  at an exponential rate. The precise result is not written in terms of the free energy but in terms of a weighted  $L^2$  norm and goes as follows.

**Theorem 6.1.** *Let  $d = 2$  or  $3$  and consider a potential  $\phi$  satisfying (C). Assume that  $n$  solves (6.2) with initial datum  $n(0, \cdot) = n_0 \in L^2_+(n_\infty^{-1} dx)$ ,  $\int_{\mathbb{R}^d} n_0 dx = M$ , and  $\mathcal{F}[n_0] < \infty$ . Then there exist two positive constants  $C$  and  $\Lambda$  such that*

$$\int_{\mathbb{R}^d} |n(t, \cdot) - n_\infty|^2 n_\infty^{-1} dx \leq C e^{-\Lambda t} \quad \forall t \geq 0.$$

In section 4, we will characterize  $\Lambda$  as the spectral gap of the linearized operator associated with (6.2) and observe, as a special case, that  $\Lambda = 2\mu$  if  $d = 2$  and  $\phi = \frac{\mu}{2}|x|^2$ , for some  $\mu > 0$ .

Beyond free energy and entropy methods, the study of the large time asymptotics of the Poisson-Nernst-Planck system involves various tools of nonlinear analysis. Proving an exponential rate of convergence is interesting for studies of Poisson-Nernst-Planck systems by methods of scientific computing. Specific methods are needed for the numerical computation of the solutions, see [15, 138]. In [127], Liu and Wang implement at the level of the free energy a finite difference method to compute the numerical solution in a bounded domain. Concerning rates of convergence from a more theoretical point of view, let us mention that the existence of special solutions and self-similar solutions is considered in [26, 27, 110]. We refer to [147] for a discussion of the evolution problem from the point of view of physics.

Variants of the Poisson-Nernst-Planck system with nonlinear diffusions have been considered, for which the sharp rate of convergence is still unknown. Some papers rely on the use of distances related to the  $L^2$ -Wasserstein distance, see [70, 118, 163]. Exponential decay rates should be natural in view of the expected gradient flow structure of the system in this framework. The simpler case of linear diffusions on a bounded domain of  $\mathbb{R}^d$  with  $d \geq 3$  was studied in [28]: the convergence to the stationary solution occurs at an exponential rate. As already

mentioned, another related model is the Keller-Segel system in dimension 2. Regularity and asymptotic estimates for this system were discussed in [33, 50] and are a source of inspiration for the present study, in particular concerning the scalar product and the coercivity estimates. For completeness, let us mention that similar ideas have been recently developed in [124] for the study of a McKean-Vlasov model of flocking, which also involves a non-local coupling.

## 6.2 Minimizers of the free energy and convergence to the stationary solution

The main goal of this section is to prove that the minimizer of the free energy  $\mathcal{F}$  is the stationary solution  $(n_\infty, c_\infty)$  considered in the introduction and that it attracts any solution of (6.2) as  $t \rightarrow +\infty$ .

### 6.2.1 Minimizers of the free energy and stationary solutions

**Lemma 6.1.** *Let  $d = 2$  or  $d = 3$  and assume that the potential  $\phi$  satisfies (C). On the set*

$$\mathcal{X} := \left\{ f \in L^1_+(\mathbb{R}^d) : \int_{\mathbb{R}^d} f(x) dx = M, f \log f \in L^1(\mathbb{R}^d), f \phi \in L^1(\mathbb{R}^d) \right\},$$

*the free energy  $\mathcal{F}$  is semi-bounded from below.*

*Proof.* According to Assumption (C1), we know that  $e^{-\phi} \in L^1(\mathbb{R}^d)$ . Set  $\rho(x) := \lambda e^{-\phi}$ , such that  $\int_{\mathbb{R}^d} \rho(x) dx = M$ . Since the function  $x \log x$  is convex, we obtain that  $\int_{\mathbb{R}^d} f \log f dx \geq \int_{\mathbb{R}^d} f \log \rho dx$  by Jensen's inequality. So

$$\begin{aligned} \mathcal{F}[f] &\geq \int_{\mathbb{R}^d} f \log \rho dx + \int_{\mathbb{R}^d} f \phi dx + \frac{1}{2} \int_{\mathbb{R}^d} f (-\Delta)^{-1} f dx \\ &= M \log \lambda + \frac{1}{2} \int_{\mathbb{R}^d} f (-\Delta)^{-1} f dx. \end{aligned}$$

If  $d = 3$ ,  $\int_{\mathbb{R}^d} f (-\Delta)^{-1} f dx \geq 0$  because the Green function  $G_3(x)$  is nonnegative. If  $d = 2$ , the result has been established in [80, Corollary 1.2] as a consequence of Assumption (C4).  $\square$

**Lemma 6.2.** *Let  $d = 2$  or  $d = 3$  and assume that the potential  $\phi$  satisfies (C). There exists a unique minimizer  $n_\infty$  of  $\mathcal{F}$  in  $\mathcal{X}$ .*

*Proof.* Standard minimization methods show that a minimizing sequence admits, up to the extraction of a subsequence, a limit which is a minimizer. From the proof above,  $\mathcal{F}$  is lower bounded and satisfies the coercivity inequality. For a fixed minimizer  $n_\infty$ , it should satisfy the Euler-Lagrange equation

$$\log n_\infty + \phi + c_\infty = \lambda, \quad c_\infty = (-\Delta)^{-1} n_\infty,$$

for some Lagrange multiplier  $\lambda$  associated with the mass constraint, which means that  $(n_\infty, c_\infty)$  solve (6.4). By direct computation, with  $c = (-\Delta)^{-1}n$ , we observe that

$$\mathcal{F}[n] - \mathcal{F}[n_\infty] = \int_{\mathbb{R}^d} n \log\left(\frac{n}{n_\infty}\right) dx + \frac{1}{2} \int_{\mathbb{R}^d} (n - n_\infty)(c - c_\infty) dx.$$

Since  $\int_{\mathbb{R}^d} n dx = \int_{\mathbb{R}^d} n_\infty dx = M$ , we obtain from Jensen's inequality that

$$\int_{\mathbb{R}^d} n \log\left(\frac{n}{n_\infty}\right) dx \geq 0$$

and, according to [33],

$$\int_{\mathbb{R}^d} (n - n_\infty)(c - c_\infty) dx = \int_{\mathbb{R}^d} |\nabla(c - c_\infty)|^2 dx \geq 0.$$

Hence  $\mathcal{F}[n] - \mathcal{F}[n_\infty] \geq 0$  for any  $n \in \mathcal{X}$ , with equality if and only if  $n = n_\infty$ . This means that the minimizer of  $\mathcal{F}$  is unique.  $\square$

We may notice that  $n_\infty$  is radially symmetric if  $\phi$  is radially symmetric, as a consequence of the uniqueness result of Lemma 6.2.

We learn from the proof of [33, Lemma 23] that

$$\max_{|x| \rightarrow \infty} \left| c_\infty + \frac{M}{2\pi} \log|x| \right| < \infty \quad \text{if } d = 2, \quad \max_{|x| \rightarrow \infty} \left| c_\infty - \frac{M}{4\pi|x|} \right| < \infty \quad \text{if } d = 3,$$

and deduce from (6.4) that, as  $|x| \rightarrow \infty$ ,

$$n_\infty \sim |x|^{\frac{M}{2\pi}} e^{-\phi} \quad \text{if } d = 2, \quad n_\infty \sim e^{-\frac{M}{4\pi|x|} - \phi} \quad \text{if } d = 3. \quad (6.7)$$

**Proposition 6.1.** *Let  $d = 2$  or  $d = 3$  and assume that the potential  $\phi$  satisfies (C). Then the solutions  $(n_\infty, c_\infty)$  of (6.4) are such that  $c_\infty$  is bounded if  $d = 3$  and  $\|\nabla c_\infty\|_{L^q(\mathbb{R}^2)}$  is bounded for any  $q \in (2, +\infty]$  if  $d = 2$ .*

*Proof.* From (6.7) and (C4), we know that  $n_\infty$  is bounded outside of a large centered ball of radius  $R > 0$ . Let us assume that  $|x| \leq R$  and recall that

$$c_\infty(x) = \kappa_3 \int_{\mathbb{R}^3} \frac{e^{-c_\infty(y) - \phi(y)}}{|x - y|} dy \quad \text{if } d = 3, \quad \text{with } \kappa_3 = \frac{M}{4\pi \int_{\mathbb{R}^3} e^{-c_\infty - \phi} dx},$$

$$|\nabla c_\infty(x)| \leq \kappa_2 \int_{\mathbb{R}^2} \frac{e^{-c_\infty(y) - \phi(y)}}{|x - y|} dy \quad \text{if } d = 2, \quad \text{with } \kappa_2 = \frac{M}{2\pi \int_{\mathbb{R}^2} e^{-c_\infty - \phi} dx}.$$

In dimension  $d = 3$ , it is enough to observe that  $c_\infty = (-\Delta)^{-1}n_\infty \leq 0$  and deduce the bound

$$0 \leq c_\infty(x) \leq \kappa_3 \int_{\mathbb{R}^3} \frac{e^{-\phi(y)}}{|x - y|} dy.$$

In dimension  $d = 2$ , we deduce from (6.7) and (C) that for  $R > 0$  large enough, there exists a constant  $\kappa > 0$  such that

$$n_\infty(x) \leq n_\infty(x) \mathbb{1}_{|x| < R} + \kappa \mathbb{1}_{|x| \geq R} |x|^{-4},$$

which allows us to write

$$c_\infty(x) \geq -\frac{M}{2\pi} \log(2R) - \frac{\kappa}{2\pi} \int_{|y| \geq R} \frac{\log|x-y|}{|y|^4} dy$$

for any  $x \in \mathbb{R}^2$  such that  $|x| \leq R$ . Reinjecting this estimate in the expression of  $|\nabla c_\infty(x)|$  completes the proof. The bound on  $\|\nabla c_\infty\|_{L^q(\mathbb{R}^2)}$  follows by observing that  $|\nabla c_\infty(x)| \sim |x|^{-1}$  as  $|x| \rightarrow +\infty$ .  $\square$

### 6.2.2 Uniform bounds on the solution of (6.2)

We establish bounds on the solution  $n(t, \cdot)$  of (6.2) which are independent of  $t$ .

**Lemma 6.3.** *Let  $d = 2$  or  $d = 3$  and assume that the potential  $\phi$  satisfies (C). For any solution  $n$  of (6.2), there exists a constant  $\mathcal{C} > 0$  and a time  $T > 0$  such that*

$$\|n(t, \cdot)\|_{L^p} \leq \mathcal{C} \quad \forall t \geq T, \quad \forall p \in (1, +\infty].$$

*Proof.* For any integer  $k$ , set  $n_{0,k} = \min(n_0, k)$ , then  $n_{0,k} \in L^p(\mathbb{R}^d)$  for any  $p \geq 1$ . The solution  $n(t, \cdot)$  of the equation (6.2) with initial data  $n_{0,k}$  is in  $L^p(\mathbb{R}^d)$  for any  $t > 0$  by the Maximum Principle. Since, by assumption,  $|\nabla\phi|$  satisfies a Lipschitz condition, there exists a constant  $C > 0$  such that  $\Delta\phi \leq C$ , and we have the estimate

$$\begin{aligned} \frac{1}{p-1} \frac{d}{dt} \int_{\mathbb{R}^d} n(t, x)^p dx &= -p \int_{\mathbb{R}^d} |\nabla n|^2 n^{p-2} dx - \int_{\mathbb{R}^d} n^{p+1} dx + \int_{\mathbb{R}^d} n^p \Delta\phi dx \\ &\leq - \int_{\mathbb{R}^d} n^{p+1} dx + C \int_{\mathbb{R}^d} n^p dx. \end{aligned}$$

Using Hölder's inequality  $(\int_{\mathbb{R}^d} n dx)^{\frac{1}{p}} (\int_{\mathbb{R}^d} n^{p+1} dx)^{\frac{p-1}{p}} \geq \int_{\mathbb{R}^d} n^p dx$ , we obtain that

$$\int_{\mathbb{R}^d} n^{p+1} dx \geq M^{-\frac{1}{p-1}} \left( \int_{\mathbb{R}^d} n^p dx \right)^{\frac{p}{p-1}}$$

With  $z(t, \cdot) = \int_{\mathbb{R}^d} n(t, \cdot)^p dx$ , the problem reduces to the differential inequality

$$\frac{1}{p-1} z' \leq -M^{-\frac{1}{p-1}} z^{\frac{p}{p-1}} + C z$$

using  $\int_{\mathbb{R}^d} n_{0,k} dx \leq M$ . It is elementary to prove that

$$z(t) \leq (2C)^{p-1} M \quad \forall t \geq 4C$$

and conclude that the bound

$$\|n(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leq (2C)^{\frac{p-1}{p}} M^{\frac{1}{p}}$$

has a uniform upper bound in the limit as  $p \rightarrow +\infty$ . See [34] for further details on a similar estimate.  $\square$

**Corollary 6.1.** *Let  $d = 2$  or  $d = 3$  and assume that the potential  $\phi$  satisfies (C). For any solution  $n$  of (6.2) with initial datum  $n_0 \in L^1_+(\mathbb{R}^d)$  such that  $\mathcal{F}[n_0] < +\infty$ , there exists a constant  $\mathcal{C} > 0$  and a time  $T > 0$  such that*

$$\|\nabla c(t, \cdot)\|_{L^q(\mathbb{R}^d)} \leq \mathcal{C} \quad \forall t \geq T, \quad \forall q \in (2, +\infty].$$

*Proof.* The method is inspired from [50, Section 3]. If  $h = (-\Delta)^{-1}\rho$ , then

$$|\nabla h(x)| \leq \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{R}^d} \frac{\rho(y)}{|x-y|^{d-1}} dx$$

can be estimated by splitting the integral into two parts corresponding to  $|x-y| \leq 1$  and  $|x-y| > 1$ . By applying twice Hölder's inequality, we deduce from

$$\begin{aligned} \frac{1}{|\mathbb{S}^{d-1}|} \int_{\|x-y\| < 1} \frac{\rho(y)}{|x-y|^{d-1}} dy &\leq d^{\frac{d}{d+1}} |\mathbb{S}^{d-1}|^{-\frac{1}{d+1}} \|\rho\|_{L^{d+1}(\mathbb{R}^d)} \\ \frac{1}{|\mathbb{S}^{d-1}|} \int_{\|x-y\| \geq 1} \frac{\rho(y)}{|x-y|^{d-1}} dy &\leq \frac{1}{|\mathbb{S}^{d-1}|} \|\rho\|_{L^1(\mathbb{R}^d)} \end{aligned}$$

that

$$\|\nabla((-\Delta)^{-1}\rho)\|_{L^\infty(\mathbb{R}^d)} \leq \|\rho\|_{L^1(\mathbb{R}^d)} + d^{\frac{d}{d+1}} |\mathbb{S}^{d-1}|^{-\frac{1}{d+1}} \|\rho\|_{L^{d+1}(\mathbb{R}^d)} \quad (6.8)$$

for any  $\rho \in L^1 \cap L^{d+1}(\mathbb{R}^d)$ . Applying it with  $\rho = n(t, \cdot)$  and  $c = (-\Delta)^{-1}n$  and using Minkowski's inequality  $\|\nabla c(t, \cdot)\|_{L^q(\mathbb{R}^d)} \leq \|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^q(\mathbb{R}^d)} + \|\nabla c_\infty\|_{L^q(\mathbb{R}^d)}$ , the result follows from the estimate  $\|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^2(\mathbb{R}^d)}^2 \leq 2\mathcal{F}[n_0]$  together with Proposition 6.1 and Lemma 6.3.  $\square$

### 6.2.3 Convergence to stationary solutions

The next step is to establish the convergence *without rate* of the solution of (6.2) to the stationary solution. For later purpose, let us recall the Aubin-Lions compactness lemma. A simple statement goes as follows (see [100] for more details).

**Lemma 6.4. (Aubin-Lions Lemma)** *Take  $T > 0$ ,  $p \in (1, \infty)$ , and let  $(f_k)_{k \in \mathbb{N}}$  be a bounded sequence of functions in  $L^p(0, T; H)$ , where  $H$  is a Banach space. If  $(f_k)_{k \in \mathbb{N}}$  is bounded in  $L^p(0, T; V)$ , where  $V$  is compactly imbedded in  $H$  and if  $(\partial f_k / \partial t)_{k \in \mathbb{N}}$  is bounded in  $L^p(0, T; V')$  uniformly with respect to  $k \in \mathbb{N}$ , where  $V'$  is the dual space of  $V$ , then  $(f_k)_{k \in \mathbb{N}}$  is relatively compact in  $L^p(0, T; H)$ .*

With this result in hand, we are in a position to prove the following result.

**Proposition 6.2.** *Suppose that  $d = 2$  or  $3$ . Let  $n$  be the solution of (6.2) and assume that the potential  $\phi$  satisfies (C). Then for any  $p \in [1, \infty)$  and any  $q \in [2, \infty)$ , we have*

$$\lim_{t \rightarrow \infty} \|n(t, \cdot) - n_\infty\|_{L^p(\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^q(\mathbb{R}^d)} = 0.$$

*Proof.* Since  $\mathcal{F}[n(t, \cdot)]$  is nonnegative and decreasing, by (6.6) we know that

$$\lim_{t \rightarrow \infty} \int_t^\infty \mathcal{F}[n(s, \cdot)] ds = 0. \quad (6.9)$$

This means that the sequence  $(n_k, c_k)_{k \in \mathbb{N}}$ , defined by  $n_k(t, \cdot) = n(t+k, \cdot)$ ,  $c_k = (-\Delta)^{-1} n_k$ , is such that  $\nabla n_k + n_k \nabla c_k + n_k \nabla \phi$  strongly converges to 0 in  $L^2(\mathbb{R}^+ \times \mathbb{R}^d)$ . By lemma 6.4, this shows that  $(n_k)_{k \in \mathbb{N}}$  is relatively compact and converges, up to the extraction of a subsequence, to a limit  $\bar{n}$ . Up to the extraction of an additional subsequence,  $(c_k)_{k \in \mathbb{N}}$  converges to  $\bar{c} = (-\Delta)^{-1} \bar{n}$  so that we may pass to the limit in the quadratic term and know that

$$\nabla \bar{n} + \bar{n} \nabla \bar{c} + \bar{n} \nabla \phi = 0, \quad -\Delta \bar{c} = \bar{n}.$$

Since mass is conserved by passing to the limit, we conclude that  $\bar{n} = n_\infty$  and  $\bar{c} = c_\infty$ . The limit is uniquely defined, so it is actually the whole family  $(n(t, \cdot))_{t > 0}$  which converges as  $t \rightarrow +\infty$  to  $n_\infty$  and  $\lim_{t \rightarrow +\infty} \mathcal{F}[n(t, \cdot)] = \mathcal{F}[n_\infty]$ , then proving by the Csiszár-Kullback inequality that  $\lim_{t \rightarrow \infty} \|n(t, \cdot) - n_\infty\|_{L^1(\mathbb{R}^d)} = 0$  (see [116]) and  $\lim_{t \rightarrow \infty} \|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^2(\mathbb{R}^d)} = 0$ . The result for any  $p \in [1, \infty)$  and any  $q \in [2, \infty)$  follows by Hölder interpolation.  $\square$

#### 6.2.4 Uniform convergence in $L^\infty$ norm in the harmonic potential case

The issue of the convergence of  $n(t, \cdot)$  to  $n_\infty$  and of  $\nabla c(t, \cdot)$  to  $\nabla c_\infty$  in  $L^\infty(\mathbb{R}^d)$  was left open in Section 6.2.3. As in the case of the Keller-Segel model, see [34], better results can be achieved in the case of the harmonic potential.

