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Cycles séparants, isopérimétrie et modifications de distances dans les grandes cartes planaires aléatoires

Thomas Lehéricy

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Cycles séparants, isopérimétrie et modifications de
distances dans les grandes cartes planaires aléatoires

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Chapitre 1

Introduction

Où je commence par une courte présentation du domaine des cartes planaires, et des différents objets qu'on peut y rencontrer. J'expose ensuite quelques enjeux et outils clés de l'étude des cartes planaires, en insistant sur ceux qui jouent un rôle central dans cette thèse. Enfin, les trois dernières sections présentent chacune les résultats principaux de mon travail de thèse : la section 1.3 porte sur les propriétés isopérimétriques d'un modèle de carte infinie, l'UIPQ, et reprend l'article [43]. La section 1.4 explore les modifications de distances dans les cartes aléatoires, suivant en cela l'article [46]. Enfin, la section 1.5 présente des résultats inédits, sur les temps de mélange de la marche aléatoire fainéante sur les cartes finies.

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Notations et définitions

$\#S$ le cardinal d'un ensemble S .

$\mathbb{N}, \mathbb{Z}, \mathbb{Z}_-, \mathbb{R}$ l'ensemble des entiers naturels, des entiers, des entiers négatifs ou nuls, et des réels.

$G = (V, E)$ un graphe, encodé par son ensemble de sommets V et son ensemble d'arêtes E . E est un multiensemble de paires $\{v, w\}$ avec v, w des sommets de G : une même paire de sommets peut être reliée par plusieurs arêtes.

adjacence : deux faces sont adjacentes lorsqu'elles partagent une arête ; deux sommets sont adjacents lorsque une arête les relie.

incidence : un sommet et une arête sont incidents lorsque le sommet est une extrémité de l'arête. Une arête (ou un sommet) et une face sont incidentes lorsque l'arête (ou le sommet) est incluse dans le bord de la face.

1.1 Cartes aléatoires et quadrangulations

1.1.1 Cartes planaires

Définition 1.1. Une *carte planaire* est un *graphe planaire enraciné*, muni d'un *plongement cellulaire propre* dans la sphère, vu à homéomorphisme préservant l'orientation près.

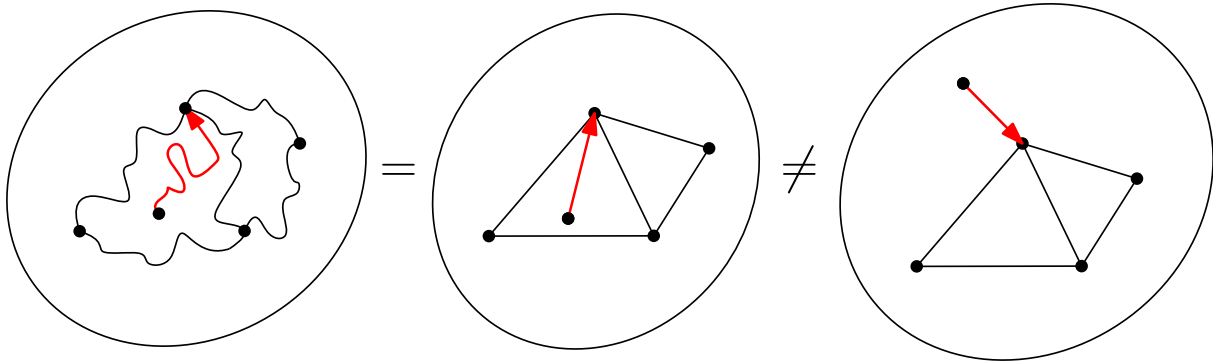


Figure 1.1 – Trois plongements cellulaires propres d'un même graphe planaire enraciné sur la sphère. L'arête racine est représentée en rouge. Les deux plongements à gauche définissent la même carte planaire, et celui de droite une carte planaire différente.

Déchiffrons cette définition. Un *graphe planaire* est un graphe pouvant être dessiné dans le plan ou la sphère sans que ses arêtes ne s'intersectent. Une *carte planaire* M est, essentiellement, le dessin d'un graphe planaire sur la sphère. Les sommets et arêtes du graphe, une fois dessinés, deviennent les arêtes et sommets de la carte ; par extension, on note $V(M)$ l'ensemble des sommets et $E(M)$ l'ensemble des arêtes de M . Pour des raisons techniques, on ne s'intéresse qu'aux cartes *enracinées*, c'est-à-dire qu'une arête est distinguée et orientée ; cette arête est appelée *arête racine* de la carte, et sa queue (le sommet vers lequel l'arête ne pointe pas) est le *sommet racine* de la carte.

On peut également définir l'ensemble $F(M)$ des faces de M : une face de M est une composante connexe du complémentaire de l'union des arêtes et des sommets de M . Le *degré* d'une face est le nombre d'arêtes auxquelles elle est incidente.

Un plongement du graphe planaire dans la sphère, qui définit une carte M , doit être *propre*, c'est-à-dire que les arêtes de M ne s'intersectent qu'à leurs extrémités. En d'autres termes, le fait que le plongement soit propre signifie simplement qu'on a “bien dessiné” le graphe planaire. Le plongement doit de plus être *cellulaire*, autrement dit toutes les faces de M doivent être homéomorphes à des disques (sur la sphère, c'est équivalent à dire que le graphe est connexe).

Enfin, les cartes sont vues à déformation près : si on peut passer d'une première carte à une seconde en déformant la sphère sur laquelle la première carte est dessinée, sans

rompre ou retourner cette sphère, alors les deux cartes sont égales.

Relation d’Euler. On dispose de trois quantités naturelles pour mesurer la taille d’une carte planaire : son nombre de sommets, son nombre d’arêtes, et son nombre de faces. Euler a établi la très célèbre relation entre ces trois quantités :

Théorème 1.1.1 (Relation d’Euler, 1752). *Pour toute carte planaire M ,*

$$\#V(M) + \#F(M) - \#E(M) = 2.$$

Anticipons un peu, et considérons une carte D dont toutes les faces ont degré d (nécessairement $d \geq 3$). D a alors $d \cdot \#F(D)$ côtés d’arêtes, avec deux côtés d’arêtes par arête, soit $2 \cdot \#E(D) = d \cdot \#F(D)$. La relation d’Euler se réécrit alors

$$\begin{cases} \#V(D) &= \left(\frac{d}{2} - 1\right) \cdot \#F(D) + 2, \\ \#E(D) &= \frac{d}{2} \cdot \#F(D). \end{cases}$$

Ainsi, il suffit de connaître une seule des quantités $\#V(D)$, $\#F(D)$ ou $\#E(D)$ pour retrouver les deux autres. Pour les cartes dont toutes les faces ont le même degré, on utilise souvent celle de ces trois quantités la plus adaptée à la situation, plutôt que le nombre d’arêtes. Dans cette thèse, ce sera le nombre de faces.

Quadrangulations. Par “modèle de cartes”, on entend ici, quelque peu informellement, la collection de toutes les cartes satisfaisant certaines contraintes. On impose en général des contraintes de structure sur les cartes planaires, comme le *type* de la carte. Les seules cartes que nous utilisons dans cette thèse sont les cartes de type I, c’est-à-dire celles qui peuvent posséder des arêtes multiples et des boucles (qui sont des arêtes dont les deux extrémités sont le même sommet). En l’absence de contrainte autre que le type, on parle de carte générale.

Une autre contrainte usuelle porte sur les degrés des faces. Une carte planaire dont toutes les faces ont degré $d \geq 3$ est appelée une d -angulation. Si $d = 3$, on parle de triangulation, et si $d = 4$ de quadrangulation. Lorsque d est pair, le graphe induit est biparti, et on peut donc colorier les sommets de la carte avec deux couleurs de sorte que deux sommets reliés par une arête soient toujours de couleur différente. On en déduit qu’une quadrangulation ne contient pas de boucle.

Les quadrangulations s’avèrent particulièrement riches en outils qui facilitent leur étude. On présentera dans la section 1.2.3 la bijection de Cori-Vauquelin et Schaeffer entre les quadrangulations de type I et les arbres planaires étiquetés, et dans la section 1.2.4 la décomposition en squelette de Krikun. Le premier outil, et le plus simple, est la bijection de Tutte.

Bijection de Tutte. La bijection de Tutte (parfois appelée la bijection triviale) est une bijection entre les quadrangulations de type I et les cartes générales de type I. Voir la figure 1.2.

- Soit Q une quadrangulation de type I. Dans chaque face de Q , on trace une “diagonale” entre les deux coins de la même couleur que le sommet racine de Q . On efface ensuite toutes les arêtes de Q , et tous les sommets qui se retrouvent ainsi isolés. Il reste une carte générale, qu’on enracine à la “diagonale” initialement contenue dans la face de Q incidente à la droite de l’arête racine de Q , en gardant le même sommet racine.
- Réciproquement, soit M une carte de type I. Dans chaque face f de M , on ajoute un sommet, qu’on relie à chaque coin de f . Ensuite, on efface toutes les arêtes de M . On obtient une quadrangulation de type I, qu’on enracine à l’arête immédiatement à gauche de l’arête racine de M , en gardant le même sommet racine.

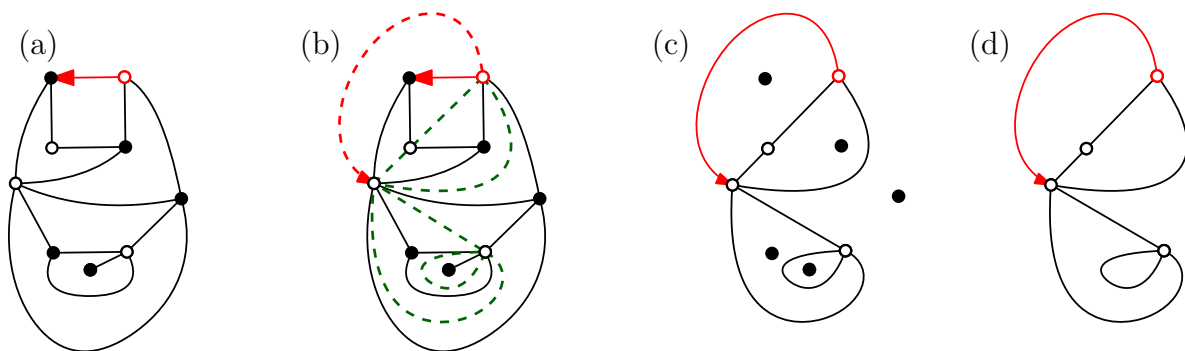


Figure 1.2 – Bijection de Tutte. En a) une quadrangulation Q ; ses sommets sont coloriés en noir et blanc, de sorte que le sommet racine soit colorié en blanc, et que deux sommets adjacents soient de couleur différente. En b), on trace une diagonale dans chaque face de Q . Puis on efface les arêtes de Q en c), et les sommets noirs en d). La carte ainsi obtenue est une carte générale.

1.1.2 Cartes aléatoires uniformes

Les cartes planaires sont des objets combinatoires : il n’y a qu’un nombre fini de cartes ayant un nombre d’arêtes fixé. On peut construire de nombreuses variables aléatoires à valeurs dans l’ensemble des cartes. Dans cette thèse, je me concentre sur les cartes aléatoires uniformes : une carte aléatoire uniforme à n arêtes est une variable aléatoire de loi uniforme sur l’ensemble des cartes planaires à n arêtes. De la même manière, une quadrangulation aléatoire uniforme à n faces est une quadrangulation tirée uniformément parmi l’ensemble des quadrangulations à n faces.

Je m’intéresse aux propriétés à grande échelle des cartes aléatoires, c’est-à-dire quand leur taille (qui peut être, suivant le choix du modèle, leur nombre de faces, de sommets, d’arêtes...) tend vers l’infini. Il faut donc une notion de limite d’une suite de cartes (éventuellement aléatoires) de taille croissante. Deux notions se distinguent dans le cadre dans lequel se place cette thèse : la limite dans la topologie de Gromov-Hausdorff, souvent

désignée comme limite d'échelle ; et la limite dans la topologie locale. Nous les présentons brièvement dans les sections 1.1.3 et 1.1.4.

De nombreux résultats établis pour les quadrangulations sont conjecturés pour d'autres modèles de cartes. En termes physiques, ces modèles appartiennent à la même classe d'universalité : on s'attend à observer des comportements similaires malgré les différences dans le réseau sous-jacent. Un résultat rigoureux qui illustre ce principe est la convergence vers la carte Brownienne, établie indépendamment par Le Gall [40] et Miermont [55], à laquelle nous reviendrons dans la section 1.1.3. Pour montrer un résultat sur les cartes, il est souvent utile de démontrer qu'il est vrai pour le type de carte dans lequel l'étude est la plus simple, avant de la généraliser à d'autres types de cartes. La richesse des outils disponibles pour l'étude des quadrangulations justifie le choix de ce modèle de cartes.

1.1.3 Limite d'échelle et carte Brownienne

Un espace métrique (E, d) est un ensemble E muni d'une distance d . Une carte planeaire M est souvent vue comme un espace métrique, en munissant l'ensemble de ses sommets $V(M)$ de la distance de graphe d_{gr}^M . Toutefois, l'espace métrique $(V(M), d_{\text{gr}}^M)$ est loin d'être le seul qu'on peut construire à partir d'une carte M ; on en rencontrera d'autres dans le chapitre 3.

La distance de Gromov-Hausdorff est une distance sur l'ensemble \mathbb{M} des espaces métriques compacts. Elle permet de mesurer la similarité entre deux espaces métriques de manière intrinsèque, indépendamment de leurs plongements éventuels. On peut se référer à [56, Section 3.3] pour une introduction brève à cet objet dans le contexte des cartes aléatoires.

Un des enjeux principaux de l'étude des cartes aléatoires depuis la fin des années 1990 est de comprendre leur structure métrique. Un résultat fondateur, indépendamment prouvé par Miermont dans l'article [55] et Le Gall dans l'article [41] (qui le montre également pour les triangulations et d -angulations pour tout entier pair $d \geq 4$, avec des constantes spécifiques à chaque modèle), établit que les quadrangulations aléatoires uniformes à n faces, munies de leurs distances de graphe renormalisées par un facteur $(\frac{9}{8n})^{1/4}$, convergent en loi quand $n \rightarrow \infty$ dans la topologie de Gromov-Hausdorff vers un espace métrique aléatoire appelé la carte Brownienne.

Cette convergence a depuis été généralisée à de nombreuses classes de cartes, avec des constantes spécifiques aux modèles, fournissant une illustration du principe d'universalité cher aux physiciens : [2] pour les d -angulations avec $d \geq 5$, [9] pour les cartes générales, [1] pour les cartes biparties, [8] pour les triangulations et quadrangulations qui ne contiennent ni boucle ni arête multiple, [50, 51] pour les cartes dont on prescrit le degré des faces... L'idée a également été appliquée pour définir d'autres objets limites, comme le plan Brownien hyperbolique [14], les disques browniens [10], etc.

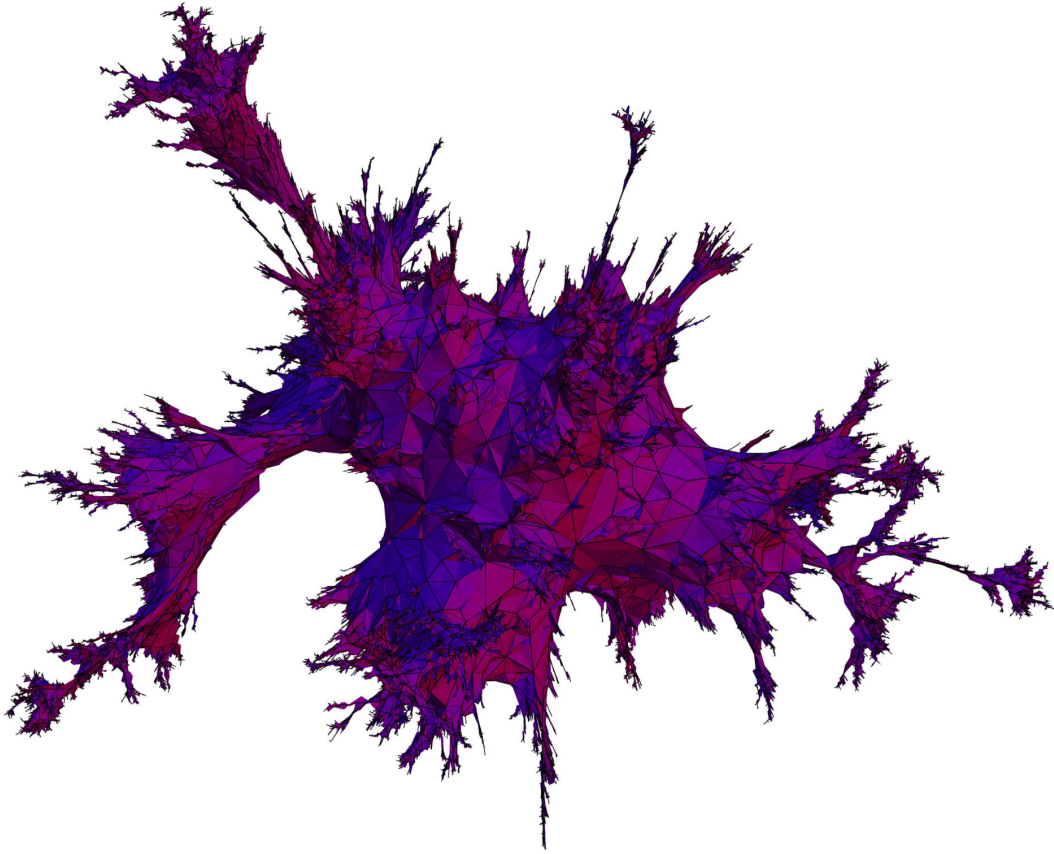


Figure 1.3 – Les quadrangulations uniformes approchent la carte Brownienne dans la topologie de Gromov-Hausdorff. Ici, une représentation en trois dimensions d’une réalisation d’une quadrangulation uniforme à 50 000 sommets. L’algorithme de représentation utilisé est choisi pour respecter, autant que possible, la métrique intrinsèque. Image de Jérémie Bettinelli.

1.1.4 Limite locale et quadrangulation infinie du plan

Boules et limite locale. Soit M une carte planaire, et r un entier strictement positif. La boule de rayon r de M , qu’on notera $B_M(r)$, est la carte planaire obtenue en ne conservant que les faces de M incidentes à un sommet à distance $r-1$ ou moins du sommet racine : tous les arêtes et sommets qui ne sont pas incidents à une de ces faces sont effacés. L’union de ces faces forme “l’intérieur” de la boule, et le complémentaire “l’extérieur”. Notons qu’en général, l’extérieur de la boule n’est pas connexe. Par convention, $B_M(0)$ est la carte composée d’un unique sommet et de zéro arête.

La distance locale d_{loc} , définie sur l’ensemble \mathcal{M} des cartes planaires, donne une mesure de la similarité entre deux cartes : si M et M' sont deux cartes planaires,

$$d_{\text{loc}}(M, M') \stackrel{\text{déf}}{=} (1 + \sup \{r \geq 0 : B_M(r) = B_{M'}(r)\})^{-1}.$$

Plus grand est le voisinage de la racine qui est identique dans les cartes M et M' , plus la

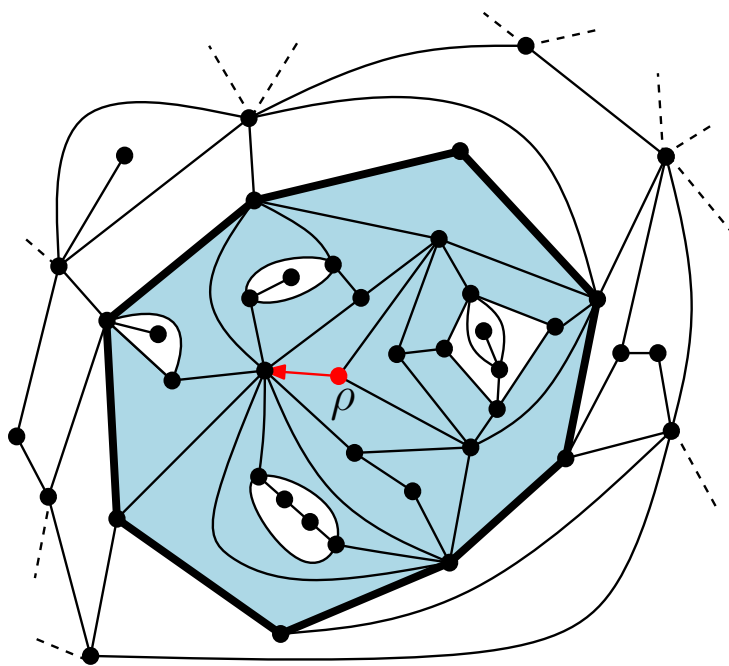


Figure 1.4 – En bleu, la boule de rayon 2 d’une grande quadrangulation. Le sommet racine est noté ρ , et l’arête racine est dessinée en rouge. Notons que la boule n’est pas simplement connexe.

distance locale entre les deux cartes est faible.

Le complété $\overline{\mathcal{M}}$ de \mathcal{M} pour d_{loc} est un espace polonais (séparable, métrique et complet). Il contient toutes les cartes planaires, mais également des “cartes planaires infinies”, contenant une infinité de faces, d’arêtes, et de sommets. Notons que toute carte de $\overline{\mathcal{M}}$ ne possède que des sommets et des faces de degrés finis.

Limite locale des quadrangulations uniformes. Pour tout entier $n \geq 1$, soit Q_n une quadrangulation uniforme à n faces. Un résultat clé pour l’étude des cartes aléatoires est le suivant, établi par Krikun dans l’article [38] :

Théorème 1.1.2. *La famille $(Q_n)_{n \geq 1}$ converge en loi pour la topologie locale.*

La quadrangulation uniforme infinie du plan (en anglais *uniform infinite plane quadrangulation*, ou UIPQ) est une variable aléatoire qui a la loi de la limite des Q_n .

Le premier résultat de ce genre a été établi par Angel et Schramm dans [5], et montre la convergence des triangulations uniformes vers une carte infinie du plan, la triangulation uniforme infinie du plan. L’UIPQ a été construite directement de deux manières différentes dans [38] et [15], réunifiées dans [52]. On peut mentionner également les modèles du demi-plan [26], les triangulations hyperboliques du plan [18], et la carte générale uniforme infinie du plan ou UIPM [54]. Remarquons que dans cette thèse, la version de l’UIPM que nous

utilisons est définie comme l'image de l'UIPQ par la bijection de Tutte. Nous n'avons pas formellement établi l'équivalence des deux approches.

Mentionnons quelques propriétés de l'UIPQ. La marche aléatoire sur l'UIPQ est récurrente par [30] ; des résultats sur la sous-diffusivité de la marche aléatoire ont été obtenus récemment dans un modèle cousin, l'UIPT [31]. Une carte M a un bout si et seulement si le complémentaire de toute union finie de faces de M contient une seule composante connexe qui intersecte une infinité de faces de M ; avec probabilité 1, l'UIPQ n'a qu'un bout, qu'on appelle *l'infini* ou *le point à l'infini*. On peut donc représenter l'UIPQ dans le plan de sorte que chaque région compacte du plan n'intersecte qu'un nombre fini de faces, de sommets et d'arêtes de l'UIPQ.

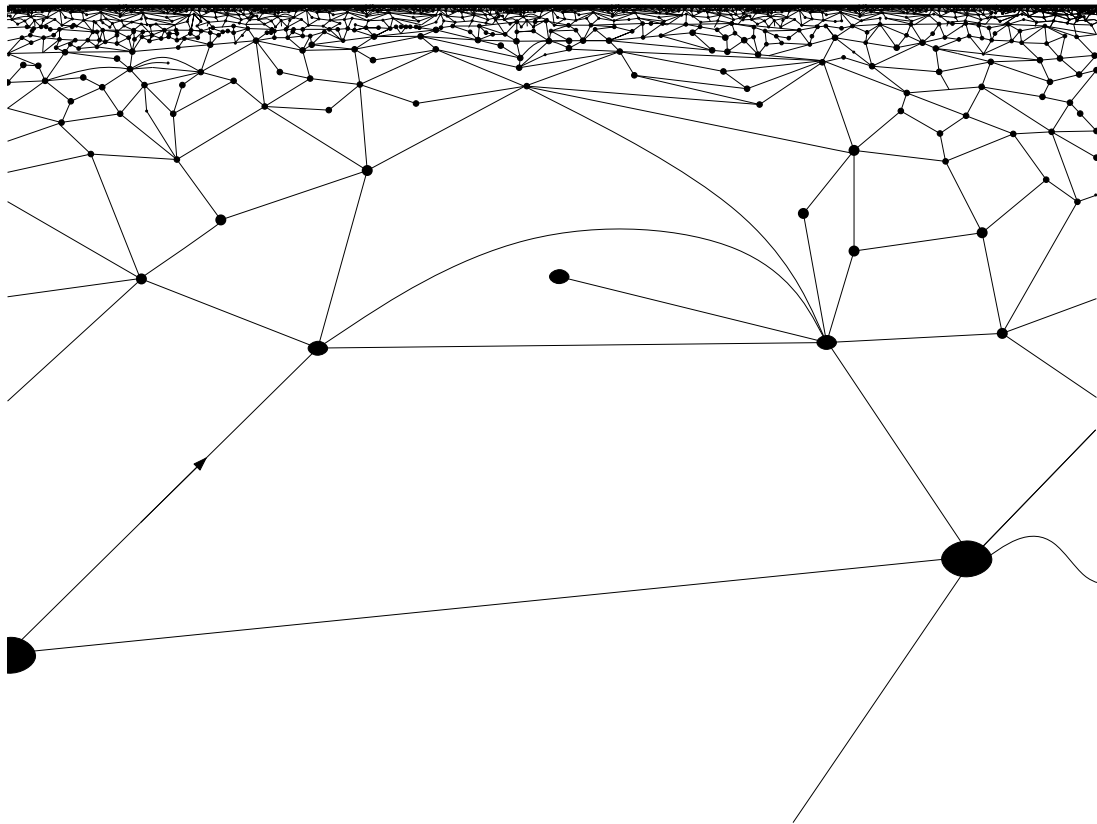


Figure 1.5 – Vue d'artiste d'une réalisation de la quadrangulation uniforme infinie du plan. L'arête racine est visible en bas à gauche de l'image.

Par sa définition même, on peut interpréter informellement l'UIPQ comme le voisinage de l'origine dans une très grande quadrangulation de la sphère. Si on dispose d'un énoncé suffisamment précis (voir le chapitre 4, section 3), on peut ainsi transférer des résultats de l'UIPQ aux quadrangulations de grande taille et réciproquement. Cela justifie la place consacrée à l'UIPQ dans cette thèse, d'autant plus que l'étude est souvent plus aisée dans l'UIPQ que dans les quadrangulations finies : voir la section 1.2.4 pour un outil (la décomposition en squelette) qui s'exprime beaucoup plus aisément dans l'UIPQ que dans les quadrangulations finies.

1.2 Outils

1.2.1 Combinatoire des cartes planaires

À l'origine de l'étude combinatoire des cartes planaires, il y avait le théorème des quatre couleurs. Ce théorème, prouvé en 1976 par Appel et Haken, stipule qu'on peut colorier les faces de n'importe quelle carte plane, de sorte qu'aucune face adjacente ne soit de la même couleur, et cela en utilisant au plus quatre couleurs.

Tutte tenta de montrer qu'il y avait autant de cartes 4-coloriables que de cartes générales. Il n'y parvint pas, mais inventa à l'occasion sa méthode quadratique, qui permit de grandes avancées en combinatoire analytique. Dans [62], membre d'une série d'articles dénombrant plusieurs modèles de cartes et autres objets combinatoires [59, 60, 61, 62], il établit que le nombre de quadrangulations de type I à n faces est donné par la formule suivante :

$$3^n \frac{2}{(n+2)(n+1)} \binom{2n}{n}. \quad (1.1)$$

L'élégance de cette formule (on voit notamment apparaître le n -ième nombre de Catalan $\frac{1}{n+1} \binom{2n}{n} \dots$) a poussé les chercheurs à chercher des bijections avec des modèles plus simples à étudier. Nous verrons dans la section 1.2.3 une telle bijection.

On considère souvent des quadrangulations à bord, c'est-à-dire dont la face à droite de l'arête racine est distinguée et peut avoir un degré quelconque. Cette face est appelée face externe, et toutes les autres sont appelées faces internes. Pour compter les quadrangulations à bord, il faut garder trace de deux paramètres : le nombre de faces internes, et le périmètre, c'est à dire le degré de la face externe. Notons que le périmètre d'une quadrangulation est nécessairement pair. Tutte a également compté les quadrangulations à bord : si on note $\mathbb{Q}_{n,p}$ l'ensemble des quadrangulations à bord à n faces internes et de périmètre $2p$,

$$\#\mathbb{Q}_{n,p} = 3^{n-p} \frac{(3p)!(2n+p-1)!}{p!(2p-1)!(n-p+1)!(n+2p)!}. \quad (1.2)$$

Voir par exemple [22]. Remarquons que la méthode de Tutte ne dérive pas directement (1.1) et (1.2), mais plutôt la *série génératrice des quadrangulations à bord* :

$$Q(x, y) \stackrel{\text{déf}}{=} \sum_{p \geq 1, n \geq p} \#\mathbb{Q}_{n,p} x^n y^p.$$

Il s'agit d'une série entière bivariée, dont le coefficient du monôme $x^n y^p$ est le cardinal de l'ensemble de quadrangulations à n faces internes et à bord de longueur $2p$. Formellement, la série génératrice contient la même information que la suite des cardinaux des ensembles de quadrangulations, tout en se prêtant davantage aux manipulations combinatoires ; on rencontre souvent les séries génératrices dans la littérature combinatoire sur les cartes planaires. C'est particulièrement le cas pour certains modèles de cartes, comme les *quadrangulations tronquées*, introduites par Krikun sous le nom un peu trompeur de *quadrangulation with simple boundary*, et que j'utilise fréquemment dans cette thèse. La série

génératrice des quadrangulations tronquées possède une écriture relativement simple, voir [38, Section 2.2], ou [46, (18)]. Toutefois, le calcul des premiers cardinaux des ensembles de quadrangulations tronquées semble indiquer qu’il n’existe pas de formule explicite comme (1.2) pour ce modèle de cartes.

1.2.2 Universalité

On observe pour beaucoup de modèles de cartes que, si $\#\mathbb{C}_{n,p}$ est le nombre de cartes de taille n et de périmètre p ,

$$\begin{cases} \#\mathbb{C}_{n,p} & \underset{n \rightarrow \infty}{\sim} c_p^{\mathbb{C}} n^{-5/2} \rho^n, \\ c_p^{\mathbb{C}} & \underset{p \rightarrow \infty}{\sim} D \alpha^p \sqrt{p}, \end{cases} \quad (1.3)$$

avec $D, \rho, \alpha \in (0, \infty)$ des constantes qui dépendent du modèle de carte. Les asymptotiques (1.3) sont vérifiées pour les quadrangulations, les quadrangulations tronquées, les triangulations, les cartes générales, etc... On devine déjà, à partir des cardinaux des ensembles de cartes, que beaucoup de modèles de cartes différents partagent des propriétés communes.

Il semblerait bien que ces asymptotiques soient un critère “presque suffisant” pour converger vers la carte brownienne : si elles sont vérifiées pour un modèle de cartes, on s’attend en général à ce qu’une suite de cartes uniformes dans ce modèle, munies de leur distance de graphe convenablement renormalisée, converge bien vers la carte brownienne. Inversement, quand (1.3) n’est plus vérifiée, c’est un signe que la limite d’échelle de ces cartes n’est pas la carte brownienne. On peut citer l’exemple des triangulations équipées d’un modèle d’Ising critique, qui ne satisfont pas (1.3), et qui sont conjecturées converger vers un objet différent, la gravité quantique de Liouville de paramètre $\sqrt{3}$.

Les asymptotiques (1.3) sont donc un premier indicateur de la convergence d’un modèle de cartes vers la carte Brownienne. Toutefois, elles sont loin d’être satisfaisantes. Tout l’enjeu est de comprendre quelles propriétés sont spécifiques à un modèle de cartes donné, et quelles propriétés sont “universelles”. C’est dans ce contexte que se placent les résultats du chapitre 3.

Mentionnons également un autre modèle de géométrie aléatoire, la gravité quantique de Liouville, qui est une famille à un paramètre de mesures sur des domaines du plan. Contrairement aux cartes, qui sont naturellement munies d’une structure d’espace métrique et d’une mesure, mais n’ont pas de “plongement canonique” dans le plan ou dans la sphère, la gravité quantique de Liouville possède une mesure et un plongement canonique, ainsi qu’une structure conforme — mais, jusqu’il y a peu, pas de métrique. Ce manque est comblé depuis des travaux récents [33], qui établissent l’existence et l’unicité d’une distance satisfaisant certaines propriétés raisonnables.

Chaque choix du paramètre de la gravité quantique de Liouville correspond à une classe d’universalité, de même que la carte Brownienne possède sa propre classe d’universalité. Les physiciens anticipent un lien fort entre cartes aléatoires et gravité quantique de Liouville : cartes aléatoires et gravité quantique de Liouville représenteraient deux visions d’un même objet, la “gravité quantique”. Ce lien est prouvé dans un cas particulier (mais

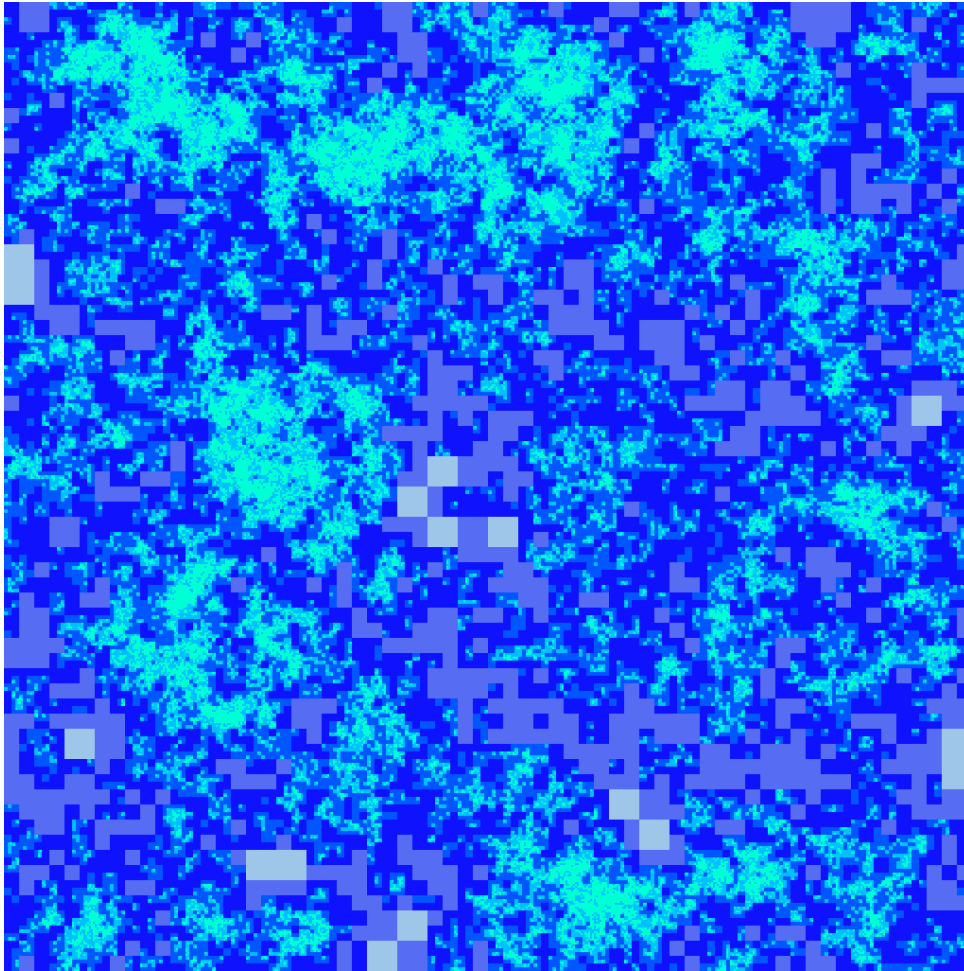


Figure 1.6 – Une réalisation de la gravité quantique de Liouville de paramètre $1,25$. La gravité quantique de Liouville est une mesure sur le plan ; sur cette illustration, les pixels plus sombres et plus gros correspondent à des zones auxquelles la mesure accorde moins de masse. A l’inverse, les pixels plus clairs et plus petits indiquent les zones où la mesure se concentre. Simulation de Jason Miller.

de grand intérêt), dit “gravité quantique sans matière”, cf. [57] : la carte Brownienne est en effet équivalente, dans un certain sens, à la gravité quantique de Liouville de paramètre $\sqrt{8/3}$. Une conjecture largement admise est que les cartes aléatoires équipées d’un modèle de physique statistique (ou “gravité quantique avec matière”, pour les différencier des cartes aléatoires uniformes) convergent vers une gravité quantique de Liouville pour un certain paramètre spécifique au modèle de physique statistique (et non au modèle de cartes), donné par les physiciens. Ces conjectures sont l’objet de recherches actives, tant au niveau numérique [7] que théorique. Il semble en tout cas que pour l’instant, le cadre des cartes aléatoires uniformes, bien que “moins général”, soit davantage accessible à l’analyse que celui de la gravité quantique, ou que celui des cartes aléatoires munies d’un modèle de physique statistique.

1.2.3 La bijection de Cori-Vauquelin-Schaeffer

La bijection de Cori-Vauquelin et Schaeffer, ou bijection CVS, est une bijection entre l'ensemble des quadrangulations et l'ensemble des arbres plans étiquetés. La découverte de cette bijection a permis de considérables avancées dans l'étude des propriétés métriques des cartes aléatoires. Elle se trouve même au cœur des résultats de convergence vers la carte Brownienne. La bijection CVS a été depuis généralisée à de nombreux autres types de cartes, voir par exemple [12].

Les objets qui interviennent dans la bijection CVS sont les arbres planaires bien étiquetés. Un arbre planaire est une carte à une seule face; c'est un arbre, au sens usuel du terme, mais dont les sommets sont ordonnés (on ne peut pas échanger deux branches voisines), et dont une arête est distinguée. Il est bien étiqueté si ses sommets portent des étiquettes à valeurs entières, positives ou non, telles que les étiquettes varient de 0, +1 ou -1 entre deux sommets adjacents, et telles que le sommet racine porte l'étiquette 0.

La bijection CVS permet de construire, à partir d'un arbre planaire bien étiqueté τ à n arêtes et d'un entier $\epsilon \in \{+1, -1\}$, une quadrangulation à n faces avec un sommet distingué, de la manière suivante. Soit (τ, ϵ) une telle paire. Un coin de τ est un coin de son unique face; on les ordonne en sens horaire. A chaque coin c de τ , on associe son successeur $S(c)$, qui est le premier coin après c dont l'étiquette est celle de c moins 1. Les coins dont l'étiquette est minimale sont traités séparément : on ajoute un sommet ∂ dans l'unique face de τ (ce sera le sommet distingué de la quadrangulation); l'unique coin de ∂ est le successeur de tous les sommets d'étiquette minimale.

Pour construire la quadrangulation, on relie tous les coins de τ à leur successeur, puis on efface les arêtes de τ . La carte ainsi obtenue est une quadrangulation Q . L'arête racine de Q est l'arête tracée à partir du coin à gauche de l'arête racine de τ . Il faut encore préciser l'orientation de l'arête racine de Q : si $\epsilon = +1$, elle pointe vers le sommet racine de τ , sinon elle s'en détourne.

Profitons de l'occasion pour expliquer d'où vient (1.1). Le nombre d'arbres planaires enracinés à n arêtes est le n -ième nombre de Catalan $\frac{1}{n+1} \binom{2n}{n}$. L'étiquetage d'un arbre planaire bien étiqueté est déterminé par la différence entre les étiquettes le long des arêtes de l'arbre, il y a donc 3^n bons étiquetages possibles pour tout arbre planaire à n arêtes. En prenant en compte les 2 choix possibles pour ϵ , on en conclut que le nombre de quadrangulations à n faces munies d'un sommet distingué est

$$3^n \frac{2}{n+1} \binom{2n}{n}. \quad (1.4)$$

Comme chaque quadrangulation à n faces possède $n+2$ sommets, (1.4) donne précisément $n+2$ fois le nombre de quadrangulations, ce qui redonne (1.1).

La bijection CVS se prête particulièrement bien à l'étude des quadrangulations : si τ_n est un arbre planaire étiqueté uniforme à n arêtes et ϵ est tiré uniformément dans $\{+1, -1\}$, alors la bijection CVS appliquée à (τ_n, ϵ) donne une quadrangulation Q_n uniforme parmi les quadrangulations à n faces, avec un sommet distingué choisi uniformément parmi ses

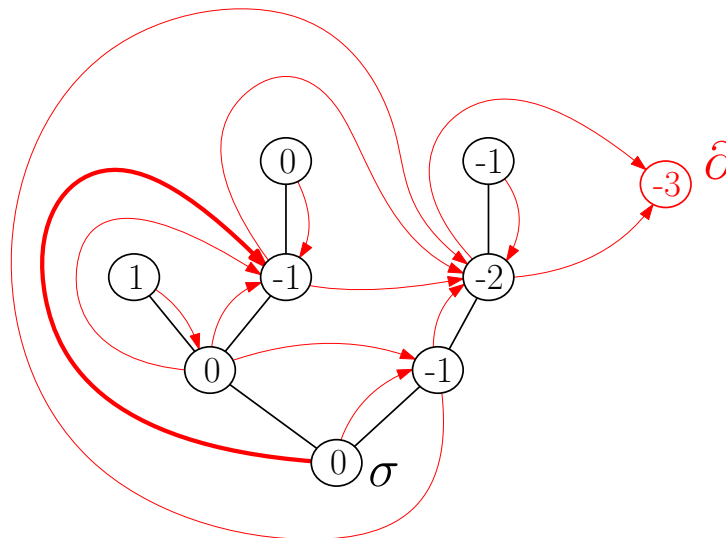


Figure 1.7 – La bijection de Cori-Vauquelin et Schaeffer relie quadrangulations et arbres planaires bien étiquetés. Ici, l’arbre τ est dessiné en noir, et les arêtes de la quadrangulation, obtenues en reliant chaque coin de l’arbre à son successeur, en rouge. L’orientation de l’arête racine de la quadrangulation (en gras) est déterminée par ϵ (ici égal à -1).

sommets. Certaines propriétés métriques de Q_n , comme la distance au sommet distingué, sont visibles dans τ_n . Quand $n \rightarrow \infty$, l’arbre planaire étiqueté τ_n , une fois ses étiquettes et ses distances convenablement renormalisées, converge vers un objet continu appelé le *serpent Brownien*, qui joue un rôle clé dans la construction de la carte Brownienne. L’UIPQ, la limite locale des quadrangulations uniforme, peut également être définie directement à partir de la limite locale de τ_n , voir [15].

On voit que l’étude des grands arbres aléatoires occupe une place centrale dans le domaine des cartes aléatoires. La section suivante présente une autre occasion de les rencontrer, sous la forme d’un des outils clé des travaux de cette thèse : la décomposition en squelette.

1.2.4 Décomposition en squelette

Confluence des géodésiques. Pour mieux connaître les propriétés métriques des cartes aléatoires, il est naturel de considérer les géodésiques reliant deux points donnés. La compréhension des géodésiques joue un rôle essentiel dans la preuve de l’unicité de la carte Brownienne et dans la construction de sa métrique à l’aide du serpent Brownien [40].

Ces géodésiques sont-elles uniques? On peut comparer la situation dans les cartes aléatoires à celle de \mathbb{Z}^2 . Si A et B sont deux points du plan, de coordonnées entières $x_A < x_B$ et $y_A < y_B$, alors tout chemin de A à B dont les coordonnées sont croissantes est une géodésique. Clairement, il n’y a pas unicité des géodésiques. Un autre exemple, au

comportement très différent, est les variétés Riemanniennes, où les géodésiques sont (au moins localement) uniques : [27, Corollaire 3.11] assure qu'entre deux points à distance inférieure à une certaine constante (dépendant de la variété), il existe une unique courbe de longueur minimale.

Il n'y a pas unicité des géodésiques dans les cartes aléatoires, même localement, comme illustré dans figure 1.8 : comme dans \mathbb{Z}^2 , il est bien souvent possible de passer d'un côté ou de l'autre d'une face ou d'un ensemble de faces sans changer la longueur du chemin. Cependant, paradoxalement, on retrouve l'unicité des géodésiques dans la limite d'échelle des cartes aléatoires : [39] assure que presque tous les points de la carte Brownienne sont reliés à la racine par une unique géodésique.

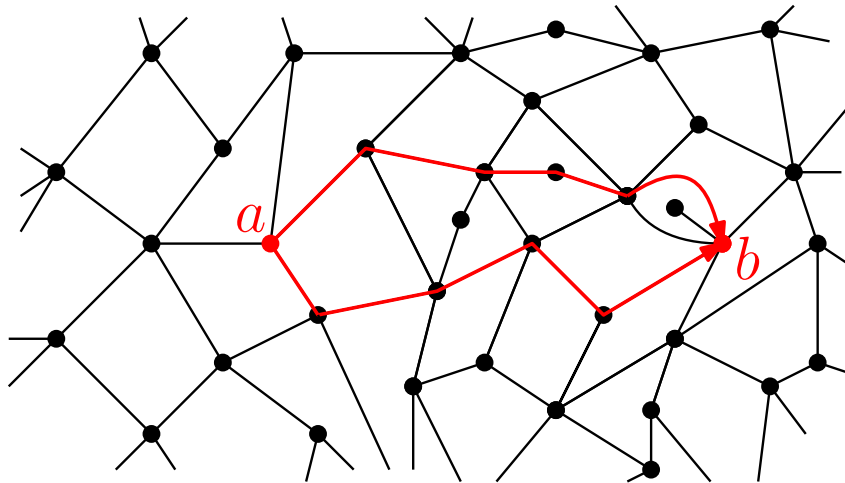


Figure 1.8 – Les géodésiques ne sont en général pas uniques dans les cartes planaires : en rouge, deux chemins de longueur minimale entre deux sommets a et b d'une grande quadrangulation.

On observe dans la carte Brownienne un phénomène de *confluence des géodésiques* : pour presque tous points x, y, z deux à deux disjoints, les géodésiques de x à y et de x à z coalescent ; c'est-à-dire, si γ_y est la géodésique de x à y et γ_z celle de x à z , alors il existe un temps $\alpha > 0$ tel que $\gamma_y([0, \alpha]) = \gamma_z([0, \alpha])$. Ceci contraste fortement avec le cas des variétés Riemanniennes, où, si deux géodésiques sont confondues sur un voisinage d'un point, alors elles sont égales.

Ce comportement est désormais bien connu. On peut citer [13] dans le cadre des grandes quadrangulations, ou [4, 39] dans la carte brownienne pour les géodésiques vers la racine. Dans les cartes infinies du plan, [25, Théorème 3.4] (voir aussi [24]) prouve l'existence d'un ensemble infini de points dans la quadrangulation uniforme infinie du plan que tout chemin géodésique infini doit visiter.

Enveloppes tronquées. Donnons-nous une quadrangulation infinie du plan à un bout Q , et étiquetons ses sommets par leur distance au sommet racine. Chaque face de Q peut être de deux types :

- soit les étiquettes de ses coins sont $r, r + 1, r, r - 1$. On dit que la face est r -simple ;
- soit les étiquettes sont $r, r - 1, r, r - 1$. La face est dite confluente.

Fixons un $r > 0$. Dans chaque face r -simple, ajoutons une arête entre les coins d'étiquette r , que nous appelons r -diagonale.

Lemme 1.2.1. *Il existe un unique cycle simple, noté $\partial_r Q$, constitué de r -diagonales tel que la composante connexe infinie du complémentaire de $\partial_r Q$ ne contient aucune r -diagonale ni aucun sommet à distance r ou moins du sommet racine.*

L'enveloppe tronquée de rayon r de Q , notée $\mathcal{H}_Q^{\text{tr}}(r)$, est la carte contenue dans l'intérieur de $\partial_r Q$, à laquelle on ajoute les arêtes de $\partial_r Q$.

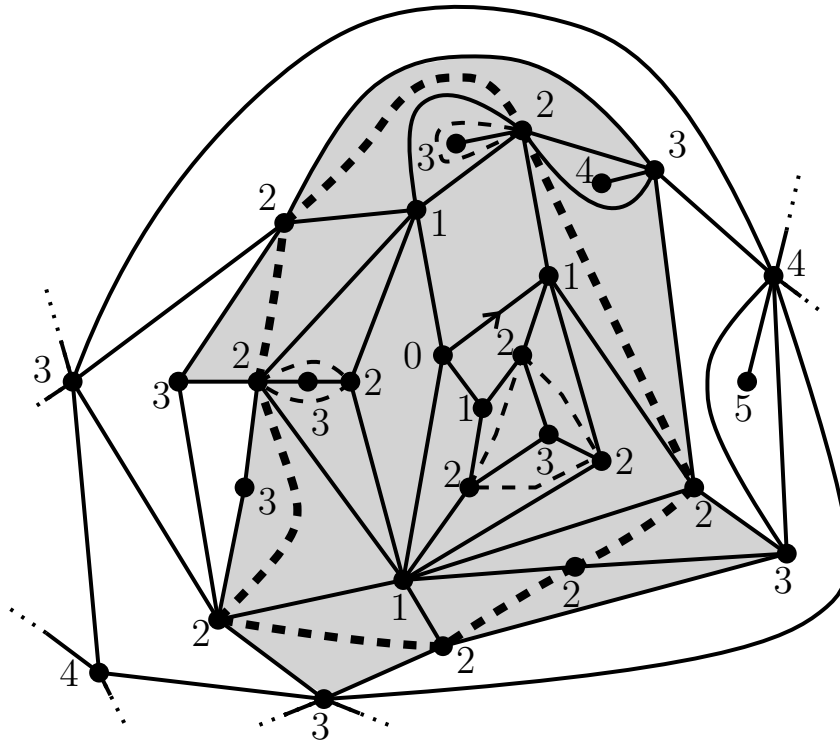


Figure 1.9 – En gris, l'enveloppe standard $B_Q^\bullet(2)$ de rayon $r = 2$ d'une quadrangulation Q . On étiquette les sommets de Q par leur distance au sommet racine, et on trace les diagonales reliant les sommets d'étiquette $r = 2$ dans chaque face avec les étiquettes $r - 1, r$ et $r + 1$ (ici 1, 2 et 3), en pointillé. Le cycle maximal, en pointillé gras, est le bord de l'enveloppe tronquée de rayon $r = 2$ de Q , notée $\mathcal{H}_Q^{\text{tr}}(2)$.

Il existe un autre voisinage naturel de l'origine, plus souvent rencontré : les *enveloppes*, ou *enveloppes standard*. L'enveloppe standard de rayon r de Q , notée $B_Q^\bullet(r)$, est obtenue en ajoutant à la boule de rayon r de Q toutes les composantes connexes finies de son

complémentaire. Les enveloppes tronquées sont proches des enveloppes standard : pour tout entier $r \geq 1$,

$$\mathcal{H}_Q^{\text{tr}}(r) \subset B_Q^\bullet(r) \subset \mathcal{H}_Q^{\text{tr}}(r+1). \quad (1.5)$$

Dans le chapitre 4, je calcule la loi du nombre de faces contenues dans les enveloppes tronquées de l'UIPQ. L'inclusion (1.5) permet de faire passer ces informations aux enveloppes standard.

Géodésiques les plus à gauche. Soit Q une quadrangulation infinie à un bout, v_0 son sommet racine. Pour tout entier $r \geq 1$ et v un sommet de $\partial_r Q$, grâce au lemme 1.2.1, on est assuré que toute géodésique γ de v à v_0 reste à l'intérieur de $\partial_r Q$; et de plus, que toute arête appartenant à une géodésique de v à v_0 relie un sommet de $\partial_k Q$ à un sommet de $\partial_{k-1} Q$ avec $k > 1$, ou un sommet de $\partial_1 Q$ à v_0 .

Soit un entier $r \geq 1$. Pour tout $v \in \partial_r Q$, on appelle *arête descendante issue de v* une arête de v à $\partial_{r-1} Q$ si $r > 1$, et de v à v_0 si $r = 1$. Ordonnons les arêtes descendantes issues de v en sens horaire, en commençant par un des coins incidents à v et à l'extérieur de $\partial_r Q$. Cet ordre sur les arêtes descendantes induit un ordre lexicographique sur les géodésiques vers le sommet racine issues d'un sommet de $\partial_s Q$ pour un entier $s \geq 1$. Les géodésiques maximales sont appelées “géodésiques les plus à gauche”.

Cette appellation provient de la convention de représenter le voisinage d'un point v de $\partial_r Q$ avec l'extérieur de $\partial_r Q$ dans le demi-plan supérieur, et le sommet racine dans le demi-plan inférieur (on rappelle la représentation en cactus des cartes aléatoires, qui place la racine “en bas” et l'infini “en haut”). De cette manière, la géodésique la plus à gauche issue de v emprunte l'arête “la plus à gauche” parmi toutes les arêtes de v à $\partial_{r-1} Q$.

Une propriété intéressante de ces géodésiques est qu'elles sont construites par récurrence, et “localement” : si γ est une géodésique la plus à gauche d'un sommet v de $\partial_r Q$ à v_0 , sa première arête est entièrement déterminée par le choix de v et la partie de Q comprise entre $\partial_{r-1} Q$ et $\partial_r Q$, et le reste est entièrement déterminé par $\mathcal{H}_Q^{\text{tr}}(r-1)$ ainsi que le premier sommet de γ dans $\partial_{r-1} Q$. En particulier, si deux géodésiques les plus à gauches se rejoignent en un $v \in \partial_r Q$ avec $r > 0$ (on dit qu'elles *coalescent* en v), alors la partie de ces géodésiques entre v et le sommet racine est exactement la géodésique la plus à gauche issue de v .

La conséquence immédiate de cette observation est que les géodésiques les plus à gauche forment un arbre dont les sommets sont $\{v_0\} \cup \bigcup_{r>0} V(\partial_r Q)$ (cf. figure 1.10). On s'intéressera plutôt à sa forêt duale : la *squelette de Q* , dont les sommets sont les arêtes de $\bigcup_{r>0} \partial_r Q$, qui s'avère avoir la loi d'un processus de branchement lorsque Q est l'UIPQ, à une densité explicite près. Conséquence de cette dualité : les géodésiques les plus à gauche issues de deux sommets v, w de $\partial_r C$ coalescent avant d'atteindre $\partial_s C$ si et seulement si, soit aucune des arêtes de $\partial_r C$ entre v et w n'a de descendant dans $\partial_s C$, soit aucune des arêtes entre w et v n'a de descendant dans $\partial_s C$ (où on parcourt $\partial_t C$ en sens horaire pour tout t).

Décomposition en squelette. Ajoutons toutes les arêtes de $\cup_{0 < s < r} \partial_s Q$ à Q . Toute arête $e \in \partial_s Q$ avec $0 < s \leq r$ est incidente à une face triangulaire dont le troisième sommet appartient à $\partial_{s-1} Q$. On appelle cette face le *triangle descendant* de bord supérieur e . Les triangles descendants héritent de la structure généalogique sur les arêtes de $\cup_{r > 0} \partial_r Q$: leur structure est contenue dans le squelette de Q .

Les triangles descendants délimitent une collection de régions disjointes de l'intérieur de Q (elles peuvent éventuellement partager un sommet de $\cup_{0 \leq s \leq r} \partial_s Q$). On appelle ces régions des alvéoles (*slots* en anglais). La donnée de l'arrangement des triangles descendants et des cartes incluses à l'intérieur des alvéoles suffit pour reconstruire Q . Cette donnée est la *décomposition en squelette*.

Krikun la décrit dans l'article [37] dans le cadre des triangulations, et l'adapte dans [38] aux quadrangulations. Dans l'UIPQ, le squelette est un arbre de Galton-Watson infini avec un conditionnement, les cartes qui remplissent les alvéoles sont indépendantes, et la loi du contenu d'une alvéole ne dépend que du périmètre de la alvéole.

1.3 Cycles séparants et isopérimétrie dans la quadrangulation uniforme infinie du plan (chapitre 2)

Mon premier travail, effectué en collaboration avec Jean-François Le Gall, a donné lieu à l'article [43]. Il porte sur l'étude de la longueur minimale des cycles qui séparent les grandes boules de l'UIPQ de l'infini. Nos résultats nous permettent d'établir des inégalités isopérimétriques sur les ensembles composés d'un nombre fini de faces de l'UIPQ et contenant l'arête racine.

Dans cette section, nous considérons uniquement les boules centrées en le sommet racine.

Cycles séparants. L'UIPQ a un seul bout : presque sûrement, tout cycle fini et simple (c'est-à-dire sans auto-intersection) découpe le plan en deux régions dont une exactement intersecte une infinité de faces de l'UIPQ. On dit qu'un cycle simple d'arêtes de l'UIPQ sépare une région S du plan (et par extension un ensemble de sommets, de faces...) de l'infini quand S est contenue dans le domaine fini du plan délimité par le cycle. La longueur d'un cycle est le nombre d'arêtes de l'UIPQ qu'il contient.

Étant donnée une région finie S du plan, on s'intéresse à la longueur minimale que peut avoir un cycle séparant S de l'infini. Notons qu'il n'y a en général pas de borne supérieure à la longueur que peut avoir un tel cycle.

Commençons par observer ce qui se passe dans \mathbb{R}^2 muni de la distance euclidienne : pour tout $r > 0$ la longueur du cycle le plus court séparant la boule de rayon r centrée en $(0, 0)$ de l'infini est $2\pi r$, et il s'agit précisément du bord de la boule euclidienne.

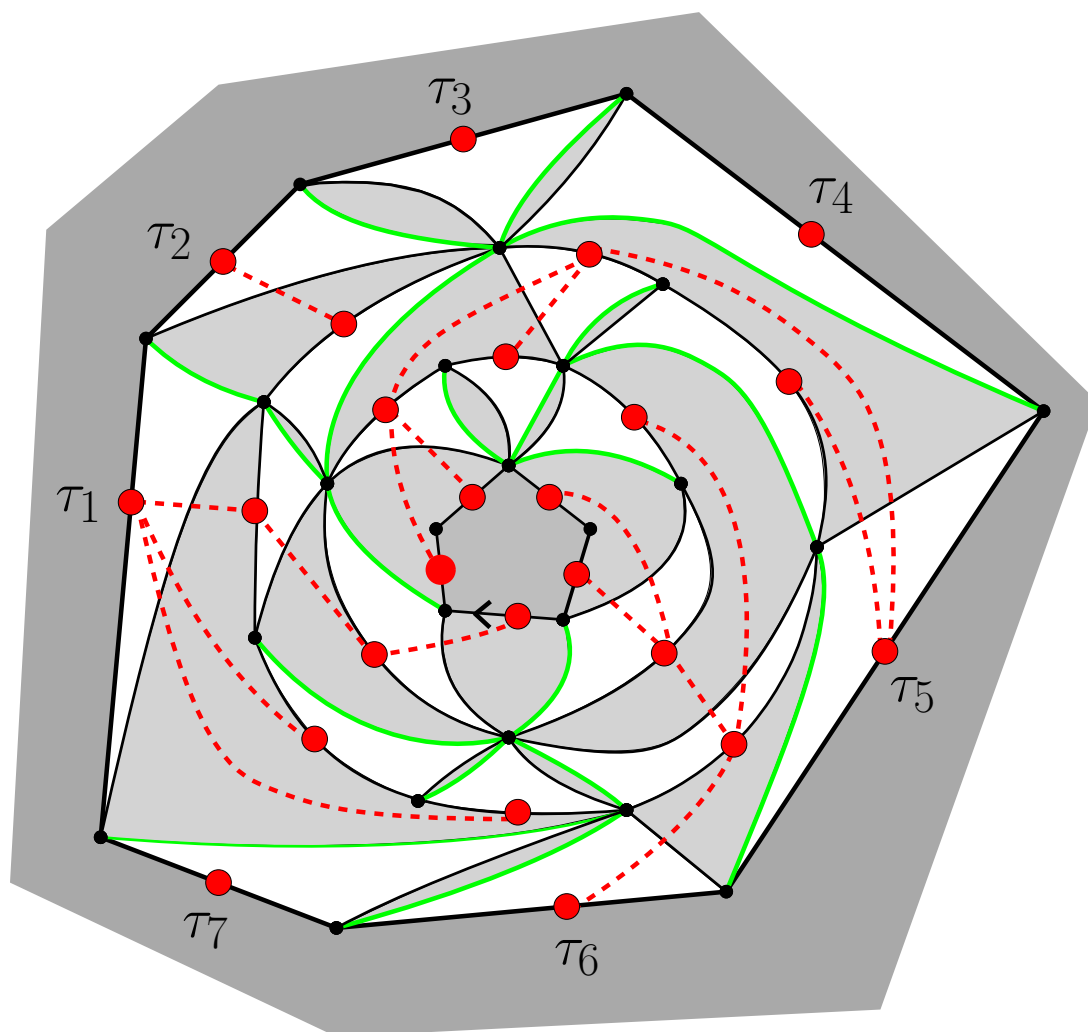


Figure 1.10 – Un anneau d'une quadrangulation infinie Q , contenu entre les cycles $\partial_r Q$ et $\partial_{r+3} Q$ pour un certain entier $r \geq 1$. Afin de simplifier le dessin, les triangles descendants sont dessinés en blanc, et le contenu des alvéole est peint en gris clair. En vert, les géodésiques les plus à gauche forment un arbre planaire. Le squelette, en rouge, encode une relation généalogique sur les arêtes des cycles $\partial_s Q$, et est la forêt duale de celle des géodésiques les plus à gauche.

Dans l'UIPQ, les boules ne sont en général pas simplement connexes. Pour tout entier $r \geq 1$, l'enveloppe de rayon r est obtenue en bouchant les trous finis de la boule de rayon r . Le bord de l'enveloppe constitue un cycle séparant la boule de l'infini. [22] établit que la longueur du bord de l'enveloppe de rayon r est d'ordre r^2 . Voir aussi [38, Theorem 1.2] pour un autre cycle (le bord des enveloppes tronquées), lui aussi d'ordre r^2 , et [3] pour le cas des triangulations.

Peut-on trouver plus court ? Dans le même article [38] où il construit l'UIPQ en tant que limite locale des quadrangulations uniformes quand leur taille tend vers l'infini, Krikun

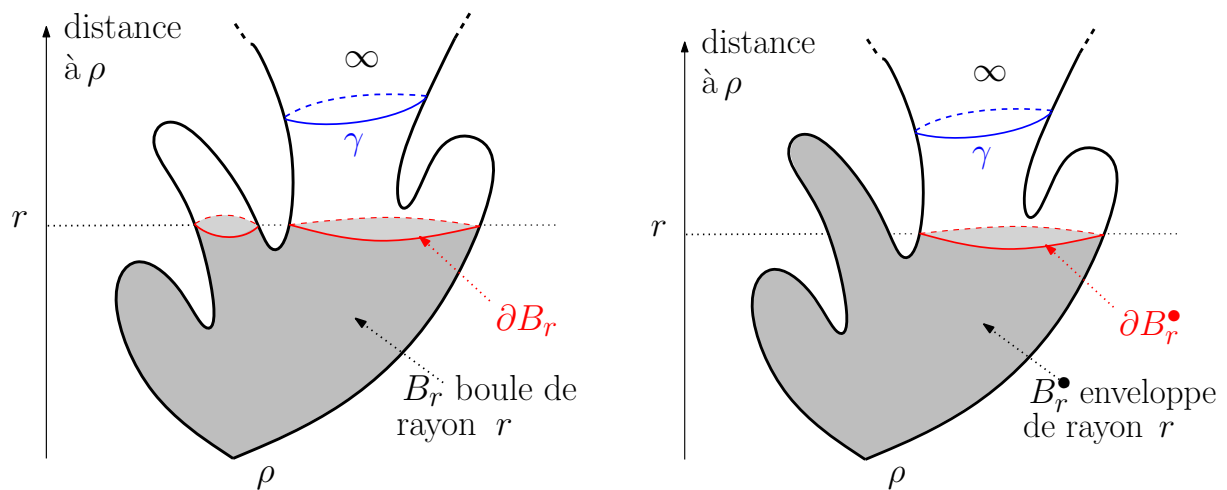


Figure 1.11 – Représentation en “cactus” de l’UIPQ : les sommets de l’UIPQ sont représentés à une hauteur proportionnelle à leur distance au sommet racine, noté ici ρ . À gauche, la boule de rayon r de l’UIPQ contient tous les sommets à distance r ou moins, et aucun sommet à distance $r + 2$ ou plus : sur cette représentation, elle correspond grosso modo à la partie de la carte en-dessous de la hauteur r . En bouchant les trous finis de la boule de rayon r , on obtient l’enveloppe de rayon r de l’UIPQ (à droite). En bleu, γ est un cycle séparant la boule et l’enveloppe de rayon r de l’infini.

construit pour tout entier $r \geq 1$ des cycles de longueur d’ordre r (c’est-à-dire plus petit que $C_\varepsilon r$ avec probabilité plus grande que $1 - \varepsilon$, pour une constante C_ε qui ne dépend que de ε) qui séparent la boule de rayon r de l’infini. Il conjecture qu’il n’est pas possible de faire mieux. [20] établit une forme faible de cette conjecture, mais qui contrôle seulement les cycles “proches de la boule” : il n’exclut pas la possibilité que des cycles courts puissent exister très loin de la racine.

Nous confirmons la conjecture de Krikun. Notre théorème l’établit sous une forme précise, avec des estimées sur la probabilité qu’il existe un cycle séparant de longueur très petite, ou qu’il n’en existe aucun de petite longueur.

Théorème 1.3.1

Pour tout entier $r \geq 1$, soit $L(r)$ la longueur du cycle le plus court qui sépare la boule de rayon r de l'infini.

- (i) Pour tout $\delta < 2$, il existe une constante C_δ telle que, pour tout entier $r \geq 1$, pour tout $\varepsilon \in (0, 1)$,

$$\mathbb{P}(L(r) \leq \varepsilon r) \leq C_\delta \varepsilon^\delta.$$

- (ii) Il existe des constantes $C, \lambda > 0$ telles que pour tout $a > 0$ et tout entier $r \geq 1$,

$$\mathbb{P}(L(r) \geq ar) \leq Ce^{-\lambda a}.$$

La partie (i) est presque optimale : nous pensons que $\mathbb{P}(L(r) \leq \varepsilon r)$ devrait être d'ordre ε^2 , possiblement avec des corrections logarithmiques. Nous prouvons que $\mathbb{P}(L(r) \leq \varepsilon r)$ est au moins $C'\varepsilon^3$ pour une certaine constante $C' > 0$ quand r est assez grand, et plus de travail permettrait vraisemblablement d'améliorer cette minoration à $C''\varepsilon^2$.

Le théorème 1.3.1 porte sur les cycles séparant la boule de rayon r de l'infini sans restriction sur la distance de ces cycles à la boule. Nous établissons un résultat plus précis, qui permet de garder trace de cette information. Remarquons que c'est de ce résultat que nous déduisons la partie (i) du théorème 1.3.1. En voici une version équivalente :

Théorème 1.3.2

Soit $\beta \in (0, 3)$. Il existe une constante C'_β telle que, pour tout entier $r \geq 1$ et tout entier $n \geq 1$, la probabilité qu'il existe un cycle de longueur au plus r , qui est contenu dans l'enveloppe de rayon $(n+2)r$ et sépare l'enveloppe de rayon nr de l'infini, est majorée par $C'_\beta n^{-\beta}$.

On déduit du théorème 1.3.1 que pour tout entier $k > 0$, il n'existe qu'un nombre fini de cycles de longueur plus petite que k qui séparent le sommet racine de l'infini : $L(r)$ est croissant et tend vers l'infini quand $r \rightarrow \infty$.

Isopérimétrie dans la quadrangulation uniforme infinie du plan. On s'intéresse à des ensembles finis et simplement connexes de faces de l'UIPQ qui contiennent l'arête racine.

Pourquoi considérer uniquement les ensembles contenant l'arête racine ? On cherche à connaître les propriétés des ensembles au voisinage d'un point "typique", et la racine de l'UIPQ représente justement un tel point typique. Dès lors qu'on s'éloigne de la racine, on rencontre des ensembles pathologiques, qu'on souhaite éviter. Remarquons que

le chapitre 4 présente une inégalité isopérimétrique sans cette restriction, valable dans les quadrangulations finies.

Le bord d'un ensemble S de faces, noté ∂S , est l'ensemble des arêtes incidentes d'un côté à une face dans S et de l'autre à une face qui n'est pas dans S . On note $|S|$ le nombre de faces de l'UIPQ que contient S (son "volume"), et $|\partial S|$ le nombre d'arêtes de ∂S (son "périmètre"). Enfin, \mathcal{K} est la collection des ensembles finis simplement connexes de faces de l'UIPQ dont au moins une face est incidente à l'arête racine.

