

Analyse asymptotique d'équations intégro-différentielles : modèles d'évolution et de dynamique des populations

Florian Patout

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Analyse asymptotique d'équations intégro-différentielles : modèles d'évolution et de dynamique des populations

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There ain't nothin'
like a friend
Who can tell you
you're just pissin'
in the wind.

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Première partie

Introduction

Le fil rouge de cette thèse est l'étude asymptotique d'équations intégro-différentielles apparaissant en dynamique de populations ou pour la modélisation de l'évolution Darwinienne. En toute généralité, ces équations peuvent s'écrire sous la forme suivante, même si on s'autorisera des libertés par rapport à cette équation étalon.

$$\partial_t f(t,x) = \mathcal{B}(f)(t,x) - \mathcal{R}(f)(t,x). \tag{I.0.1}$$

Ces équations apparaissent communément en biologie et en écologie quand on veut modéliser le comportement et les interactions entre individus. L'ensemble de ces individus forme une population dont on veut étudier le comportement global, tel que sa croissance. La fonction f représente la densité de population évoluant avec le temps $t \in \mathbb{R}^+$, et structurée par une variable "d'espace" x dont l'interprétation variera avec les modèles. Dans tous les travaux de cette thèse l'opérateur $\mathcal{B}$ sera un opérateur intégral lié à la croissance au sens large de la population. Le terme de réaction $\mathcal{R}$ représentera les interactions de la population avec son environnement ou entre individus. Il sera au gré des modèles local ou non local, parfois linéaire, souvent non linéaire.

Les équilibres entre le terme de reproduction $\mathcal{B}$ et celui d'interaction $\mathcal{R}$ donnent naissance à des formes singulières comme par exemple la propagation spécifique à certains régimes ou des solutions qui se concentrent. Cela amène à transformer l'équation (I.0.1) en une équation rééchelonnée par un paramètre ε :

$$\partial_t f_\varepsilon(t,x) = \mathcal{B}_\varepsilon(f_\varepsilon)(t,x) - \mathcal{R}_\varepsilon(f_\varepsilon)(t,x). \tag{I.0.2}$$

Pour mesurer ces phénomènes singuliers, nous serons amenés à effectuer une analyse asymptotique que nous choisirons toujours de dénoter dans cette thèse "$\varepsilon \to 0$". La limite $f_0 := \lim_{\varepsilon\to 0} f_\varepsilon$ permet de capturer le comportement écologique ou biologique qui est pertinent. Si l'asymptotique est judicieuse, il est raisonnable de croire que le comportement des solutions f_ε pour ε positif mais petit – $\varepsilon \ll 1$– est qualitativement proche du comportement de f_0 *i.e.* à $\varepsilon = 0$.

Cette thèse a été consacrée à rigoureusement justifier le passage à la limite $\varepsilon \to 0$ pour des équations de réaction-diffusion ou la dispersion prend la forme d'un noyau à queue lourde et pour une équation de sélection-mutation avec un opérateur intégral non-linéaire modélisant la reproduction sexuée. Ceci permet d'apporter un soutien théorique aux nombreuses analyses qualitatives basées sur l'approximation $\varepsilon \to 0$ que nous ferons.

Nous tentons d'illustrer cette démarche sur la figure I.1. La plupart du temps, le raisonnement rigoureux amenant à faire le lien du problème à $\varepsilon > 0$ vers le problème limite (flèche orange inférieure) se nourrit d'une compréhension fine du problème à $\varepsilon = 0$ (flèche orange supérieure).

Nous présentons tout d'abord sur des équations de propagation spatiale de type réaction-dispersion un exemple historique de mise en œuvre de cette démarche, agrémentée des premiers résultats obtenus dans cette thèse. Dans un second temps nous présenterons des travaux sur la dynamique d'une reproduction sexuée où les outils classiques ne s'appliqueront plus, mais où la démarche de la figure I.1 prend tout son sens.

I.1 DYNAMIQUES DE PROPAGATION

Le but de cette partie est d'expliquer les phénomènes de propagation dans l'équation de réaction dispersion non locale suivante :

$$\partial_t f(t,x) = \Big(J * f - f\Big)(t,x) + f(t,x)(1 - f(t,x)), \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{I.1.1}$$

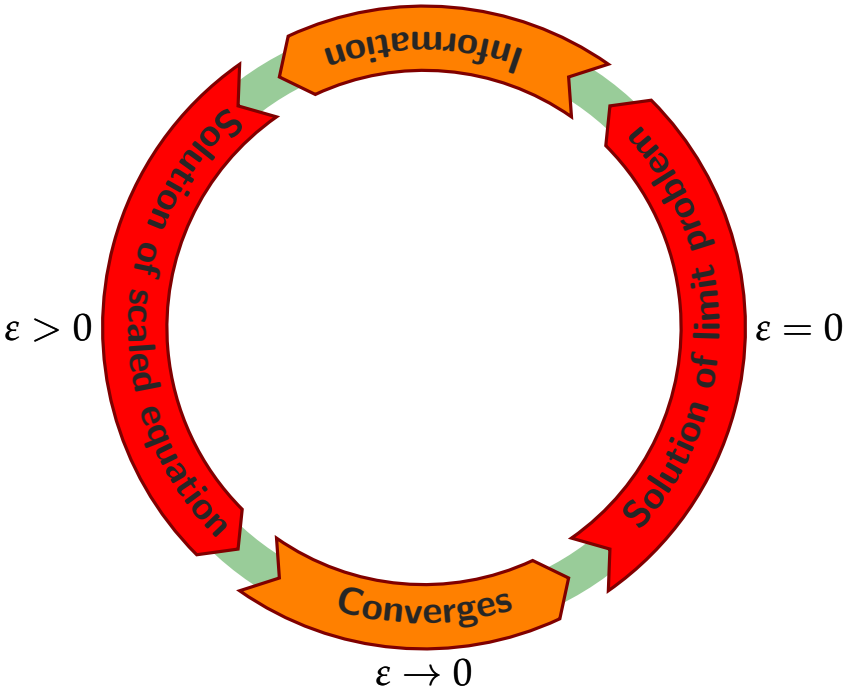


FIGURE I.1 : Math flow

où J désigne un noyau de probabilité et pour toute fonction n :

$$(J * n)(t,x) := \int_{\mathbb{R}} J(x-y)n(t,y)\,dy.$$

I.1.1 *Propagation spatiale*

On considère ici que la variable $x \in \mathbb{R}$ représente une position spatiale. Le modèle (I.1.1) permet de décrire l'invasion d'un environnement donné par une population. Cette invasion est possible grâce à la capacité de dispersion de la population caractérisée par l'opérateur de de convolution : un individu parvient à se déplacer d'une position x à une position $x+h$, et donc d'effectuer un saut de taille $h \in \mathbb{R}$ avec une fréquence $J(h)$. Ainsi la queue de distribution du noyau J permet en particulier de mesurer la fréquence des évènements de dispersion à longue distance ($h \gg 1$). Ces phénomènes de dispersion à longue distance peuvent jouer un rôle important dans la modélisation de la propagation d'une épidémie ou de la pollinisation de certaines plantes, Skellam [1951]; Turchin [1998]; Clark [1998]. Dans ces cas là ils s'expliquent souvent par un vecteur de transport extérieur à la population (certains animaux, moyens de transports humains à longue distance). Citons Roques [2013] pour une très belle introduction au sujet.

De plus, la croissance de la population qui se disperse selon le noyau J est régie par le terme $f(1-f)$. Cette non linéarité permet de modéliser les interactions des individus entre eux ou avec l'habitat. Dans notre cas il est dit "de type KPP" pour Kolmogorov, Petrovsky et Piskounov, Fisher [1937]; Kolmogorov et al. [1937], mais des choix différents peuvent être faits, menant à des comportements de solution radicalement différents, citons comme exemple le cas de l'effet Allee fort pour $f(1-f)(f-\rho)$ avec $0 < \rho < 1$ une certaine constante, voir Lewis and Kareiva [1993] pour

une motivation biologique, Aronson and Weinberger [1975]; Fife and McLeod [1977]; Xin [2000] pour des arguments mathématiques. On peut citer également une non linéarité du style ignition, par exemple Berestycki et al. [1985]; Berestycki and Larrouturou [1988]; Roques [2005] ou encore le cas de l'effet Allee faible, Aronson and Weinberger [1975]. Dans notre cas simple, il représente une compétition locale entre individus, par exemple pour accéder à une ressource.

On observe que l'équation (I.1.1) admet deux états d'équilibres homogènes : $f = 1$ et $f = 0$. Pour le premier, la population réside dans l'habitat, pour le second la population disparaît. En l'absence de dispersion, l'état 0 est instable, l'état 1 est stable, puisqu'on est réduit à l'équation logistique :

$$\frac{d\rho(t)}{dt} = \rho(t)(1-\rho(t)).$$

On cherche à savoir dans le cas général, si, partant d'une densité initiale $0 \leqslant f_0(x) \leqslant 1$, la population f solution de l'équation (I.0.2) envahit l'espace, ce que l'on écrit souvent comme "l'état 1 envahit l'état 0", ainsi que la vitesse du phénomène. Pour relier les deux états stationnaire, on cherche la solution sous forme de front progressif. Cela revient à chercher f sous la forme :

$$f(t,x) := F(x-ct), \tag{I.1.2}$$

où c, la vitesse de propagation, et F est le profil du front connectant l'état 1 à l'état 0. Le problème possède deux inconnues : c et $F \geqslant 0$. La littérature sur l'existence ou non de fronts solutions de (I.1.1) est gigantesque, notre but ici n'est pas de rappeler tous les résultats mais d'illustrer sur quelques cas simples la méthodologie que nous utiliserons plus tard.

Le cas diffusif

Dans le cas où l'opérateur $J * n - n$ est remplacé par le terme diffusif Δf dans l'équation (I.1.1), on retrouve l'équation de Fisher-KPP. Supposons que le noyau de probabilité est J_ε que l'on peut écrire sous la forme suivante :

$$J_\varepsilon(x) := \frac{1}{\varepsilon} J\left(\frac{x}{\varepsilon}\right), \quad \int_{\mathbb{R}} J(x')\,dx' = \int_{\mathbb{R}} x'^2 J(x')\,dx' = 1, \quad J(-x) = J(x) \text{ pour tout } x \in \mathbb{R}.$$

Avec ces notations ε représente la variance du noyau de propagation. Alors, quitte à négliger les termes d'ordre supérieurs en ε, quand ε est petit on a, si f est suffisamment régulière :

$$\frac{1}{\varepsilon^2}\Big(J_\varepsilon * f - f_\varepsilon\Big)(t,x) \xrightarrow[\varepsilon\to 0]{} \frac{1}{2}\partial_x^2 f_\varepsilon(t,x).$$

On utilise pour cette limite la symétrie de J. Cette égalité est la raison pour laquelle nous rappellerons quelques résultats à propos de l'équation de Fisher-KPP :

$$\partial_t f(t,x) = \Delta f(t,x) + f(t,x)(1-f(t,x)). \tag{I.1.3}$$

On sait depuis Aronson and Weinberger [1975]; Kolmogorov et al. [1937]; Fisher [1937] qu'il existe des fronts progressifs solutions de l'équation (I.1.3) si et seulement si

$$c \geqslant c^* := 2. \tag{I.1.4}$$

De plus, pour chaque $c \geqslant c^*$, le profil est unique à translation près. Pour démontrer ce résultat, on utilise en général la décroissance du front pour linéariser autour de sa limite en $+\infty$, *i.e.* de

l'état instable 0. Ceci donne un équation différentielle ordinaires (EDO) d'ordre 2, dont l'étude du plan de phase est suffisante pour obtenir la condition nécessaire et suffisante (I.1.4), voir Aronson and Weinberger [1975]; Gardner [1986]. Néanmoins nous ne détaillerons pas cette construction fondamentale, car nous nous intéresserons seulement à la vitesse de propagation des fronts. Dans le cas de Fisher-KPP, cette vitesse est constante : le front avance linéairement dans sa direction de propagation. La vitesse la plus "naturelle" est c^*, et ce pour plusieurs raisons. Tout d'abord, pour une large classe de données initiales, dont les fonctions à support compact, la solution du problème de Cauchy associé à l'équation (I.1.3) converge en temps grand vers le front de vitesse c^*, Fife and McLeod [1977]. De plus, c^* est également caractérisée par le résultat de propagation ("spreading") suivant :

$$f(t, x + ct) \xrightarrow[t\to+\infty]{} 1, \text{ localement uniformément, pour tout } c < c^*, \tag{I.1.5}$$

$$f(t, x + ct) \xrightarrow[t\to+\infty]{} 0, \text{ localement uniformément, pour tout } c > c^*. \tag{I.1.6}$$

La vitesse de propagation, ici c^*, est si elle existe la plus grande vitesse telle que pour toute vitesse inférieure la solution dans le repère mobile converge vers 1 en temps grand. De plus, c'est la plus petite vitesse dont la solution dans le repère mobile converge vers 0. Cette notion est introduite dans un contexte très général par Weinberger [1982]. Dans le cas de l'équation (I.1.3), une transition entre l'état 1 et l'état 0 s'effectue donc à vitesse c^*. Par la suite nous nous focaliserons sur cette vitesse de propagation, et nous expliquons désormais un moyen asymptotique de la caractériser, via l'approximation de l'optique géométrique. Cette méthode que nous illustrons sur le cas particulier de l'équation de Fisher-KPP est très représentative des techniques asymptotiques développées dans cette thèse.

Vitesse de propagation par l'optique géométrique

On introduit un paramètre $\varepsilon > 0$ dans l'équation (I.1.3). Le but est de capturer la vitesse de propagation grâce à un changement d'échelle approprié. Pour cela on regarde donc les solutions dans un nouveau référentiel temps/espace, parfois appelé changement d'échelle hyperbolique :

$$(t, x) \longrightarrow \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right). \tag{I.1.7}$$

Pour des ε petits, cela revient à regarder de très loin une solution de équation (I.1.3) en temps grand. Le temps et l'espace sont renormalisés de la même manière, ce qui a du sens car dans les résultats (I.1.5) et (I.1.6), x et t doivent être du même ordre. Alors (I.1.3) devient, avec un léger abus de notations :

$$\varepsilon \partial_t f_\varepsilon(t, x) = \varepsilon^2 \Delta f_\varepsilon(t, x) + f_\varepsilon(t, x)(1 - f_\varepsilon(t, x)). \tag{I.1.8}$$

Formellement, en laissant $\varepsilon \to 0$ dans la précédente équation, on voit que $f_0 := \lim_{\varepsilon\to 0} f_\varepsilon$ doit nécessairement être un des deux états stationnaires, 1 ou 0, car les termes différentiels disparaissent. De plus, toujours formellement, on retrouve cette convergence si on pense en terme de fronts, Freidlin [1985]. En effet, avec la notation de l'équation (I.1.2), on a pour les solutions particulières de type onde progressive :

$$f_\varepsilon(t, x) = F\left(\frac{x - ct}{\varepsilon}\right) \xrightarrow[\varepsilon\to 0]{} \mathbf{1}_{\mathbb{R}^-}(x - ct).$$

Dans les deux cas, si on retrouve bien les deux états stationnaires, comme attendu, la vitesse de propagation n'apparaît pas. En revanche, on constate que f_ε peut devenir singulier. En suivant l'idée de Freidlin [1985]; Evans and Souganidis [1989]; Barles et al. [1990], on effectue alors la transformation de Hopf-Cole suivante :

$$f_\varepsilon(t,x) = \exp\left(-\frac{u_\varepsilon(t,x)}{\varepsilon}\right). \tag{I.1.9}$$

On peut la justifier de plusieurs manières différentes. A posteriori, il s'avère que c'est la bonne échelle pour regarder l'équation (I.1.8). A priori, on peut remarquer que si on regarde la même équation sans le terme de réaction, on retrouve l'équation de la chaleur dont la solution fondamentale prend la forme (I.1.9). Tout ce qu'on expliquera pour la suite s'appliquera d'ailleurs à cette équation. On peut ajouter que dans l'approche "classique" de l'équation de Fisher-KPP, les fronts, qui sont solutions d'une EDO, sont cherchés sous forme exponentielle pour le problème linéarisé. En injectant l'ansatz (I.1.9) dans l'équation (I.1.8), et en divisant par f_ε on obtient :

$$\partial_t u_\varepsilon(t,x) + |\partial_x u_\varepsilon(t,x)|^2 + 1 - f_\varepsilon(t,x) = 0. \tag{I.1.10}$$

Sur cette équation en apparence plus compliquée que l'initiale, on peut en fait rigoureusement passer à la limite quand $\varepsilon \to 0$. Cela est possible grâce à la théorie des solutions de viscosité (Crandall and Lions [1983]; Crandall et al. [1984]; Barles [1994]). On obtient que u_ε converge localement uniformément vers u_0, définie comme l'unique solution de l'équation de Hamilton Jacobi contrainte suivante :

$$\min\left(\partial_t u_0(t,x) + |\partial_x u_0(t,x)|^2 + 1, u_0(t,x)\right) = 0. \tag{I.1.11}$$

Grâce à la structure des équations de Hamilton Jacobi, il existe une formule de représentation pour u_0 aux points de positivité :

$$u_0(t,x) = \inf_{y\in\mathbb{R}}\left\{\frac{|x-y|^2}{4t} - t + u_0(0,y)\right\}. \tag{I.1.12}$$

Grâce à u_0 on peut retrouver des informations à propos de la propagation de f, Fleming and Souganidis [1986]; Evans [1989]; Barles and Souganidis [1994]. La dichotomie entre 1 et 0 les deux états stationnaires de (I.1.3) est remplacée grâce à la transformation de Hopf-Cole (I.1.9) en une discussion sur le signe de u_ε. En effet, quand u_ε est nulle, f_ε vaut 1, et dès que u_0 est positive et ε petit, f_ε est très proche de 0. On essaie de représenter cette dynamique d'interface sur le dessin suivant, pour la dimension supérieure à 1 :

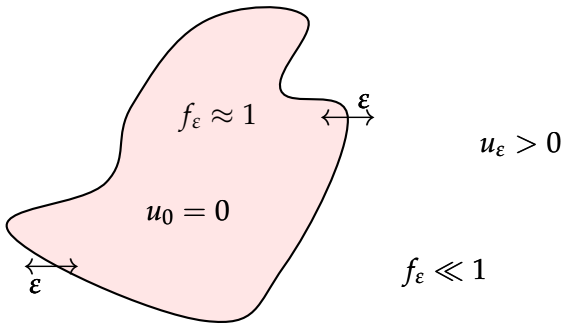


FIGURE I.2 : Lien entre le comportement de u_ε et f_ε quand ε est petit.

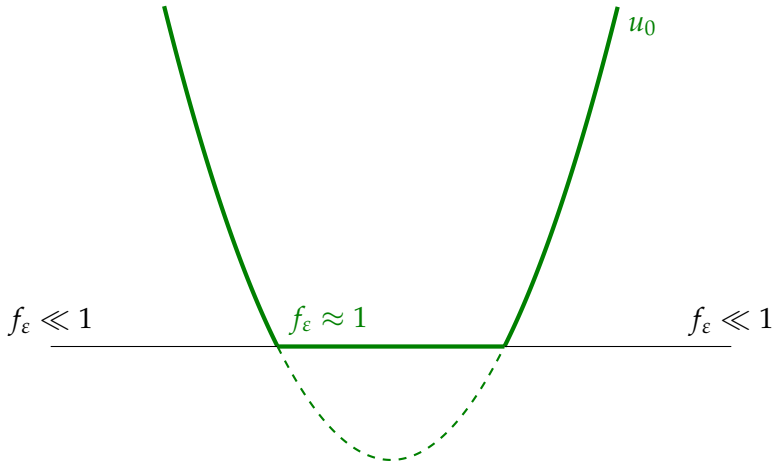


FIGURE I.3 : Illustration du comportement de la solution de l'équation de Hamilton-Jacobi (I.1.10).

Quand $\varepsilon \to 0$, on s'attend donc à capter le mouvement de la frontière entre les deux états stationnaires. C'est en effet le cas, grâce à (I.1.12), on peut formellement calculer les lignes de niveau de u_0, et les relier à la vitesse de propagation. Le lien entre v_0, u_0 et f_ε peut être résumé sur la figure I.3. Par la suite nous expliquerons comment généraliser cette approche aux équation non locales du type (I.1.1).

Vitesse de propagation dans le cas non local

La nature de l'équation (I.1.1) est complètement différente de celle de (I.1.3). Néanmoins leurs comportements respectifs de propagation peuvent être similaires ou non, selon la taille de la queue de distribution du noyau J. Il a été prouvé dans les travaux de Carr and Chmaj [2004], Coville and Dupaigne [2005] que les solutions de (I.1.1) se propagent à vitesse constante comme les solutions de l'équation de Fisher-KPP, à condition que le noyau soit exponentiellement borné, on parle également de noyau à queue de distribution légère. Cette hypothèse sur J signifie que

$$\exists \eta > 0 \text{ tel que } \int_{\mathbb{R}} J(h)\exp(\eta h)\,dh < +\infty. \tag{I.1.13}$$

Nous nous intéresserons plutôt au cas où le noyau de dispersion J est exponentiellement non borné *i.e.* "à queue lourde, c'est à dire que la condition (I.1.13) n'est pas vérifiée. Dans ce cas il n'existe pas de solution de type front progressif Yagisita [2009], et des études numériques ont mis en lumière que les solutions se propageaient à une vitesse infinie Kot et al. [1996]; Medlock and Kot [2003]. Ces résultats ont été rendus rigoureux par Garnier [2011] pour le problème de Cauchy associé à (I.1.1) avec un noyau de diffusion à queue lourde. Plus précisément l'auteur a démontré qu'il fallait suivre les lignes de niveau de la solution f, pour $\lambda \in (0,1)$

$$E_\lambda(t) := \{x \in \mathbb{R} : f(t,x) = \lambda\}. \tag{I.1.14}$$

Il a alors montré qu'il était possible de mesurer l'accélération des solutions. Il y est prouvé qu'il existe une constante $\rho > 1$ telle que pour tout $\lambda \in (0,1)$ et pour tout $\varepsilon > 0$, tout élément x_λ de l'ensemble E_λ vérifie à $\varepsilon > 0$ et t suffisamment grand :

$$J^{-1}\left(e^{-(1-\varepsilon)t}\right) \leqslant |x_\lambda(t)| \leqslant J^{-1}\left(e^{-\rho t}\right). \tag{I.1.15}$$

Ce résultat est également à mettre en parallèle avec les résultats de Hamel and Roques [2010]. Les auteurs montrent que les solutions de l'équation de Fisher-KPP partant d'une donnée initiale à queue lourde ont une vitesse de propagation infinie. Les méthodes de preuve de Garnier [2011] se rapprochent de celles de Hamel et Roques : dans le cas de Garnier [2011] la donnée initiale à support compact devient exponentiellement non bornée (comme J) en temps court, et ressemble alors à la donnée initiale exponentiellement non bornée couplée à la diffusion de Hamel and Roques [2010].

Le premier résultat de cette thèse est d'établir des bornes plus précises que celles de Garnier [2011]. On suppose que f est solution de (I.1.1), avec J à queue lourde, et quelques conditions de décroissance sur la queue de distribution, qu'on peut écrire grossièrement quand $|x|$ est grand :

$$\exp(-x) \lesssim J(x) \lesssim \frac{1}{x}. \tag{I.1.16}$$

On démontre alors le résultat suivant :

Théorème 1 (Bouin, Garnier, Henderson, Patout, 2018)**.**
Tout élément x_λ d'une ligne de niveau de (I.1.1) *définie par* (I.1.14)*, vérifient, pour tout $t > 0$, et $0 < \lambda < 1$:*

$$|x_\lambda(t)| = J^{-1}\left(e^{-t}\right) + o_{t\to\infty}\left(J^{-1}\left(e^{-t}\right)\right) \tag{I.1.17}$$

Ce résultat est plus précis que les précédentes estimations de Garnier [2011].

Un des objectifs de cette thèse a été de relier ce résultat de propagation à une équation de Hamilton Jacobi sous-jacente, en utilisant les techniques d'optique géométrique présentées plus haut, introduites par Freidlin [1985]; Evans and Souganidis [1989]. Cela a déjà été fait dans le cadre d'un noyau exponentiellement borné, voir Perthame and Souganidis [2005].

Une première étape est de proposer un re-scaling adéquat de l'équation (I.1.1), afin de définir correctement f_ε. Grâce aux estimations de Garnier (I.1.15), et au Théorème 1, n'importe quel point x d'une ligne de niveau doit vérifier quand t est grand

$$J(x) \sim e^{-t}. \tag{I.1.18}$$

On s'intéresse au comportement en temps long donc on accélère le temps : $\frac{t}{\varepsilon}$. Alors on cherche ψ_ε le changement de variable, qui doit donc d'après ce qui précède nécessairement vérifier

$$J(\psi_\varepsilon(x)) \sim e^{-\frac{t}{\varepsilon}} \sim \left(e^{-t}\right)^{\frac{1}{\varepsilon}} \sim J(x)^{\frac{1}{\varepsilon}}.$$

Pour que cela continue à être vrai le long des lignes de niveau cela impose

$$(x,t) \longrightarrow \left(\psi_\varepsilon(x), \frac{t}{\varepsilon}\right), \quad \psi_\varepsilon(x) := \operatorname{sign}(x) J^{-1}\left(J(x)^{\frac{1}{\varepsilon}}\right). \tag{I.1.19}$$

Alors, $f_\varepsilon := f(\psi_\varepsilon(x), t/\varepsilon)$ vérifie

$$\begin{cases} \varepsilon\partial_t f_\varepsilon = \displaystyle\int_{-\infty}^{\infty} J(h)\left[f_\varepsilon\left(t, \psi_\varepsilon^{-1}\left(\psi_\varepsilon(x) - h\right)\right) - f_\varepsilon\right] dh \; + \; f_\varepsilon\left(1 - f_\varepsilon\right) & \text{sur } (0,\infty)\times\mathbb{R}, \\ f_\varepsilon(0,x) = f(0, \psi_\varepsilon(|x|)), & \text{pour } x \in \mathbb{R}. \end{cases} \tag{I.1.20}$$

Alors on démontre le résultat suivant sur $u_\varepsilon := -\varepsilon \ln f_\varepsilon$, la transformée logarithmique introduite en (I.1.9) :

Théorème 2 (Bouin, Garnier, Henderson, Patout, 2018).
Si $f(0,x)$ est à support compact, et le noyau J vérifie des hypothèses techniques du type (I.1.16), *alors quand $\varepsilon \to 0$, la suite u_ε converge localement uniformément sur $(0,\infty)\times\mathbb{R}$ vers*

$$u_0(t,x) := \max\{-\log(J(x)) - t, 0\},$$

De plus, cela induit le résultat de propagation asymptotique suivant sur f_ε :

(a) Uniformément sur tous les compacts de $\{u_0 > 0\}$, on a

$$\lim_{\varepsilon\to 0} f_\varepsilon = 0;$$

(b) Pour tout sous ensemble compact $K \subset Int\,(\{u_0(t,x) = 0\})$, on a

$$\lim_{\varepsilon\to 0} f_\varepsilon(t,x) = 1,$$

la limite étant uniforme sur K.

Grâce à la formule explicite sur u_0, on peut calculer la frontière de $\{u(t,x) = 0\}$: $|x| = J^{-1}(\exp(-t))$. Ainsi quand $\varepsilon \to 0$, $f_\varepsilon \sim 1$ si et seulement si $|x| < J^{-1}(\exp(-t))$. On retrouve donc la formulation "réaction diffusion" du théorème 1, grâce aux méthodes d'optique géométrique, résumées sur la figure I.3 et la figure I.2.

La méthode de preuve usuelle des théorèmes similaires au théorème 2 dans Perthame and Barles [2008]; Perthame and Souganidis [2005] suivaient le plan de la "procédure des semis limites relaxées" (Viscosity semi relaxed procedure, introduite par Barles and Perthame [1988; 1990] voir aussi Crandall et al. [1992]; Barles [1994]; Barles and Perthame [2007]). Quitte à simplifier à l'extrême, les étapes de preuve sont les suivantes :

- ▷ On prouve d'abord que u_ε est localement uniformément bornée (temps, espace).
- ▷ On définit
$$\underline{u}(t,x) := \liminf_{\substack{\varepsilon\to 0\\(s,y)\to(t,x)}} u_\varepsilon(s,y), \qquad \overline{u}(t,x) := \limsup_{\substack{\varepsilon\to 0\\(s,y)\to(t,x)}} u_\varepsilon(s,y),$$
et on démontre grâce à l'équation sur f_ε que ce sont des sur et sous solutions.
- ▷ Par comparaison discontinue : $\overline{u} \leqslant \underline{u}$.
- ▷ Par construction, l'inverse est vraie $\underline{u} \leqslant \overline{u}$. Ainsi, on définit $u := \overline{u} = \underline{u}$. Alors *u est l'unique solution* aux sens des viscosités de l'équation limite, et $u_\varepsilon \to u$ quand $\varepsilon \to 0$.

Dans notre cas, nous n'avons pas pu appliquer cette méthodologie. Tout d'abord, les sur et sous solutions établies par Garnier [2011] n'étaient pas compatibles avec le re-scaling ψ_ε, et ne permettaient pas d'avoir des estimations suffisantes sur f_ε. Une raison est que l'opérateur de diffusion non local $J * f - f$ ne possède pas de noyau explicite *i.e.* de fonction p vérifiant $\partial_t p = J * p - p$. On peut néanmoins noter des estimations de Chasseigne et al. [2006] qui ne sont pas suffisantes pour caractériser l'accélération. Dans le cas de la diffusion, le noyau de la chaleur est bien connu, et fournit des estimations précises pouvant être utilisées pour étudier la propagation dans l'équation de

Fisher-KPP. Il existe également un noyau de la chaleur pour les solutions de l'équation non locale suivante utilisant le Laplacien fractionnaire, voir par exemple Sato et al. [1999] :

$$\partial_t f = -(-\Delta^{\frac{\alpha}{2}})f + f(1-f), \tag{I.1.21}$$

où l'opérateur fractionnaire est défini par

$$(-\Delta^{\frac{\alpha}{2}})f(t,x) = -\int_{\mathbb{R}} \frac{f(t,x+h)+f(t,x-h)-2f(t,x)}{h^{1+\alpha}}\,dh.$$

Cabré et Roquejoffre ont obtenu des résultats de propagation précis sur le comportement des lignes de niveau de ces solutions, Cabré and Roquejoffre [2013]. En particulier ils montrent que la vitesse de propagation de la solution de (I.1.21) associée à une donnée initiale bien préparée est $\exp(t/(1+\alpha))$. Il y a donc également accélération pour cette équation. Dans Méléard and Mirrahimi [2015], les auteures s'inspirent des sur et sous solutions de Cabré et Roquejoffre pour caractériser l'accélération. Nous avons voulu faire la même chose dans le cadre du théorème 2. Néanmoins, ne disposant pas d'estimations telles que celles de Cabré et Roquejoffre pour des noyaux à queue lourdes plus généraux que le Laplacien fractionnaire (mais également plus réguliers), nous avons établi une estimation plus précise qu'auparavant sur les solutions de (I.1.1) :

Théorème 3 (Bouin, Garnier, Henderson, Patout, 2018)**.**
Il existe une fonction positive θ, définie de $\mathbb{R}^+$ vers $\mathbb{R}^+$, qui dépend seulement de J, telle que $\theta(s) \to 0$ quand $s \to +\infty$, et des constantes positives $\underline{C} < 1 < \overline{C}$, telles que pour tout $t > 0$ et $x \in \mathbb{R}$, la fonction f solution de (I.1.1) vérifie

$$\underline{C}\exp\left(-\int_0^t \theta(s)\,ds\right)\phi(t,x) \leqslant f(t,x) \leqslant \overline{C}\exp\left(\int_0^t \theta(s)ds\right)\phi(t,x),$$

où la fonction ϕ est définie comme

$$\phi(t,x) := \frac{1}{1+e^{-t}/J(x)}.$$

Ainsi, le comportement de f est bien approché par les solutions de cette famille d'équations différentielles :

$$\begin{cases} \dfrac{d\phi(t,x)}{dt} = \phi(1-\phi), \\ \phi(0,\cdot) = \frac{J}{1+J} \leqslant J, \end{cases}$$

paramétrées par $x \in \mathbb{R}$, et dans laquelle la dispersion n'apparaît qu'à l'instant initial. Ceci indique que ces estimations seront compatibles avec le rescaling f_ε qui se concentre sur la propagation en oubliant la forme précise de f quand ε est petit.

I.1.2 *Modélisation des mutations phénotypiques*

Il s'agit désormais de jeter un regard neuf sur l'équation (I.1.1), dont on peut proposer une toute autre interprétation. On considère désormais que $x \in \mathbb{R}$ représente un trait phénotypique, c'est à dire un caractère quantifiable d'un individu, qui peut être morphologique, génétique ou comportemental (Losos [2013]). On considère donc à travers le modèle (I.1.1) l'évolution d'une population structurée par son trait, que l'on suppose donc pouvoir représenter par un nombre réel. La source de variation de

traits dans la population est supposée due aux mutations, modélisées par l'opérateur de convolution $J * f$. Les mutations sont des changements du phénotype. Les travaux fondateurs de Luria and Delbrück [1943] ont montré que des mutations aléatoires suffisaient à expliquer l'adaptation d'une population de bactéries *E.coli*. Dans le modèle (I.1.1) que nous considérons, les mutations sont justement tirées aléatoirement selon le noyau de mutation J, fixé.

Le but de cette partie est d'expliquer comment trouver une échelle appropriée pour observer l'effet des mutations dans la population. Cette échelle doit traduire l'hypothèse de modélisation que nous ferons, qui est que les mutations ont un petit effet. Un point de vue asymptotique a été pour la première fois introduit dans ce contexte évolutif par Diekmann et al. [2005]. De nombreux travaux existent sur une modélisation locale des mutations, ce qui se traduit par un opérateur diffusif pour obtenir l'équation de Fisher-KPP (I.1.3), voir Barles and Perthame [2007]; Lorz et al. [2011]. Ils utilisent le rescaling hyperbolique (I.1.7) pour étudier l'impact des mutations aux petits effets en temps long, ce rescaling coïncide donc avec celui correspondant à la recherche du front de propagation de la partie précédente. Le terme de réaction que nous avons utilisé pour illustrer notre propos est trop simple pour tirer des conclusions biologiques de l'équation (I.1.11), nous expliquerons ce qu'il se passe avec des modèles plus complexes (couplés à un effet de la sélection) dans la prochaine section. En revanche, l'obtention à la limite $\varepsilon \to 0$ d'une limite non triviale montre que l'asymptotique considérée a du sens pour mesurer l'effet des mutations.

Si on revient désormais à l'équation non locale (I.1.1), l'analyse asymptotique est différente dans le modèle de mutations comparé au modèle de propagation. Ainsi, dès les travaux de Diekmann et al. [2005], puis dans les travaux Perthame and Barles [2008]; Barles et al. [2009] le changement de fonction suivant $f_\varepsilon := f(t/\varepsilon, x)$ est introduit. Ainsi il n'y a pas de rescaling de l'espace des traits car le terme de réaction dépend également de x et on veut juste mesurer le régime des petits effets des mutations. L'hypothèse de départ est que la variance des mutations est de taille ε^2. On suppose donc que le noyau de diffusion prend la forme suivante :

$$J(h) \to J_\varepsilon(h) = \frac{1}{\varepsilon} J\left(\frac{h}{\varepsilon}\right). \tag{I.1.22}$$

Cela revient à supposer que les mutations sont distribuées autour d'un trait x selon $x + \varepsilon h$, avec h qui suit la loi de densité J. On observe bien un petit effet des mutations via le paramètre ε.

Cette approche ne donne pas de résultats dans le cas d'un noyau à queue lourde, comme remarqué en premier dans le cas du Laplacien fractionnaire dans l'article de Méléard and Mirrahimi [2015]. Les auteures avaient alors introduit le rescaling suivant sur la taille des mutations :

$$x + |h+1|^\varepsilon - 1.$$

Cette hypothèse de petites mutations est présentée dans leur article comme un changement de noyau, à l'image de l'équation (I.1.22).

De manière similaire, dans le cas de l'équation (I.1.1), nous avons proposé avec Emeric Bouin, Jimmy Garnier et Christopher K. Henderson le rescaling suivant des mutations :

$$x + \psi_\varepsilon^{-1}(h), \tag{I.1.23}$$

où ψ_ε est définie en (I.1.19). L'idée est donc de transformer la taille des mutations d'une manière compatible avec le noyau à queue lourde J. Or l'étude de propagation liée au Théorème 2 a mis en évidence le rôle pertinent de l'échelle "spatiale" $\psi_\varepsilon(x)$. Par le changement d'échelle (I.1.23), la taille des mutations tend bien vers 0 quand $\varepsilon \to 0$. En effet,

$$\psi_\varepsilon^{-1}(h) = J^{-1}\left(J(h)^\varepsilon\right).$$

Sous l'hypothèse nécessaire que $J(0) = 1$, et qu'il s'agit du seul point où J prend la valeur 1, on a bien :

$$\psi_\varepsilon^{-1}(h) \xrightarrow[\varepsilon\to 0]{} 0.$$

Cet argument utilise fortement la régularité en 0 du noyau J et explique la disparité apparente de scaling avec celui introduit dans Méléard and Mirrahimi [2015].

Dans la version publiée de ces travaux, Bouin et al. [2018], suite à une malencontreuse erreur de notation, il subsiste une erreur de présentation. Il faut bien lire que le changement de taille se fait selon ψ_ε^{-1}, comme dans (I.1.23), et non comme ψ_ε comme nous l'avons écrit à tort, voir (II.1.15). Par ailleurs les preuves par la suite étaient bien réalisées avec la formule pour ψ_ε^{-1}.

L'équation (I.1.23) est également équivalente à un changement de noyau peu esthétique J_ε. Enfin, nous remarquons que ce changement d'échelle ne correspond pas à supposer une variance de mutations de taille ε. Il s'agit de la quantité suivante qui est de taille ε, que l'on peut baptiser "twisted variance" :

$$\int_{\mathbb{R}} \Big(\ln J(x)\Big)^2 J_\varepsilon(x)\,dx = \varepsilon^2 \int_{\mathbb{R}} \Big(\ln J(x)\Big)^2 J(x)\,dx.$$

Cela illustre la différence entre la fréquence et la taille des mutations, toutes deux petites, et que l'on n'observe pas sur le changement d'échelle hyperbolique ou pour un noyau à queue légère. La transformation (I.1.23) conduit à considérer le problème suivant sur f_ε

$$\varepsilon\partial_t f_\varepsilon(t,x) = \int_{\mathbb{R}} \Big(f_\varepsilon(t, x-\psi_\varepsilon^{-1}(h)) - f_\varepsilon(t,x)\Big) J(|h|)dh + f_\varepsilon(1-f_\varepsilon). \tag{I.1.24}$$

Cette équation est de nature complètement différente que (I.1.20). On a vu que l'opérateur non local disparaît à la limite $\varepsilon \to 0$ dans (I.1.20). Ceci est le contraire de ce que l'on cherche à caractériser dans cette partie : le comportement asymptotique de l'opérateur de mutations. Grâce à la transformée de Hopf Cole, on prouve alors le résultat suivant :

Théorème 4 (Bouin, Garnier, Henderson, Patout, 18)**.**
Soit $u_\varepsilon := -\varepsilon \ln f_\varepsilon$, où f_ε est solution de (I.1.24), partant d'une donnée initiale suffisamment préparée. Alors, quand $\varepsilon \to 0$, la suite $(u_\varepsilon)_\varepsilon$ converge localement uniformément sur $(0,+\infty)\times\mathbb{R}$ vers u_0, une fonction Lipschitz par rapport à x et continue par rapport à t, mais aussi solution de viscosité de :

$$\begin{cases} \min\left\{\partial_t u_0 + \displaystyle\int_{\mathbb{R}} \left[e^{\frac{\mathrm{sign}(h)f(h)}{f'(0)}\partial_x u_0} - 1\right] J(h)\,dh + 1, u_0\right\} = 0, \quad \text{sur } (0,+\infty)\times\mathbb{R}, \\ u_0(0,\cdot) = u^0. \end{cases} \tag{I.1.25}$$

Pour établir ce résultat, on peut utiliser la méthode des semis limites relaxées, qui est maintenant possible, car le noyau (et la donnée initiale) ont été bien conditionnés. On obtient donc une équation non triviale à la limite pour u_ε. On peut en déduire des informations sur quels phénotypes sont réalisés dans la population f grâce au résultat suivant, qui va de pair avec le précédent :

Théorème 5 (Bouin, Garnier, Henderson, Patout, 18)**.**
Soit f_ε la solution de (I.1.24), partant d'une donnée initiale suffisamment bien préparée. Alors quand $\varepsilon \to 0$,

$$\begin{cases} f_\varepsilon \to 0 \text{ localement uniformément sur } \mathcal{A} = \{(t,x)\in(0,\infty)\times\mathbb{R} \mid u_0(t,x) > 0\}, \\ f_\varepsilon \to 1 \text{ localement uniformément sur } \mathcal{B} = Int\{(t,x)\in(0,\infty)\times\mathbb{R} \mid u_0(t,x) = 0\}. \end{cases} \tag{I.1.26}$$

Pour de petits ε, on s'attend à que $f_\varepsilon \sim 1$ sur $\mathcal{B}$, donc les individus qui au temps t, ont le phénotype x dans la population vérifient $(t,x) \in \mathcal{B}$. De même, le trait x au temps t ne sera pas présent dans la population si $(t,x) \in \mathcal{A}$. On en déduit que pour un effet de mutations suffisamment petit, les ensembles $\mathcal{A}$ et $\mathcal{B}$ déterminent les traits dans la population.

Dans un travail récent, Mirrahimi [2018], Sepideh Mirrahimi généralise ce type de résultat pour le Laplacien fractionnaire à des termes de réaction plus généraux autorisant une hétérogénéité en espace, au prix d'une notion de solution un peu plus faible que les solutions de viscosité classiques pour l'équation similaire à (I.1.25).

Les résultats présentés dans cette partie font l'objet de la partie II et ont été publiés, Bouin et al. [2018].

I.2 MODÈLES D'ÉVOLUTION DARWINIENNE

Nous avons discuté de résultats permettant de comprendre l'effet des mutations sur une population structurée en traits. Nous allons désormais utiliser des méthodes similaires pour coupler cet effet des mutations à la *sélection naturelle* afin de modéliser l'évolution. Bien qu'établie au rang de paradigme grâce au travaux de Darwin, Darwin [1859], le terme "évolution" n'y apparaît pourtant jamais dans les première éditions au profit de l'expression plus terre à terre "descent with modification". Darwin y introduit le mécanisme et la terminologie de "sélection naturelle", c'est à dire le procédé par lequel des individus avec un certain trait auront tendance à avoir plus de descendants dans la génération suivante, comparé à des individus avec un trait différent. Néanmoins la sélection naturelle n'est pas la même chose que le phénomène d'évolution c'est à dire la transformation d'une espèce au cours du temps. Pour que la sélection naturelle induise des changements évolutifs, il est nécessaire que les traits soient transmis d'une génération à l'autre. Si ce n'est pas le cas, la variance phénotypique peut s'expliquer par exemple par l'environnement et la sélection n'aura pas de conséquences évolutives. A l'inverse, il est bien établi que les mutations pouvaient changer le paysage phénotypique d'une population Luria and Delbrück [1943]. Dans cette thèse, nous avons tout particulièrement étudié à travers des modèles de mathématiques le lien entre différents mode de transmission du patrimoine génétique et l'adaptation d'une population. Il est tout à fait remarquable que Darwin n'avait pas d'idées claire sur comment se transmettait le patrimoine génétique. La (re) découverte des travaux de Mendel sur l'hérédité créa donc un schisme entre biométriciens défendant l'approche de Darwin et des modèles continus d'une part, et de l'autre les Mendeliens, croyant à des variations (discrètes) petites à l'échelle allélique, telles que décrites par les expériences de Mendel sur les pois. La synthèse fut réalisée plus tard, notamment par Ronald Fisher Fisher [1919; 1930]. Nous nous sommes essentiellement intéressés dans cette partie de la thèse à un modèle de Fisher pour la reproduction sexuée, qui a justement montré grâce à des arguments mathématiques qu'un nombre infini de petits changements alléliques pouvaient mener à une variation continue du phénotype, via son modèle infinitésimal, Fisher [1919], que l'on notera ici $\mathcal{B}_\varepsilon$, et que l'on précisera plus loin, voir (I.2.2). Dans cette partie de la thèse nous étudions alors un modèle prenant la forme suivante :

$$\partial_t f_\varepsilon(t,x) = \mathcal{B}_\varepsilon(f_\varepsilon)(t,x) - m(x) f_\varepsilon(t,x). \tag{I.2.1}$$

Cette équation représente la dynamique de la population soumise aux deux effets antagonistes impliqués dans la théorie de l'évolution, la création de diversité induite par l'opérateur infinitésimal $\mathcal{B}_\varepsilon$ et la sélection via la mortalité, créée par la fonction de sélection m qui dépend du trait. Pour les naissances, le modèle que nous considérons est un modèle sexué, ce qui signifie qu'un descendant est créé par la combinaison du patrimoine génétique de deux parents.

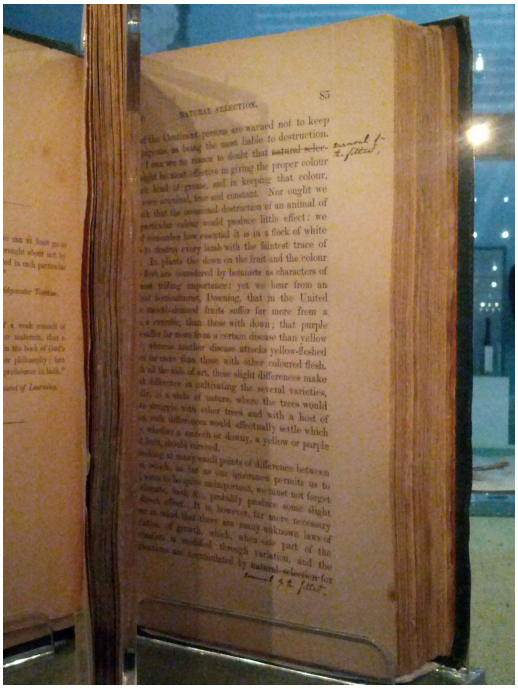


FIGURE I.4 : Première édition de *On the Origin of Species*, exemplaire de Alfred Wallace (Cambridge University). De manière cocasse, Wallace avait barré toutes les occurrences de "natural selection" et les avait remplacé par "survival of the fittest". Crédit photo et anecdote : VC.

Plus précisément, nous considérerons que chaque évènement de naissance dans la population correspond à un premier parent. Alors, un deuxième parent est choisi uniformément dans la population. Le trait du descendant est alors calculé à partir d'une déviation Gaussienne de la moyenne des traits des deux parents :

$$x_{\text{offspring}} = \frac{x_{prt1} + x_{prt2}}{2} + \varepsilon \mathcal{N}\left(0, \frac{1}{2}\right).$$

C'est le contenu de la formule suivante définissant l'opérateur infinitésimal :

$$\mathcal{B}_\varepsilon(f)(x) := \frac{1}{\varepsilon^d \pi^{\frac{d}{2}}} \iint_{\mathbb{R}^{2d}} \exp\left[-\frac{1}{\varepsilon^2}\left(x - \frac{y_1+y_2}{2}\right)^2\right] f(y_1) \frac{f(y_2)}{\int_{\mathbb{R}^d} f(y_2')\, dy_2'}\, dy_1 dy_2. \tag{I.2.2}$$

On peut légitimement s'interroger sur la motivation et la pertinence d'étudier un modèle de reproduction sexuée. Un premier constat est que la plupart des organismes eukaryotes se reproduisent de la sorte, et cela est également vrai pour certaines bactéries. Néanmoins, d'un point de vue écologique, quels bénéfices peuvent tirer deux individus ayant survécus à mélanger leur génome avec celui d'un autre ? D'autant plus que viennent se rajouter des "coûts" : devoir trouver un partenaire, le risque de rester seul, et tous les dangers liés potentiellement à la transmission de maladies ou à la prédation durant la reproduction. Enfin, du point de vue d'un parent, transmettre seulement la moitié de son patrimoine génétique à ses descendants automatiquement diminue sa fitness effective. Pour pouvoir répondre à cet apparent paradoxe du "coût du sexe", il faut comprendre finement les mécanismes biologiques qui gouvernent la reproduction sexuée. Ils sont très complexes, et constituent encore un vaste sujet de recherche Andersson [1994]. Une tentative d'explication est que la grande variance

induite par le sexe permet une réponse plus rapide à la sélection Otto [2009]. L'étude des phénomènes de recombinaison Barton [1995]; Burt [2000] semble être primordiale pour fournir une explication.

