Constant mean curvature surfaces in euclidean and hyperbolic spaces

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Surfaces à courbure moyenne constante dans les espaces euclidien et hyperbolique

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Résumé

Les surfaces à courbure moyenne constante non-nulle apparaissent en physique comme solutions à certains problèmes d'interface entre deux milieux de pressions différentes. Elles sont décrites mathématiquement par des équations aux dérivées partielles et sont constructibles à partir de données holomorphes via une représentation similaire à celle de Weierstrass pour les surfaces minimales. On présente dans cette thèse deux résultats s'appuyant sur cette représentation, dite «méthode DPW». Le premier indique que les données donnant naissance à un bout Delaunay de type onduloïde induisent encore un anneau plongé après perturbation. Cette propriété sert notamment à démontrer que certaines surfaces construites par la méthode DPW sont plongées. Le second résultat est la construction, dans l'espace hyperbolique, de n-noïdes : surfaces plongées, de genre zéro, à courbure moyenne constante et munies de n bouts de type Delaunay.

Mots clés : Surfaces à courbure moyenne constante – Représentation de Weierstrass généralisée – Méthode DPW – Bouts Delaunay – Géométrie hyperbolique.

Abstract

Non-zero constant mean curvature surfaces are mathematical models for physical interface problems with non-zero pressure difference. They are described by partial differential equations and can be constructed from holomorphic data via a Weierstrass-type representation, called "the DPW method". In this thesis, we use the DPW method and prove two main results. The first one states that perturbations of the DPW data for Delaunay unduloidal ends generate embedded annuli. This can be used to prove the embeddedness of surfaces constructed via the DPW method. The second result is the construction of n-noids in Hyperbolic space: genus 0, embedded, constant mean curvature surfaces with n Delaunay ends.

Keywords: Constant Mean Curvature Surfaces – Generalized Weierstrass Representation – DPW Method – Delaunay Ends – Hyperbolic Geometry.

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Introduction

Children (and grown-ups) enjoy enclosing pressurised volumes of air inside soap spheres. Together with the shimmering colours and the poetic fragility of the floating interface, they are undoubtedly also amazed by the mean curvature function defined at each point of their soap bubbles and often exclaim 'it is constant!' Indeed, the physical property of soap films minimising surface tension can be translated into a mathematical property of surfaces having the same mean curvature at every point. If the mean curvature is null everywhere on a surface, mathematicians call it "minimal" and physicists can use it to model soap films without pressure constraints. Planes and catenoids are examples of minimal surfaces. If the mean curvature is constant but not zero, mathematicians tend to call the surface "CMC" (short for "constant mean curvature"), and physicists can use it to model soap films with pressure constraints. Spheres and cylinders are examples of CMC surfaces but most of surfaces are not CMC nor minimal. Mathematicians have been looking for new examples and found many since the 18th century. This thesis is a contribution to the construction of CMC surfaces.

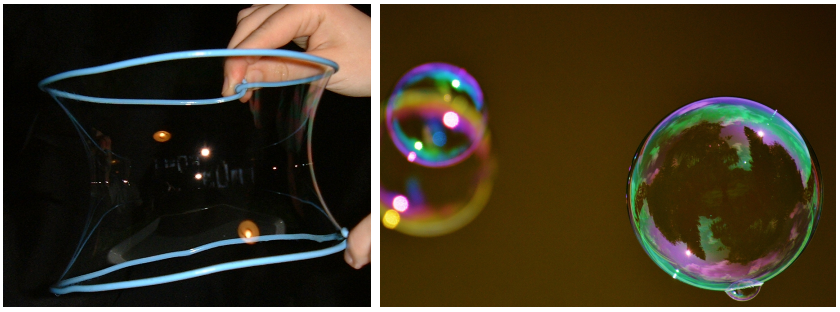

Figure 1 – *The catenoid (on the left) is a minimal surface, the sphere (on the right) is a CMC surface. The mean value between the greatest and the lowest curvatures is the same at every point of these surfaces.*

Constructing minimal surfaces is easier since the 1860s when Enneper and Weierstrass came up with a nice way of parametrising them [41]. The Enneper-Weierstrass parametrisation takes for input data a couple of meromorphic functions defining three differential

forms to integrate on a Riemann surface $\mathcal{S}$. Taking the real part of these complex integrals gives the three coordinate functions for a minimal immersion of $\mathcal{S}$ into Euclidean space. The nice point is that every minimal surface can locally be obtained this way. More than a hundred years later, Dorfmeister, Pedit and Wu came up with a Weierstrass-type representation for CMC surfaces, called the "DPW method" [7]. The DPW method takes for input data a matrix-valued holomorphic potential defining on $\mathcal{S}$ a linear Cauchy problem. Performing a matrix decomposition of the solution to the Cauchy problem induces a moving frame for a CMC immersion of $\mathcal{S}$. Again, the nice point is that every CMC surface can locally be obtained this way. This thesis uses the DPW method to construct CMC surfaces. Chapter 1 introduces the basic tools that are necessary to the exposition of the original DPW method, explained in more details in Chapter 2. Some classic extensions and choices of conventions are introduced in Chapter 3.

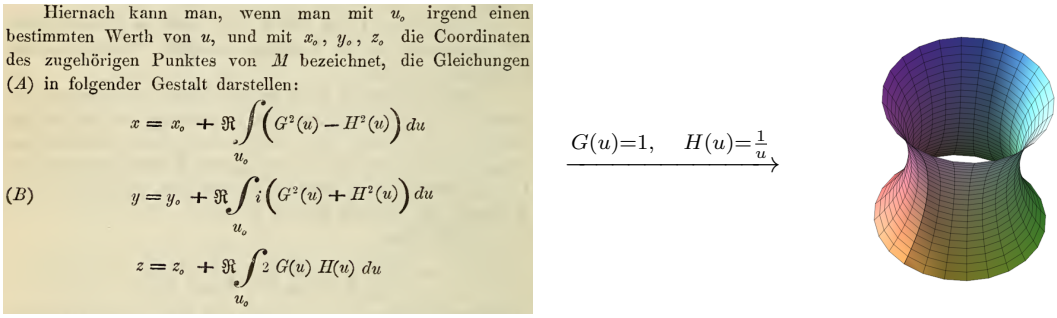

Figure 2 – *The Weierstrass parametrisation is an efficient tool for constructing minimal surfaces. Its equivalent for CMC surfaces is the DPW method.*

One key feature of children soap bubbles is that they are embedded CMC surfaces. Actually, Alexandrov proved in 1958 that any compact embedded CMC surface must be a round sphere [1]. Thus, complete and properly embedded CMC surfaces must have ends. The first examples of such surfaces (beside the cylinder) were constructed by Delaunay in 1841 [4]. The embedded Delaunay surfaces (called "unduloids") form a one-parameter family of surfaces of revolution starting from the cylinder and edging towards a chain of spheres. The family then goes on and consists in non-embedded Delaunay surfaces, called "nodoids" (see Figure 3). In 1989, Korevaar, Kusner and Solomon [24], stimulated by a previous result of Meeks [28], showed that any annular end of any complete properly embedded CMC surface must be of unduloidal type. The status of Delaunay surfaces is thus paramount in the study of embedded CMC surfaces, and Chapter 4 of this thesis is devoted to them. We consider the DPW data giving rise to an unduloid and perturb it. Under natural assumptions on the perturbation, we show that the perturbed resulting surface is still an embedding of a uniform punctured disk.

Two different techniques have been used to construct new CMC surfaces: a glu-

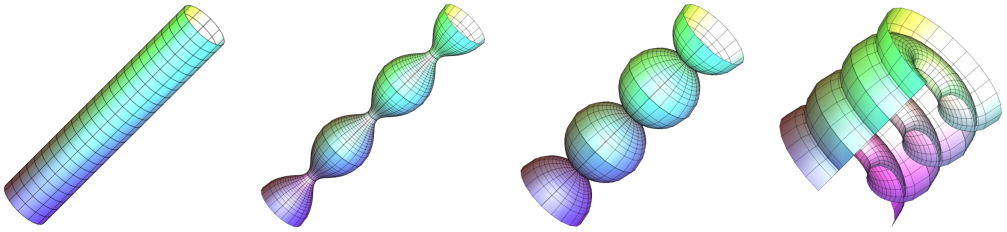

Figure 3 – *The Delaunay family: cylinder – unduloid – chain of spheres – nodoid.*

ing method and the Lawson correspondence. The gluing method was notably used by Kapouleas in 1990 and resulted in a breakthrough [16]. Starting from stacks and chains of spheres, he constructed infinitely many CMC surfaces with any number $k \geq 2$ of ends and any genus $g \in [0, \infty]$. All of these surfaces are close to a stack of spheres and some are embedded. With a slightly different gluing technique, Mazzeo and Pacard showed in 2001 how to construct CMC surfaces out of two types of building blocks: half-unduloids and minimal k-noids [26]. Another strategy has been to use the Lawson correspondence [25]: the definition of mean curvature applies not only in Euclidean space but also in Spherical and Hyperbolic spaces and it turns out that there exist some strong relationships between CMC or minimal surfaces in Euclidean space and CMC or minimal surfaces in Hyperbolic or Spherical space. For example, any simply connected CMC surface in Euclidean space possesses a minimal isometric cousin in Spherical space. This relationship allowed Grosse-Brauckmann, Kusner and Sullivan to construct and classify all the CMC triunduloids in Euclidean space [10] and all genus zero CMC k-unduloids with coplanar ends [11] by constructing their minimal cousins in the three-sphere.

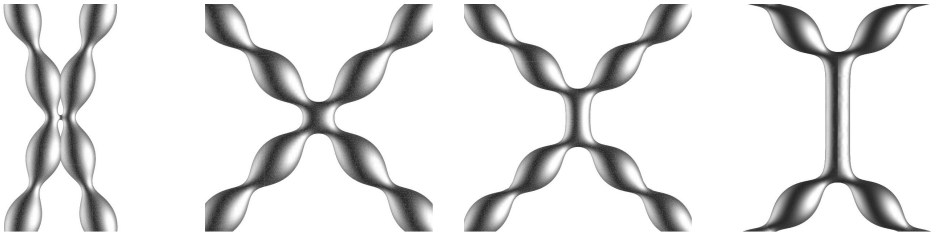

Figure 4 – *Alexandrov-embedded 4-noids with unduloidal ends (as seen in [11])*

Naturally, the DPW method gives a third way of constructing new CMC surfaces. Kilian, McIntosh and Schmitt used it in 2000 to produce new CMC cylinders [19], and had experimental evidences that perturbations of Delaunay ends in the DPW data still give Delaunay ends to the perturbed surface. In 2008, Kilian, Rossman and Schmmitt proved that it is indeed the case [20]. In 2007, Dorfmeister and Wu gave the general form of the DPW data for trinoids [8]. The resulting surfaces have been proved to have embedded ends, but until the construction of Traizet in 2017 [39], they were never proved to be entirely

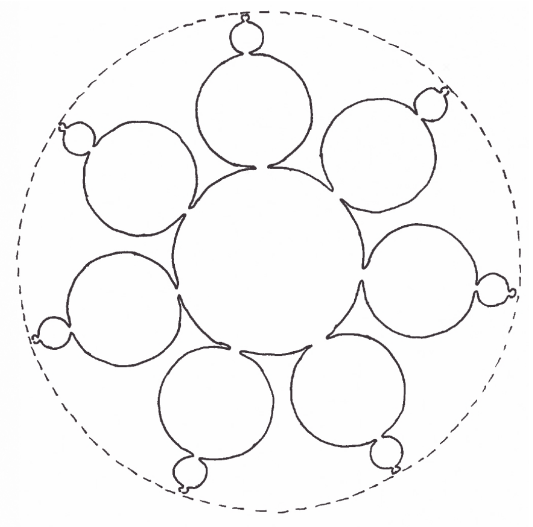

Figure 5 – *An embedded 7-noid with small necks in Hyperbolic space (ball model). This surface can be constructed using Chapter 5 of this thesis.*

embedded (or Alexandrov-embedded). Traizet perturbs the DPW data giving rise to a sphere with the data giving rise to half-unduloids and uses the Implicit Function Theorem to recover DPW data inducing a well-defined CMC k-unduloid. He then shows that the whole surface is embedded, using our contributions in Chapter 4 of the present work. This is the main achievement of this thesis: it allows, for the first time, to prove that non-trivial examples constructed via the DPW method are embedded. Extensions of the original DPW method have been made in order to construct CMC or minimal surfaces in various non-Euclidean spaces [13, 12]. We use the extension of [32] and the techniques of Chapter 4 and [39, 38] to construct in Chapter 5 CMC $H > 1$ complete, properly Alexandrov-embedded surfaces with genus zero and any number of ends in Hyperbolic three-space. These new examples constitute the second achievement of this thesis.

Part I

Introduction to the DPW method

Chapter 1

The method in a nutshell

In this chapter, we introduce the basic tools (Section 1.1) required to apply the DPW method (Section 1.2) to a simple example (Section 1.3).

1.1 Preliminaries

The DPW method constructs maps that take values in matrix loops $\mathbb{S}^1 \longrightarrow \mathrm{SU}(2)$. These loops are defined in Section 1.1.2 and are interpreted as rotations of Euclidean space. Hence the use of $\mathfrak{su}(2)$ as a model for $\mathbb{R}^3$, presented in Section 1.1.1.

1.1.1 $\mathfrak{su}(2)$ model of $\mathbb{R}^3$

Define $\Psi : \mathbb{R}^3 \to \mathfrak{su}(2)$ as

$$\Psi(x_1, x_2, x_3) := \frac{-i}{2} \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}.$$

Recall that $\mathbb{R}^3$ equipped with the cross product is a Lie algebra. Define for all $X \in \mathfrak{su}(2)$

$$\|X\|^2 := 4 \det X.$$

Thanks to Proposition 1, one can identify the three-dimensional Euclidean space with the Lie algebra $\mathfrak{su}(2)$.

Proposition 1. The map Ψ is an isometric homomorphism between Lie algebras.

Consider the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.1}$$

Then the canonical basis of $\mathbb{R}^3$ induces via Ψ the following direct orthonormal basis of $\mathfrak{su}(2)$:

$$\mathcal{B} := \left(\frac{-i}{2}\sigma_1, \frac{-i}{2}\sigma_2, \frac{-i}{2}\sigma_3 \right). \tag{1.2}$$

Under the identification of Proposition 1, the Lie group $\mathrm{SU}(2)$ acts on $\mathbb{R}^3$ as linear isometries via:

$$F \cdot X := FXF^{-1}, \qquad X \in \mathfrak{su}(2), \quad F \in \mathrm{SU}(2). \tag{1.3}$$

The action is transitive, hence the following proposition.

Proposition 2. For all $X \in \mathfrak{su}(2)$ and for all direct orthonormal bases (u, v, w) of the tangent space $T_X\mathfrak{su}(2) \simeq \mathfrak{su}(2)$, there exists $P \in \mathrm{SU}(2)$ such that

$$(u, v, w) = P \cdot \mathcal{B}$$

where $\mathcal{B}$ is as in (1.2).

1.1.2 Loop groups and algebras

In the DPW method, most of the objects dealt with are smooth maps defined for a spectral parameter $\lambda \in \mathbb{S}^1$. The standard reference is [29]. In order to take advantage of the diagonal+off-diagonal decomposition of $\mathfrak{gl}(2, \mathbb{C})$, it is convenient to define the following involution:

$$\begin{array}{rccc} \sigma & : & \mathfrak{gl}(2,\mathbb{C}) & \longrightarrow & \mathfrak{gl}(2,\mathbb{C}) \\ & & M & \longmapsto & \sigma_3 M \sigma_3 \end{array}$$

where σ_3 is defined in Equation (1.1). Formally,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies \sigma(M) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Definition 1. Let G be a Lie group and $\mathfrak{g}$ a Lie algebra. Let $\mathbb{S}^1 \subset \mathbb{C}$ denote the unit circle and let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disk.

- The group of smooth maps $\Phi : \mathbb{S}^1 \longrightarrow G$ such that for all $\lambda \in \mathbb{S}^1$, $\sigma(\Phi(\lambda)) = \Phi(-\lambda)$ is denoted ΛG^σ. Its elements are called **loops** and ΛG^σ is called a **loop group**.

- If G is complex, $\Lambda_{\geq 0}G^\sigma$ denotes the subgroup of ΛG^σ consisting of maps that holomorphically extend to $\mathbb{D}$.
- The algebra of smooth maps $\varphi : \mathbb{S}^1 \longrightarrow \mathfrak{g}$ such that for all $\lambda \in \mathbb{S}^1$, $\sigma(\varphi(\lambda)) = \varphi(-\lambda)$ is denoted $\Lambda\mathfrak{g}^\sigma$. It is called a **loop algebra**.
- If $\mathfrak{g}$ is complex, $\Lambda_{\geq 0}\mathfrak{g}^\sigma$ denotes the subalgebra of $\Lambda\mathfrak{g}^\sigma$ consisting of maps that holomorphically extend to $\mathbb{D}$.
- $\Lambda_+\mathrm{SL}(2,\mathbb{C})^\sigma$ denotes the subgroup of $\Lambda_{\geq 0}\mathrm{SL}(2,\mathbb{C})^\sigma$ consisting of maps B such that (after extension to $\mathbb{D}$) $B(0)$ is upper-triangular.
- $\Lambda^{\mathbb{R}}_+\mathrm{SL}(2,\mathbb{C})^\sigma$ denotes the subgroup of $\Lambda_+\mathrm{SL}(2,\mathbb{C})^\sigma$ consisting of loops such that $B(0)$ has positive elements on the diagonal.

Remark 1. The fact that $\sigma(B(\lambda)) = B(-\lambda)$ for $B \in \Lambda_+\mathrm{SL}(2,\mathbb{C})^\sigma$ implies that $B(0)$ is diagonal. However, we keep the above definition for reasons to be clear in Section 3.1.

An important step of the DPW method relies upon the following theorem (see [29] and [27]), which is a generalisation to loop groups of the QR decomposition.

Theorem 1. The multiplication map

$$\begin{array}{rccc} \times & : & \Lambda\mathrm{SU}(2)^\sigma \times \Lambda^{\mathbb{R}}_+\mathrm{SL}(2,\mathbb{C})^\sigma & \longrightarrow & \Lambda\mathrm{SL}(2,\mathbb{C})^\sigma \\ & & (F,B) & \longmapsto & FB \end{array}$$

is bijective. Its inverse map is called **Iwasawa decomposition** and is denoted:

$$\begin{array}{rccc} \mathrm{Iwa} & : & \Lambda\mathrm{SL}(2,\mathbb{C})^\sigma & \longrightarrow & \Lambda\mathrm{SU}(2)^\sigma \times \Lambda^{\mathbb{R}}_+\mathrm{SL}(2,\mathbb{C})^\sigma \\ & & \Phi & \longmapsto & (\mathrm{Uni}\,\Phi, \mathrm{Pos}\,\Phi). \end{array}$$

Finally, note that because of the following Proposition 3, the loops in Definition 1 are often called **twisted loops**.

Proposition 3. Let $X : \mathbb{S}^1 \longrightarrow \mathfrak{gl}(2,\mathbb{C})$ be a smooth map with Fourier series

$$X(\lambda) = \sum_{i\in\mathbb{Z}} X_i \lambda^i.$$

Then $X \in \Lambda\mathfrak{gl}(2,\mathbb{C})^\sigma$ if, and only if X_i is diagonal for even values of i and off-diagonal for odd values of i.

1.2 The recipe

We present the DPW method as a recipe to build CMC $H \neq 0$ surfaces in $\mathbb{R}^3$. The ingredients are introduced in Section 1.2.1 and the instructions are given in Section 1.2.2.

1.2.1 DPW data

The main ingredient is defined as follows.

Definition 2. Let Σ be a Riemann surface and let $\xi \in \Omega^{(1,0)}(\Sigma, \Lambda\mathfrak{sl}(2,\mathbb{C})^\sigma)$ be a holomorphic 1-form defined on Σ and taking values in $\Lambda\mathfrak{sl}(2,\mathbb{C})^\sigma$. For all $z \in \Sigma$, expanding $\xi(z)$ in its Fourier series

$$\xi(z,\lambda) = \sum_{i\in\mathbb{Z}} A_i(z)\lambda^i dz, \tag{1.4}$$

the 1-form ξ is called an **admissible DPW potential** if $A_i = 0$ for all $i < -1$.

Note the abuse of notation "$\xi(z,\lambda)$" instead of "$\xi(z)(\lambda)$". We will often identify the set of real analytic maps $f : \Sigma \times \mathbb{S}^1 \longrightarrow G$ satisfying $\sigma(f(\cdot,\lambda)) = f(\cdot,-\lambda)$ for all λ with the set of real analytic maps $\widetilde{f} : \Sigma \longrightarrow \Lambda G^\sigma$.

The ingredients of the recipe are

- A simply connected Riemann surface Σ. Note that under suitable conditions, the DPW method can also produce immersions of non-simply-connected surfaces. See Section 3.2 for more details.
- An admissible DPW potential ξ (as in Definition 2) defined on Σ.
- An initial point $z_0 \in \Sigma$.
- An initial condition $\phi_0 \in \Lambda\mathrm{SL}(2,\mathbb{C})^\sigma$.

Such a family $(\Sigma, \xi, z_0, \phi_0)$ is called **a set of DPW data**.

1.2.2 DPW method

Let $(\Sigma, \xi, z_0, \phi_0)$ be a set of DPW data. The DPW method consists of the following three steps.

1. Solve for $z \in \Sigma$ the Cauchy problem

$$\begin{cases} d\Phi = \Phi\xi, \\ \Phi(z_0) = \phi_0 \end{cases} \tag{1.5}$$

   in order to define the **holomorphic frame** $\Phi : \Sigma \longrightarrow \Lambda\mathrm{SL}(2,\mathbb{C})^\sigma$.

2. For all $z \in \Sigma$, Iwasawa decompose $\Phi(z)$ (see Theorem 1) into $F(z) := \mathrm{Uni}\,\Phi(z) \in \Lambda\mathrm{SU}(2)^\sigma$ and $B(z) := \mathrm{Pos}\,\Phi(z) \in \Lambda^{\mathbb{R}}_+\mathrm{SL}(2,\mathbb{C})^\sigma$.

3. Choose $H \neq 0$ and define $f_\lambda : \Sigma \longrightarrow \mathfrak{su}(2)$ via the **Sym-Bobenko formula**:

$$f_\lambda(z) := \frac{1}{2H}\left(-i\lambda\frac{\partial F(z,\lambda)}{\partial\lambda}F(z,\lambda)^{-1} + F(z,\lambda)\cdot\left(\frac{-i}{2}\sigma_3\right)\right) \tag{1.6}$$

where the dot denotes the action of Equation (1.3).

Then, for all $\lambda \in \mathbb{S}^1$, the map f_λ is a conformal real analytic CMC H immersion of Σ into $\mathbb{R}^3 \simeq \mathfrak{su}(2)$. (More details in Section 2, especially Theorem 3).

In order to simplify notations, let us set for any smooth map $F : \mathbb{S}^1 \longrightarrow \mathrm{SU}(2)$

$$\mathrm{Sym}_\lambda F := -i\lambda \frac{\partial F(\lambda)}{\partial \lambda} F(\lambda)^{-1} \tag{1.7}$$

and with the action of (1.3),

$$\mathrm{Nor}_\lambda F := F(\lambda) \cdot \left(\frac{-i}{2} \sigma_3 \right), \tag{1.8}$$

so that Equation (1.6) reads

$$f_\lambda(z) = \frac{1}{2H} \left(\mathrm{Sym}_\lambda F(z) + \mathrm{Nor}_\lambda F(z) \right).$$

1.3 A useful example

In this Section, we use the DPW method to construct a simple non-zero constant mean curvature surface of $\mathbb{R}^3$. It will be needed in Section 2.3.1 to produce rigid motions of $\mathbb{R}^3$ using the Sym-Bobenko formula.

Consider the following DPW data:

$$\Sigma = \mathbb{C}, \quad \xi(z, \lambda) = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} dz, \quad z_0 = 0, \quad \phi_0 = \mathrm{I}_2.$$

Solving the Cauchy problem (5.5) gives

$$\Phi(z, \lambda) = \begin{pmatrix} 1 & \lambda^{-1} z \\ 0 & 1 \end{pmatrix}.$$

Iwasawa decomposition is explicit:

$$\underbrace{\begin{pmatrix} 1 & \lambda^{-1} z \\ 0 & 1 \end{pmatrix}}_{\Phi(z) \in \Lambda \mathrm{SL}(2,\mathbb{C})^\sigma} = \underbrace{\frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & \lambda^{-1} z \\ -\lambda \bar{z} & 1 \end{pmatrix}}_{F(z) \in \Lambda \mathrm{SU}(2)^\sigma} \times \underbrace{\frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & 0 \\ \lambda \bar{z} & 1+|z|^2 \end{pmatrix}}_{B(z) \in \Lambda^{\mathbb{R}}_{+} \mathrm{SL}(2,\mathbb{C})^\sigma}.$$

One can thus compute:

$$\mathrm{Sym}_\lambda F(z) = \frac{i}{1+|z|^2}\begin{pmatrix} |z|^2 & \lambda^{-1}z \\ \lambda\bar{z} & -|z|^2 \end{pmatrix}, \tag{1.9}$$

$$\mathrm{Nor}_\lambda F(z) = \frac{-i}{2}\frac{1}{1+|z|^2}\begin{pmatrix} 1-|z|^2 & -2\lambda^{-1}z \\ -2\lambda\bar{z} & -1+|z|^2 \end{pmatrix}$$

and the induced immersion is given for any $H \neq 0$ by

$$f_\lambda = \frac{-i}{2H}\left[\frac{1}{1+|z|^2}\begin{pmatrix} 1-|z|^2 & -2\lambda^{-1}z \\ -2\lambda\bar{z} & |z|^2-1 \end{pmatrix} + \begin{pmatrix} \frac{-1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right].$$

For all $\lambda \in \mathbb{S}^1$, the identification $\mathfrak{su}(2) \simeq \mathbb{R}^3$ of Section 1.1.1 makes $f_\lambda(\mathbb{C})$ a sphere centered at $(0,0,\frac{-1}{2H})$ with radius $\frac{1}{H}$, which is indeed a CMC surface.

Remark 2. Note that $\mathrm{Sym}_\lambda F$ gives a sphere of radius 1 centered at $(0,0,-1)$. This fact will be used in Section 2.3.1.

Chapter 2

Why does it work?

Example of Section 1.3 seems to be a "twisted" way of constructing a sphere. This is because the link between the DPW data and the resulting immersion is not yet obvious. This chapter exhibit this link. Section 2.1 is a necessary digression to understand what is really constructed by the DPW method: solutions to a certain type of Lax pairs. Section 2.2 explains how the method constructs these solutions whereas Section 2.3 is devoted to the fact that all CMC surfaces can be obtained this way.

2.1 The integrable systems framework

This section recalls how the surfaces we consider can be encoded by a pair of non-linear partial differential equations (Section 2.1.1). We then introduce the moving frame method (Section 2.1.2), transforming these fundamental equations into a pair of matrix differential equations (Section 2.1.3). Finally we show how to retrieve a CMC immersion from a solution to these linear systems (Section 2.1.4).

2.1.1 Fundamental equations

In the DPW method, a whole one-parameter family of immersions is constructed, each of which corresponding to a constant mean curvature surface. These surfaces are linked by their fundamental forms. Let Σ be a simply connected domain of $\mathbb{C}$ with coordinate $z = x + iy$. Let f be a conformal real-analytic immersion of Σ into $\mathbb{R}^3$. Then the first and second fundamental forms of f can be written as follows:

$$I = 4e^{2u}|dz|^2, \qquad II = Qdz^2 + 4e^{2u}H|dz|^2 + \bar{Q}d\bar{z}^2 \tag{2.1}$$

where $u, H : \Sigma \longrightarrow \mathbb{R}$ and $Q : \Sigma \longrightarrow \mathbb{C}$ are real analytic functions. The first fundamental form is the metric of f, the function H is its mean curvature and the differential form Qdz^2 is called the **Hopf differential** of f. They satisfy the **Gauss and Codazzi equations**:

$$4u_{z\bar{z}} - |Q|^2 e^{-2u} + 4H^2 e^{2u} = 0, \quad Q_{\bar{z}} = 2e^{2u} H_z \tag{2.2}$$

where the index variables denote the partial derivatives with respect to these variables.

Conversely, let $u, H : \Sigma \longrightarrow \mathbb{R}$ and $Q : \Sigma \longrightarrow \mathbb{C}$ satisfy the Gauss and Codazzi equations (2.2). Then Bonnet's theorem states that there exists locally (and more broadly, on simply connected domains) an immersion $f : \Sigma \longrightarrow \mathbb{R}^3$ whose first and second fundamental forms read as in (2.1). This immersion is unique up to rigid motions of $\mathbb{R}^3$. Now suppose that the mean curvature H is constant on Σ. Reading back the Gauss and Codazzi equations, one can note that for all $t \in \mathbb{R}$, the triple $(u, H, e^{it}Q)$ satisfies the Gauss and Codazzi equations (2.2). Bonnet's theorem ensures that this new triple produces a one-parameter family of CMC H immersions.

Definition 3. Let $f : \Sigma \longrightarrow \mathbb{R}^3$ be a real analytic CMC H immersion with metric and Hopf differential as in (2.1). The **associated family** of f is the one parameter family $(f_\lambda)_{\lambda \in \mathbb{S}^1}$ of CMC H immersions with the same metric as f and Hopf differential $\lambda^{-2}Qdz^2$ for all $\lambda \in \mathbb{S}^1$.

The condition of f being conformal and real-analytic is no restriction because any CMC surface admits locally such a parametrisation. The Hopf differential of f_λ reads $\lambda^{-2}Qdz^2$ and not λQdz^2 for reasons that are specific to the DPW method, as we shall see in Section 2.2.

2.1.2 Moving frames

Let $f : \Sigma \longrightarrow \mathbb{R}^3$ be a real-analytic immersion. Suppose that f is conformal. Then for all $z \in \Sigma$, the tangent vectors f_x and f_y are orthogonal and the normal map of f is given by

$$N = \frac{f_x \times f_y}{\|f_x \times f_y\|}.$$

We denote by e_x, e_y the normalised tangent vectors of f, so that $\mathcal{F} := (e_x, e_y, N)$ is a direct orthonormal basis of $\mathbb{R}^3$.

Definition 4. The map $\mathcal{F} : \Sigma \longrightarrow \mathrm{SO}(3, \mathbb{R})$ defined above is called the **moving frame** of f. A **unitary frame** for f is a real-analytic map $P : \Sigma \longrightarrow \mathrm{SU}(2)$ such that $\mathcal{F} = P \cdot \mathcal{B}$ with $\mathcal{B}$ as in Equation (1.2).

As seen with Definition 3, CMC surfaces come in a one-parameter family. Hence the following definition.

Definition 5. Let $f : \Sigma \longrightarrow \mathbb{R}^3$ be a CMC H immersion with associated family (f_λ). An **extended unitary frame** for f is a real-analytic map $P : \Sigma \times \mathbb{S}^1 \longrightarrow \mathrm{SU}(2)$ such that for all $\lambda \in \mathbb{S}^1$, $P(\cdot, \lambda)$ is a unitary frame for f_λ.

2.1.3 Lax pairs

With the moving frames point of view, the Gauss and Codazzi equations (2.2) become a pair of linear differential systems, as stated by Proposition 4 below.

Definition 6. Let $n \in \mathbb{N}^*$, Ω be an open subset of $\mathbb{C}^2$ with coordinates (z, w) and $X : \Omega \longrightarrow \mathrm{SL}(n, \mathbb{C})$ holomorphic. Defining the holomorphic maps $U, V : \Omega \longrightarrow \mathfrak{sl}(n, \mathbb{C})$ as

$$\begin{cases} U := X^{-1} X_z, \\ V := X^{-1} X_w, \end{cases} \tag{2.3}$$

The system (2.3) is called the **Lax pair** for X. Symmetry of second derivatives for X are encoded into the **Maurer-Cartan equation**:

$$U_w - V_z = [U, V]. \tag{2.4}$$

An elementary proof of the following theorem can be found in [9] (Proposition 3.1.2). It states that the Maurer-Cartan equation is not only necessary, but also sufficient for the Lax pair to admit a solution.

Theorem 2. Let $U, V : \Omega \longrightarrow \mathfrak{sl}(n, \mathbb{C})$ holomorphic. The Lax pair (2.3) together with the initial condition $X(z_0, w_0) \in \mathrm{SL}(n, \mathbb{C})$ admits a solution if, and only if the Maurer-Cartan equation (2.4) holds.

Proposition 4. Let $f : \Sigma \longrightarrow \mathbb{R}^3$ be a CMC H immersion with associated family (f_λ) and $P : \Sigma \times \mathbb{S}^1 \longrightarrow \mathrm{SU}(2)$ be an extended unitary frame for f. Then P satisfies the Lax pair

$$P^{-1} dP = \frac{1}{2} \begin{pmatrix} -u_z & \lambda^{-2} Q e^{-u} \\ -2He^u & u_z \end{pmatrix} dz + \frac{1}{2} \begin{pmatrix} u_{\bar{z}} & 2He^u \\ -\lambda^2 \overline{Q} e^{-u} & -u_{\bar{z}} \end{pmatrix} d\bar{z} \tag{2.5}$$

with u, Q and H as in (2.1). Moreover, the Maurer-Cartan equation for this Lax pair is equivalent to the Gauss and Codazzi equations (2.2) for the associated family (f_λ).

Remark 3. The terminology "Lax pair" is used in Proposition 4 in the following sense:

$P(z) = X(z, \bar{z}, \lambda)$ where $X : \mathbb{C}^2 \times \mathbb{S}^1 \longrightarrow \mathrm{SU}(2)$ satisfies the Lax pair

$$X^{-1}dX = \frac{1}{2}\begin{pmatrix} -u_z & \lambda^{-2}Qe^{-u} \\ -2He^u & u_z \end{pmatrix} dz + \frac{1}{2}\begin{pmatrix} u_w & 2He^u \\ -\lambda^2\overline{Q}e^{-u} & -u_w \end{pmatrix} dw.$$

2.1.4 Retrieving the immersion

The extended frame P has been defined by differentiating the associated family (f_λ), and it is natural to think that in order to retrieve the immersions from the extend frame one has to integrate it. Fortunately, Sym [34] and Bobenko [2] have found a way to bypass this step:

Proposition 5. In the setting of Proposition 4 and if $H \neq 0$, up to a rigid motion of $\mathbb{R}^3$ and for all $\lambda \in \mathbb{S}^1$,

$$f_\lambda \simeq \frac{1}{2H}\left(\mathrm{Sym}_\lambda P - 2\mathrm{Nor}_\lambda P\right) \tag{2.6}$$

with Sym and Nor defined in Equations (1.7) and (1.8).

Corollary 1. Let $P : \Sigma \times \mathbb{S}^1 \longrightarrow \mathrm{SU}(2)$ real-analytic satisfying the Lax pair (2.5) for some $u : \Sigma \longrightarrow \mathbb{R}$, $Q : \Sigma \longrightarrow \mathbb{C}$ and $H \neq 0$. Then P is the extended frame of an associated family of CMC H immersions (f_λ).

Equation (2.6) is close to the Sym-Bobenko fomula (1.6) of the DPW method. It is not exactly the same because the map F defined in Section 1.2 does not satisfy the Lax pair (2.5). We will address this remark in Section 2.2.

2.2 Constructing solutions to Lax pairs

We are now ready to understand why the DPW method induces associated families of CMC immersions. For any admissible DPW potential ξ expanded in its Fourier series as in (1.4), write

$$A_{-1}(z) = \begin{pmatrix} 0 & \alpha(z) \\ \beta(z) & 0 \end{pmatrix}. \tag{2.7}$$

For any holomorphic frame $\Phi : \Sigma \longrightarrow \Lambda\mathrm{SL}(2, \mathbb{C})^\sigma$, write

$$\mathrm{Pos}\,\Phi(z)\,|_{\lambda=0} = \begin{pmatrix} b(z) & 0 \\ 0 & b(z)^{-1} \end{pmatrix}. \tag{2.8}$$

Theorem 3. Let $(\Sigma, \xi, z_0, \phi_0)$ be a set of DPW data and $H \neq 0$. Let Φ, F, B and f_λ defined via the DPW method applied to $(\Sigma, \xi, z_0, \phi_0)$ (see Section 1.2). Then for all $\lambda \in \mathbb{S}^1$,

f_λ is a CMC H real analytic conformal immersion with metric

$$I = \frac{4b^4|\alpha|^2}{H^2}|dz|^2 \tag{2.9}$$

and Hopf differential

$$II^{(2,0)} = \frac{-2\lambda^{-2}\alpha\beta}{H}dz^2, \tag{2.10}$$

with α, β and b as in (2.7) and (2.8). Its normal map reads $N_\lambda = -\mathrm{Nor}_\lambda F$ and its differential is given by

$$df_\lambda(z) = F(z,\lambda)\cdot\frac{ib(z)^2}{H}\begin{pmatrix}0 & \lambda^{-1}\alpha(z)dz\\ \lambda\overline{\alpha(z)}d\bar{z} & 0\end{pmatrix}. \tag{2.11}$$

Sketch of the proof: An elementary proof can be found in [9], whereas a more theoretical one can be found in [7]. The idea is to compute the Lax Pair associated to F. Writing $F = \Phi B^{-1}$ gives using (1.5)

$$F^{-1}dF = B\xi B^{-1} - (dB)B^{-1}. \tag{2.12}$$

But $F^{-1}dF \in \Lambda\mathfrak{su}(2)^\sigma$, B is holomorphic with respect to λ and ξ is an admissible potential (see Definition 2). Using these facts, one can write the series expansion of $F^{-1}dF$ as

$$F^{-1}dF = \eta_-\lambda^{-1} + \eta_0 + \eta_+\lambda \in \Omega(\Sigma, \Lambda\mathfrak{su}(2)^\sigma).$$

Compute η_- and η_0 using the series expansion of (2.12) to get the Lax pair

$$F^{-1}dF = \begin{pmatrix}b^{-1}b_z & \lambda^{-1}b^2\alpha\\ \lambda^{-1}b^{-2}\beta & -b^{-1}b_z\end{pmatrix}dz + \begin{pmatrix}-b^{-1}b_{\bar{z}} & -\lambda b^{-2}\bar{\beta}\\ -\lambda b^2\bar{\alpha} & b^{-1}b_{\bar{z}}\end{pmatrix}d\bar{z}. \tag{2.13}$$

Differentiating the Sym-Bobenko formula (1.6) with respect to z and $\bar{z}$ using this Lax pair gives the expected results.

□

Remark 4. Instead of differentiating the Sym-Bobenko formula, one could have noticed that setting

$$dw = \frac{-\alpha}{H}dz, \qquad u = 2\log b, \qquad Q = -2H\frac{\beta}{\alpha},$$

and defining

$$P(w,\lambda) := g(\lambda)^{-1}\cdot F(w,\lambda),$$

with

$$g(\lambda) := -i \begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda}} \\ \sqrt{\lambda} & 0 \end{pmatrix}, \tag{2.14}$$

the map P satisfies the Lax pair (2.5) for the coordinate w and is thus the extended frame of an associated family of CMC H immersions.

We will often say that the DPW method gives rise to a CMC immersion f, instead of a one-parameter family (f_λ). In these cases, it is assumed that the Sym-Bobenko formula (1.6) is evaluated at $\lambda = 1$.

2.3 Surjectivity

In order to explain why any CMC $H \neq 0$ surface can locally be obtained via the DPW method (section 2.3.2), we need to introduce in Section 2.3.1 two operations on the DPW data.

2.3.1 Gauging and dressing

Let $(\Sigma, \xi, z_0, \phi_0)$ be a set of DPW data giving rise to Φ, F and B via the DPW method and inducing an associated family (f_λ) of CMC H immersions. We introduce two changes in the data that result into simple geometric changes on the surface.

Definition 7. Let $G : \Sigma \longrightarrow \Lambda_+\mathrm{SL}(2,\mathbb{C})^\sigma$ holomorphic and set

$$\xi \cdot G := G^{-1}\xi G + G^{-1}dG. \tag{2.15}$$

Equation (2.15) defines a right-action called **gauging the potential** and the map

$$(\Sigma, \xi, z_0, \phi_0) \longmapsto (\Sigma, \xi \cdot G, z_0, \phi_0 G)$$

is called **gauging the data**.