L'inégalité isopérimétrique suivante établit que le périmètre de tout ensemble simplement connexe de faces de l'UIPQ dont au moins une face est incidente à l'arête racine ne peut pas être plus petit que son nombre de faces à la puissance $1/4$, à correction logarithmique près.

Théorème 1.3.3

Pour tout $\delta > 0$, presque sûrement

$$\inf_{A \in \mathcal{K}} \frac{|\partial A|}{|A|^{1/4} (\ln |A|)^{-\frac{3}{4}-\delta}} > 0. \quad (1.6)$$

On sait de plus construire des ensembles qui réalisent cet infimum (aux termes logarithmiques près) : ce sont les enveloppes standard de rayon r de l'UIPQ, agrandies jusqu'à un des cycles de longueur minimale qui les séparent de l'infini. En effet, l'enveloppe de rayon r de l'UIPQ a un volume qui est équivalent à $\frac{3}{8}r^4$ quand $r \rightarrow \infty$ (voir la section 4.4.3), et on peut trouver un cycle qui la sépare de l'infini et de longueur d'ordre r . L'intérieur d'un tel cycle est un ensemble contenant un nombre de faces d'ordre au moins r^4 , et de périmètre d'ordre r , c'est-à-dire son nombre de faces à la puissance $1/4$. Le théorème 1.3.3 affirme, en d'autres termes, qu'il n'existe pas d'ensembles de faces qui font significativement mieux.

Cette inégalité est donc quasiment optimale, au terme logarithmique près. Peut-on améliorer ce terme ? Malheureusement, il n'est pas possible de se débarrasser complètement du $\ln |A|$: on peut toujours trouver des ensembles A tel que $|\partial A| / |A|^{1/4}$ soit arbitrairement petit. Toutefois, l'exposant $3/4$ du logarithme n'est probablement pas optimal. Un meilleur contrôle sur les moments du volume des enveloppes (comme le lemme 13 du chapitre 4, établi postérieurement à l'écriture du chapitre 2) permettrait de l'améliorer ; des travaux de Riera dans le plan Brownien laissent penser qu'on pourrait descendre jusqu'à $1/2$.

Nous établissons également une autre inégalité isopérimétrique qui donne une borne inférieure sur la longueur du bord des ensembles de faces contenant l'arête racine, et dont le nombre de faces est minoré.

Théorème 1.3.4

Soit $\varepsilon > 0$. Il existe une constante $c_\varepsilon > 0$ telle que pour tout entier $n \geq 1$, avec probabilité au moins $1 - \varepsilon$:

$$|\partial A| \geq c_\varepsilon n^{1/4} \quad \text{pour tout } A \in \mathcal{K} \text{ tel que } |A| \geq n. \quad (1.7)$$

La principale différence avec le théorème 1.3.3 est l'absence du terme logarithmique, et une uniformité en n de la probabilité que la propriété (1.7) soit vérifiée.

1.4 Modifications de distance et bijection de Tutte dans les quadrangulations finies et infinies (chapitre 3)

On sait depuis les travaux de Le Gall [41] et Miermont [55] que l'espace métrique composé de l'ensemble des sommets d'une d -angulation uniforme de taille n , avec $d = 3$ ou d un entier pair, muni de la distance de graphe multipliée par $n^{-1/4}$ fois une constante dépendant de d , converge quand $n \rightarrow \infty$ vers la carte brownienne, introduite en section 1.1.3.

Pour mieux comprendre la classe d'universalité de la carte Brownienne, les mathématiciens s'efforcent de généraliser cette convergence à d'autres objets aléatoires. La plupart du temps, les efforts sont concentrés sur l'étude d'espaces métriques obtenus en munissant l'ensemble des sommets d'une carte aléatoire différente de la distance de graphe — informellement parlant, on change le réseau sous-jacent, mais on garde la même manière d'obtenir un espace métrique à partir d'une carte [1, 2, 8, 9, 50, 51] .

J'explore ici une autre approche : construire un autre espace métrique à partir d'un modèle de cartes bien connu. Plus précisément, je m'intéresse à la distance de percolation de premier passage sur les quadrangulations, ainsi que sur les cartes générales, et montre qu'elle est proche de la distance de graphe multipliée par une certaine constante. Je présente ces résultats plus en détail en section 1.4.1.

Par ailleurs, j'établis que la bijection de Tutte (présentée en section 1.1.1), entre quadrangulations et cartes générales, est asymptotiquement une isométrie ; c'est-à-dire que, si M_n est une carte générale uniforme à n arêtes et Q_n est la quadrangulation associée par la bijection de Tutte, alors les espaces métriques obtenus en munissant $V(M_n)$ de la distance de graphe sur la carte M_n , resp. en munissant $V(Q_n)$ de la distance de graphe sur Q_n , toutes les deux multipliées par $n^{-1/4}$, ont une distance de Gromov-Hausdorff qui tend vers 0 quand $n \rightarrow \infty$. Mon résultat (théorème 1.4.7), que je présente en section 1.4.2, est en réalité plus fort, car la bijection de Tutte permet un couplage très étroit entre quadrangulations et cartes générales.

1.4.1 Percolation de premier passage

La percolation de premier passage a été proposée par Hammersley et Welsh en 1965, et a mené à une riche littérature depuis. Plaçons-nous d’abord dans le cas du réseau carré. Attribuons des poids i.i.d. et p.s. positifs aux arêtes de \mathbb{Z}^d , pour $d \geq 2$. La distance de percolation de premier passage $d_{\text{fpp}}(x, y)$ entre deux points $x, y \in \mathbb{Z}^d$ est l’infimum des poids des chemins entre x et y . L’appellation “distance” est quelque peu abusive, car d_{fpp} peut ne pas satisfaire l’axiome de séparation, par exemple si les poids ont probabilité non nulle d’être nuls ; il est plus juste de considérer d_{fpp} comme une fonction bivariée sur les sommets de \mathbb{Z}^d .

Comment comparer d_{fpp} à la distance de graphe sur \mathbb{Z}^d ? En particulier, l’aspect aléatoire de la géométrie (dû au caractère aléatoire des poids) perdure-t-il quand on s’intéresse à de très grandes régions de \mathbb{Z}^d ? Que dire de la distance de percolation de premier passage, lorsque le réseau \mathbb{Z}^d est remplacé par une carte plane, elle-même aléatoire ?

Un résultat fondamental en percolation de premier passage est l’existence d’une forme limite pour les boules pour d_{fpp} . Cox et Durrett [17] établirent un tel résultat dans \mathbb{Z}^2 , sous des hypothèses faibles sur les poids, et Kesten [36] dans \mathbb{Z}^d pour $d \geq 2$. Nous donnons ici la formulation de Kesten. Notons $B_t = \{x \in \mathbb{Z}^d, d_{\text{fpp}}(0_{\mathbb{Z}^d}, x) \leq t\}$.

Théorème 1.4.1. [36, Théorème 1.7] *Supposons que $\mathbb{E}[Y^d] < \infty$, où Y est le minimum de $2d$ copies i.i.d. des poids. Alors il existe un convexe K déterministe, d’intérieur non vide, invariant par permutation des coordonnées et par réflexion selon les hyperplans orthogonaux aux vecteurs de la base canonique, qui est soit compact, soit \mathbb{R}^d entier, et tel que*

- (i) *si K est compact, alors pour tout $\varepsilon > 0$, avec probabilité qui tend vers 1 quand $t \rightarrow \infty$,*

$$(1 - \varepsilon)tK \cap \mathbb{Z}^d \subset B_t \subset (1 + \varepsilon)tK \cap \mathbb{Z}^d. \quad (1.8)$$

- (ii) *si $K = \mathbb{R}^d$, alors pour tout compact $K' \subset \mathbb{R}^d$, pour tout t assez grand,*

$$tK' \cap \mathbb{Z}^d \subset B_t.$$

Seul le cas (i) nous intéresse ; le cas (ii) est plus pathologique, et peut être éliminé sous des hypothèses assez faibles sur la loi des poids. Plaçons-nous dorénavant dans le cas (i) : le théorème 1.4.1 affirme que la boule pour la distance de percolation de premier passage, à la limite, n’est plus aléatoire. Ce théorème s’interprète comme une “loi des grands nombres”. K n’est toutefois ni une boule euclidienne, ni (en général) une boule pour la distance de graphe : la non-isotropie du réseau persiste à la limite.

Que se passe-t-il si on modifie le réseau sous-jacent ? [63] étend le théorème 1.4.1 au cas où on remplace \mathbb{Z}^d par un diagramme de Voronoï obtenu à partir d’un processus ponctuel de Poisson de mesure intensité homogène. K est alors une boule euclidienne, ce qui reflète l’invariance (en loi) par rotation du réseau sous-jacent. La distance de percolation de premier passage est également utile dans la gravité quantique ; on peut se référer par

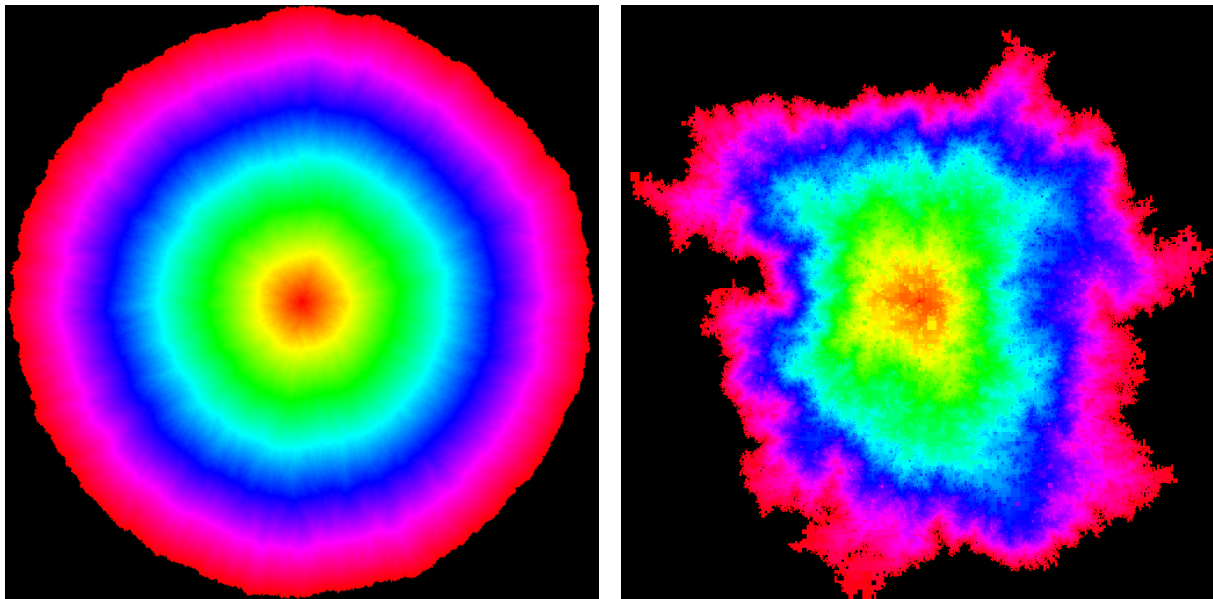


Figure 1.12 – À gauche, une réalisation de la percolation de premier passage avec poids exponentiels sur \mathbb{Z}^2 ; à droite, sur la gravité quantique de Liouville de paramètre 1. Les couleurs correspondent à la distance à l’origine. La simulation de droite est de Jason Miller.

exemple aux progrès récents obtenus dans [33], où une généralisation de la distance de percolation de premier passage à des domaines continus permet de construire une métrique “canonique” dans la gravité quantique de Liouville pour presque tous les choix de paramètre, là où un seul cas (la $\sqrt{8/3}$ -LQG, induisant la métrique de la carte Brownienne) était connu auparavant.

Forme limite dans les cartes aléatoires. L’analogie du plan pour les quadrangulations est l’UIPQ Q_∞ ; c’est donc dans ce modèle que j’énonce mes résultats de forme limite. Notons ρ le sommet racine de l’UIPQ. Pour toute carte M , la distance de percolation de premier passage, obtenue en attribuant des poids i.i.d. à chaque arête de M , est notée d_{fpp}^M .

Dans toute la suite de cette section, nous faisons l’hypothèse suivante :

Hypothèse 1.4.2. *Les poids sont p.s. dans un compact de $(0, \infty)$.*

Cette restriction découle d’une volonté de garder les preuves à un niveau de complexité aussi raisonnable que possible, et peut être affaiblie. Pour toute carte M , sous l’hypothèse 1.4.2, d_{fpp}^M est une vraie distance. De plus, sous cette même hypothèse pour \mathbb{Z}^d , le théorème 1.4.1 s’applique, et on est toujours dans le cas (i).

Rappelons que pour tout entier $r \geq 1$, la boule $B_{Q_\infty}(r)$ de rayon r est l’ensemble des faces incidentes à un sommet de Q_∞ à distance strictement moins que r du sommet racine.

Par commodité, on étend cette définition aux r réels strictement positifs ; autrement dit, pour tout $r > 0$, la boule de rayon $r > 0$ est la boule de rayon $\lceil r \rceil$, où $\lceil r \rceil$ est le plus petit entier plus grand que r .

Pour tout $r > 0$, plus nécessairement entier, on définit $B_{Q_\infty}^{\text{fpp}}(r)$ la boule de rayon r pour la distance de percolation de premier passage comme l'ensemble des faces incidentes à un sommet x de Q_∞ tel que $d_{\text{fpp}}^{Q_\infty}(\rho, x) < r$.

Théorème 1.4.3

Il existe une constante $\mathbf{c} \in (0, \infty)$ dépendant uniquement de la loi des poids, telle que

$$B_{Q_\infty}((1 - \varepsilon)r/\mathbf{c}) \subset B_{Q_\infty}^{\text{fpp}}(r) \subset B_{Q_\infty}((1 + \varepsilon)r/\mathbf{c})$$

est satisfait avec probabilité tendant vers 1 quand $r \rightarrow \infty$.

Contrairement au cas de \mathbb{Z}^d , la forme limite est une boule pour la distance de graphe. La constante est en général non explicite ; son existence découle d'un théorème sous-additif, et sans information supplémentaire, elle reste hors d'atteinte.

Nous établissons également un tel résultat dans la carte uniforme infinie du plan (ou UIPM) M_∞ , qui est la limite locale des cartes uniformes quand leur nombre d'arêtes tend vers l'infini. Les boules de M_∞ sont définies comme dans l'UIPQ.

Théorème 1.4.4

Il existe une constante $\mathbf{c}' \in (0, \infty)$ dépendant uniquement de la loi des poids, telle que

$$B_{M_\infty}((1 - \varepsilon)r/\mathbf{c}') \subset B_{M_\infty}^{\text{fpp}}(r) \subset B_{M_\infty}((1 + \varepsilon)r/\mathbf{c}')$$

est satisfait avec probabilité tendant vers 1 quand $r \rightarrow \infty$.

Comportement en tant qu'espace métrique. Considérons une carte M . Comme la distance de percolation de premier passage est une vraie distance, $(V(M), d_{\text{fpp}}^M)$ est un espace métrique. Comment se compare-t-il à l'espace métrique usuel $(V(M), d_{\text{gr}}^M)$?

Je montre que l'ajout de poids sur les arêtes de quadrangulations ou de cartes générales ne modifie pas leur géométrie à grande échelle. Je me suis pour cela placé dans le cadre de [23], qui s'intéresse aux triangulations, avec lequel ce chapitre partage de nombreux points communs — mais également des différences significatives sur les outils utilisés. Mon premier résultat dans cette direction porte sur les modèles du plan complet.

Théorème 1.4.5

Notons ρ_{Q_∞} et ρ_{M_∞} les sommets racines de Q_∞ et M_∞ . Alors pour tout $\varepsilon > 0$, avec les mêmes constantes $\mathbf{c}, \mathbf{c}' \in (0, \infty)$ que dans le théorème 1.4.3, resp. le théorème 1.4.4,

$$\lim_{r \rightarrow \infty} \mathbb{P} \left(\sup_{x, y \in V(Q_\infty), d_{\text{gr}}^{Q_\infty}(\rho_{Q_\infty}, x) \vee d_{\text{gr}}^{Q_\infty}(\rho_{Q_\infty}, y) \leq r} \left| d_{\text{fpp}}^{Q_\infty}(x, y) - \mathbf{c} d_{\text{gr}}^{Q_\infty}(x, y) \right| > \varepsilon r \right) = 0,$$

et

$$\lim_{r \rightarrow \infty} \mathbb{P} \left(\sup_{x, y \in V(M_\infty), d_{\text{gr}}^{M_\infty}(\rho_{M_\infty}, x) \vee d_{\text{gr}}^{M_\infty}(\rho_{M_\infty}, y) \leq r} \left| d_{\text{fpp}}^{M_\infty}(x, y) - \mathbf{c}' d_{\text{gr}}^{M_\infty}(x, y) \right| > \varepsilon r \right) = 0.$$

En d'autres termes, la distance de percolation de premier passage entre toute paire de sommets de l'UIPQ (resp. de l'UIPM) à distance au plus r de la racine est proche de la distance de graphe entre ces deux points, multipliée par la constante du théorème 1.4.3 (resp. du théorème 1.4.4) : de plus, ceci est vrai uniformément pour toute telle paire de sommets.

Le théorème 1.4.5 constitue un renforcement conséquent des théorème 1.4.3 et théorème 1.4.4, puisqu'il permet de contrôler les distances de percolation de premier passage entre n'importe quelle paire de sommets d'une même boule de l'UIPQ, et non plus seulement les distances au sommet racine.

Nous établissons un résultat contrôlant l'écart maximal entre la distance de graphe et la distance de percolation de premier passage dans les quadrangulations et dans les cartes finies.

Théorème 1.4.6

Pour tout entier $n \geq 1$, soit Q_n une quadrangulation uniforme parmi les quadrangulations à n faces, et M_n une carte uniforme parmi les cartes à n arêtes. Alors avec les mêmes constantes $\mathbf{c}, \mathbf{c}' \in (0, \infty)$ que dans le théorème 1.4.3, resp. le théorème 1.4.4,

$$n^{-1/4} \sup_{x, y \in V(Q_n)} \left| d_{\text{fpp}}^{Q_n}(x, y) - \mathbf{c} d_{\text{gr}}^{Q_n}(x, y) \right| \xrightarrow[n \rightarrow \infty]{} 0$$

et

$$n^{-1/4} \sup_{x, y \in V(M_n)} \left| d_{\text{fpp}}^{M_n}(x, y) - \mathbf{c}' d_{\text{gr}}^{M_n}(x, y) \right| \xrightarrow[n \rightarrow \infty]{} 0$$

en probabilité.

Le fait que les mêmes constantes interviennent dans le théorème 1.4.3 (resp. le théorème 1.4.4), le théorème 1.4.5 et le théorème 1.4.6 n'est pas un hasard. Comme l'UIPQ représente, informellement parlant, le voisinage de l'origine dans une grande quadrangulation, le théorème 1.4.5 indique que $d_{\text{fpp}}^{Q_n}$ se comporte comme $\mathbf{c} \cdot d_{\text{gr}}^{Q_n}$ quand n est grand, au moins dans un voisinage du sommet racine, ce qu'on retrouve dans le théorème 1.4.6.

Conséquence du théorème 1.4.6, l'espace métrique $(V(Q_n), \mathbf{c}^{-1} \left(\frac{9}{8n}\right)^{1/4} d_{\text{fpp}}^{Q_n})$ converge également vers la carte Brownienne. Plus fort encore, il converge p.s. vers la même carte Brownienne que l'espace métrique usuel :

$$\left(\left(V(Q_n), \left(\frac{9}{8n}\right)^{1/4} d_{\text{gr}}^{Q_n} \right), \left(V(Q_n), \left(\frac{9}{8n}\right)^{1/4} d_{\text{fpp}}^{Q_n} \right) \right) \xrightarrow[n \rightarrow \infty]{} ((m_\infty, D^*), (m_\infty, \mathbf{c}D^*)) \quad (1.9)$$

en loi, pour la topologie de Gromov-Hausdorff. (1.9) reste valable si on remplace Q_n par M_n et \mathbf{c} par \mathbf{c}' .

1.4.2 Bijections entre modèles de cartes

On a vu que les espaces métriques $\left(V(Q_n), \left(\frac{9}{8n}\right)^{1/4} d_{\text{gr}}^{Q_n} \right)$ et $\left(V(Q_n), \frac{1}{\mathbf{c}} \left(\frac{9}{8n}\right)^{1/4} d_{\text{fpp}}^{Q_n} \right)$ étaient asymptotiquement isométriques quand $n \rightarrow \infty$. Existe-t-il des liens de ce genre entre espaces métriques issus de différents modèles de cartes ? Qu'implique une bijection entre différents modèles de cartes ? [23] étudie par exemple le passage au dual dans les triangulations, et [9] la bijection d'Ambjorn-Budd, qui permet aux auteurs d'établir la convergence des cartes générales vers la carte Brownienne. Je présente ici mes résultats sur la bijection de Tutte entre cartes générales et quadrangulations.

Théorème 1.4.7

Soit Q_n une quadrangulation uniforme parmi les quadrangulations à n faces, et soit M_n l'image de Q_n par la bijection de Tutte. M_n est uniformément distribuée parmi les cartes à n arêtes, $V(M_n)$ est identifié à un sous-ensemble de $V(Q_n)$, et

$$n^{-1/4} \sup_{x,y \in V(M_n)} \left| d_{\text{gr}}^{Q_n}(x,y) - d_{\text{gr}}^{M_n}(x,y) \right| \xrightarrow[n \rightarrow \infty]{} 0$$

en probabilité.

Le théorème 1.4.7 signifie que la bijection de Tutte est asymptotiquement une isométrie. Plus fort encore, quadrangulations et cartes générales reliées par la bijection de Tutte, munies de leurs distances de graphes respectives convenablement renormalisées, convergent vers la même carte Brownienne : sous l'hypothèse 1.4.2,

$$\left(\left(V(Q_n), \left(\frac{9}{8n}\right)^{1/4} d_{\text{gr}}^{Q_n} \right), \left(V(M_n), \left(\frac{9}{8n}\right)^{1/4} d_{\text{gr}}^{M_n} \right) \right) \xrightarrow[n \rightarrow \infty]{} ((m_\infty, D^*), (m_\infty, D^*))$$

en loi, pour la topologie de Gromov-Hausdorff, avec (m_∞, D^*) la carte Brownienne. Le fait que la constante de renormalisation soit la même pour les deux modèles est remarquable, et découle des résultats de [9].

Le théorème 1.4.7, avec le théorème 1.4.6, permet de comparer les espaces métriques dérivés des cartes générales et des quadrangulations, munies de distances de percolation de premier passage. Il est également possible de comparer les distances dans une carte générale uniforme et dans son dual, avec une application judicieuse du théorème 1.4.7 : le passage au dual dans les cartes générales se traduit par une simple inversion de l'orientation de l'arête racine, qui ne modifie pas la structure métrique, sur la quadrangulation associée par la bijection de Tutte.

Enfin, le théorème suivant est un résultat analogue au théorème 1.4.5, mais comparant les distances de graphe dans les cartes infinies du plan, lorsqu'elles sont reliées par la bijection de Tutte.

Théorème 1.4.8

Supposons Q_∞ et M_∞ reliées par la bijection de Tutte généralisée aux cartes infinies, de sorte que l'ensemble des sommets de M_∞ est identifié avec un sous-ensemble des sommets de Q_∞ . Notons ρ_{Q_∞} et ρ_{M_∞} les sommets racines de Q_∞ et M_∞ . Alors pour tout $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} \mathbb{P} \left(\sup_{x, y \in V(M_\infty), d_{\text{gr}}^{M_\infty}(\rho_{M_\infty}, x) \vee d_{\text{gr}}^{M_\infty}(\rho_{M_\infty}, y) \leq r} \left| d_{\text{gr}}^{M_\infty}(x, y) - d_{\text{gr}}^{Q_\infty}(x, y) \right| > \varepsilon r \right) = 0.$$

1.5 Isopérimétrie et temps de mélange dans les quadrangulations finies (chapitre 4)

Temps de mélange sur les cartes finies. La marche aléatoire fainéante sur un graphe fini $G = (V, E)$ est une chaîne de Markov sur l'ensemble des sommets du graphe dont les probabilités de transition sont définies comme suit. Si, à un instant donné, elle se trouve en un sommet v , avec probabilité 1/2 elle reste en v , et avec probabilité 1/2 elle emprunte une arête de G choisie uniformément parmi les arêtes issues de v . C'est une chaîne de Markov irréductible et apériodique, qui possède donc une unique mesure de probabilité invariante π_G . On note $p_G^t(v, w)$ la probabilité que la marche aléatoire fainéante issue de v se trouve en w après une durée t .

Pour tout $\varepsilon > 0$, le temps de mélange ε -uniforme $\tau_G(\varepsilon)$ de la marche aléatoire fainéante est la durée après laquelle la loi du sommet auquel se trouve la marche aléatoire fainéante est proche de la mesure stationnaire, quel que soit le sommet de départ. Plus formellement,

il est défini par

$$\tau_G(\varepsilon) \stackrel{\text{déf}}{=} \inf \left\{ t : \forall v, w \in V, \left| \frac{p_G^t(v, w) - \pi_G(w)}{\pi_G(w)} \right| \leq \varepsilon \right\}.$$

Soit Q_n une quadrangulation uniforme à n faces, et Q_n^\dagger son dual.

Théorème 1.5.1

Pour tout $\varepsilon, \delta > 0$, avec probabilité qui tend vers 1 quand $n \rightarrow \infty$,

$$\tau_{Q_n}(\varepsilon) \leq n^{3/2+\delta},$$

$$\tau_{Q_n^\dagger}(\varepsilon) \leq n^{3/2+\delta}.$$

Pour compléter le théorème 1.5.1 et connaître l'asymptotique du temps de mélange, il faudrait disposer d'une borne inférieure qui corresponde. La meilleure borne inférieure que je peux établir rigoureusement, à partir des résultats du chapitre 4, est en $n^{3/4+o(1)}$. Toutefois, en supposant que les travaux récents [31], établis pour les triangulations, soient également valables dans les quadrangulations, on arrive à une borne inférieure en $n^{1+o(1)}$. A l'heure où je rédige cette introduction, je n'ai pas de conjecture sur la bonne valeur de l'asymptotique, si elle existe ; des simulations numériques devraient aider à formuler une conjecture informée.

Transformée de Laplace du volume des enveloppes tronquées de l'UIPQ. Rappelons ce que sont les enveloppes tronquées de l'UIPQ. Fixons un entier $r \geq 1$. Ajoutons à chaque sommet de l'UIPQ une étiquette portant la distance du sommet au sommet racine. Dans chaque face de l'UIPQ dont les sommets ont pour étiquettes $(r, r+1, r, r-1)$, ajoutons une arête entre les deux coins de cette face qui portent l'étiquette r . Les arêtes ainsi ajoutées sont appelées *r-diagonales*. Il existe un cycle simple $\partial_r Q_\infty$ formé de *r-diagonales*, tel que la région finie du plan qu'il délimite contient toutes les *r-diagonales* et tous les sommets à distance au plus $r-1$ du sommet racine. L'enveloppe tronquée de rayon r de l'UIPQ, notée $\mathcal{H}_{Q_\infty}^{\text{tr}}(r)$, est l'intérieur de $\partial_r Q_\infty$.

Je fournis une expression explicite pour la transformée de Laplace du volume des enveloppes tronquées :

Théorème 1.5.2

Pour tout entier $r \geq 1$, pour tout $p \in (0, 1)$,

$$\mathbb{E} \left[(1 - p^2)^{|\mathcal{H}_{Q_\infty}^{\text{tr}}(r)|} \right] = (1 - p)^{-1} (\mathcal{K} \circ \psi_r)'(0), \quad (1.10)$$

avec

$$\mathcal{K}(t) = \frac{3}{4} \sqrt{\frac{8+t}{t}},$$

$$\psi_r(u) = p + \frac{6p}{-1 + \sqrt{\frac{2(1-p)}{p+2}} \cosh \left(\cosh^{-1} \left(\sqrt{\frac{p+2}{2(1-p)}} \left[1 + \frac{6p}{(1-u)(1-p)} \right] \right) + r \cdot \cosh^{-1} \left(\frac{2p+1}{1-p} \right)}.$$

On en déduit l'espérance du volume de l'enveloppe tronquée de rayon r :

$$\mathbb{E} \left[|\mathcal{H}_{Q_\infty}^{\text{tr}}(r)| \right] = \frac{r(r+3)(6r^4 + 36r^3 + 87r^2 + 99r + 44)}{4(2r+3)^2} \underset{r \rightarrow \infty}{\sim} \frac{3}{8} r^4. \quad (1.11)$$

Seule l'asymptotique, valable aussi pour l'enveloppe standard, était connue auparavant (voir la section 1.2.4, et (1.5) pour le lien entre enveloppe tronquée et enveloppe standard).

La preuve reprend des idées de [53], où Ménard établit un analogue du théorème 1.5.2 pour la triangulation uniforme infinie du plan. J'utilise fortement la décomposition en squelette de l'UIPQ.

Inégalités isopérimétriques dans les quadrangulations finies. Soit Q_n une quadrangulation uniforme à n faces. Pour tout ensemble S de faces de Q_n , la frontière ∂S de l'ensemble S est l'ensemble des arêtes de Q_n qui ont un côté dans S et l'autre côté hors de S . Le théorème suivant établit qu'avec grande probabilité, le nombre d'arêtes de la frontière de n'importe quel ensemble de faces de Q_n est minoré par une expression qui dépend uniquement du nombre de faces de S et de n .

Théorème 1.5.3

Pour tout $\nu \in (0, 1)$:

$$\inf_{S \subset F(Q_n) : 0 < |S| \leq n/2} n^{2/3+\nu} \frac{|\partial S|^{4/3}}{|S|} \xrightarrow{n \rightarrow \infty} \infty$$

en probabilité.

Remarquons que, contrairement aux résultats du chapitre 2, il n'y a plus aucune contrainte sur les ensembles considérés : ni topologique (on n'impose plus qu'ils soient simplement connexes), ni d'ancrage (ils ne contiennent plus nécessairement la racine). En contrepartie, l'inégalité du théorème 1.5.3 apparaît moins forte que celle du théorème 1.3.3.

On peut se demander si cette borne est-elle optimale. Est-il vrai que pour tout $0 \leq k < k' \leq n/2$, pour tout $\nu \in (0, 1)$:

$$\inf_{S \subset F(Q_n) : k < |S| \leq k'} n^{2/3-\nu} \frac{|\partial S|^{4/3}}{|S|} \xrightarrow{n \rightarrow \infty} 0$$

en probabilité? Autrement dit, peut-on, avec grande probabilité, pour toute taille $0 < k < k' < n/2$, trouver des ensembles de faces de Q_n avec entre k et k' faces et proches de réaliser la borne du théorème 1.5.3? Cette affirmation est vérifiée pour $k = \delta n$ et $k' = n/2$; je conjecture qu'elle est aussi vraie pour tout choix de k, k' .

De manière analogue, pour tout ensemble A de sommets de Q_n , la frontière $E(A, A^c)$ est l'ensemble des arêtes de Q_n ayant une extrémité dans A et l'autre extrémité au-dehors de A . Le théorème suivant donne une autre inégalité isopérimétrique, portant cette fois sur les ensembles de sommets de Q_n , et pouvant être vue comme la version duale du théorème 1.5.3. On rappelle que π_{Q_n} est la mesure stationnaire de la marche aléatoire fainéante sur Q_n .

Théorème 1.5.4

Pour tout $\nu \in (0, 1)$:

$$\inf_{A \subset V(Q_n) : \pi_{Q_n}(A) \leq 1/2} n^{2/3+\nu} \frac{\#E(A, A^c)^{4/3}}{4n\pi_{Q_n}(A)} \xrightarrow{n \rightarrow \infty} \infty$$

en probabilité.

Chapitre 2

Separating cycles and isoperimetric inequalities in the uniform infinite planar quadrangulation

This chapter is taken from the article [43], written in collaboration with Jean-François Le Gall.

We study geometric properties of the infinite random lattice called the uniform infinite planar quadrangulation or UIPQ. We establish a precise form of a conjecture of Krikun stating that the minimal size of a cycle that separates the ball of radius R centered at the root vertex from infinity grows linearly in R . As a consequence, we derive certain isoperimetric bounds showing that the boundary size of any simply connected set A consisting of a finite union of faces of the UIPQ and containing the root vertex is bounded below by a (random) constant times $|A|^{1/4} (\ln |A|)^{-(3/4)-\delta}$, where the volume $|A|$ is the number of faces in A .

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2.1 Introduction

In the recent years, much work has been devoted to discrete and continuous models of random geometry in two dimensions. Two of the most popular discrete models are the uniform infinite planar triangulation (or UIPT), which was introduced by Angel and Schramm [3, 5] and in fact motivated much of the subsequent work, and the uniform infinite planar quadrangulation (or UIPQ). In the present work, we concentrate on the UIPQ, although we believe that our methods can be adapted to give similar results for the UIPT. Roughly speaking, the UIPQ is a random infinite graph embedded in the plane, such that all faces (connected components of the complement of edges) are quadrangles, possibly with two edges glued together. See Figure 2.4 below for an illustration of what the UIPQ may look like near its root vertex. We study certain geometric properties of the UIPQ, concerning the existence of “small” cycles that separate a large ball centered at the root vertex from infinity, with applications to isoperimetric inequalities.

The starting point of our work is a conjecture of Krikun in the paper [38] which provided the first construction of the UIPQ as the local limit of uniform planar quadrangulations with a fixed number of faces (another construction was suggested by Chassaing and Durhuus [15], and the equivalence between the two approaches was established by Ménard [52] — see also [25] for a third construction). Denote the UIPQ by \mathcal{P} , and, for every integer $r \geq 1$, let $B_{\mathcal{P}}(r)$ stand for the ball of radius r centered at the root vertex, which is defined as the union of all faces that are incident to at least one vertex whose graph distance from the root is at most $r - 1$. The complement of the ball $B_{\mathcal{P}}(r)$ is in general not connected, but there is a unique unbounded component, whose boundary is called the exterior boundary of the ball. The set inside the exterior boundary, which may be obtained by filling in the “bounded holes” of the ball, is called the (standard) hull of radius r and will be denoted by $B_{\mathcal{P}}^{\bullet}(r)$. It is known that the size of the exterior boundary, that is, the number of edges in this boundary, grows like r^2 when $r \rightarrow \infty$: See [22] for more precise asymptotics obtained both for the UIPT and the UIPQ. On the other hand, Krikun constructed a cycle that separates the ball $B_{\mathcal{P}}(r)$ from infinity and whose size grows linearly in r when r is large. Here we say that a cycle \mathcal{C} made of edges of the UIPQ separates a finite set A of vertices from infinity if A does not intersect \mathcal{C} but any path from a vertex of A to infinity intersects \mathcal{C} (see Fig. 2.1 for a schematic illustration). Krikun conjectured that the cycle he constructed is essentially the shortest possible, meaning that the minimal size of a cycle that separates the ball $B_{\mathcal{P}}(r)$ from infinity must be linear in r . A weak form of this conjecture was derived in [20], but the

results of this paper did not exclude the possibility that a ball could be separated from infinity by a small cycle lying “very far away” from the ball.

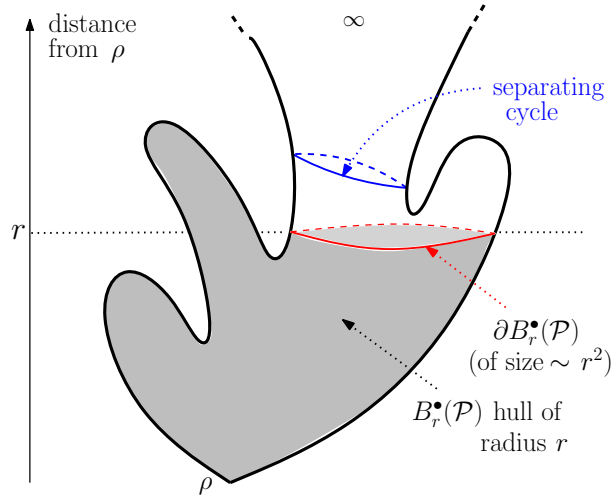


Figure 2.1 – A schematic “cactus” representation of the UIPQ. The root vertex is denoted by ρ and the vertical coordinate corresponds to the graph distance from ρ . The shaded part is the hull $B_r^*(\mathcal{P})$.

The following theorem provides quantitative estimates that confirm Krikun’s conjecture.

Theorem 2.1. *For every integer $R \geq 1$, let $L(R)$ be the smallest length of a cycle separating $B_{\mathcal{P}}(R)$ from infinity.*

- (i) *For every $\delta < 2$, there exists a constant C_δ such that, for every $R \geq 1$, for every $\varepsilon \in (0, 1)$,*

$$\mathbb{P}(L(R) \leq \varepsilon R) \leq C_\delta \varepsilon^\delta.$$

- (ii) *There exist constants C and $\lambda > 0$ such that, for every $a > 0$ and $R \geq 1$,*

$$\mathbb{P}(L(R) \geq aR) \leq C e^{-\lambda a}.$$

Part (ii) of the theorem is proved by using the separating cycle introduced by Krikun and sharpening the estimates in [38]. So the most interesting part of the theorem is part (i). We believe that our condition $\delta < 2$ is close to optimal, in the sense that, for R large, $\mathbb{P}(L(R) \leq \varepsilon R)$ should behave like ε^2 , possibly up to logarithmic corrections. At the end of Section 2.4, we provide a short argument showing that the probability $\mathbb{P}(L(R) \leq \varepsilon R)$ is bounded below by $\text{const} \cdot \varepsilon^3$ when R is large.

The proof of part (i) relies on a technical estimate which is of independent interest and that we now present. We first label vertices of the UIPQ by their distances from the root vertex, and for every integer $r \geq 1$, we say that a face of the UIPQ is r -simple if the labels

of the vertices incident to this face take the three values $r - 1, r, r + 1$ (note that there are faces such that labels of incident vertices take only two values, these faces are called confluent in [16]). In each r -simple face, we draw a “diagonal” connecting the two corners labeled r (these two corners may correspond to the same vertex), and such diagonals, which are **not** edges of the UIPQ, are called r -diagonals. Then, there is a “maximal” cycle made of r -diagonals, which is simple and such that the labels of vertices lying in the unbounded component of the complement of this cycle are at least $r + 1$. We denote this maximal cycle by \mathcal{C}_r , and, for $1 \leq r < r'$, we define the annulus $\mathcal{C}(r, r')$ as the part of the UIPQ between the cycles \mathcal{C}_r and $\mathcal{C}_{r'}$. See Section 2.2 below for more precise definitions. Note that the cycles \mathcal{C}_r are **not** made of edges of the UIPQ in contrast with the separating cycles that we consider in Theorem 2.1 and in the next proposition.

Proposition 2.2. *Let $\beta \in (0, 3)$. There exists a constant C'_β such that, for every integer $r \geq 1$ and for every integer $n \geq 1$, the probability that there exists a cycle of the UIPQ of length smaller than r , which is contained in $\mathcal{C}(nr, (n + 2)r)$, does not intersect $\mathcal{C}_{(n+2)r}$, and disconnects the root vertex from infinity, is bounded above by $C'_\beta n^{-\beta}$.*

The condition that the cycle does not intersect $\mathcal{C}_{(n+2)r}$ is included for technical convenience, and could be removed from the statement.

The proof of Proposition 2.2 relies on a “skeleton decomposition” of the UIPQ, which is already presented in the work of Krikun [38]. Our presentation is however different from the one in [38] and better suited to our purposes. We introduce and use the notion of a truncated quadrangulation, which is basically a planar map with a boundary, where all faces (distinct from the distinguished one) are quadrangles, except for those incident to the boundary, which are triangles (see Section 2.2.1 for precise definitions). The annulus $\mathcal{C}(r, r')$ can be viewed as a truncated quadrangulation of the cylinder of height $r' - r$. Our motivation for introducing truncated quadrangulations comes from the fact that they allow certain explicit calculations in the UIPQ. For every integer $r \geq 1$, we define the “truncated hull” of radius r of the UIPQ, which is basically the part of the UIPQ inside the maximal cycle \mathcal{C}_r (see Section 2.2.2 for a precise definition). This truncated hull is different from the standard hull $B_{\mathfrak{F}}^\bullet(r)$ introduced above, which had been considered in [21, 22] in particular, but it is essentially the same object as the hull defined in [38]. It turns out that it is possible to compute the law of the truncated hull in a rather explicit manner (Corollary 2.8) and in particular the law of the perimeter of the hull has a very simple form (Proposition 2.11). These calculations make heavy use of the skeleton decomposition of the UIPQ, and more generally of the similar decomposition for truncated quadrangulations of the cylinder. This decomposition involves a forest structure, which was already described by Krikun [38, Section 3.2] and is similar to the one for triangulations that was discovered in [37] and heavily used in the recent work [23] dealing with first-passage percolation on the UIPT.

Given the forest structure associated with a truncated quadrangulation of the cylinder, the idea of the proof of Proposition 2.2 is as follows. One first observes that, with high probability, there exist, for some $\delta > 0$, more than n^δ trees with maximal height in the forest coding the annulus $\mathcal{C}(nr, (n + 2)r)$. For each of these trees, one can find a vertex

on the cycle \mathcal{C}_{nr} (the interior boundary of the annulus) which is connected to the exterior boundary $\mathcal{C}_{(n+2)r}$ by a path of length $2r$. Assuming that there is a cycle of length r in the annulus that disconnects the root vertex from infinity, it follows that any two of these particular vertices of \mathcal{C}_{nr} can be connected by a path staying in the annulus with length at most $5r$. Results of Curien and Miermont [26] about the graph distances between boundary points in infinite quadrangulations with a boundary, show that this cannot occur except on a set of small probability.

Our lower bounds on the minimal size of separating cycles lead to interesting isoperimetric inequalities showing informally that the size of the boundary of a simply connected set which is a finite union of faces and contains the root vertex must be at least of the order of the volume raised to the power $1/4$. The fact that we cannot do better than the power $1/4$ follows from part (ii) in Theorem 2.1, since it is well known [15, 21, 22] that the volume of the ball, or of the standard hull, of radius r is of order r^4 . We refer to [49, Chapter 6] for a thorough discussion of isoperimetric inequalities on infinite graphs.

Let \mathcal{K} denote the collection of all simply connected compact subsets of the plane that are finite unions of faces of the UIPQ (including their boundaries) and contain the root vertex. For $A \in \mathcal{K}$, the volume of A , denoted by $|A|$, is the number of faces of the UIPQ contained in A , and the boundary size of A , denoted by $|\partial A|$, is the number of edges in the boundary of A .

Theorem 2.3. *Let $\delta > 0$. Then,*

$$\inf_{A \in \mathcal{K}} \frac{|\partial A|}{|A|^{\frac{1}{4}} (\ln |A|)^{-\frac{3}{4}-\delta}} > 0, \text{ a.s.}$$

The exponent $\frac{3}{4}$ in the statement of the theorem is presumably not the optimal one. Our method involves estimates for the tail of the distribution of the volume of the hull $B_{\varnothing}^{\bullet}(r)$, which are derived from a first moment bound (Proposition 2.15). We expect that these estimates can be improved, leading to a better value of the exponent of $\ln |A|$ (the results of Riera [58] for the Brownian plane suggest that one should be able to replace $\frac{3}{4}$ by $\frac{1}{2}$ in the statement of the theorem). On the other hand, one cannot hope to replace $|A|^{\frac{1}{4}} (\ln |A|)^{-\frac{3}{4}-\delta}$ by $|A|^{\frac{1}{4}}$ in the theorem: Simple zero-one arguments using the separating cycles introduced by Krikun [38] (see Section 2.2.4 below) show that there exist sets A such that the ratio $|\partial A| / |A|^{\frac{1}{4}}$ is arbitrarily small.

Still we can state the following proposition.

Proposition 2.4. *Let $\varepsilon > 0$. There exists a constant $c_{\varepsilon} > 0$ such that, for every integer $n \geq 1$, the property*

$$|\partial A| \geq c_{\varepsilon} n^{1/4}, \text{ for every } A \in \mathcal{K} \text{ such that } |A| \geq n,$$

holds with probability at least $1 - \varepsilon$.

As an immediate consequence of Proposition 2.4, we also get that, for every $\varepsilon > 0$ and every $M > 1$, we can find a constant $c_{\varepsilon, M} > 0$ such that, for every integer $n \geq 1$,

$$\mathbb{P} \left(\inf_{A \in \mathcal{K}, n \leq |A| \leq Mn} \frac{|\partial A|}{|A|^{\frac{1}{4}}} \geq c_{\varepsilon, M} \right) \geq 1 - \varepsilon.$$

Indeed, we just have to take $c_{\varepsilon, M} = c_{\varepsilon}/M^{1/4}$, with the notation of Proposition 2.4. But, as explained after the statement of Theorem 2.3, we cannot lift the constraint $n \leq |A| \leq Mn$ in the last display.

The proofs of both Theorem 2.3 and Proposition 2.4 rely on Theorem 2.1 and on the fact that the volume of the hull of radius r is of order r^4 . Assuming that $|\partial A|$ is small, then either the root vertex is sufficiently far from ∂A , which implies that a large ball centered at the root vertex is disconnected from infinity by the small cycle ∂A (so that we can use the estimate of Theorem 2.1) or the root vertex is close to ∂A , but then it follows that the whole set A is contained in the standard hull of radius (approximately) equal to the distance from the root vertex to ∂A , which implies that the volume of A cannot be too big (at this point of the argument, in the proof of Theorem 2.3, we need estimates for the tail of the distribution of the volume of hulls).

The paper is organized as follows. Section 2.2 presents a number of preliminaries, concerning truncated quadrangulations, their relations with the UIPQ and their skeleton decompositions, and a number of related calculations. As mentioned earlier, this section owes a lot to the work of Krikun [38], and in particular we make use of enumeration results derived in [38]. One additional motivation for deriving the results of Section 2.2 in a somewhat more precise form than in [38] is the fact that we plan to use these results in a forthcoming work [46] on local modifications of distances in the UIPQ, in the spirit of [23]. Proposition 2.2 is proved in Section 2.3, and part (i) of Theorem 2.1 easily follows from this proposition. Section 2.4 is devoted to the proof of part (ii) of Theorem 2.1. This proof relies on the explicit calculation of the distribution of the number of trees with maximal height in the forest coding the annulus $\mathcal{C}(r, r')$ (Proposition 2.14). This calculation is also used to give an easy lower bound for the probability $\mathbb{P}(L(R) \leq \varepsilon R)$. Section 2.5 contains the proof of Proposition 2.4 and Theorem 2.3. An important ingredient of the proof of Theorem 2.3 is Proposition 2.15, which provides a first moment bound for the volume of hulls. Finally, the Appendix gives the proof of a technical lemma stated at the end of Section 2.2, which plays an important role in Section 2.3.

2.2 Preliminaries

2.2.1 Truncated quadrangulations

We will consider truncated quadrangulations. Informally, these are quadrangulations with a simple boundary, where the quadrangles incident to the boundary are replaced by triangles. A more precise definition is as follows.

Definition 2.1. Let $p \geq 1$ be an integer. A truncated quadrangulation with boundary size p is a planar map \mathcal{M} having a distinguished face \mathfrak{f} with a simple boundary of size p such that:

- Each edge of the boundary of \mathfrak{f} is incident both to \mathfrak{f} and to a triangular face of \mathcal{M} and these triangular faces are distinct.
- All faces other than \mathfrak{f} and the triangular faces incident to the boundary of \mathfrak{f} have degree 4.

It will be convenient to view truncated quadrangulations as drawn in the plane in such a way that the distinguished face is the unbounded face. With this convention, we will always assume that a truncated quadrangulation is rooted and, unless otherwise specified, that the root edge lies on the boundary of the distinguished face and is oriented clockwise. See Figure 2.2 for an example. Faces distinct from the distinguished face are called inner faces, and vertices that do not lie on the boundary of the distinguished face are called inner vertices.

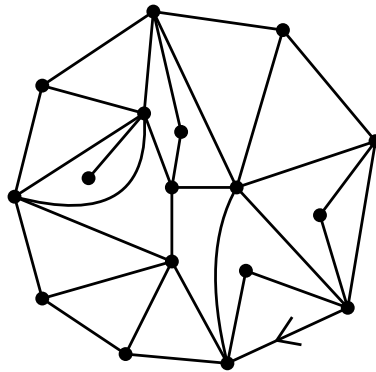


Figure 2.2 – A truncated quadrangulation with boundary size 9, 8 inner vertices and 16 inner faces.

Notice that, when $p \geq 2$, any of the triangular faces incident to the boundary of \mathfrak{f} must be nondegenerate (i.e. its boundary cannot contain a loop). Furthermore, a simple argument shows that each of these triangular faces is incident to an inner vertex. The last property clearly also holds if $p = 1$. Hence a truncated quadrangulation with boundary size $p \geq 1$ must have at least one inner vertex.

We notice that our truncated quadrangulations with boundary size p are in one-to-one correspondence with the “quadrangulations with a simple boundary” of size $2p$ considered by Krikun [38] (starting from the latter, we just “cut” the boundary quadrangles along the appropriate diagonals to get a truncated quadrangulation). If we add an extra vertex v_* inside the face \mathfrak{f} , then draw an edge from each vertex of the boundary of \mathfrak{f} to v_* , and finally remove all edges of the boundary of \mathfrak{f} , we get a plane quadrangulation and hence a bipartite graph: In particular, it follows that, if v and v' are two adjacent inner vertices of \mathcal{M} , their distances from the boundary differ by 1. This observation will be useful later.

For integers $n \geq 1$ and $p \geq 1$, we let $\mathbb{Q}_{n,p}^{\text{tr}}$ be the set of all (rooted) truncated quadrangulations with boundary size p and n inner faces.

We need another definition.

Definition 2.2. *Let $h, p, q \geq 1$ be positive integers. A truncated quadrangulation of the cylinder of height h with boundary sizes (p, q) is a planar map \mathcal{Q} having two distinguished faces f_b and f_t such that:*

- *The face f_b (called the bottom face) has a simple boundary of size p , which is called the bottom cycle, and the face f_t (called the top face) has a simple boundary of size q , which is called the top cycle.*
- *Each edge of the bottom cycle (resp. of the top cycle) is incident both to f_b (resp. to f_t) and to a triangular face of \mathcal{Q} and these triangular faces are distinct.*
- *All faces other than f_b, f_t , and the triangular faces incident to the bottom and top cycles, have degree 4.*
- *Every vertex of the top cycle is at graph distance exactly h from the bottom cycle, and every edge of the top cycle is incident to a triangular face containing a vertex at graph distance $h - 1$ from the bottom cycle.*

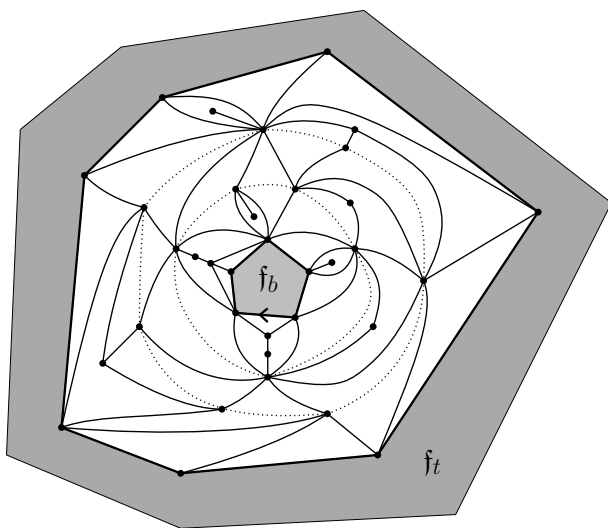


Figure 2.3 – A truncated quadrangulation \mathcal{Q} of the cylinder of height 3 with boundary sizes $(5, 7)$. The two dotted cycles represent $\partial_1\mathcal{Q}$ and $\partial_2\mathcal{Q}$ respectively (see Section 2.2.3 below for the definition of $\partial_k\mathcal{Q}$).

By definition, the inner faces of \mathcal{Q} are all faces except the two distinguished ones. The last assertion of Definition 2.2 shows that the top face and the bottom face do not play a symmetric role. We will implicitly assume that truncated quadrangulations of the cylinder of height h are drawn in the plane so that the top face is the unbounded face,

and that they are rooted in such a way that the root edge lies on the bottom cycle and is oriented clockwise. See Figure 2.3 for an example.

In a way similar to the truncated quadrangulations of Definition 2.1, the triangular face associated with an edge of the bottom cycle must contain a vertex which does not belong to this cycle. The same holds for the top cycle — this is obvious from the last assertion of Definition 2.2.

2.2.2 Truncated quadrangulations in the UIPQ

Let us now explain why the definitions of the previous section are relevant to our study of the UIPQ. We label vertices of the UIPQ by their graph distance from the root vertex. Then the labels of corners incident to a face (enumerated in cyclic order along the boundary of the face) are of the type $k, k-1, k, k-1$ or $k, k+1, k, k-1$ for some integer $k \geq 1$, and the face is called k -simple in the second case. Fix an integer $r \geq 1$. For every r -simple face, we draw a diagonal between the two corners labeled r in this face, and these diagonals are called r -diagonals. If v is a vertex incident to an r -diagonal (equivalently, if v has label r and is incident to an r -simple face), then a simple combinatorial argument shows that the number of r -diagonals incident to v is even — to be precise, we need to count this number with multiplicities, since r -diagonals may be loops. It follows that the collection of all r -diagonals can be obtained as the union of a collection of disjoint simple cycles (disjoint here means that no edge is shared by two of these cycles). See Figure 2.4 for an example.

Lemma 2.5. *There is a unique simple cycle made of r -diagonals such that the unbounded component of the complement of this cycle contains no r -diagonal and no vertex at distance less than or equal to r from the root vertex. This cycle will be called the maximal cycle made of r -diagonals and will be denoted by \mathcal{C}_r .*

Proof. It suffices to verify that the root vertex lies inside a bounded component of the complement of some cycle made of r -diagonals (this cycle may be taken to be simple and then satisfies the properties stated in the lemma). To this end, consider a geodesic γ from the root vertex to infinity and write v_r , resp. v_{r-1}, v_{r+1} , for the unique vertex of γ at distance r , resp. $r-1, r+1$, from the root vertex. Also write $v_r v_{r-1}$, resp. $v_r v_{r+1}$, for the edge of γ incident to v_r and v_{r-1} , resp. to v_r and v_{r+1} . Let k_1 , resp. k_2 , denote the number of r -diagonals incident to v_r that lie between $v_r v_{r+1}$ and $v_r v_{r-1}$, resp. between $v_r v_{r-1}$ and $v_r v_{r+1}$, when turning around v_r in clockwise order (self-loops are counted twice). An easy combinatorial argument shows that both k_1 and k_2 are odd. It follows that there must exist a cycle made of r -diagonals that starts with an edge lying between $v_r v_{r+1}$ and $v_r v_{r-1}$ (in clockwise order) and ends with an edge lying between $v_r v_{r-1}$ and $v_r v_{r+1}$. Simple topological considerations now show that the root vertex, and in fact the whole geodesic path γ up to vertex v_{r-1} must lie in a bounded component of the complement of this cycle. \square

If we now add all edges of \mathcal{C}_r to the UIPQ and then remove all edges that lie in the

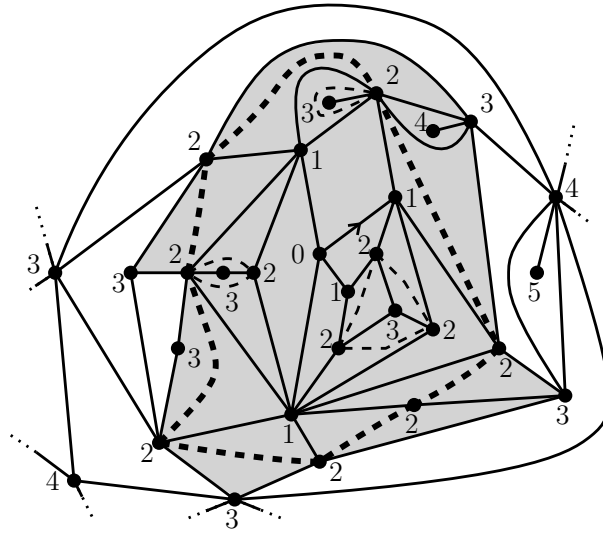


Figure 2.4 – The UIPQ near the root vertex. Figures correspond to graph distances from the root vertex. The dashed lines show the cycles made of r -diagonals, for $r = 2$, and the cycle in thick dashed lines is the maximal one. The shaded part is the standard hull of radius 2. Note that the standard hull contains the truncated hull of the same radius, which is the part delimited by the maximal cycle.

unbounded component of the complement of \mathcal{C}_r , we get a truncated quadrangulation in the sense of Definition 2.1 (with the minor difference that, assuming that we keep the same root as in the UIPQ, the root edge does not belong to the boundary of the distinguished face). This truncated quadrangulation is called the *truncated hull* of radius r and is denoted by $\mathcal{H}_r^{\text{tr}}$. Its boundary size (the length of \mathcal{C}_r) is called the perimeter of the hull and denoted by H_r . Notice that, by construction, any vertex belonging to the boundary of the distinguished face is at distance exactly r from the root vertex. Furthermore, for any vertex v of the UIPQ that does not belong to $\mathcal{H}_r^{\text{tr}}$ (equivalently, that lies in the unbounded component of the complement of \mathcal{C}_r) there exists a path going from v to infinity that visits only vertices with label at least r . This property follows from the fact that any two points of \mathcal{C}_r are connected by a path that visits only vertices with label at least r .

We may and will sometimes view the truncated hull $\mathcal{H}_r^{\text{tr}}$ as a quadrangulation of the cylinder: To this end, we just split the root edge into a double edge, and insert a loop (based on the root vertex) inside the resulting 2-gon. This yields a truncated quadrangulation of the cylinder of height r with boundary sizes $(1, H_r)$, whose top cycle is \mathcal{C}_r . The root edge is the inserted loop as required in our conventions. See Figure 2.5 for an illustration.

Similarly, if $1 \leq r < r'$, we can consider the part of the UIPQ that lies between the cycles \mathcal{C}_r and $\mathcal{C}_{r'}$. More precisely, we add all edges of \mathcal{C}_r and $\mathcal{C}_{r'}$ to the UIPQ and then remove all edges that lie either inside the cycle \mathcal{C}_r or outside the cycle $\mathcal{C}_{r'}$. This gives rise to a quadrangulation of the cylinder of height $r' - r$ whose bottom cycle and top cycle

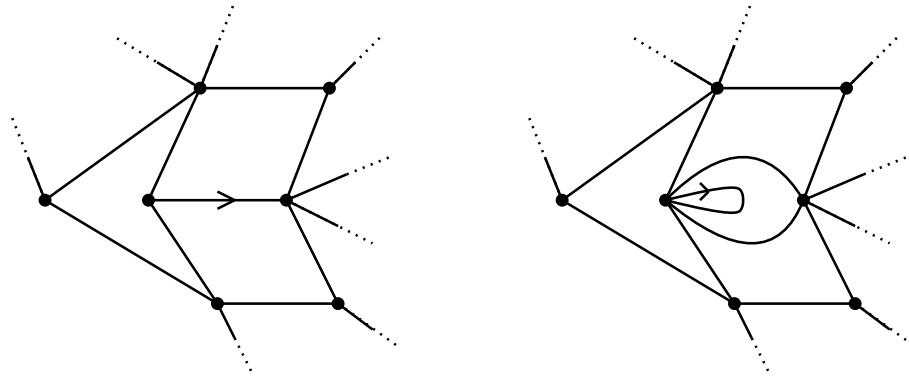


Figure 2.5 – Viewing the hull $\mathcal{H}_r^{\text{tr}}$ as a quadrangulation of the cylinder: The root edge is split in a double edge, and a loop is inserted inside the resulting 2-gon.

are \mathcal{C}_r and $\mathcal{C}_{r'}$ respectively (we in fact need to specify the root edge on the bottom cycle, but we will come back to this later). By definition, this is the *annulus* $\mathcal{C}(r, r')$. We can extend this definition to $r = 0$: The annulus $\mathcal{C}(0, r')$ is just the truncated hull $\mathcal{H}_{r'}^{\text{tr}}$ viewed as a quadrangulation of the cylinder (we can also say that it is the part of the UIPQ that lies between the cycles \mathcal{C}_0 and $\mathcal{C}_{r'}$, if \mathcal{C}_0 consists of the loop added as explained above).

As an important remark, we note that the truncated hull of radius r is quite different from the (usual) hull of radius r considered e.g. in [21, 22], which is denoted by $B_{\mathcal{D}}^{\bullet}(r)$ and is obtained by filling in the bounded holes in the ball of radius r (recall that the ball of radius $r \geq 1$ is obtained as the union of all faces incident to at least one vertex whose graph distance from the root vertex is at most $r - 1$). To avoid any ambiguity, the hull $B_{\mathcal{D}}^{\bullet}(r)$ will be called the *standard hull* of radius r . The truncated hull can be recovered from the standard hull by considering the maximal cycle made of r -diagonals as explained above. On the other hand, the standard hull is “bigger” than the truncated hull: To recover the standard hull from the truncated hull, we need to add the triangles incident to r -diagonals that have been cut when removing the unbounded component of the complement of the maximal cycle, but also to fill in the bounded holes that may appear when adding these triangles (see Figure 2.4 for an example). For future use, we notice that the boundary of the standard hull $B_{\mathcal{D}}^{\bullet}(r)$ is a simple cycle, and that the graph distances of vertices in this cycle to the root vertex alternate between the values r and $r + 1$: Those vertices at graph distance r also belong to the cycle \mathcal{C}_r , but in general there are other vertices of \mathcal{C}_r that do not belong to the boundary of $B_{\mathcal{D}}^{\bullet}(r)$ (see Figure 2.4).

2.2.3 The skeleton decomposition

We will now describe a decomposition of quadrangulations of the cylinder in layers. This is essentially due to Krikun [38] and very similar to the case of triangulations, which is treated in [37, 23]. For this reason, we will skip some details.

Let us fix a quadrangulation \mathcal{Q} of the cylinder of height $h \geq 2$ with boundary sizes

(p, q) . Assign to each vertex a label equal to its distance from the bottom boundary. Let $k \in \{1, 2, \dots, h-1\}$, and consider all diagonals connecting corners labeled k in k -simple faces (defined in exactly the same manner as in the previous section for the UIPQ). As in the case of the UIPQ described above, these diagonals form a collection of cycles, and there is a maximal cycle which is simple and has the property that the unbounded component of the complement of this cycle contains no vertex with label less than or equal to k . Define the hull $\mathcal{H}_{\mathcal{Q}}^{\text{tr}}(k)$ by first adding to \mathcal{Q} the edges of this maximal cycle and then removing all edges that lie in the unbounded component of the complement of the maximal cycle. We obtain a quadrangulation of the cylinder of height k with boundary sizes (p, q_k) , where q_k denotes the size of the maximal cycle. We write $\mathcal{H}_{\mathcal{Q}}^{\text{tr}}(k)$ for this quadrangulation of the cylinder, and $\partial_k \mathcal{Q}$ for its top cycle, so that $q_k = |\partial_k \mathcal{Q}|$. See Figure 2.3 for the cycles $\partial_k \mathcal{Q}$ in a particular example.

Suppose now that we add to \mathcal{Q} all diagonals drawn in the previous procedure, for every $1 \leq k \leq h-1$ (in other words, we add the cycles $\partial_k \mathcal{Q}$ for every $1 \leq k \leq h-1$), and write \mathcal{Q}^\bullet for the resulting planar map (whose faces, except for the two distinguished faces of \mathcal{Q} , are either quadrangles or triangles). For every $1 \leq k \leq h$, the k -th layer of \mathcal{Q} is obtained as the part of \mathcal{Q}^\bullet that lies between the cycles $\partial_{k-1} \mathcal{Q}$ and $\partial_k \mathcal{Q}$, where by convention $\partial_0 \mathcal{Q}$ is the bottom cycle of \mathcal{Q} and $\partial_h \mathcal{Q}$ is the top cycle. We can view this layer as a quadrangulation of the cylinder of height 1 with boundary sizes $(|\partial_{k-1} \mathcal{Q}|, |\partial_k \mathcal{Q}|)$ (except that we have not specified the choice of the root edge — we will come back to this later in the case of interest to us).

We will now introduce an unordered forest $\mathcal{F}(\mathcal{Q})$ of (rooted) plane trees that in some sense describes the configuration of layers. First note that, for every $1 \leq k \leq h$, each edge of $\partial_k \mathcal{Q}$ is incident to a unique triangle of \mathcal{Q}^\bullet whose third vertex lies on $\partial_{k-1} \mathcal{Q}$ (when $k = h$, this is a consequence of the last assertion of Definition 2.2, and when $k < h$ this follows from the way we constructed the triangles incident to the top boundary of $\mathcal{H}_{\mathcal{Q}}^{\text{tr}}(k)$). We call such triangles *downward triangles* of \mathcal{Q}^\bullet (see the left side of Figure 2.6). The forest $\mathcal{F}(\mathcal{Q})$ consists of exactly q trees, each tree being associated with an edge of $\partial_h \mathcal{Q}$. The vertex set of the forest is the collection of all edges of $\partial_k \mathcal{Q}$, for $0 \leq k \leq h$. The genealogical relation is specified as follows: The roots of the trees are the edges of $\partial_h \mathcal{Q}$, and, for every $k \in \{0, \dots, h-1\}$, an edge e of $\partial_k \mathcal{Q}$ is a “child” of an edge e' of $\partial_{k+1} \mathcal{Q}$ if and only if the downward triangle associated with e' (i.e., containing e' in its boundary) is the first one that one encounters when turning around $\partial_{k-1} \mathcal{Q}$ in clockwise order, starting from the middle of the edge e . This definition should be clear from the right side of Figure 2.6. Notice that edges of $\partial_k \mathcal{Q}$ correspond to vertices of the forest $\mathcal{F}(\mathcal{Q})$ at generation $h-k$, for every $0 \leq k \leq h$. The planar structure of each tree in the forest is obviously induced by the planar structure of \mathcal{Q} , see again Figure 2.6.

We note that the root edge of \mathcal{Q} is a vertex of $\mathcal{F}(\mathcal{Q})$ at generation h and belongs to one of the trees of $\mathcal{F}(\mathcal{Q})$, which we denote by τ_1 . We may then write τ_2, \dots, τ_q for the other trees of $\mathcal{F}(\mathcal{Q})$ listed in clockwise order from τ_1 . Without risk of confusion, we keep the notation $\mathcal{F}(\mathcal{Q})$ for the ordered forest (τ_1, \dots, τ_q) .

The ordered forest $\mathcal{F}(\mathcal{Q})$ characterizes the combinatorial structure of the downward triangles in \mathcal{Q} . To determine \mathcal{Q} completely, one also needs to specify the way “slots”

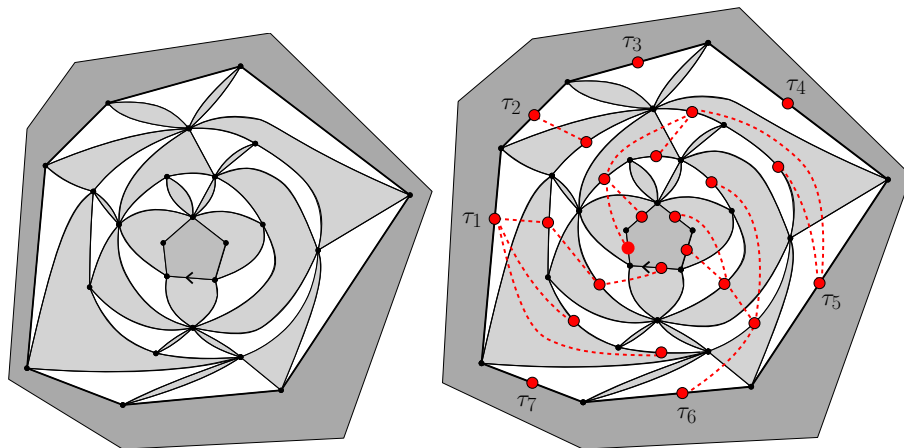


Figure 2.6 – On the left side, the downward triangles, in white, and the slots, in light grey, in the truncated quadrangulation of Figure 2.3. Notice that each edge incident to two downward triangles has been split in a double edge, to emphasize the fact that this creates a slot which is a two-gon (whose filling leads to gluing the two sides of the two-gon in Figure 2.3). On the right side, the red dashed lines are the edges of the trees τ_1, \dots, τ_7 of the forest coding the configuration of downward triangles (notice that τ_3, τ_4, τ_7 are trivial trees consisting only of their root vertex). The roots of the trees in the forest are the edges of the top cycle, and the trees grow “toward” the bottom cycle.

between two successive downward triangles in a given layer are filled in. More precisely, let e be an edge of $\partial_k \mathcal{Q}$, for some $1 \leq k \leq h$, and let \tilde{e} be the edge of $\partial_k \mathcal{Q}$ preceding e in clockwise order (we discuss below the case when there is only one edge in $\partial_k \mathcal{Q}$). The part of the k -th layer of \mathcal{Q} between the downward triangle associated with \tilde{e} and the downward triangle associated with e produces a slot with perimeter $c_e + 2$, where c_e is the number of children of e in the forest $\mathcal{F}(\mathcal{Q})$. This slot is said to be associated with e (it is also incident to a unique vertex v of $\partial_k \mathcal{Q}$). See the left side of Figure 2.7 for an illustration. If $c_e = 0$, it may happen that the slot is empty, if the downward triangles associated with \tilde{e} and e are adjacent. Also notice that when $|\partial_k \mathcal{Q}| = 1$, the only edge of $\partial_k \mathcal{Q}$ is a loop, but there is still an associated slot, which is bounded by the double edge in the boundary of the downward triangle associated with the unique edge of $\partial_k \mathcal{Q}$, and the edges of $\partial_{k-1} \mathcal{Q}$.