**Proposition 6.3.** *Set  $d = 2$ ,  $\phi = \frac{\mu}{2} |x|^2$ , for some  $\mu > 0$ . Then for any solution  $n$  of (6.2) is such that*

$$\lim_{t \rightarrow +\infty} \|n(t, \cdot) - n_\infty\|_{L^\infty(\mathbb{R}^d)} = 0.$$

*Proof.* The main tool is the Duhamel formula: see [50] for more details. We have

$$n(t, x) = \int_{\mathbb{R}^2} K(t, x, y) n_0(y) dy - \int_0^t \int_{\mathbb{R}^2} \nabla K(t-s, x, y) \cdot n(s, y) \nabla c(s, y) dy ds$$

where  $K(t, x, y)$  is the Green function of the Fokker-Planck equation

$$\frac{\partial n}{\partial t} = \Delta n + \mu \nabla(n x)$$

which is

$$K(t, x, y) := \frac{\mu}{2\pi(1-e^{-2t})} e^{-\frac{\mu|x-e^{-t}y|^2}{2(1-e^{-2t})}}$$

and from the semi-group property we get that

$$\begin{aligned} n(t+1, x) &= \int_{\mathbb{R}^2} K(t, x, y) n(t, y) dy \\ &\quad - \int_t^{t+1} \int_{\mathbb{R}^2} \nabla K(t+1-s, x, y) \cdot n(s, y) \nabla c(s, y) dy ds. \end{aligned} \quad (6.10)$$



Notice that the stationary solution  $n_\infty$  is a fixed-point of the evolution map, that is,

$$n_\infty(x) = \int_{\mathbb{R}^2} K(t, x, y) n_\infty(y) dy - \int_t^{t+1} \int_{\mathbb{R}^2} \nabla K(t+1-s, x, y) \cdot n_\infty(y) \nabla c_\infty(y) dy ds. \quad (6.11)$$

Buy doing the difference between (6.10) and (6.11), we have

$$\begin{aligned} n(t+1, x) - n_\infty(x) &= \int_{\mathbb{R}^2} K(t, x, y) (n(t, y) - n_\infty(y)) dy \\ &\quad - \int_t^{t+1} \int_{\mathbb{R}^2} \nabla K(t+1-s, x, y) (n(s, y) \nabla c(s, y) dy - n_\infty(y) \nabla c_\infty(y)) ds. \end{aligned}$$

Hence

$$\begin{aligned} \|n(t+1, x) - n_\infty(x)\|_{L^\infty(\mathbb{R}^2)} &\leq \|K(t, x, y)\|_{L^\infty(\mathbb{R}_x^2; L^r(\mathbb{R}_y^2))} \|n(t, x) - n_\infty\|_{L^1(\mathbb{R}^2)} \\ &\quad + \int_0^1 \|\nabla K(s, x, y)\|_{L^\infty(\mathbb{R}_x^2; L^r(\mathbb{R}_y^2))} ds \mathcal{R}(t) \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$  with  $p \in (2, \infty)$ ,  $q \in [2, \infty)$ ,  $r \in (1, 2)$ , and

$$\begin{aligned} \mathcal{R}(t) := \sup_{s \in (t, t+1)} &\left( \|n(s, \cdot)\|_{L^p(\mathbb{R}^2)} \|\nabla c(s, \cdot) - \nabla c_\infty\|_{L^q(\mathbb{R}^2)} \right. \\ &\left. + \|\nabla c_\infty\|_{L^q(\mathbb{R}^2)} \|n(s, \cdot) - n_\infty\|_{L^p(\mathbb{R}^2)} \right). \quad (6.12) \end{aligned}$$

Notice that

$$\nabla K = \frac{\mu^2 (e^{-t} y - x)}{2\pi (1 - e^{-2t})} e^{-\frac{\mu|x - e^{-t}y|^2}{2(1 - e^{-2t})}}$$

allows us to compute

$$\|\nabla K\|_{L^r(\mathbb{R}_y^2)} = \frac{\mu^2}{2\pi (1 - e^{-2t})} \left( \int_{\mathbb{R}^2} |x|^r e^{-\frac{\mu r |x|^2}{2(e^{2t}-1)}} dx \right)^{\frac{1}{r}} = \kappa(r) e^{3t} \left( \frac{e^{2t} - 1}{\mu} \right)^{-\frac{3}{2} + \frac{1}{r}}$$

where  $\kappa(r) = \left( \int_0^\infty x^r e^{-\frac{1}{2}x^2} dx \right)^{\frac{1}{r}}$ . So  $\|\nabla K\|_{L^r(\mathbb{R}_y^2)}$  is integrable in  $t \in (0, 1)$  if and only if  $1 \leq r < 2$ . From Proposition 6.2,  $\mathcal{R}(t)$  converges to 0, which completes the proof.  $\square$

### 6.3 Coercivity result of quadratic forms

In this section, we study the quadratic forms associated with the free energy  $\mathcal{F}$  and the Fisher information  $\mathcal{I}$  when we Taylor expand these functionals around

the stationary solution  $(n_\infty, c_\infty)$  defined by (6.4). Let us consider a smooth perturbation  $n = f n_\infty$  of  $n_\infty$  such that  $\int_{\mathbb{R}^d} f n_\infty dx = 0$  and suppose that  $g c_\infty := (-\Delta)^{-1}(f n_\infty)$ . We define

$$Q_1[f] := \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^2} \mathcal{F}[n_\infty(1 + \varepsilon f)] = \int_{\mathbb{R}^d} f^2 n_\infty dx + \int_{\mathbb{R}^d} |\nabla(g c_\infty)|^2 dx,$$

$$Q_2[f] := \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^2} \mathcal{G}[n_\infty(1 + \varepsilon f)] = \int_{\mathbb{R}^d} |\nabla(f + g c_\infty)|^2 n_\infty dx.$$

### 6.3.1 A spectral gap inequality

According to [3, Section 3.2], if the potential  $\phi$  satisfies (C1), (C2) and (C3), then there exists a positive constant  $\mathcal{C}_\star$ , such that

$$\int_{\mathbb{R}^d} |\nabla h|^2 n_\infty dx \geq \mathcal{C}_\star \int_{\mathbb{R}^d} h^2 n_\infty dx$$

$$\forall f \in H^1(\mathbb{R}^d, n_\infty dx) \quad \text{such that} \quad \int_{\mathbb{R}^d} h n_\infty dx = 0. \quad (6.13)$$

Here  $n_\infty$  is the stationary solution given by (6.4).

**Proposition 6.4.** *Let  $d = 2$  or  $d = 3$  and assume that the potential  $\phi$  satisfies (C). Then for any  $f \in H^1(\mathbb{R}^d, n_\infty dx)$  such that  $\int_{\mathbb{R}^d} f n_\infty dx = 0$ , we have*

$$Q_2[f] \geq \mathcal{C}_\star Q_1[f].$$

*Proof.* We apply (6.13) to  $h(x) = f(x) + g c_\infty(x) - \frac{1}{M} \int_{\mathbb{R}^d} g c_\infty n_\infty dx$ . Notice that  $\int_{\mathbb{R}^d} h(x) n_\infty dx = 0$  from  $\int_{\mathbb{R}^d} f n_\infty dx = 0$  and  $\int_{\mathbb{R}^d} n_\infty(x) dx = M$ . So we obtain that

$$\begin{aligned} Q_2[f] &= \int_{\mathbb{R}^d} |\nabla(f + g c_\infty)|^2 n_\infty dx \\ &\geq \mathcal{C}_\star \int_{\mathbb{R}^d} (f + g c_\infty)^2 n_\infty dx - \frac{\mathcal{C}_\star}{M} \left( \int_{\mathbb{R}^d} g c_\infty n_\infty dx \right)^2 \\ &= \mathcal{C}_\star \int_{\mathbb{R}^d} f(f + g c_\infty) n_\infty dx + \mathcal{C}_\star \int_{\mathbb{R}^d} g c_\infty (f + g c_\infty) n_\infty dx \\ &\quad - \frac{\mathcal{C}_\star}{M} \left( \int_{\mathbb{R}^d} g c_\infty n_\infty dx \right)^2 \\ &= \mathcal{C}_\star Q_1[f] + \mathcal{C}_\star \int_{\mathbb{R}^d} f n_\infty g c_\infty dx + \mathcal{C}_\star \int_{\mathbb{R}^d} (g c_\infty)^2 n_\infty dx \\ &\quad - \frac{\mathcal{C}_\star}{M} \left( \int_{\mathbb{R}^d} g c_\infty n_\infty dx \right)^2. \end{aligned}$$

Let us study the term  $\int_{\mathbb{R}^d} f n_\infty g c_\infty dx$ . Obviously  $f n_\infty$  is in  $L^2(\mathbb{R}^d)$  because  $n_\infty$  is bounded. Moreover, for any  $p \in (1, 2)$ , from Hölder's inequality, we infer that

$$\int_{\mathbb{R}^d} |f|^p n_\infty^p dx \leq \left( \int_{\mathbb{R}^d} f^2 dx \right)^{\frac{p}{2}} \left( \int_{\mathbb{R}^d} n_\infty^{\frac{2p}{2-p}} dx \right)^{\frac{2-p}{2}} < \infty$$

because  $n_\infty \in L^1 \cap L^\infty(\mathbb{R}^d)$ . When  $d = 3$ , we directly obtain from the Hardy-Littlewood-Sobolev inequality that  $\int_{\mathbb{R}^d} f n_\infty g c_\infty dx$  is well defined and equal to  $\int_{\mathbb{R}^d} |\nabla g c_\infty|^2 dx$ . When  $d = 2$ , by log-Hölder interpolation,  $|f n_\infty| \log|f n_\infty|$  is integrable. From the logarithmic Hardy-Littlewood-Sobolev inequality (see [52]), we also know that  $\int_{\mathbb{R}^d} f n_\infty g c_\infty dx$  is well defined and learn from [33] that the function  $\nabla(g c_\infty)$  is bounded in  $L^2(\mathbb{R}^2)$  using the fact that  $\int_{\mathbb{R}^d} f n_\infty dx = 0$ . In a word, this means that

$$\int_{\mathbb{R}^d} f n_\infty g c_\infty dx = \int_{\mathbb{R}^d} |\nabla g c_\infty|^2 dx$$

for  $d = 2$  or  $3$ . Next, let us notice that

$$\begin{aligned} \mathcal{C}_\star \int_{\mathbb{R}^d} (g c_\infty)^2 n_\infty dx - \frac{\mathcal{C}_\star}{M} \left( \int_{\mathbb{R}^d} g c_\infty n_\infty dx \right)^2 \\ = \frac{\mathcal{C}_\star}{M} \int_{\mathbb{R}^d} (g c_\infty)^2 n_\infty dx \int_{\mathbb{R}^d} n_\infty dx - \frac{\mathcal{C}_\star}{M} \left( \int_{\mathbb{R}^d} g c_\infty n_\infty dx \right)^2 \end{aligned}$$

is nonnegative by Hölder's inequality. Altogether, we conclude that

$$Q_2[f] \geq \mathcal{C}_\star Q_1[f] + \mathcal{C}_\star \int_{\mathbb{R}^d} |\nabla(g c_\infty)|^2 dx \geq \mathcal{C}_\star Q_1[f].$$

□

### 6.3.2 Optimal spectral gap in a special case.

As a conclusion, let us give the optimal coercivity constant in the special case that the dimension  $d = 2$  and the harmonic function  $\phi = \frac{\mu}{2}|x|^2$ ,  $\mu > 0$ ,

**Lemma 6.5.** *Suppose that  $d = 2$ ,  $\phi = \frac{\mu}{2}|x|^2$ , where  $\mu > 0$ . Then for any  $f \in H^1(\mathbb{R}^2, n_\infty dx)$  such that  $\int_{\mathbb{R}^2} f n_\infty dx = 0$ , we have*

$$Q_2[f] \geq \mu Q_1[f].$$

*Proof.* We establish the proof into three steps.

**Step 1.** *Radially symmetric functions and cumulated densities.* We first consider the case of a spherically symmetric function  $f$ . The problem is reduced to solving an ordinary differential equation, for which we use a reformulation in terms of *cumulated densities*. Let

$$\Phi(s) := \frac{1}{2\pi} \int_{B(0, \sqrt{s})} n_\infty(x) dx, \quad \phi(s) := \frac{1}{2\pi} \int_{B(0, \sqrt{s})} (f n_\infty)(x) dx$$

and

$$\Psi(s) := \frac{1}{2\pi} \int_{B(0, \sqrt{s})} c_\infty(x) dx, \quad \psi(s) := \frac{1}{2\pi} \int_{B(0, \sqrt{s})} (g c_\infty)(x) dx.$$

Notice that  $n_\infty$  and  $c_\infty$  are both radial, so they can be regarded as functions of  $r = |x|$ . We can easily infer that

$$n_\infty(\sqrt{s}) = 2\Phi'(s), \quad n'_\infty(\sqrt{s}) = 4\sqrt{s}\Phi''(s)$$

and

$$c_\infty(\sqrt{s}) = 2\Psi'(s), \quad c'_\infty(\sqrt{s}) = 4\sqrt{s}\Psi''(s).$$

The Poisson equation  $-\sqrt{s}c'_\infty(\sqrt{s}) = \Phi(s)$  can henceforth be rephrased as

$$-4s\Psi'' = \Phi \tag{6.14}$$

while the equation for the density,

$$n'_\infty(\sqrt{s}) + \mu\sqrt{s}n_\infty(\sqrt{s}) + n_\infty(\sqrt{s})c'_\infty(\sqrt{s}) = 0,$$

is now equivalent to

$$\Phi'' + \frac{\mu}{2}\Phi' + 2\Phi'\Psi'' = 0. \tag{6.15}$$

After eliminating  $\Psi''$  from (6.14) and (6.15), we can get that  $\Phi$  satisfies the ordinary differential equation

$$\Phi'' + \frac{\mu}{2}\Phi' - \frac{1}{2s}\Phi\Phi' = 0 \tag{6.16}$$

with initial data  $\Phi(0) = 0$  and  $\Phi'(0) = a$ . The solutions of the ODE are parameterized in terms of  $a > 0$ .

Let us consider the linearized operator

$$\mathcal{L}f := \frac{1}{n_\infty} \nabla \cdot [f n_\infty \nabla (g c_\infty)].$$

If  $f$  solves  $-\mathcal{L}f = \lambda f$ , computations similar to the above ones show that

$$(n_\infty f)(\sqrt{s}) = 2\phi'(s), \quad (n_\infty f')(\sqrt{s}) = 4\sqrt{s}\phi''(s) - 2\frac{n'_\infty}{n_\infty}\phi'(s)$$

which is equivalent to

$$(g c_\infty)(\sqrt{s}) = 2\psi'(s), \quad (g c_\infty)'(\sqrt{s}) = 4\sqrt{s}\psi''(s). \tag{6.17}$$

Using (6.17), we find that

$$-\sqrt{s}(g c_\infty)'(\sqrt{s}) = \phi(s), \quad \sqrt{s}((n_\infty f')(\sqrt{s}) + n_\infty(g c_\infty)'(\sqrt{s})) + \lambda\phi(s) = 0.$$

After eliminating  $\Psi$  and  $\psi$ , we get that  $\Phi$  and  $\phi$  satisfy the equation

$$\phi'' + \frac{\mu s - \Phi}{2s}\phi' + \frac{\lambda - 2\Phi'}{4s}\phi = 0. \tag{6.18}$$

Next we check that  $\phi = s\Phi'(s)$  is a nonnegative solution of (6.18) with  $\lambda = 2\mu$ . In fact, (6.18) is equivalent to

$$2s\phi'' + (\mu s - \Phi)\phi' + (\mu - \Phi')\phi = 0$$

which is

$$(2(s\phi' - \phi) + (\mu s - \Phi)\phi)' = 0.$$

notice that when  $\phi = s\phi'$ ,

$$2(s\phi' - \phi) + (\mu s - \Phi)\phi = s(2s\Phi'' + (\mu s - \Phi)\phi') = 0.$$

Hence  $\lambda = 2\mu$  is an eigenvalue of the linearized operator  $\mathcal{L}f$ .

**Step 2.** *Characterization of the radial ground state.* Let us prove that  $2\mu$  is the lowest positive eigenvalue corresponding to a radial eigenfunction. Assume by contradiction that  $\mathcal{L}$  admits an eigenvalue  $\lambda \in (0, 2\mu)$  with eigenfunction  $f_1$  and define the corresponding function  $\phi_1$  that satisfy (6.18). Let us consider various cases depending on the zeros of  $\phi$ .

- Assume that  $\phi_1$  is always strictly positive or strictly negative in  $(0, \infty)$ . Suppose without losing generality that  $\phi_1(s) > 0$  in  $(0, \infty)$ . On the one hand, if we multiply (6.18) written for the eigenvalue  $2\mu$  and for the eigenvalue  $\lambda$  respectively by  $\phi_1$  and  $\phi$ , we obtain that

$$\phi_1\phi'' - \frac{\Phi''}{\Phi'}\phi_1\phi' + \frac{2\mu - 2\Phi'}{4s}\phi_1\phi = 0,$$

$$\phi\phi_1'' - \frac{\Phi''}{\Phi'}\phi\phi_1' + \frac{\lambda - 2\Phi'}{4s}\phi\phi_1 = 0.$$

By subtracting the second identity from the first one, we have

$$\frac{\phi_1'\phi(s) - \phi_1\phi'(s)}{\Phi'(s)} \Big|_0^\infty = \int_{(0, \infty)} \frac{2\mu - \lambda}{4s}\phi\phi_1 ds > 0. \quad (6.19)$$

On the other hand, define

$$h(s) := \frac{1}{2\pi} \int_{B(0, \sqrt{s})} f_1^2 n_\infty(r) dr.$$

From the cumulated mass formulation of Step 1, we find that

$$h'(s) = \frac{1}{2} f_1^2 n_\infty(\sqrt{s}) = \frac{2\Phi'(s)^2}{n_\infty(\sqrt{s})}$$

is in  $L^1(0, \infty)$ . So, for some constant  $\kappa > 0$ , we have

$$\begin{aligned} \phi_1(s)^2 &= \left( \int_{(s, \infty)} \phi_1'(s) ds \right)^2 \leq \left( \int_{(s, \infty)} \frac{\phi_1'(s)^2}{n_\infty(\sqrt{s})} ds \right) \left( \int_{(s, \infty)} n_\infty(\sqrt{s}) ds \right) \\ &\leq \kappa \int_{(s, \infty)} s^{-\frac{\alpha}{2}} e^{-\frac{s}{2}} ds \leq \kappa e^{-\frac{\mu s}{4}} \end{aligned}$$

when  $s$  is large enough. As a consequence, we know that

$$\lim_{s \rightarrow \infty} \phi_1(s) = \lim_{s \rightarrow \infty} \phi(s) = \lim_{s \rightarrow \infty} s \phi(s) = 0.$$

We also claim that

$$\lim_{s \rightarrow \infty} s \phi_1'(s) = 0. \quad (6.20)$$

In fact, for any large enough  $x_1, x_2$ , by integrating on  $(x_1, x_2)$ , we have

$$\begin{aligned} \phi_1'(x_2) - \phi_1'(x_1) + \frac{\mu}{2} (\phi_1(x_2) - \phi_1(x_1)) - \frac{\Phi}{2s} (\phi_1(x_2) - \phi_1(x_1)) \\ - \int_{x_1}^{x_2} \phi_1 \frac{s\phi_1' - \Phi}{2s^2} ds + \int_{(x_1, x_2)} \frac{\lambda - \Phi'}{4s} \phi_1 ds = 0. \end{aligned}$$

Using again that  $\phi_1(s) \leq \kappa e^{-\frac{\mu s}{4}}$ , we get that there exists a constant  $c_2$  which is independent of  $x_1$  and  $x_2$ , such that  $|\phi_1'(x_2) - \phi_1'(x_1)| \leq c_2$ . So  $\phi_1'(s)$  is bounded. As a result,  $\phi_1''(s)$  is also bounded, with a bound  $c_3$ . If (6.20) is not true, then there exists a constant  $c_1$  and a strictly increasing, diverging sequence  $(s_k)_{k \in \mathbb{N}}$  such that  $s_k \phi_1'(s_k) \geq c_1$ . For any interval  $(s_k, \infty)$ , we have that

$$\frac{c_1}{s_k} \leq C \sqrt{e^{-\frac{\mu s_k}{4}}}$$

which is impossible as  $k \rightarrow \infty$ . So from (6.20), we obtain that

$$\lim_{s \rightarrow \infty} \frac{\phi_1' \phi(s) - \phi_1 \phi'(s)}{\Phi'(s)} = \lim_{s \rightarrow \infty} s \phi_1' - \phi_1 \left( 1 + \frac{s \phi_1''}{\Phi'} \right) = \lim_{s \rightarrow \infty} s \phi_1' - \phi_1 \left( 1 - \frac{\mu s - \Phi}{2} \right) = 0. \quad (6.21)$$

From (6.19), (6.21), we have

$$0 = \frac{\phi_1' \phi(s) - \phi_1 \phi'(s)}{\Phi'(s)} \Big|_0^\infty = \int_{(0, \infty)} \frac{2\mu - \lambda}{4s} \phi \phi_1 ds > 0$$

a contradiction.

• Assume that  $\phi_1$  has a zero in  $(0, \infty)$ . By Sturm comparison theorem (see [69]), we get that

$$\phi(s) = s \Phi'(s)$$

has a zero in  $(0, \infty)$ . It means that

$$n_\infty(\sqrt{s}) = 2 \Phi'(s)$$

has a zero between  $(0, \infty)$ . But according to the definition of  $n_\infty$ , it is impossible. Hence we have shown that  $2\mu$  is the best constant.

**Step 3.** *Spherical harmonics decomposition.*

We now deal with the non-radial modes of  $\mathcal{L}$ . Notice that  $n_\infty$  and  $c_\infty$  are radial functions: we can use a spherical harmonics decomposition as in [50]. In

dimension  $d = 2$ , we use radial coordinates and a Fourier decomposition for the angular variables. On the  $k^{\text{th}}$  mode we can write the operator  $\mathcal{L}$  corresponding to the radial functions  $f$  and  $g$  as

$$\begin{aligned} -f'' - \frac{f'}{r} + \frac{k^2 f}{r^2} + (\mu r + c'_\infty)(f' + (g c_\infty)') - n_\infty f &= \lambda f, \\ -(g c_\infty)'' - \frac{(g c_\infty)'}{r} + \frac{k^2 g c_\infty}{r^2} &= n_\infty f, \end{aligned}$$

for any integer  $k \geq 1$ , It is obvious that in non-radial functions,  $k = 1$  realizes the infimum of the spectrum of  $\mathcal{L}$ . We now check that when  $k = 1$ ,  $\lambda = \mu$  and  $f = -n'_\infty/n_\infty$  is an eigenstate. In fact, we can choose  $g c_\infty = -c'_\infty$ , so that  $f = \mu r + c'_\infty$ , and notice that

$$-c''_\infty - \frac{c'_\infty}{r} = n_\infty, \quad f' + \frac{f}{r} = 2\mu + c''_\infty + \frac{c'_\infty}{r} = 2\mu - n_\infty,$$

for the first equation, and

$$\begin{aligned} -f'' - \frac{f'}{r} + \frac{k^2 f}{r^2} + (\mu r + c'_\infty)(f' + (g c_\infty)') + n_\infty f \\ = -\left(f' + \frac{f}{r}\right)' - n'_\infty + \mu(\mu r + c'_\infty) = \mu f \end{aligned}$$

for the second equation, while

$$-(g c_\infty)'' - \frac{(g c_\infty)'}{r} + \frac{k^2 g c_\infty}{r^2} = -\left(c''_\infty + \frac{c'_\infty}{r}\right) = -n'_\infty = n_\infty f.$$

It is easy to prove that  $f$  is nonnegative and that  $f_1(r) := r f(r)$  solves  $-\mathcal{L} f_1 = (\lambda + \mu) f_1$  among the radial functions: we are back to the Step 2 and find that  $\lambda = \mu$ .