Note that G is not necessary in $\Lambda_+^{\mathbb{R}}\mathrm{SL}(2,\mathbb{C})^\sigma$. Thus, using Remark 1, it has the form

$$G = \begin{pmatrix} \rho e^{i\theta} & 0 \\ 0 & \rho^{-1}e^{-i\theta} \end{pmatrix} + \mathcal{O}(\lambda) \quad (\rho, \theta : \Sigma \longrightarrow \mathbb{R}_+^*). \tag{2.16}$$

Proposition 6. Let $(\Sigma, \xi \cdot G, z_0, \phi_0 G)$ be a set of gauged data. Then the induced holo-

morphic frame is ΦG and the induced unitary frame is

$$F\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

(with θ defined in (2.16)). The induced CMC H immersions are unchanged: they equal f_λ for all $\lambda \in \mathbb{S}^1$.

Definition 8. Let $J \in \Lambda\mathrm{SL}(2,\mathbb{C})^\sigma$. The map

$$(\Sigma, \xi, z_0, \phi_0) \longmapsto (\Sigma, \xi, z_0, J\phi_0)$$

is called **dressing the data**. The **dressed holomorphic frame** is the frame $J\Phi$ induced by the dressed data.

Proposition 7. Let $(\Sigma, \xi, z_0, J\phi_0)$ be a set of dressed data. Then the induced CMC H immersions $\widehat{f_\lambda}$ have the same Hopf differential as f_λ. Moreover, if $J \in \Lambda\mathrm{SU}(2)^\sigma$, then for all $\lambda \in \mathbb{S}^1$, f_λ and $\widehat{f_\lambda}$ differ by a rigid motion of $\mathbb{R}^3$:

$$\widehat{f_\lambda} = J(\lambda) \cdot f_\lambda + \mathrm{Sym}_\lambda J. \tag{2.17}$$

In order to apply a rotation to f_λ, it then suffices to dress the data $(\Sigma, \xi, z_0, \phi_0)$ by a λ-independent factor that can be explicitly computed (see Lemma 3.3. in [18]). Translations can be explicitly computed using the example of the sphere in Section 1.3 and the following Proposition.

Proposition 8. Let $(\Sigma, \xi, z_0, \phi_0)$ be a set of DPW data. Let $Q : \Omega \longrightarrow \Lambda\mathrm{SU}(2)^\sigma$ and $p \in \Omega \cap \Sigma$. Define

$$J(\lambda) := Q(p,\lambda)Q(p,1)^{-1}.$$

Then the immersion $\widehat{f} = \widehat{f_1}$ induced by the dressed data $(\Sigma, \xi, z_0, J\phi_0)$ satisfies for all $z \in \Sigma$

$$\widehat{f}(z) = f(z) + \mathrm{Sym}_1 Q(p).$$

2.3.2 Constructing all CMC immersions

Given a mean curvature $H \neq 0$, the DPW method induces a map from the set of admissible DPW data to the set of CMC immersions in $\mathbb{R}^3$. This map is surjective: any CMC H immersion can be locally obtained this way.

Theorem 4. For any $f : \Sigma \longrightarrow \mathbb{R}^3$ conformal real analytic CMC $H \neq 0$ immersion, there exist DPW data $(\Sigma, \xi, z_0, \phi_0)$ inducing f via the DPW method.

Sketch of the proof: Without loss of generality, one can suppose by applying a homothety that $H = 1$. Define $P : \Sigma \times \mathbb{S}^1 \longrightarrow \mathrm{SU}(2)$ as the solution to the Lax pair (2.5) with initial condition $P(z_0) = \mathrm{I}_2$ and set $F := g \cdot P : \Sigma \longrightarrow \Lambda\mathrm{SU}(2)^\sigma$ with g as in Equation (2.14). Then F satisfies the Lax pair

$$F^{-1}dF = \frac{1}{2}\begin{pmatrix} u_z & -2\lambda^{-1}He^u \\ \lambda^{-1}Qe^{-u} & -u_z \end{pmatrix} dz + \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -\lambda\bar{Q}e^{-u} \\ 2\lambda He^u & u_{\bar{z}} \end{pmatrix} d\bar{z}. \tag{2.18}$$

Now solve for $B : \Sigma \longrightarrow \Lambda_+\mathrm{SL}(2,\mathbb{C})^\sigma$ the $\bar{\partial}$ problem:

$$(FB)_{\bar{z}} = 0,$$

so that $\Phi := FB$ is holomorphic on Σ and takes values in $\Lambda\mathrm{SL}(2,\mathbb{C})^\sigma$. Then $\xi := \Phi^{-1}d\Phi$ is a holomorphic 1-form on Σ and the series expansion of $F^{-1}dF$ shows that ξ is an admissible DPW potential. Let $\phi_0 := \Phi(z_0)$ and apply the DPW method to the DPW data $(\Sigma, \xi, z_0, \phi_0)$ to retrieve Φ. Iwasawa decomposition gives

$$\mathrm{Iwa}\,\Phi = (F\,\mathrm{Uni}\,B, \mathrm{Pos}\,B)$$

where

$$\mathrm{Uni}\,B = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta : \Sigma \longrightarrow \mathbb{R}.$$

Thus,

$$\mathrm{Sym} \circ \mathrm{Uni}\,\Phi = \mathrm{Sym}F \quad \text{and} \quad \mathrm{Nor} \circ \mathrm{Uni}\,\Phi = \mathrm{Nor}F.$$

Compute

$$\mathrm{Sym}g = \frac{i}{2}\sigma_3$$

to get

$$\begin{aligned} \mathrm{Sym}F &= \mathrm{Sym}(g \cdot P) \\ &= \mathrm{Sym}g + g \cdot \mathrm{Sym}P - (g \cdot P) \cdot \mathrm{Sym}g \\ &= \frac{i}{2}\sigma_3 + g \cdot \mathrm{Sym}P + \mathrm{Nor}F. \end{aligned}$$

Compute

$$g(\lambda)^{-1}\sigma_3 g(\lambda) = -\sigma_3 \qquad \forall \lambda \in \mathbb{S}^1$$

to get

$$\mathrm{Nor}F = -g \cdot \mathrm{Nor}P.$$

The induced immersion is then

$$\begin{aligned}\widehat{f_\lambda} &= \frac{1}{2H}\left(\mathrm{Sym}_\lambda F + \mathrm{Nor}_\lambda F\right)\\ &= g(\lambda)\cdot\left[\frac{1}{2H}\left(\mathrm{Sym}_\lambda P - 2\mathrm{Nor}_\lambda P\right)\right] + \frac{i}{4H}\sigma_3\end{aligned}$$

which gives at $\lambda = 1$, up to a rigid motion, the immersion f (Use Equation (2.6) to identify f_λ). Apply a well chosen isometry by dressing the data (see Section 2.3.1) to retrieve f.

□

Chapter 3

Further options

Now that the basic tools of the DPW method are introduced, one can adjust it in many ways, depending on what one wants to do with it. For example, untwisting the loops (Section 3.1) makes the Sym-Bobenko formula and the loop groups simpler to deal with, without changing the fundamental idea behind the method. One can also build surfaces with topology and assume that Σ is not simply connected, provided that an extra condition on the data is satisfied (Section 3.2). Finally, the Lawson correspondence between CMC surfaces in Euclidean and Hyperbolic spaces provides a DPW method for constructing CMC $H > 1$ surfaces in $\mathbb{H}^3$ (Section 3.3).

3.1 Untwisting the loops

Twistedness of loops is not essential to the DPW method, and some papers (such as [20, 21, 39, 38]) use loop groups that are not twisted. In their case, untwisting the loops simplify the proofs and formulae. For any group or algebra G, let ΛG denote the set of smooth maps $\mathbb{S}^1 \longrightarrow G$.

3.1.1 Definition

Let $M \in \Lambda\mathfrak{gl}(2,\mathbb{C})^\sigma$ be a twisted loop and define for all λ in the double cover of $\mathbb{S}^1$

$$\widetilde{M}(\lambda) := \begin{pmatrix} \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} M(\lambda) \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}.$$

Then $\widetilde{M}$ is well-defined on $\mathbb{S}^1$ and $\widetilde{M} \in \Lambda\mathfrak{gl}(2,\mathbb{C})$. Moreover, $\widetilde{M}$ is an even function of λ. One can thus define

$$\widehat{M}(\lambda) := \widetilde{M}(\sqrt{\lambda}).$$

Definition 9. Applying the map

$$\begin{array}{rccc} \kappa & : & \Lambda\mathfrak{gl}(2,\mathbb{C})^{\sigma} & \longrightarrow & \Lambda\mathfrak{gl}(2,\mathbb{C}) \\ & & M & \longmapsto & \widehat{M} \end{array}$$

defined above is called **untwisting the loop** M.

Formally,

$$M(\lambda) = \begin{pmatrix} a(\lambda^2) & \lambda b(\lambda^2) \\ \lambda^{-1}c(\lambda^2) & d(\lambda^2) \end{pmatrix} \implies \kappa(M)(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}.$$

The definition of untwisted loop groups and algebras is exactly the same as in Section 1 without the condition $\sigma(X(\lambda)) = X(-\lambda)$ (and Remark 1 should now make sense). We denote them by dropping the "σ" and write, for example, $\Lambda\mathfrak{sl}(2,\mathbb{C})$ for the untwisted version of $\Lambda\mathfrak{sl}(2,\mathbb{C})^{\sigma}$.

3.1.2 Untwisting the recipe

One can adapt the DPW method so that it works with untwisted loop groups. The important changes are in the definition of admissible potentials and in the Sym-Bobenko formula (which is simpler in the untwisted framework).

Definition 10. Let Σ be a Riemann surface and let $\xi \in \Omega^{(1,0)}(\Sigma, \Lambda\mathfrak{sl}(2,\mathbb{C}))$ be a holomorphic 1-form defined on Σ and taking values in $\Lambda\mathfrak{sl}(2,\mathbb{C})$. For all $z \in \Sigma$, expanding $\xi(z)$ in its Fourier series as in (1.4), the 1-form ξ is called an **admissible DPW potential** (in the untwisted setting) if $A_i = 0$ for all $i < -1$ and if A_{-1} is of the form

$$A_{-1}(z) = \begin{pmatrix} 0 & \alpha(z) \\ 0 & 0 \end{pmatrix}. \tag{3.1}$$

A **set of DPW data in the untwisted setting** is a family $(\Sigma, \xi, z_0, \phi_0)$ where ξ is an admissible potential in the untwisted setting and $\phi_0 \in \Lambda\mathrm{SL}(2,\mathbb{C})$.

The three steps of the DPW method in the untwisted setting are basically the same provided that one changes twisted loop groups into untwisted loop groups. First, solve for $\Phi(z) \in \Lambda\mathrm{SL}(2,\mathbb{C})$ the Cauchy problem (5.5). Then Iwasawa decompose Φ into (F, B) using the following untwisted Iwasawa decomposition.

Theorem 5. The multiplication map

$$\begin{array}{rccc} \times & : & \Lambda\mathrm{SU}(2) \times \Lambda^{\mathbb{R}}_{+}\mathrm{SL}(2,\mathbb{C}) & \longrightarrow & \Lambda\mathrm{SL}(2,\mathbb{C}) \\ & & (F, B) & \longmapsto & FB \end{array}$$

is bijective. Its inverse map is called **Iwasawa decomposition** and is denoted:

$$\begin{array}{rccc} \text{Iwa} & : & \Lambda\mathrm{SL}(2,\mathbb{C}) & \longrightarrow & \Lambda\mathrm{SU}(2)\times\Lambda^{\mathbb{R}}_{+}\mathrm{SL}(2,\mathbb{C}) \\ & & \Phi & \longmapsto & (\mathrm{Uni}\,\Phi, \mathrm{Pos}\,\Phi). \end{array}$$

Finally, for any $H \neq 0$, define the induced CMC H immersions f_λ via the Sym-Bobenko formula

$$f_\lambda := \frac{1}{H}\mathrm{Sym}_\lambda F. \tag{3.2}$$

Theorem 6. Let $(\Sigma, \xi, z_0, \phi_0)$ be a set of untwisted DPW data and $H \neq 0$. Let Φ, F, B and f_λ be defined via the DPW method applied to $(\Sigma, \xi, z_0, \phi_0)$. Then for all $\lambda \in \mathbb{S}^1$, f_λ is a CMC H real analytic conformal immersion with metric given by (2.9) and Hopf differential

$$II^{(2,0)} = \frac{-2\lambda^{-1}\alpha\beta}{H}dz^2,$$

where α is as in (3.1), β is the lower-left entry of A_0 and b is the upper-left entry of $B\,|_{\lambda=0}$. Its normal map reads $N_\lambda = -\mathrm{Nor}_\lambda F$ and its differential is as in (2.11).

Note that dressing and gauging with untwisted loops give the same results as in the twisted setting (see Section 2.3.1). In particular, Equation (2.17) describing the dressing action with unitary loops as isometries is the same in the untwisted setting.

As stated before, some authors work in the untwisted framework and the others in the twisted framework. Whenever the choice is not explicitly stated, a simple look at the loop groups definitions makes it clear. Moreover, one can translate one setting into another. Let $(\Sigma, \xi, z_0, \phi_0)$ be a set of admissible DPW data in the twisted setting, giving rise to the twisted maps Φ, F, B, and to the CMC $H \neq 0$ immersion $f = f_1$. Use Definition 9 to untwist the potential and the initial condition, giving rise to a set $(\Sigma, \widehat{\xi}, z_0, \widehat{\phi_0})$ of DPW untwisted data, inducing the maps $\widehat{\Phi}$, $\widehat{F}$, $\widehat{B}$ and the CMC H immersion $\widehat{f} = \widehat{f_1}$.

Proposition 9. With the above notations and κ as in Definition 9,

$$\widehat{\Phi} = \kappa(\Phi), \quad \widehat{F} = \kappa(F), \quad \widehat{B} = \kappa(B)$$

and

$$\widehat{f} = f - \frac{1}{2H}\left(\frac{-i}{2}\sigma_3\right).$$

A good way to test Proposition 9 is to apply it to the example of Section 1.3 in which untwisting the data does not change it, thus simplifying the computations.

3.2 Non-trivial fundamental group: the monodromy problem

Let $\xi : \Sigma \longrightarrow \Lambda\mathfrak{sl}(2,\mathbb{C})$ be an admissible DPW potential, $z_0 \in \Sigma$ and $\phi_0 \in \Lambda\mathrm{SL}(2,\mathbb{C})$. Suppose that Σ is not simply connected and let $\widetilde{\Sigma}$ be its universal cover with $\widetilde{z}_0 \in \widetilde{\Sigma}$ above z_0. One can use the DPW method on the data $(\widetilde{\Sigma}, \xi, \widetilde{z}_0, \phi_0)$ because $\widetilde{\Sigma}$ is simply connected: let Φ, F, B and f be the induced maps. The immersion f is a CMC immersion of $\widetilde{\Sigma}$ into $\mathbb{R}^3$, but nothing ensures that it descends to a well-defined immersion of Σ.

Definition 11. For any deck transformation $\tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$, the loop

$$\mathcal{M}_\tau(\Phi) := \Phi(\tau(\widetilde{z})) \times \Phi(\widetilde{z})^{-1} \in \Lambda\mathrm{SL}(2,\mathbb{C}) \tag{3.3}$$

does not depend on $\widetilde{z} \in \widetilde{\Sigma}$ (but only on the choice of $\widetilde{z}_0$ above z_0) and is called the **monodromy of Φ with respect to τ**.

Proposition 10. Suppose that

$$\forall \tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma), \qquad \begin{cases} \mathcal{M}_\tau(\Phi) & \in & \Lambda\mathrm{SU}(2), \\ \mathcal{M}_\tau(\Phi)(1) & = & \pm \mathrm{I}_2, \\ \frac{d\mathcal{M}_\tau(\Phi)}{d\lambda}(1) & = & 0. \end{cases} \tag{3.4}$$

Then the immersion $f : \widetilde{\Sigma} \longrightarrow \mathbb{R}^3$ induced by Φ descends to a well-defined immersion of Σ.

Equation (3.4) is called **the monodromy problem in $\mathbb{R}^3$**. Equation (3.3) shows that deck transformations act on the DPW data as dressing (see Section 2.3.1), and Proposition 10 states that if this dressing represents the identity isometry, then the immersion is well-defined on Σ.

3.3 Hyperbolic space

The DPW method can be used to construct CMC $H > 1$ immersions into Hyperbolic space. One first need to extend the definition of loops to annuli, and this is the chance to equip the groups with a simple Banach structure, as in Section 3.3.1. After introducing a suitable matrix model of $\mathbb{H}^3$ in Section 3.3.2, we briefly describe the method in the untwisted framework (Section 3.3.3). The ideas are the same as in Euclidean space, which is why we do not give proofs.

3.3.1 Extending the loops

In order to define differentiable maps between loop groups and algebras, one might want to equip them with a Banach structure. Let $|\cdot|$ be a sub-multiplicative norm on $\mathfrak{gl}(2,\mathbb{C})$. Let $R>1$ and define for all $f\in\Lambda\mathfrak{gl}(2,\mathbb{C})$

$$\|f\|_R := \sum_{i\in\mathbb{Z}} |f_i| R^{|i|} \tag{3.5}$$

where $\sum f_i\lambda^i$ is the Fourier expansion of f. For any loop group or algebra ΛG defined in Section 5.1.2, define

$$\Lambda G_R := \{f\in\Lambda G \mid \|f\|_R<\infty\}.$$

Then $\Lambda\mathfrak{gl}(2,\mathbb{C})_R$ is a Banach algebra. Moreover, $\Lambda\mathrm{SL}(2,\mathbb{C})_R$, $\Lambda\mathrm{SU}(2)_R$ and $\Lambda^{\mathbb{R}}_+\mathrm{SL}(2,\mathbb{C})_R$ are Banach manifolds and Iwasawa decomposition is a smooth diffeomorphism:

Theorem 7. [38] For all $R>1$, the multiplication map $\Lambda\mathrm{SU}(2)_R\times\Lambda^{\mathbb{R}}_+\mathrm{SL}(2,\mathbb{C})_R\longrightarrow\Lambda\mathrm{SL}(2,\mathbb{C})_R$ is a smooth diffeomorphism between Banach manifolds.

This Banach structure is used to construct CMC immersions in Hyperbolic space via the DPW method. It has been introduced in [39] and differs from the one used in [7], which would be less convenient to use for the purposes of this thesis.

3.3.2 Matrix model

The matrix model we use is the same as in [3]. Let $\mathbb{R}^{1,3}$ denote the vector space $\mathbb{R}^4$ equipped with the Lorentzian metric

$$\langle x,x\rangle = -x_0^2+x_1^2+x_2^2+x_3^2, \qquad x=(x_0,x_1,x_2,x_3)\in\mathbb{R}^4.$$

The Hyperbolic space $\mathbb{H}^3$ is the Riemannian sub-manifold of $\mathbb{R}^{1,3}$ defined by

$$\mathbb{H}^3 = \left\{x\in\mathbb{R}^{1,3} \mid \langle x,x\rangle=-1\right\}.$$

The point $(1,0,0,0)\in\mathbb{H}^3$ will play the role of an origin and is thus denoted by $\mathbf{0}$. The tangent space $T_{\mathbf{0}}\mathbb{H}^3=\{0\}\times\mathbb{R}^3$ is identified with its projection on $\mathbb{R}^3$ equipped with the cross product, which makes $T_{\mathbf{0}}\mathbb{H}^3$ a real Lie algebra.

Definition 12. For all $p,q\in\mathbb{H}^3$, the map $\Gamma_p^q : T_p\mathbb{H}^3\longrightarrow T_q\mathbb{H}^3$ denotes the **parallel transport** of vectors from p to q along the geodesic joining p to q. It is an isomorphism between vector spaces and it is the one we use to identify $T_p\mathbb{H}^3$ with $T_{\mathbf{0}}\mathbb{H}^3$ for all $p\in\mathbb{H}^3$.

Let $\mathcal{H}$ denote the real Lie algebra of 2×2 Hermitian matrices equipped with the Lie bracket and Lorentzian metric

$$[X, Y] := \frac{i}{2}(XY - YX), \qquad \langle X, X \rangle := -\det X.$$

Let $\Psi : \mathbb{R}^{1,3} \longrightarrow \mathcal{H}$ defined by

$$\Psi(x_0, x_1, x_2, x_3) = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}.$$

Proposition 11. The map Ψ is an isometric isomorphism between real vector spaces and

$$\Psi(\mathbb{H}^3) = \{FF^* \mid F \in \mathrm{SL}(2, \mathbb{C})\}$$

where $M^* := \overline{M}^t$ for all $M \in \mathfrak{gl}(2, \mathbb{C})$. Moreover,

$$\begin{array}{rccc} d\Psi_{\mathbf{0}} & : & T_{\mathbf{0}}\mathbb{H}^3 & \longrightarrow & \mathcal{H} \cap \mathfrak{sl}(2, \mathbb{C}) \\ & & v & \longmapsto & \Psi(v) \end{array}$$

is an isometric homomorphism between Lie algebras.

Proposition 11 provides an identification between $\mathbb{H}^3$ and $\Psi(\mathbb{H}^3)$. Under this identification, the Lie group $\mathrm{SL}(2, \mathbb{C})$ acts on $\mathbb{R}^{1,3}$ as linear Lorentzian isometries via

$$F \cdot X := FXF^*, \qquad X \in \mathcal{H}, \quad F \in \mathrm{SL}(2, \mathbb{C}). \tag{3.6}$$

The following proposition relies upon the polar decomposition in $\mathrm{SL}(2, \mathbb{C})$ and is proved in Chapter 5.

Proposition 12. For all $x \in \mathbb{H}^3$ and for all direct orthonormal bases (u, v, w) of the tangent space $T_x\mathbb{H}^3 \simeq T_{\mathbf{0}}\mathbb{H}^3$, there exists $F \in \mathrm{SL}(2, \mathbb{C})$ such that

$$(x, u, v, w) = F \cdot \mathcal{B}$$

where $\mathcal{B} = \Psi(\mathcal{B}_0)$ is the image of the canonical basis of $\mathbb{R}^4$.

Inspired by the Euclidean case, one can get the following definitions and propositions for Hyperbolic space.

Definition 13. The **moving frame** of $f : \Sigma \longrightarrow \mathbb{H}^3$ is defined as $\mathcal{F} := (f, e_x, e_y, N)$. A **special frame** for f is a map $P : \Sigma \longrightarrow \mathrm{SL}(2, \mathbb{C})$ such that $\mathcal{F} = P \cdot \mathcal{B}$. An **extended frame** for f is a map $P : \Sigma \longrightarrow \Lambda\mathrm{SL}(2, \mathbb{C})$ such that for all $\lambda \in \mathbb{S}^1$, $P(\cdot, \lambda)$ is a special frame for the associated CMC immersion f_λ.

Proposition 13. Let P be a special frame for $f : \Sigma \longrightarrow \mathbb{H}^3$. Then P satisfies the Lax pair:

$$P^{-1}dP = \frac{1}{2}\begin{pmatrix} u_z & 2(H+1)e^u \\ -Qe^{-u} & -u_z \end{pmatrix} dz + \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & \bar{Q}e^{-u} \\ -2(H-1)e^u & u_{\bar{z}} \end{pmatrix} d\bar{z} \tag{3.7}$$

with u, Q and H defined via (2.1). Moreover, the Maurer-Cartan equation for this Lax pair is equivalent to the Gauss and Codazzi equations in $\mathbb{H}^3$:

$$4u_{z\bar{z}} - |Q|^2e^{-2u} + 4(H^2-1)e^{2u} = 0, \quad Q_{\bar{z}} = 2e^{2u}H_z. \tag{3.8}$$

3.3.3 The (untwisted) DPW method for CMC $H > 1$ surfaces in $\mathbb{H}^3$

In this section, we introduce a DPW method with untwisted data that leads to CMC $H > 1$ immersions of a simply connected domain Σ into Hyperbolic space. Let $q = \operatorname{arcoth} H > 0$ and $R = e^q$.

Definition 14. In Hyperbolic space and untwisted setting, a set of **admissible DPW data** is given by $(\Sigma, \xi, z_0, \phi_0)$ admissible in the sense of Definition 10 and such that $\phi_0 \in \Lambda\mathrm{SL}(2,\mathbb{C})_R$ and ξ takes values in $\Lambda\mathfrak{sl}(2,\mathbb{C})_R$.

Remark 5. In Chapter 5, R is taken greater than e^q. This is not essential to the method, but allows us to get control formulas that are essential to our purpose.

Given a set of admissible data in the sense of Definition 14, the three steps of the DPW method are the following. First, solve for $\Phi(z) \in \Lambda\mathrm{SL}(2,\mathbb{C})_R$ the Cauchy problem (5.5). Then Iwasawa decompose Φ into (F, B) using Theorem 7. Finally, define f via the Sym-Bobenko formula

$$F(\cdot, e^{-q}) \cdot \mathrm{I}_2 \tag{3.9}$$

with the action of (3.6).

Proposition 14. The map $f : \Sigma \longrightarrow \mathbb{H}^3$ defined above is a CMC H immersion. With α as in (3.1), β the lower-left entry of A_0 and b the upper-left entry of $B\,|_{\lambda=0}$, the metric of f is

$$I = \frac{4b^4|\alpha|^2}{H^2-1}|dz|^2,$$

its Hopf differential is

$$II^{(2,0)} = \frac{-2\alpha\beta}{\sqrt{H^2-1}}dz^2$$

and its normal map is given by

$$N = F(\cdot, e^{-q}) \cdot \sigma_3.$$

Moreover, its differential reads

$$df = \frac{2b^2}{\sqrt{H^2-1}} F(\cdot, e^{-q}) \cdot \begin{pmatrix} 0 & \alpha dz \\ \bar{\alpha} d\bar{z} & 0 \end{pmatrix}.$$

Remark 6. Proposition 14 only constructs one member of the associated family (f_λ). This constraint is not essential to the DPW method in $\mathbb{H}^3$ and has been introduced only to simplify the notation. Furthermore, Chapter 5 only deals with single members of the associated families.

As in $\mathbb{R}^3$, every CMC $H > 1$ immersion can be obtained locally and up to an isometry via the DPW method.

3.3.4 Gauging, dressing, isometries, monodromy

Definitions 7 and 8 of gauging and dressing (adapted to the untwisted framework) are the same in $\mathbb{H}^3$, and gauging the potential does not change the immersion induced by formula (3.9). Therefore, Proposition 6 holds in $\mathbb{H}^3$. The effect of dressing is exactly the same: some dressing act as rigid motions.

Proposition 15. Let $(\Sigma, \xi, z_0, J\phi_0)$ be a set of dressed data (in the untwisted setting). Then the dressed CMC $H > 1$ immersion $\widehat{f} : \Sigma \longrightarrow \mathbb{H}^3$ has the same Hopf differential as the original immersion f. Moreover, if $J(z) \in \Lambda\mathrm{SU}(2)$ for all $z \in \Sigma$, then for all $\lambda \in \mathbb{S}^1$, f and $\widehat{f}$ differ by a rigid motion of $\mathbb{H}^3$:

$$\widehat{f} = J(e^{-q}) \cdot f \tag{3.10}$$

for the action defined in (3.6).

Finally, monodromy is defined in $\mathbb{H}^3$ as in Definition 11, but the expression of the monodromy problem differs from $\mathbb{R}^3$, because the isometric action of dressing is not the same in $\mathbb{H}^3$. Thus, **the monodromy problem in** $\mathbb{H}^3$ reads:

$$\forall \tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma), \qquad \begin{cases} \mathcal{M}_\tau(\Phi) & \in & \Lambda\mathrm{SU}(2)_R, \\ \mathcal{M}_\tau(\Phi)(e^{-q}) & = & \pm \mathrm{I}_2. \end{cases}$$

Part II

Constant Mean Curvature Surfaces in Euclidean and Hyperbolic Spaces

Chapter 4

On Delaunay Ends in the DPW method

[1]We consider constant mean curvature 1 surfaces in $\mathbb{R}^3$ arising via the DPW method from a holomorphic perturbation of the standard Delaunay potential on the punctured disk. Kilian, Rossman and Schmitt have proven that such a surface is asymptotic to a Delaunay surface. We consider families of such potentials parametrised by the necksize of the model Delaunay surface and prove the existence of a uniform disk on which the surfaces are close to the model Delaunay surface and are embedded in the unduloid case.

Introduction

Beside the sphere, the simplest non-zero constant mean curvature (CMC) surface is the cylinder, which belongs to a one-parameter family of surfaces generated by the revolution of an elliptic function: the Delaunay surfaces, first described in [4]. Like the cylinder, Delaunay surfaces have two annular type ends, and Delaunay ends are the only possible embedded annular ends for a non-zero CMC surface. Indeed, as proven in [24] by Korevaar, Kusner and Solomon, if $\mathcal{M} \subset \mathbb{R}^3$ is a proper, embedded, non-zero CMC surface of finite topological type, then every annular end of $\mathcal{M}$ is asymptotic to a Delaunay surface and if $\mathcal{M}$ has exactly two ends which are of annular type, then $\mathcal{M}$ is a Delaunay surface. Thus, the status of Delaunay surfaces for non-zero CMC surfaces is very much alike the catenoid position in the study of minimal surfaces (see the result of Schoen in [33]), and one has to understand the behaviour of Delaunay ends in order to construct examples of non-compact CMC surfaces with annular ends, as Kapouleas did in 1990 [16].

1. This chapter is the first paper of the thesis. It has been accepted in *Indiana University Mathematics Journal* and is available at `www.iumj.indiana.edu/IUMJ/Preprints/8123.pdf`

For an immersion, having a constant mean curvature and having a harmonic Gauss map are equivalent. This is why the Weierstrass type representation of Dorfmeister, Pedit and Wu [7] has been used since the publication of their article to construct CMC surfaces. The DPW method can construct any conformal non-zero CMC immersion in $\mathbb{R}^3$, $\mathbb{H}^3$ or $\mathbb{S}^3$ with three ingredients: a holomorphic potential which takes its values in a loop algebra, a loop group factorisation, and a Sym-Bobenko formula. Several examples of CMC surfaces with annular ends, like n-noids and bubbletons, have been made by Dorfmeister, Wu, Kilian, Kobayashi, McIntosh, Rossman, Schmitt and Sterling [8, 32, 17, 19, 22, 31]. These constructions often rely on a holomorphic perturbation of the holomorphic potential giving rise to a Delaunay surface via the DPW method, and Kilian, Rossman and Schmitt [20] have proven that such perturbations always induce asymptotically a Delaunay end.

More precisely, any Delaunay embedding can be obtained with a holomorphic potential of the form $\xi^{\mathcal{D}} = Az^{-1}dz$ where

$$A = \begin{pmatrix} 0 & r\lambda^{-1} + s \\ r\lambda + s & 0 \end{pmatrix}.$$

The main result of [20] states that any immersion obtained from a perturbed potential of the form $\xi = \xi^{\mathcal{D}} + \mathcal{O}(z^0)$ is asymptotic to an embedded half-Delaunay surface in a neighbourhood of $z = 0$, provided that the monodromy problem is solved. In this paper, we allow the perturbed potential to move in the family of Delaunay potentials by introducing a real parameter t, proportional to the weight (or neck-size) of the model Delaunay surface, and consider $\xi_t = \xi_t^{\mathcal{D}} + \mathcal{O}_t(z^0)$ where $\xi_t^{\mathcal{D}}$ is a Delaunay potential of weight $8\pi t$. The main theorem of [20] states that for every $t > 0$, there exists a small neighbourhood of the origin on which the surface produced by the potential ξ_t is embedded and asymptotic to a half Delaunay surface. Unfortunately, without further hypotheses, this neighbourhood vanishes into a single point as t tends to zero. Adding a few assumptions, we prove here that there exists a uniform neighbourhood of the origin upon which the surfaces induced by the family ξ_t are all embedded and asymptotic to a half Delaunay surface for $t > 0$ small enough.

Hence, the point of our paper is not to show that the ends of the perturbed unduloid family are embedded (which is what [20] does), but that all the immersions of this family are embedded on a uniform punctured disk. Equipped with our result, Martin Traizet (in [39] and [38]) showed for the first time how the DPW method can be used to both construct CMC n-noids without symmetries and prove that they are Alexandrov embedded.

The theorem we prove is the following one (definitions and notations are clarified in Section 4.1):

Theorem 8. Let Φ_t be a holomorphic frame arising from a perturbed Delaunay potential

ξ_t defined on a punctured neighbourhood of $z = 0$. Suppose that $\Phi_0(1, \lambda) = \mathrm{I}_2$ and that the monodromy of Φ_t is unitary. Then, if f_t denotes the immersion obtained via the DPW method,

- There exists a family $f_t^{\mathcal{D}}$ of Delaunay immersions such that for all $\alpha < 1$ and $|t|$ small enough,
$$\|f_t(z) - f_t^{\mathcal{D}}(z)\|_{\mathbb{R}^3} \leq C_\alpha |t||z|^\alpha$$
on a uniform neighbourhood of $z = 0$.
- If $t > 0$ is small enough, then f_t is an embedding of a uniform neighbourhood of $z = 0$.
- The limit axis of $f_t^{\mathcal{D}}$ as t tends to 0 can be made explicit.

An outline of the proof is given in Section 4.1.9, together with an explanation of why the convergence of t to 0 forbids us from using several key results of [20].

4.1 The DPW method

4.1.1 Loop groups

Our maps will often depend on a spectral parameter λ that can be in one of the following subsets of $\mathbb{C}$ ($R > 1$):

$$\mathcal{D}_R = \{\lambda \in \mathbb{C},\ |\lambda| < R\}, \quad \mathcal{A}_R = \left\{\lambda \in \mathbb{C},\ \tfrac{1}{R} < |\lambda| < R\right\},$$
$$\mathcal{D}_1 = \{\lambda \in \mathbb{C},\ |\lambda| < 1\}, \quad \mathcal{A}_1 = \{\lambda \in \mathbb{C},\ |\lambda| = 1\}.$$

For the coordinate z, we will note ($\epsilon > 0$):

$$\mathbb{D}_\epsilon = \{z \in \mathbb{C},\ |z| < \epsilon\}, \quad \mathbb{S}_\epsilon = \{z \in \mathbb{C},\ |z| = \epsilon\}.$$

Let us define the following (untwisted) loop groups and algebras:

- $\Lambda \mathrm{SL}_2\mathbb{C}$ is the set of smooth maps $\Phi : \mathcal{A}_1 \longrightarrow \mathrm{SL}_2\mathbb{C}$.
- $\Lambda \mathrm{SU}_2 \subset \Lambda \mathrm{SL}_2\mathbb{C}$ is the set of maps $F \in \Lambda \mathrm{SL}_2\mathbb{C}$ such that $F(\lambda) \in \mathrm{SU}_2$ for all $\lambda \in \mathcal{A}_1$.
- $\Lambda_+ \mathrm{SL}_2\mathbb{C} \subset \Lambda \mathrm{SL}_2\mathbb{C}$ is the set of maps $G \in \Lambda \mathrm{SL}_2\mathbb{C}$ that can be holomorphically extended to $\mathcal{D}_1$ and such that $G(0)$ is upper triangular.
- $\Lambda_+^{\mathbb{R}} \mathrm{SL}_2\mathbb{C} \subset \Lambda_+ \mathrm{SL}_2\mathbb{C}$ is the set of maps $B \in \Lambda_+ \mathrm{SL}_2\mathbb{C}$ such that $B(0)$ has positive elements on the diagonal.
- $\Lambda \mathfrak{sl}_2\mathbb{C}$ is the set of smooth maps $A : \mathcal{A}_1 \longrightarrow \mathfrak{sl}_2\mathbb{C}$.
- $\Lambda \mathfrak{su}_2$ is the set of maps $m \in \Lambda \mathfrak{sl}_2\mathbb{C}$ such that $m(\lambda) \in \mathfrak{su}_2$ for all $\lambda \in \mathcal{A}_1$.

- $\Lambda_+\mathfrak{sl}_2\mathbb{C} \subset \Lambda\mathfrak{sl}_2\mathbb{C}$ is the set of maps $g \in \Lambda\mathfrak{sl}_2\mathbb{C}$ that can be holomorphically extended to $\mathcal{D}_1$ and such that $g(0)$ is upper triangular.
- $\Lambda_+^{\mathbb{R}}\mathfrak{sl}_2\mathbb{C} \subset \Lambda_+\mathfrak{sl}_2\mathbb{C}$ is the set of maps $b \in \Lambda_+\mathfrak{sl}_2\mathbb{C}$ such that $b(0)$ has real elements on the diagonal.

We also use the following notation:

$$\mathcal{O}(t^\alpha, z^\beta, \lambda^\gamma) = t^\alpha z^\beta \lambda^\gamma f(t, z, \lambda)$$

where f, on its domain of definition, is continuous with respect to (t, z, λ) and holomorphic with respect to (z, λ) for any t. If one variable is not specified, its exponent is assumed to be 0.

One step of the DPW method relies on the following Iwasawa decomposition (Theorem 8.1.1. in [29] and Proposition 6.2. in [27]):

Theorem 9 (Iwasawa decomposition)**.** Any element $\Phi \in \Lambda\mathrm{SL}_2\mathbb{C}$ can be uniquely factorised into a product

$$\Phi = F \times B$$

where $F \in \Lambda\mathrm{SU}_2$ and $B \in \Lambda_+^{\mathbb{R}}\mathrm{SL}_2\mathbb{C}$. Moreover, the map $\Lambda\mathrm{SL}_2\mathbb{C} \longrightarrow \Lambda\mathrm{SU}_2 \times \Lambda_+^{\mathbb{R}}\mathrm{SL}_2\mathbb{C}$ is a $\mathcal{C}^\infty$ diffeomorphism for the intersection of the $\mathcal{C}^k$ topologies (see [20]).

The Iwasawa decomposition of a map Φ will often be written:

$$\Phi = \mathrm{Uni}\,(\Phi) \times \mathrm{Pos}\,(\Phi)\,,$$

where $\mathrm{Uni}\,(\Phi)$ is called "the unitary factor" of Φ and $\mathrm{Pos}\,(\Phi)$ is "the positive factor" of Φ. Using Corollary 4 of Section 4.6, note that if Φ is holomorphic on $\mathcal{A}_R$, then its unitary factor holomorphically extends to $\mathcal{A}_R$ and its positive factor holomorphically extends to $\mathcal{D}_R$.

4.1.2 The $\mathfrak{su}_2$ model of $\mathbb{R}^3$

In the DPW method, immersions are given in a matrix model. The euclidean space $\mathbb{R}^3$ is thus identified with the Lie algebra $\mathfrak{su}_2$ by

$$x = (x_1, x_2, x_3) \simeq X = \frac{-i}{2}\begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix}.$$

The canonical basis of $\mathbb{R}^3$ identified as $\mathfrak{su}_2$ is denoted (e_1, e_2, e_3). In this model, the euclidean norm is given by

$$\|x\|^2 = 4\det(X). \tag{4.1}$$

Linear isometries are represented by the conjugacy action of SU_2 on $\mathfrak{su}_2$:

$$H \cdot X = HXH^{-1}.$$

4.1.3 The recipe

The DPW method takes for input data:

- A Riemann surface Σ;
- A $\Lambda\mathfrak{sl}_2\mathbb{C}$-valued holomorphic 1-form $\xi = \xi(z,\lambda)$ on Σ called "the DPW potential" which extends meromorphically to $\mathcal{D}_1$ with a pole only at $\lambda = 0$, and which must be of the form
$$\xi(z,\lambda) = \sum_{j=-1}^{\infty} \xi_j(z)\lambda^j$$
where each matrix $\xi_j(z)$ depends holomorphically on z and all the entries of $\xi_{-1}(z)$ are zero except for the upper right entry which must never vanish;
- A base point $z_0 \in \Sigma$;
- An initial condition $\Phi_{z_0} \in \Lambda\mathrm{SL}_2\mathbb{C}$.