The boundary of the slot associated with e is of the type pictured in the left side of Figure 2.7, where there are c_e horizontal edges and the two non-horizontal edges are incident to the downward triangles associated with \tilde{e} and e . Strictly speaking, the random planar map consisting of the part of \mathcal{Q}^\bullet in the slot is not a truncated quadrangulation with a boundary, but a simple transformation allows us to view it as a truncated quadrangulation with a boundary of size $c_e + 1$: this transformation, which involves adding an extra edge, is illustrated in Figure 2.7 (see also Fig.6 in [38]) — to be precise, one should notice that the two vertices a and b in Figure 2.7 may be the same if all edges of $\partial_{k-1} \mathcal{Q}$ have the same “parent” in $\partial_k \mathcal{Q}$, but our interpretation still goes through. There

is therefore a one-to-one correspondence between possible fillings of the slot and such truncated quadrangulations. To make this correspondence precise, we need a convention for the position of the root: we can declare that in the filling of the slot, the root edge of the truncated quadrangulation corresponds to the added extra edge. We notice that, in the special case where $c_e = 0$, if the truncated quadrangulation used to fill in the slot is the unique truncated quadrangulation with boundary size 1 and no quadrangle, this means that the slot is empty so that two sides of the downward triangles associated with \tilde{e} and e are glued together.

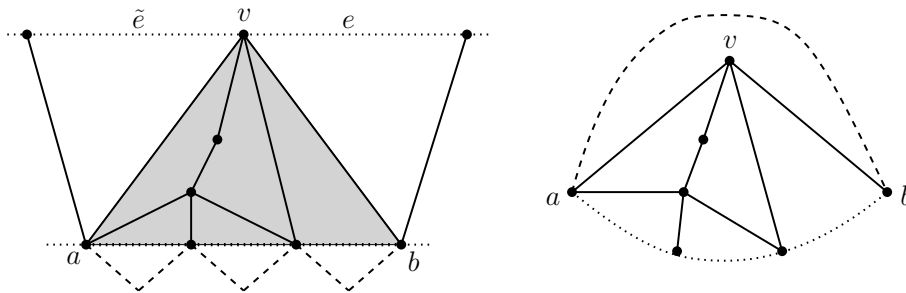


Figure 2.7 – On the left side, the shaded part corresponds to the slot associated with an edge e of $\partial_k \mathcal{Q}$, such that $c_e = 3$. This slot is bounded by the two “vertical edges” av and bv (which are incident to the downward triangles associated with \tilde{e} and e respectively) and by three diagonals (in dotted lines between a and b). On the right side, this slot is viewed as a truncated triangulation with boundary size 4 by adding the edge between a and b in dashed lines.

Following [23], we say that a forest \mathcal{F} with a distinguished vertex is (h, p, q) -admissible if

- (i) the forest consists of an ordered sequence $(\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_q)$ of q (rooted) plane trees,
- (ii) the maximal height of these trees is h ,
- (iii) the total number of vertices of the forest at generation h is p ,
- (iv) the distinguished vertex has height h and belongs to \mathbb{T}_1 .

If \mathcal{F} is a (h, p, q) -admissible forest, we write \mathcal{F}^* for the set of all vertices of \mathcal{F} at height strictly less than h . We write $\mathbb{F}_{h,p,q}^\circ$ for the set of all (h, p, q) -admissible forests.

The preceding discussion yields a bijection between, on the one hand, truncated quadrangulations \mathcal{Q} of the cylinder of height h with boundary sizes (p, q) , and, on the other hand, pairs consisting of a (h, p, q) -admissible forest \mathcal{F} and a collection $(M_v)_{v \in \mathcal{F}^*}$ such that, for every $v \in \mathcal{F}^*$, M_v is a truncated quadrangulation with boundary size $c_v + 1$, if c_v stands for the number of children of v in \mathcal{F} . We call this bijection the *skeleton decomposition* and we say that \mathcal{F} is the *skeleton* of the quadrangulation \mathcal{Q} .

It will also be convenient to use the notation $\mathbb{F}_{h,p,q}$ for the set of all (ordered) forests, with **no** distinguished vertex, that satisfy properties (i),(ii),(iii) above. If $\mathcal{F} \in \mathbb{F}_{h,p,q}$, we keep the notation \mathcal{F}^* for the set of all vertices of \mathcal{F} at height strictly less than h .

We also set, for every $p \geq 1$, $h \geq 1$,

$$\mathbb{F}_{h,p}^\circ = \bigcup_{q \geq 1} \mathbb{F}_{h,p,q}^\circ, \quad \mathbb{F}_{h,p} = \bigcup_{q \geq 1} \mathbb{F}_{h,p,q}.$$

We conclude this section with a useful observation about connections between the truncated hull and the standard hull of the UIPQ. Consider two integers u and r with $1 \leq u < r$. Recall that the truncated hull $\mathcal{H}_r^{\text{tr}}$ is viewed as a truncated quadrangulation of the cylinder of height r , whose top cycle is \mathcal{C}_r . Write $\mathcal{F}_{(r)}^\circ$ for the skeleton of this truncated quadrangulation, and also consider the cycle $\partial_u \mathcal{H}_r^{\text{tr}}$, which by construction coincides with \mathcal{C}_u . Vertices of this cycle are at distance u from the root vertex of the UIPQ, and may or may not belong to the boundary of the standard hull of radius u . However, assuming that $|\mathcal{C}_u| > 1$, if a vertex v of the cycle \mathcal{C}_u is such that the parents (in the forest $\mathcal{F}_{(r)}^\circ$) of the two edges of $\mathcal{C}_u = \partial_u \mathcal{H}_r^{\text{tr}}$ incident to v are different edges of the cycle $\mathcal{C}_{u+1} = \partial_{u+1} \mathcal{H}_r^{\text{tr}}$, then v must belong to the boundary of the standard hull of radius u . We leave the easy verification of this combinatorial fact to the reader. Notice that this is only a sufficient condition and that vertices of \mathcal{C}_u that do not satisfy this condition may also belong to the boundary of the standard hull of radius u .

2.2.4 Geodesics in the skeleton decomposition

Consider again a quadrangulation \mathcal{Q} of the cylinder of height $h \geq 2$ with boundary sizes (p, q) . Let v be a vertex of $\partial_k \mathcal{Q}$ for some $k \in \{1, \dots, h\}$. We assume that $|\partial_k \mathcal{Q}| \geq 2$. Then v is incident to two downward triangles which both contain an edge of $\partial_k \mathcal{Q}$ and a vertex of $\partial_{k-1} \mathcal{Q}$. Each of these triangles has an edge incident both to v and to a vertex of $\partial_{k-1} \mathcal{Q}$, and these two edges (which may be the same if the slot incident to v in the k -th layer of \mathcal{Q} is empty) are called downward edges from v . If the slot incident to v is nonempty we can in fact define the left downward edge by declaring that it is the first (downward) edge visited when exploring the boundary of the slot in clockwise order starting from a point of $\partial_{k-1} \mathcal{Q}$, and the other downward edge is called the right downward edge (of course if the slot is empty, the left and right downward edges coincide). We leave it to the reader to adapt these definitions in the case $|\partial_k \mathcal{Q}| = 1$ — in that case the left and right downward edges form a double edge.

We then define the left downward geodesic from v by saying that we first follow the left downward edge from v to arrive at a vertex v' of $\partial_{k-1} \mathcal{Q}$, then the left downward edge from v' to a vertex v'' of $\partial_{k-2} \mathcal{Q}$, and so on until we reach the bottom cycle $\partial_0 \mathcal{Q}$. Similarly we define the right downward geodesic from v by choosing at the first step the right downward edge from v , but then, as previously, following left downward edges from the visited vertices. See Figure 2.8 for an illustration.

Let N be the number of trees with maximal height in the skeleton decomposition of

\mathcal{Q} . Assume that $N \geq 2$, which implies that $|\partial_k \mathcal{Q}| \geq 2$ for every $k \in \{0, 1, \dots, h\}$. Let e be an edge of $\partial_h \mathcal{Q}$ corresponding to a tree with maximal height, and let v be the first vertex incident to e in clockwise order around $\partial_h \mathcal{Q}$. Then the left downward geodesic (resp. the right downward geodesic) from v hits the bottom cycle at a vertex v_1 (resp. at v_2) such that the edges of the bottom cycle lying between v_1 and v_2 in clockwise order are exactly the descendants of e at generation h in the skeleton decomposition. See Figure 2.8 for an example. The concatenation of these two geodesic paths gives a path from v_1 to v_2 with length $2h$. If we vary the edge e among all roots of trees with maximal height, we can concatenate the resulting paths to get a cycle \mathcal{C} with length $2Nh$, such that any path from the bottom cycle to the top cycle must visit a vertex of \mathcal{C} . In particular, if \mathcal{Q} is the annulus $\mathcal{C}(R, R+h)$ in the UIPQ, with $R \geq 3$, the cycle \mathcal{C} disconnects the ball $B_{\mathcal{Q}}(R-2)$ (or the hull $B_{\mathcal{Q}}^{\bullet}(R-2)$) from infinity.

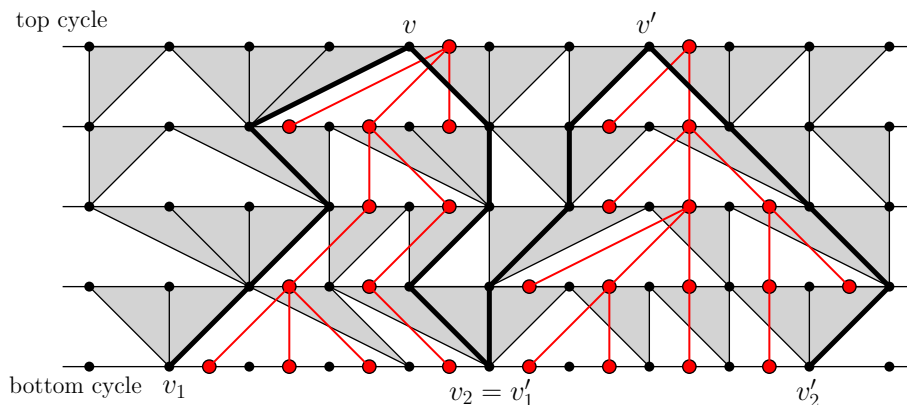


Figure 2.8 – A portion of a truncated quadrangulation of the cylinder of height 4. The downward triangles are colored in grey, and the associated slots are in white (they should of course be “filled in” by truncated quadrangulations as explained in the text). Two successive trees with maximal height in the coding forest are represented in red. The thick black lines are the (left and right) downward geodesics from the vertices v and v' of the top cycle associated with the two trees.

2.2.5 Enumeration

We rely on the results of Krikun [38]. Recall that $\mathcal{Q}_{n,p}^{\text{tr}}$ is the set of all truncated quadrangulations with boundary size p and n inner faces (this set is empty if $n < p$).

Section 2.2 of Krikun [38] provides an explicit formula for the generating function

$$U(x, y) = \sum_{p=1}^{\infty} \sum_{n=1}^{\infty} \#\mathcal{Q}_{n,p}^{\text{tr}} x^n y^p.$$

We will not need this formula, but we record the special case

$$U\left(\frac{1}{12}, y\right) = \frac{1}{24} \sqrt{(18-y)(2-y)^3} - \frac{1}{2} + \frac{y}{2} - \frac{y^2}{24}, \quad (2.1)$$

for $0 \leq y < 2$.

As a consequence of the explicit formula for the generating function U , we have, for every fixed $p \geq 1$,

$$\#\mathbb{Q}_{n,p}^{\text{tr}} \underset{n \rightarrow \infty}{\sim} \kappa_p n^{-5/2} 12^n, \quad (2.2)$$

where the constants κ_p are determined by the generating function

$$\sum_{p=1}^{\infty} \kappa_p y^p = \frac{128\sqrt{3}}{\sqrt{\pi}} \frac{y}{\sqrt{(18-y)(2-y)^3}}. \quad (2.3)$$

for $0 \leq y < 2$. We again refer to [38, Section 2.2] for these results. From (2.3) and standard singularity analysis [28, Corollary VI.1], we get

$$\kappa_p \underset{p \rightarrow \infty}{\sim} \frac{64\sqrt{3}}{\pi\sqrt{2}} \sqrt{p} 2^{-p}. \quad (2.4)$$

We also note that

$$\kappa_1 = \frac{32}{\sqrt{3\pi}}.$$

2.2.6 The distribution of hulls

Fix integers n and p with $n \geq p$. Let $\mathcal{Q}_p^{(n)}$ be uniformly distributed over $\mathbb{Q}_{n,p}^{\text{tr}}$ and given with a distinguished vertex chosen uniformly at random. Let $r \geq 1$. If the height (distance from the boundary) of this distinguished vertex is at least $r+1$, we can make sense of the hull $\mathcal{H}_{\mathcal{Q}_p^{(n)}}^{\text{tr}}(r)$. To this end, we label each vertex by its graph distance from the boundary of the distinguished face, and we proceed in a way very similar to the case of the UIPQ discussed in Section 2.2.2. We consider all diagonals connecting corners labeled r in r -simple faces (of type $r-1, r, r+1, r$), and the maximal cycle made of these diagonals, which has the property that the connected component of the complement of this cycle containing the distinguished vertex contains only vertices whose label is greater than r . We then add to $\mathcal{Q}_p^{(n)}$ the edges of this maximal cycle, and remove all edges lying in the connected component of the complement of this cycle containing the distinguished vertex. In this way, we obtain the hull $\mathcal{H}_{\mathcal{Q}_p^{(n)}}^{\text{tr}}(r)$, and it is easy to verify that $\mathcal{H}_{\mathcal{Q}_p^{(n)}}^{\text{tr}}(r)$ is a quadrangulation of the cylinder of height r (the size of its bottom cycle is p). If the height of the distinguished vertex is smaller than or equal to r , the preceding definition no longer makes sense, but by convention we define $\mathcal{H}_{\mathcal{Q}_p^{(n)}}^{\text{tr}}(r) = \dagger$ to be some ‘‘cemetery point’’ added to the set of all quadrangulations of the cylinder of height r .

The next lemma, which is an analog of Lemma 2 in [23], shows that the distribution of $\mathcal{H}_{\mathcal{Q}_p^{(n)}}^{\text{tr}}(r)$ has a limit when $n \rightarrow \infty$. We let \mathcal{Q} be a fixed quadrangulation of the cylinder of height r with boundary sizes (p, q) . This quadrangulation is coded by an (r, p, q) -admissible forest \mathcal{F} and a collection $(M_v)_{v \in \mathcal{F}^*}$, such that, for every $v \in \mathcal{F}^*$, M_v is a truncated quadrangulation with boundary size $c_v + 1$. Let $\text{Inn}(M_v)$ denote the number of inner faces of M_v .

Lemma 2.6. *We have*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{H}_{\mathcal{Q}_p^{(n)}}^{\text{tr}}(r) = \mathcal{Q}) = \frac{2^q \kappa_q}{2^p \kappa_p} \prod_{v \in \mathcal{F}^*} \left(\theta(c_v) \frac{12^{-\text{Inn}(M_v)}}{Z(c_v + 1)} \right) \quad (2.5)$$

where, for every $k \geq 1$,

$$Z(k) = \sum_{n=k}^{\infty} \#\mathcal{Q}_{n,k}^{\text{tr}} 12^{-n},$$

and θ is the critical offspring distribution defined by

$$\theta(k) = 6 \cdot 2^k Z(k + 1).$$

The generating function of θ is given for $0 \leq y < 1$ by

$$g_{\theta}(y) = 1 - \frac{8}{\left(\sqrt{\frac{9-y}{1-y}} + 2 \right)^2 - 1}. \quad (2.6)$$

We note that the property $Z(k) < \infty$ follows from (2.2).

Proof. We proceed in a very similar way to the proof of Lemma 2 in [23]. Let N be the number of inner faces of \mathcal{Q} , which is also the total number of vertices of \mathcal{Q} (by Euler's formula). We observe that the property $\mathcal{H}_{\mathcal{Q}_p^{(n)}}^{\text{tr}}(r) = \mathcal{Q}$ holds if and only if $\mathcal{Q}_p^{(n)}$ is obtained from \mathcal{Q} by gluing on the top boundary of \mathcal{Q} an arbitrary truncated quadrangulation with boundary size q and $n - (N - q)$ inner faces (for this gluing to make sense we need to specify an edge of the top boundary of \mathcal{Q} , which can be the root of the first tree in the forest \mathcal{F}), and if the distinguished vertex of $\mathcal{Q}_p^{(n)}$ is chosen among the inner vertices of this truncated quadrangulation. Noting that $\mathcal{Q}_p^{(n)}$ has $n + 1$ vertices, it follows that

$$\mathbb{P}(\mathcal{H}_{\mathcal{Q}_p^{(n)}}^{\text{tr}}(r) = \mathcal{Q}) = \frac{\#\mathcal{Q}_{n-(N-q),q}^{\text{tr}}}{\#\mathcal{Q}_{n,p}^{\text{tr}}} \times \frac{n + 1 - N}{n + 1}.$$

Using (2.2), we get

$$\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{H}_{\mathcal{Q}_p^{(n)}}^{\text{tr}}(r) = \mathcal{Q}) = \frac{\kappa_q}{\kappa_p} 12^{-N+q}. \quad (2.7)$$

Simple combinatorics shows that the number of inner faces of \mathcal{Q} can be written as

$$N = p + \sum_{v \in \mathcal{F}^*} (\text{Inn}(M_v) - c_v). \quad (2.8)$$

So the right-hand side of (2.7) is also equal to

$$\frac{\kappa_q}{\kappa_p} 12^{q-p} \prod_{v \in \mathcal{F}^*} (12^{c_v} 12^{-\text{Inn}(M_v)}).$$

It is now straightforward to verify that the last quantity is equal to the right-hand side of (2.5). Just observe that

$$12^{c_v} Z(c_v + 1) = 6^{c_v-1} \theta(c_v)$$

and notice that

$$\sum_{v \in \mathcal{F}^*} (c_v - 1) = p - q.$$

To see that θ is an offspring distribution, we rely on (2.1), which shows that the generating function of θ is

$$\begin{aligned} g_\theta(y) &:= \sum_{k=0}^{\infty} \theta(k) y^k = 6 \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} y^k 2^k 12^{-n} \#\mathbb{Q}_{n,k+1}^{\text{tr}} \\ &= \frac{6}{2y} U\left(\frac{1}{12}, 2y\right) \\ &= \frac{1}{2y} \left(\sqrt{(9-y)(1-y)^3} - 3 + 6y - y^2 \right), \end{aligned}$$

in agreement with Theorem 2 of [38]. Since $g_\theta(1) = 1$, θ is a probability distribution, and the fact that θ is critical is obtained by checking that $g'_\theta(1) = 1$.

Finally, a somewhat tedious calculation shows that the formula for g_θ in the last display is equivalent to the one given in the statement of the lemma. The latter is more convenient for computing iterates of g_θ , as we will see below in formula (2.15). \square

From the explicit form of g_θ , we have

$$g_\theta(1-x) = 1 - x + \sqrt{2} x^{3/2} + O(x^2)$$

as $x \downarrow 0$. By singularity analysis, it follows that

$$\theta(k) \underset{k \rightarrow \infty}{\sim} \frac{3\sqrt{2}}{4\sqrt{\pi}} k^{-5/2}. \quad (2.9)$$

Remark 2.1. *The offspring distribution θ appears in the seemingly different context of labeled trees. Consider a critical Galton-Watson tree with geometric offspring distribution with parameter $1/2$. Given the tree, assign labels to vertices by declaring that the label of the root is 0 and that label increments on different edges are independent and uniformly distributed over $\{-1, 0, 1\}$. Let N be the number of vertices labeled -1 whose (strict) ancestors all have nonnegative labels. Then N is distributed according to θ (see [25, Proof of Theorem 5.2]). Via Schaeffer's bijection relating plane quadrangulations to labeled trees, this interpretation of θ is in fact closely related to Lemma 2.6.*

We define, for every $p \geq 1$,

$$h(p) := \frac{1}{p} 2^p \kappa_p. \quad (2.10)$$

Lemma 2.7. *Let $p \geq 1$. The formula*

$$\mu_{r,p}(\mathcal{F}) := \frac{h(q)}{h(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v), \quad \mathcal{F} \in \mathbb{F}_{r,p,q}, \quad q \geq 1$$

defines a probability measure on $\mathbb{F}_{r,p}$. Consequently, the formula

$$\mu_{r,p}^\circ(\mathcal{F}) := \frac{2^q \kappa_q}{2^p \kappa_p} \prod_{v \in \mathcal{F}^*} \theta(c_v), \quad \mathcal{F} \in \mathbb{F}_{r,p,q}^\circ, \quad q \geq 1$$

defines a probability measure on $\mathbb{F}_{r,p}^\circ$.

Proof. Let Π be the generating function of the sequence $(h(k))_{k \geq 1}$,

$$\Pi(x) := \sum_{k=1}^{\infty} h(k) x^k.$$

To verify that $\mu_{r,p}$ defines a probability distribution on the set $\mathbb{F}_{r,p}$, it is enough to check that $(h(k))_{k \geq 1}$ is an (infinite) stationary measure for the branching process with offspring distribution θ , or equivalently that, for every $0 < y < 1$,

$$\Pi(g_\theta(y)) - \Pi(g_\theta(0)) = \Pi(y). \quad (2.11)$$

From (2.3), we get by integration that

$$\sum_{p=1}^{\infty} \kappa_p \frac{x^p}{p} = \frac{48}{\sqrt{\pi}} \left(\sqrt{\frac{18-x}{3(2-x)}} - \sqrt{3} \right),$$

and, for $0 < x < 1$,

$$\Pi(x) = \frac{48}{\sqrt{3\pi}} \left(\sqrt{\frac{9-x}{1-x}} - 3 \right).$$

From this explicit formula and (2.6), the desired identity (2.11) follows at once.

Once we know that $\mu_{r,p}$ is a probability distribution on $\mathbb{F}_{r,p}$, the fact that $\mu_{r,p}^\circ$ is a probability distribution on $\mathbb{F}_{r,p}^\circ$ follows easily. First note that

$$\tilde{\mu}((\mathcal{F}, v)) := \frac{1}{p} \mu_{r,p}(\mathcal{F})$$

defines a probability distribution on the set of all pairs (\mathcal{F}, v) consisting of a forest $\mathcal{F} \in \mathbb{F}_{r,p}$ and a distinguished vertex v of \mathcal{F} at generation r . Then notice that $\mu_{r,p}^\circ$ is just the push forward of $\tilde{\mu}$ under the mapping $(\mathcal{F}, v) \mapsto \mathcal{F}'$, where \mathcal{F}' is obtained by circularly permuting the trees of \mathcal{F} so that v belongs to the first tree of the forest. This completes the proof. \square

Let us consider now the UIPQ. Recall our notation $\mathcal{H}_r^{\text{tr}}$ for the truncated hull of radius r , and H_r for the perimeter of $\mathcal{H}_r^{\text{tr}}$, which is also the length of the cycle \mathcal{C}_r . For every $r, p \geq 1$, let $\mathbb{C}_{r,p}$ be the set of all truncated quadrangulations of the cylinder of height r with bottom boundary size p and arbitrary top boundary size. If $\mathcal{Q} \in \mathbb{C}_{r,p}$ and the size of the top boundary of \mathcal{Q} is q , we set

$$\Delta_{r,p}(\mathcal{Q}) = \frac{2^q \kappa_q}{2^p \kappa_p} \prod_{v \in \mathcal{F}^*} \left(\theta(c_v) \frac{12^{-\text{Inn}(M_v)}}{Z(c_v + 1)} \right),$$

where $(\mathcal{F}, (M_v)_{v \in \mathcal{F}^*})$ is the skeleton decomposition of \mathcal{Q} .

Corollary 2.8. $\Delta_{r,p}$ is a probability measure on $\mathbb{C}_{r,p}$. Furthermore, the distribution of $\mathcal{H}_r^{\text{tr}}$ is $\Delta_{r,1}$.

Proof. The fact that $\Delta_{r,p}$ is a probability measure on $\mathbb{C}_{r,p}$ readily follows from the second assertion of Lemma 2.7, noting that, by the very definition of $Z(k)$, we have

$$\sum_{n=1}^{\infty} \#\mathbb{Q}_{n,k+1}^{\text{tr}} \frac{12^{-n}}{Z(k+1)} = 1.$$

To get the second assertion of the corollary, let \mathbb{Q}_n stand for the set of all (rooted) planar quadrangulations with n faces. Via the transformation that consists in splitting the root edge to get a double edge, and then inserting a loop inside the resulting 2-gon (as in Figure 2.5), the set \mathbb{Q}_n is canonically identified to $\mathbb{Q}_{n+1,1}^{\text{tr}}$. From the local convergence of planar quadrangulations to the UIPQ [38], we deduce that the distribution of the hull of radius r in a uniformly distributed quadrangulation in $\mathbb{Q}_{n,1}^{\text{tr}}$ (equipped with a distinguished uniformly distributed vertex) converges to the distribution of the hull of radius r in the UIPQ. The second assertion of the corollary now follows from Lemma 2.6. \square

Corollary 2.9. The distribution of H_r is given by

$$\mathbb{P}(H_r = p) = \frac{h(p)}{h(1)} \mathbb{P}_p(Y_r = 1), \quad p \geq 1,$$

where $(Y_n)_{n \geq 0}$ denotes a Galton-Watson branching process with offspring distribution θ that starts from p under the probability measure \mathbb{P}_p .

Proof. Let $\mathcal{F}_{(r)}^\circ$ be the skeleton of the hull $\mathcal{H}_r^{\text{tr}}$ viewed as a quadrangulation of the cylinder of height r . As a direct consequence of the second assertion of Corollary 2.8, $\mathcal{F}_{(r)}^\circ$ is distributed as $\mu_{r,1}^\circ$. Define $\mathcal{F}_{(r)}$ from $\mathcal{F}_{(r)}^\circ$ by “forgetting” the distinguished vertex and applying a uniform random circular permutation to the trees in the sequence. Arguing as in the end of the proof of Lemma 2.7, it follows that $\mathcal{F}_{(r)}$ is distributed according to $\mu_{r,1}$.

Since H_r is just the number of trees in the forest $\mathcal{F}_{(r)}$, we have

$$\mathbb{P}(H_r = p) = \sum_{\mathcal{F} \in \mathbb{F}_{r,1,p}} \mathbb{P}(\mathcal{F}_{(r)} = \mathcal{F}) = \frac{h(p)}{h(1)} \sum_{\mathcal{F} \in \mathbb{F}_{r,1,p}} \prod_{v \in \mathcal{F}^*} \theta(c_v),$$

and the desired result follows. \square

Remark 2.2. *One can interpret the distribution of H_r as the limit when $T \rightarrow \infty$ of the distribution at time $T - r$ of a Galton-Watson process X with offspring distribution θ started from $X_0 = 1$ and conditioned on the event $\{X_T = 1\}$. This suggests that one may code the combinatorial structure of downward triangles in the whole UIPQ (and not only in a hull of fixed radius) by an infinite tree, which could be viewed as the genealogical tree for a Galton-Watson process with offspring distribution θ , indexed by nonpositive integer times and conditioned to be equal to 1 at time 0. This interpretation will not be needed in the present work and we omit the details.*

Let us now fix integers $0 \leq u < r$. As explained earlier, the annulus $\mathcal{C}(u, r)$ is the part of the UIPQ that lies between the cycles \mathcal{C}_u and \mathcal{C}_r — recall our convention for \mathcal{C}_0 from Section 2.2.2 — and $\mathcal{C}(u, r)$ is viewed as a truncated quadrangulation of the cylinder of height $r - u$ with boundary sizes (H_u, H_r) . We now specify the root edge of $\mathcal{C}(u, r)$, by declaring that it corresponds to the root of the tree, in the skeleton decomposition of $\mathcal{H}_u^{\text{tr}}$, that carries the root edge of the UIPQ (of course when $u = 0$, the root edge is the unique edge of \mathcal{C}_0).

Let $\mathcal{F}_{u,r}^\circ$ be the skeleton of $\mathcal{C}(u, r)$, which is a random element of $\cup_{p \geq 1} \mathbb{F}_{r-u,p}^\circ$. It will be convenient to introduce also the forest $\mathcal{F}_{u,r}$ (in $\cup_{p \geq 1} \mathbb{F}_{r-u,p}$) obtained from $\mathcal{F}_{u,r}^\circ$ by first “forgetting” the distinguished vertex and then applying a uniform random circular permutation to the trees in the sequence.

Corollary 2.10. *Let $p \geq 1$. The conditional distribution of $\mathcal{F}_{u,r}$ knowing that $H_u = p$ is $\mu_{r-u,p}$.*

Proof. Recall the notation $\mathcal{F}_{(r)}^\circ, \mathcal{F}_{(r)}$ introduced in the previous proof. We notice that, if $(\tau_1, \dots, \tau_{H_r})$ are the trees in the forest $\mathcal{F}_{(r)}^\circ$, the trees in the forest $\mathcal{F}_{u,r}^\circ$ are just $(\tau_1^{[r-u]}, \dots, \tau_{H_r}^{[r-u]})$, where the notation $\tau_i^{[r-u]}$ refers to the tree τ_i truncated at generation $r - u$. It follows that $\mathcal{F}_{u,r}$ can be assumed to be equal to the forest $\mathcal{F}_{(r)}$ truncated at generation $r - u$. Note that H_u is just the number of vertices of $\mathcal{F}_{(r)}$ at generation $r - u$.

Let $q \geq 1$ and $\mathcal{G} \in \mathbb{F}_{r-u,p,q}$. We have

$$\mathbb{P}(\mathcal{F}_{u,r} = \mathcal{G}) = \sum_{\mathcal{F} \in \mathbb{F}_{r,1}: \mathcal{F}^{[r-u]} = \mathcal{G}} \mathbb{P}(\mathcal{F}_{(r)} = \mathcal{F}),$$

using the notation $\mathcal{F}^{[r-u]}$ for the forest \mathcal{F} truncated at generation $r - u$. It follows that

$$\begin{aligned} \mathbb{P}(\mathcal{F}_{u,r} = \mathcal{G}) &= \frac{h(q)}{h(1)} \sum_{\mathcal{F} \in \mathbb{F}_{r,1}: \mathcal{F}^{[r-u]} = \mathcal{G}} \prod_{v \in \mathcal{F}^*} \theta(c_v) \\ &= \frac{h(q)}{h(1)} \prod_{v \in \mathcal{G}^*} \theta(c_v) \sum_{\tilde{\mathcal{F}} \in \mathbb{F}_{u,1,p}} \prod_{v \in \tilde{\mathcal{F}}^*} \theta(c_v), \end{aligned}$$

where we just use the fact that a forest $\mathcal{F} \in \mathbb{F}_{r,1}$ such that $\mathcal{F}^{[r-u]} = \mathcal{G}$ is obtained by “gluing” a forest of $\mathbb{F}_{u,1,p}$ to the p vertices of \mathcal{G} at generation $r - u$. As in Corollary 2.9

and its proof, we have

$$\sum_{\tilde{\mathcal{F}} \in \mathbb{F}_{u,1,p}} \prod_{v \in \tilde{\mathcal{F}}^*} \theta(c_v) = \mathbb{P}_p(Y_u = 1) = \frac{h(1)}{h(p)} \mathbb{P}(H_u = p).$$

So we get

$$\mathbb{P}(\mathcal{F}_{u,r} = \mathcal{G}) = \frac{h(q)}{h(p)} \prod_{v \in \mathcal{G}^*} \theta(c_v) \times \mathbb{P}(H_u = p) = \mathbb{P}(H_u = p) \mu_{r-u}(\mathcal{G}).$$

This completes the proof. \square

2.2.7 The law of the perimeter of hulls

We give a more explicit formula for the distribution of H_r .

Proposition 2.11. *We have, for every $r \geq 1$ and $p \geq 1$,*

$$\mathbb{P}(H_r = p) = K_r \kappa_p (2\pi_r)^p \quad (2.12)$$

where

$$\begin{aligned} \pi_r &= 1 - \frac{8}{(3+2r)^2 - 1} = \frac{r(r+3)}{(r+1)(r+2)} \\ K_r &= \frac{32}{3\kappa_1} \frac{3+2r}{((3+2r)^2 - 1)^2} \frac{1}{\pi_r} = \frac{2}{3\kappa_1} \frac{2r+3}{r(r+1)(r+2)(r+3)}. \end{aligned}$$

Consequently, there exist positive constants M_1, M_2 and ρ such that, for every $a > 0$, for every integer $r \geq 1$,

$$\mathbb{P}(H_r \geq ar^2) \leq M_1 e^{-\rho a} \quad (2.13)$$

and

$$\mathbb{P}(H_r \leq ar^2) \leq M_2 a^{3/2}. \quad (2.14)$$

We notice that $K_r \sim (4/(3\kappa_1)) r^{-3}$ as $r \rightarrow \infty$ and recall that $\kappa_1 = \frac{32}{\sqrt{3\pi}}$.

Proof. We rely on the formula of Corollary 2.9. Recalling that $(Y_n)_{n \geq 0}$ denotes a Galton-Watson branching process with offspring distribution θ that starts from p under the probability measure \mathbb{P}_p , and using formula (2.6), we obtain that the generating function of Y_r under \mathbb{P}_1 is

$$g_\theta^{(r)}(y) = \underbrace{g_\theta \circ \dots \circ g_\theta}_{r \text{ times}}(y) = 1 - \frac{8}{\left(\sqrt{\frac{9-y}{1-y}} + 2r\right)^2 - 1}. \quad (2.15)$$

It follows that

$$\mathbb{P}_1(Y_r = 0) = g_\theta^{(r)}(0) = 1 - \frac{8}{(3+2r)^2 - 1} = \pi_r, \quad (2.16)$$

and

$$\begin{aligned}
\mathbb{P}_p(Y_r = 1) &= \lim_{x \downarrow 0} \frac{1}{x} \left(\mathbb{E}_p[x^{Y_r}] - \mathbb{P}_p(Y_r = 0) \right) \\
&= \lim_{x \downarrow 0} \frac{1}{x} \left(g_\theta^{(r)}(x)^p - g_\theta^{(r)}(0)^p \right) \\
&= p g_\theta^{(r)}(0)^{p-1} \times \frac{64}{3} \frac{3+2r}{((3+2r)^2 - 1)^2} \\
&= \frac{64}{3} p \frac{3+2r}{((3+2r)^2 - 1)^2} \left(1 - \frac{8}{(3+2r)^2 - 1} \right)^{p-1}. \tag{2.17}
\end{aligned}$$

Since

$$\frac{h(p)}{h(1)} = \frac{1}{p} \frac{2^p \kappa_p}{2 \kappa_1},$$

Corollary 2.9 and (2.17) lead to formula (2.12). Finally, the bounds (2.13) and (2.14) are simple consequences of this explicit formula and the asymptotics (2.4) for the constants κ_p . To derive (2.13), we observe that we can find a constant $\eta > 0$ such that $\mathbb{P}(H_r \geq ar^2)$ is bounded above by a constant times

$$r^{-3} \sum_{p > ar^2} \sqrt{p} \times e^{-\eta p/r^2} \leq \text{Cst.} \cdot r^{-3} \int_{ar^2/2}^{\infty} \sqrt{x} e^{-\eta x/r^2} dx = \text{Cst.} \times \int_{a/2}^{\infty} \sqrt{y} e^{-\eta y} dy,$$

and the proof of (2.14) is even easier just bounding π_r by 1. \square

2.2.8 A conditional limit for branching processes

We keep the notation $(Y_n)_{n \geq 0}$ for a branching process with offspring distribution θ , which starts at p under the probability measure \mathbb{P}_p .

Lemma 2.12. *We have*

$$\mathbb{P}_1(Y_r \neq 0) \underset{r \rightarrow \infty}{\sim} \frac{2}{r^2}$$

and the distribution of $r^{-2}Y_r$ under $\mathbb{P}_1(\cdot \mid Y_r \neq 0)$ converges to the distribution with Laplace transform

$$1 - \left(1 + \sqrt{\frac{2}{\lambda}} \right)^{-2}.$$

Proof. The first assertion is immediate from (2.16). Next, from (2.15) and (2.16), we have

$$\mathbb{E}_1 \left[e^{-\lambda r^{-2} Y_r} \mathbf{1}_{\{Y_r \neq 0\}} \right] = \frac{8}{(3+2r)^2 - 1} - \frac{8}{\left(\sqrt{1 + \frac{8}{1 - e^{-\lambda r^{-2}}}} + 2r \right)^2 - 1}$$

and it easily follows that

$$\frac{r^2}{2} \mathbb{E}_1 \left[e^{-\lambda r^{-2} Y_r} \mathbf{1}_{\{Y_r \neq 0\}} \right] \xrightarrow{r \rightarrow \infty} 1 - \left(1 + \sqrt{\frac{2}{\lambda}} \right)^{-2},$$

giving the desired result. \square

2.2.9 An estimate on discrete bridges

In this short section, which is independent of the previous ones, we state an estimate for discrete bridges, which plays an important role in the proof of Proposition 2.2 in the next section.

Let $K \geq 1$ be an integer, and let $(b(0), b(1), \dots, b(2K))$ be a discrete bridge of length $2K$. This means that $(b(0), b(1), \dots, b(2K))$ is uniformly distributed over sequences $(x_0, x_1, \dots, x_{2K})$ such that $x_0 = x_{2K} = 0$ and $|x_i - x_{i-1}| = 1$ for every $i = 1, \dots, 2K$. It will be convenient to define intervals on $\{0, 1, \dots, 2K - 1\}$ in a cyclic manner: If $i, j \in \{0, 1, \dots, 2K - 1\}$, $[i, j] = \{i, i + 1, \dots, j\}$ as usual if $i \leq j$, but $[i, j] = \{i, i + 1, \dots, 2K - 1, 0, 1, \dots, j\}$ if $i > j$.

Let $c > 0$ be a fixed constant and let $r \geq 1$ be an integer. For every integer $k \geq 2$, we let $\mathcal{P}_{k,K}(r)$ stand for the event where there exist integers $0 \leq m_1 < m_2 < \dots < m_k < 2K$, such that $m_i - m_{i-1} \geq cr^2$ for every $2 \leq i \leq k$, and $m_1 + 2K - m_k \geq cr^2$, and, for every $i, j \in \{1, \dots, k\}$,

$$b(m_i) + b(m_j) - 2 \max \left(\min_{\ell \in [m_i, m_j]} b(\ell), \min_{\ell \in [m_j, m_i]} b(\ell) \right) \leq 5r.$$

Lemma 2.13. *There exist constants $C > 0$ and $\gamma \in (0, 1)$, which only depend on c , such that, for every $r \geq 1$ and every $k \geq 2$,*

$$\mathbb{P}(\mathcal{P}_{k,K}(r)) \leq \frac{C}{k} \left(\frac{K}{r^2} \right)^2 \gamma^k.$$

We postpone the proof to the Appendix.

2.3 Lower bound on the size of the separating cycle

In this section, we prove Proposition 2.2, and then explain how part (i) of Theorem 2.1 follows from this result.

Proof of Proposition 2.2. As a preliminary observation, we note that it is enough to prove that the stated bound holds for n large enough (and for every $r \geq 1$). Let u and w be two integers with $0 \leq u < w$. Recall the notation $\mathcal{F}_{u,w}$ for the forest obtained from the skeleton of $\mathcal{C}(u, w)$ by forgetting the distinguished vertex and then applying a uniform random circular permutation to the trees in the forest. By Corollary 2.10, we have for every $p, q \geq 1$ and $\mathcal{F} \in \mathbb{F}_{w-u,p,q}$,

$$\mathbb{P}(\mathcal{F}_{u,w} = \mathcal{F} \mid H_u = p) = \frac{h(q)}{h(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v),$$

where h is defined in (2.10). By Corollary 2.9, we have

$$\frac{h(q)}{h(p)} = \frac{\mathbb{P}(H_w = q) \mathbb{P}_p(Y_u = 1)}{\mathbb{P}(H_u = p) \mathbb{P}_q(Y_w = 1)}$$

where we must take $p = 1$ if $u = 0$. It follows that

$$\mathbb{P}(\mathcal{F}_{u,w} = \mathcal{F}) = \mathbb{P}(H_w = q) \frac{\mathbb{P}_p(Y_u = 1)}{\mathbb{P}_q(Y_w = 1)} \prod_{v \in \mathcal{F}^*} \theta(c_v)$$

and therefore

$$\mathbb{P}(\mathcal{F}_{u,w} = \mathcal{F} \mid H_w = q) = \frac{\mathbb{P}_p(Y_u = 1)}{\mathbb{P}_q(Y_w = 1)} \prod_{v \in \mathcal{F}^*} \theta(c_v) = \frac{\varphi_u(p)}{\varphi_w(q)} \prod_{v \in \mathcal{F}^*} \theta(c_v), \quad (2.18)$$

with the notation

$$\varphi_u(p) = \mathbb{P}_p(Y_u = 1) = \frac{64}{3} p \frac{3 + 2u}{((3 + 2u)^2 - 1)^2} \pi_u^{p-1}, \quad (2.19)$$

by (2.17). We will apply formula (2.18) with $u = nr$ and $w = (n+2)r$ for integers $n, r \geq 1$.

Let us fix $\alpha > 0, \beta > 0$ with $\alpha < 2 < \beta$ and $\beta - 2 < \alpha$. Let $c_0 > 0$ be a constant whose value will be specified later. Say that a plane tree satisfies property $(P)_r$ if it has at least $c_0 r^2$ vertices of generation $2r - 1$ that have at least one child at generation $2r$. Thanks to Lemma 2.12, we can choose the constant $c_0 > 0$ small enough so that, for every r large enough, the probability for a Galton-Watson tree with offspring distribution θ to satisfy property $(P)_r$ is greater than $a_0 r^{-2}$, for some other constant $a_0 > 0$. Let δ be another constant, with $0 < \delta < \alpha$. We write $\Theta_{n,r}$ for the collection of all forests in

$$\bigcup_{p,q \geq 1} \mathbb{F}_{2r,p,q}$$

having at least n^δ trees that satisfy property $(P)_r$. Our first goal is to find an upper bound for

$$\mathbb{P}(\mathcal{F}_{nr,(n+2)r} \notin \Theta_{n,r}).$$

From (2.13), we have

$$\sup_{r \geq 1} \left(\max \left(\mathbb{P}(H_{nr} > n^\beta r^2), \mathbb{P}(H_{(n+2)r} > n^\beta r^2) \right) \right) = O(e^{-n^\varepsilon}) \quad (2.20)$$

as $n \rightarrow \infty$, with a constant $\varepsilon > 0$ that depends only on β . On the other hand, by (2.14), we have also

$$\mathbb{P}(H_{(n+2)r} < n^\alpha r^2) \leq M_2 n^{\frac{3}{2}(\alpha-2)}, \quad (2.21)$$

and the constant M_2 does not depend on r .

We then restrict our attention to

$$\begin{aligned} & \mathbb{P}(\mathcal{F}_{nr,(n+2)r} \notin \Theta_{n,r}, n^\alpha r^2 \leq H_{(n+2)r} \leq n^\beta r^2, H_{nr} \leq n^\beta r^2) \\ &= \sum_{n^\alpha r^2 \leq q \leq n^\beta r^2} \mathbb{P}(H_{(n+2)r} = q) \mathbb{P}(H_{nr} \leq n^\beta r^2, \mathcal{F}_{nr,(n+2)r} \notin \Theta_{n,r} \mid H_{(n+2)r} = q). \end{aligned} \quad (2.22)$$

Fix q such that $n^\alpha r^2 \leq q \leq n^\beta r^2$. We have

$$\begin{aligned} & \mathbb{P}(H_{nr} \leq n^\beta r^2, \mathcal{F}_{nr, (n+2)r} \notin \Theta_{n,r} \mid H_{(n+2)r} = q) \\ &= \sum_{p \leq n^\beta r^2} \mathbb{P}(H_{nr} = p, \mathcal{F}_{nr, (n+2)r} \notin \Theta_{n,r} \mid H_{(n+2)r} = q) \\ &= \sum_{p \leq n^\beta r^2} \frac{\varphi_{nr}(p)}{\varphi_{(n+2)r}(q)} \sum_{\mathcal{F} \in \mathbb{F}_{p, q, 2r} \setminus \Theta_{n,r}} \prod_{v \in \mathcal{F}^*} \theta(c_v) \end{aligned} \quad (2.23)$$

by formula (2.18).

For $p \leq n^\beta r^2$, we first bound the quantity

$$\frac{\varphi_{nr}(p)}{\varphi_{(n+2)r}(q)} \leq M \frac{p}{q} \frac{(\pi_{nr})^{p-1}}{(\pi_{(n+2)r})^{q-1}}$$

where M is a constant and the quantities π_r were defined in Proposition 2.11. Since $\pi_{nr} \leq 1$, we obtain that

$$\frac{\varphi_{nr}(p)}{\varphi_{(n+2)r}(q)} \leq M \frac{p}{q} (\pi_{(n+2)r})^{-q} \leq M n^{\beta-\alpha} \exp\left(\frac{Aq}{n^2 r^2}\right) \leq M n^{\beta-\alpha} \exp(A n^{\beta-2}) \quad (2.24)$$

with some constant A . On the other hand, the quantity

$$\sum_{p \leq n^\beta r^2} \sum_{\mathcal{F} \in \mathbb{F}_{p, q, 2r} \setminus \Theta_{n,r}} \prod_{v \in \mathcal{F}^*} \theta(c_v) \quad (2.25)$$

is bounded above by the probability that a forest of q independent Galton-Watson trees with offspring distribution θ (truncated at level $2r$) is not in $\Theta_{n,r}$. For each tree in this forest, the probability that it satisfies property $(P)_r$ is at least a_0/r^2 . The quantity (2.25) is thus bounded above by

$$\mathbb{P}(\varepsilon_1 + \dots + \varepsilon_q < n^\delta)$$

where the random variables $\varepsilon_1, \varepsilon_2, \dots$ are i.i.d., with $\mathbb{P}(\varepsilon_1 = 1) = 1 - \mathbb{P}(\varepsilon_1 = 0) = a_0 r^{-2}$. Since $q \geq n^\alpha r^2$ and $\alpha > \delta$, standard estimates on the binomial distribution show that the quantity in the last display is bounded above by $\exp(-\tilde{a}_0 n^\alpha)$ for all n sufficiently large and for every $r \geq 1$, with some other constant $\tilde{a}_0 > 0$. Recalling (2.24), we get that the quantity (2.23) is bounded above for n large by

$$M n^{\beta-\alpha} \exp(A n^{\beta-2}) \exp(-\tilde{a}_0 n^\alpha).$$

Since $\beta - 2 < \alpha$, this shows that the left-hand side of (2.22) goes to 0 faster than any negative power of n , uniformly in $r \geq 1$.

By combining this observation with (2.20) and (2.21), we obtain that, for n large enough,

$$\mathbb{P}(\mathcal{F}_{nr, (n+2)r} \notin \Theta_{n,r}) \leq C' n^{\frac{3}{2}(\alpha-2)} \quad (2.26)$$

with a constant C' independent of r .

Let us argue on the event $\{\mathcal{F}_{nr,(n+2)r} \in \Theta_{n,r}\}$. If τ is a tree of $\mathcal{F}_{nr,(n+2)r}$ with height $2r$, the vertices of τ at height $2r$ correspond to consecutive edges of \mathcal{C}_{nr} , and if $v = v(\tau)$ is the last vertex in clockwise order that is incident to these edges, we know from Section 2.2.4 that there is a downward geodesic path from a vertex \tilde{v} of $\mathcal{C}_{(n+2)r}$ (incident to the edge which is the root of τ) to v , which has length exactly $2r$. Also, by the comments of the end of Section 2.2.3, we know that $v(\tau)$ belongs to the boundary of the standard hull of radius nr . Moreover, let τ and τ' be two distinct trees of $\mathcal{F}_{nr,(n+2)r}$ with height $2r$, and assume that they both satisfy property $(P)_r$. Then the part of the boundary of the standard hull of radius nr between $v(\tau)$ and $v(\tau')$, in clockwise or in counterclockwise order, must contain at least $c_0 r^2$ vertices: This follows from the definition of property $(P)_r$ and the fact that, with each vertex a of τ (or of τ') at generation $2r - 1$ having at least one child at generation $2r$ we can associate a vertex of the UIPQ—namely, the last vertex (in clockwise order) incident to the edges that are children of a —which belongs to the boundary of the standard hull of radius nr , as explained at the end of Section 2.2.1.

Write \mathcal{C}_{nr}^\bullet for the boundary of the standard hull $B_{\mathcal{P}}^\bullet(nr)$ and H_{nr}^\bullet for its perimeter (note that $H_{nr}^\bullet \leq 2H_{nr}$). Also let \mathcal{P}_{nr}^\bullet stand for the complement of $B_{\mathcal{P}}^\bullet(nr)$ in the UIPQ, viewed as an infinite quadrangulation with (simple) boundary \mathcal{C}_{nr}^\bullet . By the preceding observations, the property $\mathcal{F}_{nr,(n+2)r} \in \Theta_{n,r}$ implies that there are vertices u_1, u_2, \dots, u_k of \mathcal{C}_{nr}^\bullet , with $k \geq n^\delta$, such that, for every $1 \leq i \leq k$, u_i is at graph distance nr from the root vertex of the UIPQ and is connected to a vertex \tilde{u}_i of $\mathcal{C}_{(n+2)r}$ by a path of length $2r$, and moreover, if $i \neq j$, u_i and u_j are separated by at least $c_0 r^2$ edges of \mathcal{C}_{nr}^\bullet . Furthermore, write $\mathcal{E}_{n,r}$ for the event considered in Proposition 2.2: $\mathcal{E}_{n,r}$ is the event where there exists a cycle γ of length smaller than r that stays in $\mathcal{C}(nr, (n+2)r)$, does not intersect $\mathcal{C}_{(n+2)r}$, and disconnects the root vertex of \mathcal{P} from infinity. On the event $\mathcal{E}_{n,r} \cap \{\mathcal{F}_{nr,(n+2)r} \in \Theta_{n,r}\}$, for every $i \in \{1, \dots, k\}$, the path from u_i to \tilde{u}_i must intersect the latter cycle: Indeed, if we concatenate this path with a geodesic from the root vertex to u_i and then with a path which goes from the vertex \tilde{u}_i to infinity and does not visit vertices at distance smaller than $(n+2)r$ from the root (see Section 2.2.2), we get a path Γ_i from the root vertex to infinity, which must intersect γ . It follows that, for every $i, j \in \{1, \dots, k\}$, we can construct a path of length at most $5r$ between u_i and u_j that stays in $\mathcal{C}(nr, (n+2)r)$. Since u_i and u_j both belong to \mathcal{C}_{nr}^\bullet , a simple combinatorial argument shows that this path can be required to stay in \mathcal{P}_{nr}^\bullet .

We then use the fact that, conditionally on H_{nr}^\bullet , \mathcal{P}_{nr}^\bullet is an infinite planar quadrangulation with a simple boundary of size H_{nr}^\bullet , which is independent of $B_{\mathcal{P}}^\bullet(nr)$: This follows from the spatial Markov property of the UIPQ (we refer to Theorem 5.1 in [5] for the UIPT, and the argument for the UIPQ is exactly the same). For every even integer $m \geq 2$, write $\mathcal{P}^{(m)}$ for the UIPQ with simple boundary of length m (see [26]) and $\partial\mathcal{P}^{(m)}$ for the collection of its boundary vertices. Also denote the graph distance on the vertex set of $\mathcal{P}^{(m)}$ by $d_{\text{gr}}^{(m)}$. Let $\mathcal{E}_1^{(m,n,r)}$ stand for the event where there are at least $k = \lceil n^\delta \rceil$ vertices v_1, \dots, v_k of $\partial\mathcal{P}^{(m)}$ such that, if $i \neq j$, v_i and v_j are separated by at least $c_0 r^2$ edges of $\partial\mathcal{P}^{(m)}$, and moreover

$$d_{\text{gr}}^{(m)}(v_i, v_j) \leq 5r.$$

We will verify that $\sup\{\mathbb{P}(\mathcal{E}_1^{(m,n,r)}) : m \leq 2n^\beta r^2\}$ decays exponentially in n uniformly in

r . Since by previous observations, we know that

$$\mathbb{P}(\mathcal{E}_{n,r} \cap \{\mathcal{F}_{nr,(n+2)r} \in \Theta_{n,r}\}) \leq \sup\{\mathbb{P}(\mathcal{E}_1^{(m,n,r)}) : m \leq 2n^\beta r^2\}$$

Proposition 2.2 will follow from the bound (2.26) (observe that we can choose $\alpha > 0$ small so that $\frac{3}{2}(2 - \alpha)$ is as close to 3 as desired).

In order to get the preceding exponential decay, we first replace the UIPQ with simple boundary $\mathcal{P}^{(m)}$ by the UIPQ with general boundary of the same size, which we denote by $\widetilde{\mathcal{P}}^{(m)}$, and without risk of confusion, we keep the notation $d_{\text{gr}}^{(m)}$ for the graph distance (see again [26] for the definition of the UIPQ with general boundary). We write $\mathbf{c}_0, \mathbf{c}_1, \dots, \mathbf{c}_{m-1}$ for the “exterior” corners of the boundary of $\widetilde{\mathcal{P}}^{(m)}$ enumerated in clockwise order starting from the root corner. Consider the event $\mathcal{E}_2^{(m,n,r)}$ where one can find integers $0 \leq p_1 < p_2 < \dots < p_k < m$, with $k = \lceil n^\delta \rceil$, such that $p_{i+1} - p_i \geq c_0 r^2$ for every $1 \leq i < k$, and $p_1 + m - p_k \geq c_0 r^2$, and furthermore

$$d_{\text{gr}}^{(m)}(\mathbf{c}_{p_i}, \mathbf{c}_{p_j}) \leq 5r$$

whenever $i \neq j$. In order to bound the probability of $\mathcal{E}_2^{(m,n,r)}$, recall that, from the results of [26] about infinite planar quadrangulations with a boundary, we can assign labels $\ell(0), \ell(1), \dots, \ell(m-1)$ to the corners $\mathbf{c}_0, \mathbf{c}_1, \dots, \mathbf{c}_{m-1}$, which correspond to “renormalized” distances from infinity, and are such that $\ell(0) = 0$. Moreover, the sequence $(\ell(0), \ell(1), \dots, \ell(m))$ (with $\ell(m) = 0$) is a discrete bridge of length m , and we have, for every $i, j \in \{0, 1, \dots, m-1\}$,

$$d_{\text{gr}}^{(m)}(\mathbf{c}_i, \mathbf{c}_j) \geq \ell(i) + \ell(j) - 2 \max \left(\min_{m \in [i,j]} \ell(m), \min_{m \in [j,i]} \ell(m) \right) \quad (2.27)$$

with the same convention for intervals as in Section 2.2.9. The bound (2.27) follows from the “treed bridge” representation of $\widetilde{\mathcal{P}}^{(m)}$ in [26, Theorem 2] as an instance of the so-called cactus bound (see [44, Proposition 5.9(ii)] for the case of finite planar quadrangulations, and [32, Lemma 3.7] for the case of finite quadrangulations with a boundary — the proof in our infinite setting is exactly the same). We can thus apply the bound of Lemma 2.13 to the bridge $(\ell(0), \ell(1), \dots, \ell(m))$ to get that $\mathbb{P}(\mathcal{E}_2^{(m,n,r)})$ decays exponentially as $n \rightarrow \infty$ uniformly in $m \leq 10n^\beta r^2$ and in $r \geq 1$.

We still need to verify that a similar exponential decay holds if we replace $\mathbb{P}(\mathcal{E}_2^{(m,n,r)})$ by $\mathbb{P}(\mathcal{E}_1^{(m,n,r)})$. To this end, we rely on Theorem 4 of [26], which states that conditionally on the event where the size of its boundary is equal to m' , the core of $\widetilde{\mathcal{P}}^{(m)}$ is distributed as $\mathcal{P}^{(m')}$ (the core of $\widetilde{\mathcal{P}}^{(m)}$ is obtained informally by removing the finite “components” of $\widetilde{\mathcal{P}}^{(m)}$ that can be disconnected from the infinite part by removing just one vertex, see Fig.2 in [26]). Furthermore, the probability that the boundary size of the core of $\widetilde{\mathcal{P}}^{(3m)}$ is equal to m is bounded below by $c_1 m^{-2/3}$ for some constant $c_1 > 0$, as shown in the proof of Theorem 1 of [26]. Noting that the graph distance between two vertices of the core only depends on the core itself (and not on the finite components hanging off the core), we easily conclude that

$$\mathbb{P}(\mathcal{E}_1^{(m,n,r)}) \leq (c_1 m^{-2/3})^{-1} \mathbb{P}(\mathcal{E}_2^{(3m,n,r)}),$$

and this completes the proof of Proposition 2.2. □

Let us now explain how part (i) of Theorem 2.1 is derived from Proposition 2.2.

Proof of Theorem 2.1 (i). Let $R \geq 1$ and let $\varepsilon \in (0, 1/2)$. Without loss of generality we can assume that $\varepsilon R \geq 1$. Suppose that γ is a cycle separating $B_{\mathcal{F}}(R)$ from infinity, with length smaller than $r := \lfloor \varepsilon R \rfloor$. Also let $n = \lfloor 1/\varepsilon \rfloor$ so that $nr \leq R$. Then the cycle γ disconnects $B_{\mathcal{F}}(nr)$ from infinity, which also implies that it disconnects $B_{\mathcal{F}}^{\bullet}(nr)$ from infinity (in particular, γ does not intersect $B_{\mathcal{F}}^{\bullet}(nr)$). Let $k \geq 0$ be the first integer such that γ intersects the annulus $\mathcal{C}((n+k)r, (n+k+1)r)$. Then γ is contained in $\mathcal{C}((n+k)r, (n+k+2)r)$ and does not intersect $\mathcal{C}_{(n+k+2)r}$ (otherwise this would contradict the fact that γ has length smaller than r). These considerations show that the event $\{L(R) \leq \varepsilon R\}$ is contained in the union over $k \geq 0$ of the events where there exists a cycle of length smaller than r that is contained in $\mathcal{C}((n+k)r, (n+k+2)r)$, does not intersect $\mathcal{C}_{(n+k+2)r}$, and disconnects the root vertex from infinity. Hence, if $\beta \in (1, 3)$ is given, we deduce from Proposition 2.2 that

$$\mathbb{P}(L(R) \leq \varepsilon R) \leq \sum_{k=0}^{\infty} C'_{\beta} (n+k)^{-\beta} \leq \tilde{C}_{\beta} n^{-\beta+1} = \tilde{C}_{\beta} (\lfloor 1/\varepsilon \rfloor)^{-\beta+1}.$$

The result of Theorem 2.1 (i) follows. □

2.4 Upper bound on the size of the separating cycle

In this section, we prove Theorem 2.1 (ii). We recall the notation $\mathcal{F}_{u,w}$ introduced before Corollary 2.10, for integers $0 \leq u < w$. We write $N_{u,w}$ for the number of trees of $\mathcal{F}_{u,w}$ that have maximal height $w - u$. As explained in Section 2.2.4, for every integer $R \geq 3$, one can find a cycle of the UIPT that disconnects the ball $B_{\mathcal{F}}(R-2)$ from infinity and whose length is bounded above by $2R N_{R,2R}$. In order to get bounds on $N_{R,2R}$, we determine more generally the distribution of $N_{u,w}$ for any $1 \leq u < w$.

Proposition 2.14. *The generating function of $N_{u,w}$ is given by*

$$\mathbb{E}[a^{N_{u,w}}] = a \left(\frac{9 - \pi_w}{9 - a\pi_w - (1-a)\pi_{w-u}} \right)^{1/2} \left(\frac{1 - \pi_w}{1 - a\pi_w - (1-a)\pi_{w-u}} \right)^{3/2},$$

where we recall the notation

$$\pi_k = g_{\theta}^{(k)}(0) = 1 - \frac{8}{(3+2k)^2 - 1}.$$

Consequently, $N_{u,w}$ has the same distribution as $1+U+V$, where U and V are independent, U follows the negative binomial distribution with parameters $(\frac{1}{2}, \frac{\pi_w - \pi_{w-u}}{9 - \pi_{w-u}})$ and V follows the negative binomial distribution with parameters $(\frac{3}{2}, \frac{\pi_w - \pi_{w-u}}{1 - \pi_{w-u}})$.

Proof. Let $q \geq 1$ and first condition on the event $H_w = q$. By formula (2.18),

$$\mathbb{E}[a^{N_{u,w}} \mid H_w = q] = \sum_{\mathcal{F} \in \cup_{p \geq 1} \mathbb{F}_{w-u,p,q}} a^{N_{w-u}(\mathcal{F})} \frac{\varphi_u(Y_{w-u}(\mathcal{F}))}{\varphi_w(q)} \prod_{v \in \mathcal{F}^*} \theta(c_v),$$

where $Y_{w-u}(\mathcal{F})$ is the number of vertices of the forest \mathcal{F} at generation $w-u$, and $N_{w-u}(\mathcal{F})$ denotes the number of trees of this forest with maximal height $w-u$. Using the explicit formula (2.19) for $\varphi_u(p)$, we get

$$\mathbb{E}[a^{N_{u,w}} \mid H_w = q] = \frac{f(u)}{\varphi_w(q)} \mathbb{E}_q \left[Y_{w-u} \pi_u^{Y_{w-u}-1} a^{N_{w-u}} \right], \quad (2.28)$$

where

$$f(u) = \frac{64}{3} \frac{3+2u}{((3+2u)^2-1)^2},$$

and under the probability measure \mathbb{P}_q , we consider a forest of q independent Galton-Watson trees with offspring distribution θ , $(Y_k)_{k \geq 1}$ denoting the associated Galton-Watson process.

Let us compute the quantity $\mathbb{E}_q \left[Y_{w-u} \pi_u^{Y_{w-u}-1} a^{N_{w-u}} \right]$. To simplify notation, we set $m = w-u$. It is convenient to write ξ for a random variable distributed as the number of vertices at generation m in a Galton-Watson tree with offspring distribution θ conditioned to be non-extinct at generation m . Then a simple argument shows that

$$\mathbb{E}_q \left[Y_m \pi_u^{Y_m-1} a^{N_m} \right] = \sum_{k=1}^{\infty} \mathbb{P}_q(N_m = k) k \mathbb{E}[\xi \pi_u^{\xi-1}] \left(\mathbb{E}[\pi_u^\xi] \right)^{k-1} a^k, \quad (2.29)$$

and furthermore,

$$E[\pi_u^\xi] = \frac{g_\theta^{(m)}(\pi_u) - g_\theta^{(m)}(0)}{1 - g_\theta^{(m)}(0)} = \frac{\pi_w - \pi_m}{1 - \pi_m}, \quad E[\xi \pi_u^{\xi-1}] = \frac{\dot{g}_\theta^{(m)}(\pi_u)}{1 - g_\theta^{(m)}(0)},$$

where $\dot{g}_\theta^{(m)}$ stand for the derivative of $g_\theta^{(m)}$. We are thus led to the calculation of

$$\sum_{k=1}^{\infty} \mathbb{P}_q(N_m = k) k \left(\frac{\pi_w - \pi_m}{1 - \pi_m} \right)^{k-1} a^k = a \mathbb{E}_q \left[N_m \left(a \left(\frac{\pi_w - \pi_m}{1 - \pi_m} \right) \right)^{N_m-1} \right].$$

Since $\mathbb{E}_q[a^{N_m}] = (a(1 - \pi_m) + \pi_m)^q$ and so $\mathbb{E}_q[N_m a^{N_m-1}] = q(1 - \pi_m)(a(1 - \pi_m) + \pi_m)^{q-1}$, we easily obtain that the quantities in (2.29) are equal to

$$q a \dot{g}_\theta^{(m)}(\pi_u) \left(a(\pi_w - \pi_m) + \pi_m \right)^{q-1}.$$

We substitute this in (2.28) and then use the formula for the distribution of H_w in Proposition 2.11. It follows that

$$\mathbb{E}[a^{N_{u,w}}] = f(u) a \dot{g}_\theta^{(m)}(\pi_u) \sum_{q=1}^{\infty} \frac{K_w}{\varphi_w(q)} \kappa_q q \left(a(\pi_w - \pi_m) + \pi_m \right)^{q-1} \left(2\pi_w \right)^q.$$

To compute the sum of the series in the last display, first note that

$$q \frac{K_w}{\varphi_w(q)} \left(2\pi_w\right)^q = \frac{1}{\kappa_1} 2^{q-1}.$$

It follows that

$$\begin{aligned} \mathbb{E}[a^{N_{u,w}}] &= \frac{f(u) a \dot{g}_\theta^{(m)}(\pi_u)}{\kappa_1} \sum_{q=1}^{\infty} \kappa_q \left(2(a(\pi_w - \pi_m) + \pi_m)\right)^{q-1} \\ &= \frac{32\sqrt{3}}{\sqrt{\pi}} \frac{f(u) a \dot{g}_\theta^{(m)}(\pi_u)}{\kappa_1 \sqrt{(9 - (a(\pi_w - \pi_m) + \pi_m))(1 - (a(\pi_w - \pi_m) + \pi_m))}^3} \end{aligned}$$

by (2.3). A simple calculation shows that

$$\dot{g}_\theta^{(m)}(\pi_u) = \frac{f(w)}{f(u)}.$$

Recalling that $\kappa_1 = \frac{32}{\sqrt{3\pi}}$, we get

$$\mathbb{E}[a^{N_{u,w}}] = 3 f(w) a \frac{1}{\sqrt{(9 - (a(\pi_w - \pi_m) + \pi_m))(1 - (a(\pi_w - \pi_m) + \pi_m))}^3}$$

and we just have to note that

$$f(w) = \frac{1}{3} \sqrt{(9 - \pi_w)(1 - \pi_w)^3},$$

by an easy calculation. □

Remark 2.3. *Certain details of the previous proof could be simplified by using the interpretation suggested in the remark following Corollary 2.9.*

Proof of Part (ii) of Theorem 2.1. As we already noticed, we have $L(R - 2) \leq 2R N_{R,2R}$ for every $R \geq 3$, and we also note that $R \mapsto L(R)$ is nondecreasing. Then the proof boils down to verifying that there exist constants C' and $\lambda' > 0$ such that, for every $R \geq 1$, $\mathbb{E}[\exp(\lambda' N_{R,2R})] \leq C'$. Noting that

$$\frac{\pi_{2R} - \pi_R}{1 - \pi_R}$$

is bounded above by a constant $\eta < 1$, this follows from Proposition 2.14 and the form of the negative binomial distribution. □

Remark 2.4. *We can also use Proposition 2.14 to get a lower bound on the probability that is bounded above in Theorem 2.1(i). From Proposition 2.14, we have immediately*

$$\mathbb{P}(N_{u,w} = 1) = \left(\frac{9 - \pi_w}{9 - \pi_{w-u}}\right)^{1/2} \left(\frac{1 - \pi_w}{1 - \pi_{w-u}}\right)^{3/2},$$

and taking $u = R$ and $w = \lfloor (1 + \varepsilon)R \rfloor$, we get

$$\mathbb{P}(N_{R, \lfloor (1+\varepsilon)R \rfloor} = 1) \xrightarrow{R \rightarrow \infty} \left(\frac{\varepsilon}{1 + \varepsilon} \right)^3.$$

Since we know that $L(R - 2) \leq 2\varepsilon R N_{R, \lfloor (1+\varepsilon)R \rfloor}$, we conclude that

$$\liminf_{R \rightarrow \infty} \mathbb{P}(L(R) \leq 2\varepsilon R) \geq \left(\frac{\varepsilon}{1 + \varepsilon} \right)^3,$$

which should be compared with Theorem 2.1(i). Note that we get a lower bound by (a constant times) ε^3 , whereas the upper bound of Theorem 2.1(i) gives ε^δ for every $\delta < 2$. It is very plausible that one can refine the preceding argument, by considering all annuli of the form $\mathcal{C}_{\lfloor (1+n\varepsilon)R \rfloor, \lfloor (1+(n+1)\varepsilon)R \rfloor}$, and get also a lower bound of order ε^2 (of course one should deal with the lack of independence of the variables $N_{\lfloor (1+n\varepsilon)R \rfloor, \lfloor (1+(n+1)\varepsilon)R \rfloor}$).

2.5 Isoperimetric inequalities

In this section, we prove Theorem 2.3 and Proposition 2.4. We start with the proof of Proposition 2.4, which is easier.