Let us summarize: the spectral gap  $\lambda$  associated with the operator  $\mathcal{L}$  is achieved either among radial functions and  $\lambda = 2\mu$  in this sense, or it is achieved among the functions in one of the non-radial components (in the sense of harmonics decomposition), which has to be the  $k = 1$  component, and in that case we have found that  $\lambda + \mu = 2\mu$ , that is  $\lambda = \mu$ . Obviously  $\lambda = \mu$  is optimal, which completes the proof of Lemma 6.5.  $\square$

## 6.4 Linearized equation and the large time behaviour

This section is primarily devoted to the proof Theorem 6.1 but also collects some additional results.

### 6.4.1 The scalar product and the linearized operator.

We adapt the strategy of [50]. Notice that

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^d} f_1 f_2 n_\infty dx + \int_{\mathbb{R}^d} n_\infty f_1 (G_d * (f_2 n_\infty)) dx \quad (6.22)$$

is a scalar product on the admissible set

$$\mathcal{A} := \left\{ f \in L^2(\mathbb{R}^d, n_\infty dx) : \int_{\mathbb{R}^d} f n_\infty dx = 0 \right\}$$

because  $Q_1[f] = \langle f, f \rangle$ . Now come back to the Poisson-Nernst-Planck system with confinement (6.2). For any  $x \in \mathbb{R}^d$  and  $t \geq 0$ , let us set

$$n(t, x) = n_\infty(x) (1 + f(t, x)), \quad c(t, x) = c_\infty (1 + g(t, x))$$

and rewrite the evolution problem in terms of  $f$  and  $g$  as

$$n_\infty \frac{\partial f}{\partial t} = \Delta(n_\infty f) + \nabla \cdot (n_\infty f \nabla \phi) + \nabla \cdot (n_\infty \nabla(c_\infty g) + n_\infty f \nabla c_\infty + n_\infty f \nabla(c_\infty g)).$$

After observing that

$$\Delta(n_\infty f) + \nabla \cdot (n_\infty f \nabla \phi) + \nabla \cdot (n_\infty f \nabla c_\infty) = \nabla \cdot (n_\infty \nabla f),$$

it turns out that

$$n_\infty \frac{\partial f}{\partial t} = \nabla \cdot (n_\infty \nabla f) + \nabla \cdot (n_\infty \nabla(c_\infty g)) + \nabla \cdot (n_\infty f \nabla(c_\infty g)).$$

Hence  $(f, g)$  solves

$$\begin{cases} \frac{\partial f}{\partial t} - \mathcal{L}f = \frac{1}{n_\infty} \nabla \cdot [f n_\infty \nabla(g c_\infty)] & x \in \mathbb{R}^d, t > 0 \\ -\Delta(g c_\infty) = f n_\infty \end{cases} \quad (6.23)$$

for any  $x \in \mathbb{R}^d$ ,  $t \geq 0$ , where the linear operator  $\mathcal{L}$  is defined by

$$\mathcal{L}f := \frac{1}{n_\infty} \nabla [n_\infty \nabla(f + g c_\infty)].$$

**Lemma 6.6.** *The linearized operator  $\mathcal{L}$  is self-adjoint on  $\mathcal{A}$  with the scalar product defined in (6.22), which means that  $\langle f_1, \mathcal{L}f_2 \rangle = \langle \mathcal{L}f_1, f_2 \rangle$  for any  $f_1, f_2 \in \mathcal{A}$ , and moreover,*

$$-\langle f, \mathcal{L}f \rangle = Q_2[f]$$

for any  $f \in \mathcal{A}$ .

*Proof.* Set  $g_1 c_\infty = (-\Delta)^{-1}(f_1 n_\infty)$ ,  $g_2 c_\infty = (-\Delta)^{-1}(f_2 n_\infty)$ . By direct computation, we obtain that

$$\begin{aligned} \langle \mathcal{L}f_1, f_2 \rangle &= \int_{\mathbb{R}^d} f_2 \nabla \cdot (n_\infty \nabla(f_1 + g_1 c_\infty)) dx + \int_{\mathbb{R}^d} g_2 c_\infty \nabla \cdot (n_\infty \nabla(f_1 + g_1 c_\infty)) dx \\ &= - \int_{\mathbb{R}^d} n_\infty \nabla(f_1 + g_1 c_\infty) \cdot \nabla(f_2 + g_2 c_\infty) dx, \end{aligned}$$

which proves the lemma.  $\square$



### 6.4.2 Proof of Theorem 6.1

*Proof.* For the equations (6.23), we find that

$$\frac{d}{dt} Q_1[f] = -2 Q_2[f] - 2 \lambda(t) \quad \text{with} \quad \lambda(t) := \int_{\mathbb{R}^d} \nabla(f + g c_\infty) \cdot f n_\infty \nabla(g c_\infty) dx.$$

According to the Cauchy-Schwarz inequality, we have that

$$(\lambda(t))^2 \leq Q_2[f] \int_{\mathbb{R}^d} f^2 n_\infty dx \|\nabla(g c_\infty)\|_{L^\infty(\mathbb{R}^d)}^2 \leq Q_2[f] Q_1[f] \|\nabla(g c_\infty)\|_{L^\infty(\mathbb{R}^d)}^2.$$

So we obtain

$$\frac{d}{dt} Q_1[f] \leq -2 \left( 1 - \frac{\|\nabla(g c_\infty)\|_{L^\infty(\mathbb{R}^d)}}{\sqrt{\mathcal{C}_*}} \right) Q_2[f] \leq -2 \mathcal{C}_* \left( 1 - \frac{\|\nabla(g c_\infty)\|_{L^\infty(\mathbb{R}^d)}}{\sqrt{\mathcal{C}_*}} \right) Q_1[f].$$

We know from Proposition 6.2 that  $\lim_{t \rightarrow +\infty} \|\nabla(g c_\infty)\|_{L^\infty(\mathbb{R}^d)} = 0$ , which proves that

$$\limsup_{t \rightarrow \infty} e^{2(\mathcal{C}_* - \varepsilon)t} Q_1[f(t, \cdot)] < \infty$$

for any  $\varepsilon \in (0, \mathcal{C}_*)$ . It remains to prove that we can also obtain this estimate with  $\varepsilon = 0$ .

Suppose that  $\mathcal{C}_*$  is the optimal constant without losing generality. Let us give a more accurate estimate of  $\lambda(t)$ . If  $d = 2$ , according to (6.8) applied to  $\rho = f n_\infty$ , we have

$$\|\nabla(g c_\infty)\|_{L^\infty} \leq C(\|f n_\infty\|_{L^1} + \|f n_\infty\|_{L^3})$$

where

$$\|f n_\infty\|_{L^1} \leq \sqrt{M} \|f \sqrt{n_\infty}\|_{L^2}, \quad \|f n_\infty\|_{L^3} \leq \|f \sqrt{n_\infty}\|_{L^2}^{\frac{2}{3}} \|f n_\infty\|_{L^\infty}^{\frac{1}{3}} \|n_\infty\|_{L^\infty}^{\frac{1}{3}}.$$

Notice that from  $\|f \sqrt{n_\infty}\|_{L^2}^2 \leq Q_1[f]$ , we deduce that

$$\|\nabla(g c_\infty)\|_{L^\infty} = O(Q_1[f(t, \cdot)])^{\frac{1}{3}} \tag{6.24}$$

which leads to

$$\lambda(t) \leq O(Q_1[f(t, \cdot)]^{\frac{4}{3}}) \quad \text{as} \quad t \rightarrow +\infty.$$

As a result, we read from

$$\frac{d}{dt} Q_1[f] \leq -2 \mathcal{C}_* Q_1[f] + O(Q_1[f]^{\frac{4}{3}})$$

that

$$\limsup_{t \rightarrow \infty} e^{2 \mathcal{C}_* t} Q_1[f(t, \cdot)] < \infty.$$

When  $d = 3$ , we have the estimate

$$\|\nabla(g c_\infty)\|_{L^\infty} \leq C(\|f n_\infty\|_{L^1} + \|f n_\infty\|_{L^4})$$

and similarly obtain that

$$\|f n_\infty\|_{L^1} \leq \sqrt{M} \|f \sqrt{n_\infty}\|_{L^2}, \quad \|f n_\infty\|_{L^4} \leq \|f \sqrt{n_\infty}\|_{L^2}^{\frac{1}{2}} \|f n_\infty\|_{L^\infty}^{\frac{1}{2}} \|n_\infty\|_{L^\infty}^{\frac{1}{4}}.$$

Using again  $\|f \sqrt{n_\infty}\|_{L^2}^2 \leq Q_1[f]$ , we have

$$\|\nabla(g c_\infty)\|_{L^\infty} = O(Q_1[f(t, \cdot)]^{\frac{1}{4}}) \quad (6.25)$$

which allows us to write that

$$\lambda(t) \leq O\left(Q_1[f(t, \cdot)]^{\frac{5}{4}}\right) \quad \text{as } t \rightarrow +\infty.$$

We conclude as above, which completes the proof of Theorem 6.1.  $\square$

### 6.4.3 Uniform rate of convergence

Let us give additional results on the convergence in various norms of the solution of (6.2) to the stationary solution.

**Corollary 6.2.** *Under the assumptions of Theorem 6.1, if  $\phi(x) = \frac{1}{2}|x|^2$ , the solution  $n$  of (6.2) is such that*

$$\|n(t, \cdot) - n_\infty\|_{L^p} = O\left(e^{-\frac{t}{p}}\right) \quad \text{and} \quad \|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^q} = O\left(e^{-\frac{t(q+2d)}{(d+1)q}}\right)$$

as  $t \rightarrow +\infty$ , for any  $p \in (1, \infty)$  and any  $q \in (2, \infty)$ . Additionally, if  $d = 2$ , then

$$\|n(t, \cdot) - n_\infty\|_{L^\infty} = O\left(e^{-\lambda t}\right)$$

as  $t \rightarrow +\infty$ , for any  $\lambda < 1$ .

*Proof.* From the Cauchy-Schwarz inequality, we read that

$$\|n(t, \cdot) - n_\infty\|_{L^1(\mathbb{R}^d)} \leq \left( \|n_\infty\|_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{|n(t, \cdot) - n_\infty|^2}{n_\infty} dx \right)^{\frac{1}{2}} \leq \sqrt{CM} e^{-t}$$

for some  $C > 0$  if  $t$  is taken large enough, and we also know also that

$$\|n(t, \cdot) - n_\infty\|_{L^p(\mathbb{R}^d)} = O\left(e^{-\frac{t}{p}}\right) \quad (6.26)$$

for any  $p \in [1, \infty)$ . By definition of  $Q_1[f]$ , we have

$$\|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^2(\mathbb{R}^d)} \leq \sqrt{C} e^{-t}$$

for some  $C > 0$  if  $t$  is taken large enough, according to Lemma 6.5. Moreover, according to (6.24), (6.25) and Theorem 6.1, we obtain that

$$\|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^\infty(\mathbb{R}^d)} = O\left(e^{-\frac{t}{d+1}}\right). \quad (6.27)$$

This proves that

$$\|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^q(\mathbb{R}^d)} = O\left(e^{-\frac{t(q+2d)}{(d+1)q}}\right)$$

for any  $q \in [2, \infty)$  by interpolating between (6.26) and (6.27).

The proof of the case  $d = 2$  is inspired by [50, Remark 5]. We reconsider  $\mathcal{R}(t)$  defined in (6.12) in Section 6.2 with  $p = \frac{7r}{5r-4}$ ,  $q = \frac{7r}{2r-3}$ . We obtain from Corollary 6.2 that

$$\|n(t, \cdot) - n_\infty\|_{L^\infty(\mathbb{R}^2)} = O\left(e^{-\frac{5r-4}{7r}t}\right).$$

This is the first step of a proof by induction. If

$$\|n(t, \cdot) - n_\infty\|_{L^\infty(\mathbb{R}^2)} = O\left(e^{-at}\right),$$

then one has

$$\|n(t, \cdot) - n_\infty\|_{L^\infty(\mathbb{R}^2)} = O\left(e^{-\frac{5r-4+(2r+4)a}{7r}t}\right).$$

By iterating this estimate infinitely many times, we finally have

$$\|n(t, \cdot) - n_\infty\|_{L^\infty(\mathbb{R}^2)} = O\left(e^{-\lambda t}\right)$$

for any  $\lambda < 1$ . The proof of the corollary is complete.  $\square$

#### 6.4.4 Intermediate asymptotics of the Poisson-Nernst-Planck system

Let us come back to the Poisson-Nernst-Planck equation (6.1). The self-similar solution of (6.1) has the expression

$$u_\infty(x, t) = \frac{1}{1+2t} n_\infty\left(\frac{1}{2} \log(1+2t), \frac{x}{\sqrt{1+2t}}\right), \quad (6.28)$$

$$v_\infty(x, t) = c_\infty\left(\frac{1}{2} \log(1+2t), \frac{x}{\sqrt{1+2t}}\right), \quad (6.29)$$

where  $(n_\infty, c_\infty)$  are the stationary solutions of (6.2) given by (6.4) with the harmonic potential  $\phi(x) = \frac{1}{2}|x|^2$ . Using Theorem 6.1 and Corollary 6.2, we achieve a result on the *intermediate asymptotics* for the solutions of the Poisson-Nernst-Planck system in absence of any external potential of confinement.

**Theorem 6.2.** *Assume that  $u$  solves (6.1) with initial datum  $u(0, \cdot) = n_0 \in L^2_+(n_\infty^{-1} dx)$ ,  $\int_{\mathbb{R}^d} n_0 dx = M$ , and  $\mathcal{F}[n_0] < \infty$ . Let us consider the self-similar solution defined by (6.28) and (6.29) of mass  $M$ . Then, as  $t \rightarrow +\infty$ , we have*

(i) *for any  $p \in (1, \infty)$  and any  $\lambda < 1$ ,*

$$\|u(t, \cdot) - u_\infty\|_{L^1(\mathbb{R}^2)} = O\left((1+2t)^{-\frac{1}{2}}\right),$$

$$\|u(t, \cdot) - u_\infty\|_{L^p(\mathbb{R}^2)} = O\left((1+2t)^{-\frac{\lambda}{2} - \frac{d(p-1)}{2p}}\right),$$

(ii) for any  $q \in (2, \infty)$  and any  $\lambda < 1$ ,

$$\begin{aligned}\|\nabla v(t, \cdot) - \nabla v_\infty\|_{L^2(\mathbb{R}^2)} &= O\left((1+2t)^{-1+\frac{d}{4}}\right), \\ \|\nabla v(t, \cdot) - \nabla v_\infty\|_{L^q(\mathbb{R}^2)} &= O\left((1+2t)^{-\frac{\lambda+1}{2}+\frac{d}{2q}}\right).\end{aligned}$$



## Chapter 7

# Linearized Vlasov-Poisson-Fokker-Planck system

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This chapter is devoted to the linearized Vlasov-Poisson-Fokker-Planck system in presence of an external potential of confinement. We investigate the large time behaviour of the solutions using hypocoercivity methods and a notion of scalar product adapted to the presence of a Poisson coupling. Our framework provides estimates which are uniform in the diffusion limit. As an application, we study the one-dimensional case and prove the exponential convergence of the nonlinear Vlasov-Poisson-Fokker-Planck system without small mass assumption.

## 7.1 Introduction and main results

The *Vlasov-Poisson-Fokker-Planck system* in presence of an external potential  $V$  is

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f - (\nabla_x V + \nabla_x \phi) \cdot \nabla_v f &= \Delta_v f + \nabla_v \cdot (v f), \\ -\Delta_x \phi &= \rho_f = \int_{\mathbb{R}^d} f \, dv. \end{aligned} \quad (\text{VPFP})$$

In this paper, we shall assume that  $\phi$  is a *self-consistent potential* corresponding to repulsive electrostatic forces and that  $V$  is a *confining potential* in the sense that the system has a unique nonnegative integrable stationary solution  $f_\star$  with associated potential  $\phi_\star$ . We shall denote by  $M = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_\star \, dx \, dv > 0$  the mass. System (VPFP) is of interest for understanding the evolution of a system of charged particles with interactions of two different natures: a self-consistent, nonlinear interaction through the mean field potential  $\phi$  and collisions with a background inducing a diffusion and a friction represented by a Fokker-Planck operator acting on velocities. System (VPFP) describes for instance the dynamics of a plasma of Coulomb particles in a thermal reservoir: see for instance [30], but it has also been derived in stellar dynamics for gravitational models, as in [59], in the case of an attractive mean field Newton-Poisson equation. Here we shall focus on the repulsive, electrostatic case. Applications range from plasma physics to semi-conductor modelling. A key open question is to get estimates on the rate of convergence to equilibrium in dimensions  $d = 2$  and  $d = 3$ . As a step in this direction, we will establish a constructive estimate of the decay rate of the linearized problem, which provides us with an upper bound for the convergence rate of the nonlinear (VPFP) problem. A technical but important issue is to decide how one should measure such a rate of relaxation. For this purpose, we introduce a norm which is adapted to the linearized problem and consistent with the *diffusion limit*.

Let us consider the linearized problem around  $f_\star$ . Let  $h$  be a function such that  $f = f_\star (1 + \eta h)$  with  $\iint_{\mathbb{R}^d \times \mathbb{R}^d} f \, dx \, dv = M$ , that is, such that  $\iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star \, dx \, dv = 0$ . The system (VPFP) can be rewritten as

$$\begin{aligned} \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h &= \eta \nabla_x \psi_h \cdot \nabla_v h, \\ -\Delta_x \psi_h &= \int_{\mathbb{R}^d} h f_\star \, dv. \end{aligned}$$

At formal level, by dropping the  $\mathcal{O}(\eta)$  term, we obtain the *linearized Vlasov-Poisson-Fokker-Planck system*

$$\begin{aligned} \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h &= 0, \\ -\Delta_x \psi_h &= \int_{\mathbb{R}^d} h f_\star \, dv, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star \, dx \, dv = 0. \end{aligned} \quad (7.1)$$

Let us define the norm

$$\|h\|^2 := \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^2 f_\star \, dx \, dv + \int_{\mathbb{R}^d} |\nabla_x \psi_h|^2 \, dx.$$

Our main result is devoted to the large time behaviour of a solution of *the linearized system* (7.1) on  $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \ni (t, x, v)$  with given initial datum  $h_0$  at  $t = 0$ . For simplicity, we shall state a result for a simple specific potential, but an extension to more general potentials will be given to the price of a rather long list of technical assumptions that are detailed in Section 7.3.

**Theorem 7.1.** *Let us assume that  $d \geq 1$ ,  $V(x) = |x|^\alpha$  for some  $\alpha > 1$  and  $M > 0$ . Then there exist two positive constants  $\mathcal{C}$  and  $\lambda$  such that any solution  $h$  of (7.1) with an initial datum  $h_0$  of zero average with  $\|h_0\|^2 < \infty$  is such that*

$$\|h(t, \cdot, \cdot)\|^2 \leq \mathcal{C} \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0. \quad (7.2)$$

Our analysis is consistent with the *diffusion limit* of the linearized system, as we shall explain below. For any  $\varepsilon > 0$ , if we consider the solution of the *linearized problem in the parabolic scaling* given by

$$\begin{aligned} \varepsilon \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \frac{1}{\varepsilon} (\Delta_v h - v \cdot \nabla_v h) &= 0, \\ -\Delta_x \psi_h = \int_{\mathbb{R}^d} h f_\star dv, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star dx dv &= 0, \end{aligned} \quad (7.3)$$

then we obtain a decay estimate which is uniform with respect to  $\varepsilon \rightarrow 0_+$ . The result goes as follows.

**Theorem 7.2.** *Let us assume that  $d \geq 1$ ,  $V(x) = |x|^\alpha$  for some  $\alpha > 1$  and  $M > 0$ . For any  $\varepsilon > 0$  small enough, there exist two positive constants  $\mathcal{C}$  and  $\lambda$ , which do not depend on  $\varepsilon$ , such that any solution  $h$  of (7.3) with an initial datum  $h_0$  of zero average and such that  $\|h_0\|^2 < \infty$  satisfies (7.2).*

The result of Theorem 7.1 will be extended in Theorem 7.3 to a larger class of external potentials  $V$ : in the technical part of the proof of Theorem 7.1, we will specify precise but more general conditions under which the same result holds. A similar extension applies in the case of Theorem 7.2. As an application of our method, we establish the exponential rate of convergence of the solution of *the non-linear system (VPFP)* when  $d = 1$ . For sake of simplicity, we state the result for the same potential  $V$  as in Theorem 7.2.