Nous ne nous interrogerons pas ici sur la pertinence du modèle infinitésimal vis à vis de l'évolution, citons toutefois Barton et al. [2017] qui utilise des arguments rigoureux pour dériver le modèle infinitésimal à partir d'un modèle microscopique. De nombreux travaux de biologie utilisent ce modèle, voir Slatkin [1970]; Roughgarden [1972]; Slatkin and Lande [1976]; Bulmer [1980]; Turelli and Barton [1994]; Tufto [2000]; Barfield et al. [2011]; Huisman and Tufto [2012]; Cotto and Ronce [2014]; Turelli [2017] Les travaux de Mirrahimi and Raoul [2013]; Raoul [2017] se concentrent autour d'une échelle différente de celle définie par (I.2.2), mais toujours pour le modèle infinitésimal. La variance est d'ordre un, mais le taux de reproduction est très importante. Une structure spatiale est ajoutée dans Mirrahimi and Raoul [2013]; Raoul [2017] comparée au modèle (I.2.1), et utilisée pour dériver des équations macroscopiques.

Ce modèle de reproduction sexuée contraste avec la plupart des travaux existant qui s'intéressaient à un mode de reproduction asexué. Les modèles mathématiques pour étudier l'évolution ont d'abord été tournés vers la théorie des jeux Smith [1974]; Hofbauer and Sigmund [1998; 2003]. On peut s'attendre à ce que la population présente une concentration autour de certains traits particuliers favorisés par la sélection naturelle. Cela se traduit par l'apparition de solutions proches de masses de Dirac dans un régime de faible variance qui joue le même rôle que le paramètre ε dans l'équation (I.2.2). Dans ce contexte évolutif ce point de vue asymptotique a été introduit pour la première fois par Diekmann, Jabin, Mischler et Perthame Diekmann et al. [2005] pour une population asexuée. De nombreux travaux se sont engouffrés à sa suite et ont utilisé ces techniques pour différents modèles Carrillo et al. [2007]; Barles et al. [2009]; Barles and Perthame [2007]; Perthame and Barles [2008]. Citons également Desvillettes et al. [2008]; Jabin and Raoul [2011] qui ont étudié des états d'équilibre singuliers apparaissant en temps long. Ces modèles ont été rigoureusement dérivés de modèles individus-centrés stochastiques par Champagnat et al. [2006; 2007]. Le point commun de tous ces travaux est l'étude d'un mode de reproduction asexué. Rappelons brièvement le type de résultats obtenus dans ce cadre. Tout d'abord, par souci de continuité avec le modèle (I.2.1) que nous étudions, nous supposons que l'équation étudiée est :

$$\varepsilon \partial_t f_\varepsilon(t,z) = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} J\left(\frac{z-z'}{\varepsilon}\right) f_\varepsilon(t,z')\, dz' - m(z) f_\varepsilon(t,z). \tag{I.2.3}$$

Le noyau est bien préparé similairement à (I.1.22), et d'autres termes de réaction peuvent remplacer celui que nous présentons, voir Perthame [2007] et les références à l'intérieur. On utilise alors la transformation logarithmique (I.1.9) :

$$f_\varepsilon(t,z) := \exp\left(-\frac{U_\varepsilon(t,z)}{\varepsilon}\right). \tag{I.2.4}$$

On s'attend à ce que f_ε se concentre comme une masse de Dirac quand $\varepsilon \to 0$ car il y a peu de diversité créée. Alors U_ε permet de mesurer les queues de distributions. La fonction U_ε vérifie par définition

$$\partial_t U_\varepsilon(t,z) = \int_{\mathbb{R}^d} J(y) \exp\left(\frac{U_\varepsilon(z) - U_\varepsilon(z-\varepsilon y)}{\varepsilon}\right) dy - m(z). \tag{I.2.5}$$

Il a été établi, voir Barles et al. [2009], que quand $\varepsilon \to 0$, les solutions de U_ε convergent au sens des viscosités vers U_0 solution de

$$\partial_t U_0(t,z) = L(J)\Big(\partial_z U_0(z)\Big) - m(z), \tag{I.2.6}$$

où L définit la transformée de Laplace du noyau J :

$$L(J)(p) = \int_{\mathbb{R}^d} J(y)\exp(py)dy.$$

On peut montrer formellement que cela fixe la dynamique du trait dominant z_* dans la population :

$$\frac{d\,z_*(t)}{dt} = -\partial_z^2(u_0(t,z_*))^{-1}\partial_z m(z_*(t)).$$

On retrouve l'équation "canonique" de la dynamique adaptative, qui stipule que la dynamique dominante dans une population est le produit de la variance et du gradient de sélection, Fisher [1937].

Les travaux effectués lors de cette thèse contrastent avec le reste de la littérature que nous avons présenté. Nous nous intéressons au régime dans lequel ε est petit et suivons alors les idées de Diekmann et al. [2005] pour faire une étude asymptotique. Néanmoins l'utilisation de ces méthodes asymptotiques est une nouveauté dans le cadre de l'opérateur $\mathcal{B}_\varepsilon$. Il est non linéaire, non monotone et n'admet pas de principes de comparaison, ainsi les outils présentés plus haut dans le cas de la reproduction asexuée ne s'appliquent pas. Dès lors nous n'en retenons que les idées. Nous expliquons par la suite comment la structure de l'opérateur infinitésimal nous aide à comble le vide laissé par l'absence de méthodes précises.

Nous nous sommes appliqués à démontrer plusieurs choses vis à vis du modèle (I.2.1) :

- ▽ L'existence de solution particulières de la forme $\exp(\lambda_\varepsilon t)F_\varepsilon(x)$, que nous appellerons souvent "solutions stationnaires", malgré qu'elles proviennent plutôt d'une étude spectrale de (I.2.1).
- ▽ L'étude du problème de Cauchy lié à l'équation (I.2.1).

I.2.1 *Solutions stationnaires*

On étudie ici le problème suivant :

$$\lambda_\varepsilon F_\varepsilon(z) = \mathcal{B}_\varepsilon(F_\varepsilon)(z) - m(z)F_\varepsilon(z)\,, \quad z \in \mathbb{R}^d. \tag{I.2.7}$$

La première étape est d'identifier le problème limite quand $\varepsilon \to 0$. Faire tendre naïvement le paramètre ε vers 0 dans (I.2.1) donne un résultat singulier. Pour comprendre la concentration de F_ε, on effectue donc la transformation logarithmique suivante

$$F_\varepsilon(z) := \exp\left(-\frac{U_\varepsilon(z)}{\varepsilon^2}\right).$$

La fonction U_ε permet de comprendre les queues de distributions de la distribution F_ε. Le choix de l'échelle est différent de la précédente transformation (I.2.4), elle provient de la nature de l'opérateur infinitésimal. On peut le justifier a priori ainsi : en l'absence de sélection, les gaussiennes de variance ε^2 laissent invariantes l'opérateur $\mathcal{B}_\varepsilon$. Plus précisément, pour tout z_0, la famille de solution

$$G_\varepsilon := \frac{1}{(2\pi)^{\frac{d}{2}}\varepsilon^d}\exp\left(-\frac{(z-z_0)^2}{2\varepsilon^2}\right),$$

vérifie

$$\mathcal{B}_\varepsilon(G_\varepsilon) = G_\varepsilon.$$

On s'attend donc à ce que l'ajout de la fonction de sélection m permette de déterminer la moyenne de ces gaussiennes, et que le profil F_ε ne soit pas trop éloigné d'un profil Gaussien quand ε est petit. L'analyse formelle que nous présentons permet de retrouver ces deux résultats, elle est l'œuvre de Bouin et al. [2019]. Pour cela, cherchons des solutions $(\lambda_\varepsilon, U_\varepsilon)$ sous forme de développements successifs en puissance de ε :

$$U_\varepsilon(z) := U_0(z) + \varepsilon^2 U_1(z) + o(\varepsilon^2), \tag{I.2.8}$$

$$\lambda_\varepsilon := \lambda_0 + \varepsilon^2 \lambda_1 + o(\varepsilon^2). \tag{I.2.9}$$

C'est une hypothèse forte sur la structure des solutions de (I.2.7), mais nous verrons que contrairement au cas asexué, il est obligatoire de garder le premier terme correctif pour déterminer le profil U_ε. De plus, remarquons que les solutions de (I.2.7) sont déterminées à une constant multiplicative près, donc U_ε sera défini à une constante additive près. En divisant par F_ε l'équation (I.2.7), et en injectant le contenu de (I.2.8), on obtient le problème suivant pour U_0 et U_1, en négligeant les contributions d'ordre supérieur :

$$\lambda_0 + m(z) = \frac{\displaystyle\iint_{\mathbb{R}^2} \exp\left(-\frac{1}{\varepsilon^2}\left[\left(z - \frac{z_1+z_2}{2}\right)^2 + U_0(z_1) + U_0(z_2) - U_0(z)\right] - U_1(z_1) - U_1(z_2) + U_1(z)\right) dz_1 dz_2}{\varepsilon\sqrt{\pi}\displaystyle\int_{\mathbb{R}} \exp\left(-\frac{U_0(z')}{\varepsilon^2} - U_1(z')\right) dz'} \tag{I.2.10}$$

Pour que l'équation à la limite $\varepsilon \to 0$ ne soit ni singulière ni triviale, il est nécessaire que le quotient d'intégrale ait une limite finie, afin d'être du même ordre que le terme de gauche de (I.2.10). On s'attend à ce que ces intégrales se concentrent autour d'un minimum par le principe de Laplace, donc par souci de clarté on soustrait $\min U_0$ de part et d'autres de la fraction :

$$\lambda_0 + m(z) = \frac{\displaystyle\iint_{\mathbb{R}^2} E_\varepsilon(z_1, z_2, z) dz_1 dz_2}{\varepsilon\sqrt{\pi}\displaystyle\int_{\mathbb{R}} \exp\left(-\frac{U_0(z') - \min U_0}{\varepsilon^2} - U_1(z')\right) dz'}. \tag{I.2.11}$$

avec

$$E_\varepsilon(z_1, z_2, z) = \exp\left(-\frac{1}{\varepsilon^2}\left[\left(z - \frac{z_1+z_2}{2}\right)^2 + U_0(z_1) + U_0(z_2) - U_0(z) - \min U_0\right] - U_1(z_1) - U_1(z_2) + U_1(z)\right).$$

Pour que l'intégrale de E_ε dans (I.2.11) ne soit ni 0 ni singulière, il est nécessaire que U_0 soit solution de l'identité suivante, pour tout $z \in \mathbb{R}$:

$$\inf_{(z_1,z_2)\in\mathbb{R}^2}\left[\left(z - \frac{z_1+z_2}{2}\right)^2 + U_0(z_1) + U_0(z_2) - U_0(z) - \min U_0\right] = 0. \tag{I.2.12}$$

Par des arguments d'analyse convexe, on montre alors que seuls les polynômes de degré deux sont solutions de cette formule. On choisit donc d'écrire les solutions sous la forme générique suivant, ou z_0^* est a priori inconnu :

$$U_0(z) = \frac{(z - z_0^*)^2}{2}.$$

Ainsi, à l'ordre principal, les gaussiennes avec une variance ε^2 sont une première approximation de la solution. On retrouve que sans sélection, les gaussiennes laissent invariantes l'opérateur infinitésimal. Néanmoins seule la sélection permettra de déterminer la moyenne de la gaussienne z_0^*. L'équation sur U_1 fera elle intervenir la sélection. Ainsi, dans le régime de petite variance pour l'opérateur infinitésimal, la reproduction est dominante. Le terme U_1 peut donc s'interpréter comme une correction au premier ordre des gaussiennes, induite par la sélection.

Étude du correcteur

Maintenant que nous avons déterminé la première approximation de la solution U_ε de (I.2.7), nous pouvons déterminer quelle équation est vérifiée par le correcteur U_1. La dérivation de cette équation n'étant pas triviale, nous la détaillons ici. Il va s'agir de comprendre comment et en quel point les intégrales de (I.2.10) se concentrent. Nous illustrons ceci par l'exemple simple du dénominateur de (I.2.10).

$$\frac{1}{\varepsilon\sqrt{\pi}}\int_{\mathbb{R}}\exp\left(-\frac{(z'-z_0^*)^2}{\varepsilon^2}\right)\exp\Big(-U_1(z')\Big)dz' = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\exp\left(-\frac{y'^2}{\varepsilon^2}\right)\exp\Big(-U_1(z_0^*+\varepsilon y')\Big)dz',$$
$$\xrightarrow[\varepsilon\to 0]{} \sqrt{2}\exp\Big(-U_1(z_0^*)\Big). \tag{I.2.13}$$

Nous pouvons faire de même pour le numérateur, mais le calcul est légèrement plus compliqué car l'intégrale est double, et il est moins clair quel est le point autour duquel l'intégrale se concentre, tandis que pour le dénominateur il était transparent : z_0^*. Déterminer ce point équivaut à résoudre l'équation suivante :

$$\operatorname*{argmin}_{(z_1,z_2)\in\mathbb{R}^2}\left[\left(z-\frac{z_1+z_2}{2}\right)^2+U_0(z_1)+U_0(z_2)-U_0(z)-\min U_0\right]=0, \quad \text{avec } U_0(z)=\frac{(z-z_0^*)^2}{2}. \tag{I.2.14}$$

On se souvient que le minimum est 0 par définition de U_0, voir (I.2.12). Un calcul direct de (I.2.14) montre que le minimum est atteint au point

$$(\bar{z},\bar{z}) \quad \text{avec } \bar{z} := \frac{z+z_0^*}{2}.$$

On effectue alors un changement de variable affine pour se recentrer autour du point de minimum :

$$z_1 = \bar{z}+\varepsilon y_1, \quad z_2 = \bar{z}+\varepsilon y_2.$$

Alors l'opérateur que l'on minimise se retrouve réduit à la forme quadratique Q suivante

$$\left[\left(z-\frac{z_1+z_2}{2}\right)^2+U_0(z_1)+U_0(z_2)-U_0(z)-\min U_0\right]=\frac{1}{2}y_1y_2+\frac{3}{4}(y_1^2+y_2^2):=Q(y_1,y_2).$$

Ceci constitue le résidu de l'opérateur infinitésimal à cette échelle. Finalement, le numérateur se concentre ainsi :

$$\left(\frac{1}{\sqrt{\pi}}\right)\iint_{\mathbb{R}^2}\exp\left(-Q(y_1,y_2)-U_1(\bar{z}+\varepsilon y_1)-U_1(\bar{z}+\varepsilon y_2)+U_1(z)\right)dy_1dy_2$$
$$\xrightarrow[\varepsilon\to 0]{} \sqrt{2}\exp\Big(-2U_1(\bar{z})+U_1(z)\Big).$$

Cela nous permet de conclure sur la nature du problème vérifié par U_1 en combinant ce calcul avec (I.2.13) :

$$\lambda_0 + m(z) = \exp\Big(U_1(z_0^*) - 2U_1(\bar{z}) + U_1(z)\Big). \tag{I.2.15}$$

Cette équation non locale est véritablement la signature de l'opérateur infinitésimal dans la limite de petite variance $\varepsilon \to 0$. On y remarque l'occurrence de deux points caractéristiques au delà du point courant z. Le centre de la première approximation Gaussienne z_0^* autour duquel va se concentrer la solution apparaît, ainsi que $\bar{z}$ le point moitié entre cet optimum et z.

Mathématiquement, nous pouvons déduire des informations quantitatives de l'équation (I.2.15), et même déterminer z_0^*. En évaluant en z_0^* et en dérivant puis en évaluant en z_0^* on obtient les relations suivantes :

$$\begin{cases} \lambda_0 + m(z_0^*) = 1, \\ m'(z_0^*) = 0. \end{cases} \tag{I.2.16}$$

Ces conditions nécessaires caractérisent λ_0 et z_0^*. La première traduit l'équilibre phénotypique à cette échelle entre la mortalité à l'optimum, le taux de croissance et le taux de naissance. La seconde indique que nécessairement les gaussiennes doivent être centrées autour d'un point critique de la fonction de sélection. A priori, comme indiqué par (I.2.16) on peut construire une solution autour de chaque point critique de m, avec une condition d'écartement des minima. Notre méthode de preuve nous garantira que c'est bien le cas.

Cette équation limite contraste fortement avec le reste de la littérature pour une reproduction sexuée, Barles et al. [2009] par exemple, qui obtiennent une équation de type Hamilton-Jacobi à la limite voir équation (I.2.6), similairement à (I.1.11). La structure riche de ce type d'équation leur permet alors de déduire des informations qualitatives et quantitatives comparables à (I.2.16), d'une manière qui n'est pas possible dans notre cas.

Résultats théoriques de convergence

Cette thèse a rendu ces résultats formels rigoureux. Il faut noter qu'on peut rajouter toute partie affine à une solution U_1 de (I.2.15). Nous travaillerons dans des espaces fonctionnels mesurant les dérivées (jusqu'à l'ordre 3) et qui ne voient donc pas les fonctions affines. De plus, les solutions de (I.2.15) ont une formulation explicite :

$$U_1(z_0^* + h) = \gamma_0 h + \sum_{k=0}^{+\infty} 2^k \log\Big(\lambda_0 + m\big(2^{-k}(z_0^* + h)\big)\Big), \quad \gamma_0 \text{ indéterminé.} \tag{I.2.17}$$

Les conditions nécessaires déduites du problème initial (I.2.16) sont suffisantes pour que cette série soit convergente, associées à des hypothèses peu contraignantes sur la fonction m. Ici, la constante γ_0 peut être choisie quelconque, mais le problème de départ pour $\varepsilon > 0$ n'est, lui, pas transparent vis à vis des fonctions affines. On va donc chercher à construire des solutions

$$U_\varepsilon(z) := U_0(z) + \varepsilon^2 \gamma_\varepsilon (z - z_0^*) + \varepsilon^2 V_\varepsilon(z).$$

On espère que la fonction U_1 correspondant à l'analyse formelle est une bonne approximation. La convergence rigoureuse de V_ε et γ_ε a été démontrée dans Calvez et al. [2019], on se reportera à la

partie III. Nous rappelons brièvement les étapes clés. Tout d'abord, il existe une fonctionnelle I_ε, résidu de l'opérateur infinitésimal telle que l'équation sur V_ε s'écrive :

$$\lambda_0 + m(z) = I_\varepsilon(V_\varepsilon)(z) \exp\Big(V_\varepsilon(z_0^*) - 2V_\varepsilon(\bar{z}) + V_\varepsilon(z)\Big). \tag{I.2.18}$$

Les estimations de I_ε sont cruciales, puisqu'elles font le lien entre le problème à $\varepsilon > 0$ et le problème pour $\varepsilon = 0$ (I.2.15). Si on maîtrise les dérivées de V_ε -en réalité des estimations Lipschitz suffisent- on peut alors montrer que

$$I_\varepsilon(V_\varepsilon) \xrightarrow[\varepsilon \to 0]{\|\cdot\|_\infty} 1.$$

Ceci explicite le lien avec l'équation (I.2.15). Nous n'écrivons pas la forme précise de I_ε, mais elle peut se lire sur le terme intégral de (I.2.10), voir la partie III pour plus de détails. Les paramètres λ_ε et γ_ε peuvent s'exprimer comme des fonctions de V_ε. On montre alors le théorème suivant, :

Théorème 6 (Calvez, Garnier, Patout, 19)**.**

(i) Il existe $\varepsilon_0 > 0$ tel que pour tout $\varepsilon \leqslant \varepsilon_0$, il existe une unique solution V_ε à (I.2.18) sur une petite boule d'un espace fonctionnel $\mathcal{E}$.

(ii) La famille $(\lambda_\varepsilon, \gamma_\varepsilon, V_\varepsilon)_\varepsilon$ converge quand $\varepsilon \to 0$, vers

$$\lambda_0 = 1,$$
$$U_1(z) = \gamma_0(z - z_0^*) + V_0(z),$$

avec

$$\gamma_0 = \frac{\partial_z^3 m(z_0^*)}{2\partial_z^2 m(z_0^*)} (d = 1) \text{ et } V_0 = \sum_{k=0}^{+\infty} 2^k \log\Big(\lambda_0 + m\big(2^{-k}(z_0^* + h)\big)\Big).$$

$V_\varepsilon \to V_0$ converge localement uniformément pour la première et seconde dérivée.

Pour la preuve, on reformule (I.2.18) comme solution d'un point fixe qui utilise une généralisation de la solutions sous forme de série du type (I.2.17).

$$\mathcal{H}_\varepsilon(V_\varepsilon) = V_\varepsilon.$$

Des estimations fines de la fonctionnelle $\mathcal{H}_\varepsilon$ via de l'analyse perturbative en ε permettent d'établir un résultat de contraction ressemblant à

$$\|\mathcal{H}_\varepsilon(V_1) - \mathcal{H}_\varepsilon(V_2)\|_{\mathcal{E}} \leqslant \Big(\kappa + O(\varepsilon)\Big) \|V_1 - V_2\|_{\mathcal{E}}, \quad \kappa < 1. \tag{I.2.19}$$

L'espace $\mathcal{E}$ contrôle la norme L^∞ de trois dérivées de fonctions, agrémentées d'un poids pour les dérivées seconde et troisième. Cela découle de la quête d'une contraction dans l'équation (I.2.19). Pour la convergence $\varepsilon \to 0$, on réutilise la formulation de point fixe, car les boules de notre espace fonctionnel ne sont pas compactes pour pouvoir en extraire une sous suite convergente. Ce théorème permet de construire une solution autour d'un minimum (et même maximum) local de m. Cela répond par la négative à l'unicité des solutions dans le cas où la fonction de sélection m admet plusieurs minima locaux (défaut de convexité). Néanmoins, la condition de positivité du terme dans

la série définie par l'équation (I.2.17) impose la condition suivante sur les minima locaux z_0^* autour desquels on peut construire une solution par notre procédure :

$$m(z_0^*) < 1 + \min m.$$

Remarquons que la limite de γ_ε n'étant pas déterminée par l'analyse formelle, il aurait donc fallu a priori pour la fixer calculer un approximation à un ordre supplémentaire en ε, mais cela n'est pas le cas, on arrive à exprimer γ_ε en fonction de V_ε et à rendre cela stable à la limite $\varepsilon \to 0$.

L'étude de ces solutions stationnaires fait l'objet de la partie III, et a été publié, Calvez et al. [2019].

I.2.2 *Etude du problème de Cauchy*

On se consacre désormais au problème compagnon de l'existence de solutions stationnaire, à savoir le comportement quand ε est petit du système de Cauchy associée à l'équation (I.2.1) :

$$\begin{cases} \varepsilon^2 \partial_t f_\varepsilon(t,z) = \mathcal{B}_\varepsilon(f_\varepsilon)(t,z) - m(x) f_\varepsilon(t,z), \\ f_\varepsilon(0,z) = f_\varepsilon^0(z). \end{cases} \tag{I.2.20}$$

On a vu auparavant qu'il existait des solutions spéciales qui étaient des perturbations de Gaussiennes. La stratégie pour étudier ce problème est donc de chercher des solutions qui vont rester proches des Gaussiennes pour tout temps. On introduit donc la transformation suivante :

$$f_\varepsilon(t,z) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left(\frac{\lambda(t)}{\varepsilon^2} - \frac{(z - \mathrm{z}_*(t))^2}{2\varepsilon^2} - V_\varepsilon(t,z) \right).$$

En injectant dans (I.2.20), on obtient l'équation suivante :

$$-\varepsilon^2 \partial_t V_\varepsilon(t,z) + \dot\lambda(t) + (z - \mathrm{z}_*)\dot{\mathrm{z}}_* + m(z) = I_\varepsilon(V_\varepsilon)(t,z) \exp\Big(V_\varepsilon(t,z) - 2V_\varepsilon(t,\bar z) + V_\varepsilon(t,\mathrm{z}_*) \Big).$$

Cette fois ci, il n'est plus possible de reformuler la solution sous forme de point fixe, la structure du problème étant perturbée par le terme de dérivée en temps. Ce terme disparaît formellement pour $\varepsilon = 0$, ce qui donne lieu à un problème de type limite singulière pour passer à la la limite $\varepsilon \to 0$. En revanche, c'est un gain considérable de facilité pour l'analyse du problème $\varepsilon \to 0$. En particulier on retrouve les relations suivantes, en évaluant en z_* puis en dérivant et évaluant en z_* puis en laissant $\varepsilon \to 0$:

$$\begin{cases} \dot\lambda(t) + m(\mathrm{z}_*(t)) = 1, \\ \dot{\mathrm{z}}_*(t) + \partial_z m(\mathrm{z}_*(t)) = 0. \end{cases} \tag{I.2.21}$$

De manière similaire à l'étude stationnaire, la première équation traduit l'équilibre phénotypique à cette échelle. La deuxième équation s'interprète comme l'équation "canonique" de la dynamique adaptative liant variance dans la population et le gradient de sélection. Comme nous l'avons vu, le modèle infinitésimal prescrit la variance des solutions à l'ordre principal (ε^2, mais unitaire ici, après changement d'échelle), et donc cette équation "canonique" nous permet de déterminer le centre des gaussiennes z^*. On définit alors une fonction de sélection "translatée" autour de z^*

$$\Gamma(t,z) := 1 + m(z) - m(\mathrm{z}_*(t)) - \partial_z m(\mathrm{z}_*(t)(z - \mathrm{z}_*(t)).$$

Ceci permet de retrouver formellement à $\varepsilon = 0$ le problème suivant :

$$\Gamma(t,z) = \exp\left(V_0(t,z) - 2V_0\left(t,\bar{z}(t)\right) + V_0(\mathrm{z}_*(t))\right). \tag{I.2.22}$$

Ce problème est l'exact analogue de celui trouvé dans le cas stationnaire (I.2.15). La croissance et sélection s'équilibrent dans le terme de gauche, la somme est égale au même opérateur non local provenant de l'opérateur infinitésimal où apparaissent le point courant z, l'optimum gaussien z_* et le point milieu $\frac{1}{2}(z+\mathrm{z}_*)$. On s'attend que V_ε soit qualitativement proche de V_0 pour de petits ε. On démontre alors le résultat suivant :

Théorème 7 (Patout, 19)**.**
Dans un certain espace fonctionnel $\mathcal{F}$, on suppose qu'il existe une constante ε_0 et une constante C_0 telles que pour tout $\varepsilon \leqslant \varepsilon_0$, on parte initialement proche de U_0 la solution de l'équation (I.2.22) :

$$\|U_\varepsilon(0,\cdot) - U_0(0,\cdot)\|_{\mathcal{F}} \leqslant \ C_0\varepsilon^2.$$

Alors, ce contrôle reste vrai uniformément en temps pour $\varepsilon \leqslant \varepsilon_0$, au prix d'une autre constante K_0 qui ne dépend que des conditions initiales :

$$\sup_{t\in\mathbb{R}} \|U_\varepsilon(t,\cdot) - U_0\|_{\mathcal{F}} \leqslant K_0\varepsilon^2.$$

Pour montrer ce résultat, on utilise des arguments d'analyse perturbative, visant à montrer que la perturbation R_ε est uniformément bornée :

$$U_\varepsilon := U_0 + \varepsilon^2 R_\varepsilon.$$

Formellement, R_ε est solution du problème linéarisé suivant :

$$\varepsilon^2\partial_t R_\varepsilon(t,z) = \Gamma(t,z)\Big(R_\varepsilon(t,z) - 2R_\varepsilon\left(t,\bar{z}(t)\right) + R_\varepsilon(t,\mathrm{z}_*(t))\Big).$$

Il se trouve qu'on connaît les valeurs propres et les vecteurs propres duaux de cette équation :

eigenvalues :	0	0	$-\frac{1}{2}$	$-\frac{3}{4}$	...
Dual eigenvector :	$\delta_{\mathrm{z}_*(t)}$	$\delta'_{\mathrm{z}_*(t)}$	$\delta^{(2)}_{\mathrm{z}_*(t)}$	$\delta^{(3)}_{\mathrm{z}_*(t)}$	...

On s'aperçoit que les deux premières valeurs propres nulles correspondent aux fonctions affines. Le reste du spectre est négatif, donc nous décomposons R_ε en séparant les fonctions affines :

$$R_\varepsilon(t,z) = p_\varepsilon + q_\varepsilon(t)(z - \mathrm{z}_*(t)) + W_\varepsilon(t,z).$$

On s'attend à ce que W_ε soit bornée, puisque les valeurs propres correspondant sont strictement négatives. On s'attend même à ce que W_ε suive une décroissance exponentielle. En revanche, la dynamique est complètement différente pour la partie affine. On peut montrer que q_ε est solution de l'équation suivante à la limite $\varepsilon \to 0$:

$$\dot{q}^*(t) = -\partial_z^2 m(\mathrm{z}_*(t))q^*(t) + \frac{\partial_z^3 m(\mathrm{z}_*(t))}{2} - 2\partial_z^2 m(\mathrm{z}_*(t))\partial_z m(\mathrm{z}_*(t)).$$

Ainsi, pour que q reste uniformément borné, il est nécessaire que le signe de $\partial_z^2 m(z_*(t))$ soit positif asymptotiquement. On fera les hypothèses suffisantes sur m et $z^*(0)$ pour que ce soit le cas, sinon le théorème 7 est en défaut. On imagine de plus que q_ε a une dynamique lente en temps, correspondant à la deuxième valeur propre nulle.

Enfin pour être clair, nos résultats ne nous permettent pas de déterminer la convergence en temps long de U_ε vers le profil stationnaire de la partie précédente. Néanmoins, toutes nos estimations sont uniformes en temps, et la limite formelle en temps long de chacune des quantités est compatible avec la solution du problème stationnaire. L'étude de ce problème de Cauchy fait l'objet de la partie IV, et a été soumis pour publication.

I.3 LIGNÉES PHÉNOTYPIQUES

L'étude de la réponse phénotypique d'une population à un changement d'environnement est un thème saisi par la communauté scientifique : Parmesan [2006]; Hoffmann and Sgro [2011]. Le but du modèle que nous allons présenter est de prendre en compte des traits quantitatifs en présence de mutation, sélection et variation environnementale, voir par exemple Kopp and Matuszewski [2014].

Plus précisément, nous supposons que la sélection favorise un unique trait, et que ce trait change linéairement au cours du temps. On peut vouloir chercher à savoir quelle est la vitesse critique de l'environnement pour que la population s'adapte, le lien avec la variance phénotypique dans la population, et le taux de croissance par exemple. Des travaux préliminaires existent déjà *e.g.* Bürger and Lynch [1995]. Leur analyse est néanmoins limitée au cas où toutes les distributions sont des gaussiennes. Dans le modèle que nous présentons, qui est l'œuvre de Bouin et al. [2019], aucune supposition n'est faite a priori sur la forme de la distribution. Des travaux récents dans ce sens existent, sans changement d'environnement, citons Gandon and Mirrahimi [2017] pour le cas d'une population pouvant migrer entre deux habitats, ou Martin and Roques [2016] pour un modèle ou la population est structurée en valeur sélective ou taux de croissance par individu..

Un léger changement du modèle présenté dans la partie précédente permet de prendre en compte ces facteurs environnementaux, et nous étudierons ici l'équation ,

$$\partial_t f_\varepsilon(t,x) - c_\varepsilon \partial_z f_\varepsilon(t,z) = \mathcal{B}_\varepsilon(f_\varepsilon)(t,x) - m(x) f_\varepsilon(t,x). \tag{I.3.1}$$

On a seulement rajouté un terme de transport décrivant l'évolution à vitesse constante c_ε de l'environnement et donc de l'optimum phénotypique. Précisons que dans cette section nous englobons deux modes de reproduction via $\mathcal{B}_\varepsilon$, l'opérateur infinitésimal (I.2.2) pour la reproduction sexuée d'une part, et de l'autre la reproduction asexuée via un terme de convolution modélisant les mutations :

$$\mathcal{B}_\varepsilon(F)(x) = \frac{1}{\varepsilon}\int_{\mathbb{R}} K\left(\frac{x-x'}{\varepsilon}\right) F(x')\,dx', \tag{I.3.2}$$

Une analyse formelle de l'équation (I.3.1) a été conduite dans Bouin et al. [2019] dans le régime de petite variance. Les auteurs montrent formellement comment déduire des informations quantitatives à partir du problème limite $\varepsilon = 0$. En particulier, ils montrent que l'environnement ne doit pas varier trop vite pour que la population puisse s'adapter, et proposent un échelle en $\varepsilon^\alpha C$ pour la vitesse de changement d'environnement, avec $\alpha \in \{1,2\}$ selon le mode de reproduction. Pour le cas sexué il s'agit essentiellement des équilibres explicités précédemment, tandis que pour le cas asexué, les auteurs reprennent la méthodologie "classique" de Diekmann et al. [2005]; Barles et al. [2009] qui s'adapte bien quand on rajoute un terme de transport puisque par exemple le Hamiltonien

associé reste convexe. Notons bien que la preuve de convergence présentée précédemment pour le cas du modèle infinitésimal ne se généralise pas pour (I.3.1), le terme de transport ne permet pas de démontrer l'existence d'un point fixe pour le cas stationnaire par exemple.

Dans cette thèse nous nous sommes intéressés à la dynamique interne de ces états d'équilibre, pour les deux modes de reproduction. Tout d'abord nous avons lancé des simulations numériques à partir d'un modèle individu centré lié à l'équation, voir Champagnat et al. [2006]. En gardant en mémoire les lignées phénotypiques des individus, c'est à dire l'histoire des traits des parents de chaque individu au cours du temps dans la population, on peut regarder la contribution de chaque trait à l'équilibre de la population. En particulier dans le cas du modèle asexué, on observe que la survie de la population est possible grâce aux quelques individus fondateurs qui ont le trait optimal vis à vis de la sélection. Ce résultat est paradoxal car la persistance de la population dépend de quelques individus "rares", on se reportera à la figure V.10, ou bien figure V.5 pour une représentation schématique.

Nous expliquons à la partie V de nombreuses heuristiques pour se convaincre de ce résultat, ainsi que des pistes mathématiques pour prouver ce résultat de manière rigoureuse. Ainsi, nous proposons des modèles permettant d'avoir de l'information sur une lignée complète au cours du temps dans le régime $\varepsilon = 0$, grâce à la structure Hamilton-Jacobi, en s'inspirant de Barles and Roquejoffre [2006]. Nous étudions également ces lignées avec le point de vues des fractions neutres, voir Garnier et al. [2012]. On y trouvera également les résultats des simulations numériques que nous avons évoquées.

Pour le modèle infinitésimal, il est intéressant de remarquer que nous observons la dynamique contraire du modèle asexué vis vis de l'origine des ancêtres. La variance de la population est amenée par tous les individus qui sont au trait dominant. Néanmoins il faut noter que dans ce cadre de reproduction avec *deux* parents, définir une lignée est plus difficile parmi l'arbre généalogique entier. Nous proposerons enfin des simulations numériques pour un autre mode de reproduction sexuée, basée sur le mécanisme de la méiose. Le modèle prend la forme suivante dans le régime stationnaire :

$$-\sigma C\cdot\nabla F(x,y)+\Big(\lambda+m(x,y)\Big)F(x,y)= \\ \frac{1}{\iint F(x',y')dx'dy'}\left(\iint K_\sigma(x-x')F(x',y^\dagger)\,dx'dy^\dagger\right)\left(\iint K_\sigma(y-y')F(x^\dagger,y')\,dx^\dagger dy'\right). \tag{I.3.3}$$

Ici, x et y désignent la valeur prise par un allèle sur chacun des chromosomes d'un individu, respectivement maternels et paternels. La fonction de sélection m agit sur une combinaison de ces allèles, qu'on peut voir comme le trait ou la fitness exprimée. Des recombinaisons peuvent être ajoutées à la modélisation, ce qui correspondrait à l'étude de plusieurs loci, Barton [1995]; Burt [2000]. Nous expliquons brièvement comment réaliser une analyse asymptotique de petite variance $\sigma \to 0$ pour obtenir une équation de Hamilton-Jacobi. Nous essayons ensuite de donner un sens aux lignées alléliques dans ce cas là, et justifions nos résultats par des calculs exacts (gaussiens) sur un modèle jouet correspondant au modèle équivalent à (I.3.3) pour des temps discrets.

I.4 PERSPECTIVES

L'étude de la transmission génétique pour les populations sexuées est complexe. Ceci amène naturellement à considérer d'autres modèles que le modèle infinitésimal. Un apport de cette thèse a été de montrer la robustesse des méthodes basées sur l'analyse asymptotique "petite variance" qu'on peut espérer appliquer à d'autres modèles :

- ▷ Un premier pas serait d'envisager ce qu'il se passe si on n'impose plus la déviation autour de la moyenne des traits des parents à être une Gaussienne. Bien sûr le prix à payer est la perte de la justification microscopique du modèle effectuée par Barton et al. [2017], mais on peut imaginer un comportement qualitatif autre.

- ▷ L'hypothèse d'uniformité dans le choix des parents peut sembler peu réaliste. On espère pouvoir rajouter un biais de préférence sexuelle pour déterminer le choix du partenaire, voir Kokko et al. [2006]; Kelly and Jennions [2011] ou Coron et al. [2019] pour une approche mathématique. Cela implique nécessairement de se placer dans le cas d'une équation pour la distribution mâle et une autre pour la distribution des femelles, ce qui dans le cadre du modèle infinitésimal n'existait pas, car le modèle fait l'hypothèse d'un "sex ratio" de $\frac{1}{2}$.

- ▷ On veut pouvoir rajouter une structure spatiale au modèle (I.2.1), ce qui a par exemple été fait dans le cas des mutations Bouin and Mirrahimi [2013]. Le lien entre le modèle infinitésimal et le modèle d'écologie de Kirkpatrick et Barton Kirkpatrick and Barton [1997] a été établi pour un certain régime dans les travaux de Gaël Raoul Raoul [2017]; Mirrahimi and Raoul [2013]. Il serait intéressant de pouvoir établir également dans notre asymptotique une limite rigoureuse.

- ▷ Enfin, un autre objectif est de construire un schéma numérique robuste adapté au modèle infinitésimal. Il devrait converger pour tout ε et posséder des propriétés de stabilité quand $\varepsilon \to 0$. Un tel schéma est dit Asymptotic Preserving (AP), introduit par Jin [1999]; Klar [1998]. Des estimations précises des queues de distribution Gaussiennes fut conduite par Hélène Hivert pour des modèles de dynamique adaptative asexués. Ceci est l'objet d'un travail embryonnaire avec Hélène Hivert.

De plus, les méthodes présentées dans cette thèse ouvrent la voie à une analyse d'autres modèles avec le modèle infinitésimal. Par exemple dans l'esprit des travaux de Mirrahimi and Gandon [2018] on peut adapter les techniques pour étudier une population avec un mode de reproduction sexuée pouvant migrer entre deux habitats. Léonard Dekens étudie actuellement ce modèle.

Formellement, l'étude des états d'équilibre correspondant à équation (I.3.1) a été conduite par Bouin et al. [2019]. Néanmoins le cas $c_\varepsilon > 0$ ne rentre pas dans notre cadre de preuve pour démontrer rigoureusement la convergence $\varepsilon \to 0$, car il brise la structure de point fixe que nous utilisons dans le cas stationnaire, voir la partie III. Une généralisation naturelle est donc d'essayer de pouvoir adapter les preuves des parties III et IV avec une environnement changeant. De plus la question de la nature des fronts de propagation avec optimum mobile est également à notre connaissance une question ouverte. Il peut être important, afin de comprendre la dynamique des équilibres de déterminer si ils sont "tirés" ou "poussés". Cette interrogation découle naturellement de l'étude des lignées phénotypiques à la partie V. Les autres perspectives liées à l'étude de la partie V, qui sont un travail toujours en cours sont rassemblées à la fin de ce chapitre.

Deuxième partie

Diffusion non locale avec un noyau à queue lourde et formalisme Hamilton-Jacobi

Dans cette partie nous étudions le comportement asymptotique des solutions d'une équation intégro-différentielle de type Fisher-KPP, où la dispersion opère par un terme de convolution avec un noyau à queue lourde. Nous étudions deux régimes en temps long. Tout d'abord le comportement asymptotique en espace, en introduisant un rééchelonnement de l'espace et des sur et sous solutions qui permettent de capturer précisément la propagation avec le formalisme Hamilton-Jacobi. Dans un second temps, nous proposons un changement d'échelle approprié au petites mutations dans le cadre de la "dynamique adaptative", puis nous montrons la convergence vers une équation de Hamilton-Jacobi. Ce travail a été publié dans le SIAM Journal of Mathematical Analysis : Bouin et al. [2018].

II.1 INTRODUCTION

The model

In this paper, we focus on the asymptotic behavior of solutions to the following integro–differential equation

$$\begin{cases} n_t = J * n - n + n(1-n), & \text{in } (0,+\infty)\times\mathbb{R}, \\ n(t=0,\cdot) = n^0. \end{cases} \tag{II.1.1}$$

where the *dispersal kernel* J, or *mutation kernel* depending on the ecological context, is a given function and

$$(J * n)(t,x) := \int_{\mathbb{R}} J(x-y)n(t,y)\,dy.$$

When the term $J * n - n$ is replaced by Δn, this is the well-known Fisher-KPP equation Fisher [1937]; Kolmogorov et al. [1937].

This equation arises naturally in population dynamics to model systems with non–local effects Fife [2013]; Medlock and Kot [2003]. In this context, the unknown function n represents a density of individuals at time t and at position x. One of the most interesting features of this model, compared to the classical Fisher-KPP equation, is that it allows for long range dispersal events. Here, $t \in (0,+\infty)$ is a time variable, and $x \in \mathbb{R}$ is a position variable that could represent space or phenotype. The function J is called a dispersal kernel and is given from now on. Moreover, The existence of these events depends critically on the tail of the kernel J. To differentiate between the two regimes that arise, we introduce the following notation. Roughly, we say that the kernel J is *thin–tailed*, or *exponentially bounded*, if there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}} J(h)e^{\lambda h}\mathrm{d}h < \infty, \tag{II.1.2}$$

Otherwise we may say the kernel is *fat–tailed*. We make mathematically precise what we call a *fat–tailed* kernel in (II.1). When the kernel is thin–tailed, solutions exhibit the same behavior as solutions to the Fisher–KPP equation in that travelling wave solutions exist Aronson and Weinberger [1978]; Diekmann [1979]. This regime can be used to model a biological invasion scenario in which a population invades a homogeneous landscape at constant speed. There is an extensive literature about models similar to (II.1.1) investigating the existence and stability of traveling waves solutions, see Carr and Chmaj [2004]; Schumacher [1980]; Weinberger [1982] along with the work and references contained in the habilitation thesis of Coville Coville [2007].

On the other hand, super-linear in time propagation phenomena can occur in ecology. A classical example is Reid's paradox of rapid plant migration Clark et al. [1998]; Clark [1998] that is usually resolved using fat–tailed kernels. Indeed, when the kernel is fat–tailed, the solutions of (II.1.1) do not propagate at constant speed but accelerate with a rate that depends on the thickness of the tail of the kernel J Medlock and Kot [2003]; Garnier [2011]. This acceleration phenomenon results from the combination of fat–tailed dispersion events and the Fisher–KPP non–linearity $n(1-n)$ that makes the solution grow almost exponentially when small. In particular, when this cooperation between the kernel and the non–linear term is broken, acceleration can be stopped. Recently, Alfaro and Coville Alfaro and Coville [2017] have proved that traveling wave solutions may exist with fat–tailed kernels when a weak Allee effect is present, that is a non–linearity of the form $n^\beta(1-n)$ with $\beta > 1$, if the tail of the kernel is not too fat.

We shall now be more precise on the type of dispersal kernels that we will consider. We emphasize that the following assumptions on the kernel J hold true throughout this work, even when not explicitly stated:

Hypothesis II.1 (Fat–tailed kernel)**.** *The kernel J is a symmetric probability density, that is, for all $x \in \mathbb{R}$,*

$$\int_{\mathbb{R}} J(x)\,dx = 1, \qquad J(x) = J(|x|) \qquad \text{and} \qquad J(x) > 0. \tag{II.1.3}$$

The decay of J is encoded in the function

$$f := -\ln(J), \tag{II.1.4}$$

We further assume the following three properties:

Monotonicity and asymptotic convexity of J. *The function $f \in \mathcal{C}^2(\mathbb{R})$ is strictly increasing on $(0, +\infty)$ and asymptotically concave, that is, there exists $x_{conc} > 0$ such that*

$$\begin{cases} f(x) > f(y), & \text{if } x > y \geqslant 0 \qquad \text{and} \\ f''(x) \leqslant 0 & \text{if } x \geqslant x_{conc}. \end{cases} \tag{II.1.5}$$

Without loss of generality, we suppose that $f(0) = 0$, or $J(0) = 1$, since otherwise we may re-scale the equation. This implies that $J(\mathbb{R}) = (0, 1]$. Moreover, J is invertible on $\mathbb{R}^+$, this inverse from $(0, 1]$ to $\mathbb{R}^+$ is what we denote J^{-1} in the sequel. Similarly, f is invertible on $\mathbb{R}^+$, this inverse from $\mathbb{R}^+$ to $\mathbb{R}^+$ is what we denote J^{-1} in the sequel.

Lower bound on the tail of J. *The kernel J decays slower than any exponential in the sense that*

$$\limsup_{x\to\infty} \frac{x\,f'(x)}{f(x)} < 1. \tag{II.1.6}$$

Roughly speaking, this implies that f grows sub-linearly and that $J'(x) = o_{x\to\infty}(J(x))$.

Upper bound on the tail of J. *The tail of J is thinner than $|x|^{-1}$, in the sense that*

$$\liminf_{x\to\infty} x\,f'(x) > 1. \tag{II.1.7}$$

For ease of notation and since it will play a role in our analysis, we define $\mu := \liminf_{x\to\infty} x\,f'(x)$.

These kernels are called *fat–tailed* kernels. The main examples of kernels J that satisfy (II.1) are either sub–exponential kernels where $f(x) = (1 + |x|^2)^{\alpha/2}$ with $\alpha < 1$ or polynomial kernels where $f(x) = \alpha \ln(1 + |x|^2)/2$ with $\alpha > 0$. Our technical assumptions (II.1.5)–(II.1.7) do not cover borderline kernels such as $f(x) = |x|/\ln(1 + |x|^2)$ that were considered by Garnier in Garnier [2011]. Moreover, we restrict our focus to the effects of the tails of J on the rate of propagation. As a consequence, we do not include potential singularities at the origin, which is the case for a fractional Laplacian operator, for example. We expect however that our results also hold for these cases.

The author of Garnier [2011], proved that the acceleration propagation of the solution of (II.1.1) can be measured by tracking the level sets $E_\lambda(t) := \{x \in \mathbb{R} : n(t, x) = \lambda\}$ of the solution n, where $\lambda \in (0, 1)$. Under the fat–tailed kernel hypothesis, these level sets move super-linearly in time. More precisely, he proved that there exists a constant $\rho > 1$ such that for any $\lambda \in (0, 1)$ and any $\varepsilon > 0$, any element x_λ of the level set E_λ satisfies for all $\varepsilon > 0$ and t large enough:

$$J^{-1}\left(e^{-(1-\varepsilon)t}\right) \leqslant |x_\lambda(t)| \leqslant J^{-1}\left(e^{-\rho t}\right). \tag{II.1.8}$$

The propagation problem has been recently considered with non–symmetric kernels in Finkelshtein and Tkachov [2017] and in the multi-dimensional case in Finkelshtein et al. [2016]. There Finkelshtein, Kondratiev, and Tkachov present a technical argument improving the precision in Garnier's bounds. As will be made clear below, one goal of this paper is to connect these results to an underlying Hamilton-Jacobi equation, giving a new interpretation to the speed of propagation. In doing so, we aim to provide a simpler proof of propagation that is, in some regimes, more precise than those above. The approach we use comes from the seminal paper of Evans and Souganidis Evans and Souganidis [1989], where the authors applied the long time/long range limit to the Fisher-KPP equation and showed convergence, after applying the Hopf-Cole transform, to a Hamilton–Jacobi equation. Further, they show that the solution to this Hamilton–Jacobi equation determines the propagation rate of the original Fisher-KPP equation. In the first part of our work, we first describe the scaling corresponding to the long time/long range limit for (II.1.1). Then we show convergence to a Hamilton–Jacobi equation, which gives information on the spreading properties of the solution to the original (II.1.1).

The same approach has been used extensively to understand propagation in various physical systems as well as more general qualitative behavior of solutions of parabolic equations. In the context of adaptative dynamics, Diekmann, Jabin, Mischler, and Perthame in Diekmann et al. [2005] have provided an example of this approach. They have derived, by a limiting procedure, a Hamilton–Jacobi equation from a mutation–selection equation with small mutations.

The literature on this topic is enormous, but we mention a few closely related works. Perthame and Souganidis studied (II.1.1) with thin–tailed kernels in Perthame and Souganidis [2005]. Barles, Mirrahimi and Perthame focused on Dirac concentration in integro-differential equations with local and non–local non–linearities in Barles et al. [2009]. Recently, Mirrahimi and Méléard extended this approach to a fractional diffusion Méléard and Mirrahimi [2015]. The second part of our work is inspired by and closely follows this last work, extending their technique to the case with a general dispersal kernel J, the main difficulties coming from the fact that, in contrast to the fractional Laplacian, the kernel J is not explicit and has no natural scaling. As a consequence, the scalings we use are not always easy to read but we give some heuristics that make them appear naturally. Moreover, our scalings allows to characterize the small mutation regime when the mutation kernel is fat–tailed.