Proof of Proposition 2.4. We first observe that we can assume that n is larger than some fixed integer, by then adjusting the constant c_ε if necessary. In agreement with the notation of the introduction, write $|B_{\mathcal{P}}^\bullet(r)|$ for the number of faces of the UIPQ contained in the standard hull $B_{\mathcal{P}}^\bullet(r)$.

By Theorem 5.1 in [21] (or [22, Section 6.2]), we know that $r^{-4}|B_{\mathcal{P}}^\bullet(r)|$ converges in distribution as $r \rightarrow \infty$ to a limit which is finite a.s. It follows that we can fix an integer $M > 0$ such that, for every $r \geq 1$,

$$\mathbb{P}(|B_{\mathcal{P}}^\bullet(r)| < M r^4) \geq 1 - \frac{\varepsilon}{2}. \quad (2.30)$$

On the other hand, for every $r \geq 1$ and every integer $N \geq 1$, let $E_{r,N}$ stand for the event where the minimal length of a cycle separating $B_{\mathcal{P}}(r)$ from infinity is greater than r/N . By Theorem 2.1, we can fix $N \geq 1$ large enough so that $\mathbb{P}(E_{r,N}) > 1 - \varepsilon/2$ for every $r \geq 1$.

We fix $c > 0$ such that $M(N + 1)^4 c^4 < 1$. Then, let $n \geq 1$ large enough so that $cn^{1/4} \geq 1$ and $M((N + 1)\lceil cn^{1/4} \rceil)^4 < n$. We argue on the event

$$\{|B_{\mathcal{P}}^\bullet((N + 1)\lceil cn^{1/4} \rceil)| < n\} \cap E_{N\lceil cn^{1/4} \rceil, N}$$

which has probability at least $1 - \varepsilon$ by the preceding observations (we use (2.30) and our choice of c).

We claim that, on the latter event, we have $|\partial A| \geq cn^{1/4}$ for every $A \in \mathcal{K}$ such that $|A| \geq n$. Indeed, writing Δ for the root vertex of \mathcal{P} , we distinguish two cases:

- If $d_{\text{gr}}(\Delta, \partial A) > N \lceil cn^{1/4} \rceil + 1$, we note that the ball $B_{\mathcal{F}}(N \lceil cn^{1/4} \rceil)$ is separated from infinity by the cycle ∂A , which implies that $|\partial A| \geq cn^{1/4}$ by the very definition of the event $E_{N \lceil cn^{1/4} \rceil, N}$.
- If $d_{\text{gr}}(\Delta, \partial A) \leq N \lceil cn^{1/4} \rceil + 1$, then we argue by contradiction assuming that $|\partial A| < cn^{1/4}$ and in particular the diameter of ∂A is bounded above by $\lceil cn^{1/4} \rceil - 1$. To simplify notation, we set $r_n = (N + 1) \lceil cn^{1/4} \rceil$. The property $d_{\text{gr}}(\Delta, \partial A) \leq N \lceil cn^{1/4} \rceil + 1$ ensures that any vertex of ∂A is at distance at most r_n from Δ , and therefore any edge of ∂A is incident to a vertex at distance at most $r_n - 1$ from Δ . It follows that any face incident to an edge of ∂A is contained in the hull $B_{\mathcal{F}}^{\bullet}(r_n)$. Consequently, the whole boundary ∂A is contained in $B_{\mathcal{F}}^{\bullet}(r_n)$, and so is the set A . In particular, $|A| \leq |B_{\mathcal{F}}^{\bullet}(r_n)| < n$, which is a contradiction with our assumption $|A| \geq n$.

This completes the proof of Proposition 2.4. \square

Before we proceed to the proof of Theorem 2.3, we state a proposition which is a key ingredient of this proof.

Proposition 2.15. *There exists a constant M_0 such that, for every $r \geq 1$,*

$$\mathbb{E}[|B_{\mathcal{F}}^{\bullet}(r)|] \leq M_0 r^4.$$

We postpone the proof of Proposition 2.15 to the end of the section.

Proof of Theorem 2.3. Thanks to Proposition 2.15 and Markov's inequality, we have for every $r \geq 1$, for every $a > 0$,

$$\mathbb{P}(|B_{\mathcal{F}}^{\bullet}(r)| \geq a) \leq \frac{M_0 r^4}{a}. \quad (2.31)$$

We then proceed in a way very similar to the proof of Proposition 2.4. Let $\delta \in (0, 1/4)$ and $\delta' = \delta/2$. For every integer $p \geq 1$, set

$$c_p = p^{-\frac{3}{4} - \delta}$$

and

$$N_p = \lceil p^{\frac{1}{2} + \delta'} \rceil.$$

Recalling the notation $E_{r, N}$ in the proof of Proposition 2.4, we first observe that the bound of Theorem 2.1 (i) implies

$$\sum_{p=1}^{\infty} \mathbb{P}(E_{N_p \lceil c_p 2^{p/4} \rceil, N_p}^c) < \infty.$$

Similarly, the bound (2.31) gives

$$\sum_{p=1}^{\infty} \mathbb{P}(|B_{\mathcal{F}}^{\bullet}((N_p + 1) \lceil c_p 2^{p/4} \rceil)| \geq 2^p) < \infty.$$

The Borel-Cantelli lemma then shows that a.s. there exists an integer p_0 such that the event

$$\{|B_{\mathcal{D}}^{\bullet}((N_p + 1)\lceil c_p 2^{p/4} \rceil) | < 2^p\} \cap E_{N_p \lceil c_p 2^{p/4} \rceil, N_p}$$

holds for every $p \geq p_0$. However, we can now reproduce the same arguments as in the end of the proof of Proposition 2.4 (replacing n by 2^p , N by N_p and c by c_p) to get that, if the latter event holds for some $p \geq 1$, then for every $A \in \mathcal{K}$ such that $2^p \leq |A| \leq 2^{p+1}$, we have

$$|\partial A| \geq \lceil c_p 2^{p/4} \rceil \geq 2^{-1/4} (\ln 2) |A|^{\frac{1}{4}} (\ln |A|)^{-\frac{3}{4}-\delta}.$$

This completes the proof. \square

We still have to prove Proposition 2.15. We note that the continuous analog of the UIPQ, or of the UIPT, is the Brownian plane introduced in [20]. For every $r > 0$ the definition of the hull of radius r makes sense in the Brownian plane and the explicit distribution of the volume of the hull was computed in [21], showing in particular that the expected volume is finite and (by scaling invariance) equal to a constant times r^4 . On the other hand, in the case of the UIPT, Ménard [53] was recently able to compute the exact distribution of the volume of hulls. Such an exact expression is not yet available in the case of the UIPQ *Note from the PhD candidate: an exact expression is computed in section 4.4.2.*, and so we use a different method based on the skeleton decomposition.

Before we proceed to the proof of Proposition 2.15, we state a lemma concerning truncated quadrangulations with a Boltzmann distribution. Let $p \geq 1$. We say that a random truncated quadrangulation \mathbf{M} with boundary size $p \geq 1$ is Boltzmann distributed if, for every integer $n \geq 1$, for every $\mathcal{M} \in \mathbb{Q}_{n,p}^{\text{tr}}$, $\mathbb{P}(\mathbf{M} = \mathcal{M}) = Z(p)^{-1} 12^{-n}$.

Lemma 2.16. *There exists a constant $L_0 > 0$ such that, for every $p \geq 1$, if \mathbf{M}_p is a Boltzmann distributed truncated quadrangulation with boundary size p ,*

$$\mathbb{E}[\#\text{Inn}(\mathbf{M}_p)] \underset{p \rightarrow \infty}{\sim} L_0 p^2.$$

Remark 2.5. *By analogy with the case of triangulations [22, Proposition 9], one expects that $p^{-2} \#\text{Inn}(\mathbf{M}_p)$ converges in distribution to (a scaled version of) the distribution with density $(2\pi)^{-1/2} x^{-5/2} \exp(-\frac{1}{2x})$.*

Proof. By definition,

$$\mathbb{E}[\#\text{Inn}(\mathbf{M}_p)] = Z(p)^{-1} \sum_{n=p}^{\infty} n 12^{-n} \#\mathbb{Q}_{n,p}^{\text{tr}}.$$

In particular, (2.2) shows that $\mathbb{E}[\#\text{Inn}(\mathbf{M}_p)] < \infty$. From the definition of U in Section 2.2.5, we have

$$Z(p) = \sum_{n=p}^{\infty} 12^{-n} \#\mathbb{Q}_{n,p}^{\text{tr}} = [y^p] U\left(\frac{1}{12}, y\right),$$

where as usual $[y^p]U(\frac{1}{12}, y)$ denotes the coefficient of y^p in the series expansion of $U(\frac{1}{12}, y)$. Then the explicit formula (2.1), and standard singularity analysis [28, Corollary VI.1], show that

$$Z(p) \underset{p \rightarrow \infty}{\sim} c' p^{-5/2} 2^{-p} \quad (2.32)$$

for some constant $c' > 0$, whose exact value is unimportant for our purposes. Similarly,

$$\sum_{n=p}^{\infty} n 12^{-n} \#\mathbb{Q}_{n,p}^{\text{tr}} = [y^p] \frac{\partial U}{\partial x} \left(\frac{1}{12}, y \right).$$

where the partial derivative is a left derivative at $x = 1/12$. Formula (3) in [38] gives

$$\frac{\partial U}{\partial x} \left(\frac{1}{12}, y \right) = -\frac{y^2}{2} - \frac{y}{2} \frac{y^2 - 10y - 32}{\sqrt{(18-y)(2-y)}}.$$

and again singularity analysis leads to

$$\sum_{n=p}^{\infty} n 12^{-n} \#\mathbb{Q}_{n,p}^{\text{tr}} \underset{p \rightarrow \infty}{\sim} c_j p^{-1/2} 2^{-p} \quad (2.33)$$

for some other constant $c_j > 0$. The lemma now follows from (2.32) and (2.33). \square

Proof of Proposition 2.15. We first observe that in the statement of the proposition we may replace the standard hull $B_{\mathcal{F}}^{\bullet}(r)$ by the truncated hull $\mathcal{H}_r^{\text{tr}}$. Indeed, the standard hull $B_{\mathcal{F}}^{\bullet}(r)$ is contained in the truncated hull $\mathcal{H}_{r+1}^{\text{tr}}$. So let $N_{(r)}$ be the number of inner faces in the truncated hull $\mathcal{H}_r^{\text{tr}}$. We aim at proving that $\mathbb{E}[N_{(r)}] \leq M_0 r^4$ for some constant M_0 .

Recall our notation $\mathcal{F}_{(r)}^{\circ}$ for skeleton of $\mathcal{H}_r^{\text{tr}}$. As we already noticed in the proof of Corollary 2.9, $\mathcal{F}_{(r)}^{\circ}$ is distributed according to $\mu_{r,1}^{\circ}$. The fact that the distribution of $\mathcal{H}_r^{\text{tr}}$ is $\Delta_{r,1}$ (Corollary 2.8) yields that, conditionally on $\mathcal{F}_{(r)}^{\circ} = \mathcal{F}$, the truncated quadrangulations M_v , $v \in \mathcal{F}^*$ associated with the ‘‘slots’’ are independent, and M_v is Boltzmann distributed with boundary size $c_v + 1$, with the notation c_v for the number of offspring of v in \mathcal{F} . We then observe that, still on the event $\{\mathcal{F}_{(r)}^{\circ} = \mathcal{F}\}$,

$$N_{(r)} \leq \sum_{v \in \mathcal{F}^*} \#\text{Inn}(M_v)$$

(see formula (2.8) in the proof of Lemma 2.6). Using Lemma 2.16, it follows that

$$\mathbb{E}[N_{(r)} | \mathcal{F}_{(r)}^{\circ} = \mathcal{F}] \leq L_0 \mathbb{E} \left[\sum_{v \in \mathcal{F}^*} (1 + c_v)^2 \right].$$

As previously, it is convenient to use the notation $\mathcal{F}_{(r)}$ for the forest obtained by forgetting the distinguished vertex of $\mathcal{F}_{(r)}^{\circ}$ and applying a uniform circular permutation to the trees of $\mathcal{F}_{(r)}^{\circ}$. From the last display, we have also

$$\mathbb{E}[N_{(r)}] \leq L_0 \mathbb{E} \left[\sum_{v \in \mathcal{F}_{(r)}^*} (1 + c_v)^2 \right], \quad (2.34)$$

where we abuse notation by still writing c_v for the number of offspring of the vertex v of $\mathcal{F}_{(r)}$.

In order to bound the expectation in the last display, we first consider vertices v that are roots of trees in $\mathcal{F}_{(r)}^*$ (or equivalently which correspond to edges of \mathcal{C}_r). In the forthcoming calculations, we also assume that $r \geq 2$. Let $c_{(1)}, c_{(2)}, \dots, c_{(H_r)}$ denote the offspring numbers of the roots of the successive trees in $\mathcal{F}_{(r)}^*$. Using formula (2.18) applied with $u = r - 1$ and $w = r$, we get, for every $k \geq 1$,

$$\mathbb{E} \left[\sum_{i=1}^{H_r} (1 + c_{(i)})^2 \mid H_r = k \right] = \mathbb{E} \left[\frac{\varphi_{r-1}(\xi_1 + \dots + \xi_k)}{\varphi_r(k)} \sum_{i=1}^k (1 + \xi_i)^2 \right],$$

where ξ_1, ξ_2, \dots are i.i.d. with distribution θ . Recall formula (2.12) for the distribution of H_r , and the definition (2.19) of $\varphi_r(k)$. Using also the estimate (2.4) for asymptotics of the constants κ_p , we get, with some constant L_1 ,

$$\frac{\mathbb{P}(H_r = k)}{\varphi_r(k)} \leq \frac{L_1}{\sqrt{k}}.$$

We have therefore

$$\mathbb{E} \left[\sum_{i=1}^{H_r} (1 + c_{(i)})^2 \mathbf{1}_{\{H_r=k\}} \right] \leq \frac{L_1}{\sqrt{k}} \mathbb{E} \left[\varphi_{r-1} \left(\sum_{i=1}^k \xi_i \right) \sum_{i=1}^k (1 + \xi_i)^2 \right].$$

At this point, we again use (2.19) to see that there exist positive constants L_2, L_3, a_1 such that, for every $\ell \geq 1$,

$$\varphi_{r-1}(\ell) \leq L_2 \frac{\ell}{r^3} \left(1 - \frac{a_1}{r^2}\right)^\ell \leq L_2 \frac{\ell}{r^3} \exp\left(-\frac{a_1 \ell}{r^2}\right) \leq \frac{L_3}{r} \exp\left(-\frac{a_1 \ell}{2r^2}\right).$$

It follows that

$$\begin{aligned} \mathbb{E} \left[\sum_{i=1}^{H_r} (1 + c_{(i)})^2 \mathbf{1}_{\{H_r=k\}} \right] &\leq \frac{L_1 L_3}{r \sqrt{k}} \mathbb{E} \left[\left(\sum_{i=1}^k (1 + \xi_i)^2 \right) \exp \left(-\frac{a_1}{2r^2} \sum_{i=1}^k \xi_i \right) \right] \\ &= \frac{L_1 L_3 \sqrt{k}}{r} \mathbb{E} \left[(1 + \xi_1)^2 \exp\left(-\frac{a_1}{2r^2} \xi_1\right) \right] \left(\mathbb{E} \left[\exp\left(-\frac{a_1}{2r^2} \xi_1\right) \right] \right)^{k-1}. \end{aligned}$$

Using the asymptotics (2.9) for $\theta(k)$, it is elementary to verify that

$$\mathbb{E} \left[(1 + \xi_1)^2 \exp\left(-\frac{a_1}{2r^2} \xi_1\right) \right] \leq L_4 r$$

for some constant L_4 . Moreover, we can also find a constant $a_2 > 0$ such that

$$\mathbb{E} \left[\exp\left(-\frac{a_1}{2r^2} \xi_1\right) \right] \leq 1 - \frac{a_2}{r^2}.$$

We then conclude that

$$\mathbb{E} \left[\sum_{i=1}^{H_r} (1 + c_{(i)})^2 \mathbf{1}_{\{H_r=k\}} \right] \leq L_1 L_2 L_4 \sqrt{k} \left(1 - \frac{a_2}{r^2}\right)^{k-1}.$$

By summing this estimate over $k \geq 1$, we get

$$\mathbb{E} \left[\sum_{i=1}^{H_r} (1 + c_{(i)})^2 \right] \leq L_5 r^3$$

with some other constant L_5 .

A similar estimate holds if instead of summing over the roots of trees in the forest $\mathcal{F}_{(r)}$ we sum over vertices at generation $r - j$, for every $1 \leq j \leq r - 1$, as this amounts to replacing $\mathcal{F}_{(r)}$ by the forest $\mathcal{F}_{(j)}$ (the case $j = 1$ requires a slightly different argument since we assumed $r \geq 2$ in the above calculations). By summing over j , recalling (2.34), we conclude that

$$\mathbb{E}[N_{(r)}] \leq L_0 \mathbb{E} \left[\sum_{v \in \mathcal{F}_{(r)}^*} (1 + c_v)^2 \right] \leq L_6 \sum_{j=1}^r j^3$$

with some constant L_6 . This completes the proof of Proposition 2.15. \square

Appendix. Proof of Lemma 2.13

We first note that $\mathcal{P}_{k,K}(r)$ is trivially empty if $k > 2K/(cr^2)$. If $k \leq 2K/(cr^2)$ and if we restrict our attention to $k \leq k_0$ for some constant k_0 , the bound of the lemma holds for any choice of $\gamma \in (0, 1)$ by choosing the constant C large enough. So we may assume that $k \geq k_0$ where k_0 can be taken large (but fixed).

Recall that $(b(0), b(1), \dots, b(2K))$ stands for a discrete bridge of length $2K$. We first observe that, for every $\ell \in \{0, 1, \dots, 2K - 1\}$, we can “re-root” the bridge $b(\cdot)$ at ℓ by setting, for every $j \in \{0, 1, \dots, 2K\}$,

$$b_\ell(j) := \begin{cases} b(\ell + j) - b(\ell) & \text{if } \ell + j \leq 2K, \\ b(\ell + j - 2K) - b(\ell) & \text{if } \ell + j > 2K. \end{cases}$$

Then $b_\ell(\cdot)$ is again a discrete bridge of length $2K$. Moreover the property defining the event $\mathcal{P}_{k,K}(r)$ holds for $b(\cdot)$ with the sequence of times (m_1, \dots, m_k) if and only if it holds for $b_\ell(\cdot)$ with the sequence $(m_1^{(\ell)}, \dots, m_k^{(\ell)})$ which is obtained by ordering the representatives in $\{0, 1, \dots, 2K - 1\}$ of $m_1 - \ell, \dots, m_k - \ell$ modulo $2K$.

We start with a trivial observation. Let $0 \leq m_1 < m_2 < \dots < m_k < 2K$ be integers. If $i_0 \in \{1, \dots, k\}$ is an index such that the minimal value of $b(\cdot)$ is attained in the interval $[m_{i_0}, m_{i_0+1}]$ (there is at least one such value i_0), then, for every $i, j \in \{1, \dots, k\}$, $i \neq j$, the maximum

$$\max \left(\min_{\ell \in [m_i, m_j]} b(\ell), \min_{\ell \in [m_j, m_i]} b(\ell) \right)$$

is attained for the one among the two intervals $[m_i, m_j]$ and $[m_j, m_i]$ that does not contain $[m_{i_0}, m_{i_0+1}]$.

Suppose that $0 \leq m_1 < m_2 < \dots < m_k < 2K$ are such that the property of the definition of $\mathcal{P}_{k,K}(r)$ holds, and that $k \geq 16$ is an integer multiple of 4 (we can make

the latter assumption without loss of generality). As already mentioned, the property defining $\mathcal{P}_{k,K}(r)$ still holds if $b(\cdot)$ is replaced by the re-rooted bridge $b_\ell(\cdot)$, for every $\ell \in \{0, 1, \dots, 2K - 1\}$, with the sequence $0 \leq m_1^{(\ell)} < m_2^{(\ell)} < \dots < m_k^{(\ell)}$ defined as explained above. Moreover, a simple argument shows that there are at least $\frac{k}{2}$ values of $i \in \{1, \dots, k\}$ such that, if $\ell \in (m_i, m_{i+1}]$, the minimum of $b_\ell(\cdot)$ is attained in an interval $[m_j^{(\ell)}, m_{j+1}^{(\ell)}]$ with $\frac{k}{4} \leq j < \frac{3k}{4}$. Suppose that ℓ is chosen uniformly at random in $\{0, 1, \dots, 2K - 1\}$, conditionally given $b(\cdot)$: the conditional probability for the minimum of $b_\ell(\cdot)$ to be attained in an interval $[m_j^{(\ell)}, m_{j+1}^{(\ell)}]$ with $\frac{k}{4} \leq j < \frac{3k}{4}$ is thus at least

$$\frac{\frac{k}{2} \times cr^2}{2K}.$$

It follows that

$$\frac{ckr^2}{4K} \mathbb{P}(\mathcal{P}_{k,K}(r)) \leq \mathbb{P}(\mathcal{P}_{k,K}^*(r))$$

where $\mathcal{P}_{k,K}^*(r)$ is defined as $\mathcal{P}_{k,K}(r)$ but imposing the additional constraint that the minimum of $b(\cdot)$ is attained in an interval $[m_{i_0}, m_{i_0+1}]$ with $\frac{k}{4} \leq i_0 < \frac{3k}{4}$.

If $\mathcal{P}_{k,K}^*(r)$ holds with the sequence (m_1, \dots, m_k) , at least one of the two properties $m_{k/4} < K$ or $m_{3k/4} > K$ holds. We write $\mathcal{P}_{k,K}^{**}(r)$ for the event where $\mathcal{P}_{k,K}^*(r)$ holds and $m_{k/4} < K$ and we will bound the probability of $\mathcal{P}_{k,K}^{**}(r)$ (the other case where $m_{3k/4} > K$ can be treated by time-reversal and leads to the same bound).

Let us argue on the event $\mathcal{P}_{k,K}^{**}(r)$. Using the definition of $\mathcal{P}_{k,K}(r)$ and the trivial observation made at the beginning of the proof, we note that, if $1 \leq i \leq k/4$,

$$b(m_k) + b(m_i) - 2 \min_{\ell \in [m_k, m_i]} b(\ell) < 5r$$

and in particular

$$b(m_i) < \min_{\ell \in [m_k, m_i]} b(\ell) + 5r \leq \underline{b}(m_i) + 5r, \quad (2.35)$$

using the notation $\underline{b}(\ell) = \min\{b(j) : 0 \leq j \leq \ell\}$.

We fix an integer $n \geq 1$ such that, if $S(0), S(1), \dots$ is a simple random walk on \mathbb{Z} started from 0, the quantity

$$\mathbb{P}\left(\min_{0 \leq \ell \leq \lfloor ncr^2 \rfloor} S(\ell) \geq -10r\right)$$

is bounded above by a constant $\alpha < 1$ independent of $r \geq 1$. Notice that the choice of n only depends on c . In what follows we assume that k is large enough so that $(\frac{k}{4} - 1)/n \geq 1$ (recall the first observation of the proof).

We then define by induction $T_1 = 0$ and, for every integer $p \geq 1$,

$$T_{p+1} := \inf\{\ell \geq T_p + ncr^2 : b(\ell) \leq \underline{b}(\ell) + 5r\} \wedge K,$$

where $\inf \emptyset = \infty$ as usual. Recalling that we argue on $\mathcal{P}_{k,K}^{**}(r)$, we notice that there are at least $\lfloor (\frac{k}{4} - 1)/n \rfloor$ (consecutive) values of $p \geq 1$ such that $m_1 \leq T_p \leq m_{k/4}$. Indeed, the

first time T_p that exceeds m_1 must be smaller than m_{n+1} (by (2.35) and our assumption $m_{i+1} - m_i > cr^2$), the next one must be smaller than m_{2n+1} if $2n + 1 \leq k/4$, and so on. Moreover, if $p \geq 1$ is such that $m_1 \leq T_p < T_{p+1} \leq m_{k/4}$, we have

$$b(T_p) \leq \underline{b}(T_p) + 5r \leq b(m_1) + 5r \leq \min_{\ell \in [m_1, m_{k/4}]} b(\ell) + 10r \leq \min_{\ell \in [T_p, T_{p+1}]} b(\ell) + 10r,$$

where in the third inequality we use the fact that $b(m_1) \leq \min\{b(\ell) : \ell \in [m_1, m_{k/4}]\} + 5r$, from the definition of $\mathcal{P}_{k,K}(r)$. Set $N_k = \lfloor (\frac{k}{4} - 1)/n \rfloor - 1$. We have obtained that $\mathcal{P}_{k,K}^{**}(r)$ is contained in the event

$$\mathcal{E}_{k,K}(r) := \bigcup_{j=0}^{\infty} \left\{ T_{j+N_k+1} < K \right. \\ \left. \text{and } b(T_{j+p}) \leq \min_{\ell \in [T_{j+p}, T_{j+p+1}]} b(\ell) + 10r, \text{ for every } 1 \leq p \leq N_k \right\}.$$

Note that in the last display we can restrict the union to values of $j \in \{0, 1, \dots, \lfloor \frac{K}{ncr^2} \rfloor - 1\}$, since by construction $T_p \geq (p-1)ncr^2$ if $T_p < K$.

Recall that $S(0), S(1), \dots$ is a simple random walk on \mathbb{Z} started from 0, and let $\tilde{T}_1, \tilde{T}_2, \dots$ be the stopping times defined like T_1, T_2, \dots by replacing $(b(0), \dots, b(2K))$ by $(S(0), \dots, S(2K))$ and removing “ $\wedge K$ ”. We know that the distribution of $(b(0), b(1), \dots, b(K))$ is absolutely continuous with respect to that of $(S(0), S(1), \dots, S(K))$, with a Radon-Nikodym derivative that is bounded by a constant M independent of K . It follows that

$$\begin{aligned} \mathbb{P}(\mathcal{E}_{k,K}(r)) &\leq M \sum_{j=0}^{\lfloor \frac{K}{ncr^2} \rfloor - 1} \mathbb{P}\left(S(\tilde{T}_{j+p}) \leq \min_{\ell \in [\tilde{T}_{j+p}, \tilde{T}_{j+p+1}]} S(\ell) + 10r, \forall 1 \leq p \leq N_k\right) \\ &\leq M \times \lfloor \frac{K}{ncr^2} \rfloor \times \mathbb{P}\left(\min_{0 \leq \ell \leq ncr^2} S(\ell) \geq -10r\right)^{N_k} \\ &\leq M \times \lfloor \frac{K}{ncr^2} \rfloor \times \alpha^{N_k} \end{aligned}$$

using the strong Markov property of S in the second line, and our choice of n in the last one. We conclude that

$$\mathbb{P}(\mathcal{P}_{k,K}^{**}(r)) \leq \mathbb{P}(\mathcal{E}_{k,K}(r)) \leq M \times \lfloor \frac{K}{ncr^2} \rfloor \times \alpha^{N_k}$$

and since we have

$$\mathbb{P}(\mathcal{P}_{k,K}(r)) \leq \left(\frac{ckr^2}{4K}\right)^{-1} \times 2 \mathbb{P}(\mathcal{P}_{k,K}^{**}(r))$$

we get the bound of the lemma.

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Chapitre 3

First-passage percolation in random planar maps and Tutte's bijection

This chapter is taken from the article [46].

We consider large random planar maps and study the first-passage percolation distance obtained by assigning independent identically distributed lengths to the edges. We consider the cases of quadrangulations and of general planar maps. In both cases, the first-passage percolation distance is shown to behave in large scales like a constant times the usual graph distance. We apply our method to the metric properties of the classical Tutte bijection between quadrangulations with n faces and general planar maps with n edges. We prove that the respective graph distances on the quadrangulation and on the associated general planar map are in large scales equivalent when $n \rightarrow \infty$.

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3.1 Introduction

A planar map is a finite planar graph embedded in the sphere and considered up to orientation-preserving homeomorphisms. In this work, we only consider rooted planar maps, meaning that we distinguish an oriented edge called the root edge, whose origin is the root vertex. There exist many different models of random maps depending on the conditions one imposes on the degrees of faces, the existence or non-existence of multiple edges and loops, etc. In the following, we always allow loops and multiple edges. A particular case that will play a central role in this article is the case of quadrangulations, where all faces have degree 4. For any map M , we denote its vertex set by $V(M)$ and the graph distance on the map M by d_{gr}^M . The root vertex is usually denoted by ρ .

Several recent developments (see in particular [1, 8, 9, 41, 55]) have established that, for a wide range of models of random maps, the vertex set viewed as a metric space for the (suitably rescaled) graph distance, converges in distribution when the size of the map tends to infinity towards a random compact metric space called the Brownian map. This convergence holds in the sense of the Gromov-Hausdorff convergence for compact metric spaces. The convergence to the Brownian map gives a unified approach to asymptotic properties of different models of random planar maps.

A natural question is to ask what can be said when the graph distance is replaced by other choices of distances on the vertex set $V(M)$. A simple way to get other distances is to assign positive weights (or lengths) to the edges: The distance between two vertices will then be the minimal total weight of a path connecting these two vertices. When the weights of the different edges are chosen to be independent and identically distributed given the planar map M in consideration, this leads to the so-called first-passage percolation distance, which we denote here by d_{fpp}^M . Of course, when weights are all equal to 1 we recover the graph distance. The recent paper [23] has investigated the asymptotic properties of the first-passage percolation distance in large triangulations. Roughly speaking, the main results of [23] show that, in large scales, the first-passage percolation distance behaves like a constant times the graph distance. The relevant constant is found via a sub-additivity argument and cannot be computed in general. Interestingly, this behavior for random planar maps is quite different from the one observed in deterministic lattices such as \mathbb{Z}^d , where the first-passage percolation distance is not believed to be asymptotically proportional to the graph distance (nor to the Euclidean distance).

One of the main goals of the present work is to show that results similar to those of [23] hold both for quadrangulations and for general planar maps. Recall that the diameter of a typical quadrangulation with n faces, or of a typical planar map with n edges is known to be of order $n^{1/4}$.

Theorem 3.1. *For every integer $n \geq 1$, let Q_n be uniformly distributed over the set of all rooted quadrangulations with n faces, and let M_n be uniformly distributed over the set of all rooted planar maps with n edges. Define the first-passage percolation distances $d_{\text{fpp}}^{Q_n}$ and $d_{\text{fpp}}^{M_n}$ by assigning independent and identically distributed weights to the edges of Q_n and M_n . Assume that the common distribution of the weights is supported on a compact subset of $(0, \infty)$. Then there exist two positive constants \mathbf{c} and \mathbf{c}' such that*

$$n^{-1/4} \sup_{x,y \in V(Q_n)} \left| d_{\text{fpp}}^{Q_n}(x,y) - \mathbf{c} d_{\text{gr}}^{Q_n}(x,y) \right| \xrightarrow[n \rightarrow \infty]{} 0$$

and

$$n^{-1/4} \sup_{x,y \in V(M_n)} \left| d_{\text{fpp}}^{M_n}(x,y) - \mathbf{c}' d_{\text{gr}}^{M_n}(x,y) \right| \xrightarrow[n \rightarrow \infty]{} 0$$

where both convergences hold in probability.

As an immediate consequence of the theorem, we get that the convergence to the Brownian map still holds for both models in consideration if the graph distance is replaced by the first-passage percolation distance. More precisely, under the assumptions of Theorem 3.1 and with the same constants \mathbf{c} and \mathbf{c}' , we have

$$\left(V(Q_n), \left(\frac{9}{8n} \right)^{1/4} d_{\text{fpp}}^{Q_n} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathfrak{m}_\infty, \mathbf{c} D^*), \quad (3.1)$$

and

$$\left(V(M_n), \left(\frac{9}{8n} \right)^{1/4} d_{\text{fpp}}^{M_n} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathfrak{m}_\infty, \mathbf{c}' D^*), \quad (3.2)$$

where (m_∞, D^*) is the Brownian map, and both convergences hold in distribution in the Gromov-Hausdorff space. Indeed, this follows from Theorem 3.1 and the known convergences for the graph distance which have been established in [41, 55] for Q_n and in [9] for M_n . It is remarkable that the same constant $(9/8)^{1/4}$ appears in both (3.1) and (3.2). This will be better understood in the next theorem.

Another major goal of the present article is to have a better understanding of the metric properties of Tutte's bijection. Recall that Tutte's bijection, also called the *trivial bijection*, gives for every $n \geq 1$ a one-to-one correspondence between the set of all rooted quadrangulations with n faces and the set of all rooted planar maps with n edges. The definition of this correspondence should be clear from Figure 3.1. The following theorem can be interpreted by saying that Tutte's transformation acting on a large quadrangulation is nearly isometric with respect to the graph distances.

Theorem 3.2. *Let Q_n be uniformly distributed over the set of all rooted quadrangulations with n faces and let M_n be the image of Q_n under Tutte's bijection, so that M_n is uniformly distributed over the set of all rooted planar maps with n edges and $V(M_n)$ is identified to a subset of $V(Q_n)$. Then,*

$$n^{-1/4} \sup_{x,y \in V(M_n)} \left| d_{\text{gr}}^{Q_n}(x,y) - d_{\text{gr}}^{M_n}(x,y) \right| \xrightarrow[n \rightarrow \infty]{} 0$$

where the convergence holds in probability.

We can combine Theorems 3.1 and 3.2 to get that, if Q_n and M_n are linked by Tutte's bijection, the convergences in distribution (3.1) and (3.2) hold jointly, with the *same* space (m_∞, D^*) in the limit. This is reminiscent of Theorem 1.1 in [9], which gives a similar joint convergence, but in the case where Q_n and M_n are linked by a different bijection (the Ambjørn-Budd bijection) which is more faithful to graph distances.

We can also give versions of the preceding results for the infinite random lattices that arise as local limits (in the Benjamini-Schramm sense) of large quadrangulations or general planar maps. We write Q_∞ for the UIPQ or uniform infinite planar quadrangulation, and M_∞ for the UIPM or uniform infinite planar map. As was observed by Ménard and Nolin [54], the UIPM can be obtained by applying (a generalized version of) Tutte's correspondence to the UIPQ.

Theorem 3.3. *Let $d_{\text{fpp}}^{Q_\infty}$ and $d_{\text{fpp}}^{M_\infty}$ be the first-passage percolation distances defined on the vertex sets $V(Q_\infty)$ and $V(M_\infty)$, respectively, by assigning edge weights satisfying the same assumptions as in Theorem 3.1. Write ρ_{Q_∞} and ρ_{M_∞} for the respective root vertices of Q_∞ and M_∞ . Then, for every $\varepsilon > 0$,*

$$\lim_{r \rightarrow \infty} \mathbb{P} \left(\sup_{x,y \in V(Q_\infty), d_{\text{gr}}^{Q_\infty}(\rho_{Q_\infty}, x) \vee d_{\text{gr}}^{Q_\infty}(\rho_{Q_\infty}, y) \leq r} |d_{\text{fpp}}^{Q_\infty}(x,y) - c d_{\text{gr}}^{Q_\infty}(x,y)| > \varepsilon r \right) = 0,$$

and similarly,

$$\lim_{r \rightarrow \infty} \mathbb{P} \left(\sup_{x,y \in V(M_\infty), d_{\text{gr}}^{M_\infty}(\rho_{M_\infty}, x) \vee d_{\text{gr}}^{M_\infty}(\rho_{M_\infty}, y) \leq r} |d_{\text{fpp}}^{M_\infty}(x,y) - c' d_{\text{gr}}^{M_\infty}(x,y)| > \varepsilon r \right) = 0,$$

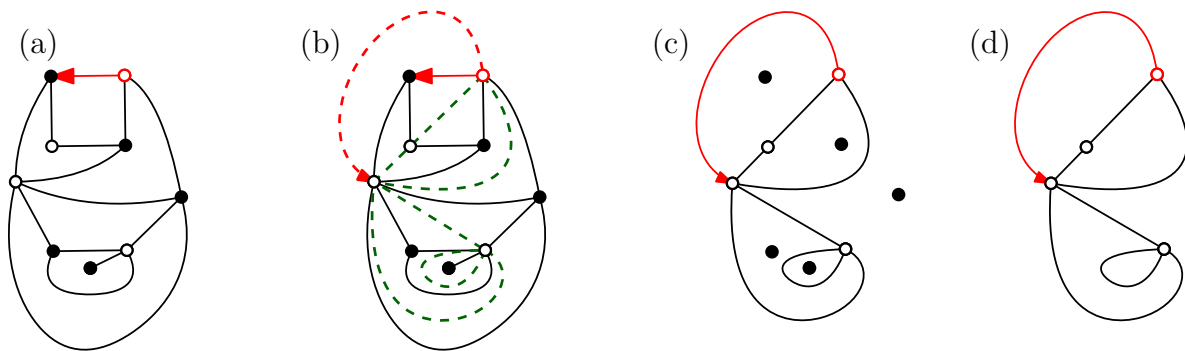


Figure 3.1 – Illustration of Tutte’s bijection. (a) On the left, a quadrangulation with 7 faces. Color the tail of its root vertex in white, and every other vertex in black and white so that adjacent vertices have a different color. (b) In every face of the quadrangulation, add a diagonal between its white corners. (c) Erase the edges of the quadrangulation. (d) The black vertices now have degree zero and are also erased. We obtain a map with 7 edges, which is rooted at the edge corresponding to the diagonal drawn in the face to the right of the root edge of the quadrangulation, oriented so that the root vertex remains the same.

with the same constants \mathbf{c} and \mathbf{c}' as in Theorem 3.1. Moreover, if the UIPQ Q_∞ and the UIPM M_∞ are linked by Tutte’s correspondence, we have also

$$\lim_{r \rightarrow \infty} \mathbb{P} \left(\sup_{x, y \in V(M_\infty), d_{\text{gr}}^{M_\infty}(\rho_{M_\infty}, x) \vee d_{\text{gr}}^{M_\infty}(\rho_{M_\infty}, y) \leq r} |d_{\text{gr}}^{M_\infty}(x, y) - d_{\text{gr}}^{Q_\infty}(x, y)| > \varepsilon r \right) = 0.$$

A consequence of the first assertions of the theorem is the fact that balls (centered at the root vertex) for the first-passage percolation distance are asymptotically the same as for the graph distance, both in Q_∞ and in M_∞ . More precisely, in Q_∞ for definiteness, the (metric) ball of radius r for the first-passage percolation distance will be contained in the graph distance ball of radius $(1 + \varepsilon)r/\mathbf{c}$ and will contain the graph distance ball of radius $(1 - \varepsilon)r/\mathbf{c}$, with high probability when r is large. This is in sharp contrast with the behavior expected for deterministic lattices.

In the same way as in [23], our proofs rely on a “skeleton decomposition” which in the case of quadrangulations appeared first in the work of Krikun [38], and has been used extensively in [43]. Recall that, in the UIPQ Q_∞ , the hull of radius r is defined as the complement of the infinite connected component of the complement of the ball of radius r centered at the root vertex (informally, the hull is obtained by filling in the finite holes in the ball of radius r , see Section 3.2 for a more precise definition). The skeleton decomposition provides a detailed description of the joint distribution of layers of the UIPQ, where, roughly speaking, a layer corresponds to the part of the map between the boundary of the hull of radius r and the boundary of the hull of radius $r + 1$. This description allows us to compare the UIPQ near the boundary of a hull with another infinite model which we call the LHPQ for lower half-plane quadrangulation (Section 3.3).

The point is then that a subadditive ergodic theorem can be applied to evaluate first-passage percolation distances in the LHPQ. Quite remarkably, this method carries over to the study of the first-passage percolation distance in the general planar maps that are obtained from quadrangulations by Tutte's correspondence, with the minor difference that we must restrict to hulls of even radius in the quadrangulation.

Even though the idea of using the skeleton decomposition already appeared in [23] in the setting of triangulations, there are important differences between the present work and [23], and our proofs are by no means straightforward extensions of those of [23]. In particular, a very important ingredient of our method involves bounds on graph distances along the boundary in the LHPQ. To derive these bounds we use a completely different approach from that developed in [23] for the model called the lower half-plane triangulation. Our approach, which relies on certain ideas of [19], is simpler and avoids the heavy combinatorial analysis of [23]. Similarly, the application of the subadditive ergodic theorem gives information about the first-passage percolation distance between points of the boundary of a hull of the UIPQ and the root vertex (Proposition 3.17 below) but a key step is then to derive information about the distance between a typical point and the root vertex in the finite quadrangulation Q_n (Proposition 3.18): For this purpose, the lack of certain explicit combinatorial expressions did not allow us to use the same approach as in [23], and we had to develop a different method based on a coupling between Q_n and the UIPQ. Finally, the treatment of a general planar map M_n given as the image of the quadrangulation Q_n under Tutte's bijection also required a number of new tools, in particular because the graph distance on M_n is not easily controlled in terms of the graph distance on Q_n .

The paper is organized as follows. Section 3.2 gives a number of preliminaries about planar maps and the skeleton decomposition. We introduce the notion of a quadrangulation of the cylinder, which already played an important role in [43], and we define the so-called truncated hulls, which are variants of the (standard) hulls considered in earlier work. Section 3.3 discusses the lower half-plane quadrangulation. In particular, we derive the important bounds controlling distances along the boundary (Proposition 3.8). Section 3.4 gives two technical propositions. The first one (Proposition 3.14) provides bounds for the distribution of the skeleton of a (truncated) hull of the UIPQ in terms of the skeleton associated with the LHPQ. This is the key to transfer results obtained in the LHPQ (by subadditive arguments) to the UIPQ. Section 3.5 derives our main results about first-passage percolation distances in quadrangulations. We start by proving Proposition 3.16, which estimates the $d_{\text{fpp}}^{Q_\infty}$ -distance between a vertex of the boundary of the (truncated) hull of radius r and the boundary of the hull of radius $r - \lfloor \eta r \rfloor$, for $\eta > 0$ small. This key proposition is then used to get Proposition 3.17 concerning the distance between a point of the boundary of a hull and the root vertex. Then the hard work is to prove Proposition 3.18 controlling the distance between a uniformly distributed vertex of Q_n and the root vertex. From this proposition, it is not too hard to derive Theorem 3.22, which gives the part of Theorem 3.1 dealing with quadrangulations. Section 3.6 contains certain technical results concerning graph distances in the general maps associated with quadrangulations via Tutte's correspondence. We introduce the so-called downward paths, which are closely

related to the skeleton decomposition of the associated quadrangulation, and we derive important bounds on the length of these paths (Lemma 3.25). Finally, Section 3.7 is devoted to the proof of the results concerning general maps. In particular, Theorem 3.34 shows that, if M_n and Q_n are linked by Tutte's bijection, the first-passage percolation distance in M_n is asymptotically proportional to the graph distance in Q_n . In the case where weights are equal to 1, the proportionality constant has to be equal to 1 (because of the known results about scaling limits of Q_n and M_n), which gives Theorem 3.2 and then the part of Theorem 3.1 concerning general maps. Several proofs in this section are very similar to the proofs of Section 3.5. For this reason, we have only sketched certain arguments, but we emphasize the places where new ingredients are required.

3.2 Preliminaries

A (finite) *planar map* is a planar graph embedded in the sphere and seen up to orientation-preserving homeomorphisms. We allow multiple edges and loops. Since we consider only planar maps we often say *map* instead of planar map. If M is a map, we denote the set of vertices, edges, and faces of M by $V(M)$, $E(M)$, $F(M)$ respectively. We write d_{gr}^M for the *graph distance* on $V(M)$. Given i.i.d. random weights $(\omega_e)_{e \in E(M)}$ assigned to the edges of M , we also define the associated *first-passage percolation distance* d_{fpp}^M as follows. We define the weight of a path γ as the sum of the weights of its edges, and the first-passage percolation distance $d_{\text{fpp}}^M(x, y)$ between two vertices x and y of M is the infimum of the weights of paths starting at x and ending at y . Note that, if $\omega_e = 1$ for every edge e , we recover the graph distance on $V(M)$.

A *rooted map* is a map with a distinguished oriented edge called the *root edge*. The tail of the root edge is called the *root vertex* and is usually denoted by ρ . The face lying to the right of the root edge is called the root face. We say that a rooted map is *pointed* if in addition it has a distinguished vertex ∂ .

Models of quadrangulations. A *quadrangulation* is a rooted map whose faces all have degree 4. Quadrangulations are bipartite maps, so we may and will color the vertices of a quadrangulation in black and white so that two adjacent vertices have different colors and the root vertex is white.

A *truncated quadrangulation* is a rooted map such that

- the root face has a simple boundary and an arbitrary degree,
- every edge incident to the root face is also incident to another face which has degree 3, and these triangular faces are distinct,
- all the other faces have degree 4.

The root face is also called the external face, and faces other than the external face are

called inner faces. The length of the external boundary (the boundary of the external face) is called the perimeter of the truncated quadrangulation.

We will also consider infinite (rooted but not pointed) quadrangulations which we assume — except in the case of the LHPQ discussed in Section 3.3 — are embedded in the plane in such a way that all faces are bounded subsets of the plane, and furthermore every compact subset of the plane intersects only finitely many faces (and again infinite quadrangulations are viewed up to orientation preserving homeomorphisms). These properties hold a.s. for the UIPQ.

Hulls and truncated hulls. Let Q be a (finite or infinite) quadrangulation with root vertex ρ . For every integer $r \geq 1$, we denote the *ball of radius r in Q* by $B_Q(r)$. This ball is the map obtained by taking the union of all faces that are incident to a vertex at graph distance at most $r - 1$ from ρ . Suppose in addition that Q is finite and pointed, and let $R = d_{\text{gr}}^Q(\rho, \partial)$ be the graph distance between the root vertex and the distinguished vertex. Then for every integer $1 \leq r \leq R - 2$, the *standard hull of radius r of Q* , denoted by $B_Q^\bullet(r)$, is the union of $B_Q(r)$ and of the connected components of its complement that do not contain ∂ . If Q is an infinite quadrangulation, then, for every $r \geq 1$, the standard hull of radius r of Q is defined as the union of $B_Q(r)$ and the finite connected components of its complement, and is also denoted by $B_Q^\bullet(r)$. In both the finite and the infinite case, the standard hull $B_Q^\bullet(r)$ is a quadrangulation with a simple boundary (meaning that all faces are quadrangles, except for one distinguished face, which has a simple boundary). If $r > 1$ is not an integer, we will agree that $B_Q(r) = B_Q(\lfloor r \rfloor)$ and $B_Q^\bullet(r) = B_Q^\bullet(\lfloor r \rfloor)$.

We also need to define *truncated hulls*. To this end, consider first the case where Q is finite and pointed. We label the vertices of Q with their graph distance to ρ , and we consider an integer r such that $0 < r < d_{\text{gr}}^Q(\rho, \partial)$. Inside every face such that the labels of its incident corners are $r, r - 1, r, r + 1$, we draw a “diagonal” between the corners of label r . The added edges form a collection of cycles, from which we extract a “maximal” simple cycle $\partial_r Q$. This cycle is maximal in the sense that the connected component of the complement of $\partial_r Q$ that contains the distinguished vertex contains no vertex with label less than or equal to r . See [43, Section 2.2] for more details. The exterior of $\partial_r Q$ is the connected component of the complement of $\partial_r Q$ that contains the marked vertex of Q . If Q is an infinite quadrangulation, the cycles $\partial_r Q$ can be defined in exactly the same way, now for every integer $r > 0$ (the exterior of $\partial_r Q$ is now the the unbounded connected component of the complement of $\partial_r Q$).

In both the finite and the infinite case, the *truncated hull* of radius r of Q is the map $\mathcal{H}_Q^{\text{tr}}(r)$ made of $\partial_r Q$ and of the edges of Q inside $\partial_r Q$, and is rooted at the “same” edge as Q . Then we may view $\mathcal{H}_Q^{\text{tr}}(r)$ as a truncated quadrangulation (for which the external face corresponds to the exterior of $\partial_r Q$) provided we re-root $\mathcal{H}_Q^{\text{tr}}(r)$ at an edge of its external boundary.

Quadrangulations of the cylinder. A *quadrangulation of the cylinder of height $R > 0$* is a rooted map Q with two distinguished faces, called the *top* and *bottom* faces (the other



Figure 3.2 – We can split the root edge of a truncated hull, add a loop inside the newly created face, and see the map as a quadrangulation of the cylinder of bottom cycle the added loop.

faces are called *inner faces*), such that

- (i) the boundary of the top (resp. bottom) face, called the *top boundary* (resp. the *bottom boundary*) is a simple cycle,
- (ii) Q is rooted at an oriented edge of its bottom boundary so that the bottom face lies on the right of the root edge (so the bottom face is the root face),
- (iii) every edge of the top (resp. bottom) boundary is incident both to the top (resp. bottom) face and to a triangular face, these triangular faces are distinct, and all other inner faces have degree 4,
- (iv) any vertex of the top boundary is at graph distance R from the bottom boundary, and the inner triangular face incident to any edge of the top boundary is also incident to a vertex at graph distance $R - 1$ from the bottom boundary.

Let Q be a quadrangulation of the cylinder of height R . Label every vertex of Q by its distance from the bottom boundary. For $0 < r < R$ we can define $\partial_r Q$ and the truncated hull of radius r in a way very similar to what we did for pointed quadrangulations ($\partial_r Q$ is the “maximal” cycle made of diagonals between corners labeled r in faces whose corners are labeled $r - 1, r, r + 1, r$, and the exterior of $\partial_r Q$ now contains the top cycle). See [43, Section 2.3] for details. Note that the truncated hull of radius r of Q is itself a quadrangulation of the cylinder of height r . By convention, we agree that $\partial_0 Q$ denotes the bottom boundary, and $\partial_R Q$ stands for the top boundary of Q . We will assume that quadrangulations of the cylinder are drawn in the plane in such a way that the top face is the unbounded face (see Figure 3.3). Then we will orient the cycles $\partial_r Q$ clockwise by convention.

We may view the truncated hull of radius r of a pointed quadrangulation Q as a quadrangulation of the cylinder of height r , by splitting the root edge of Q into a double edge and adding a loop inside the newly created face as in Figure 3.2. In this way, we get a quadrangulation of the cylinder of height r whose bottom cycle is a loop.

Left-most geodesics. Let Q be a quadrangulation of the cylinder of height R and let $0 < r \leq R$. We now explain a “canonical” choice of a geodesic between a vertex of $\partial_r Q$

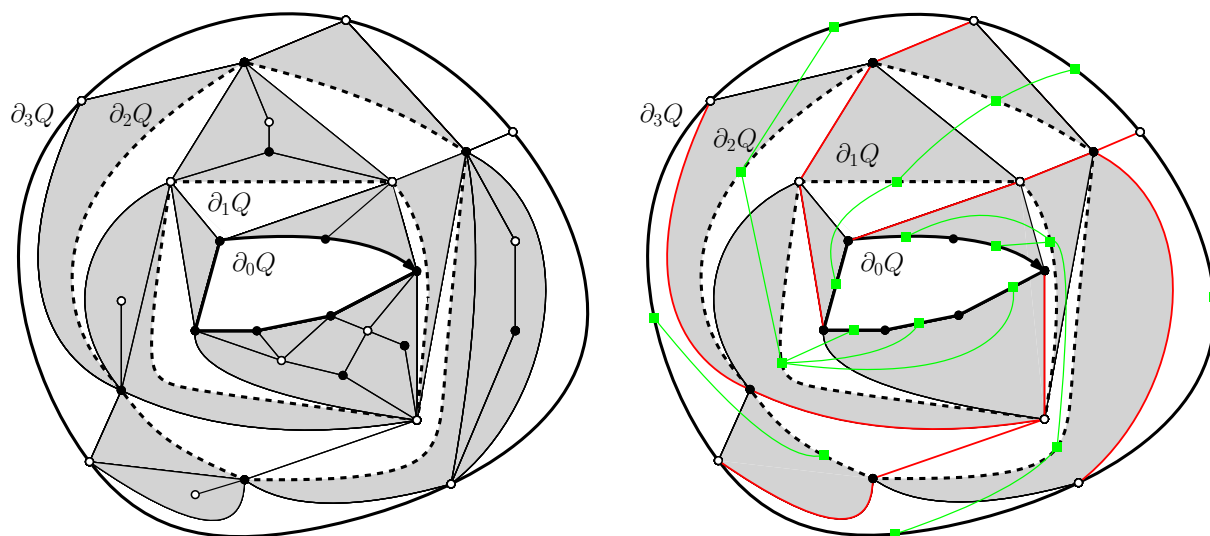


Figure 3.3 – Left, a quadrangulation of the cylinder of height 3, with its cycles ∂_1Q , ∂_2Q in dashed lines. We always draw the top face as the infinite face. Downward triangles are in white and the slots in grey. Right, we erased the content of the slots; in green, the genealogical relation on edges of ∂_rQ for $0 \leq r \leq 3$, in red, the left-most geodesics to the bottom cycle follow the “right” side of downward triangles (assuming their top edge is “up”), or equivalently the “left” side of slots.

and the bottom cycle ∂_0Q . So let v be a vertex on ∂_rQ , and let e be the edge of ∂_rQ with tail v (recall our convention for the orientation of ∂_rQ). Then list all edges incident to v in clockwise order around v , starting from e . The first step of the *left-most geodesic* from v to ∂_0Q is the last edge in this enumeration that connects v to $\partial_{r-1}Q$ (Property (iv) above ensures that there is at least one such edge). We define the next steps of the geodesic by the obvious induction.

Skeleton decomposition. Let Q be a quadrangulation of the cylinder of height R . Following ideas in [38], the article [43] gives a representation of Q by a forest of planar trees of height at most R and a collection of truncated quadrangulations indexed by the vertices of this forest. We refer to [43] for more details, and give a brief presentation.

We first add the edges of ∂_rQ to Q for every $0 < r \leq R$ and recall that these edges are oriented clockwise in each cycle ∂_rQ . Let $0 < r \leq R$, and let e be an edge of ∂_rQ . Then e is incident to exactly one triangular face in Q whose third vertex is at distance $r - 1$ from the bottom boundary. We call this face the *downward triangle with top edge e* . Furthermore, if v is the aim of e , the downward triangle with top edge e is also incident to the first edge of the left-most geodesic from v to the bottom boundary.

The downward triangles disconnect Q into a collection of *slots*, which are filled in by finite maps with a simple boundary. See Figure 3.3. Any slot is contained in the region between ∂_rQ and $\partial_{r-1}Q$ for some $1 \leq r \leq R$, and there is a unique vertex v of ∂_rQ that

is incident to the slot. We then say that the slot is associated with the edge of $\partial_r Q$ whose tail is v . We equip the set of edges of $\cup_{r=0}^R \partial_r Q$ with the following genealogical relation: for every $0 < r \leq R$, an edge $e \in \partial_r Q$ is the parent of all the edges of $\partial_{r-1} Q$ that are incident to the slot associated with e , provided that this slot exists. See the right side of Figure 3.3.

We now explain how we can define the truncated quadrangulation associated with a slot, or rather with an edge $e \in \partial_r Q$, $1 \leq r \leq R$. Suppose first that there is a slot associated with e . The part of Q inside this slot, including its boundary, defines a planar map with a simple boundary and a distinguished vertex on this boundary. Adding one edge in the way explained in Figure 3.4 turns this map into a truncated quadrangulation M , which is rooted at the added edge as shown on Figure 3.4. If there is no slot associated with e we let M be the unique truncated quadrangulation with perimeter one and one inner face (rooted at its boundary edge so that the external face lies on the right of the root edge).

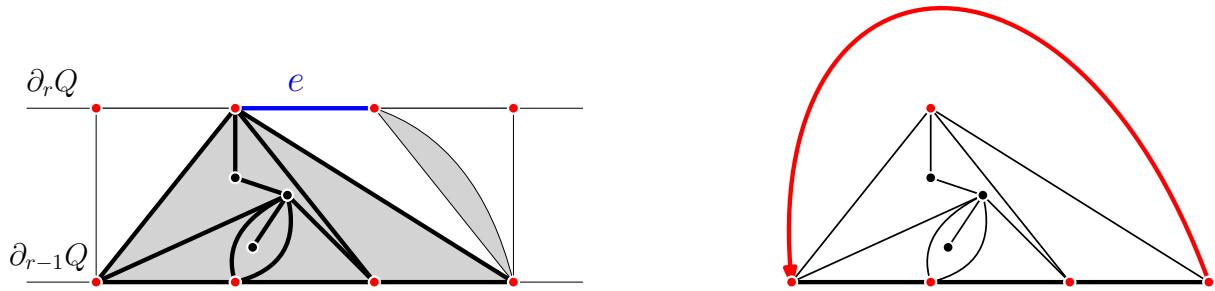


Figure 3.4 – Left (in grey and thick black lines), the map with simple boundary filling the slot of an edge e (in blue) of $\partial_r Q$ with 3 offspring. Right, the truncated quadrangulation we obtain by adding a root edge “over the top vertex of the slot”.

Recalling the definition in [43], we say that a forest \mathcal{F} with one marked vertex is (R, p, q) -admissible if

- (i) the forest consists of an ordered sequence of q rooted plane trees,
- (ii) these trees have height at most R ,
- (iii) exactly p vertices of the forest are at height R ,
- (iv) the marked vertex is at height R and belongs to the first tree.

Write $\mathbb{F}_{R,p,q}^0$ for the set of all (R, p, q) -admissible forests. We will also need the set of all forests with no marked vertex that satisfy properties (i) to (iii) above, and we denote this set by $\mathbb{F}_{R,p,q}$.

For $\mathcal{F} \in \mathbb{F}_{R,p,q}^0$ or $\mathcal{F} \in \mathbb{F}_{R,p,q}$ we let \mathcal{F}^* denote the set of all vertices of \mathcal{F} at height at most $R - 1$. For every $e \in \mathcal{F}^*$, c_e is the offspring number of e in \mathcal{F} .

The construction described above and illustrated in Figure 3.3 provides a bijection that, with every quadrangulation Q of the cylinder of height R , with top perimeter q and bottom perimeter p , associates an (R, p, q) -admissible forest \mathcal{F} and a collection $(S_e)_{e \in \mathcal{F}^*}$ of truncated quadrangulations, such that S_e has perimeter $c_e + 1$ for every $e \in \mathcal{F}^*$. The forest encodes the genealogical relation of edges of $\cup_{r=0}^R \partial_r Q$: each tree in \mathcal{F} corresponds to the descendants of an edge of the top boundary, the first tree is the one that contains the root edge, and the other trees are then listed by following the clockwise order on the top boundary. We call the forest \mathcal{F} the *skeleton* of Q .

On the other hand, the union of all left-most geodesics forms a forest of trees made of edges of Q . This forest can be viewed as dual to the skeleton of Q , and the two forests do not cross, as suggested in Figure 3.3. This has the following important consequence. Let v, w be two distinct vertices of the top boundary of Q , let \mathcal{F}' be the forest consisting of the trees of the skeleton that are rooted on the part of the top boundary between v and w (in clockwise order), and let \mathcal{F}'' consist of the other trees in the skeleton. Then for every $0 \leq r < R$, the left-most geodesics from v and w coalesce before reaching $\partial_r Q$ or when hitting $\partial_r Q$ iff either \mathcal{F}' or \mathcal{F}'' has height strictly smaller than $R - r$.

Law of the skeleton decomposition of the UIPQ. The following formulas are derived by singularity analysis from the generating series of truncated quadrangulations, computed in [38]. See [43, Section 2.5].

For every $1 \leq p \leq n$, let $\mathbb{Q}_{n,p}^{\text{tr}}$ be the set of all truncated quadrangulations with a boundary of length p and n inner faces. There exists a sequence $(\kappa_p)_{p \geq 1}$ of positive reals such that for every $p \geq 1$,

$$\#\mathbb{Q}_{n,p}^{\text{tr}} \underset{n \rightarrow \infty}{\sim} \kappa_p n^{-5/2} 12^{-n}.$$

Furthermore,

$$\kappa_p \underset{p \rightarrow \infty}{\sim} \frac{64\sqrt{3}}{\pi\sqrt{2}} \sqrt{p} 2^{-p}. \quad (3.3)$$

For every $p \geq 1$, we define

$$h(p) \stackrel{\text{def}}{=} \frac{1}{p} 2^p \kappa_p,$$

$$Z(p) \stackrel{\text{def}}{=} \sum_{n=p}^{\infty} \#\mathbb{Q}_{n,p}^{\text{tr}} 12^{-n},$$

and we define the Boltzmann probability measure Γ_p on $\cup_{n=p}^{\infty} \mathbb{Q}_{n,p}^{\text{tr}}$ by setting

$$\Gamma_p(Q) \stackrel{\text{def}}{=} \frac{12^{-n}}{Z(p)}.$$

for every $Q \in \mathbb{Q}_{n,p}^{\text{tr}}$, $n \geq p$. We also set, for every $p \geq 0$,

$$\theta(p) \stackrel{\text{def}}{=} 6 \cdot 2^p Z(p+1).$$

Then [43, Lemma 6], $(\theta(p))_{p \geq 0}$ is a critical offspring distribution with generating function

$$g_\theta(y) = 1 - \frac{8}{\left(\sqrt{\frac{9-y}{1-y}} + 2\right)^2 - 1}.$$

Let $g_\theta^{(p)} = g_\theta \circ \dots \circ g_\theta$ denote the p -th iterate of g_θ . If $(Y_p)_{p \geq 0}$ is a Bienaymé-Galton-Watson process with offspring distribution θ started at $Y_0 = 1$, then for every $p \geq 0$,

$$\mathbb{E}[y^{Y_p}] = g_\theta^{(p)}(y) = 1 - \frac{8}{\left(\sqrt{\frac{9-y}{1-y}} + 2p\right)^2 - 1}, \quad 0 \leq y < 1.$$

Note that $Z(p) \sim 2^{-p}(8\pi)^{-1/2}p^{-5/2}$ as $p \rightarrow \infty$, and thus θ defines an offspring distribution in the domain of attraction of a stable law of index $3/2$.

We now consider the uniform infinite planar quadrangulation Q_∞ to which we can apply the preceding definition to get the cycles $\partial_r Q_\infty$ for every $r \geq 0$ (when $r = 0$ we split the root edge as explained in Figure 3.2). For $r \geq 1$, we distinguish the edge of $\partial_r Q_\infty$ that corresponds to the first tree of the skeleton of the truncated hull of radius r , viewed as a quadrangulation of the cylinder of height r . Then, for every $0 \leq r < s$ we define the annulus $\mathcal{C}(r, s)$ as the quadrangulation of the cylinder of height $s - r$ that corresponds to the part of Q_∞ between $\partial_r Q_\infty$ and $\partial_s Q_\infty$, rooted at the distinguished edge of $\partial_r Q_\infty$.

Let $(\mathcal{F}_{r,s}^0, (S_e)_{e \in \mathcal{F}_{r,s}^{0,*}})$ be the skeleton decomposition of $\mathcal{C}(r, s)$. Conditionally on the skeleton $\mathcal{F}_{r,s}^0$, the truncated quadrangulations S_e , $e \in \mathcal{F}_{r,s}^{0,*}$, are independent and the conditional distribution of S_e is Γ_{c_e+1} , where we recall that c_e is the number of offspring of e in $\mathcal{F}_{r,s}^0$. See [43, Corollary 8].

Let H_r be the length of $\partial_r Q_\infty$.

Proposition 3.4 ([43], Proposition 11). *For every $r \geq 1$ and $p \geq 1$,*

$$\mathbb{P}(H_r = p) = K_r \kappa_p (2\pi_r)^p, \quad (3.4)$$

where

$$\pi_r = g_\theta^{(r)}(0) = 1 - \frac{8}{(3 + 2r)^2 - 1} \quad (3.5)$$

is the probability that a Bienaymé-Galton-Watson process of offspring distribution θ started at 1 becomes extinct before generation r , and

$$K_r = \frac{32}{3\kappa_1} \frac{3 + 2r}{((3 + 2r)^2 - 1)^2}.$$

Consequently, we can find positive constants M_1, M_2 and ρ such that for every $a > 0$ and $r \geq 1$,

$$\begin{aligned} \mathbb{P}(H_r \geq ar^2) &\leq M_1 e^{-\rho a}, \\ \mathbb{P}(H_r \leq ar^2) &\leq M_2 a^{3/2}. \end{aligned}$$

From the skeleton $\mathcal{F}_{r,s}^0$, we define a new forest $\mathcal{F}_{r,s}$ by “forgetting” the marked vertex and applying a uniform random circular permutation to the trees of $\mathcal{F}_{r,s}^0$. Then $\mathcal{F}_{r,s}$ is a random element of $\cup_{p \geq 1, q \geq 1} \mathbb{F}_{s-r,p,q}$.

Proposition 3.5 ([43], Corollary 10). *Let $p \geq 1$. The conditional distribution of $\mathcal{F}_{r,s}$ knowing that $H_r = p$ is as follows: for every $q > 0$, for every $\mathcal{F} \in \mathbb{F}_{s-r,p,q}$,*

$$\mathbb{P}(\mathcal{F}_{r,s} = \mathcal{F} \mid H_r = p) = \frac{h(q)}{h(p)} \prod_{e \in \mathcal{F}^*} \theta(c_e). \quad (3.6)$$

Proposition 3.6. *Let $q \geq 1$. The conditional distribution of $\mathcal{F}_{r,s}$ knowing that $H_s = q$ is as follows: for every $p > 0$, for every $\mathcal{F} \in \mathbb{F}_{s-r,p,q}$,*

$$\mathbb{P}(\mathcal{F}_{r,s} = \mathcal{F} \mid H_s = q) = \frac{\varphi_r(p)}{\varphi_s(q)} \prod_{e \in \mathcal{F}^*} \theta(c_e), \quad (3.7)$$

where

$$\varphi_r(p) = \frac{64}{3} p \frac{3 + 2r}{((3 + 2r)^2 - 1)^2} \pi_r^{p-1}. \quad (3.8)$$

We refer to formulas (18) and (19) in [43] for the last proposition.

3.3 The lower half-plane quadrangulation

3.3.1 Definition of the model

We construct an infinite quadrangulation with an infinite truncated boundary, which will be denoted by \mathcal{L} and called the *lower half plane quadrangulation* or LHPQ. Roughly speaking, \mathcal{L} is what we see near a uniformly chosen random edge of the boundary of a very large truncated hull of Q_∞ (see Proposition 3.7 below for a more precise statement). Our construction relies on the skeleton decomposition.

It will be convenient to use a particular embedding of \mathcal{L} in the plane (see Figure 3.5). In the rest of this paper, we denote the set of non-negative integers by \mathbb{N} , and the set of non-positive integers by $-\mathbb{N} = \{0, -1, -2, \dots\}$. Every point of $\mathbb{Z} \times -\mathbb{N}$ will be a vertex of \mathcal{L} ; the edges of the form $((i, 0), (i + 1, 0))$ for $i \in \mathbb{Z}$ will be the edges of the boundary of \mathcal{L} ; and the upper half-plane will correspond to an “external face” of \mathcal{L} . Furthermore, \mathcal{L} will be rooted at the edge $((0, 0), (1, 0))$.

In order to construct \mathcal{L} , we start from a forest $(\tau_i)_{i \in \mathbb{Z}}$ of i.i.d. Bienaymé-Galton-Watson trees with offspring distribution θ (as usual these trees are random plane trees). This forest will be the *skeleton* of the LHPQ. A.s. each generation has an infinite number of individuals: we can thus embed vertices of the skeleton at generation $r \geq 0$ bijectively on $\{(j + \frac{1}{2}, -r), j \in \mathbb{Z}\}$, in a way that is consistent with the order on vertices at generation r of the forest, so that vertices in trees with non-negative indices $(\tau_i)_{i \geq 0}$ fill the lower right quadrant $(1/2, 0) + (\mathbb{N} \times -\mathbb{N})$, and vertices in trees with negative indices $(\tau_i)_{i < 0}$ fill the

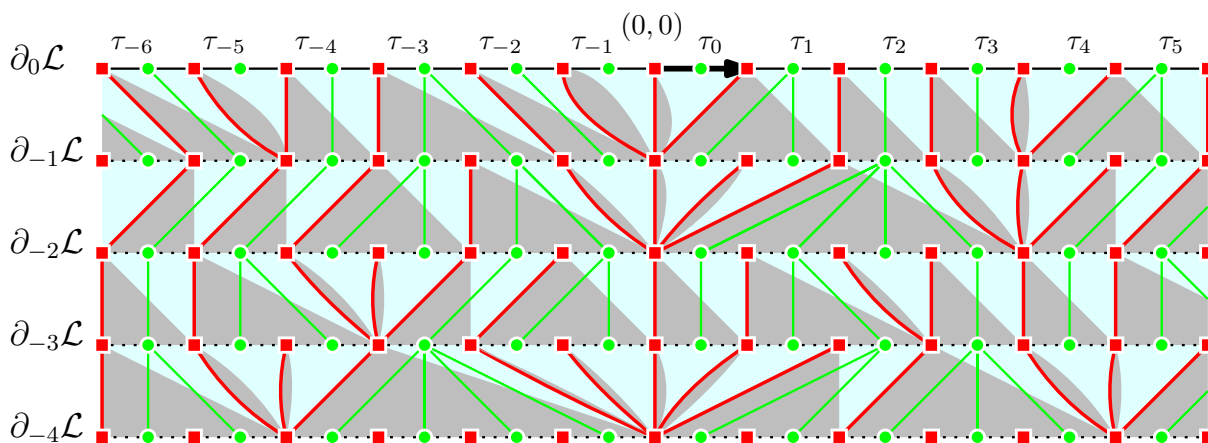


Figure 3.5 – Embedding of the LHPQ and its skeleton. The vertices of the lattice $\mathbb{Z} \times -\mathbb{N}$ are the red dots, the vertices of the forest are the green dots, the trees are drawn in green. We drew the downward triangles in cyan, and the slots in grey. Notice that the slot associated with an edge having no offspring may be empty though it is represented here as the “inside” of a double edge (this simply means that this double edge may be glued to form a single edge). In red, left-most geodesics in the LHPQ, named so because they follow the left-most edge going downward. They follow the left side of slots, or the right side of downward triangles. The left-most geodesics form the dual tree of the skeleton.

lower left quadrant $(-1/2, 0) + (-\mathbb{N} \times -\mathbb{N})$. In particular the root of τ_i will be $(i + 1/2, 0)$. See the green trees on Figure 3.5. In what follows, we identify vertices of the skeleton and the points where they are embedded.