Given such data, here are the three steps of the DPW method for constructing CMC-1 surfaces in $\mathbb{R}^3$ (in the untwisted setting):

1. Solve for Φ the Cauchy problem with parameter $\lambda \in \mathcal{A}_1$:
$$\begin{cases} d_z\Phi(z,\lambda) &= \Phi(z,\lambda)\xi(z,\lambda), \\ \Phi(z_0,\lambda) &= \Phi_{z_0}(\lambda). \end{cases}$$
The solution $\Phi(z,\cdot) \in \Lambda\mathrm{SL}_2\mathbb{C}$ is called the "holomorphic frame" of the surface. In general, $\Phi(\cdot,\lambda)$ is only defined on the universal cover $\widetilde{\Sigma}$ of Σ (see Section 4.1.6). Note that if $\xi(z,\cdot)$ can be holomorphically extended to $\mathcal{A}_R$ $(R > 1)$, then $\Phi(z,\cdot)$ can also be holomorphically extended to $\mathcal{A}_R$ provided that Φ_{z_0} is holomorphic on $\mathcal{A}_R$.
2. For all $z \in \widetilde{\Sigma}$, Iwasawa decompose $\Phi(z,\lambda) = F(z,\lambda)B(z,\lambda)$. The decomposition is done pointwise in z, but $F(z,\lambda)$ and $B(z,\lambda)$ depend real-analytically on z. The map F is called the "unitary frame" of the surface.
3. Define $f : \widetilde{\Sigma} \longrightarrow \mathfrak{su}_2$ by the Sym-Bobenko formula:
$$f(z) = \mathrm{Sym}(F) = i\frac{\partial F}{\partial \lambda}(z,1)F(z,1)^{-1}.$$

The map f is then a conformal CMC-1 immersion whose normal map is given by

$$\mathcal{N}(z) = \frac{-i}{2} F(z,1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F(z,1)^{-1}. \tag{4.2}$$

Its metric and Hopf differential are

$$ds = 2\rho^2 |\xi_{-1}^{12}||dz|,$$

$$Q = -2\xi_{-1}^{12}\xi_0^{21} dz^2$$

where ξ_j^{kl} is the (k,l)-entry of the matrix $\xi_j(z)$ and ρ is the upper-left entry of $B(z,0)$.

The theory states that every conformal CMC-1 immersion can be obtained this way.

4.1.4 Rigid motions of the surface

Let ξ be a DPW potential and $\Phi \in \Lambda \mathrm{SL}_2\mathbb{C}$ a solution of $d\Phi = \Phi\xi$. Take a loop $H \in \Lambda \mathrm{SU}_2$ that does not depend on z. Then $\widetilde{\Phi} = H\Phi$ also satisfies $d\widetilde{\Phi} = \widetilde{\Phi}\xi$ and gives rise to a rigid motion of the original surface given by Φ. Let $f = \mathrm{Sym} \circ \mathrm{Uni}(\Phi)$ and $\widetilde{f} = \mathrm{Sym} \circ \mathrm{Uni}(\widetilde{\Phi})$. Then,

$$\widetilde{f}(z) = H(1) \cdot f(z) + \mathrm{Sym}(H).$$

This leads us to extend the action of section 4.1.2 to affine isometries by

$$H(\lambda) \cdot X = H(1) X H(1)^{-1} + i \frac{\partial H}{\partial \lambda}(1) H(1)^{-1}.$$

Note that $\Lambda \mathrm{SU}_2$ also acts on the tangent bundle of $\mathbb{R}^3$ via:

$$H \cdot (p, \vec{v}) = (H \cdot p, H(1) \cdot \vec{v}). \tag{4.3}$$

This action will be useful to follow the axis of our surfaces: oriented affine lines are generated by pairs $(p, \vec{v})$ and the action of $\Lambda \mathrm{SU}_2$ on a given oriented affine line corresponds to the action (4.3) on its generators.

4.1.5 Gauging

Let $(\Sigma, \xi, z_0, \Phi_{z_0})$ be a set of DPW data with $d\Phi = \Phi\xi$. Let $G(z,\lambda)$ be a holomorphic map with respect to $z \in \Sigma$ such that $G(z,\cdot) \in \Lambda_+ \mathrm{SL}_2\mathbb{C}$ (such a map is called an "admissible gauge"). If we define $\widetilde{\Phi} = \Phi G$, then Φ and $\widetilde{\Phi}$ give rise to the same immersion f. This

operation is called "gauging" and one can retrieve $\widetilde{\Phi}$ by applying the DPW method to the data $(\Sigma, \xi \cdot G, z_0, \Phi_{z_0} G(z_0, \cdot))$ where

$$\xi \cdot G = G^{-1}\xi G + G^{-1}dG$$

is the action of gauges on potentials.

4.1.6 The monodromy problem

Since Φ is defined as the solution of a Cauchy problem on Σ, it is only defined on the universal cover $\widetilde{\Sigma}$ of Σ. For any deck transformation τ of $\widetilde{\Sigma}$, we define the monodromy matrix $\mathcal{M}_\tau(\Phi) \in \Lambda\mathrm{SL}_2\mathbb{C}$ as follow:

$$\Phi(\tau(z), \lambda) = \mathcal{M}_\tau(\Phi)(\lambda)\Phi(z, \lambda).$$

Note that $\mathcal{M}_\tau(\Phi)$ does not depend on z. The standard sufficient condition for the immersion f to be be well-defined on Σ is the following set of equations, called the monodromy problem in $\mathbb{R}^3$:

$$\left\{\begin{array}{rcll} \mathcal{M}_\tau(\Phi) & \in & \Lambda\mathrm{SU}_2, & (i) \\ \mathcal{M}_\tau(\Phi)(1) & = & \pm \mathrm{I}_2, & (ii) \\ \frac{\partial}{\partial\lambda}\mathcal{M}_\tau(\Phi)(1) & = & 0. & (iii) \end{array}\right.$$

Remark 7. In this paper, the Riemann surface Σ will always be a punctured neighbourhood $\mathbb{D}^*_\epsilon$ of $z = 0$. Thus, all the deck transformations τ will be associated to a closed loop around $z = 0$ and we will write $\mathcal{M}(\Phi)$ instead of $\mathcal{M}_\tau(\Phi)$.

Remark 8. Let $\Phi : \mathbb{C}^* \longrightarrow \Lambda\mathrm{SL}_2\mathbb{C}$ such that $\mathcal{M}(\Phi) \in \Lambda\mathrm{SU}_2$. Let $\widetilde{\Phi} = H\,(h^*\Phi) \cdot G$ where $H \in \Lambda\mathrm{SL}_2\mathbb{C}$, $G : \mathbb{C} \longrightarrow \Lambda_+\mathrm{SL}_2\mathbb{C}$ is holomorphic at $z = 0$ and $h : \mathbb{C} \longrightarrow \mathbb{C}$ is a Möbius transformation that leaves $z = 0$ invariant. Then

$$\mathcal{M}(\widetilde{\Phi}) = H\mathcal{M}(\Phi)\,H^{-1}.$$

Thus, if the monodromy problem for Φ is solved, a sufficient condition for the monodromy problem for $\widetilde{\Phi}$ to be solved is that $H \in \Lambda\mathrm{SU}_2$.

4.1.7 The Delaunay family

Delaunay surfaces come in a one-parameter family: for all $t \in \left(-\infty, \frac{1}{16}\right] \setminus \{0\}$, there exists a unique Delaunay surface, whose weight (as defined in [24]) is $8\pi t$. As shown in

[32], the DPW method can retrieve these surfaces using the following data:

$$\Sigma = \mathbb{C}^*, \quad \xi_t(z,\lambda) = A_t(\lambda) z^{-1} dz, \quad z_0 = 1, \quad \Phi_{z_0} = \mathrm{I}_2,$$

where

$$A_t(\lambda) = \begin{pmatrix} 0 & r\lambda^{-1} + s \\ r\lambda + s & 0 \end{pmatrix}$$

and r, s are functions of $t \in \left(-\infty, \frac{1}{16}\right]$ satisfying

$$\begin{cases} r, s \in \mathbb{R}, \\ r + s = \frac{1}{2}, \\ rs = t. \end{cases} \tag{4.4}$$

Note that the system (4.4) admits two solutions, whether $r \geq s$ or $r \leq s$. For a fixed value of t, these two solutions give two different parametrisations of the same surface (up to a translation). If $r \geq s$, the unit circle of $\mathbb{C}^*$ is mapped onto a parallel circle of maximal radius: a bulge of the Delaunay surface. If $r \leq s$, the unit circle of $\mathbb{C}^*$ is mapped onto a parallel circle of minimal radius: a neck of the Delaunay surface. As t tends to 0 and in the case $r \geq s$, the immersions tend towards the parametrisation of a sphere on every compact subset of $\mathbb{C}^*$, which is why we call this setting the "spherical case". On the other hand, when $r \leq s$ and t tends to 0, the immersions degenerate into a point on every compact subset of $\mathbb{C}^*$. Nevertheless, we call this setting the "catenoidal case" because applying a blowup to the immersions makes them converge towards a catenoid on every compact subset of $\mathbb{C}^*$ (see [38] for further details).

In any case, the corresponding holomorphic frame is explicit:

$$\Phi_t(z,\lambda) = z^{A_t(\lambda)}$$

as is its monodromy around $z = 0$:

$$\mathcal{M}\left(\Phi_t\right)(\lambda) = \exp\left(2i\pi A_t(\lambda)\right) = \cos\left(2\pi\mu_t(\lambda)\right)\mathrm{I}_2 + \frac{i\sin\left(2\pi\mu_t(\lambda)\right)}{\mu_t(\lambda)} A_t(\lambda) \tag{4.5}$$

where

$$\mu_t(\lambda)^2 = -\det A_t(\lambda) = \frac{1}{4} + t\lambda^{-1}(\lambda - 1)^2. \tag{4.6}$$

Note that the conditions (4.4) have been chosen in order for the monodromy problem of Section 4.1.6 to be solved. The axis of the surface is given by $\{(x, 0, -2r),\ x \in \mathbb{R}\}$ and its weight is $8\pi t$. Thus, the induced surface is an unduloid if $t > 0$ and a nodoid if $t < 0$.

Remark 9. In order to deal with a single-valued square root of $\mu_t(\lambda)^2$ and to avoid some

resonance cases in Section 4.3, we set $T > 0$ and $R > 1$ small enough for

$$\left|\mu_t(\lambda)^2 - \frac{1}{4}\right| < \frac{1}{4}$$

to hold for all $(t, \lambda) \in (-T, T) \times \mathcal{A}_R$.

4.1.8 Perturbed Delaunay DPW data

We take a Delaunay potentials family as in section 4.1.7 and we perturb it for z in a small uniform neighbourhood of 0:

Definition 15 (Perturbed Delaunay potential)**.** Let $\epsilon > 0$. A perturbed Delaunay potential is a one-parameter family $\{\xi_t\}_{t\in(-T,T)}$ of DPW potentials, holomorphic on $\mathbb{D}^*_\epsilon \times \mathcal{A}_R$ and of the form

$$\xi_t(z, \lambda) = A_t(\lambda)z^{-1}dz + R_t(z, \lambda)dz$$

where A_t is a Delaunay residue as in Section 4.1.7 and $R_t(z, \lambda) \in \mathcal{C}^2$ with respect to (t, z, λ), is holomorphic on $\mathbb{D}_\epsilon \times \mathcal{A}_R$ for all t and satisfies $R_0(z, \lambda) = 0$.

The following set of hypotheses will be used to make sure that our holomorphic frames have a $\mathcal{C}^0$ regularity, are holomorphic with respect to (z, λ), and solve the monodromy problem:

Hypotheses 1. Let ξ_t be a perturbed Delaunay potential. Let Φ_t be a holomorphic frame associated to it. We suppose that

- For all $t \in (-T, T)$ and $z \in \mathbb{D}^*_\epsilon$, $\Phi_t(z, \cdot)$ is holomorphic on $\mathcal{A}_R$,
- $\Phi_t(z, \lambda)$ is continous with respect to (t, z, λ),
- The monodromy is unitary: $\mathcal{M}(\Phi_t) \in \Lambda\mathrm{SU}_2$.

Remark 10. When needed, one can replace $R > 1$ by a smaller value in order for Φ_t to be holomorphic on $\mathcal{A}_R$ and continuous on $\overline{\mathcal{A}_R}$.

The theorem we prove in this paper is the following:

Theorem 10. Let ξ_t be a perturbed Delaunay potential and Φ_t a holomorphic frame associated to ξ_t satisfying Hypotheses 1 and such that $\Phi_0(1, \lambda) = \mathrm{I}_2$. Let $f_t = \mathrm{Sym}\,(\mathrm{Uni}(\Phi_t))$. Then,

1. For all $\alpha < 1$ there exist constants $\epsilon > 0$, $T > 0$ and $C > 0$ such that for all $0 < |z| < \epsilon$ and $|t| < T$,

$$\|f_t(z) - f_t^{\mathcal{D}}(z)\|_{\mathbb{R}^3} \leq C|t||z|^\alpha$$

where $f_t^{\mathcal{D}}$ is a Delaunay immersion of weight $8\pi t$.

2. There exist $T' > 0$ and $\epsilon' > 0$ such that for all $0 < t < T'$, f_t is an embedding of $\{0 < |z| < \epsilon'\}$.
3. If $r \geq s$, the limit axis as t tends to 0 of $f_t^{\mathcal{D}}$ is the oriented line generated by $(-e_3, -\vec{e_1})$.

   If $r \leq s$, the limit axis as t tends to 0 of $f_t^{\mathcal{D}}$ is the oriented line generated by $(0, -\vec{e_1})$.

Remark 11. We do not have to assume that $1 \in \mathbb{D}_\epsilon$ for Φ_0 to be defined at $z = 1$. This only comes from the fact that ξ_0 is defined on $\mathbb{C}^*$, which implies that Φ_0 is defined on the universal cover $\widetilde{\mathbb{C}^*}$.

4.1.9 Outline of the proof and comparison with [20]

In Section 4.3 we start the proof of Theorem 10 by gauging the potential and changing coordinates. Starting from

$$\xi_t = A_t z^{-1} dz + \mathcal{O}(t, z^0) dz$$

we gain an order on z and obtain the following new potential:

$$\widetilde{\xi}_t = A_t z^{-1} dz + \mathcal{O}(t, z) dz.$$

We then use the Fröbenius method and the new holomorphic frame is

$$\widetilde{\Phi}_t = \widetilde{M}_t z^{A_t} \left(\mathrm{I}_2 + \mathcal{O}(t, z^2) \right).$$

In Section 4.4, we use the above estimate on $\widetilde{\Phi}_t$ to prove the convergence of the immersions:

$$\left\| \widetilde{f}_t(z) - \widetilde{f}_t^{\mathcal{D}}(z) \right\|_{\mathbb{R}^3} \leq C|t||z|^\alpha, \quad \alpha < 1$$

where $\widetilde{f}_t^{\mathcal{D}}$ is a Delaunay immersion whose axis can be explicitly computed. To do so, we need to know the asymptotic behaviour of the positive part $\mathrm{Pos}(\widetilde{\Phi}_t)$, which we compute using the fact that $\widetilde{f}_t^{\mathcal{D}}(\mathbb{C}^*)$ is a surface of revolution.

Finally, Section 4.5 proves that perturbations of unduloids are embedded on a uniform neighbourhood of the origin.

Sections 4.6 to 4.8 are appendices.

Although the method of this paper is inspired by what Kilian, Rossman and Schmitt did in [20], their results cannot be used to prove our theorem. This is mainly because the asymptotics given in [20] for a fixed value of our parameter t do not hold as t tends to 0. As an example, consider the proof of Lemma 2.5 in [20]: with our hypotheses, the constant

they call κ becomes a function of t such that (with our notation of Section 4.3.2)

$$\kappa\mid_{t=0}=\frac{c_{12}(0,0)}{4}\neq 0.$$

Later in the proof, computing the determinant of the linear map $\mathcal{L}_1$ gives

$$\det\mathcal{L}_1=\mathcal{O}(t)$$

and their gauged potential is then of the form

$$\widehat{\xi}_t=A_t z^{-1}dz+\mathcal{O}(t^{-1},z)dz,$$

the corresponding holomorphic frame being

$$\widehat{\Phi}_t=\widehat{M}_t z^{A_t}\left(\mathrm{I}_2+\mathcal{O}(t^{-1},z^2)\right).$$

Applying the Sym-Bobenko formula would give at best

$$\left\|\widehat{f}_t(z)-\widehat{f}_t^{\mathcal{D}}(z)\right\|_{\mathbb{R}^3}\leq C\frac{1}{|t|}|z|^{\alpha},\quad \alpha<1 \tag{4.7}$$

which is not enough to show the convergence of the immersions on the compact sets of $\mathbb{C}^*$ as t tends to 0. Note that gaining one order on $|t|$ in the estimate (4.7) is still not enough to show the embeddedness of $\widehat{f}_t$, since the first catenoidal neck of $\widehat{f}_t^{\mathcal{D}}$, which has a size of the order of t, is attained for $|z|\sim|t|$ as t tends to 0.

Finally, some bounds used in [20] such as (see Lemma 1.11 in [20])

$$c_1(\lambda)=\max_{x\in[0,\rho)}\|B(x,\lambda)\|$$

depend on t in our framework and may explode as t tends to 0.

4.2 An application

Before proving Theorem 10, we must take account of the fact that one of its hypotheses is too restrictive. Indeed, $\Phi_0(1,\lambda)=\mathrm{I}_2$ has no reason to hold when one wants to construct examples, as Martin Traizet did in [39] and [38]. We thus show here on a specific example how to ensure this hypothesis by gauging the potential and changing coordinates.

In all the section, ξ_t is a perturbed Delaunay potential with $r\geq s$ and Φ_t is a holomorphic frame associated to ξ_t, satisfying Hypotheses 1 and such that $\Phi_0(1,\lambda)=M(\lambda)$

where

$$M(\lambda) = \begin{pmatrix} a & b\lambda^{-1} \\ c\lambda & d \end{pmatrix} \in \Lambda \mathrm{SL}_2\mathbb{C} \ (a, b, c, d \in \mathbb{C}). \tag{4.8}$$

After some simplification, we will be able to apply Theorem 10 even though $\Phi_0(1, \lambda) \neq \mathrm{I}_2$. The only difference in the conclusion will be in the third point: as t tends to 0, the limit axis of the model Delaunay surface $f_t^{\mathcal{D}}$ will be the oriented line generated by $Q \cdot (0, \vec{e_3})$ where

$$Q = \mathrm{Uni}\,[MH] \tag{4.9}$$

with

$$H(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix}. \tag{4.10}$$

The method involves gauging, changing coordinates and applying an isometry, and relies on the fact that one can explicitly compute the Iwasawa decomposition of MH. Indeed, for all $a, b, c, d \in \mathbb{C}$ such that $ad - bc = 1$,

$$\begin{pmatrix} a & b\lambda^{-1} \\ c\lambda & d \end{pmatrix} = \frac{1}{\sqrt{|b|^2 + |d|^2}} \begin{pmatrix} \overline{d} & b\lambda^{-1} \\ -\overline{b}\lambda & d \end{pmatrix} \times \frac{1}{\sqrt{|b|^2 + |d|^2}} \begin{pmatrix} 1 & 0 \\ \left(a\overline{b} + c\overline{d}\right)\lambda & |b|^2 + |d|^2 \end{pmatrix} \tag{4.11}$$

is the Iwasawa decomposition of the left-hand side term. Note that if the matrix M is explicit, then this formula makes both the matrix Q in Equation (4.9) and the limit axis of $f_t^{\mathcal{D}}$ explicit because MH and M have the same form.

Lemma 1. Let ξ_t be a perturbed Delaunay potential as in Definition 15 with $r \geq s$. Let Φ_t be a holomorphic frame associated to it, satisfying Hypotheses 1 and such that $\Phi_0(1, \lambda) = M(\lambda)$ as in (4.8). Then there exists a Möbius transformation that leaves $z = 0$ invariant and a gauge G such that:

1. the new potential $\widetilde{\xi}_t = (h^*\xi_t) \cdot G$ is also a perturbed Delaunay potential with the same residue as ξ_t,
2. the holomorphic frame $\widetilde{\Phi}_t$ associated to $\widetilde{\xi}_t$ satisfies Hypotheses 1 with $\widetilde{\Phi}_0(1, \lambda) \in \Lambda \mathrm{SU}_2$.

Proof. Let A_t and R_t be as in Definition 15. Then

$$\widetilde{\xi}_t = G^{-1}\left(A_t h^{-1} dh + (h^* R_t) dh\right) G + G^{-1} dG.$$

The Möbius transformation we are looking for satisfies $h(0) = 0$ and thus

$$h^{-1} dh = z^{-1} dz + \mathcal{O}(z) dz.$$

Wanting $\widetilde{\xi}_t$ to have a simple pole at $z = 0$, we look for a gauge G that is holomorphic at $z = 0$. Wanting the residue of $\widetilde{\xi}_t$ to be A_t, we suppose that $G(0, \lambda) = \mathrm{I}_2$. These two conditions together with $\widetilde{\xi}_0 = A_0 z^{-1} dz$ lead us to solve the following Cauchy problem:

$$\begin{cases} dG &= GA_0 z^{-1} dz - A_0 G h^{-1} dh \\ G(0) &= \mathrm{I}_2. \end{cases} \tag{4.12}$$

If we write

$$h(z) = \frac{z}{pz + q},\ p \in \mathbb{C},\ q \in \mathbb{C}^*$$

so that

$$h^{-1} dh = z^{-1} dz - \frac{p dz}{pz + q},$$

then the only solution of (4.12) is given (by Maple) by:

$$G(z, \lambda) = \begin{pmatrix} \sqrt{\frac{q}{pz+q}} & 0 \\ \frac{\lambda pz}{\sqrt{q(pz+q)}} & \sqrt{\frac{pz+q}{q}} \end{pmatrix}$$

and a straightforward computation allows us to check that G satisfies (4.12). Setting $0 < \epsilon' < \epsilon$ with $\epsilon' < \frac{|q|}{|p|}$ if necessary, this proves the first point of the lemma.

In order to prove the second point, diagonalise $A_0 = HDH^{-1}$ with H as in (4.10) and compute

$$\widetilde{\Phi}_0(1, \lambda) = M(\lambda) H(\lambda) \left(h(1)^D H(\lambda)^{-1} G(1, \lambda) H(\lambda) \right) H(\lambda)^{-1} \tag{4.13}$$

where

$$D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix}.$$

Hence $\widetilde{\Phi}_0(1, \cdot)$ is holomorphic on $\mathcal{A}_R$. Moreover, the fact that $\widetilde{\xi}_t$ is $\mathcal{C}^2$ in (t, z, λ) together with remark 8 imply that $\widetilde{\Phi}_t$ satisfies Hypotheses 1. Finally, compute

$$h(1)^D H(\lambda)^{-1} G(1, \lambda) H(\lambda) = \begin{pmatrix} \frac{1}{\sqrt{q}} & 0 \\ \lambda \frac{p}{\sqrt{q}} & \sqrt{q} \end{pmatrix}$$

and, using Equation (4.11),

$$\mathrm{Pos}\,(MH) = \begin{pmatrix} \rho & 0 \\ \lambda\mu & \rho^{-1} \end{pmatrix}$$

where

$$\rho = \frac{\sqrt{2}}{\sqrt{|b-a|^2 + |d-c|^2}}, \qquad \mu = \frac{1}{\sqrt{2}} \times \frac{(a+b)(\bar{b}-\bar{a}) + (c+d)(\bar{d}-\bar{c})}{\sqrt{|b-a|^2 + |d-c|^2}}.$$

Then, setting

$$p = -\rho\mu, \quad q = \rho^2,$$

Equation (4.13) becomes (Q is defined in (4.9))

$$\widetilde{\Phi}_0(1,\lambda) = QH^{-1} \in \Lambda \mathrm{SU}_2$$

because $H \in \Lambda \mathrm{SU}_2$. □

If one wants to apply Theorem 10, it then suffices to set

$$\widehat{\Phi}_t = HQ^{-1}\widetilde{\Phi}_t$$

where $\widetilde{\Phi}_t$ is constructed by Lemma 1. Let $\widehat{f}_t^{\mathcal{D}}$ be the model Delaunay immersion towards which the immmersion $\mathrm{Sym}\left(\mathrm{Uni}(\widehat{\Phi}_t)\right)$ converges. Theorem 10 then states that the limit axis as t tends to 0 of $\widehat{f}_t^{\mathcal{D}}$ is the oriented line generated by $(-e_3, -\vec{e_1})$. Compute

$$H^{-1} \cdot (-e_3, -\vec{e_1}) = (-e_3, \vec{e_3}) \simeq (0, \vec{e_3})$$

to prove that $\mathrm{Sym}\,(\mathrm{Uni}(\Phi_t))$ converges to a model Delaunay surface whose limit axis as t tends to 0 is $Q \cdot (0, \vec{e_3})$. The following corollary summarises this section:

Corollary 2. Let ξ_t be a perturbed Delaunay potential with $r \geq s$ and Φ_t a holomorphic frame associated to ξ_t satisfying Hypotheses 1 and such that $\Phi_0(1,\lambda)$ is of the form given by (4.8). Let $f_t = \mathrm{Sym}\,(\mathrm{Uni}(\Phi_t))$. Then,

1. For all $\alpha < 1$ there exist constants $\epsilon > 0$, $T > 0$ and $C > 0$ such that for all $0 < |z| < \epsilon$ and $|t| < T$,

$$\|f_t(z) - f_t^{\mathcal{D}}(z)\|_{\mathbb{R}^3} \leq C|t||z|^\alpha$$

   where $f_t^{\mathcal{D}}$ is a Delaunay immersion of weight $8\pi t$.
2. There exist $T' > 0$ and $\epsilon' > 0$ such that for all $0 < t < T'$, f_t is an embedding of $\{0 < |z| < \epsilon'\}$.
3. The limit axis as t tends to 0 of $f_t^{\mathcal{D}}$ is the oriented line generated by $Q \cdot (0, \vec{e_3})$ where Q is given by Equation (4.9).

4.3 The $z^A P$ form of Φ_t

Let us start the proof of Theorem 10: let ξ_t be a perturbed Delaunay potential and Φ_t a holomorphic frame associated to ξ_t satisfying Hypotheses 1 and such that $\Phi_0(1,\lambda) = \mathrm{I}_2$.

In this section, we want to apply the Fröbenius method and write Φ_t in a $z^A P$ form. Unfortunately, the underlying Fuchsian system seems to admit resonance points. Our goal is to avoid them and to gain an order of convergence in the matrix P of the $z^A P$ form. We will obtain the following result:

Proposition 16. There exist a change of coordinate h_t and a gauge G_t such that, denoting

$$\widetilde{\Phi}_t = h_t^* \left(\Phi_t G_t\right)$$

and

$$\widetilde{\xi}_t = h_t^* \left(\xi_t \cdot G_t\right),$$

$\widetilde{\xi}_t$ is a perturbed Delaunay potential and $\widetilde{\Phi}_t$ is a holomorphic frame associated to $\widetilde{\xi}_t$ satisfying Hypotheses 1 and such that $\widetilde{\Phi}_0(1,\lambda) = \mathrm{I}_2$. Moreover,

$$\widetilde{\Phi}_t(z,\lambda) = \widetilde{M}_t(\lambda) z^{A_t(\lambda)} \widetilde{P}_t(z,\lambda) \tag{4.14}$$

where $\widetilde{M}_t \in \Lambda\mathrm{SL}_2\mathbb{C}$ is continuous and holomorphic on $\mathcal{A}_R$ for all t and $\widetilde{P}_t : \mathbb{D}_{\epsilon'} \longrightarrow \Lambda\mathrm{SL}_2\mathbb{C}$ is $\mathcal{C}^2$, holomorphic on $\mathbb{D}'_\epsilon \times \mathcal{A}_R$ for all t and satisfies $\widetilde{P}_t(z,\lambda) = \mathrm{I}_2 + \mathcal{O}(t, z^2)$.

4.3.1 Extending to the resonance points

In this section, we use the Fröbenius method to write Φ_t in a $z^A P$ form, and extend this form to the resonance points. We will thus prove:

Proposition 17. There exist $M_t \in \Lambda\mathrm{SL}_2\mathbb{C}$ continuous and holomorphic on $\mathcal{A}_R$ for all t and $P_t : \mathbb{D}_\epsilon \longrightarrow \Lambda\mathrm{SL}_2\mathbb{C}$ continuous and holomorphic on $\mathbb{D}_\epsilon \times \mathcal{A}_R$ for all t satisfying $P_t(0,\lambda) = \mathrm{I}_2$ and

$$\Phi_t(z,\lambda) = M_t(\lambda) z^{A_t(\lambda)} P_t(z,\lambda).$$

Let us first recall the Fröbenius method in the non-resonant case (see [36] and [35]). Let $\epsilon > 0$ and ξ be a holomorphic 1-form from $\mathbb{D}^*_\epsilon$ to $\mathcal{M}_2(\mathbb{C})$ defined by

$$\xi(z) = Az^{-1}dz + \sum_{k\in\mathbb{N}} C_k z^k dz.$$

For all $k \in \mathbb{N}$, let P_k solve

$$\left\{ \begin{array}{rcl} P_0 & = & \mathrm{I}_2, \\ \mathcal{L}_{k+1}(P_{k+1}) & = & \displaystyle\sum_{i+j=k} P_i C_j \end{array} \right. \tag{4.15}$$

where for all $n \in \mathbb{N}$,

$$\begin{array}{rccc} \mathcal{L}_n & : & \mathcal{M}_2(\mathbb{C}) & \longrightarrow & \mathcal{M}_2(\mathbb{C}) \\ & & X & \longmapsto & [A,X]+nX. \end{array}$$

Then $P(z)=\sum_{k\in\mathbb{N}} P_k z^k$ is holomorphic on $\mathbb{D}_\epsilon$ and $\Phi(z)=z^AP(z)$ is holomorphic on the universal cover $\widetilde{\mathbb{D}_\epsilon^*}$ of $\mathbb{D}_\epsilon^*$ and solves $d\Phi=\Phi\xi$.

Let us now recall Lemma 2.2 of [20] in our framework:

Lemma 2. Let $A \in \mathfrak{sl}_2\mathbb{C}$ such that $A^2=\mu^2\mathrm{I}_2$. Then for all $n\in\mathbb{N}$,

$$\det \mathcal{L}_n = n^2\left(n^2-4\mu^2\right) \tag{4.16}$$

and

$$\mathcal{L}_n^{-1}(X) = \frac{1}{n}\left(X - \frac{1}{n^2-4\mu^2}\left(n\mathrm{I}_2-2A\right)[A,X]\right) \tag{4.17}$$

Corollary 3 follows from Remark 9 and Equation (4.16).

Corollary 3. Let $\mathcal{L}_{t,n}(X)=[A_t(\lambda),X]+nX$.

- For all $n\geq 2$, $\mathcal{L}_{t,n}$ is invertible on $(t,\lambda)\in(-T,T)\times\mathcal{D}_R^*$.
- For $n=1$, $\mathcal{L}_{t,1}$ is invertible on $(t,\lambda)\in(-T,T)\setminus\{0\}\times\mathcal{D}_R^*\setminus\{1\}$.

Remark 12. If we use the Ansatz given by the Fröbenius method and write

$$\Phi_t(z,\lambda)=M_t(\lambda)z^{A_t(\lambda)}P_t(z,\lambda) \tag{4.18}$$

where

$$P_t(z,\lambda)=\sum_{k=0}^{\infty}P_{t,k}(\lambda)z^k,$$

note that the resonance points only occur in the computation of $P_{t,1}(\lambda)$ because $\mathcal{L}_{t,n}$ is invertible on $(t,\lambda)\in(-T,T)\times\mathcal{A}_R$ for all $n\geq 2$. Thus, we only need to extend $P_{t,1}(\lambda)$ at $t=0$ and $\lambda=1$ to extend the z^AP form of Φ_t. Let us write

$$tC_t(\lambda)dz := \left(\xi_t(z,\lambda)-A_t(\lambda)z^{-1}dz\right)|_{z=0}\,.$$

According to (4.15),

$$P_{t,1}(\lambda)=\mathcal{L}_{t,1}^{-1}(tC_t(\lambda)) \tag{4.19}$$

and the form of $\det\mathcal{L}_{t,1}$ shows that $P_{t,1}$ has at most a pole of order 2 at $\lambda=1$. Moreover, $\det\mathcal{L}_{t,1}=\mathcal{O}(t)$ and $tC_t=\mathcal{O}(t)$, so we already know that P_t (and as a consequence, M_t) extends to $t=0$.

It remains to extend the z^AP form (4.18) to $\lambda=1$. To do this, we adapt the techniques used in Lemma 2.5 of [31] to prove the following unitary $\times$ commutator lemma:

Lemma 3. Let $M : \mathcal{A}_R\backslash\{1\} \longrightarrow \mathrm{SL}_2\mathbb{C}$ holomorphic on $\mathcal{A}_R\backslash\{1\}$ with at most a pole at $\lambda = 1$. Let $t \neq 0$, $\mathcal{Q} = \exp(2i\pi A_t) \in \Lambda\mathrm{SU}_2$ and suppose that for all $\lambda \in \mathcal{A}_1\backslash\{1\}$, $M\mathcal{Q}M^{-1} \in \mathrm{SU}_2$. Then there exist $U \in \Lambda\mathrm{SU}_2$ and $K : \mathcal{A}_R\backslash\{1\} \longrightarrow \mathrm{SL}_2\mathbb{C}$ holomorphic such that

$$\begin{cases} M = UK \\ [A_t, K] = 0. \end{cases}$$

Proof. We first apply Lemma 2.5 of [31] to construct U and K satisfying $M = UK$ and $[\mathcal{Q}, K] = 0$ on $\mathcal{A}_1\backslash\{1\}$. The map U is holomorphic on a small neighbourhood of $\mathcal{A}_1$. Without loss of generality, let this neighbourhood be $\mathcal{A}_R$. Then, K is meromorphic on $\mathcal{A}_R\backslash\{1\}$ with at most a pole at $\lambda = 1$. Hence the map $\lambda \longmapsto [\mathcal{Q}(\lambda), K(\lambda)]$ is holomorphic on $\mathcal{A}_R\backslash\{1\}$ and vanishes on $\mathcal{A}_1\backslash\{1\}$. Thus, for all $\lambda \in \mathcal{A}_R\backslash\{1\}$,

$$[\mathcal{Q}(\lambda), K(\lambda)] = 0. \tag{4.20}$$

Recalling Equation (4.5),

$$\mathcal{Q} = \cos(2\pi\mu_t)\mathrm{I}_2 + \frac{i\sin(2\pi\mu_t)}{\mu_t} A_t.$$

Hence Equation (4.20) implies that $[A_t, K] = 0$ wherever $\mu_t(\lambda)^2 \neq \frac{1}{4}$. Using (4.6), $[A_t(\lambda), K(\lambda)] = 0$ for all $(t, \lambda) \in (-T, T)\backslash\{0\} \times \mathcal{A}_R\backslash\{1\}$. □

We can now extend the $z^A P$ form of Φ_t to $\lambda = 1$. For $t \neq 0$ and $\lambda \in \mathcal{A}_1\backslash\{1\}$, use Lemma 3 to write

$$\Phi_t(z, \lambda) = U_t(\lambda) z^{A_t(\lambda)} K_t(\lambda) P_t(z, \lambda).$$

Let $\epsilon > 0$ small enough for $P_t(\cdot, \lambda)$ to be defined on $\overline{\mathbb{D}}_\epsilon$. On $\mathbb{S}_\epsilon \times \mathcal{A}_1\backslash\{1\}$, Φ_t and z^{A_t} are bounded. Then the map $(z, \lambda) \longmapsto K_t P_t$ is bounded on $\mathbb{S}_\epsilon \times \mathcal{A}_1\backslash\{1\}$ and holomorphic on $\mathbb{D}_\epsilon \times \mathcal{A}_1\backslash\{1\}$, so it is bounded on $\mathbb{D}_\epsilon \times \mathcal{A}_1\backslash\{1\}$. But $P_t(0, \lambda) = \mathrm{I}_2$, so K_t is bounded on $\mathcal{A}_1\backslash\{1\}$. Thus, P_t is bounded on $\mathbb{D}_\epsilon \times \mathcal{A}_1\backslash\{1\}$. But P_t is holomorphic on $\mathbb{D}_\epsilon \times \mathcal{A}_R\backslash\{1\}$ with at most a pole at $\lambda = 1$, so P_t is holomorphic on $\mathbb{D}_\epsilon \times \mathcal{A}_R$ and M_t is holomorphic on $\mathcal{A}_R$. This ends the proof of Proposition 17.

4.3.2 A property of ξ_t

The fact that there exists a holomorphic frame Φ_t associated to ξ_t such that $\mathcal{M}(\Phi_t) \in \Lambda\mathrm{SU}_2$ and $\Phi_0(1, \lambda) = \mathrm{I}_2$ gives us a piece of information on the potential ξ_t. Let $C_t(\lambda) \in \mathfrak{sl}_2\mathbb{C}$ so that

$$\xi_t(z, \lambda) = A_t(\lambda) z^{-1} dz + tC_t(\lambda) dz + \mathcal{O}(t, z) dz$$

and write

$$C_t(\lambda) = \begin{pmatrix} c_{11}(t,\lambda) & \lambda^{-1}c_{12}(t,\lambda) \\ c_{21}(t,\lambda) & -c_{11}(t,\lambda) \end{pmatrix}. \tag{4.21}$$

Define

$$p_t = \frac{sc_{12}(t,0) + rc_{21}(t,0)}{2}. \tag{4.22}$$

Lemma 4. The quantity p_t vanishes at $t = 0$.

Proof. First, note that $\Phi_0(1,\lambda) = \mathrm{I}_2$ implies that $\Phi_0(z,\lambda) = z^{A_0(\lambda)}$, and thus $\mathcal{M}(\Phi_0) = -\mathrm{I}_2$. Let $\gamma \subset \mathbb{D}^*_\epsilon$ be a closed loop around 0. Apply Proposition 20 of Section 4.7 to get (X' denotes the derivative of X at $t = 0$ and R_t is the holomorphic part of ξ_t)

$$\begin{aligned} \mathcal{M}(\Phi_t)' &= \int_\gamma z^{A_0}\xi' z^{-A_0} \times \mathcal{M}(\Phi_0) \\ &= -\int_\gamma z^{A_0}\left(A'z^{-1}\right)z^{-A_0}dz - \int_\gamma z^{A_0}R'z^{-A_0}dz \\ &= \mathcal{M}(z^{A_t})' - \int_\gamma z^{A_0}R'z^{-A_0}dz. \end{aligned}$$

But $\mathcal{M}(\Phi_t), \mathcal{M}(z^{A_t}) \in \Lambda\mathrm{SU}_2$ and $\mathcal{M}(\Phi_0) = \mathcal{M}(z^{A_0}) = -\mathrm{I}_2$. Thus, $\mathcal{M}(\Phi_t)', \mathcal{M}(z^{A_t})' \in \Lambda\mathfrak{su}_2$ and

$$\int_\gamma z^{A_0}R'z^{-A_0}dz \in \Lambda\mathfrak{su}_2. \tag{4.23}$$

Diagonalise $A_0 = HDH^{-1}$ with

$$D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix}$$

and $H \in \Lambda\mathrm{SU}_2$ to be expressed later. Then

$$z^D = \frac{1}{\sqrt{z}}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} \int_\gamma z^{A_0}R'z^{-A_0}dz &= \int_\gamma Hz^DH^{-1}\left(C_0 + \mathcal{O}(z)\right)Hz^{-D}H^{-1} \\ &= H\left(\mathrm{Res}_{z=0}z^DH^{-1}C_0Hz^{-D}\right)H^{-1}. \end{aligned}$$

Equation (4.23) and $H \in \Lambda\mathrm{SU}_2$ imply that

$$\mathrm{Res}_{z=0}\left(z^DH^{-1}C_0Hz^{-D}\right) \in \Lambda\mathfrak{su}_2. \tag{4.24}$$

Denoting by $c(\lambda)$ the bottom-left entry of $H^{-1}C_0H$ and looking at the product $z^D(H^{-1}C_0H)z^{-D}$, Equation (4.24) gives

$$\begin{pmatrix} 0 & 0 \\ c(\lambda) & 0 \end{pmatrix} \in \Lambda\mathfrak{su}_2$$

and thus,

$$c(\lambda) = \left(H^{-1}C_0H\right)_{21} \equiv 0 \tag{4.25}$$

Two cases can occur:

- If $r \geq s$,

  $$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix} \in \Lambda\mathrm{SU}_2$$

  and computation gives

  $$c(\lambda) = -\lambda\left(c_{11}(0,\lambda) + \frac{c_{12}(0,\lambda)}{2}\right) + \frac{c_{21}(0,\lambda)}{2}.$$

  Using Equation (4.25), $c_{21}(0,0) = 0$ and $p_0 = 0$.

- If $r \leq s$, the same reasoning applies with

  $$H(\lambda) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } c(\lambda) = -\lambda^{-1}\frac{c_{12}(0,\lambda)}{2} + \frac{c_{21}(0,\lambda)}{2} - c_{11}(0,\lambda).$$

  Thus, $c_{12}(0,0) = 0$ and $p_0 = 0$.