In the present work, we only focus on the local Fisher–KPP nonlinearity $n(1-n)$ but our results can be generalized to a non–local non–linearity of the form $n(r - \int_{\mathbb{R}} n(x)dx)$, exactly as in Méléard and Mirrahimi [2015]. This non–local term arises also naturally in the context of mutation–selection model or structured population models. In this context, x denotes a quantitative trait and $n(t,\cdot)$ describes the distribution of this trait inside the population. Thus, the parameter r describes the fitness of the population and the integral term $\int_{\mathbb{R}} n(t,x)dx$ is a mean–competition term. The model (II.1.1) with this non–local nonlinearity can be derived rigorously from an individual–based model where mutations are described by a fat–tailed kernel without jump (see for instance Baeumer et al. [2007]; Gurney and Nisbet [1975]). In general, the growth rate r depends also on the trait parameter x Jourdain et al. [2012]. However, this general form induces more technical difficulties that we do not tackle in this paper.

The propagation regime

In order to capture the accelerated propagation phenomenon that occurs with fat–tailed kernels, we look at the behavior of n in the long time/long range limit. Indeed, we first rescale the time $t \mapsto t/\varepsilon$ by a small parameter ε and then we need to find an accurate rescaling in space that captures the propagation regime. We thus look for a space rescaling of the form $x \mapsto \psi_\varepsilon(x)$. The seminal paper Evans and Souganidis [1989] used the hyperbolic scaling $(t/\varepsilon, x/\varepsilon)$ for the asymptotic study of the Fisher-KPP equation. The precise shape of ψ_ε will be given below, but we first start with heuristic explanation of this expression.

Using the results of Garnier (II.1.8), it is reasonable to say that the position x of any level sets satisfies

$$J(x) \sim e^{-t}.$$

Our aim is to find a rescaling ψ_ε that follows the level sets in the long time rescaling t/ε. So we want

$$J(\psi_\varepsilon(x)) \sim e^{-\frac{t}{\varepsilon}} \sim \left(e^{-t}\right)^{\frac{1}{\varepsilon}} \sim J(x)^{\frac{1}{\varepsilon}}.$$

As a consequence, for any $\varepsilon > 0$, it is natural to set

$$\psi_\varepsilon(x) = \operatorname{sign}(x) J^{-1}\left(J(x)^{\frac{1}{\varepsilon}}\right) \quad \text{for all } x \in \mathbb{R}. \tag{II.1.9}$$

The rescaling ψ_ε transforms functions that looks like J into functions that looks like $\exp\left\{\left(\ln J(x)\right)/\varepsilon\right\}$. Indeed, notice that $J(\psi_\varepsilon(x)) = J(x)^{1/\varepsilon} = \exp\left\{\left(\ln J(x)\right)/\varepsilon\right\}$. Since the solution n of the Cauchy problem (II.1.1) is expected to behave like $e^t J$ for large x, we can heuristically say that the rescaled function $n_\varepsilon = n(t/\varepsilon, \psi_\varepsilon)$ should behave like $e^{t/\varepsilon} J(\psi_\varepsilon(x)) = \exp\left\{(t + \ln J(x))/\varepsilon\right\} \sim \exp\left\{(\ln n(t,x))/\varepsilon\right\}$. The last expression is the logarithmic Hopf–Cole transformation of $n(t,x)$. Our rescaling is thus compatible with this transformation.

The scaling ψ_ε can also be rewritten in terms of the function f introduced in II.1. Indeed,

$$\psi_\varepsilon(x) = \operatorname{sign}(x) f^{-1}\left(f(x)/\varepsilon\right).$$

We derive a precise formula for this scaling for our two main examples: the sub–exponential kernels and the polynomial kernels.

Example II.2 (Sub–exponential kernels). *Consider $f(x) := (1+|x|^2)^{\alpha/2} - 1$ with $\alpha \in (0,1)$. Then*

$$\psi_\varepsilon(x) = \operatorname{sign}(x)\left[\left(1 + \frac{1}{\varepsilon}\left[\left(1+|x|^2\right)^{\frac{\alpha}{2}} - 1\right]\right)^{\frac{2}{\alpha}} - 1\right]^{\frac{1}{2}}.$$

Observe that $\psi_\varepsilon(x) \sim \varepsilon^{-\frac{1}{\alpha}} x$ when $|x| \to +\infty$ at fixed ε and $\psi_\varepsilon(x) \sim \varepsilon^{-\frac{1}{\alpha}} \operatorname{sign}(x)\left[\left(1+|x|^2\right)^{\frac{\alpha}{2}} - 1\right]^{\frac{1}{\alpha}}$ when $\varepsilon \to 0$ and $x \neq 0$.

Example II.3 (Polynomial kernels). *Consider $f(x) := (1+\alpha)\ln\left((1+|x|^2)^{\frac{1}{2}}\right)$ for $\alpha > 0$. In this case, the scaling becomes*

$$\psi_\varepsilon(x) = \operatorname{sign}(x)\sqrt{(1+x^2)^{1/\varepsilon} - 1}.$$

One can observe that, for any fixed ε and α, $\psi_\varepsilon(x) \sim \operatorname{sign}(x)|x|^{\frac{1}{\varepsilon}}$ as $|x| \to +\infty$. In addition, $\psi_\varepsilon(x) \sim \operatorname{sign}(x)\left(1+|x|^2\right)^{\frac{1}{2\varepsilon}}$, when $\varepsilon \to 0$ at fixed $x \neq 0$. We point out that when $\alpha \in (0,2)$, the

kernel J decays at the same rate as the kernel in the definition of the fractional Laplacian $(-\Delta)^{\alpha/2}$ as $|x| \to \infty$. This suggests that the behaviour of the solution of (II.1.11) with f as above and $\alpha \in (0,2)$ and the behaviour of the solution of (II.1.1) $(-\Delta)^{\alpha/2} n$ in the place of $J * n - n$ are the same. We verify this below. We also point out that in the limit $|x| \to +\infty$, $\psi_\varepsilon(x) \sim \operatorname{sign}(x)|x|^{\frac{1}{\varepsilon}}$, which is the rescaling chosen in Méléard and Mirrahimi [2015] for the fractional Laplacian.

As far as the initial data is concerned, we assume without lost of generality that there exists two positive constants $\underline{C}$ and $\overline{C}$ such that $\underline{C} < 1 < \overline{C}$ and

$$\underline{C} J \leqslant n^0 \leqslant \overline{C} J. \tag{II.1.10}$$

Moreover, we assume that the initial data is symmetric, and since the kernel J is also assumed to be symmetric (see (II.1.3)), the solution n thus remains symmetric for all times.

Remark II.4. *The lower bound in assumption (II.1.10) is not restrictive, though it allows us to avoid discussion of a boundary layer at $t = 0$. Indeed, assuming (II.1), any solution of (II.1.1) starting with initial data that decays faster than J at infinity satisfies (II.1.10) after at most time 1. More precisely, for any n^0 decaying faster than J, there exist constants $\underline{C}$ and $\overline{C}$ such that $\underline{C} J(x) \leqslant n(1,x) \leqslant \overline{C} J(x)$ for any $x \in \mathbb{R}$ (see, e.g., [Garnier 2011, Section 4.2]). After translating in time, our argument applies with initial data $n(1,x)$. From the uniqueness of solutions of the Cauchy problem (II.1.1), our conclusions hold for n as well.*

If, on the other hand, the upper bound in (II.1.10) does not hold, i.e. n^0 decays slower than J, then we expect different behavior. Indeed, by analogy with Alfaro [2017]; Hamel and Roques [2010]; Henderson [2016], we expect faster propagation depending only on the rate of decay of the initial data.

In view of the above, the assumption (II.1.10) is quite general for the regimes that we wish to understand.

Let us now rescale time and space as follows: $t \mapsto t/\varepsilon$ and $x \mapsto \psi_\varepsilon(x)$ and define the solution n_ε in the new variables: $n_\varepsilon(t,x) = n(t/\varepsilon, \psi_\varepsilon(x))$ where n solves (II.1.1) with initial condition n^0 satisfying (II.1.10). Plugging this quantity into (II.1.1), we obtain the following equation:

$$\begin{cases} \varepsilon \partial_t n_\varepsilon = \displaystyle\int_{-\infty}^{\infty} J(h) \left[n_\varepsilon \left(t, \psi_\varepsilon^{-1}\left(\psi_\varepsilon(x) - h\right)\right) - n_\varepsilon \right] dh \; + \; n_\varepsilon \left(1 - n_\varepsilon\right) & \text{in } (0,\infty) \times \mathbb{R}, \\ n_\varepsilon(0,x) = n^0(\psi_\varepsilon(|x|)), & \text{for } x \in \mathbb{R}. \end{cases} \tag{II.1.11}$$

We know from Garnier [2011] that the solutions of (II.1.11) will propagate and converge to one as $t \to \infty$. In the large scale limit $\varepsilon \to 0$ with our change of variables, we expect this propagation to be transformed into dynamics of an interface moving with time. To capture this phenomenon, we use the logarithmic Hopf–Cole transform Evans and Souganidis [1989]; Freidlin [1985] as follows:

$$u_\varepsilon := -\varepsilon \ln n_\varepsilon. \tag{II.1.12}$$

Notice that this is equivalent to $n_\varepsilon = \exp\left(-\frac{u_\varepsilon}{\varepsilon}\right)$. Then, the function u_ε solves:

$$\begin{cases} \partial_t u_\varepsilon + 1 = \displaystyle\int_{-\infty}^{\infty} J(h) \left[1 - e^{-\frac{1}{\varepsilon}\left(u_\varepsilon\left(t, \psi_\varepsilon^{-1}(\psi_\varepsilon(x) - h)\right) - u_\varepsilon(t,x)\right)} \right] dh \; + n_\varepsilon, & \text{in } (0,\infty) \times \mathbb{R} \\ u_\varepsilon(0,x) = -\varepsilon \ln\left(n^0(\psi_\varepsilon(x))\right), \; x \in \mathbb{R}. \end{cases} \tag{II.1.13}$$

Note that assumptions (II.1.10) imply that $u_\varepsilon(0,\cdot) \to f$ uniformly in $\mathbb{R}$ as $\varepsilon \to 0$. Our aim is to compute the limit $\varepsilon \to 0$ of u_ε and then deduce the behavior of n_ε. The result is the following.

Theorem II.5. *Let u_ε be the solution of (II.1.13) with initial condition satisfying (II.1.10). If the kernel J satisfies Hypothesis (II.1), then as $\varepsilon \to 0$, the sequence u_ε converges locally uniformly on $(0,\infty)\times\mathbb{R}$ to*

$$u(t,x) := \max\{f(x) - t, 0\}.$$

From this convergence result, we may deduce the asymptotics of n_ε.

Theorem II.6. *Let n_ε be the solution of (II.1.11) with the initial data satisfying (II.1.10). If the kernel J satisfies Hypothesis II.1, then*

(a) uniformly on compact subsets of $\{u > 0\}$,

$$\lim_{\varepsilon\to 0} n_\varepsilon = 0;$$

(b) for every compact subset $K \subset \mathrm{Int}\,(\{u(t,x) = 0\})$,

$$\lim_{\varepsilon\to 0} n_\varepsilon(t,x) = 1,$$

where the limit is uniform in K.

Since f is a continuous and increasing function of $|x|$, the boundary of $\{u(t,x) = 0\}$ is given by $|x| = f^{-1}(t)$. Hence, as $\varepsilon \to 0$, $n_\varepsilon \sim 1$ if and only if $|x| < f^{-1}(t)$. Since $n(t,x) \sim n_\varepsilon(1, \psi_\varepsilon^{-1}(x))$ with $\varepsilon = 1/t$, then as $t \to \infty$ we see that $n(t,x) \sim 1$ if and only if $|x| < f^{-1}(t)$. As such, Theorem II.5 and Theorem II.6 imply that the location of the front of n is $\sim f^{-1}(t)$.

Let us apply our two main results Theorem II.5 and Theorem II.6 to our basic examples.

Example II.7. *When f is a sub–exponential kernel of the form $f(x) = (1+|x|^2)^{\alpha/2}$ with $\alpha \in (0,1)$, we see that the front is located at $\sim t^{1/\alpha}$. In the thin-tailed limit $\alpha \to 1$ see recover constant speed propagation.*

On the other hand, when f is a polynomial kernel of the form $f(x) = (1+\alpha)\ln(1+|x|^2)/2$, with $\alpha > 0$, we see that the front is located at $\sim e^{t/(1+\alpha)}$.

In Theorem II.5 the dispersion kernel J disappears when we pass to the limit $\varepsilon \to 0$, in the sense that the constrained Hamilton–Jacobi equation satisfied by the limit function u is simply $\min\{\partial_t u + 1, u\} = 0$ in which J is absent. Solutions to this equation are given by $\max\{u_0(\cdot) - t, 0\}$, so that the effect of the kernel is felt only through the initial data. Without the assumption (II.1.10), a boundary layer at $t = 0$ would develop during the limit $\varepsilon \to 0$. See Remark (II.4).

This is quite different from the case of a thin tailed kernel, for which the Hamiltonian would typically contain a term of the form $\widehat{J} - 1$ Diekmann et al. [2005]; Perthame and Souganidis [2005]; Barles et al. [2009]. This is explained by our rescaling ψ_ε which focuses on the behavior at infinity of the solution, and thus mainly ignores the precise dynamics of the dispersion. This phenomenon was already observed in Méléard and Mirrahimi [2015] for the fractional Laplacian. Despite this, Theorem II.6 states that our rescaling is sharp enough to capture the interface at infinity.

One way to understand intuitively why the kernel disappears in the limiting equation is to investigate the integral term in (II.1.13). Due to the fat–tailed assumption in Hypothesis (II.1), the quantity $\psi_\varepsilon^{-1}(\psi_\varepsilon(x) + h) - \psi_\varepsilon^{-1}(\psi_\varepsilon(x))$ is likely to go to zero faster than ε. Hence, the integral disappears in the limit. While this is formally clear, it is difficult to make this intuition rigorous.

We comment momentarily on the method of proof. We construct explicit sub- and super-solutions of u using the kernel J and the general solution to the logistic equation. While the most natural

thing to do would be to use half relaxed limits along with the limiting Hamilton-Jacobi equation (see Barles and Perthame [1990]), the non–locality of the kernel makes this very difficult because the non-local term in the equation "sees" all of $\mathbb{R}$ but the half-relaxed limits only provide convergence locally. Thus, as in Méléard and Mirrahimi [2015], we construct sharp sub- and super–solutions of (II.1.13) to conclude. The construction of these sub- and super–solutions also provides sharp sub- and super–solutions for equation (II.1.1). We point out that Theorem II.6 improves the existing bounds in Garnier [2011].

To illustrate the results of Theorem II.5, we provide the results of some numerical simulations in Figure II.1 for four choices of kernels J: a Gaussian kernel for which linear spreading is expected Coville and Dupaigne [2005], two sub-exponential kernels $J \sim \exp(-|\cdot|^{1/2})$, $J \sim \exp(-|\cdot|^{3/4})$ and a polynomial one $J \sim \left(1+|\cdot|^5\right)^{-1}$.

The small mutations limit

Our main equation (II.1.1) also arises naturally in the context of population genetics, to capture the effect of genetic mutations Barles et al. [2009]; Perthame and Barles [2008]. Under this perspective, the variable x now corresponds to a phenotypic trait and the convolution term describes the mutation process during which an individual with trait x can give birth to an individual with trait $x+h$ with probability $J(h)$.

We are interested in a situation where large–effect mutations, while still uncommon, are *relatively* frequent. This is exactly what is encoded in a mutation kernel with fat–tails. The aim of this section is to understand the effect of these large mutation events on the adaptive dynamics when the mean effect of mutations is small. This regime of small mean–effect of mutations will be referred as the small mutation regime. Note that even in the small mutation regime, mutation events with a large effect can occur. The main difficulty is to identify the appropriate scaling of this small mutation regime when the mutation kernel has fat tails.

To work in the small mutation regime, we introduce a small parameter ε, such that ε^{-1} typically represents the time-scale on which these mutations accumulate. This time scale being given, one needs to scale the size of the mutations in a relevant way to capture the expected (non-trivial) dynamics.

In the case of a thin–tailed mutation kernel, the small mutation regime corresponds to mutation kernel with small variance of order ε^2. Thus, it is natural to rewrite the mutation kernel $J(h)$ as $J_\varepsilon(h) := J(h/\varepsilon)/\varepsilon$ where now J is of variance 1. With such transformation, the jump $x \mapsto x+h$ with probability $J(h)$ is replaced by the jump $x \mapsto x+\varepsilon h$ with the same probability $J(h)$. In this case, the asymptotic behavior of the population is described by a Hamilton–Jacobi equation Barles et al. [2009]; Diekmann et al. [2005]; Perthame and Barles [2008].

In the fat–tailed setting, however, the large mutation events modify the dynamics. As a consequence, it is necessary to rescale the jump size non-linearly to take this into account as the scaling above does not contract the kernel enough. Indeed, if the size of jumps is of order ε it does not go to zero fast enough to be comparable with the contraction of the kernel. Inspired by our first part about propagation, we replace the jumps $x \mapsto x+h$ by $x \mapsto x+\psi_\varepsilon(h)$ with the same probability, where

$$\psi_\varepsilon(h) = \operatorname{sign}(h) f^{-1}\left(\frac{f(h)}{\varepsilon}\right).$$

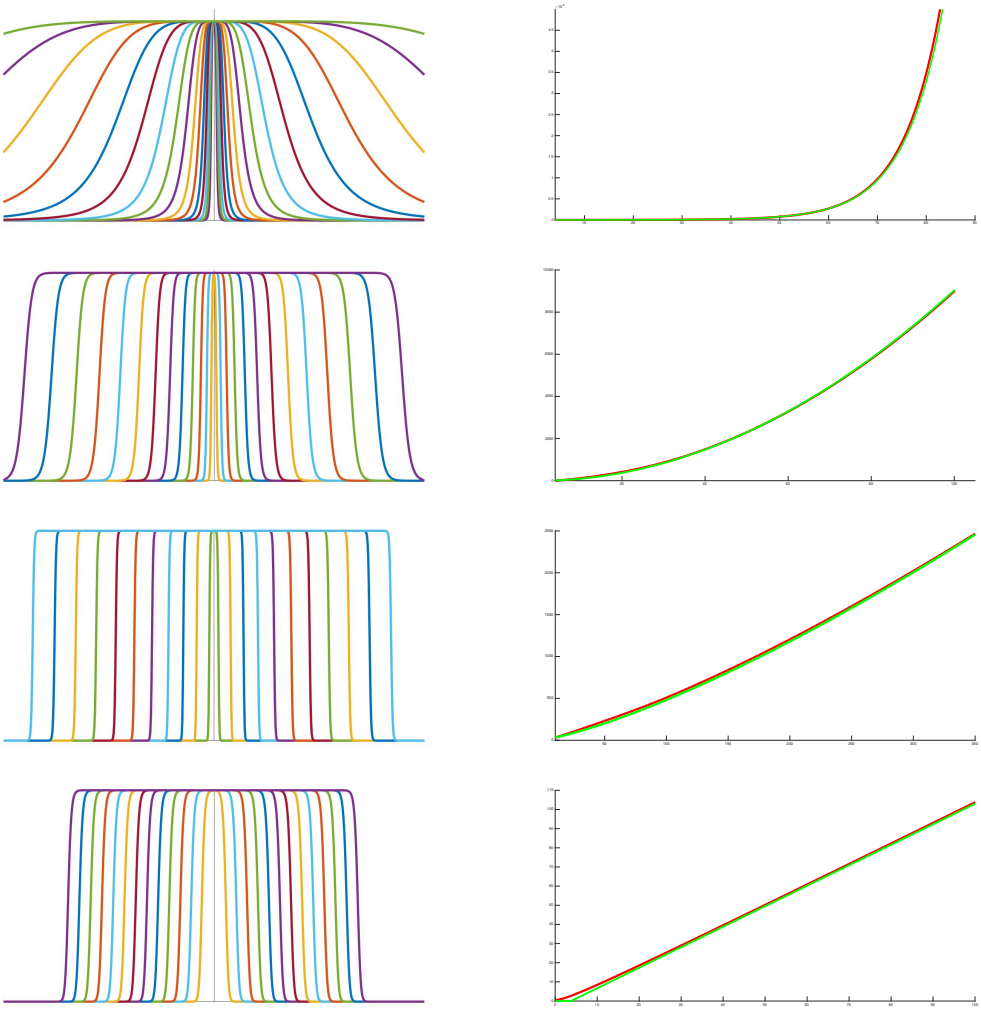


Figure II.1: Numerical simulations of the Cauchy problem (II.1.1), for various kernels J. The first line corresponds to $J \sim |x|^{-5}$, the second line to $J \sim \exp\left(-|x|^{1/2}\right)$, the third line to $J \sim \exp\left(-|x|^{3/4}\right)$ and the last line to a Gaussian kernel $J \sim \exp\left(-|x|^2\right)$. For each kernel, we present in the left column the evolution of the solution by plotting it on the same figure for various successive (linearly chosen) values of time t. To quantify this and recover and illustrate Theorem (II.6), we present in the right column the time evolution of the level set $\{x \in \mathbb{R} : n(t,x) = 1/2\}$ for each kernel. The red bold curve is the numerical simulation, starting from an initial condition of the form J. The green curve is the expected asymptotic rate of expansion predicted by Theorem (II.5), that is $J^{-1}(e^{-t})$, except for the Gaussian kernel, in which case it is a line. Each kernel is successively more thin–tailed, yielding an interpolation between the obvious acceleration for the first kernel and the linear propagation for the last kernel.

Our jump size scaling procedure is actually equivalent to rewrite the mutation kernel J. The mutation kernel J is transformed into J_ε defined by

$$J(h) = e^{-f(h)} \longmapsto J_\varepsilon(h) = \frac{1}{\varepsilon} \frac{f'(h)}{f'\left(\psi_\varepsilon^{-1}(h)\right)} J(h)^{\frac{1}{\varepsilon}}. \tag{II.1.14}$$

where now J is a fat–tailed kernel of mass 1 and it satisfies

$$\int_{\mathbb{R}} f(x)^2 J(x)dx = 1.$$

Thus, with fat–tailed mutation kernel, the small mutation regime corresponds to mutation kernel with the following quantity is small, that is of order ε^2:

$$\int_{\mathbb{R}} f(x)^2 J_\varepsilon(x)dx = \varepsilon^2.$$

Observe that in the case of a thin tailed kernel, this quantity would be exactly the variance of J_ε, which is exactly what happens in the small mutation regime in this case. From the formula (II.1.14), we can observe that J_ε is a contraction of J.

Due to the small size of the mutations, their effect can only be seen after many mutations accumulate. Hence, we want to capture the long time behavior of the population, or, equivalently, the setting where the rate of mutation is large. This suggests that we rescale the time by the parameter ε as $t \mapsto t/\varepsilon$. Under this rescaling and the rewriting of the mutation (II.1.14), (II.1.1) becomes:

$$\varepsilon \partial_t n_\varepsilon(t,x) = \int_{\mathbb{R}} \left(n_\varepsilon(t, x - \psi_\varepsilon(h)) - n_\varepsilon(t,x)\right) J(|h|)dh + n_\varepsilon(1 - n_\varepsilon). \tag{II.1.15}$$

As in the propagation regime, the scaled size of jumps goes to 0 as $\varepsilon \to 0$. So we expect the solution n_ε to concentrate. In order to capture this concentration phenomenon, we perform the logarithmic Hopf–Cole transformation, $u_\varepsilon := -\varepsilon \ln(n_\varepsilon)$ and u_ε satisfies the following equation:

$$\begin{cases} \partial_t u_\varepsilon(t,x) + \int_{\mathbb{R}} e^{-\frac{1}{\varepsilon}\left(u_\varepsilon(t,x-\psi_\varepsilon(h)) - u_\varepsilon(t,x)\right)} J(h)\mathrm{d}h = n_\varepsilon(t,x), & \text{on } \ (0,+\infty)\times\mathbb{R} \\ u_\varepsilon(t,\cdot) = u_\varepsilon^0. \end{cases} \tag{II.1.16}$$

Before stating our main results, we need the following additional technical assumption on the derivative of $f = -\ln(J)$ at $x = 0$.

Hypothesis II.8. *Assume that $f \in C^2(0,\infty)$ and that f satisfies $\lim_{x\searrow 0} f(x)/x \in (0,\infty)$. We abuse notation by denoting it $f'(0)$. Additionally, f satisfies all the assumptions of Hypothesis II.1 except for regularity at zero. Namely, $f \in C^0(\mathbb{R})$ but f is not C^2 at 0.*

We also require additional assumptions on the initial data $u_\varepsilon^0 = -\varepsilon \ln(n_\varepsilon^0)$. We assume that u_ε^0 is a positive sequence of Lipschitz continuous functions which converges locally uniformly to u^0 as $\varepsilon \to 0$ and there exists $A \in (0, 1 - 1/\mu)$ where, we recall, $\mu := \liminf_{x\to+\infty} |x| f'(|x|) > 1$ such that for all $x, h \in \mathbb{R}$:

$$u_\varepsilon^0(x+h) - u_\varepsilon^0(x) \geqslant -Af(h). \tag{II.1.17}$$

Note that $(u_\varepsilon^0)_\varepsilon$ is thus uniformly locally bounded and $n_\varepsilon^0 = \exp(-u_\varepsilon^0/\varepsilon)$ satisfies, for all $x \in \mathbb{R}$ and $\varepsilon > 0$,

$$0 \leqslant n_\varepsilon^0(x) \leqslant 1. \tag{II.1.18}$$

From the maximum principle, we have that $0 < n_\varepsilon(t,x) < 1$ for all $x \in \mathbb{R}$ and $t > 0$. Moreover, the property (II.1.17) propagates for any positive time – see the following lemma.

Lemma II.9. *Let f satisfy Hypothesis II.8. Then any solution u_ε of (II.1.16) starting with initial condition u_ε^0 satisfying (II.1.17), satisfies the following properties:*

1. *the sequence $(u_\varepsilon)_\varepsilon$ is locally uniformly bounded. In particular, there exists $r > 0$ such that, for all $x \in \mathbb{R}$ and $t \in (0, \infty)$,*
$$-rt \leqslant u_\varepsilon(t,x) - u_\varepsilon^0(x) \leqslant t;$$

2. *for all $t \geqslant 0$ and all $x, h \in \mathbb{R}$, we have:*
$$u_\varepsilon(t, x+h) - u_\varepsilon(t,x) \geqslant -Af(|h|). \tag{II.1.19}$$
In particular u_ε is uniformly Lipschitz with respect to x with the bound
$$\|\partial_x u_\varepsilon\|_{L^\infty((0,+\infty)\times\mathbb{R})} \leqslant Af'(0). \tag{II.1.20}$$

The local uniform estimates of Lemma II.9 allows us to define the following upper- and lower–half–relaxed limits of u_ε by the following formulas:
$$\underline{u}(t,x) := \liminf_{\substack{\varepsilon\to 0\\(s,y)\to(t,x)}} u_\varepsilon(s,y) \qquad \text{and} \qquad \overline{u}(t,x) := \limsup_{\substack{\varepsilon\to 0\\(s,y)\to(t,x)}} u_\varepsilon(s,y)\,. \tag{II.1.21}$$

From the properties of half-relaxed limits, the estimates (II.1.19) and (II.1.20) hold true for the functions $\underline{u}$ and $\overline{u}$. In addition, it is apparent that $\underline{u} \leqslant \overline{u}$ in $(0,\infty)\times\mathbb{R}$ by construction. With this sub- and super–solution in hands, we can state our main results on the asymptotics of u_ε and n_ε as $\varepsilon \to 0$.

Theorem II.10. *Let f satisfy Hypothesis II.8 and let u_ε be the solution of (II.1.16) starting with initial condition u_ε^0 satisfying (II.1.17). Then as $\varepsilon \to 0$ we have the following:*

i) the upper (resp. lower) half–relaxed limit $\overline{u}$ (resp. $\underline{u}$) is a sub- (resp. super-) solution to the following constrained Hamilton–Jacobi equation:
$$\min\left\{\partial_t u + \int_{\mathbb{R}}\left[e^{\frac{\operatorname{sign}(h)f(h)}{f'(0)}\partial_x u} - 1\right] J(h)\,\mathrm{d}h + 1, u\right\} = 0, \quad \text{on } (0,+\infty)\times\mathbb{R};$$

ii) the sequence $(u_\varepsilon)_\varepsilon$ converges locally uniformly on $(0,+\infty)\times\mathbb{R}$ to a function u that is Lipschitz continuous with respect to x and continuous in time, and which is a viscosity solution to
$$\begin{cases} \min\left\{\partial_t u + \int_{\mathbb{R}}\left[e^{\frac{\operatorname{sign}(h)f(h)}{f'(0)}\partial_x u} - 1\right] J(h)\,\mathrm{d}h + 1, u\right\} = 0, \quad \text{on } (0,+\infty)\times\mathbb{R}, \\ u(0,\cdot) = u^0. \end{cases} \tag{II.1.22}$$

We first note that the integral term in (II.1.22) is well defined due to the inequality (II.1.20). It is also related, up to a change of variables, to the analogous equation obtained by Méléard and Mirrahimi [Méléard and Mirrahimi 2015, equation (27)]. The proof of Theorem II.10, appearing in Section II.4, uses the half–relaxed limits method introduced by Barles and Perthame Barles and Perthame [1990]. It relies heavily upon Lemma II.9, which is proved last in Section II.4.3. Under the small mutation regime, we do not use explicit sub- and super–solutions. In particular, the sub- and super–solutions introduced in the propagation regime are not relevant in this situation.

The previous result Theorem II.10 on the behavior of $u_\varepsilon = -\varepsilon\ln(n_\varepsilon)$ as $\varepsilon \to 0$ allows us to study the convergence of n_ε as $\varepsilon \to 0$. Heuristically, when ε is small, we expect that $n_\varepsilon \simeq \exp(u(t,x)/\varepsilon)$. Thus, the solution u gives an indication on where the solution n_ε is concentrated in the regime of small mutations at least at a first order of approximation. More precisely we obtain the following result:

Theorem II.11. *Let f satisfy Hypothesis II.8 and let n_ε be the solution of (II.1.15) starting with initial condition n_ε^0 such that $u_\varepsilon^0 = -\varepsilon \ln(n_\varepsilon^0)$ satisfies (II.1.17). Then as $\varepsilon \to 0$,*

$$\begin{cases} n_\varepsilon \to 0 \text{ locally uniformly in } \mathcal{A} = \{(t,x) \in (0,\infty)\times\mathbb{R} \mid u(t,x) > 0\}, \\ n_\varepsilon \to 1 \text{ locally uniformly in } \mathcal{B} = Int\{(t,x) \in (0,\infty)\times\mathbb{R} \mid u(t,x) = 0\}. \end{cases} \tag{II.1.23}$$

We now discuss, heuristically, Theorem II.11. When ε is small, we expect that $n_\varepsilon \sim 1$ in $\mathcal{B}$, implying that, at time t, the phenotype x is realized in the population if $(t,x) \in \mathcal{B}$. On the other hand, $n_\varepsilon \sim 0$ in $\mathcal{A}$. Similarly, the trait x at time t will not be realized in the population if $(t,x) \in \mathcal{A}$. Hence, if the mutations are sufficiently small, i.e. if ε is small enough, the sets $\mathcal{A}$ and $\mathcal{B}$ determine which phenotypes are realized in the population. We can thus deduce the form of the solution n_ε when ε is small. This final result is proved in Section II.4.2.

To illustrate Theorem II.10, we discuss the following example which is also discussed in the paper of Mirrahimi and Méléard Méléard and Mirrahimi [2015] in the case of the fractional Laplacian. We show how our results Theorem II.10 and Theorem II.11 gives an approximation of the behaviour of the solution n_ε of the problem (II.1.15).

Example II.12. *Let n_ε be the solution of (II.1.15) starting with $n_\varepsilon^0(x) = J(x)^{A/\varepsilon}$, then the initial condition $u_\varepsilon^0(x) = -\varepsilon \ln(n_\varepsilon^0(x)) = Af(x)/\varepsilon$. Notice that u_ε^0 satisfies (II.1.17) because f is concave. It follows from the Theorem II.10 that u_ε converge locally uniformly to the unique viscosity solution of*

$$\begin{cases} \min\left(\partial_t u + H(\partial_x u), u\right) = 0, \quad \text{on } (0,+\infty)\times\mathbb{R}, \\ u(0,\cdot) = Af. \end{cases}$$

where the Hamiltonian H is defined by

$$H(p) = \int_{\mathbb{R}} \left[e^{\frac{\text{sign}(h) f(h)}{f'(0)} p} - 1 \right] J(h)\, \mathrm{d}h + 1.$$

From a Taylor expansion, one can check that there exists two positive constants $\overline{\kappa}$ and $\underline{\kappa}$ such that

$$\begin{aligned} 1 + \underline{\kappa} p^2 &= 1 + p^2 \int_0^\infty \left(\frac{f(h)}{f'(0)}\right)^2 e^{-\frac{f(h)}{f'(0)} A} J(h)\, \mathrm{d}h \\ &\leqslant H(p) \leqslant 1 + p^2 \int_0^\infty \left(\frac{f(h)}{f'(0)}\right)^2 e^{\frac{f(h)}{f'(0)} A} J(h)\, \mathrm{d}h = 1 + \overline{\kappa} p^2. \end{aligned}$$

Using these inequalities, we obtain the following estimates of u:

$$\max\left(\inf_{y\in\mathbb{R}}\left(Af(y) + \frac{|x-y|^2}{4\underline{\kappa} t} - t\right), 0\right) \leqslant u(t,x) \leqslant \max\left(\inf_{y\in\mathbb{R}}\left(Af(y) + \frac{|x-y|^2}{4\overline{\kappa} t} - t\right), 0\right)$$

We then deduce that

$$\begin{aligned} &\left\{(t,x) \in (0,+\infty)\times\mathbb{R} : \ |x| \leqslant \max_{r\in[0,1]} \left(2\sqrt{\underline{\kappa}} r t + f^{-1}(t(1-r^2)/A)\right)\right\} \subset \{u = 0\} \\ &\qquad \subset \left\{(t,x) \in (0,+\infty)\times\mathbb{R} : \ |x| \leqslant \max_{r\in[0,1]} \left(2\sqrt{\overline{\kappa}} r t + f^{-1}(t(1-r^2)/A)\right)\right\} \end{aligned}$$

Combining these estimates with Theorem II.11, we conclude that the population propagates in the phenotype space to be order $f^{-1}(t/A)$ for large time. Roughly, when the mutations in (II.1.15)

are sufficiently small and time is sufficiently large, we see that the phenotypes x realized in the population are those for which $x \lesssim f^{-1}(t/A)$ and those that have not been realized are those for which $x \gtrsim f^{-1}(t/A)$. Moreover, we can deduce the following approximation at the first order of ε. Formally, using the convergence Theorem II.10, we can say that the solutions $n_\varepsilon = \exp(-u_\varepsilon/\varepsilon)$ of (II.1.15) can be approximated by $\exp(-u/\varepsilon)$. Roughly, we conclude that $n_\varepsilon(t,x) \simeq 1$ when $|x| \lesssim f^{-1}(t/A)$ and that $n_\varepsilon(t,x) \simeq 0$ when $|x| \gtrsim f^{-1}(t/A)$, up to some additional small error depending on ε.

Remark II.13. *Throughout the paper, we use C to refer to any constant depending only on the kernel J. This constant may change line-by-line.*

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II.2 THE PROPAGATION RESULT: PROOF OF THEOREM II.5.

To prove Theorem II.5, we construct sharp explicit sub- and super–solutions to the non-rescaled problem (II.1.1). Our sub- and super–solutions are sharp enough to converge after rescaling to the same solution. This construction is defined in the next proposition.

Proposition II.14. *Let the kernel J satisfy Hypothesis II.1 and n be a solution of (II.1.1) with initial data n^0 satisfying (II.1.10). Then, there exists a bounded positive function θ, which only depends on J, such that $\theta(s) \to 0$ as $s \to +\infty$, and positive constants $\underline{C} < 1 < \overline{C}$, such that, for all $t > 0$ and $x \in \mathbb{R}$,*

$$\frac{\underline{C}\exp\left(-\int_0^t \theta(s)ds\right)}{1+e^{-t}/J(x)} \leqslant n(t,x) \leqslant \frac{2\overline{C}\exp\left(\int_0^t \theta(s)ds\right)}{1+e^{-t}/J(x)}. \tag{II.2.1}$$

The function θ could, for a specific kernel, be computed explicitly; however, such a computation would be quite involved. For our purposes, we only need to know that it converges to 0 as $t \to +\infty$. Next let us give some interpretation of this result and an insight into the underlying ideas of the proofs. Let us first rewrite the estimate as

$$\underline{C}\exp\left(-\int_0^t \theta(s)\,ds\right)\phi(t,x) \leqslant n(t,x) \leqslant 2\overline{C}\exp\left(\int_0^t \theta(s)ds\right)\phi(t,x).$$

where we define the function $\phi(t,x) := (1+e^{-t}/J(x))^{-1}$. We observe that the behaviour of n is well approximated by the solution of the family of decoupled ODEs:

$$\begin{cases} \dfrac{d\phi(t,x)}{dt} = \phi(1-\phi), \\ \phi(0,\cdot) = \frac{J}{1+J} \leqslant J, \end{cases}$$

parametrized by $x \in \mathbb{R}$. In other words, the behaviour of n at large time is dominated by the reaction term; that is to say that the dispersion term plays a negligible role, in some sense, compared to the growth by reaction.

Before embarking on the proof of this proposition, we explain how Proposition II.14 implies Theorem II.5.

Proof of Theorem II.5. Let us assume that the estimate (II.2.1) holds true for a solution n of (II.1.1) with initial data n^0 satisfying (II.1.10). Then, for any $\varepsilon > 0$, the rescaled solution $n_\varepsilon(t,x) = n(t/\varepsilon, \psi_\varepsilon(x))$ with ψ_ε defined in (II.1.9), satisfies for all $t > 0$ and $x \in \mathbb{R}$:

$$\frac{\underline{C}\exp\left(-\int_0^{\frac{t}{\varepsilon}}\theta(s)ds\right)}{1+e^{-t/\varepsilon}/J(x)^{\frac{1}{\varepsilon}}} \leqslant n_\varepsilon(t,x) \leqslant \frac{2\overline{C}\exp\left(\int_0^{\frac{t}{\varepsilon}}\theta(s)ds\right)}{1+e^{-t/\varepsilon}/J(x)^{\frac{1}{\varepsilon}}},$$

Thus, its Hopf–Cole transformation $u_\varepsilon(t,x) = -\varepsilon\ln(n_\varepsilon(t,x))$ satisfies for all $t > 0$ and $x \in \mathbb{R}$

$$\begin{aligned} -\varepsilon\ln(\underline{C}) + \varepsilon\int_0^{\frac{t}{\varepsilon}}\theta(s)ds + \varepsilon\ln\left(1+\left(e^{-t}/J(x)\right)^{\frac{1}{\varepsilon}}\right) &\geqslant u_\varepsilon(t,x) \\ \geqslant -\varepsilon\ln(2\overline{C}) - \varepsilon\int_0^{\frac{t}{\varepsilon}}\theta(s)ds + \varepsilon\ln\left(1+\left(e^{-t}/J(x)\right)^{\frac{1}{\varepsilon}}\right)&. \end{aligned} \tag{II.2.2}$$

We point out that $t^{-1}\int_0^t\theta(s)ds \to 0$ as $t \to \infty$ since $\theta(t) \to 0$ as $t \to +\infty$. It follows that, locally uniformly in t,

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^{\frac{t}{\varepsilon}}\theta(s)ds = 0. \tag{II.2.3}$$

In addition, it is easy to see that, locally uniformly in x and t,

$$\lim_{\varepsilon\to 0}\varepsilon\ln\left(1+\left(e^{-t}/J(x)\right)^{\frac{1}{\varepsilon}}\right) = \max\left(f(|x|)-t,0\right). \tag{II.2.4}$$

Hence, using (II.2.2) and (II.2.3) in (II.2.4), we see that, locally uniformly in x and t,

$$\lim_{\varepsilon\to 0} u_\varepsilon(t,x) = \max\{f(x)-t,0\},$$

which concludes the proof of Theorem II.5. □

Remark II.15. *Using the work below, one could, in practice, compute θ and determine for which kernels J θ is integrable. When this is the case, the estimate given by (II.2.1) is more precise since $\int_0^t\theta(s)ds$ could be replaced by a constant on both sides of the equation. One could then quantify and compare more precisely the expansion of the λ-level lines of the solution n for various values of λ. Further, by plotting the function $x \mapsto \left(1+e^{-t}/J(|x|)\right)n(t,x)$ for various values of time (results not shown), one can investigate the accuracy of the upper and lower bounds given by (II.2.1).*

The threshold for integrability of θ appears to be kernels like $\exp\{\sqrt{|x|}\}$: those which are fatter yield an integrable θ. In this case, (II.2.1) gives a sharp estimate, up to the constants, of n and, in turn, on the expansion of the level sets of n. On the other hand, when kernels are thinner θ is not integrable and (II.2.1) is no longer an accurate point-wise bound, though it is good enough for our purposes. This is consistent with the fact that when the kernel is thin-tailed the qualitative behavior of n is quite different.

II.2.1 *The existence of sub- and super-solutions: Proof of Proposition II.14*

We now come back to the proof of Proposition II.14. We will show that the left hand side and the right hand side of (II.2.1) are respectively a sub- and a super–solution of (II.1.1). As already observed these sub- and super–solution are constructed from the family of solutions of ODEs of the form

$$\phi(t,x) = \frac{1}{1+e^{-t}/J(x)}. \tag{II.2.5}$$

Then, we may write our (potential) sub- and super-solutions as

$$\underline{\phi}(t,x) = \underline{C}\phi(t,x)\exp\left(-\int_0^t \theta(s)ds\right) \quad \text{and} \quad \overline{\phi}(t,x) = 2\,\overline{C}\phi(t,x)\exp\left(\int_0^t \theta(s)ds\right),$$

for all $t>0$ and $x\in\mathbb{R}$. Note that, for all $x\in\mathbb{R}$

$$\underline{\phi}(0,x) = \frac{\underline{C}J(x)}{1+J(x)} \leqslant \underline{C}J(x) \leqslant n^0(x) \leqslant \overline{C}J(x) \leqslant \frac{2\,\overline{C}J(x)}{1+J(x)} = \overline{\phi}(0,x).$$

Moreover, a direct computation shows that the function $\overline{\phi}$ satisfies, for all $t>0$ and $x\in\mathbb{R}$,

$$\partial_t\overline{\phi}(t,x) = \overline{\phi}(t,x)\left(1-\phi(t,x)\right) + \theta(t)\overline{\phi}(t,x) \geqslant \overline{\phi}(t,x)\left(1-\overline{\phi}(t,x)\right) + \theta(t)\overline{\phi}(t,x).$$

Then,

$$\partial_t\overline{\phi}(t,x) - \overline{\phi}(t,x)\left(1-\overline{\phi}(t,x)\right) - \left(J*\overline{\phi}-\overline{\phi}\right)(t,x) \geqslant \\ \left(\theta(t)\phi(t,x) - \left(J*\phi-\phi\right)(t,x)\right)2\,\overline{C}\exp\left(\int_0^t\theta(s)ds\right).$$

Thus, if $\theta\phi-(J*\phi-\phi)\geqslant 0$ in $(0,\infty)\times\mathbb{R}$, the function $\overline{\phi}$ is a super–solution to (II.1.1). Similarly, if $-\theta\phi-(J*\phi-\phi)\leqslant 0$ in $(0,\infty)\times\mathbb{R}$, the function $\underline{\phi}$ is a sub–solution to (II.1.1). The proof of Proposition II.14 then reduces to proving that

$$-\theta\phi \leqslant J*\phi-\phi \leqslant \theta\phi. \tag{II.2.6}$$

In the following sections, we obtain upper and lower bounds on this convolution term, completing the proof of Proposition II.14. To do so, we will split the space into two regions depending on time; the large range region $\mathcal{E}_\ell(t)$ and the short range region $\mathcal{E}_s(t)$ defined, for all $t>0$, by

$$\mathcal{E}_\ell(t) = \left\{x\in\mathbb{R} : |x| \geqslant f^{-1}(t)\right\} \qquad \text{and} \qquad \mathcal{E}_s(t) = \mathbb{R}\setminus\mathcal{E}_\ell(t). \tag{II.2.7}$$

We immediately notice that both regions are preserved by the scaling $(t/\varepsilon, \psi_\varepsilon(x))$. We shall estimate $J*\phi-\phi$ in both regions $\mathcal{E}_\ell(t)$ and $\mathcal{E}_s(t)$ separately.

II.2.1.1 *Establishing Proposition II.14, the proof of the bound* (II.2.6)

Proof of Proposition II.14. To estimate the convolution term, regardless the region in which x lies, we split the domain of integration of the convolution term as follows

$$\left(J*\phi-\phi\right)(t,x) = \int_{|x-y|\leqslant\gamma(t)} J(x-y)\left(\phi(t,y)-\phi(t,x)\right)dy \\ + \int_{|x-y|\geqslant\gamma(t)} J(x-y)\left(\phi(t,y)-\phi(t,x)\right)dy = I_1(t,x)+I_2(t,x),$$

for a function $\gamma(t)$ to be determined that localizes the integral around x. In what follows, we choose $\gamma(t)$ to be a positive and increasing function of time t such that $\gamma(t) \leqslant f^{-1}(t)$. Note that by symmetry of the problem, we can assume that $x \geqslant 0$, which we do from now on.

The existence of θ in (II.2.6) is equivalent to showing that, for $k = 1, 2$,

$$\lim_{t\to\infty} \frac{|I_k(t,x)|}{\phi(t,x)} = 0, \tag{II.2.8}$$

where the limit holds uniformly in x.

Estimation of the integral I_1

In the region where y is close to x, that is $|x-y| \leqslant \gamma(t)$, we estimate the difference $\phi(t,y) - \phi(t,x)$ by using the Taylor expansion of ϕ around location x.

More precisely, for any $t > 0$, $x \in \mathbb{R}$ and y such that $|x-y| \leqslant \gamma(t)$, there exists $\xi_{t,x,y} \in (0,1)$ such that

$$\phi(t,y) - \phi(t,x) = (y-x)\partial_x\phi(t, x\xi_{t,x,y} + (1-\xi_{t,x,y})y)\,. \tag{II.2.9}$$

For notational ease, we omit the subscripts (t,x,y) for $\xi_{t,x,y}$ in the sequel. Plugging this expression into I_1 we obtain

$$I_1 = \int_{|x-y|\leqslant\gamma(t)} J(x-y)(y-x)\partial_x\phi(t, x\xi + (1-\xi)y)\, dy.$$

To estimate the first derivative of ϕ, we use the following.

Lemma II.16. *There exists a positive function $\theta_1(t)$, depending only on J, such that, for any $z \in \mathbb{R}$ and $t > 0$,*

$$|\partial_x\phi(t,z)| \leqslant \theta_1(t)\,\phi(t,z),$$

and such that $\theta_1(s) \to 0$ as $s \to \infty$.

Proof. Using the form of J, we can rewrite $\phi(t,z) = (1+e^{f(z)-t})^{-1}$. A direct computation shows that, for all $t > 0$ and $z \in \mathbb{R}$,

$$|\partial_x\phi(t,z)| = \left|\phi(t,z)f'(z)\frac{e^{f(z)-t}}{1+e^{f(z)-t}}\right|,$$

so that

$$\left|\frac{\partial_x\phi}{\phi}\right|(t,z) \leqslant \frac{e^{f(z)-t}}{1+e^{f(z)-t}}f'(z)\,.$$

To estimate the right hand side of the inequality above, fix $\alpha \in (0,1)$. First, consider the case when $|z| \leqslant f^{-1}(\alpha t)$. Then $e^{f(z)-t} \leqslant e^{-(1-\alpha)t}$, and, hence

$$\left|\frac{\partial_x\phi}{\phi}\right|(t,z) \leqslant e^{-(1-\alpha)t}\|f'\|_\infty.$$

On the other hand if $|z| \geqslant f^{-1}(\alpha t)$, then $\frac{e^{f(z)-t}}{1+e^{f(z)-t}} \leqslant 1$ and, consequently,

$$\left|\frac{\partial_x\phi}{\phi}\right|(t,z) \leqslant \sup_{|z|\geqslant f^{-1}(\alpha t)} |f'(z)|.$$

Defining

$$\theta_1(t) = \max\left\{ \|f'\|_\infty e^{-(1-\alpha)t}, \sup_{|z|\geqslant f^{-1}(\alpha t)} |f'(z)| \right\}, \tag{II.2.10}$$

we conclude that, for all $t > 0$ and $z \in \mathbb{R}$

$$\left| \frac{\partial_x \phi}{\phi} \right| (t,z) \leqslant \theta_1(t).$$

The convergence of $\theta_1(s)$ to zero as s tends to infinity is clear from the definition and the assumptions on f in Hypothesis II.1. This concludes the proof of Lemma II.16. □

We deduce from Lemma II.16, an estimate on I_1

$$|I_1| \leqslant \theta_1(t) \int_{|x-y|\leqslant\gamma(t)} J(x-y)|y-x|\phi(t, x\xi + (1-\xi)y)\, dy.$$

Changing variables, we must estimate

$$\int_{|h|\leqslant\gamma(t)} J(h)|h|\phi(t, x + (1-\xi)h)\, dh. \tag{II.2.11}$$

To do so, we bound $\phi(t, x + (1-\xi)h)$ with $\phi(t,x)$ by choosing the function $\gamma(t)$ carefully.