**Corollary 7.1.** *Assume that  $d = 1$ ,  $V(x) = |x|^\alpha$  for some  $\alpha > 1$  and  $M > 0$ . If  $h$  solves (VPFP) with an initial datum  $h_0$  of zero average and such that  $\|h_0\|^2 < \infty$  and  $(1 + h_0) \geq 0$ , then (7.2) holds for some positive constants  $\mathcal{C}$  and  $\lambda$ .*

The *diffusion limit* of systems of kinetic equations in presence of electrostatic forces has been studied in many papers. The mathematical results go back at least to the study of a model for semi-conductors involving a linear Boltzmann kernel by F. Poupaud in [143]. The case of a Fokker-Planck operator in dimension  $d = 2$  was later studied by F. Poupaud and J. Soler in [144], and by T. Goudon in [101], on the basis of the existence results of [137, 158]. With a self-consistent



Poisson coupling, we refer to [42] for existence results in dimension  $d = 3$  and to [91, 72] for steady states, confinement and related issues. Based on free energy considerations introduced in [44, 72], N. El Ghani and N. Masmoudi were able in [93] to establish diffusion limits also when  $d = 3$ . Altogether, it is proved in dimensions  $d = 2$  and  $d = 3$  that the Vlasov–Poisson–Fokker–Planck system with parameters corresponding to the parabolic scaling,

$$\varepsilon \partial_t f + v \cdot \nabla_x f - (\nabla_x V + \nabla_x \phi) \cdot \nabla_v f = \frac{1}{\varepsilon} (\Delta_v f + \nabla_v \cdot (v f)), \quad -\Delta_x \phi = \rho_f = \int_{\mathbb{R}^d} f \, dv. \quad (7.4)$$

has a weak solution  $(f^\varepsilon, \phi^\varepsilon)$  which converges as  $\varepsilon \rightarrow 0_+$  to  $(f^0 = \rho \mathcal{M}, \phi)$  where

$$\mathcal{M}(v) = \frac{e^{-\frac{1}{2}|v|^2}}{(2\pi)^{d/2}} \quad \forall v \in \mathbb{R}^d$$

is the normalized Maxwellian function and where the charge density  $\rho = \int_{\mathbb{R}^d} f^0 \, dv$  is a weak solution of the *drift-diffusion-Poisson* system

$$\frac{\partial \rho}{\partial t} = \nabla_x \cdot (\nabla_x \rho + \rho \nabla_x (V + \phi)), \quad -\Delta_x \phi = \rho. \quad (7.5)$$

Another piece of information is the asymptotic behavior of the solutions of (7.5) for large times. As  $t \rightarrow +\infty$ , it is well known (see for instance [23] in the case of a bounded domain, [9] in the Euclidean case when  $V(x) = |x|^2$ , and [28] in  $\mathbb{R}^d$  with a confining external potential  $V$  for any  $d \geq 3$ ) that the solution of (7.5) converges a steady state  $(\rho_\star, \phi_\star)$  given by

$$-\Delta_x \phi_\star = \rho_\star = e^{-V - \phi_\star} \quad (7.6)$$

at an exponential rate. The optimal asymptotic rates have been characterized recently in [123] using the linearized drift-diffusion-Poisson system and a norm which involves the Poisson potential. Apart the difficulty arising from the self-consistent potential, the technique is based on relative entropy methods, which are by now standard in the study of large time asymptotics of drift-diffusion equations.

Our motivation is to study both regimes  $\varepsilon \rightarrow 0_+$  and  $t \rightarrow +\infty$  simultaneously. More precisely, we aim at proving that each solution  $(f^\varepsilon, \phi^\varepsilon)$  of (7.4) converges to  $(f_\star, \phi_\star)$  as  $t \rightarrow +\infty$  in a weighted  $L^2$  sense at an exponential rate which is uniform in  $\varepsilon > 0$ , small. In the present paper, we will focus on a linearized regime in any dimension and obtain an estimate of the decay rate in the asymptotic regime. This allows us to obtain an asymptotic decay rates in the non-linear regime when  $d = 1$ , but so far not in higher dimensions because we are still lacking of some key estimates. Compared to the large time asymptotics of (7.5), the study of the convergence rate of the solution of (7.4) or, in the case  $\varepsilon = 1$ , of the

decay rate of the solution of (7.1), is much more difficult because the diffusion only acts on the velocities and requires the use of hypocoercive methods.

T. Gallay coined the word *hypocoercivity* in the context of convergence without regularization as opposed to hypoellipticity where both properties arise simultaneously. It is well adapted to kinetic equations with general collision kernels and C. Villani made the *hypocoercivity* very popular in kinetic theory: see [159, 161]. Understanding the large time behavior of the kinetic Fokker-Planck equation (without Poisson coupling) is an interesting problem which has a long history: see [119, 113, 115, 92, 108] for some earlier contributions. C. Villani [161] proved convergence results in various senses: in  $H^1$  [161, Theorem 35], in  $L^2$  [161, Theorem 37], and in entropy [161, Theorem 39] when  $\text{Hess}(V)$  is bounded. His approach is however inspired by hypoelliptic methods, as in [106, 107, 133]. The method of [7] is based on a spectral decomposition and produces an exponential decay in relative entropy with a sharp rate. In a somewhat similar spirit, we can also quote [45], which is based on a Fourier decomposition. Due to the Fokker-Planck operator, smoothing effects in (7.4) can be expected as was proved in [43], consistently with hypoelliptic methods: this will not be exploited in the present paper.

In presence of a Poisson coupling, several papers deal, without any rate, with the large time behavior of the solutions of (7.4), in presence of or without an external potential: cf. [44, 56, 72]. When  $d = 2$  and  $d = 3$ , F. Hérau and L. Thomann [109] proved the trend to the equilibrium for the Vlasov-Poisson-Fokker-Planck system with a small nonlinear term but with a possibly large exterior confining potential. More recently, M. Herda and M. Rodrigues considered in [111] the double limit as  $\varepsilon \rightarrow 0_+$  and  $t \rightarrow +\infty$ . All these approaches are however essentially of perturbative nature.

In [83], J. Dolbeault, C. Mouhot, and C. Schmeiser studied the exponential decay in a modified  $L^2$  norm for the Vlasov-Fokker-Planck equation (and also for a larger class of linear kinetic equations). The method was motivated by the results of [106] but the main source of inspiration came from the analysis of the diffusion limit, as in [24, 128, 81] (also see [150] in presence of an oscillating external force field): the idea was to build a norm which reflects the spectral gap that determines the rate of convergence in (7.5). Inspired by [31, 38, 88], another idea emerged that asymptotic rates of convergence should be measured in a norm induced by a Taylor expansion of the entropy around the asymptotic state and that, in presence of a Poisson coupling, this norm should involve a non-local term: see [50, 123, 124]. The goal of this paper is to mix these two ideas.

This paper is organized as follows. In Section 7.2, we expose the strategy for the  $L^2$ -hypocoercivity method of [83] in the abstract setting of a general Hilbert space. The notion of Hilbert space adapted to (7.1) is exposed in Section 7.3

with some fundamental considerations on confinement by an external potential and adapted Poincaré inequalities. Section 7.4 is devoted to the proof of Theorem 5.1: we have to check that the assumptions of Section 7.2 hold in the functional setting of Section 7.3, with the special scalar product for Poisson coupling involving a non-local term associated with the norm defined by (7.1). In Section 7.5, we prove Theorem 7.2: our estimates are compatible with the diffusion limit as  $\varepsilon \rightarrow 0$ . Coming back to the non-linear problem (VPFP) in dimension  $d = 1$ , we prove in this latter case that an exponential rate of convergence as  $t \rightarrow +\infty$  can be measured in the hypocoercive norm, that is, we prove Corollary 7.1.

To make *notation* slightly lighter, we adopt the convention that  $\nabla$  and  $\Delta$  denote respectively  $\nabla_x$  and  $\Delta_x$  unless the variable is  $v$  or when there is a possible ambiguity, and in that case, it will be explicitly specified. We shall also adopt the following conventions. If  $\mathbf{a} = (a_i)_{i=1}^d$  and  $\mathbf{b} = (b_i)_{i=1}^d$  are two vectors with values in  $\mathbb{R}^d$ , then  $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^d a_i b_i$  and  $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$ . If  $\mathbf{A} = (A_{ij})_{i,j=1}^d$  and  $\mathbf{B} = (B_{ij})_{i,j=1}^d$  are two matrices with values in  $\mathbb{R}^d \times \mathbb{R}^d$ , then  $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^d A_{ij} B_{ij}$  and  $|\mathbf{A}|^2 = \mathbf{A} : \mathbf{A}$ . We shall use the tensor convention that  $\mathbf{a} \otimes \mathbf{b}$  is the matrix of elements  $a_i b_j$ . By extension to functions,  $\nabla w$  is the gradient of a scalar function  $w$  while  $\nabla \cdot \mathbf{u}$  denotes the divergence of a vector valued function  $\mathbf{u} = (u_i)_{i=1}^d$  and  $\nabla \otimes \mathbf{u}$  is the matrix valued function of elements  $\partial u_i / \partial x_j$ . Hence  $\text{Hess}(w) = (\nabla \otimes \nabla) w = (\partial^2 w / \partial x_i \partial x_j)_{i,j=1}^d$  denotes the Hessian of  $w$  and, for instance,  $\mathbf{u} \otimes \mathbf{u} : \text{Hess}(w) = \sum_{i,j=1}^d u_i u_j (\text{Hess}(w))_{ij}$ . We shall also write that  $|\text{Hess}(w)|^2 = \text{Hess}(w) : \text{Hess}(w)$ .

## 7.2 Hypocoercivity result and decay rates

This section is devoted to the abstract hypocoercivity method in general Hilbert spaces and it is inspired from [83, 45]. Since the methods sets the overall strategy of proof of our main results, we expose it for the convenience of the reader.

Let us consider the evolution equation

$$\frac{dF}{dt} + \mathbb{T}F = \mathbb{L}F \quad (7.7)$$

on a Hilbert space  $\mathcal{H}$ . In view of the applications, we shall call  $\mathbb{T}$  and  $\mathbb{L}$  the *transport* and the *collision* operators and assume without further notice that they are respectively antisymmetric and symmetric. On  $\mathcal{H}$ , we shall denote by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$  the scalar product and the norm. As in [83], we assume that there are positive constants  $\lambda_m$ ,  $\lambda_M$ , and  $C_M$  such that, for any  $F \in \mathcal{H}$ , the following properties hold:

▷ *microscopic coercivity*

$$- \langle \mathbb{L}F, F \rangle \geq \lambda_m \|(\text{Id} - \Pi)F\|^2, \quad (\text{H1})$$

▷ *macroscopic coercivity*

$$\|\mathbb{T}\Pi F\|^2 \geq \lambda_M \|\Pi F\|^2, \quad (\text{H2})$$

▷ *parabolic macroscopic dynamics*

$$\Pi\mathbb{T}\Pi F = 0, \quad (\text{H3})$$

▷ *bounded auxiliary operators*

$$\|\mathbb{A}\mathbb{T}(\text{Id} - \Pi)F\| + \|\mathbb{A}LF\| \leq C_M \|(\text{Id} - \Pi)F\|. \quad (\text{H4})$$

Here  $\text{Id}$  is the identity,  $\Pi$  is the orthogonal projection onto the null space of  $L$ ,  $*$  denotes the adjoint with respect to  $\langle \cdot, \cdot \rangle$  and as in [82, 83], the operator  $\mathbb{A}$  is defined by

$$\mathbb{A} := (\text{Id} + (\mathbb{T}\Pi)^* \mathbb{T}\Pi)^{-1} (\mathbb{T}\Pi)^*.$$

Since a solution  $F$  of (7.7) obeys to

$$\frac{1}{2} \frac{d}{dt} \|F\|^2 = \langle LF, F \rangle \leq -\lambda_m \|(\text{Id} - \Pi)F\|^2,$$

this is not enough to conclude that  $\|F(t, \cdot)\|^2$  decays exponentially with respect to  $t \geq 0$  and this is why we shall consider the Lyapunov functional

$$\mathbb{H}_\delta[F] := \frac{1}{2} \|F\|^2 + \delta \langle \mathbb{A}F, F \rangle$$

for some  $\delta > 0$  to be determined later. If  $F$  solves (7.7), then

$$\begin{aligned} -\frac{d}{dt} \mathbb{H}_\delta[F] &= \mathbb{D}_\delta[F] := -\langle LF, F \rangle + \delta \langle \mathbb{A}\mathbb{T}\Pi F, F \rangle \\ &\quad - \delta \langle \mathbb{T}\mathbb{A}F, F \rangle + \delta \langle \mathbb{A}\mathbb{T}(\text{Id} - \Pi)F, F \rangle - \delta \langle \mathbb{A}LF, F \rangle. \end{aligned}$$

Let us define

$$\delta_\star = \min \left\{ 2, \lambda_m, \frac{4\lambda_m\lambda_M}{4\lambda_M + C_M^2(1 + \lambda_M)} \right\}.$$

We recall that the two main properties of the *hypocoercivity* method of [83] for real valued operators and later extended in [45] to complex Hilbert spaces go as follows.

**Proposition 7.1.** *Assume that (H1)–(H4) hold and take  $\delta \in (0, \delta_\star)$ . Then we have:*

(i)  $\mathbb{H}_\delta$  and  $\|\cdot\|^2$  are equivalent in the sense that

$$\frac{2 - \delta}{4} \|F\|^2 \leq \mathbb{H}_\delta[F] \leq \frac{2 + \delta}{4} \|F\|^2 \quad \forall F \in \mathcal{H}. \quad (7.8)$$

(ii) For some  $\lambda > 0$  depending on  $\delta$ ,  $\mathbb{H}_\delta$  and  $\mathbb{D}_\delta$  are related by the entropy–entropy production inequality

$$\lambda \mathbb{H}_\delta[F] \leq \mathbb{D}_\delta[F] \quad \forall F \in \mathcal{H}. \quad (7.9)$$

As a straightforward consequence, we have that a solution  $F$  of (7.7) with initial datum  $F_0$  obeys to

$$H_\delta[F(t, \cdot)] \leq H_\delta[F_0] e^{-\lambda t}$$

and

$$\|F(t, \cdot)\|^2 \leq \frac{4}{2-\delta} H_\delta[F(t, \cdot)] \leq \frac{4}{2-\delta} H_\delta[F_0] e^{-\lambda t} \leq \frac{2+\delta}{2-\delta} \|F_0\|^2 e^{-\lambda t} \quad \forall t \geq 0. \quad (7.10)$$

*Proof.* For completeness, we sketch the main steps of the proof, with slightly improved estimates compared to [45, Theorem 3]. Since  $A\Pi$  can be viewed as  $z \mapsto (1+z)^{-1}z$  applied to  $(\Pi)^*\Pi$ , (H1) and (H2) imply that

$$-\langle LF, F \rangle + \delta \langle A\Pi F, F \rangle \geq \lambda_m \|(\text{Id} - \Pi)F\|^2 + \frac{\delta \lambda_M}{1 + \lambda_M} \|\Pi F\|^2.$$

Our goal is to prove that the r.h.s. controls the other terms in the expression of  $D_\delta[F]$ . By (H4), we know that

$$|\langle A\Pi(\text{Id} - \Pi)F, F \rangle + \langle ALF, F \rangle| \leq C_M \|\Pi F\| \|(\text{Id} - \Pi)F\|.$$

As in [83, Lemma 1], if  $G = AF$ , i.e., if  $(\Pi)^*F = G + (\Pi)^*\Pi G$ , then

$$\langle TAF, F \rangle = \langle G, (\Pi)^*F \rangle = \|G\|^2 + \|\Pi G\|^2 = \|AF\|^2 + \|TAF\|^2.$$

By the Cauchy-Schwarz inequality, we know that

$$\begin{aligned} \langle G, (\Pi)^*F \rangle &= \langle TAF, (\text{Id} - \Pi)F \rangle \\ &\leq \|TAF\| \|(\text{Id} - \Pi)F\| \leq \frac{1}{2\mu} \|TAF\|^2 + \frac{\mu}{2} \|(\text{Id} - \Pi)F\|^2 \end{aligned}$$

for any  $\mu > 0$ . Hence

$$2\|AF\|^2 + \left(2 - \frac{1}{\mu}\right) \|TAF\|^2 \leq \mu \|(\text{Id} - \Pi)F\|^2,$$

which, by taking either  $\mu = 1/2$  or  $\mu = 1$ , proves that

$$\|AF\| \leq \frac{1}{2} \|(\text{Id} - \Pi)F\| \quad \text{and} \quad \|TAF\| \leq \|(\text{Id} - \Pi)F\|. \quad (7.11)$$

This establishes (7.8) and, as a side result, also proves that

$$|\langle TAF, F \rangle| = |\langle TAF, (\text{Id} - \Pi)F \rangle| \leq \|(\text{Id} - \Pi)F\|^2.$$

Collecting terms in the expression of  $D_\delta[F]$ , we find that

$$D_\delta[F] \geq (\lambda_m - \delta) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta C_M X Y$$

with  $X := \|(\text{Id} - \Pi)F\|$  and  $Y := \|\Pi F\|$ . We know that  $H_\delta[F] \leq \frac{1}{2}(X^2 + Y^2) + \frac{\delta}{2}XY$ , so that the largest value of  $\lambda$  for which  $D_\delta[F] \geq \lambda H_\delta[F]$  can be estimated by the largest value of  $\lambda$  for which

$$(X, Y) \mapsto (\lambda_m - \delta)X^2 + \frac{\delta \lambda_M}{1 + \lambda_M}Y^2 - \delta C_M XY - \frac{\lambda}{2}(X^2 + Y^2) - \frac{\lambda}{2}\delta XY$$

is a nonnegative quadratic form, as a function of  $(X, Y)$ . It is characterized by the discriminant condition

$$h(\delta, \lambda) := \delta^2 \left( C_M + \frac{\lambda}{2} \right)^2 - 4 \left( \lambda_m - \delta - \frac{\lambda}{2} \right) \left( \frac{\delta \lambda_M}{1 + \lambda_M} - \frac{\lambda}{2} \right) \leq 0$$

and the sign condition  $\lambda_m - \delta - \lambda/2 > 0$ . For any  $\delta \in (0, \delta_*)$ , the sign condition is always satisfied by any  $\lambda > 0$  and we also have that  $h(\delta, 0) > 0$ . Since  $\lambda \mapsto h(\delta, \lambda)$  is a second order polynomial, the largest possible value of  $\lambda$  can be estimated by the positive root of  $h(\delta, \lambda) = 0$ .  $\square$

Notice that the proof of Proposition 7.1 provides us with a constructive estimate of the decay rate  $\lambda$ , as a function of  $\delta \in (0, \delta_*)$ . We refer to [46] for a discussion of the best estimate of the decay rate of  $H_\delta$ , *i.e.*, the largest possible estimate of  $\lambda$  when  $\delta$  varies in the admissible range  $(0, \delta_*)$ .

### 7.3 Functional setting

In this section, we collect a number of observations on the external potential  $V$  and on estimates based on the stationary solution obtained by solving the Poisson-Boltzmann equation. Depending on growth conditions on  $V$ , we establish a notion of *confinement* (which guarantees that (VPFP) admits an integrable stationary solution) and some coercivity properties (which amount to Poincaré type inequalities). Our goal is to give sufficient conditions in order that:

- 1) there exists a nonnegative stationary solution  $f_\star$  of (VPFP) of arbitrary given mass  $M > 0$ : see Section 7.3.2;
- 2) there is a Poincaré inequality associated with the measure  $e^{-V - \phi_\star} dx$  on  $\mathbb{R}^d$ , and variants of it, with weights: see Section 7.3.3;
- 3) there is a Hilbert space structure on which we can study the evolution equation (7.1): see Section 7.3.6. These conditions on  $V$  determine a functional setting which is adapted to implement the method of Section 7.2. The reader is invited to check that  $V(x) = |x|^\alpha$  with  $\alpha > 1$  is an *admissible potential* in that perspective.