□

4.3.3 Gaining an order of convergence

We can now prove Proposition 16 by following the method used in Section 2.2 of [20]: gauging the potential. The gauge we will use is of the following form:

$$G_t(z,\lambda) = \exp\left(g_t(\lambda)z\right) \tag{4.26}$$

which is an admissible gauge provided that $g_t \in \Lambda_+\mathfrak{sl}_2\mathbb{C}$. This is why we need the following lemma:

Lemma 5. Let

$$g_t(\lambda) = p_tA_t(\lambda) - P_{t,1}(\lambda)$$

where $P_{t,1}$ is defined in Equation (4.19). Then

1. The map g_t is in $\Lambda_+\mathfrak{sl}_2\mathbb{C}$.

2. The map g_t extends to $t = 0$ with $g_0 = 0$.

Proof. To prove the first point, let $t \neq 0$ and use Equations (4.19), (4.21), (4.17) and (4.22) to compute (this is a tedious calculation)

$$P_{t,1}(\lambda) = \lambda^{-1} \begin{pmatrix} 0 & rp_t \\ 0 & 0 \end{pmatrix} + \lambda^0 \begin{pmatrix} \star & \star \\ sp_t & \star \end{pmatrix} + \mathcal{O}(\lambda).$$

Thus,

$$g_t(\lambda) = p_t A_t(\lambda) - P_{t,1}(\lambda) = \lambda^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \lambda^0 \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} + \mathcal{O}(\lambda).$$

For the second point, use Equations (4.19) and (4.17) to write for $t \neq 0$:

$$P_{t,1} = t\mathcal{L}_{t,1}^{-1}(C_t) = t\left(C_t - \frac{1}{1-4\mu_t^2}\left(\mathrm{I}_2 - 2A_t\right)[A_t, C_t]\right).$$

Note that C_t is continuous at $t = 0$ because $\xi_t \in \mathcal{C}^2$ and that $1 - 4\mu_t^2 = \mathcal{O}(t)$ to extend $P_{t,1}$ to $t = 0$. Moreover, recall Lemma 4, Equation (4.6) and diagonalise $A_0 = HDH^{-1}$ to get:

$$g_0 = \frac{-\lambda}{4(\lambda-1)^2} H\left(\mathrm{I}_2 - 2D\right)\left[D, H^{-1}C_0H\right]H^{-1}.$$

A straightforward computation gives

$$\left(\mathrm{I}_2 - 2D\right)\left[D, H^{-1}C_0H\right] = \begin{pmatrix} 0 & 0 \\ -2c(\lambda) & 0 \end{pmatrix}$$

with $c(\lambda)$ as in Equation (4.25). Hence $g_0 = 0$. □

Let G_t be the gauge defined by (4.26). Then the gauged potential has the form

$$\begin{aligned} \xi_t \cdot G_t(z,\lambda) &= A_t(\lambda)z^{-1}dz + \left([A_t(\lambda), g_t(\lambda)] + g_t(\lambda) + tC_t(\lambda)\right)dz + \mathcal{O}(t,z)dz + \mathcal{O}(g_t^2 z)dz \\ &= A_t(\lambda)z^{-1}dz + \left(\mathcal{L}_{t,1}(g_t(\lambda)) + tC_t(\lambda)\right)dz + \mathcal{O}(t,z)dz \\ &= A_t(\lambda)z^{-1}dz + p_t A_t(\lambda)dz + \mathcal{O}(t,z)dz, \end{aligned}$$

because of Equation (4.19). This gauge has been chosen to fit with the following change of coordinate:

$$h_t(z) = \frac{z}{1 + p_t z}.$$

The resulting potential (defined in Proposition 16) is then

$$\widetilde{\xi}_t = A_t \frac{dz}{1+p_t z} + p_t A_t \frac{dz}{(1+p_t z)^2} + \mathcal{O}(t,z)dz = A_t z^{-1} dz + \mathcal{O}(t,z)dz$$

because $p_0 = 0$. Apply the Fröbenius method to $\widetilde{\xi}_t$ to obtain (4.14) and choose $\epsilon' \leq \epsilon$ such that for all $t \neq 0$, $\epsilon' < |p_t|^{-1}$ to end the proof of Proposition 16.

4.4 Convergence of immersions

In this section, we prove the first and third points of Theorem 10. In the end, we want to compare $\Phi_t(z,\lambda) = M_t(\lambda) z^{A_t(\lambda)} \left(\mathrm{I}_2 + \mathcal{O}(t,z^2)\right)$ to

$$\Phi_t^{\mathcal{D}}(z,\lambda) = M_t(\lambda) z^{A_t(\lambda)}.$$

We will denote

$$F_t^{\mathcal{D}} = \mathrm{Uni}(\Phi_t^D)$$

and

$$f_t^{\mathcal{D}} = \mathrm{Sym}(F_t^{\mathcal{D}}).$$

We first want to make sure that $\Phi_t^{\mathcal{D}}$ induces a Delaunay surface for all t. For this purpose, recall Lemma 1.12 in [20], which implies that $f_t^{\mathcal{D}}$ is a Delaunay surface of weight $8\pi t$. Hence, there exists a rigid motion ϕ of $\mathbb{R}^3$ such that $\phi \circ f_t^{\mathcal{D}}$ has the following parametrisation:

$$\begin{array}{rccc} \phi \circ f_t^{\mathcal{D}} & : & \Sigma & \longrightarrow & \mathbb{R}^3 \\ & & z = e^{x+iy} & \longmapsto & (\tau_t(x), \sigma_t(x)\cos y, \sigma_t(x)\sin y) \end{array}$$

where $(\tau_t(x), \sigma_t(x))$ is the profile curve of the surface. Recalling that the coordinates are isothermal gives the following metric:

$$ds_t^2 = \sigma_t^2 \frac{|dz|^2}{|z|^2}. \tag{4.27}$$

Let us compare the asymptotic behaviours of the unitary parts of Φ_t and $\Phi_t^{\mathcal{D}}$ for $\lambda \in \mathcal{A}_1$ using, as in [20], a Cauchy formula. We will use the following norms:

— For $v = (v_1, v_2) \in \mathbb{C}^2$, $|v| = \left(|v_1|^2 + |v_2|^2\right)^{\frac{1}{2}}$.

— For $M \in \mathcal{M}_2(\mathbb{C})$, $\|M\| = \sup_{|v|=1} |Mv|$.

— For $\Psi : \mathcal{E} \longrightarrow \mathcal{M}_2(\mathbb{C})$, $\|\Phi\|_{\mathcal{E}} = \sup_{\lambda \in \mathcal{E}} \|\Psi(\lambda)\|$.

Lemma 6. For all $\alpha < 1$ there exist constants $\epsilon > 0$, $T > 0$ and $C > 0$ such that for all $0 < |z| < \epsilon$ and $|t| < T$,

$$\left\| \left(F_t^{\mathcal{D}}\right)^{-1} F_t - \mathrm{I}_2 \right\|_{\mathcal{A}_1} \leq C|t||z|^{\alpha} \tag{4.28}$$

and

$$\left\| \frac{\partial}{\partial \lambda} \left[\left(F_t^{\mathcal{D}}\right)^{-1} F_t \right] \right\|_{\mathcal{A}_1} \leq C|t||z|^{\alpha}. \tag{4.29}$$

Proof. The first step is to estimate the norm of the positive part $B_t^{\mathcal{D}}$ of $\Phi_t^{\mathcal{D}}$. We first estimate $\Phi_t^{\mathcal{D}}$ for $|z| < 1$: noting that A_t is diagonalisable, that its eigenvalues tend to $\pm 1/2$ as $t \to 0$, and recalling that M_t is continuous at $t = 0$ ensure that for all $\alpha < 1$ there exists (T, R) and $C_1 > 1$ such that for all $|t| < T$,

$$\left\| \Phi_t^{\mathcal{D}}(z, \lambda) \right\|_{\mathcal{A}_R} \leq C_1 |z|^{-\frac{1}{2} - \frac{1-\alpha}{4}}.$$

We then estimate $F_t^{\mathcal{D}}$: let $\gamma \subset \mathbb{C}^*$ be a path from z to 1, use Equation (4.40) of Section 4.8 and Equation (4.27) to get

$$\left\| F_t^{\mathcal{D}}(z, \lambda) \right\|_{\mathcal{A}_R} \leq C_2 \left\| F_t^{\mathcal{D}}(1, \lambda) \right\|_{\mathcal{A}_R} \times \exp\left(\frac{(R-1)}{2} \int_{\gamma} \frac{|\sigma_t(\log|z|)|}{|z|} \right).$$

But σ_t is uniformly bounded because so is the distance between the profile curve and the axis of a Delaunay surface. Moreover, the unitary frame at $z = 1$ is also bounded. Hence the existence, for $R > 1$ small enough, of a constant $C_3 \geq 1$ such that

$$\left\| F_t^{\mathcal{D}}(z, \lambda) \right\|_{\mathcal{A}_R} \leq C_3 |z|^{-\frac{1-\alpha}{4}}.$$

We can now estimate the positive factor: for all $\alpha < 1$ there exist $T > 0$, $R > 1$ and $C_4 \geq 1$ such that for all $|t| < T$ and $|z| < 1$

$$\left\| B_t^{\mathcal{D}}(z, \lambda) \right\|_{\mathcal{A}_R} \leq \left\| F_t^{\mathcal{D}}(z, \lambda)^{-1} \right\|_{\mathcal{A}_R} \times \left\| \Phi_t^{\mathcal{D}}(z, \lambda) \right\|_{\mathcal{A}_R} \leq C_4 |z|^{\frac{\alpha}{2} - 1}.$$

We then define

$$\begin{array}{rcccl} \widetilde{\Phi}_t & := & \left(\left(F_t^{\mathcal{D}}\right)^{-1} F_t\right) & \times & \left(B_t \left(B_t^{\mathcal{D}}\right)^{-1}\right) \quad = B_t^{\mathcal{D}} \left(\Phi_t^{\mathcal{D}}\right)^{-1} \Phi_t \left(B_t^{\mathcal{D}}\right)^{-1} \\ & =: & \widetilde{F}_t & \times & \widetilde{B}_t \end{array}$$

with $\widetilde{F}_t \in \Lambda \mathrm{SU}_2$ and $\widetilde{B}_t \in \Lambda^{\mathbb{R}}_+ \mathrm{SL}_2\mathbb{C}$ and thus have

$$\begin{aligned}\left\|\widetilde{\Phi}_t(z,\lambda) - \mathrm{I}_2\right\|_{\mathcal{A}_R} &= \left\|B_t^{\mathcal{D}}(z,\lambda)\left(P_t(z,\lambda) - \mathrm{I}_2\right)\left(B_t^{\mathcal{D}}(z,\lambda)\right)^{-1}\right\|_{\mathcal{A}_R} \\ &\leq \left\|B_t^{\mathcal{D}}(z,\lambda)\right\|^2_{\mathcal{A}_R} \mathcal{O}(t,|z|^2) \\ &\leq C|t||z|^\alpha.\end{aligned}$$

Let n_k denote the seminorms

$$n_k(X) = \sum_{j=0}^{k} \left\| \frac{\partial^k X}{\partial \lambda^k} \right\|_{\mathcal{A}_1}.$$

Apply Cauchy formula with $\lambda \in \partial\mathcal{A}_R$ to get

$$n_k\left(\widetilde{\Phi}_t - \mathrm{I}_2\right) \leq c_k|t||z|^\alpha, \ \forall k \in \mathbb{N}$$

where $c_k > 0$ are uniform constants. But $\mathrm{Uni}(\widetilde{\Phi}_t) = \widetilde{F}_t = \left(F_t^{\mathcal{D}}\right)^{-1} F_t$ and Iwasawa decomposition is a $\mathcal{C}^1$-diffeomorphism, so $n_0\left(\widetilde{F}_t - \mathrm{I}_2\right) \leq C|t||z|^\alpha$ and $n_1\left(\widetilde{F}_t - \mathrm{I}_2\right) \leq C|t||z|^\alpha$. We then have (4.28) and (4.29). □

The asymptotic behaviour of $\frac{\partial \widetilde{F}_t}{\partial \lambda}$ allows us to prove the convergence of immersions as stated in the first point of Theorem 10. The Sym-Bobenko formula for $\mathbb{R}^3$ implies that (we omit the index t)

$$\begin{aligned}iF(z,1)\frac{\partial(F^{-1}F^{\mathcal{D}})}{\partial\lambda}(z,1)F^{\mathcal{D}}(z,1)^{-1} &= i\frac{\partial F^{\mathcal{D}}}{\partial\lambda}(z,1)F^{\mathcal{D}}(z,1)^{-1} - i\frac{\partial F}{\partial\lambda}(z,1)F(z,1)^{-1} \\ &= f^{\mathcal{D}}(z) - f(z).\end{aligned}$$

We can then compute

$$\begin{aligned}\left\|f_t(z) - f_t^{\mathcal{D}}(z)\right\|^2_{\mathbb{R}^3} &= 4\det\left(f_t(z) - f_t^{\mathcal{D}}(z)\right) \\ &= -4\det\frac{\partial(F_t^{-1}F_t^{\mathcal{D}})}{\partial\lambda}(z,1) \\ &\leq C_2^2 t^2|z|^{2\alpha}.\end{aligned}$$

And then for all $\alpha < 1$ there exist constants $\epsilon > 0$, $T > 0$ and $C > 0$ such that for all $0 < |z| < \epsilon$ and $|t| < T$,

$$\|f_t(z) - f_t^{\mathcal{D}}(z)\|_{\mathbb{R}^3} \leq C|t||z|^\alpha. \tag{4.30}$$

To prove the third point of Theorem 10, use (4.4) and note that $M_0 = \mathrm{I}_2$. So the axis

of $f_t^{\mathcal{D}}$ as $t \to 0$ is the same that the axis of the unperturbed Delaunay surface induced by z^{A_t}.

In order to prove that the surface is embedded, we will need the convergence of the normal maps:

Proposition 18. For all $\alpha < 1$ there exist constants $\epsilon > 0$, $T > 0$ and $C > 0$ such that for all $0 < |z| < \epsilon$ and $|t| < T$,

$$\left\|\mathcal{N}_t(z) - \mathcal{N}_t^{\mathcal{D}}(z)\right\|_{\mathbb{R}^3} \leq C|t||z|^{\alpha}$$

Proof. Use the definition of the normal maps in Equation (4.2) to write

$$\mathcal{N}_t(z) - \mathcal{N}_t^{\mathcal{D}}(z) = \frac{-i}{2} F_t^{\mathcal{D}}(z,1)\left[AM\widetilde{A} + AM + M\widetilde{A}\right] F_t^{\mathcal{D}}(z,1)^{-1}$$

where

$$A = F_t^{\mathcal{D}}(z,1)^{-1} F_t(z,1) - \mathrm{I}_2 = \mathcal{O}(t, |z|^{\alpha}),$$
$$\widetilde{A} = F_t(z,1)^{-1} F_t^{\mathcal{D}}(z,1) - \mathrm{I}_2 = \mathcal{O}(t, |z|^{\alpha})$$

and

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Use equation (4.1) to get the conclusion. □

It remains to show that the surface is embedded if $t > 0$.

4.5 Embeddedness

We suppose in this section that $0 < t < T$. The asymptotic behaviour of f_t and the fact that $f_t^{\mathcal{D}}$ is an embedding for all t allow us to show that f_t is an embedding of a sufficiently small uniform neighbourhood of $z = 0$ for t small enough. We first give a general result of embeddedness and then apply this result to show that our surfaces are embedded.

Proposition 19. Let $f_n^{\mathcal{R}} : \mathbb{C}^* \longrightarrow \mathcal{M}_n^{\mathcal{R}} = f_n^{\mathcal{R}}(\mathbb{C}^*) \subset \mathbb{R}^3$ be a sequence of complete immersions with normal maps $\mathcal{N}_n^{\mathcal{R}}$ and an end at $z = 0$. Suppose that for all n there exists $r_n > 0$ such that the tubular neighbourhood $\mathrm{Tub}_{r_n}\mathcal{M}_n^{\mathcal{R}}$ of $\mathcal{M}_n^{\mathcal{R}}$ is embedded. Suppose that for all $\epsilon > 0$ there exists $0 < \epsilon' < \epsilon$ such that for all $n \in \mathbb{N}$, $x \in \mathbb{S}_{\epsilon}$ and $y \in \mathbb{D}^*_{\epsilon'}$,

$$\left\|f_n^{\mathcal{R}}(x) - f_n^{\mathcal{R}}(y)\right\|_{\mathbb{R}^3} > 2r_n. \tag{4.31}$$

Let $U^* \subset \mathbb{C}^*$ be a punctured neighbourhood of $z = 0$ and $f_n : U^* \longrightarrow \mathbb{R}^3$ a sequence of immersions with normal maps $\mathcal{N}_n$ satisfying

$$\sup_{n\in\mathbb{N}} \frac{\left\|f_n(z) - f_n^{\mathcal{R}}(z)\right\|_{\mathbb{R}^3}}{r_n} \underset{z\to 0}{\longrightarrow} 0 \tag{4.32}$$

and

$$\sup_{z\in U^*} \left\|\mathcal{N}_n(z) - \mathcal{N}_n^{\mathcal{R}}(z)\right\|_{\mathbb{R}^3} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.33}$$

Then there exist $\epsilon' > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$, f_n is an embedding of $\mathbb{D}^*_{\epsilon'}$.

Proof. Let us split the proof in several steps.

- *Claim* 1: there exists $\epsilon > 0$ such that the map

$$\begin{array}{rccc} \varphi_n & : & \mathbb{D}^*_\epsilon & \longrightarrow & \mathcal{M}_n^{\mathcal{R}} \\ & & z & \longmapsto & \pi_n \circ f_n(z) \end{array}$$

  (where π_n is the projection from $\mathrm{Tub}_{r_n}\mathcal{M}_n^{\mathcal{R}}$ onto $\mathcal{M}_n^{\mathcal{R}}$) is well-defined and satisfies

$$\left\|\varphi_n(z) - f_n^{\mathcal{R}}(z)\right\|_{\mathbb{R}^3} < r_n \tag{4.34}$$

  for all $z \in \mathbb{D}^*_\epsilon$.

To prove this first claim, use Hypothesis (4.32): there exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$ and $z \in \mathbb{D}^*_\epsilon$

$$\left\|f_n(z) - f_n^{\mathcal{R}}(z)\right\|_{\mathbb{R}^3} < \frac{r_n}{2}. \tag{4.35}$$

So $f_n(\mathbb{D}^*_\epsilon) \subset \mathrm{Tub}_{\frac{r_n}{2}}\mathcal{M}_n^{\mathcal{R}}$ and φ_n is well-defined. Moreover, using (4.35) and the triangle inequality, for all $z \in \mathbb{D}^*_\epsilon$

$$\left\|\varphi_n(z) - f_n^{\mathcal{R}}(z)\right\|_{\mathbb{R}^3} \leq \left\|\varphi_n(z) - f_n(z)\right\|_{\mathbb{R}^3} + \left\|f_n(z) - f_n^{\mathcal{R}}(z)\right\|_{\mathbb{R}^3} < r_n$$

and Equation (4.34) holds. We fix ϵ and ϵ' so that Equation (4.31) is satisfied.

- *Claim 2*: there exists $N \in \mathbb{N}$ such that for all $n \geq N$, φ_n is a local diffeomorphism on $\mathbb{D}^*_\epsilon$.

Let $z \in \mathbb{D}^*_\epsilon$. In order to show that φ_n is a local diffeomorphism, we show that

$$\langle \mathcal{N}_{\varphi_n}(z), \mathcal{N}_n(z)\rangle > 0 \tag{4.36}$$

where $\mathcal{N}_{\varphi_n}$ is defined by

$$\begin{array}{rccc} \mathcal{N}_{\varphi_n} & : & \mathbb{D}_\epsilon^* & \longrightarrow & \mathbb{S}^2 \subset \mathbb{R}^3 \\ & & z & \longmapsto & \eta_n^{\mathcal{R}}(\varphi_n(z)) \end{array}$$

and $\eta_n^{\mathcal{R}}$ is the Gauss map of $\mathcal{M}_n^{\mathcal{R}}$. First, let $\gamma \subset \mathcal{M}_n^{\mathcal{R}}$ be a path joining $\varphi_n(z)$ to $f_n^{\mathcal{R}}(z)$. Using the fact that $\mathrm{Tub}_{r_n}\mathcal{M}_n^{\mathcal{R}}$ is embedded, one has

$$\left\|d\eta_n^{\mathcal{R}}\right\| \leq \frac{1}{r_n}$$

and

$$\left\|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n^{\mathcal{R}}(z)\right\|_{\mathbb{R}^3} \leq \frac{1}{r_n} \times |\gamma|.$$

Let $\sigma(t) = (1-t)f_n(z) + tf_n^{\mathcal{R}}(z)$, $t \in [0,1]$. Then,

$$\left\|\sigma(t) - f_n^{\mathcal{R}}(z)\right\|_{\mathbb{R}^3} \leq (1-t)\left\|f_n(z) - f_n^{\mathcal{R}}(z)\right\|_{\mathbb{R}^3} < \frac{r_n}{2} \tag{4.37}$$

because of Equation (4.35). Let $\gamma = \pi_n \circ \sigma$. Note that Equation (4.37) implies that $\sigma \subset \mathrm{Tub}_{\frac{r_n}{2}}\mathcal{M}_n^{\mathcal{R}}$ and restricting π_n to $\mathrm{Tub}_{\frac{r_n}{2}}\mathcal{M}_n^{\mathcal{R}}$ gives

$$\|d\pi_n\| \leq \frac{r_n}{r_n - \frac{r_n}{2}} = 2$$

and thus $|\gamma| < r_n$. Hence,

$$\left\|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n^{\mathcal{R}}(z)\right\| < 1.$$

Use Hypothesis (4.33) to choose a uniform $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n(z)\| \leq \left\|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n^{\mathcal{R}}(z)\right\| + \left\|\mathcal{N}_n^{\mathcal{R}}(z) - \mathcal{N}_n(z)\right\| < \sqrt{2},$$

which proves Equation (4.36) and this second claim. We fix such N and n.

- *Claim 3*: the restriction

$$\begin{array}{rccc} \widetilde{\varphi}_n & : & \varphi_n^{-1}\left(\varphi_n(\mathbb{D}_{\epsilon'}^*)\right) \cap \mathbb{D}_\epsilon^* & \longrightarrow & \varphi_n\left(\mathbb{D}_{\epsilon'}^*\right) \\ & & z & \longmapsto & \varphi_n(z) \end{array}$$

  is a covering map.

It sufices to show that $\widetilde{\varphi}_n$ is a proper map. Let $(x_i)_{i\in\mathbb{N}} \subset \varphi_n^{-1}\left(\varphi_n(\mathbb{D}_{\epsilon'}^*)\right) \cap \mathbb{D}_\epsilon^*$ such that $(\widetilde{\varphi}_n(x_i))_{i\in\mathbb{N}}$ converges to $p \in \varphi_n\left(\mathbb{D}_{\epsilon'}^*\right)$. Then $(x_i)_i$ converges to $x \in \overline{\mathbb{D}}_\epsilon$. Using Equation (4.34) and the fact that $f_n^{\mathcal{R}}$ has an end at 0, $x \neq 0$. If $x \in \partial\mathbb{D}_\epsilon$, denoting $\widetilde{x} \in \mathbb{D}_{\epsilon'}^*$ such that

$\widetilde{\varphi}_n(\widetilde{x}) = p$, one has

$$\left\|f_n^{\mathcal{R}}(x) - f_n^{\mathcal{R}}(\widetilde{x})\right\|_{\mathbb{R}^3} < \left\|f_n^{\mathcal{R}}(x) - p\right\|_{\mathbb{R}^3} + \left\|f_n^{\mathcal{R}}(\widetilde{x}) - \widetilde{\varphi}_n(\widetilde{x})\right\|_{\mathbb{R}^3} < 2r_n$$

which contradicts the definition of ϵ'. Thus, $\widetilde{\varphi}_n$ is a proper local diffeomorphism between locally compact spaces, i.e. a covering map.

- *Claim 4*: this covering map is one-sheeted.

To compute the number of sheets, let $\gamma : [0,1] \longrightarrow \mathbb{D}^*_{\epsilon'}$ be a loop of winding number 1 around 0, $\Gamma = f_n^{\mathcal{R}}(\gamma)$ and $\widetilde{\Gamma} = \widetilde{\varphi}_n(\gamma) \subset \mathcal{M}_n^{\mathcal{R}}$ and let us construct a homotopy between Γ and $\widetilde{\Gamma}$. Let

$$\begin{array}{rccc} \sigma_t & : & [0,1] & \longrightarrow & \mathbb{R}^3 \\ & & s & \longmapsto & (1-s)\widetilde{\Gamma}(t) + s\Gamma(t). \end{array}$$

For all $t, s \in [0,1]$,

$$\|\sigma_t(s) - \Gamma(t)\|_{\mathbb{R}^3} < r_n$$

which implies that $\sigma_t(s) \in \mathrm{Tub}_{r_n}\mathcal{M}_n^{\mathcal{R}}$ because $\mathcal{M}_n^{\mathcal{R}}$ is complete. One can thus define the following homotopy between Γ and $\widetilde{\Gamma}$

$$\begin{array}{rccc} H & : & [0,1]^2 & \longrightarrow & \mathcal{M}_n^{\mathcal{R}} \\ & & (s,t) & \longmapsto & \pi_n \circ \sigma_t(s) \end{array}$$

where π_n is the projection from $\mathrm{Tub}_{r_n}\mathcal{M}_n^{\mathcal{R}}$ to $\mathcal{M}_n^{\mathcal{R}}$. Using the fact that $f_n^{\mathcal{R}}$ is an embedding, the degree of Γ is one, and the degree of $\widetilde{\Gamma}$ is also one. Hence, $\widetilde{\varphi}_n$ is one-sheeted.

- *Conclusion*: the map $\widetilde{\varphi}$ is a diffeomorphism, so $f_n\left(\mathbb{D}^*_{\epsilon'}\right)$ is a graph over $\mathcal{M}_n^{\mathcal{R}}$ contained in its embedded tubular neighbourhood and $f_n\left(\mathbb{D}^*_{\epsilon'}\right)$ is thus embedded.

□

We can now apply Proposition 19 to each case. Let (t_n) be any sequence in $(-T, T)$ such that $t_n \to 0$.

- If $r \geq s$, we set $\widehat{f}_n^{\mathcal{R}} = f_{t_n}^{\mathcal{D}}$ and $\widehat{f}_n = f_{t_n}$. We aim to apply Proposition 19 on $\widehat{f}_n^{\mathcal{R}}$ and $\widehat{f}_n$. The tubular radius r_n is of the order of $4t_n$ and Hypothesis (4.31) is satisfied because $\widehat{f}_n^{\mathcal{R}}$ tends to an immersion of a sphere. Equation (4.30) and Proposition 18 ensure that Hypotheses (4.32) and (4.33) hold.
- If $r \leq s$, we set $\widehat{f}_n^{\mathcal{R}} = \frac{1}{t_n} f_{t_n}^{\mathcal{D}}$ and $\widehat{f}_n = \frac{1}{t_n} f_{t_n}$. We aim to apply Proposition 19 on $\widehat{f}_n^{\mathcal{R}}$ and $\widehat{f}_n$. The tubular radius r_n is of the order of 4 and Hypothesis (4.31) is satisfied

because $\widehat{f}_n^{\mathcal{R}}$ tends to an immersion of a catenoid (see [38]). Equation (4.30) and Proposition 18 ensure that Hypotheses (4.32) and (4.33) hold.

The second point of our theorem is then proved.

4.6 Iwasawa extended

In this section, we set $\mathcal{A}_{\frac{1}{R},1} = \left\{\lambda \in \mathbb{C} \ : \ \frac{1}{R} < |\lambda| < 1\right\}$.

Lemma 7. Let $F : \mathcal{A}_{\frac{1}{R},1} \longrightarrow \mathrm{SL}_2\mathbb{C}$ be a holomorphic map that can be continuously extended to the circle $\mathcal{A}_1$ and such that $F(\lambda) \in \mathrm{SU}_2$ for all $\lambda \in \mathcal{A}_1$. Then F holomorphically extends to $\mathcal{A}_R$ into a map that satisfies

$$\overline{{}^tF\left(\frac{1}{\overline{\lambda}}\right)} = F(\lambda)^{-1} \qquad \forall \lambda \in \mathcal{A}_R. \tag{4.38}$$

Proof. Apply Schwarz reflexion principle on each coefficient of the matrix

$$\widetilde{F}(\lambda) = \begin{pmatrix} F_{11}(\lambda) + F_{22}(\lambda) & F_{12}(\lambda) - F_{21}(\lambda) \\ i\left(F_{12}(\lambda) + F_{21}(\lambda)\right) & i\left(F_{11}(\lambda) - F_{22}(\lambda)\right) \end{pmatrix}$$

where F_{ij} denote the entries of F. The fact that $F(\lambda) \in \mathrm{SU}_2$ for all $\lambda \in \mathcal{A}_1$ ensures that $\mathrm{Im}\widetilde{F} = 0$ on $\mathcal{A}_1$. Thus, $\widetilde{F}$ holomorphically extends to $\mathcal{A}_R$ and satisfies for all $\lambda \in \mathcal{A}_R$

$$\widetilde{F}\left(\frac{1}{\overline{\lambda}}\right) = \overline{\widetilde{F}(\lambda)}.$$

Hence, F holomorphically extends to $\mathcal{A}_R$ and satisfies

$$F_{11}\left(\frac{1}{\overline{\lambda}}\right) = \overline{F_{22}(\lambda)}, \qquad F_{12}\left(\frac{1}{\overline{\lambda}}\right) = -\overline{F_{21}(\lambda)}$$

which implies Equation (4.38) because $F(\lambda) \in \mathrm{SL}_2\mathbb{C}$. □

Corollary 4. Let $\Phi : \mathcal{A}_R \longrightarrow \mathrm{SL}_2\mathbb{C}$ be a holomorphic map and let FB be the Iwasawa decomposition of its restriction to $\mathcal{A}_1$. Then F holomorphically extends to $\mathcal{A}_R$, satisfies Equation (4.38), and B holomorphically extends to $\mathcal{D}_R$.

Proof. Write $F = \Phi B^{-1}$ to holomorphically extend F to $\mathcal{A}_{\frac{1}{R},1}$. Apply Lemma 7 to holomorphically extend F to $\mathcal{A}_R$, and write $B = F^{-1}\Phi$ to holomorphically extend B to $\mathcal{D}_R$. □

4.7 Derivative of the monodromy

The following proposition, used in Section 4.3, is derived from Proposition 8 in [39].

Proposition 20. Let ξ_t be a $\mathcal{C}^1$ family of matrix-valued 1-forms on a Riemann surface Σ, defined for t in a neighbourhood of $t_0 \in \mathbb{R}$. Let $\widetilde{\Sigma}$ be the universal cover of Σ. Fix a

point z_0 in Σ and let $\widetilde{z}_0$ be a lift of z_0 to $\widetilde{\Sigma}$. Let Φ_t be a continuous family of solutions of $d\Phi_t = \Phi_t\xi_t$ on $\widetilde{\Sigma}$ such that for all t,

$$\left[\mathcal{M}(t_0), \Phi_{t_0}(z_0)\Phi_t(z_0)^{-1}\right] = 0, \tag{4.39}$$

where $\mathcal{M}(t)$ is the monodromy of Φ_t with respect to some $\gamma \in \pi_1(\Sigma, z_0)$. Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{\Sigma}$ such that $\widetilde{\gamma}(0) = \widetilde{z}_0$. Then $\mathcal{M}$ is differentiable at t_0 and

$$\mathcal{M}'(t_0) = \left(\int_\gamma \Phi_{t_0}\frac{\partial \xi_t}{\partial t} \mid_{t=t_0} \Phi_{t_0}^{-1}\right) \times \mathcal{M}(t_0).$$

In particular, if $\mathcal{M}(t_0) = \pm \mathrm{I}_2$ or if $\Phi_t(z_0)$ is constant, then (4.39) is satisfied.

Proof. Proposition 8 in [39] is proved in the case where $\Phi_t(z_0)$ is constant. Let $\widetilde{\Phi}_t(z) = \Phi_t(z_0)^{-1}\Phi_t(z)$, so that $d\widetilde{\Phi}_t = \widetilde{\Phi}_t\xi_t$ and $\widetilde{\Phi}_t(z_0) = \mathrm{I}_n$. Let $\widetilde{\mathcal{M}}(t)$ be the monodromy of $\widetilde{\Phi}_t$ along γ. Then Proposition 5 of [39] applies and

$$\widetilde{\mathcal{M}}'(t_0) = \left(\int_\gamma \widetilde{\Phi}_{t_0}(z)\frac{\partial \xi_t(z)}{\partial t} \mid_{t=t_0} \widetilde{\Phi}_{t_0}(z)^{-1}\right) \times \widetilde{\mathcal{M}}(t_0).$$

On the other hand,

$$\mathcal{M}(t) = \Phi_t(z_0)\widetilde{\mathcal{M}}(t)\Phi_t(z_0)^{-1}$$

and because of Equation (4.39),

$$\mathcal{M}(t_0) = \Phi_t(z_0)\widetilde{\mathcal{M}}(t_0)\Phi_t(z_0)^{-1}.$$

Thus, $\mathcal{M}$ is differentiable at t_0 and

$$\mathcal{M}'(t_0) = \Phi_{t_0}(z_0)\widetilde{\mathcal{M}}'(t_0)\Phi_{t_0}(z_0)^{-1}$$

which proves the proposition. □

4.8 A control formula on the unitary frame

The following proposition is used in Section 4.4.

Proposition 21. Let $(\Sigma, \xi, z_0, \Phi_{z_0})$ be a set of untwisted DPW data, holomorphic for $\lambda \in \mathcal{A}_R$ with $R \geq 1$. Then for all $z_1, z_2 \in \Sigma$ and $\gamma \subset \Sigma$ joining z_1 to z_2,

$$\|F(z_1, \lambda)\|_{\mathcal{A}_R} \leq C \|F(z_2, \lambda)\|_{\mathcal{A}_R} \times \exp\left((R-1)\int_\gamma \rho^2(w)|a_{-1}(w)||dw|\right)$$

where C is a uniform positive constant, $a_{-1}(z)dz$ is the λ^{-1} factor of ξ and $\rho(z)$ is the upper-left entry of $\mathrm{Pos}(\Phi)(z,0)$.

Proof. Write

$$\xi(z,\lambda) = \lambda^{-1}\begin{pmatrix} 0 & a_{-1}(z) \\ 0 & 0 \end{pmatrix} dz + \lambda^0 \begin{pmatrix} c_0(z) & a_0(z) \\ b_0(z) & -c_0(z) \end{pmatrix} dz + \mathcal{O}(\lambda).$$

Let $\Phi = FB$ be the Iwasawa decomposition of Φ. Computing formula (4.40) of [7] in an explicit form and in the untwisted setting gives $dF = FL$ where

$$L(z,\lambda) = \begin{pmatrix} \rho^{-1}\rho_z & \lambda^{-1}\rho^2 a_{-1} \\ b_0\rho^{-2} & -\rho^{-1}\rho_z \end{pmatrix} dz + \begin{pmatrix} -\rho^{-1}\rho_{\bar{z}} & -\bar{b}_0\rho^{-2} \\ -\lambda\rho^2\overline{a}_{-1} & \rho^{-1}\rho_{\bar{z}} \end{pmatrix} d\bar{z}.$$

Let

$$\widetilde{F}(z,\lambda) = F\left(z, \frac{\lambda}{|\lambda|}\right)$$

so that $\widetilde{F}(z,\lambda) \in \mathrm{SU}_2$ for all $\lambda \in \mathcal{A}_R$. Then $d\widetilde{F} = \widetilde{F}\widetilde{L}$ where

$$\widetilde{L}(z,\lambda) = L\left(z, \frac{\lambda}{|\lambda|}\right).$$

Using the variation of constants method, for all $z_1, z_2 \in \Sigma$ (we ommit the variable λ),

$$F(z_1) = F(z_2)\widetilde{F}(z_2)^{-1}\widetilde{F}(z_1) + \left(\int_{z_2}^{z_1} F(w)\left(L(w) - \widetilde{L}(w)\right)\widetilde{F}(w)^{-1}\right)\widetilde{F}(z_1).$$

But

$$L(w,\lambda) - \widetilde{L}(w,\lambda) = \rho^2(w)\begin{pmatrix} 0 & a_{-1}(w)\lambda^{-1}\left(1-|\lambda|\right)dw \\ -\overline{a}_{-1}(w)\lambda\left(1-|\lambda|^{-1}\right)d\bar{w} & 0 \end{pmatrix}$$

so there exists a uniform constant $\widetilde{C}$ such that

$$\left\|L(w,\lambda) - \widetilde{L}(w,\lambda)\right\|_{\mathcal{A}_R} \leq \widetilde{C}(R-1)\rho^2(w)|a_{-1}(w)||dw|$$

and the result follows from Gronwall's inequality (Lemma 2.7 in [36]) using the fact that $\widetilde{F} \in \mathrm{SU}_2$ for all $\lambda \in \mathcal{A}_R$. □

As an application, recall that in the untwisted $\mathbb{R}^3$ setting, if $f = \mathrm{Sym}(F)$, then f is a CMC 1 conformal immersion whose metric is given by

$$ds = 2\rho^2|a_{-1}||dz|.$$

So let $z_1, z_2 \in \Sigma$ and $\gamma \subset \Sigma$ be a path joining $f(z_1)$ to $f(z_2)$. Then,

$$\|F(z_1,\lambda)\|_{\mathcal{A}_R} \leq C \|F(z_2,\lambda)\|_{\mathcal{A}_R} \exp\left(\frac{(R-1)}{2}|\gamma|\right). \tag{4.40}$$

Chapter 5

Constant mean curvature n-noids in Hyperbolic space

[1] Using the DPW method, we construct genus zero Alexandrov-embedded constant mean curvature (greater than one) surfaces with any number of Delaunay ends in hyperbolic space.

Introduction

In [7], Dorfmeister, Pedit and Wu introduced a loop group method (the DPW method) for constructing harmonic maps from a Riemann surface into a symmetric space. As a consequence, their method provides a Weierstrass-type representation of constant mean curvature surfaces (CMC) in Euclidean space $\mathbb{R}^3$, three-dimensional sphere $\mathbb{S}^3$, or hyperbolic space $\mathbb{H}^3$. Many examples have been constructed (see for example [19, 8, 32, 6, 13, 14]). Among them, Traizet [39, 38] showed how the DPW method in $\mathbb{R}^3$ can construct genus zero n-noids with Delaunay ends (as Kapouleas did with partial differential equations techniques in [16]) and glue half-Delaunay ends to minimal surfaces (as did Mazzeo and Pacard in [26], also with PDE techniques). A natural question is whether these constructions can be carried out in $\mathbb{H}^3$. Although properly embedded CMC annuli of mean curvature $H > 1$ in $\mathbb{H}^3$ are well-known since the work of Korevaar, Kusner, Meeks and Solomon [23], no construction similar to [16] or [26] can be found in the literature. This paper uses the DPW method in $\mathbb{H}^3$ to construct these surfaces. The two resulting theorems are as follows.

Theorem 11. Given a point $p \in \mathbb{H}^3$, $n \geq 3$ distinct unit vectors $u_1, \cdots, u_n$ in the tangent

1. This chapter is the second paper of the thesis. It has been assigned the arXiv number 1905.09096 but has not been submitted yet.

space of $\mathbb{H}^3$ at p and n non-zero real weights $\tau_1, \cdots, \tau_n$ satisfying the balancing condition

$$\sum_{i=1}^{n} \tau_i u_i = 0 \tag{5.1}$$

and given $H > 1$, there exists a smooth 1-parameter family of CMC H surfaces $(M_t)_{0<t<T}$ with genus zero, n Delaunay ends and the following properties:

1. Denoting by $w_{i,t}$ the weight of the i-th Delaunay end,

$$\lim_{t\to 0} \frac{w_{i,t}}{t} = \tau_i.$$

2. Denoting by $\Delta_{i,t}$ the axis of the i-th Delaunay end, $\Delta_{i,t}$ converges to the oriented geodesic through the point p in the direction of u_i.
3. If all the weights τ_i are positive, then M_t is Alexandrov-embeddedd.
4. If all the weights τ_i are positive and if for all $i \neq j \in [1, n]$, the angle θ_{ij} between u_i and u_j satisfies

$$\left|\sin \frac{\theta_{ij}}{2}\right| > \frac{\sqrt{H^2-1}}{2H}, \tag{5.2}$$

then M_t is embedded.