We first prove that we may choose the positive function $\gamma(t)$ such that, for all $x \in \mathcal{E}_\ell(t)$, $h \in (-\gamma(t), \gamma(t))$, and $\xi \in (0,1)$,

$$\phi(t, x + (1-\xi)h) \leqslant e\phi(t,x). \tag{II.2.12}$$

Indeed, we have

$$\begin{aligned} \ln\left(\frac{\phi(t, x + (1-\xi)h)}{\phi(t,x)}\right) &\leqslant |\ln(\phi(t, x + (1-\xi)h)) - \ln(\phi(t,x))| \\ &\leqslant \sup_{z\in[x-(1-\xi)|h|, x+(1-\xi)|h|]} |\partial_x(\ln\phi)(t,z)|\, |h|. \end{aligned}$$

Using the bound on $\partial_x(\ln\phi)$ of Lemma II.16, we have

$$\ln\left(\frac{\phi(t, x + (1-\xi)h)}{\phi(t,x)}\right) \leqslant \exp(\theta_1(t)\gamma(t))$$

Fixing γ such that $\theta_1\gamma \leqslant 1$ yields (II.2.12). For technical reasons, discussed in the proof of the estimate of I_2, we define

$$\gamma(t) = \min\left\{ f'\left(\frac{f^{-1}(t)}{2}\right)^{-1}, \theta_1(t)^{-1} \right\} \tag{II.2.13}$$

It is clear that $\theta_1\gamma \leqslant 1$, as desired. We point out that, from (II.2.10),

$$\theta_1(t)^{-1} = \min\left(e^{(1-\alpha)t}, \left[f'(f^{-1}(\alpha t)) \right]^{-1} \right).$$

Our choice of γ implies

$$|I_1| \leqslant e\theta_1(t) \left(\int_{|h|\leqslant\gamma(t)} J(h)|h|\, dy \right) \phi(t,x).$$

We now show that

$$\lim_{t\to+\infty} \theta_1(t) \int_{|h|<\gamma(t)} J(h)|h|dh = 0. \tag{II.2.14}$$

From our assumption (II.1.7) in Hypothesis II.1 on J, there exists $H > 0$ such that if $|h| > H$, then $J(h) \leqslant |h|^{-\mu}$. We deduce that there exists a constant C, depending only on J such that, for all $t > 0$,

$$\begin{aligned}\theta_1(t) \int_{|h|<\gamma(t)} J(h)|h|dh &\leqslant \theta_1(t) \int_{|h|<H} J(h)|h|dh + \theta_1(t) \int_{H<|h|<\gamma(t)} J(h)|h|dh \\ &\leqslant \theta_1(t)C + \theta_1(t) \int_{H<|h|<\gamma(t)} |h|^{1-\mu}dh,\end{aligned}$$

where we interpret the last integral to be zero in the case when $\gamma(t) \leqslant H$. Thus we get

$$\theta_1(t) \int_{|h|<\gamma(t)} J(h)|h|dh \leqslant \begin{cases} C\left(\theta_1(t) + \theta_1(t)|\gamma(t)|^{2-\mu}\right), & \text{if } \mu \neq 2, \\ C\left(\theta_1(t) + \theta_1(t)\ln(\gamma(t))\right), & \text{if } \mu = 2. \end{cases}$$

Using the definition of γ, (II.2.13), this inequality becomes

$$\theta_1(t) \int_{|h|<\gamma(t)} J(h)|h|dh \leqslant \begin{cases} C\left(\theta_1(t) + \theta_1(t)^{\mu-1}\right), & \text{if } \mu \neq 2, \\ C\theta_1(t)\left(1 + \ln\left(\theta_1(t)^{-1}\right)\right), & \text{if } \mu = 2. \end{cases}$$

In both cases, we have established that $|I_1|/\phi(t,x) \to 0$ as $t \to \infty$, as claimed. This establishes (II.2.8) for $k = 1$.

Estimation of the integral I_2.

The arguments for the upper and lower bounds are different. As it is simpler, we prove the lower bound first. Since ϕ is positive,

$$I_2(t,x) \geqslant -\int_{|x-y|\geqslant\gamma(t)} J(x-y)\phi(t,x)dx = -\phi(t,x) \int_{|h|\geqslant\gamma(t)} J(h)dh.$$

Clearly $\liminf_{t\to\infty} I_2/\phi(t,x) \geqslant 0$.

This finishes the proof of the lower bound for I_1 and I_2. To conclude, we need only obtain a matching upper bound of I_2 in order to obtain (II.2.8) and thus Proposition II.14. The proof of this bound is somewhat involved. We break up our estimates based on whether x is in the short-range or long-range regime.

THE LONG RANGE REGION We handle first the case when $x \in \mathcal{E}_\ell(t)$, that is $x \geqslant f^{-1}(t)$. We first split the integral I_2 as

$$\begin{aligned} I_2 &= \int_{\substack{|x-y|\geqslant\gamma(t)\\ |y|\geqslant|x|}} J(x-y)\big(\phi(t,y) - \phi(t,x)\big)\,dy + \int_{\substack{|x-y|\geqslant\gamma(t)\\ |y|\leqslant|x|}} J(x-y)\big(\phi(t,y) - \phi(t,x)\big)\,dy, \\ &\leqslant \left(\int_{\substack{|x-y|\geqslant\gamma(t)\\ |y|\geqslant|x|}} J(x-y)\frac{\phi(t,y)}{\phi(t,x)}\,dy + \int_{\substack{|x-y|\geqslant\gamma(t)\\ |y|\leqslant|x|}} J(x-y)\frac{\phi(t,y)}{\phi(t,x)}\,dy \right) \phi(t,x), \\ &\leqslant (II_1 + II_2)\,\phi(t,x). \end{aligned} \tag{II.2.15}$$

The first part of the integral, II_1, is estimated using the monotonicity of ϕ in the spatial variable to obtain

$$II_1 \leqslant \int_{\substack{|x-y|\geqslant\gamma(t)\\ |y|\geqslant|x|}} J(x-y)\,dy \leqslant \int_{|h|\geqslant\gamma(t)} J(h)\,dh.$$

We now turn to the second part term in (II.2.15), II_2. Recall that we are assuming, by the symmetry of the problem that x is positive. We decompose the integral in four pieces, one integral close to $y=-x$, two integrals close to $y=0$ and the last one centered around $|y|=x/2$. More precisely,

$$\begin{aligned} II_2 = \int_{\substack{|x-y|\geqslant\gamma(t)\\ |y|\leqslant|x|}} J(x-y)\frac{\phi(t,y)}{\phi(t,x)}dy &\leqslant \int_{-x}^{-\gamma(t)} J(x-y)\frac{\phi(t,y)}{\phi(t,x)}\,dy + \int_{-\gamma(t)}^{0} J(x-y)\frac{\phi(t,y)}{\phi(t,x)}\,dy \\ + \int_{0}^{\gamma(t)} J(x-y)\frac{\phi(t,y)}{\phi(t,x)}\,dy + \int_{\gamma(t)}^{x-\gamma(t)} J(x-y)\frac{\phi(t,y)}{\phi(t,x)}\,dy &= III_1 + III_2 + III_3 + III_4. \end{aligned}$$

We now estimate each of the four integrals $III_1, III_2, III_3, III_4$ in turn and show that each tends to 0 in the limit $t\to\infty$.

Let us first notice that for all $x,y\in\mathbb{R}$ and $t>0$,

$$J(x-y)\frac{\phi(t,y)}{\phi(t,x)} = \frac{J(x-y)J(y)}{J(x)}\frac{J(x)+e^{-t}}{J(y)+e^{-t}}. \tag{II.2.16}$$

In addition, for all $x\in\mathcal{E}_\ell(t)$ and we have on the one hand

$$J(x)\leqslant e^{-t},$$

and since $y\in(-x,x)$,

$$\frac{J(x)+e^{-t}}{J(y)+e^{-t}}\leqslant 1.$$

To proceed further, we require the following useful fact, in which we see the need for the intricate description in (II.2.13). From its definition, it is clear that if $t>0$ and $x\in\mathcal{E}_\ell(t)$, then $2\gamma(t)\leqslant x$.

ESTIMATION OF III_1 First we estimate III_1. Due to the limits of integration, we have that $|x-y|\geqslant|x|$, giving

$$III_1 = \int_{-x}^{-\gamma(t)} \frac{J(x-y)J(y)}{J(x)}\frac{J(x)+e^{-t}}{J(y)+e^{-t}}\,dy \leqslant \int_{-x}^{-\gamma(t)} J(y)\,dy \leqslant \int_{\gamma(t)}^{\infty} J(y)\,dy.$$

Since J is integrable and since $\gamma(t)\to\infty$ as $t\to\infty$, this tends to zero as t tends to infinity.

Next we estimate III_2. Due to the limits of integration $J(x-y)\leqslant J(x)$. Using this and (II.2.16), we have

ESTIMATION OF III_2.

$$\begin{aligned} III_2 &= \int_{-\gamma(t)}^{0} \frac{J(x-y)J(y)}{J(x)}\frac{J(x)+e^{-t}}{J(y)+e^{-t}}\,dy \leqslant \int_{-\gamma(t)}^{0} \frac{J(x-y)}{J(x)}(J(x)+e^{-t})\,dy \\ &\leqslant \int_{-\gamma(t)}^{0} (J(x)+e^{-t})\,dy \leqslant \left(J(x)+e^{-t}\right)\gamma(t) \leqslant 2e^{-t}\gamma(t)\,. \end{aligned}$$

In the last step we used that $x\geqslant f^{-1}(t)$ so that $J(x)\leqslant e^{-t}$. Then III_2 tends to zero as $t\to\infty$ because, by construction, $\gamma(t)\leqslant e^{(1-\alpha)t}$.

ESTIMATION OF III_3. To estimate the third integral, III_3, we first notice that $J(x-y) \leqslant J(x-\gamma(t))$ for all y in the domain of integration $[0, \gamma(t)]$. Here we are using the definition of γ (II.2.13) and the fact that $x \geqslant 2\gamma(t)$, observed above. Using this inequality and (II.2.16), again, we obtain

$$III_3 = \int_0^{\gamma(t)} \frac{J(x-y)J(y)}{J(x)} \frac{J(x)+e^{-t}}{J(y)+e^{-t}}\, dy \leqslant \int_0^{\gamma(t)} \frac{J(x-\gamma(t))}{J(x)} (J(x)+e^{-t})\, dy. \tag{II.2.17}$$

The main difficulty is in obtaining a bound, independent of time, of $J(x-\gamma(t))/J(x)$. We obtain this bound now. From the definition of $\gamma(t)$ (II.2.13), we see that

$$\begin{aligned} \frac{J(x-\gamma(t))}{J(x)} = \exp\{f(x) - f(x-\gamma(t))\} &\leqslant \exp\left\{ \max_{z\in[x-\gamma(t),x]} f'(z)\gamma(t) \right\} \\ &\leqslant \exp\left\{ \max_{z\in[f^{-1}(t)/2,x]} f'(z)\gamma(t) \right\}. \end{aligned} \tag{II.2.18}$$

The last step follows from the fact that $x \geqslant f^{-1}(t)$ and that $x - \gamma(t) \geqslant x/2$, which follows from the inequality $x \geqslant 2\gamma(t)$.

When t is small, $\gamma(t)$ is bounded; hence, $\max_{z\in[\gamma(t),x]} f'(z)\gamma(t)$ is bounded. From this and (II.2.18), it follows that $J(x-\gamma(t))/J(x) \leqslant C$, for some constant C independent of time. When t is sufficiently large, the concavity of f, given in Hypothesis II.1, implies that f' is monotonic so that (II.2.18) becomes

$$\frac{J(x-\gamma(t))}{J(x)} \leqslant \exp\left\{ f'\left(\frac{f^{-1}(t)}{2}\right)\gamma(t) \right\}. \tag{II.2.19}$$

From the definition of γ, (II.2.13), we have that $\gamma(t) \leqslant f'(f^{-1}(t)/2)^{-1}$. Putting this together with (II.2.19) yields

$$\frac{J(x-\gamma(t))}{J(x)} \leqslant e.$$

Hence, there is a constant C, indepedent of time, such that $J(x-\gamma(t))/J(x) \leqslant C$. Using this in our estimate (II.2.17), we get the bound

$$III_3 \leqslant C(J(x)+e^{-t})\gamma(t) \leqslant C(J(2\gamma(t)) + e^{-t}\gamma(t)).$$

Using Hypothesis II.1 and our arguments above, we see that the right hand side tends to zero, as desired.

ESTIMATION OF III_4. For all y in the domain of integration $[\gamma(t), x-\gamma(t)]$, we have that $y \leqslant x$ so that $J(x) \leqslant J(y)$. Re-arranging the fourth integral with this inequality yields

$$\begin{aligned} III_4 &= \int_{\gamma(t)}^{x-\gamma(t)} \frac{J(x-y)J(y)}{J(x)} \frac{J(x)+e^{-t}}{J(y)+e^{-t}}\, dy \leqslant \int_{\gamma(t)}^{x-\gamma(t)} \frac{J(x-y)J(y)}{J(x)}\, dy \\ &= \int_{\gamma(t)}^{x/2} \frac{J(x-y)J(y)}{J(x)} dy + \int_{x/2}^{x-\gamma(t)} \frac{J(x-y)J(y)}{J(x)} dy = 2\int_{\gamma(t)}^{x/2} \frac{J(x-y)J(y)}{J(x)}\, dy, \end{aligned} \tag{II.2.20}$$

where we have changed the variable $y \mapsto x-y$ to obtain the last equality. Notice that $\gamma(t) \leqslant x/2 \leqslant x-\gamma(t)$ due to our observation that $x \geqslant 2\gamma(t)$. To estimate the last term in (II.2.20), we need to distinguish between the cases when $xf'(x)$ is bounded and when $xf'(x)$ is unbounded.

If $xf'(x)$ is bounded, we have

$$III_4 \leqslant 2\frac{J(x/2)}{J(x)}\int_{\gamma(t)}^{x/2} J(y)dy \leqslant 2\frac{J(x/2)}{J(x)}\int_{\gamma(t)}^{\infty} J(y)dy,$$

because $y \in (\gamma(t), x/2)$ and thus $J(x-y) \leqslant J(x/2)$. By Taylor's theorem, we have that

$$\frac{J(x/2)}{J(x)} = \exp\{f(x) - f(x/2)\} \leqslant \exp\left\{\sup_{\xi\in(x/2,x)} f'(\xi)\frac{x}{2}\right\}.$$

Using the eventual concavity of f along with the boundedness of $f'(x/2)x/2$, we have that $J(x/2)/J(x) \leqslant C$. Hence

$$III_4 \leqslant C\int_{\gamma(t)}^{\infty} J(y)dy,$$

which tends to 0 as t tends to infinity.

If $xf'(x)$ tends to infinity, we require a different argument. First, notice that, if $J(x-y)J(y)/J(x)$ is bounded above by an integrable function $\overline{J}$ uniformly in x, then we are finished because

$$III_4 \leqslant 2\int_{\gamma(t)}^{\infty} \overline{J}(y)dy,$$

which tends to 0 as t tends to infinity. We prove this now.

When t is small, the domain of integration in III_4 is bounded as is $J(x-y)J(y)/J(x)$. Hence, we may restrict to considering only the case when t is large enough that f' is decreasing on $(\gamma(t), \infty)$.

First, notice that

$$\frac{J(x-y)J(y)}{J(x)} = \exp\{f(x) - f(y) - f(x-y)\} = \exp\left\{y\int_0^1 f'(x-ys)ds - f(y)\right\}.$$

Since f' is decreasing, we have that

$$\frac{J(x-y)J(y)}{J(x)} \leqslant \exp\{yf'(y) - f(y)\}.$$

From Hypothesis II.1, there exists $\varepsilon_0 \in (0,1)$ such that $yf'(y)/f(y) \leqslant 1-\varepsilon_0$ for $y \geqslant \gamma(t)$ and t sufficiently large. Hence, we obtain that

$$\frac{J(x-y)J(y)}{J(x)} \leqslant \exp\{-\varepsilon_0 f(y)\}.$$

On the other hand, since $yf'(y)$ tends to infinity, it is clear that $e^{-\varepsilon_0 f}$ is integrable, finishing the proof that III_4 tends to 0 as t tends to infinity.

In conclusion, we have show that when $x \in \mathcal{E}_\ell(t)$,

$$I_2(t,x) = (II_1 + (III_1 + III_2 + III_3 + III_4))\,\phi(t,x)$$

where II_1 and III_k tend to zero as $t \to \infty$ uniformly in x for $k \in \{1,2,3,4\}$. This concludes the proof in the long range region $x \in \mathcal{E}_\ell(t)$.

THE SHORT RANGE REGION We now turn to the simpler case where x is in the short range region $\mathcal{E}_s(t)$, that is $|x| \leqslant f^{-1}(t)$. In this region, notice that the function ϕ is bounded from below $\phi(t,x) \geqslant 1/2$. We can estimate directly as follows

$$I_2 = \int_{|x-y|\geqslant\gamma(t)} J(x-y)\big(\phi(t,y)-\phi(t,x)\big)\,dy \leqslant 2\left(\int_{|h|\geqslant\gamma(t)} J(h)\,dh\right)\phi(t,x).$$

Of course, $\int_{|h|\geqslant\gamma(t)} J(h)dh$ tends to 0 as t tends to infinity.

This concludes the proof of (II.2.8) and, thus, the proof of Proposition II.14.

□

II.2.1.2 *The lower–bound estimate: proof of the sub–solution.*

To construct a sub–solution, we estimate the convolution again by splitting the integral with the γ defined in the super–solution part,

$$(J*\phi-\phi)(t,x) = \int_{|x-y|\leqslant\gamma(t)} J(x-y)\,(\phi(t,y)-\phi(t,x))\,dy + \int_{|x-y|\geqslant\gamma(t)} J(x-y)\,(\phi(t,y)-\phi(t,x))\,dy \;= I_1 + I_2.$$

By the positivity of ϕ, we immediately observe that

$$I_2 \geqslant -\left(\int_{|h|\geqslant\gamma(t)} J(h)\,dy\right)\phi(t,x).$$

Moreover, using Lemma II.16 and arguing as in the estimate of I_1, above, we obtain

$$I_1 \geqslant -C\theta_1(t)\left(\int_{|h|\leqslant\gamma(t)} |h|J(h)\,dy\right)\phi(t,x).$$

Using (II.2.14), we see that $I_1/\phi(t,x)$ tends to 0 as t tends to infinity uniformly in $x \in \mathcal{E}_s(t)$.

This completes the proof of Proposition II.14.

II.3 THE PROPAGATION REGIME: THE PROOF OF THEOREM II.6.

We now deduce from Theorem II.5 the asymptotic behaviour of n_ε as ε tends to 0.

Proof of Theorem II.6. We first look at the limit in any compact subset of $\{u>0\}$, and then we focus on the limit in any compact subset of $\mathrm{Int}\{u=0\}$.

Part (a): convergence of n_ε in $\mathrm{Int}\{u>0\}$

Fix any compact subset $K \subset \{u>0\}$, there exists a positive constant α such that for all $(t,x)\in K$, we have $u(t,x) > \alpha$. Due to Theorem II.5, we know that u_ε converges locally uniformly on $(0,+\infty)\times\mathbb{R}$ to $u(t,x) = \max(f(x)-t,0)$. Hence, for all ε sufficiently small, $u_\varepsilon(t,x) \geqslant \alpha/2$ for $(t,x)\in K$. Then for ε sufficiently small, we have that, for $(t,x)\in K$,

$$n_\varepsilon(t,x) = e^{-\frac{u_\varepsilon(t,x)}{\varepsilon}} \leqslant e^{-\frac{\alpha}{2\varepsilon}}.$$

Taking the limit $\varepsilon\to 0$ yields the uniform convergence of n_ε to zero on K. This proves point (a) of Theorem II.6.

Part (b): convergence of n_ε in $\mathrm{Int}\{u=0\}$

First, we use the maximum principle to show that, locally uniformly on $\{t>0\}$,

$$\limsup_{\varepsilon\to 0} \|n_\varepsilon(t,\cdot)\|_\infty \leqslant 1. \tag{II.3.1}$$

Indeed, it follows from (II.1.10) that there exists $\overline{C}$, independent of ε such that $n_\varepsilon(0,x)\leqslant \overline{C}$ for all $\varepsilon>0$. From (II.1.11), $\overline{n}(t):=1+\overline{C}e^{-t/\varepsilon}$ is a super-solution to n_ε. Hence, we have that $n_\varepsilon(t,x)\leqslant 1+\overline{C}e^{-t/\varepsilon}$, which establishes the estimate (II.3.1).

Let K be a compact subset of $\mathrm{Int}\{u=0\}$. Our goal is to show that $n_\varepsilon=\exp(-u_\varepsilon/\varepsilon)$ converges to 1 uniformly in K. Recall that u_ε is a solution to (II.1.13); that is, for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$n_\varepsilon(t,x)=1+\partial_t u_\varepsilon(t,x)-\int_{-\infty}^{\infty} J(h)\left[1-e^{-\frac{1}{\varepsilon}\left(u_\varepsilon\left(t,\psi_\varepsilon^{-1}(\psi_\varepsilon(x)-h)\right)-u_\varepsilon(t,x)\right)}\right]dh. \tag{II.3.2}$$

We need a lower–bound for the right hand side to conclude. To do so, we follow the approach of Evans and Souganidis [1989]; Méléard and Mirrahimi [2015] which consists of replacing u_ε by a well chosen test function to obtain a sharp estimate. Fix any $(t_0,x_0)\in K$. Then, since $K\subset\{u<0\}$, we may fix $\delta_0>0$ such that $f(x_0)<t_0-\delta_0$. For all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, let

$$\chi(t,x)=\max\{0,f(x)-(t_0-\delta_0)\}+(t-t_0)^2.$$

Since χ is nonnegative and $u=0$ on K, $u-\chi$ has a maximum at (t_0,x_0), strict and local in t, global in x. In order to follow the convergence, we also define the following perturbed test function χ_ε as follows. For any $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, let

$$\chi_\varepsilon(t,x):=\varepsilon\ln\left(1+e^{-\frac{t_0-\delta_0}{\varepsilon}}J(x)^{-\frac{1}{\varepsilon}}\right)+(t-t_0)^2.$$

Observe that we may reformulate χ_ε in term of ϕ, defined in (II.2.5), and the space scaling ψ_ε as

$$\chi_\varepsilon(t,x)=-\varepsilon\ln\left(\phi\left(\frac{t_0-\delta_0}{\varepsilon},\psi_\varepsilon(x)\right)\right)+(t-t_0)^2. \tag{II.3.3}$$

Observe that $\chi_\varepsilon(t,x)\to\chi(t,x)$ as $\varepsilon\to 0$ locally uniformly on $(0,\infty)\times\mathbb{R}$. Then there exists a sequence $(t_\varepsilon,x_\varepsilon)$ such that $u_\varepsilon-\chi_\varepsilon$ has a maximum on $(t_0-r,t_0+r)\times\mathbb{R}$ for some small $r>0$ at $(t_\varepsilon,x_\varepsilon)$ that is strict in t and such that $t_\varepsilon\to t_0$ as $\varepsilon\to 0$. We note that the fact that $u_\varepsilon-\chi_\varepsilon$ has a maximum that is global in x is not immediate from the locally uniform convergence of u_ε to u; however, it follows easily from Proposition II.14.

We now plug our test functions χ_ε into (II.3.2) and obtain

$$\begin{aligned} n_\varepsilon(t,x)&=1+\partial_t u_\varepsilon(t,x)\\ &\quad-\int_{\mathbb{R}} J(h)\left[1-e^{-\frac{1}{\varepsilon}\left((u_\varepsilon-\chi_\varepsilon)\left(t,\psi_\varepsilon^{-1}(|\psi_\varepsilon(x)-h|)\right)-(u_\varepsilon-\chi_\varepsilon)(t,x)+\chi_\varepsilon\left(t,\psi_\varepsilon^{-1}(|\psi_\varepsilon(x)-h|)\right)-\chi_\varepsilon(t,x)\right)}\right]dh.\end{aligned}$$

From the maximum property of $u_\varepsilon-\chi_\varepsilon$ at $(t_\varepsilon,x_\varepsilon)$, we know that $\partial_t\chi_\varepsilon(t_\varepsilon,x_\varepsilon)=\partial_t u_\varepsilon(t_\varepsilon,x_\varepsilon)=2(t_\varepsilon-t_0)$ and that for all $h\in\mathbb{R}$,

$$(u_\varepsilon-\chi_\varepsilon)\left(t_\varepsilon,\psi_\varepsilon^{-1}\left(|\psi_\varepsilon(x_\varepsilon)-h|\right)\right)-(u_\varepsilon-\chi_\varepsilon)\left(t_\varepsilon,x_\varepsilon\right)\leqslant 0.$$

Thus, we obtain, at $(t_\varepsilon, x_\varepsilon)$,

$$n_\varepsilon(t_\varepsilon, x_\varepsilon) \geqslant 1 + 2(t_\varepsilon - t_0) - \int_{\mathbb{R}} J(h) \left[1 - e^{-\frac{1}{\varepsilon}\left(\chi_\varepsilon\left(t_\varepsilon, \psi_\varepsilon^{-1}(\psi_\varepsilon(x)-h)\right) - \chi_\varepsilon(t_\varepsilon, x_\varepsilon)\right)}\right] dh.$$

Then, the link between χ_ε and ϕ yields

$$\begin{aligned} n_\varepsilon(t_\varepsilon, x_\varepsilon) &\geqslant 1 + 2(t_\varepsilon - t_0) - \int_{\mathbb{R}} J(h) \left[1 - \frac{\phi\left(\frac{t_0-\delta_0}{\varepsilon}, \psi_\varepsilon(x_\varepsilon) - h\right)}{\phi\left(\frac{t_0-\delta_0}{\varepsilon}, \psi_\varepsilon(x_\varepsilon)\right)}\right] dh \\ &= 1 + 2(t_\varepsilon - t_0) + \frac{J * \phi - \phi}{\phi}\left(\frac{t_0 - \delta_0}{\varepsilon}, \psi_\varepsilon(x_\varepsilon)\right). \end{aligned}$$

Using (II.2.6), this implies

$$n_\varepsilon(t_\varepsilon, x_\varepsilon) \geqslant 1 + 2(t_\varepsilon - t_0) - \theta\left(\frac{t_0 - \delta_0}{\varepsilon}\right),$$

where we recall that $\theta(s) \to 0$ as $s \to \infty$. Since $t_\varepsilon \to t_0$ as $\varepsilon \to \infty$ and since $t_0 - \delta_0 > f(x_0) \geqslant 0$, we obtain

$$\liminf_{\varepsilon \to 0} n_\varepsilon(t_\varepsilon, x_\varepsilon) \geqslant 1. \tag{II.3.4}$$

To conclude, we must bootstrap (II.3.4) to deduce information about $n_\varepsilon(t_0, x_0)$. By construction of $(t_\varepsilon, x_\varepsilon)$,

$$u_\varepsilon(t_\varepsilon, x_\varepsilon) - \chi_\varepsilon(t_\varepsilon, x_\varepsilon) \geqslant u_\varepsilon(t_0, x_0) - \chi_\varepsilon(t_0, x_0),$$

which implies that

$$u_\varepsilon(t_0, x_0) - u_\varepsilon(t_\varepsilon, x_\varepsilon) \leqslant \chi_\varepsilon(t_0, x_0) - \chi_\varepsilon(t_\varepsilon, x_\varepsilon) \leqslant \chi_\varepsilon(t_0, x_0) = \varepsilon \ln\left(1 + e^{-\frac{t_0-\delta_0}{\varepsilon}} \left[J(\psi_\varepsilon(x_0))\right]^{-1}\right).$$

Since $n_\varepsilon = \exp(-u_\varepsilon/\varepsilon)$, we obtain

$$n_\varepsilon(x_\varepsilon, t_\varepsilon) \leqslant n_\varepsilon(x_0, t_0)\left(1 + \left(e^{-(t_0-\delta_0)} J(x_0)^{-1}\right)^{\frac{1}{\varepsilon}}\right).$$

Using that $f(x_0) < t_0 + \delta_0$ along with (II.3.4), we obtain

$$1 \leqslant \liminf_{\varepsilon \to 0} n_\varepsilon(t_\varepsilon, x_\varepsilon) \leqslant \liminf_{\varepsilon \to 0} n_\varepsilon(t_0, x_0). \tag{II.3.5}$$

The combination of (II.3.1) and (II.3.5) concludes the proof. □

II.4 THE SMALL MUTATION REGIME

This section is devoted to the proof of Theorem II.10. We obtain some *a priori* estimates on u_ε and $\partial_x u_\varepsilon$ in order to take the half–relaxed limits of u_ε to obtain u, the solution of (II.1.22). We then use this limit u to estimate the level sets of n_ε as $\varepsilon \to 0$. With the strategy in mind, we proceed with the proof of Theorem II.10.

II.4.1 *Proof of Theorem II.10*

We start with the proof of Theorem II.10 (i), which we rephrase into the following lemma for legibility.

Lemma II.17. *Let $\underline{u}$ and $\overline{u}$ be defined by (II.1.21). Then $\underline{u}$ and $\overline{u}$ satisfy*

$$\min\left\{\partial_t \underline{u} + \int_{\mathbb{R}} \left[e^{\frac{\operatorname{sign}(h) f(h)}{f'(0)} \partial_x \underline{u}} - 1\right] J(h)\,\mathrm{d}h + 1, \underline{u}\right\} \geqslant 0, \quad \textit{and}$$
$$\min\left\{\partial_t \overline{u} + \int_{\mathbb{R}} \left[e^{\frac{\operatorname{sign}(h) f(h)}{f'(0)} \partial_x \overline{u}} - 1\right] J(h)\,\mathrm{d}h + 1, \overline{u}\right\} \leqslant 0$$

on $(0,+\infty)\times\mathbb{R}$.

Proof. We first prove that the lower half–relaxed limit $\underline{u}$ of u_ε is a viscosity super–solution to (II.1.22), where we recall that $\underline{u}$ is defined by

$$\underline{u}(t,x) := \liminf_{\substack{\varepsilon\to 0\\ (s,y)\to(t,x)}} u_\varepsilon(s,y).$$

First, $\underline{u} \geqslant 0$ because $u_\varepsilon \geqslant 0$ for all ε.

Let φ be a test function in $\mathcal{C}^1((0,+\infty)\times\mathbb{R})$ such that $\underline{u} - \varphi$ has a strict global minimum equal to 0 at some point (t_0, x_0) with $t_0 > 0$. Our goal is to show that

$$\partial_t \varphi(t_0,x_0) + \int_{\mathbb{R}} \left[e^{\frac{\operatorname{sign}(h) f(h)}{f'(0)} \partial_x \varphi(t_0,x_0)} - 1\right] J(h)\,\mathrm{d}h + 1 \geqslant 0. \tag{II.4.1}$$

Fix any $M > 0$. We eventually take the limit $M \to \infty$. Using the definition of $\underline{u}$ and classical arguments (see Barles [1994]), we find $r > 0$ and a sequence $(t_\varepsilon, x_\varepsilon)$ such that $u_\varepsilon - \varphi$ has a minimum at $(t_\varepsilon, x_\varepsilon)$ in $(t_0 - r, t_0 + r)\times B_{2M}(x_0)$ and such that $(t_\varepsilon, x_\varepsilon)$ converges to (t_0, x_0) as $\varepsilon \to 0$ after passing to a sub-sequence, which we denote the same way, if necessary. Since u_ε is a solution of (II.1.16),

$$\partial_t \varphi(t_\varepsilon, x_\varepsilon) + \int_{\mathbb{R}} \left[e^{-\frac{1}{\varepsilon}\left(u_\varepsilon\left(t_\varepsilon, x_\varepsilon - \operatorname{sign}(h) f^{-1}(\varepsilon f(h))\right) - u_\varepsilon(t_\varepsilon,x_\varepsilon)\right)} - 1\right] J(h)\mathrm{d}h + 1 \geqslant 0. \tag{II.4.2}$$

The proof now hinges on estimating the integral in (II.4.2). By construction of $(t_\varepsilon, x_\varepsilon)$ we have

$$u_\varepsilon(t_\varepsilon,x_\varepsilon) - \varphi(t_\varepsilon,x_\varepsilon) \leqslant u_\varepsilon(t,x) - \varphi(t,x)$$

for all $(t,x) \in (t_0 - r, t_0 + r)\times B_{2M}(x_0)$. Also, notice that $|x_0 - x_\varepsilon| \leqslant M$ and $|\operatorname{sign}(h) f^{-1}(\varepsilon f(h))| \leqslant M$ for all $h \in [0,M]$ for ε small enough. Hence, we have

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-\frac{1}{\varepsilon}\left(u_\varepsilon\left(t_\varepsilon, x_\varepsilon - \operatorname{sign}(h) f^{-1}(\varepsilon f(h))\right) - u_\varepsilon(t_\varepsilon,x_\varepsilon)\right)} J(h)\mathrm{d}h \\
&\qquad \leqslant \int_{[-M,M]} e^{-\frac{1}{\varepsilon}\left(\varphi\left(t_\varepsilon, x_\varepsilon - \operatorname{sign}(h) f^{-1}(\varepsilon f(h))\right) - \varphi(t_\varepsilon,x_\varepsilon)\right)} J(h)\mathrm{d}h \\
&\qquad\qquad + \int_{\mathbb{R}\setminus[-M,M]} e^{-\frac{1}{\varepsilon}\left(u_\varepsilon\left(t_\varepsilon, x_\varepsilon - \operatorname{sign}(h) f^{-1}(\varepsilon f(h))\right) - u_\varepsilon(t_\varepsilon,x_\varepsilon)\right)} J(h)\mathrm{d}h. \\
&\qquad := I_1^{(M,\varepsilon)} + I_2^{(M,\varepsilon)}.
\end{aligned} \tag{II.4.3}$$

First, we address the integral set on $[-M, M]$, which we denote $I_1^{(M,\varepsilon)}$. Since h lies in a bounded set in this integral, φ is $\mathcal{C}^1$, and $\lim_{\varepsilon\to 0}(t_\varepsilon, x_\varepsilon) = (t_0, x_0)$, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\varphi\left(t_\varepsilon, x_\varepsilon - \operatorname{sign}(h) f^{-1}(\varepsilon f(h))\right) - \varphi(t_\varepsilon, x_\varepsilon)\right) = \frac{\operatorname{sign}(h) f(h)}{f'(0)} \partial_x \varphi(t_0, x_0)$$

uniformly in $[-M, M]$. Hence, we obtain

$$\lim_{\varepsilon\to 0} I_1^{(M,\varepsilon)} = \int_{[-M,M]} e^{\frac{\operatorname{sign}(h) f(h)}{f'(0)} \partial_x \varphi} J(h)\, \mathrm{d}h. \tag{II.4.4}$$

Next, we address the integral set on $\mathbb{R} \setminus [-M, M]$, which we denote $I_2^{(M,\varepsilon)}$. Using estimate (II.1.19) from Lemma II.9 on u_ε, we have that, for all $h \in \mathbb{R}$,

$$u_\varepsilon\left(t_\varepsilon, x_\varepsilon - \operatorname{sign}(h) f^{-1}(\varepsilon f(h))\right) - u_\varepsilon(t_\varepsilon, x_\varepsilon) \geqslant -A f(\operatorname{sign}(h) f^{-1}(\varepsilon f(h))) = -A\varepsilon f(h),$$

so that

$$I_2^{(M,\varepsilon)} \leqslant \int_{\mathbb{R}\setminus[-M,M]} e^{Af(h)} J(h) \mathrm{d}h = 2\int_M^\infty e^{-(1-A)f(h)} \mathrm{d}h.$$

Recall, from (II.1.17), that $A < 1 - 1/\mu$ where $\mu = \liminf_{x\to\infty} x f'(x) > 1$. This implies that the integrand above is integrable. Indeed, fix $\alpha_A := \mu/2 - 1/(2-2A)$. An easy computation using only that $A < 1 - 1/\mu$ shows that $\alpha_A > 0$. Then from Hypothesis II.1 on the kernel J, we have that, if M is sufficiently large, $f(h) \geqslant (\mu - \alpha_A)\ln(|h|) - C$ for some constant $C > 0$, depending only on f and A, and all $|h| > M$. Then

$$\begin{aligned} I_2^{(M,\varepsilon)} \leqslant 2\int_M^\infty e^{-(1-A)f(h)} \mathrm{d}h &\leqslant 2\int_M^\infty e^{-(1-A)(\mu-\alpha_A)\ln(|h|)+(1-A)C} \mathrm{d}h \\ &\leqslant 2e^{(1-A)C}\int_M^\infty |h|^{-(1-A)(\mu-\alpha_A)} \mathrm{d}h. \end{aligned}$$

By our choice of α_A, it follows that $(1-A)(\mu-\alpha_A) > 1$. Hence the right hand side tends to 0 as $M \to \infty$, uniformly in ε.

From the estimates (II.4.3) and (II.4.4), we can pass to the limit $\varepsilon \to 0$ and then $M \to \infty$ in (II.4.2) to obtain the inequality (II.4.1). This concludes the proof that $\underline{u}$ is a viscosity super–solution to (II.1.22) as desired.

In order to show that $\overline{u}$ is a viscosity sub–solution to (II.1.22), the steps are almost identical. The only difference being that one must deal with the term $\exp(-u_\varepsilon/\varepsilon)$. However, this is easily dealt with by splitting into cases when $u = 0$ and when $u > 0$. As such, we omit the proof.

□

We now move on to the proof of Theorem II.10 (ii).

Proof of Theorem II.10(ii). The first step is to state and prove that $\underline{u}$ and $\overline{u}$ satisfy related initial conditions so that we may apply the comparison principle to conclude that $\overline{u} \leqslant \underline{u}$. As before, we detail the proof for $\underline{u}$ but the proof for $\overline{u}$ is very similar. The initial condition is

$$\max\left\{\min\left\{\partial_t \underline{u} - \left(1 - \int_{\mathbb{R}} e^{\frac{\operatorname{sign}(h) f(h)}{f'(0)} \partial_x \underline{u}} J(h)\, \mathrm{d}h\right) + 1, \underline{u}\right\}, \underline{u} - u^0\right\} \geqslant 0, \tag{II.4.5}$$

on $\{t=0\}\times\mathbb{R}$ in the viscosity sense, where u^0 is the limit as $\varepsilon\to 0$ of the initial data sequence $(u_\varepsilon^0)_\varepsilon$. To prove the inequality (II.4.5), let $\phi\in\mathcal{C}^1\left((0,+\infty)\times\mathbb{R}\right)$ be a test function such that $\underline{u}-\phi$ has a strict global minimum at $(t_0=0,x_0)$. We now prove that either

$$\underline{u}(0,x_0)\geqslant u^0(x_0)$$

or

$$\partial_t\phi-\left(1-\int_{\mathbb{R}}e^{\frac{\operatorname{sign}(h)f(h)}{f'(0)}\partial_x\phi}J(h)\,\mathrm{d}h\right)+1\geqslant 0\ \text{ on }\ \{t=0\}\times\mathbb{R}\qquad\text{and}\qquad\underline{u}(0,x_0)\geqslant 0.$$

Suppose that $\underline{u}(0,x_0)<u^0(x_0)$. The argument now starts similarly as in the proof above. By the definition of the lower half–relaxed limit, there exists a sequence $(t_\varepsilon,x_\varepsilon)$ of minimum points of $u_\varepsilon-\psi$ satisfying $(t_\varepsilon,x_\varepsilon)\to(0,x_0)$ as $\varepsilon\to 0$. We first claim that there exists a sub–sequence $(t_{\varepsilon_k},x_{\varepsilon_k})_k$ of the above sequence, with $\varepsilon_k\to 0$ as $k\to\infty$, such that $t_{\varepsilon_k}>0$, for all k.

Suppose that this is not true. Then, for ε small enough, $t_\varepsilon=0$ and thus $u_\varepsilon-\phi$ has a local minimum at $(0,x_\varepsilon)$. It follows that, for all (t,x) in some neighborhood of $(0,x_\varepsilon)$, we have

$$u_\varepsilon^0(x_\varepsilon)-\phi(0,x_\varepsilon)\leqslant u_\varepsilon(0,x_\varepsilon)-\phi(0,x_\varepsilon)\leqslant u_\varepsilon(t,x)-\phi(t,x).$$

Taking the lower half–relaxed limit as $\varepsilon\to 0$ and $(t,x)\to(0,x_0)$ on the right hand side of the above inequality, we obtain

$$u^0(x_0)-\phi(0,x_0)\leqslant\underline{u}(0,x_0)-\phi(0,x_0).$$

This contradicts our assumption $\underline{u}(0,x_0)<u^0(x_0)$.

Hence, there exists a sub–sequence $(t_{\varepsilon_k},x_{\varepsilon_k})_k$ such that $t_{\varepsilon_k}>0$, for all $k>0$. We can reproduce the same argument as in the proof of Lemma II.17 above to conclude that (II.4.5) holds.

We are now ready to conclude the proof of (ii). Due to standard arguments of viscosity solutions, see Barles and Perthame [1990]; Barles et al. [1990], we know that equation (II.1.22) has a comparison principle for possibly discontinuous viscosity solutions. As such, Lemma II.17 implies that $\overline{u}\leqslant\underline{u}$. On the other hand, we recall that, $\overline{u}\geqslant\underline{u}$ by construction. It follows that $\overline{u}=\underline{u}$, which in turn implies that u_ε converges locally uniformly to a function u satisfying the equation (II.1.22). Moreover u inherits the gradient bound (II.9)

$$\|\partial_x u\|_{L^\infty((0,\infty)\times\mathbb{R})}\leqslant Af'(0). \tag{II.4.6}$$

This concludes the proof. □

II.4.2 *Proof of Theorem II.11: convergence of n_ε*

We now return to the behavior of n_ε as $\varepsilon\to 0$.

Proof of Theorem II.11.

 # Convergence on the positive set. Fix any $(t_0,x_0)\in\{(t,x)\in\mathbb{R}\times(0,\infty):u(t,x)>0\}$. Since u_ε converges locally uniformly to u, then on a small ball around (t_0,x_0), there exists $\alpha>0$, such that $u_\varepsilon(t,x)\geqslant\alpha>0$ for all ε sufficiently small. Using the Hopf–Cole transform, we see that for all (t,x) in a small ball around (t_0,x_0),

$$n_\varepsilon(t,x)=e^{-\frac{u_\varepsilon(t,x)}{\varepsilon}}\leqslant e^{-\frac{\alpha}{\varepsilon}}.$$

Taking the limit $\varepsilon \to 0$ clearly yields the convergence of n_ε to zero. Hence $n_\varepsilon(t,x)$ converges to 0 locally uniformly on $\{(t,x) \in \mathbb{R} \times (0,\infty) : u(t,x) > 0\}$.

 # Convergence on the null set. We next consider the case when (t_0, x_0) is an element of the interior of $\{(t,x) \in \mathbb{R} \times (0,\infty) : u(t,x) = 0\}$. Take r sufficiently small so that u vanishes on the ball $B_r(t_0, x_0)$. Consider the test function

$$\phi(t,x) = \frac{2Af'(0)}{r}|x - x_0|^2 + |t - t_0|^2, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R};$$

Due to the finite difference bound that u inherits from Lemma II.9, it is easy to check that $(u - \phi)$ has a strict local maximum at (t_0, x_0). In addition, the function $x \mapsto (u - \phi)(t_0, x)$ has a strict global maximum at x_0. Indeed, we have that

$$u(t_0,x) = u(t_0,x) - u(t_0,x_0) \leqslant Af(x - x_0) \leqslant A|x - x_0|f'(\xi),$$

for some $\xi \in [0, |x - x_0|]$. Since f is concave, then $f'(\xi) \leqslant f'(0)$. Consider first x such that $|x - x_0| > r$. In this case, we have that

$$u(t_0,x) \leqslant A|x - x_0|f'(0) \leqslant \frac{Af'(0)}{r}|x - x_0|^2 < \phi(t_0,x).$$

Hence we have that $u(t_0,x) - \phi(t_0,x) < 0 = u(t_0,x_0) - \phi(t_0,x_0)$ for all x such that $|x - x_0| > r$. On the other hand, if $0 < |x - x_0| \leqslant r$ then $u(t_0,x) = 0$ and we have that $u(t_0,x) - \phi(t_0,x) < 0 = u(t_0,x_0) - \phi(t_0,x_0)$. In both cases, we see that x_0 is a strict global maximum in x of $u - \phi$ at time t_0.

Since u_ε converges locally uniformly to u, for ε small enough, we can construct sequence of points $(t_\varepsilon, x_\varepsilon)$ such that $(t_\varepsilon, x_\varepsilon)$ is the location of a maximum of $u_\varepsilon - \phi$ in $B_r(t_0,x_0)$ and $(t_\varepsilon, x_\varepsilon) \to (t_0, x_0)$ as $\varepsilon \to 0$. In addition, arguing as above, the function $x \mapsto (u_\varepsilon - \phi)(t_\varepsilon, x)$ has a global maximum in x_ε. This gives us the inequalities, for all $x \in \mathbb{R}$,

$$u_\varepsilon(t_\varepsilon,x) - u_\varepsilon(t_\varepsilon,x_\varepsilon) \leqslant \phi(t_\varepsilon,x) - \phi(t_\varepsilon,x_\varepsilon), \qquad \text{and} \qquad \partial_t u_\varepsilon(t_\varepsilon,x_\varepsilon) = 2(t_\varepsilon - t_0). \tag{II.4.7}$$

Since u_ε solves (II.1.16), we deduce from (II.4.7) that for $(t_\varepsilon, x_\varepsilon)$,

$$\begin{aligned} n_\varepsilon(t_\varepsilon,x_\varepsilon) - 1 &= \partial_t u_\varepsilon(t_\varepsilon,x_\varepsilon) - \left(1 - \int_{\mathbb{R}} e^{-\frac{1}{\varepsilon}\left(u_\varepsilon\left(t_\varepsilon, x_\varepsilon - \mathrm{sign}(h) f^{-1}(\varepsilon f(|h|))\right) - u_\varepsilon(t_\varepsilon,x_\varepsilon)\right)} J(h)\mathrm{d}h\right) \\ &= 2(t_\varepsilon - t_0) - \left(1 - \int_{\mathbb{R}} e^{-\frac{1}{\varepsilon}\left(u_\varepsilon\left(t_\varepsilon, x_\varepsilon - \mathrm{sign}(h) f^{-1}(\varepsilon f(|h|))\right) - u_\varepsilon(t_\varepsilon,x_\varepsilon)\right)} J(h)\mathrm{d}h\right). \end{aligned} \tag{II.4.8}$$

Arguing as above and using (II.4.7) with the integral term in (II.4.8), it follows that

$$\lim_{\varepsilon\to 0}\left(1 - \int_{\mathbb{R}} e^{-\frac{1}{\varepsilon}\left(u_\varepsilon\left(t_\varepsilon, x_\varepsilon - \mathrm{sign}(h) f^{-1}(\varepsilon f(|h|))\right) - u_\varepsilon(t_\varepsilon,x_\varepsilon)\right)} J(h)\mathrm{d}h\right) \leqslant 1 - \int_{\mathbb{R}} e^{\frac{\mathrm{sign}(h) f(|h|)}{f'(0)}\partial_x\phi(t_0,x_0)} J(h)\,\mathrm{d}h = 0.$$

Here, the last equality used the explicit expression of ϕ, which gives $\partial_x\phi(t_0,x_0) = 0$. The above yields, along with (II.4.8)

$$\liminf_{\varepsilon\to 0} n_\varepsilon(t_\varepsilon,x_\varepsilon) \geqslant 1. \tag{II.4.9}$$

Since $t_\varepsilon \to t_0$ as $\varepsilon \to 0$, we may conclude that $\liminf_{\varepsilon\to 0} n_\varepsilon(t_\varepsilon,x_\varepsilon) \geqslant 1$. On the other hand, we have that

$$\liminf_{\varepsilon\to 0} n_\varepsilon(t_0,x_0) \geqslant \liminf_{\varepsilon\to 0} n_\varepsilon(t_\varepsilon,x_\varepsilon) e^{\frac{\phi(t_\varepsilon,x_\varepsilon)}{\varepsilon}} \geqslant 1,$$

where the first inequality is due to (II.4.7) and the second is due to the non-negativity of ϕ along with (II.4.9). Using that $n_\varepsilon \leqslant 1$, we conclude that $\lim_{\varepsilon\to 0} n_\varepsilon(t_0,x_0) = 1$ as claimed.

□

II.4.3 *Proof of Lemma II.9: the* a priori *bounds*

The only remaining ingredient is to prove the *a priori* bounds on u^ε. We proceed by constructing explicit sub- and super-solutions.