We denote by $\partial_{-r}\mathcal{L}$ the infinite line $\mathbb{Z} \times \{-r\}$ viewed as a linear graph. The skeleton of \mathcal{L} induces a genealogical relation on the edges of $\cup_{r \geq 0} \partial_{-r}\mathcal{L}$, if we identify each edge with its middle point.

To each edge e of $\partial_{-r}\mathcal{L}$, we associate a downward triangle with “top boundary” e and “bottom vertex” the vertex v of $\partial_{-r-1}\mathcal{L}$, chosen as follows. If e has at least one offspring, then v is the right-most vertex of the edge which is the right-most offspring of e . If not, let e' be the first edge of $\partial_r\mathcal{L}$ on the left of e having at least one offspring. Then the downward triangles of e and e' have the same bottom vertex.

Downward triangles delimit a collection of *slots* (see Figure 3.5), and each slot is associated with an edge of $\cup_{r \geq 0} \partial_{-r}\mathcal{L}$ as in the UIPQ. By construction, the edges of the lower boundary of a slot are exactly the offspring of the associated edge. The last step to get the LHPQ is to fill in the slots, and we do so exactly as in the UIPQ, see Figure 3.6. We note that, for $r > 0$, edges of $\partial_{-r}\mathcal{L}$ do not belong to \mathcal{L} : these edges are removed in order to get quadrangles by the gluing of two triangles.

Left-most geodesics in \mathcal{L} are defined as in the case of quadrangulations of the cylinder, but are now infinite paths on \mathcal{L} that start at a vertex of $\partial_{-r}\mathcal{L}$ and then visit each line

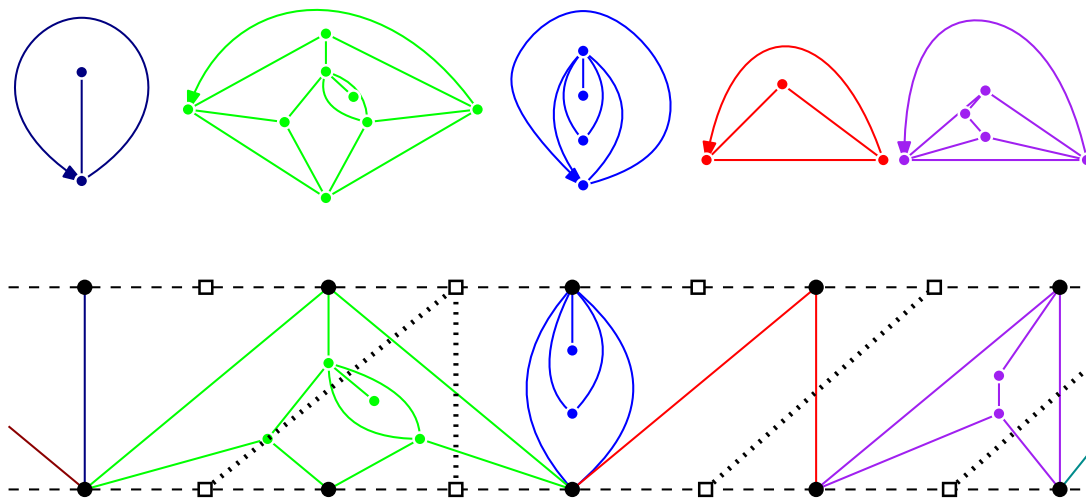


Figure 3.6 – Constructing a part of the LHPQ from truncated quadrangulations. In the lower part, the dotted lines are trees of the skeleton and the dashed lines are $\partial_{-r-1}\mathcal{L}$ and $\partial_{-r}\mathcal{L}$ for some $r \geq 0$. The upper part of the figure shows 5 truncated quadrangulations that fill in successive slots as shown in the lower part. Note that any edge of a truncated quadrangulation that is glued to some edge of $\partial_{-r}\mathcal{L}$ is removed in the LHPQ.

$\partial_{-r'}\mathcal{L}$, $r' \geq r$, exactly once. Left-most geodesics form a forest whose vertex set is the lattice $\mathbb{Z} \times -\mathbb{N}$. This forest and the skeleton never intersect and can be seen as “dual” to each other. Furthermore, the two left-most geodesics started at $(i, 0)$ and $(i', 0)$ with $i < i'$ coalesce before height $j < 0$ or exactly at height j if and only if every τ_k with $i < k + \frac{1}{2} < i'$ becomes extinct before generation $-j$. See Figure 3.5 for an illustration.

Note that the left-most geodesic started from any point $(i, 0)$ of the top boundary breaks the LHPQ into two halves. Any path whose endpoints are on different sides of this geodesic must cross it through one of its vertices. Note finally that left-most geodesics never cross any tree of the skeleton. As an example, the left-most geodesic started from $(0, 0)$ is the vertical line $((0, -n))_{n \geq 0}$.

As another useful observation, we note that the graph distance between any point of $\partial_r\mathcal{L}$ and $\partial_{r'}\mathcal{L}$ (with $r' < r$) is exactly $r - r'$.

3.3.2 The lower-half plane quadrangulation is the local limit of large hulls

The following proposition explains why we consider the model of the LHPQ.

Proposition 3.7. *For every $r > 0$, let \mathcal{H}_r be the truncated hull of radius r of the UIPQ, re-rooted at a uniformly chosen edge of its boundary. Then*

$$\mathcal{H}_r \xrightarrow[r \rightarrow \infty]{(d)} \mathcal{L}$$

for the local topology on rooted planar maps.

We omit the proof as we will not need this result in the remaining part of the paper. See Proposition 7 in [23] for the analogous statement in the case of triangulations.

3.3.3 Control of distances along the boundary

The main estimate

The following proposition shows that $d_{\text{gr}}^{\mathcal{L}}((0, 0), (j, 0))$ grows at least like \sqrt{j} .

Proposition 3.8. *For every $\epsilon > 0$, there exists an integer $K \geq 1$ such that for every $r \geq 1$, for every integer $A > 0$,*

$$\mathbb{P} \left(\min_{|i| \geq A+Kr^2} \min_{-A \leq i' \leq A} d_{\text{gr}}^{\mathcal{L}}((i', 0), (i, 0)) \geq r \right) \geq 1 - \epsilon.$$

In order to prove this proposition, we adapt the proof of [19, theorem 5]. This result does not apply directly to our settings, but to another model also constructed from a Bienaymé Galton-Watson forest.

Let us first define slices, half-slices, and blocks of the LHPQ.

Slice. Let $-\infty < j' < j \leq 0$. Consider all vertices and edges of \mathcal{L} contained in $\mathbb{R} \times [j', j]$ and add all edges of the form $((i, j), (i+1, j))$ and $((i, j'), (i+1, j'))$ for $i \in \mathbb{Z}$. The resulting map is called the slice $\mathcal{L}_{j'}^j$. By convention it is rooted at $((0, j), (1, j))$. The skeleton of $\mathcal{L}_{j'}^j$ is the planar forest $(\tau_n^{(j, j')})_{n \in \mathbb{Z}}$ corresponding to the part of the skeleton of \mathcal{L} between generation $-j$ and $-j'$ (these trees are numbered as previously, so that $\tau_n^{(j, j')}$ is rooted at the vertex $(n + \frac{1}{2}, j)$).

Half-slice. The half-slice $\mathcal{HL}_{j'}^j$ is the part of the slice $\mathcal{L}_{j'}^j$, that is contained in $\mathbb{R}_+ \times [j', j]$. Its skeleton consists of the trees of the skeleton of $\mathcal{L}_{j'}^j$ with nonnegative indices.

Blocks. Cut the half-slice $\mathcal{HL}_{j'}^j$ along the left-most geodesics that follow the right boundary of trees of maximal height in its skeleton. We obtain a sequence of finite maps, which we will call blocks as in [19].

Let us give a precise definition of these blocks (see also Figure 3.7). Let $(\xi_n)_{n>0}$ be the sequence of all indices (in increasing order) of trees that reach height $j - j'$ in the skeleton of $\mathcal{HL}_{j'}^j$, and add the convention that $\xi_0 = -1$. Let $n > 0$ be an integer. The n -th block $\mathcal{HL}_{j'}^j(n)$ is the part of $\mathcal{HL}_{j'}^j$ contained between the left-most geodesics started at $(\xi_{n-1} + 1, j)$ and at $(\xi_n + 1, j)$ respectively. The left boundary of a block is the left-most geodesic on its left, its right boundary is the left-most geodesic on its right.

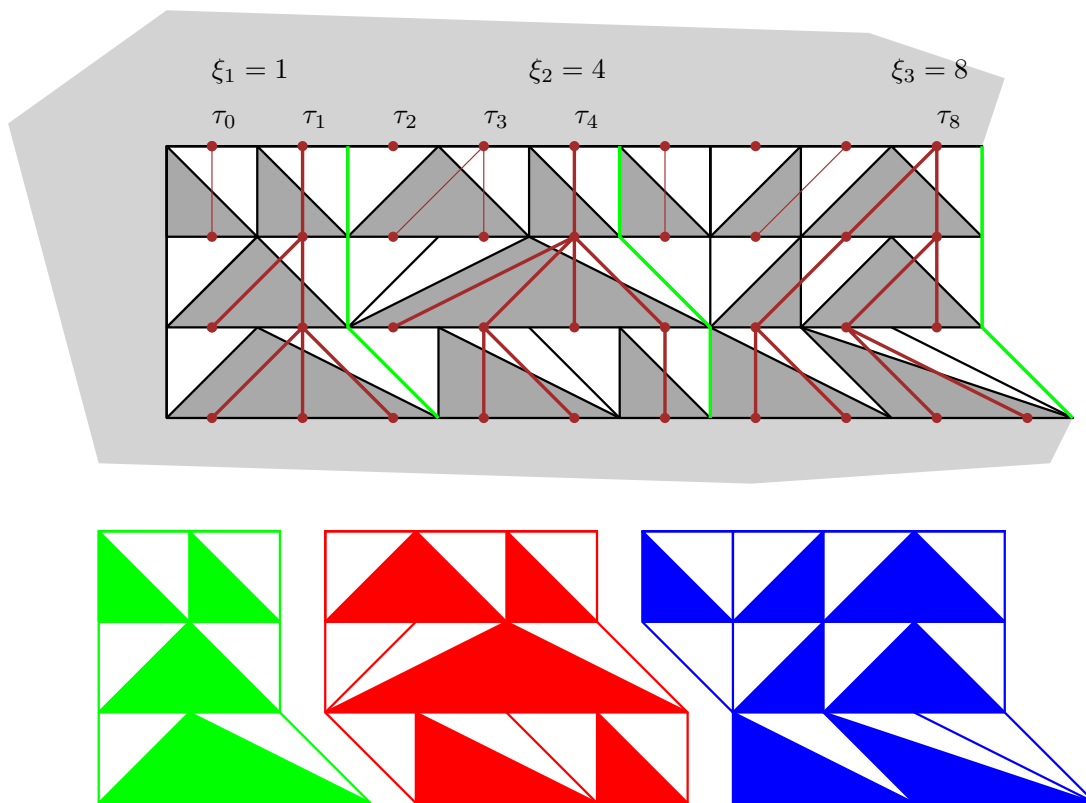


Figure 3.7 – Block decomposition of the half-slice \mathcal{HL}_{-3}^0 . The first block is pictured in green, the second one in red, the third one in blue. In the upper figure, we represented the skeleton in brown (with trees of height 3 in thick lines), slots in dark gray, and the left-most geodesics following the right boundary of trees of height 3 in green. In this example, $\xi_1 = 1$, $\xi_2 = 4$, $\xi_3 = 8$.

The skeleton of \mathcal{L} is made of i.i.d. trees, thus all the blocks $\mathcal{HL}_{j'}^j(n)$, viewed as planar maps with a boundary, are independent and share the same law, which only depends on $j - j'$.

The *thickness* of the n -th block $\text{Thickness}(\mathcal{HL}_{j'}^j(n))$ (called diameter in [19]) is the minimal graph distance in this block between a point of its left boundary and a point of its right boundary. Note that the thickness of a block cannot be 0, since the presence of a tree of maximal height implies that the left-most geodesics corresponding to the left and right boundary of the block do not coalesce (by a previous remark).

Furthermore, the thickness of $\mathcal{HL}_{j'}^j(n)$ is always smaller than $j - j' + 1$. Indeed, the left-most geodesic started at $(\xi_{n-1} + 1, j)$ (i.e. the left boundary) and the left-most geodesic started at (ξ_n, j) coalesce before height j' (i.e. after at most $j - j'$ steps), since no tree rooted between $\xi_{n-1} + 1$ and ξ_n reaches height $j - j'$. In this way, we get a path of length at most $j - j'$ that connects the left boundary of the block to the vertex (ξ_n, j) , and we just have to add the edge $((\xi_n, j), (\xi_n + 1, j))$ to get the desired bound.

It will be useful to note the following simple fact: any path that stays in the half-slice $\mathcal{HL}_{j'}^j$ with one endpoint on the left side of the left boundary of some $\mathcal{HL}_{j'}^j(n)$, and its other endpoint on the right side of its right boundary, has a length which is at least the thickness of the block.

Let us outline the key idea of the proof of Proposition 3.8. A path of length r between two vertices of the boundary of \mathcal{L} cannot exit the slice \mathcal{L}_{-h}^0 for any $h \geq r$. We apply this observation with $h = \lceil Cr \rceil$ with some constant $C \geq 1$. If we fix $K > 0$ large enough, then with high probability we will find a block of \mathcal{L}_{-h}^0 with top boundary included in $[A, A + Kr^2] \times \{0\}$, and any path in \mathcal{L}_{-h}^0 that goes from the left side to the right side of $[A, A + Kr^2] \times \{0\}$ must cross this block. All we need to conclude is the fact that we can choose h such this block has thickness at least r with high probability.

The latter fact is derived from the following result, which is adapted from [19, theorem 5]. For every integer $h > 0$, let $\mathcal{G}(h)$ be a random variable with the law of a block of height h . Fix $\varepsilon \in (0, 1)$. The ε -quantile $f_\varepsilon(h)$ of the thickness of $\mathcal{G}(h)$ is the largest integer n such that

$$\mathbb{P}(\text{Thickness}(\mathcal{G}(h)) \geq n) \geq 1 - \varepsilon. \quad (3.9)$$

Note that $1 \leq f_\varepsilon(h) \leq h + 1$ by previous observations.

Proposition 3.9. *For every $\varepsilon \in (0, 1)$, there exists $C_\varepsilon \in (0, 1)$ such that for every $h \geq 1$, $f_\varepsilon(h) \geq C_\varepsilon h$.*

We postpone the proof of Proposition 3.9 to the next section and complete the proof of Proposition 3.8.

Proof of Proposition 3.8. Let $r \geq 1$, and set $h = \lceil r/C_{\varepsilon/4} \rceil$, where $C_{\varepsilon/4}$ is given by Proposition 3.9, and consider the first block of the half-slice \mathcal{L}_{-h}^0 , that is, $\mathcal{HL}_{-h}^0(1)$ with the previous notation. By Proposition 3.9, we have

$$\begin{aligned} \mathbb{P}(\text{Thickness}(\mathcal{HL}_{-h}^0(1)) \geq r) &\geq \mathbb{P}(\text{Thickness}(\mathcal{HL}_{-h}^0(1)) \geq C_{\varepsilon/4}h) \\ &\geq \mathbb{P}(\text{Thickness}(\mathcal{HL}_{-h}^0(1)) \geq f_{\varepsilon/4}(h)) \\ &\geq 1 - \varepsilon/4. \end{aligned}$$

Let \mathcal{E}_1 denote the event $\{\text{Thickness}(\mathcal{HL}_{-h}^0(1)) \geq r\}$.

The right-most point of the top boundary of $\mathcal{HL}_{-h}^0(1)$ is $(\xi_1 + 1, 0)$, where ξ_1 follows a geometric law with parameter $1 - \pi_h \geq c/r^2$ for some constant $c > 0$ independent of r (cf. Proposition 3.4). We can take $K > 0$ large enough so that $\xi_1 + 1 \leq Kr^2$ holds with probability larger than $1 - \varepsilon/4$. Let us call \mathcal{E}_2 the event where $\xi_1 + 1 \leq Kr^2$.

On the event $\mathcal{E}_1 \cap \mathcal{E}_2$ of probability at least $1 - \varepsilon/2$ the block $\mathcal{HL}_{-h}^0(1)$ has thickness at least r and its top boundary is contained in $[0, Kr^2] \times \{0\}$. Then any two points $(i, 0)$ and $(i', 0)$ with $i \leq 0$ and $i' \geq Kr^2$ are at $d_{\text{gr}}^{\mathcal{L}}$ -distance at least r , since any path of length smaller than r linking them necessarily crosses the block $\mathcal{HL}_{-h}^0(1)$.

From an obvious argument of translation invariance, we obtain that for any integer $A > 0$, with probability larger than $1 - \varepsilon/2$, any point in $(\infty, A] \times \{0\}$ and any point

in $[A + Kr^2, +\infty) \times \{0\}$ are at $d_{\text{gr}}^{\mathcal{L}}$ -distance at least r . Similarly, with probability larger than $1 - \varepsilon/2$ the two half-lines $(\infty, -A - Kr^2] \times \{0\}$ and $[-A, +\infty) \times \{0\}$ are also at $d_{\text{gr}}^{\mathcal{L}}$ -distance larger than r . The statement of the proposition follows. \square

Proof of Proposition 3.9

For $h \geq 6$ and $m \in \{1, 2, \dots, \lfloor \frac{h}{6} \rfloor\}$, we set

$$J_m = \left\{ 0, -m, -2m, \dots, - \left(\left\lfloor \frac{h}{m} \right\rfloor - 3 \right) m, -h + 3m \right\}.$$

We write $\mathcal{G}(h)$ for a block of height h , and $\mathcal{G}_{j'}^j(h)$ for the slice of $\mathcal{G}(h)$ contained between heights j' and j , for $-h \leq j' < j \leq 0$. We recall that $\varepsilon \in (0, 1)$ is fixed.

Lemma 3.10. *There exists $C \in (0, 1/6)$ s.t. for all h large enough and $0 < m \leq Ch$, the following property holds with probability at least $1 - \varepsilon$: for every $j \in J_m$, the length of any path connecting the left boundary of $\mathcal{G}(h)$ to its right boundary and staying in $\mathcal{G}_{j-3m}^j(h)$ is at least $C \left(\frac{h}{m}\right)^2 f_\varepsilon(m)$.*

As a technical ingredient of the proof of Lemma 3.10, we need a uniform lower bound on the size of the block at every generation. For $0 \leq k \leq h$, let $X_k(h)$ denote the number of vertices of the skeleton of $\mathcal{G}(h)$ at generation k .

Lemma 3.11. *There exists a constant $C_1 > 0$ which does not depend of h , such that*

$$\mathbb{P} \left(\inf_{0 \leq k \leq h} X_k(h) > C_1 h^2 \right) \geq 1 - \varepsilon/2. \tag{3.10}$$

See [19, Lemma 2] for a proof of Lemma 3.11.

Proof of Lemma 3.10. The idea is to choose C' small enough so that with high probability, one can find at least $C'(h/m)^2$ blocks of thickness at least $f_\varepsilon(m)$ inside the slice $\mathcal{G}_{j-3m}^j(h)$, for every $j \in J_m$. Any path connecting the left and right boundaries of this slice will then have length at least $C'(h/m)^2 f_\varepsilon(m)$.

We argue in the half-slice \mathcal{HL}_{-h}^0 . Note that the first block of this half-slice has the same law as $\mathcal{G}(h)$.

Let C_1 be chosen as in Lemma 3.11 so that (3.10) holds. Consider the half-slice \mathcal{HL}_{j-3m}^j for $j \in J_m$, and let k be a positive integer. The number of blocks of this slice whose top boundary lies in $[0, \lceil C_1 h^2 \rceil] \times \{j\}$ is distributed as the number of trees with height at least $3m$ in a forest of $\lceil C_1 h^2 \rceil$ independent Bienaymé-Galton-Watson trees with offspring distribution θ . Each block has a probability greater than $1 - \varepsilon$ of having thickness at least $f_\varepsilon(3m)$. The number N_j of blocks with thickness at least $f_\varepsilon(3m)$ and top boundary in $[0, \lceil C_1 h^2 \rceil] \times \{j\}$ is then bounded below in distribution by a binomial variable with parameters $(\lceil C_1 h^2 \rceil, (1 - \varepsilon)(1 - \pi_{3m}))$, where π_s is defined in Proposition

3.4. By standard large deviation estimates for the binomial distribution and the bound $1 - \pi_s \geq c/s^2$, we can find $C, C', C'' > 0$ such that for all large enough h , for every $m \leq Ch$, for every $j \in J_m$

$$\mathbb{P} \left(N_j < C' \left(\frac{h}{m} \right)^2 \right) < \exp \left(-C'' \left(\frac{h}{m} \right)^2 \right).$$

Summing over $j \in J_m$ and taking C even smaller if necessary, we get

$$\mathbb{P} \left(\forall j \in J_m : N_j \geq C \left(\frac{h}{m} \right)^2 \right) \geq 1 - \varepsilon/2. \quad (3.11)$$

On the event of probability at least $1 - \varepsilon/2$ considered in Lemma 3.11, the first $\lceil C_1 r^2 \rceil$ vertices of the skeleton of \mathcal{HL}_{-r}^0 at generation j (i.e. the first $\lceil C_1 r^2 \rceil$ vertices of the skeleton of \mathcal{HL}_{j-3m}^j) belong to the first block $\mathcal{HL}_{-r}^0(1)$, thus the N_j blocks of thickness at least $f_\varepsilon(3m)$ considered above are contained in $\mathcal{HL}_{-r}^0(1)$. The property of Lemma 3.10 then holds on the intersection of the event in Lemma 3.11 with the event in (3.11). \square

Proposition 3.9 will be proved via the following functional inequality on f_ε :

Proposition 3.12. *Let $C \in (0, 1/6)$ be as in Lemma 3.10. Then for all h large enough,*

$$f_\varepsilon(h) \geq C \max_{1 \leq m \leq Ch} \min \left(m, \left(\frac{h}{m} \right)^2 f_\varepsilon(3m) \right). \quad (3.12)$$

Proof. The idea is to cut the block into slices of height $3m$, and to consider separately the cases where the shortest path crossing the block from left to right stays inside such a slice or not. See Figure 3.8 for an illustration.

Let m be an integer with $1 \leq m \leq Ch$. Consider a path γ in $\mathcal{G}(h)$ with length equal to the thickness of this block, and let x be its starting point on the left boundary. We assume for simplicity that x is at distance at least m from the top and bottom boundaries (the case where x is at distance smaller than m from the top or bottom boundaries is treated similarly). By our choice of J_m , there is always an index $j \in J_m$ such that x is in the slice $\mathcal{G}_{j-2m}^{j-m}(h)$. Then either γ leaves the slice $\mathcal{G}_{j-3m}^j(h)$, which takes at least m steps; or γ stays in $\mathcal{G}_{j-3m}^j(h)$, but then by Lemma 3.10 its length is at least $C \left(\frac{h}{m} \right)^2 f_\varepsilon(m)$ with probability at least $1 - \varepsilon$.

We conclude that the thickness of $\mathcal{G}(h)$ is at least $C \min \left(m, \left(\frac{h}{m} \right)^2 f_\varepsilon(m) \right)$ with probability at least $1 - \varepsilon$. Hence $f_\varepsilon(m) \geq C \min \left(m, \left(\frac{h}{m} \right)^2 f_\varepsilon(m) \right)$. Since this holds for all m with $1 \leq m \leq Ch$, this gives the result of Proposition 3.12. \square

Proof of Proposition 3.9. First note that by taking C smaller if necessary we may assume that the bound of Proposition 3.12 holds for every $h \geq \lfloor 6/C^2 \rfloor$. We then prove by

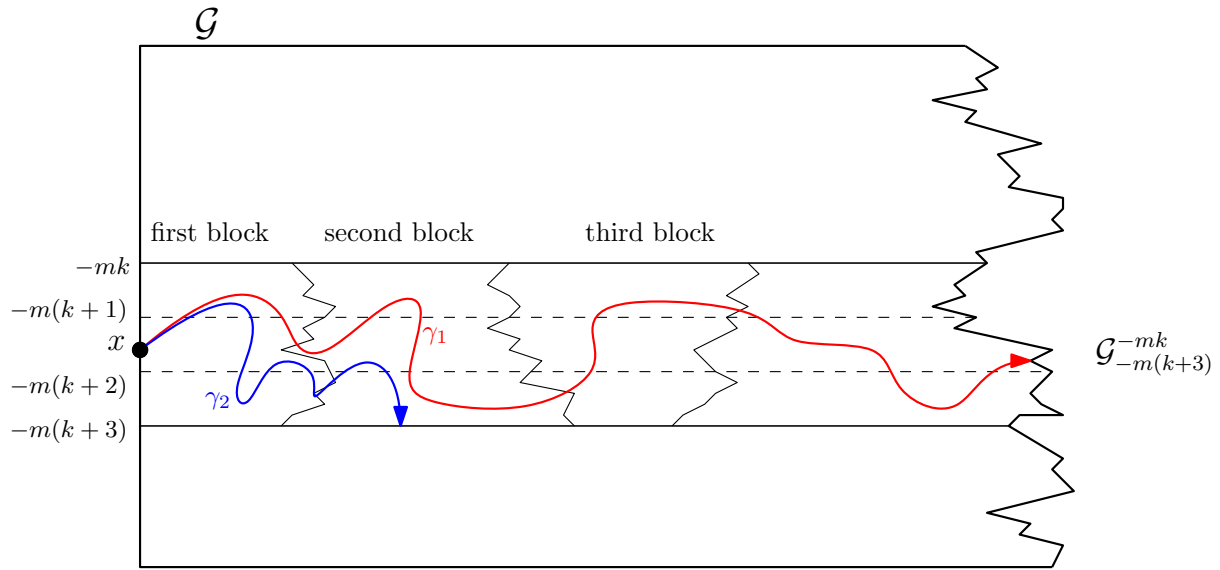


Figure 3.8 – Paths started from a point x on the left boundary can either stay in a slice of height $3m$ around x (and have length at least $C\left(\frac{h}{m}\right)^2 f_\varepsilon(m)$ by Lemma 3.10 w.h.p.), or leave it and have length at least m a.s..

induction that $f_\varepsilon(h) \geq \frac{C^2}{6}h$ for every $h \geq 1$. If $h \leq \lfloor 6/C^2 \rfloor$ this bound is trivial. So let $h_0 \geq \lfloor 6/C^2 \rfloor$ and assume that $f_\varepsilon(h) \geq \frac{C^2}{6}h$ for every $1 \leq h \leq h_0$. Take $h = h_0 + 1$ and $m = \lfloor Ch/3 \rfloor$. One verifies that $3m \leq Ch < h$, hence $3m \leq h_0$, so by our assumption $f_\varepsilon(3m) \geq C^2m/2$.

We note as well that $\frac{Ch}{3} > \frac{C}{3} \frac{6}{C^2} = \frac{2}{C} > 12$, so that $\lfloor \frac{Ch}{3} \rfloor \geq \frac{Ch}{6}$, hence $m \geq Ch/6$.

By Proposition 3.12,

$$f_\varepsilon(h) \geq C \min \left(m, \left(\frac{h}{m} \right)^2 f_\varepsilon(3m) \right) \geq C \min \left(\frac{Ch}{6}, \left(\frac{3}{C} \right)^2 \frac{C^2 Ch}{2 \cdot 6} \right)$$

This completes the proof. □

3.3.4 Subadditivity

It will be convenient to consider the map $\widetilde{\mathcal{L}}$ which is derived from the LHPQ \mathcal{L} by removing all “horizontal edges” $((j, 0), (j + 1, 0))$ for $j \in \mathbb{Z}$. For every integer $j < 0$, we also denote by $\widetilde{\mathcal{L}}_{-\infty}^j$ the submap of $\widetilde{\mathcal{L}}$ (or of \mathcal{L}) contained in the half-plane below ordinate j . If $-\infty < j < j' \leq 0$, $\widetilde{\mathcal{L}}_j^{j'}$ is the submap of $\widetilde{\mathcal{L}}$ contained in $\mathbb{R} \times [j, j']$. We equip the vertex sets of these graphs with the first-passage percolation distance induced by i.i.d. weights to the edges (the common distribution of these weights is supported on $[1, \kappa]$). Recall our notation $\rho = (0, 0)$ for the root vertex of \mathcal{L} (or of $\widetilde{\mathcal{L}}$), and $\partial_j \mathcal{L}$ for the line at vertical coordinate $j \leq 0$ (viewed here as a collection of vertices).

Proposition 3.13. *There exists a constant $\mathbf{c}_p \in [1, \kappa]$ such that*

$$r^{-1} d_{\text{fpp}}^{\tilde{\mathcal{L}}}(\rho, \partial_{-r}\mathcal{L}) \xrightarrow[r \rightarrow \infty]{\text{a.s.}} \mathbf{c}_p.$$

Proof. We derive this proposition from the subadditive ergodic theorem. Let $-\infty < j' < j < 0$, and let x_j be the left-most vertex of $\partial_j\mathcal{L}$ such that $d_{\text{fpp}}^{\tilde{\mathcal{L}}}(\rho, \partial_j\mathcal{L}) = d_{\text{fpp}}^{\tilde{\mathcal{L}}}(\rho, x_j)$. Then,

$$d_{\text{fpp}}^{\tilde{\mathcal{L}}^{j'}}(\rho, \partial_{j'}\mathcal{L}) \leq d_{\text{fpp}}^{\tilde{\mathcal{L}}^j}(\rho, \partial_j\mathcal{L}) + d_{\text{fpp}}^{\tilde{\mathcal{L}}^{j'}}(x_j, \partial_{j'}\mathcal{L}).$$

Note that x_j is a function of $\tilde{\mathcal{L}}_j^0$ and of the weights on edges of $\tilde{\mathcal{L}}_j^0$. Thanks to the independence of layers of the map, $d_{\text{fpp}}^{\tilde{\mathcal{L}}^{j'}}(x_j, \partial_{j'}\mathcal{L})$ is independent of $d_{\text{fpp}}^{\tilde{\mathcal{L}}^j}(\rho, \partial_j\mathcal{L})$ and has the same distribution as $d_{\text{fpp}}^{\tilde{\mathcal{L}}^{j'-j}}(\rho, \partial_{j'-j}\mathcal{L})$. We then apply Liggett's version of Kingman's subadditive ergodic theorem [48, theorem 1.10] to conclude that

$$r^{-1} d_{\text{fpp}}^{\tilde{\mathcal{L}}^0}(\rho, \partial_{-r}\mathcal{L}) \xrightarrow[r \rightarrow \infty]{\text{a.s.}} \mathbf{c}_p$$

for some constant \mathbf{c}_p . The fact that $\mathbf{c}_p \in [1, \kappa]$ is immediate since weights belong to $[1, \kappa]$ and the graph distance from ρ to $\partial_{-r}\mathcal{L}$ (in $\tilde{\mathcal{L}}_{-r}^0$) is equal to r . The lemma follows by noting that $d_{\text{fpp}}^{\tilde{\mathcal{L}}}(\rho, \partial_j\mathcal{L}) = d_{\text{fpp}}^{\tilde{\mathcal{L}}^j}(\rho, \partial_j\mathcal{L})$. □

3.4 Technical tools

3.4.1 Density between the LHPQ and truncated hulls of the UIPQ

Proposition 3.7 suggests that the neighborhood of a vertex chosen uniformly on the boundary of a large hull in the UIPQ looks like the LHPQ. We will need a quantitative version of this property; this is provided by Proposition 3.14, whose proof does not depend on Proposition 3.7.

Let $a \in (0, 1)$, let $(\tau_i)_{i \in \mathbb{Z}}$ be an i.i.d. Bienaymé-Galton-Watson forest with offspring distribution θ , and for every integer $r \geq 1$, let $N_r^{(a)}$ be a random variable distributed uniformly over $\{\lfloor ar^2 \rfloor + 1, \dots, \lfloor r^2/a \rfloor\}$, and independent of $(\tau_i)_{i \in \mathbb{Z}}$. We denote the tree τ_i truncated at height r by $[\tau_i]_r$ (we only keep vertices at generation at most r). For every $0 \leq r < s$, let $\mathcal{F}_{r,s}$ be the forest defined from the skeleton of the annulus $\mathcal{C}(r, s)$ in the UIPQ as explained at the end of Section 3.2.

Proposition 3.14. *For every $a \in (0, 1)$, we can find $C_a > 0$ such that for every large enough integer r , for every choice of the integers $s > r$ and $\lfloor ar^2 \rfloor + 1 \leq p, q \leq \lfloor r^2/a \rfloor$, for every forest $\mathcal{F} \in \mathbb{F}_{s-r,p,q}$,*

$$\mathbb{P}(\mathcal{F}_{r,s} = \mathcal{F}) \leq C_a \mathbb{P}\left([\tau_1]_{s-r}, \dots, [\tau_{N_r^{(a)}}]_{s-r} = \mathcal{F}\right). \quad (3.13)$$

Proof. By Proposition 3.5, for $\mathcal{F} \in \mathbb{F}_{s-r,p,q}$,

$$\mathbb{P}(\mathcal{F}_{s,r} = \mathcal{F} \mid H_r = p) = \frac{p}{q} \frac{2^q \kappa_q}{2^p \kappa_p} \prod_{v \in \mathcal{F}^*} \theta(c_v). \quad (3.14)$$

where we recall that \mathcal{F}^* is the set of vertices at generation at most $s - r - 1$ in the forest \mathcal{F} .

Let us consider the right-hand side of (3.13). Using the asymptotics (3.3), we find $C > 0$ that only depends on a such that for every large enough r , for every $\lfloor ar^2 \rfloor < p, q < \lfloor r^2/a \rfloor$, we have $\frac{p}{q} \frac{2^q \kappa_q}{2^p \kappa_p} \leq C$. On the other hand, Proposition 3.4 and (3.3) allow us to find $C' > 0$ such that $\mathbb{P}(H_r = p) \leq C'/r^2$ for every $r \geq 1$ and $p \geq 1$. From (3.14), we now get

$$\mathbb{P}(\mathcal{F}_{r,s} = \mathcal{F}) \leq \frac{CC'}{r^2} \prod_{v \in \mathcal{F}^*} \theta(c_v). \quad (3.15)$$

On the other hand,

$$\mathbb{P}\left(\left([\tau_1]_{s-r}, \dots, [\tau_{N_r^{(a)}}]_{s-r}\right) = \mathcal{F}\right) = \mathbb{P}(N_r^{(a)} = q) \mathbb{P}\left(\left([\tau_1]_{s-r}, \dots, [\tau_q]_{s-r}\right) = \mathcal{F}\right) = \frac{1}{\lfloor r^2/a \rfloor - \lfloor ar^2 \rfloor} \prod_{v \in \mathcal{F}^*} \theta(c_v). \quad (3.16)$$

The desired result follows by comparing (3.15) and (3.16). \square

3.4.2 Coalescence of left-most geodesics in the UIPQ

Left-most geodesics in the UIPQ coalesce quickly, in the following sense. Consider the set of all left-most geodesics started from the boundary of the hull of radius $r \geq 1$, and let $\gamma \in (0, 1)$. Then the number of vertices at distance $\lfloor \gamma r \rfloor$ from the root that belong to one of these geodesics is bounded in distribution when r is large. The next proposition (which is inspired from [23, Proposition 17]) gives a precise version of this property, which will be particularly useful in the proof of Proposition 3.16 below.

Recall that $\mathcal{H}_{Q_\infty}^{\text{tr}}(r)$ is the truncated hull of radius r of the UIPQ Q_∞ and that $\partial_r Q_\infty$ is the external boundary of this hull, which has length H_r . Pick a vertex $u_0^{(r)}$ on $\partial_r Q_\infty$ uniformly at random, and write $u_0^{(r)}, u_1^{(r)}, \dots, u_{H_r-1}^{(r)}$ for all vertices of the boundary listed in clockwise order starting from $u_0^{(r)}$. We extend the definition of $u_j^{(r)}$ to all $j \in \mathbb{Z}$ by periodicity, so that $u_j^{(r)} = u_{j+H_r}^{(r)}$ for every j .

Proposition 3.15. *Let $\gamma \in (0, 1/2)$ and $\delta > 0$. For every integer $A > 0$, let $\mathcal{X}_{r,A}$ be the event where any left-most geodesic to the root starting from a vertex of $\partial_r Q_\infty$ coalesces before time $\lfloor \gamma r \rfloor$ with one of the left-most geodesics started from $u_{\lfloor kr^2/A \rfloor}^{(r)}$, $0 \leq k \leq \lfloor \frac{AH_r}{r^2} \rfloor$. Then we can choose A large enough such that, for every sufficiently large r ,*

$$\mathbb{P}(\mathcal{X}_{r,A}) \geq 1 - \delta.$$

Proof. The vertices $u_{\lfloor kr^2/A \rfloor}$, $0 \leq k \leq \lfloor \frac{AH_r}{r^2} \rfloor$ divide $\partial_r Q_\infty$ into a collection of “intervals” made of consecutive edges of the boundary. We call an interval bad if at least two trees of the skeleton of $\mathcal{H}_{Q_\infty}^{\text{tr}}(r)$ rooted in this interval have height at least $\lfloor \gamma r \rfloor$ and good otherwise.

Now recall the observations made in Section 3.2 before discussing the law of the skeleton of the UIPQ. It follows that, if an interval S is good, then the left-most geodesic started from any vertex of S coalesces with one of the two left-most geodesics started from the endpoints of S . The proposition then reduces to proving that we can choose $A > 0$ such that, for all r large enough, the probability of having no bad interval is greater than $1 - \delta$.

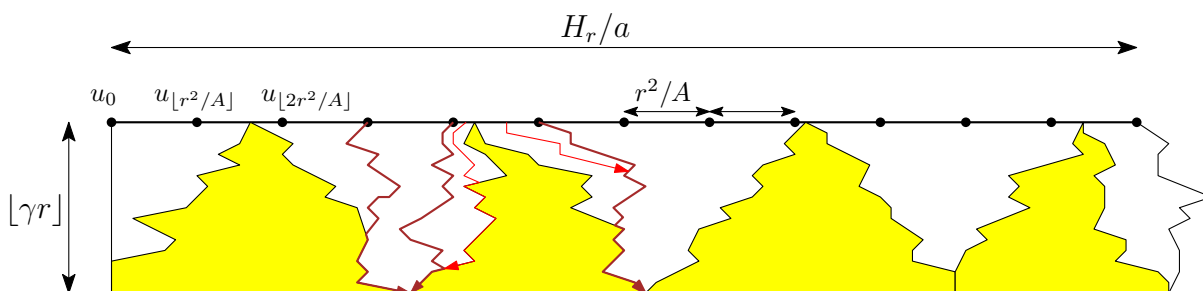


Figure 3.9 – As soon as there are no bad intervals, every left-most geodesic started at a vertex v of the top boundary coalesces before time γr with the left-most geodesic started at one of the endpoints of the interval that contains v . By choosing A large enough, we ensure that with high probability there is no bad interval.

From the explicit law of the perimeter of truncated hulls in Proposition 3.4, we get that there exists $a \in (0, 1)$ such that for all large enough r ,

$$\mathbb{P} \left(H_r \notin \left[\lfloor ar^2 \rfloor + 1, \lfloor r^2/a \rfloor \right] \text{ or } H_{\lfloor \gamma r \rfloor} \notin \left[\lfloor ar^2 \rfloor + 1, \lfloor r^2/a \rfloor \right] \right) < \delta/2. \quad (3.17)$$

Consider first a forest made of $\lfloor r^2/A \rfloor + 1$ independent Bienaymé-Galton-Watson trees with offspring distribution θ . Simple estimates show that the probability that at least two trees of the forest have height greater than or equal to $\lfloor \gamma r \rfloor$ is bounded by $C(A\gamma)^{-2}$ (use Proposition 3.4) independently of r . If we now consider $\lfloor A/a \rfloor + 1$ independent such forests, the probability that at least one of these forests satisfies the preceding property is bounded above by $C(\lfloor A/a \rfloor + 1)(A\gamma)^{-2}$, with a constant C that does not depend on r nor on A . By choosing A large, the latter quantity can be made smaller than $\delta/(2C_a)$, where C_a is the constant in Proposition 3.14. The proof is completed by using Proposition 3.14 and (3.17). \square

3.5 Main results for the first-passage percolation distance on quadrangulations

3.5.1 Distance through a thin annulus

Recall the constant \mathbf{c}_p introduced in Proposition 3.13.

Proposition 3.16. *Let $\varepsilon \in (0, 1)$ and $\delta > 0$. For every $\eta > 0$ small enough, for all sufficiently large n , the property*

$$(1 - \varepsilon)\mathbf{c}_p\eta n \leq d_{\text{fpp}}^{Q_\infty}(v, \partial_{n - \lfloor \eta n \rfloor} Q_\infty) \leq (1 + \varepsilon)\mathbf{c}_p\eta n \quad (3.18)$$

holds for every $v \in \partial_n Q_\infty$, with probability at least $1 - \delta$.

The proof of this result is technical but very similar to the proof of [23, Proposition 19], to which we refer for additional details. Let us start by an outline of the main ideas of the proof. Recalling the absolute continuity relations stated in Proposition 3.14, we observe that a sufficiently thin slice of the UIPQ (of the form $\mathcal{C}(n - \lfloor \eta n \rfloor, n)$), seen from a uniformly chosen vertex of its outer boundary, looks like a slice of the LHPQ. This in turn allows us to use Proposition 3.13.

In order to implement the latter observation, we need to make sure that with high probability, distances from a point v of the top boundary of the annulus $\mathcal{C}(n - \lfloor \eta n \rfloor, n)$ to the bottom boundary are determined by a “small” neighborhood of v in the annulus. This essentially follows from the control of distances along the boundary discussed in Section 3.3.3.

Finally, we need (3.18) to hold simultaneously for all v on the top boundary. Proposition 3.15 ensures that with high probability, the left-most geodesic started at a vertex v of the top boundary coalesces quickly with one of the left-most geodesics started from a bounded number of points on the top boundary. Thanks to this observation, it is enough to verify that (3.18) holds for a *bounded* number of vertices $v \in \partial_n Q_\infty$.

Proof of Proposition 3.16. In a way similar to Section 3.3.4, we let $\widetilde{\mathcal{H}}_{Q_\infty}(n)$ denote the map obtained from $\mathcal{H}_{Q_\infty}^{\text{tr}}(n)$ by removing the edges of the external boundary. It is then convenient to write $d_{\text{gr}}^{(n)}$ for the graph distance on $\widetilde{\mathcal{H}}_{Q_\infty}(n)$, and similarly $d_{\text{fpp}}^{(n)}$ for the first-passage percolation distance on $\widetilde{\mathcal{H}}_{Q_\infty}(n)$ (in both cases we allow only paths made of edges of $\widetilde{\mathcal{H}}_{Q_\infty}(n)$). Similarly as in the proof of Proposition 3.15, we pick a vertex $u_0^{(n)}$ uniformly at random on $\partial_n Q_\infty$, and denote the vertices of $\partial_n Q_\infty$ in clockwise order starting from $u_0^{(n)}$ by $(u_j^{(n)})_{0 \leq j < H_n}$. We extend the definition of $u_j^{(n)}$ to $j \in \mathbb{Z}$ by periodicity. Let $\delta \in (0, 1)$.

First, we use Proposition 3.4 to fix $a \in (0, 1)$ small enough such that, for every $\eta \in (0, 1/2)$, the top and bottom perimeters of the annulus $\mathcal{C}(n - \lfloor \eta n \rfloor, n)$ are both within the range $[an^2, a^{-1}n^2]$ with probability at least $1 - \delta/4$. In the remaining part of the proof, we implicitly argue on the event $\mathcal{E}_\eta^{(n)}$ where the latter properties hold. We also set $N = \lceil 9a^{-2} \rceil$.

Proposition 3.14 allows us to bound the probability of any event concerning the forest encoding the skeleton of $\mathcal{C}(n - \lfloor \eta n \rfloor, n)$ by a constant times the probability of the same event concerning an i.i.d. forest of Bienaymé-Galton-Watson trees with offspring distribution θ . In particular, by taking η small enough, one can ensure that the left-most geodesics started at $u_{-\lfloor an^2/4 \rfloor}^{(n)}$ and $u_{\lfloor an^2/4 \rfloor}^{(n)}$ do not coalesce before reaching $\partial_{n-\lfloor \eta n \rfloor} Q_\infty$ except on a set of probability at most $\delta/(8N)$. On this event, the complement in the annulus $\mathcal{C}(n - \lfloor \eta n \rfloor, n)$ of the union of the left-most geodesics started at $u_{-\lfloor an^2/4 \rfloor}^{(n)}$ and at $u_{\lfloor an^2/4 \rfloor}^{(n)}$ has two components, and we call $\mathcal{G}_0^{(n)}$ the one containing the part of $\partial_n Q_\infty$ between $u_{-\lfloor an^2/4 \rfloor}^{(n)}$ and $u_{\lfloor an^2/4 \rfloor}^{(n)}$ in clockwise order. The lateral boundary $\partial^l \mathcal{G}_0^{(n)}$ consists of the two left-most geodesics bounding $\mathcal{G}_0^{(n)}$, and the bottom boundary $\partial^b \mathcal{G}_0^{(n)}$ is defined in an obvious way.

Let us argue on the event where $\mathcal{G}_0^{(n)}$ is well-defined. Using Proposition 3.8 and Proposition 3.14, and taking η even smaller if necessary, we can ensure that the following holds except on a set of probability at most $\delta/(8N)$: any point $u_k^{(n)}$ with $|k| \leq an^2/16$ is at $d_{\text{gr}}^{(n)}$ -distance at least $(4\kappa + 1)\eta n$ from $u_{-\lfloor an^2/4 \rfloor}^{(n)}$ and $u_{\lfloor an^2/4 \rfloor}^{(n)}$. By the triangle inequality, we thus obtain that on this event, the $d_{\text{gr}}^{(n)}$ -distance between any point $u_k^{(n)}$ with $|k| \leq an^2/16$ and $\partial^l \mathcal{G}_0^{(n)}$ is at least $4\kappa\eta n$. Any path in the annulus $\mathcal{C}(n - \lfloor \eta n \rfloor, n)$ with one endpoint in $\{u_k^{(n)}, |k| \leq an^2/16\}$ and the other one in $\partial_{n-\lfloor \eta n \rfloor} Q_\infty$ that crosses $\partial^l \mathcal{G}_0^{(n)}$ will have length at least $4\eta\kappa n$, and thus first-passage percolation weight at least $4\eta\kappa n$. On the other hand, the left-most geodesic started at any $u_k^{(n)}, |k| \leq an^2/16$, gives a path of length at most ηn between $u_k^{(n)}$ and $\partial_{n-\lfloor \eta n \rfloor} Q_\infty$, that is thus of first-passage-percolation weight at most $\kappa\eta n$. It follows that no $d_{\text{fpp}}^{(n)}$ -shortest path between a vertex of the form $u_k^{(n)}, |k| \leq an^2/16$, and $\partial_{n-\lfloor \eta n \rfloor} Q_\infty$ reaches $\partial^l \mathcal{G}_0^{(n)}$, except on an event of probability at most $\delta/(8N)$.

The previous considerations apply as well if we replace $u_0^{(n)}$ by $u_j^{(n)}$ for any j (possibly depending on n). Let $\mathcal{G}_j^{(n)}$ stand for the analog of $\mathcal{G}_0^{(n)}$ when $u_0^{(n)}$ is replaced by $u_j^{(n)}$. We obtain that, except possibly on an event of probability at most $\delta/4$, for every j of the form $j = i\lfloor an^2/8 \rfloor$, $0 \leq i \leq N - 1$, the set $\mathcal{G}_j^{(n)}$ is well-defined and for every integer k with $j - \lfloor an^2/16 \rfloor \leq k \leq j + \lfloor an^2/16 \rfloor$, any $d_{\text{fpp}}^{(n)}$ -shortest path from $u_k^{(n)}$ to $\partial_{n-\lfloor \eta n \rfloor} Q_\infty$ reaches the bottom boundary of $\mathcal{G}_j^{(n)}$ before its lateral boundary. We write $\mathcal{D}_\eta^{(n)}$ for the event of probability at least $1 - \delta/4$ where the preceding properties hold. On the intersection $\mathcal{E}_\eta^{(n)} \cap \mathcal{D}_\eta^{(n)}$, for any choice of j and k as previously, the $d_{\text{fpp}}^{(n)}$ -distance from $u_k^{(n)}$ to $\partial_{n-\lfloor \eta n \rfloor} Q_\infty$ can be computed from the information given by $\mathcal{G}_j^{(n)}$ and the weights on edges of $\mathcal{G}_j^{(n)}$. From our choice of N , we also see that the vertices $u_k^{(n)}$ with $j - \lfloor an^2/16 \rfloor \leq k \leq j + \lfloor an^2/16 \rfloor$ and j of the form $j = i\lfloor an^2/8 \rfloor$, $0 \leq i \leq N - 1$, cover the whole boundary $\partial_n Q_\infty$ (provided $\mathcal{E}_\eta^{(n)}$ holds).

At this stage, we use the absolute continuity relations in Proposition 3.14 in connection with Proposition 3.13. Let j and k be as previously (possibly depending on n). On the event $\mathcal{E}_\eta^{(n)} \cap \mathcal{D}_\eta^{(n)}$, the $d_{\text{fpp}}^{(n)}$ -distance from $u_k^{(n)}$ to $\partial_{n-\lfloor \eta n \rfloor} Q_\infty$ is determined as a function of the skeleton of $\mathcal{G}_j^{(n)}$ (meaning the forest consisting of the trees of the skeleton of

$\mathcal{C}(n - \lfloor \eta n \rfloor, n)$ rooted at edges between $u_{j - \lfloor an^2/4 \rfloor}^{(n)}$ and $u_{j + \lfloor an^2/4 \rfloor}^{(n)}$ in clockwise order) and the quadrangulations that fill in the slots — and of course of the weights on edges. But the same function determines the first passage percolation distance in the LHPQ (which is estimated by Proposition 3.13) and one just has to compare the distributions of skeletons, for which one may use Proposition 3.14. It follows that, on the event $\mathcal{E}_\eta^{(n)} \cap \mathcal{D}_\eta^{(n)}$, we have

$$d_{\text{fpp}}^{(n)}(u_k^{(n)}, \partial_{n - \lfloor \eta n \rfloor} Q_\infty) \in [(1 - \varepsilon/2)\mathbf{c}_p \eta n, (1 + \varepsilon/2)\mathbf{c}_p \eta n]. \quad (3.19)$$

except possibly on an event of probability tending to 0 as $n \rightarrow \infty$.

We now want to argue that (3.19) holds simultaneously for all k outside a set of small probability. To this end, we rely on the coalescence of geodesics (Proposition 3.15). Let A be chosen as in Proposition 3.15, replacing γ by $\mathbf{c}_p \eta \varepsilon / (4\kappa)$ and δ by $\delta/4$. As in Proposition 3.15, consider indices k of the form $\lfloor in^2/A \rfloor$, $0 \leq i \leq \lfloor A/a \rfloor$. Then, for n large enough, (3.19) holds simultaneously for all these values of k , on the event $\mathcal{E}_\eta^{(n)} \cap \mathcal{D}_\eta^{(n)}$, except possibly on event of probability less than $\delta/4$. Furthermore, thanks to Proposition 3.15, we know on the event $\mathcal{E}_\eta^{(n)} \cap \mathcal{D}_\eta^{(n)}$ that, outside an event of probability at most $\delta/4$, every vertex $v \in \partial_n Q_\infty$ is at $d_{\text{gr}}^{(n)}$ -distance at most $\varepsilon \mathbf{c}_p \eta n / (2\kappa)$ (thus at $d_{\text{fpp}}^{(n)}$ -distance at most $\varepsilon \mathbf{c}_p \eta n / 2$) from one of these vertices $u_k^{(n)}$. We conclude that we have $d_{\text{fpp}}^{(n)}(v, \partial_{n - \lfloor \eta n \rfloor} Q_\infty) \in [(1 - \varepsilon)\mathbf{c}_p \eta n, (1 + \varepsilon)\mathbf{c}_p \eta n]$ for every vertex v of $\partial_n Q_\infty$, outside an event of probability at most δ . This is the desired result, except that we need to replace $d_{\text{fpp}}^{(n)}$ by $d_{\text{fpp}}^{Q_\infty}$. This is however easy since on one hand $d_{\text{fpp}}^{Q_\infty} \leq d_{\text{fpp}}^{(n)}$ and on the other hand the minimal values of $d_{\text{fpp}}^{(n)}(v, \partial_{n - \lfloor \eta n \rfloor} Q_\infty)$ and $d_{\text{fpp}}^{Q_\infty}(v, \partial_{n - \lfloor \eta n \rfloor} Q_\infty)$ on $\partial_n Q_\infty$ are the same. This completes the proof. \square

3.5.2 Distance from the boundary of a hull to its center

The next step is to show that the distance from the root vertex of the UIPQ Q_∞ to an arbitrary vertex of the boundary of a hull is close to a constant times the radius. Recall that ρ is the root vertex of Q_∞ .

Proposition 3.17. *For every $\varepsilon \in (0, 1)$,*

$$\mathbb{P} \left((\mathbf{c}_p - \varepsilon)n \leq d_{\text{fpp}}^{Q_\infty}(\rho, v) \leq (\mathbf{c}_p + \varepsilon)n, \text{ for every } v \text{ in } \partial_n Q_\infty \right) \xrightarrow{n \rightarrow \infty} 1.$$

Proof. Fix $\varepsilon \in (0, 1)$, and take $\delta = \varepsilon^2 / (5\kappa |\ln(\varepsilon / (5\kappa))|)$. For every $0 < m < n$, we say that the annulus $\mathcal{C}(m, n)$ is good if, for every $v \in \partial_n Q_\infty$,

$$(1 - \varepsilon/2)\mathbf{c}_p \eta m \leq d_{\text{fpp}}^{Q_\infty}(v, \partial_m Q_\infty) \leq (1 + \varepsilon/2)\mathbf{c}_p \eta m, \quad (3.20)$$

and it is bad otherwise. Proposition 3.16 ensures that we can fix $\eta \in (0, 1)$ small enough such that for all n large enough, the annulus $\mathcal{C}(n - \lfloor \eta n \rfloor, n)$ is good with probability at least $1 - \delta$.

Let $n_0 = n$, and define by induction $n_{k+1} = n_k - \lfloor \eta n_k \rfloor$ for every $k \geq 0$. Let $q = \lfloor \frac{\ln(\varepsilon/(5\kappa))}{\ln(1-\eta)} \rfloor$. Note that $n_q \geq \frac{\varepsilon}{5\kappa}n$ and $n_q \leq \frac{\varepsilon}{4\kappa}n$ for n large. Using Markov's inequality we get that for n large enough,

$$\mathbb{P}\left(\#\{k \in \{0, 1, \dots, q-1\} : \mathcal{C}(n_{k+1}, n_k) \text{ is bad}\} > \frac{\varepsilon}{5\kappa|\ln(1-\eta)|}\right) \leq \frac{5\kappa|\ln(1-\eta)|}{\varepsilon}q\delta \leq \varepsilon$$

by our choice of δ and q . Let $\mathcal{D}_n^\varepsilon$ denote the event whose probability appears in the previous display. We will show that the property $(\mathbf{c}_p - \varepsilon)n \leq d_{\text{fpp}}^{Q_\infty}(\rho, v) \leq (\mathbf{c}_p + \varepsilon)n$ for every $v \in \partial_n Q_\infty$ holds on the complement of $\mathcal{D}_n^\varepsilon$. Since $\mathbb{P}(\mathcal{D}_n^\varepsilon) \leq \varepsilon$ for n large, this will complete the proof.

Suppose that $\mathcal{D}_n^\varepsilon$ does not hold. Then the fpp-distance between any vertex $v \in \partial_n Q_\infty$ and ρ is larger than the cost one must pay to cross the good annuli, that is

$$\begin{aligned} d_{\text{fpp}}^{Q_\infty}(v, \rho) &\geq \sum_{k=0}^{q-1} (1 - \varepsilon/2)\mathbf{c}_p(n_k - n_{k+1}) - \sum_{k : \mathcal{C}(n_{k+1}, n_k) \text{ bad}} (1 - \varepsilon/2)\mathbf{c}_p(n_k - n_{k+1}) \\ &\geq (1 - \varepsilon/2)\mathbf{c}_p(n_0 - n_q) - \frac{\varepsilon}{5\kappa|\ln(1-\eta)|}\mathbf{c}_p \max_{0 \leq k < q} (n_k - n_{k+1}) \\ &\geq \mathbf{c}_p n \left[1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4|\ln(1-\eta)|}\eta \right] \\ &\geq \mathbf{c}_p n (1 - \varepsilon), \end{aligned}$$

using the properties $\kappa \geq 1$ and $\eta/|\ln(1-\eta)| < 1$ for $\eta \in (0, 1)$. Conversely, we can build a path from v to ρ that crosses the good annuli in the d_{fpp} -shortest way, and the bad annuli or the hull $\mathcal{H}_{Q_\infty}^{\text{tr}}(n_q)$ along left-most geodesics. Using the properties $d_{\text{gr}} \leq d_{\text{fpp}} \leq \kappa d_{\text{gr}}$ and $1 \leq \mathbf{c}_p \leq \kappa$, we can bound its fpp-weight by

$$\begin{aligned} &\sum_{k=0}^{q-1} (1 + \varepsilon/2)\mathbf{c}_p(n_k - n_{k+1}) + \kappa n_q + \sum_{k : \mathcal{C}(n_{k+1}, n_k) \text{ bad}} \kappa(n_k - n_{k+1}) \\ &\leq (1 + \varepsilon/2)\mathbf{c}_p(n_0 - n_q) + \kappa \frac{\varepsilon}{4\kappa}n + \frac{\varepsilon}{5\kappa|\ln(1-\eta)|}\kappa \max_{0 \leq k < q} (n_k - n_{k+1}) \\ &\leq \mathbf{c}_p n \left[1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4|\ln(1-\eta)|}\eta \right] \\ &\leq \mathbf{c}_p n (1 + \varepsilon), \end{aligned}$$

giving $d_{\text{fpp}}^{Q_\infty}(v, \rho) \leq \mathbf{c}_p n (1 + \varepsilon)$. This completes the proof. \square

3.5.3 Distance between two uniform points in finite quadrangulations

In this section, we consider a uniformly distributed rooted and pointed quadrangulation with n faces, which we denote by Q_n^\bullet . The associated (unpointed) rooted quadrangulation

is simply denoted by Q_n . We will write ρ_n for the root vertex of Q_n (or of Q_n^\bullet) and ∂_n for the distinguished vertex of Q_n^\bullet . This notation will be in force throughout the remaining part of this work. We note that, conditionally on the unpointed map Q_n , ∂_n is uniformly distributed over $V(Q_n)$.

Our next goal is to control the fpp-distance between ρ_n and ∂_n .

Proposition 3.18. *For every $\varepsilon \in (0, 1)$,*

$$\mathbb{P} \left(|d_{\text{fpp}}^{Q_n}(\rho_n, \partial_n) - \mathbf{c}_p d_{\text{gr}}^{Q_n}(\rho_n, \partial_n)| > \varepsilon n^{1/4} \right) \xrightarrow[n \rightarrow \infty]{} 0.$$

We postpone the proof of this result to Section 3.5.4, and first give some technical tools that are needed in this proof.

The idea is to transfer the results that we established in the UIPQ to the setting of finite quadrangulations. The core tool that we establish in this section compares the law of a neighborhood of the root in the UIPQ and in a finite quadrangulation. In this direction, Proposition 3.20 gives a result valid for neighborhoods of diameter smaller than a constant times the typical diameter of the quadrangulation. This is closely related to [20, Lemma 8 and Proposition 9], but we need sharper results.

Let us briefly introduce the objects we need. Our proofs in this section and the next one make use of the (now classical) Cori-Vauquelin-Schaeffer bijection [44, Section 5.4] between rooted and pointed quadrangulations with n faces, and labeled rooted plane trees with n edges. For more details, we refer to [20, Section 4].

The Cori-Vauquelin-Schaeffer correspondence. Consider a rooted plane tree τ , with root vertex σ , together with a labeling $Z = (Z_x)_{x \in V(\tau)}$ of its vertices by integers such that $Z_\sigma = 0$, and $|Z_x - Z_y| \leq 1$ if x and y are adjacent. We fix $\varepsilon \in \{0, 1\}$ and explain how to get a pointed and rooted quadrangulation Q from (τ, Z, ε) . To this end, we suppose that τ is embedded in the plane as shown on Figure 3.10. Then, firstly, we add a vertex ∂ , and extend the labeling to ∂ so that $Z_\partial = -1 + \min_{x \in V(\tau)} Z_x$. The vertex set of the quadrangulation Q is $V(\tau) \cup \{\partial\}$, and ∂ is its distinguished vertex. We also extend the labeling to corners of τ , by declaring that the label of a corner is the label of the incident vertex of τ . Secondly, we order the corners of τ in clockwise order around τ , starting from the bottom corner of σ . For every $n > Z_\partial + 1$, we draw an edge of Q from each corner labeled n to the next corner labeled $n - 1$; and for each corner of index $Z_\partial + 1$, we draw an edge from this corner to ∂ . This defines the edges of Q . Finally, we root Q at the edge drawn from the bottom corner of σ and use ε to determine its orientation: the root vertex is σ iff $\varepsilon = 1$. The construction should be clear from Figure 3.10. We mention the following important property relating labels on τ to distances from ∂ in Q : For every vertex $v \in V(Q)$, $d_{\text{gr}}^Q(\partial, v) = Z_v - Z_\partial$.

The CVS correspondence allows us to code uniform rooted and pointed quadrangulations by uniform rooted labeled plane trees. More precisely, let T_n be a uniform rooted plane tree with n edges. Given T_n , assign i.i.d. weights on its edges, with uniform law over $\{-1, 0, +1\}$. For every $x \in V(T_n)$, define the label Z_x^n as the sum of the weights of

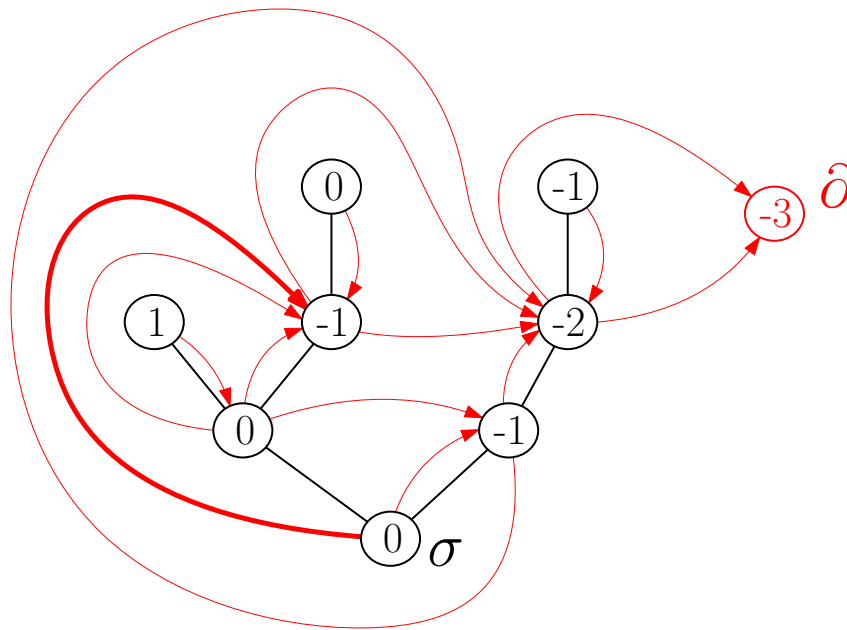


Figure 3.10 – The CVS correspondence applied to a finite labeled tree (in black). The thick red edge is the root edge of the quadrangulation, and its orientation is determined by ϵ (here $\epsilon = 1$).

edges along the geodesic from the root to x in T_n . Pick $\epsilon \in \{0, 1\}$ uniformly at random. The CVS correspondence applied to (T_n, Z^n, ϵ) then gives a uniform rooted and pointed quadrangulation with n faces.

Pruned trees. Let τ be a (finite) rooted plane tree. For every vertex v of τ and for every $h > 0$ such that $d_{\text{gr}}^r(\sigma, v) \geq [h]$, we denote the ancestor of v at height $[h]$ in τ by $[v]_h$. We construct the *pruned tree* $\mathcal{P}((\tau, v), h)$ by removing all strict descendants of $[v]_h$ in τ (see [20, Fig. 5]) and we see $\mathcal{P}((\tau, v), h)$ as a rooted plane tree pointed at $[v]_h$.

Note that the vertex set of $\mathcal{P}((\tau, v), h)$ is a subset of the set of all vertices of τ . It follows that, if τ is labeled by Z , we can construct a labeling of $\mathcal{P}((\tau, v), h)$ by restricting Z to $V(\mathcal{P}((\tau, v), h))$. We will do so implicitly, keeping the same notation for the labeling on τ and on the pruned tree.

The case where τ is infinite is similar. In that case, we always assume that the tree has one end: there is only one infinite injective path started at its root, called the spine. We use the notation $[\infty]_h$ for the unique vertex of the spine at distance h from the root, and remove its strict descendants in τ to get the pruned tree $\mathcal{P}(\tau, h)$. Informally, ∞ plays the role of the distinguished vertex.

We now state the first result of this section. Let (τ, Z) be a finite rooted labeled plane tree with root vertex σ . Let $\epsilon \in \{0, 1\}$, and let Q be the rooted and pointed quadrangulation constructed from (τ, Z, ϵ) via the CVS correspondence (we denote the

root vertex of Q by ρ). Let $\xi \in V(\tau)$, and let h be an integer such that $0 < h < d_{\text{gr}}^{\tau}(\sigma, \xi)$. Set

$$r = -\min_{0 \leq i \leq h} Z_{[\xi]_i} \geq 0.$$

Proposition 3.19. *Assume that $r \geq 4$. Then $B_Q^{\bullet}(r-3)$ is a function of the pruned tree $\mathcal{P}((\tau, \xi), h)$, its labeling, and ϵ .*

Proof. The idea is to prove that, if Q' is the quadrangulation obtained by applying the CVS correspondence to the pruned tree, then $B_Q^{\bullet}(r-3) = B_{Q'}^{\bullet}(r-3)$ (considered as an equality between rooted quadrangulations with a boundary).

Let us state some useful observations. Without loss of generality, by taking h smaller we can assume that $Z_{[\xi]_h} = \min_{0 \leq i \leq h} Z_{[\xi]_i} = -r$. We note that $d_{\text{gr}}^Q(\sigma, \rho) = 0$ or 1 depending on ϵ . If $v \in V(\tau)$,

$$d_{\text{gr}}^Q(\sigma, v) \geq |Z_{\sigma} - Z_v| = |Z_v|,$$

and by the triangle inequality $d_{\text{gr}}^Q(\rho, v) \geq |Z_v| - 1$.

Our first step is to prove that vertices of $B_Q^{\bullet}(r-3)$ “belong” to the pruned tree, and that their labels are at least $-r+1$.

Let v be a vertex of τ such that $Z_v \leq -r$. Starting from any corner of v , the construction of edges in the CVS correspondence yields a path starting from v that visits vertices with strictly decreasing labels. This path ultimately connects v to ∂ by visiting only vertices with labels less than $-r$, thus at distance at least $r-1$ from ρ in Q . By construction, any vertex of $B_Q^{\bullet}(r-3)$ is such that any path from this vertex to ∂ visits a vertex w with $d_{\text{gr}}^Q(\rho, w) \leq r-2$. It follows that v does not belong to $B_Q^{\bullet}(r-3)$.