Theorem 12. Let $M_0 \subset \mathbb{R}^3$ be a non-degenerate minimal n-noid with $n \geq 3$ and let $H > 1$. There exists a smooth family of CMC H surfaces $(M_t)_{0<|t|<T}$ in $\mathbb{H}^3$ such that

1. The surfaces M_t have genus zero and n Delaunay ends.
2. After a suitable blow-up, M_t converges to M_0 as t tends to 0.
3. If M_0 is Alexandrov-embedded, then all the ends of M_t are of unduloidal type if $t > 0$ and of nodoidal type if $t < 0$. Moreover, M_t is Alexandrov-embedded if $t > 0$.

Following the proofs of [39, 38] gives an effective strategy to construct the desired CMC surfaces M_t. This is done in Sections 5.3 and 5.4. However, showing that M_t is Alexandrov-embedded requires a precise knowledge of its ends. This is the purpose of the main theorem (Section 5.2, Theorem 14). We consider a family of holomorphic perturbations of the data giving rise via the DPW method to a half-Delaunay embedding $f_0 : \mathbb{D}^* \subset \mathbb{C} \longrightarrow \mathbb{H}^3$ and show that the perturbed induced surfaces $f_t(\mathbb{D}^*)$ are also embedded. Note that the domain on which the perturbed immersions are defined does not depend on the parameter t, which is stronger than f_t having an embedded end, and is critical for showing that the surfaces M_t are Alexandrov-embedded. The essential hypothesis on the perturbations is that they do not occasion a period problem on the domain $\mathbb{D}^*$ (which is not simply connected). The proof relies on the Fröbenius method for linear differential systems with regular singular

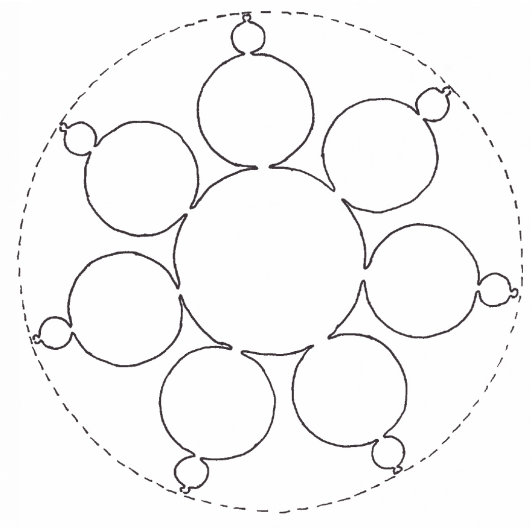

Figure 5.1 – *Theorem 11 ensures the existence of n-noids with small necks. For $H > 1$ small enough ($H \simeq 1.5$ on the picture), there exist coplanar embedded n-noids with more than six ends.*

points. Although this idea has been used in $\mathbb{R}^3$ by Kilian, Rossman, Schmitt [20] and [30], the case of $\mathbb{H}^3$ generates two extra resonance points that are unavoidable and make their results inapplicable. Our solution is to extend the Fröbenius method to loop-group-valued differential systems.

5.1 Delaunay surfaces in $\mathbb{H}^3$ via the DPW method

This Section fixes the notation and recalls the DPW method in $\mathbb{H}^3$.

5.1.1 Hyperbolic space

Matrix model. Let $\mathbb{R}^{1,3}$ denote the space $\mathbb{R}^4$ with the Lorentzian metric $\langle x, x\rangle = -x_0^2 + x_1^2 + x_2^2 + x_3^2$. Hyperbolic space is the subset $\mathbb{H}^3$ of vectors $x \in \mathbb{R}^{1,3}$ such that $\langle x, x\rangle = -1$ and $x_0 > 0$, with the metric induced by $\mathbb{R}^{1,3}$. The DPW method constructs CMC immersions into a matrix model of $\mathbb{H}^3$. Consider the identification

$$x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^{1,3} \simeq X = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in \mathcal{H}_2$$

where $\mathcal{H}_2 := \{M \in \mathcal{M}(2, \mathbb{C}) \mid M^* = M\}$ denotes the Hermitian matrices. In this model, $\langle X, X\rangle = -\det X$ and $\mathbb{H}^3$ is identified with the set $\mathcal{H}_2^{++} \cap \mathrm{SL}(2, \mathbb{C})$ of Hermitian positive

definite matrices with determinant 1. This fact leads us to write

$$\mathbb{H}^3 = \{FF^* \mid F \in \mathrm{SL}(2,\mathbb{C})\}.$$

Setting

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.3}$$

gives us an orthonormal basis $(\sigma_1, \sigma_2, \sigma_3)$ of the tangent space $T_{\mathrm{I}_2}\mathbb{H}^3$ of $\mathbb{H}^3$ at the identity matrix. We choose the orientation of $\mathbb{H}^3$ induced by this basis.

Rigid motions. In the matrix model of $\mathbb{H}^3$, $\mathrm{SL}(2,\mathbb{C})$ acts as rigid motions: for all $p \in \mathbb{H}^3$ and $A \in \mathrm{SL}(2,\mathbb{C})$, this action is denoted by

$$A \cdot p := ApA^* \in \mathbb{H}^3.$$

This action extends to tangent spaces: for all $v \in T_p\mathbb{H}^3$, $A \cdot v := AvA^* \in T_{A\cdot p}\mathbb{H}^3$. The DPW method takes advantage of this fact and contructs immersions in $\mathbb{H}^3$ with the moving frame method.

Geodesics. Let $p \in \mathbb{H}^3$ and $v \in UT_p\mathbb{H}^3$. Define the map

$$\begin{array}{rccc} \mathrm{geod}(p,v) & : & \mathbb{R} & \longrightarrow & \mathbb{H}^3 \\ & & t & \longmapsto & p\cosh t + v \sinh t. \end{array} \tag{5.4}$$

Then $\mathrm{geod}(p,v)$ is the unit speed geodesic through p in the direction v. The action of $\mathrm{SL}(2,\mathbb{C})$ extends to oriented geodesics via:

$$A \cdot \mathrm{geod}(p,v) := \mathrm{geod}(A \cdot p, A \cdot v).$$

Parallel transport. Let $p, q \in \mathbb{H}^3$ and $v \in T_p\mathbb{H}^3$. We denote the result of parallel transporting v from p to q along the geodesic of $\mathbb{H}^3$ joining p to q by $\Gamma_p^q v \in T_q\mathbb{H}^3$. The parallel transport of vectors from the identity matrix is easy to compute with Proposition 22.

Proposition 22. For all $p \in \mathbb{H}^3$ and $v \in T_{\mathrm{I}_2}\mathbb{H}^3$, there exists a unique $S \in \mathcal{H}_2^{++} \cap \mathrm{SL}(2,\mathbb{C})$ such that $p = S \cdot \mathrm{I}_2$. Moreover, $\Gamma_{\mathrm{I}_2}^p v = S \cdot v$.

Proof. The point p is in $\mathbb{H}^3$ identified with $\mathcal{H}_2^{++} \cap \mathrm{SL}(2,\mathbb{C})$. Define S as the unique square

root of p in $\mathcal{H}_2^{++} \cap \mathrm{SL}(2,\mathbb{C})$. Then $p = S \cdot \mathrm{I}_2$. Define for $t \in [0,1]$:

$$S(t) := \exp(t \log S), \qquad \gamma(t) := S(t) \cdot \mathrm{I}_2, \qquad v(t) := S(t) \cdot v.$$

Then $v(t) \in T_{\gamma(t)}\mathbb{H}^3$ because

$$\langle v(t), \gamma(t) \rangle = \langle S(t) \cdot \mathrm{I}_2, S(t) \cdot v \rangle = \langle \mathrm{I}_2, v \rangle = 0$$

and $S \cdot v = v(1) \in T_p\mathbb{H}^3$.

Suppose that S is diagonal. Then

$$S(t) = \begin{pmatrix} e^{\frac{at}{2}} & 0 \\ 0 & e^{\frac{-at}{2}} \end{pmatrix} \qquad (a \in \mathbb{R})$$

and using equations (5.3) and (5.4),

$$\gamma(t) = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix} = \mathrm{geod}(\mathrm{I}_2, \sigma_3)(at)$$

is a geodesic curve. Write $v = v^1\sigma_1 + v^2\sigma_2 + v^3\sigma_3$ and compute $S(t) \cdot \sigma_i$ to find

$$v(t) = v^1\sigma_1 + v^2\sigma_2 + v^3 \begin{pmatrix} e^{at} & 0 \\ 0 & -e^{-at} \end{pmatrix}.$$

Compute in $\mathbb{R}^{1,3}$

$$\frac{Dv(t)}{dt} = \left(\frac{dv(t)}{dt}\right)^T = av^3 \left(\gamma(t)\right)^T = 0$$

to see that $v(t)$ is the parallel transport of v along the geodesic γ.

If S is not diagonal, write $S = QDQ^{-1}$ where $Q \in \mathrm{SU}(2)$ and $D \in \mathcal{H}_2^{++} \cap \mathrm{SL}(2,\mathbb{C})$ is diagonal. Then,

$$S \cdot v = Q \cdot \left(D \cdot \left(Q^{-1} \cdot v\right)\right) = Q \cdot \Gamma_{\mathrm{I}_2}^{D \cdot \mathrm{I}_2}(Q^{-1} \cdot v).$$

But for all $A \in \mathrm{SL}(2,\mathbb{C})$, $p, q \in \mathbb{H}^3$ and $v \in T_p\mathbb{H}^3$,

$$A \cdot \Gamma_p^q v = \Gamma_{A \cdot p}^{A \cdot q} A \cdot v$$

and thus

$$S \cdot v = \Gamma_{\mathrm{I}_2}^p v.$$

□

5.1.2 The DPW method for CMC $H > 1$ surfaces in $\mathbb{H}^3$

Loop groups. In the DPW method, a whole family of surfaces is constructed, depending on a spectral parameter λ. This parameter will always be in one of the following subsets of $\mathbb{C}$ ($\rho > 1$):

$$\mathbb{S}^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}, \quad \mathbb{A}_\rho = \left\{\lambda \in \mathbb{C} \mid \rho^{-1} < |\lambda| < \rho\right\}, \quad \mathbb{D}_\rho = \{\lambda \in \mathbb{C} \mid |\lambda| < \rho\}.$$

Any smooth map $f : \mathbb{S}^1 \longrightarrow \mathcal{M}(2, \mathbb{C})$ can be decomposed into its Fourier series

$$f(\lambda) = \sum_{i \in \mathbb{Z}} f_i \lambda^i.$$

Let $|\cdot|$ denote a norm on $\mathcal{M}(2, \mathbb{C})$. Fix some $\rho > 1$ and consider

$$\|f\|_\rho := \sum_{i \in \mathbb{Z}} |f_i| \rho^{|i|}.$$

Let G be a Lie group or algebra of $\mathcal{M}(2, \mathbb{C})$. We define

- ΛG as the set of smooth functions $f : \mathbb{S}^1 \longrightarrow G$.
- $\Lambda G_\rho \subset \Lambda G$ as the set of functions f such that $\|f\|_\rho$ is finite. If G is a group (or an algebra) then $(\Lambda G_\rho, \|\cdot\|_\rho)$ is a Banach Lie group (or algebra).
- $\Lambda G_\rho^{\geq 0} \subset \Lambda G_\rho$ as the set of functions f such that $f_i = 0$ for all $i < 0$.
- $\Lambda_+ G_\rho \subset \Lambda G_\rho^{\geq 0}$ as the set of functions such that f_0 is upper-triangular.
- $\Lambda_+^{\mathbb{R}} \mathrm{SL}(2, \mathbb{C})_\rho \subset \Lambda_+ \mathrm{SL}(2, \mathbb{C})_\rho$ as the set of functions that have positive elements on the diagonal.

We also define $\Lambda\mathbb{C}$ as the set of smooth maps from $\mathbb{S}^1$ to $\mathbb{C}$, and $\Lambda\mathbb{C}_\rho$ and $\Lambda\mathbb{C}_\rho^{\geq 0}$ as above. Note that every function of ΛG_ρ holomorphically extends to $\mathbb{A}_\rho$ and that every function of $\Lambda G_\rho^{\geq 0}$ holomorphically extends to $\mathbb{D}_\rho$.

We will use the Fröbenius norm on $\mathcal{M}(2, \mathbb{C})$:

$$|A| := \left(\sum_{i,j} |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

Recall that this norm is sub-multiplicative. Therefore, the norm $\|\cdot\|_\rho$ is sub-multiplicative. Moreover, for all $A \in \Lambda \mathrm{SL}(2, \mathbb{C})_\rho$,

$$\left\|A^{-1}\right\|_\rho = \|A\|_\rho$$

and for all $A \in \Lambda\mathcal{M}(2,\mathbb{C})_\rho$ and $\lambda \in \mathbb{A}_\rho$,

$$|A(\lambda)| \leq \|A\|_\rho .$$

The DPW method relies on the Iwasawa decomposition. The following theorem is proved in [27] and is cited with our notation in [38].

Theorem 13. The multiplication map $\Lambda\mathrm{SU}(2)_\rho \times \Lambda^{\mathbb{R}}_+\mathrm{SL}(2,\mathbb{C})_\rho \longmapsto \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ is a smooth diffeomorphism between Banach manifolds. Its inverse map is called "Iwasawa decomposition" and is denoted for $\Phi \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$:

$$\mathrm{Iwa}(\Phi) = (\mathrm{Uni}(\Phi), \mathrm{Pos}(\Phi)) .$$

The ingredients. Let $H > 1$, $q = \operatorname{arcoth} H > 0$ and $\rho > e^q$. The DPW method takes for input data:

- A Riemann surface Σ.
- A holomorphic 1-form on Σ with values in $\Lambda\mathfrak{sl}(2,\mathbb{C})_\rho$ of the following form:

$$\xi = \begin{pmatrix} \alpha & \lambda^{-1}\beta \\ \gamma & -\alpha \end{pmatrix}$$

  where α, β, γ are holomorphic 1-forms on Σ with values in $\Lambda\mathbb{C}^{\geq 0}_\rho$. The 1-form ξ is called "the potential".
- A base point $z_0 \in \Sigma$.
- An initial condition $\phi \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$.

The recipe. The DPW method consists in the following steps:

1. Let $\widetilde{z}_0$ be any point above z_0 in the universal cover $\widetilde{\Sigma}$ of Σ. Solve on $\widetilde{\Sigma}$ the following Cauchy problem:
$$\begin{cases} d\Phi = \Phi\xi \\ \Phi(\widetilde{z}_0) = \phi. \end{cases} \tag{5.5}$$
   Then $\Phi : \widetilde{\Sigma} \longrightarrow \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ is called "the holomorphic frame".
2. Compute pointwise on $\widetilde{\Sigma}$ the Iwasawa decomposition of Φ:
$$(F(z), B(z)) := \mathrm{Iwa}\,\Phi(z).$$
   The unitary part F of this decomposition is called "the unitary frame".

3. Define $f : \widetilde{\Sigma} \longrightarrow \mathbb{H}^3$ via the Sym-Bobenko formula:

$$f(z) = F(z, e^{-q})F(z, e^{-q})^* =: \mathrm{Sym}_q F(z)$$

where $F(z, \lambda_0) := F(z)(\lambda_0)$.

Then f is a CMC $H > 1$ ($H = \coth q$) conformal immersion from $\widetilde{\Sigma}$ to $\mathbb{H}^3$. Its Gauss map (in the direction of the mean curvature vector) is given by

$$N(z) = F(z, e^{-q})\sigma_3 F(z, e^{-q})^* =: \mathrm{Nor}_q F(z)$$

where σ_3 is defined in (5.3). The differential of f is given by

$$df(z) = 2\sinh(q)b(z)^2 F(z, e^{-q}) \begin{pmatrix} 0 & \beta(z,0) \\ \overline{\beta(z,0)} & 0 \end{pmatrix} F(z, e^{-q})^* \tag{5.6}$$

where $b(z) > 0$ is the upper-left entry of $B(z)\,|_{\lambda=0}$. The metric of f is given by

$$ds_f(z) = 2\sinh(q)b(z)^2\,|\beta(z,0)|$$

and its Hopf differential reads

$$-2\beta(z,0)\gamma(z,0)\sinh q\; dz^2. \tag{5.7}$$

Remark 13. The results of this paper hold for any $H > 1$. We thus fix now $H > 1$ and $q = \operatorname{arcoth} H$. Hence,

$$e^{-q} = \sqrt{\frac{H-1}{H+1}}.$$

Rigid motions. Let $H \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ and define the new holomorphic frame $\widetilde{\Phi} = H\Phi$ with unitary part $\widetilde{F}$ and induced immersion $\widetilde{f} = \mathrm{Sym}_q \widetilde{F}$. If $H \in \Lambda\mathrm{SU}(2)_\rho$, then $\widetilde{F} = HF$ and $\widetilde{\Phi}$ gives rise to the same immersion as Φ up to an isometry of $\mathbb{H}^3$:

$$\widetilde{f}(z) = H(e^{-q}) \cdot f(z).$$

If $H \notin \Lambda\mathrm{SU}(2)_\rho$, this transformation is called a "dressing" and may change the surface.

Gauging. Let $G : \widetilde{\Sigma} \longrightarrow \Lambda_+\mathrm{SL}(2,\mathbb{C})_\rho$ and define the new potential:

$$\widehat{\xi} = \xi \cdot G := G^{-1}\xi G + G^{-1}dG.$$

The potential $\widehat{\xi}$ is a DPW potential and this operation is called "gauging". The data (Σ, ξ, z_0, ϕ) and $\left(\Sigma, \widehat{\xi}, z_0, \phi\, G(z_0)\right)$ give rise to the same immersion.

The monodromy problem. Since the immersion f is only defined on the universal cover $\widetilde{\Sigma}$, one might ask for conditions ensuring that it descends to a well-defined immersion on Σ. For any deck transformation $\tau \in \mathrm{Deck}\left(\widetilde{\Sigma}/\Sigma\right)$, define the monodromy of Φ with respect to τ as:

$$\mathcal{M}_\tau(\Phi) := \Phi(\tau(z))\Phi(z)^{-1} \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho.$$

This map is independent of $z \in \widetilde{\Sigma}$. The standard sufficient conditions for the immersion f to be well-defined on Σ is the following set of equations, called the monodromy problem in $\mathbb{H}^3$:

$$\forall \tau \in \mathrm{Deck}\left(\widetilde{\Sigma}/\Sigma\right), \quad \begin{cases} \mathcal{M}_\tau(\Phi) \in \Lambda\mathrm{SU}(2)_\rho, \\ \mathcal{M}_\tau(\Phi)(e^{-q}) = \pm \mathrm{I}_2. \end{cases} \tag{5.8}$$

Use the point $\widetilde{z}_0$ defined in step 1 of the DPW method to identify the fundamental group $\pi_1(\Sigma, z_0)$ with $\mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$. Let $\{\gamma_i\}_{i\in I}$ be a set of generators of $\pi_1(\Sigma, z_0)$. Then the problem (5.8) is equivalent to

$$\forall i \in I, \quad \begin{cases} \mathcal{M}_{\gamma_i}(\Phi) \in \Lambda\mathrm{SU}(2)_\rho, \\ \mathcal{M}_{\gamma_i}(\Phi)(e^{-q}) = \pm \mathrm{I}_2. \end{cases} \tag{5.9}$$

Example: the standard sphere. The DPW method can produce spherical immersions of $\Sigma = \mathbb{C} \cup \{\infty\}$ with the potential

$$\xi_{\mathbb{S}}(z,\lambda) = \begin{pmatrix} 0 & \lambda^{-1} dz \\ 0 & 0 \end{pmatrix}$$

and initial condition $\Phi_{\mathbb{S}}(0,\lambda) = \mathrm{I}_2$. The potential is not regular at $z = \infty$ because it has a double pole there. However, the immersion will be regular at this point because $\xi_{\mathbb{S}}$ is gauge-equivalent to a regular potential at $z = \infty$. Indeed, consider on $\mathbb{C}^*$ the gauge

$$G(z,\lambda) = \begin{pmatrix} z & 0 \\ -\lambda & \frac{1}{z} \end{pmatrix}.$$

The gauged potential is then

$$\xi_{\mathbb{S}} \cdot G(z,\lambda) = \begin{pmatrix} 0 & \lambda^{-1} z^{-2} dz \\ 0 & 0 \end{pmatrix}$$

which is regular at $z = \infty$. The holomorphic frame is

$$\Phi_{\mathbb{S}}(z, \lambda) = \begin{pmatrix} 1 & \lambda^{-1} z \\ 0 & 1 \end{pmatrix} \tag{5.10}$$

and its unitary factor is

$$F_{\mathbb{S}}(z, \lambda) = \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 & \lambda^{-1} z \\ -\lambda \overline{z} & 1 \end{pmatrix}.$$

The induced CMC-H immersion is

$$f_{\mathbb{S}}(z) = \frac{1}{1 + |z|^2} \begin{pmatrix} 1 + e^{2q}|z|^2 & 2z \sinh q \\ 2\overline{z} \sinh q & 1 + e^{-2q}|z|^2 \end{pmatrix}.$$

It is not easy to see that $f_{\mathbb{S}}(\Sigma)$ is a sphere because it is not centered at I_2. To solve this problem, notice that $F_{\mathbb{S}}(z, e^{-q}) = R(q) \widetilde{F}_{\mathbb{S}}(z) R(q)^{-1}$ where

$$R(q) := \begin{pmatrix} e^{\frac{q}{2}} & 0 \\ 0 & e^{\frac{-q}{2}} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) \tag{5.11}$$

and

$$\widetilde{F}_{\mathbb{S}}(z) := \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 & z \\ -\overline{z} & 1 \end{pmatrix} \in \mathrm{SU}(2).$$

Apply an isometry by setting

$$\widetilde{f}_{\mathbb{S}}(z) := R(q)^{-1} \cdot f_{\mathbb{S}}(z)$$

and compute

$$\widetilde{f}_{\mathbb{S}}(z) = \frac{1}{1 + |z|^2} \begin{pmatrix} e^{-q} + e^{q}|z|^2 & 2z \sinh q \\ 2\overline{z} \sinh q & e^{q} + e^{-q}|z|^2 \end{pmatrix} = (\cosh q)\mathrm{I}_2 + \frac{\sinh q}{1 + |z|^2} \begin{pmatrix} |z|^2 - 1 & 2z \\ 2\overline{z} & 1 - |z|^2 \end{pmatrix}$$

i.e.

$$\widetilde{f}_{\mathbb{S}}(z) = \mathrm{geod}(\mathrm{I}_2, v_{\mathbb{S}}(z))(q) \tag{5.12}$$

with geod defined in (5.4) and where in the basis $(\sigma_1, \sigma_2, \sigma_3)$ of $T_{\mathrm{I}_2}\mathbb{H}^3$,

$$v_{\mathbb{S}}(z) := \frac{1}{1 + |z|^2} \left(2 \operatorname{Re} z, 2 \operatorname{Im} z, |z|^2 - 1\right) \tag{5.13}$$

describes a sphere of radius one in the tangent space of $\mathbb{H}^3$ at I_2 (it is the inverse stereographic projection from the north pole). Hence, $\widetilde{f}_{\mathbb{S}}(\Sigma)$ is a sphere centered at I_2 of hyperbolic radius q and $f_{\mathbb{S}}(\Sigma)$ is a sphere of same radius centered at $\mathrm{geod}(\mathrm{I}_2, \sigma_3)(q)$.

One can compute the normal map of $f_{\mathbb{S}}$:

$$N_{\mathbb{S}}(z) := \mathrm{Nor}_q F_{\mathbb{S}}(z) = R(q) \cdot \widetilde{N}_{\mathbb{S}}(z)$$

where

$$\begin{aligned}\widetilde{N}_{\mathbb{S}}(z) := \mathrm{Nor}_q\left(\widetilde{F}_{\mathbb{S}}(z)\right) &= \frac{1}{1+|z|^2}\begin{pmatrix} e^{-q}-e^q|z|^2 & -2z\cosh q \\ -2\overline{z}\cosh q & e^{-q}|z|^2 - e^q \end{pmatrix} \\ &= -(\sinh q)\mathrm{I}_2 - (\cosh q) v_{\mathbb{S}}(z) = -\dot{\mathrm{geod}}\left(\mathrm{I}_2, v_{\mathbb{S}}(z)\right)(q).\end{aligned}$$

Note that this implies that the normal map Nor_q is oriented by the mean curvature vector.

5.1.3 Delaunay surfaces

Constant mean curvature $H > 1$ surfaces of revolution in $\mathbb{H}^3$ have been described in the DPW framework in [32]. We recall here the basic facts needed to our purpose.

The data. Let $\Sigma = \mathbb{C}^*$, $\xi_{r,s}(z,\lambda) = A_{r,s}(\lambda) z^{-1} dz$ where

$$A_{r,s}(\lambda) := \begin{pmatrix} 0 & r\lambda^{-1} + s \\ r\lambda + s & 0 \end{pmatrix}, \qquad r, s \in \mathbb{R}, \quad \lambda \in \mathbb{S}^1, \tag{5.14}$$

and initial condition $\Phi_{r,s}(1) = \mathrm{I}_2$. With these data, the holomorphic frame reads

$$\Phi_{r,s}(z) = z^{A_{r,s}}.$$

The unitary frame $F_{r,s}$ can be expressed in terms of elliptic functions (see [32]) and the DPW method states that the map $f_{r,s} = \mathrm{Sym}_q(F_{r,s})$ is a CMC H immersion from the universal cover $\widetilde{\mathbb{C}^*}$ of $\mathbb{C}^*$ into $\mathbb{H}^3$.

Monodromy. Computing the monodromy along $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0, 2\pi]$ gives

$$\mathcal{M}\left(\Phi_{r,s}\right) := \mathcal{M}_\gamma\left(\Phi_{r,s}\right) = \exp\left(2i\pi A_{r,s}\right).$$

Recall that $r, s \in \mathbb{R}$ to see that $iA_{r,s} \in \Lambda\mathfrak{su}(2)_\rho$, and thus $\mathcal{M}\left(\Phi_{r,s}\right) \in \Lambda\mathrm{SU}(2)_\rho$: the first equation of (5.9) is satisfied. To solve the second one, one can determine r and s such that

$A_{r,s}(e^{-q})^2 = \frac{1}{4}\mathrm{I}_2$, which will imply that $\mathcal{M}\left(\Phi_{r,s}\right)\left(e^{-q}\right) = -\mathrm{I}_2$. This condition is equivalent to

$$r^2 + s^2 + 2rs\cosh q = \frac{1}{4}. \tag{5.15}$$

Seeing this equation as a polynomial in r and computing its discriminant ($1+4s^2\sinh^2 q > 0$) ensures the existence of an infinite number of solutions: given a couple $(r,s) \in \mathbb{R}^2$ solution to (5.15), $f_{r,s}$ is a well-defined CMC H immersion from $\mathbb{C}^*$ into $\mathbb{H}^3$.

Surface of revolution. Let $(r,s) \in \mathbb{R}^2$ satisfying (5.15) and let $\theta \in \mathbb{R}$. Then,

$$\Phi_{r,s}\left(e^{i\theta}z\right) = \exp\left(i\theta A_{r,s}\right)\Phi_{r,s}(z).$$

Using $iA_{r,s} \in \Lambda\mathfrak{su}(2)_\rho$ and diagonalising $A_{r,s}(e^{-q})$ gives

$$\begin{aligned} f_{r,s}\left(e^{i\theta}z\right) &= \exp\left(i\theta A_{r,s}(e^{-q})\right)\cdot f_{r,s}(z) \\ &= \left(H_{r,s}\exp\left(i\theta D\right)H_{r,s}^{-1}\right)\cdot f_{r,s}(z) \end{aligned}$$

where

$$H_{r,s} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -2\left(re^{-q}+s\right) \\ 2\left(re^{q}+s\right) & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix}.$$

Noting that $\exp\left(i\theta D\right)$ acts as a rotation of angle θ around the axis $\mathrm{geod}(\mathrm{I}_2, \sigma_3)$ and that $H_{r,s}$ acts as an isometry of $\mathbb{H}^3$ independent of θ shows that $\exp\left(i\theta A_{r,s}(e^{-q})\right)$ acts as a rotation around the axis $H_{r,s}\cdot\mathrm{geod}(\mathrm{I}_2,\sigma_3)$ and that $f_{r,s}$ is CMC $H > 1$ immersion of revolution of $\mathbb{C}^*$ into $\mathbb{H}^3$ and by definition (as in [23]) a Delaunay immersion.

The weight as a parameter. For a fixed $H > 1$, CMC H Delaunay surfaces in $\mathbb{H}^3$ form a family parametrised by the weight. This weight can be computed in the DPW framework: given a solution (r,s) of (5.15), the weight w of the Delaunay surface induced by the DPW data $(\mathbb{C}^*, \xi_{r,s}, 1, \mathrm{I}_2)$ reads

$$w = 2\pi \times 4rs\sinh q \tag{5.16}$$

(see [32] or [9] for details).

Lemma 8. Writing $t := \frac{w}{2\pi}$ and assuming $t \neq 0$, equations (5.15) and (5.16) imply that

$$\begin{cases} t \leq T_1, \\ r^2 = \frac{1}{8}\left(1 - 2Ht \pm 2\sqrt{T_1 - t}\sqrt{T_2 - t}\right), \\ s^2 = \frac{1}{8}\left(1 - 2Ht \pm 2\sqrt{T_1 - t}\sqrt{T_2 - t}\right) \end{cases} \tag{5.17}$$

with

$$T_1 = \frac{\tanh\frac{q}{2}}{2} < \frac{1}{2\tanh\frac{q}{2}} = T_2.$$

Proof. First, note that (5.15) and (5.16) imply

$$r^2 + s^2 = \frac{1}{4}\left(1 - 2t\coth q\right) = \frac{1}{4}(1 - 2Ht)$$

and thus

$$t \leq \frac{H}{2} < T_2. \tag{5.18}$$

If $r = 0$, then $t = 0$. Thus $r \neq 0$ and

$$s = \frac{t}{4r\sinh q}.$$

Equation (5.15) is then equivalent to

$$r^2 + \frac{t^2}{16r^2\sinh^2 q} + \frac{Ht}{2} = \frac{1}{4} \iff r^4 - \frac{1 - 2Ht}{4}r^2 + \frac{t^2}{16\sinh^2 q} = 0.$$

Using $\coth q = H$, the discriminant of this quadratic polynomial in r^2 is

$$\Delta(t) = \frac{1}{16}\left(1 - 4Ht + 4t^2\right)$$

which in turn is a quadratic polynomial in t with discriminant

$$\widetilde{\Delta} = \frac{H^2 - 1}{16} > 0$$

because $H > 1$. Thus

$$\Delta(t) = \frac{(T_1 - t)(T_2 - t)}{4}$$

because $H = \coth q$. Using (5.18), $\Delta(t) \geq 0$ if, and only if $t \leq T_1$ and

$$r^2 = \frac{1}{8}\left(1 - 2Ht \pm 2\sqrt{(T_1 - t)(T_2 - t)}\right).$$

By symmetry of equations (5.15) and (5.16), s^2 is as in (5.17). □

We consider the two continuous parametrisations of r and s for $t \in (-\infty, T_1)$ such that (r, s) satisfies equations (5.15) and (5.16) with $w = 2\pi t$:

$$\begin{cases} r(t) := \frac{\pm 1}{2\sqrt{2}}\left(1 - 2Ht + 2\sqrt{T_1 - t}\sqrt{T_2 - t}\right)^{\frac{1}{2}}, \\ s(t) := \frac{t}{4r(t)\sinh q}. \end{cases} \tag{5.19}$$

Choosing the parametrisation satisfying $r > s$ maps the unit circle of $\mathbb{C}^*$ onto a parallel circle of maximal radius, called a "bulge" of the Delaunay surface. As t tends to 0, the immersions tend towards a parametrisation of a sphere on every compact subset of $\mathbb{C}^*$, which is why we call this family of immersions "the spherical family". When $r < s$, the unit circle of $\mathbb{C}^*$ is mapped onto a parallel circle of minimal radius, called a "neck" of the Delaunay surface. As t tends to 0, the immersions degenerate into a point on every compact subset of $\mathbb{C}^*$. Nevertheless, we call this family the "catenoidal family" because applying a blowup to the immersions makes them converge towards a catenoidal immersion of $\mathbb{R}^3$ on every compact subset of $\mathbb{C}^*$ (see Section 5.4.1 for more details). In both cases, the weight of the induced surfaces is given by $w = 2\pi t$.

5.2 Perturbed Delaunay immersions

In this section, we study the immersions induced by a perturbation of Delaunay DPW data with small non-vanishing weights in a neighbourhood of $z = 0$. Our results are the same whether we choose the spherical or the catenoidal family of immersions. We thus drop the index r, s in the Delaunay DPW data and replace it by a small value of $t = 4rs\sinh q$ in a neighbourhood of $t = 0$ such that

$$t < T_{\max} := \frac{\tanh\frac{q}{2}}{2} = \frac{1}{2}\left(H - \sqrt{H^2 - 1}\right).$$

For all $\epsilon > 0$, we denote

$$D_\epsilon := \{z \in \mathbb{C} \mid |z| < \epsilon\}, \qquad D_\epsilon^* := D_\epsilon \backslash \{0\}.$$

Definition 16 (Perturbed Delaunay potential)**.** Let $\rho > e^q$, $0 < T < T_{\max}$ and $\epsilon > 0$. A perturbed Delaunay potential is a continuous one-parameter family $(\xi_t)_{t\in(-T,T)}$ of DPW potentials defined for $(t, z) \in (-T, T) \times D_\epsilon^*$ by

$$\xi_t(z) = A_t z^{-1} dz + C_t(z) dz$$

where $A_t \in \Lambda\mathfrak{sl}(2,\mathbb{C})_\rho$ is a Delaunay residue as in (5.14) satisfying (5.19) and $C_t(z) \in \mathfrak{sl}(2,\mathbb{C})_\rho$ is $\mathcal{C}^1$ with respect to (t, z), holomorphic with respect to z for all t and satisfies $C_0(z) = 0$ for all z.

Theorem 14. Let $\rho > e^q$, $0 < T < T_{\max}$, $\epsilon > 0$ and ξ_t be a perturbed Delaunay potential $\mathcal{C}^2$ with respect to (t, z). Let Φ_t be a holomorphic frame associated to ξ_t for all t via the DPW method. Suppose that the family of initial conditions ϕ_t is $\mathcal{C}^2$ with respect to t,

with $\phi_0 = \widetilde{z}_0^{A_0}$, and that the monodromy problem (5.9) is solved for all $t \in (-T, T)$. Let $f_t = \mathrm{Sym}_q(\mathrm{Uni}\,\Phi_t)$. Then,

1. For all $\delta > 0$, there exist $0 < \epsilon' < \epsilon$, $T' > 0$ and $C > 0$ such that for all $z \in D^*_{\epsilon'}$ and $t \in (-T', T')\backslash\{0\}$,

$$d_{\mathbb{H}^3}\left(f_t(z), f_t^{\mathcal{D}}(z)\right) \leq C|t||z|^{1-\delta}$$

   where $f_t^{\mathcal{D}}$ is a Delaunay immersion of weight $2\pi t$.
2. There exist $T' > 0$ and $\epsilon' > 0$ such that for all $0 < t < T'$, f_t is an embedding of $D^*_{\epsilon'}$.
3. The limit axis as t tends to 0 of the Delaunay immersion $f_t^{\mathcal{D}}$ oriented towards the end at $z = 0$ is given by:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -e^q \\ e^{-q} & 1 \end{pmatrix} \cdot \mathrm{geod}\,(\mathrm{I}_2, -\sigma_3) \quad \text{in the spherical family } (r > s),$$

$$\mathrm{geod}\,(\mathrm{I}_2, -\sigma_1) \quad \text{in the catenoidal family } (r < s).$$

Let ξ_t and Φ_t as in Theorem 14 with ρ, T and ϵ fixed. This Section is dedicated to the proof of Theorem 14.

The $\mathcal{C}^2$-regularity of ξ_t essentially means that $C_t(z)$ is $\mathcal{C}^2$ with respect to (t, z). Together with the $\mathcal{C}^2$-regularity of ϕ_t, it implies that Φ_t is $\mathcal{C}^2$ with respect to (t, z). Thus $\mathcal{M}(\Phi_t)$ is also $\mathcal{C}^2$ with respect to t. These regularities and the fact that there exists a solution Φ_t solving the monodromy problem are used in Section 5.2.1 to deduce an essential piece of information about the potential ξ_t (Proposition 24). This step then allows us to write in Section 5.2.2 the holomorphic frame Φ_t in a Mz^AP form given by the Fröbenius method (Proposition 25), and to gauge this expression, in order to gain an order of convergence with respect to z (Proposition 26). During this process, the holomorphic frame will loose one order of regularity with respect to t, which is why Theorem 14 asks for a $\mathcal{C}^2$-regularity of the data. Section 5.2.3 is devoted to the study of dressed Delaunay frames Mz^A in order to ensure that the immersions f_t will converge to Delaunay immersions as t tends to 0, and to estimate the growth of their unitary part around the end at $z = 0$. Section 5.2.4 proves that these immersions do converge, which is the first point of Theorem 14. Before proving the embeddedness in Section 5.2.6, Section 5.2.5 is devoted to the convergence of the normal maps. Finally, we compute the limit axes in Section 5.2.7.

5.2.1 A property of ξ_t

We begin by diagonalising A_t in a unitary basis (Proposition 23) in order simplify the computations in Proposition 24, in which we use the Fröbenius method for a fixed value of

$\lambda = e^{\pm q}$. This will ensure the existence of the $\mathcal{C}^1$ map $P_1 \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ that will be used in Section 5.2.2 to define the factor P in the Mz^AP form of Φ_t.

Proposition 23. There exist $e^q < R < \rho$ and $0 < T' < T$ such that for all $t \in (-T', T')$, $A_t = H_t D_t H_t^{-1}$ with $H_t \in \Lambda\mathrm{SU}(2)_R$ and $iD_t \in \Lambda\mathfrak{su}(2)_R$. Moreover, H_t and D_t are smooth with respect to t.

Proof. For all $\lambda \in \mathbb{S}^1$,

$$\begin{aligned} -\det A_t(\lambda) &= \frac{1}{4} + \frac{t\lambda^{-1}(\lambda - e^q)(\lambda - e^{-q})}{4\sinh q} \\ &= \frac{1}{4} + \frac{t}{4\sinh q}\left(\lambda + \lambda^{-1} - 2\cosh q\right) \in \mathbb{R}. \end{aligned} \tag{5.20}$$

Extending this determinant as a holomorphic function on $\mathbb{A}_\rho$, there exists $T' > 0$ such that

$$\left| -\det A_t(\lambda) - \frac{1}{4} \right| < \frac{1}{4} \quad \forall (t,\lambda) \in (-T', T') \times \mathbb{A}_\rho$$

With this choice of T', the function $\mu_t : \mathbb{A}_\rho \longrightarrow \mathbb{C}$ defined as the positive-real-part square root of $(-\det A_t)$ is holomorphic on $\mathbb{A}_\rho$ and real-valued on $\mathbb{S}^1$. Note that μ_t is also the positive-real-part eigenvalue of A_t and thus $A_t = H_t D_t H_t^{-1}$ with

$$H_t(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{-(r\lambda^{-1}+s)}{\mu_t(\lambda)} \\ \frac{r\lambda+s}{\mu_t(\lambda)} & 1 \end{pmatrix}, \qquad D_t(\lambda) = \begin{pmatrix} \mu_t(\lambda) & 0 \\ 0 & -\mu_t(\lambda) \end{pmatrix}. \tag{5.21}$$

Let $e^q < R < \rho$. For all $t \in (-T', T')$, $\mu_t \in \Lambda\mathbb{C}_R$ and the map $t \mapsto \mu_t$ is smooth on $(-T', T')$. Moreover, $H_t \in \Lambda\mathrm{SU}(2)_R$, $iD_t \in \Lambda\mathfrak{su}(2)_R$ and these functions are smooth with respect to t. □

Remark 14. The bound $t < T'$ ensures that that $4\det A_t(\lambda)$ is an integer only for $t = 0$ and $\lambda = e^{\pm q}$. These points make the Fröbenius system resonant, but they are precisely the points that bear an extra piece of information due to the hypotheses on $\mathcal{M}(\Phi_t)(e^q)$ and Φ_0. Allowing the parameter t to leave the interval $(-T', T')$ would bring other resonance points and make Section 5.2.2 invalid. This is why Theorem 14 does not state that the end of the immersion f_t is a Delaunay end for all t.