Proof of Lemma II.9. To estimate u_ε from above, we first observe that n_ε is positive and bounded by 1 and u_ε solves (II.1.16). Thus

$$\begin{cases} \partial_t u_\varepsilon(t,x) \leqslant 1, & \text{on } (0,\infty)\times\mathbb{R} \\ u_\varepsilon(0,\cdot) = u_\varepsilon^0. \end{cases}$$

As a consequence, for all $t \geqslant 0$ and $x \in \mathbb{R}$,

$$u_\varepsilon(t,x) \leqslant u_\varepsilon^0(x) + t.$$

To get a bound from below, we define $\underline{s}(t,x) = u_\varepsilon^0(x) - rt$, where r is chosen below. We prove that $\underline{s}$ is a super–solution. Using assumption (II.1.17), for all $\varepsilon > 0$ and $x \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon}\left(u_\varepsilon^0(x-\psi_\varepsilon(h))-u_\varepsilon^0(x)\right)} J(h)\mathrm{d}h \leqslant \int_{-\infty}^{\infty} e^{\frac{A}{\varepsilon}f(|\psi_\varepsilon(h)|)} J(h)\mathrm{d}h = \int_{-\infty}^{\infty} e^{Af(h)} J(h)\mathrm{d}h < +\infty.$$

Define $r := \int_{\mathbb{R}} e^{Af(h)}dh$. We deduce that $\underline{s}$ satisfies

$$\begin{cases} \partial_t \underline{s} - \left(1 - \displaystyle\int_{\mathbb{R}} e^{-\frac{1}{\varepsilon}(\underline{s}(t,x-\psi_\varepsilon(h))-\underline{s}(t,x))} J(h)\mathrm{d}h\right) + 1 \leqslant 0, & \text{on } (0,\infty)\times\mathbb{R}, \\ \underline{s}(0,x) \leqslant u_\varepsilon^0(x), & x \in \mathbb{R}. \end{cases}$$

The function $\underline{s}$ is a sub–solution to (II.1.16). Hence, for all $t \geqslant 0$ and $x \in \mathbb{R}$,

$$-rt \leqslant u_\varepsilon(t,x) - u_\varepsilon^0(x) \leqslant t.$$

To conclude the proof of the lemma, we now prove the inequality on the finite difference of u_ε, namely (II.1.19). To this end, we define for all $t \geqslant 0$, $x \in \mathbb{R}$ and $h \in \mathbb{R}$,

$$w_{\varepsilon,h}(t,x) := u_\varepsilon(t,x) - u_\varepsilon(t,x+h).$$

Then, using (II.1.16) we see that $w_{\varepsilon,h}$ satisfies the equation

$$\begin{cases} \partial_t w_{\varepsilon,h} + \displaystyle\int_{\mathbb{R}} \left(e^{-\frac{1}{\varepsilon}(u_\varepsilon(t,x-\psi_\varepsilon(h))-u_\varepsilon(t,x))} - e^{-\frac{1}{\varepsilon}(u_\varepsilon(t,x-\psi_\varepsilon(h)+h)-u_\varepsilon(t,x+h))}\right) J(h)\mathrm{d}h \\ \qquad = \left(1 - e^{\frac{w_{\varepsilon,h}}{\varepsilon}}\right) n_\varepsilon, & \text{on } (0,\infty)\times\mathbb{R}, \\ w_{\varepsilon,h}(0,x) = u_\varepsilon^0(x) - u_\varepsilon^0(x+h), & \text{for all } x \in \mathbb{R}. \end{cases}$$

This reduces to

$$\begin{cases} \partial_t w_{\varepsilon,h} + \displaystyle\int_{\mathbb{R}} e^{-\frac{1}{\varepsilon}(u_\varepsilon(t,x-\psi_\varepsilon(h))-u_\varepsilon(t,x))} \left(1 - e^{\frac{1}{\varepsilon}(w_{\varepsilon,h}(t,x-\psi_\varepsilon(h))-w_{\varepsilon,h}(t,x))}\right) J(h)\mathrm{d}h \\ \qquad = \left(1 - e^{\frac{w_{\varepsilon,h}}{\varepsilon}}\right) n_\varepsilon, & \text{on } (0,\infty)\times\mathbb{R}, \\ w_{\varepsilon,h}(0,x) = u_\varepsilon^0(x) - u_\varepsilon^0(x+h), & \text{for all } x \in \mathbb{R}. \end{cases}$$

We apply the maximum principle to deduce that

$$\sup_{x\in\mathbb{R}} w_{\varepsilon,h}(t,x) \leqslant \max\left\{0, \sup_{x\in\mathbb{R}} w_{\varepsilon,h}(0,x)\right\}.$$

This implies that for all $x \in \mathbb{R}$ and $h \in \mathbb{R}$,

$$u_\varepsilon(t,x+h) - u_\varepsilon(t,x) \geqslant \min\left\{0, \inf_{y\in\mathbb{R}} \left(u_\varepsilon^0(y+h) - u_\varepsilon^0(y)\right)\right\} \geqslant \min\{0, -Af(h)\} = -Af(h).$$

This finishes the proof. □

Troisième partie

Analyse asymptotique d'un modèle de génétique quantitative pour un modèle de reproduction sexuée stationnaire

Nous étudions le comportement asymptotique des solutions stationnaires d'un modèle de génétique quantitative qui comprend une fonction de mortalité qui dépend du trait et un opérateur de reproduction intégral non-linéaire avec un petit paramètre. Celui ci mesure la déviation du trait du descendant par rapport à la moyenne du trait des parents. Notre analyse asymptotique traite du régime où le paramètre est petit. Sous des hypothèses de régularité suffisantes sur la croissance et la sélection, nous prouvons existence et unicité locale d'un profil stationnaire qui prend une forme Gaussienne avec une petite variance. Notre approche est basée sur des techniques d'analyse perturbative qui exigent une description précise des correcteurs de l'ordre principal : les profils Gaussiens. Nos résultats étendent des résultats précédent obtenus avec un opérateur de reproduction asexué en utilisant des techniques différentes. Ce travail a été publié dans le Journal de l'Ecole Polytechnique : Calvez et al. [2019].

We investigate solutions $(\lambda_\varepsilon, F_\varepsilon) \in \mathbb{R} \times L^1(\mathbb{R})$ of the following stationary problem:

$$\lambda_\varepsilon F_\varepsilon(z) + m(z)F_\varepsilon(z) = \mathcal{B}_\varepsilon(F_\varepsilon)(z)\,, \quad z \in \mathbb{R}^d\,, \tag{PF_ε}$$

where $\mathcal{B}_\varepsilon(f)$ is the following non linear, homogeneous integral operator associated to the infinitesimal model Fisher [1918]; Barton et al. [2017]:

$$\mathcal{B}_\varepsilon(f)(z) := \frac{1}{\varepsilon^d \pi^{\frac{d}{2}}} \iint_{\mathbb{R}^{2d}} \exp\left[-\frac{1}{\varepsilon^2}\left(z - \frac{z_1+z_2}{2}\right)^2\right] f(z_1) \frac{f(z_2)}{\int_{\mathbb{R}^d} f(z_2')\,dz_2'}\,dz_1 dz_2. \tag{III.1.1}$$

In the context of quantitative genetics, the variable z denotes a multi-dimensional phenotypic trait, $F_\varepsilon(z)$ is the phenotypic distribution of the population and $m(z)$ is the (trait-dependent) mortality rate which results in the selection of the fittest individuals.

The mixing operator $\mathcal{B}_\varepsilon$ acts as a simple model for the inheritance of quantitative traits in a population with a sexual mode of reproduction. As formulated in (III.1.1), it is assumed that offspring traits are distributed normally around the mean of the parental traits $(z_1 + z_2)/2$, with a variance which remains constant accross generations, here $\varepsilon^2/2$.

We are interested in the asymptotic behavior of the trait distribution F_ε as ε^2 vanishes.

This asymptotic regime was investigated thoroughly for various linear operators $\mathcal{B}_\varepsilon$ associated with asexual reproduction such as for instance the diffusion operator $F_\varepsilon(z) + \varepsilon^2 \Delta F_\varepsilon(z)$, or the convolution operator $\frac{1}{\varepsilon}K(\frac{z}{\varepsilon}) * F_\varepsilon(z)$ where K is a probability kernel with unit variance, see Diekmann et al. [2005]; Perthame [2007]; Barles and Perthame [2007]; Barles et al. [2009]; Lorz et al. [2011] for the earliest investigations, see further Méléard and Mirrahimi [2015]; Mirrahimi [2018]; Bouin et al. [2018] for the case of a fractional diffusion operator (or similarly a fat-tailed kernel K), and see further Mirrahimi [2013]; Mirrahimi and Perthame [2015]; Bouin and Mirrahimi [2015]; Lam and Lou [2017]; Gandon and Mirrahimi [2017]; Mirrahimi [2017]; Mirrahimi and Gandon [2018]; Calvez et al. [2018] for the interplay between evolutionary dynamics and a spatial structure. In the linear case, the asymptotic analysis usually leads to a Hamilton-Jacobi equation for the Hopf-Cole transform $U_\varepsilon = -\varepsilon \log F_\varepsilon$. This yields an original problem with non-negativity constraint that requires a careful well-posedness analysis Mirrahimi and Roquejoffre [2015]; Calvez and Lam [2018].

Much less is known about the non linear equation (PF_ε), although this model is widely used in theoretical evolutionary biology to describe sexual reproduction, see *e.g.* Slatkin [1970]; Roughgarden [1972]; Slatkin and Lande [1976]; Bulmer [1980]; Turelli and Barton [1994]; Tufto [2000]; Barfield et al. [2011]; Huisman and Tufto [2012]; Cotto and Ronce [2014]; Barton et al. [2017]; Turelli [2017]. From a mathematical viewpoint, the model (III.1.1) received recent attention in the field of probability theory Barton et al. [2017] and integro-differential equations Mirrahimi and Raoul [2013]; Raoul [2017]. In the latter couple of articles, a scaling different from (III.1.1) is studied: the variance is of order one, but there is a large reproduction rate that enforces the relaxation of the phenotypic distribution towards a Gaussian local equilibrium. Macroscopic equations are rigorously derived in Raoul [2017], in the case of an additional spatial structure, in the spirit of hydrodynamic limits for kinetic equations.

In a different context, a similar collisional operator as $\mathcal{B}_\varepsilon$ (III.1.1) was introduced in the modelling of self-propelled particles with alignment interactions, see for instance Bertin et al. [2006]; Degond et al. [2014]. When two particles interact they tend to align with the mean velocity, with some possible noise. However, there are some discrepancies with our case study, since the operator is

not conservative in our case, by definition of a reproduction operator. Moreover, it is normalized by the total mass of the phenotypic distribution: $\int f(z_2')\,dz_2'$. The rationale behind this choice is that during the mating process, the first parent chooses the trait of its partner depending on its frequency in the population. This is the neutral case without any assumption about assortative mating. Moreover, this dependency upon the frequency rather than the density discards any small population effects that could arise from a quadratic collisional operator. Such homogeneity of degree one is a key ingredient in our analysis.

The problem (PF_ε) is equivalent to the existence of special solutions of the form $\exp(\lambda_\varepsilon t)F_\varepsilon(z)$, for the following non-linear but one-homogeneous equation which will be the subject of future work:

$$\partial_t f(t,z) + m(z)f(t,z) = \mathcal{B}_\varepsilon(f)(t,z)\,, \quad t>0\,,\ z\in\mathbb{R}^d. \tag{III.1.2}$$

Alternatively speaking, the problem (PF_ε) expresses the balance between selection via trait-dependent mortality $m(z)$, and the generation of diversity through reproduction $\mathcal{B}_\varepsilon$. The scalar λ_ε is analogous to the principal eigenvalue of the operator $\mathcal{B}_\varepsilon - m$. However, it might not be unique, as in the Krein-Rutman theory, see Corollary III.5. It measures the global fitness of the population: the population grows exponentially fast $\lambda_\varepsilon > 0$ when the reproduction term $\mathcal{B}_\varepsilon$ dominates, while it declines exponentially fast $\lambda_\varepsilon < 0$ when the mortality m out-competes the reproduction.

This preliminary analysis on the stationary profile paves the way for a systematic analysis of various quantitative genetics models, including time marching problems and the combination of multiple effects.

Our work is inspired by similar asymptotics in the case of linear operator $\mathcal{B}_\varepsilon$, see the seminal work by Diekmann et al. [2005] and references cited above. Accordingly, our goal is to analyze problem (PF_ε) in the limit of vanishing variance $\varepsilon^2 \to 0$. As there is few diversity generated in this asymptotic regime, we expect that the variance of the distribution solution F_ε vanishes as well. Actually, there is strong evidence that the leading order profile of F_ε is a Gaussian distribution with variance ε^2. As a matter of fact, any Gaussian distribution with variance ε^2 is invariant by the infinitesimal operator $\mathcal{B}_\varepsilon$ in the absence of selection $(m\equiv 0,\ \lambda_\varepsilon = 1)$ Turelli and Barton [1994]; Mirrahimi and Raoul [2013]. This motivates the following decomposition of the solution:

$$F_\varepsilon(z) = \frac{1}{(2\pi)^{\frac{d}{2}}\varepsilon^d}\exp\left(-\frac{(z-z_0)^2}{2\varepsilon^2} - U_\varepsilon(z)\right). \tag{III.1.3}$$

The latter (III.1.3) is similar to the Hopf-Cole transform used in the asymptotic analysis of adaptative evolutionary dynamics in asexual populations. In our case U_ε is a corrector term that measures the deviation from the leading Gaussian distribution of variance ε^2. Our analysis reveals that selection determines the center of the distribution z_0, as expected, and also reshapes the distribution F_ε via the corrector U_ε.

The operator $\mathcal{B}_\varepsilon$ is invariant by translation. Up to a translation of m, we may assume that the leading order Gaussian distribution is centered at the origin, *i.e.* $z_0 = 0$. Next, up to a change of $\lambda_\varepsilon \leftarrow \lambda_\varepsilon + m(0)$, we may assume that $m(0) = 0$. Note that we may also assume $U_\varepsilon(0) = 0$ without loss of generality, as the original problem is homogeneous.

Plugging the transformation of (III.1.3) into (PF_ε) yields the following equivalent problem for U_ε :

$$\lambda_\varepsilon + m(z) = I_\varepsilon(U_\varepsilon)(z)\exp\left(U_\varepsilon(z) - 2U_\varepsilon\left(\frac{z}{2}\right) + U_\varepsilon(0)\right), \quad z\in\mathbb{R}^d. \tag{PU_ε}$$

The residual term from the integral contribution is the following non-local term $I_\varepsilon(U_\varepsilon)$, see Section III.2.1 for the details of the derivation:

$$I_\varepsilon(U_\varepsilon)(z) = \frac{\iint_{\mathbb{R}^{2d}} \exp\left[-\frac{1}{2}y_1\cdot y_2 - \frac{3}{4}\left(|y_1|^2+|y_2|^2\right) + 2U_\varepsilon\left(\frac{z}{2}\right) - U_\varepsilon\left(\frac{z}{2}+\varepsilon y_1\right) - U_\varepsilon\left(\frac{z}{2}+\varepsilon y_2\right)\right] dy_1 dy_2}{\pi^{d/2}\int_{\mathbb{R}^d}\exp\left[-\frac{1}{2}|y|^2 + U_\varepsilon(0) - U_\varepsilon(\varepsilon y)\right] dy}. \tag{III.1.4}$$

This decomposition appears to be relevant because a formal computation shows that $I_\varepsilon(U_\varepsilon)\to 1$ as $\varepsilon\to 0$. Establishing uniform convergence is actually a cornerstone of our analysis. Thus for small ε, the problem (PU_ε) is presumably close to the following corrector equation, obtained formally at $\varepsilon = 0$:

$$\lambda_0 + m(z) = \exp\left(U_0(z) - 2U_0\left(\frac{z}{2}\right) + U_0(0)\right), \quad z\in\mathbb{R}^d. \tag{PU_0}$$

Interestingly, this finite difference equation admits explicit solutions by means of an infinite series:

$$U_0(z) = \gamma_0\cdot z + \sum_{k\geqslant 0} 2^k \log\left(\lambda_0 + m(2^{-k}z)\right),$$

However, two difficulty remains: identify *(i)* the linear part $\gamma_0\in\mathbb{R}^d$ and *(ii)* the unknown $\lambda_0\in\mathbb{R}$. On the one hand, the linear part γ_0 cannot be recovered from (PU_0) because linear contributions cancel in the right-hand-side of (PU_0). Thus, identifying the coefficient γ_0 will be a milestone of our analysis. On the other hand, two important conditions must be fulfilled to guarantee that the series above converges, namely:

$$\lambda_0 + m(0) = 1, \quad \text{and} \quad \partial_z m(0) = 0.$$

The latter is a constraint on the possible translations that can be operated: the origin must be located at a critical point of m. The former prescribes the value of λ_0 accordingly. These two conditions are necessary conditions for the resolvability of Problem (PU_0). Indeed, evaluating Problem (PU_0) at $z=0$, we get the first identity. Next, differentiating and evaluating again at $z=0$, we get the second identity.

In the sequel we make this formal discussion rigourous, following a perturbative approach for ε small enough. Before stating our main result, we need to prescribe the appropriate functional space for the corrector U_ε.

Definition III.1 (Functional space for U_ε)**.**
For any positive parameter $\alpha\leqslant 2/5$, we define the functional space

$$\mathcal{E}^\alpha = \left\{ u\in\mathcal{C}^3(\mathbb{R}^d) \,:\, u(0)=0, \text{ and } \left| \begin{array}{r} |Du(z)| \\ (1+|z|)^\alpha \left|D^2u(z)\right| \\ (1+|z|)^\alpha \left|D^3u(z)\right| \end{array} \right. \in L^\infty(\mathbb{R}^d) \right\},$$

equipped with the norm

$$\|u\|_\alpha = \max\left(\sup_{z\in\mathbb{R}^d} |Du(z)|, \sup_{z\in\mathbb{R}^d} (1+|z|)^\alpha \left\{\left|D^2u(z)\right|, \left|D^3u(z)\right|\right\}\right). \tag{III.1.5}$$

For any bounded set K of $\mathcal{E}^\alpha$, we use the notation $\|K\|_\alpha = \sup_{u\in K}\|u\|_\alpha$. Occasionally we use the notation φ_α for the weight function $\varphi_\alpha(z) = (1+|z|)^\alpha$. Although $2/5$ is not the critical threshold, it happens that the exponent α cannot be taken too large in our approach. We set implicitly $\alpha = 2/5$ in the following results, however we leave it as a parameter to emphasize its role in the analysis, and to pinpoint the apparition of the threshold. Note that $\alpha > 0$ is required in our approach, as one constant collapses in the limit $\alpha \to 0$ (see estimate (III.5.7) below).

Then, we detail the assumptions on the selection function m.

Definition III.2 (Assumptions on m)**.**
The function m is a $\mathcal{C}^3(\mathbb{R}^d)$ function, bounded below, that admits a local *non-degenerate minimum at 0 such that $m(0) = 0$, and there exists $\mu_0 > 0$ such that $D^2m(0) \geqslant \mu_0\,\mathrm{Id}$ in the sense of symmetric matrices. Furthermore we suppose that $(\forall z)\ 1+m(z) > 0$ and*

$$(1+|z|)^\alpha \frac{D^k m(z)}{1+m(z)} \in L^\infty(\mathbb{R}^d)\,, \quad \text{for } k = 1,2,3\,. \tag{III.1.6}$$

Remark III.3. *Our result is insensitive to the sign of the local extremum. Indeed, one can replace the hypothesis that m admits a "local non degenerate* minimum *" at 0 with a "local non degenerate* maximum *" at 0, and that there exists $\mu_0 < 0$ such that $D^2m(0) \leqslant \mu_0\,\mathrm{Id}$. However, we leave our main assumption as in Definition III.2 as it is the most natural one from the point of view of stability analysis for the time-marching problem* (III.1.2).

The condition (III.1.6) is clearly verified if m is a polynomial function. It would be tempting to write, in short, that $\log(1+m) \in \mathcal{E}^\alpha$, which is indeed a consequence of (III.1.6). However, the latter condition also contains the decay of the first order derivative $D\log(1+m)$ with rate $|z|^{-\alpha}$, which is not contained in the definition of $\mathcal{E}^\alpha$ (III.1.5) for good reasons.

We also introduce the subset $\mathcal{E}^\alpha_0$:

$$\mathcal{E}^\alpha_0 = \left\{ v \in \mathcal{E}^\alpha \,:\, Dv(0) = 0,\ D^2v(0) \geqslant D^2m(0) \geqslant \mu_0\,\mathrm{Id} \right\}, \tag{III.1.7}$$

Then, our assumption on m in fact guarantees that

$$\log(1+m) \in \mathcal{E}^\alpha_0. \tag{III.1.8}$$

The main result of this article is the following theorem :

Theorem III.4 (Existence and convergence)**.**

(i) There exist K_0 a ball of $\mathcal{E}^\alpha$, and ε_0 a positive constant, such that for any $\varepsilon \leqslant \varepsilon_0$, the problem ($PU_\varepsilon$) admits a unique solution $(\lambda_\varepsilon, U_\varepsilon) \in \mathbb{R}\times K_0$.

(ii) The family $(\lambda_\varepsilon, U_\varepsilon)_\varepsilon$ converges to (λ_0, U_0) as $\varepsilon \to 0$, with

$$\lambda_0 = 1, \tag{III.1.9}$$

$$U_0(z) = \gamma_0 \cdot z + V_0(z), \tag{III.1.10}$$

where

$$\gamma_0 = \begin{cases} \dfrac{\partial_z^3 m(0)}{2\partial_z^2 m(0)}\,, & \text{if } \ d = 1 \\[2ex] \dfrac{1}{2}\left(D^2m(0)\right)^{-1} D(\Delta m)(0)\,, & \text{if } \ d > 1 \end{cases} \qquad \text{and} \qquad V_0 = \sum_{k\geqslant 0} 2^k \log\left(1 + m(2^{-k}z)\right). \tag{III.1.11}$$

Moreover, the convergence $U_\varepsilon \to U_0$ is locally uniform up to the second derivative.

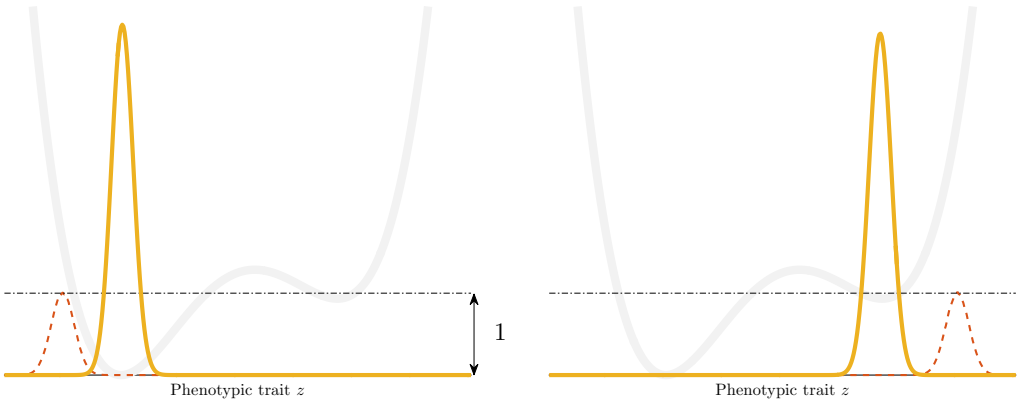


Figure III.1: Numerical simulations of the stationary problem (PF_ε) with $\varepsilon = 0.1$ in an asymmetric double-well mortality rate (grey line). The numerical equilibrium is in yellow plain line. The only difference between the two simulations is the initial data (red dashed line). The simulations illustrate the lack of uniqueness for problem (PF_ε).

An immediate remark is that the regularity required by (III.1.6), and particularly the $\mathcal{C}^3$ regularity of m, is consistent with formula (III.1.10) which involves the pointwise value of third derivatives of m. Alternatively speaking we think that our result is close to optimal in terms of regularity.

It is important to notice that our result holds true for *any* local mimimum z_0 such that

$$m(z_0) < 1 + \inf m. \tag{III.1.12}$$

One should define the functional spaces $\mathcal{E}^\alpha$ and $\mathcal{E}_0^\alpha$ accordingly (and particularly replace the conditions $u(0) = 0$ and $Du(0) = 0$ by the conditions $u(z_0) = 0$ and $Du(z_0) = 0$), and then adapt (III.1.9)–(III.1.10) as follows, for the one-dimensional case:

$$\begin{aligned} &\lambda_0 = 1 - m(z_0), \\ &U_0(z_0 + h) = \gamma_0 \cdot h + \sum_{k \geqslant 0} 2^k \log\left(1 + m(2^{-k}(z_0 + h)) - m(z_0)\right), \end{aligned} \tag{III.1.13}$$

where γ_0 is defined by the same formula as in (III.1.11) but evaluated at z_0. Immediately, one sees that the compatibility condition (III.1.12) is necessary to have the positivity of the term inside the $\log$ in (III.1.13). As a consequence, we have:

Corollary III.5 (Lack of uniqueness)**.**
If the selection function m has at least two different local non-degenerate minima that verify the compatibility condition (III.1.12)*, there exists at least two pairs $(\lambda_\varepsilon, F_\varepsilon)$ solutions of problem (PF_ε) for ε small enough.*

We performed numerical simulations to illustrate this phenomenon (see Figure III.1). The function m is an asymmetric double well function. We solved the time marching problem (III.1.2) but on the renormalized density $F_\varepsilon / \int F_\varepsilon$ in order to catch a stationary profile. We clearly observed the co-existence of two equilibria for the same set of parameters, that were obtained for two different initializations of the scheme. However, let us mention that the question of uniqueness in the case of a convex selection function m is an open question, to the extent of our knowledge.

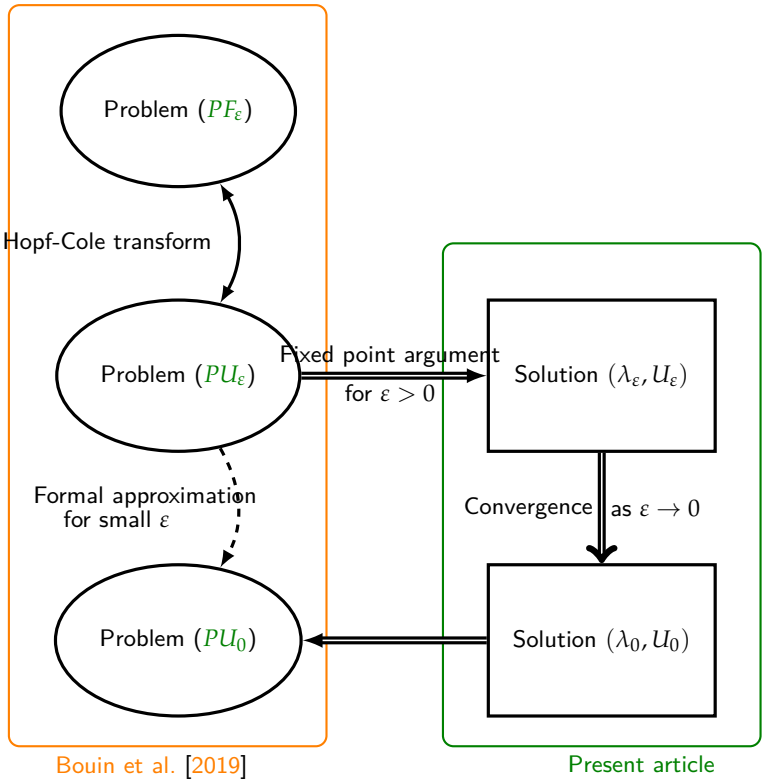


Figure III.2: Scope of our paper compared to precedent work

This result is in contrast with analogous eigenvalue problems where $\mathcal{B}_\varepsilon$ is replaced with a linear operator, say $F_\varepsilon + \varepsilon^2 \Delta F_\varepsilon$ as in various quantitative genetics models with asexual mode of reproduction, see *e.g.* Barles et al. [2009] and references mentioned above, or in the semi-classical analysis of the Schrödinger equation, see *e.g.* Dimassi et al. [1999]. In the linear case, $\lambda_\varepsilon \in \mathbb{R}$ and $F_\varepsilon \geqslant 0, F_\varepsilon \not\equiv 0$ are uniquely determined (up to a multiplicative constant for F_ε) under mild assumptions on the potential m. This is the signature that $\mathcal{B}_\varepsilon$ (III.1.1) is genuinely non-linear and non-monotone, so that possible extensions of the Krein-Rutman theorem for one-homogeneous operators, as in Mahadevan [2007], are not applicable.

The existence part (i) has already been investigated in Bourgeron et al. [2017] using the Schauder fixed point theorem and very loose variance estimates. But the approach was not designed to catch the asymptotic regime $\varepsilon \to 0$. The current methodology gives much more precise information on the behavior of the solutions of the problem (PF_ε) in the regime of vanishing variance.

Theorem 1 provides a rigorous background for the connection between problem (PU_ε) and problem (PU_0) in a perturbative setting. It justifies that the problem (PF_ε) is well approximated by the solution (λ_0, U_0) of the problem (PU_0). Quite surprisingly, the value γ_0 of the linear part of the corrector function U_0 is resolved during the asymptotic analysis although it cannot be obtained readily from problem (PU_0) as mentioned above. It coincides with the heuristics of Bouin et al. [2019] where the same coefficient was obtained by studying the formal expansion up to the next order in ε^2: $U_\varepsilon = U_0 + \varepsilon^2 U_1 + o(\varepsilon^2)$, and by identifying the equation on U_1 in which the value of γ_0 appears as another compatibility condition. Here the value of γ_0 is obtained directly as a by-product of the perturbative analysis.

As mentioned above, our approach is very much inspired, yet different to most of the current literature about asymptotic analysis of asexual models, where the limiting problem is a Hamilton-Jacobi equation, see Perthame [2007] for a comprehensive introduction, and references above. To draw a parallel with our problem, let us consider the case where $\mathcal{B}_\varepsilon(f)$ is replaced with the (linear) convolution operator $K_\varepsilon * f$, where the kernel has the scaling property $K_\varepsilon = \frac{1}{\varepsilon}K\left(\frac{\cdot}{\varepsilon}\right)$, and K is a probability distribution kernel. There, the small parameter ε measures the typical size of the deviation between the offspring trait and the sole parental trait. In this context, it is natural to introduce the Hopf-Cole transform $U_\varepsilon = -\varepsilon \log F_\varepsilon$. Then, the problem is equivalent to the asymptotic analysis of the following equation as $\varepsilon \to 0$:

$$\lambda_\varepsilon + m(z) = \int_{\mathbb{R}^d} K(y) \exp\left(\frac{U_\varepsilon(z) - U_\varepsilon(z-\varepsilon y)}{\varepsilon}\right) dy, \tag{III.1.14}$$

For this model, it is known that U_ε converges towards the viscosity solution of a Hamilton-Jacobi equation Barles et al. [2009]:

$$\lambda_0 + m(z) = H(DU_0(z)) = \int_{\mathbb{R}^d} K(y) \exp\left(DU_0(z)\cdot y\right) dy\,. \tag{III.1.15}$$

Note that the limiting equation on U_0 (III.1.15) can be derived formally from (III.1.14) by a first order Taylor expansion on U_ε.

There are two noticeable discrepancies between the asexual case (III.1.14)–(III.1.15) and our problem involving the infintesimal model with small variance. Firstly, ε plays a similar role in both cases, *i.e.* measuring typical deviations between offspring and parental traits. However, the appropriate normalization differs by a factor ε: it is $-\varepsilon \log F_\varepsilon$ in the asexual case, whereas it is $-\varepsilon^2 \log F_\varepsilon$ in our context, see (III.1.3). This scaling difference is the signature of major differences between the two problems (asexual vs. sexual). Secondly, the two limiting problems (III.1.15) and (PU_0) have completely different natures: a Hamilton-Jacobi PDE in the asexual case, vs. a finite difference equation in the sexual case. Moreover, due to the lack of a comparison principle in the original problem (PF_ε), we could not envision a similar notion of viscosity solutions for (PU_0). Instead, we use rigid contraction properties and a suitable perturbative analysis to construct a unique strong solution near the limiting problem, as depicted in Figure III.2.

Mirrahimi and Raoul [2013] observed that the infinitesimal operator $\mathcal{B}_\varepsilon$ alone enjoys a uniform contraction property with respect to the quadratic Wasserstein distance, with a factor of contraction $1/2$. Recently, this was used by Magal and Raoul [2015] to perform a hydrodynamic limit in a different regime than the one under consideration here. However, the combination of $\mathcal{B}_\varepsilon$ with a zeroth-order heterogeneous mortality $m(z)$ seems to destroy this nice structure (details not shown).

The next section is devoted to the reformulation of problem (PU_ε) into a fixed point problem, introducing a set of notation and the strategy to prove Theorem III.4. The organization of the paper is postponed to the end of the next Section.

Up until the last part of the article we implicitly work in dimension $d = 1$, for the readers' convenience. In section III.7 we pinpoint the few elements of the proof that are specific to the one-dimensional case and give an extension to the higher-dimensional case in order to complete the proof of Theorem III.4.

Acknowledgement.

The authors are grateful to Laure Saint-Raymond for stimulating discussions at the early stage of this work. They are thankful to Sepideh Mirrahimi for pointing out the extension of the result to

local maxima of the selection function, see Remark III.3. Part of this work was completed when VC was on temporary leave to the PIMS (UMI CNRS 3069) at the University of British Columbia. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 639638) and from the French National Research Agency with the project NONLOCAL (ANR-14-CE25-0013) and GLOBNETS (ANR-16-CE02-0009).

III.2 REFORMULATION OF THE PROBLEM AS A FIXED POINT

III.2.1 *Looking for problem* (PU_ε)

The equivalence between problem (PF_ε) and problem (PU_ε) through the transform (III.1.3) is not immediate. It is detailed in Bouin et al. [2019], but we recall here the key steps for the sake of completeness. Plugging (III.1.3) into problem (PF_ε) yields, with the notation $q(z) = \frac{z^2}{2}$:

$$\lambda_\varepsilon + m(z) = \frac{\iint_{\mathbb{R}^2} \exp\left[-\frac{1}{\varepsilon^2}\left(2q\left(z - \frac{z_1+z_2}{2}\right) + q(z_1) + q(z_2) - q(z)\right) - U_\varepsilon(z_1) - U_\varepsilon(z_2) + U_\varepsilon(z)\right] dz_1 dz_2}{\varepsilon\sqrt{\pi}\int_{\mathbb{R}} \exp\left(-\frac{q(z')}{\varepsilon^2} - U_\varepsilon(z')\right) dz'}.$$

When $\varepsilon \to 0$, we expect the numerator integral to concentrate around the minimum of the principal term that is :

$$\operatorname*{argmin}_{(z_1,z_2)} \left[2q\left(z - \frac{z_1+z_2}{2}\right) + q(z_1) + q(z_2) - q(z)\right] = \left(\frac{z}{2}, \frac{z}{2}\right).$$

We introduce the notation

$$\bar{z} = \frac{z}{2}.$$

Using the change of variable $(z_1, z_2) = (\bar{z} + \varepsilon y_1, \bar{z} + \varepsilon y_2)$, we obtain the following equation :

$$\lambda_\varepsilon + m(z) = \frac{\iint_{\mathbb{R}^2} \exp\left(-Q(y_1,y_2) - U_\varepsilon(\bar{z}+\varepsilon y_1) - U_\varepsilon(\bar{z}+\varepsilon y_2) + U_\varepsilon(z)\right) dy_1 dy_2}{\sqrt{\pi}\int_{\mathbb{R}} \exp\left(-y^2/2 - U_\varepsilon(\varepsilon y)\right) dy}, \quad \text{(III.2.1)}$$

where

$$\frac{1}{\varepsilon^2}\left[2q\left(z - \frac{z_1+z_2}{2}\right) + q(z_1) + q(z_2) - q(z)\right] = \frac{1}{2}y_1y_2 + \frac{3}{4}(y_1^2 + y_2^2) = Q(y_1, y_2).$$

Definition III.6.
We denote by Q the following quadratic form :

$$Q(y_1, y_2) = \frac{1}{2}y_1y_2 + \frac{3}{4}(y_1^2 + y_2^2).$$

It is the residual quadratic form after our change of variable. We notice that $\frac{1}{\sqrt{2\pi}}\exp(-Q)$ is the density of a bivariate normal random variable with covariance matrix

$$\Sigma = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}. \quad \text{(III.2.2)}$$

At the denominator of (III.2.1) naturally arises N the density function of a $\mathcal{N}(0,1)$ random variable.

Finally, (III.2.1) is equivalent to problem (PU_ε):

$$\lambda_\varepsilon + m(z) = I_\varepsilon(U_\varepsilon)(z) \exp\left(U_\varepsilon(z) - 2U_\varepsilon\left(\tfrac{z}{2}\right) + U_\varepsilon(0)\right),$$

simply by conjuring $2U_\varepsilon(z/2)$ at the numerator and $U_\varepsilon(0)$ at the denominator, resulting into the defintion of the remainder $I_\varepsilon(U_\varepsilon)$ (III.1.4) that will be controlled uniformly close to 1 in all our analysis.

In the next section we explain how we reformulate the problem (PU_ε) into a fixed point argument in order to use a Banach-Picard fixed point theorem which prove our results rigorously.

III.2.2 *Some auxiliary functionals and the fixed point mapping*

This section is devoted to the derivation of an alternative formulation for problem (PU_ε). Let $(\lambda_\varepsilon, U_\varepsilon)$ be a solution of problem (PU_ε) in $\mathbb{R} \times \mathcal{E}^\alpha$.

The first step is to dissociate the study of λ_ε and U_ε. We first evaluate the problem (PU_ε) at $z = 0$. It yields the following condition on λ_ε, since $m(0) = 0$:

$$\lambda_\varepsilon = I_\varepsilon(U_\varepsilon)(0). \tag{III.2.3}$$

Considering the terms $\mathcal{I}_\varepsilon$ as a perturbation, we divide problem (PF_ε) by $\mathcal{I}_\varepsilon(U_\varepsilon)(z)$ which is positive, and we take the logarithm on each side. Then we obtain the following equation, considering (III.2.3) :

$$U_\varepsilon(z) - 2U_\varepsilon\left(\tfrac{z}{2}\right) + U_\varepsilon(0) = \log\left(\frac{I_\varepsilon(U_\varepsilon)(0) + m(z)}{I_\varepsilon(U_\varepsilon)(z)}\right) \tag{III.2.4}$$

It would be tempting to transform (III.2.4) into a fixed point problem by inverting the linear operator in the left-hand-side. However, the latter is not invertible as it contains linear functions in its kernel. Therefore we are led to consider linear contributions separately.

Our main strategy is to decompose the unknown U_ε under the form

$$\boxed{U_\varepsilon(z) = \gamma_\varepsilon z + V_\varepsilon(z),} \tag{III.2.5}$$

with $V_\varepsilon \in \mathcal{E}_0^\alpha$. This is consistent with the analytic shape of our statement in (III.1.10), where γ_0 and V_0 have quite different features with respect to the function m.

Next, it is natural to differentiate (III.2.4). One ends up with the following recursive equation for every $z \in \mathbb{R}$

$$\partial_z U_\varepsilon(z) - \partial_z U_\varepsilon\left(\tfrac{z}{2}\right) = \partial_z \left[\log\left(\frac{I_\varepsilon(U_\varepsilon)(0) + m}{I_\varepsilon(U_\varepsilon)(z)}\right)\right](z). \tag{III.2.6}$$

One simply deduces that, if U_ε exists and is regular, then we must have:

$$\partial_z U_\varepsilon(z) = \partial_z U_\varepsilon(0) + \sum_{k\geqslant 0} \partial_z \left[\log\left(\frac{I_\varepsilon(U_\varepsilon)(0) + m}{I_\varepsilon(U_\varepsilon)(z)}\right)\right](2^{-k}z). \tag{III.2.7}$$

One can formally integrate back the previous equation to obtain

$$U_\varepsilon(z) = U_\varepsilon(0) + \partial_z U_\varepsilon(0) z + \sum_{k\geqslant 0} 2^k \log\left(\frac{I_\varepsilon(U_\varepsilon)(0) + m}{I_\varepsilon(U_\varepsilon)(z)}\right)(2^{-k}z). \tag{III.2.8}$$

At this stage we formally identify :

▷ $U_\varepsilon(0) = 0$, since $U_\varepsilon \in \mathcal{E}^\alpha$. This is not a loss of generality by homogeneity since F_ε is itself defined up to a multiplicative constant in problem (PF_ε).

▷ $\gamma_\varepsilon = \partial_z U_\varepsilon(0)$. In fact this is part of the decomposition (III.2.5) since $V_\varepsilon \in \mathcal{E}_0^\alpha$.

The real number γ_ε is unknown at this stage, but it needs to verify some compatibility condition to make the series converging in (III.2.6)–(III.2.8). In particular, if we evaluate (III.2.6) at $z = 0$ we obtain that γ_ε must satisfy

$$0 = \partial_z I_\varepsilon(\gamma_\varepsilon \cdot + V_\varepsilon)(0). \tag{III.2.9}$$

We will solve (III.2.9) using an implicit function theorem in order to recover the value γ_ε associated with a given V. Beforehand, we introduce the following notation:

Definition III.7 (Finite differences operator $\mathcal{D}_\varepsilon$)**.**
We define the finite differences functional $\mathcal{D}_\varepsilon$ as

$$\mathcal{D}_\varepsilon(V)(y_1, y_2, z) = V(\bar{z}) - \frac{1}{2}V(\bar{z} + \varepsilon y_1) - \frac{1}{2}V(\bar{z} + \varepsilon y_2)\,, \quad \bar{z} = \frac{z}{2}\,.$$

We introduce the following auxiliary functional which makes the link between γ_ε and V.

Definition III.8 (Auxiliary function $\mathcal{J}_\varepsilon$)**.**
We define the functional $\mathcal{J}_\varepsilon : \mathbb{R} \times \mathcal{E}_0^\alpha \to \mathbb{R}$ as follows

$$\mathcal{J}_\varepsilon(g, V) = \frac{1}{\varepsilon^2\sqrt{2}\pi} \iint_{\mathbb{R}^2} \exp\left[-Q(y_1, y_2) - \varepsilon g(y_1 + y_2) + 2\mathcal{D}_\varepsilon(V)(y_1, y_2, 0)\right] \mathcal{D}_\varepsilon(\partial_z V)(y_1, y_2, 0)\, dy_1 dy_2. \tag{III.2.10}$$

The implicit relationship (III.2.9) is equivalent to $\mathcal{J}_\varepsilon(\gamma_\varepsilon, V_e) = 0$. From this perspective, the following result is an important preliminary step.

Proposition III.9 (Existence and uniqueness of γ_ε)**.**
For any ball $K \subset \mathcal{E}_0^\alpha$, there exists ε_K, such that for all $\varepsilon \leqslant \varepsilon_K$ and for any $V \in K$, there exists a unique solution $\gamma_\varepsilon(V)$ to the equation :

$$\text{Find } \gamma \in (-R_K, R_K) \text{ such that:} \quad \mathcal{J}_\varepsilon(\gamma, V) = 0,$$

where the bound $|\gamma_\varepsilon(V)| \leqslant R_K$ is defined as

$$R_K = \max\left(\frac{\|K\|_\alpha \iint_{\mathbb{R}^2} \exp(-Q(y_1, y_2))\left(y_1^2 + y_2^2\right) dy_1 dy_2 + 8}{2\partial_z^2 m(0)}; \|K\|_\alpha\right). \tag{III.2.11}$$

Next we define the main quantity we will work with: the double integral I_ε which is the rescaled infinitesimal operator. For convenience we define it as a mapping on $\mathcal{E}_0^\alpha$. It is compatible with (III.1.4) because of the decomposition (III.2.5).

Definition III.10 (Auxiliary functional $\mathcal{I}_\varepsilon$)**.**
We define the functional $\mathcal{I}_\varepsilon : \mathcal{E}_0^\alpha \to \mathcal{C}^3(\mathbb{R})$ as follows

$$\mathcal{I}_\varepsilon(V)(z) = \frac{\iint_{\mathbb{R}^2} \exp\left(-Q(y_1, y_2) - \varepsilon\gamma_\varepsilon(V)(y_1 + y_2) + 2\mathcal{D}_\varepsilon(V)(y_1, y_2, z)\right) dy_1 dy_2}{\sqrt{\pi}\int_{\mathbb{R}} \exp\left(-y^2/2 - \varepsilon\gamma_\varepsilon(V)y + V(0) - V(\varepsilon y)\right) dy}. \tag{III.2.12}$$

Finally, in view of (III.2.8) and (III.2.5), we see that V_ε must be a solution of this implicit equation :

$$V_\varepsilon(z) = \sum_{k\geqslant 0} 2^k \log\left(\frac{\mathcal{I}_\varepsilon(V_\varepsilon)(0)+m}{\mathcal{I}_\varepsilon(V_\varepsilon)(z)}\right)(2^{-k}z), \quad \text{for every } z\in\mathbb{R}. \tag{III.2.13}$$

This justifies the introduction of our central mapping, upon which our fixed point argument will be based.

Definition III.11 (Fixed point mapping)**.**
We define the mapping $\mathcal{H}_\varepsilon : \mathcal{E}_0^\alpha \to \mathcal{E}_0^\alpha$ as follows

$$\mathcal{H}_\varepsilon(V)(h) = \sum_{k\geqslant 0} 2^k \log\left(\frac{\mathcal{I}_\varepsilon(V)(0)+m(2^{-k}h)}{\mathcal{I}_\varepsilon(V)(2^{-k}h)}\right). \tag{III.2.14}$$

III.2.3 *Reformulation of the problem*

We are now in position to write our main result for this Section:

Theorem III.12 (Existence and uniqueness of the fixed point)**.**
There is a ball $K_0 \subset \mathcal{E}_0^\alpha$ and a positive constant ε_0 such that for every $\varepsilon \leqslant \varepsilon_0$, the mapping $\mathcal{H}_\varepsilon$ admits a unique fixed point in K_0.

To conclude, it is sufficient to check that solving problem (PU_ε), on the ball K_0, and seeking a fixed point for $\mathcal{H}_\varepsilon$ in K_0 are equivalent problems for $\varepsilon \leqslant \varepsilon_0$ small enough.

Proposition III.13 (Reformulation of the problem (PU_ε))**.**
There is a ball K_0' of $\mathcal{E}^\alpha$, and a positive constant ε_0' such that for every $\varepsilon \leqslant \varepsilon_0'$, the following statements are equivalent:

- ▷ *$(\lambda_\varepsilon, U_\varepsilon)$ is a solution of problem (PU_ε) in $\mathbb{R}\times K_0'$.*
- ▷ *$U_\varepsilon = \gamma_\varepsilon(V_\varepsilon)\cdot + V_\varepsilon$, with $V_\varepsilon \in \mathcal{E}_0^\alpha \cap K_0'$, $\mathcal{H}_\varepsilon(V_\varepsilon) = V_\varepsilon$, and $\lambda_\varepsilon = \mathcal{I}_\varepsilon(V_\varepsilon)(0)$.*

Moreover, the statement of Theorem III.12 holds true in the set $\mathcal{E}_0^\alpha \cap K_0'$.

The main mathematical difficulties are stacked into Theorem III.12. The rest of the article is organized as follows :

- ▷ In section III.3, we justify why the function γ_ε is well defined in Proposition III.9.
- ▷ Then in section III.4, we provide the main properties and the key estimates of the nonlocal operator $\mathcal{I}_\varepsilon$. We point out why this term plays the role of a perturbation between problem (PU_ε) and problem (PU_0). In section III.4.2 we prove crucial contraction estimates.
- ▷ Those estimates are the main ingredients of the proof of properties of $\mathcal{H}_\varepsilon$ in section III.5: most notably the finiteness of $\mathcal{H}_\varepsilon(V)$, and the fact that $\mathcal{H}_\varepsilon$ is a contraction mapping.
- ▷ This allows us to establish the proof of Theorem III.12 and Proposition III.13, and finally to come back to the proof of our main result Theorem III.4 in the sections III.6.1 and III.6.2.
- ▷ Section III.7 is devoted to those specific arguments that require an extension to the higher dimensional case $d > 1$.

III.3 WELL-POSEDNESS OF THE IMPLICIT FUNCTION γ_ε

III.3.1 *Heuristics on finding γ_ε*

We consider $V \in \mathcal{E}_0^\alpha$, and we look for solutions γ_ε of $\mathcal{J}_\varepsilon(\gamma_\varepsilon, V) = 0$, or equivalently :

$$0 = \frac{1}{\varepsilon^2\sqrt{2}\pi}\iint_{\mathbb{R}^2} \exp\left[-Q(y_1,y_2) - \varepsilon\gamma_\varepsilon(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,0)\right]\Big(\mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,0)\Big)dy_1dy_2, \tag{III.3.1}$$

in accordance with (III.2.10). We will see here how a Taylor expansion of the right-hand-side around $\varepsilon = 0$ helps to understand why it defines a unique γ_ε in a given interval for small ε. We will show formally why $\mathcal{J}_\varepsilon(\cdot, V)$ can be uniformly approximated by a non-degenerate linear function for small ε.