Let us now check that vertices in $V(\tau) \setminus V(\mathcal{P}((\tau, \xi), h))$ do not belong to $B_Q^{\bullet}(r-3)$ either. Let v be such a vertex with $Z_v > -r$ (the case $Z_v \leq -r$ was already considered above). Then $[\xi]_h$ is an ancestor of v , and the cactus bound [44, Proposition 5.9 (ii)] shows that

$$d_{\text{gr}}^Q(\sigma, v) \geq Z_{\sigma} - Z_{[\xi]_h} = r,$$

and thus $d_{\text{gr}}^Q(\rho, v) \geq r-1$.

Let c be any corner of v . Order the corners of τ in clockwise order starting at c , and let c'' be the first corner of $[\xi]_h$ that appears in this enumeration: every corner between c and c'' is incident to a vertex of $V(\tau) \setminus V(\mathcal{P}((\tau, \xi), h))$. Since labels change by at most 1 in this enumeration, there will be a corner c' with label $Z_v - 1$ between c and c'' (possibly $c' = c''$). This ensures that the edge drawn from c in the CVS correspondence ends at a vertex in $V(\tau) \setminus V(\mathcal{P}((\tau, \xi), h))$ with label $Z_v - 1$, or possibly at $[\xi]_h$. Therefore, we can inductively construct a path from v that stays in $V(\tau) \setminus V(\mathcal{P}((\tau, \xi), h))$ until it reaches a vertex w of label $-r$, and we extend this path to a path from v to ∂ as we did previously. Every vertex of this path is at distance at least $r-1$ from ρ , so it follows again that v does not belong to $B_Q^{\bullet}(r-3)$.

Note that by the same reasoning, if c is a corner of label at least $-r+1$ that belongs to the pruned tree, then the edge drawn from c will reach a vertex of the pruned tree (possibly $[\xi]_h$), and thus will be present in Q' as well as in Q .

As our second and last step, we now verify that $B_Q^\bullet(r-3) = B_{Q'}^\bullet(r-3)$. By the first step, any edge belonging to $B_Q^\bullet(r-3)$ is drawn from a corner c of τ (not incident to $[\xi]_h$) with label at least $-r+1$ that “belongs” to the pruned tree, so it appears both in Q and Q' . It follows that we have both

$$\begin{aligned} d_{\text{gr}}^{Q'}(\rho, v) &\leq d_{\text{gr}}^Q(\rho, v) && \text{if } v \in B_Q^\bullet(r-3), \\ d_{\text{gr}}^Q(\rho, v) &\leq d_{\text{gr}}^{Q'}(\rho, v) && \text{if } v \in B_{Q'}^\bullet(r-3). \end{aligned}$$

Next, any edge of the boundary of $B_Q^\bullet(r-3)$ is incident to a face of Q containing a vertex at d_{gr}^Q -distance (hence $d_{\text{gr}}^{Q'}$ -distance) less than or equal to $r-4$ from ρ . This face must then also be contained in $B_{Q'}^\bullet(r-3)$. It easily follows that $B_Q^\bullet(r-3) \subset B_{Q'}^\bullet(r-3)$, and the converse is also true by a symmetric argument. \square

The CVS correspondence can be extended to infinite labeled trees. Precisely, we consider the set \mathcal{S} of all infinite rooted labeled trees with one end, such that the infimum of the labels on the spine is $-\infty$. With every $(\tau, Z) \in \mathcal{S}$ and $\varepsilon \in \{0, 1\}$, one can associate an infinite planar quadrangulation Q , which is defined via a direct extension of the rules of the CVS correspondence (see [25, Proposition 2.5]). Furthermore, the preceding proposition is immediately extended to that case, with the same proof: for every integer $h > 0$, if $r := -\min\{Z_{[\infty]_j} : 0 \leq j \leq h\} \geq 4$, the hull $B_Q^\bullet(r-3)$ only depends on the pruned tree $\mathcal{P}(\tau, h)$, its labeling, and ε .

The following proposition is closely related to [20, Proposition 9], but deals with hulls instead of balls. Recall that Q_n^\bullet is uniformly distributed over the set of rooted and pointed quadrangulations with n faces.

Proposition 3.20. *For every $\varepsilon > 0$, there exists $\chi > 0$ such that for every sufficiently large n , we can construct Q_n^\bullet and Q_∞ on the same probability space in such a way that the equality*

$$B_{Q_n^\bullet}^\bullet(\chi n^{1/4}) = B_{Q_\infty}^\bullet(\chi n^{1/4})$$

holds with probability at least $1 - \varepsilon$.

Proof. The proof is very similar to that of [20, Proposition 9], using our Proposition 3.19 instead of [20, Proposition 8]. Let us only outline the argument. We may assume that Q_n^\bullet is obtained via the CVS correspondence from a uniform labeled tree with n edges (T_n, Z^n) , and we consider a uniformly distributed vertex ξ_n of T_n . From [25, Theorem 2.8], we also get that Q_∞ can be constructed as the image under the extended CVS correspondence of the so-called uniform infinite labeled tree (T_∞, Z^∞) (see [25, Definition 2.6] for the definition of the latter object). Using Proposition 3.19 (and its analog in the infinite case), the desired result follows once we know that we can couple the labeled trees (T_n, Z^n) and (T_∞, Z^∞) so that the (labeled) pruned trees $\mathcal{P}((T_n, \xi_n), \chi n^{1/4} + 3)$ and $\mathcal{P}(T_\infty, \chi n^{1/4} + 3)$ are equal with probability at least $1 - \varepsilon$. We refer to the proof of [20, Proposition 9] for additional details. \square

3.5.4 Proof of Proposition 3.18

Recall the notation of Section 3.5.3. In particular, Q_n^\bullet is a uniformly distributed rooted and pointed quadrangulation with n faces. We denote its root vertex by ρ_n , and its distinguished vertex by ∂_n . The triplet associated with Q_n^\bullet via the CVS correspondence is denoted by (T_n, Z^n, ϵ_n) . We will also use the uniform infinite labeled tree (T_∞, Z^∞) (cf. [25, Definition 2.6]) from which one constructs the UIPQ Q_∞ via the extended CVS correspondence.

First step: Pruning finite trees and infinite trees

For any rooted plane tree τ , any vertex $v \in V(\tau)$, and $h \leq d_{\text{gr}}^\tau(\sigma, v)$, we set

$$\Theta(\tau, v, h) = \#V(\tau) - \#V(\mathcal{P}((\tau, v), h)),$$

which is the number of vertices that are removed from the tree when pruning it. Denote the first (in lexicographical order) vertex with minimal label in T_n by η_n , and consider the pointed tree (T_n, η_n) . Let $\beta > 0$ and $b \in (0, 1)$. We claim that we can find a constant C that depends only on b such that, for all large enough n , for every nonnegative function F on the space of all rooted and pointed labeled plane trees,

$$\mathbb{E} \left[F \left(\mathcal{P}((T_n, \eta_n), \lfloor \beta\sqrt{n} \rfloor) \right) \mathbb{1}_{\{d_{\text{gr}}^{T_n}(\rho_n, \eta_n) > \beta\sqrt{n}, \Theta(T_n, \eta_n, \lfloor \beta\sqrt{n} \rfloor) \geq b(n+1)\}} \right] \leq C \mathbb{E} \left[F \left(\mathcal{P}(T_\infty, \lfloor \beta\sqrt{n} \rfloor) \right) \right].$$

In the previous display, we slightly abuse notation by viewing both $\mathcal{P}((T_n, \eta_n), \lfloor \beta\sqrt{n} \rfloor)$ and $\mathcal{P}(T_\infty, \lfloor \beta\sqrt{n} \rfloor)$ as *labeled trees* — we obviously keep the labels of the original trees.

Proof of the claim. To simplify notation, we write A_n for the event

$$A_n := \left\{ d_{\text{gr}}^{T_n}(\rho_n, \eta_n) > \beta\sqrt{n}, \Theta(T_n, \eta_n, \lfloor \beta\sqrt{n} \rfloor) \geq b(n+1) \right\}.$$

Let ξ_n be uniformly distributed over $V(T_n)$. We note that on the event A_n , the conditional probability that ξ_n is not in $V(\mathcal{P}((T_n, \eta_n), \lfloor \beta\sqrt{n} \rfloor))$ knowing T_n is at least b , because on A_n ,

$$\#V(T_n) - \#V(\mathcal{P}((T_n, \eta_n), \lfloor \beta\sqrt{n} \rfloor)) \geq b(n+1).$$

It follows that

$$\mathbb{E} \left[F \left(\mathcal{P}((T_n, \eta_n), \lfloor \beta\sqrt{n} \rfloor) \right) \mathbb{1}_{A_n} \right] \leq \frac{1}{b} \mathbb{E} \left[F \left(\mathcal{P}((T_n, \eta_n), \lfloor \beta\sqrt{n} \rfloor) \right) \mathbb{1}_{A_n} \mathbb{1}_{\{\xi_n \notin V(\mathcal{P}((T_n, \eta_n), \lfloor \beta\sqrt{n} \rfloor))\}} \right] \quad (3.21)$$

If $\xi_n \notin V(\mathcal{P}((T_n, \eta_n), \lfloor \beta\sqrt{n} \rfloor))$, we have $\lfloor \xi_n \rfloor_{\beta\sqrt{n}} = \lfloor \eta_n \rfloor_{\beta\sqrt{n}}$ and

$$\mathcal{P}((T_n, \xi_n), \lfloor \beta\sqrt{n} \rfloor) = \mathcal{P}((T_n, \eta_n), \lfloor \beta\sqrt{n} \rfloor).$$

It follows that the expectation in the right-hand side of (3.21) is bounded above by

$$\mathbb{E} \left[F \left(\mathcal{P}((T_n, \xi_n), \lfloor \beta\sqrt{n} \rfloor) \right) \mathbb{1}_{A_n'} \right] \quad (3.22)$$

where $A'_n := \{d_{\text{gr}}^{T_n}(\rho_n, \xi_n) > \beta\sqrt{n}, \Theta(T_n, \xi_n, \lfloor \beta\sqrt{n} \rfloor) \geq b(n+1)\}$.

Next let τ be a rooted and pointed labeled plane tree such that $|\tau| < n$ (here $|\tau|$ denotes the number of edges of τ) and the distinguished vertex is at generation $\lfloor \beta\sqrt{n} \rfloor$ and has no strict descendants. Formulas (19) and (21) in [20] show that, for n large enough (independently of the choice of τ),

$$\frac{\mathbb{P}(\mathcal{P}((T_n, \xi_n), \lfloor \beta\sqrt{n} \rfloor)) = \tau)}{\mathbb{P}(\mathcal{P}(T_\infty, \lfloor \beta\sqrt{n} \rfloor) = \tau)} \leq 2 \left(1 - \frac{|\tau|}{n}\right)^{-1/2}.$$

For n large enough, the condition $\Theta(T_n, \xi_n, \lfloor \beta\sqrt{n} \rfloor) \geq b(n+1)$ ensures that $\mathcal{P}((T_n, \xi_n), \lfloor \beta\sqrt{n} \rfloor)$ has less than $(1 - \frac{b}{2})n$ edges. It follows that the quantity in (3.22) is bounded above by

$$2 \left(\frac{b}{2}\right)^{-1/2} \mathbb{E} \left[F \left(\mathcal{P}(T_\infty, \lfloor \beta\sqrt{n} \rfloor) \right) \right].$$

This completes the proof of the claim. □

Second step: Hulls in finite quadrangulations and in the UIPQ

Recall that, for every integer $r \geq 1$, the hull $B_{Q_n^\bullet}(r)$ is well defined under the condition $d_{\text{gr}}^{Q_n^\bullet}(\rho_n, \partial_n) > r + 1$.

Let $\alpha > 0$, and set $\alpha_{(n)} = \lfloor \alpha n^{1/4} \rfloor - 1$ to simplify notation. Recall the notation A_n introduced in the first step above, and set

$$E_n = A_n \cap \{Z_{[\eta_n]_{\beta\sqrt{n}}}^n < -\alpha n^{1/4}\},$$

We note that (on A_n) the condition $Z_{[\eta_n]_{\beta\sqrt{n}}}^n < -\lfloor \alpha n^{1/4} \rfloor$ implies a fortiori $Z_{\eta_n}^n \leq Z_{[\eta_n]_{\beta\sqrt{n}}}^n - 1 \leq -\alpha_{(n)} - 3$ and $d_{\text{gr}}^{Q_n}(\rho_n, \eta_n) \geq \alpha_{(n)} + 2$ so that $d_{\text{gr}}^{Q_n}(\rho_n, \partial_n) \geq \alpha_{(n)} + 3$. In particular, the hull $B_{Q_n^\bullet}(\alpha_{(n)})$ is well defined on the event E_n .

By Proposition 3.19, on the event E_n , the hull $B_{Q_n^\bullet}(\alpha_{(n)})$ is equal to a deterministic function of the pruned tree $\mathcal{P}((T_n, \eta_n), \lfloor \beta\sqrt{n} \rfloor)$ (and labels of this tree and ϵ_n). Furthermore on the event $\{Z_{[\infty]_{\beta\sqrt{n}}}^\infty < -\alpha_{(n)}\}$, the hull $B_{Q_\infty^\bullet}(\alpha_{(n)})$ is equal to the same deterministic function of $\mathcal{P}(T_\infty, \lfloor \beta\sqrt{n} \rfloor)$ (and its labels and ϵ_∞). As a consequence of this fact and the first step, we have also, for every nonnegative function G on the space of rooted planar maps,

$$\mathbb{E} \left[G(B_{Q_n^\bullet}(\alpha_{(n)})) \mathbb{1}_{E_n} \right] \leq C \mathbb{E} \left[G(B_{Q_\infty^\bullet}(\alpha_{(n)})) \mathbb{1}_{\{Z_{[\infty]_{\beta\sqrt{n}}}^\infty < -\alpha n^{1/4}\}} \right]. \quad (3.23)$$

Final step

Let $\delta > 0$ to be fixed later, and for every integers $j, l \geq 1$ set

$$\begin{aligned} \alpha_j &= j\delta^2, & \alpha'_j &= (j+1)\delta^2, & \alpha''_j &= (j+2)\delta^2, \\ \beta_l &= l\delta^5, & \beta'_l &= (l+1)\delta^5, & \beta''_l &= (l+2)\delta^5. \end{aligned}$$

Lemma 3.21. *For every integers $j, l \geq 1$, set*

$$H_{n,\delta}^{j,l} = \{Z_{\eta_n}^n \in [-\alpha_j'' n^{1/4}, -\alpha_j' n^{1/4}], Z_{[\eta_n]_{\beta_l \sqrt{n}}}^n < -\alpha_j n^{1/4}\} \\ \cap \{\beta_l' \sqrt{n} \leq d_{\text{gr}}^{T_n}(\rho_n, \eta_n) \leq \beta_l'' \sqrt{n}\} \cap \{\Theta(T_n, \eta_n, \lfloor \beta_l \sqrt{n} \rfloor) > \delta^{11} n\}.$$

Let $\varepsilon > 0$. For all $\delta \in (0, 1)$ small enough, for all large enough n , the event

$$H_{n,\delta} := \bigcup_{j=\lfloor \delta^{-1} \rfloor}^{\lfloor \delta^{-3} \rfloor} \bigcup_{l=\lfloor \delta^{-4} \rfloor}^{\lfloor \delta^{-6} \rfloor} H_{n,\delta}^{j,l}$$

has probability at least $1 - \varepsilon$.

Let us postpone the proof of this lemma and complete that of Proposition 3.18.

Each set $H_{n,\delta}^{j,l}$ is contained in a set of the type E_n (with $\alpha = \alpha_j$, $\beta = \beta_l$ and $b = \delta^{11}$). Using (3.23) and Proposition 3.17, we get that, on the event $H_{n,\delta}^{j,l}$, except on a set of probability tending to 0 as $n \rightarrow \infty$, the fpp-distance between any point of the boundary of $B_{Q_n^\bullet}(\lfloor \alpha_j n^{1/4} \rfloor - 1)$ and the root vertex ρ_n is close to $\mathbf{c}_0 \alpha_j n^{1/4}$, up to an error bounded by $\varepsilon n^{1/4}$. Note that Proposition 3.17 considers vertices of the boundary of the truncated hull, whereas here we want to deal with the boundary of the standard hull of the same radius: This makes no difference since it is easily checked that any vertex of the boundary of the standard hull either belongs to the boundary of the truncated hull or is adjacent to a vertex of the latter boundary. Moreover, when applying (3.23), we should consider the “intrinsic fpp-distance” on $B_{Q_n^\bullet}(\lfloor \alpha_j n^{1/4} \rfloor - 1)$ in order to compare it with the similar intrinsic distance on the corresponding hull of Q_∞ . However, similarly as in the proof of Proposition 3.16, we can use the fact that the fpp-distance is bounded above by the intrinsic fpp-distance, and the minimal fpp-distance from a point of the hull boundary is equal to the minimal intrinsic fpp-distance.

We also know that, still on the event $H_{n,\delta}^{j,l}$, the graph distance (in Q_n^\bullet) between ∂_n and the boundary of the hull $B_{Q_n^\bullet}(\lfloor \alpha_j n^{1/4} \rfloor)$ is bounded above by $2\delta^2 n^{1/4}$ (to see this, recall that labels Z_a^n correspond to distances from ∂_n , up to a shift by $-Z_{\eta_n}^n + 1$, and consider a geodesic from ρ_n to ∂_n).

Recalling that the fpp-weights are bounded above by κ , we then obtain, on the event $H_{n,\delta}^{j,l}$ except on a set of probability tending to 0 as $n \rightarrow \infty$, that the fpp-distance (in Q_n) between ∂_n and ρ_n is close to $\mathbf{c}_0 d_{\text{gr}}^{Q_n}(\rho_n, \partial_n)$, up to an error bounded by $(2\delta^2 + \varepsilon)\kappa n^{1/4}$.

We can apply the previous property to each set $H_{n,\delta}^{j,l}$, and we obtain that on the event $H_{n,\delta}$, except on a set of probability tending to 0 when $n \rightarrow \infty$, we have

$$|d_{\text{fpp}}^{Q_n}(\rho_n, \partial_n) - \mathbf{c}_0 d_{\text{gr}}^{Q_n}(\rho_n, \partial_n)| \leq (2\delta^2 + \varepsilon)\kappa n^{1/4}.$$

This completes the proof of Proposition 3.18.

Proof of Lemma 3.21. We need to introduce some notation. We write $u_0^n, u_1^n, \dots, u_{2n}^n$ for the contour sequence of T_n : u_0^n is the root of T_n and, for $1 \leq j \leq 2n$, u_j^n is either

the first child of u_{j-1}^n that does not appear in u_0^n, \dots, u_{j-1}^n , or the parent of u_{j-1}^n if there is no such child. We write $(C_k^n)_{0 \leq k \leq 2n}$ for the contour function (so that $C_k^n = |u_k^n|$). The discrete snake associated with T_n is denoted by $(W_k^n)_{0 \leq k \leq 2n}$: $W_k^n = (W_k^n(j))_{0 \leq j \leq C_k^n}$, where $W_k^n(j)$ is the label of the ancestor of u_k^n at generation j . For simplicity, we write $Y_k^n = W_k^n(C_k^n) = Z_{u_k^n}^n$. By the results of Janson and Marckert [34], we have

$$\left(\frac{1}{\sqrt{2n}} C_{\lfloor 2nt \rfloor}^n, \left(\frac{9}{8} \right)^{1/4} n^{-1/4} W_{\lfloor 2nt \rfloor}^n \left(\lfloor \sqrt{2n} \cdot \rfloor \right) \right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_t, W_t)_{0 \leq t \leq 1}, \quad (3.24)$$

where \mathbf{e} is a normalized Brownian excursion, and W is the Brownian snake driven by \mathbf{e} . The convergence in (3.24) holds in the topology of uniform convergence. By the Skorokhod representation theorem, we may and will assume that the latter convergence holds a.s. We write $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ for the tree coded by \mathbf{e} and we also set $Y_t = \widehat{W}_t$. The process Y (or W) can be viewed as indexed by $\mathcal{T}_{\mathbf{e}}$. Fix $\alpha \in (0, 1/100)$. We observe that

$$\sup_{a, b \in \mathcal{T}_{\mathbf{e}}, a \neq b} \frac{|Y_a - Y_b|}{d_{\mathbf{e}}(a, b)^{\frac{1}{2} - \alpha}} =: C_{\omega} < \infty, \quad \text{a.s.}$$

Since conditionally given \mathbf{e} , Y can be interpreted as Brownian motion indexed by $\mathcal{T}_{\mathbf{e}}$, this follows from standard chaining arguments (using metric entropy bounds) and we omit the details.

It follows that

$$\sup_{0 \leq s \leq 1} \sup_{0 \leq r \leq \delta^5 \wedge \mathbf{e}_s} |\widehat{W}_s - W_s(\mathbf{e}_s - r)| \leq C_{\omega} (\delta^5)^{\frac{1}{2} - \alpha}$$

and the right-hand side is smaller than $\delta^2/10$ except on a set of small probability when δ is small. On the other hand, it follows from the (a.s.) convergence (3.24) that

$$\sup_{0 \leq k \leq 2n} \sup_{0 \leq j \leq 2\delta^5 \sqrt{n} \wedge C_k^n} \left(\frac{9}{8} \right)^{1/4} n^{-1/4} |W_k^n(C_k^n) - W_k^n(C_k^n - j)| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \sup_{0 \leq s \leq 1} \sup_{0 \leq r \leq \sqrt{2}\delta^5 \wedge \mathbf{e}_s} |\widehat{W}_s - W_s(\mathbf{e}_s - r)|.$$

Hence, by the preceding observations, we can find $\delta_0 > 0$ such that, for every $\delta \in (0, \delta_0]$, for all n large enough,

$$\sup_{0 \leq k \leq 2n} \sup_{0 \leq j \leq 2\delta^5 \sqrt{n} \wedge C_k^n} \left(\frac{9}{8} \right)^{1/4} n^{-1/4} |W_k^n(C_k^n) - W_k^n(C_k^n - j)| < \frac{\delta^2}{10} \quad (3.25)$$

except possibly on a set of probability bounded above by $\varepsilon/4$.

Write k_n for the first index such that $u_{k_n}^n = \eta_n$ (so with our notation $Y_{k_n}^n = Z_{\eta_n}^n$). We note that, by (3.24), we have $k_n/\sqrt{2n} \rightarrow t_*$ as $n \rightarrow \infty$, a.s., where t_* is the unique value such that $Y_{t_*} = \min\{Y_t : 0 \leq t \leq 1\}$.

Outside a set of small probability when δ is small (uniformly in n) we can find $j \in \{[\delta^{-1}], \dots, [\delta^{-3}]\}$ and $l \in \{[\delta^{-4}], \dots, [\delta^{-6}]\}$ such that

$$-\alpha_j'' n^{1/4} \leq Z_{\eta_n}^n < -\alpha_l' n^{1/4}, \quad \beta_l' \sqrt{n} \leq C_{k_n}^n = d_{\text{gr}}^{T_n}(\rho_n, \eta_n) < \beta_l'' \sqrt{n}.$$

We must now justify the fact that we have also $Z_{[\eta_n]_{\beta_l\sqrt{n}}}^n < -\alpha_j n^{1/4}$ and $\Theta(T_n, \eta_n, \lfloor \beta_l\sqrt{n} \rfloor) > \delta^{11}n$. The first property follows from (3.25) since

$$|Z_{\eta_n}^n - Z_{[\eta_n]_{\beta_l\sqrt{n}}}^n| = |W_{k_n}^n(C_{k_n}^n) - W_{k_n}^n(\lfloor \beta_l\sqrt{n} \rfloor)|,$$

and $C_{k_n}^n - \lfloor \beta_l\sqrt{n} \rfloor \leq 2\delta^5\sqrt{n}$. For the second property, we note that $\Theta(T_n, \eta_n, \lfloor \beta_l\sqrt{n} \rfloor) \geq \frac{1}{2}(k'_n - k_n)$ where $k'_n \stackrel{\text{def}}{=} \min\{j \geq k_n : C_j^n \leq \beta_l\sqrt{n}\}$, and using (3.24), we have

$$\liminf_{n \rightarrow \infty} (2n)^{-1}(k'_n - k_n) \geq \inf\{s > t_* : \mathbf{e}_s = (\mathbf{e}_{t_*} - \delta^5/\sqrt{2})^+\} - t_*.$$

For $\delta > 0$ small enough, for n large, the Hölder continuity properties of the Brownian excursion show that the right-hand side of the last display is greater than δ^{11} except on a set of probability bounded above by $\varepsilon/4$. This completes the proof. \square

3.5.5 Distances between two arbitrary points in a finite quadrangulation

The next statement gives the part of Theorem 3.1 concerning quadrangulations.

Theorem 3.22. *For every $\varepsilon > 0$, we have*

$$\mathbb{P}\left(\sup_{x,y \in V(Q_n)} \left|d_{\text{fpp}}^{Q_n}(x,y) - \mathbf{c}_p d_{\text{gr}}^{Q_n}(x,y)\right| > \varepsilon n^{1/4}\right) \xrightarrow[n \rightarrow \infty]{} 0.$$

Proof. The proof follows the same pattern as that of [23, Theorem 1], and we refer to [23] for more details. We first claim that Proposition 3.18 remains valid if ρ_n is replaced by a uniformly distributed vertex of Q_n . In other words, if ∂'_n is uniformly distributed on $V(Q_n)$ conditionally on Q_n^\bullet ,

$$\mathbb{P}\left(\left|d_{\text{fpp}}^{Q_n}(\partial_n, \partial'_n) - \mathbf{c}_p d_{\text{gr}}^{Q_n}(\partial_n, \partial'_n)\right| > \varepsilon n^{1/4}\right) \xrightarrow[n \rightarrow \infty]{} 0.$$

Let us explain this. The law of Q_n is invariant under uniform re-rooting, so that the statement of Proposition 3.18 still holds if we replace ρ_n by the tail of an oriented edge chosen uniformly on Q_n . Let $\vec{E}(Q_n)$ be the set of all oriented edges of Q_n (with cardinality $4n$), and, for every $e \in \vec{E}(Q_n)$, write $t(e)$ for the tail of e . Then Proposition 3.18 and the invariance under uniform re-rooting give

$$\frac{1}{4n} \mathbb{E} \left[\sum_{e \in \vec{E}(Q_n)} \mathbb{1}_{\{|d_{\text{fpp}}^{Q_n}(\partial_n, t(e)) - \mathbf{c}_p d_{\text{gr}}^{Q_n}(\partial_n, t(e))| > \varepsilon n^{1/4}\}} \right] \xrightarrow[n \rightarrow \infty]{} 0.$$

Every vertex of Q_n appears at least once as the tail $t(e)$ of an oriented edge e , and thus it also follows that

$$\mathbb{E} \left[\frac{1}{n+2} \sum_{v \in V(Q_n)} \mathbb{1}_{\{|d_{\text{fpp}}^{Q_n}(\partial_n, v) - \mathbf{c}_p d_{\text{gr}}^{Q_n}(\partial_n, v)| > \varepsilon n^{1/4}\}} \right] \xrightarrow[n \rightarrow \infty]{} 0.$$

This proves our claim.

Let $\alpha > 0$. We know that $V(Q_n)$ equipped with the graph distance rescaled by $\left(\frac{9}{8n}\right)^{1/4}$ and with the uniform probability measure converges in distribution in the Gromov-Hausdorff-Prokhorov topology towards the Brownian map equipped with its volume measure (see [42, Theorem 7]). Since the Brownian map is a compact metric space, it follows that for every $\varepsilon > 0$, we can fix N large enough (not depending on n) such that, if $(\partial_n^i)_{1 \leq i \leq N}$ are i.i.d. uniformly distributed random vertices of Q_n , then the $d_{\text{gr}}^{Q_n}$ -balls of radius $\varepsilon n^{1/4}$ centered at the vertices ∂_n^i , $1 \leq i \leq N$, cover Q_n with probability at least $1 - \alpha$ (see the end of Appendix A1 in [42] for a detailed justification).

The preceding assertion ensures that, on an event of probability at least $1 - \alpha$, the $d_{\text{gr}}^{Q_n}$ -distance (respectively the $d_{\text{fpp}}^{Q_n}$ -distance) between any pair of points is well approximated by the distance between a certain pair of vertices in $(\partial_n^i)_{1 \leq i \leq N}$, up to a difference bounded by $2\varepsilon n^{1/4}$ (resp. by $2\kappa\varepsilon n^{1/4}$). On the other hand, the first part of the proof shows that, for n large enough, we have $|d_{\text{fpp}}^{Q_n}(\partial_n^i, \partial_n^j) - \mathbf{c}_p d_{\text{gr}}^{Q_n}(\partial_n^i, \partial_n^j)| \leq \varepsilon n^{1/4}$ for all $1 \leq i, j \leq N$ with probability at least $1 - \alpha$. We conclude that, except on a set of probability at most 2α , $\mathbf{c}_p d_{\text{gr}}^{Q_n}$ and $d_{\text{fpp}}^{Q_n}$ differ by at most $(1 + 4\kappa)\varepsilon n^{1/4}$. This completes the proof. \square

The next result is very similar to [23, Theorem 2]. Stating the result for hulls instead of balls is a minor improvement that could also be achieved in the framework of [23]. Balls and hulls with respect to the first-passage percolation distance are defined in the same way as for the graph distance: For every $r \in (0, \infty)$, the fpp-ball $B_{Q_\infty}^{\text{fpp}}(r)$ is the union of all faces of Q_∞ that are incident to a vertex at fpp-distance strictly less than r from the root vertex of Q_∞ , and the fpp-hull $B_{Q_\infty}^{\bullet, \text{fpp}}(r)$ is the union of $B_{Q_\infty}^{\text{fpp}}(r)$ and of the finite connected components of its complement.

Theorem 3.23. *Let $\varepsilon \in (0, 1)$. We have*

$$\lim_{r \rightarrow \infty} \mathbb{P} \left(\sup_{x, y \in V(B_{Q_\infty}^\bullet(r))} |d_{\text{fpp}}^{Q_\infty}(x, y) - \mathbf{c}_p d_{\text{gr}}^{Q_\infty}(x, y)| > \varepsilon r \right) = 0. \quad (3.26)$$

Consequently,

$$\begin{aligned} \mathbb{P} \left(B_{Q_\infty}((1 - \varepsilon)r / \mathbf{c}_p) \subset B_{Q_\infty}^{\text{fpp}}(r) \subset B_{Q_\infty}((1 + \varepsilon)r / \mathbf{c}_p) \right) &\xrightarrow{r \rightarrow \infty} 1, \\ \mathbb{P} \left(B_{Q_\infty}^\bullet((1 - \varepsilon)r / \mathbf{c}_p) \subset B_{Q_\infty}^{\bullet, \text{fpp}}(r) \subset B_{Q_\infty}^\bullet((1 + \varepsilon)r / \mathbf{c}_p) \right) &\xrightarrow{r \rightarrow \infty} 1. \end{aligned}$$

We will need the following lemma, where we use the same notation Q_n^\bullet as in the preceding sections. We make the convention that, if $r \geq d_{\text{gr}}^{Q_n}(\rho_n, \partial_n) - 1$, then $B_{Q_n}^\bullet(r) = Q_n^\bullet$.

Lemma 3.24. *For every $\varepsilon > 0$, we can choose $K' > 1$ s.t. for every $\beta > 0$, for all n large enough, we have with probability greater than $1 - \varepsilon$,*

$$B_{Q_n}^\bullet(\beta n^{1/4}) \subset B_{Q_n}(K' \beta n^{1/4}),$$

and for all r large enough, with probability greater than $1 - \varepsilon$,

$$B_{Q_\infty}^\bullet(r) \subset B_{Q_\infty}(K'r).$$

Proof of Lemma 3.24. Let us begin with an observation that will be useful later in the proof. Fix $\varepsilon > 0$. Let (\mathcal{P}, D_∞) stand for the Brownian plane of [20]. Recall that the Brownian plane comes with a distinguished point, which we denote by x_0 . We write $\mathcal{B}_\mathcal{P}(\beta)$ for the closed ball of radius β centered at x_0 in \mathcal{P} . For every $\beta > 0$, the hull of radius β in (\mathcal{P}, D_∞) , denoted by $\mathcal{B}_\mathcal{P}^\bullet(\beta)$, is the complement of the unbounded connected component of the complement of $\mathcal{B}_\mathcal{P}(\beta)$. Then,

$$\sup\{D_\infty(x_0, x) : x \in \mathcal{B}_\mathcal{P}^\bullet(1)\} < \infty \text{ a.s.}$$

and thus we can find $K > 1$ such that the latter supremum is smaller than K with probability at least $1 - \varepsilon/4$. By the scaling invariance of the Brownian plane, it follows that for every $\beta > 0$,

$$\mathbb{P}(\mathcal{B}_\mathcal{P}^\bullet(\beta) \subset \mathcal{B}_\mathcal{P}(K\beta)) \geq 1 - \varepsilon/4. \quad (3.27)$$

Consider then the Brownian map (m_∞, D^*) , which also comes with a distinguished point x_* (in the construction of the Brownian motion from the CRT indexed by Brownian labels, x_* is the point with minimal label). We write $\mathcal{B}_{m_\infty}(\beta)$ for the closed ball of radius β centered at x_* . We let ∂ be another distinguished point uniformly distributed over m_∞ , and, if $0 < \beta < D^*(x_*, \partial)$, we define $\mathcal{B}_{m_\infty}^\bullet(\beta)$ as the complement of the connected component of the complement of the ball $\mathcal{B}_{m_\infty}(\beta)$ that contains ∂ . If $\beta \geq D^*(x_*, \partial)$, we take $\mathcal{B}_{m_\infty}^\bullet(\beta) = m_\infty$. Using the coupling between the Brownian map (m_∞, D^*) and the Brownian plane found in [20, Theorem 1], one gets from (3.27) that there exists $\delta > 0$ such that

$$\mathbb{P}(\mathcal{B}_{m_\infty}^\bullet(\beta) \subset \mathcal{B}_{m_\infty}(K\beta)) \geq 1 - \varepsilon/2, \quad (3.28)$$

for every $0 < \beta < \delta$. Let us briefly justify this. We note that [20, Theorem 1] allows us to couple \mathcal{P} and m_∞ so that there exists $\alpha_0 > 0$ such that, with high probability, we have $\mathcal{B}_{m_\infty}(\alpha) = \mathcal{B}_\mathcal{P}(\alpha)$ for every $\alpha \in (0, \alpha_0]$. Then, if $K\beta < \alpha \leq \alpha_0$, the property $\mathcal{B}_\mathcal{P}^\bullet(\beta) \subset \mathcal{B}_\mathcal{P}(K\beta) = \mathcal{B}_{m_\infty}(K\beta)$ also implies that $\mathcal{B}_{m_\infty}^\bullet(\beta) = \mathcal{B}_\mathcal{P}^\bullet(\beta)$, provided that ∂ does not belong to $\mathcal{B}_{m_\infty}(\alpha)$, which holds with high probability if α has been taken small enough.

In fact, taking the constant K larger if necessary, we may even assume that the bound in (3.28) holds for *every* $\beta > 0$. Indeed, we just have to take K so large that $\mathbb{P}(\mathcal{B}_{m_\infty}(K\delta) = m_\infty) \geq 1 - \varepsilon/2$.

In order to deduce the first assertion of the lemma from the preceding considerations, we use the convergence of $(V(Q_n^\bullet), (8/9)^{1/4}d_{\text{gr}}^{Q_n^\bullet})$ towards the Brownian map in the bipointed Gromov-Hausdorff topology (see [42, Theorem 7]). Note $V(Q_n^\bullet)$ is viewed as a bipointed space with distinguished points ρ_n and ∂_n (in this order) and similarly m_∞ is a bipointed space with distinguished points x_* and ∂ — at this point we note that [42, Theorem 7] considers a seemingly different choice of distinguished points in the Brownian map, but the re-rooting invariance properties of [39, Section 8] show that this makes no difference. It follows from the preceding convergence of bipointed spaces that, for any choice of $0 < \beta < \beta' < \gamma' < \gamma$,

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(B_{Q_n^\bullet}^\bullet((8/9)^{1/4}\beta n^{1/4}) \subset B_{Q_n}((8/9)^{1/4}\gamma n^{1/4})\right) \geq \mathbb{P}\left(\mathcal{B}_{m_\infty}^\bullet(\beta') \subset \mathcal{B}_{m_\infty}(\gamma')\right). \quad (3.29)$$

The derivation of (3.29) is a simple exercise on the Gromov-Hausdorff convergence and we omit the details.

The first assertion of the lemma now follows from (3.28) and (3.29): just take $K' > K$ to obtain the desired statement for n large enough. The second assertion of the lemma can be derived by similar arguments using now the fact that the Brownian plane is the scaling limit of the UIPQ in the local Gromov-Hausdorff sense [20, Theorem 2]. \square

Proof of Theorem 3.23. Let us focus on the first statement (the second one follows easily). Let $\delta > 0$.

By Lemma 3.24 applied to Q_∞ , we can find $K' > 1$ such that, for r large enough, $B_{Q_\infty}^\bullet(r) \subset B_{Q_\infty}(K'r)$ with probability at least $1 - \delta/4$. An elementary argument allows one to find a large enough constant $C > 1$ such that, for every $r \geq 1$, the $d_{\text{gr}}^{Q_\infty}$ and $d_{\text{fpp}}^{Q_\infty}$ -distances between vertices of $B_{Q_\infty}(K'r)$ are determined by $B_{Q_\infty}(CK'r)$ and the weights on the edges of $B_{Q_\infty}(CK'r)$. In particular, outside of an event of probability smaller than $\delta/4$, the event whose probability is considered in (3.26) can be expressed in terms of the ball $B_{Q_\infty}(CK'r)$ (and weights in this ball). Similarly it follows from Lemma 3.24 that for any $\beta > 0$, for n large enough, the $d_{\text{gr}}^{Q_n}$ -distance and the $d_{\text{fpp}}^{Q_n}$ -distance between two vertices of $B_{Q_n}^\bullet(\beta n^{1/4})$ are determined by $B_{Q_n}(CK'\beta n^{1/4})$, except on a set of probability smaller than $\delta/4$.

On the other hand, by Proposition 3.20, we can find $\chi > 0$ such that for all n large, we can couple Q_n^\bullet and Q_∞ in such a way that $B_{Q_n}(\chi n^{1/4}) = B_{Q_\infty}(\chi n^{1/4})$ except on a set of probability at most $\delta/4$.

Let $\varepsilon > 0$. For n large we have

$$\begin{aligned} & \mathbb{P} \left(\sup_{x,y \in V(B_{Q_\infty}^\bullet(\frac{\chi n^{1/4}}{CK'}))} |d_{\text{fpp}}^{Q_\infty}(x,y) - \mathbf{c}_p d_{\text{gr}}^{Q_\infty}(x,y)| > \varepsilon n^{1/4} \right) \\ & \leq \frac{3\delta}{4} + \mathbb{P} \left(\sup_{x,y \in V(Q_n^\bullet)} |d_{\text{fpp}}^{Q_n^\bullet}(x,y) - \mathbf{c}_p d_{\text{gr}}^{Q_n^\bullet}(x,y)| > \varepsilon n^{1/4} \right) \\ & \leq \delta. \end{aligned}$$

The second inequality follows from Theorem 3.22. For the first one, we observe that both the $d_{\text{fpp}}^{Q_\infty}$ -distance and the $d_{\text{gr}}^{Q_\infty}$ -distance on $V(B_{Q_\infty}^\bullet(\frac{\chi n^{1/4}}{CK'}))$ only depend on the ball $B_{Q_\infty}(\chi n^{1/4})$ (except on a set of probability at most $\delta/4$ in each case) and we know that we can couple Q_∞ and Q_n so that the balls $B_{Q_\infty}(\chi n^{1/4})$ and $B_{Q_n}(\chi n^{1/4})$ are equal except on a set of probability less than $\delta/4$. This completes the proof. \square

3.6 Technical lemmas for distances in the general map

We now proceed to prove that Tutte's bijection is asymptotically an isometry. In order to do so, we first prove a handful of lemmas that control the distance in the map obtained from a quadrangulation Q via Tutte's bijection in terms of the graph distance in Q . The key object is an analog of left-most geodesics, which we call downward paths, and which we define in Section 3.6.1.

Let us briefly recall the definition of Tutte's bijection (see Figure 3.1). Let Q be a quadrangulation with n faces, and color its vertices in black and white so that adjacent vertices have a different color and the root vertex is white (this is always possible since Q is bipartite). In every face of Q , draw an edge between the two white corners of this face. Then erase the edges of Q and all black vertices. We denote by $\mathcal{T}(Q)$ the map with n edges obtained in this way.

The preceding construction of a (general) planar map from a quadrangulation can also be applied to the UIPQ, and yields an infinite (random) planar map called the UIPM for Uniform Infinite Planar Map. Indeed, it was observed in [54] that the UIPM is the local limit of uniformly distributed (general) planar maps with n edges.

We can extend Tutte's correspondence to truncated hulls of even radius: The white vertices are those whose graph distance from the root vertex is even, then we draw diagonals between the two white corners of any quadrangular inner face, and we also keep the edges of the external boundary (indeed this external boundary was made of diagonals in the construction of the truncated hull). By definition, the (new) root edge is the diagonal drawn in the face to the left of the original root edge, and the root vertex remains the same. See Figure 3.11 for an example. Similarly, we can extend Tutte's correspondence to quadrangulations of a cylinder of even height, in such a way that we keep edges of both the top and the bottom boundary. The root edge then remains the same.

Finally, Tutte's correspondence is also extended in an obvious manner to the LHPQ, in such a way that all edges of the boundary of the LHPQ remain present in the resulting infinite map (the latter also contains all edges of the form $((i, j), (i + 1, j))$ for even values of $j \leq 0$).

3.6.1 Downward paths

In this section, we define certain special paths called downward paths, in the image of a quadrangulation of the cylinder under Tutte's correspondence. These special paths will be used to derive upper bounds for the distances in the UIPM.

Let $R > 0$ be an integer, let Q be a quadrangulation of the cylinder of height $2R$ with top boundary length p , and let u_0 be a vertex of its top boundary. We write $(u_i)_{0 \leq i < p}$ for the vertices of the top boundary in clockwise order, and extend this numbering to $i \in \mathbb{Z}$ by periodicity (recall that the top face is drawn as the infinite face). Recall that for every

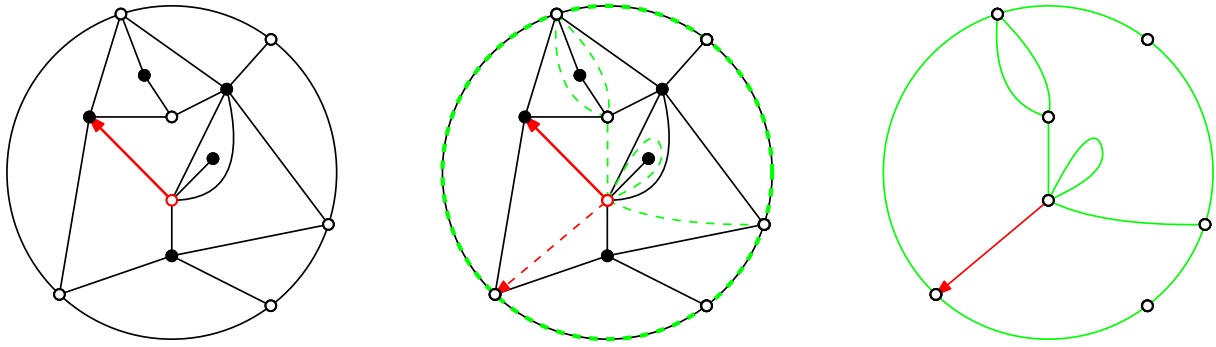


Figure 3.11 – Tutte’s bijection applied to a truncated hull of even radius, here of radius 2. In every face of Q of degree 4 but the external one, draw a “diagonal” between the two white corners. Then erase the edges of the original map and all black vertices, keeping however the edges of the boundary. The map we obtain is rooted at the edge drawn in the face of Q on the left of the root edge of Q , oriented in such a way that its tail vertex is the root vertex of Q .

$i \in \mathbb{Z}$, the edge $\{u_i, u_{i+1}\}$ is also an edge of $\mathcal{T}(Q)$.

Recall the skeleton decomposition from Section 3.2: Q is encoded by a forest $(\tau_i)_{0 \leq i < p}$, whose vertices are identified with the edges of $\partial_r q$ for $0 \leq r \leq 2R$, and a collection of truncated quadrangulations. We extend the numbering of the forest to \mathbb{Z} by periodicity, and shift the indices so that for all $i \in \mathbb{Z}$, the left-most vertex of the root of τ_i is u_i .

We say that the vertex u_i is good if the slot associated with the edge (u_i, u_{i+1}) is filled in by the truncated quadrangulation represented in the right side of Figure 3.12 (in particular the edge (u_i, u_{i+1}) must have exactly one child in the skeleton), and bad otherwise. Assume that at least one of the u_j ’s is labeled good, and let u_i , with $-p \leq i \leq -1$, be the first good vertex visited when exploring the top boundary in counterclockwise order starting from u_{-1} . We define the downward path $DP(u_0, 2R - 2)$ from u_0 to $\partial_{2R-2}Q$ as follows. We first move along $\partial_{2R}Q$ in counterclockwise order from u_0 to u_i . Then, we follow the unique edge of $\mathcal{T}(Q)$ from u_i to $\partial_{2R-2}q$ inside the slot associated with (u_i, u_{i+1}) . See Figure 3.12 for an illustration.

This path can be extended by induction to a path in $\mathcal{T}(Q)$ from u_0 to $\partial_{2s}Q$ for every $0 \leq s < 2R$, provided we can find good vertices at height $2k$ for all $s < k \leq R$. If not, the downward path is not defined, but we still define its length to be $+\infty$.

We can extend this definition to downward paths in $\mathcal{T}(\mathcal{L})$ where \mathcal{L} is the LHPQ with truncated boundary. There will a.s. be good vertices at all heights, thus downward paths are always well defined.

The following lemma provides an upper bound on the length of downward paths in annuli of the UIPQ (recall that these annuli are quadrangulations of the cylinder). Roughly speaking, this upper bound shows that the graph distance (in $\mathcal{T}(Q_\infty)$) between a vertex of $\partial_{2s}Q_\infty$ and the cycle $\partial_{2r}Q_\infty$ is not larger than a constant times $s - r$, with high probability

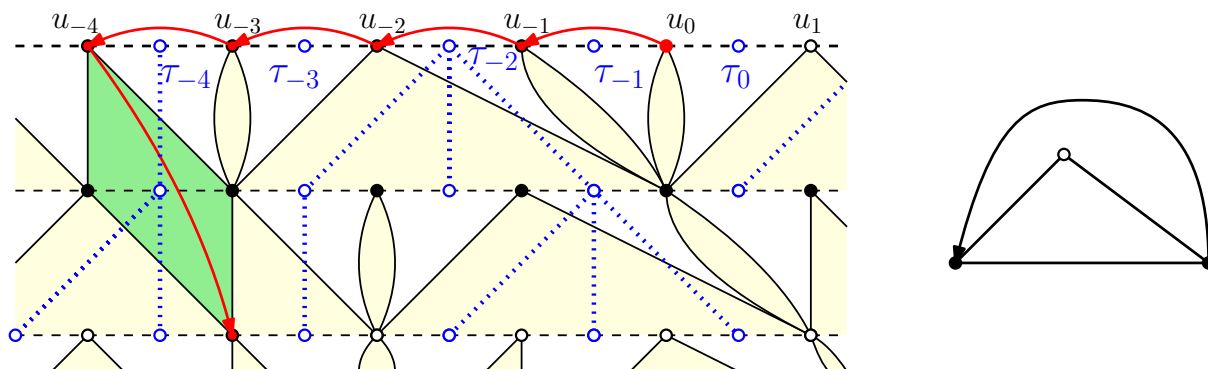


Figure 3.12 – Left, a part of the annulus $\mathcal{C}(2R-2, 2R)$ with slots in pale yellow, the skeleton in dotted blue lines, and the special slot in green. We have not drawn the edges and vertices inside the other slots. The downward path (in red) visits $u_0, u_{-1}, u_{-2}, \dots$ until it meets a “good” vertex (here u_{-4}), whose corresponding slot is filled by the truncated quadrangulation on the right side.

uniformly in $r < s < R$.

Lemma 3.25. *We can find $\alpha > 0$ such that the following holds. For every $\varepsilon, \delta > 0$ and every integer $R > 0$, let $A_R(\delta, \varepsilon)$ be the event where for all $\delta \frac{R}{\ln R} \leq r < s \leq R$ such that $s - r \geq \varepsilon r$, for every $v \in \partial_{2s}Q_\infty$, the downward path from v to $\partial_{2r}Q_\infty$ is well defined and has length smaller than $\alpha(s - r)$. Then*

$$\mathbb{P}(A_R(\delta, \varepsilon)) \xrightarrow{R \rightarrow \infty} 1.$$

Proof. We fix an integer $R > 0$. Let us first consider a forest $(\tilde{\tau}_1, \tilde{\tau}_2, \dots, \tilde{\tau}_p)$ of p independent Bienaymé-Galton-Watson trees with offspring distribution θ , where $R < p \leq R^3$. We truncate this forest at generation $2R$ (we remove all vertices whose height is greater than $2R$). We can view this forest as the skeleton of a quadrangulation \tilde{Q} of the cylinder whose height is the maximal generation in the truncated forest (in the skeleton decomposition of \tilde{Q} , slots are filled in independently according to the same distribution as in the UIPQ). We again say that a vertex u of the skeleton is good if it has a unique child and the slot corresponding to u is filled as shown in the right side of Figure 3.12. We let $a > 0$ be the product of $\theta(1)$ with the probability that a slot with lower boundary of size 1 is filled in as just explained. Informally, a represents the probability that a vertex is good.

Let T stand for the smallest $i \geq 0$ such that generation $2i$ has no good vertex. For every $i \geq 0$, let Y_i be the number of vertices at generation $2i$, and

$$\zeta = \min\{i : Y_i \leq R \text{ or } Y_i > R^3\}.$$

Note that, on the event $\{\zeta > R\}$, the height of the truncated forest is $2R$ and \tilde{Q} is a quadrangulation of the cylinder of height $2R$. For every $i \geq 0$, let \mathcal{F}_i be the σ -field generated by the trees truncated at generation $2i$ and the labels good or bad up to generation $2i - 2$ (the labels at generation $2i$ are not \mathcal{F}_i -measurable).

Fix a vertex u at generation 0 in the forest. For every $1 \leq i \leq T$, we can construct the downward path from u , or rather from the vertex v of \tilde{Q} which is the left end of the edge associated with u , to generation $2i$ of the forest (more precisely to the cycle whose edges form generation $2i$ of the forest), and we define $L_u(i)$ as the length of this downward path. By convention, $L_u(0) = 0$.

Let us observe the following key fact: we can construct a sequence G_0, G_1, \dots of i.i.d. geometric random variables with parameter a such that, for every $0 \leq i \leq R-1$,

$$\mathbf{1}_{\{T \geq i\}} \mathbb{P}(L_u(i+1) - L_u(i) \neq G_i + 2 \mid \mathcal{F}_i) \leq \mathbf{1}_{\{T \geq i\}} a^{Y_i} \quad (3.30)$$

(note that $\{T \geq i\} \in \mathcal{F}_i$). This bound holds because \mathcal{F}_i gives no information on whether vertices at generation $2i$ are good or not: if these vertices are enumerated in clockwise order starting from a random vertex measurable w.r.t. \mathcal{F}_i the index of the first good one will follow a geometric distribution “truncated” at Y_i .

Let us now bound, for $1 \leq k \leq R$,

$$\begin{aligned} & \mathbb{P}\left(T \geq k, \zeta \geq k, L_u(k) \neq \sum_{i=0}^{k-1} (G_i + 2)\right) \\ & \leq \mathbb{P}\left(T \geq k, \zeta \geq k, L_u(k-1) \neq \sum_{i=0}^{k-2} (G_i + 2)\right) + \mathbb{P}(T \geq k, \zeta \geq k, L_u(k) - L_u(k-1) \neq G_{k-1} + 2) \\ & \leq \mathbb{P}\left(T \geq k-1, \zeta \geq k-1, L_u(k-1) \neq \sum_{i=0}^{k-2} (G_i + 2)\right) + \mathbb{E}\left[\mathbf{1}_{\{\zeta \geq k\}} a^{Y_{k-1}}\right], \end{aligned}$$

using (3.30) and the property that $\{\zeta \geq k\}$ is \mathcal{F}_{k-1} -measurable. By induction, we get

$$\mathbb{P}\left(T \geq k, \zeta \geq k, L_u(k) \neq \sum_{i=0}^{k-1} (G_i + 2)\right) \leq \sum_{i=0}^{k-1} \mathbb{E}\left[\mathbf{1}_{\{\zeta > i\}} a^{Y_i}\right] \leq ka^R. \quad (3.31)$$

We can similarly bound

$$\begin{aligned} \mathbb{P}(\zeta \geq k, T < k) &= \sum_{i=0}^{k-1} \mathbb{P}(T = i, \zeta \geq k) \\ &\leq \sum_{i=0}^{k-1} \mathbb{P}(T = i, \zeta > i) \\ &= \sum_{i=0}^{k-1} \mathbb{E}\left[\mathbf{1}_{\{T > i-1\}} \mathbf{1}_{\{\zeta > i\}} a^{Y_i}\right] \end{aligned}$$

where the last equality is obtained by conditioning with respect to \mathcal{F}_i (on the event $\{T > i-1\}$, we have $\mathbb{P}(T = i \mid \mathcal{F}_i) = a^{Y_i}$). It follows that

$$\mathbb{P}(\zeta \geq k, T < k) \leq ka^R. \quad (3.32)$$

On the other hand, by elementary large deviations estimates, there exist $\alpha, A > 0$ such that for every $k \geq 0$,

$$\mathbb{P}\left(\sum_{i=0}^{k-1} (G_i + 2) > \alpha k\right) \leq e^{-Ak}. \quad (3.33)$$

By combining (3.31) and (3.33), we arrive at

$$\mathbb{P}(T \geq k, \zeta \geq k, L_u(k) > \alpha k) \leq ka^R + e^{-Ak}.$$

We apply this to $k = R - r$ for all values of r such that $r \geq \delta \frac{R}{\ln R}$ and $R - r > \varepsilon r$, to get

$$\begin{aligned} \mathbb{P}\left(T \geq R - \left\lfloor \delta \frac{R}{\ln R} \right\rfloor, \zeta \geq R - \left\lfloor \delta \frac{R}{\ln R} \right\rfloor, L_u(R - r) > \alpha(R - r) \text{ for some } r \text{ s.t. } \delta \frac{R}{\ln R} \leq r < \frac{R}{1 + \varepsilon}\right) \\ \leq R^2 a^R + R e^{-A\delta \frac{R}{\ln R}}. \end{aligned}$$

We can then consider the union over all vertices u at generation 0 of the events appearing in the last display. Since generation 0 has at most R^3 vertices, the probability of this union is trivially bounded by $R^3(R^2 a^R + R e^{-A\delta R / \ln R})$.

Fix an integer s such that $\frac{R}{\ln R} \leq r < s \leq R$. For every vertex u at generation $2(R - s)$ in the forest and every $R - s \leq k \leq R$, we use the same notation $L_u(k)$ for the length of the downward path in \tilde{Q} from u (or rather from the vertex v of \tilde{Q} which is the left end of the edge associated with u) to $\partial_{2R-2k}\tilde{Q}$ (corresponding to generation $2k$ in the forest coding \tilde{Q}), provided this downward path exists. The same argument we used in the case $s = R$ shows that the probability of the event where $T \wedge \zeta \geq R - \left\lfloor \delta \frac{R}{\ln R} \right\rfloor$ and there exists a vertex u at generation $2(R - s)$ such that $L_u(s - r) > \alpha(s - r)$ for some r such that $\delta \frac{R}{\ln R} \leq r < s$ and $s - r \geq \varepsilon r$ is bounded above by

$$R^3(R^2 a^R + R e^{-A\delta \varepsilon R / \ln R}).$$

We then sum this estimate over possible values of s . To simplify notation, set

$$\mathcal{R}(R, \delta, \varepsilon) = \left\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \delta \frac{R}{\ln R} \leq r < s \leq R, s - r \geq \varepsilon r \right\},$$

and also write \mathcal{D} for the event where $L_u(R - r) < \alpha(s - r)$ for every $(r, s) \in \mathcal{R}(R, \delta, \varepsilon)$ and every vertex u at generation $2(R - s)$ in the forest. Then using also (3.32) we get

$$\mathbb{P}\left(\left\{\zeta \geq R - \delta \frac{R}{\ln R}\right\} \cap \mathcal{D}^c\right) \leq Ra^R + R^6 a^R + R^5 e^{-A\varepsilon \delta R / \ln R}. \quad (3.34)$$

Consider now the UIPQ Q_∞ and fix $\eta > 0$. Using the tail estimates in Proposition 3.4, we can easily verify that for R large enough, the event

$$\mathcal{E} \stackrel{\text{def}}{=} \bigcap_{\left\lfloor \delta \frac{R}{\ln R} \right\rfloor \leq r \leq R} \left\{ R < H_{2r} \leq R^3 \right\}.$$

has probability at least $1 - \eta/2$. Let $A_R(\delta, \varepsilon)$ be the event considered in the statement of the lemma: $A_R(\delta, \varepsilon)$ is the event where, for every $(r, s) \in \mathcal{R}(R, \delta, \varepsilon)$, for every vertex v of $\partial_{2s}Q_\infty$ the downward path from v to $\partial_{2r}Q_\infty$ (exists and) has length smaller than $\alpha(s - r)$. We observe that the event

$$\mathcal{E} \cap A_R(\delta, \varepsilon)^c$$

is a function of the skeleton of the annulus $\mathcal{C}(2\lceil\delta\frac{R}{\ln R}\rceil, 2R)$ (and the quadrangulations filling in the slots). The point is now that the event in the left-hand side of (3.34) is the same function of the forest $(\tilde{\tau}_1, \tilde{\tau}_2, \dots, \tilde{\tau}_p)$ of independent trees truncated at height $2R - 2\lceil\delta\frac{R}{\ln R}\rceil$ (and of the quadrangulations used to construct \tilde{Q} from its skeleton). This means that we can use the relations between the law of the skeleton of the annulus and that of a forest of independent trees to compare $\mathbb{P}(\mathcal{E} \cap A_R(\delta, \varepsilon)^c)$ with the probability in (3.34). More precisely, Proposition 3.6 gives for every $R < p' \leq R^3$,

$$\begin{aligned} & \mathbb{P}\left(\mathcal{E} \cap A_R(\delta, \varepsilon)^c \cap \left\{H_{2\lceil\delta\frac{R}{\ln R}\rceil} = p'\right\} \mid H_{2R} = p\right) \\ &= \frac{\varphi_{2\lceil\delta\frac{R}{\ln R}\rceil}(p')}{\varphi_{2R}(p)} \mathbb{P}\left(\left\{\zeta > R - \delta\frac{R}{\ln R}\right\} \cap \mathcal{D}^c \cap \left\{Y_{R-\lceil\delta\frac{R}{\ln R}\rceil} = p'\right\}\right), \end{aligned} \quad (3.35)$$

where we recall that Y_i is the number of vertices at generation $2i$ in the forest $(\tilde{\tau}_1, \tilde{\tau}_2, \dots, \tilde{\tau}_p)$. Using the explicit formula (3.8), we find a constant $C > 0$ such that for every sufficiently large R and $p' \leq R^3$,

$$\varphi_{2\lceil\delta\frac{R}{\ln R}\rceil}(p') \leq CR^3 \left(\delta\frac{R}{\ln R}\right)^{-3} \leq C\delta^{-3}(\ln R)^3.$$

On the other hand, (3.8) and Proposition 3.4 give for $p' > 0$

$$\frac{\mathbb{P}(H_{2R} = p)}{\varphi_{2R}(p)} = \frac{32}{3\kappa_1} \frac{3 + 2R}{((3 + 2R)^2 - 1)^2} \kappa_p (2\pi_{2R})^p \left(\frac{64}{3} p \frac{3 + 2R}{((3 + 2R)^2 - 1)^2} \pi_{2R}^{p-1}\right)^{-1} = \frac{2^p \kappa_p}{p(2\kappa_1)} \pi_{2R} \leq C'$$

for a suitable constant C' independent of $p > 0$. Multiplying (3.35) by $\mathbb{P}(H_{2R} = p)$ and summing over all choices of $R < p, p' \leq R^3$ (using (3.34)), we get

$$\mathbb{P}\left(\mathcal{E} \cap A_R(\delta, \varepsilon)^c\right) \leq \left(C\delta^{-3}(\ln R)^3\right) \left(C'R^3\right) \left(Ra^R + R^6 a^R + R^5 e^{-A\varepsilon\delta R/\ln R}\right),$$

which is smaller than $\eta/2$ for R large. We already saw that the probability of \mathcal{E} is larger than $1 - \eta/2$ for R large enough, so we get that for all $\eta > 0$, for R large enough $\mathbb{P}(A_R(\delta, \varepsilon)) \geq 1 - \eta$. This completes the proof. \square

3.6.2 Coalescence in the UIPM

Downward paths do not coalesce as nicely as left-most geodesics, but we can still get an ersatz of Proposition 3.15. Let $R > 0$ an integer. Choose a vertex $u_0^{(R)}$ uniformly distributed over $\partial_{2R}Q_\infty$, write $(u_j^{(R)})_{0 \leq j \leq H_{2R}-1}$ for the vertices of $\partial_{2R}Q_\infty$ enumerated in clockwise order, and extend the definition of $u_j^{(R)}$ to all $j \in \mathbb{Z}$ by periodicity.

Corollary 3.26 (Coalescence of downward paths). *For every $\varepsilon > 0$ and $\eta \in (0, 1)$, we can find a constant $A > 1$ such that, for every large enough integer R , the following holds with probability at least $1 - \varepsilon$: any vertex of $\partial_{2R}Q_\infty$ is connected to one of the vertices $u_{\lfloor kR^2/A \rfloor}^{(R)}$, $0 \leq k \leq \lfloor AH_{2R}/R^2 \rfloor$, by a path in $\mathcal{T}(\mathcal{C}(2R - 2\lceil\eta R\rceil, 2R))$ of length at most ηR .*

Proof. Consider two integers $0 < s < R$. In the same way as we defined the downward path from a vertex v of $\partial_{2R}Q_\infty$ to $\partial_{2s}Q_\infty$, we can define the dual notion of the right downward path from v to $\partial_{2s}Q_\infty$: If $v = u_j^{(R)}$, the first step goes from v to $u_{j+1}^{(R)}$, then we move clockwise along $\partial_{2R}Q_\infty$ until we meet a good vertex which allows us to go in one step from $\partial_{2R}Q_\infty$ to $\partial_{2(R-2)}Q_\infty$, and we continue by induction. As in the case of downward paths, the existence of the right downward path requires the existence of at least one good vertex on every cycle $\partial_{2j}Q_\infty$, $s < j \leq R$.

Let v, w be two distinct vertices of $\partial_{2R}Q_\infty$ and let $0 < s < R$. Assume that the downward paths (and right downward paths) from v and w to $\partial_{2s}Q_\infty$ are well defined, and furthermore that the left-most geodesics from v and from w coalesce (strictly) before reaching $\partial_{2s}Q_\infty$. Write \mathcal{L} for the union of these two left-most geodesics up to their coalescence time, and \mathcal{C}_1 , resp. \mathcal{C}_2 , for the path from v to w along $\partial_{2R}Q_\infty$ in clockwise order, resp. in counterclockwise order. Let \mathcal{R}_1 , resp. \mathcal{R}_2 , be the (closed) bounded region delimited by the closed path which is the union of \mathcal{L} and \mathcal{C}_1 , resp. the union of \mathcal{L} and \mathcal{C}_2 . Then either \mathcal{R}_1 or \mathcal{R}_2 does not intersect $\partial_{2s}Q_\infty$. Consider the first case for definiteness, so that \mathcal{R}_1 is contained in the annulus between $\partial_{2R}Q_\infty$ and $\partial_{2s}Q_\infty$. Then our construction shows that the downward path starting from w can only exit \mathcal{R}_1 after hitting the left-most geodesic started from v (informally a downward path cannot cross a left-most geodesic “from left to right” — for this property to hold, it is important that we started our downward paths with a “horizontal” step). Similarly, the right downward path started from v can only exit \mathcal{R}_1 after hitting the left-most geodesic started from w . A planarity argument now shows that the downward path started from w and the right downward path started from v intersect before exiting \mathcal{R}_1 , and therefore before hitting $\partial_{2s}Q_\infty$. Consequently, v and w are connected by a path in $\mathcal{T}(\mathcal{C}(2R - 2\lceil\eta R\rceil, 2R))$ whose length is bounded by the sum of the lengths of the downward path from w to $\partial_{2s}Q_\infty$ and the right downward path from v to $\partial_{2s}Q_\infty$.

Let α be as given in Lemma 3.25. Without loss of generality we can assume $\alpha \geq 1$. We apply the preceding considerations with $s = s(R) = R - \lceil\eta R/(3\alpha)\rceil$. Using also Lemma 3.25, we get that, if R is large enough, the following holds with probability at least $1 - \varepsilon/2$: Whenever v and w are two vertices of $\partial_{2R}Q_\infty$ such that the left-most geodesics from v and from w coalesce before reaching $\partial_{2s}Q_\infty$, v and w are connected by a path of $\mathcal{T}(\mathcal{C}(2s, 2R))$ of length at most $2\alpha(R - s) \leq \eta R$.

On the other hand, Proposition 3.15 yields $A > 0$ such that with probability at least $1 - \varepsilon/2$, any left-most geodesics starting from a vertex of $\partial_{2R}Q_\infty$ coalesces before reaching $\partial_{2s(R)}Q$ with one of the left-most geodesics started from $u_{\lfloor kR^2/A \rfloor}$, $0 \leq k \leq \lfloor AH_{2R}/(2R)^2 \rfloor$. By combining this with the preceding paragraph, we get that, with probability at least $1 - \varepsilon$, any vertex of $\partial_{2R}Q_\infty$ is connected to one of these vertices $u_{\lfloor kR^2/A \rfloor}$ by a path of $\mathcal{T}(\mathcal{C}(2s, 2R))$ of length at most ηR . This completes the proof. \square

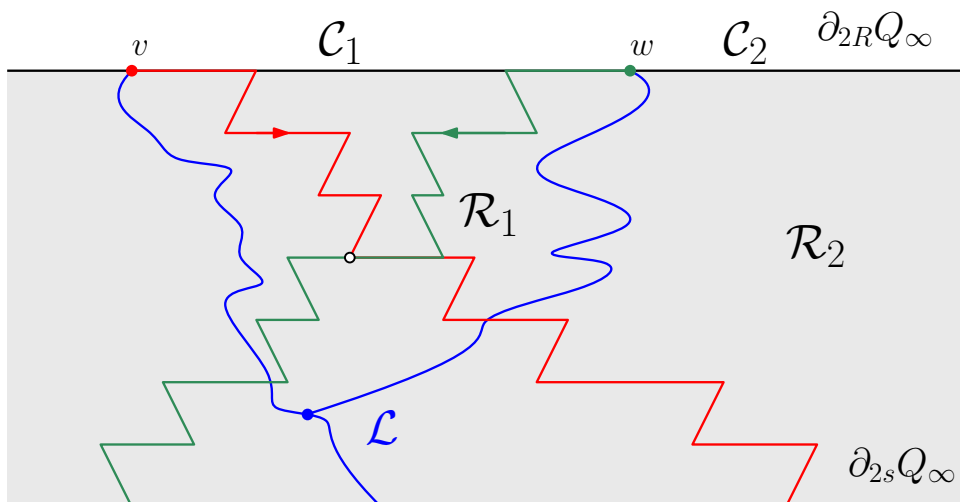


Figure 3.13 – In blue, the left-most geodesics (in Q_∞) started from v, w two vertices of $\partial_{2R}Q_\infty$, that coalesce before reaching $\partial_{2s}Q_\infty$. The downward path in $\mathcal{T}(Q_\infty)$ started at w (in green) cannot cross the left-most geodesic in Q_∞ from w , and the right downward path in $\mathcal{T}(Q_\infty)$ started at v (in red) cannot cross the left-most geodesic in Q_∞ from v : by planarity, they must cross (and thus intersect) before reaching $\partial_{2s}Q_\infty$. We get a path in $\mathcal{T}(Q_\infty)$ between v and w by first following the right downward path from v up to the intersection (white point), then the (reverse) downward path from w .

3.6.3 Two technical lemmas

We prove an estimate on the maximum degree of an inner face in the image of a large truncated hull of Q_∞ by Tutte’s correspondence. We then obtain a bound on the first-passage percolation distance in $\mathcal{T}(Q_\infty)$ between a vertex of $\partial_{2n}Q_\infty$ and the root vertex, for n large enough.