Remark 15. At $t = 0$, the change of basis H_t in the diagonalisation of A_t takes different values whether $r > s$ (spherical family) or $r < s$ (catenoidal family). One has:

$$H_0(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix} \qquad \text{in the spherical case,} \tag{5.22}$$

$$H_0(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \text{in the catenoidal case.} \tag{5.23}$$

In both cases, $\mu_0 = \frac{1}{2}$, and thus D_0 is the same.

A basis of $\Lambda\mathcal{M}(2,\mathbb{C})_\rho$. Let R and T' given by Proposition 23. Identify $\Lambda\mathcal{M}(2,\mathbb{C})_\rho$ with the free $\Lambda\mathbb{C}_\rho$-module $\mathcal{M}(2,\Lambda\mathbb{C}_\rho)$ and define for all $t \in (-T', T')$ the basis

$$\mathcal{B}_t = H_t \left(E_1, E_2, E_3, E_4\right) H_t^{-1} =: \left(X_{t,1}, X_{t,2}, X_{t,3}, X_{t,4}\right)$$

where

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For all $t \in (-T', T')$, write

$$C_t(0) = \begin{pmatrix} tc_1(t) & \lambda^{-1} tc_2(t) \\ tc_3(t) & -tc1(t) \end{pmatrix} = \sum_{j=1}^{4} t\widehat{c}_j(t) X_{t,j}. \tag{5.24}$$

The functions $c_j, \widehat{c}_j$ are $\mathcal{C}^1$ with respect to $t \in (-T', T')$ and take values in $\Lambda\mathbb{C}_R$. Moreover, the functions $c_i(t)$ holomorphically extend to $\mathbb{D}_\rho$.

Proposition 24. There exists a continuous function $\widetilde{c}_3 : (-T', T') \longrightarrow \Lambda\mathbb{C}_R$ such that for all $\lambda \in \mathbb{S}^1$ and $t \in (-T', T')$,

$$\widehat{c}_3(t) = t(\lambda - e^q)(\lambda - e^{-q})\widetilde{c}_3(t).$$

Proof. It suffices to show that $\widehat{c}_3(0) = 0$ and that the holomorphic extension of $\widehat{c}_3(t)$ satisfies $\widehat{c}_3(t, e^{\pm q}) = 0$ for all t.

To show that $\widehat{c}_3(0) = 0$, recall that the monodromy problem (5.9) is solved for all t and note that $\mathcal{M}(\Phi_0) = -\mathrm{I}_2$, which implies that, as a function of t, the derivative of $\mathcal{M}(\Phi_t)$ at $t = 0$ is in $\Lambda\mathfrak{su}(2)_\rho$. On the other hand, Proposition 20 in Chapter 4 ensures that

$$\frac{d\mathcal{M}(\Phi_t)}{dt} \mid_{t=0} = \left(\int_\gamma \Phi_0 \frac{d\xi_t}{dt} \mid_{t=0} \Phi_0^{-1} \right) \mathcal{M}(\Phi_0)$$

where γ is a generator of $\pi_1(D_\epsilon^*, z_0)$. Expanding the right-hand side gives

$$-\int_\gamma z^{A_0} \frac{dA_t}{dt} \mid_{t=0} z^{-A_0} z^{-1} dz - \int_\gamma z^{A_0} \frac{dC_t(z)}{dt} \mid_{t=0} z^{-A_0} dz \in \Lambda\mathfrak{su}(2)_\rho.$$

Using Proposition 20 of Chapter 4 once again, note that the first term is the derivative

of $\mathcal{M}(z^{A_t})$ at $t=0$, which is in $\Lambda\mathfrak{su}(2)_\rho$ because $\mathcal{M}(z^{A_t}) \in \Lambda\mathrm{SU}(2)_\rho$ and $\mathcal{M}(z^{A_0}) = -\mathrm{I}_2$. Therefore, the second term is also in $\Lambda\mathfrak{su}(2)_\rho$. Diagonalising A_0 with Proposition 23 and using $H_0 \in \Lambda\mathrm{SU}(2)_R$ gives

$$2i\pi \operatorname*{Res}_{z=0}\left(z^{D_0}H_0^{-1}\frac{d}{dt}C_t(z)\mid_{t=0} H_0 z^{-D_0}\right) \in \Lambda\mathfrak{su}(2)_R.$$

But using Equation (5.24),

$$\begin{aligned} z^{D_0}H_0^{-1}\frac{d}{dt}C_t(z)\mid_{t=0} H_0 z^{-D_0} &= z^{D_0}H_0^{-1}\left(\sum_{j=1}^{4}\widehat{c}_j(0)X_{0,j}\right)H_0 z^{-D_0} \\ &= \sum_{j=1}^{4}\widehat{c}_j(0)z^{D_0}E_j z^{-D_0} \\ &= \begin{pmatrix} \widehat{c}_1(0) & z\widehat{c}_2(0) \\ z^{-1}\widehat{c}_3(0) & \widehat{c}_4(0)\end{pmatrix}. \end{aligned}$$

Thus

$$2i\pi\begin{pmatrix} 0 & 0 \\ \widehat{c}_3(0) & 0\end{pmatrix} \in \Lambda\mathfrak{su}(2)_R$$

which gives $\widehat{c}_3(0) = 0$.

Let $\lambda_0 \in \{e^q, e^{-q}\}$ and $t \neq 0$. Using the Fröbenius method (Theorem 4.11 of [36] and Lemma 11.4 of [35]) at the resonant point λ_0 ensures the existence of $\epsilon' > 0$, $B, M \in \mathcal{M}(2,\mathbb{C})$ and a holomorphic map $P : D_{\epsilon'} \longrightarrow \mathcal{M}(2,\mathbb{C})$ such that for all $z \in D^*_{\epsilon'}$,

$$\begin{cases} \Phi_t(z,\lambda_0) = Mz^B z^{A_t(\lambda_0)}P(z), \\ B^2 = 0, \\ P(0) = \mathrm{I}_2, \\ [A_t(\lambda_0), d_zP(0)] + d_zP(0) = C_t(0,\lambda_0) - B. \end{cases}$$

Compute the monodromy of Φ_t at $\lambda = \lambda_0$:

$$\mathcal{M}(\Phi_t)(\lambda_0) = M\exp(2i\pi B)z^B\exp(2i\pi A_t(\lambda_0))z^{-B}M^{-1} = -M\exp(2i\pi B)M^{-1}.$$

Since the monodromy problem (5.9) is solved, this quantity equals $-\mathrm{I}_2$. Use $B^2 = 0$ to show that $B = 0$ and thus

$$\begin{cases} \Phi_t(z,\lambda_0) = Mz^{A_t(\lambda_0)}P(z), \\ P(0) = \mathrm{I}_2, \\ [A_t(\lambda_0), d_zP(0)] + d_zP(0) = C_t(0,\lambda_0). \end{cases}$$

Diagonalise $A_t(\lambda_0)$ with Proposition 23 and write $d_zP(0) = \sum p_j X_{t,j}$ to get for all $1 \leq j \leq 4$

$$p_j\left([D_t(\lambda_0), E_j] + E_j\right) = t\widehat{c}_j(t, \lambda_0)E_j.$$

In particular, using $\mu_t(\lambda_0) = 1/2$,

$$t\widehat{c}_3(t, \lambda_0) = p_3\left([D_t(\lambda_0), E_3] + E_3\right) = 0.$$

□

Note that with the help of equations (5.24) and (5.22) or (5.23), and one can compute the series expansion of $\widehat{c}_3(0)$:

$$\widehat{c}_3(0) = \frac{-\lambda^{-1}}{2}c_2(0,0) + \mathcal{O}(\lambda^0) \quad \text{if} \quad r < s,$$

$$\widehat{c}_3(0) = \frac{1}{2}c_3(0,0) + \mathcal{O}(\lambda) \quad \text{if} \quad r > s.$$

Hence,

$$sc_2(t,0) + rc_3(t,0) \underset{t\to 0}{\longrightarrow} 0. \tag{5.25}$$

The following map will be useful in the next section:

$$t \in (-T', T') \longmapsto P^1(t) := t\widehat{c}_1(t)X_{t,1} + \frac{t\widehat{c}_2(t)}{1+2\mu_t}X_{t,2} + \frac{t\widehat{c}_3(t)}{1-2\mu_t}X_{t,3} + t\widehat{c}_4(t)X_{t,4}. \tag{5.26}$$

For all t, Proposition 24 ensures that the map $P^1(t,\lambda)$ holomorphically extends to $\mathbb{A}_R$. Taking a smaller value of R if necessary, $P^1(t) \in \Lambda\mathcal{M}(2,\mathbb{C})_R$ for all t. Moreover,

$$\operatorname{tr} P^1(t) = t\widehat{c}_1(t) + t\widehat{c}_4(t) = \operatorname{tr} C_t(0) = 0.$$

Thus $P^1 \in \mathcal{C}^1((-T', T'), \Lambda\mathfrak{sl}(2,\mathbb{C})_R)$.

5.2.2 The z^AP form of Φ_t

The map P^1 defined above allows us to use the Fröbenius method in a loop group framework and in the non-resonant case, that is, for all t (Proposition 25). The techniques used in [30] will then apply in order to gauge the Mz^AP form and gain an order on z (Proposition 26).

Proposition 25. There exists $\epsilon' > 0$ such that for all $t \in (-T', T')$ there exist $M_t \in$

$\Lambda\mathrm{SL}(2,\mathbb{C})_R$ and a holomorphic map $P_t : D_{\epsilon'} \longrightarrow \Lambda\mathrm{SL}(2,\mathbb{C})_R$ such that for all $z \in D^*_{\epsilon'}$,

$$\Phi_t(z) = M_t z^{A_t} P_t(z).$$

Moreover, M_t is $\mathcal{C}^1$ with respect to t, $M_0 = \mathrm{I}_2$, $P_t(z)$ is $\mathcal{C}^1$ with respect to (t,z), $P_0(z) = \mathrm{I}_2$ for all z and $P_t(0) = \mathrm{I}_2$ for all t.

Proof. For all $k \in \mathbb{N}^*$ and $t \in (-T', T')$, define the linear map

$$\begin{array}{rccc} \mathcal{L}_{t,k} & : & \Lambda\mathcal{M}(2,\mathbb{C})_\rho & \longrightarrow & \Lambda\mathcal{M}(2,\mathbb{C})_\rho \\ & & X & \longmapsto & [A_t, X] + kX. \end{array}$$

Use the bases $\mathcal{B}_t$ and restrict $\mathcal{L}_{t,k}$ to $\Lambda\mathcal{M}(2,\mathbb{C})_R$ to get

$$\mathrm{Mat}_{\mathcal{B}_t}\mathcal{L}_{t,k} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k+2\mu_t & 0 & 0 \\ 0 & 0 & k-2\mu_t & 0 \\ 0 & 0 & 0 & k \end{pmatrix}.$$

Note that

$$\det \mathcal{L}_{t,k} = k^2(k^2 - 4\mu_t^2).$$

Thus for all $k \geq 2$, $\det \mathcal{L}_{t,k}$ is an invertible element of $\Lambda\mathbb{C}_R$, which implies that $\mathcal{L}_{t,k}$ is invertible for all $t \in (-T', T')$ and $k \geq 2$.

Write

$$C_t(z) = \sum_{k\in\mathbb{N}} C_{t,k} z^k.$$

With $P^0 := \mathrm{I}_2$ and P^1 as in Equation (5.26), define for all $k \geq 1$:

$$P^{k+1}(t) := \mathcal{L}^{-1}_{t,k+1} \left(\sum_{i+j=k} P^i(t) C_{t,j} \right).$$

so that the sequence $(P^k)_{k\in\mathbb{N}} \subset \mathcal{C}^1\left((-T', T'), \Lambda\mathfrak{sl}(2,\mathbb{C})_\rho\right)$ satisfies the following recursive system for all $t \in (-T', T')$:

$$\begin{cases} P^0(t) = \mathrm{I}_2, \\ \mathcal{L}_{t,k+1}(P^{k+1}(t)) = \sum_{i+j=k} P^i(t) C_{t,j}. \end{cases}$$

With $P_t(z) := \sum P^k(t) z^k$, the Fröbenius method ensures convergence for all t (see [36]). Restricting to a compact interval in $(-T', T')$ if necessary, there exists $\epsilon' > 0$ such that for

all $z \in D^*_{\epsilon'}$ and $t \in (-T', T')$,

$$\Phi_t(z, \lambda) = M_t z^{A_t} P_t(z)$$

where $M_t \in \Lambda\mathrm{SL}(2, \mathbb{C})_R$ is $\mathcal{C}^1$ with respect to t, $P_t(z)$ is $\mathcal{C}^1$ with respect to t and z, and for all t, $P_t : D_{\epsilon'} \longrightarrow \Lambda\mathrm{SL}(2, \mathbb{C})_R$ is holomorphic and satisfies $P_t(0) = \mathrm{I}_2$. Moreover, the map P^1 defined in (5.26) vanishes at $t = 0$ and thus $P_0(z) = \mathrm{I}_2$ for all $z \in D_{\epsilon'}$, which implies that $M_0 = \mathrm{I}_2$. □

Proposition 26. There exists $\epsilon' > 0$ such that for all $t \in (-T', T')$ there exist an admissible gauge $G_t : D_\epsilon \longrightarrow \Lambda_+\mathrm{SL}(2, \mathbb{C})_R$, a change of coordinates $h_t : D_{\epsilon'} \longrightarrow D_\epsilon$, a holomorphic map $\widetilde{P}_t : D_{\epsilon'} \longrightarrow \Lambda\mathrm{SL}(2, \mathbb{C})_R$ and $\widetilde{M}_t \in \Lambda\mathrm{SL}(2, \mathbb{C})_R$ such that for all $z \in D^*_{\epsilon'}$,

$$h_t^* \left(\Phi_t G_t\right)(z) = \widetilde{M}_t z^{A_t} \widetilde{P}_t(z).$$

Moreover, $\widetilde{M}_t$ is $\mathcal{C}^1$ with respect to t, $\widetilde{M}_0 = \mathrm{I}_2$ and there exists a uniform $C > 0$ such that for all t and z,

$$\left\| \widetilde{P}_t(z) - \mathrm{I}_2 \right\|_\rho \leq C|t||z|^2.$$

Proof. The proof goes as in Section 4.3.3 of Chapter 4. Expand $P^1(t)$ given by Equation (5.26) as a series to get (this is a tedious but simple computation):

$$P_1(t, \lambda) = \begin{pmatrix} 0 & \frac{stc_2(t,0)+rtc_3(t,0)}{2s}\lambda^{-1} \\ \frac{stc_2(t,0)+rtc_3(t,0)}{2r} & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\lambda^0) & \mathcal{O}(\lambda^0) \\ \mathcal{O}(\lambda) & \mathcal{O}(\lambda^0) \end{pmatrix}.$$

Define

$$p_t := 2 \sinh q(sc_2(t, 0) + rc_3(t, 0))$$

so that

$$g_t := p_t A_t - P^1(t) \in \Lambda_+ \mathfrak{sl}(2, \mathbb{C})_R$$

and recall Equation (5.25) together with $P_0 = \mathrm{I}_2$ to show that $g_0 = 0$. Thus

$$G_t := \exp\left(g_t z\right) \in \Lambda_+\mathrm{SL}(2, \mathbb{C})_R$$

is an admissible gauge. Let $\epsilon' < |p_t|^{-1}$ for all $t \in (-T', T')$. Define

$$\begin{array}{rccc} h_t & : & D_{\epsilon'} & \longrightarrow & D_\epsilon \\ & & z & \longmapsto & \frac{z}{1+p_t z}. \end{array}$$

Then,

$$\widetilde{\xi}_t := h_t^* \left(\xi_t \cdot G_t\right) = A_t z^{-1} dz + \widetilde{C}_t(z) dz$$

is a perturbed Delaunay potential as in Definition 16 such that $\widetilde{C}_t(0) = 0$ for all $t \in (-T', T')$. The holomorphic frame

$$\widetilde{\Phi}_t := h_t^* \left(\Phi_t G_t\right)$$

satisfies $d\widetilde{\Phi}_t = \widetilde{\Phi}_t \widetilde{\xi}_t$. With $\widetilde{C}_t(0) = 0$, one can apply the Fröbenius method on $\widetilde{\xi}_t$ to get

$$\widetilde{\Phi}_t(z) = \widetilde{M}_t z^{A_t} \widetilde{P}_t(z)$$

with $\widetilde{M}_0 = \mathrm{I}_2$ and

$$\left\| \widetilde{P}_t(z) - \mathrm{I}_2 \right\|_R \leq C|t||z|^2.$$

□

Conclusion. The new frame $\widetilde{\Phi}_t$ is associated to a perturbed Delaunay potential $(\widetilde{\xi}_t)_{t\in(-T',T')}$, defined for $z \in D^*_{\epsilon'}$, with values in $\Lambda\mathfrak{sl}(2,\mathbb{C})_R$ and of the form

$$\widetilde{\xi}_t(z) = A_t z^{-1} dz + \widehat{C}_t(z) z dz.$$

Note that $\widetilde{C}_t(z) \in \mathfrak{sl}(2,\mathbb{C})_R$ is now $\mathcal{C}^1$ with respect to (t,z). The monodromy problem (5.9) is solved for $\widetilde{\Phi}_t$ and for any $\widetilde{z}_0$ in the universal cover $\widetilde{D}^*_{\epsilon'}$, $\widetilde{\Phi}_0(\widetilde{z}_0) = \widetilde{z}_0^{A_0}$. Moreover, writing $\widetilde{f}_t := \mathrm{Sym}_q(\mathrm{Uni}\,\widetilde{\Phi}_t)$ and $f_t := \mathrm{Sym}_q(\mathrm{Uni}\,\Phi_t)$, then $\widetilde{f}_t = h_t^* f_t$ with $h_0(z) = z$. Hence in order to prove Theorem 14 it suffices to prove the following proposition.

Proposition 27. Let $\rho > e^q$, $0 < T < T_{\max}$, $\epsilon > 0$ and ξ_t be a perturbed Delaunay potential as in Definition 16. Let Φ_t be a holomorphic frame associated to ξ_t for all t via the DPW method. Suppose that the monodromy problem (5.9) is solved for all $t \in (-T, T)$ and that

$$\Phi_t(z) = M_t z^{A_t} P_t(z)$$

where $M_t \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ is $\mathcal{C}^1$ with respect to t, satisfies $M_0 = \mathrm{I}_2$, and $P_t : D_\epsilon \longrightarrow \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ is a holomorphic map such that for all t and z,

$$\|P_t(z) - \mathrm{I}_2\|_\rho \leq C|t||z|^2$$

where $C > 0$ is a uniform constant. Let $f_t = \mathrm{Sym}_q(\mathrm{Uni}\,\Phi_t)$. Then the three points of Theorem 14 hold for f_t.

We now reset the values of ρ, T and ϵ and suppose that we are given a perturbed Delaunay frame ξ_t and a holomorphic frame Φ_t associated to it and satisfying the hypotheses of Proposition 27.

5.2.3 Dressed Delaunay frames

In this section we study dressed Delaunay frames arising from the DPW data $(\widetilde{\mathbb{C}}^*, \xi_t^{\mathcal{D}}, 1, M_t)$, where $\widetilde{\mathbb{C}}^*$ is the universal cover of $\mathbb{C}^*$ and

$$\xi_t^{\mathcal{D}}(z) := A_t z^{-1} dz$$

with A_t as in (5.14) satisfying (5.19), and M_t as in Proposition 26. The induced holomorphic frame is

$$\Phi_t^{\mathcal{D}}(z) = M_t z^{A_t}.$$

Note that the fact that the monodromy problem (5.9) is solved for Φ_t implies that it is solved for $\Phi_t^{\mathcal{D}}$ because P_t is holomorphic on D_ϵ. Let $\widetilde{D}_1^*$ be the universal cover of D_1^* and let

$$F_t^{\mathcal{D}} := \operatorname{Uni} \Phi_t^{\mathcal{D}}, \qquad B_t^{\mathcal{D}} := \operatorname{Pos} \Phi_t^{\mathcal{D}}, \qquad f_t^{\mathcal{D}} := \operatorname{Sym}_q F_t^{\mathcal{D}}.$$

In this section, our goal is to prove the following proposition.

Proposition 28. The immersion $f_t^{\mathcal{D}}$ is a CMC H Delaunay immersion of weight $2\pi t$ for $|t|$ small enough. Moreover, for all $\delta > 0$ and $e^q < R < \rho$ there exists $C, T' > 0$ such that

$$\left\| F_t^{\mathcal{D}}(z) \right\|_R \leq C|z|^{-\delta}.$$

for all $(t, z) \in (-T', T') \times \widetilde{D}_1^*$.

Delaunay immersion. We will need the following lemma, inspired by [31].

Lemma 9. Let $M \in \mathrm{SL}(2, \mathbb{C})$ and $\mathcal{A} \in \mathfrak{su}(2)$ such that

$$M \exp(\mathcal{A}) M^{-1} \in \mathrm{SU}(2). \tag{5.27}$$

Then there exist $U \in \mathrm{SU}(2)$ and $K \in \mathrm{SL}(2, \mathbb{C})$ such that $M = UK$ and $[K, \mathcal{A}] = 0$.

Proof. Let

$$K = \sqrt{M^* M}, \qquad U = MK^{-1}$$

be a polar decomposition of M. The matrix K is then hermitian and positive-definite because $\det M \neq 0$. Moreover, $U \in \mathrm{SU}(2)$ and Equation (5.27) is then equivalent to

$$K \exp(\mathcal{A}) K^{-1} \in \mathrm{SU}(2). \tag{5.28}$$

Let us diagonalise $K = QDQ^{-1}$ where $Q \in \mathrm{SU}(2)$ and

$$D = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \quad x > 0.$$

Hence Equation (5.28) now reads

$$D \exp\left(Q^{-1}\mathcal{A}Q\right) D^{-1} \in \mathrm{SU}(2). \tag{5.29}$$

But $Q \in \mathrm{SU}(2)$ and $\mathcal{A} \in \mathfrak{su}(2)$, so $Q^{-1}\mathcal{A}Q \in \mathfrak{su}(2)$ and $\exp\left(Q^{-1}\mathcal{A}Q\right) \in \mathrm{SU}(2)$. Let us write

$$\exp\left(Q^{-1}\mathcal{A}Q\right) = \begin{pmatrix} p & -\overline{q} \\ q & \overline{p} \end{pmatrix}, \quad |p|^2 + |q|^2 = 1$$

so that Equation (5.29) is now equivalent to

$$x = 1 \text{ or } q = 0.$$

If $x = 1$ then $K = \mathrm{I}_2$ and $[K, \mathcal{A}] = 0$. If $q = 0$ then $Q^{-1}\mathcal{A}Q$ is diagonal and $\mathcal{A}$ is Q-diagonalisable. Thus K and $\mathcal{A}$ are simultaneously diagonalisable and $[K, \mathcal{A}] = 0$. □

Corollary 5. There exists $T' > 0$ such that for all $t \in (-T', T')$,

$$\Phi_t(z) = U_t z^{A_t} K_t$$

where $U_t \in \Lambda\mathrm{SU}(2)_R$ and $K_t \in \Lambda\mathrm{SL}(2,\mathbb{C})_R$ for any $e^q < R < \rho$.

Proof. Write $\mathcal{M}\left(\Phi_t^{\mathcal{D}}\right) = M_t \exp\left(\mathcal{A}_t\right) M_t^{-1}$ with $\mathcal{A}_t := 2i\pi A_t \in \Lambda\mathfrak{su}(2)_\rho$ continuous on $(-T, T)$. The map

$$M \longmapsto \sqrt{M^*M} = \exp\left(\frac{1}{2}\log M^*M\right)$$

is a diffeomorphism from a neighbourhood of $\mathrm{I}_2 \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ to another neighbourhood of I_2. Using the convergence of M_t towards I_2 as t tends to 0, this allows us to use Lemma 9 pointwise on $\mathbb{A}_\rho$ and thus construct $K_t := \sqrt{M_t^* M_t} \in \Lambda\mathrm{SL}(2,\mathbb{C})_R$ for all $t \in (-T', T')$ and any $e^q < R < \rho$. Let $U_t := M_t K_t^{-1} \in \Lambda\mathrm{SL}(2,\mathbb{C})_R$ and compute $U_t U_t^*$ to show that $U_t \in \Lambda\mathrm{SU}(2)_R$. Use Lemma 9 to show that $[K_t(\lambda), \mathcal{A}_t(\lambda)] = 0$ for all $\lambda \in \mathbb{S}^1$. Hence $[K_t, \mathcal{A}_t] = 0$ and thus $\Phi_t^{\mathcal{D}} = U_t z^{A_t} K_t$. □

Returning to the proof of Proposition 28, let $\theta \in \mathbb{R}$, $z \in \mathbb{C}^*$ and $e^q < R < \rho$. Apply Corollary 5 to get

$$\Phi_t^{\mathcal{D}}(e^{i\theta} z) = U_t \exp(i\theta A_t) U_t^{-1} \Phi_t^{\mathcal{D}}(z)$$

and note that $U_t \in \Lambda\mathrm{SU}(2)_R$, $iA_t \in \Lambda\mathfrak{su}(2)_R$ imply

$$R_t(\theta) := U_t \exp(i\theta A_t) U_t^{-1} \in \Lambda\mathrm{SU}(2)_R. \tag{5.30}$$

Hence

$$F_t^{\mathcal{D}}(e^{i\theta z}) = R_t(\theta) F_t^{\mathcal{D}}(z)$$

and

$$f_t^{\mathcal{D}}(e^{i\theta} z) = R_t(\theta, e^{-q}) \cdot f_t^{\mathcal{D}}(z).$$

Use Section 5.1.3 and note that U_t does not depend on θ to see that $f_t^{\mathcal{D}}$ is a CMC immersion of revolution and hence a Delaunay immersion. Its weight can be read from its Hopf diffferential, which in turn can be read from the potential $\xi_t^{\mathcal{D}}$ (see Equation (5.7)). Thus $f_t^{\mathcal{D}}$ is a CMC H Delaunay immersion of weight $2\pi t$, which proves the first point of Proposition 28.

Restricting to a meridian. Note that for all $t \in (-T', T')$ and $z \in \mathbb{C}^*$,

$$\left\| F_t^{\mathcal{D}}(z) \right\|_R \leq C \left\| F_t^{\mathcal{D}}(|z|) \right\|_R$$

where

$$C = \sup \left\{ \|R_t(\theta)\|_R \mid (t, \theta) \in (-T', T') \times [0, 2\pi] \right\}$$

depends only on R. We thus restrict $F_t^{\mathcal{D}}$ to $\mathbb{R}_+^*$ with $\widehat{F}_t^{\mathcal{D}}(x) := F_t^{\mathcal{D}}(|z|)$ $(x = |z|)$.

Grönwall over a period. Recalling the Lax Pair associated to $F_t^{\mathcal{D}}$ (see Section 4.8 in Chapter 4), the restricted map $\widehat{F}_t^{\mathcal{D}}$ satisfies $d\widehat{F}_t^{\mathcal{D}} = \widehat{F}_t^{\mathcal{D}} \widehat{W}_t dx$ with

$$\widehat{W}_t(x, \lambda) = \frac{1}{x} \begin{pmatrix} 0 & \lambda^{-1} r b^2(x) - s b^{-2}(x) \\ s b^{-2}(x) - \lambda r b^2(x) & 0 \end{pmatrix}$$

where $b(x)$ is the upper-left entry of $B_t^{\mathcal{D}}(x) \mid_{\lambda=0}$. Recall Section 5.1.2 and define

$$g_t(x) = 2 \sinh q |r| b(x)^2 x^{-1}$$

so that the metric of $f_t^{\mathcal{D}}$ reads $g_t(x)|dz|$. Let $\widetilde{f}_t^{\mathcal{D}} := \exp^* f_t^{\mathcal{D}}$. Then the metric of $\widetilde{f}_t^{\mathcal{D}}$ satisfies

$$d\widetilde{s}^2 = 4r^2 (\sinh q)^2 b^4(e^u)(du^2 + d\theta^2)$$

at the point $u + i\theta = \log z$. Using Proposition 33 of Section 5.5 gives

$$\int_0^{S_t} 2|r|b^2(e^u)du = \pi \quad \text{and} \quad \int_0^{S_t} \frac{du}{2\sinh q|r|b^2(e^u)} = \frac{\pi}{|t|}$$

where $S_t > 0$ is the period of the profile curve of $\widetilde{f}_t$. Thus

$$\int_1^{e^{S_t}} |rb^2(x)x^{-1}|dx = \frac{\pi}{2} = \int_1^{e^{S_t}} |sb^{-2}(x)x^{-1}|dx.$$

Using

$$\left\|\widehat{W}_t(x)\right\|_R = \sqrt{2}\left|sb^{-2}(x)x^{-1}\right| + 2R\left|rb^2(x)x^{-1}\right|,$$

we deduce

$$\int_1^{e^{S_t}} \left\|\widehat{W}_t(x)\right\|_R dx = \frac{\pi}{2}(2R + \sqrt{2}) < C \tag{5.31}$$

where $C > 0$ is a constant depending only on ρ and T. Applying Grönwall's lemma to the inequality

$$\left\|\widehat{F}_t^{\mathcal{D}}(x)\right\|_R \leq \left\|\widehat{F}_t^{\mathcal{D}}(1)\right\|_R + \int_1^x \left\|\widehat{F}_t^{\mathcal{D}}(u)\right\|_R \left\|\widehat{W}_t(u)\right\|_R |du|$$

gives

$$\left\|\widehat{F}_t^{\mathcal{D}}(x)\right\|_R \leq \left\|\widehat{F}_t^{\mathcal{D}}(1)\right\|_R \exp\left(\int_1^x \left\|\widehat{W}_t(u)\right\|_R |du|\right).$$

Use Equation (5.31) together with the fact that $\widetilde{F}_0^{\mathcal{D}}(0) = F_0^{\mathcal{D}}(1) = \mathrm{I}_2$ and the continuity of Iwasawa decomposition to get $C, T > 0$ such that for all $t \in (-T', T')$ and $x \in [1, e^{S_t}]$

$$\left\|\widehat{F}_t^{\mathcal{D}}(x)\right\|_R \leq C. \tag{5.32}$$

Control over the periodicity matrix. Let $t \in (-T', T')$ and $\Gamma_t := \widehat{F}_t^{\mathcal{D}}(xe^{S_t})\widehat{F}_t^{\mathcal{D}}(x)^{-1} \in \Lambda\mathrm{SU}(2)_R$ for all $x > 0$. The periodicity matrix Γ_t does not depend on x because $\widehat{W}_t(xe^{S_t}) = \widehat{W}_t(x)$ (by periodicity of the metric in the log coordinate). Moreover,

$$\|\Gamma_t\|_R = \left\|\widehat{F}_t^{\mathcal{D}}(e^{S_t})\widehat{F}_t^{\mathcal{D}}(1)^{-1}\right\|_R \leq \left\|\widehat{F}_t^{\mathcal{D}}(e^{S_t})\right\|_R \left\|\widehat{F}_t^{\mathcal{D}}(1)\right\|_R,$$

and using Equation (5.32),

$$\|\Gamma_t\|_R \leq C. \tag{5.33}$$

Conclusion. Let $x < 1$. Then there exist $k \in \mathbb{N}^*$ and $\zeta \in \left[1, e^{S_t}\right)$ such that $x = \zeta e^{-kS_t}$. Thus using equations (5.32) and (5.33),

$$\left\|\widehat{F}_t^{\mathcal{D}}(x)\right\|_R \leq \left\|\Gamma_t^{-k}\right\|_R \left\|\widehat{F}_t^{\mathcal{D}}(\zeta)\right\|_R \leq C^{k+1}.$$

Writing

$$k = \frac{\log \zeta}{S_t} - \frac{\log x}{S_t},$$

one gets

$$C^k = \exp\left(\frac{\log \zeta}{S_t} \log C\right) \exp\left(\frac{-\log C}{S_t} \log x\right) \leq C x^{-\delta_t}$$

where $\delta_t = \frac{\log C}{S_t}$ does not depend on x and tends to 0 as t tends to 0 (because $S_t \underset{t\to 0}{\longrightarrow} +\infty$). Returning back to $F_t^{\mathcal{D}}$, we showed that for all $\delta > 0$ there exist $T' > 0$ and $C > 0$ such that for all $t \in (-T', T')$ and $0 < |z| < 1$,

$$\left\| F_t^{\mathcal{D}}(z) \right\|_R \leq C|z|^{-\delta}$$

and Proposition 28 is proved.

5.2.4 Convergence of the immersions

In this section, we prove the first point of Theorem 14: the convergence of the immersions f_t towards the immersions $f_t^{\mathcal{D}}$. Our proof relies on the Iwasawa decomposition being a diffeomorphism in a neighbourhood of I_2.

Behaviour of the Delaunay positive part. Let $z \in \widetilde{D}_1^*$. The Delaunay positive part satisfies

$$\left\| B_t^{\mathcal{D}}(z) \right\|_\rho = \left\| F_t^{\mathcal{D}}(z)^{-1} M_t z^{A_t} \right\|_\rho \leq \left\| F_t^{\mathcal{D}}(z) \right\|_\rho \left\| M_t \right\|_\rho \left\| z^{A_t} \right\|_\rho .$$

Diagonalise $A_t = H_t D_t H_t^{-1}$ as in Proposition 23. By continuity of H_t and M_t, and according to Proposition 28, there exists $C, T' > 0$ such that for all $t \in (-T', T')$

$$\left\| B_t^{\mathcal{D}}(z) \right\|_R \leq C|z|^{-\delta} \left\| z^{-\mu_t} \right\|_R .$$

Recall Equation (5.20) and extend $\mu_t^2 = -\det A_t$ to $\mathbb{A}_{\widetilde{R}}$ with $\rho > \widetilde{R} > R$. One can thus assume that for $t \in (-T', T')$ and $\lambda \in \mathbb{A}_{\widetilde{R}}$,

$$|\mu_t(\lambda)| < \frac{1}{2} + \delta,$$

which implies that

$$\left| z^{-\mu_t(\lambda)} \right| \leq |z|^{\frac{-1}{2}-\delta}.$$

This gives us the following estimate in the $\Lambda\mathbb{C}_R$ norm (using Cauchy formula and $\widetilde{R} > R$):

$$\left\| z^{-\mu_t} \right\|_R \leq C|z|^{\frac{-1}{2}-\delta}$$

and

$$\left\|B_t^{\mathcal{D}}(z)\right\|_R \leq C|z|^{\frac{-1}{2}-2\delta}. \tag{5.34}$$

Behaviour of a holomorphic frame. Let

$$\widetilde{\Phi}_t := B_t^{\mathcal{D}} \left(\Phi_t^{\mathcal{D}}\right)^{-1} \Phi_t \left(B_t^{\mathcal{D}}\right)^{-1}.$$

Recall Proposition 26 and use Equation (5.34) to get for all $t \in (-T', T')$ and $z \in D_\epsilon^*$:

$$\left\|\widetilde{\Phi}_t(z) - \mathrm{I}_2\right\|_R = \left\|B_t^{\mathcal{D}}(z)\left(P_t(z) - \mathrm{I}_2\right) B_t^{\mathcal{D}}(z)^{-1}\right\|_R \leq C|t||z|^{1-4\delta}.$$

Behaviour of the perturbed immersion. Note that

$$\begin{aligned}\widetilde{\Phi}_t &= B_t^{\mathcal{D}} \left(\Phi_t^{\mathcal{D}}\right)^{-1} \Phi_t \left(B_t^{\mathcal{D}}\right)^{-1} \\ &= \left(F_t^{\mathcal{D}}\right)^{-1} F_t \times B_t \left(B_t^{\mathcal{D}}\right)^{-1}\end{aligned}$$

and recall that the Iwasawa decomposition is differentiable at the identity to get

$$\left\|F_t^{\mathcal{D}}(z)^{-1} F_t(z) - \mathrm{I}_2\right\|_R = \left\|\mathrm{Uni}\,\widetilde{\Phi}_t(z) - \mathrm{Uni}\,\mathrm{I}_2\right\|_R \leq C|t||z|^{1-4\delta}.$$

The map

$$\widetilde{F}_t(z) := F_t^{\mathcal{D}}(z, e^{-q})^{-1} F_t(z, e^{-q}) \in \mathrm{SL}(2,\mathbb{C})$$

satisfies

$$\left|\widetilde{F}_t(z) - \mathrm{I}_2\right| \leq \left\|F_t^{\mathcal{D}}(z)^{-1} F_t(z) - \mathrm{I}_2\right\|_R \leq C|t||z|^{1-4\delta}. \tag{5.35}$$

Lemma 10. There exists a neighbourhood $V \subset \mathrm{SL}(2,\mathbb{C})$ of I_2 and $C > 0$ such that for all $A \in \mathrm{SL}(2,\mathbb{C})$,

$$A \in V \implies \left|\mathrm{tr}\left(AA^*\right) - 2\right| \leq C\left|A - \mathrm{I}_2\right|^2.$$

Proof. Consider $\exp : U \subset \mathfrak{sl}(2,\mathbb{C}) \longrightarrow V \subset \mathrm{SL}(2,\mathbb{C})$ as a local chart of $\mathrm{SL}(2,\mathbb{C})$ around I_2. Let $A \in V$. Write

$$\begin{array}{rccc} f & : & \mathrm{SL}(2,\mathbb{C}) & \longrightarrow & \mathbb{R} \\ & & X & \longmapsto & \mathrm{tr}\left(XX^*\right) \end{array}$$

and $a = \log A \in U$ to get

$$|f(A) - f(\mathrm{I}_2)| \leq df(\mathrm{I}_2) \cdot a + C|a|^2.$$

Notice that for all $a \in \mathfrak{sl}(2,\mathbb{C})$,

$$df(\mathrm{I}_2) \cdot a = \mathrm{tr}(a + a^*) = 0$$

to end the proof. □

Corollary 6. There exists a neighbourhood $V \subset \mathrm{SL}(2,\mathbb{C})$ of I_2 and $C > 0$ such that for all $F_1, F_2 \in \mathrm{SL}(2,\mathbb{C})$,

$$F_2^{-1}F_1 \in V \implies d_{\mathbb{H}^3}(f_1, f_2) < C\left|F_2^{-1}F_1 - \mathrm{I}_2\right|,$$

where $f_i = F_i \cdot \mathrm{I}_2 \in \mathbb{H}^3$.

Proof. Just note that

$$d_{\mathbb{H}^3}(f_1, f_2) = \cosh^{-1}\left(-\langle f_1, f_2\rangle\right) = \cosh^{-1}\left(\frac{1}{2}\mathrm{tr}(f_2^{-1}f_1)\right)$$

and that

$$\mathrm{tr}\left(f_2^{-1}f_1\right) = \mathrm{tr}\left(\left(F_2{F_2}^*\right)^{-1}F_1{F_1}^*\right) = \mathrm{tr}\,\widetilde{F}\widetilde{F}^*$$

where $\widetilde{F} = F_2^{-1}F_1$. Apply Lemma 10 and use $\cosh^{-1}(1+x) \sim \sqrt{2x}$ as $x \to 0$ to end the proof. □

Without loss of generality, we can suppose from (5.35) that $C|t||z|^{1-4\delta}$ is small enough for $\widetilde{F}_t(z)$ to be in V for all t and z. Apply Corollary 6 to end the proof of the first point in Theorem 14:

$$d_{\mathbb{H}^3}\left(f_t(z), f_t^{\mathcal{D}}(z)\right) \leq C|t||z|^{1-4\delta}.$$

5.2.5 Convergence of the normal maps

Before starting the proof of the second point of Theorem 14, we will need to compare the normal maps of our immersions. Let $N_t := \mathrm{Nor}_q F_t$ and $N_t^{\mathcal{D}} := \mathrm{Nor}_q F_t^{\mathcal{D}}$ be the normal maps associated to the immersions f_t and $f_t^{\mathcal{D}}$. This section is devoted to the proof of the following proposition.

Proposition 29. For all $\delta > 0$ there exist $\epsilon', T', C > 0$ such that for all $t \in (-T', T')$ and $z \in D_\epsilon^*$,

$$\left\|\Gamma_{f_t(z)}^{f_t^{\mathcal{D}}(z)} N_t(z) - N_t^{\mathcal{D}}(z)\right\|_{T\mathbb{H}^3} \leq C|t||z|^{1-\delta}.$$

The following lemma measures the lack of euclideanity in the parallel transportation of unitary vectors.