We expand the right-hand-side with respect to ε:

$$\begin{aligned}
&\frac{1}{\varepsilon^2\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp\left[-Q(y_1,y_2)\right]\exp\Big[-\varepsilon\gamma_\varepsilon(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,0)\Big]\Big(\mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,0)\Big)dy_1dy_2\\
&= -\frac{1}{\varepsilon^2\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp\left[-Q(y_1,y_2)\right]\left[1-\varepsilon\gamma_\varepsilon(y_1+y_2)+o\left(\varepsilon\right)\right]\\
&\qquad\qquad\times\left(\frac{\varepsilon}{2}\left(y_1+y_2\right)\partial_z^2V(0) + \frac{\varepsilon^2}{4}\left(y_1^2+y_2^2\right)\partial_z^3V(0) + o(\varepsilon^2)\right)dy_1dy_2\\
&= \frac{1}{\varepsilon^2}\left(\frac{\varepsilon^2}{2}\gamma_\varepsilon\partial_z^2V(0) - \varepsilon^2\frac{3\partial_z^3V(0)}{8} + o(\varepsilon^2)\right).
\end{aligned}$$

Then solving

$$0 = -\frac{3\partial_z^3V(0)}{8} + \frac{1}{2}\gamma_\varepsilon\partial_z^2V(0) + o(1),$$

we get the expression :

$$\gamma_\varepsilon \underset{\varepsilon\to 0}{\sim} \frac{3}{4}\frac{\partial_z^3V(0)}{\partial_z^2V(0)}. \tag{III.3.2}$$

These heuristics are consistent with the statement in Theorem III.4, up to the relation between V_0 and m that can be easily read out from (III.1.11). Note that the denominator involves $\partial_z^2V(0)$, so that the local convexity of V should be controlled uniformly during our construction. This is the purpose of the restriction in $\mathcal{E}_0^\alpha$ (III.1.7). In the following, we provide estimates that turn these heuristics into a rigorous proof.

III.3.2 *Proof of Proposition III.9*

The aim of this section is to prove the existence and uniqueness of $\gamma_\varepsilon(V)$ stated in Proposition III.9. We first start with a Lemma providing some useful estimates on the function $\mathcal{J}_\varepsilon$. Combining these estimates with a continuity and monotonicity arguments, we will be able to prove the Proposition III.9.

Lemma III.14 (Estimates of $\mathcal{J}_\varepsilon$)**.**
For any ball $K \subset \mathcal{E}_0^\alpha$, there exists $\varepsilon_K > 0$, such that for all $\varepsilon \leqslant \varepsilon_K$ and $V \in K$, the following estimate holds true for all g in the interval $(-R_K, R_K)$:

$$\mathcal{J}_\varepsilon(0,V) = -\frac{1}{4\sqrt{2}\pi}\iint_{\mathbb{R}^2} \exp(-Q(y_1,y_2))\left[y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2})\right] dy_1 dy_2 + \mathcal{O}(\varepsilon), \tag{III.3.3}$$

$$\partial_g \mathcal{J}_\varepsilon(g,V) = \frac{\partial_z^2 V(0)}{2} + \mathcal{O}(\varepsilon), \tag{III.3.4}$$

where, in the former expansion, the variable $\widetilde{y_i}$ is a by-product of Taylor expansions and is such that $|\widetilde{y_i}| \leqslant |y_i| + 1$.

Remark III.15.
We prove the uniqueness of γ_ε on a uniformly bounded interval. One may think it is a strong restriction not to look at large γ_ε. It is in fact a natural restriction as we have by definition $\gamma_\varepsilon = \partial_z U_\varepsilon(0)$, and $\partial_z U_\varepsilon \in L^\infty$ in our perturbative setting.

We postpone the proof of the technical Lemma III.14 at the end of this section and we first use it to prove the Proposition III.9:

Proof of Proposition III.9. Let K be a ball of $\mathcal{E}_0^\alpha$ and $V \in K$. We deduce from Lemma III.14 that $|\mathcal{J}_\varepsilon(0,V)| \leqslant G_K + 1$, where

$$G_K = \frac{\|K\|_\alpha}{4\sqrt{2}\pi}\iint_{\mathbb{R}^2} \exp(-Q(y_1,y_2))\left(y_1^2+y_2^2\right) dy_1 dy_2\,,$$

for ε small enough. Integrating (III.3.4) with respect to g, we obtain

$$\mathcal{J}_\varepsilon(g,V) = \mathcal{J}_\varepsilon(0,V) + \frac{\partial_z^2 V(0)}{2} g + \mathcal{O}(\varepsilon)\,,$$

where it is important to notice that the perturbation $\mathcal{O}(\varepsilon)$ is uniform with respect to ε for $g \in (-R_K, R_K)$ and $V \in K$. Since $V \in \mathcal{E}_0^\alpha$, we know that $\partial_z^2 V(0) \geqslant \partial_z^2 m(0) > 0$. Therefore, $\mathcal{J}_\varepsilon$ is uniformly increasing with respect to g on $(-R_K, R_K)$. Moreover, the choice of R_K is such that

$$\mathcal{J}_\varepsilon(R_K,V) \geqslant -1 - G_K + \frac{\partial_z^2 m(0)}{2} R_K + \mathcal{O}(\varepsilon) = 1 + \mathcal{O}(\varepsilon) > 0\,,$$

for ε small enough, and similarly, $\mathcal{J}_\varepsilon(-R_K,V) < 0$. Finally, there exists a unique $\gamma_\varepsilon(V)$ satisfying $\mathcal{J}_\varepsilon(\gamma_\varepsilon(V),V) = 0$ because $\mathcal{J}_\varepsilon$ is continuous with repect to g for $V \in \mathcal{E}_0^\alpha$. □

Proof of Lemma III.14. Let K be a ball of $\mathcal{E}_0^\alpha$ of radius $\|K\|_\alpha$. In section III.3.1, we have used formal Taylor expansions to get a formula for $\gamma_\varepsilon(V)$, morally valid when $\varepsilon = 0$. The idea here is to write exact rests to broaden the formula for small but positive ε.

▷ **Proof of expansion** (III.3.3)**.** Let us pick $V \in K$ and $\varepsilon > 0$. Recall the expression of $\mathcal{J}_\varepsilon(0,V)$:

$$\mathcal{J}_\varepsilon(0,V) = \frac{1}{\varepsilon^2\sqrt{2}\pi}\iint_{\mathbb{R}^2} \exp\left[-Q(y_1,y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,0)\right]\Big(\mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,0)\Big) dy_1 dy_2.$$

We perform two Taylor expansions, namely:

$$\begin{cases} 2\mathcal{D}_\varepsilon(V)(y_1,y_2,0) = -\dfrac{\varepsilon^2}{2}\left(y_1^2\partial_z^2 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^2 V(\varepsilon\widetilde{y_2})\right) \\ \mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,0) = -\dfrac{\varepsilon(y_1+y_2)}{2}\partial_z^2 V(0) - \dfrac{\varepsilon^2}{4}(y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2})), \end{cases} \tag{III.3.5}$$

where $\widetilde{y_i}$ denote some generic number such that $|\widetilde{y_i}| \leqslant |y_i|$ for $i = 1, 2$. Moreover, we can write

$$\exp(-\varepsilon^2 P) = 1 - \varepsilon^2 P \exp(-\theta\varepsilon^2 P)\,, \quad P = \frac{1}{2}\left(y_1^2\partial_z^2 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^2 V(\varepsilon\widetilde{y_2})\right)$$
$$|P| \leqslant \frac{1}{2}\left(y_1^2 + y_2^2\right)\|V\|_\alpha\,, \tag{III.3.6}$$

for some $\theta = \theta(y_1, y_2) \in (0, 1)$. Combining the expansions, we find:

$$\mathcal{J}_\varepsilon(0,V) = \frac{1}{\varepsilon^2\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp\left[-Q(y_1,y_2)\right]\left(1-\varepsilon^2 P\exp(-\theta\varepsilon^2 P)\right)$$
$$\times\left(-\frac{\varepsilon(y_1+y_2)}{2}\partial_z^2 V(0) - \frac{\varepsilon^2}{4}(y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2}))\right) dy_1 dy_2\,.$$

The crucial point is the cancellation of the $O(\varepsilon^{-1})$ contribution due to the symmetry of Q:

$$\iint_{\mathbb{R}^2}\exp(-Q(y_1,y_2))(y_1+y_2)dy_1dy_2 = 0\,. \tag{III.3.7}$$

So, it remains

$$\begin{aligned}\mathcal{J}_\varepsilon(0,V) = &-\frac{1}{4\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp(-Q(y_1,y_2))\left[y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2}))\right]dy_1dy_2 \\ &+\frac{\varepsilon}{2\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp(-Q(y_1,y_2))P\exp(-\theta\varepsilon^2 P)(y_1+y_2)\partial_z^2 V(0)dy_1dy_2 \\ &+\frac{\varepsilon^2}{4\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp(-Q(y_1,y_2))P\exp(-\theta\varepsilon^2 P)\left(y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2})\right)dy_1dy_2\end{aligned}$$

Clearly the last two contributions are uniform $\mathcal{O}(\varepsilon)$ for $V \in K$ and $\varepsilon \leqslant \varepsilon_K$ small enough. Indeed, the term P is at most quadratic with respect to y_i (III.3.6), so $Q + \theta\varepsilon^2 P$ is uniformly bounded below by a positive quadratic form for ε small enough.

▷ **Proof of expansion** (III.3.4)**.** The first step is to compute the derivative of J with respect to g:

$$\partial_g\mathcal{J}_\varepsilon(g,V) = -\frac{1}{\varepsilon\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp\left[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,0)\right]$$
$$\times(y_1+y_2)\left[\mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,0)\right]dy_1dy_2.$$

Similar Taylor expansions as above yields:

$$\partial_g\mathcal{J}_\varepsilon(g,V) = -\frac{1}{\varepsilon\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp\left[-Q(y_1,y_2)\right]\left(1-\varepsilon P'\exp(-\theta\varepsilon P')\right)$$
$$\times(y_1+y_2)\left(-\frac{\varepsilon(y_1+y_2)}{2}\partial_z^2 V(0) - \frac{\varepsilon^2}{4}(y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2}))\right)dy_1dy_2,$$

where $P' = g(y_1+y_2) + y_1\partial_z V(\varepsilon\widetilde{y_1}) + y_2\partial_z V(\varepsilon\widetilde{y_2})$. Interestingly, the leading order term does not cancel anymore, and it remains:

$$\partial_g \mathcal{J}_\varepsilon(g,V) = \frac{1}{2\sqrt{2}\pi}\left(\iint_{\mathbb{R}^2} \exp\left[-Q(y_1,y_2)\right](y_1+y_2)^2\, dy_1 dy_2\right)\partial_z^2 V(0) + \mathcal{O}(\varepsilon)\,.$$

The justification that the remainder is a uniform $\mathcal{O}(\varepsilon)$ is similar as above, except that now P' has a linear part depending on g, but the latter is assumed to be bounded a priory by R_K. □

III.4 ANALYSIS OF THE PERTURBATIVE TERM I_ε

III.4.1 *Lispchitz continuity of some auxiliary functionals*

The function $\mathcal{I}_\varepsilon$ is crucially involved in the definition of the mapping $\mathcal{H}_\varepsilon$. Thus to prove any contraction property on this mapping we will need Lipschitz estimates about $\mathcal{I}_\varepsilon$ and the three first derivatives of its logarithm. But first we show that $\mathcal{I}_\varepsilon$ really plays the role of a perturbation between problem (PU_ε) and problem (PU_0) that converges to 1 uniformly as $\varepsilon \to 0$.

Proposition III.16 (Estimation of $\mathcal{I}_\varepsilon$)**.**
For every K ball of $\mathcal{E}_0^\alpha$, for every $\delta > 0$, there exists a constant ε_δ that depends only on K and δ, such that for every $\varepsilon \leqslant \varepsilon_\delta$ and for every $V \in K$:

$$(\forall z \in \mathbb{R}) \quad 1-\delta \leqslant \mathcal{I}_\varepsilon(V)(z) \leqslant 1+\delta\,.$$

Proof. Let V in K. For $\varepsilon \leqslant \varepsilon_K$, one can apply Proposition III.9 which gives $|\gamma_\varepsilon(V)| \leqslant R_K$. Next it is enough to write that :

$$\underline{C}_\varepsilon \doteqdot \frac{\displaystyle\iint_{\mathbb{R}^2} \exp\left[-Q(y_1,y_2) - 2\varepsilon R_K(|y_1|+|y_2|)\right] dy_1 dy_2}{\displaystyle\sqrt{2}\pi\int_{\mathbb{R}} \exp\left(-y^2/2 - 2\varepsilon R_K\,|y|\right) dy} \leqslant \mathcal{I}_\varepsilon(V)(z), \text{ and}$$

$$\mathcal{I}_\varepsilon(V)(z) \leqslant \frac{\displaystyle\iint_{\mathbb{R}^2} \exp\left(-Q(y_1,y_2) + 2\varepsilon R_K(|y_1|+|y_2|)\right) dy_1 dy_2}{\displaystyle\sqrt{2}\pi\int_{\mathbb{R}} \exp\left(-y^2/2 - 2\varepsilon R_K\,|y|\right) dy} \doteqdot \overline{C}_\varepsilon.$$

We deduce from this lower and upper estimates that the whole $\mathcal{I}_\varepsilon(V)$ converges uniformly to 1 as $\varepsilon \to 0$. □

Next, we show Lipschitz continuity of various quantities of interest.

Proposition III.17 (Lipschitz continuity of γ_ε)**.**
For every ball $K \subset \mathcal{E}_0^\alpha$, there exist constants $L_K(\gamma)$, and ε_K, depending only on K, such that for all $\varepsilon \leqslant \varepsilon_K$, $V_1, V_2 \in K$

$$|\gamma_\varepsilon(V_1) - \gamma_\varepsilon(V_2)| \leqslant L_K(\gamma)\,\|V_1 - V_2\|_\alpha\,.$$

Proof. Let K be a ball of $\mathcal{E}_0^\alpha$, and let $V_1, V_2 \in K$. Let denote $\Gamma_\varepsilon^I = \gamma_\varepsilon(V_i)$ for $i = 1,2$. We argue by means of Fréchet derivatives: let $s \in (0,1)$, $\gamma_s = s\gamma_1 + (1-s)\gamma_2$, $V_s = sV_1 + (1-s)V_2$, and consider the following computation:

$$\frac{d}{ds}\mathcal{J}_\varepsilon(\gamma_s, V_s) = \partial_\gamma \mathcal{J}_\varepsilon(\gamma_s, V_s)(\gamma_1 - \gamma_2) + D_V \mathcal{J}_\varepsilon(\gamma_s, V_s)\cdot(V_1 - V_2)\,, \tag{III.4.1}$$

where the Fréchet derivative of J with respect to V is:

$$D_V\mathcal{J}_\varepsilon(\gamma,V)\cdot H=\frac{1}{\varepsilon^2\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp\left[-Q(y_1,y_2)-\varepsilon\gamma(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(y_1,y_2,0)\right]\\ \times\Big(2\mathcal{D}_\varepsilon(H)(y_1,y_2,0)\Big)\Big(\mathcal{D}_\varepsilon(\partial_z V)(y_1,y_2,0)\Big)dy_1dy_2\\ +\frac{1}{\varepsilon^2\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp\left[-Q(y_1,y_2)-\varepsilon\gamma(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(y_1,y_2,0)\right]\Big(\mathcal{D}_\varepsilon(\partial_z H)(y_1,y_2,0)\Big)dy_1dy_2$$

We perform similar Taylor expansions as in (III.3.5),

$$2\mathcal{D}_\varepsilon(W)(y_1,y_2,0)=\begin{cases}-\varepsilon(y_1+y_2)O\left(\|\partial_z W\|_\infty\right)\\ -\varepsilon(y_1+y_2)\partial_z W(0)-(\varepsilon^2/2)\left(y_1^2+y_2^2\right)O\left(\|\partial_z^2 W\|_\infty\right)\end{cases}$$

either for $W=V, H\in\mathcal{E}_0^\alpha$, or $W=\partial_z V,\partial_z H$. We deduce that

$$D_V\mathcal{J}_\varepsilon(\gamma,V)\cdot H=\frac{1}{\varepsilon^2\sqrt{2}\pi}\iint_{\mathbb{R}^2}\exp\left[-Q(y_1,y_2)-\varepsilon\gamma(y_1+y_2)-\varepsilon(y_1+y_2)O\left(\|\partial_z V\|_\infty\right)\right]\\ \times\Bigg[\left(-\frac{\varepsilon^2}{2}\left(y_1^2+y_2^2\right)O\left(\|\partial_z^2 H\|_\infty\right)\right)\left(-\varepsilon(y_1+y_2)O\left(\|\partial_z^2 V\|_\infty\right)\right)\\ +\left(-\varepsilon(y_1+y_2)\partial_z^2 H(0)-\frac{\varepsilon^2}{2}\left(y_1^2+y_2^2\right)O\left(\|\partial_z^3 H\|_\infty\right)\right)\Bigg]dy_1dy_2 \quad \text{(III.4.2)}$$

We proceed as in the previous section for the exponential term: there exists $\theta=\theta(y_1,y_2)\in(0,1)$ such that

$$\exp(-\varepsilon P')=1-\varepsilon P'\exp(-\theta\varepsilon P'),\quad \text{where}\quad P'=\gamma(y_1+y_2)+(y_1+y_2)O\left(\|\partial_z V\|_\infty\right)$$

Again, the crucial point is the cancellation of the $O(\varepsilon^{-1})$ contribution in (III.4.2), as in (III.3.7) What remains is of order one or below, and one can easily show that there exists C_K such that

$$\begin{aligned}|D_V\mathcal{J}_\varepsilon(\gamma,V)\cdot H|&\leqslant C_K\left(\|\partial_z^3 H\|_\infty+|\partial_z^2 H(0)|\left(|\gamma+\|\partial_z V\|_\infty|\right)\right.\\ &\quad\left.+\varepsilon\|\partial_z^2 H\|_\infty\|\partial_z^2 V\|_\infty+\varepsilon\|\partial_z^3 H\|_\infty\left(|\gamma+\|\partial_z V\|_\infty|\right)\right)\\ &\leqslant C_K\|H\|_\alpha,\end{aligned}$$

provided $\varepsilon\leqslant\varepsilon_K$ is small enough.

On the other hand, we have already established that $\partial_\gamma\mathcal{J}_\varepsilon(\gamma,V)=\partial_z^2V(0)/2+\mathcal{O}(\varepsilon)$ in Lemma III.14. Consequently, integrating (III.4.1) from $s=0$ to 1, we find:

$$0=\mathcal{J}_\varepsilon(\gamma_1,V_1)-\mathcal{J}_\varepsilon(\gamma_2,V_2)=\left(\frac{\partial_z^2V(0)}{2}+\mathcal{O}(\varepsilon)\right)(\gamma_1-\gamma_2)+\left(\int_0^1 D_V\mathcal{J}_\varepsilon(\gamma_s,V_s)\cdot(V_1-V_2)\,ds\right).$$

We deduce from the previous estimates and the local convexity condition in (III.1.7) that

$$|\gamma_1-\gamma_2|\leqslant C_K\left(\frac{2}{\partial_z^2m(0)}+C_K\varepsilon\right)\|V_1-V_2\|_\alpha,$$

for some C_K and $\varepsilon\leqslant\varepsilon_K$ small enough. □

In turn, Proposition III.17 implies the Lipschitz continuity of $\mathcal{I}_\varepsilon$ as a function of V.

Proposition III.18 (Lipschitz continuity of $\mathcal{I}_\varepsilon$)**.**
For every ball K of $\mathcal{E}_0^\alpha$, there exist constants ε_K, C_K depending only on K, such that for all $\varepsilon \leqslant \varepsilon_K$, $V_1, V_2 \in K$,

$$\sup_{z\in\mathbb{R}} |\mathcal{I}_\varepsilon(V_1)(z) - \mathcal{I}_\varepsilon(V_2)(z)| \leqslant \varepsilon C_K \|V_1 - V_2\|_\alpha . \tag{III.4.3}$$

Proof. The Lipschitz continuity of $\mathcal{I}_\varepsilon$ with respect to V can be proven by composition of Lipschitz functions. With the same notations as in the proof of Proposition III.17, and with the shortcut notation $\mathcal{I}_\varepsilon = A_\varepsilon / B_\varepsilon$ to separate the numerator from the denominator in (III.2.12) we have,

$$\begin{aligned}\frac{d}{ds}A_\varepsilon(V_s)(z) = -\iint_{\mathbb{R}^2} G_\varepsilon^V(y_1,y_2,z)\Big(\varepsilon\frac{d}{ds}\gamma_\varepsilon(V_s)(y_1+y_2)\\ + \int_{\overline{z}}^{\overline{z}+\varepsilon y_1}\partial_z(V_1-V_2)(z')dz' - \int_{\overline{z}}^{\overline{z}+\varepsilon y_2}\partial_z(V_1-V_2)(z')dz'\Big)\,dy_1dy_2\,,\end{aligned}$$

where we have simply written $V(\overline{z}+\varepsilon y_1) - V(\overline{z}) = \int_{\overline{z}}^{\overline{z}+\varepsilon y_1}\partial_z V(z')\,dz'$, and where G_ε^V denotes the exponential weight:

$$G_\varepsilon^V(y_1,y_2,z) = \frac{1}{\sqrt{2\pi}}\exp\left[-Q(y_1,y_2) - \varepsilon\gamma_\varepsilon(V)(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,z)\right].$$

We deduce that A_ε is such that:

$$(\forall z)\quad \left|\frac{d}{ds}A_\varepsilon(V_s)(z)\right| \leqslant \varepsilon\iint_{\mathbb{R}^2} G_\varepsilon^V(y_1,y_2,z)\left(L_K(\gamma)\|V_1-V_2\|_\alpha + \|V_1-V_2\|_\alpha\right)\left(|y_1|+|y_2|\right)\,dy_1dy_2\,.$$

As the weight G_ε^V is uniformly close to a positive quadratic form for small ε, we find that the numerator has a Lipschitz constant of order ε uniformly with respect to z:

$$\sup_{z\in\mathbb{R}} |A_\varepsilon(V_1)(z) - A_\varepsilon(V_2)(z)| \leqslant \varepsilon C_K \|V_1 - V_2\|_\alpha .$$

The same holds true for the denominator B_ε. In addition, a direct by-product of the proof of Proposition III.16 is that A_ε and B_ε are uniformly bounded above and below by positive constants for ε small enough. Consequently, the quotient $\mathcal{I}_\varepsilon = A_\varepsilon / B_\varepsilon$ is Lipschitz continuous. □

It is useful to introduce the probability measure dG_ε^V induced by the exponential weight G_ε^V:

$$\begin{aligned}dG_\varepsilon^V(y_1,y_2,z) &= \frac{G_\varepsilon^V(y_1,y_2,z)}{\iint_{\mathbb{R}^2} G_\varepsilon^V(\cdot,\cdot,z)}\\ &= \frac{\exp\left[-Q(y_1,y_2) - \varepsilon\gamma_\varepsilon(V)(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,z)\right]}{\iint_{\mathbb{R}^2}\exp\left[-Q(y_1,y_2) - \varepsilon\gamma_\varepsilon(V)(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(y_1,y_2,z)\right]dy_1dy_2}.\end{aligned}$$

As a consequence of the previous estimates, we obtain the following one:

Lemma III.19 (Lipschitz continuity of dG_ε^V)**.**
For every ball K of $\mathcal{E}_0^\alpha$, there exist constants ε_K, C_K depending only on K, such that for all $\varepsilon \leqslant \varepsilon_K$, $V_1, V_2 \in K$,

$$\begin{aligned}\sup_{z\in\mathbb{R}}\left|dG_\varepsilon^{V_1}(y_1,y_2,z) - dG_\varepsilon^{V_2}(y_1,y_2,z)\right|\\ \leqslant \varepsilon C_K\|V_1-V_2\|_\alpha\left(1+|y_1|+|y_2|\right)\exp\Big(-Q(y_1,y_2) + 2\varepsilon R_K(|y_1|+|y_2|)\Big).\end{aligned} \tag{III.4.4}$$

Furthermore, under the same conditions, we have the following bound, uniform with respect to $z \in \mathbb{R}$*:*

$$dG_\varepsilon^V(y_1,y_2,z) \leqslant \frac{1}{4}\exp\left[-Q(y_1,y_2)+2\varepsilon R_K(|y_1|+|y_2|)\right]. \tag{III.4.5}$$

Proof. We first prove (III.4.5): the function G_ε^V is such that

$$G_\varepsilon^V(y_1,y_2,z) \geqslant \frac{1}{\sqrt{2}\pi}\exp\left[-Q(y_1,y_2)\mp 2\varepsilon R_K(|y_1|+|y_2|)\right].$$

Therefore, its integral over $(y_1,y_2)\in\mathbb{R}^2$ converges to 1 as $\varepsilon\to 0$, and there exists ε_K depending on K such that $\iint G_\varepsilon^V(y_1,y_2,z)\,dy_1dy_2 \geqslant 4/\sqrt{2}\pi$ for $\varepsilon\leqslant\varepsilon_K$. This leads to (III.4.5).

In order to obtain (III.4.4), we proceed as in the proof of Proposition III.18, as the denominator of dG_ε^V is the numerator A_ε of $\mathcal{I}_\varepsilon$. For the Lipschitz continuity of the numerator of dG_ε^V, we find:

$$(\forall z)\quad \left|\frac{d}{ds}G_\varepsilon^{V_s}(y_1,y_2,z)\right| \leqslant \varepsilon G_\varepsilon^{V_s}(y_1,y_2,z)\left(L_K(\gamma)\|V_1-V_2\|_\alpha + \|V_1-V_2\|_\alpha\right)(|y_1|+|y_2|)\,.$$

We deduce that the quotient $dG_\varepsilon^V = G_\varepsilon^V/A_\varepsilon(V)$ is also Lipschitz continuous:

$$\begin{aligned}\left|dG_\varepsilon^{V_1}-dG_\varepsilon^{V_2}\right| &\leqslant \left|\frac{G_\varepsilon^{V_1}-G_\varepsilon^{V_2}}{A_\varepsilon(V_1)}+\frac{A_\varepsilon(V_2)-A_\varepsilon(V_1)}{A_\varepsilon(V_1)A_\varepsilon(V_2)}G_\varepsilon^{V_2}\right| \\ &\leqslant \varepsilon C_K\|V_1-V_2\|_\alpha\,(|y_1|+|y_2|)\exp\Big(-Q(y_1,y_2)+2\varepsilon R_K(|y_1|+|y_2|)\Big) \\ &\quad+\varepsilon C_K\|V_1-V_2\|_\alpha\exp\Big(-Q(y_1,y_2)+2\varepsilon R_K(|y_1|+|y_2|)\Big).\end{aligned}$$

This concludes the proof of (III.4.4). □

To conclude, we have established in this section that $\mathcal{I}_\varepsilon$ is a small perturbation around 1, both in the uniform sense $\mathcal{I}_\varepsilon(V)\to 1$, and in the Lipschitz sense: $\mathrm{Lip}_V\mathcal{I}_\varepsilon = \mathcal{O}(\varepsilon)$. In addition, we have proven a similar Lipschitz smallness property for a probabilty distribution dG_ε^V that will appear frequently in our contraction estimates.

III.4.2 *Contraction properties (first part)*

On the way to estimating the fixed point mapping $\mathcal{H}_\varepsilon$ (III.2.14), we need good estimates on the logarithmic derivatives of
I_ε. For that purpose, we introduce the following quantities for $i=1,2,3$:

$$W_\varepsilon^{(i)}(V)(z) = \frac{\partial_z^i\mathcal{I}_\varepsilon(V)(z)}{\mathcal{I}_\varepsilon(V)(z)}\,. \tag{III.4.6}$$

For the sake of conciseness, we omit sometimes the dependency with respect to y_1,y_2 in the notations, as for instance: $dG_\varepsilon^V(y_1,y_2,z) = dG_\varepsilon^V(z)$. The following notation with a duality bracket is useful:

$$\left\langle dG_\varepsilon^V(z), f\right\rangle = \iint_{R^2} dG_\varepsilon^V(y_1,y_2,z)f(y_1,y_2)\,dy_1dy_2.$$

Indeed, for any $V\in\mathcal{E}_0^\alpha$, we have:

$$W_\varepsilon^{(1)}(V)(z) = \left\langle dG_\varepsilon^V(z), \mathcal{D}_\varepsilon(\partial_z V)(z)\right\rangle. \tag{III.4.7}$$

Similarly:

$$W_\varepsilon^{(2)}(V)(z) = \left\langle dG_\varepsilon^V(z), \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V)(z) + (\mathcal{D}_\varepsilon(\partial_z V)(z))^2 \right\rangle.$$

And finally :

$$W_\varepsilon^{(3)}(V)(z) = \left\langle dG_\varepsilon^V(z), \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)(z) + (\mathcal{D}_\varepsilon(\partial_z V)(z))^3 + 3\mathcal{D}_\varepsilon(\partial_z V)(z)\left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V)(z)\right) \right\rangle. \tag{III.4.8}$$

In order to obtain estimates on $W^{(i)}$ it seems natural from the previous pattern of differentiation to begin with estimates on the symmetric difference of the derivatives of V.

Lemma III.20. *For any $V \in \mathcal{E}^\alpha$, and $(y_1, y_2) \in \mathbb{R}^2$, we have:*

$$\sup_z\ (1+|z|)^\alpha \left|\mathcal{D}_\varepsilon(\partial_z V)(y_1, y_2, z)\right| \leqslant \varepsilon 2^\alpha \|V\|_\alpha \left[|y_1| + |y_2| + \varepsilon^\alpha |y_1|^{1+\alpha} + \varepsilon^\alpha |y_2|^{1+\alpha}\right], \tag{III.4.9}$$

$$\sup_z\ (1+|z|)^\alpha \left|\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V)(y_1, y_2, z)\right| \leqslant \varepsilon 2^{\alpha-1} \|V\|_\alpha \left[|y_1| + |y_2| + \varepsilon^\alpha |y_1|^{1+\alpha} + \varepsilon^\alpha |y_2|^{1+\alpha}\right], \tag{III.4.10}$$

$$\sup_z\ (1+|z|)^\alpha \left|\frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)(y_1, y_2, z)\right| \leqslant 2^{\alpha-1} \|V\|_\alpha \left(1 + \frac{\varepsilon^\alpha}{4}\left[|y_1|^\alpha + |y_2|^\alpha\right]\right). \tag{III.4.11}$$

It is important to notice that the first two right-hand-sides (resp. first and second derivatives) are of order ε. The third one is larger but controlled by $2^{\alpha-1} < 1$. This is the first occurrence of the contraction property we are seeking. This is the main reason why we make the analysis up to the third derivatives.

Proof. We introduce the additional notation $\varphi_\alpha(z) = (1+|z|)^\alpha$. First, since $\bar{z} = z/2$, we have $\varphi_\alpha(z) \leqslant 2^\alpha \varphi_\alpha(\bar{z})$.

▷ **Proof of** (III.4.9)**.** By Taylor expansions, we have:

$$\varphi_\alpha(z)\left|\mathcal{D}_\varepsilon(\partial_z V)(y_1, y_2, z)\right| \leqslant 2^\alpha \varphi_\alpha(\bar{z}) \left|\frac{\varepsilon y_1}{2}\partial_z^2 V(\bar{z} + \varepsilon\widetilde{y}_1) + \frac{\varepsilon y_2}{2}\partial_z^2 V(\bar{z} + \varepsilon\widetilde{y}_2)\right|,$$

where $|\widetilde{y}_i| \leqslant |y_i|$. Using the definition of $\|\cdot\|_\alpha$ (III.1.5), we obtain

$$\varphi_\alpha(\bar{z})\left|\varepsilon y_1 \partial_z^2 V(\bar{z} + \varepsilon\widetilde{y}_1)\right| \leqslant \frac{\varepsilon|y_1|\varphi_\alpha(\bar{z})}{\varphi_\alpha(\bar{z} + \varepsilon\widetilde{y}_1)}\|V\|_\alpha \leqslant \frac{\varepsilon|y_1|(1 + |\varepsilon\widetilde{y}_1| + |\bar{z} + \varepsilon\widetilde{y}_1|)^\alpha}{(1 + |\bar{z} + \varepsilon\widetilde{y}_1|)^\alpha}\|V\|_\alpha$$

Since we chose $\alpha < 1$, $|\cdot|^\alpha$ is sub-additive. Thus, we get

$$\begin{aligned}\varphi_\alpha(\bar{z})\left|\varepsilon y_1 \partial_z^2 V(\bar{z} + \varepsilon\widetilde{y}_1)\right| &\leqslant \varepsilon|y_1|\left(1 + \frac{|\varepsilon\widetilde{y}_1|^\alpha}{(1 + |\bar{z} + \varepsilon\widetilde{y}_1|)^\alpha}\right)\|V\|_\alpha \\ &\leqslant \varepsilon|y_1|(1 + |\varepsilon y_1|^\alpha)\|V\|_\alpha \leqslant \varepsilon(|y_1| + |y_1|^{1+\alpha}).\end{aligned}$$

By symmetry of the role played by y_1 and y_2, we have proven equation (III.4.9).

▷ **Proof of** (III.4.10)**.** The second estimate is a consequence of the first one, applied to the derivative of V. Notice that it is allowed as $\mathcal{E}_0^\alpha$ enables control of derivatives up to the third order.

▷ **Proof of** (III.4.11)**.** We must be a little more careful in the estimations of the third estimate (III.4.11), because we cannot go up to the fourth derivative in the Taylor expansions. This is why we do not have an ε bound, but we gain a contraction factor instead. We have

$$\begin{aligned}\varphi_\alpha(z)\left|\frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)(y_1,y_2,z)\right| &\leqslant 2^\alpha\varphi_\alpha(\bar z)\left|\frac{1}{4}\partial_z^3 V(\bar z)-\frac{1}{8}\partial_z^3 V(\bar z+\varepsilon y_1)-\frac{1}{8}\partial_z^3 V(\bar z+\varepsilon y_2)\right|\\ &\leqslant \frac{2^\alpha}{4}\|V\|_\alpha + 2^\alpha\varphi_\alpha(\bar z)\left|\frac{1}{8}\partial_z^3 V(\bar z+\varepsilon y_1)+\frac{1}{8}\partial_z^3 V(\bar z+\varepsilon y_2)\right|\end{aligned}$$

We bound separately each term using again the sub-additivity of $|\cdot|^\alpha$. For $\varepsilon\leqslant 1$:

$$\begin{aligned}\varphi_\alpha(\bar z)\left|\frac{1}{8}\partial_z^3 V(\bar z+\varepsilon y_1)\right| &\leqslant \frac{\varphi_\alpha(\bar z)}{8\varphi_\alpha(\bar z+\varepsilon y_1)}\|V\|_\alpha\\ &\leqslant \frac{\|V\|_\alpha}{8}\left(1+\frac{(|\varepsilon y_1|)^\alpha}{(1+|\bar z+\varepsilon y_1|)^\alpha}\right)\leqslant (1+|\varepsilon y_1|^\alpha)\frac{\|V\|_\alpha}{8}.\end{aligned}$$

Summing it all up, one ends up with:

$$\varphi_\alpha(z)\left|\frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)(y_1,y_2,z)\right|\leqslant 2^{\alpha-1}\|V\|_\alpha\left(1+\frac{1}{4}\varepsilon^\alpha\left[|y_1|^\alpha+|y_2|^\alpha\right]\right).$$

This is precisely equation (III.4.11). □

The following proposition is a first step towards contraction properties that will be established in section III.5. For convenience, we introduce the following notation:

$$\begin{cases}\triangle W_\varepsilon^{(i)} = W_\varepsilon^{(i)}(V_1)-W_\varepsilon^{(i)}(V_2)\\ \triangle V = V_1-V_2\end{cases}\tag{III.4.12}$$

Proposition III.21 (Lipschitz continuity of W_ε with respect to V)**.**
Let K a ball of $\mathcal{E}_0^\alpha$, and $V_1, V_2\in K$. There exists constants ε_K, C_K depending only on K such that for all $\varepsilon\leqslant\varepsilon_K$, we have:

$$\sup_z\,(1+|z|)^\alpha|\triangle W_\varepsilon^{(1)}(z)|\leqslant\varepsilon C_K\|\triangle V\|_\alpha\tag{III.4.13}$$

$$\sup_z\,(1+|z|)^\alpha|\triangle W_\varepsilon^{(2)}(z)|\leqslant\varepsilon C_K\|\triangle V\|_\alpha,\tag{III.4.14}$$

$$\sup_z\,(1+|z|)^\alpha|\triangle W_\varepsilon^{(3)}(z)|\leqslant\left(2^{\alpha-1}+\varepsilon^\alpha C_K\right)\|\triangle V\|_\alpha.\tag{III.4.15}$$

It is also possible to get estimates on $W_\varepsilon^{(i)}(V)$ itself, with the same hypotheses. This is useful to prove the invariance of certain subsets of $\mathcal{E}_0^\alpha$.

Proposition III.22.
With the same setting as in Proposition III.21, we also have:

$$\sup_z\,(1+|z|)^\alpha|W_\varepsilon^{(1)}(V)(z)|\leqslant\varepsilon C_K\|V\|_\alpha,$$

$$\sup_z\,(1+|z|)^\alpha|W_\varepsilon^{(2)}(V)(z)|\leqslant\varepsilon C_K\|V\|_\alpha,$$

$$\sup_z\,(1+|z|)^\alpha|W_\varepsilon^{(3)}(V)(z)|\leqslant\left(2^{\alpha-1}+\varepsilon^\alpha C_K\right)\|V\|_\alpha.$$

We do not give the details of the proof of the latter Proposition, since it is a straightforward adaptation of Proposition III.21. Actually, we cannot readily apply Proposition III.21 to $(V_1, V_2) = (V, 0)$ as $0 \notin \mathcal{E}_0^\alpha$, because of the additional condition on $\partial_z^2 V(0)$ (III.1.7) which is required to prove boundedness and Lipschitz continuity of γ_ε.

Proof of Proposition III.21. The proof of theses inequalities is quite tedious because of the numerous non-linear calculations. However, the technique is similar for each inequality, and consists in separating the fully non linear behavior from the quasi-linear parts of the left-hand-sides of equations (III.4.13) to (III.4.15).

▷ **Proof of** (III.4.13)**.** This is the easiest part, because it is quasi-linear with respect to V. Indeed, we have

$$\triangle W_\varepsilon^{(1)}(z) = \left\langle dG_\varepsilon^{V_1}(z), \mathcal{D}_\varepsilon(\partial_z V_1)(z) \right\rangle - \left\langle dG_\varepsilon^{V_2}(z), \mathcal{D}_\varepsilon(\partial_z V_2)(z) \right\rangle .$$

We reformulate it in two parts, one involving $V_1 - V_2$, and the other involving $dG_\varepsilon^{V_1} - dG_\varepsilon^{V_2}$:

$$\triangle W_\varepsilon^{(1)}(z) = \left\langle dG_\varepsilon^{V_2}(z), \mathcal{D}_\varepsilon(\partial_z \triangle V)(z) \right\rangle + \left\langle dG_\varepsilon^{V_1}(z) - dG_\varepsilon^{V_2}(z), \mathcal{D}_\varepsilon(\partial_z V_1)(z) \right\rangle . \qquad \text{(III.4.16)}$$

For the first contribution in (III.4.16), we apply directly Lemma III.20 to $V_1 - V_2$:

$$\begin{aligned}(1+|z|)^\alpha \left| \left\langle dG_\varepsilon^{V_2}(z), \mathcal{D}_\varepsilon(\partial_z \triangle V)(z) \right\rangle \right| \\ \leqslant \varepsilon 2^\alpha \left\| \triangle V \right\|_\alpha \left\langle dG_\varepsilon^{V_2}(z), \left(|y_1| + |y_2| + \varepsilon^\alpha |y_1|^{1+\alpha} + \varepsilon^\alpha |y_2|^{1+\alpha} \right) \right\rangle \leqslant \varepsilon C_K \left\| \triangle V \right\|_\alpha .\end{aligned}$$

For the last inequality we used equation (III.4.5), which enables to bound uniformly the measure dG_ε^V with respect to z. From Lemmas III.19 and III.20, there exists ε_K and C_K such that for $\varepsilon \leqslant \varepsilon_K$, the second contribution in the right-hand-side (III.4.16) satisfies

$$\begin{aligned}(1+|z|)^\alpha \left| \left\langle dG_\varepsilon^{V_1}(z) - dG_\varepsilon^{V_2}(z), \mathcal{D}_\varepsilon(\partial_z V_1)(z) \right\rangle \right| \\ \leqslant \varepsilon^2 C_K \left\| \triangle V \right\|_\alpha \left\| V_1 \right\|_\alpha \Big\langle (1+|y_1|+|y_2|) \exp(-Q(y_1,y_2) + 2\varepsilon R_K(|y_1|+|y_2|)), \\ \left(|y_1| + |y_2| + \varepsilon^\alpha |y_1|^{1+\alpha} + \varepsilon^\alpha |y_2|^{1+\alpha} \right) \Big\rangle . \qquad \text{(III.4.17)}\end{aligned}$$

The last integral is uniformly bounded for ε small enough, involving moments of a Gaussian distribution. Therefore, the whole quantity is bounded by $\varepsilon^2 C_K \left\| \triangle V \right\|_\alpha$, uniformly with respect to z. This concludes the proof of equation (III.4.13).

▷ **Proof of** (III.4.14)**.** To begin with, we have

$$\begin{aligned}\triangle W_\varepsilon^{(2)}(z) = \left\langle dG_\varepsilon^{V_1}(z), \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_1)(z) + (\mathcal{D}_\varepsilon(\partial_z V_1)(z))^2 \right\rangle \\ - \left\langle dG_\varepsilon^{V_2}(z), \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_2)(z) + (\mathcal{D}_\varepsilon(\partial_z V_2)(z))^2 \right\rangle .\end{aligned}$$

We split the difference into two, as in the previous part,

$$\begin{aligned}\triangle W_\varepsilon^{(2)}(z) &= \left\langle dG_\varepsilon^{V_2}(z), \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_1)(z) + (\mathcal{D}_\varepsilon(\partial_z V_1)(z))^2 - \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_2)(z) - (\mathcal{D}_\varepsilon(\partial_z V_2)z))^2 \right\rangle \\ &\quad + \left\langle dG_\varepsilon^{V_1}(z) - dG_\varepsilon^{V_2}(z), \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_1)(z) + (\mathcal{D}_\varepsilon(\partial_z V_1)(z))^2 \right\rangle \\ &= A + B\end{aligned}$$

The first contribution can be rearranged as follows, by factorizing the difference of squares:

$$A = \left\langle dG_\varepsilon^{V_2}(z), \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 \triangle V)(z) + \mathcal{D}_\varepsilon(\partial_z \triangle V)(z)\mathcal{D}_\varepsilon(\partial_z(V_1+V_2))(z) \right\rangle.$$

The term involving V_1+V_2 is bounded uniformly in a crude way: $\|\mathcal{D}_\varepsilon(\partial_z(V_1+V_2))\|_\infty \leqslant 2\|V_1+V_2\|_\alpha$ (in fact it is bounded by a $\mathcal{O}(\varepsilon)$ uniformly with respect to z, but this detail is omitted here). Then, we apply Lemma III.20 twice with $V_1 - V_2$ to obtain:

$$(1+|z|)^\alpha |A| \leqslant \varepsilon C_K \|\triangle V\|_\alpha \left\langle dG_\varepsilon^{V_2}(z), \left(|y_1| + |y_2| + \varepsilon^\alpha |y_1|^{1+\alpha} + \varepsilon^\alpha |y_2|^{1+\alpha}\right) \right\rangle$$

To estimate B, the term involving the difference of measures dG_ε^V, we apply (III.4.4) and Lemma III.20:

$$\begin{aligned}&(1+|z|)^\alpha |B| \leqslant \\ &\left\langle \left| dG_\varepsilon^{V_1}(z) - dG_\varepsilon^{V_2}(z) \right|, \varepsilon C \left(\|V_1\|_\alpha^2 + \|V_2\|_\alpha \right) \left(|y_1| + |y_2| + \varepsilon^\alpha |y_1|^{1+\alpha} + \varepsilon^\alpha |y_2|^{1+\alpha} \right) \right\rangle. \quad \text{(III.4.18)}\end{aligned}$$

We find, exactly as above, that the quantity $(1+|z|)^\alpha |B|$ is bounded by $\varepsilon^2 C_K \|\triangle V\|_\alpha$. Combining both estimates on A, B, we deduce equation (III.4.14).

▷ **Proof of** (III.4.15)**.** The full expression for $\triangle W_\varepsilon^{(3)}$ is as follows:

$$\begin{aligned}\triangle W_\varepsilon^{(3)}(z) = &\left\langle dG_\varepsilon^{V_1}(z), \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V_1)(z) + (\mathcal{D}_\varepsilon(\partial_z V_1)(z))^3 + 3\mathcal{D}_\varepsilon(\partial_z V_1)(z)\left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_1)(z)\right)\right\rangle \\ &- \left\langle dG_\varepsilon^{V_2}(z), \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V_2)(z) + (\mathcal{D}_\varepsilon(\partial_z V_2)(z))^3 + 3\mathcal{D}_\varepsilon(\partial_z V_2)(z)\left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_2)(z)\right)\right\rangle.\end{aligned}$$

We split again in two pieces, one involving $V_1 - V_2$, and the other involving $dG_\varepsilon^{V_1} - dG_\varepsilon^{V_2}$:

$$\triangle W_\varepsilon^{(3)}(V)(z) = \left\langle dG_\varepsilon^{V_2}(z), A_1 + A_2 + A_3 \right\rangle + \left\langle dG_\varepsilon^{V_1}(z) - dG_\varepsilon^{V_2}(z), B \right\rangle,$$

with

$$\begin{aligned}A_1 &= \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 \triangle V)(z) \\ A_2 &= (\mathcal{D}_\varepsilon(\partial_z V_1)(z))^3 - (\mathcal{D}_\varepsilon(\partial_z V_2)(z))^3 \\ &= (\mathcal{D}_\varepsilon(\partial_z \triangle V)(z)) \left[(\mathcal{D}_\varepsilon(\partial_z V_1)(z))^2 + (\mathcal{D}_\varepsilon(\partial_z V_2)(z))^2 + (\mathcal{D}_\varepsilon(\partial_z V_1)(z))(\mathcal{D}_\varepsilon(\partial_z V_2)(z)) \right] \\ A_3 &= 3\mathcal{D}_\varepsilon(\partial_z^2 V_1)(z)\left(\frac{1}{2}\right)\mathcal{D}_\varepsilon(\partial_z V_1)(z) - 3\mathcal{D}_\varepsilon(\partial_z V_2)(z)\left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_2)(z)\right) \\ &= 3\mathcal{D}_\varepsilon(\partial_z V_1)(z)\left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 \triangle V)(z)\right) + 3\mathcal{D}_\varepsilon(\partial_z \triangle V)(z)\left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_2)(z)\right) \\ B &= \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V_1)(z) + (\mathcal{D}_\varepsilon(\partial_z V_1)(z))^3 + 3\mathcal{D}_\varepsilon(\partial_z V_1)(z)\left(\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V_1)(z)\right).\end{aligned}$$

We shall estimate all the contributions separately. Firstly, A_1 yields the contraction factor:

$$\begin{aligned}(1+|z|)^\alpha \left\langle dG_\varepsilon^{V_2}(z), |A_1| \right\rangle &\leqslant 2^{\alpha-1} \|\triangle V\|_\alpha \left\langle dG_\varepsilon^{V_2}(z), 1 + \frac{\varepsilon^\alpha}{4}\left[|y_1|^\alpha + |y_2|^\alpha\right] \right\rangle \\ &\leqslant \left(2^{\alpha-1} + \varepsilon^\alpha C_K\right) \|\triangle V\|_\alpha.\end{aligned}$$

The latter is the main contribution in (III.4.15). The remaining terms are lower-order contributions with respect to ε. For A_2, we have

$$(1+|z|)^\alpha \left\langle dG_\varepsilon^{V_2}(z), |A_2| \right\rangle \leqslant \varepsilon 2^\alpha \|\triangle V\|_\alpha \left\langle dG_\varepsilon^{V_2}(z), \left(\|V_1\|_\alpha^2 + \|V_2\|_\alpha^2 + \|V_1\|_\alpha \|V_2\|_\alpha \right) \right.$$
$$\left. \times \left[|y_1| + |y_2| + \varepsilon^\alpha |y_1|^{1+\alpha} + \varepsilon^\alpha |y_2|^{1+\alpha} \right] \right\rangle$$
$$\leqslant \varepsilon C_K \|\triangle V\|_\alpha .$$

For A_3, we have similarly

$$(1+|z|)^\alpha \left\langle dG_\varepsilon^{V_2}(z), |A_3| \right\rangle \leqslant \varepsilon C_K \|\triangle V\|_\alpha .$$

It remains to control the term involving B. We argue as in (III.4.17) and (III.4.18):

$$\left\langle \left| dG_\varepsilon^{V_1}(z) - dG_\varepsilon^{V_2}(z) \right|, (1+|z|)^\alpha |B| \right\rangle$$
$$\leqslant \varepsilon C_K \|\triangle V\|_\alpha \left\langle (1+|y_1|+|y_2|) \exp(-Q(y_1,y_2) + 2\varepsilon R_K(|y_1|+|y_2|)), \right.$$
$$\left. 2^{\alpha-1} \|V_1\|_\alpha \left(1 + \frac{\varepsilon^\alpha}{4} \left[|y_1|^\alpha + |y_2|^\alpha \right] \right) + C\varepsilon \left(\|V_1\|_\alpha^3 + \|V_1\|_\alpha^2 \right) \left(|y_1| + |y_2| + \varepsilon^\alpha |y_1|^{1+\alpha} + \varepsilon^\alpha |y_2|^{1+\alpha} \right) \right\rangle.$$

The latter is controlled by $\varepsilon C_K \|\triangle V\|_\alpha$ for the same reasons as usual.