Lemma 3.27. *For every integer $r \geq 1$, let $\Delta^\circ(\mathcal{T}(\mathcal{H}_{Q_\infty}^{tr}(2r)))$ denote the maximal degree of internal faces of $\mathcal{T}(\mathcal{H}_{Q_\infty}^{tr}(2r))$.*

$$\mathbb{P}(\Delta^\circ(\mathcal{T}(\mathcal{H}_{Q_\infty}^{tr}(2r))) > 5 \ln r) \xrightarrow{r \rightarrow \infty} 0.$$

Proof. For any map M , let $\Delta(M)$ be the maximal degree of a face of M . Let M_n be a uniform rooted map with n edges. It follows from [29, Theorem 3] that

$$\mathbb{P}(\Delta(M_n) > \ln n) \xrightarrow{n \rightarrow \infty} 0. \tag{3.36}$$

This result is actually stated for the maximal degree of a vertex of M_n in [29], but (3.36) then follows by self-duality of M_n , see [9, Lemma 3.2].

By Proposition 3.20 we can fix an integer $A > 0$ large enough such that, for every r large enough, we can couple $Q_{\lfloor Ar^4 \rfloor}^\bullet$ and Q_∞ so that $B_{Q_{\lfloor Ar^4 \rfloor}^\bullet}^\bullet(2r) = B_{Q_\infty}^\bullet(2r)$ except on a

set of probability at most $\varepsilon/2$. Note that this equality of hulls also implies $\mathcal{H}_{Q_{\lfloor Ar^4 \rfloor}}^{\text{tr}}(2r) = \mathcal{H}_{Q_\infty}^{\text{tr}}(2r)$ (the truncated hull is determined by the “standard” hull). Thus on the latter event,

$$\Delta^\circ \left(\mathcal{T} \left(\mathcal{H}_{Q_\infty}^{\text{tr}}(2r) \right) \right) = \Delta^\circ \left(\mathcal{T} \left(\mathcal{H}_{Q_{\lfloor Ar^4 \rfloor}}^{\text{tr}}(2r) \right) \right) \leq \Delta \left(\mathcal{T} \left(Q_{\lfloor Ar^4 \rfloor} \right) \right).$$

To get the last inequality, we observe that the degree of an internal face of $\mathcal{T}(\mathcal{H}_{Q_{\lfloor Ar^4 \rfloor}}^{\text{tr}}(2r))$ is exactly the degree of the black vertex of $\mathcal{H}_{Q_{\lfloor Ar^4 \rfloor}}^{\text{tr}}(2r)$ that is contained in this face, and this vertex has the same degree in $\mathcal{H}_{Q_{\lfloor Ar^4 \rfloor}}^{\text{tr}}(2r)$ and in $Q_{\lfloor Ar^4 \rfloor}$. See Figure 3.11.

We know that $\mathcal{T}(Q_{\lfloor Ar^4 \rfloor})$ is distributed as $M_{\lfloor Ar^4 \rfloor}$. To complete the proof, we note that (3.36) ensures that for r large enough, $\Delta(M_{\lfloor Ar^4 \rfloor}) \leq 5 \ln r$ with probability at least $1 - \varepsilon/2$. \square

Lemma 3.28. *For every $\varepsilon > 0$, there exists $A > 0$ such that for all n large enough,*

$$\mathbb{P} \left(\max_{v \in \partial_{2n} Q_\infty} d_{\text{fpp}}^{\mathcal{T}(Q_\infty)}(\rho, v) \leq 2An \right) \geq 1 - \varepsilon.$$

Proof. Since weights are bounded, it is enough to prove this statement with $d_{\text{fpp}}^{\mathcal{T}(Q_\infty)}$ replaced by $d_{\text{gr}}^{\mathcal{T}(Q_\infty)}$. Let $\eta \in (0, 1)$. To simplify notation, in the remaining part of the proof, we write $H_n = \mathcal{H}_{Q_\infty}^{\text{tr}}(2\lfloor \eta \frac{n}{\ln n} \rfloor)$ so that $\partial H_n = \partial_{2\lfloor \eta \frac{n}{\ln n} \rfloor} Q_\infty$.

By Lemma 3.25, if n is large enough, we have with probability at least $1 - \varepsilon/2$

$$d_{\text{gr}}^{\mathcal{T}(Q_\infty)}(v, \partial H_n) \leq \alpha n,$$

for every $v \in \partial_{2n} Q_\infty$. On the other hand, by Lemma 3.27 the bound $\Delta^\circ(\mathcal{T}(H_n)) \leq 5 \ln n$ holds with probability at least $1 - \varepsilon/2$ when n is large. On this event, the simple bound

$$d_{\text{gr}}^{\mathcal{T}(H_n)}(x, y) \leq \Delta^\circ(\mathcal{T}(H_n)) d_{\text{gr}}^{H_n}(x, y),$$

valid for all $x, y \in V(\mathcal{T}(H_n))$, ensures that for all $v \in \partial H_n$, $d_{\text{gr}}^{\mathcal{T}(Q_\infty)}(\rho, v) \leq 10\eta n$. The statement of the lemma follows by combining these observations. \square

3.6.4 Continuity properties of the Tutte correspondence

Let us use the notation $M_n = \mathcal{T}(Q_n)$ for the image of Q_n under Tutte's bijection. Note that M_n is uniformly distributed over (rooted) planar maps with n edges. For every $r > 0$, we write $\mathcal{B}_{M_n}(r)$ for the (closed) metric ball of radius r centered at the root vertex ρ_n in $V(M_n)$. We may view this ball (resp. its complement) as a graph by keeping only those edges incident to a face of the ball (resp. of its complement).

Proposition 3.29. *For every $\varepsilon, \eta > 0$, there exists $\delta > 0$ s.t. for all n large enough,*

$$\mathbb{P} \left(\sup_{x \in V(M_n), d_{\text{gr}}^{Q_n}(\rho_n, x) \leq \delta n^{1/4}} d_{\text{gr}}^{M_n}(\rho_n, x) \leq \varepsilon n^{1/4} \right) \geq 1 - \eta.$$

Proof. We again use the convergence in distribution to the Brownian map (m_∞, D^*) . It follows from (3.28) that we can choose a constant $K > 2$ large enough so that, for every $\varepsilon > 0$, the probability of the event where at least two connected components of the complement of the ball $\mathcal{B}_{m_\infty}(\varepsilon)$ intersect the complement of $\mathcal{B}_{m_\infty}(K\varepsilon/2)$ is bounded above by $\eta/4$. From [9, Corollary 1.2], we know that the random compact metric spaces $(V(M_n), (\frac{9}{8n})^{1/4}d_{\text{gr}}^{M_n})$ converge in distribution in the Gromov-Hausdorff sense to the Brownian map. Although this is not stated in [9], it follows from the proof that this convergence also holds in the pointed Gromov-Hausdorff sense, if M_n is pointed at the root vertex ρ_n (and m_∞ is pointed at x_*). From this pointed convergence, and the properties of the Brownian map stated above, we can now deduce that, for every $\varepsilon > 0$, for all sufficiently large n , the probability that at least two components of the complement of the ball $\mathcal{B}_{M_n}(\varepsilon n^{1/4})$ intersect the complement of the ball $\mathcal{B}_{M_n}(K\varepsilon n^{1/4})$ is bounded above by $\eta/2$.

Let us fix $\varepsilon > 0$ and set $\beta = \varepsilon/K$. We can assume that ε is so small that $\mathbb{P}(d_{\text{gr}}^{Q_n}(\rho_n, \partial_n) > 4K\beta n^{1/4}) > 1 - \eta/4$. Using the coupling between Q_n^\bullet and the UIPQ Q_∞ in Proposition 3.20, we get from Lemma 3.28 that there exists $\mu > 0$ such that

$$\max_{v \in \partial_2 \lfloor \mu n^{1/4} \rfloor Q_n^\bullet} d_{\text{gr}}^{M_n}(\rho_n, v) < \frac{\beta}{2} n^{1/4}$$

with probability at least $1 - \eta/4$. Recall that the edges of the cycle $\partial_2 \lfloor \mu n^{1/4} \rfloor Q_n^\bullet$ are also edges of M_n . Argue on the event where both the bound in the last display holds and $d_{\text{gr}}^{Q_n}(\rho_n, \partial_n) > 4K\beta n^{1/4}$. Then, except on an event of probability at most $\eta/2$, at most one of the two components bounded by the cycle $\partial_2 \lfloor \mu n^{1/4} \rfloor Q_n^\bullet$ can intersect the complement of the ball $\mathcal{B}_{M_n}(K\beta n^{1/4})$, and this must be the component that contains the distinguished vertex ∂_n . We conclude that, except on a set of probability at most η , the truncated hull $\mathcal{H}_{Q_n^\bullet}^{\text{tr}}(2 \lfloor \mu n^{1/4} \rfloor)$ does not intersect the complement of the ball $\mathcal{B}_{M_n}(K\beta n^{1/4})$, and in particular the bound

$$\sup_{x \in V(M_n), d_{\text{gr}}^{Q_n}(\rho_n, x) \leq 2 \lfloor \mu n^{1/4} \rfloor} d_{\text{gr}}^{M_n}(\rho_n, x) \leq K\beta n^{1/4} = \varepsilon n^{1/4}$$

holds with probability at least $1 - \eta$. This completes the proof. \square

3.7 Main results for general maps

3.7.1 Subadditivity in the LHPQ

Recall from the beginning of Section 3.6 that we can apply Tutte's bijection to the LHPQ \mathcal{L} , and that $\mathcal{T}(\mathcal{L})$ denotes the resulting infinite map. We observe that every edge of the form $((i, -2r), (i+1, -2r))$ for $r \geq 0$ and $i \in \mathbb{Z}$ appear in $\mathcal{T}(\mathcal{L})$ because vertices of the type $(i, -2r)$ are white, and every edge of the preceding form is a diagonal of some quadrangle. It follows that we can define slices of $\mathcal{T}(\mathcal{L})$ for even coordinates in a way

similar to the case of the LHPQ: for even $j \leq j' \leq 0$, $\mathcal{T}(\mathcal{L})_j^{j'}$ is the part of $\mathcal{T}(\mathcal{L})$ contained in $\mathbb{R} \times [j, j']$. Furthermore, disjoint slices are independent.

We write $d_{\text{fpp}}^{\mathcal{T}(\mathcal{L})}(x, y)$, for $x, y \in V(\mathcal{T}(\mathcal{L}))$, for the first-passage percolation distance on $V(\mathcal{T}(\mathcal{L}))$ (recall that weights belong to $[1, \kappa]$).

Proposition 3.30. *There exists a constant $c_T \in [\frac{1}{2}, \infty)$ such that*

$$(2r)^{-1} d_{\text{fpp}}^{\mathcal{T}(\mathcal{L})}(\rho, \partial_{-2r}\mathcal{L}) \xrightarrow[r \rightarrow \infty]{a.s.} c_T. \quad (3.37)$$

Proof. The proof uses the same subadditivity argument as that of Proposition 3.13, with the minor difference that we restrict our attention to even heights.

We first note that, for $x, y \in V(\mathcal{T}(\mathcal{L}))$, $d_{\text{fpp}}^{\mathcal{T}(\mathcal{L})}(x, y) \geq d_{\text{gr}}^{\mathcal{T}(\mathcal{L})}(x, y) \geq \frac{1}{2} d_{\text{gr}}^{\mathcal{L}}(x, y)$, so that the limit c_T in (3.37), if it exists, has to be greater than or equal to $1/2$. The only new part is that we have to check that $\mathbb{E}[d_{\text{gr}}^{\mathcal{T}(\mathcal{L})}(\rho, \partial_{-2}\mathcal{L})] < \infty$.

As we already noticed in Section 3.6.1, the downward path started from the root is well defined in the LHPQ, and provides an upper bound on $d_{\text{gr}}^{\mathcal{T}(\mathcal{L})}(\rho, \partial_{-2}\mathcal{L})$. The number of steps of this downward path before it reaches a vertex of $\partial_{-2}\mathcal{L}$ is distributed as $G + 2$, where G is a geometric random variable, thus has a finite expectation, giving the desired result. \square

3.7.2 Distance through a thin annulus

We use the notation $M_\infty = \mathcal{T}(Q_\infty)$, so that M_∞ may be called the uniform infinite planar map or UIPM.

Proposition 3.31. *Let $\varepsilon \in (0, 1)$ and $\delta > 0$. For every $\eta > 0$ small enough, for all sufficiently large n , the property*

$$2(1 - \varepsilon) c_T \eta n \leq d_{\text{fpp}}^{M_\infty}(v, \partial_{2(n - \lfloor \eta n \rfloor)} Q_\infty) \leq 2(1 + \varepsilon) c_T \eta n, \quad (3.38)$$

holds for every $v \in \partial_{2n} Q_\infty$, with probability at least $1 - \delta$.

Proof. The proof is patterned after that of Proposition 3.16 using Proposition 3.30 instead of Proposition 3.13. The following minor adaptations are required.

Let $u_0^{(n)}$ be a uniformly distributed vertex of $\partial_{2n} Q_\infty$. Then Lemma 3.25 ensures that we have with high probability,

$$d_{\text{fpp}}^{M_\infty}(u_0^{(n)}, \partial_{2(n - \lfloor \eta n \rfloor)} Q_\infty) \leq 2\alpha\kappa \lfloor \eta n \rfloor.$$

Let $\mathcal{G}_0^{(n)}$ be defined as in the proof of Proposition 3.16 (with n replaced by $2n$). As in the latter proof, we know with high probability that the length of the minimal path (in Q_∞) between $u_0^{(n)}$ and the lateral boundary of $\mathcal{G}_0^{(n)}$ that stays in $\widetilde{\mathcal{H}}_{Q_\infty}^{\text{tr}}(2n)$ is bounded below by cn with some constant $c > 0$. Trivially, two vertices of $\mathcal{H}_{Q_\infty}^{\text{tr}}(2n)$ that are linked by

an edge of $\mathcal{T}(Q_\infty)$ are also connected by a Q_∞ -path of length two in $\widetilde{\mathcal{H}}_{Q_\infty}^{\text{tr}}(2n)$. Taking η smaller if necessary, it follows that the $d_{\text{fpp}}^{M_\infty}$ -shortest path between $u_0^{(n)}$ and $\partial_{2(n-\lfloor \eta n \rfloor)}Q_\infty$ that stays in $\mathcal{H}_{Q_\infty}^{\text{tr}}(2n)$ does not leave $\mathcal{G}_0^{(n)}$ on an event of high probability. We can then use the same density arguments as in the proof of Proposition 3.16.

In the last step of the proof, we need to verify that it suffices to obtain (3.38) for a bounded number of vertices v of $\partial_{2n}Q_\infty$. The argument is the same as in the proof of Proposition 3.16, but we use Corollary 3.26 in place of Proposition 3.15. \square

3.7.3 Distance from the boundary of a hull to its center

The next proposition is analogous to Proposition 3.17.

Proposition 3.32. *For every $\varepsilon \in (0, 1)$,*

$$\mathbb{P}\left(2(\mathbf{c}_T - \varepsilon)n \leq d_{\text{fpp}}^{M_\infty}(\rho, v) \leq 2(\mathbf{c}_T + \varepsilon)n \text{ for every } v \text{ in } \partial_{2n}Q_\infty\right) \xrightarrow{n \rightarrow \infty} 1.$$

Proof. The proof is very similar to that of Proposition 3.17. Consider the annuli $\mathcal{C}(2n_{k+1}, 2n_k)$ for every $0 \leq k < q$, where $n_0 = n$ and $n_{k+1} = n_k - \lfloor \eta n_k \rfloor$, and q is as defined in Proposition 3.17. By Proposition 3.31, we can find $\eta > 0$ such that “most” of these annuli will satisfy the analog of (3.38), except possibly on a set of probability at most ε . We then use Lemma 3.25 to bound the $d_{\text{fpp}}^{M_\infty}$ -distance through the annuli where (3.38) does not hold, and Lemma 3.28 to control the $d_{\text{fpp}}^{M_\infty}$ -distance between ρ and $\partial_{2n_q}Q_\infty$. \square

3.7.4 Distance between two uniformly sampled points in finite maps

Recall that $M_n = \mathcal{T}(Q_n)$ in such a way that $V(M_n)$ can be viewed as the subset of $V(Q_n)$ consisting of the “white” vertices. Also recall that $\#V(Q_n) = n + 2$. We observe that

$$\frac{\#V(M_n)}{\#V(Q_n)} \xrightarrow{n \rightarrow \infty} \frac{1}{2}$$

in probability (see e.g. the proof of Proposition 3.1 in [9]). Then, since conditionally on Q_n the distinguished vertex ∂_n is uniformly distributed over $V(Q_n)$, we have also

$$\mathbb{P}(\partial_n \in V(M_n)) \xrightarrow{n \rightarrow \infty} \frac{1}{2}.$$

We also observe that the result of Proposition 3.29 remains valid if we replace the root vertex ρ_n by ∂_n . More precisely, for every $\varepsilon, \eta > 0$, we can find $\beta > 0$ such that, for all n large enough, we have

$$\mathbb{P}\left(\sup_{x \in V(M_n), d_{\text{gr}}^{Q_n}(\partial_n, x) \leq \beta n^{1/4}} d_{\text{gr}}^{M_n}(\partial_n, x) \leq \varepsilon n^{1/4} \mid \partial_n \in V(M_n)\right) \geq 1 - \eta. \quad (3.39)$$

This follows from the invariance of Q_n under uniform re-rooting, by an argument very similar to the one used in the proof of Theorem 3.22.

Proposition 3.33. *For every $\gamma \in (0, 1)$,*

$$\mathbb{P} \left(\left| d_{\text{fpp}}^{M_n}(\rho_n, \partial_n) - \mathbf{c}_T d_{\text{gr}}^{Q_n}(\rho_n, \partial_n) \right| > \gamma n^{1/4} \mid \partial_n \in V(M_n) \right) \xrightarrow[n \rightarrow \infty]{} 0.$$

Proof. The proof is based on the same ingredients as that of Proposition 3.18, but we use Proposition 3.32, or rather (3.39), instead of Proposition 3.17. We argue on the event where $\partial_n \in V(M_n)$. Let $\gamma > 0$, $\eta > 0$ and choose $\beta > 0$ so that (3.39) holds with $\varepsilon = \gamma$.

Fix $\delta > 0$ small enough so that $3\delta^2 < \beta$ and the event $H_{n,\delta}$ of Lemma 3.21 has probability at least $1 - \eta$ when n is large. For integers $j, l \geq 1$, let $H_{n,\delta}^{j,l}$ be defined as in Lemma 3.21. If n is large, the $d_{\text{gr}}^{Q_n}$ -distance between ∂_n and $\partial_{2\lfloor \alpha_j n^{1/4}/2 \rfloor} Q_n^\bullet$ is smaller than $\beta n^{1/4}$ on $H_{n,\delta}^{j,l}$. Thus, on the intersection of $H_{n,\delta}^{j,l}$ with the event considered in (3.39) (with $\varepsilon = \gamma$), the $d_{\text{gr}}^{M_n}$ -distance between ∂_n and $\partial_{2\lfloor \alpha_j n^{1/4}/2 \rfloor} Q_n^\bullet$ is smaller than $\gamma n^{1/4}$.

On the other hand, from Proposition 3.32 and using (3.23) as in the proof of Proposition 3.18, we get that, outside of a set of probability going to 0 as $n \rightarrow \infty$, the $d_{\text{fpp}}^{M_n}$ -distance between any vertex of $\partial_{2\lfloor \alpha_j n^{1/4}/2 \rfloor} Q_n^\bullet$ and ρ_n is close to $\mathbf{c}_T \alpha_j n^{1/4}$, up to an error term bounded by $\gamma n^{1/4}$.

Finally, on the intersection of $H_{n,\delta}$ with $\{\partial_n \in V(M_n)\}$ and with the event considered in (3.39), we have

$$\left| d(\rho_n, \partial_n) - \mathbf{c}_T d_{\text{gr}}^{Q_n}(\rho_n, \partial_n) \right| \leq (1 + \kappa) \gamma n^{1/4},$$

except on a set of probability tending to 0 as $n \rightarrow \infty$. Using Lemma 3.21 we obtain that the latter intersection has probability larger than $\mathbb{P}(\partial_n \in V(M_n)) - 2\eta$ for all n large enough. This completes the proof. \square

3.7.5 Distances between any pair of points of finite maps

The next statement gives both Theorem 3.2 and the part of Theorem 3.1 concerning general planar maps.

Theorem 3.34. *For every $\varepsilon > 0$, we have*

$$\mathbb{P} \left(\sup_{x,y \in V(M_n)} \left| d_{\text{fpp}}^{M_n}(x, y) - \mathbf{c}_T d_{\text{gr}}^{Q_n}(x, y) \right| > \varepsilon n^{1/4} \right) \xrightarrow[n \rightarrow \infty]{} 0$$

If all weights are equal to 1 (that is, $d_{\text{fpp}}^{M_n} = d_{\text{gr}}^{M_n}$), we have $\mathbf{c}_T = 1$.

Before we prove Theorem 3.34, we state and prove a lemma.

Lemma 3.35. *Let $\eta \in (0, 1)$, and, for every $n \geq 1$, conditionally on M_n , let $\partial_n^1, \partial_n^2, \dots$ be independent random vertices uniformly distributed over $V(M_n)$. Then, for every $\varepsilon > 0$, we can find an integer $N \geq 1$ such that, for every sufficiently large n , we have*

$$\mathbb{P} \left(\max_{v \in V(M_n)} \left(\min_{1 \leq \ell \leq N} d_{\text{gr}}^{M_n}(\partial_n^\ell, v) \right) < \varepsilon n^{1/4} \right) > 1 - \eta.$$

Proof. We first note that the statement would follow if we knew the convergence in the Gromov-Hausdorff-Prokhorov sense of $(V(M_n), (9/8n)^{1/4} d_{\text{gr}}^{M_n})$ equipped with the uniform probability measure to the Brownian map — cf. the analogous statement for Q_n used in the proof of Theorem 3.22. Unfortunately, [9] does not give the Gromov-Hausdorff-Prokhorov convergence, and so we will provide a direct proof, which still relies much on the arguments of [9]. We start by observing that [9, Proposition 3.1] allows us to replace M_n by a random *pointed* planar map M_n^\bullet which is uniformly distributed over pointed planar maps with n edges (this replacement needs to be justified because, in contrast with the case of quadrangulations, forgetting the distinguished vertex of M_n^\bullet does not give a map distributed as M_n). Then, as in [9, Section 4], we can construct a finite sequence $\tilde{v}_0^n, \tilde{v}_1^n, \dots, \tilde{v}_{2n}^n$ such that every vertex v of M_n^\bullet appears at least once in this sequence, and, if we set $\tilde{D}_n(i, j) = d_{\text{gr}}^{M_n^\bullet}(\tilde{v}_i^n, \tilde{v}_j^n)$ for $i, j \in \{0, 1, \dots, 2n\}$ and interpolate linearly to get a function $\tilde{D}(s, t)$ defined on $[0, 2n]^2$, we have

$$\left(\left(\frac{9}{8n} \right)^{1/4} \tilde{D}_n(2ns, 2nt) \right)_{0 \leq s, t \leq 1} \xrightarrow{n \rightarrow \infty} (D^*(s, t))_{0 \leq s, t \leq 1},$$

in distribution in the space of continuous functions on $[0, 1]^2$. Here $D^*(s, t)$ is the random pseudo-metric on $[0, 1]^2$ that defines the Brownian map. Since D^* vanishes on the diagonal, we can fix an integer $A \geq 1$ such that, writing $\delta = 1/A$, the property

$$D^*(s, s') < \frac{\varepsilon}{4}, \quad \forall s, s' \in [(k-1)\delta, k\delta], \quad \forall k \in \{1, \dots, A\}$$

holds with probability greater than $1 - \eta/2$. Using the preceding convergence, it follows that, for n large enough, the property

$$d_{\text{gr}}^{M_n^\bullet}(\tilde{v}_i^n, \tilde{v}_j^n) < \frac{\varepsilon}{2} n^{1/4}, \quad \forall i, j \in [2n(k-1)\delta, 2nk\delta] \cap \mathbb{Z}, \quad \forall k \in \{1, \dots, A\}$$

also holds with probability greater than $1 - \eta/2$. We claim that we can find an integer N large enough so that, for every n large enough, with probability greater than $1 - \eta/2$, there exists for each $k \in \{1, \dots, A\}$ an index $\ell \in \{1, \dots, N\}$ and an integer $i \in [2n(k-1)\delta, 2nk\delta]$ such that

$$d_{\text{gr}}^{M_n^\bullet}(\partial_n^\ell, \tilde{v}_i^n) \leq \frac{\varepsilon}{2} n^{1/4}.$$

If we combine the claim with the preceding considerations, we get that, with probability at least $1 - \eta$, any vertex of M_n^\bullet is at distance smaller than $\varepsilon n^{1/4}$ from one of the vertices ∂_n^ℓ , $\ell \in \{1, \dots, N\}$, which was the desired result.

It remains to prove our claim. To this end, we need more information about the sequence \tilde{v}_i^n (we refer to [9] for more details). Via the Ambjørn-Budd bijection, the pointed

planar map M_n^\bullet is associated with a (uniformly distributed) pointed quadrangulation Q_n^\bullet with n faces, in such a way that $V(M_n^\bullet)$ is identified to a subset of $V(Q_n^\bullet)$, and in particular $\#V(M_n^\bullet) \leq \#V(Q_n^\bullet) = n+2$. In the CVS correspondence, $V(Q_n^\bullet)$ corresponds to a labeled tree T_n , and the contour sequence $v_0^n, v_1^n, \dots, v_{2n}^n$ of the tree T_n (defined as in the proof of Lemma 3.21) can also be viewed as a sequence of vertices of Q_n^\bullet . Then, for every $i \in \{1, \dots, 2n\}$, v_i^n and \tilde{v}_i^n are linked by an edge of Q_n^\bullet (see [9]). Moreover, in the case when $v_i^n \in V(M_n^\bullet)$, one has

$$d_{\text{gr}}^{M_n^\bullet}(v_i^n, \tilde{v}_i^n) \leq \Delta(M_n^\bullet). \quad (3.40)$$

This bound follows directly from the construction of the Ambjørn-Budd bijection (the point is that any edge of Q_n^\bullet is contained in a face of M_n^\bullet). Recalling (3.36), and using again [9, Proposition 3.1], we know that we have $\Delta(M_n^\bullet) < \frac{\varepsilon}{2}n^{1/4}$ with probability greater than $1 - \eta/8$. For every integer $p \in \{0, 1, \dots, n\}$, let $\mathcal{N}_p^{(n)}$ be the number of distinct vertices v_i^n with $i \in \{0, 1, \dots, p\}$ that belong to $V(M_n^\bullet)$. Then, from the end of [9, Section 5], we have for every $t \in [0, 1]$,

$$\frac{1}{n} \mathcal{N}_{\lfloor 2nt \rfloor}^{(n)} \xrightarrow{n \rightarrow \infty} \frac{t}{2}$$

in probability. It follows that, for n large enough, we have

$$\#\{v_i^n : i \in [2n(k-1)\delta, 2nk\delta] \text{ and } v_i^n \in V(M_n^\bullet)\} \geq \mathcal{N}_{\lfloor 2nk\delta \rfloor}^{(n)} - \mathcal{N}_{\lfloor 2n(k-1)\delta \rfloor}^{(n)} \geq \frac{\delta}{4}n, \quad \forall k \in \{0, 1, \dots, A\}, \quad (3.41)$$

with probability at least $1 - \eta/8$. We can choose N large enough so that, on the event (3.41), the conditional probability given M_n^\bullet that each set $\{v_i^n : i \in [2n(k-1)\delta, 2nk\delta] \text{ and } v_i^n \in M_n^\bullet\}$, for $1 \leq k \leq A$, contains at least one of the vertices ∂_n^ℓ , $1 \leq \ell \leq N$, is greater than $1 - \eta/4$. Summarizing the preceding considerations and using (3.40), we get that, with probability at least $1 - \eta/2$, for every $k \in \{1, \dots, A\}$, there exist an index $\ell \in \{1, \dots, N\}$ and an integer $i \in [2n(k-1)\delta, 2nk\delta]$ such that $\partial_n^\ell = v_i^n$ and $d_{\text{gr}}^{M_n^\bullet}(\partial_n^\ell, \tilde{v}_i^n) = d_{\text{gr}}^{M_n^\bullet}(v_i^n, \tilde{v}_i^n) < \frac{\varepsilon}{2}n^{1/4}$. This completes the proof of the claim and of the lemma. \square

Proof of Theorem 3.34. By the same re-rooting invariance argument as in the proof of Theorem 3.22, the statement of Proposition 3.33 remains valid if the pair (ρ_n, ∂_n) is replaced by $(\partial'_n, \partial''_n)$, where, conditionally on Q_n , ∂'_n and ∂''_n are independent and uniformly distributed over $V(Q_n)$: more precisely, we have, for every $\varepsilon > 0$,

$$\mathbb{P} \left(|d_{\text{fpp}}^{M_n}(\partial'_n, \partial''_n) - \mathbf{c}_T d_{\text{gr}}^{Q_n}(\partial'_n, \partial''_n)| > \varepsilon n^{1/4} \mid \partial'_n \in V(M_n), \partial''_n \in V(M_n) \right) \xrightarrow{n \rightarrow \infty} 0.$$

Let us fix $\varepsilon > 0$ and $\eta > 0$. Thanks to Lemma 3.35, we can fix an integer N large enough so that, if $\partial_n^1, \partial_n^2, \dots, \partial_n^N$ are independent and uniformly distributed over $V(M_n)$, then, with probability at least $1 - \eta$, the metric balls of radius $\varepsilon n^{1/4}$ in $(V(M_n), d_{\text{gr}}^{M_n})$ centered at $\partial_n^1, \dots, \partial_n^N$ cover $V(M_n)$. Let us call \mathcal{H}_n the event where this covering property holds.

On the other hand, consider the event

$$\mathcal{X}_n := \{|d_{\text{fpp}}^{M_n}(\partial_n^i, \partial_n^j) - \mathbf{c}_T d_{\text{gr}}^{Q_n}(\partial_n^i, \partial_n^j)| \leq \varepsilon n^{1/4}, \quad \forall i, j \in \{1, \dots, N\}\}.$$

By the first observation of the proof, we have also $\mathbb{P}(\mathcal{K}_n) \geq 1 - \eta$ for n large enough.

For n large, the event $\mathcal{H}_n \cap \mathcal{K}_n$ has probability at least $1 - 2\eta$. Let us argue on this event in the remaining part of the proof. Let $x, y \in V(M_n)$, we can find $i, j \in \{1, \dots, N\}$ such that $d_{\text{gr}}^{M_n}(\partial_n^i, x) \leq \varepsilon n^{1/4}$ and $d_{\text{gr}}^{M_n}(\partial_n^j, y) \leq \varepsilon n^{1/4}$. Note that this implies $d_{\text{gr}}^{Q_n}(\partial_n^i, x) \leq 2\varepsilon n^{1/4}$ and $d_{\text{gr}}^{Q_n}(\partial_n^j, y) \leq 2\varepsilon n^{1/4}$. It follows that we have

$$|d_{\text{fpp}}^{M_n}(x, y) - d_{\text{fpp}}^{M_n}(\partial_n^i, \partial_n^j)| \leq d_{\text{fpp}}^{M_n}(\partial_n^i, x) + d_{\text{fpp}}^{M_n}(\partial_n^j, y) \leq 2\kappa\varepsilon n^{1/4}$$

and

$$|d_{\text{gr}}^{Q_n}(x, y) - d_{\text{gr}}^{Q_n}(\partial_n^i, \partial_n^j)| \leq 4\varepsilon n^{1/4}.$$

Hence, from the definition of \mathcal{K}_n ,

$$|d_{\text{fpp}}^{M_n}(x, y) - \mathbf{c}_T d_{\text{gr}}^{Q_n}(x, y)| \leq (1 + 4\mathbf{c}_T + 2\kappa)\varepsilon n^{1/4}.$$

This completes the proof of the first assertion.

As for the second one, we observe that the first assertion, together with the known convergence of rescaled quadrangulations to the Brownian map, implies that

$$\left(V(M_n), \left(\frac{9}{8n} \right)^{1/4} d_{\text{fpp}}^{M_n} \right) \xrightarrow[n \rightarrow \infty]{(d)} (m_\infty, \mathbf{c}_T D^*)$$

in distribution in the Gromov-Hausdorff sense. In the case where all weights are equal to 1, comparing this convergence with [9, Corollary 1.2] gives $\mathbf{c}_T = 1$. \square

3.7.6 Distances in the UIPM

Recall that $M_\infty = \mathcal{T}(Q_\infty)$ is the UIPM.

Theorem 3.36. *Let $\varepsilon \in (0, 1)$. We have*

$$\lim_{r \rightarrow \infty} \mathbb{P} \left(\sup_{x, y \in V(M_\infty), d_{\text{gr}}^{M_\infty}(\rho, x) \vee d_{\text{gr}}^{M_\infty}(\rho, y) \leq r} |d_{\text{fpp}}^{M_\infty}(x, y) - \mathbf{c}_T d_{\text{gr}}^{M_\infty}(x, y)| > \varepsilon r \right) = 0, \quad (3.42)$$

and

$$\lim_{r \rightarrow \infty} \mathbb{P} \left(\sup_{x, y \in V(M_\infty), d_{\text{gr}}^{M_\infty}(\rho, x) \vee d_{\text{gr}}^{M_\infty}(\rho, y) \leq r} |d_{\text{gr}}^{M_\infty}(x, y) - d_{\text{gr}}^{Q_\infty}(x, y)| > \varepsilon r \right) = 0. \quad (3.43)$$

We only sketch the proof, as it is very similar to that of Theorem 3.23. Since $d_{\text{gr}}^{M_\infty}(x, y) \geq \frac{1}{2} d_{\text{gr}}^{Q_\infty}(x, y)$ for every $x, y \in V(M_\infty)$, the condition $d_{\text{gr}}^{M_\infty}(\rho, x) \leq r$ implies $d_{\text{gr}}^{Q_\infty}(\rho, x) \leq 2r$. By the same argument, we can find a constant K large enough so that, for every $r \geq 1$ and for every $x, y \in V(M_\infty)$ such that $d_{\text{gr}}^{M_\infty}(\rho, x) \leq r$ and $d_{\text{gr}}^{M_\infty}(\rho, y) \leq r$, the quantities $d_{\text{fpp}}^{M_\infty}(x, y)$, $d_{\text{gr}}^{M_\infty}(x, y)$ and $d_{\text{gr}}^{Q_\infty}(x, y)$ are determined by the hull $B_{Q_\infty}^\bullet(Kr)$ (and of course weights on edges in the case of $d_{\text{fpp}}^{M_\infty}$). We then use Proposition 3.20 that allows us to find a large constant C such that the hulls $B_{Q_\infty}^\bullet \big|_{[C(Kr)^4]}(Kr)$ and $B_{Q_\infty}^\bullet(Kr)$ are equal with probability close to 1. We conclude by using Theorem 3.34.

Theorem 3.3 stated in the introduction follows from Theorem 3.23 and Theorem 3.36.

Chapitre 4

Uniform mixing time and bottlenecks in uniform finite quadrangulations

We prove a lower bound on the size of bottlenecks in uniform quadrangulations, valid at all scales simultaneously. We use it to establish upper bounds on the uniform mixing time of the lazy random walk on uniform quadrangulations, as well as on their dual. The proofs involve an explicit computation of the Laplace transform of the number of faces in truncated hulls of the uniform infinite plane quadrangulation.

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4.1 Introduction

Uniform mixing time in uniform quadrangulations. A *rooted planar map* is an embedding of a planar graph on the sphere with no edge-crossing, seen up to orientation-preserving homeomorphisms, equipped with a distinguished oriented edge called the *root edge*. A quadrangulation is a rooted planar map such that all its faces have degree 4. In this paper, we will be interested in type I quadrangulations, where multiple edges are allowed.

If Q is a quadrangulation, denote its vertex set, resp. edge set, face set, by $V(Q)$, resp. $E(Q), F(Q)$, and write $|Q| \stackrel{\text{def}}{=} |F(Q)|$ for its number of faces. The degree of a vertex x of Q is denoted by $\deg_Q(x)$, and the number of edges with endpoints x and y is denoted by $\text{mult}_Q(x, y)$. We are interested in the lazy random walk on Q , which is a reversible Markov chain on $V(Q)$ with transition probabilities

$$p_Q(x, y) = \begin{cases} 1/2 & \text{if } x = y, \\ \frac{\text{mult}_Q(x, y)}{2\deg_Q(x)} & \text{if } x \neq y, \end{cases}$$

and stationary distribution

$$\pi_Q(x) = \frac{\deg_Q(x)}{4|Q|}. \quad (4.1)$$

The main advantage of the lazy random walk over the simple random walk is that it is aperiodic. One may check using Euler's relation that $\sum_{y \in V(Q)} \deg_Q(y) = 2|E(Q)| = 4|Q|$, so π_Q is a probability distribution. We write $p_Q^k(x, y)$ for the k -step transition probabilities of the lazy random walk and define the ε -uniform mixing time of the lazy random walk

$$\tau_Q(\varepsilon) \stackrel{\text{def}}{=} \inf \left\{ k : \forall x, y \in V(Q), \left| \frac{p_Q^k(x, y) - \pi_Q(y)}{\pi_Q(y)} \right| \leq \varepsilon \right\}.$$

Let Q_n be a uniform quadrangulation with n faces. Our first theorem provides an upper bound on the mixing time of the lazy random walk in Q_n .

Theorem 4.1. *For every $\varepsilon, \delta > 0$, with probability going to 1 as $n \rightarrow \infty$,*

$$\tau_{Q_n}(\varepsilon) \leq n^{3/2+\delta}.$$

Our bound relies on a known result that relates the uniform mixing time to the size of ‘‘bottlenecks’’ [35]. The narrower the bottlenecks, the harder it is for the random walk to cross them, and the longer the mixing time. Conversely, if there is no very narrow bottleneck then the mixing time will not be too large. The bulk of this article is thus dedicated to showing that bottlenecks cannot be too narrow, see Theorem 4.3 and Corollary 4.6.

We could derive from Corollary 4.6 and [47, Theorem 7.4] a lower bound of the form $n^{3/4+o(1)}$ on the mixing time in total variation for the lazy random walk. However, $3/4$ is

in all likelihood not the optimal exponent: [31] proved that the simple random walk on the UIPT (the local limit of uniform triangulations as their size goes to infinity) travels a distance $t^{1/4+o(1)}$ after time t . We thus expect that the mixing time should be of order at least $n^{1+o(1)}$. Indeed, roughly speaking, since a map with n faces has diameter $n^{1/4+o(1)}$, we need to wait for a time $n^{1+o(1)}$ before the random walk has a chance to explore the whole map.

If Q is a quadrangulation, the dual Q^\dagger of Q is the planar graph whose vertices are the faces of Q , where two faces are adjacent if they share an edge in Q . We prove a similar upper bound on the uniform mixing time of the lazy random walk on Q^\dagger , which is a reversible Markov chain on $F(Q)$ with the following transition probability: at each time step, the walk has probability $1/2$ of staying at the same face, and probability $1/2$ of crossing one of the four sides of the current face, chosen uniformly at random. Note that Q is of type I, so both sides of a given edge may be incident to the same face; crossing such an edge results in staying at the current face. We denote the transition kernel of the lazy random walk by p_{Q^\dagger} . Its stationary distribution π_{Q^\dagger} is the uniform probability measure on $F(Q)$. The ε -uniform mixing time of the lazy random walk is defined as before:

$$\tau_{Q^\dagger}(\varepsilon) \stackrel{\text{def}}{=} \inf \left\{ k : \forall x, y \in F(Q), \left| \frac{p_{Q^\dagger}^k(x, y) - \pi_{Q^\dagger}(y)}{\pi_{Q^\dagger}(y)} \right| \leq \varepsilon \right\}.$$

Our second theorem provides an upper bound on the mixing time of the lazy random walk in the dual of Q_n .

Theorem 4.2. *For every $\varepsilon, \delta > 0$, with probability going to 1 as $n \rightarrow \infty$,*

$$\tau_{Q_n^\dagger}(\varepsilon) \leq n^{3/2+\delta}.$$

Note of the PhD candidate: I do not expect the exponent $3/2$ to be optimal. At the time of writing, I am conducting simulations to get a conjecture on the correct exponent.

Bottlenecks in finite quadrangulations. We now state our lower bounds on the size of bottlenecks in Q_n . The first bound considers sets of faces of Q_n . For every $S \subset F(Q_n)$, we denote the set of all edges of Q_n incident on one side to a face of S and on the other side to a face outside of S by ∂S .

Theorem 4.3. *For every $\nu \in (0, 1)$:*

$$\mathbb{P} \left(\inf_{S \subset F(Q_n) : 0 < |S| \leq n/2} \frac{|\partial S|^{4/3}}{|S|} \geq n^{-2/3-\nu} \right) \xrightarrow{n \rightarrow \infty} 1.$$

This theorem is instrumental in the proof of Theorems 4.1, 4.2 and 4.4. Section 3 to 6 are devoted to its proof.

An interesting feature of Theorem 4.3 is that the bound holds for all scales simultaneously: S can have any size, and is not restricted to contain a macroscopic fraction of

faces of Q_n . We conjecture that the bound of the Theorem is the best possible, in the sense that for every $0 < k < n/4$, we can find an S with $k \leq |S| \leq 2k$ and $\frac{|\partial S|^{4/3}}{|S|} \approx n^{-2/3}$.

In order to establish our results, we will heavily study the local limit of quadrangulations, the uniform infinite plane quadrangulation or UIPQ [38].

Let us mention earlier results in this direction. [43] established a lower bound on the size of bottlenecks in the UIPQ, in the form of an isoperimetric inequality. More precisely, it is proved in [43] that any subset A of the UIPQ with n faces that contains the root vertex must have boundary size at least $n^{1/4}(\ln n)^{-(3/4)-\delta}$. However, this result is not sufficient for our purpose: firstly because it applies to the infinite-volume limit of uniform quadrangulations, secondly because it only controls the size of bottlenecks that separate the root vertex from infinity. Our results are established independently from those in [43].

The convergence of uniform random quadrangulations towards the Brownian map [41, 55] gives a rough lower bound on the size of macroscopic bottlenecks in finite quadrangulations. The Brownian map being homeomorphic to the sphere [45], with probability close to 1, for n large enough, any cycle in Q_n that separates Q_n in two subsets, each with at least δn faces, must have length at least $\varepsilon n^{1/4}$. On the other hand, it is easy to find a set of size roughly $n/2$ and perimeter no larger than some large constant times $n^{1/4}$. However, this result only gives a lower bound on the size of bottlenecks at large scales (where the infimum holds over subsets S of $F(Q_n)$ with $\delta n \leq |S| \leq n/2$).

Let us give the intuition why this bound is optimal, focusing on large and small scales only. Our previous remark ensures that $\inf_{S \subset F(Q_n) : \delta n < |S| \leq n/2} \frac{|\partial S|^{4/3}}{|S|} \approx n^{-2/3+o(1)}$, so the bound of Theorem 4.3 is indeed optimal for sets containing a macroscopic proportion of faces of Q_n . For small scales, [6, Proposition 5] states that the supremum over all cycles of length 2 of the number of faces contained in the smallest component of the complement of the cycle is at most $n^{2/3+o(1)}$. We expect the supremum to be indeed $n^{2/3+o(1)}$; if this is true, then the bound of Theorem 4.3 is also optimal for sets containing at most $n^{2/3+o(1)}$ faces.

We now state our second bound on the size of bottlenecks Q_n , which holds for sets of vertices of Q_n . For every $A, B \subset V(Q_n)$, we denote the set of all edges of Q_n with one endpoint in A and the other in B by $E(A, B)$, and $A^c \stackrel{\text{def}}{=} V(Q_n) \setminus A$.

Theorem 4.4. *For every $\nu \in (0, 1)$:*

$$\mathbb{P} \left(\inf_{A \subset V(Q_n) : \pi_{Q_n}(A) \leq 1/2} \frac{|E(A, A^c)|^{4/3}}{4n\pi_{Q_n}(A)} \geq n^{-2/3-\nu} \right) \xrightarrow{n \rightarrow \infty} 1.$$

The similarity between Theorem 4.3 and Theorem 4.4 can be highlighted by noticing that $E(A, A^c)$ is the set of all edges with one end in A and another in A^c , while ∂S is the set of all edges with one side in S and another in S^c . The denominator in Theorem 4.3 may also be rewritten in a way closer to Theorem 4.4: $|S| = n\pi_{Q_n}^\dagger(S)$.

Hull volume. For every integer $r > 0$ and every vertex v of the UIPQ, the r -ball of the UIPQ Q_∞ centered at v is the union of faces of the UIPQ that are incident to a vertex at

distance $\leq r - 1$ from v . The standard r -hull centered at v , denoted by $B_{Q_\infty}^\bullet(v, r)$ is the union of the ball and of the finite connected components of its complement.

In order to prove Theorem 4.3, we compute the Laplace transform of the volume of truncated hulls centered at the root vertex of the UIPQ, and derive an upper bound on the probability that their volume is large. Truncated hulls is a type of hulls that is particularly adapted to the decomposition of the UIPQ into layers, see [43, 46] and Section 4.4 for a precise definition. They are also closely related to the standard hulls in the following way: if $\mathcal{H}_{Q_\infty}^{\text{tr}}(v, r)$ is the truncated r -hull of the UIPQ centered at v , then the following inclusions are verified for every integer $r > 0$:

$$\mathcal{H}_{Q_\infty}^{\text{tr}}(v, r) \subset B_{Q_\infty}^\bullet(v, r) \subset \mathcal{H}_{Q_\infty}^{\text{tr}}(v, r + 1). \quad (4.2)$$

It follows that bounds on the volume can be transferred from truncated hulls to standard hulls, up to negligible terms. Our methods are reminiscent of the paper [53] dealing with triangulations, although the formulas are more involved. To keep the technicalities to a minimum, we only establish an analog of Theorem 1 in [53].

Let us state an interesting consequence of our computations, which is a key tool in Section 5.

Lemma 4.5. *For every $\delta > 0$, there exists $C(\delta) \in (0, \infty)$ such that for every $r > 0$, for every $t > 0$,*

$$\mathbb{P}\left(|B_{Q_\infty}^\bullet(r)| > tr^4\right) \leq C(\delta)t^{-3/2+\delta}.$$

This bound is not surprising; the volume of the unit hull of the Brownian Plane has a known distribution that exhibits such a tail, without the δ in the exponent. However, we need a tail estimate that holds simultaneously for every r and every t , and getting such an estimate turns out to be quite involved.

Plan of the paper. We explain in Section 4.2 how to derive the bound on the mixing time in Theorem 4.2, as well as Theorem 4.4 and Theorem 4.1, from the bound on the size of bottlenecks in Theorem 4.3. The rest of the article is devoted to the proof of Theorem 4.3. At the core of the proof lies a fine control of the volume of standard hulls, and more precisely of the probability that their volume is large. Such a control is easier to establish in the UIPQ. Section 4.3 derives Lemma 4.9, which allows us to transfer results from the UIPQ to finite quadrangulations. We compute in Section 4.4 the Laplace transform of the volume of truncated hulls in the UIPQ, and derive the required tail estimates from its Taylor expansion near 0. This section makes heavy use of the so-called skeleton decomposition of the UIPQ. Section 4.5 uses the two previous sections to establish Proposition 4.15, stating that we can cover a uniform finite quadrangulation with a “small” number of hulls whose volumes are “controlled”. Finally, we prove Theorem 4.3 in Section 4.6.

4.2 Proof of the mixing time theorem

We derive Theorem 4.2 from Theorem 4.3. The first step is a bound on the Cheeger constant of Q_n^\dagger , which is a straightforward consequence of Theorem 4.3.

Corollary 4.6. *For every $\delta > 0$, the Cheeger constant $\mathfrak{H}^\dagger \stackrel{\text{def}}{=} \inf \left\{ \frac{|\partial S|}{|S|} : S \subset F(Q_n), 0 < |S| \leq n/2 \right\}$ of Q_n^\dagger is larger than $n^{-3/4-\delta}$ with probability going to 1 as $n \rightarrow \infty$.*

Proof. If $0 < |S| \leq n/2$,

$$\frac{|\partial S|}{|S|} \geq \left(\frac{|\partial S|^{4/3}}{|S|} \right)^{3/4} \left(\frac{2}{n} \right)^{1/4}$$

and we just need to apply Theorem 4.3. □

Proof of Theorem 4.2. For every $x, y \in F(Q_n)$, let $w_{x,y} \stackrel{\text{def}}{=} p_{Q_n^\dagger}(x, y) \pi_{Q_n^\dagger}(x)$. The conductance of Q_n^\dagger is

$$\Phi^\dagger \stackrel{\text{def}}{=} \inf \left\{ \frac{\sum_{x \in S, y \notin S} w_{x,y}}{\pi_{Q_n^\dagger}(S)} : S \subset F(Q_n), \pi_{Q_n^\dagger}(S) \leq 1/2 \right\}. \quad (4.3)$$

For every adjacent and distinct $x, y \in F(Q_n)$, $p_{Q_n^\dagger}(x, y) \geq 1/8$ and $w_{x,y} \geq 1/8n$, thus

$$\sum_{x \in S, y \notin S} w_{x,y} \geq \frac{|\partial S|}{8n},$$

and $\Phi^\dagger \geq \mathfrak{H}^\dagger/8$. We then apply [35, Corollary 2.3]:

$$\tau_{Q_n^\dagger}(\varepsilon) \leq \frac{128}{\mathfrak{H}^{\dagger 2}} (\ln(n) + \ln(1/\varepsilon)). \quad (4.4)$$

The theorem then follows from Corollary 4.6. □

We now derive Theorem 4.4 and Theorem 4.1 from Theorem 4.3. Let us first prove an upper bound on the maximum degree of a vertex of Q_n .

Lemma 4.7. *Let $\Delta_n \stackrel{\text{def}}{=} \max\{\deg_{Q_n}(x) : x \in V(Q_n)\}$. Then*

$$\mathbb{P}(\Delta_n > \ln n) \xrightarrow[n \rightarrow \infty]{} 0. \quad (4.5)$$

Proof. We recall the so-called “trivial” bijection between the set of all quadrangulations with n faces and the set of all rooted planar maps with n edges: let Q be a quadrangulation with n faces, and color its vertices so that the tail of the root vertex is white, and every two adjacent vertices have different colors. In each face of Q , draw a diagonal between the two white corners of the face. The map obtained by keeping only the added diagonals,

together with the white vertices of Q , is a planar map M with n edges, that we root at the edge contained in the root face of Q in such a way that the root vertex is the same as that of Q .

the set of all white vertices of Q is exactly the set of all vertices of M , and every face of M contains exactly one black vertex of Q . An easy observation is that, if x is a white vertex of Q , then $\deg_Q(x) = \deg_M(x)$. If x is a black vertex of Q , then $\deg_Q(x)$ is the degree of the face of M that contains x .

Denote the image of Q_n under the trivial bijection by M_n . M_n is uniformly distributed over the set of all rooted planar maps with n edges. [29, Theorem 3] ensures that, writing Δ^{M_n} for the maximum degree of a vertex of M_n ,

$$\mathbb{P}(\Delta^{M_n} > \ln n) \xrightarrow[n \rightarrow \infty]{} 0.$$

By self-duality of M_n , the same holds when replacing Δ^{M_n} by the maximum degree of a face of M_n . The lemma follows by our previous observation. \square

Proof of Theorem 4.4. Let $\nu > 0$. Fix $N(\nu)$ so that $8n^{-\nu/3} \ln n < 1$ for every $n \geq N(\nu)$. Let Q be a quadrangulation of size $n \geq N(\nu)$ such that

$$\inf_{S \subset F(Q) : 0 < |S| \leq n/2} \frac{|\partial S|^{4/3}}{|S|} \geq n^{-2/3-\nu}, \quad (4.6)$$

$$\max_{x \in V(Q)} \deg_Q(x) \leq \ln n. \quad (4.7)$$

Recall that the probability that Q_n satisfies (4.6) goes to 1 as $n \rightarrow \infty$ by Theorem 4.3, and the probability that Q_n satisfies (4.7) goes to 1 as well by Lemma 4.7.

For every $A \subset V(Q)$, we let $E = E(A, A^c)$ be the set of all edges of Q with one end in A and one end in $A^c = V(Q) \setminus A$. Assume first that $|E| \leq n^{1/4-\nu}$.

Let $S(A)$ be the set of all faces that are incident to a vertex of A . We can see that $|\partial S(A)| \leq 2|E| \leq 2n^{1/4-\nu}$ (observe that every face of $S(A)$ that is incident to at least one edge of $\partial S(A)$ is incident to at least two edges of E). It follows by (4.6) that either $|S(A)| \leq 2^{4/3}|E|^{4/3}n^{2/3+\nu} \leq 3n^{1-\nu/3}$, or $|S(A)| > n/2$.

Let us show by contradiction that the second case is not possible whenever $\pi_Q(A) \leq 1/2$, where π_Q is the stationary measure of the lazy random walk on $V(Q)$. If $|S(A)| > n/2$, then $|S(A)^c| < n/2$, and since $|\partial S(A)^c| = |\partial S(A)| \leq 2n^{1/4-\nu}$, (4.6) applied to $S(A)^c$ gives $|S(A)^c| \leq 3n^{1-\nu/3}$. Consider now $S(A^c)$. It is immediate that $S(A) \cup S(A^c) = F(Q_n)$; on the other hand, any face of $S(A^c) \cap S(A)$ is incident to an edge of E , so $|S(A^c) \cap S(A)| \leq 2|E|$. Consequently, $|S(A^c)| = |S(A)^c| + |S(A^c) \cap S(A)| \leq 4n^{1-\nu/3}$, and additionally $|A^c| \leq 4|S(A^c)| \leq 16n^{1-\nu/3}$. By the expression of π_Q and (4.7), $\pi_Q(A^c) \leq \frac{|A^c|}{4n} \max_{x \in V(Q)} \deg_Q(x) < 1/2$. It follows that $\pi_Q(A) > 1/2$, which contradicts our assumption.

It follows that $|S(A)| \leq 2^{4/3}|E|^{4/3}n^{2/3+\nu}$ as soon as $\pi_Q(A) \leq 1/2$. The bounds $|A| \leq 4|S(A)|$ and $\pi_Q(A) \leq \frac{|A|}{4n} \max_{x \in V(Q)} \deg_Q(x)$ give that for every $A \subset V(Q)$ with $\pi_Q(A) \leq$

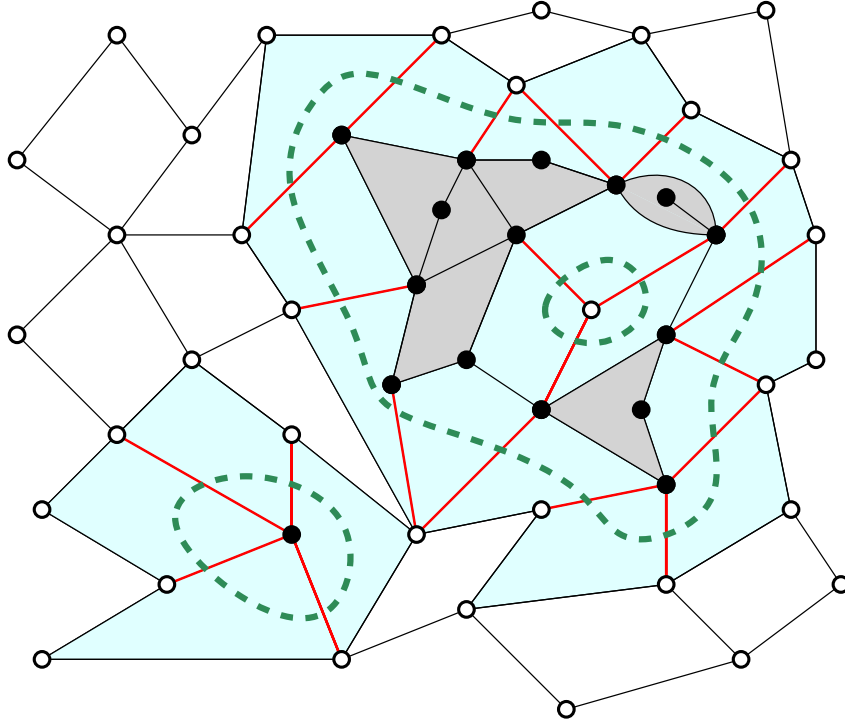


Figure 4.1 – The set A is the set of all black vertices. Edges in $E(A, A^c)$, that is with one end in A and the other in $A^c = V(Q) \setminus A$, are red. Faces in $S(A) \cap S(A^c)$ are colored in light blue, faces that are only in $S(A)$ are colored in grey, and faces that are only in $S(A^c)$ are colored in white. $S(A) \cap S(A^c)$ can be interpreted as the set of all faces crossed by the interfaces between the set of all white and the set of all black vertices (in dashed green).

$1/2$ and $|E(A, A^c)| \leq n^{1/4-\nu}$,

$$\pi_Q(A) \leq 2^{4/3} |E(A, A^c)|^{4/3} n^{-1/3+\nu} \ln n.$$

Considering separately every $A \subset V(Q)$ with $\pi_Q(A) \leq 1/2$ and $|E(A, A^c)| > n^{1/4-\nu}$, we have shown that

$$\inf \left\{ \frac{|E(A, A^c)|^{4/3}}{4n\pi_Q(A)} : A \subset V(Q), \pi_Q(A) \leq 1/2 \right\} \geq n^{-2/3-2\nu}.$$

This holds as soon as (4.6) and (4.7) are satisfied. The probability that Q_n satisfies both conditions goes to 1 as $n \rightarrow \infty$, so Theorem 4.4 follows. □

Proof of Theorem 4.1. For $x, y \in V(Q)$, $x \neq y$, set $w_{x,y} \stackrel{\text{def}}{=} p_Q(x, y)\pi_Q(x) \geq 1/(4n)$. Note that for every $A \subset V(Q_n)$, $\sum_{x \in A, x \notin A} w_{x,y} = \frac{|E(A, A^c)|}{4n}$. By Theorem 4.4, with probability going to 1 as $n \rightarrow \infty$,

$$\Phi \stackrel{\text{def}}{=} \inf \left\{ \frac{\sum_{x \in A, y \notin A} w_{x,y}}{\pi_{Q_n}(A)} : A \subset V(Q_n), \pi_{Q_n}(A) \leq 1/2 \right\} \geq n^{-3/4+\nu}.$$

We then apply [35, Corollary 2.3]: with probability going to 1 as $n \rightarrow \infty$,

$$\tau_{Q_n}(\varepsilon) \leq n^{3/2+2\nu}(\ln(4n) + \ln(1/\varepsilon)).$$

□

4.3 Standard hulls in finite quadrangulations and density with the UIPQ

In this section, we establish Lemma 4.9, which is stated below. Roughly speaking, this Lemma states that the probability of observing a given (simply connected) neighborhood of the root vertex in a uniform quadrangulation of finite size is smaller than the probability of observing the same neighborhood in the UIPQ, times a factor that only involves the size of the neighborhood and the size of the finite quadrangulation.

Recall that a quadrangulation is a rooted planar map such that all its faces have degree 4. We denote the set of all quadrangulations with n faces by \mathbb{Q}_n , and the cardinality of \mathbb{Q}_n by $\#\mathbb{Q}_n$. A quadrangulation Q with a simple boundary is a rooted planar map such that all its faces but (possibly) the face on the right of its root edge have degree 4, and the boundary of this face is a simple cycle. The face on the right of its root edge is called the external face, and its boundary is called the boundary of Q ; the other faces of Q are called inner faces. We write $\mathbb{Q}_{n,p}$ for the set of all rooted quadrangulations with a simple boundary of length $2p$ and n inner faces. Note that $\mathbb{Q}_{n,p} = \emptyset$ if $p > n + 1$ as an easy consequence of Euler's formula. By convention, we fix that \mathbb{Q}_0 contains one quadrangulation, the "edge-quadrangulation", and $\mathbb{Q}_{0,1}$ contains the unique map with one face and one edge. By [22, Section 6.2], for every $n \geq 0$ and $1 \leq p \leq n + 1$,

$$\#\mathbb{Q}_n = 3^n \frac{2(2n)!}{n!(n+2)!}, \quad (4.8)$$

$$\#\mathbb{Q}_{n,p} = 3^{n-p} \frac{(3p)!(2n+p-1)!}{(n+1-p)!p!(2p-1)!(n+2p)!}. \quad (4.9)$$

The following asymptotics come from (4.8) and (4.9):

$$\#\mathbb{Q}_n \underset{n \rightarrow \infty}{\sim} \frac{2}{\sqrt{\pi}} n^{-5/2} 12^n \quad (4.10)$$

and for every $1 \leq p \leq n + 1$,

$$\#\mathbb{Q}_{n,p} \underset{n \rightarrow \infty}{\sim} C_p n^{-5/2} 12^n \quad (4.11)$$

with

$$C_p = 2^p 3^{-p} \frac{(3p)!}{p!(2p-1)!} \frac{1}{2\sqrt{\pi}}. \quad (4.12)$$

Lemma 4.8. *There exists a constant $c > 0$ such that for every $n > 0, 1 \leq p \leq n + 1$,*

$$\#\mathbb{Q}_{n,p} \leq c \cdot C_p n^{-5/2} 12^n. \quad (4.13)$$

Proof. If $p = n + 1$, then (4.13) is verified directly. Using Stirling's formula, we can find a positive constant c such that for every $n > 0$, for every $1 \leq p \leq n$,

$$\begin{aligned} \#\mathbb{Q}_{n,p} &= 3^{n-p} \frac{(3p)!}{p!(2p-1)!} \frac{(2n+p-1)!}{(n+1-p)!(n+2p)!} \\ &\leq c \cdot 2\sqrt{\pi} C_p 2^{-p} 3^n \frac{(2n+p+1)^{2n+p+1}}{(n+1-p)^{n+1-p} (n+2p)^{n+2p}} \frac{1}{(2n+p+1)(2n+p)} \sqrt{\frac{2\pi(2n+p+1)}{4\pi^2(n+1-p)(n+2p)}} \\ &= c \cdot C_p n^{-5/2} 12^n \frac{(1 + \frac{p+1}{2n})^{2n+p+1/2}}{(1 - \frac{p-1}{n})^{n+1-p+1/2} (1 + \frac{2p}{n})^{n+2p+1/2}} \frac{1}{1 + \frac{p}{2n}} \\ &= c \cdot C_p n^{-5/2} 12^n \exp\left(n f_n\left(\frac{p-1}{n}\right)\right) \frac{1}{1 + \frac{p}{2n}}, \end{aligned} \quad (4.14)$$

where for every $n > 0$ and $x \in [0, 1)$,

$$f_n(x) \stackrel{\text{def}}{=} \left(2 + x + \frac{3}{2n}\right) \ln\left(1 + \frac{x}{2} + \frac{1}{n}\right) - \left(1 - x + \frac{1}{2n}\right) \ln(1 - x) - \left(1 + 2x + \frac{5}{2n}\right) \ln\left(1 + 2x + \frac{2}{n}\right).$$

If $\chi_n(u) = \left(u + \frac{1}{2n}\right) \ln u$,

$$f_n(x) = 2\chi_n\left(\frac{a+b}{2}\right) - \chi_n(a) - \chi_n(b) - \frac{1}{2n} \ln\left(1 + \frac{x}{2} + \frac{1}{n}\right)$$

for $a = 1 - x$ and $b = 1 + 2x + 2/n$. Since $\frac{1}{2n} \ln\left(1 + \frac{x}{2} + \frac{1}{n}\right) \geq 0$, and χ_n is convex on $[1/(2n), \infty)$, taking $x = (p-1)/n$ (ensuring $a \geq 1/n$) gives $f_n\left(\frac{p-1}{n}\right) \leq 0$. \square

For every quadrangulation Q and $v \in V(Q)$, we write $B_Q(v, r)$ for the r -ball (or ball of radius r) centered at v , defined as the union of faces that are incident to a vertex at distance at most $r - 1$ from v . We define the standard r -hull $B_Q^\bullet(v, r)$ as the union of the ball with the connected components of its complement, excluding the one containing the most faces (if there is an ambiguity, we lift it by a deterministic rule). We view $B_Q^\bullet(v, r)$ as a quadrangulation with a simple boundary (the external face corresponds to the excluded component of the complement). We say a map b is an *admissible standard r -hull* if there exists a quadrangulation Q and $v \in V(Q)$ such that $b = B_Q^\bullet(v, r)$.

Let Q_n be uniformly distributed over \mathbb{Q}_n , and Q_∞ be the UIPQ. We denote the root vertex of Q_n , resp. Q_∞ by ρ_n , resp. ρ_∞ .

Lemma 4.9. *There exists $c' > 0$ such that for every $r > 0$, for every $n > 0$, for every admissible standard r -hull b with N inner faces, $0 < N < n$,*

$$\frac{\mathbb{P}(B_{Q_n}^\bullet(\rho_n, r) = b)}{\mathbb{P}(B_{Q_\infty}^\bullet(\rho_\infty, r) = b)} \leq c' \left(\frac{n - N}{n}\right)^{-5/2}. \quad (4.15)$$

Proof. Let b be a rooted planar map with a distinguished face, such that all faces but the distinguished one have degree 4. We assume that b has $N < n$ inner faces (not counting the distinguished face), and that its distinguished face has simple boundary of length $2p$. Finally, we mark an oriented edge on the boundary of the distinguished face of b by some deterministic procedure that only involves b .

Let Q be a quadrangulation with n faces and root vertex ρ . Suppose that $n > N$. Then $b \subset Q$ holds if and only if Q is obtained by gluing a quadrangulation Q' with simple boundary of length $2p$ and $n - N$ inner faces inside the distinguished face of b , so that the root edge of Q' is glued on the marked edge of b . From (4.10), we can find c' large enough such that for every $n > 0$, we have $(\#Q_n)^{-1} \leq c'n^{5/2}12^{-n}$. It follows that for every $n > N$, using Lemma 4.8,

$$\begin{aligned} \mathbb{P}(b \subset Q_n) &= \frac{\#Q_{n-N,p}}{\#Q_n} \\ &\leq \frac{c c'' \cdot C_p (n - N)^{-5/2} 12^{n-N}}{n^{-5/2} 12^n} \\ &\leq c c'' \cdot C_p \left(\frac{n - N}{n} \right)^{-5/2} 12^{-N}. \end{aligned}$$

On the other hand, since the UIPQ is the local limit of Q_n as n goes to infinity,

$$\begin{aligned} \mathbb{P}(b \subset Q_\infty) &= \lim_{n \rightarrow \infty} \mathbb{P}(b \subset Q_n) \\ &= \lim_{n \rightarrow \infty} \frac{\#Q_{n-N,p}}{\#Q_n} \\ &= \lim_{n \rightarrow \infty} \frac{C_p (n - N)^{-5/2} 12^{n-N}}{\frac{2}{\sqrt{\pi}} n^{-5/2} 12^n} \\ &= \frac{\sqrt{\pi}}{2} C_p 12^{-N}. \end{aligned}$$

If b is an admissible standard r -hull, then $\mathbb{P}(B_{Q_n}^\bullet(\rho_n, r) = b) \leq \mathbb{P}(b \subset Q_n)$, and $\mathbb{P}(B_{Q_\infty}^\bullet(\rho_\infty, r) = b) = \mathbb{P}(b \subset Q_\infty)$. Combining the two bounds yields the lemma. \square

4.4 Laplace transform and tail estimates of the volume of hulls in the UIPQ

The goal of this section is to establish a bound on the probability that the volume of a standard hull of the UIPQ is large. We establish such a bound for truncated hulls, for which the analysis is simpler thanks to the skeleton decomposition of the UIPQ. The bound is then transferred to standard hulls using the inclusion (4.2).

4.4.1 Preliminaries

Following the definition in [43, Section 2], a truncated quadrangulation is a rooted planar map with a distinguished face, called the external face, such that

- (i) the external face f has simple boundary,
- (ii) every edge incident to f is also incident to a triangular face, and these triangular faces are all distinct,
- (iii) every other face has degree 4.

The faces that are not the external face are called inner faces. Consider a one-ended infinite quadrangulation Q of the plane and assume that it is drawn on the plane in such a way that any compact set of the plane intersects only a finite number of faces of Q (the UIPQ can be drawn in this way). Label every vertex of Q by its distance to the root vertex. Fix $r > 0$, and in every face whose incident vertices have label (in clockwise order) $r, r - 1, r, r + 1$, draw an edge between the two corners labeled r . The collection of added edges forms a union of (not necessarily disjoint) simple cycles, one of which is “maximal” in the sense that the connected component of its complement containing the root vertex of Q also contains every other added edge (see [43, Lemma 5]). We denote this maximal cycle by $\partial_r Q$. Adding the edges of $\partial_r Q$ to Q and removing the infinite connected component of the complement of $\partial_r Q$ gives a truncated quadrangulation, which we call the *truncated r -hull* centered at the root vertex of Q , or in short “the truncated r -hull of Q ”, and denote by $\mathcal{H}_Q^{\text{tr}}(r)$.

Let us explain why (4.2) holds. [43, Lemma 5] ensures that every vertex of Q at distance less than or equal to $r + 1$ is contained in $\mathcal{H}_Q^{\text{tr}}(r + 1)$. Since the standard r -hull of Q is bounded by a cycle of edges of Q that only visits vertices at distance r or $r + 1$ of the root vertex, it follows that $B_Q^\bullet(r) \subset \mathcal{H}_Q^{\text{tr}}(r + 1)$. On the other hand, $\partial_r Q$ is entirely contained in faces of $B_Q(r)$, so when adding the finite connected components of its complement one gets $\mathcal{H}_Q^{\text{tr}}(r) \subset B_Q^\bullet(r)$.