Lemma 11. Let $a, b, c \in \mathbb{H}^3$, $v_a \in T_a\mathbb{H}^3$ and $v_b \in T_b\mathbb{H}^3$ both unitary. Let $\mathcal{A}$ be the hyperbolic area of the triangle (a, b, c). Then

$$\left\|\Gamma_a^b v_a - v_b\right\|_{T_b\mathbb{H}^3} \leq \mathcal{A} + \|\Gamma_a^c v_a - \Gamma_b^c v_b\|_{T_c\mathbb{H}^3}\,.$$

Proof. Just use the triangular inequality and Gauss-Bonnet formula in $\mathbb{H}^2$ to write:

$$\begin{aligned}\left\|\Gamma_a^b v_a - v_b\right\|_{T_b\mathbb{H}^3} &= \left\|\Gamma_c^a\Gamma_b^c\Gamma_a^b v_a - \Gamma_c^a\Gamma_b^c v_b\right\|_{T_a\mathbb{H}^3}\\ &\leq \left\|\Gamma_c^a\Gamma_b^c\Gamma_a^b v_a - v_a\right\|_{T_a\mathbb{H}^3} + \|v_a - \Gamma_c^a\Gamma_b^c v_b\|_{T_a\mathbb{H}^3}\\ &\leq \mathcal{A} + \|\Gamma_a^c v_a - \Gamma_b^c v_b\|_{T_c\mathbb{H}^3}\,.\end{aligned}$$

□

Lemma 12 below clarifies how the unitary frame encodes the immersion and the normal map.

Lemma 12. Let $f = \mathrm{Sym}_q F$ and $N = \mathrm{Nor}_q F$. Denoting by $(S(z), Q(z)) \in \mathcal{H}_2^{++} \cap \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SU}(2)$ the polar decomposition of $F(z, e^{-q})$,

$$f = S^2 \quad \text{and} \quad N = \Gamma_{\mathrm{I}_2}^f(Q \cdot \sigma_3).$$

Proof. The formula for f is straightforward after noticing that $QQ^* = \mathrm{I}_2$ and $S^* = S$. The formula for N is a direct consequence of Proposition 22. □

Proof of Proposition 29. Let $\delta > 0$, $t \in (-T', T')$ and $z \in D_\epsilon^*$. Using Lemma 11,

$$\left\|\Gamma_{f_t(z)}^{f_t^{\mathcal{D}}(z)} N_t(z) - N_t^{\mathcal{D}}(z)\right\| \leq \mathcal{A} + \left\|\Gamma_{f_t(z)}^{\mathrm{I}_2} N_t(z) - \Gamma_{f_t^{\mathcal{D}}(z)}^{\mathrm{I}_2} N_t^{\mathcal{D}}(z)\right\|$$

where $\mathcal{A}$ is the hyperbolic area of the triangle $\left(\mathrm{I}_2, f_t(z), f_t^{\mathcal{D}}(z)\right)$. Using Heron's formula in $\mathbb{H}^2$ (see [40], p.66), Proposition 28 and the first point of Theorem 14,

$$\mathcal{A} \leq d_{\mathbb{H}^3}\left(f_t(z), f_t^{\mathcal{D}}(z)\right) \times d_{\mathbb{H}^3}\left(\mathrm{I}_2, f_t^{\mathcal{D}}(z)\right) \leq C|t||z|^{1-\delta}.$$

Moreover, denoting by Q_t and $Q_t^{\mathcal{D}}$ the unitary parts of $F_t(e^q)$ and $F_t^{\mathcal{D}}(e^q)$ in their polar decomposition and using Lemma 12 together with Corollary 7 and Equation (5.35),

$$\begin{aligned}\left\|\Gamma^{\mathrm{I}_2}_{f_t(z)}N_t(z) - \Gamma^{\mathrm{I}_2}_{f_t^{\mathcal{D}}(z)}N_t^{\mathcal{D}}(z)\right\| &= \left\|Q_t(z)\cdot\sigma_3 - Q_t^{\mathcal{D}}(z)\cdot\sigma_3\right\|_{T_{\mathrm{I}_2}\mathbb{H}^3}\\ &\leq C\left\|F_t^{\mathcal{D}}(z)\right\|_R^2\left\|F_t^{\mathcal{D}}(z)^{-1}F_t^{\mathcal{D}}(z) - \mathrm{I}_2\right\|_R\\ &\leq C|t||z|^{1-3\delta}.\end{aligned}$$

5.2.6 Embeddedness

In this section, we prove the second point of Theorem 14. We thus assume that $t > 0$. We suppose that $C, \epsilon, T, \delta > 0$ satisfy Proposition 29 and the first point of Theorem 14.

Lemma 13. Let $r_t > 0$ such that the tubular neighbourhood of $f_t^{\mathcal{D}}(\mathbb{C}^*)$ with hyperbolic radius r_t is embedded. There exists $T > 0$ and $0 < \epsilon' < \epsilon$ such that for all $0 < t < T$, $x \in \partial D_\epsilon$ and $y \in D^*_{\epsilon'}$,

$$d_{\mathbb{H}^3}\left(f_t^{\mathcal{D}}(x), f_t^{\mathcal{D}}(y)\right) > 4r_t.$$

Proof. The convergence of $f_t^{\mathcal{D}}(\mathbb{C}^*)$ towards a chain of spheres implies that r_t tends to 0 as t tends to 0. If $f_t^{\mathcal{D}}$ does not degenerate into a point, then it converges towards the parametrisation of a sphere, and for all $0 < \epsilon' < \epsilon$ there exists $T > 0$ satisfying the inequality. If $f_t^{\mathcal{D}}$ does degenerate into a point, then a suitable blow-up makes it converge towards a catenoidal immersion on the punctured disk D^*_ϵ (see Section 5.4.1). This implies that for $\epsilon' > 0$ small enough, there exists $T > 0$ satisfying the inequality. □

We can now prove embeddedness with the same method than in Chapter 4. Let $\mathcal{D}_t := f_t^{\mathcal{D}}(\mathbb{C}^*) \subset \mathbb{H}^3$ be the image Delaunay surface of $f_t^{\mathcal{D}}$. We denote by $\eta_t^{\mathcal{D}} : \mathcal{D}_t \longrightarrow T\mathbb{H}^3$ the Gauss map of $\mathcal{D}_t$. We also write $\mathcal{M}_t = f_t(D^*_\epsilon)$ and $\eta_t : \mathcal{M}_t \longrightarrow T\mathbb{H}^3$. Let r_t be the maximal value of α such that the following map is a diffeomorphism:

$$\begin{array}{rccc}\mathcal{T} & : & (-\alpha,\alpha)\times\mathcal{D}_t & \longrightarrow & \mathrm{Tub}_\alpha\,\mathcal{D}_t \subset \mathbb{H}^3\\ & & (s,p) & \longmapsto & \mathrm{geod}\left(p, \eta_t^{\mathcal{D}}(p)\right)(s).\end{array}$$

According to Lemma 18, the maximal tubular radius satisfies $r_t \sim t$ as t tends to 0. Using the first point of Theorem 14, we thus assume that on D^*_ϵ,

$$d_{\mathbb{H}^3}\left(f_t(z), f_t^{\mathcal{D}}(z)\right) \leq \alpha r_t$$

where $\alpha < 1$ is given by Lemma 19 of Section 5.5.

Let π_t be the projection from $\mathrm{Tub}_{r_t}\mathcal{D}_t$ to $\mathcal{D}_t$. Then the map

$$\begin{array}{rccc} \varphi_t & : & D_\epsilon^* & \longrightarrow & \mathcal{D}_t \\ & & z & \longmapsto & \pi_t \circ f_t(z) \end{array}$$

is well-defined and satisfies

$$d_{\mathbb{H}^3}\left(\varphi_t(z), f_t^{\mathcal{D}}(z)\right) \leq 2\alpha r_t \tag{5.36}$$

because of the triangular inequality.

Lemma 14. For $t > 0$ small enough, φ_t is a local diffeomorphism on D_ϵ^*.

Proof. It suffices to show that for all $z \in D_\epsilon^*$,

$$\left\| \Gamma_{\varphi_t(z)}^{f_t(z)} \eta_t^{\mathcal{D}}(\varphi_t(z)) - N_t(z) \right\| < 1. \tag{5.37}$$

Using Lemma 11 (we drop the variable z to ease the notation),

$$\left\| \Gamma_{\varphi_t}^{f_t} \eta_t^{\mathcal{D}}(\varphi_t) - N_t \right\| \leq \mathcal{A} + \left\| \Gamma_{\varphi_t}^{f_t^{\mathcal{D}}} \eta_t^{\mathcal{D}}(\varphi_t) - \Gamma_{f_t}^{f_t^{\mathcal{D}}} N_t \right\|$$

where $\mathcal{A}$ is the area of the triangle $\left(f_t, f_t^{\mathcal{D}}, \varphi_t\right)$. Recall the isoperimetric inequality in $\mathbb{H}^2$ (see [37]):

$$\mathcal{P}^2 \geq 4\pi\mathcal{A} + \mathcal{A}^2$$

from which we deduce

$$\mathcal{A} \leq \mathcal{P}^2 \leq \left(2d_{\mathbb{H}^3}\left(f_t, f_t^{\mathcal{D}}\right) + 2d_{\mathbb{H}^3}\left(\varphi_t, f_t^{\mathcal{D}}\right)\right)^2 \leq (6\alpha r_t)^2$$

which uniformly tends to 0 as t tends to 0. Using the triangular inequality and Proposition 29,

$$\left\| \Gamma_{\varphi_t}^{f_t^{\mathcal{D}}} \eta_t^{\mathcal{D}}(\varphi_t) - \Gamma_{f_t}^{f_t^{\mathcal{D}}} N_t \right\| \leq \left\| \Gamma_{\varphi_t}^{f_t^{\mathcal{D}}} \eta_t^{\mathcal{D}}(\varphi_t) - N_t^{\mathcal{D}} \right\| + C|t||z|^{1-\delta}$$

and the second term of the right-hand side uniformly tends to 0 as t tends to 0. Because α satisfies Lemma 19 in Section 5.5,

$$\left\| \Gamma_{\varphi_t}^{f_t^{\mathcal{D}}} \eta_t^{\mathcal{D}}(\varphi_t) - N_t^{\mathcal{D}} \right\| < 1$$

which implies Equation (5.37). □

Let $\epsilon' > 0$ given by Lemma 13. The restriction

$$\begin{array}{rccccc} \widetilde{\varphi}_t & : & \varphi_t^{-1}\left(\varphi_t(D_{\epsilon'}^*)\right) \cap D_\epsilon^* & \longrightarrow & \varphi_t\left(D_{\epsilon'}^*\right) \\ & & z & \longmapsto & \varphi_t(z) \end{array}$$

is a covering map because it is a proper local diffeomorphism between locally compact spaces. To show this, proceed by contradiction as in $\mathbb{R}^3$ (see [30]): let $(x_i)_{i\in\mathbb{N}} \subset \varphi_t^{-1}\left(\varphi_t(D^*_{\epsilon'})\right) \cap D^*_\epsilon$ such that $(\widetilde{\varphi}_t(x_i))_{i\in\mathbb{N}}$ converges to $p \in \varphi_t\left(D^*_{\epsilon'}\right)$. Then $(x_i)_i$ converges to $x \in \overline{D}_\epsilon$. Using Equation (5.36) and the fact that $f_t^{\mathcal{D}}$ has an end at 0, $x \neq 0$. If $x \in \partial D_\epsilon$, denoting $\widetilde{x} \in D^*_{\epsilon'}$ such that $\widetilde{\varphi}_t(\widetilde{x}) = p$, one has

$$d_{\mathbb{H}^3}\left(f_t^{\mathcal{D}}(x), f_t^{\mathcal{D}}(\widetilde{x})\right) < d_{\mathbb{H}^3}\left(f_t^{\mathcal{D}}(x), p\right) + d_{\mathbb{H}^3}\left(f_t^{\mathcal{D}}(\widetilde{x}), \widetilde{\varphi}_t(\widetilde{x})\right) < 4\alpha r_t < 4r_t$$

which contradicts the definition of ϵ'.

Let us now prove as in Chapter 4 that $\widetilde{\varphi}_t$ is a one-sheeted covering map. Let $\gamma : [0,1] \longrightarrow D^*_{\epsilon'}$ be a loop of winding number 1 around 0, $\Gamma = f_t^{\mathcal{D}}(\gamma)$ and $\widetilde{\Gamma} = \widetilde{\varphi}_t(\gamma) \subset \mathcal{D}_t$ and let us construct a homotopy between Γ and $\widetilde{\Gamma}$. For all $s \in [0,1]$, let $\sigma_s : [0,1] \longrightarrow \mathbb{H}^3$ be a geodesic arc joining $\sigma_s(0) = \widetilde{\Gamma}(s)$ to $\sigma_s(1) = \Gamma(s)$. For all $s, r \in [0,1]$, $d_{\mathbb{H}^3}\left(\sigma_s(r), \Gamma(s)\right) \leq \alpha r_t$ which implies that $\sigma_s(r) \in \mathrm{Tub}_{r_t}\,\mathcal{D}_t$ because $\mathcal{D}_t$ is complete. One can thus define the following homotopy between Γ and $\widetilde{\Gamma}$

$$\begin{array}{rccc} H & : & [0,1]^2 & \longrightarrow & \mathcal{D}_t \\ & & (r,s) & \longmapsto & \pi_t \circ \sigma_s(r) \end{array}$$

where π_t is the projection from $\mathrm{Tub}_{r_t}\,\mathcal{D}_t$ to $\mathcal{D}_t$. Using the fact that $f_t^{\mathcal{D}}$ is an embedding, the degree of Γ is one, and the degree of $\widetilde{\Gamma}$ is also one. Hence, $\widetilde{\varphi}_t$ is one-sheeted.

Finally, the map $\widetilde{\varphi}_t$ is a one-sheeted covering map and hence a diffeomorphism, so $f_t\left(D^*_{\epsilon'}\right)$ is a normal graph over $\mathcal{D}_t$ contained in its embedded tubular neighbourhood and $f_t\left(D^*_{\epsilon'}\right)$ is thus embedded, which proves the second point of Theorem 14.

5.2.7 Limit axis

In this section, we prove the third point of Theorem 14 and compute the limit axis of $f_t^{\mathcal{D}}$ as t tends to 0. Recall that $f_t^{\mathcal{D}} = \mathrm{Sym}_q\left(\mathrm{Uni}\left(M_t z^{A_t}\right)\right)$ where M_t tends to I_2 as t tends to 0. Hence, the limit axis of $f_t^{\mathcal{D}}$ and $\widetilde{f}_t^{\mathcal{D}} := \mathrm{Sym}_q\left(\mathrm{Uni}\left(z^{A_t}\right)\right)$ are the same. Two cases can occur, whether $r > s$ or $r < s$.

Spherical family. At $t = 0$, $r = \frac{1}{2}$ and $s = 0$. The limit potential is thus

$$\xi_0(z,\lambda) = \begin{pmatrix} 0 & \frac{\lambda^{-1}}{2} \\ \frac{\lambda}{2} & 0 \end{pmatrix} z^{-1} dz.$$

Consider the gauge

$$G(z,\lambda) = \frac{1}{\sqrt{2z}} \begin{pmatrix} 1 & 0 \\ \lambda & 2z \end{pmatrix}.$$

The gauged potential is then

$$\xi_0 \cdot G(z,\lambda) = \begin{pmatrix} 0 & \lambda^{-1} dz \\ 0 & 0 \end{pmatrix} = \xi_{\mathbb{S}}(z,\lambda)$$

where $\xi_{\mathbb{S}}$ is the spherical potential as in Section 5.1.2. Let $\widetilde{\Phi} := z^{A_0} G$ be the gauged holomorphic frame and compute

$$\begin{aligned} \widetilde{\Phi}(1,\lambda) &= G(1,\lambda) \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ \lambda & 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix} \\ &= H(\lambda) \Phi_{\mathbb{S}}(1,\lambda) \end{aligned}$$

where $\Phi_{\mathbb{S}}$ is defined in (5.10) and $H = H_0$ as in (5.22). This means that $\widetilde{\Phi} = H\Phi_{\mathbb{S}}$, $\operatorname{Uni}\widetilde{\Phi} = HF_{\mathbb{S}}$ and $\operatorname{Sym}_q(\operatorname{Uni}\widetilde{\Phi}) = H(e^{-q}) \cdot f_{\mathbb{S}}$ because $H \in \Lambda\mathrm{SU}(2)_R$. Thus using equations (5.12) and (5.13),

$$\begin{aligned} \widetilde{f}_0^{\mathcal{D}}(\infty) = \operatorname{Sym}_q(\operatorname{Uni}\widetilde{\Phi})(\infty) &= H(e^{-q}) \cdot f_{\mathbb{S}}(\infty) \\ &= \left(H(e^{-q}) R(q)\right) \cdot \operatorname{geod}(\mathrm{I}_2, \sigma_3)(q) \\ &= H(e^{-q}) \cdot \operatorname{geod}(\mathrm{I}_2, \sigma_3)(2q). \end{aligned}$$

And with the same method,

$$\widetilde{f}_0^{\mathcal{D}}(0) = H(e^{-q}) \cdot \operatorname{geod}(\mathrm{I}_2, \sigma_3)(0).$$

This means that the limit axis of $\widetilde{f}_t^{\mathcal{D}}$ as $t \to 0$, oriented from $z = \infty$ to $z = 0$ is given in the spherical family by

$$H(e^{-q}) \cdot \operatorname{geod}(\mathrm{I}_2, -\sigma_3).$$

Catenoidal family. We cannot use the same method as above, as the immersion $\widetilde{f}_t^{\mathcal{D}}$ degenerates into the point I_2. Use Proposition 32 of Section 5.4.1 to get

$$\widehat{f} := \lim_{t \to 0} \frac{1}{t} (f_t - \mathrm{I}_2) = \psi \subset T_{\mathrm{I}_2} \mathbb{H}^3$$

where ψ is the immersion of a catenoid of axis oriented by $-\sigma_1$ as $z \to 0$. This suffices to show that the limit axis oriented from the end at ∞ to the end at 0 of the catenoidal family $\widetilde{f}_t^{\mathcal{D}}$ converges as t tends to 0 to the oriented geodesic $\mathrm{geod}(\mathrm{I}_2, -\sigma_1)$.

5.3 Gluing Delaunay ends to hyperbolic spheres

In this section, we follow step by step the method Martin Traizet used in $\mathbb{R}^3$ ([39]) to construct CMC $H > 1$ n-noids in $\mathbb{H}^3$ and prove Theorem 11. This method relies on the Implicit Function Theorem and aims to find a pair (ξ_t, Φ_t) satisfying the hypotheses of Theorem 14 around each pole of an n-punctured sphere. More precisely, the Implicit Function Theorem is used to solve the monodromy problem around each pole and to ensure that the potential is regular at $z = \infty$. The set of equations characterising this problem at $t = 0$ is the same as in [39], and the partial derivative with respect to the parameters is the same as in [39] at $t = 0$. Therefore, the Implicit Function Theorem can be used exactly as in [39] and we do not repeat it here. Showing that the surface has Delaunay ends involves slightly different computations, but the method is the same as in [39], namely, find a suitable gauge and change of coordinates around each pole of the potential in order to retrieve a perturbed Delaunay potential as in Definition 16. One can then apply Theorem 14. Finally, we show that the surface is Alexandrov-embedded (and embedded in some cases) by adapting the arguments of [38] to the case of $\mathbb{H}^3$.

5.3.1 The DPW data

Let $H > 1$, $q = \operatorname{arcoth} H$ and $\rho > e^q$. Let $n \geq 3$ and $u_1, \cdots, u_n$ unitary vectors of $T_{\mathrm{I}_2}\mathbb{H}^3$. Suppose, by applying a rotation, that $u_i \neq \pm\sigma_3$ for all $i \in [1, n]$. Let $v_{\mathbb{S}} : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{S}^2$ defined as in Equation (5.13) and $\pi_i := v_{\mathbb{S}}^{-1}(u_i) \in \mathbb{C}^*$. Consider $3n$ parameters a_i, b_i, $p_i \in \Lambda\mathbb{C}_{\rho}^{\geq 0}$ assembled into a vector $\mathbf{x}$ which stands in a neighbourhood of a central value $\mathbf{x}_0$ so that the central values of a_i and p_i are τ_i and π_i. Introduce a real parameter t in a neighbourhood of 0 and define

$$\beta_t(\lambda) := t\left(\lambda - e^q\right)\left(\lambda - e^{-q}\right).$$

The potential we use is

$$\xi_{t,\mathbf{x}}(z,\lambda) := \begin{pmatrix} 0 & \lambda^{-1}dz \\ \beta_t(\lambda)\omega_{\mathbf{x}}(z,\lambda) & 0 \end{pmatrix}$$

where

$$\omega_{\mathbf{x}}(z,\lambda) := \sum_{i=1}^{n} \left(\frac{a_i(\lambda)}{(z-p_i(\lambda))^2} + \frac{b_i(\lambda)}{z-p_i(\lambda)} \right) dz.$$

The initial condition is the identity matrix, taken at the point $z_0 = 0 \in \Omega$ where

$$\Omega = \{z \in \mathbb{C} \mid \forall i \in [1,n], |z-\pi_i| > \epsilon\}$$

and $\epsilon > 0$ is a fixed constant such that the disks $D(\pi_i, 2\epsilon) \subset \mathbb{C}$ are disjoint and do not contain 0. Although the poles $p_1, \ldots, p_n$ of the potential $\xi_{t,\mathbf{x}}$ are functions of λ, $\xi_{t,\mathbf{x}}$ is well-defined on Ω for $\mathbf{x}$ sufficiently close to $\mathbf{x}_0$. We thus define $\Phi_{t,\mathbf{x}}$ as the solution to the Cauchy problem (5.5) with data $(\Omega, \xi_{t,\mathbf{x}}, 0, \mathrm{I}_2)$.

The main properties of this potential are the same as in [39], namely: it is a perturbation of the spherical potential $\xi_{0,\mathbf{x}}$ and the factor $(\lambda - e^{-q})$ in β_t ensures that the second equation of the monodromy problem (5.8) is solved.

Let $\{\gamma_1, \cdots, \gamma_{n-1}\}$ be a set of generators of the fundamental group $\pi_1(\Omega, 0)$ and define for all $i \in [1, n-1]$

$$M_i(t,\mathbf{x}) := \mathcal{M}_{\gamma_i}(\Phi_{t,\mathbf{x}}).$$

Noting that

$$\lambda \in \mathbb{S}^1 \implies \lambda^{-1}(\lambda - e^{q})\left(\lambda - e^{-q}\right) = -2(\cosh q - \operatorname{Re}\lambda) \in \mathbb{R},$$

the unitarity of the monodromy is equivalent to

$$\widetilde{M}_i(t,\mathbf{x})(\lambda) := \frac{\lambda}{\beta_t(\lambda)} \log M_i(t,\mathbf{x})(\lambda) \in \Lambda\mathfrak{su}(2)_\rho.$$

Note that at $t = 0$, the expression above takes the same value as in [39], and so does the regularity conditions. One can thus apply Propositions 2 and 3 of [39] which we recall in Proposition 30 below.

Proposition 30. For t in a neighbourhood of 0, there exists a unique smooth map $t \mapsto \mathbf{x}(t) = (a_{i,t}, b_{i,t}, p_{i,t})_{1\le i\le n} \in (\mathcal{W}_R^{\geq 0})^3$ such that $\mathbf{x}(0) = \mathbf{x}_0$, the monodromy problem and the regularity problem are solved at $(t, \mathbf{x}(t))$ and the following normalisations hold:

$$\forall i \in [1, n-1], \qquad \operatorname{Re}(a_{i,t})\,|_{\lambda=0} = \tau_i \quad \text{and} \quad p_{i,t}\,|_{\lambda=0} = \pi_i.$$

Moreover, at $t = 0$, $\mathbf{x}_0$ is a constant with a_i real and such that

$$b_i = \frac{-2a_i\overline{p_i}}{1+|p_i|^2} \quad \text{and} \quad \sum_{i=1}^{n} a_i v_{\mathbb{S}}(p_i) = 0.$$

Now write $\omega_t := \omega_{\mathbf{x}(t)}$, $\xi_t := \xi_{t,\mathbf{x}(t)}$ and apply the DPW method to define the holomorphic frame Φ_t associated to ξ_t on the universal cover $\widetilde{\Omega}$ of Ω with initial condition $\Phi_t(0) = \mathrm{I}_2$. Let $F_t := \mathrm{Uni}\,\Phi_t$ and $f_t := \mathrm{Sym}_q F_t$. The monodromy problem for Φ_t being solved, f_t descends to a well-defined CMC H immersion on Ω. Use Theorem 3 and Corollary 1 of [39] to extend f_t to $\Sigma_t := \mathbb{C} \cup \{\infty\} \setminus \{p_{1,t}(0), \dots, p_{n,t}(0)\}$ and define $M_t = f_t(\Sigma_t)$. Moreover, with the same proof as in [39] (Proposition 4, point (2)), $a_{i,t}$ is a real constant with respect to λ for all i and t.

5.3.2 Delaunay ends

Perturbed Delaunay potential. Let $i \in [1, n]$. We are going to gauge ξ_t around its pole $p_{i,t}(0)$ and show that the gauged potential is a perturbed Delaunay potential as in Definition 16. Let $(r, s) : (-T, T) \longrightarrow \mathbb{R}^2$ be the continuous solution to (see Section 5.1.3 for details)

$$\begin{cases} rs = ta_{i,t}, \\ r^2 + s^2 + 2rs\cosh q = \frac{1}{4}, \\ r > s. \end{cases}$$

For all t and λ, define $\psi_{i,t,\lambda}(z) := z + p_{i,t}(\lambda)$ and

$$G_t(z,\lambda) := \begin{pmatrix} \frac{\sqrt{z}}{\sqrt{r+s\lambda}} & 0 \\ \frac{-\lambda}{2\sqrt{z}\sqrt{r+s\lambda}} & \frac{\sqrt{r+s\lambda}}{\sqrt{z}} \end{pmatrix}.$$

For T small enough, one can thus define on a uniform neighbourhood of 0 the potential

$$\widetilde{\xi}_{i,t}(z,\lambda) := ((\psi_{i,t,\lambda}^* \xi_t) \cdot G_t)(z,\lambda) = \begin{pmatrix} 0 & r\lambda^{-1} + s \\ \frac{\beta_t(\lambda)}{r+s\lambda}(\psi_{i,t,\lambda}^* \omega_t(z)) z^2 + \frac{\lambda}{4(r+s\lambda)} & 0 \end{pmatrix} z^{-1} dz.$$

Note that by definition of r, s and β_t,

$$(r + s\lambda)(r\lambda + s) = \frac{\lambda}{4} + \beta_t(\lambda) a_{i,t}$$

and thus

$$\widetilde{\xi}_{i,t}(z,\lambda) = A_t(\lambda) z^{-1} dz + C_t(z,\lambda) dz$$

with A_t as in Equation (5.14) satisfies Equation (5.19) and C_t as in Definition 16. The potential $\widetilde{\xi}_{i,t}$ is thus a perturbed Delaunay potential as in Definition 16. Moreover, using Theorem 3 of [39], the induced immersion $\widetilde{f}_{i,t}$ satisfies

$$\widetilde{f}_{i,t} = \psi_{i,t,0}^* f_t.$$

Applying Theorem 14. The holomorphic frame $\widetilde{\Phi}_{i,t} := \Phi_t G_{i,t}$ associated to $\widetilde{\xi}_{i,t}$ satisfies the regularity and monodromy hypotheses of Theorem 14, but at $t = 0$ and $z = 1$,

$$\widetilde{\Phi}_{i,0}(1,\lambda) = \begin{pmatrix} 1 & (1+\pi_i)\,\lambda^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ \frac{-\lambda}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1-\pi_i & (1+\pi_i)\lambda^{-1} \\ -\lambda & 1 \end{pmatrix} =: M_i(\lambda),$$

and thus $\widetilde{\Phi}_{i,0}(z) = M_i z^{A_0}$. Recall (5.21) and let

$$H := H_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix} \in \Lambda \mathrm{SU}(2)_\rho$$

and $Q_i := \mathrm{Uni}\,(M_i H)$. Using Lemma 1 in Chapter 4, Q_i can be made explicit and one can find a change of coordinates h and a gauge G such that $\widehat{\Phi}_{i,t} := HQ_i^{-1}(h^*\widetilde{\Phi}_{i,t})G$ solves $d\widehat{\Phi}_{i,t} = \widehat{\Phi}_{i,t}\widehat{\xi}_{i,t}$ where $\widehat{\xi}_{i,t}$ is a perturbed Delaunay potential and $\widehat{\Phi}_{i,0}(z) = z^{A_0}$. Explicitely,

$$Q_i(\lambda) = \frac{1}{\sqrt{1+|\pi_i|^2}} \begin{pmatrix} 1 & \lambda^{-1}\pi_i \\ -\lambda\overline{\pi}_i & 1 \end{pmatrix}$$

and

$$h(z) = \frac{(1+|\pi_i|^2)z}{1-\overline{\pi}_i z}, \qquad G(z,\lambda) = \frac{1}{\sqrt{1-\overline{\pi}_i z}} \begin{pmatrix} 1 & 0 \\ -\lambda\overline{\pi}_i z & 1-\overline{\pi}_i z \end{pmatrix}.$$

One can thus apply Theorem 14 on $\widehat{\xi}_{i,t}$ and $\widehat{\Phi}_{i,t}$, which proves the existence of the family $(M_t)_{0<t<T}$ of CMC H surfaces of genus zero and n Delaunay ends, each of weight (according to Equation (5.16))

$$w_{i,t} = 8\pi rs \sinh q = \frac{8\pi t a_{i,t}}{\sqrt{H^2-1}},$$

which proves the first point of Theorem 11 (after a normalisation on t). Let $\widehat{f}_{i,t} := \mathrm{Sym}_q(\mathrm{Uni}\,\widehat{\Phi}_{i,t})$ and $\widehat{f}^{\mathcal{D}}_{i,t}$ the Delaunay immersion given by Theorem 14.

Limit axis. In order to compute the limit axis of f_t at the end around $p_{i,t}$, let $\widehat{\Delta}_{i,t}$ be the oriented axis of $\widehat{f}^{\mathcal{D}}_{i,t}$ at $w = 0$. Then, using Theorem 14,

$$\widehat{\Delta}_{i,0} = H(e^{-q}) \cdot \mathrm{geod}\,(\mathrm{I}_2, -\sigma_3)\,.$$

And using $\widehat{f}_{i,t}(w) = H(e^{-q})Q_i(e^{-q})^{-1} \cdot (h^* f_t(z))$,

$$\widehat{\Delta}_{i,0} = H(e^{-q})Q_i(e^{-q})^{-1} \cdot \Delta_{i,0}$$

and thus

$$\Delta_{i,0} = Q_i(e^{-q}) \cdot \mathrm{geod}(\mathrm{I}_2, -\sigma_3).$$

Computing $M_iH = \Phi_{\mathbb{S}}(\pi_i)$ as in (5.10), one has $Q_i = F_{\mathbb{S}}(\pi_i)$. Hence

$$\Delta_{i,0} = \mathrm{geod}\left(f_{\mathbb{S}}(\pi_i), -N_{\mathbb{S}}(\pi_i)\right)$$

where $N_{\mathbb{S}}$ is the normal map associated to $\Phi_{\mathbb{S}}$. Using Equation (5.11), $f_{\mathbb{S}}(z) = R(q)\cdot\widetilde{f}_{\mathbb{S}}(z)$ and $N_{\mathbb{S}}(z) = R(q)\cdot\widetilde{N}_{\mathbb{S}}(z)$ where $\widetilde{N}_{\mathbb{S}}$ is the normal map of $\widetilde{f}_{\mathbb{S}}$. Using Equation (5.12) and the fact that $\widetilde{f}_{\mathbb{S}}$ is a spherical immersion gives

$$\widetilde{N}_{\mathbb{S}}(z) = \Gamma_{\mathrm{I}_2}^{\widetilde{f}_{\mathbb{S}}(z)}\left(-\upsilon_{\mathbb{S}}(z)\right)$$

and thus

$$\begin{aligned}\Delta_{i,0} &= \mathrm{geod}\left(R(q)\cdot\widetilde{f}_{\mathbb{S}}(\pi_i), -R(q)\cdot\widetilde{N}_{\mathbb{S}}(\pi_i)\right)\\ &= R(q)\cdot\mathrm{geod}\left(\widetilde{f}_{\mathbb{S}}(\pi_i), \Gamma_{\mathrm{I}_2}^{\widetilde{f}_{\mathbb{S}}(\pi_i)}\upsilon_{\mathbb{S}}(\pi_i)\right)\\ &= R(q)\cdot\mathrm{geod}\left(\mathrm{I}_2, u_i\right).\end{aligned}$$

Apply the isometry given by $R(q)^{-1}$ and note that $R(q)$ does not depend on i to prove point 2 of Theorem 11.

5.3.3 Embeddedness

We suppose that $t > 0$ and that all the weights τ_i are positive, so that the ends of f_t are embedded. Recall the definition of Alexandrov-embeddedness (as stated in [38]):

Definition 17. A surface $M^2 \subset \mathcal{M}^3$ of finite topology is Alexandrov-embedded if M is properly immersed, if each end of M is embedded, and if there exists a compact 3-manifold $\overline{W}$ with boundary $\partial\overline{W} = \overline{S}$, n points $p_1, \cdots, p_n \in \overline{S}$ and a proper immersion $F : W = \overline{W}\backslash\{p_1, \cdots, p_n\} \longrightarrow \mathcal{M}$ whose restriction to $S = \overline{S}\backslash\{p_1, \cdots, p_n\}$ parametrises M.

The following lemma is proved in [38] in $\mathbb{R}^3$ and for surfaces with catenoidal ends, but the proof is the same in $\mathbb{H}^3$ for surfaces with any type of embedded ends. For any oriented surface M with Gauss map N and any $r > 0$, the tubular map of M with radius r is defined by

$$\begin{array}{rccc}\mathcal{T} & : & (-r,r)\times M & \longrightarrow & \mathrm{Tub}_r\, M\\ & & (s,p) & \longmapsto & \mathrm{geod}(p, N(p))(s).\end{array}$$

Lemma 15. Let M be an oriented Alexandrov-embedded surface of $\mathbb{H}^3$ with n embedded ends. Let $r > 0$ and suppose that the tubular map of M with radius r is a local diffeomorphism. With the notations of Definition 17, there exist a hyperbolic 3-manifold W'

containing W and a local isometry $F' : W' \longrightarrow \mathbb{H}^3$ extending F such that the tubular neighbourhood $\mathrm{Tub}_r\, S$ is embedded in W'.

In order to show that M_t is embedded, we will use the techniques of [38]. We thus begin by lifting M_t to $\mathbb{R}^3$ with the exponential map at the identity $\exp_{\mathrm{I}_2} : \mathbb{R}^3 \longrightarrow \mathbb{H}^3$. This map is a diffeomorphism, so M_t is Alexandrov-embedded if, and only if its lift $\widehat{M_t}$ to $\mathbb{R}^3$ given by the immersion

$$\widehat{f_t} := \exp_{\mathrm{I}_2}^{-1} \circ f_t : \Sigma_t \longrightarrow \mathbb{R}^3$$

is Alexandrov-embedded.

Let $T, \epsilon > 0$ such that f_t (and hence $\widehat{f_t}$) is an embedding of $D^*(p_{i,t}, \epsilon)$ for all $i \in [1, n]$ and let $f_{i,t}^{\mathcal{D}} : \mathbb{C}\backslash\{p_{i,t}\} \longrightarrow \mathbb{H}^3$ be the Delaunay immersion approximating f_t in $D^*(p_{i,t}, \epsilon)$. Let $\widehat{f}_{i,t}^{\mathcal{D}} := \exp_{\mathrm{I}_2}^{-1} \circ f_{i,t}^{\mathcal{D}}$. Apply an isometry of $\mathbb{H}^3$ so that the limit immersion f_0 maps Σ_0 to a n-punctured geodesic sphere of hyperbolic radius q centered at I_2. Then $\widehat{f_0}(\Sigma_0)$ is a Euclidean sphere of radius q centered at the origin. Define

$$\begin{array}{rccc} \widehat{N_t} & : & \Sigma_t & \longrightarrow & \mathbb{S}^2 \\ & & z & \longmapsto & d(\exp_{\mathrm{I}_2}^{-1})(f_t(z))N_t(z). \end{array}$$

At $t = 0$, $\widehat{N}_0$ is the normal map of $\widehat{f_0}$ (by Gauss Lemma), but not for $t > 0$ because the Euclidean metric of $\mathbb{R}^3$ is not the metric induced by $\exp_{\mathrm{I}_2}$.

Let

$$\begin{array}{rccc} h_i & : & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & \left\langle x, -\widehat{N}_0(p_{i,0}) \right\rangle \end{array}$$

be the height function in the direction of the limit axis.

As in [38], one can show that

Claim 1. There exist $\delta < \delta'$ and $0 < \epsilon' < \epsilon$ such that for all $i \in [1, n]$ and $0 < t < T$,

$$\max_{C(p_{i,t},\epsilon)} h_i \circ \widehat{f_t} < \delta < \min_{C(p_{i,t},\epsilon')} h_i \circ \widehat{f_t} \leq \max_{C(p_{i,t},\epsilon')} h_i \circ \widehat{f_t} < \delta'.$$

Define for all i and t:

$$\gamma_{i,t} := \left\{ z \in D^*_{p_{i,t},\epsilon} \mid h_i \circ \widehat{f_t}(z) = \delta \right\}, \quad \gamma'_{i,t} := \left\{ z \in D^*_{p_{i,t},\epsilon'} \mid h_i \circ \widehat{f_t}(z) = \delta' \right\}.$$

From their convergence as t tends to 0,

Claim 2. The regular curves $\gamma_{i,t}$ and $\gamma'_{i,t}$ are topological circles around $p_{i,t}$.

Define $D_{i,t}, D'_{i,t}$ as the topological disks bounded by $\gamma_{i,t}, \gamma'_{i,t}$, and $\Delta_{i,t}, \Delta'_{i,t}$ as the topo-

logical disks bounded by $\widehat{f}_t(\gamma_{i,t}), \widehat{f}_t(\gamma'_{i,t})$. Let $\mathcal{A}_{i,t} := D_{i,t} \backslash D'_{i,t}$. Then $\widehat{f}_t(\mathcal{A}_{i,t})$ is a graph over the plane $\{h_i(x) = \delta\}$. Moreover, for all $z \in D^*_{i,t}$, $h_i \circ \widehat{f}_t(z) \geq \delta' > \delta$. Thus

Claim 3. The intersection $\widehat{f}_t(D^*_{i,t}) \cap \Delta_{i,t}$ is empty.

Define a sequence $(R_{i,t,k})$ such that $\widehat{f}_t(D^*_{i,t})$ transversally intersects the planes $\{h_i(x) = R_{i,t,k}\}$. Define

$$\gamma_{i,t,k} := \left\{ z \in D^*_{i,t} \mid h_i \circ \widehat{f}_{i,t}(z) = R_{i,t,k} \right\},$$

and the topological disks $\Delta_{i,t,k} \subset \{h_i(x) = R_{i,t,k}\}$ bounded by $\widehat{f}_t(\gamma_{i,t,k})$. Define $\mathcal{A}_{i,t,k}$ as the annuli bounded by $\gamma_{i,t}$ and $\gamma_{i,t,k}$. Define $W_{i,t,k} \subset \mathbb{R}^3$ as the interior of $\widehat{f}_t(\mathcal{A}_{i,t,k}) \cup \Delta_{i,t} \cup \Delta_{i,t,k}$ and

$$W_{i,t} := \bigcup_{k \in \mathbb{N}} W_{i,t,k}.$$

Hence,

Claim 4. The union $\widehat{f}_t(D^*_{i,t}) \cup \Delta_{i,t}$ is the boundary of a topological punctured ball $W_{i,t} \subset \mathbb{R}^3$.

The union

$$\widehat{f}_t \left(\Sigma_t \backslash \left(D_{1,t} \cup \cdots D_{n,t} \right) \right) \cup \Delta_{1,t} \cup \cdots \cup \Delta_{n,t}$$

is the boundary of a topological ball $W_{0,t} \subset \mathbb{R}^3$. Take

$$W_t := W_{0,t} \cup W_{1,t} \cup \cdots \cup W_{n,t}$$

to show that $\widehat{M}_t$, and hence M_t is Alexandrov-embedded for $t > 0$ small enough.