In [83], without Poisson coupling, sufficient conditions were given on  $V$  which were inspired by the *carré du champ* method and the Holley-Stroock perturbation lemma. These conditions are not well adapted to handle an additional Poisson coupling. Here we adopt a slightly different approach, which amounts

to focus on sufficient growth conditions of the external potential  $V$  and on tools of spectral theory like Persson's lemma. For sake of simplicity, we require some basic regularity properties of  $V$  (which are not optimal but avoid technicalities) and assume that

$$V \in L_{loc}^\infty \cap W_{loc}^{2,1}(\mathbb{R}^d) \quad \text{and} \quad \liminf_{|x| \rightarrow +\infty} V(x) = +\infty. \quad (V1)$$

### 7.3.1 Preliminary considerations on the Poisson equation and conventions

Let us consider the Green function  $G_d$  associated with  $-\Delta$ . We shall write  $\phi = (-\Delta)^{-1} \rho$  as a generic notation for  $\phi = G_d * \rho$  with  $G_d(x) = c_d |x|^{2-d}$ ,  $c_d^{-1} = (d-2) |\mathbb{S}^{d-1}|$  if  $d \geq 3$ . With no restriction, using integrations by parts, we have that

$$\int_{\mathbb{R}^d} \rho \phi \, dx = \int_{\mathbb{R}^d} (-\Delta \phi) \phi \, dx = \int_{\mathbb{R}^d} |\nabla \phi|^2 \, dx.$$

If  $d = 2$ , we use  $G_2(x) = -\frac{1}{2\pi} \log|x|$ . It is a standard observation that  $\phi = (-\Delta)^{-1} \rho$  is such that  $\nabla \phi(x) = -\frac{1}{2\pi} \left( \int_{\mathbb{R}^2} \rho \, dx \right) \frac{x}{|x|^2}$  as  $|x| \rightarrow +\infty$  is not square integrable unless  $\int_{\mathbb{R}^2} \rho \, dx = 0$ . If  $\int_{\mathbb{R}^2} \rho \, dx = 0$ , one can prove that

$$\int_{\mathbb{R}^2} \rho \phi \, dx = \int_{\mathbb{R}^2} |\nabla \phi|^2 \, dx < +\infty.$$

If  $d = 1$ , we have  $G_1(x) = -|x|/2$ , but it is sometimes more convenient to rely on the equivalent representation

$$\phi(x) = \frac{M}{2} x - \int_{-\infty}^x dy \int_{-\infty}^y \rho(z) \, dz \quad (7.12)$$

for some integration constant  $\phi_0 \in \mathbb{R}$  and we shall still write  $\phi - \phi_0 = (-d^2/dx^2)^{-1} \rho$  whenever we use (7.12). We can moreover notice that  $\phi - \phi_0 = (-d^2/dx^2)^{-1} \rho$  is such that  $\phi' = -m$  where  $m(x) := \int_{-\infty}^x \rho(y) \, dy$  if  $M = \int_{\mathbb{R}} \rho \, dx = 0$ . In that case, if we further assume that  $\rho$  is compactly supported or has a sufficient decay at infinity, an integration by parts shows that

$$\int_{\mathbb{R}} \phi \rho \, dx = - \int_{\mathbb{R}} \phi' m \, dx = \int_{\mathbb{R}} |\phi'|^2 \, dx = \int_{\mathbb{R}} m^2 \, dx \geq 0. \quad (7.13)$$

Altogether, whenever  $\int_{\mathbb{R}^d} \rho \, dx = 0$ , we shall write  $\int_{\mathbb{R}^d} \rho \phi \, dx = \int_{\mathbb{R}^d} |\nabla \phi|^2 \, dx \geq 0$  without any further precaution, for any  $d \geq 1$ .

### 7.3.2 The Poisson-Boltzmann equation

According to [91, 158, 72], stationary solutions of the (VPFP) system are given by

$$f_\star(x, v) = \rho_\star(x) \mathcal{M}(v)$$

where  $\mathcal{M}(v) = (2\pi)^{-d/2} e^{-|v|^2/2}$  is the normalized Maxwellian function (or Gaussian function) and the spatial density  $\rho_\star$  is determined by the *Poisson-Boltzmann equation*

$$-\Delta\phi_\star = \rho_\star = M \frac{e^{-V-\phi_\star}}{\int_{\mathbb{R}^d} e^{-V-\phi_\star} dx}.$$

It is obvious that  $\phi_\star$  is defined up to an additive constant which can be chosen such that  $M = \int_{\mathbb{R}^d} e^{-V-\phi_\star} dx$  and therefore solves (7.6). Here  $M = \|\rho_\star\|_{L^1(\mathbb{R}^d)} = \|f_\star\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = M$  is the mass, which is a free parameter of the problem. The critical growth of  $V$  needed to obtain solutions  $\rho_\star \in L^1(\mathbb{R}^d)$  of (7.6) which minimize the free energy strongly depends on the dimension. It is characterized as follows.

**Lemma 7.1.** *Let  $M > 0$ . Assume that  $V$  satisfies (V1) and*

$$\begin{aligned} |v| e^{-V} &\in L^1(\mathbb{R}^d) && \text{if } d \geq 3, \\ \liminf_{|x| \rightarrow +\infty} \frac{V(x)}{\log|x|} &> 4 + \frac{M}{2\pi} && \text{if } d = 2, \\ \liminf_{|x| \rightarrow +\infty} \frac{V(x) - M|x|/2}{\log|x|} &> 2 && \text{if } d = 1. \end{aligned} \quad (\text{V2})$$

Then (7.6) has a unique solution  $\rho_\star \in L^1(\mathbb{R}^d)$  such that  $\int_{\mathbb{R}^d} \rho_\star dx = M$  and  $\phi_\star$  is the unique solution of (7.6). Moreover  $\phi_\star$  is of class  $C^2$  and  $\liminf_{|x| \rightarrow +\infty} W_\star(x) = +\infty$ , where

$$W_\star = V + \phi_\star \quad \text{and} \quad \rho_\star = e^{-W_\star}.$$

As a straightforward consequence of Lemma 7.1, we learn that under Assumptions (V1) and (V2), the potential  $W_\star$  also satisfies (V1).

*Proof.* The case  $d \geq 3$  is covered by [72, p. 123]. The free energy

$$\mathcal{F}[\rho] := \int_{\mathbb{R}^d} \rho \log \rho dx + \int_{\mathbb{R}^d} \rho V dx + \frac{1}{2} \int_{\mathbb{R}^d} \rho \phi dx$$

is bounded from below under the mass constraint  $\int_{\mathbb{R}^d} \rho dx = M$  using the fact that

$$\int_{\mathbb{R}^d} \rho \phi dx = \int_{\mathbb{R}^d} |\nabla \phi|^2 dx \geq 0$$

and Jensen's inequality

$$\begin{aligned} \mathcal{F}[\rho] &\geq \int_{\mathbb{R}^d} \rho \log \rho dx + \int_{\mathbb{R}^d} \rho V dx = \int_{\mathbb{R}^d} (u \log u) e^{-V} dx \\ &\geq \left( \int_{\mathbb{R}^d} u e^{-V} dx \right) \log \left( \int_{\mathbb{R}^d} u e^{-V} dx \right) = M \log M \end{aligned}$$

applied to  $u := \rho e^V$ . Here we assume with no loss of generality that  $\int_{\mathbb{R}^d} e^{-V} dx = 1$ .



The existence follows by a minimization method. As noticed in , the uniqueness is a consequence of the convexity of  $\mathcal{F}$ . Finally, by standard elliptic regularity,  $\phi_\star = (-\Delta)^{-1} \rho_\star$  is continuous and has a limit as  $|x| \rightarrow +\infty$ .

In dimension  $d = 1$  or  $d = 2$ , the same scheme can be adapted after proving that  $\mathcal{F}$  is bounded from below. This has been established in [80, Theorem 3.5] (also see [123]) when  $d = 2$  under Assumption (V2). The case  $d = 1$  can be dealt with by elementary methods. Let us consider the potential

$$V_0(x) = \frac{M}{2} \left( (x+1) \mathbb{1}_{(-\infty, -1)}(x) + (x+1)(x-1) \mathbb{1}_{(-1, +1)}(x) - (x-1) \mathbb{1}_{(+1, +\infty)}(x) - 3 \right)$$

such that  $-V_0'' = \frac{M}{2} \mathbb{1}_{(-1, +1)} =: \rho_0$  and let  $\psi = V - V_0$ . We claim that

$$\mathcal{F}[\rho] = \int_{\mathbb{R}} \rho \log \rho \, dx + \int_{\mathbb{R}} \rho (V + V_0) \, dx - \frac{1}{2} \int_{\mathbb{R}} \psi'' \psi \, dx + \frac{1}{2} \int_{\mathbb{R}} \rho_0 \psi \, dx - \frac{1}{2} \int_{\mathbb{R}} \rho V_0 \, dx$$

is bounded from below because the first two integrals can be bounded using Jensen's inequality,  $\int_{\mathbb{R}} \psi'' \psi \, dx = - \int_{\mathbb{R}} |\psi'|^2 \, dx$ ,  $\rho_0$  has compact support and  $\int_{\mathbb{R}} \rho |V_0| \, dx$  provides a moment bound. Combining these estimates provides us with the lower bound we need.  $\square$

### 7.3.3 Some non-trivial Poincaré inequalities

Assume that  $V$  is such that (V1)-(V2) hold. Before considering the case of the measure  $e^{-W_\star} dx$  on  $\mathbb{R}^d$ , with  $W_\star = V + \phi_\star$ , we may ask under which conditions on  $V$  the *Poincaré inequality*

$$\int_{\mathbb{R}^d} |\nabla u|^2 e^{-V} \, dx \geq \mathcal{C}_P \int_{\mathbb{R}^d} |u|^2 e^{-V} \, dx \quad \forall u \in H^1(\mathbb{R}^d) \quad \text{such that} \quad \int_{\mathbb{R}^d} u e^{-V} \, dx = 0 \quad (7.14)$$

is true for some constant  $\mathcal{C}_P > 0$ . Let us define  $w = u e^{-V/2}$  and observe that (7.14) is equivalent to

$$\int_{\mathbb{R}^d} |\nabla w|^2 \, dx + \int_{\mathbb{R}^d} \Phi |w|^2 \, dx \geq \mathcal{C}_P \int_{\mathbb{R}^d} |w|^2 \, dx$$

under the condition that  $\int_{\mathbb{R}^d} w e^{-V/2} \, dx = 0$ . Here  $\Phi = \frac{1}{4} |\nabla V|^2 - \frac{1}{2} \Delta V$  is obtained by expanding the square in  $\int_{\mathbb{R}^d} |\nabla w + \frac{1}{2} w \nabla V|^2 \, dx$  and integrating by parts the cross-term. It is also straightforward to observe that the kernel of the Schrödinger operator  $-\Delta + \Phi$  on  $L^2(\mathbb{R}^d, dx)$  is generated by  $e^{-V/2}$ . According to Persson's result [140, Theorem 2.1], the lower end  $\sigma$  of the continuous spectrum of the Schrödinger operator  $-\Delta + \Phi$  is such that

$$\sigma \geq \liminf_{r \rightarrow +\infty} \inf_{x \in B_r^c} \Phi(x) =: \sigma_0.$$

As a consequence, if  $\sigma$  is positive, either there is no eigenvalue in the interval  $(0, \sigma)$  and  $\mathcal{C}_P = \sigma$ , or  $\mathcal{C}_P$  is the lowest positive eigenvalue, and it is positive by

construction. In both cases, we know that (7.14) holds for some  $\mathcal{C}_P > 0$  if  $\sigma_0 > 0$ . In order to prove (7.14), it is enough to check that

$$\sigma_V := \lim_{r \rightarrow +\infty} \inf_{x \in B_r^c} \left( \frac{1}{4} |\nabla V|^2 - \frac{1}{2} \Delta V \right) > 0 \quad \text{and} \quad \lim_{r \rightarrow +\infty} \inf_{x \in B_r^c} |\nabla V| > 0. \quad (\text{V3a})$$

Now let us consider the measure  $\rho_\star dx = e^{-W_\star} dx$  on  $\mathbb{R}^d$  and establish the corresponding Poincaré inequality.

**Lemma 7.2.** *Assume that  $d \geq 1$  and consider  $V$  such that (V1), (V2) and (V3a) hold. We further assume that*

$$\lim_{r \rightarrow +\infty} \inf_{|x| > r} \left( (M - 2V')^2 - 8V'' + M^2 \right) > 0 \quad \text{if } d = 1. \quad (\text{V4})$$

If  $\phi_\star$  solves (7.6) and  $W_\star = V + \phi_\star$ , then there is a positive constant  $\mathcal{C}_\star$  such that

$$\int_{\mathbb{R}^d} |\nabla u|^2 \rho_\star dx \geq \mathcal{C}_\star \int_{\mathbb{R}^d} |u|^2 \rho_\star dx \quad \forall u \in H^1(\mathbb{R}^d) \quad \text{s.t.} \quad \int_{\mathbb{R}^d} u \rho_\star dx = 0. \quad (\text{7.15})$$

*Proof.* It is enough to prove that

$$\sigma_{W_\star} := \lim_{r \rightarrow +\infty} \inf_{x \in B_r^c} \left( \frac{1}{4} |\nabla \phi_\star + \nabla V|^2 - \frac{1}{2} (\Delta \phi_\star + \Delta V) \right) > 0.$$

We observe that, by (V3a),  $|\Delta \phi_\star| = \rho_\star = o(|x|^{-d}) = o(|\nabla V|^2 - 2\Delta V)$  and  $|\nabla \phi_\star| = \mathcal{O}(|x|^{1-d})$  is negligible compared to  $|\nabla V|$  if  $d \geq 2$ . If  $d = 1$ , the result follows from (V4) using the fact that  $\phi_\star'(\mp x) \sim \pm M/2$  as  $x \rightarrow +\infty$ .  $\square$

We shall now replace (V3a) by the slightly stronger assumption that for some  $\theta \in [0, 1)$ ,

$$\lim_{r \rightarrow +\infty} \inf_{x \in B_r^c} \left( \frac{\theta}{4} |\nabla V|^2 - \frac{1}{2} \Delta V \right) \geq 0 \quad \text{and} \quad \lim_{r \rightarrow +\infty} \inf_{x \in B_r^c} |\nabla V| > 0. \quad (\text{V3b})$$

**Corollary 7.2.** *Assume that  $d \geq 1$  and consider  $V$  such that (V1), (V2), (V3b) and (V4) hold. If  $\phi_\star$  solves (7.6) and  $W_\star = V + \phi_\star$ , then there is a positive constant  $\mathcal{C}$  such that*

$$\int_{\mathbb{R}^d} |\nabla u|^2 \rho_\star dx \geq \mathcal{C} \int_{\mathbb{R}^d} |u|^2 |\nabla W_\star|^2 \rho_\star dx \quad \forall u \in H^1(\mathbb{R}^d) \quad \text{s.t.} \quad \int_{\mathbb{R}^d} u \rho_\star dx = 0. \quad (\text{7.16})$$

*Proof.* By expanding  $|\nabla(u\sqrt{\rho_\star})|^2$ , using  $\nabla\sqrt{\rho_\star} = -\frac{1}{2}\nabla W_\star \rho_\star$  and integrating by parts, we obtain that

$$0 \leq \int_{\mathbb{R}^d} |\nabla(u\sqrt{\rho_\star})|^2 dx = \int_{\mathbb{R}^d} |\nabla u|^2 \rho_\star dx - \int_{\mathbb{R}^d} \left( \frac{1}{4} |\nabla W_\star|^2 - \frac{1}{2} \Delta W_\star \right) |u|^2 \rho_\star dx.$$

Combined with (7.15), this shows that

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla u|^2 \rho_\star dx &\geq \int_{\mathbb{R}^d} \left[ (1-\eta) \mathcal{C}_\star + \eta \left( \frac{\theta}{4} |\nabla W_\star|^2 - \frac{1}{2} \Delta W_\star \right) \right] |u|^2 \rho_\star dx \\ &\quad + \frac{\eta\theta}{4} \int_{\mathbb{R}^d} |u|^2 |\nabla W_\star|^2 \rho_\star dx \end{aligned}$$

for any  $\eta \in (0, 1)$ . With  $\eta$  chosen small enough so that  $(1-\eta) \mathcal{C}_\star + \eta \left( \frac{\theta}{4} |\nabla W_\star|^2 - \frac{1}{2} \Delta W_\star \right)$  is nonnegative a.e., the conclusion holds with  $\mathcal{C} = \eta\theta/4$ .  $\square$

In the same spirit as for Corollary 7.2, we shall assume that for some  $\theta \in [0, 1)$ ,

$$\begin{aligned} \lim_{r \rightarrow +\infty} \inf_{x \in B_r^c} \left( \frac{\theta}{4} |\nabla V|^4 - \frac{1}{2} \Delta V |\nabla V|^2 - \text{Hess}(V) : \nabla V \otimes \nabla V \right) &\geq 0 \\ \text{and } \lim_{r \rightarrow +\infty} \inf_{x \in B_r^c} |\nabla V| &> 0. \end{aligned} \quad (\text{V5})$$

**Corollary 7.3.** *Assume that  $d \geq 1$  and consider  $V$  such that (V1), (V2), (V3b) and (V5) hold. If  $\phi_\star$  solves (7.6) and  $W_\star = V + \phi_\star$ , then there is a positive constant  $\mathcal{C}_\circ$  such that*

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla u|^2 |\nabla W_\star|^2 \rho_\star dx &\geq \mathcal{C}_\circ \int_{\mathbb{R}^d} |u|^2 |\nabla W_\star|^4 \rho_\star dx \\ \text{for any } u \in H^1(\mathbb{R}^d) \text{ satisfies } \int_{\mathbb{R}^d} u \rho_\star dx &= 0. \end{aligned}$$

The proof is based on the expansion of the square in  $|\nabla(u\sqrt{\rho_\star})|^2 |\nabla W_\star|^2$ , integrations by parts and an IMS truncation argument in order to use Lemma 7.2 in a finite centered ball of radius  $2R$ , on which  $\nabla W_\star$  is bounded and Assumption (V5) outside of the centered ball of radius  $R$ . Details are left to the reader. See [132, 148] or section 2 in [39] for details on the IMS (for Ismagilov, Morgan, Morgan-Simon, Sigal) truncation method.

### 7.3.4 Further inequalities based on pointwise estimates

If  $\mathfrak{M}$  is a  $d \times d$  symmetric real valued matrix, let us denote by  $\Lambda(\mathfrak{M})$  the largest eigenvalue of  $\mathfrak{M}$ . With this notation, let us assume that

$$\Lambda_V := \lim_{r \rightarrow +\infty} \sup_{x \in B_r^c} \frac{1}{|\nabla V(x)|^2} \Lambda \left( e^{V(x)} \left( \text{Hess}(e^{-V(x)}) - \frac{1}{2} \Delta(e^{-V(x)}) \text{Id} \right) \right) < +\infty. \quad (\text{V6})$$

In other words, Assumption (V6) means that for any  $\varepsilon > 0$ , there exists some  $R > 0$  such that

$$\begin{aligned} e^{V(x)} \left( \text{Hess}(e^{-V(x)}) - \frac{1}{2} \Delta(e^{-V(x)}) \text{Id} \right) &\leq (\Lambda_V - \varepsilon) |\nabla V(x)|^2 \text{Id}, \\ &x \in \mathbb{R}^d \text{ a.e. such that } |x| > R, \end{aligned}$$

where the inequality holds in the sense of positive matrices.

**Lemma 7.3.** *Assume that  $d \geq 1$  and consider  $V$  such that (V1), (V2) and (V6) hold. If  $\phi_\star$  solves (7.6) and  $W_\star = V + \phi_\star$ , then there is a positive constant  $\Lambda_\star$  such that*

$$\int_{\mathbb{R}^d} \left( \text{Hess}(\rho_\star) - \frac{1}{2} \Delta \rho_\star \text{Id} \right) : \nabla w \otimes \nabla w \, dx \leq \Lambda_\star \int_{\mathbb{R}^d} |\nabla w|^2 |\nabla W_\star|^2 \rho_\star \, dx \quad (7.17)$$

for any function  $w \in H_{\text{loc}}^1(\mathbb{R}^d)$ .

*Proof.* An elementary computation shows that

$$\text{Hess}(\rho_\star) = (\nabla W_\star \otimes \nabla W_\star - \text{Hess}(W_\star)) \rho_\star \quad \text{and} \quad \Delta \rho_\star = (|\nabla W_\star|^2 - \Delta W_\star) \rho_\star.$$

The proof is then similar to the above estimates. Details are left to the reader.  $\square$

Similarly, let us assume that

$$\limsup_{r \rightarrow +\infty} \sup_{x \in B_r^c} \left| \nabla (\log(|\nabla V(x)|^2)) \right| < +\infty. \quad (V7)$$

**Lemma 7.4.** *Assume that  $d \geq 1$  and consider  $V$  such that (V1), (V2) and (V7) hold. If  $\phi_\star$  solves (7.6) and  $W_\star = V + \phi_\star$ , then there is a positive constant  $\Lambda_\circ$  such that*

$$\left| \nabla (|\nabla W_\star(x)|^2) \nabla W_\star(x) \right| \leq \Lambda_\circ |\nabla W_\star(x)|^3, \quad x \in \mathbb{R}^d \text{ a.e. such that } |x| > R. \quad (7.18)$$

Here we mean that  $\nabla (|\nabla W_\star|^2) \nabla W_\star = 2 \text{Hess}(W_\star) : \nabla W_\star \otimes \nabla W_\star$  and a straightforward consequence of (7.18) is that

$$\left| \nabla (|\nabla W_\star|^2) \nabla w \right| = 2 \left| \text{Hess}(W_\star) : \nabla W_\star \otimes \nabla w \right| \leq \Lambda_\circ |\nabla W_\star(x)|^2 |\nabla w|.$$

The inequality follows from the regularity and decay estimates of  $\phi_\star$ . Details are left to the reader.