Combining all the pieces together, we obtain finally (III.4.15). □

III.5 ANALYSIS OF THE FIXED POINT MAPPING H_ε

In this section we focus on the fixed point mapping $\mathcal{H}_\varepsilon$ (III.2.14), which is defined through an infinite series. We are first concerned with the convergence of the series for $V \in \mathcal{E}_0^\alpha$.

III.5.1 *Well-posedness of H_ε on balls*

Consider the following decomposition of each term of the series (III.2.14) in two parts, with the corresponding notations:

$$\Gamma_\varepsilon(z) = \log\left(\frac{\mathcal{I}_\varepsilon(V)(0) + m(z)}{\mathcal{I}_\varepsilon(V)(0)} \right) - \log\left(\frac{\mathcal{I}_\varepsilon(V)(z)}{\mathcal{I}_\varepsilon(V)(0)} \right) = \Gamma_\varepsilon^m(z) - \Gamma_\varepsilon^I(z).$$

They have the following properties :

Lemma III.23.
For every ball $K \subset \mathcal{E}_0^\alpha$, there exists ε_K such that for any $\varepsilon \leqslant \varepsilon_K$, and $V \in K$, we have $\Gamma_\varepsilon^m \in \mathcal{E}_0^\alpha$. Moreover, we have $(1+|z|)^\alpha \partial_z \Gamma_\varepsilon^m \in L^\infty$.

The proof of Lemma III.23 is a straightforward consequence of Proposition III.16 and the assumptions on m made in definition III.2, particularly (III.1.6).

Lemma III.24.
For every ball $K \subset \mathcal{E}_0^\alpha$, there exists ε_K such that for any $\varepsilon \leqslant \varepsilon_K$, and $V \in K$, we have $\Gamma_\varepsilon^I \in \mathcal{E}^\alpha$, and $\partial_z \Gamma_\varepsilon^I(0) = 0$. Moreover, we have $(1+|z|)^\alpha \partial_z \Gamma_\varepsilon^I \in L^\infty$.

Proof. We begin by verifying the condition $\partial_z \Gamma_\varepsilon^I(0) = 0$. This is in fact equivalent to the choice of $\gamma_\varepsilon(V)$, as can be seen on the following computation:

$$\partial_z \Gamma_\varepsilon^I(0) = \frac{\partial_z \mathcal{I}_\varepsilon(V)(0)}{\mathcal{I}_\varepsilon(V)(0)} = W_\varepsilon^{(1)}(V)(0).$$

Now, comparing (III.3.1) with (III.4.7), we see that $\partial_z \Gamma_\varepsilon^I(0) = 0$ is equivalent to $J(\gamma_\varepsilon(V), V) = 0$, provided ε is small enough (for the quantities to be well defined).

Secondly, we need to get uniform bounds on the derivatives of Γ_ε^I to prove that it belongs to $\mathcal{E}^\alpha$. The following formulas relate the successive logarithmic derivatives of $\mathcal{I}_\varepsilon(V)$ to the $W_\varepsilon^{(i)}(V)$ introduced in equation (III.4.6):

$$\partial_z \Gamma_\varepsilon^I(z) = W_\varepsilon^{(1)}(V)(z) \tag{III.5.1}$$

$$\partial_z^2 \Gamma_\varepsilon^I(z) = W_\varepsilon^{(2)}(V)(z) - \left[W_\varepsilon^{(1)}(V)(z)\right]^2, \tag{III.5.2}$$

$$\partial_z^3 \Gamma_\varepsilon^I(z) = W_\varepsilon^{(3)}(V)(z) + 3W_\varepsilon^{(1)}(V)(z)W_\varepsilon^{(2)}(V)(z) + 2\left[W_\varepsilon^{(1)}(V)(z)\right]^3. \tag{III.5.3}$$

We can use directly the weighted estimates in Proposition III.22, which include the algebraic decay of the first order derivative. Algebraic combinations are compatible with those estimates because $W_\varepsilon^{(i)}(V) \in L^\infty(\mathbb{R})$. A fortiori those terms are all uniformly bounded and so we obtain that $\Gamma_\varepsilon^I \in \mathcal{E}^\alpha$. □

The main result of this section is the following one:

Proposition III.25 (Convergence of the series $\mathcal{H}_\varepsilon(V)$)**.**
For every ball $K \subset \mathcal{E}_0^\alpha$, there exists ε_K such that for any $\varepsilon \leqslant \varepsilon_K$, and $V \in K$, the sum $\mathcal{H}_\varepsilon(V)$ is finite.

Before proving this statement, we first establish an auxiliary technical lemma about the following summation operator S:

$$\mathcal{S} : \Lambda \longmapsto \left(h \mapsto \sum_{k \geqslant 0} 2^k \Lambda(2^{-k}h)\right).$$

Lemma III.26 (Existence of the sum)**.**
Take any function $\Lambda \in \mathcal{E}^\alpha$ such that $\partial_z \Lambda(0) = 0$. Then $\mathcal{S}(\Lambda)(h)$ is well-defined for every $h \in \mathbb{R}$.

Proof. We perform a Taylor expansion: there exists $\widetilde{h}_k$, such that $\Lambda(2^{-k}h) = \frac{1}{2}(2^{-k}h)^2 \partial_z^2 \Lambda(2^{-k}\widetilde{h_k})$. Therefore, we have immediately

$$\left|\sum_{k \geqslant 0} 2^k \Lambda(2^{-k}h)\right| \leqslant \left(h^2 \sum_{k \geqslant 0} 2^{-k}\right) \left\|\partial_z^2 \Lambda\right\|_\infty < \infty.$$

□

One can now proceed to the proof of the finiteness of the sum of $\mathcal{H}_\varepsilon$ in definition III.11.

Proof of Proposition III.25. Let K be the ball of $\mathcal{E}_0^\alpha$ of radius $\|K\|_\alpha$ and take $V \in K$, $z \in \mathbb{R}$. To use the previous lemma, we first notice the identity by definition:

$$\mathcal{H}_\varepsilon(V) = \mathcal{S}(\Gamma_\varepsilon). \tag{III.5.4}$$

There are two conditions to verify in order to apply Lemma III.26:

$$\partial_z \Gamma_\varepsilon(0) = 0, \text{ and } \Gamma_\varepsilon \in \mathcal{E}^\alpha.$$

Those properties are verified thanks to Lemmas III.23 and III.24. The Proposition III.25 immediately follows. □

So far, we have not used the algebraic decay condition which is part of the definition of $\mathcal{E}^\alpha$. In the following lemma, we refine the estimate on $\mathcal{S}(\Lambda) \in \mathcal{E}^\alpha$. This foreshadows the same result for the function $\mathcal{H}_\varepsilon(V)$, as stated in the next section.

Lemma III.27 (Better control of the series)**.**
Assume that $\Lambda \in \mathcal{E}^\alpha$, that $\partial_z \Lambda(0) = 0$, and that $(1+|z|)^\alpha \partial_z \Lambda \in L^\infty$. Then, $\mathcal{S}(\Lambda)$ belongs to $\mathcal{E}^\alpha$, with a uniform estimate:

$$\|\mathcal{S}(\Lambda)\|_\alpha \leqslant C \max\left(\|\Lambda\|_\alpha, \sup_{z\in\mathbb{R}} (1+|z|)^\alpha |\partial_z \Lambda(z)| \right) \tag{III.5.5}$$

There is some subtlety hidden here. In fact, we were not able to propagate the algebraic decay at first order from Λ to $\mathcal{S}(\Lambda)$. What saves the day is that we gain some algebraic decay of the first order derivatives somewhere in our procedure (see *e.g.* Proposition III.22).

Proof. Recall the notation $\varphi_\alpha(h) = (1+|h|)^\alpha$. We begin with the uniform bound on the first derivative, which is the main reason why we have to impose algebraic decay in our functional spaces.

▷ **Step 1: $\partial_z \mathcal{S}(\Lambda)$ is uniformly bounded.** We split the sum in two parts. Let $h \in \mathbb{R}$, and let $N_h \in \mathbb{N}$ be the lowest integer such that $|h| \leqslant 2^{N_h}$. We consider the two regimes: $k > N_h$ and $k \leqslant N_h$. In the former regime, a simple Taylor expansion yields

$$\left| \sum_{k>N_h} \partial_z \Lambda(2^{-k}h) \right| \leqslant \sum_{k>N_h} 2^{-k}|h| \left\|\partial_z^2 \Lambda\right\|_\infty \leqslant \left\|\partial_z^2 \Lambda\right\|_\infty, \tag{III.5.6}$$

by definition of N_h. In the regime $k \leqslant N_h$, we use the algebraic decay which is encoded in the space $\mathcal{E}^\alpha$. If $|h| > 1$, we have $N_h \geqslant 1$, and

$$\begin{aligned}\left| \sum_{k\leqslant N_h} \partial_z \Lambda(2^{-k}h) \right| &\leqslant \sum_{k\leqslant N_h} \frac{\|\varphi_\alpha \partial_z \Lambda\|_\infty}{(1+2^{-k}|h|)^\alpha} \\ &\leqslant \left(\sum_{k\leqslant N_h} \frac{2^{k\alpha}}{|h|^\alpha} \right) \|\varphi_\alpha \partial_z \Lambda\|_\infty = \left(\frac{1}{|h|^\alpha} \frac{2^{(N_h+1)\alpha}-1}{2^\alpha - 1} \right) \|\varphi_\alpha \partial_z \Lambda\|_\infty.\end{aligned}$$

By definition of N_h, we have $2^{N_h-1} < |h|$, so that the right-hand-side above is bounded by a constant that get arbitrarily large as $\alpha \to 0$ (hence, the restriction on $\alpha > 0$):

$$\left| \sum_{k\leqslant N_h} \partial_z \Lambda(2^{-k}h) \right| \leqslant \left(\frac{4^\alpha}{2^\alpha - 1} \right) \|\varphi_\alpha \partial_z \Lambda\|_\infty. \tag{III.5.7}$$

The case $|h| \leqslant 1$ is trivial as the sum is reduced to a single term $\partial_z \Lambda(h)$ since $N_h = 0$.

▷ **Step 2:** $\varphi_\alpha \left|\partial_z^2 \mathcal{S}(\Lambda)\right|$ **is uniformly bounded.** This bound and the next one are easier. For any $h \in \mathbb{R}$, we have

$$\varphi_\alpha(h) \left| \sum_{k\geqslant 0} 2^{-k} \partial_z^2 \Lambda(2^{-k}h) \right| \leqslant \left(\sum_{k\geqslant 0} 2^{-k} \frac{\varphi_\alpha(h)}{\varphi_\alpha(2^{-k}h)} \right) \|\varphi_\alpha \partial_z^2 \Lambda\|_\infty.$$

Since $1 \geqslant 2^{-k}$, one obtains

$$\varphi_\alpha(h) \left| \sum_{k\geqslant 0} 2^{-k} \partial_z^2 \Lambda(2^{-k}h) \right| \leqslant \left(\sum_{k\geqslant 0} 2^{k(\alpha-1)} \right) \|\varphi_\alpha \partial_z^2 \Lambda\|_\infty = \left(\frac{2}{2-2^\alpha} \right) \|\varphi_\alpha \partial_z^2 \Lambda\|_\infty.$$

The latter sum is finite since $\alpha < 1$.

▷ **Step 3:** $\varphi_\alpha \left|\partial_z^3 \mathcal{S}(\Lambda)\right|$ **is uniformly bounded.** The proof is similar to the previous argument. □

III.5.2 *Contraction properties (second part)*

In this section we prove that $\mathcal{H}_\varepsilon$ stabilizes some subset of $\mathcal{E}_0^\alpha$. We first show that $\mathcal{H}_\varepsilon$ maps balls into balls with incremental radius that do not depends on the initial ball (Proposition III.30). This property immediately implies the existence of an invariant subset for $\mathcal{H}_\varepsilon$ (corollary III.31). Finally, we prove that the mapping $\mathcal{H}_\varepsilon$ is a contraction mapping for ε small enough (Theorem III.32). To completely justify the definition of $\mathcal{H}_\varepsilon$, it remains to show that $\mathcal{H}_\varepsilon(V) \in \mathcal{E}_0^\alpha$. We begin with the lower bound on the second derivative, which is for free.

Lemma III.28 (Lower bound on $\partial_z^2 \mathcal{H}_\varepsilon(V)(0)$)**.**
For every ball $K \subset \mathcal{E}_0^\alpha$, there exists ε_K such that for any $\varepsilon \leqslant \varepsilon_K$, and $V \in K$, we have:

$$\partial_z \mathcal{H}_\varepsilon(V)(0) = 0, \quad \partial_z^2 \mathcal{H}_\varepsilon(V)(0) \geqslant \partial_z^2 m(0).$$

Proof. The identity $\partial_z \mathcal{H}_\varepsilon(V)(0) = 0$, and more particularly $\partial_z \Gamma_\varepsilon^I(0) = 0$ is a consequence of the choice of $\gamma_e(V)$ in Proposition III.9. Indeed, we have, by (III.5.4),

$$\partial_z \mathcal{H}_\varepsilon(V)(0) = \sum_{k\geqslant 0} \partial_z \Gamma_\varepsilon(V)(0) = 0.$$

For the second estimate, a simple computation yields, using $m(0) = \partial_z m(0) = 0$:

$$\partial_z^2 \mathcal{H}_\varepsilon(V)(0) = \sum_{k\geqslant 0} 2^{-k} \left[\frac{\partial_z^2 m(0)}{\mathcal{I}_\varepsilon(V)(0)} - W_\varepsilon^{(2)}(V)(0) - W_\varepsilon^{(1)}(V)(0)^2 \right]$$

But since $V \in \mathcal{E}_0^\alpha$, one can use again the uniform estimates of Proposition III.22 to write that for $\varepsilon \leqslant \varepsilon_K$, that depends only on the ball K :

$$\partial_z^2 \mathcal{H}_\varepsilon(V) = 2 \frac{\partial_z^2 m(0)}{\mathcal{I}_\varepsilon(V)(0)} + \mathcal{O}(\varepsilon), \tag{III.5.8}$$

where $\mathcal{O}(\varepsilon)$ that depends only on the ball K. Then, we use Proposition III.16 with $\delta = 1/3$ to deduce that for ε small enough, we have $\mathcal{I}_\varepsilon(V) \leqslant 4/3$. Then (III.5.8) can be simplified into

$$\partial_z^2 \mathcal{H}_\varepsilon(V)(0) \geqslant \frac{3\partial_z^2 m(0)}{2} + \mathcal{O}(\varepsilon).$$

Recall that $\partial_z^2 m(0) > 0$ by assumption. Therefore, for ε small enough, we get as claimed

$$\partial_z^2 \mathcal{H}_\varepsilon(V)(0) \geqslant \partial_z^2 m(0).$$

□

Remark III.29. *Considering the proof, another way to interpret the result is that automatically for any function $V \in \mathcal{E}^\alpha$ such that $\partial_z V(0) = 0$, the function $\mathcal{H}_\varepsilon$ prescribes a lower bound on $\partial_z^2 V(0)$. Since we are seeking a fixed point $\mathcal{H}_\varepsilon(V) = V$, we may as well put this condition in the subspace $\mathcal{E}_0^\alpha$ without loss of generality.*

Finally, we can establish a first useful estimate on $\|\mathcal{H}_\varepsilon(V)\|_\alpha$, showing more than just its finiteness:

Proposition III.30 (Contraction in the large)**.**
For every ball $K \in \mathcal{E}_0^\alpha$, there exists an explicit constant $\kappa(\alpha) < 1$, as well as C_m, C_K and ε_K that depend only on K such that, for all $\varepsilon \leqslant \varepsilon_K$, and for every $V \in K$,

$$\|\mathcal{H}_\varepsilon(V)\|_\alpha \leqslant C_m + (\kappa(\alpha) + \varepsilon^\alpha C_K) \|V\|_\alpha . \tag{III.5.9}$$

Proof. Let K be the ball of $\mathcal{E}_0^\alpha$, and take $V \in K$. For clarity we write respectively $\mathcal{I}_\varepsilon(h)$ and $W_\varepsilon^{(i)}(h)$ instead of $\mathcal{I}_\varepsilon(V)(h)$ and $W_\varepsilon^{(i)}(V)(h)$. Combining various estimates derived in Section III.5.1, and particularly Lemma III.27 together with Lemmas III.23 and III.24, we find that $\mathcal{H}_\varepsilon(V) = \mathcal{S}(\Gamma_\varepsilon) = \mathcal{S}(\Gamma_\varepsilon^m) - \mathcal{S}(\Gamma_\varepsilon^I)$ belongs to $\mathcal{E}^\alpha$. However, the associated estimate (III.5.5) is not satisfactory, at least for the $\mathcal{S}(\Gamma_\varepsilon^I)$ and we need to re-examine the dependency of the constants upon ε and α.

The first and second derivatives of Γ_ε^I involve $W_\varepsilon^{(1)}$ and $W_\varepsilon^{(2)}$ which are both of order $\varepsilon C_K \|V\|_\alpha$ thanks to Proposition III.22. Back to the proof of Lemma III.27, the quantities $\|\varphi_\alpha \partial_z \Gamma_\varepsilon^I\|_\infty$ and $\|\varphi_\alpha \partial_z^2 \Gamma_\varepsilon^I\|_\infty$ are in fact of order $\varepsilon \|\Lambda\|_\alpha$, and so are $\|\partial_z \mathcal{S}(\Gamma_\varepsilon^I)\|_\infty$ and $\|\varphi_\alpha \partial_z^2 \mathcal{S}(\Gamma_\varepsilon^I)\|_\infty$.

This cannot be extended readily to the third derivative as we lose the order ε at this stage. However, Proposition III.22 provides an explicit constant that is going to be used. From (III.5.3), we have:

$$\partial_z^3 \mathcal{S}\left(\Gamma_\varepsilon^I\right)(h) = \sum_{k \geqslant 0} 4^{-k} \left[W_\varepsilon^{(3)}(2^{-k}h) + 3 W_\varepsilon^{(2)}(2^{-k}h) W_\varepsilon^{(1)}(2^{-k}h) + 2 W_\varepsilon^{(1)}(2^{-k}h)^3 \right].$$

The contributions involving $W_\varepsilon^{(1)}$ and $W_\varepsilon^{(2)}$ are of order ε, and can be handled exactly as above. However, the linear term involving $W_\varepsilon^{(3)}$ requires a careful attention. We obtain from Proposition III.22 that $\varphi_\alpha W_\varepsilon^{(3)}$ is bounded uniformly by $\left(2^{\alpha-1} + \varepsilon^\alpha C_K\right) \|V\|_\alpha$. Therefore,

$$\begin{aligned} \varphi_\alpha(h) \left| \sum_{k \geqslant 0} 4^{-k} W_\varepsilon^{(3)}(2^{-k}h) \right| &\leqslant \left(2^{\alpha-1} + \varepsilon^\alpha C_K\right) \left(\sum_{k \geqslant 0} 4^{-k} \frac{\varphi_\alpha(h)}{\varphi_\alpha(2^{-k}h)} \right) \|V\|_\alpha \\ &\leqslant \left(2^{\alpha-1} + \varepsilon^\alpha C_K\right) \left(\sum_{k \geqslant 0} 2^{k(\alpha-2)} \right) \|V\|_\alpha = \left(\frac{2^{\alpha+1}}{4 - 2^\alpha} + \varepsilon^\alpha C_K \right) \|V\|_\alpha . \end{aligned} \tag{III.5.10}$$

In view of the latter estimate, we define the explicit constant $\kappa(\alpha)$ as

$$\kappa(\alpha) = \frac{2^{1+\alpha}}{4 - 2^\alpha}. \tag{III.5.11}$$

A simple calculation shows that $\kappa(\alpha) < 1$ if and only if $\alpha < 2 - \log_2(3) \approx 0.415$. The choice $\alpha < 2/5$ gives some room below this threshold. We conclude that $\left\|\mathcal{S}(\Gamma_\varepsilon^I)\right\|_\alpha \leqslant (\kappa(\alpha) + \varepsilon^\alpha C_K) \|V\|_\alpha$.

The other contribution to $\mathcal{H}_\varepsilon(V)$, namely $\mathcal{S}(\Gamma_\varepsilon^m)$ can be bounded in an easier way. Indeed, we have

$$\Gamma_\varepsilon^m = \log(1+m) + \log\left(1+\frac{m}{\mathcal{I}_\varepsilon(0)}\right) - \log(1+m) = \log(1+m) + \log\left(1+\frac{m}{1+m}\left(\frac{1}{\mathcal{I}_\varepsilon(0)}-1\right)\right). \tag{III.5.12}$$

We define accordingly

$$C_m = \max_{k=1,2,3}\left(\left\|\varphi_\alpha \frac{\partial_z^k m}{1+m}\right\|_\infty\right), \tag{III.5.13}$$

Moreover, Proposition III.16 can be easily refined into $|\mathcal{I}_\varepsilon(0)-1| \leqslant \varepsilon C_m \|V\|_\alpha$, using the definition of R_K in (III.2.11). Straightforward computations show that the last contribution in (III.5.12) can be estimated by $\varepsilon C_m \|V\|_\alpha$.

Combining the estimates obtained for $\mathcal{S}(\Gamma_\varepsilon^m)$ and $\mathcal{S}(\Gamma_\varepsilon^I)$, we come to the conclusion:

$$\|\mathcal{H}_\varepsilon(V)\|_\alpha \leqslant C_m + (\kappa(\alpha)+\varepsilon^\alpha C_K)\|V\|_\alpha,$$

□

Proposition III.30 calls an immediate corollary.

Corollary III.31 (Invariant subset)**.**
There exist K_0 a ball of $\mathcal{E}_0^\alpha$, and ε_0 a positive constant such that for all $\varepsilon \leqslant \varepsilon_0$ the set K_0 is invariant by $\mathcal{H}_\varepsilon$:

$$\mathcal{H}_\varepsilon(K_0) \subset K_0.$$

Proof. Let K_0 be the ball of radius $R_0 = 2C_m/(1-\kappa(\alpha))$. We deduce from Proposition III.30 that, for all $V \in K_0$,

$$\begin{aligned}\|\mathcal{H}_\varepsilon(V)\|_\alpha \leqslant C_m + (\kappa(\alpha)+\varepsilon^\alpha C_{K_0})R_0 &= C_m\left(1+\frac{2\kappa(\alpha)}{1-\kappa(\alpha)}\right) + \varepsilon^\alpha C_{K_0}R_0\\ &= C_m\left(\frac{2}{1-\kappa(\alpha)}-1\right)+\varepsilon^\alpha C_{K_0}R_0\\ &= R_0 + C_m\left(-1+\frac{2\varepsilon^\alpha C_{K_0}}{1-\kappa(\alpha)}\right).\end{aligned}$$

Therefore, the choice $\varepsilon_0 = \left(\frac{1-\kappa(\alpha)}{2C_{K_0}}\right)^{\frac{1}{\alpha}}$ guarantees that K_0 is left invariant by $\mathcal{H}_\varepsilon$. □

We are now in position to state the more important result of this section:

Theorem III.32 (Contraction mapping)**.** *There exists a constant C_{K_0} such that for any $\varepsilon \leqslant \varepsilon_0$, and every function V_1, $V_2 \in K_0$, the following estimate holds true*

$$\|\mathcal{H}_\varepsilon(V_1) - \mathcal{H}_\varepsilon(V_2)\|_\alpha \leqslant (\kappa(\alpha)+\varepsilon^\alpha C_K)\|V_1-V_2\|_\alpha. \tag{III.5.14}$$

Proof. We denote by $\triangle V$ the difference $V_1 - V_2$, again. The proof is analogous to Proposition III.30. For clarity we write respectively $\mathcal{I}_\varepsilon^i(h)$ instead of $\mathcal{I}_\varepsilon(V_i)(h)$ and $\triangle W_\varepsilon^{(i)}(h)$ instead of $W_\varepsilon^{(i)}(V_1)(h) - W_\varepsilon^{(i)}(V_2)(h)$. We decompose $\triangle\mathcal{H}_\varepsilon(V)$ as above:

$$\triangle\mathcal{H}_\varepsilon = \triangle\left(\mathcal{S}(\Gamma_\varepsilon^m) - \mathcal{S}(\Gamma_\varepsilon^I)\right) = \triangle\mathcal{H}_\varepsilon^m - \triangle\mathcal{H}_\varepsilon^I. \tag{III.5.15}$$

We deal with $\triangle\mathcal{H}_\varepsilon^m$ in the following lemma :

Lemma III.33. *There exists a constant C_0 such that for any $\varepsilon \leqslant \varepsilon_0$, and every function V_1, $V_2 \in K_0$, we have*

$$\|\triangle \mathcal{H}_\varepsilon^m\|_\alpha \leqslant \varepsilon C_0 \|\triangle V\|_\alpha .$$

Proof. Recall the following definition:

$$\triangle \Gamma_\varepsilon^m = \log\left(\mathcal{I}_\varepsilon^1(0) + m\right) - \log\left(\mathcal{I}_\varepsilon^2(0) + m\right) - \log\left(\frac{\mathcal{I}_\varepsilon^1(0)}{\mathcal{I}_\varepsilon^2(0)}\right) . \tag{III.5.16}$$

The first derivative has the following expression,

$$\partial_z \triangle \Gamma_\varepsilon^m = -\frac{\partial_z m}{(\mathcal{I}_\varepsilon^1(0) + m)(\mathcal{I}_\varepsilon^2(0) + m)} \triangle \mathcal{I}_\varepsilon(0) . \tag{III.5.17}$$

Clearly, $\mathcal{I}_\varepsilon^2(0) + m$ is bounded below, uniformly for ε small enough. Therefore, we can repeat the arguments of Lemma III.27, with $\Lambda = \log(\mathcal{I}_\varepsilon^1(0) + m)$ in order to get

$$\|\partial_z \mathcal{S}(\triangle \Gamma_\varepsilon^m)\|_\infty \leqslant C_m |\triangle \mathcal{I}_\varepsilon(0)|. \tag{III.5.18}$$

However, Proposition III.18 yields that $|\triangle \mathcal{I}_\varepsilon(0)| \leqslant \varepsilon C_0 \|\triangle V\|_\alpha$.

The next order derivatives can be handled similarly. Indeed, the following quantities must be bounded uniformly by $\varepsilon C_0 \|\triangle V\|_\alpha$:

$$\begin{aligned}
\varphi_\alpha(h) \left| \sum_{k\geqslant 0} 2^{-k} \left[\frac{\partial_z^2 m(2^{-k}h)}{\mathcal{I}_\varepsilon^1(0) + m(2^{-k}h)} - \frac{\partial_z^2 m(2^{-k}h)}{\mathcal{I}_\varepsilon^2(0) + m(2^{-k}h)} \right] \right| &\leqslant \varepsilon C_0 \|\triangle V\|_\alpha \\
\varphi_\alpha(h) \left| \sum_{k\geqslant 0} 2^{-k} \left[\frac{\partial_z m(2^{-k}h)^2}{(\mathcal{I}_\varepsilon^1(0) + m(2^{-k}h))^2} - \frac{\partial_z m(2^{-k}h)^2}{(\mathcal{I}_\varepsilon^2(0) + m(2^{-k}h))^2} \right] \right| &\leqslant \varepsilon C_0 \|\triangle V\|_\alpha \\
\varphi_\alpha(h) \left| \sum_{k\geqslant 0} 4^{-k} \left[\frac{\partial_z^3 m(2^{-k}h)}{\mathcal{I}_\varepsilon^1(0) + m(2^{-k}h)} - \frac{\partial_z^3 m(2^{-k}h)}{\mathcal{I}_\varepsilon^2(0) + m(2^{-k}h)} \right] \right| &\leqslant \varepsilon C_0 \|\triangle V\|_\alpha \\
\varphi_\alpha(h) \left| \sum_{k\geqslant 0} 4^{-k} \left[\frac{\partial_z m(2^{-k}h)^3}{(\mathcal{I}_\varepsilon^1(0) + m(2^{-k}h))^3} - \frac{\partial_z m(2^{-k}h)^3}{(\mathcal{I}_\varepsilon^2(0) + m(2^{-k}h))^3} \right] \right| &\leqslant \varepsilon C_0 \|\triangle V\|_\alpha , \\
\varphi_\alpha(h) \left| \sum_{k\geqslant 0} 4^{-k} \left[\frac{\partial_z^2 m(2^{-k}h)\partial_z m(2^{-k}h)}{(\mathcal{I}_\varepsilon^1(0) + m(2^{-k}h))^2} - \frac{\partial_z^2 m(2^{-k}h)\partial_z m(2^{-k}h)}{(\mathcal{I}_\varepsilon^2(0) + m(2^{-k}h))^2} \right] \right| &\leqslant \varepsilon C_0 \|\triangle V\|_\alpha .
\end{aligned}$$

The first and the third items are handled similarly as for the first derivative. The three other items are handled analogously. For the sake of concision, we focus on the second line: we have,

$$\begin{aligned}
&\frac{\partial_z m(z)^2}{(\mathcal{I}_\varepsilon^1(0) + m(z))^2} - \frac{\partial_z m(z)^2}{(\mathcal{I}_\varepsilon^2(0) + m(z))^2} \\
&\qquad = \left[\frac{\partial_z m(z)}{\mathcal{I}_\varepsilon^1(0) + m(z)} + \frac{\partial_z m(z)}{\mathcal{I}_\varepsilon^2(0) + m(z)} \right] \left[\frac{-\partial_z m(z) \triangle \mathcal{I}_\varepsilon(0)}{(\mathcal{I}_\varepsilon^1(0) + m(z))(\mathcal{I}_\varepsilon^2(0) + m(z))} \right]
\end{aligned}$$

The first factor is uniformly bounded by assumption (III.1.6), for ε small enough. The second factor is the same as above, so we can conclude directly. □

It remains to handle $\triangle\mathcal{H}^I_\varepsilon$. We have the following formulas for the two first derivatives (III.5.1)–(III.5.3):

$$\partial_z\triangle\mathcal{H}^I_\varepsilon(h) = \sum_{k\geqslant 0}\triangle W^{(1)}_\varepsilon(2^{-k}h),$$
$$\partial^2_z\triangle\mathcal{H}^I_\varepsilon(h) = \sum_{k\geqslant 0}2^{-k}\left[\triangle W^{(2)}_\varepsilon(2^{-k}h) - \triangle\left(W^{(1)}_\varepsilon(2^{-k}h)^2\right)\right]$$

Finally the formula for the third derivative is:

$$\partial^3_z\triangle\mathcal{H}^I_\varepsilon(h) = \sum_{k\geqslant 0}4^{-k}\left[\triangle W^{(3)}_\varepsilon(2^{-k}h) + 3\triangle\left(W^{(2)}_1 W^{(1)}_1(2^{-k}h)\right) + 2\triangle\left(W^{(1)}_\varepsilon(2^{-k}h)^3\right)\right]. \tag{III.5.19}$$

The combination of Proposition III.21 and Lemma III.27 yields

$$\|\partial_z\triangle\mathcal{H}^I_\varepsilon\|_\infty \leqslant \varepsilon C_0\,\|\triangle V\|_\alpha\,. \tag{III.5.20}$$

In the same way, we get the bound for the second derivative, using the factorization

$$\triangle\left(W^{(1)}_\varepsilon(z)^2\right) = \left(W^{(1)}_\varepsilon(V_1)(z) + W^{(1)}_\varepsilon(V_2)(z)\right)\triangle\left(W^{(1)}_\varepsilon(z)\right), \tag{III.5.21}$$

together with the uniform bound in Proposition III.22.

As in the proof of Proposition III.30, the third order derivative must be handled with care, as it does not yield a $\mathcal{O}(\varepsilon)$ bound.

Exactly as above, the contribution involving $\triangle W^{(3)}_\varepsilon$ in (III.5.19) is the one that yields the contraction factor, the remaining part being of order $\mathcal{O}(\varepsilon)\,\|\triangle V\|_\alpha$. Actually, we have precisely:

$$\varphi_\alpha(h)\left|\sum_{k\geqslant 0}4^{-k}\triangle W^{(3)}_\varepsilon(2^{-k}h)\right| \leqslant (\kappa(\alpha) + \varepsilon^\alpha C_0)\,\|\triangle V\|_\alpha$$

as in (III.5.10). This concludes the proof of the main contraction estimate. □

III.6 Existence of a (locally) unique U_ε, and convergence as $\varepsilon \to 0$.

III.6.1 *Solving problem (PU_ε) – Theorem III.4(i)*

First of all, Theorem III.32 immediately implies Theorem III.12, that is the existence of a unique fixed point $\mathcal{H}_\varepsilon(V_\varepsilon) = V_\varepsilon$ in the invariant subset K_0, for $\varepsilon \leqslant \varepsilon_0$. Note that ε_0 could possibly be reduced to meet the requirement of the last estimate in (III.5.14).

However, due to the peculiar role played by the linear part $\gamma_\varepsilon(V_\varepsilon)$, it is convenient to enlarge slightly the set K_0. More precisely, after corollary III.31 we define K'_0 the ball of radius

$$R'_0 = R_0 + \sup_{V\in K_0}|\gamma_\varepsilon(V)|\,. \tag{III.6.1}$$

It is clear that, up to reducing further ε_0 to ε'_0 in order to control the new constant $C_{K'_0}$, the set K'_0 is also invariant for $\varepsilon \leqslant \varepsilon'_0$. The same contraction estimate as in Theorem III.32 holds, obviously. Furthermore, the fixed point on K'_0 coincides with the fixed point on the smaller ball K_0, by uniqueness.

Next, we show that finding this fixed point is equivalent to solving problem (PU_ε), as claimed in Proposition III.13. We prove in fact the two sides of the equivalence.

▷ The easy part consists in saying that, being given V_ε the unique fixed point in K_0, the function $U_\varepsilon = \gamma_\varepsilon(V_\varepsilon)\cdot + V_\varepsilon$ belongs to K_0' by definition of K_0' (III.6.1), and it solves problem (PU_ε) by construction.

▷ On the other side, suppose that $(\lambda_\varepsilon, U_\varepsilon) \in \mathbb{R}\times K_0'$ is a solution of the problem (PU_ε). As in section III.2, evaluating (PU_ε) at $z=0$ yields the following necessary condition on λ_ε, since $m(0)=0$:

$$\lambda_\varepsilon = I_\varepsilon(U_\varepsilon)(0).^1$$

Then, we focus on U_ε. We decompose it as $U_\varepsilon = \gamma_U\cdot + V_U$, with $\gamma_U = \partial_z U_\varepsilon(0)$, and $\partial_z V_U(0)=0$. Our purpose is threefold: (i) first, we show that $\gamma_U = \gamma_\varepsilon(V_U)$, then (ii) we prove that $V_U \in \mathcal{E}_0^\alpha$, and finally (iii), we prove that $\mathcal{H}_\varepsilon(V_U) = V_U$.

We can reformulate problem (PU_ε) as follows:

$$I_\varepsilon(\gamma_U\cdot + V_U)(0) + m(z) = I_\varepsilon(\gamma_U\cdot + V_U)\exp\left(V_U(z) - 2V_U(\bar{z}) + V_U(0)\right). \tag{III.6.2}$$

Since we assume $U_\varepsilon \in \mathcal{E}^\alpha$, we can differentiate the previous equation, and evaluate it at $z=0$ to get :

$$\partial_z I_\varepsilon(\gamma_U\cdot + V_U)(0) = 0.$$

As in Section III.2, a direct computation shows that γ_U and V_U are linked by the following relation:

$$0 = \mathcal{J}_\varepsilon(\gamma_U, V_U), \tag{III.6.3}$$

In order to invert this relationship, and deduce that $\gamma_U = \gamma_\varepsilon(V_U)$, it is important to prove that $V_\varepsilon \in \mathcal{E}_0^\alpha$, which amounts to showing that $\partial_z^2 V_\varepsilon(0) \geqslant \partial_z^2 m(0)$, the other conditions being clearly verified.

Differentiating the problem (PU_ε) twice, and evaluating at $z=0$, we get:

$$\begin{aligned}\partial_z^2 m(0) &= \partial_z^2 I_\varepsilon(U_\varepsilon)(0) + I_\varepsilon(U_\varepsilon)(0)\frac{\partial_z^2 U_\varepsilon(0)}{2}.\\ &= I_\varepsilon(U_\varepsilon)(0)\left(\frac{\partial_z^2 I_\varepsilon(U_\varepsilon)(0)}{I_\varepsilon(U_\varepsilon)(0)} + \frac{\partial_z^2 U_\varepsilon(0)}{2}\right).\end{aligned}$$

Then, using straightforward adaptations of Propositions III.16 and III.22, where V should be replaced with $V_U \in \mathcal{E}^\alpha$ and $\gamma_\varepsilon(V)$ should be replaced by γ_U, we find that

$$\partial_z^2 m(0) \leqslant \frac{3}{2}\left(\varepsilon C_{K_0'} + \frac{\partial_z^2 U_\varepsilon(0)}{2}\right).$$

for ε sufficiently small. We deduce that the missing condition is in fact a consequence of the formulation (PU_ε):

$$\partial_z^2 U_\varepsilon(0) \geqslant \partial_z^2 m(0).$$

By definition, $\partial_z^2 U_\varepsilon(0) = \partial_z^2 V_\varepsilon(0)$, so we have established that $V_\varepsilon \in \mathcal{E}_0^\alpha$.

1 We use the notation $I_\varepsilon(U_\varepsilon)$ introduced in equation (III.1.4), that should not be confused with $\mathcal{I}_\varepsilon(V_\varepsilon)$. It is the purpose of the present argument to show that the two quantities do coincide.

Hence, we can legitimately invert (III.6.3), so as to find $\gamma_U = \gamma_\varepsilon(V_U)$, where the function γ_ε is defined in Proposition III.9. Since $U_\varepsilon \in K_0'$ by assumption, we have in particular $\|V_U\|_\alpha \leqslant R_0'$. Of course, V_U is the candidate of being the unique fixed point of $\mathcal{H}_\varepsilon$ in K_0' (but also in K_0). The proof of this claim follows the lines of section III.2.2, checking that all manipulations are justified.

First, we divide (III.6.2) by $I_U = I_\varepsilon(\gamma_U \cdot + V_U) = I_\varepsilon(\gamma_\varepsilon(V_U) \cdot + V_U) = \mathcal{I}_\varepsilon(V_U)$. According to Proposition III.16, this quantity is uniformly close to 1 for ε small, so it does not vanish. Taking the logarithm on both sides, we get for all $z \in \mathbb{R}$:

$$V_U(z) - 2V_U(\bar{z}) + V_U(0) = \log\left(\frac{I_U(0) + m(z)}{I_U(z)}\right).$$

We differentiate the last equation to end up with the following recursive equation for every $z \in \mathbb{R}$

$$\partial_z V_U(z) - \partial_z V_U(\bar{z}) = \partial_z \log\left(\frac{I_U(0) + m}{I_U(z)}\right)(z).$$

One simply deduces, that for all $z \in \mathbb{R}$, we necessarily have:

$$\partial_z V_U(z) = \partial_z V_U(0) + \sum_{k \geqslant 0} \log\left(\frac{I_U(0) + m(2^{-k}z)}{I_U(2^{-k}z)}\right).$$

Note that the $\mathcal{C}^1$ continuity at $z = 0$ is used here. Moreover, $\partial_z V_U(0) = 0$ by definition of V_U. The analysis performed in Proposition III.25 guarantees that this sum is indeed finite. Finally, integrating back the previous identity yields

$$V_U(z) = \sum_{k \geqslant 0} 2^k \log\left(\frac{I_U(0) + m(2^{-k}z)}{I_U(2^{-k}z)}\right) = \sum_{k \geqslant 0} 2^k \log\left(\frac{\mathcal{I}_\varepsilon(V_U)(0) + m(2^{-k}z)}{\mathcal{I}_\varepsilon(V_U)(2^{-k}z)}\right).$$

The last expression is nothing but $\mathcal{H}_\varepsilon(V_U)$, by definition (III.2.14). Therefore, $V_U = \mathcal{H}_\varepsilon(V_U)$ is the unique fixed point of $\mathcal{H}_\varepsilon$ in K_0'.

III.6.2 *Convergence of $(\lambda_\varepsilon, U_\varepsilon)$ towards (λ_0, U_0) – Theorem III.4(ii)*

As previously, we decompose $U_\varepsilon = \gamma_\varepsilon \cdot + V_\varepsilon$, where γ_ε stands for $\gamma_\varepsilon(V_\varepsilon)$. Firstly, we have $\lambda_\varepsilon = \mathcal{I}_\varepsilon(V_\varepsilon)(0) \to 1$, using Proposition III.16. Secondly, using an argument of diagonal extraction, there exists a subsequence ε_n, and a limit function V_0 such that

$$\lim_{\varepsilon \to 0} \partial_z V_\varepsilon = \partial_z V_0\,, \quad \text{in } L^\infty_{\text{loc}}, \tag{III.6.4}$$

$$\lim_{\varepsilon \to 0} \partial_z^2 V_\varepsilon = \partial_z^2 V_0\,, \quad \text{in } L^\infty_{\text{loc}}. \tag{III.6.5}$$

We have used the Arzela-Ascoli theorem and the uniform $\mathcal{C}^3$ bound in order to get the convergence up to the second derivative. However, there is no reason why the convergence should hold for the third derivative, due to the lack of compactness.

Looking at (PU_ε), we see that $\mathcal{I}_\varepsilon(U_\varepsilon)$ converges uniformly to 1, and, for every given $z \in \mathbb{R}$, (III.6.4) implies that

$$U_\varepsilon(z) - 2U_\varepsilon(\bar{z}) + U_\varepsilon(0) = V_\varepsilon(z) - 2V_\varepsilon(\bar{z}) + V_\varepsilon(0) \xrightarrow[\varepsilon \to 0]{} V_0(z) - 2V_0(\bar{z}) + V_0(0). \tag{III.6.6}$$

Passing to the pointwise limit in problem (PU_ε), we get that V_0 solves the following problem:

$$1 + m(z) = \exp\left(V_0(z) - 2V_0(\bar{z}) + V_0(0)\right).$$

Then, we have necessarily:

$$V_0(z) = \sum_{k\geqslant 0} 2^k \log\left(1 + m(2^{-k}z)\right). \tag{III.6.7}$$

This completes the proof of Theorem III.4(ii), up to the identification of the limit of γ_ε, if it exists. In our approach, this goes through the characterization of the functional $\mathcal{J}_\varepsilon$ (III.2.10). This was indeed the purpose of Lemma III.14. Here comes an important difficulty, as compactness estimates are not sufficient to pass to the limit in $\mathcal{J}_\varepsilon(0, V_\varepsilon)$ as $\varepsilon \to 0$ (III.3.3), as it would formally involve the pointwise value $\partial_z^3 V_0(0)$ which is beyond what our compactness estimates can provide. Note that passing to the limit in $\partial_g \mathcal{J}_\varepsilon(g, V_\varepsilon)$ as $\varepsilon \to 0$ is not an issue, as it can be encompassed by (III.6.5), see (III.3.4).

It remains to prove that the following limit holds true

$$\lim_{\varepsilon\to 0} \frac{1}{4\sqrt{2}\pi} \iint_{\mathbb{R}^2} \exp(-Q(y_1, y_2)) \left[y_1^2 \partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2 \partial_z^3 V(\varepsilon\widetilde{y_2})\right] dy_1 dy_2 = \frac{1}{2}\partial_z^3 m(0). \tag{III.6.8}$$

Indeed, this would directly imply that

$$\lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon(g, V_\varepsilon) = -\frac{1}{2}\partial_z^3 m(0) + g\partial_z^2 m(0), \tag{III.6.9}$$

as $\partial_z^2 V_0(0) = 2\partial_z^2 m(0)$ as a consequence of (III.6.7). We could deduce immediately that the root $\gamma_\varepsilon(V_\varepsilon)$ converges to the expected value (III.1.11).

In the absence of compactness, we call the contraction argument, in order to prove the following key result:

Lemma III.34. *For every $\delta > 0$, there exists $R_1(\delta) > 0$, such that, for every $R \geqslant R_1(\delta)$, there exists $\varepsilon_1(\delta, R)$ such that for all $\varepsilon \leqslant \varepsilon_1(\delta, R)$, we have:*

$$\sup_{|z|\leqslant \varepsilon R} \left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right| \leqslant \delta. \tag{III.6.10}$$

Proof. To begin with, we differentiate the problem (PU_ε) three times:

$$\partial_z^3 V_\varepsilon(z) - \frac{1}{4}\partial_z^3 V_\varepsilon(\bar{z}) = \partial_z^3 \log\left(\lambda_\varepsilon + m(z)\right) - \partial_z^3 \log\left(\mathcal{I}_\varepsilon(U_\varepsilon)(z)\right).$$

We expand the right hand side as usual:

$$\partial_z^3 V_\varepsilon(z) - \frac{1}{4}\partial_z^3 V_\varepsilon(\bar{z}) = \frac{\partial_z^3 m(z)}{\lambda_\varepsilon + m(z)} + \frac{3\partial_z^2 m(z)\partial_z m(z)}{(\lambda_\varepsilon + m(z))^2} + \frac{2\partial_z m(z)^3}{(\lambda_\varepsilon + m(z))^3} - W_\varepsilon^{(3)}(z) - 3W_\varepsilon^{(2)}(z)W_\varepsilon^{(1)}(z) - 2W_\varepsilon^{(1)}(z)^3.$$

We subtract $\partial_z^3 m(z)$ on each side, and we reorganize the terms in order to conjure the difference $\partial_z^3 V_\varepsilon(z) - (4/3)\partial_z^3 m(z)$ we are interested in:

$$\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z) - \frac{1}{4}\left(\partial_z^3 V_\varepsilon(\bar{z}) - \frac{4}{3}\partial_z^3 m(\bar{z})\right) + \frac{1}{3}\left(\partial_z^3 m(z) - \partial_z^3 m(\bar{z})\right) = \partial_z^3 m(z)\left(\frac{1}{\lambda_\varepsilon + m(z)} - 1\right) + \frac{3\partial_z^2 m(z)\partial_z m(z)}{(\lambda_\varepsilon + m(z))^2} + \frac{2\partial_z m(z)^3}{(\lambda_\varepsilon + m(z))^3} - W_\varepsilon^{(3)}(z) - 3W_\varepsilon^{(2)}(z)W_\varepsilon^{(1)}(z) - 2W_\varepsilon^{(1)}(z)^3. \tag{III.6.11}$$

We estimate below each term of (III.6.11). First, the terms involving m and its derivatives on the right hand side of (III.6.11) converge to zero, uniformly for $|z| \leqslant \varepsilon R$, as $\varepsilon \to 0$, simply because $m(0) = \partial_z m(0) = 0$, and $\lambda_\varepsilon \to 1$. Actually, the same holds true for the difference of $\partial_z^3 m(z) - \partial_z^3 m(\bar{z})$ by continuity of $\partial_z^3 m$ at the origin.

Second, from Proposition III.22, we know that

$$\max\left(\left\|W_\varepsilon^{(1)}\right\|_\infty, \left\|W_\varepsilon^{(2)}\right\|_\infty\right) = \mathcal{O}(\varepsilon). \tag{III.6.12}$$

The remaining term, $W_\varepsilon^{(3)}(z)$ is more delicate to handle. In fact, it will result in a contraction estimate, exactly as in section III.5. We recall the expression of $W_\varepsilon^{(3)}$ (III.4.8):

$$W_\varepsilon^{(3)}(z) = \left\langle dG_\varepsilon^{V_\varepsilon}(z), \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V_\varepsilon)(z) + (\mathcal{D}_\varepsilon(\partial_z V_\varepsilon)(z))^3 + \frac{3}{2}\mathcal{D}_\varepsilon(\partial_z V_\varepsilon)(z)\mathcal{D}_\varepsilon(\partial_z^2 V_\varepsilon)(z)\right\rangle.$$

As in the proof of equations (III.4.13) to (III.4.15), we get that the last two contributions involving the non-linear and lower order terms $(\mathcal{D}_\varepsilon(\partial_z V_\varepsilon))^3$ and $\mathcal{D}_\varepsilon(\partial_z V_\varepsilon)\mathcal{D}_\varepsilon(\partial_z^2 V_\varepsilon)$ are $\mathcal{O}(\varepsilon)$. It remains the term $\left\langle dG_\varepsilon^{V_\varepsilon}, (1/4)\mathcal{D}_\varepsilon(\partial_z^3 V_\varepsilon)\right\rangle$, which is a double integral in variables (y_1, y_2) that we split in two regions of integration: $\Omega = \{|y_1| \leqslant R/2, \text{ and } |y_2| \leqslant R/2\}$ and $\Omega^c = \{|y_1| > R/2, \text{ or } |y_2| > R/2\}$.