Lemma 16. Let $S \subset \mathbb{H}^3$ be a sphere of hyperbolic radius q centered at $p \in \mathbb{H}^3$. Let $n \geq 2$ and $\{u_i\}_{i \in [1,n]} \subset T_p \mathbb{H}^3$. Let $\{p_i\}_{i \in [1,n]}$ defined by $p_i = S \cap \mathrm{geod}(p, u_i)(\mathbb{R}_+)$. For all $i \in [1, n]$, let $S_i \subset \mathbb{H}^3$ be the sphere of hyperbolic radius q such that $S \cap S_i = \{p_i\}$. For all $(i, j) \in [1, n]^2$, let θ_{ij} be the angle between u_i and u_j.

If for all $i \neq j$,

$$\theta_{ij} > 2 \arcsin \left(\frac{1}{2 \cosh q} \right)$$

then $S_i \cap S_j = \emptyset$ for all $i \neq j$.

Proof. Without loss of generality, we assume that $p = \mathrm{I}_2$. We use the ball model of $\mathbb{H}^3$ equipped with its metric

$$ds^2_{\mathbb{B}}(x) = \frac{4 ds^2_E}{\left(1 - \|x\|^2_E\right)^2}$$

where ds_E is the euclidean metric and $\|x\|_E$ is the euclidean norm. In this model, the sphere S is centered at the origin and has euclidean radius $R = \tanh\frac{q}{2}$. For all $i \in [1, n]$, the sphere S_i has euclidean radius

$$r = \frac{1}{2}\left(\tanh\frac{3q}{2} - \tanh\frac{q}{2}\right) = \frac{\tanh\frac{q}{2}}{2\cosh q - 1}.$$

Let $j \neq i$. In order to have $S_i \cap S_j = \emptyset$, one must solve

$$(R + r)\sin\frac{\theta_{ij}}{2} \geq r,$$

which gives the expected result. □

In order to prove the last point of Theorem 11, just note that

$$H = \coth q \implies \frac{1}{2\cosh q} = \frac{\sqrt{H^2 - 1}}{2H}.$$

Suppose that the angle θ_{ij} between u_i and u_j satisfies Equation (5.2) for all $i \neq j$. Then for $t > 0$ small enough, the proper immersion F_t given by Definition 17 is injective (because of the convergence towards a chain of spheres) and hence M_t is embedded.

Remark 16. This means for example that in hyperbolic space, one can construct embedded CMC n-noids with seven coplanar ends or more.

5.4 Gluing Delaunay ends to minimal n-noids

Again, this section is an adaptation of Traizet's work in [38] applied to the proof of Theorem 12. We first give in Section 5.4.1 a blow-up result for CMC $H > 1$ surfaces in Hyperbolic space. We then introduce in Section 5.4.2 the DPW data giving rise to the surface M_t of Theorem 12 and prove the convergence towards the minimal n-noid. Finally, using the same arguments as in [38], we prove Alexandrov-embeddedness in Section 5.4.3.

5.4.1 A blow-up result

As in $\mathbb{R}^3$ (see [38]), the DPW method accounts for the convergence of CMC $H > 1$ surfaces in $\mathbb{H}^3$ towards minimal surfaces of $\mathbb{R}^3$ (after a suitable blow-up). We work with the following Weierstrass parametrisation:

$$W(z) = W(z_0) + \mathrm{Re}\int_{z_0}^{z}\left(\frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega\right) \tag{5.38}$$

Proposition 31. Let Σ be a Riemann surface, $(\xi_t)_{t\in I}$ a family of DPW potentials on Σ and $(\Phi_t)_{t\in I}$ a family of solutions to $d\Phi_t = \Phi_t\xi_t$ on the universal cover $\widetilde{\Sigma}$ of Σ, where $I \subset \mathbb{R}$ is a neighbourhood of 0. Fix a base point $z_0 \in \widetilde{\Sigma}$ and $\rho > e^q > 1$. Assume that

1. $(t, z) \mapsto \xi_t(z)$ and $t \mapsto \Phi_t(z_0)$ are $\mathcal{C}^1$ maps into $\Omega^1(\Sigma, \Lambda\mathfrak{sl}(2,\mathbb{C})_\rho)$ and $\Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ respectively.
2. For all $t \in I$, Φ_t solves the monodromy problem (5.8).
3. $\Phi_0(z, \lambda)$ is independent of λ:

$$\Phi_0(z,\lambda) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}.$$

Let $f_t = \mathrm{Sym}_q(\mathrm{Uni}(\Phi_t)) : \Sigma \longrightarrow \mathbb{H}^3$ be the CMC $H = \coth q$ immersion given by the DPW method. Then, identifying $T_{\mathrm{I}_2}\mathbb{H}^3$ with $\mathbb{R}^3$ via the basis $(\sigma_1, \sigma_2, \sigma_3)$ defined in (5.3),

$$\lim_{t\to 0} \frac{1}{t}(f_t - \mathrm{I}_2) = W$$

where W is a (possibly branched) minimal immersion with the following Weierstrass data:

$$g(z) = \frac{a(z)}{c(z)}, \quad \omega(z) = -4(\sinh q)c(z)^2 \frac{\partial \xi^{(-1)}_{t,12}(z)}{\partial t}\,|_{t=0}\,.$$

The limit is for the uniform $\mathcal{C}^1$ convergence on compact subsets of Σ.

Proof. With the same arguments as in [38], $(t, z) \mapsto \Phi_t(z)$, $(t, z) \mapsto F_t(z)$ and $(t, z) \mapsto B_t(z)$ are $\mathcal{C}^1$ maps into $\Lambda\mathrm{SL}(2,\mathbb{C})_\rho$, $\Lambda\mathrm{SU}(2)_\rho$ and $\Lambda^{\mathbb{R}}_+\mathrm{SL}(2,\mathbb{C})_\rho$ respectively. At $t = 0$, Φ_0 is constant. Thus F_0 and B_0 are constant with respect to λ:

$$F_0 = \frac{1}{\sqrt{|a|^2+|c|^2}}\begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix}, \quad B_0 = \frac{1}{\sqrt{|a|^2+|c|^2}}\begin{pmatrix} |a|^2+|c|^2 & \bar{a}b+\bar{c}d \\ 0 & 1 \end{pmatrix}.$$

Thus $F_0(z, e^{-q}) \in \mathrm{SU}(2)$ and $f_0(z)$ degenerates into the identity matrix. Let $b_t := B_{t,11}\,|_{\lambda=0}$ and β_t the upper-right residue at $\lambda = 0$ of the potential ξ_t. Recalling Equation (5.6),

$$df_t(z) = 2b_t(z)^2 \sinh q F_t(z, e^{-q}) \begin{pmatrix} 0 & \beta_t(z) \\ \overline{\beta}_t(z) & 0 \end{pmatrix} F_t(z, e^{-q})^*.$$

Hence $(t, z) \mapsto df_t(z)$ is a $\mathcal{C}^1$ map. At $t = 0$, $\xi_0 = \Phi_0^{-1}d\Phi_0$ is constant with respect to λ, so $\beta_0 = 0$ and $df_0(z) = 0$. Define $\widetilde{f}_t(z) := \frac{1}{t}(f_t(z) - \mathrm{I}_2)$ for $t \neq 0$. Then $d\widetilde{f}_t(z)$ extends at

$t = 0$, as a continuous function of (t, z) by

$$d\widetilde{f}_0 = \frac{d}{dt} df_t \mid_{t=0} = 2\sinh q \begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix} \begin{pmatrix} 0 & \beta' \\ \overline{\beta'} & 0 \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ -c & a \end{pmatrix}$$
$$= 2\sinh q \begin{pmatrix} -ac\beta' - \overline{ac\beta'} & a^2\beta' - \overline{c^2\beta'} \\ \overline{a^2\beta'} - c^2\beta' & ac\beta' + \overline{ac\beta'} \end{pmatrix}$$

where $\beta' = \frac{d}{dt}\beta_t \mid_{t=0}$. In $T_{\mathrm{I}_2}\mathbb{H}^3$, this gives

$$d\widetilde{f}_0 = 4\sinh q \operatorname{Re}\left(\frac{1}{2}\beta'(a^2 - c^2), \frac{-i}{2}\beta'(a^2 + c^2), -ac\beta'\right).$$

Writing $g = \frac{a}{c}$ and $\omega = -4c^2\beta' \sinh q$ gives:

$$\widetilde{f}_0(z) = \widetilde{f}_0(z_0) + \operatorname{Re}\int_{z_0}^{z}\left(\frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega\right).$$

□

As a useful example for Proposition 31, one can show the convergence of Delaunay surfaces in $\mathbb{H}^3$ towards a minimal catenoid.

Proposition 32. Let $q > 0$, $A_t = A_{r,s}$ as in (5.14) with $r \leq s$ satisfying (5.19). Let $\Phi_t(z) := z^{A_t}$ and $f_t := \operatorname{Sym}_q(\operatorname{Uni}\Phi_t)$. Then

$$\widetilde{f} := \lim_{t\to 0}\frac{1}{t}(f_t - \mathrm{I}_2) = \psi$$

where $\psi : \mathbb{C}^* \longrightarrow \mathbb{R}^3$ is the immersion of a catenoid centered at $(0, 0, 1)$, of neck radius 1 and of axis orientd by the positive x-axis in the direction from $z = 0$ to $z = \infty$.

Proof. Compute

$$\Phi_0(z, \lambda) = \begin{pmatrix} \cosh\left(\frac{\log z}{2}\right) & \sinh\left(\frac{\log z}{2}\right) \\ \sinh\left(\frac{\log z}{2}\right) & \cosh\left(\frac{\log z}{2}\right) \end{pmatrix}$$

and

$$\frac{\partial\xi_{t,12}^{(-1)}(z)}{\partial t} \mid_{t=0} = \frac{z^{-1}dz}{2\sinh q}$$

in order to apply Proposition 31 and get

$$\widetilde{f}(z) = \widetilde{f}(1) + \operatorname{Re}\int_1^z \left(\frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega\right)$$

where

$$g(z) = \frac{z+1}{z-1} \qquad \text{and} \qquad \omega(z) = \frac{-1}{2}\left(\frac{z-1}{z}\right)^2 dz.$$

Note that $\Phi_t(1) = \mathrm{I}_2$ for all t to show that $\widetilde{f}(1) = 0$ and get

$$\widetilde{f}(z) = \mathrm{Re}\int_1^z \left(w^{-1}dw, \frac{-i}{2}(1+w^2)w^{-2}dw, \frac{1}{2}(1-w^2)w^{-2}dw\right).$$

Integrating gives for $(x, y) \in \mathbb{R} \times [0, 2\pi]$:

$$\widetilde{f}(e^{x+iy}) = \psi(x, y)$$

where

$$\begin{array}{rccc} \psi & : & \mathbb{R}\times[0,2\pi] & \longrightarrow & \mathbb{R}^3 \\ & & (x,y) & \longmapsto & (x, \cosh(x)\sin(y), 1-\cosh(x)\cos(y)) \end{array}$$

and hence the result. □

5.4.2 The DPW data

In this Section, we introduce the DPW data inducing the surface M_t of Theorem 12. The method is very similar to Section 5.3 and to [38], which is why we omit the details.

The data. Let (g, ω) be the Weierstrass data (for the parametrisation defined in (5.38)) of the minimal n-noid $M_0 \subset \mathbb{R}^3$. If necessary, apply a Möbius transformation so that $g(\infty) \notin \{0, \infty\}$, and write

$$g(z) = \frac{A(z)}{B(z)}, \qquad \omega(z) = \frac{B(z)^2 dz}{\prod_{i=1}^n (z - p_{i,0})^2}.$$

Let $H > 1$, $q > 0$ so that $H = \coth q$ and $\rho > e^q$. Consider $3n$ parameters $a_i, b_i, p_i \in \Lambda\mathbb{C}_\rho$ ($i \in [1, n]$) assembled into a vector $\mathbf{x}$. Let

$$A_{\mathbf{x}}(z, \lambda) = \sum_{i=1}^n a_i(\lambda) z^{n-1}, \qquad B_{\mathbf{x}}(z, \lambda) = \sum_{i=1}^n b_i(\lambda) z^{n-1}$$

and

$$g_{\mathbf{x}}(z, \lambda) = \frac{A_{\mathbf{x}}(z, \lambda)}{B_{\mathbf{x}}(z, \lambda)}, \qquad \omega_{\mathbf{x}}(z, \lambda) = \frac{B_{\mathbf{x}}(z, \lambda)^2 dz}{\prod_{i=1}^n (z - p_i(\lambda))^2}.$$

The vector $\mathbf{x}$ is chosen in a neighbourhood of a central value $\mathbf{x}_0 \in \mathbb{C}^{3n}$ so that $A_{\mathbf{x}_0} = A$, $B_{\mathbf{x}_0} = B$ and $\omega_{\mathbf{x}_0} = \omega$. Let $p_{i,0}$ denote the central value of p_i. Introduce a real parameter

t in a neighbourhood of 0 and write

$$\beta_t(\lambda) := \frac{t(\lambda - e^q)(\lambda - e^{-q})}{4\sinh q}.$$

The potential we use is

$$\xi_{t,\mathbf{x}}(z,\lambda) = \begin{pmatrix} 0 & \lambda^{-1}\beta_t(\lambda)\omega_{\mathbf{x}}(z,\lambda) \\ d_z g_{\mathbf{x}}(z,\lambda) & 0 \end{pmatrix}$$

defined for $(t,\mathbf{x})$ sufficiently close to $(0,\mathbf{x}_0)$ on

$$\Omega = \{z \in \mathbb{C} \mid \forall i \in [1,n], |z - p_{i,0}| > \epsilon\} \cup \{\infty\}$$

where $\epsilon > 0$ is a fixed constant such that the disks $D(p_{i,0}, 2\epsilon)$ are disjoint. The initial condition is

$$\phi(\lambda) = \begin{pmatrix} i g_{\mathbf{x}}(z_0,\lambda) & i \\ i & 0 \end{pmatrix}$$

taken at $z_0 \in \Omega$ away from the poles and zeros of g and ω. Let $\Phi_{t,\mathbf{x}}$ be the holomorphic frame arising from the data $(\Omega, \xi_{t,\mathbf{x}}, z_0, \phi)$ via the DPW method and $f_{t,\mathbf{x}} := \mathrm{Sym}_q(\mathrm{Uni}\,\Phi_{t,\mathbf{x}})$.

Follow Section 6 of [38] to show that the potential $\xi_{t,\mathbf{x}}$ is regular at the zeros of $B_{\mathbf{x}}$ and to solve the monodromy problem around the poles at $p_{i,0}$ for $i \in [1, n-1]$. The Implicit Function Theorem allows us to define $\mathbf{x} = \mathbf{x}(t)$ in a small neighbourhood $(-T,T)$ of $t = 0$ satisfying $\mathbf{x}(0) = \mathbf{x}_0$ and such that the monodromy problem is solved for all t. We can thus drop from now on the index $\mathbf{x}$ in our data. As in [38], f_t descends to Ω and analytically extends to $\mathbb{C} \cup \{\infty\} \setminus \{p_{1,0}, \ldots, p_{n,0}\}$. This defines a smooth family $(M_t)_{-T<t<T}$ of CMC H surfaces of genus zero with n ends in $\mathbb{H}^3$.

The convergence of $\frac{1}{t}(M_t - \mathrm{I}_2)$ towards the minimal n-noid M_0 (point 2 of Theorem 12) is a straightforward application of Proposition 31 together with

$$\frac{\Phi_{0,11}(z)}{\Phi_{0,21}(z)} = g(z), \quad -4(\sinh q)(\Phi_{0,21}(z))^2 \frac{\partial \xi^{(-1)}_{t,12}(z)}{\partial t} = \omega(z).$$

Delaunay residue. To show that ξ_t is a perturbed Delaunay potential around each of its poles, let $i \in [1,n]$ and follow Section 5.3.2 with

$$\psi_{i,t,\lambda}(z) = g_t^{-1}\left(z + g_t(p_{i,t}(\lambda))\right).$$

Define

$$\widetilde{\omega}_{i,t}(z,\lambda) := \psi^*_{i,t,\lambda}\omega_t(z)$$

and

$$\alpha_{i,t}(\lambda) := \operatorname*{Res}_{z=0}(z\widetilde{\omega}_{i,t}(z,\lambda)).$$

Use Proposition 5, Claim 1 of [38] to show that for T small enough, $\alpha_{i,t}$ is real and does not depend on λ. Set

$$\begin{cases} rs = \frac{t\alpha_{i,t}}{4\sinh q}, \\ r^2 + s^2 + 2rs\cosh q = \frac{1}{4}, \\ r < s \end{cases}$$

and

$$G_t(z,\lambda) = \begin{pmatrix} \frac{\sqrt{r\lambda+s}}{\sqrt{z}} & \frac{-1}{2\sqrt{r\lambda+s}\sqrt{z}} \\ 0 & \frac{\sqrt{z}}{\sqrt{r\lambda+s}} \end{pmatrix}.$$

Define the gauged potential

$$\widetilde{\xi}_{i,t}(z,\lambda) := \left((\psi_{i,t,\lambda}^* \xi_t)\cdot G_t\right)(z,\lambda)$$

and compute its residue to show that it is a perturbed Delaunay potential as in Definition 16.

Applying Theorem 14. At $t=0$ and $z=1$, writing $\pi_i := g(p_{i,0})$ to ease the notation,

$$\widetilde{\Phi}_{i,0}(1,\lambda) = \begin{pmatrix} i\,(1+\pi_i) & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} 1+\pi_i & 1-\pi_i \\ 1 & -1 \end{pmatrix} =: M_i,$$

and thus $\widetilde{\xi}_{i,0}(z) = M_i z^{A_0}$. Recall (5.21) and let

$$H := H_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \Lambda\mathrm{SU}(2)_\rho$$

and $Q_i := \mathrm{Uni}\left(M_i H^{-1}\right)$. Using Lemma 1 in [38], Q_i can be made explicit and one can find a change of coordinates h and a gauge G such that $\widehat{\Phi}_{i,t} := (Q_i H)^{-1}\left(h^*\widetilde{\Phi}_{i,t}\right)G$ solves $d\widehat{\Phi}_{i,t} = \widehat{\Phi}_{i,t}\widehat{\xi}_{i,t}$ where $\widehat{\xi}_{i,t}$ is a perturbed Delaunay potential and $\widehat{\Phi}_{i,0}(z) = z^{A_0}$. One can thus apply Theorem 14 on $\widehat{\xi}_{i,t}$ and $\widehat{\Phi}_{i,t}$, which proves the existence of the family $(M_t)_{-T<t<T}$ of CMC H surfaces of genus zero and n Delaunay ends, each of weight (according to Equation (5.16))

$$w_{i,t} = 8\pi rs\sinh q = 2\pi t\alpha_{i,t},$$

which proves the first point of Theorem 12. Let $\widehat{f}_{i,t} := \mathrm{Sym}_q\left(\mathrm{Uni}\,\widehat{\Phi}_{i,t}\right)$ and let $\widehat{f}_{i,t}^{\mathcal{D}}$ be the Delaunay immersion given by Theorem 14.

Limit axis. In order to compute the limit axis of f_t at the end around $p_{i,t}$, let $\widehat{\Delta}_{i,t}$ be the oriented axis of $\widehat{f}^{\mathcal{D}}_{i,t}$ at $z=0$. Then, using Theorem 14,

$$\widehat{\Delta}_{i,0} = \mathrm{geod}\left(\mathrm{I}_2, -\sigma_1\right).$$

And using $\widehat{f}_{i,t}(z) = (Q_i H)^{-1} \cdot (h^* f_t(z))$,

$$\widehat{\Delta}_{i,0} = (Q_i H)^{-1} \cdot \Delta_{i,0},$$

and thus

$$\Delta_{i,0} = (QH) \cdot \mathrm{geod}(\mathrm{I}_2, -\sigma_1).$$

Compute $H \cdot (-\sigma_1) = \sigma_3$ and note that $M_i H^{-1} = \Phi_0(\pi_i)$ to get

$$\Delta_{i,0} = \mathrm{geod}\left(\mathrm{I}_2, N_0(p_{i,0})\right)$$

where N_0 is the normal map of the minimal immersion.

Type of the ends. Suppose that t is positive. Then the end at $p_{i,t}$ is unduloidal if, and only if its weight is positive; that is, $\alpha_{i,t}$ is positive. Use Proposition 5 of [38] to show that if the normal map N_0 of M_0 points toward the inside, then $\alpha_{i,0} = \tau_i$ where $2\pi\tau_i N_0(p_{i,0})$ is the flux of M_0 around the end at $p_{i,0}$ ($\alpha_{i,0} = -\tau_i$ for the other orientation). Thus if M_0 is Alexandrov-embedded, then the ends of M_t are of unduloidal type for $t > 0$ and of nodoidal type for $t < 0$.

5.4.3 Alexandrov-embeddedness

In order to show that M_t is Alexandrov-embedded for $t > 0$ small enough, one can follow the proof of Proposition 6 in [38]. Note that this proposition does not use the fact that M_t is CMC H, but relies on the fact that the ambient space is $\mathbb{R}^3$. This leads us to lift f_t to $\mathbb{R}^3$ via the exponential map at the identity, hence defining an immersion $\widehat{f}_t : \Sigma_t \longrightarrow \mathbb{R}^3$ which is not CMC anymore, but is Alexandrov-embedded if, and only if f_t is Alexandrov-embedded. Let $\psi : \Sigma_0 \longrightarrow M_0 \subset \mathbb{R}^3$ be the limit minimal immersion. In order to adapt the proof of [38] and show that M_t is Alexandrov-embedded, one will need the following Lemma.

Lemma 17. Let $\widetilde{f}_t := \frac{1}{t}\widehat{f}_t$. Then $\widetilde{f}_t$ converges to ψ on compact subsets of Σ_0.

Proof. For all z,

$$\exp_{\mathrm{I}_2}(\widehat{f}_0(z)) = f_0(z) = \mathrm{I}_2,$$

so $\widehat{f}_0(z) = 0$. Thus

$$\lim_{t\to 0} \widetilde{f}_t(z) = \frac{d}{dt}\widehat{f}_t(z) \mid_{t=0} .$$

Therefore, using Proposition 32,

$$\begin{aligned}
\psi(z) &= \lim_{t\to 0} \frac{1}{t}\left(f_t(z) - \mathrm{I}_2\right) \\
&= \lim_{t\to 0} \frac{1}{t}\left(\exp_{\mathrm{I}_2}(\widehat{f}_t(z)) - \exp_{\mathrm{I}_2}(\widehat{f}_0(z))\right) \\
&= \frac{d}{dt}\exp_{\mathrm{I}_2}(\widehat{f}_t(z)) \mid_{t=0} \\
&= d\exp_{\mathrm{I}_2}(0)\cdot \frac{d}{dt}\widehat{f}_t(z) \mid_{t=0} \\
&= \lim_{t\to 0} \widetilde{f}_t(z).
\end{aligned}$$

□

5.5 Appendix: CMC surfaces of revolution in $\mathbb{H}^3$

Following Sections 2.2 and 2.3 of [15],

Proposition 33. Let $X : \mathbb{R} \times [0, 2\pi] \longrightarrow \mathbb{H}^3$ be a conformal immersion of revolution with metric $g^2(s)\left(ds^2 + d\theta^2\right)$. If X is CMC $H > 1$, then g is periodic and denoting by S its period,

$$\sqrt{H^2-1}\int_0^S g(s)ds = \pi \quad \text{and} \quad \int_0^S \frac{ds}{g(s)} = \frac{2\pi^2}{|w|}$$

where w is the weight of X, as defined in [23].

Proof. According to Equation (11) in [15], writing $\tau = \frac{\sqrt{|w|}}{\sqrt{2\pi}}$ and $g = \tau e^{\sigma}$,

$$\left(\sigma'\right)^2 = 1 - \tau^2\left(\left(He^{\sigma} + \iota e^{-\sigma}\right)^2 - e^{2\sigma}\right) \tag{5.39}$$

where $\iota \in \{\pm 1\}$ is the sign of w. The solutions σ are periodic with period $S > 0$. Apply an isometry and a change of the variable $s \in \mathbb{R}$ so that

$$\sigma'(0) = 0 \quad \text{and} \quad \sigma(0) = \min_{s\in\mathbb{R}} \sigma(s).$$

By symmetry of Equation (5.39), one can thus define

$$a := e^{2\sigma(0)} = \min_{s\in\mathbb{R}} e^{2\sigma(s)} \quad \text{and} \quad b := e^{2\sigma\left(\frac{S}{2}\right)} = \max_{s\in\mathbb{R}} e^{2\sigma(s)}.$$

With these notations, Equation (5.39) can be written in a factorised form as

$$\left(\sigma'\right)^2 = \tau^2(H^2-1)e^{-2\sigma}\left(b - e^{2\sigma}\right)\left(e^{2\sigma} - a\right) \tag{5.40}$$

with

$$a = \frac{1 - 2\iota\tau^2 H - \sqrt{1 - 4\tau^2(\iota H - \tau^2)}}{2\tau^2(H^2-1)} \tag{5.41}$$

and

$$b = \frac{1 - 2\iota\tau^2 H + \sqrt{1 - 4\tau^2(\iota H - \tau^2)}}{2\tau^2(H^2-1)}.$$

In order to compute the first integral, change variables $v = e^{\sigma}$, $y = \sqrt{b - v^2}$ and $x = \frac{y}{\sqrt{b-a}}$

and use Equation (5.40) to get

$$\begin{aligned}\sqrt{H^2-1}\int_0^S \tau e^{\sigma(s)}ds &= 2\sqrt{H^2-1}\int_{\sqrt{a}}^{\sqrt{b}} \frac{\tau v dv}{\tau\sqrt{H^2-1}\sqrt{b-v^2}\sqrt{v^2-a}} \\ &= -2\int_{\sqrt{b-a}}^{0} \frac{dy}{\sqrt{b-a-y^2}} \\ &= 2\int_0^1 \frac{dx}{\sqrt{1-x^2}} \\ &= \pi.\end{aligned}$$

In the same manner with the changes of variables $v = e^{-\sigma}$, $y = \sqrt{a^{-1}-v^2}$ and $x = \frac{y}{\sqrt{a^{-1}-b^{-1}}}$,

$$\begin{aligned}\int_0^S \frac{ds}{\tau e^{\sigma(s)}} &= \frac{-2}{\tau\sqrt{H^2-1}}\int_{a^{-1/2}}^{b^{-1/2}} \frac{dv}{v\sqrt{b-v^{-2}}\sqrt{v^{-2}-a}} \\ &= \frac{2}{\tau^2\sqrt{H^2-1}}\int_0^{\sqrt{a^{-1}-b^{-1}}} \frac{dy}{\sqrt{b-a-aby^2}} \\ &= \frac{2}{\tau^2\sqrt{H^2-1}\sqrt{ab}}\int_0^1 \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{\pi}{\tau^2}\end{aligned}$$

because $ab = \frac{1}{H^2-1}$. □

Lemma 18. Let $\mathcal{D}_t$ be a Delaunay surface in $\mathbb{H}^3$ of constant mean curvature $H > 1$ and weight $2\pi t > 0$ with Gauss map η_t. Let r_t be the maximal value of R such that the map

$$\begin{array}{rccc} T & : & (-R,R)\times\mathcal{D}_t & \longrightarrow & \mathrm{Tub}_{r_t}\subset\mathbb{H}^3 \\ & & (r,p) & \longmapsto & \mathrm{geod}(p,\eta_t(p))(r)\end{array}$$

is a diffeomorphism. Then $r_t \sim t$ as t tends to 0.

Proof. The quantity r_t is the inverse of the maximal geodesic curvature of the surface. This maximal curvature is attained for small values of t on the points of minimal distance between the profile curve and the axis. Checking the direction of the mean curvature vector at this point, the maximal curvature curve is not the profile curve but the parallel curve. Hence r_t is the minimal hyperbolic distance between the profile curve and the axis. A study of the profile curve's equation as in Proposition 33 shows that

$$r_t = \sinh^{-1}\left(\tau\exp\left(\sigma_{\min}\right)\right) = \sinh^{-1}\left(\tau\sqrt{a(\tau)}\right).$$

But using Equation (5.41), as τ tends to 0, $a \sim \tau^2 = |t|$, which gives the expected result. □

Lemma 19. Let $\mathcal{D}_t$ be a Delaunay surface in $\mathbb{H}^3$ of weight $2\pi t > 0$ with Gauss map η_t and maximal tubular radius r_t. There exist $T > 0$ and $\alpha < 1$ such that for all $0 < t < T$ and $p, q \in \mathcal{D}_t$ satisfying $d_{\mathbb{H}^3}(p, q) < \alpha r_t$,

$$\left\|\Gamma_p^q \eta_t(p) - \eta_t(q)\right\| < 1.$$

Proof. Let $t > 0$. Then for all $p, q \in \mathcal{D}_t$,

$$\left\|\Gamma_p^q \eta_t(p) - \eta_t(q)\right\| \leq \sup_{s \in \gamma_t} \|II_t(s)\| \times \ell(\gamma_t)$$

where II_t is the second fundamental form of $\mathcal{D}_t$, $\gamma_t \subset \mathcal{D}_t$ is any path joining p to q and $\ell(\gamma_t)$ is the hyperbolic length of γ_t. Using the fact that the maximal geodesic curvature κ_t of $\mathcal{D}_t$ satisfies $\kappa_t \sim \coth r_t$ as t tends to zero, there exists a uniform constant $C > 0$ such that

$$\sup_{s \in \mathcal{D}_t} \|II_t(s)\| < C \coth r_t.$$

Let $0 < \alpha < (1 + C)^{-1} < 1$ and suppose that $d_{\mathbb{H}^3}(p, q) < \alpha r_t$. Let $\sigma_t : [0, 1] \to \mathbb{H}^3$ be the geodesic curve of $\mathbb{H}^3$ joining p to q. Then $\sigma_t([0, 1]) \subset \mathrm{Tub}_{\alpha r_t}$ and thus the projection $\pi_t : \sigma_t([0, 1]) \to \mathcal{D}_t$ is well-defined. Let $\gamma_t := \pi_t \circ \sigma_t$. Then

$$\begin{aligned}
\left\|\Gamma_p^q \eta_t(p) - \eta_t(q)\right\| &\leq C \coth r_t \times \sup_{s \in \sigma_t} \|d\pi_t(s)\| \\
&\leq C \coth r_t \times \sup_{s \in \mathrm{Tub}_{\alpha r_t}} \|d\pi_t(s)\| \times d_{\mathbb{H}^3}(p, q) \\
&\leq C \coth r_t \times \frac{\tanh r_t}{\tanh r_t - \tanh(\alpha r_t)} \times \alpha r_t \\
&\leq \frac{C \alpha r_t}{\tanh r_t - \tanh(\alpha r_t)} \sim \frac{C\alpha}{1 - \alpha} < 1
\end{aligned}$$

as t tends to zero. □

5.6 Appendix: Remarks on the polar decomposition

Let $\mathrm{SL}(2, \mathbb{C})^{++}$ be the subset of $\mathrm{SL}(2, \mathbb{C})$ whose elements are hermitian positive definite. Let

$$\begin{array}{rccc}
\mathrm{Pol} & : & \mathrm{SL}(2, \mathbb{C}) & \longrightarrow & \mathrm{SL}(2, \mathbb{C})^{++} \times \mathrm{SU}(2) \\
 & & A & \longmapsto & (\mathrm{Pol}_1(A), \mathrm{Pol}_2(A))
\end{array}$$

be the polar decomposition on $\mathrm{SL}(2,\mathbb{C})$. This map is differentiable and satisfies the following proposition.

Proposition 34. For all $A \in \mathrm{SL}(2,\mathbb{C})$, $\|d\mathrm{Pol}_2(A)\| \leq |A|$.

Proof. We first write the differential of Pol_2 at the identity in an explicit form. Writing

$$\begin{array}{rccc} d\mathrm{Pol}_2(\mathrm{I}_2) & : & \mathfrak{sl}(2,\mathbb{C}) & \longrightarrow & \mathfrak{su}(2) \\ & & M & \longmapsto & \mathrm{pol}_2(M) \end{array}$$

gives

$$\mathrm{pol}_2\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} i\,\mathrm{Im}\,a & \frac{b-\bar{c}}{2} \\ \frac{c-\bar{b}}{2} & -i\,\mathrm{Im}\,a \end{pmatrix}.$$

Note that for all $M \in \mathfrak{sl}(2\mathbb{C})$,

$$\begin{aligned} |\mathrm{pol}_2(M)|^2 &= 2\,(\mathrm{Im}\,a)^2 + \frac{1}{4}\left(|b-\bar{c}|^2 + \left|c-\bar{b}\right|^2\right) \\ &\leq |M|^2 - \frac{1}{2}\,|b+\bar{c}|^2 \\ &\leq |M|^2\,. \end{aligned}$$

We then compute the differential of Pol_2 at any point of $\mathrm{SL}(2,\mathbb{C})$. Let $(S_0, Q_0) \in \mathrm{SL}\,(2,\mathbb{C})^{++} \times \mathrm{SU}(2)$. Consider the differentiable maps

$$\begin{array}{rccc} \phi & : & \mathrm{SL}(2,\mathbb{C}) & \longrightarrow & \mathrm{SL}(2,\mathbb{C}) \\ & & A & \longmapsto & S_0 A Q_0 \end{array} \quad \text{and} \quad \begin{array}{rccc} \psi & : & \mathrm{SU}(2) & \longrightarrow & \mathrm{SU}(2) \\ & & Q & \longmapsto & Q Q_0. \end{array}$$

Then $\psi \circ \mathrm{Pol}_2 \circ \phi^{-1} = \mathrm{Pol}_2$ and for all $M \in T_{S_0Q_0}\mathrm{SL}(2,\mathbb{C})$,

$$d\mathrm{Pol}_2\,(S_0Q_0)\cdot M = \mathrm{pol}_2\left(S_0^{-1} M Q_0^{-1}\right) Q_0.$$

Finally, let $A \in \mathrm{SL}(2,\mathbb{C})$ with polar decomposition $\mathrm{Pol}(A) = (S, Q)$. Then for all $M \in T_A\mathrm{SL}(2,\mathbb{C})$,

$$|d\mathrm{Pol}_2(A)\cdot M| = \left|\mathrm{pol}_2\left(S^{-1} M Q^{-1}\right) Q\right| \leq |S| \times |M|$$

and thus using

$$S = \exp\left(\frac{1}{2}\log\left(AA^*\right)\right)$$

gives

$$\|d\mathrm{Pol}_2(A)\| \leq |S| \leq |A|\,.$$

□

Corollary 7. Let $0 < q < \log \rho$ and $F_1, F_2 \in \Lambda\mathrm{SU}(2)_\rho$ with unitary parts $Q_i = \mathrm{Pol}_2(F_i(e^{-q}))$. Let $\epsilon > 0$ such that

$$\left\| F_2^{-1} F_1 - \mathrm{I}_2 \right\|_\rho < \epsilon.$$

If ϵ is small enough, then there exists a uniform $C > 0$ such that for all $v \in T_{\mathrm{I}_2}\mathbb{H}^3$,

$$\|Q_2 \cdot v - Q_1 \cdot v\|_{T_{\mathrm{I}_2}\mathbb{H}^3} \leq C \, \|F_2\|_\rho^2 \, \epsilon.$$

Proof. Let $v \in T_{\mathrm{I}_2}\mathbb{H}^3$ and consider the following differentiable map

$$\begin{array}{rccc} \phi & : & \mathrm{SU}(2) & \longrightarrow & T_{\mathrm{I}_2}\mathbb{H}^3 \\ & & Q & \longmapsto & Q \cdot v. \end{array}$$

Then

$$\begin{aligned} \|Q_2 \cdot v - Q_1 \cdot v\|_{T_{\mathrm{I}_2}\mathbb{H}^3} &= \|\phi(Q_2) - \phi(Q_1)\|_{T_{\mathrm{I}_2}\mathbb{H}^3} \\ &\leq \sup_{t \in [0,1]} \|d\phi(\gamma(t))\| \times \int_0^1 |\dot{\gamma}(t)| \, dt \end{aligned}$$

where $\gamma : [0,1] \longrightarrow \mathrm{SU}(2)$ is a path joining Q_2 to Q_1. Recalling that $\mathrm{SU}(2)$ is compact gives

$$\|Q_2 \cdot v - Q_1 \cdot v\|_{T_{\mathrm{I}_2}\mathbb{H}^3} \leq C \, |Q_2 - Q_1| \tag{5.42}$$

where $C > 0$ is a uniform constant. But writing $A_i = F_i(e^{-q}) \in \mathrm{SL}(2,\mathbb{C})$,

$$\begin{aligned} |Q_2 - Q_1| &= |\mathrm{Pol}_2(A_2) - \mathrm{Pol}_2(A_1)| \\ &\leq \sup_{t \in [0,1]} \|d\mathrm{Pol}_2(\gamma(t))\| \times \int_0^1 |\dot{\gamma}(t)| \, dt \end{aligned}$$

where $\gamma : [0,1] \longrightarrow \mathrm{SL}(2,\mathbb{C})$ is a path joining A_2 to A_1. Take for example

$$\gamma(t) := A_2 \, \exp\left(t \log\left(A_2^{-1} A_1\right)\right).$$

Suppose now that ϵ is small enough for log to be a diffeomorphism from $D(\mathrm{I}_2, \epsilon) \cap \mathrm{SL}(2,\mathbb{C})$ to $D(0,\epsilon') \cap \mathfrak{sl}(2,\mathbb{C})$. Then

$$\left\| A_2^{-1} A_1 - \mathrm{I}_2 \right\| \leq \left\| F_2^{-1} F_1 - \mathrm{I}_2 \right\|_\rho < \epsilon$$

implies

$$|\gamma(t)| \leq \widetilde{C} \, |A_2| \quad \text{and} \quad |\dot{\gamma}(t)| \leq \widetilde{C}\widehat{C} \, |A_2| \, \epsilon$$

where $\widetilde{C}, \widehat{C} > 0$ are uniform constants. Using Proposition 34 gives

$$|Q_2 - Q_1| \leq \widehat{C}\widetilde{C}^2 \, |A_2|^2 \, \epsilon$$

and inserting this inequality into (5.42) gives

$$\|Q_2 \cdot v - Q_1 \cdot v\|_{T_{\mathrm{I}_2}\mathbb{H}^3} \leq C\widehat{C}\widetilde{C}^2 \, |A_2|^2 \, \epsilon \leq C\widehat{C}\widetilde{C}^2 \, \|F_2\|_\rho^2 \, \epsilon.$$

□

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Résumé :

Les surfaces à courbure moyenne constante non-nulle apparaissent en physique comme solutions à certains problèmes d'interface entre deux milieux de pressions différentes. Elles sont décrites mathématiquement par des équations aux dérivées partielles et sont constructibles à partir de données holomorphes via une représentation similaire à celle de Weierstrass pour les surfaces minimales. On présente dans cette thèse deux résultats s'appuyant sur cette représentation, dite «méthode DPW». Le premier indique que les données donnant naissance à un bout Delaunay de type onduloïde induisent encore un anneau plongé après perturbation. Cette propriété sert notamment à démontrer que certaines surfaces construites par la méthode DPW sont plongées. Le second résultat est la construction, dans l'espace hyperbolique, de n-noïdes : surfaces plongées, de genre zéro, à courbure moyenne constante et muni de n bouts de type Delaunay.

Mots clés :

Surfaces à courbure moyenne constante – Représentation de Weierstrass généralisée – Méthode DPW – Bouts Delaunay – Géométrie hyperbolique.

Abstract :

Non-zero constant mean curvature surfaces are mathematical models for physical interface problems with non-zero pressure difference. They are described by partial differential equations and can be constructed from holomorphic data via a Weierstrass-type representation, called "the DPW method". In this thesis, we use the DPW method and prove two main results. The first one states that perturbations of the DPW data for Delaunay unduloidal ends generate embedded annuli. This can be used to prove the embeddedness of surfaces constructed via the DPW method. The second result is the construction of n-noids in Hyperbolic space: genus 0, embedded, constant mean curvature surfaces with n Delaunay ends.

Keywords:

Constant Mean Curvature Surfaces – Generalized Weierstrass Representation – DPW Method – Delaunay Ends – Hyperbolic Geometry.