In the same vein, let us assume that

$$\| |\nabla V|^2 e^{-V} \|_{L^\infty(\mathbb{R}^d, dx)} < +\infty \quad \text{and} \quad \| |\nabla (|\nabla V|^2)|^2 e^{-V} \|_{L^\infty(\mathbb{R}^d, dx)} < +\infty. \quad (V8)$$

**Lemma 7.5.** *Assume that  $d \geq 1$  and consider  $V$  such that (V1), (V2) and (V8) hold. If  $\phi_\star$  solves (7.6) and  $W_\star = V + \phi_\star$ , then  $\| |\nabla W_\star|^2 \rho_\star \|_{L^\infty(\mathbb{R}^d, dx)}$  and  $\| |\nabla (|\nabla W_\star|^2)|^2 \rho_\star \|_{L^\infty(\mathbb{R}^d, dx)}$  are finite.*

### 7.3.5 A Bochner-Lichnerowicz-Weitzenböck identity and second order estimates

Algebraic computations and a few integrations by parts provide us with the following estimate.

**Lemma 7.6.** *Let  $M > 0$  and  $\rho_\star = e^{-W_\star} \in L_{\text{loc}}^\infty \cap W^{1,2}(\mathbb{R}^d)$ . Then for any smooth function  $w$  on  $\mathbb{R}^d$  with compact support, we have the identity*

$$\int_{\mathbb{R}^d} |\text{Hess}(w)|^2 \rho_\star \, dx \leq 3 \int_{\mathbb{R}^d} \frac{1}{\rho_\star} |\nabla \cdot (\rho_\star \nabla w)|^2 \, dx + \frac{7}{2} \int_{\mathbb{R}^d} (\nabla W_\star \cdot \nabla w)^2 \rho_\star \, dx.$$

Notice that if  $V$  satisfies (VI)–(V2) and  $W_\star = V + \phi_\star$  where  $\phi_\star$  is the unique solution of (7.6), then  $\rho_\star$  is an admissible function for Lemma 7.6.

*Proof.* Let us start by establishing a Bochner-Lichnerowicz-Weitzenböck identity as follows:

$$\begin{aligned} \frac{1}{2} \Delta (\rho_\star |\nabla w|^2) &= \nabla \cdot (\rho_\star \text{Hess}(w) \nabla w) + \frac{1}{2} \nabla \cdot (|\nabla w|^2 \nabla \rho_\star) \\ &= \rho_\star |\text{Hess}(w)|^2 + \rho_\star \nabla w \cdot \nabla (\Delta w) + \frac{1}{2} \Delta \rho_\star |\nabla w|^2 + \text{Hess}(w) : \nabla w \otimes \nabla \rho_\star \\ &= \rho_\star |\text{Hess}(w)|^2 + \nabla w \cdot \nabla (\rho_\star \Delta w) \\ &\quad - (\nabla w \cdot \nabla \rho_\star) \Delta w + \frac{1}{2} \Delta \rho_\star |\nabla w|^2 + \text{Hess}(w) : \nabla w \otimes \nabla \rho_\star. \end{aligned}$$

We obtain after a few integrations by parts on  $\mathbb{R}^d$  that

$$\begin{aligned} \int_{\mathbb{R}^d} \Delta (\rho_\star |\nabla w|^2) dx &= 0, \quad \int_{\mathbb{R}^d} \nabla w \cdot \nabla (\rho_\star \Delta w) dx = - \int_{\mathbb{R}^d} (\Delta w)^2 \rho_\star dx, \\ \frac{1}{2} \int_{\mathbb{R}^d} \Delta \rho_\star |\nabla w|^2 dx + \int_{\mathbb{R}^d} \text{Hess}(w) : \nabla w \otimes \nabla \rho_\star dx &= 0, \end{aligned}$$

which proves that

$$\int_{\mathbb{R}^d} |\text{Hess}(w)|^2 \rho_\star dx = \int_{\mathbb{R}^d} (\Delta w)^2 \rho_\star dx + \int_{\mathbb{R}^d} (\nabla w \cdot \nabla \rho_\star) \Delta w dx. \quad (7.19)$$

We deduce from

$$\begin{aligned} \int_{\mathbb{R}^d} (\nabla w \cdot \nabla \rho_\star) \Delta w dx &= - \int_{\mathbb{R}^d} \Delta w (\nabla w \cdot \nabla W_\star) \rho_\star dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^d} (\Delta w)^2 \rho_\star dx + \frac{1}{2} \int_{\mathbb{R}^d} (\nabla W_\star \cdot \nabla w)^2 \rho_\star dx \end{aligned}$$

that

$$\int_{\mathbb{R}^d} |\text{Hess}(w)|^2 \rho_\star dx = \frac{3}{2} \int_{\mathbb{R}^d} (\Delta w)^2 \rho_\star dx + \frac{1}{2} \int_{\mathbb{R}^d} (\nabla W_\star \cdot \nabla w)^2 \rho_\star dx.$$

Since  $\nabla \rho_\star = -\nabla W_\star \rho_\star$  and  $\Delta w \rho_\star = \nabla \cdot (\rho_\star \nabla w) + (\nabla W_\star \cdot \nabla w) \rho_\star$ , we have the estimate

$$\int_{\mathbb{R}^d} (\Delta w)^2 \rho_\star dx \leq 2 \int_{\mathbb{R}^d} \frac{1}{\rho_\star} |\nabla \cdot (\rho_\star \nabla w)|^2 dx + 2 \int_{\mathbb{R}^d} (\nabla W_\star \cdot \nabla w)^2 \rho_\star dx,$$

which completes the proof.  $\square$

### 7.3.6 The scalar product

Let us define the measure  $d\mu := f_\star(x, v) dx dv$  and consider the functional space

$$\mathcal{H} := \left\{ h \in L^1 \cap L^2(\mathbb{R}^d \times \mathbb{R}^d, d\mu) : \iint_{\mathbb{R}^d \times \mathbb{R}^d} h d\mu = 0 \text{ and } \int_{\mathbb{R}^d} |\nabla \psi_h|^2 dx < \infty \right\},$$

where we use the notation  $\rho_h = \int_{\mathbb{R}^d} h f_\star dv$  and  $\psi_h = (-\Delta)^{-1} \rho_h$ . We also define

$$\langle h_1, h_2 \rangle := \iint_{\mathbb{R}^d \times \mathbb{R}^d} h_1 h_2 d\mu + \int_{\mathbb{R}^d} \rho_{h_1} (-\Delta)^{-1} \rho_{h_2} dx \quad \forall h_1, h_2 \in \mathcal{H}.$$

**Lemma 7.7.** *Let  $M > 0$ . If  $V$  satisfies (V1)–(V2), then  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  is a Hilbert space for any  $d \geq 1$ .*

*Proof.* Up to an integration by parts, we can rewrite  $\langle h_1, h_2 \rangle$  as

$$\begin{aligned} \langle h_1, h_2 \rangle &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} h_1 h_2 d\mu + \int_{\mathbb{R}^d} (-\Delta \psi_{h_1}) \psi_{h_2} dx \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} h_1 h_2 d\mu + \int_{\mathbb{R}^d} \nabla \psi_{h_1} \cdot \nabla \psi_{h_2} dx \end{aligned}$$

and observe that this determines a scalar product. This computation has to be justified. Let us distinguish three cases depending on the dimension  $d$ .

Let us assume first that  $d \geq 3$ . We know that  $\psi_\star = G_d \star \rho_\star$  is nonnegative and deduce  $\rho_\star$  is bounded because

$$0 \leq e^{-V - \psi_\star} \leq e^{-V} \in L^\infty(\mathbb{R}^d).$$

Hence, for any  $p \in (1, 2]$ , we have

$$\|\rho_h\|_{L^p(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} h f_\star dv \right|^p dx \leq \|\rho_\star\|_{L^\infty(\mathbb{R}^d)}^{p-1} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |h|^p d\mu.$$

According to [125], we know by the Hardy-Littlewood-Sobolev inequality that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\rho_1(x)| |\rho_2(x)|}{|x - y|^{d-a}} dx dy \leq \mathcal{C}_{\text{HLS}} \|\rho_1\|_{L^p(\mathbb{R}^d)} \|\rho_2\|_{L^q(\mathbb{R}^d)}$$

if  $a \in (0, d)$  and  $p, q \in (1, +\infty)$  are such that  $1 + \frac{a}{d} = \frac{1}{p} + \frac{1}{q}$ . This justifies the fact that  $\int_{\mathbb{R}^d} \rho_h (-\Delta)^{-1} \rho_h dx$  is well defined if  $h \in L^1 \cap L^2(\mathbb{R}^d \times \mathbb{R}^d, d\mu)$ . With  $a = 2$ ,  $p < 3/2$  if  $d = 3$ ,  $p < 2$  if  $d = 4$  and  $p \leq 2$  if  $d \geq 5$ , we deduce that  $\psi_h \in L^{q'}(\mathbb{R}^d)$  where  $q' = q/(q-1) = dp/(d-2p)$ . A simple Hölder estimate shows the Gagliardo-Nirenberg type estimate

$$\|\nabla \psi\|_{L^2(\mathbb{R}^d)}^2 \leq \|\Delta \psi\|_{L^{p_1}(\mathbb{R}^d)} \|\psi\|_{L^{q_1}(\mathbb{R}^d)}$$

and proves for an appropriate choice of  $(p_1, q_1) \in (1, 2) \times (2, +\infty)$  with  $\frac{1}{p_1} + \frac{1}{q_1} = 1$  that  $\nabla \psi_h$  is bounded in  $L^2(\mathbb{R}^d)$ .

The case  $d = 2$  is well known. The boundedness of  $\|\rho_h\|_{L^p(\mathbb{R}^d)}$  for any  $p \in (1, 2]$  follows by the same argument as in the case  $d \geq 3$  and we learn that  $|\rho_h| \log|\rho_h|$  is integrable by log-Hölder interpolation. The boundedness from below of  $\int_{\mathbb{R}^2} \rho_h (-\Delta)^{-1} \rho_h$  is then a consequence of the logarithmic Hardy-Littlewood-Sobolev inequality, see [52, 80]. Using the fact that  $\int_{\mathbb{R}^d} \rho_h dx = 0$ , we also know from [33] that  $\nabla \psi_h$  is bounded in  $L^2(\mathbb{R}^2)$ .

When  $d = 1$ , the nonnegativity of the scalar product is a consequence of (7.13) and holds without additional condition by a simple density argument.  $\square$

The condition  $\iint_{\mathbb{R}^d \times \mathbb{R}^d} h d\mu = 0$  in the definition of  $h$  is simply an orthogonality condition with the constant functions, with respect to the usual scalar product in  $L^2(\mathbb{R}^d \times \mathbb{R}^d, d\mu)$ . By taking the completion of smooth compactly supported functions with zero average with respect to the norm defined by  $h \mapsto \langle h, h \rangle$ , we recover  $\mathcal{H}$ , which is therefore a Hilbert space. In the next sections, we shall denote by  $\|\cdot\|$  the norm on  $\mathcal{H}$  associated with the scalar product so that

$$\|h\|^2 = \langle h, h \rangle \quad \forall h \in \mathcal{H}.$$

## 7.4 Proof of the main result

In this section, we prove Theorem 7.1. Our task is to check that the assumptions of Section 7.2 hold in the functional setting of Section 7.3.

### 7.4.1 Definitions and elementary properties

On the space  $\mathcal{H}$ , let us consider the transport and the collision operators respectively defined by

$$\mathbb{T}h := v \cdot \nabla_x h - \nabla_x W_\star \cdot \nabla_v h + v \cdot \nabla_x \psi_h, \quad \mathbb{L}h := \Delta_v h - v \cdot \nabla_v h \quad (7.20)$$

where  $\nabla_x W_\star = \nabla_x V + \nabla_x \phi_\star$ . In the literature,  $\mathbb{L}$  is known as the *Ornstein-Uhlenbeck operator*.

**Lemma 7.8.** *With the above notation,  $\mathbb{L}$  and  $\mathbb{T}$  are respectively self-adjoint and anti-self-adjoint.*

*Proof.* If  $h_1$  and  $h_2$  are two functions in  $L^2(\mathbb{R}^d, \mathcal{M} dv)$ , then  $\mathbb{L}$  is such that

$$\int_{\mathbb{R}^d} (\mathbb{L}h_1) h_2 \mathcal{M} dv = - \int_{\mathbb{R}^d} \nabla_v h_1 \cdot \nabla_v h_2 \mathcal{M} dv$$

and as a special case corresponding to  $h_1 = h$ ,  $h_2 = 1$ , we find that  $\rho_{\mathbb{L}h} = \int_{\mathbb{R}^d} (\mathbb{L}h) f_\star dv = 0$  and also  $\psi_{\mathbb{L}h} = 0$  for any  $h \in \mathcal{H}$ . As a straightforward consequence, we have that

$$\langle (\mathbb{L}h_1), h_2 \rangle = - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v h_1 \cdot \nabla_v h_2 d\mu = \langle h_1, (\mathbb{L}h_2) \rangle \quad \forall h_1, h_2 \in \mathcal{H}.$$

Concerning the transport operator, we know that  $\mathbb{T}f_\star = 0$ . Hence an integration by parts shows that

$$\langle (\mathbb{T}h_1), h_2 \rangle = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v \cdot \nabla_x h_1 - \nabla_x W_\star \cdot \nabla_v h_1) h_2 d\mu = -\langle h_1, (\mathbb{T}h_2) \rangle \quad \forall h_1, h_2 \in \mathcal{H}$$

because  $\rho_{\mathbb{T}h} = \int_{\mathbb{R}^d} (\mathbb{T}h) f_\star dv = \nabla_x \psi_h \cdot \int_{\mathbb{R}^d} v f_\star dv = 0$  and  $\psi_{\mathbb{T}h} = 0$  for any  $h \in \mathcal{H}$ .  $\square$

## 7.4.2 Microscopic coercivity

By the Gaussian Poincaré inequality, we know that

$$\int_{\mathbb{R}^d} |\nabla_v g|^2 \mathcal{M} dv \geq \int_{\mathbb{R}^d} |g - \Pi g|^2 \mathcal{M} dv \quad \forall g \in H^1(\mathbb{R}^d, \mathcal{M} dv),$$

where  $\Pi g = \int_{\mathbb{R}^d} g \mathcal{M} dv$  denotes the average of  $g$  with respect to the Gaussian probability measure  $\mathcal{M} dv$ . By extension, we shall consider  $\Pi$  as an operator on  $\mathcal{H}$  and observe that

$$\Pi h = u_h := \frac{\rho_h}{\rho_\star} = \frac{\int_{\mathbb{R}^d} h f_\star dv}{\int_{\mathbb{R}^d} f_\star dv} = \int_{\mathbb{R}^d} h \mathcal{M} dv \quad \forall h \in \mathcal{H}.$$

Let us notice first that  $\Pi$  is an orthogonal projector.

**Lemma 7.9.**  $\Pi$  is a self-adjoint operator and  $\Pi \circ \Pi = \Pi$ .

*Proof.* It is straightforward to check that

$$(\Pi \circ \Pi) h = \Pi u_h = u_h, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\Pi h_1) h_2 d\mu = \int_{\mathbb{R}^d} u_{h_1} u_{h_2} \rho_\star dx$$

and

$$\int_{\mathbb{R}^d} \rho_{\Pi h_1} (-\Delta)^{-1} \rho_{h_2} dx = \int_{\mathbb{R}^d} \rho_{h_1} (-\Delta)^{-1} \rho_{h_2} dx$$

because  $\rho_{h_1} = \rho_\star u_{h_1} = \rho_\star u_{\Pi h_1} = \rho_{\Pi h_1}$ .  $\square$

**Lemma 7.10.** Microscopic coercivity (H1) holds with  $\lambda_m = 1$ .

*Proof.* We already know that  $-\langle (\mathbb{L}h), h \rangle = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla_v h|^2 d\mu$  and  $\rho_{h-\Pi h} = \rho_h - \rho_{\Pi h} = 0$  so that

$$\|h - \Pi h\|^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |h - \Pi h|^2 d\mu.$$

The conclusion is then a straightforward consequence of the Gaussian Poincaré inequality.  $\square$



### 7.4.3 Macroscopic coercivity

**Lemma 7.11.** *Assume that  $d \geq 1$  and consider  $V$  such that (V1), (V2), (V3a) and (V4) hold. With the notations of Lemma 7.2, macroscopic coercivity (H2) holds with  $\lambda_M = \mathcal{C}_\star$ .*

*Proof.* Using  $\mathbb{T}\Pi h = v \cdot (\nabla_x u_h + \nabla_x \psi_h)$ ,  $\int_{\mathbb{R}^d} (v \cdot e)^2 \mathcal{M} dv = 1$  for any given  $e \in \mathbb{S}^{d-1}$  and (7.15), we find that

$$\begin{aligned} M \|\mathbb{T}\Pi h\|^2 &= \int_{\mathbb{R}^d} |\nabla_x u_h + \nabla_x \psi_h|^2 \rho_\star dx \\ &\geq \mathcal{C}_\star \left[ \int_{\mathbb{R}^d} |u_h + \psi_h|^2 \rho_\star dx - \frac{1}{M} \left( \int_{\mathbb{R}^d} \psi_h \rho_\star dx \right)^2 \right] \end{aligned}$$

because  $\int_{\mathbb{R}^d} u_h \rho_\star dx = \int_{\mathbb{R}^d} \rho_h dx = 0$ . We know from Lemma 7.7 that  $\int_{\mathbb{R}^d} u_h \psi_h \rho_\star dx = \int_{\mathbb{R}^d} \rho_h \psi_h dx \geq 0$  and by the Cauchy-Schwarz inequality, we get that

$$\left( \int_{\mathbb{R}^d} \psi_h \rho_\star dx \right)^2 \leq M \int_{\mathbb{R}^d} |\psi_h|^2 \rho_\star dx.$$

Altogether, we collect these estimates into

$$\int_{\mathbb{R}^d} |\nabla_x u_h + \nabla_x \psi_h|^2 \rho_\star dx \geq \mathcal{C}_\star \left[ \int_{\mathbb{R}^d} |u_h|^2 \rho_\star dx + \int_{\mathbb{R}^d} \rho_h \psi_h dx \right] = \mathcal{C}_\star M \|u_h\|^2$$

which concludes the proof.  $\square$

### 7.4.4 Parabolic macroscopic dynamics

**Lemma 7.12.** *The transport operator  $\mathbb{T}$  satisfies the parabolic macroscopic dynamics (H3).*

*Proof.* Since  $\mathbb{T}\Pi h = v \cdot (\nabla_x u_h + \nabla_x \psi_h)$ , we obtain that

$$\Pi \mathbb{T}\Pi h = (\nabla_x u_h + \nabla_x \psi_h) \cdot \int_{\mathbb{R}^d} v f_\star dv = 0.$$

$\square$

### 7.4.5 Bounded auxiliary operators

The point is to prove that (H4) holds, *i.e.*, that for any  $F \in \mathcal{H}$ ,  $\|\mathbb{A}\mathbb{T}(\text{Id} - \Pi)F\|$  and  $\|\mathbb{A}\mathbb{L}F\|$  are bounded up to a constant by  $\|(\text{Id} - \Pi)F\|$ . This is the purpose of Lemma 7.13 and Lemma 7.14. The two quantities,  $\|\mathbb{A}\mathbb{T}(\text{Id} - \Pi)F\|$  and  $\|\mathbb{A}\mathbb{L}F\|$ , are needed to control the bad terms in the expression of  $D_\delta$ , in the abstract formulation of Proposition 7.1, namely  $\langle \mathbb{T}\mathbb{A}F, F \rangle$ ,  $\langle \mathbb{A}\mathbb{T}(\text{Id} - \Pi)F, F \rangle$  and  $\langle \mathbb{A}\mathbb{L}F, F \rangle$  (which have no definite sign), by the two good terms,  $-\langle \mathbb{L}F, F \rangle$  and  $\langle \mathbb{A}\mathbb{T}\Pi F, F \rangle$  (which are both positive).

**Lemma 7.13.** *The operators  $\mathbb{T}\mathbb{A}$  and  $\mathbb{A}\mathbb{L}$  satisfy: for all  $h \in L^2(\mathbb{R}^d \times \mathbb{R}^d, d\mu)$*

$$\|\mathbb{A}\mathbb{L}h\| \leq \frac{1}{2} \|(1 - \Pi)h\| \quad \text{and} \quad \|\mathbb{T}\mathbb{A}h\| \leq \|(1 - \Pi)h\|.$$

*Proof.* If we denote the flux by  $j_h := \int_{\mathbb{R}^d} v h f_\star dv$ , we remark that  $j_{\mathbb{L}h} = -j_h$  and

$$\Pi\mathbb{T}h = \nabla \cdot j_h - (\nabla_x V + \nabla \phi_\star) \cdot j_h$$

Since  $\mathbb{A}h = g$  means  $g + (\mathbb{T}\Pi)^*(\mathbb{T}\Pi)g = (\mathbb{T}\Pi)^*h = -\Pi\mathbb{T}h$ , this implies that

$$\mathbb{A}\mathbb{L}h = -\mathbb{A}h.$$

The same computation as for (7.11) shows that  $\|\mathbb{A}\mathbb{L}h\|^2 = \|\mathbb{A}h\|^2 = \|g\|^2 \leq \frac{1}{4} \|(1 - \Pi)h\|^2$  and  $\|\mathbb{T}\mathbb{A}h\| = \|\mathbb{T}\Pi g\| \leq \|(1 - \Pi)h\|$ , which completes the proof.  $\square$

**Lemma 7.14.** *Assume that  $d \geq 1$  and consider  $V$  such that (V1), (V2), (V3b), (V4), (V5), (V6), (V7) and (V8) hold. There exists a constant  $\mathcal{C} > 0$  such that*

$$\|\mathbb{A}\mathbb{T}(1 - \Pi)h\| \leq \mathcal{C} \|(1 - \Pi)h\| \quad \forall h \in \mathcal{H}.$$

*Proof.* In order to get an estimate of  $\|\mathbb{A}\mathbb{T}(1 - \Pi)h\|$ , we will compute  $\|(\mathbb{A}\mathbb{T}(1 - \Pi))^*h\|$ .