We now describe the *skeleton decomposition* of Q , which encodes the structure of every $\mathcal{H}_Q^{\text{tr}}(r)$ using a forest of plane trees and a collection of truncated quadrangulations. The ideas for this decomposition stem from [38]; see [43, Section 2.3] or [46, Section 2] for a more detailed explanation that is compatible with our notations. Let Q' be the following modification of Q : split the root edge of Q into a face of degree two, add a loop inside this face that is incident to the root vertex of Q , and root Q' at the added edge so that the face of degree 1 lies on the right of the new root edge, see Figure 4.2. Let $\partial_0 Q$ be the cycle of Q' made of the single root edge of Q' . For every $r > 0$ and every edge e of $\partial_r Q$, e splits a face f of Q' into two triangular faces; the one that is contained in $\mathcal{H}_Q^{\text{tr}}(r)$ is called the *downward triangle with top edge e* . The downward triangles cut the part of Q' outside $\partial_0 Q$ into a collection of *slots*, filled with finite maps (possibly reduced to a single edge), see Figure 4.3. Every slot contained between $\partial_{r-1} Q$ and $\partial_r Q$ is incident to a unique vertex v of $\partial_r Q$: we say that the slot is associated with the edge of $\partial_r Q$ with tail vertex v , where the edges of $\partial_r Q$ are oriented clockwise.

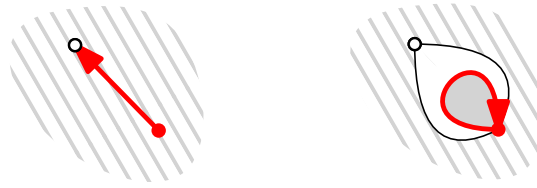


Figure 4.2 – We split the root edge of Q (left, in red), and add an edge in the manner illustrated here to obtain Q' . Q' is rooted at the new edge. The external face of Q' is in grey.

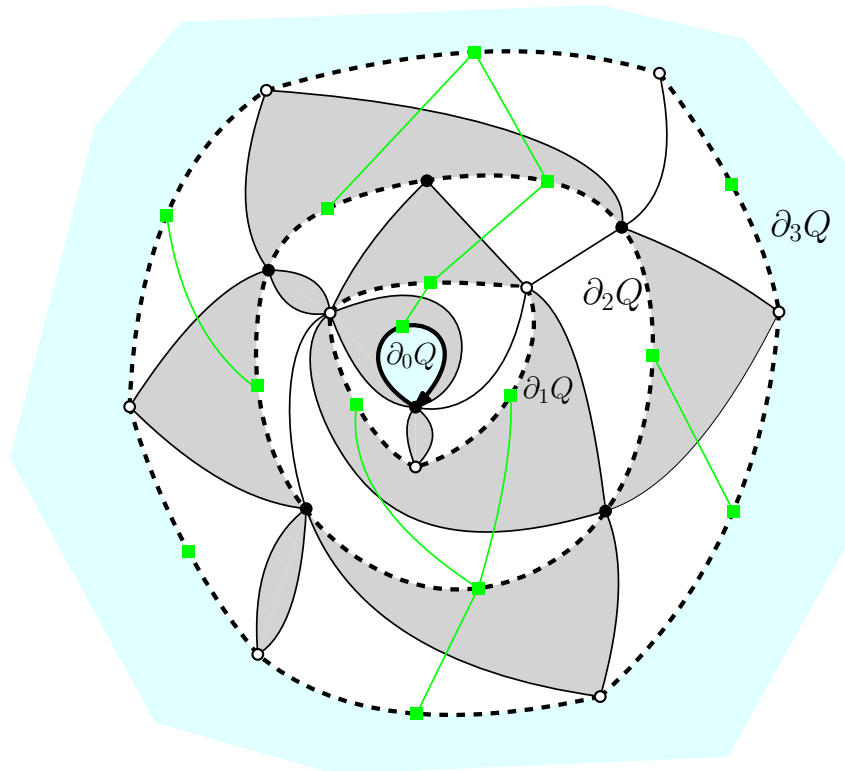


Figure 4.3 – The part of some quadrangulation Q' contained between ∂_0Q and ∂_3Q . Downward triangles are in white, slots in grey, edges of ∂_sQ in dashed thick lines (for $s \geq 1$, when they are not edges of Q') or in thick lines (for $s=0$). The outside of this part of the map is in cyan. Also included is the genealogical relation, represented by the green forest (rooted at edges of ∂_3Q).

We define the following genealogical relation on edges of $\cup_{r \geq 0} \partial_r Q$: for every $r > 0$ and every edge e of $\partial_r Q$, e is the parent of every edge of $\partial_{r-1} Q$ that is incident to the slot associated to e . The unique edge of $\partial_0 Q$ has no child. For every $r > 0$, write \mathcal{F}_r for the collection of all edges in $\partial_s Q$, $0 \leq s \leq r$ together with its genealogical relation, seen as a planar forest, and number its trees according to the clockwise order on their roots in such a way that the tree with index 1 contains the unique edge of $\partial_0 Q$. Then $\mathcal{F}_r \in \mathbb{F}_{r,q}^o$

the set of all (r, q) -admissible forests, where we say (slightly modifying [43, Section 2.3]) that a forest is (r, q) -admissible if

- (i) it consists of q rooted plane trees,
- (ii) there is exactly one vertex at generation r , and no vertex at generation $r + 1$ or more,
- (iii) the vertex at generation r is contained in the first tree.

Let $r > 0$, and let e be an edge of $\partial_r Q$. Consider the map M_e filling the slot associated to e , and let v be the vertex of M_e that belongs to $\partial_r Q$ (it is the tail of e). M_e can be considered as a truncated quadrangulation, after the following modification: add an edge inside the unbounded face of M_e to create a triangular face, in such a way that v is not anymore incident to the unbounded face, and root the M_e to the added edge in such a way that the unbounded face lies on its right. See Figure 4.4. We note that, if c_e is the number of offspring of the edge e , then M_e has perimeter $c_e + 1$.

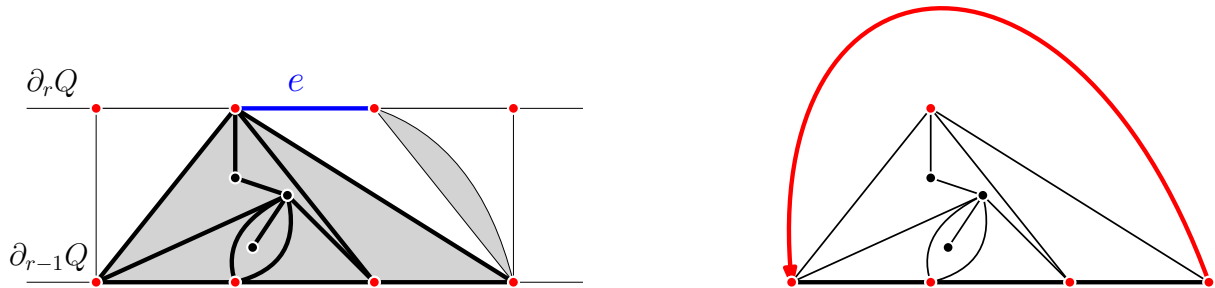


Figure 4.4 – The content of the slot associated to some edge e of $\partial_r Q$ (left, thick black lines) can be seen as a truncated quadrangulation M_e after adding one edge atop the tail of e , and rooting M_e at that new edge (right).

An admissible truncated r -hull H is a truncated quadrangulation such that there exists an infinite quadrangulation Q with $H = \mathcal{H}_Q^{\text{tr}}(r)$. Consider an infinite quadrangulation Q , and let \mathcal{F}_r^* be the collection of all vertices of \mathcal{F}_r at generation at most $r - 1$. The data of the “skeleton” \mathcal{F}_r and the truncated quadrangulations filling the slots $(M_e)_{e \in \mathcal{F}_r^*}$ is a function of H ; it is called the *skeleton decomposition of H* . Conversely, given a forest $\mathcal{F} \in \mathbb{F}_{q,r}^o$ and a collection $(M_e)_{e \in \mathcal{F}^*}$ of truncated quadrangulations such that M_e has perimeter c_e for every $e \in \mathcal{F}^*$, we can recover a unique admissible truncated r -hull: the skeleton decomposition is bijective.

4.4.2 Computing the Laplace transform of the hull volume

If M is a truncated quadrangulation, we denote the number of inner faces of M by $|M|$.

Theorem 4.10 (Laplace transform of the volume of truncated hulls). *For every $r > 0$, $p \in (0, 1)$,*

$$\mathbb{E} \left[(1 - p^2)^{|\mathcal{H}_{Q_\infty^{\text{tr}}(r)}^{\text{tr}}|} \right] = (1 - p)^{-1} (\mathcal{K} \circ \psi_r)'(0), \quad (4.16)$$

where

$$\begin{aligned} \mathcal{K}(t) &= \frac{3}{4} \sqrt{\frac{8+t}{t}}, \\ \psi_r(u) &= p + \frac{6p}{-1 + \sqrt{\frac{2(1-p)}{p+2}} \cosh(B(u) + ry)}, \\ y &= \cosh^{-1} \left(\frac{2p+1}{1-p} \right), \\ B(u) &= \cosh^{-1} \left(\sqrt{\frac{p+2}{2(1-p)}} \left[1 + \frac{6p}{(1-u)(1-p)} \right] \right). \end{aligned} \quad (4.17)$$

Proof. Let us briefly recall some results about the enumeration of truncated quadrangulations. Denote the set of all truncated quadrangulations with n inner faces and boundary length p by $\mathbb{Q}_{n,p}^{\text{tr}}$. [38, Section 2.2] gives an explicit expression for the generating function of truncated quadrangulations:

$$U(x, y) \stackrel{\text{def}}{=} \sum_{n>0, p>0} \#\mathbb{Q}_{n,p}^{\text{tr}} x^n y^p = \frac{1}{2} \left(y - xy^2 - 1 + \sqrt{y^2 - 2xy^3 - 2y + 4xyq(x) + (xy^2 - 1)^2} \right), \quad (4.18)$$

with q the generating function of quadrangulations:

$$q(x) \stackrel{\text{def}}{=} \sum_{n \geq 0} \#\mathbb{Q}_n x^n = \frac{4}{3} \frac{2\sqrt{1-12x} + 1}{(\sqrt{1-12x} + 1)^2}.$$

Singularity analysis gives the asymptotics of the number of truncated quadrangulations:

$$\#\mathbb{Q}_{n,p}^{\text{tr}} \underset{n \rightarrow \infty}{\sim} \kappa_p n^{-5/2} 12^n,$$

with

$$\sum_{p \geq 1} \kappa_p y^p = \frac{128\sqrt{3}}{\sqrt{\pi}} \frac{y}{\sqrt{(18-y)(2-y)^3}}. \quad (4.19)$$

Let us note that

$$\kappa_1 = \frac{32}{\sqrt{3\pi}}. \quad (4.20)$$

We now proceed with the proof. Let H be an admissible truncated r -hull with perimeter q , let \mathcal{F} be its skeleton. For every $e \in \mathcal{F}^*$, let c_e be the number of offspring of e and let M_e be the truncated quadrangulation filling the slot associated to e . From the proof of [43, Lemma 6]:

$$\sum_{v \in \mathcal{F}^*} (c_v - 1) = 1 - q, \quad (4.21)$$

$$|H| = q - 1 + \sum_{e \in \mathcal{F}^*} (|M_e| - 1). \quad (4.22)$$

[43, (7)] and the convergence of finite quadrangulations towards the UIPQ [38, Theorem 1] ensure that

$$\mathbb{P}(\mathcal{H}_{Q_\infty}^{\text{tr}}(r) = H) = \frac{\kappa_q}{\kappa_1} 12^{q-|H|-1} = \frac{\kappa_q}{\kappa_1} \prod_{e \in \mathcal{F}^*} 12^{-|M_e|+1}. \quad (4.23)$$

Let us introduce two variables s and t . We will later choose their value in a suitable way. For every $s, t > 0$, multiply (4.23) by $t^{q-1+\sum_{e \in \mathcal{F}^*} (c_e-1)} s^{q-1+\sum_{e \in \mathcal{F}^*} (|M_e|-1)} = s^{|H|}$ (by (4.21) and (4.22)):

$$s^{|H|} \mathbb{P}(\mathcal{H}_{Q_\infty}^{\text{tr}}(r) = H) = \frac{(st)^q \kappa_q}{(st)^1 \kappa_1} \prod_{e \in \mathcal{F}^*} t^{c_e-1} \left(\frac{s}{12}\right)^{|M_e|-1}.$$

Summing over all admissible truncated r -hulls, we get for every $s, t > 0$:

$$\begin{aligned} \mathbb{E} \left[s^{|\mathcal{H}_{Q_\infty}^{\text{tr}}(r)|} \right] &= \sum_H s^{|H|} \mathbb{P}(\mathcal{H}_{Q_\infty}^{\text{tr}}(r) = H) \\ &= \sum_{q>0} \frac{(st)^q \kappa_q}{(st)^1 \kappa_1} \sum_{\mathcal{F} \in \mathbb{F}_{r,q}^o} \prod_{e \in \mathcal{F}^*} t^{c_e-1} \sum_{|\partial M_e|=c_e+1} \left(\frac{s}{12}\right)^{|M_e|-1} \\ &= \sum_{q>0} \frac{(st)^q \kappa_q}{(st)^1 \kappa_1} \sum_{\mathcal{F} \in \mathbb{F}_{r,q}^o} \prod_{e \in \mathcal{F}^*} \Theta(c_e), \end{aligned} \quad (4.24)$$

where $\Theta(c) = t^{c-1} \sum_{|\partial M|=c+1} \left(\frac{s}{12}\right)^{|M|-1}$, where the sum is over the set of all truncated quadrangulations with perimeter $c+1$. We recognize the generating series of truncated quadrangulations:

$$\sum_{|\partial M|=c+1} \left(\frac{s}{12}\right)^{|M|-1} = \frac{12}{s} [y^{c+1}] U\left(\frac{s}{12}, y\right).$$

The generating function of Θ is

$$\Phi(u) \stackrel{\text{def}}{=} \sum_{c \geq 0} \Theta(c) u^c = \frac{12}{st} \sum_{c \geq 0} (tu)^c [y^{c+1}] U\left(\frac{s}{12}, y\right) = \frac{12}{st^2 u} \left(U\left(\frac{s}{12}, tu\right) - U\left(\frac{s}{12}, 0\right) \right).$$

We claim that if $s \in (0, 1]$ and $t = 2/(1+\sqrt{1-s})$, then Θ defines a probability distribution on nonnegative integers. Let us justify this claim. For convenience, we fix $p \in [0, 1)$ such that $s = 1 - p^2$; we then have $t = 2/(1+p)$. Using (4.18) we can rewrite Φ as follows:

$$\begin{aligned} \Phi(u) &= \frac{1}{2u(1-p)} \left(u^2 p - u^2 - 3p + 6u - 3 \right. \\ &\quad \left. + (1-u) \sqrt{p^2 u^2 + 2p^2 u - 2pu^2 + 9p^2 + 8pu + u^2 + 18p - 10u + 9} \right). \end{aligned} \quad (4.25)$$

Our claim is equivalent to $\Phi(1) = 1$, which is immediate.

Let $\mathbb{F}_{r,q}$ be the set of all planar forests satisfying conditions (i) and (ii) of the definition of admissible forests. Every forest $\mathcal{F} \in \mathbb{F}_{r,q}^o$ gives rise to q forests of $\mathbb{F}_{r,q}$ by applying one

of the q circular permutations of trees of \mathcal{F} . For every $\mathcal{F}_\bullet \in \mathbb{F}_{r,q}$, the probability that a Bienaymé-Galton-Watson forest with offspring distribution Θ is equal to \mathcal{F}_\bullet is $\prod_{e \in \mathcal{F}_\bullet} \Theta(c_e)$. It follows that

$$q \sum_{\mathcal{F} \in \mathbb{F}_{r,q}^\circ} \prod_{e \in \mathcal{F}^*} \Theta(c_e) = \sum_{\mathcal{F}_\bullet \in \mathbb{F}_{r,q}} \prod_{e \in \mathcal{F}_\bullet^*} \Theta(c_e) = \mathbb{P}(Y_r = 1) = [u] \left(\Phi^{(r)}(u) \right)^q,$$

where $(Y_r)_{r \geq 0}$ is a Bienaymé-Galton-Watson process with offspring distribution Θ such that $Y_0 = q$ and $\Phi^{(r)}$ denotes the r -th iterate of Φ . Let us rewrite (4.24) with our special choice of s and t :

$$\begin{aligned} \mathbb{E} \left[(1-p^2)^{|\mathcal{H}_{Q_\infty}^{\text{tr}}(r)|} \right] &= \frac{1}{2(1-p)\kappa_1} \sum_{q \geq 1} \frac{1}{q} \sum_{\mathcal{F}_\bullet \in \mathbb{F}_{r,q}} (2(1-p))^q \kappa_q \prod_{e \in \mathcal{F}_\bullet^*} \Theta(c_e) \\ &= \frac{1}{2(1-p)\kappa_1} \sum_{q \geq 1} \frac{1}{q} (2(1-p))^q \kappa_q [u] \left(\Phi^{(r)}(u) \right)^q \\ &= \frac{1}{\kappa_1} \left(\Phi^{(r)} \right)'(0) \sum_{q \geq 1} \kappa_q \left(2(1-p)\Phi^{(r)}(0) \right)^{q-1} \\ &= \left(\Phi^{(r)} \right)'(0) K(2(1-p)\Phi^{(r)}(0)), \end{aligned}$$

with

$$K(x) = \frac{1}{\kappa_1} \sum_{k \geq 1} \kappa_k x^{k-1} = 12 \cdot (18-x)^{-1/2} (2-x)^{-3/2}$$

by (4.19) and (4.20). Let $\mathcal{K}(t) = \frac{3}{4} \sqrt{\frac{8+t}{t}}$, then $K(x) = (2\mathcal{K}(1-x/2))' = -\mathcal{K}'(1-x/2)$, thus

$$\begin{aligned} \mathbb{E} \left[(1-p^2)^{|\mathcal{H}_{Q_\infty}^{\text{tr}}(r)|} \right] &= - \left(\Phi^{(r)} \right)'(0) \mathcal{K}'(1 - (1-p)\Phi^{(r)}(0)) \\ &= (1-p)^{-1} \psi_r'(0) \mathcal{K}'(\psi_r(0)) \\ &= (1-p)^{-1} (\mathcal{K} \circ \psi_r)'(0), \end{aligned} \tag{4.26}$$

where $\psi_r(u) = 1 - (1-p)\Phi^{(r)}(u)$. The closed formula for ψ_r is provided by the next Lemma. This finishes the proof of the Theorem. \square

Lemma 4.11. *Let $\Phi^{(r)} = \Phi \circ \dots \circ \Phi$ be the r^{th} iterate of Φ . The following holds for every $r \geq 0$, for $u \in [0, 1)$ and $p \in (0, 1)$:*

$$\Phi^{(r)}(u) = 1 - \frac{6p}{(1-p) \left(-1 + \sqrt{\frac{2(1-p)}{p+2}} \cosh(B(u) + ry) \right)}$$

with y and $B(u)$ as in Theorem 4.10. Furthermore, (4.17) holds.

Proof. Let $T_r(u) = \frac{2}{1-\Phi^{(r)}(u)}$. Writing $g(t) \stackrel{\text{def}}{=} \frac{2}{1-\Phi(1-2/t)}$, we check that for every integer $r \geq 0$, $T_{r+1}(u) = g(T_r(u))$. Let us compute a more explicit expression for g : using (4.25),

$$\Phi(1-2/t) = \frac{1}{t(t-2)(1-p)} \left(t^2 - pt^2 - 2pt + 2p - 4t - 2 + 2\sqrt{3p(p+2)t^2 + 2(1-p)(p+2)t + (1-p)^2} \right).$$

Then

$$\begin{aligned} g(t) &= \frac{2t(t-2)(1-p) \left(2t + 4pt - 2p + 2 + 2\sqrt{3p(p+2)t^2 + 2(1-p)(p+2)t + (1-p)^2} \right)}{4(1-p)^2 t(t-2)} \\ &= \frac{(2p+1)t + (1-p) + \sqrt{3p(p+2)t^2 + 2(1-p)(p+2)t + (1-p)^2}}{1-p}. \end{aligned}$$

In summary, for every $r \geq 0$

$$T_{r+1}(u) = \frac{(2p+1)T_r(u) + (1-p) + \sqrt{3p(p+2)T_r(u)^2 + 2(1-p)(p+2)T_r(u) + (1-p)^2}}{1-p}. \quad (4.27)$$

We prove by induction that for every $r \geq 0$, for every $p \in (0, 1)$ and $u \in [0, 1)$,

$$T_r(u) = \frac{1-p}{3p} \left(-1 + \sqrt{\frac{2(1-p)}{p+2}} \cosh(B(u) + ry) \right). \quad (4.28)$$

For $r = 0$,

$$\begin{aligned} \frac{1-p}{3p} \left(-1 + \sqrt{\frac{2(1-p)}{p+2}} \cosh(B(u)) \right) &= \frac{1-p}{3p} \left(-1 + 1 + \frac{6p}{(1-u)(1-p)} \right) \\ &= \frac{2}{1-u} \\ &= T_0(u). \end{aligned}$$

Let $r \geq 0$, and assume that (4.28) holds. Substituting in (4.27) gives

$$T_{r+1}(u) = \cosh(y)T_r(u) + 1 + \sqrt{1 + \frac{2(p+2)}{1-p}T_r(u) + \sinh(y)^2T_r(u)^2} \quad (4.29)$$

One checks that

$$\begin{aligned} \sqrt{1 + \frac{2(p+2)}{1-p}T_r(u) + \sinh(y)^2T_r(u)^2} &= \frac{1-p}{3p} \sqrt{\frac{2(1-p)}{p+2}} \sinh(y) \sinh(B(u) + ry), \\ \cosh(y)T_r(u) + 1 &= \frac{1-p}{3p} \sqrt{\frac{2(1-p)}{p+2}} \cosh(y) \cosh(B(u) + ry) - \frac{1-p}{3p}. \end{aligned}$$

Substituting in (4.29) gives

$$T_{r+1}(u) = -\frac{1-p}{3p} + \frac{1-p}{3p} \sqrt{\frac{2(1-p)}{p+2}} \cosh(B(u) + (r+1)y).$$

This proves (4.28). The Lemma follows from (4.28). □

4.4.3 Tail estimates for the volume of truncated hulls

The goal of this section is to get a Taylor expansion of $\mathbb{E} \left[e^{-\lambda |\mathcal{H}_{Q_\infty}^{\text{tr}}(r)|} \right]$, with a remainder term that we control simultaneously for every $r \geq 0$.

Proposition 4.12. *There exists $\lambda_0, c > 0$ such that for every $r > 0$, for every $0 \leq \lambda \leq \lambda_0$:*

$$\left| \mathbb{E} \left[e^{-\lambda |\mathcal{H}_{Q_\infty}^{\text{tr}}(r)|/r^4} \right] - \left(1 - \frac{(r+3)(6r^4 + 36r^3 + 87r^2 + 99r + 44)}{4r^3(2r+3)^2} \lambda + \frac{(r+3)(r+1)^3(r+2)^3}{2r^5(2r+3)^2} \lambda^{3/2} \right) \right| \leq c\lambda^2. \quad (4.30)$$

Proof. The Proposition is proved by computing Taylor expansions of $\psi'_r(0)$ and of $\psi_r(0)$ near $p = 0$, and then using (4.26). For readability we omitted the detailed computations of the coefficients; a maple sheet is provided. By a careful development in p of each individual term in $T_r(0)$, one can find $C_1 > 0$ and $x_1 > 0$ (that do not depend on r) such that for every $r > 0$ and $p = xr^{-2}$ with $0 < x < x_1$,

$$\left| \frac{r}{(r+1)(r+2)(r+3)} T_r(0) - \left(\frac{r}{r+3} + \frac{1}{2}x + \frac{r^2 + 3r + 1}{10r^2}x^2 + \frac{3r^4 + 18r^3 + 41r^2 + 42r + 36}{280r^4}x^3 \right) \right| \leq C_1x^4,$$

Given that $\psi_r(u) = 1 - (1-p) \left(1 - \frac{2}{T_r(u)} \right) = p + \frac{2(1-p)}{T_r(u)}$, similar expansions hold for $\psi_r(0)$, as well as for $T'_r(0)$ and $\psi'_r(0)$. We then compute the expansion of

$$\mathbb{E} \left[(1-p^2)^{|\mathcal{H}_{Q_\infty}^{\text{tr}}(r)|} \right] = -\frac{3}{1-p} \frac{\psi'_r(0)}{(8 + \psi_r(0))^{1/2} \psi_r(0)^{3/2}}$$

given by (4.26): there exists $x_2, C_2 > 0$ such that for every $r > 0$ and $0 < x < x_2$,

$$\left| \mathbb{E} \left[(1-x^2r^{-4})^{|\mathcal{H}_{Q_\infty}^{\text{tr}}(r)|} \right] - \left(1 - \frac{(r+3)(6r^4 + 36r^3 + 87r^2 + 99r + 44)}{4r^3(2r+3)^2} x^2 + \frac{(r+3)(r+1)^3(r+2)^3}{2r^5(2r+3)^2} x^3 \right) \right| \leq C_2x^4.$$

The Proposition follows by choosing x such that $e^{-\lambda/r^4} = 1 - x^2r^{-4}$. \square

We read from (4.30) that

$$\mathbb{E} \left[|\mathcal{H}_{Q_\infty}^{\text{tr}}(r)| \right] = \frac{r(r+3)(6r^4 + 36r^3 + 87r^2 + 99r + 44)}{4(2r+3)^2} \underset{r \rightarrow \infty}{\sim} \frac{3}{8}r^4,$$

which is consistent with the formula for the mean volume of the Brownian plane hull, which can be derived from [21, Theorem 1.4]. Proposition 4.12, together with [11, Theorem 8.1.6], implies that for every $r > 0$,

$$\mathbb{P} \left(|\mathcal{H}_{Q_\infty}^{\text{tr}}(r)| > t \right) \underset{t \rightarrow \infty}{\sim} \frac{r(r+3)(r+1)^3(r+2)^3}{4\sqrt{\pi}(2r+3)^2} t^{-3/2}. \quad (4.31)$$

Since we need a non-asymptotic upper bound valid for all $r > 0$ and $t > 0$, we prove the following weaker result:

Lemma 4.13. *For every $\delta > 0$, there exists $C(\delta) \in (0, \infty)$ such that for every $r > 0$, for every $t > 0$,*

$$\mathbb{P}\left(|B^\bullet(r)| > tr^4\right) \leq C(\delta)t^{-3/2+\delta}.$$

Proof. Fix $r > 0$, and write $X_r = \frac{|\mathcal{H}^{\text{tr}}(r)|}{r^4}$. By Fubini's theorem,

$$\int_0^\infty \lambda^{-5/2+\delta} \mathbb{E}\left[e^{-\lambda X_r} - 1 + \lambda X_r\right] d\lambda = \mathbb{E}\left[X_r^{3/2-\delta}\right] \int_0^\infty u^{-5/2+\delta} (e^{-u} + u - 1) du. \quad (4.32)$$

Split the left-hand side integral at λ_0 :

$$\int_0^\infty \lambda^{-5/2+\delta} \mathbb{E}\left[e^{-\lambda X_r} - 1 + \lambda X_r\right] d\lambda \leq \int_0^{\lambda_0} \lambda^{-5/2+\delta} \mathbb{E}\left[e^{-\lambda X_r} - 1 + \lambda X_r\right] d\lambda \quad (4.33)$$

$$+ \mathbb{E}\left[X_r\right] \int_{\lambda_0}^\infty \lambda^{-3/2+\delta} d\lambda. \quad (4.34)$$

(4.34) is bounded by a constant that does not depend on r . On the other hand, it follows from Proposition 4.12 that for every $\lambda \leq \lambda_0$,

$$\mathbb{E}\left[e^{-\lambda X_r} - 1 + \lambda X_r\right] \leq \frac{(r+3)(r+1)^3(r+2)^3}{2r^5(2r+3)^2} \lambda^{3/2} + c\lambda^2,$$

thus

$$\begin{aligned} & \int_0^{\lambda_0} \lambda^{-5/2+\delta} \mathbb{E}\left[e^{-\lambda X_r} - 1 + \lambda X_r\right] d\lambda \\ & \leq \int_0^{\lambda_0} c\lambda^{-1/2+\delta} d\lambda + \left(\sup_{r \geq 1} \frac{(r+3)(r+1)^3(r+2)^3}{2r^5(2r+3)^2}\right) \int_0^{\lambda_0} \lambda^{-1+\delta} d\lambda, \end{aligned}$$

so (4.33) is bounded by a constant that depends only on δ . We then have by (4.32):

$$\mathbb{E}\left[X_r^{3/2-\delta}\right] \leq \frac{\int_0^\infty \lambda^{-5/2+\delta} \mathbb{E}\left[e^{-\lambda X_r} - 1 + \lambda X_r\right] d\lambda}{\int_0^\infty u^{-5/2+\delta} (e^{-u} + u - 1) du}$$

which is smaller than a constant that depends only on δ . The Lemma follows using Markov's inequality, and recalling that $|B^\bullet(r)| \leq |\mathcal{H}^{\text{tr}}(r+1)|$. \square

4.5 Coverings of finite quadrangulations by balls

We first prove that with high probability, we can cover the quadrangulation Q_n with balls of volume uniformly bounded from below.

Lemma 4.14. *Let $\varepsilon > 0$. For every integer R with $n^\varepsilon \leq R \leq n^{1/4}$, we can find a sequence $(e_i^R)_{0 \leq i < R^{-4}n^{1+\varepsilon}}$ of oriented edges of Q_n , such that Q_n re-rooted at e_i^R has the same law as Q_n , and the following holds with probability going to 1 as $n \rightarrow \infty$:*

For every integer R such that $n^\varepsilon \leq R \leq n^{1/4}$, if z_i^R denotes the tail vertex of e_i^R ,

1. the R -balls $B_{Q_n}(z_i^R, R)$ cover Q_n ,
2. for every i with $0 \leq i < R^{-4}n^{1+\varepsilon}$, the ball $B_{Q_n}(z_i^R, R)$ contains at least $\frac{R^4}{n^\varepsilon}$ vertices.

In order to prove this lemma, we use the classical Cori-Vauquelin-Schaeffer bijection (or CVS bijection) between rooted and pointed quadrangulations and labeled trees. We briefly recall it and present our notation. Let \mathbb{T}_n be the set of all rooted labeled plane trees with n edges, i.e. plane trees whose vertices bear labels in \mathbb{Z} , such that the labels of two adjacent vertices differ by at most 1 and the root vertex has label 0. Let $t \in \mathbb{T}_n$, and l its label function. A corner of t is an angular sector incident to one of its vertices. We order the corners cyclically according to the clockwise route around t .

The CVS bijection allows us to get a rooted quadrangulation with n faces from t and from an integer $\vartheta \in \{-1, 1\}$, as follows. First, add a vertex ∂ to the tree, and extend the labeling to ∂ such that $l(\partial) = -1 + \inf_t l$. Then, for each corner c , let $S(c)$ be the first corner after c in the contour sequence with label $l(c) - 1$ (if $l(c) = \inf_t l$, we fix $S(c) = \partial$), and draw an edge between c and $S(c)$ in such a way that it does not intersect the previously drawn edges. Finally, erase the edges of t . We obtain a quadrangulation q with n faces and a distinguished vertex ∂ , and we need to specify its root edge. We root q at the edge drawn from the bottom corner of the root vertex of t , and specify its direction using ϑ : if $\vartheta = +1$ it points towards the root vertex of t , if $\vartheta = -1$ it points away from the root vertex of t .

Proof of Lemma 4.14. Let T_n be uniformly chosen over \mathbb{T}_n , and ϑ a uniform integer over $\{-1, +1\}$, then the quadrangulation obtained from (T_n, ϑ) is uniformly distributed over the set of all rooted and pointed quadrangulations with n faces. Forgetting the distinguished vertex, we get a uniform quadrangulation Q_n with n faces.

Denote the corners of T_n enumerated in clockwise order around the contour starting from the root corner by $(c_i)_{0 \leq i < 2n}$. We extend the numbering to \mathbb{Z} by periodicity, and write L_i for the label of c_i . [44, Lemma 4.4] ensures that for every $p \geq 1$, there exists a constant K_p such that for every $0 \leq j, j' \leq 2n$,

$$\mathbb{E} \left[|L_j - L_{j'}|^{4p} \right] \leq K_p |j - j'|^p.$$

Using Markov's inequality,

$$\mathbb{P}(|L_j - L_{j'}| \geq u) \leq K_p \left(\frac{|j - j'|}{u^4} \right)^p. \quad (4.35)$$

Define

$$\mathcal{A} \stackrel{\text{def}}{=} \left\{ \forall i, j \in \{0, \dots, 2n-1\}, i \neq j : |L_i - L_j| < n^{1/p} |i - j|^{1/4} \right\}. \quad (4.36)$$

Then

$$\begin{aligned} \mathbb{P}(\mathcal{A}^c) &\leq \sum_{i, j \in \{0, \dots, 2n-1\}, i \neq j} K_p \left(\frac{|i - j|}{n^{4/p} |i - j|} \right)^p \\ &\leq 4K_p n^{-2}. \end{aligned}$$

Fix $p > 5/\varepsilon$, and let us argue on \mathcal{A} . Write v_i for the vertex of Q_n that is incident to c_i . [44, Proposition 5.9 (i)] allows us to bound the distance in Q_n between v_i and v_j for $i \leq j$:

$$d_{\text{gr}}^{Q_n}(v_i, v_j) \leq L_i + L_j - 2 \min_{k \in [i, j]} L_k + 2 \leq 2n^{1/p} |i - j|^{1/4} + 2.$$

For every $0 \leq i, j < 2n$, v_j thus belongs to the ball of radius $2 + 2n^{1/p} |i - j|^{1/4}$ centered at v_i . For every $n^\varepsilon \leq R \leq n^{1/4}$, define $k(R) > 0$ as the largest integer such that $2 + 2n^{1/p} k(R)^{1/4} < R - 2$, and fix $y_i^R = v_{k(R)i}$ for every $0 \leq i < \lceil 2n/k(R) \rceil$. Every vertex of Q_n (except possibly ∂) is at distance strictly less than $R - 2$ from at least one y_i^R , and ∂ is at distance at most $R - 2$ from one of y_i^R , thus the balls $B_{Q_n}(y_i^R, R - 1)$ cover Q_n . For n large enough, this covering of Q_n contains at most $2n/k(R) \leq n^{1+5/p}/R^4$ balls.

Note that for every $n^\varepsilon \leq R \leq n^{1/4}$ and $0 \leq i < \lceil 2n/k(R) \rceil$, if we re-root the tree at the corner $c_{k(R)i}$, the map we obtain by the CVS bijection is exactly Q_n , re-rooted at the edge e_i^R drawn from corner $c_{k(R)i}$ (oriented towards y_i^R if $\vartheta = +1$, and away from it if $\vartheta = -1$). In particular, it has the same law as Q_n . We complete the sequences (e_i^R) and (y_i^R) by taking e_i^R equal to the root edge and y_i^R equal to the root vertex for every $\lceil 2n/k(R) \rceil \leq i < n^{1+\varepsilon}/R^4$. Recall that z_i^R is the tail vertex of e_i^R . Since for every $0 \leq i < n^{1+\varepsilon}/R^4$, z_i^R is at distance at most 1 from y_i^R , the first property of the lemma holds.

It remains to prove that the volume of every $B_{Q_n}(y_i^R, R)$ is bounded from below by R^4/n^ε for every $n^\varepsilon \leq R \leq n^{1/4}$ and $0 \leq i < n^{1+\varepsilon}/R^4$. From now on, we argue on the intersection of \mathcal{A} with the event where the maximum degree of Q_n is at most $\ln n$, which has probability going to 1 as $n \rightarrow \infty$ by Lemma 4.7. The number of corners of a vertex v_i in T_n is at most its degree in Q_n , so it is smaller than $\ln n$. For every $n^\varepsilon \leq R \leq n^{1/4}$ and $0 \leq i < \lceil 2n/k(R) \rceil$, every v_j with $k(R)i \leq j \leq k(R)(i + 1)$ belongs to the R -ball $B_{Q_n}(z_i^R, R)$. Each v_j appears at most $\ln n$ times in this sequence, so that $B_{Q_n}(z_i^R, R)$ contains at least $\frac{k(R)+1}{\ln n} \geq \frac{R^4}{n^\varepsilon}$ distinct vertices. Since this also holds for $\lceil 2n/k(R) \rceil \leq i < n^{1+\varepsilon}/R^4$, the second point of the lemma is proven. \square

The following proposition is the key ingredient of the proof of Theorem 4.3. Recall that r -hulls are obtained from r -balls by adding every connected component of the complement but the one containing the largest number of faces. Proposition 4.15 exhibits a covering of Q_n by balls, such that the volumes of the corresponding hulls are bounded from above and below.

Proposition 4.15. *For every $\delta \in (0, 1/8)$, the following holds with probability going to 1 as n goes to ∞ :*

For every R of the form 2^k with $n^\delta \leq R \leq n^{1/4-\delta}$, we can find a sequence z_i^R , $0 \leq i < \frac{n^{1+\delta}}{R^4}$ of vertices of Q_n such that

1. *the balls $B_{Q_n}(z_i^R, R/2)$ cover Q_n ,*
2. *for every i , $\frac{R^4}{n^\delta} \leq |B_{Q_n}^\bullet(z_i^R, R)| \leq R^{4/3} n^{2/3+\delta}$.*

Proof. Let $\varepsilon = \delta/40$. By [16], the probability that the diameter of Q_n is at least $8n^{1/4-\varepsilon}$ goes to 1 as n goes to infinity. From now on we argue on the intersection of this event with the event of Lemma 4.14, whose probability also goes to 1 as n goes to infinity. We consider the sequence (z_R^i) given by Lemma 4.14, so the property 1 and the minoration in the property 2 of Proposition 4.15 already hold.

For every $0 < R \leq n^{1/4-\varepsilon}$, for every $x \in V(Q_n)$, one may find $z \in V(Q_n)$ at distance at least $4n^{1/4-\varepsilon}$ from x . By Lemma 4.14, this vertex is contained in a $\lceil n^{1/4-\varepsilon} \rceil$ -ball B containing at least $n^{1-5\varepsilon}$ vertices. Consider now the ball $B_{Q_n}(x, R)$. One of the connected component of its complement contains the ball B , thus it contains at least $n^{1-5\varepsilon}$ inner vertices. Since this connected component is a quadrangulation with simple boundary, Euler's formula gives that its number of faces is also larger than $n^{1-5\varepsilon}$. It follows that the R -hull $B_{Q_n}^\bullet(x, R)$ contains at most $n - n^{1-5\varepsilon}$ faces.

Now take $n^\varepsilon \leq R \leq n^{1/4-\varepsilon}$ and $0 \leq i < \frac{n^{1+\varepsilon}}{R^4}$. The event $\{|B_{Q_\infty}^\bullet(\rho_\infty, R)| > tR^4\}$ has probability at most $C(\varepsilon)t^{-\frac{3}{2}+\varepsilon}$ by Lemma 4.13. Together with Lemma 4.9, and using the fact that Q_n re-rooted at e_R^i has the same law as Q_n , we get

$$\mathbb{P}(|B_{Q_n}^\bullet(z_i^R, R)| > tR^4) \leq c(n^{-5\varepsilon})^{-5/2} \mathbb{P}(|B_{Q_\infty}^\bullet(\rho_\infty, R)| > tR^4) \leq cC(\varepsilon)n^{13\varepsilon}t^{-\frac{3}{2}+\varepsilon}.$$

Fix $t = \left(\frac{R^4}{n^{1+15\varepsilon}}\right)^{-\frac{1}{\frac{3}{2}+\varepsilon}}$ and sum over every $0 \leq i < n^{1+\varepsilon}/R^4$:

$$\begin{aligned} \mathbb{P}\left(\exists 0 \leq i < \frac{n^{1+\varepsilon}}{R^4} : |B_{Q_n}^\bullet(z_i^R, R)| > tR^4\right) &\leq \frac{n^{1+\varepsilon}}{R^4} cC(\varepsilon)n^{13\varepsilon} \left(\frac{R^4}{n^{1+15\varepsilon}}\right) \\ &\leq cC(\varepsilon)n^{-\varepsilon}. \end{aligned} \quad (4.37)$$

Finally, we sum over all possible choices of $n^\varepsilon \leq R \leq n^{1/4-\varepsilon}$ of the form 2^k for some integer k . There are at most $(\ln n)/(\ln 2)$ such R , so the probability that the event in (4.37) holds for one of these R goes to 0 as n goes to infinity. Since $tR^4 \leq R^{4/3}n^{2/3+\delta}$, this gives the majoration in property 2. □

4.6 Proof of the bound on the size of bottlenecks

We now use the results of Section 4.5 to prove Theorem 4.3. We first establish the following lemma, which is a particular case of Theorem 4.3. We say $S \subset F(Q_n)$ is simply connected if ∂S is a connected, non-empty, simple cycle.

Lemma 4.16. *For every $\nu \in (0, 3/8)$, the probability that*

$$\inf \frac{|\partial S|^{4/3}}{|S|} \geq n^{-2/3-\nu}$$

goes to 1 as $n \rightarrow \infty$, where the infimum holds over all simply connected subsets S of $F(Q_n)$ such that $|S| \leq n/2$.

Proof. Let us argue on the event of Proposition 4.15 with $\delta = \nu/3$. Consider S a simply connected subset of $F(Q_n)$ with $|S| \leq n/2$.

If $|\partial S| \geq n^{1/4-\delta}$, since we necessarily have $|S| \leq n$,

$$\frac{|\partial S|^{4/3}}{|S|} \geq \frac{n^{\frac{4}{3}(\frac{1}{4}-\delta)}}{n} = n^{-2/3-4\delta/3} \geq n^{-2/3-\nu}.$$

If $|\partial S| < n^\delta$, ∂S is contained in an R -ball centered at one of z_i^R , with R the smallest power of two larger than $2n^\delta$. Then either S is contained in $B_{Q_n}^\bullet(z_i^R, R)$, or contains the complement of $B_{Q_n}^\bullet(z_i^R, R)$. We are on the event of Proposition 4.15, so (by the second point in the proposition) the complement of $B_{Q_n}^\bullet(z_i^R, R)$ has volume strictly larger than $n/2$. Since $|S| \leq n/2$, S is contained in $B_{Q_n}^\bullet(z_i^R, R)$, and thus its number of faces is less than $n^{\frac{2}{3}+\delta+\frac{4}{3}\delta}$ (by property 2 of Proposition 4.15), which gives

$$\frac{|\partial S|^{4/3}}{|S|} \geq n^{-2/3-3\delta} \geq n^{-2/3-\nu}.$$

If $n^\delta \leq |\partial S| < n^{1/4-\delta}$, consider $R = 2^k$ with $R/2 \leq |\partial S| < R$. Since the balls $B_{Q_n}(z_i^R, R/2)$ cover Q_n , we can find i such that $d(z_i^R, \partial S) \leq R/2$, hence by connexity $\partial S \subset B_{Q_n}(z_i^R, R)$. By the same argument as in the case $|\partial S| < n^\delta$, we have $S \subset B_{Q_n}^\bullet(z_i^R, R)$, and thus

$$\frac{|\partial S|^{4/3}}{|S|} \geq \frac{(R/2)^{4/3}}{R^{4/3}n^{2/3+\delta}} \geq n^{-2/3-\nu}.$$

□

Proof of Theorem 4.3. Let $\delta > 0$. Fix $\nu = 3\delta/4$, and let n be taken large enough so that $n^{\nu/3} > 2$.

Say a subset $S \subset F(Q_n)$ is connected if its corresponding subgraph in the dual graph of Q_n is connected. We first extend the result of Lemma 4.16 to connected subsets of $F(Q_n)$, and then to generic subsets of $F(Q_n)$. Let us argue on the event of Lemma 4.16.

Consider S a connected subset of $F(Q_n)$ with $|S| \leq n/2$ and $|\partial S| \leq n^{1/4-\nu}$. Let S_1, \dots, S_k be the connected components of the complement of S . Note that S_1, \dots, S_k are simply connected and $\partial S = \sqcup_{1 \leq i \leq k} \partial S_i$. We claim that if n is large enough, then exactly one of the S_i has volume $> n/2$. Note that it is enough to show that at least one has volume $> n/2$. We prove this claim by contradiction: assume that every S_i has

volume at most $n/2$. By Lemma 4.16, for every $1 \leq i \leq k$, $|S_i| \leq n^{2/3+\nu} |\partial S_i|^{4/3}$, thus

$$\begin{aligned} n = |Q_n| &= |S| + \sum_{i=1}^k |S_i| \\ &\leq n/2 + \sum_{i=1}^k n^{2/3+\nu} |\partial S_i|^{4/3} \\ &\leq n/2 + n^{2/3+\nu} |\partial S|^{4/3} \\ &\leq n/2 + n^{1-\nu/3} \\ &< n \end{aligned}$$

which is impossible, proving our claim. Without loss of generality, we assume that $|S_1| > n/2$. Define $S' \stackrel{\text{def}}{=} S \cup \bigcup_{2 \leq i \leq k} S_i$. Then S' is simply connected and $|S'| \leq n/2$, thus

$$|S| \leq |S'| \leq n^{2/3+\nu} |\partial S'|^{4/3} \leq n^{2/3+\nu} |\partial S|^{4/3}.$$

It remains to consider the case of a connected $S \subset F(Q_n)$ with $|S| \leq n/2$ and $|\partial S| > n^{1/4-\nu}$. It is immediate that for such a set,

$$|S| \leq n \leq n^{2/3+4\nu/3} |\partial S|^{4/3}.$$

To sum up, on the event of Lemma 4.16, for every connected set $S \subset F(Q_n)$ with $|S| \leq n/2$,

$$|S| \leq n^{2/3+\delta} |\partial S|^{4/3}. \quad (4.38)$$

Now consider a generic $T \subset F(Q_n)$ with $|T| \leq n/2$. Let T_1, \dots, T_j be its connected components. By (4.38), for every $1 \leq i \leq j$, $|T_i| \leq n^{2/3+\delta} |\partial T_i|^{4/3}$, and since $\partial T = \sqcup_{i=1}^j \partial T_i$, by convexity

$$|T| = \sum_{i=1}^j |T_i| \leq \sum_{i=1}^j n^{2/3+\delta} |\partial T_i|^{4/3} \leq n^{2/3+\delta} |\partial T|^{4/3}.$$

Theorem 4.3 follows. □

Annex: maple sheet

Section 4: Laplace transform of the volume of truncated hulls

[> restart

Proof of Theorem 7

$$\begin{aligned} > q := \frac{4}{3} \frac{2 \cdot \sqrt{1 - 12 \cdot x} + 1}{(\sqrt{1 - 12 \cdot x} + 1)^2} \\ q &:= \frac{4 (2 \sqrt{1 - 12 x} + 1)}{3 (\sqrt{1 - 12 x} + 1)^2} \end{aligned} \quad (1.1.1)$$

$$\begin{aligned} > U := \frac{1}{2} \cdot \left(y - x \cdot y^2 - 1 + \sqrt{y^2 - 2 \cdot x \cdot y^3 - 2 \cdot y + 4 \cdot x \cdot y \cdot q + (x \cdot y^2 - 1)^2} \right) \\ U &:= \frac{y}{2} - \frac{x y^2}{2} - \frac{1}{2} \end{aligned} \quad (1.1.2)$$

$$+ \frac{\sqrt{9 y^2 - 18 x y^3 - 18 y + \frac{48 x y (2 \sqrt{1 - 12 x} + 1)}{(\sqrt{1 - 12 x} + 1)^2} + 9 (x y^2 - 1)^2}}{6}$$

$$\begin{aligned} > PHI := \text{simplify} \left(\text{subs} \left(s = 1 - p^2, t = \frac{2}{1 + p}, \frac{12}{t^2 \cdot s \cdot u} \cdot \left(\text{subs} \left(x = \frac{s}{12}, y = t \cdot u, U \right) \right. \right. \right. \right. \\ \left. \left. \left. - \text{subs} \left(x = \frac{s}{12}, y = 0, U \right) \right) \right) \right) \text{ assuming } p < 1, p > 0, u < 1 \\ PHI &:= \frac{1}{2 u (p - 1)} \left((-1 + u) \sqrt{(p - 1)^2 u^2 + (2 p^2 + 8 p - 10) u + 9 (1 + p)^2} \right. \\ &\quad \left. + (1 - p) u^2 - 6 u + 3 p + 3 \right) \end{aligned} \quad (1.1.3)$$

$$\begin{aligned} > \text{simplify} \left(PHI - \frac{1}{2 u \cdot (1 - p)} \cdot \left(u^2 \cdot p - u^2 - 3 \cdot p + 6 \cdot u - 3 + (1 - u) \right. \right. \\ \left. \left. \cdot \sqrt{p^2 \cdot u^2 + 2 \cdot p^2 \cdot u - 2 \cdot p \cdot u^2 + 9 p^2 + 8 \cdot p \cdot u + u^2 + 18 \cdot p - 10 \cdot u + 9} \right) \right) \\ 0 \end{aligned} \quad (1.1.4)$$

Check that \$Phi\$ indeed defines a probability distribution.

$$\begin{aligned} > \text{simplify}(\text{subs}(u = 1, PHI)) \\ 1 \end{aligned} \quad (1.1.5)$$

Proof of Lemma 8

First, we derive the recurrence expression on \$T_r(u)\$ from the expression for \$Phi\$. If \$t = 2/(1-u)\$, then \$u = 1 - 2/t\$.

$$\begin{aligned} > Tr := \frac{(2 \cdot p + 1) \cdot t + (1 - p) + \sqrt{(1 - p)^2 + 2 \cdot (1 - p) \cdot (p + 2) \cdot t + 3 \cdot p \cdot (p + 2) \cdot t^2}}{1 - p} \\ Tr &:= \frac{(2 p + 1) t + 1 - p + \sqrt{(1 - p)^2 + 2 (1 - p) (p + 2) t + 3 p (p + 2) t^2}}{1 - p} \end{aligned} \quad (1.2.1)$$

Let us check that the two expressions match.

$$\begin{aligned} &> \text{simplify}\left(\text{Tr} - \text{simplify}\left(\text{subs}\left(u = 1 - \frac{2}{t}, \frac{2}{1 - PHI}\right)\right)\right) \text{ assuming } p > 0, p < 1, t > 0, t < 1 \\ & \qquad \qquad \qquad 0 \end{aligned} \tag{1.2.2}$$

Checking the two simplifications required in the proof.

$$\begin{aligned} &> B := \text{arccosh}\left(\sqrt{\frac{p+2}{2 \cdot (1-p)}} \cdot \left(1 + \frac{6p}{(1-u) \cdot (1-p)}\right)\right) \\ & \qquad \qquad \qquad B := \text{arccosh}\left(\sqrt{\frac{p+2}{2-2p}} \left(1 + \frac{6p}{(1-u)(1-p)}\right)\right) \end{aligned} \tag{1.2.3}$$

$$\begin{aligned} &> y := \text{arccosh}\left(\frac{2 \cdot p + 1}{1-p}\right) \\ & \qquad \qquad \qquad y := \text{arccosh}\left(\frac{2p+1}{1-p}\right) \end{aligned} \tag{1.2.4}$$

$$\begin{aligned} &> \text{TrTest} := \frac{1-p}{3 \cdot p} \cdot \left(-1 + \sqrt{\frac{2 \cdot (1-p)}{2+p}} \cdot \cosh(B + r \cdot y)\right) \\ \text{TrTest} &:= \frac{1}{3p} \left((1-p) \left(-1 + \sqrt{2} \sqrt{\frac{1-p}{p+2}} \cosh\left(r \text{arccosh}\left(\frac{2p+1}{1-p}\right)\right) \right. \right. \\ & \qquad \qquad \qquad \left. \left. + \text{arccosh}\left(\sqrt{\frac{p+2}{2-2p}} \left(1 + \frac{6p}{(1-u)(1-p)}\right)\right)\right) \right) \end{aligned} \tag{1.2.5}$$

$$\begin{aligned} &> \text{simplify}\left(\left(1 + \frac{2 \cdot (p+2)}{1-p} \cdot \text{TrTest} + \sinh(y)^2 \cdot \text{TrTest}^2\right) - \left(\sinh(y)^2 \cdot \left(\frac{1-p}{3p} \sqrt{\frac{2 \cdot (1-p)}{p+2}}\right)^2 \cdot \sinh(B + r \cdot y)^2\right)\right) \\ & \qquad \qquad \qquad 0 \end{aligned} \tag{1.2.6}$$

$$\begin{aligned} &> \text{simplify}\left(\left(\cosh(y) \cdot \text{TrTest} + 1\right) - \left(-\frac{1-p}{3 \cdot p} + \frac{1-p}{3p} \sqrt{\frac{2 \cdot (1-p)}{p+2}} \cdot \cosh(y) \cdot \cosh(B + r \cdot y)\right)\right) \\ & \qquad \qquad \qquad 0 \end{aligned} \tag{1.2.7}$$

Taylor expansion of the Laplace transform

The fact that the remainder terms in $O(x^4)$ can be bounded by Cx^4 for some $C > 0$ as soon as $x < x_0$, with C and x_0 that do not depend on r , is not a direct consequence of the following computations.

It can be checked by computing Taylor expansions of the function involved with a bound of the form Cx^4 (possibly multiplied by some power of r) on the remainder. We let the interested reader check it by themselves.

The only difficulty in doing so is the tediousness of the computations. We provide the Taylor expansion of some key steps below.

$$\begin{aligned} &> \text{simplify}\left(\text{subs}\left(u = 0, p = \frac{x}{r^2}, \text{TrTest}\right)\right) \end{aligned}$$

$$\frac{1}{3x} \left((r^2 - x) \left(-1 + \sqrt{2} \sqrt{\frac{r^2 - x}{2r^2 + x}} \cosh \left(r \operatorname{arccosh} \left(\frac{r^2 + 2x}{r^2 - x} \right) \right. \right. \right. \right. \right. \\ \left. \left. \left. \left. + \operatorname{arccosh} \left(\frac{\sqrt{2} \sqrt{\frac{2r^2 + x}{r^2 - x}} (r^2 + 5x)}{2r^2 - 2x} \right) \right) \right) \right) \right) \quad (1.3.1)$$

> *factor*(*simplify*(*series*($\frac{r}{(r+1) \cdot (r+2) \cdot (r+3)}$ (1.3.1), $x=0, 5$)) assuming $x > 0$)

$$\frac{r}{r+3} + \frac{1}{2}x + \frac{1}{10} \frac{r^2 + 3r + 1}{r^2} x^2 + \frac{1}{280} \frac{3r^4 + 18r^3 + 41r^2 + 42r + 36}{r^4} x^3 + \quad (1.3.2)$$

$$O(x^4)$$

> *simplify*(*subs*($u=0, p=\frac{x}{r^2}, \frac{\partial}{\partial u} \operatorname{TrTest}$))

$$\left(4 \sqrt{\frac{r^2 - x}{2r^2 + x}} \sqrt{\frac{2r^2 + x}{r^2 - x}} \sinh \left(r \operatorname{arccosh} \left(\frac{r^2 + 2x}{r^2 - x} \right) \right. \right. \right. \right. \quad (1.3.3)$$

$$\left. \left. \left. \left. + \operatorname{arccosh} \left(\frac{\sqrt{2} \sqrt{\frac{2r^2 + x}{r^2 - x}} (r^2 + 5x)}{2r^2 - 2x} \right) \right) \right) \right) /$$

$$\left(\sqrt{\frac{\sqrt{2} \sqrt{\frac{2r^2 + x}{r^2 - x}} (r^2 + 5x) - 2r^2 + 2x}{r^2 - x}} \right)$$

$$\sqrt{\frac{\sqrt{2} \sqrt{\frac{2r^2 + x}{r^2 - x}} (r^2 + 5x) + 2r^2 - 2x}{r^2 - x}}$$

> *factor*(*simplify*(*series*($\frac{3r}{2(r+3) \cdot (2 \cdot r + 3)}$ (1.3.3), $x=0, 4$)) assuming $x > 0$)

$$\frac{r}{r+3} + x + \frac{1}{10} \frac{3r^2 + 9r - 4}{r^2} x^2 + \frac{1}{70} \frac{3r^4 + 18r^3 + 21r^2 - 18r + 46}{r^4} x^3 + \quad (1.3.4)$$

$$O(x^4)$$

> $P_{sir} := 1 - (1-p) \cdot \left(1 - \frac{2}{\operatorname{TrTest}} \right)$

$$P_{sir} := 1 - (1-p) \left(1 - (6p) / \left((1-p) \left(-1 \right. \right. \right. \right. \quad (1.3.5)$$

$$\left. \left. \left. \left. + \sqrt{2} \sqrt{\frac{1-p}{p+2}} \cosh \left(r \operatorname{arccosh} \left(\frac{2p+1}{1-p} \right) \right) + \operatorname{arccosh} \left(\sqrt{\frac{p+2}{2-2p}} \right) \right) \right) \right)$$

Bibliography

- [1] Céline Abraham. Rescaled bipartite planar maps converge to the Brownian map. In *Ann. Inst. Henri Poincaré Probab. Stat.*, volume 52, pages 575–595. Institut Henri Poincaré, 2016.
- [2] Louigi Addario-Berry and Marie Albenque. Convergence of odd-angulations via symmetrization of labeled trees. *arXiv preprint arXiv:1904.04786*, 2019.
- [3] Omer Angel. Growth and percolation on the Uniform Infinite Planar Triangulation. *Geom. Funct. Anal.*, 13(5):935–974, 2003.
- [4] Omer Angel, Brett Kolesnik, and Grégory Miermont. Stability of geodesics in the Brownian map. *Ann. Probab.*, 45(5):3451–3479, 2017.
- [5] Omer Angel and Oded Schramm. Uniform infinite planar triangulations. *Comm. Math. Phys.*, 241(2-3):191–213, 2003.
- [6] Cyril Banderier, Philippe Flajolet, Gilles Schaeffer, and Michele Soria. Random maps, coalescing saddles, singularity analysis, and Airy phenomena. *Random Structures Algorithms*, 19(3-4):194–246, 2001.
- [7] Jerome Barkley and Timothy Budd. Precision measurements of Hausdorff dimensions in two-dimensional quantum gravity. *arXiv preprint arXiv:1908.09469*, 2019.
- [8] Louigi Addario Berry and Marie Albenque. The scaling limit of random simple triangulations and random simple quadrangulations. *Ann. Probab.*, 45(5):2767–2825, 2017.
- [9] Jérémie Bettinelli, Emmanuel Jacob, and Grégory Miermont. The scaling limit of uniform random plane maps, via the Ambjørn–Budd bijection. *Electron. J. Probab.*, 19, 2014.
- [10] Jérémie Bettinelli and Grégory Miermont. Compact Brownian surfaces i: Brownian disks. *Probab. Theory Related Fields*, 167(3-4):555–614, 2017.
- [11] Nicholas H Bingham, Charles M Goldie, and Jef L Teugels. *Regular variation*, volume 27. Cambridge university press, 1989.

- [12] Jérémie Bouttier, Philippe Di Francesco, and Emmanuel Guitter. Planar maps as labeled mobiles. *Electron. J. Combin.*, 11(1):69, 2004.
- [13] Jérémie Bouttier and Emmanuel Guitter. Confluence of geodesic paths and separating loops in large planar quadrangulations. *J. Stat. Mech. Theory Exp.*, 2009(03):P03001, 2009.
- [14] Thomas Budzinski. The hyperbolic Brownian plane. *Probab. Theory Related Fields*, 171(1-2):503–541, 2018.
- [15] Philippe Chassaing and Bergfinnur Durhuus. Local limit of labeled trees and expected volume growth in a random quadrangulation. *Ann. Probab.*, 34(3):879–917, 2006.
- [16] Philippe Chassaing and Gilles Schaeffer. Random planar lattices and integrated superbrownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.
- [17] J Theodore Cox and Richard Durrett. Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.*, 9(4):583–603, 1981.
- [18] Nicolas Curien. Planar stochastic hyperbolic triangulations. *Probab. Theory Related Fields*, 165(3-4):509–540, 2016.
- [19] Nicolas Curien, Tom Hutchcroft, and Asaf Nachmias. Geometric and spectral properties of causal maps. *arXiv preprint arXiv:1710.03137*, 2017.
- [20] Nicolas Curien and Jean-François Le Gall. The Brownian plane. *J. Theoret. Probab.*, 27(4):1249–1291, 2014.
- [21] Nicolas Curien and Jean-François Le Gall. The hull process of the Brownian plane. *Probab. Theory Related Fields*, 166(1-2):187–231, 2016.
- [22] Nicolas Curien and Jean-François Le Gall. Scaling limits for the peeling process on random maps. In *Ann. Inst. Henri Poincaré Probab. Stat.*, volume 53, pages 322–357. Institut Henri Poincaré, 2017.
- [23] Nicolas Curien and Jean-François Le Gall. First-passage percolation and local modifications of distances in random triangulations. *Ann. Sci. Éc. Norm. Supér.*, 52:631–701, 2019.
- [24] Nicolas Curien and Laurent Ménard. The skeleton of the UIPT, seen from infinity. *Ann. H. Lebesgue*, 2018.
- [25] Nicolas Curien, Laurent Ménard, and Grégory Miermont. A view from infinity of the Uniform Infinite Planar Quadrangulation. *Lat. Am. J. Probab. Math. Stat.*, 10(1):45–88, 2013.
- [26] Nicolas Curien and Grégory Miermont. Uniform Infinite Planar Quadrangulations with a boundary. *Random Structures Algorithms*, 47(1):30–58, 2015.

- [27] Sorin Dumitrescu and Pierre Pansu. *Cours de Géométrie Différentielle*.
- [28] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.
- [29] Zhicheng Gao and Nicholas C Wormald. The distribution of the maximum vertex degree in random planar maps. *J. Combin. Theory Ser. A*, 89(2):201–230, 2000.
- [30] Ori Gurel-Gurevich and Asaf Nachmias. Recurrence of planar graph limits. *Ann. of Math. (2)*, pages 761–781, 2013.
- [31] Ewain Gwynne and Tom Hutchcroft. Anomalous diffusion of random walk on random planar maps. *arXiv preprint arXiv:1807.01512*, 2018.
- [32] Ewain Gwynne and Jason Miller. Scaling limit of the uniform infinite half-plane quadrangulation in the Gromov-Hausdorff-Prokhorov-uniform topology. *Electron. J. Probab.*, 22, 2017.
- [33] Ewain Gwynne and Jason Miller. Existence and uniqueness of the Liouville quantum gravity metric for γ in $(0, 2)$. *arXiv preprint arXiv:1905.00383*, 2019.
- [34] Svante Janson and Jean-François Marckert. Convergence of discrete snakes. *J. Theoret. Probab.*, 18(3):615–645, 2005.
- [35] Mark Jerrum and Alistair Sinclair. Approximating the permanent. *SIAM J. Comput.*, 18(6):1149–1178, 1989.
- [36] Harry Kesten. Aspects of first passage percolation. In *École d’été de probabilités de Saint Flour XIV-1984*, pages 125–264. Springer, 1986.
- [37] Maxim A Krikun. Uniform infinite planar triangulation and related time-reversed critical branching process. *J. Math. Sci.*, 131(2):5520–5537, 2005.
- [38] Maxim A Krikun. Local structure of random quadrangulations. *arXiv:math/0512304v2*, 2008.
- [39] Jean-François Le Gall. Geodesics in large planar maps and in the Brownian map. *Acta Math.*, 205(2):287–360, 2010.
- [40] Jean-François Le Gall. Uniqueness and universality of the Brownian map. *Ann. Probab.*, 41(4):2880–2960, 2013.
- [41] Jean-François Le Gall. Uniqueness and universality of the Brownian map. *Ann. Probab.*, 41(4):2880–2960, 2013.
- [42] Jean-François Le Gall. Brownian disks and the Brownian snake. 55(1):237–313, 2019.
- [43] Jean-François Le Gall and Thomas Lehericy. Separating cycles and isoperimetric inequalities in the Uniform Infinite Planar Quadrangulation. *Ann. Probab.*, 47(3):1498–1540, 2019.

- [44] Jean-François Le Gall and Grégory Miermont. Scaling limits of random trees and planar maps. *Probability and statistical physics in two and more dimensions*, 15:155–211, 2012.
- [45] Jean-François Le Gall and Frédéric Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.*, 18(3):893–918, 2008.
- [46] Thomas Lehericy. First-passage percolation in random planar maps and Tutte’s bijection. *arXiv preprint arXiv:1906.10079*, 2019.
- [47] David A Levin and Yuval Peres. *Markov chains and mixing times*, volume 107. American Mathematical Soc., 2017.
- [48] Thomas M Liggett. An improved subadditive ergodic theorem. *Ann. Probab.*, pages 1279–1285, 1985.
- [49] Russell Lyons and Yuval Peres. *Probability on trees and networks*, volume 42. Cambridge University Press, 2017.
- [50] Cyril Marzouk. Scaling limits of random bipartite planar maps with a prescribed degree sequence. *Random Structures Algorithms*, 53(3):448–503, 2018.
- [51] Cyril Marzouk. Brownian limits of planar maps with a prescribed degree sequence. *arXiv preprint arXiv:1903.06138*, 2019.
- [52] Laurent Ménard. The two uniform infinite quadrangulations of the plane have the same law. In *Ann. Inst. Henri Poincaré Probab. Stat.*, volume 46, pages 190–208. Institut Henri Poincaré, 2010.
- [53] Laurent Ménard. Volumes in the Uniform Infinite Planar Triangulation: from skeletons to generating functions. *Combin. Probab. Comput.*, pages 1–28, 2018.
- [54] Laurent Ménard and Pierre Nolin. Percolation on uniform infinite planar maps. *Electron. J. Probab.*, 19, 2014.
- [55] Grégory Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.*, 210(2):319–401, 2013.
- [56] Grégory Miermont. Aspects of random maps. *Saint-Flour lecture notes*, 2014.
- [57] Jason Miller and Scott Sheffield. Liouville quantum gravity and the Brownian map III: the conformal structure is determined. *arXiv preprint arXiv:1608.05391*, 2016.
- [58] Armand Riera. Isometric inequalities in the brownian map and the brownian plane. in preparation.
- [59] William Thomas Tutte. A census of hamiltonian polygons. *Canad. J. Math*, 14:402–417, 1962.

- [60] William Thomas Tutte. A census of planar triangulations. *Canad. J. Math*, 14:21–38, 1962.
- [61] William Thomas Tutte. A census of slicings. *Canad. J. Math*, 14:708–722, 1962.
- [62] William Thomas Tutte. A census of planar maps. *Canad. J. Math*, 15(2):249–271, 1963.
- [63] Mohammad Q Vahidi-Asl and John C Wierman. A shape result for first-passage percolation on the Voronoi tessellation and Delaunay triangulation. In *Random graphs*, volume 2, pages 247–262, 1992.

Titre : Cycles séparants, isopérimétrie et modifications de distances dans les grandes cartes planaires aléatoires

Mots Clefs : Géométrie brownienne, quadrangulations, cartes planaires, décomposition en squelette

Résumé : Les cartes planaires sont des graphes planaires dessinés sur la sphère et vus à déformation près. De nombreuses propriétés des cartes sont supposées universelles, dans le sens où elles ne dépendent pas des détails du modèle choisi. Nous commençons par établir une inégalité isopérimétrique dans la quadrangulation infinie du plan. Nous confirmons également une conjecture de Krikun portant sur la longueur des cycles les plus courts séparant la boule de rayon r de l'infini. Dans un deuxième temps, nous nous intéressons à l'effet de modifications de distances sur la géométrie à grande échelle des quadrangulations uniformes, élargissant la classe d'universalité de la carte brownienne. Nous montrons également que la bijection de Tutte, entre quadrangulations et cartes planaires, est asymptotiquement une isométrie. Enfin, nous établissons une borne supérieure sur le temps de mélange de la marche aléatoire dans les cartes aléatoires.

Title : Separating cycles, isoperimetry and modification of distances in large random planar maps

Keys words : Brownian geometry, quadrangulations, planar maps, skeleton decomposition

Abstract : Planar maps are planar graphs drawn on the sphere and seen up to deformation. Many properties of maps are conjectured to be universal, in the sense that they do not depend on the details of the model. We begin by establishing an isoperimetric inequality in the infinite quadrangulation of the plane. We also confirm a conjecture by Krikun concerning the length of the shortest cycles separating the ball of radius r from infinity. We then consider the effect of modifications of distances on the large-scale geometry of uniform quadrangulations, extending the universality class of the Brownian map. We also show that the Tutte bijection, between quadrangulations and planar maps, is asymptotically an isometry. Finally, we establish an upper bound on the mixing time of the random walk in random maps.