*Step 1: Reformulation of the inequality as an elliptic regularity estimate.* We claim that

$$\|(\mathbb{A}\mathbb{T}(1 - \Pi))^*h\|^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |(\mathbb{A}\mathbb{T}(1 - \Pi))^*h|^2 d\mu \leq 3 \int_{\mathbb{R}^d} |\text{Hess}(w_g)|^2 \rho_\star dx, \quad (7.21)$$

where  $w_g := u_g + \psi_g$  and  $-\Delta \psi_g = \rho_g$  is computed in terms of the solution  $g$  of (7.22).

Let  $u_h = \Pi h$  and  $w_h := u_h + \psi_h$ . We observe that  $\mathbb{T}\Pi h = v \cdot \nabla_x w_h$ ,  $\rho_{\mathbb{T}\Pi h} = 0$  and, as a consequence

$$(\mathbb{T}\Pi)^*(\mathbb{T}\Pi)h = -\Pi\mathbb{T}(\mathbb{T}\Pi)h = -\Delta w_h + \nabla W_\star \cdot \nabla w_h = -e^{W_\star} \nabla (e^{W_\star} \nabla w_h)$$

where  $W_\star = V + \phi_\star$  is such that  $\rho_\star = e^{-W_\star}$ . For any  $h \in \mathcal{H}$ , the function

$$g = (1 + (\mathbb{T}\Pi)^*(\mathbb{T}\Pi))^{-1}h$$

is obtained by solving the elliptic equation

$$g - \Delta w_g + \nabla W_\star \cdot \nabla w_g = h \quad (7.22)$$

and we compute

$$\begin{aligned} (\mathbb{A}\mathbb{T}(1 - \Pi))^*h &= -(1 - \Pi)\mathbb{T}\mathbb{A}^*h = -(1 - \Pi)\mathbb{T}(\mathbb{T}\Pi)(1 + (\mathbb{T}\Pi)^*(\mathbb{T}\Pi))^{-1}h \\ &= -(1 - \Pi)\mathbb{T}(\mathbb{T}\Pi)g = -(1 - \Pi)(v \otimes v : \text{Hess}(w_g)) \end{aligned}$$

where  $\text{Hess}(w) = (\nabla \otimes \nabla)w$  denotes the Hessian of  $w$ . Hence, with  $|\text{Hess}(w)|^2 = \text{Hess}(w) : \text{Hess}(w)$ , we obtain (7.21). A bound on  $\int_{\mathbb{R}^d} |\text{Hess}(w_g)|^2 \rho_\star dx$  will now be obtained by elliptic regularity estimates based on (7.22).

*Step 2: Some  $H^1$ -type estimates.* By integrating (7.22) against  $\mathcal{M}(v) dv$ , we notice that

$$u_g - \frac{1}{\rho_\star} \nabla \cdot (\rho_\star \nabla w_g) = u_h \quad (7.23)$$

so that

$$\int_{\mathbb{R}^d} u_g \rho_\star dx = \int_{\mathbb{R}^d} u_h \rho_\star dx = \iint_{\mathbb{R}^d \times \mathbb{R}^d} h d\mu = 0. \quad (7.24)$$

If we multiply (7.23) by  $w_g \rho_\star$  and integrate over  $\mathbb{R}^d$ , we get after an integration by parts that

$$\int_{\mathbb{R}^d} u_g (u_g + \psi_g) \rho_\star dx + \int_{\mathbb{R}^d} |\nabla w_g|^2 \rho_\star dx \leq \int_{\mathbb{R}^d} u_h (u_g + \psi_g) \rho_\star dx.$$

Using  $\int_{\mathbb{R}^d} u_g \psi_g \rho_\star dx = \int_{\mathbb{R}^d} |\nabla \psi_g|^2 dx$  and  $\int_{\mathbb{R}^d} u_h \psi_g \rho_\star dx = \int_{\mathbb{R}^d} \nabla \psi_h \cdot \nabla \psi_g dx$  on the one hand, and the elementary estimates

$$\begin{aligned} \left| \int_{\mathbb{R}^d} u_h u_g \rho_\star dx \right| &\leq \frac{1}{2} \int_{\mathbb{R}^d} (|u_g|^2 + |u_h|^2) \rho_\star dx, \\ \left| \int_{\mathbb{R}^d} \nabla \psi_h \cdot \nabla \psi_g dx \right| &\leq \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla \psi_h|^2 + |\nabla \psi_g|^2) dx, \end{aligned}$$

on the other hand, we obtain that

$$\int_{\mathbb{R}^d} |u_g|^2 \rho_\star dx + \int_{\mathbb{R}^d} |\nabla \psi_g|^2 dx + 2 \int_{\mathbb{R}^d} |\nabla w_g|^2 \rho_\star dx \leq \|\Pi h\|^2 \quad (7.25)$$

where

$$\|\Pi h\|^2 = \int_{\mathbb{R}^d} |u_h|^2 \rho_\star dx + \int_{\mathbb{R}^d} |\nabla \psi_h|^2 dx.$$

Using  $|\nabla u_g|^2 = |\nabla w_g - \nabla \psi_g|^2 \leq 2(|\nabla w_g|^2 + |\nabla \psi_g|^2)$ , we deduce from (7.25) that

$$\int_{\mathbb{R}^d} |\nabla u_g|^2 \rho_\star dx \leq 2 \int_{\mathbb{R}^d} |\nabla w_g|^2 \rho_\star dx + 2 \int_{\mathbb{R}^d} |\nabla \psi_g|^2 \rho_\star dx \leq \mathcal{K} \|\Pi h\|^2 \quad (7.26)$$

with  $\mathcal{K} = 1 + 2 \|\rho_\star\|_{L^\infty(\mathbb{R}^d, dx)}$ .

*Step 3: Weighted Poincaré inequalities and weighted  $H^1$ -type estimates.* The solution  $u_g$  of (7.23) has zero average according to (7.24). We deduce from Corollary 7.2 that

$$\int_{\mathbb{R}^d} |\nabla u_g|^2 \rho_\star dx \geq \mathcal{C} \int_{\mathbb{R}^d} |u_g|^2 |\nabla W_\star|^2 \rho_\star dx,$$

from which we get that

$$X_1^2 := \int_{\mathbb{R}^d} |u_g|^2 |\nabla W_\star|^2 \rho_\star dx \leq \frac{\mathcal{K}}{\mathcal{C}} \|\Pi h\|^2. \quad (7.27)$$

Next, we look for a similar estimate for  $\int_{\mathbb{R}^d} |\psi_g|^2 |\nabla W_\star|^2 \rho_\star dx$ . The potential  $\psi_g$  has generically a non-zero average  $\bar{\psi}_g := \frac{1}{M} \int_{\mathbb{R}^d} \psi_g \rho_\star dx$  which can be estimated by

$$\begin{aligned} M^2 |\bar{\psi}_g|^2 &= \left( \int_{\mathbb{R}^d} \psi_g \rho_\star dx \right)^2 = \left( \int_{\mathbb{R}^d} \psi_g (-\Delta \phi_\star) dx \right)^2 = \left( \int_{\mathbb{R}^d} (-\Delta \psi_g) \phi_\star dx \right)^2 \\ &= \left( \int_{\mathbb{R}^d} u_g \phi_\star \rho_\star dx \right)^2 \leq \int_{\mathbb{R}^d} |\phi_\star|^2 \rho_\star dx \int_{\mathbb{R}^d} |u_g|^2 \rho_\star dx \leq \kappa_1 \|\Pi h\|^2 \end{aligned}$$

with  $\kappa_1 := \int_{\mathbb{R}^d} |\phi_\star|^2 \rho_\star dx$ , using (7.25). Since  $\nabla \rho_\star = -\nabla W_\star \rho_\star$ , we also have

$$\int_{\mathbb{R}^d} \psi_g |\nabla W_\star|^2 \rho_\star dx = - \int_{\mathbb{R}^d} \psi_g \nabla W_\star \cdot \nabla \rho_\star dx = \int_{\mathbb{R}^d} (\psi_g \Delta W_\star + \nabla \psi_g \cdot \nabla W_\star) \rho_\star dx$$

and, using the Cauchy-Schwarz inequality,

$$\begin{aligned} \left( \int_{\mathbb{R}^d} \psi_g |\nabla W_\star|^2 \rho_\star dx \right)^2 &\leq \int_{\mathbb{R}^d} |\psi_g|^2 \rho_\star dx \int_{\mathbb{R}^d} (\Delta W_\star)^2 \rho_\star dx \\ &\quad + \int_{\mathbb{R}^d} |\nabla \psi_g|^2 dx \|\rho_\star\|_{L^\infty(\mathbb{R}^d, dx)} \int_{\mathbb{R}^d} |\nabla W_\star|^2 \rho_\star dx \end{aligned}$$

By Lemma 7.2 applied to  $\psi_g - \bar{\psi}_g$ ,

$$\mathcal{E}_\star \int_{\mathbb{R}^d} |\psi_g|^2 \rho_\star dx \leq \|\rho_\star\|_{L^\infty(\mathbb{R}^d, dx)} \int_{\mathbb{R}^d} |\nabla \psi_g|^2 dx + \mathcal{E}_\star |\bar{\psi}_g|^2,$$

and (7.25), we conclude that

$$\left( \int_{\mathbb{R}^d} \psi_g |\nabla W_\star|^2 \rho_\star dx \right)^2 \leq \kappa_2 \|\Pi h\|^2$$

where

$$\begin{aligned} \kappa_2 &:= \left( \frac{1}{\mathcal{E}_\star} \int_{\mathbb{R}^d} (\Delta W_\star)^2 \rho_\star dx + \int_{\mathbb{R}^d} |\nabla W_\star|^2 \rho_\star dx \right) \|\rho_\star\|_{L^\infty(\mathbb{R}^d, dx)} \\ &\quad + \int_{\mathbb{R}^d} \phi_\star^2 \rho_\star dx \int_{\mathbb{R}^d} (\Delta W_\star)^2 \rho_\star dx. \end{aligned}$$

By applying Corollary 7.2 to  $\psi_g - \bar{\psi}_g$ , we deduce from

$$\mathcal{E} \int_{\mathbb{R}^d} |\psi_g - \bar{\psi}_g|^2 |\nabla W_\star|^2 \rho_\star dx \leq \int_{\mathbb{R}^d} |\nabla \psi_g|^2 \rho_\star dx$$

that

$$\begin{aligned} \mathcal{E} \int_{\mathbb{R}^d} |\psi_g|^2 |\nabla W_\star|^2 \rho_\star dx &\leq \int_{\mathbb{R}^d} |\nabla \psi_g|^2 \rho_\star dx + 2 \mathcal{E} \bar{\psi}_g \int_{\mathbb{R}^d} \psi_g \rho_\star |\nabla W_\star|^2 \rho_\star dx \\ X_2^2 &:= \int_{\mathbb{R}^d} |\psi_g|^2 |\nabla W_\star|^2 \rho_\star dx \leq \left( \frac{\|\rho_\star\|_{L^\infty(\mathbb{R}^d, dx)}}{\mathcal{E}} + 2 \frac{\sqrt{\kappa_1 \kappa_2}}{M} \right) \|\Pi h\|^2. \quad (7.28) \end{aligned}$$

Now we use (7.27) and (7.28) to estimate the weighted  $H^1$ -type quantity

$$X^2 := \int_{\mathbb{R}^d} |\nabla u_g|^2 |\nabla W_\star|^2 \rho_\star dx.$$

Let us multiply (7.23) by  $u_g |\nabla W_\star|^2 \rho_\star$  and integrate by parts in order to obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} |u_g|^2 |\nabla W_\star|^2 \rho_\star dx + \int_{\mathbb{R}^d} |\nabla u_g|^2 |\nabla W_\star|^2 \rho_\star dx \\ & + \int_{\mathbb{R}^d} (\nabla u_g \cdot \nabla \psi_g) |\nabla W_\star|^2 \rho_\star dx + \int_{\mathbb{R}^d} u_g \nabla (|\nabla W_\star|^2) (\nabla u_g + \nabla \psi_g) \rho_\star dx \\ & = \int_{\mathbb{R}^d} u_h u_g |\nabla W_\star|^2 \rho_\star dx. \end{aligned}$$

Using Lemma 7.4 and Lemma 7.5, we obtain that

$$\left| \int_{\mathbb{R}^d} u_g \nabla (|\nabla W_\star|^2) \nabla u_g \rho_\star dx \right| \leq \Lambda_\circ \int_{\mathbb{R}^d} |u_g| |\nabla W_\star|^2 |\nabla u_g| \rho_\star dx \leq \Lambda_\circ X_1 X$$

and

$$\begin{aligned} & \int_{\mathbb{R}^d} (\nabla u_g \cdot \nabla \psi_g) |\nabla W_\star|^2 \rho_\star dx \leq \kappa_3 X \|\Pi h\| \\ & \int_{\mathbb{R}^d} u_g \nabla (|\nabla W_\star|^2) \nabla \psi_g \rho_\star dx \leq \kappa_4 X_1 \|\Pi h\| \end{aligned}$$

with  $\kappa_3 := \|\nabla W_\star\|^2 \rho_\star\|_{L^\infty(\mathbb{R}^d, dx)}^{1/2}$  and  $\kappa_4 := \|\nabla (|\nabla W_\star|^2)\|^2 \rho_\star\|_{L^\infty(\mathbb{R}^d, dx)}^{1/2}$ , because we know from (7.25) that  $\int_{\mathbb{R}^d} |\nabla \psi_g|^2 dx \leq \|\Pi h\|^2$ . Using Corollary 7.3,

$$\left( \int_{\mathbb{R}^d} u_h u_g |\nabla W_\star|^2 \rho_\star dx \right)^2 \leq \int_{\mathbb{R}^d} |u_h|^2 \rho_\star dx \int_{\mathbb{R}^d} |u_g|^2 |\nabla W_\star|^4 \rho_\star dx \leq \|\Pi h\|^2 \frac{X^2}{\mathcal{C}_\circ}$$

Summarizing, we have shown that

$$X_1^2 + X^2 - \kappa_3 X \|\Pi h\| - \Lambda_\circ X_1 X - \kappa_4 X_1 \|\Pi h\| \leq X \frac{\|\Pi h\|}{\sqrt{\mathcal{C}_\circ}}.$$

Since  $X_1^2$  and  $X_2^2$  are bounded by  $\|\Pi h\|^2$ , we conclude that

$$X^2 = \int_{\mathbb{R}^d} |\nabla u_g|^2 |\nabla W_\star|^2 \rho_\star dx \leq \kappa \|\Pi h\|^2 \quad (7.29)$$

for some  $\kappa > 0$ , which has an explicit form in terms quantities involving  $\rho_\star$  and its derivatives, as well as all constants in the inequalities of Sections 7.3.3 and 7.3.4.

*Step 4: Second order estimates.* After multiplying (7.23) by  $\nabla \cdot (\rho_\star \nabla w_g)$ , we have

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{1}{\rho_\star} |\nabla \cdot (\rho_\star \nabla w_g)|^2 dx &= \int_{\mathbb{R}^d} (u_h - u_g) \nabla \cdot (\rho_\star \nabla w_g) dx \\ &= \int_{\mathbb{R}^d} u_h \sqrt{\rho_\star} \frac{1}{\sqrt{\rho_\star}} \nabla \cdot (\rho_\star \nabla w_g) dx + \int_{\mathbb{R}^d} \nabla u_g \cdot \nabla w_g \rho_\star dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^d} \left( |u_h|^2 \rho_\star + \frac{1}{\rho_\star} |\nabla \cdot (\rho_\star \nabla w_g)|^2 \right) dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla u_g|^2 + |\nabla w_g|^2) \rho_\star dx \end{aligned}$$

and after using (7.25) and (7.26), we obtain that

$$\int_{\mathbb{R}^d} \frac{1}{\rho_\star} |\nabla \cdot (\rho_\star \nabla w_g)|^2 dx \leq \left( \mathcal{K} + \frac{3}{2} \right) \|\Pi h\|^2. \quad (7.30)$$

Let  $Y = \left( \int_{\mathbb{R}^d} (\nabla w_g \cdot \nabla W_\star)^2 \rho_\star dx \right)^{1/2}$ . After multiplying (7.23) by  $(\nabla w_g \cdot \nabla W_\star) \rho_\star$ , we have that

$$Y^2 - \int_{\mathbb{R}^d} \Delta w_g (\nabla w_g \cdot \nabla W_\star) \rho_\star dx = \int_{\mathbb{R}^d} (u_h - u_g) (\nabla w_g \cdot \nabla W_\star) \rho_\star dx.$$

Using the Cauchy-Schwarz inequality, we know that the right-hand side can be estimated by  $Y \left( \int_{\mathbb{R}^d} |u_g|^2 \rho_\star dx \right)^{1/2} + Y \left( \int_{\mathbb{R}^d} |u_h|^2 \rho_\star dx \right)^{1/2} \leq 2Y \|\Pi h\|$  according to (7.25) and obtain that

$$Y^2 - 2Y \|\Pi h\| \leq \int_{\mathbb{R}^d} \Delta w_g (\nabla w_g \cdot \nabla W_\star) \rho_\star dx.$$

Let us notice that

$$\begin{aligned} \int_{\mathbb{R}^d} \Delta w_g (\nabla w_g \cdot \nabla W_\star) \rho_\star dx &= - \int_{\mathbb{R}^d} \Delta w_g \nabla w_g \cdot \nabla \rho_\star dx \\ &= \int_{\mathbb{R}^d} \left( \text{Hess}(\rho_\star) - \frac{1}{2} \Delta \rho_\star \text{Id} \right) : \nabla w_g \otimes \nabla w_g dx. \end{aligned}$$

As a consequence, by Lemma 7.3 and (7.25), we arrive at

$$Y^2 - 2Y \|\Pi h\| \leq \frac{\Lambda_\star}{2} \int_{\mathbb{R}^d} |\nabla w_g|^2 |\nabla W_\star|^2 \rho_\star dx = \frac{\Lambda_\star}{2} X^2$$

where  $X^2$  is the quantity that has been estimated in Step 4. Altogether, after taking (7.29) into account and with  $\lambda = \kappa \Lambda_\star / 2$ , this proves that

$$\int_{\mathbb{R}^d} (\nabla w_g \cdot \nabla W_\star)^2 \rho_\star dx \leq \left( \sqrt{1 + \lambda} - 1 \right)^2 \|\Pi h\|^2. \quad (7.31)$$

*Step 5: Conclusion of the proof.* We read from Lemma 7.6, (7.21) and (7.30)-(7.31) that

$$\|(\text{AT}(1-\Pi))^* h\|^2 \leq 3 \int_{\mathbb{R}^d} |\text{Hess}(w_g)|^2 \rho_\star dx \leq 3 \left( 3 \left( \mathcal{K} + \frac{3}{2} \right) + \frac{7}{2} \left( \sqrt{1 + \lambda} - 1 \right)^2 \right) \|\Pi h\|^2,$$

which concludes the proof of Lemma 7.14.  $\square$

#### 7.4.6 Proof of Theorem 7.1

The reader is invited to check that the potential  $V(x) = |x|^\alpha$  satisfies the assumptions (V1), (V2), (V3b), (V4) and (V6) if  $\alpha > 1$ . The result is then a straightforward consequence of Proposition 7.1 and Lemmas 7.8-7.14. A slightly more general result goes as follows.

**Theorem 7.3.** *Let us assume that  $d \geq 1$  and  $M > 0$ . If  $V$  satisfies the assumptions (V1), (V2), (V3b), (V4), (V5), (V6), (V7) and (V8), then there exist two constants  $\mathcal{C}$  and  $\lambda$  such that any solution  $h$  of (7.1) with an initial datum  $h_0$  of zero average such that  $\|h_0\|^2 < \infty$  satisfies*

$$\|h(t, \cdot, \cdot)\|^2 \leq \mathcal{C} \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0.$$

## 7.5 Uniform estimates in the diffusion limit

The hypocoercivity method of [82, 83] is directly inspired by the drift-diffusion limit, as it relies on micro/macro decomposition in which the relaxation in the velocity direction is given by the microscopic coercivity property (H1) while the relaxation in the position direction arises from the macroscopic coercivity property (H2) which governs the relaxation of the solution of the drift-diffusion equation obtained as a limit.

### 7.5.1 Formal macroscopic limit.

Let us start with a formal analysis in the framework of Section 7.2, when (7.7) is replaced by the scaled evolution equation

$$\varepsilon \frac{dF}{dt} + \mathbb{T}F = \frac{1}{\varepsilon} \mathbb{L}F \quad (7.32)$$

on the Hilbert space  $\mathcal{H}$ . We assume that a solution  $F_\varepsilon$  of (7.32) can be expanded as

$$F_\varepsilon = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \mathcal{O}(\varepsilon^3)$$

in the asymptotic regime corresponding to  $\varepsilon \rightarrow 0_+$  and, at formal level, that (7.32) can be solved order by order:

$$\begin{aligned} \varepsilon^{-1}: \quad & \mathbb{L}F_0 = 0, \\ \varepsilon^0: \quad & \mathbb{T}F_0 = \mathbb{L}F_1, \\ \varepsilon^1: \quad & \frac{dF_0}{dt} + \mathbb{T}F_1 = \mathbb{L}F_2. \end{aligned}$$

The first equation reads as  $F_0 = \Pi F_0$ , that is,  $F_0$  is in the kernel of  $\mathbb{L}$ . Assume for simplicity that  $\mathbb{L}(\mathbb{T}\Pi) = -\mathbb{L}$ , so that the second equation is simply solved by  $F_1 = -(\mathbb{T}\Pi)F_0$ . Let us consider the projection on the kernel of the  $\mathcal{O}(\varepsilon^1)$  equation:

$$\frac{d}{dt}(\Pi F_0) - \Pi \mathbb{T}(\mathbb{T}\Pi)F_0 = \Pi \mathbb{L}F_2 = 0.$$

If we denote by  $u$  the quantity  $F_0 = \Pi F_0$  and use (H3), then  $-\Pi \mathbb{T}(\mathbb{T}\Pi) = (\mathbb{T}\Pi)^*(\mathbb{T}\Pi)$  and the equation becomes

$$\partial_t u + (\mathbb{T}\Pi)^*(\mathbb{T}\Pi)u = 0,$$

which is our *drift-diffusion* limit equation. Notice that if  $u$  solves this equation, then

$$\frac{d}{dt} \|u\|^2 = -2 \|(\mathbb{T}\Pi) u\|^2 \leq -2 \lambda_M \|u\|^2$$

according to (H2). This program applies in the case of the scaled evolution equation (7.3). Let us give a few details.

Let us assume that a solution  $h_\varepsilon$  of (7.3) can be expanded as  $h_\varepsilon = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \mathcal{O}(\varepsilon^3)$ , in the asymptotic regime as  $\varepsilon \rightarrow 0_+$ . Solving (7.3) order by order in  $\varepsilon$ , we find the equations

$$\begin{aligned} \varepsilon^{-2}: \quad & \Delta_v h_0 - v \cdot \nabla_v h_0 = 0, \\ \varepsilon^0: \quad & v \cdot \nabla_x h_0 - \nabla_x W_\star \cdot \nabla_v h_0 + v \cdot \nabla_x \psi_{h_0} = \Delta_v h_1 - v \cdot \nabla_v h_1, \\ \varepsilon^1: \quad & \partial_t h_0 + v \cdot \nabla_x h_1 - \nabla_x W_\star \cdot \nabla_v h_1 = \Delta_v h_2 - v \cdot \nabla_v h_2. \end{aligned}$$

Let us define  $u = \Pi h_0$ ,  $\psi = \psi_{h_0}$  such that  $-\Delta \psi = u \rho_\star$ ,  $w = u + \psi$  and observe that the first two equations simply mean

$$u = h_0, \quad v \cdot \nabla_x w = \Delta_v h_1 - v \cdot \nabla_v h_1,$$

from which we deduce that  $h_1 = -v \cdot \nabla_x w$ . After projecting with  $\Pi$ , the third equation is

$$\partial_t u - \Delta w + \nabla_x W_\star \cdot \nabla u = 0,$$

using  $\int_{\mathbb{R}^d} v \otimes v \mathcal{M}(v) dv = \text{Id}$ . If we define  $\rho = u \rho_\star$ , we have formally obtained that it solves

$$\partial_t \rho = \Delta \rho + \nabla \cdot \left( \rho (\nabla_x V + \nabla_x \phi_\star) \right) + \nabla \cdot (\rho_\star \nabla \psi), \quad -\Delta \psi = \rho.$$

At this point, we can notice that the solution  $\rho$  converges to  $\rho_\star$  according to the results of, e.g., [?] at an exponential rate which is independent of  $\varepsilon$ .

### 7.5.2 Hypocoercivity

Let us adapt the computations of Section 7.2 to the case  $\varepsilon < 1$  as in [45]. If  $F$  solves (7.32), then

$$-\varepsilon \frac{d}{dt} H_\delta[F] = D_{\delta, \varepsilon}[F],$$

$$D_{\delta, \varepsilon}[F] := -\frac{1}{\varepsilon} \langle LF, F \rangle + \delta \langle \mathbb{A}\mathbb{T}\Pi F, F \rangle - \delta \langle \mathbb{T}\mathbb{A}F, F \rangle + \delta \langle \mathbb{A}\mathbb{T}(\text{Id} - \Pi)F, F \rangle - \frac{\delta}{\varepsilon} \langle \mathbb{A}LF, F \rangle.$$

The estimates are therefore exactly the same as in Proposition 7.1, up to the replacement of  $\lambda_m$  by  $\lambda_m/\varepsilon$  and  $C_M$  by  $C_M/\varepsilon$ . Hence, for  $\varepsilon > 0$  small enough, we have that

$$\delta(\varepsilon) := \min \left\{ 2, \frac{\lambda_m}{\varepsilon}, \varepsilon \lambda_\star(\varepsilon) \right\} = \frac{4 \lambda_m \lambda_M \varepsilon}{4 \lambda_M \varepsilon^2 + C_M^2 (1 + \lambda_M)}.$$



We may notice that  $\lim_{\varepsilon \rightarrow 0_+} \frac{\delta(\varepsilon)}{\varepsilon} = 2\zeta$  with

$$\zeta := \frac{2\lambda_m \lambda_M}{C_M^2 (1 + \lambda_M)}$$

and, for  $\varepsilon > 0$  small enough,

$$\frac{2 - \zeta \varepsilon}{4} \|F\|^2 \leq H_{\zeta \varepsilon}[F] \leq \frac{2 + \zeta \varepsilon}{4} \|F\|^2 \quad \forall F \in \mathcal{H}.$$

By revisiting the proof of Proposition 7.1, we find that with  $\delta = \zeta \varepsilon$  and  $\lambda = \eta \varepsilon$  with

$$\eta := \frac{\lambda_m \lambda_M^2}{C_M^2 (1 + \lambda_M)^2},$$

the quadratic form

$$(X, Y) \mapsto \left( \frac{\lambda_m}{\varepsilon} - \delta \right) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta \frac{C_M}{\varepsilon} X Y - \frac{\lambda}{2} (X^2 + Y^2) - \frac{\lambda}{2} \delta X Y$$

is nonnegative quadratic form for  $\varepsilon > 0$  small enough. In the regime as  $\varepsilon \rightarrow 0_+$ , the result of Proposition 7.1 can be adapted as follows.

**Corollary 7.4.** *Assume that (H1)–(H4) hold and take  $\zeta$  as above. Then for  $\varepsilon > 0$  small enough,*

$$\eta \varepsilon H_{\zeta \varepsilon}[F] \leq D_{\zeta \varepsilon, \varepsilon}[F] \quad \forall F \in \mathcal{H}.$$

*Proof.* The range for which the quadratic form is negative is given by the condition

$$\lambda_m^2 K^4 \varepsilon^4 + K C_M^3 (4K \lambda_m + 3C_M(K+4)) \varepsilon^2 - 2C_M^6 < 0.$$

Further details are left to the reader.  $\square$

As an easy consequence, if  $F_\varepsilon$  solves (7.32), we have that

$$H_{\zeta \varepsilon}[F(t, \cdot)] \leq H_{\zeta \varepsilon}[F(0, \cdot)] e^{-\eta t} \quad \forall t \geq 0.$$

*Proof of Theorem 7.2.* With the abstract result on (7.32) applied to (7.3), the estimate (7.10) applies with  $\delta = \zeta \varepsilon$ . Hence the conclusion holds with  $\lambda = \eta$  and  $\mathcal{C}$  which can be chosen arbitrarily close to 4 as  $\varepsilon \rightarrow 0_+$ .  $\square$

## 7.6 The nonlinear system in dimension $d = 1$

With the notation (7.20), we can rewrite the Vlasov-Poisson-Fokker-Planck system (VPFP) as

$$\partial_t h + \mathbb{T}h = \mathbb{L}h + \mathbb{Q}[h], \quad -\Delta_x \psi_h = \int_{\mathbb{R}^d} h f_\star dv, \quad \text{with } \mathbb{Q}[h] := \nabla_x \psi_h \cdot \nabla_v h.$$

Here we assume that  $d = 1$  and prove Corollary 7.1. Using the representation (7.12) and  $\iint_{\mathbb{R} \times \mathbb{R}} h f_\star dx dv = 0$ , we know that

$$\psi'_h(x) = - \int_{-\infty}^x u_h \rho_\star dx \quad \forall x \in \mathbb{R}.$$

**Lemma 7.15.** *Assume  $V$  satisfies (V1) and (V2) and let  $\rho_\star \in L^1(\mathbb{R}^d)$  be the solution of (7.6) such that  $\int_{\mathbb{R}^d} \rho_\star dx = M$ . Let  $f = (1+h)f_\star \in L^1_+(\mathbb{R}^d \times \mathbb{R}^d)$  such that  $\iint_{\mathbb{R} \times \mathbb{R}} f \log(f/f_\star) dx dv < \infty$ . Under the assumption  $\iint_{\mathbb{R} \times \mathbb{R}} h f_\star dx dv = 0$ ,  $\psi'_h$  as defined above satisfies the estimate*

$$\|\psi'_h\|_{L^\infty(\mathbb{R})}^2 \leq 4M \iint_{\mathbb{R} \times \mathbb{R}} f \log\left(\frac{f}{f_\star}\right) dx dv.$$

Additionally, under the assumptions of Corollary 7.1, if  $h$  solves (VPFP), then

$$\lim_{t \rightarrow +\infty} \|\psi'_h(t, \cdot)\|_{L^\infty(\mathbb{R})} = 0.$$

*Proof.* We deduce from Jensen's inequality

$$\int_{\mathbb{R}} f \log\left(\frac{f}{\mathcal{M}}\right) dv \geq \rho_h \log \rho_h$$

that

$$\iint_{\mathbb{R} \times \mathbb{R}} f \log\left(\frac{f}{f_\star}\right) dx dv \geq \int_{\mathbb{R}} \rho_h \log\left(\frac{\rho_h}{\rho_\star}\right) dx = \int_{\mathbb{R}} (1+u_h) \log(1+u_h) \rho_\star dx$$

and get according to [60, 120, 141] from the *Csiszár-Kullback-Pinsker inequality* that

$$\int_{\mathbb{R}} (1+u_h) \log(1+u_h) \rho_\star dx \geq \frac{1}{4M} \left( \int_{\mathbb{R}} |u_h| \rho_\star dx \right)^2 \geq \frac{\|\psi'_h\|_{L^\infty(\mathbb{R})}^2}{4M}.$$

Concerning the evolution problem (VPFP), we recall that

$$\frac{d}{dt} \left( \iint_{\mathbb{R} \times \mathbb{R}} f \log\left(\frac{f}{f_\star}\right) dx dv + \frac{1}{2} \int_{\mathbb{R}} |\psi'_h|^2 dx \right) = - \iint_{\mathbb{R} \times \mathbb{R}} f \left| \nabla_v \log\left(\frac{f}{f_\star}\right) \right|^2 dx dv,$$

as noticed in [44], shows that  $\lim_{t \rightarrow +\infty} \iint_{\mathbb{R} \times \mathbb{R}} f(t, x, v) \log\left(\frac{f(t, x, v)}{f_\star(x, v)}\right) dx dv = 0$ , which concludes the proof of Lemma 7.15.  $\square$

*Proof of Corollary 7.1.* With the notations of Section 7.3.6 and the functional  $H_\delta$  defined as in the linear case by

$$H_\delta[h] := \frac{1}{2} \|h\|^2 + \delta \langle Ah, h \rangle$$

we obtain that

$$\begin{aligned} \frac{d}{dt} H_\delta[h] + \langle Lh, h \rangle - \delta \langle AT\Pi h, h \rangle + \delta \langle TA h, h \rangle - \delta \langle AT(\text{Id} - \Pi)h, h \rangle + \delta \langle ALh, h \rangle \\ = \langle Q[h], h \rangle + \delta \langle AQ[h], h \rangle + \delta \langle Q[h], Ah \rangle. \end{aligned}$$

Let us give an estimate of the three terms of the right hand side.

1) In order to estimate

$$\langle Q[h], h \rangle = \iint_{\mathbb{R} \times \mathbb{R}} (\psi'_h \partial_v h) h f_\star dx dv + \int_{\mathbb{R}} \psi'_h \rho_\star \left( \int_{\mathbb{R}} \partial_v h \mathcal{M} dv \right) \psi_h dx,$$

we notice that  $\iint_{\mathbb{R} \times \mathbb{R}} |\partial_v h|^2 f_\star dx dv = -\langle Lh, h \rangle$  and  $\left( \int_{\mathbb{R}} \partial_v h \mathcal{M} dv \right)^2 \leq \int_{\mathbb{R}} |\partial_v h|^2 \mathcal{M} dv$ . Simple Cauchy-Schwarz inequalities show that

$$|\langle Q[h], h \rangle| \leq \|\psi'_h\|_{L^\infty(\mathbb{R})} |\langle Lh, h \rangle|^{1/2} \left[ \|h\| + \left( \int_{\mathbb{R}} |\psi_h|^2 \rho_\star dx \right)^{1/2} \right].$$

Since

$$\int_{\mathbb{R}} \psi_h \rho_\star dx = \int_{\mathbb{R}} \psi_h (-\phi_\star)'' dx = \int_{\mathbb{R}} (-\psi_h)'' \phi_\star dx = \int_{\mathbb{R}} u_h \phi_\star \rho_\star dx$$

we deduce from Lemma 7.2 that

$$\begin{aligned} \int_{\mathbb{R}} |\psi_h|^2 \rho_\star dx &\leq \mathcal{E}_\star^{-1} \int_{\mathbb{R}} |\psi'_h|^2 \rho_\star dx + \left( \int_{\mathbb{R}} \psi_h \rho_\star dx \right)^2 \\ &\leq \frac{\|\rho_\star\|_{L^\infty(\mathbb{R})}}{\mathcal{E}_\star} \int_{\mathbb{R}} |\psi'_h|^2 dx + \int_{\mathbb{R}} |u_h|^2 \rho_\star dx \int_{\mathbb{R}} |\phi_\star|^2 \rho_\star dx \end{aligned}$$

and finally that

$$|\langle Q[h], h \rangle| \leq \kappa \|\psi'_h\|_{L^\infty(\mathbb{R})} |\langle Lh, h \rangle|^{1/2} \|\Pi h\|$$

with

$$\kappa = 1 + \max \left\{ \|\rho_\star\|_{L^\infty(\mathbb{R})} \mathcal{E}_\star^{-1}, \int_{\mathbb{R}} |\phi_\star|^2 \rho_\star dx \right\}.$$

2) Let us consider  $g = Ah = u_g$  given by

$$u_g - \frac{1}{\rho_\star} \nabla \cdot (\rho_\star \nabla w_g) = -\frac{1}{\rho_\star} \nabla \cdot j_h \quad \text{with} \quad j_h := \int_{\mathbb{R}^d} v h f_\star dv.$$

With  $\psi_g$  such that  $-\psi_g'' = u_g \rho_\star$ , we have to estimate

$$\langle Q[h], Ah \rangle = \iint_{\mathbb{R} \times \mathbb{R}} (\psi'_h \partial_v h) u_g f_\star dx dv + \int_{\mathbb{R}} \psi'_h \rho_\star \left( \int_{\mathbb{R}} \partial_v h \mathcal{M} dv \right) \psi_g dx.$$

Exactly as above, we have on the one hand that

$$\left| \iint_{\mathbb{R} \times \mathbb{R}} (\psi'_h \partial_v h) u_g f_\star dx dv \right| \leq \|\psi'_h\|_{L^\infty(\mathbb{R})} \|g\| \|h\| \leq \|\psi'_h\|_{L^\infty(\mathbb{R})} \|(\text{Id} - \Pi)h\| \|h\|$$

because  $\|Ah\| \leq \|(\text{Id} - \Pi)h\|$ , and on the other hand that

$$\begin{aligned} \int_{\mathbb{R}} |\psi_g|^2 \rho_\star dx &\leq \mathcal{E}_\star^{-1} \int_{\mathbb{R}} |\psi'_g|^2 \rho_\star dx + \left( \int_{\mathbb{R}} \psi_g \rho_\star dx \right)^2 \\ &\leq \frac{\|\rho_\star\|_{L^\infty(\mathbb{R})}}{\mathcal{E}_\star} \int_{\mathbb{R}} |\psi'_g|^2 dx + \int_{\mathbb{R}} |u_g|^2 \rho_\star dx \int_{\mathbb{R}} |\phi_\star|^2 \rho_\star dx \end{aligned}$$

by Lemma 7.2 again, from which we conclude that

$$|\langle Q[h], Ah \rangle| \leq \kappa \|\psi'_h\|_{L^\infty(\mathbb{R})} |\langle Lh, h \rangle|^{1/2} \|(\text{Id} - \Pi)h\|.$$

3) With  $g$  given in terms of  $h$  by (7.22),  $A^*h = \nu w'_g$  and we learn from (7.25) that  $\|A^*h\| \leq \|\Pi h\|$ . Hence

$$|\langle AQ[h], h \rangle| = |\langle Q[h], A^*h \rangle| \leq \kappa \|\psi'_h\|_{L^\infty(\mathbb{R})} |\langle Lh, h \rangle|^{1/2} \|\Pi h\|.$$

Summing up all these estimates, we obtain as in the proof of Proposition 7.1 that

$$\frac{d}{dt} H_\delta[h] \leq -\lambda H_\delta[h]$$

for the largest value of  $\lambda$  for which

$$(X, Y) \mapsto (\lambda_m - \delta) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta C_M X Y - \frac{\lambda}{2} (X^2 + Y^2) - \frac{\lambda}{2} \delta X Y - \epsilon X(X + 2Y)$$

is a nonnegative quadratic form, as a function of  $(X, Y)$ . Here  $X := \|(\text{Id} - \Pi)h\|$ ,  $Y := \|\Pi h\|$ , and

$$\epsilon := \kappa \|\psi'_h\|_{L^\infty(\mathbb{R})}$$

can be taken as small as we wish, if we assume that  $t > 0$  is large enough. This completes the proof of Corollary 7.1.  $\square$

Let us conclude this section by a couple of remarks.

(i) It is clear from the proof of Corollary 7.1 that the optimal rate is as close as desired of the optimal rate in the linearized problem (7.1) obtained in Theorem 5.1. Up to a change of the constant  $\mathcal{C}$ , we can actually establish that these rates are equal because we read from the above proof that  $\epsilon(t) = \mathcal{O}(e^{-\lambda t})$  and the result follows from a simple ODE argument. This is a standard observation in entropy methods, which has been used on many occasions: see for instance [31].

(ii) Corollary 7.1 is written for  $V(x) = |x|^\alpha$  but it is clear that it can be extended to the setting of Theorem 7.3. Similarly, the reader is invited to check that our estimates are compatible with the diffusion limit, as in Section 7.5.



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## RÉSUMÉ

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Cette thèse est consacrée à l'étude du comportement asymptotique dans le temps et de l'hypocoercivité des EDP d'évolution. Nous montrons que pour l'équation de Nernst-Planck et dans les cas spécial de modèle de flockage, il existe des taux de convergence exponentiels optimaux vers des solutions stationnaires pour un temps  $t$  long, et que ces taux sont déterminés par le trou spectral du problème linéarisé autour des solutions stables. De plus, pour le modèle de flockage, nous prouvons qu'il existe une valeur seuil qui commande la transition de phase et classe toutes les solutions stationnaires et les propriétés de stabilité linéaire. Ensuite, pour l'équation cinétique de Fokker-Planck, nous prouvons que, pour son entropie, le taux de décroissance exponentiel est plus rapide que le taux optimal jusqu'à une mesure nulle définie dans le temps. Et pour l'équation linéarisée de Vlasov-Poisson-Fokker-Planck avec un potentiel de confinement externe, nous étudions le comportement dans le temps long des solutions en utilisant des méthodes d'hypocoercivité et une notion de produit scalaire adaptée à la présence d'un couplage de Poisson.

## MOTS CLÉS

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Hypocoercivité; équation cinétique linéaire; intervalle spectral; transition de phase; comportement asymptotique; énergie gratuite; intervalle spectral; convergence vers équilibre; limite de diffusion.

## ABSTRACT

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This thesis is devoted to study the large time asymptotic behaviour and hypocoercivity of evolution PDEs. We prove that for Nernst-Planck equation and in several cases of flocking model, there exist optimal exponential rates of convergence to stationary solutions for large time, and the rates are determined by spectral gap of the linearized problem around the stable solutions. Moreover, for the flocking model, we prove that there exists a threshold value that drives phase transition and classify all stationary solutions and the linear stability properties. Then for kinetic Fokker-Planck equation, we prove that for its  $\phi$ -entropy, the exponential rate of decay is faster than the optimal rate up to a zero-measure set in time  $t$ . And for linearized Vlasov-Poisson-Fokker-Planck equation with an external potential of confinement, we study the large time behaviour of the solutions using hypocoercivity methods and a notion of scalar product adapted to the presence of a Poisson coupling.

## KEYWORDS

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Hypocoercivity; linear kinetic equation; spectral gap; phase transition; asymptotic behaviour; free energy; spectral gap; convergence to equilibrium; diffusion limit.