Let $\delta > 0$. We can choose $R_1(\delta)$ large enough so that, for all $R \geqslant R_1(\delta)$, we have

$$\frac{1}{4}\left|\left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_{\Omega^c}(y_1, y_2), \mathcal{D}_\varepsilon(\partial_z^3 V_\varepsilon)(z)\right\rangle\right| \leqslant \frac{1}{2}\left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_{\Omega^c}(y_1, y_2), 1\right\rangle \|K_0\|_\alpha \leqslant \frac{\delta}{10}. \tag{III.6.13}$$

In the region where y_1 and y_2 are both below $R/2$, we introduce the difference with $\partial_z^3 m$, as in (III.6.10):

$$\left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_\Omega, \frac{1}{4}\partial_z^3 V_\varepsilon(\bar{z}) - \frac{1}{8}\partial_z^3 V_\varepsilon(\bar{z} + \varepsilon y_1) - \frac{1}{8}\partial_z^3 V_\varepsilon(\bar{z} + \varepsilon y_2)\right\rangle = A + B,$$

where

$$A = \left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_\Omega, \frac{1}{4}\left(\partial_z^3 V_\varepsilon(\bar{z}) - \frac{4}{3}\partial_z^3 m(\bar{z})\right)\right.$$
$$\left. -\frac{1}{8}\left(\partial_z^3 V_\varepsilon(\bar{z} + \varepsilon y_1) - \frac{4}{3}\partial_z^3 m(\bar{z} + \varepsilon y_1)\right) - \frac{1}{8}\left(\partial_z^3 V_\varepsilon(\bar{z} + \varepsilon y_2) - \frac{4}{3}\partial_z^3 m(\bar{z} + \varepsilon y_2)\right)\right\rangle$$

and

$$B = \frac{1}{6}\left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_\Omega, \left(\partial_z^3 m(\bar{z}) - \partial_z^3 m(\bar{z} + \varepsilon y_1)\right) + \left(\partial_z^3 m(\bar{z}) - \partial_z^3 m(\bar{z} + \varepsilon y_2)\right)\right\rangle.$$

By construction, we have $|\bar{z} + \varepsilon y_i| \leqslant \varepsilon R/2 + \varepsilon R/2 \leqslant \varepsilon R$. Therefore, we have

$$|A| \leqslant \left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_\Omega, \frac{1}{4}\sup_{|z|\leqslant\varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right| + \frac{2}{8}\sup_{|z|\leqslant\varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right|\right\rangle$$
$$\leqslant \frac{1}{2}\sup_{|z|\leqslant\varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right|.$$

As for B we find:

$$|B| \leqslant \frac{1}{3}\left\langle dG_\varepsilon^{V_\varepsilon}(z)\mathbf{1}_\Omega, \operatorname*{osc}_{|z|\leqslant\varepsilon R}(\partial_z^3 m)\right\rangle \leqslant \frac{1}{3}\operatorname*{osc}_{|z|\leqslant\varepsilon R}(\partial_z^3 m) \xrightarrow[\varepsilon\to 0]{} 0.$$

Going back to (III.6.11), we have shown that for $R \geqslant R_1$, there exists $\varepsilon_1 > 0$ small enough such that for all $\varepsilon \leqslant \varepsilon_1$ we have:

$$\sup_{|z|\leqslant \varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right| \leqslant \frac{\delta}{4} + \frac{1}{4}\sup_{|z|\leqslant \varepsilon R}\left|\partial_z^3 V_\varepsilon(\bar{z}) - \frac{4}{3}\partial_z^3 m(\bar{z})\right| + \frac{1}{2}\sup_{|z|\leqslant \varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right|$$
$$\leqslant \frac{\delta}{4} + \frac{3}{4}\sup_{|z|\leqslant \varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right|.$$

As a consequence, we find that

$$\sup_{|z|\leqslant \varepsilon R}\left|\partial_z^3 V_\varepsilon(z) - \frac{4}{3}\partial_z^3 m(z)\right| \leqslant \delta.$$

This completes the proof of Lemma III.34. □

Back to (III.6.8), we recall that $|\widetilde{y_i}| \leqslant |y_i| + 1$, as a by-product of Taylor expansions. Let $\delta > 0$, and take R sufficiently large such that

$$\frac{1}{4\sqrt{2}\pi}\iint_{\Omega^c} \exp(-Q(y_1,y_2))\left(\|K_0\|_\alpha + \frac{1}{2}\partial_z^3 m(0)\right)\left[y_1^2 + y_2^2\right] dy_1 dy_2 \leqslant \frac{\delta}{10}, \tag{III.6.14}$$

where $\Omega = \{|y_1| \leqslant R-1, \text{ and } |y_2| \leqslant R-1\}$. The other part of the double integral is:

$$\frac{1}{4\sqrt{2}\pi}\iint_{\Omega} \exp(-Q(y_1,y_2))\left[y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2})\right] dy_1 dy_2. \tag{III.6.15}$$

Using Lemma III.34 and the continuity of $\partial_z^3 m$ at $z = 0$, we can find $\varepsilon_1 > 0$ such that for all $\varepsilon \leqslant \varepsilon_1$,

$$\left|\frac{1}{4\sqrt{2}\pi}\iint_{\Omega} \exp(-Q(y_1,y_2))\left[y_1^2\left(\partial_z^3 V(\varepsilon\widetilde{y_1}) - \frac{4}{3}\partial_z^3 m(0)\right) + y_2^2\left(\partial_z^3 V(\varepsilon\widetilde{y_2}) - \frac{4}{3}\partial_z^3 m(0)\right)\right] dy_1 dy_2\right|$$
$$\leqslant \left(\frac{1}{4\sqrt{2}\pi}\iint_{\Omega} \exp(-Q(y_1,y_2))\left[y_1^2 + y_2^2\right] dy_1 dy_2\right)\frac{\delta}{10}.$$

Putting all the pieces together, and using that $\frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2} \exp(-Q(y_1,y_2))\left[y_1^2 + y_2^2\right] dy_1 dy_2 = \frac{3}{2}$ see (III.2.2), we deduce that:

$$\left|\frac{1}{4\sqrt{2}\pi}\iint_{\mathbb{R}^2} \exp(-Q(y_1,y_2))\left[y_1^2\partial_z^3 V(\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V(\varepsilon\widetilde{y_2})\right] dy_1 dy_2 - \frac{1}{2}\partial_z^3 m(0)\right| \leqslant \delta. \tag{III.6.16}$$

Hence, the limit announced in (III.6.8) holds true. This completes the proof of the asymptotic behavior $(\lambda_\varepsilon, U_\varepsilon) \to (\lambda_0, U_0)$ as described in Theorem III.4(ii).

III.7 EXTENSION TO HIGHER DIMENSIONS

Our methodology can be extended to higher dimension, without too much effort. This section is devoted to the generalization of the elements of proof that were specific to the one-dimensional case.

All the estimates on the operator $\mathcal{H}_\varepsilon$ and its constitutive pieces are still operational in higher dimension. The only part of our proof that requires some specific attention is the construction of the linear part $\gamma_\varepsilon(V_\varepsilon)$ which was performed in Section III.3. Indeed, we used a monotonicity argument to show that $\gamma_\varepsilon(V_\varepsilon)$ can be defined in a unique way.

We proceed as in Section III.3. First we show formally how to obtain the expression of the vector γ_0 (III.1.11) via suitable Taylor expansions. Then, we justify these Taylor expansions, and we exhibit a monotonic function that enables to conclude, exactly as in dimension 1.

III.7.1 *The formal expression of the linear part γ_0*

Following the very same heuristics as in section III.3.1, but being careful during the Taylor expansions, we formally end up with the following matrix valued identity:

$$D^2V(0)\left(\frac{1}{(\sqrt{2}\pi)^d}\iint_{\mathbb{R}^{2d}} e^{-Q(y_1,y_2)}(y_1\otimes y_1 + y_1\otimes y_2)dy_1dy_2\right)\gamma_0 \\ = \frac{1}{2}D^3V(0)\left(\frac{1}{(\sqrt{2}\pi)^d}\iint_{\mathbb{R}^{2d}} e^{-Q(y_1,y_2)}y_1\otimes y_1 dy_1dy_2\right). \quad \text{(III.7.1)}$$

The quadratic form Q yields the multivariate centered gaussian distribution associated with the following covariance matrix $\Sigma \in \mathcal{M}_{2d}(\mathbb{R})$:

$$\Sigma = \frac{1}{4}\begin{pmatrix} 3\,\mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & 3\,\mathrm{Id}\end{pmatrix}.$$

The Kronecker product $y_1\otimes y_1$ yields a matrix of moments, and so the relation (III.7.1) can be simplified, similarly to the one dimensional case, so as to obtain:

$$\left(D^2V(0)\left(\frac{3}{4}-\frac{1}{4}\right)\mathrm{Id}\right)\gamma_0 = \frac{1}{2}D^3V(0)\frac{3}{4}\,\mathrm{Id}, \\ \frac{1}{2}D^2V(0)\gamma_0 = \frac{3}{8}D^3V(0)\,\mathrm{Id}\,.$$

The righ hand side is a tensor applied to a matrix yields a vector that can be simplified even further using tensorial properties: $D^3V(0)\,\mathrm{Id} = D(\Delta V)(0)$. Then, provided that $D^2V(0)$ is non degenerate, we obtain the limited expected value of γ_0 in dimension higher than 1, that is a generalization of (III.3.2):

$$\gamma_0(V) = \frac{3}{4}\left(D^2V(0)\right)^{-1}D(\Delta V)(0).$$

In the case where V_0 is given by (III.1.11) through the fixed point procedure, we obtain

$$\gamma_0(V_0) = \frac{1}{2}\left(D^2m(0)\right)^{-1}D(\Delta m)(0). \quad \text{(III.7.2)}$$

III.7.2 *Extension of the proof of Proposition III.9 (section III.3.2)*

We now fix $V\in K$, where K is a ball of $\mathcal{E}_0^\alpha$. The purpose is to prove that there is a unique solution in $\mathbb{R}^d$ of the following problem:

$$\mathcal{J}_\varepsilon(\gamma, V) = 0. \quad \text{(III.7.3)}$$

We insist upon the fact that the variable g belongs to $\mathbb{R}^d$ and the function $\mathcal{J}_\varepsilon(\cdot, V)$ is now defined as a vector field on $\mathbb{R}^d$, $\mathcal{J}_\varepsilon : \mathbb{R}^d\times\mathcal{E}^\alpha \to \mathbb{R}^d$.

As in section III.3.2, we can obtain the following estimate

$$\mathcal{J}_\varepsilon(g, V) = \mathcal{J}_\varepsilon(0, V) + \frac{1}{2}D^2V(0)g + \mathcal{O}(\varepsilon)\,, \quad \text{(III.7.4)}$$

by means of refined Taylor expansions, where $\mathcal{J}_\varepsilon(0,V)$ is bounded *a priori*, independently upon $\varepsilon > 0$ for $V \in K$. To prove the existence of a root γ_ε, we used the mean value theorem in the proof of Proposition III.9. The analogous statement in higher dimension is the Brouwer fixed point theorem. Indeed, (III.7.3) can be recast as follows:

$$g = \left(\mathrm{Id} + \frac{1}{2}D^2V(0)\right)^{-1}(g - \mathcal{J}_\varepsilon(0,V) + \mathcal{O}(\varepsilon)) = \mathcal{T}(g).$$

Thus, we are led to finding a fixed point of a continuous function. As in the one-dimensional case, thanks to the lower bounded $D^2V(0) \geqslant \mu_0 \mathrm{Id}$ encoded in the definition of $\mathcal{E}_0^\alpha$ (III.1.7), we can show easily that there exists R_K such that the ball of radius R_K in $\mathbb{R}^d$ is left invariant by $\mathcal{T}$. Brouwer's fixed point theorem guarantees that there exists a fixed point γ_ε to $\mathcal{T}$, which is also a root of (III.7.3).

For the uniqueness part, we can use strict monotonicity, similarly as in the one dimensional case. This is possible, thanks to (III.3.4) :

$$D_g\mathcal{J}_\varepsilon(g,V) = \frac{1}{2}D^2V(0) + \mathcal{O}(\varepsilon). \tag{III.7.5}$$

We deduce from this strong estimate that the vector field $\mathcal{J}_\varepsilon(\cdot,V)$ is locally uniformly monotonic, in the sense that there exists μ_K such that the following inequality holds true for all ε sufficient small, and every $g_1, g_2 \in B(0,R_K)$:

$$(\mathcal{J}_\varepsilon(g_1,V) - \mathcal{J}_\varepsilon(g_2,V)) \cdot (g_1 - g_2) \geqslant \frac{1}{2}\mu_K \|g_1 - g_2\|^2. \tag{III.7.6}$$

This monotonicity condition is clearly satisfied, as it is equivalent to the following first order condition,

$$\frac{1}{2}\left(D_g\mathcal{J}_\varepsilon(g,V) + D_g\mathcal{J}_\varepsilon(g,V)^\top\right) \geqslant \mu_K Id, \tag{III.7.7}$$

It is immediate that any strictly monotonic vector field admits at most one root. This completes the proof of uniqueness of $\gamma_\varepsilon(V)$.

Quatrième partie

Limite asymptotique du problème de Cauchy pour le modèle infinitésimal dans le régime de petite variance

Nous étudions le comportement asymptotique des solutions du problème de Cauchy associées au modèle de génétique quantitative décrit dans la partie précédente avec l'opérateur infinitésimal. Nous considérons toujours le régime de petite déviation du trait des descendants par rapport à la moyenne des traits des parents. Alors, uniformément et pour tout temps, les solutions restent arbitrairement proches d'une profil Gaussien avec un optimum mobile déterminé et une petite variance. La preuve nécessite des contrôles uniformes en temps et plus précis que ceux fournis dans la partie précédente. Nous réalisons une analyse de stabilité du problème linéarisé pour aboutir à la conclusion souhaitée. Ce travail a été soumis pour publication.

We investigate solutions $f_\varepsilon \in L^1(\mathbb{R}^+ \times \mathbb{R})$ of the following Cauchy problem:

$$\begin{cases} \varepsilon^2 \partial_t f_\varepsilon(t,z) + m(z) f_\varepsilon(t,z) = \mathcal{B}_\varepsilon(f_\varepsilon)(t,z), \quad t > 0,\ z \in \mathbb{R}, \\ f_\varepsilon(0,z) = f_\varepsilon^0(z). \end{cases} \tag{$P_t f_\varepsilon$}$$

where $\mathcal{B}_\varepsilon(f)$ is the following non linear, homogeneous blending operator associated with the infinitesimal model Fisher [1918]; Barton et al. [2017]:

$$\mathcal{B}_\varepsilon(f)(z) := \frac{1}{\varepsilon\sqrt{\pi}} \iint_{\mathbb{R}^2} \exp\left[-\frac{1}{\varepsilon^2}\left(z - \frac{z_1+z_2}{2}\right)^2\right] f(z_1) \frac{f(z_2)}{\int_{\mathbb{R}^d} f(z_2')\, dz_2'} dz_1 dz_2. \tag{IV.1.1}$$

This problem originates from quantitative genetics in the context of evolutionary biology. The variable z denotes a phenotypic trait, f_ε is the distribution of the population and m is the trait dependent mortality rate. As a result, well-fitted individuals are selected into the population, in the long term.

The mixing operator $\mathcal{B}_\varepsilon$ models the inheritance of quantitative traits in the population, under the assumption of a sexual mode of reproduction. As formulated in equation (IV.1.1), it is assumed that offspring traits are distributed normally around the mean of the parental traits $(z_1+z_2)/2$. The variance of diversity generated at birth is constant, here $\varepsilon^2/2$.

In theoretical evolutionary biology, a broad literature uses this model to describe sexual reproduction, see *e.g.* Slatkin [1970]; Roughgarden [1972]; Slatkin and Lande [1976]; Bulmer [1980]; Turelli and Barton [1994]; Tufto [2000]; Barfield et al. [2011]; Huisman and Tufto [2012]; Cotto and Ronce [2014]; Barton et al. [2017]; Turelli [2017]. We are interested in the asymptotic behavior of the trait distribution f_ε as ε^2 vanishes. It is expected that the profile concentrates around some traits under the influence of selection.

The asymptotic description of concentration around some particular trait(s) has been extensively investigated for various linear operators $\mathcal{B}_\varepsilon$ associated with asexual reproduction such as, for instance, the diffusion operator $f_\varepsilon(t,z) + \varepsilon^2 \Delta f_\varepsilon(t,z)$, or the convolution operator $\frac{1}{\varepsilon}K(\frac{z}{\varepsilon}) * f_\varepsilon(t,z)$ where K is a probability kernel with unit variance, see Diekmann et al. [2005]; Perthame [2007]; Barles and Perthame [2007]; Barles et al. [2009]; Lorz et al. [2011] for the earliest investigations, and Méléard and Mirrahimi [2015]; Mirrahimi [2018]; Bouin et al. [2018] for the case of fat-tailed kernel K. In those linear cases, the asymptotic analysis usually leads to a Hamilton-Jacobi equation after performing the Hopf-Cole transform $u_\varepsilon = -\varepsilon \log f_\varepsilon$. Those problems require a careful well-posedness analysis for uniqueness and convergence as $\varepsilon \to 0$ see: Mirrahimi and Roquejoffre [2015]; Calvez and Lam [2018].

Much less is known about the operator $\mathcal{B}_\varepsilon$ defined by (IV.1.1). From a mathematical viewpoint, in the field of probability theory Barton et al. [2017] derived the model from a microscopic framework. In Mirrahimi and Raoul [2013]; Raoul [2017], the authors use a different scaling than the current small variance assumption $\varepsilon^2 \ll 1$, and add a spatial structure to derive the celebrated Kirkpatrick and Barton system Kirkpatrick and Barton [1997].

The Gaussian distributions are known to be left invariant the infinitesimal operator $\mathcal{B}_\varepsilon$, see Turelli and Barton [1994]; Mirrahimi and Raoul [2013]. In Calvez et al. [2019], the authors studied special "stationary" solutions, having the form :

$$\exp\left(\frac{\lambda_\varepsilon t}{\varepsilon^2}\right) F_\varepsilon(z), \quad \text{with } F_\varepsilon(z) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left(-\frac{(z-z_*^s)^2}{2\varepsilon^2} - U_\varepsilon^s(z)\right).$$

In particular, $U_\varepsilon^s = 0(1)$. In this paper we tackle the Cauchy Problem $(P_t f_\varepsilon)$, and we hereby look for solutions that are close to Gaussian distributions at all times :

$$f_\varepsilon(t,z) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left(\frac{\lambda(t)}{\varepsilon^2} - \frac{(z-z_*(t))^2}{2\varepsilon^2} - U_\varepsilon(t,z)\right). \tag{IV.1.2}$$

The unknown λ measures the exponential growth (or decay according to its sign) of the population. The mean of the Gaussian density, z_*, that is also the trait at which the population concentrates when $\varepsilon \to 0$, will be determined by the analysis. The function U_ε measures the deviation from Gaussian profiles induced by the selection function m. Plugging the transformation (IV.1.2) into Problem $(P_t f_\varepsilon)$ yields the following equivalent one:

$$-\varepsilon^2 \partial_t U_\varepsilon(t,z) + \dot\lambda(t) + (z-z_*(t))\dot z_*(t) + m(z) = I_\varepsilon(U_\varepsilon)(t,z) \exp\Big(U_\varepsilon(t,z) - 2U_\varepsilon\left(t,\bar z(t)\right) + U_\varepsilon(t,z_*(t))\Big), \quad (P_t U_\varepsilon)$$

where $\bar z(t)$ is the midpoint between z and $z_*(t)$:

$$\bar z(t) = \frac{z + z_*(t)}{2},$$

and the functional I_ε is defined by

$$I_\varepsilon(U_\varepsilon)(t,z) = \frac{\iint_{\mathbb{R}^2} \exp\left[-\frac{1}{2}y_1 y_2 - \frac{3}{4}\left(y_1^2+y_2^2\right) + 2U_\varepsilon\left(t,\bar z\right) - U_\varepsilon\left(t,\bar z+\varepsilon y_1\right) - U_\varepsilon\left(t,\bar z+\varepsilon y_2\right)\right] dy_1 dy_2}{\sqrt{\pi}\int_{\mathbb{R}} \exp\left[-\frac{1}{2}y^2 + U_\varepsilon(t,z_*) - U_\varepsilon(t,z_* + \varepsilon y)\right] dy}. \tag{IV.1.3}$$

This functional, albeit esoteric at first glance, is the residual shape of the infinitesimal operator (IV.1.1) after suitable transformations. It was first introduced in the formal analysis of Bouin et al. [2019] and in the study of the corresponding stationary problem in Calvez et al. [2019]. Suppose that U_ε is uniformly Lipschitz, then $\mathcal{I}_\varepsilon(U_\varepsilon) \to 1$ when $\varepsilon \to 0$. Thus for small ε, we expect that the Problem $(P_t f_\varepsilon)$ is well approximated by the following one :

$$\dot\lambda(t) + (z-z_*(t))\dot z_*(t) + m(z) = \exp\Big(U_0(t,z) - 2U_0\left(t,\bar z(t)\right) + U_0(t,z_*(t))\Big). \tag{IV.1.4}$$

Interestingly, playing upon the fact that it does not depend on ε, this characterizes the value of $z_*(t)$ and $\lambda(t)$. By differentiating equation (IV.1.4) and evaluating at the point $z = z_*(t)$, then just evaluating equation (IV.1.4) at $z = z_*(t)$, we find the following pair of relationships:

$$\dot z_*(t) + m'(z_*(t)) = 0, \tag{IV.1.5}$$

$$\dot\lambda(t) + m(z_*(t)) = 1. \tag{IV.1.6}$$

Associated to initial data $z_*(0), \lambda(0)$, this prescribes the dynamics of z_* and λ. A more compact way to write the limit problem for $\varepsilon = 0$ is

$$\Gamma(t,z) = \exp\Big(U_0(t,z) - 2U_0\left(t,\bar z(t)\right) + U_0(t,z_*(t))\Big), \quad (P_t U_0)$$

with the notation

$$\Gamma(t,z) := 1 + m(z) - m(\mathrm{z}_*(t)) - m'(\mathrm{z}_*(t))(z - \mathrm{z}_*(t)). \tag{IV.1.7}$$

It verifies

$$\Gamma(t,\mathrm{z}_*(t)) = 1, \quad \partial_z\Gamma(t,\mathrm{z}_*(t)) = 0.$$

An explicit solution of Problem (P_tU_0) exists under the form of an infinite series :

$$V^*(t,z) := \sum_{k\geqslant 0} 2^k \log\left(\Gamma\left(t, \mathrm{z}_*(t) + 2^{-k}(z - \mathrm{z}_*(t))\right)\right). \tag{IV.1.8}$$

The function V^* is a solution of Problem (P_tU_0), but not the only one. There are two degrees of freedom when solving Problem (P_tU_0), since adding any affine function leaves the right hand side unchanged. Therefore, a more general expression of solutions is the following, where the constants p_0 and q_0 are arbitrary :

$$U_0(t,z) = p_0(t) + q_0(t)(z - \mathrm{z}_*(t)) + V^*(t,z). \tag{IV.1.9}$$

To ensure the convergence of the series, one proceed with Taylor expansions around z_*. The compatibility conditions are exactly given by equations (IV.1.5) and (IV.1.6). To prove further properties of V^*, such as Lipschitz regularity, one needs some assumptions of asymptotic decay of the selection function m, when $|z| \to \infty$. The precise computations are detailed in Calvez et al. [2019]. Therefore, after those preliminary considerations, we make the following assumptions on m.

Definition IV.1 (Assumptions on m)**.**
We suppose that the function m is a $\mathcal{C}^4(\mathbb{R})$ function, bounded below. We define the function of time z_* as the following gradient flow:

$$\dot{z}_*(t) = -m'(\mathrm{z}_*(t)) \quad t > 0,$$

associated to an initial data $\mathrm{z}_*(0)$ prescribed. Next, we make the following assumptions :

- ▷ *We suppose that $\mathrm{z}_*(0)$ lies next to a* local *non-degenerate minimum of m, $\widehat{z_0}$ such that*

$$\mathrm{z}_*(t) \xrightarrow[t\to\infty]{} \widehat{z_0}. \tag{IV.1.10}$$

- ▷ *We also require that there exists an uniform positive lower bound on Γ. We suppose that there exists $g_0 > 0$ such that for all $(t,z) \in \mathbb{R}^+ \times \mathbb{R}$:*

$$\Gamma(t,z) > g_0. \tag{IV.1.11}$$

- ▷ *Finally we make decay assumptions on Γ :*

$$\text{for } k = 1,2,3,4: \quad (1 + |z - \mathrm{z}_*|)\frac{\partial_z^k\Gamma(t,z)}{\Gamma(t,z)} \in L^\infty(\mathbb{R}^+ \times \mathbb{R}). \tag{IV.1.12}$$

Our assumptions on m and z_* guarantees the following local convexity property, at least for time t large enough :

$$\exists\, \mu_0 > 0,\ \exists t_0 > 0,\ \text{such that } m''(\mathrm{z}_*(t)) \geqslant \mu_0, \quad \forall t > t_0. \tag{IV.1.13}$$

Remark IV.2. *Based on the formulation of Problem $(P_t U_0)$, the function Γ must be positive. We require a uniform bound in equation (IV.1.11) for technical reasons. It corresponds to a global assumption on the behavior of z_* and m, that further reduces the choice of $z_*(0)$. This condition holds true for globally convex functions m. However we do not want to restrict our analysis to that case, so we suppose more generally that equation (IV.1.11) is verified. Moreover, the decay assumption* (IV.1.12) *holds true if m behaves asymptotically as $|z| \to +\infty$ like a polynomial function.*

We make and additional assumption upon the behavior at infinity of m, that is roughly that it has polynomial tails :

$$\limsup_{z\to\infty} \left|\frac{\Gamma(t,\bar{z})}{\Gamma(t,z)}\right| := a < \frac{1}{2}, \quad \limsup_{z\to\infty} \left|\frac{\partial_z\Gamma(t,\bar{z})}{\partial_z\Gamma(t,z)}\right| < \infty. \tag{IV.1.14}$$

We will see that this assumption is intrinsic to the methodology of proof we used, see section IV.7.

The purpose of this work is to rigorously prove the convergence of the solutions of Problem $(P_t U_\varepsilon)$ towards the solutions of Problem $(P_t U_0)$. Similarly, in Calvez et al. [2019], it was proven that the solutions of the following stationary problem U_ε^s :

$$\lambda_\varepsilon^s + \Gamma(z) = I_\varepsilon(U_\varepsilon^s)(z) \exp\left(U_\varepsilon^s(z) - 2U_\varepsilon^s\left(\frac{z+\widehat{z_0}}{2}\right) + U_\varepsilon^s(\widehat{z_0})\right), \quad z \in \mathbb{R}. \tag{PU_ε stat}$$

are converging when $\varepsilon \to 0$ towards solutions of Problem $(P_t U_0)$, in a functional space involving uniform bounds on three derivatives.

Given the specific shape of U_0, see equation (IV.1.9), we decompose U_ε by separating the affine part from the rest:

$$U_\varepsilon(t,z) = p_\varepsilon(t) + q_\varepsilon(t)(z - z_*(t)) + V_\varepsilon(t,z). \tag{IV.1.15}$$

We also require that at all times $t > 0$,

$$V_\varepsilon(t,z_*) = \partial_z V(t,z_*) = 0.$$

The pair $(q_\varepsilon, V_\varepsilon)$ is the main unknown of this problem, as we will see that the number p_ε can be chosen straightforwardly. One expects that V_ε converges to V^* when $\varepsilon \to 0$. Our analysis will be able determine the limit of q_ε even if it is not given by the problem at $\varepsilon = 0$. Indeed in Problem $(P_t U_0)$, the affine part q_0 can be any constant. Our limit candidate for q_ε is q^*, that we define as the solution of the following differential equation

$$\dot{q}^*(t) = -m''(z_*(t))q^*(t) + \frac{m^{(3)}(z_*(t))}{2} - 2m''(z_*(t))m'(z_*(t)), \tag{IV.1.16}$$

corresponding to an initial $q^*(0)$. Moreover we define p^* as the function which verifies for a given $p^*(0)$,

$$\dot{p}^*(t) = -m'(z_*(t))q^*(t). \tag{IV.1.17}$$

Finally, the function

$$U^*(t,z) := p^*(t) + q^*(t)(z - z_*(t)) + V^*(t,z) \tag{IV.1.18}$$

will be our main candidate for the limit of U_ε when $\varepsilon \to 0$. The problem for V_ε equivalent to Problem $(P_t U_\varepsilon)$ is

$$\Gamma(t,z) - \varepsilon^2\Big(\dot{p}^*(t) + \dot{q}_\varepsilon(t)(z - z_*(t)) + m'(z_*(t))q_\varepsilon(t)\Big) - \varepsilon^2\partial_t V_\varepsilon(t,z) \\ = \mathcal{I}_\varepsilon(q_\varepsilon, V_\varepsilon)(t,z)\exp\Big(V_\varepsilon(t,z) - 2V_\varepsilon(t,\bar{z}(t)) + V_\varepsilon(t,z_*(t))\Big). \quad (P_t V_\varepsilon)$$

We choose to write $\mathcal{I}_\varepsilon(q_\varepsilon, V_\varepsilon)(t,z) := I_\varepsilon(U_\varepsilon)(t,z)$ as a functional of both unknowns because we will study variations in both directions. One can notice that thanks to cancellations the functional $\mathcal{I}_\varepsilon(U_\varepsilon)$ does not depend on p_ε, which explains for the most part why we focus upon $(q_\varepsilon, V_\varepsilon)$. One of the main difficulties to prove the link between Problems $(P_t V_\varepsilon)$ and $(P_t U_0)$ is th at formally, the term with the time derivative V_ε vanishes when $\varepsilon \to 0$. This makes our study belong to the clan of singular limit problems.

Before stating our main result we need to make clear which functional spaces we will be working on. We first define a reference space $\mathcal{E}$, similar to the one introduced in Calvez et al. [2019] for the study of the stationary equation. However, compared to that case we will need more precise controls, which is why we introduce a subspace $\mathcal{F}$ with more stringent conditions.

Definition IV.3 (Functional spaces)**.**
We define the functional space

$$\mathcal{E} = \Big\{ v \in \mathcal{C}^3(\mathbb{R}^+ \times \mathbb{R}) \text{ s.t. } \forall t > 0,\ v(t, z_*(t)) = \partial_z v(t, z_*(t)) = 0 \text{ and } |\partial_z v(t,z)| \in L^\infty(\mathbb{R}^+ \times \mathbb{R}), \\ \Big|\Big(1 + |z - z_*(t)|\Big)\partial_z^2 v(t,z)\Big| \in L^\infty(\mathbb{R}^+ \times \mathbb{R}), \Big|\Big(1 + |z - z_*(t)|\Big)\partial_z^3 v(t,z)\Big| \in L^\infty(\mathbb{R}^+ \times \mathbb{R})\Big\},$$

equipped with the norm

$$\|v\|_{\mathcal{E}} = \max\Bigg(\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} |\partial_z v(t,z)|, \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \Big|\Big(1 + |z - z_*(t)|\Big)\partial_z^2 v(t,z)\Big|, \\ \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \Big|\Big(1 + |z - z_*(t)|\Big)\partial_z^3 v(t,z)\Big|\Bigg).$$

We also define the subspace :

$$\mathcal{F} := \mathcal{E} \cap \left\{ v \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \text{ s.t. } \left| \begin{array}{c} |2v(t,\bar{z}(t)) - v(t,z)| \\ |\partial_z v(t,\bar{z}(t)) - \partial_z v(t,z)| \end{array} \right. \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \right\}$$

and we associate the corresponding norm :

$$\|v\|_{\mathcal{F}} = \max\left(\|v\|_{\mathcal{E}}, \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} |2v(t,\bar{z}(t)) - v(t,z)|, \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} |\partial_z v(t,\bar{z}(t)) - \partial_z v(t,z)| \right).$$

Most of this paper is focused on the pair $(q_\varepsilon, V_\varepsilon) \in \mathbb{R} \times \mathcal{F}$, so we will use the notation $\|(q,V)\| := \max\Big(|q|, \|V\|_{\mathcal{F}} \Big)$. Our main theorem is the following convergence result :

Theorem IV.4 (Convergence result)**.**
There exists K_0 and $\varepsilon_0 > 0$ such that if we make the following assumptions on the initial condition, $\forall \varepsilon \leqslant \varepsilon_0$:

$$\begin{aligned} \|V^*(0,\cdot) - V_\varepsilon(0,\cdot)\|_{\mathcal{F}} \leqslant\ & \varepsilon^2 K_0, \text{ and} \\ |q^*(0) - q_\varepsilon(0)| \leqslant \varepsilon^2 K_0, \quad & |p^*(0) - p_\varepsilon(0)| \leqslant \varepsilon^2 K_0, \end{aligned}$$

then the following convergence hold true :

$$\begin{aligned} \sup_{t>0} \|V_\varepsilon - V^*\|_{\mathcal{F}} &\xrightarrow[\varepsilon\to 0]{} 0, \\ \sup_{t>0} |q_\varepsilon(t) - q^*(t)| \xrightarrow[\varepsilon\to 0]{} 0, \quad \sup_{t>0} |p_\varepsilon(t) - p^*(t)| &\xrightarrow[\varepsilon\to 0]{} 0, \end{aligned}$$

where q^* is defined as the solution of equation (IV.1.16) associated to $q^*(0)$ and p^* as the solution of equation (IV.1.17) associated to $p^*(0)$.

Therefore, as predicted, the limit of U_ε when $\varepsilon \to 0$ is the function $p^*(t) + q^*(t)(z - z_*(t)) + V^*(t,z)$. The Theorem IV.4 yields the stability of the Cauchy Problem $(P_t f_\varepsilon)$ with respect to ε.

In Calvez et al. [2019] a fixed point argument was used to build convergent solutions of the stationary problem when $\varepsilon \to 0$. However, this method can no longer be applied in this case since the derivative in time breaks the structure that made the stationary problem equivalent to a fixed point. The fact that this structure holds at $\varepsilon = 0$ in Problem $(P_t U_0)$ illustrates the main difficulty of this work : the asymptotic problem has a different structure than the initial one. For this reason we replace the fixed point argument by a perturbative analysis method. We introduce the following corrector terms, $\kappa_\varepsilon, W_\varepsilon$, our aim is to bound them uniformly :

$$V_\varepsilon(t,z) = V^*(t,z) + \varepsilon^2 W_\varepsilon(t,z) \tag{IV.1.19}$$

$$q_\varepsilon(t) = q^*(t) + \varepsilon^2 \kappa_\varepsilon(t). \tag{IV.1.20}$$

The scalar q^* , perturbed by $\varepsilon^2\kappa_\varepsilon$, will tune the affine part of the solution. The function W_ε measures the error made when approximating Problem $(P_t U_\varepsilon)$ by Problem $(P_t U_0)$. We choose not to perturb p_ε because we will see in section IV.5.2 that it can be straightforwardly deduced from the analysis.

Coming back to the initial Problem $(P_t f_\varepsilon)$, the order of precision is quite high since we are investigating the error made when approximating f_ε by Gaussian distributions, see (IV.1.2) : W_ε is of order ε^2, while U_ε is of order 1 in ε. The objective of this article is to show that κ_ε and W_ε are uniformly bounded with respect to time and ε. The decomposition of equations (IV.1.19) and (IV.1.20) highlights the role of the functional spaces defined in definition IV.3. While $V_\varepsilon \in \mathcal{E}$, similarly to the stationary case, we ask more for the perturbation $W_\varepsilon \in \mathcal{F}$, that gives an higher order of precision.

IV.2 SOME HEURISTICS AND METHOD OF PROOF

For this section only, we focus on the function U_ε instead of V_ε to get another heuristic argument about the decomposition of equation (IV.1.15) and some elements supporting the Theorem IV.4. We will denote R_ε the perturbation such that we look for solutions of Problem $(P_t U_\varepsilon)$ under the following form:

$$U_\varepsilon(t,z) = U^*(t,z) + \varepsilon^2 R_\varepsilon(t,z).$$

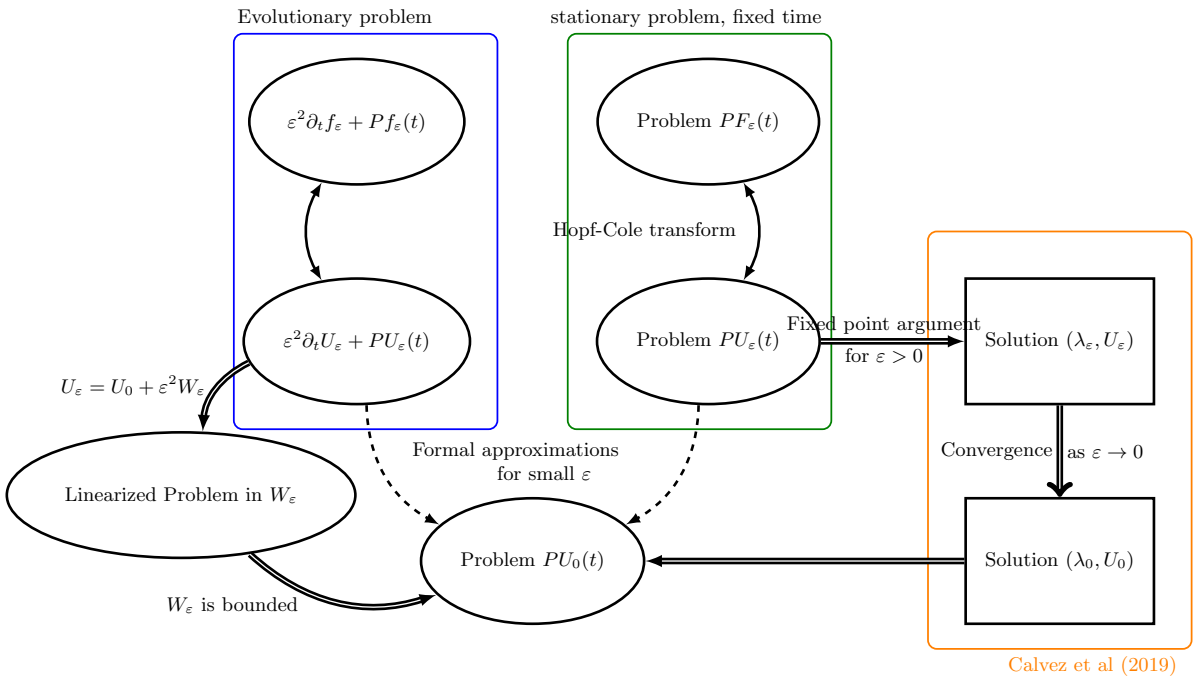


Figure IV.1: Scope of our paper compared to precedent work

The function U^*, defined in equation (IV.1.18) also solves Problem $(P_t U_0)$. Naively plugging this perturbation into Problem $(P_t U_\varepsilon)$ yields the following perturbed equation for R_ε:

$$\Gamma(t,z) - \varepsilon^2\partial_t U^*(t,z) - \varepsilon^4\partial_t R_\varepsilon(t,z) = \mathcal{I}_\varepsilon(U^* + \varepsilon^2 R_\varepsilon)(t,z)\times$$
$$\exp\Big(U^*(t,z) - 2U^*\left(t,\bar{z}(t)\right) + U^*(t,z_*(t))\Big)\exp\Big(\varepsilon^2\Big(R_\varepsilon(t,z) - 2R_\varepsilon\left(t,\bar{z}(t)\right) + R_\varepsilon(t,z_*(t))\Big)\Big).$$

By using Problem $(P_t U_0)$, one gets that R_ε solves the following :

$$\Gamma(t,z) - \varepsilon^2\partial_t U^*(t,z) - \varepsilon^4\partial_t R_\varepsilon(t,z) =$$
$$I_\varepsilon(U^* + \varepsilon^2 R_\varepsilon)(t,z)\Gamma_\varepsilon(t,z)\exp\Big(\varepsilon^2\Big(R_\varepsilon(t,z) - 2R_\varepsilon\left(t,\bar{z}(t)\right) + R_\varepsilon(t,z_*(t))\Big)\Big).$$

To prove the boundedness of R_ε solution to this non linear equation, we shall linearize it and show a stability result on the linearized problem (see Theorem IV.30). We explain here the heuristics about the linearization. We have already said that $\mathcal{I}_\varepsilon$ is expected to converge to 1. Therefore by linearizing the exponential, a natural linearized equation would be :

$$\varepsilon^2\partial_t\widetilde{R_\varepsilon}(t,z)) = \Gamma(t,z)\Big(-\widetilde{R_\varepsilon}(t,z) + 2\widetilde{R_\varepsilon}\left(t,\bar{z}(t)\right) - \widetilde{R_\varepsilon}(t,z_*(t))\Big), \tag{IV.2.1}$$

For clarity we denote T the linear operator :

$$T(R)(t,z) := \Gamma(t,z)\Big(2R(t,\bar{z}(t)) - R(t,z) + R(t,z_*(t))\Big).$$

We know precisely what are the eigen-elements of this linear operator. The eigenvalue 0 has two eigen vectors, given by the affine functions. More generally one can get every eigen-value by differentiating iteratively the operator and evaluating at $z = z_*$. This corresponds to the following table :

eigenvalues :	0	0	$-\frac{1}{2}$	$-\frac{3}{4}$	...
Dual eigenvector :	$\delta_{z_*(t)}$	$\delta'_{z_*(t)}$	$\delta^{(2)}_{z_*(t)}$	$\delta^{(3)}_{z_*(t)}$	...

This explains why R_ε should be decomposed between affine parts and the rest, and as a consequence, also the solution U_ε we are investigating. The numbers p_ε and q_ε of the decomposition in (IV.1.15) correspond to the projection of U_ε upon the first eigen-space, associated to the eigenvalue 0. On the other hand the rest is expected to be bounded since its corresponding eigenvalues are negative.

Beyond the heuristics about the stability, this linear analysis also illustrates the discrepancy between W_ε and q_ε in Theorem IV.4. While W_ε can be expected to be bounded very quickly in time, since V^* depends directly on z_* (fast dynamics), this is not true for q^* that solves a differential equation (slow dynamics), where z_* is a parameter:

$$\dot{q}^*(t) = -m''(z_*(t))q^*(t) + \frac{m^{(3)}(z_*(t))}{2} - 2m''(z_*(t))m'(z_*(t)).$$

We can infer that the eigenvalue 0 driving the dynamics of q^* is in fact equivalent to $-\varepsilon^2 m''(z_*(t))$, which explains that we can read q^* at this order.

The technique we will use in the following sections to bound W_ε in $\mathcal{F}$ will seem more natural in the light of this formal analysis. The first step will be to work around z_*, the base point of the eigen-elements see the table above. We will use uniform bounds up to the second derivative to estimate W_ε, in the Theorem IV.30 about stability.

Therefore our main objective will be to linearize the following equation on $W_\varepsilon, \kappa_\varepsilon$:

$$\begin{aligned}\Gamma(t,z) - \varepsilon^2\Big(\dot{p}_\varepsilon(t) + \dot{q}^*(t)(z-z_*) + m'(z_*)q^*(t) + \partial_t V^*(t,z)\Big)\\ - \varepsilon^4\Big(\dot{\kappa}_\varepsilon(t)(z-z_*) + m'(z_*)\kappa_\varepsilon(t) + \partial_t W_\varepsilon(t,z)\Big) = \Gamma(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)\times\\ \exp\left(\varepsilon^2\Big(W_\varepsilon(t,z) - 2W_\varepsilon\left(t,\bar{z}(t)\right) + W_\varepsilon(z_*(t))\Big)\right). \quad \text{(IV.2.2)}\end{aligned}$$

We will then deduce the boundedness of these unknowns by working on the linearized equations. We will see that we will need to investigate different scales (in ε) to capture the different behavior of each contribution.

A key element of our analysis will then be to be able to propagate the boundedness shown on the linear problem to the non linear problem. This will be possible because of the careful estimations of the remaining terms. In particular, we will write $\|(g,W)\|\, O(\varepsilon)$ to illustrate when the constants of $O(\varepsilon)$ depends linearly on the bound on (g,W). We also define a refinement of the classical notation $O(\varepsilon)$:

Definition IV.5 ($O^*(\varepsilon^\alpha)$)**.**
For $\alpha \in \mathbb{N}$, we say that a function $g(\varepsilon,t,z)$ is such that $g(\varepsilon,t,z) = O^*(\varepsilon^\alpha)$ if there exists ε^* such that for all $\varepsilon \leqslant \varepsilon^*$ it verifies :

$$|g(\varepsilon,t,z)| \leqslant C^*\varepsilon^\alpha,$$

and the constant C^* depends only on the pair (q^*, V^*).

Our intent is to make the dependency of the constants clear when we linearize. This will prove to be a crucial point when we will go back to the non-linear equation (IV.2.2). We will see that all the

terms that do not have a sufficient order in ε to be negligible will be $O^*(1)$, and therefore uniformly bounded independently of $(\kappa_\varepsilon, W_\varepsilon)$. A key point of our analysis is to segregate those terms when doing the linearization.

The rest of the paper is organized as follows :

- ▷ First we prove some properties upon the reference pair (q^*, V^*) around which we linearize. In particular Proposition IV.8, states that there is a ball of $\mathcal{E}^*$ that does not depend of time to which q^* and V^* belong. Next we give precise estimate of the integral $\mathcal{I}_\varepsilon(q^*, V^*)$ and its derivatives.
- ▷ A key part of our perturbative analysis is to be able to linearize I_ε, which we do in section IV.4 thanks to cautious and rigorous estimates upon the directional derivatives.
- ▷ We derive an equation on κ_ε in section IV.5.1, and later a linear approximation for W_ε, and more importantly all of its derivatives in section IV.6.
- ▷ We are able to show the boundedness of the solutions of the linear problem in the space $\mathcal{F}$, see section IV.7, mainly through maximum principles and a dyadic division of the space to take into account the non local behavior of the infinitesimal operator. This is the content of the Theorem IV.30.
- ▷ Finally, we tackle the proof of Theorem IV.4 in the final section IV.8. Our main ingredient is a stability argument to propagate uniform boundedness along time.

IV.3 Preliminary results : estimates of I_ε^* and V^*

IV.3.1 *Control of* (q^*, V^*)

Before tackling the main difficulties of this article, we first state some controls on the function V^*, solution of Problem $(P_t U_0)$. Most of them use the explicit expression of equation (IV.1.8) and were proved in Calvez et al. [2019]. To be able to measure this function we introduce another functional space.

Definition IV.6 (Subspace of V^*)**.**
We define $\mathcal{E}^*$ as the following subspace of $\mathcal{E}$:

$$\mathcal{E}^* = \mathcal{E} \cap \left\{ g \in \mathcal{C}^4(\mathbb{R}^+ \times \mathbb{R}) \text{ s.t. } \left(1 + |z - z_*(t)|\right) \left|\partial_z^4 g(t,z)\right| \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \right\},$$

We equip it with the norm $\|\cdot\|_*$:

$$\|g\|_* = \max\left(\|g\|_{\mathcal{E}}, \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \left(1 + |z - z_*(t)|\right) \left|\partial_z^4 g(t,z)\right| \right).$$

The intention with the successive definitions of the functional spaces is to be able to measure each term of the decomposition :

$$\underbrace{V_\varepsilon}_{\mathcal{E}} = \underbrace{V^*}_{\mathcal{E}^*} + \varepsilon^2 \underbrace{W_\varepsilon}_{\mathcal{F}}. \tag{IV.3.1}$$

The fact that $V^* \in \mathcal{E}^*$ is part of the claim of the following lemma.

Lemma IV.7 (Properties of V^*).
The function V^* belongs to the space $\mathcal{E}^*$. Moreover,

$$\partial_z^2 V^*(t, z_*(t)) = 2m''(z_*(t)), \qquad \partial_z^3 V^*(t, z_*(t)) = \frac{4}{3} m^{(3)}(z_*(t)). \tag{IV.3.2}$$

Proof.
Precise estimates of the summation operator that defines V^* in equation (IV.1.8) are studied in Calvez et al. [2019]. They can be applied there thanks to the decay assumptions about Γ : (IV.1.12). The only difference here is that we ask an uniform bound for the fourth derivative as well. The proof for this bound relies also solely upon the assumption made in equation (IV.1.12), for the fourth derivative. This shows that $V^* \in \mathcal{E}^*$. A rapid explicit computation based on equation (IV.1.8) proves equation (IV.3.2). □

A consequence of (IV.7) is that since $m''(z_*(t)) > 0$ for $t > t_0$, thanks to equation (IV.1.13), V^* has a local convex shape around $z_*(t)$. Information upon the space that V^* belongs to is not enough. We will need to bound (q^*, V^*) independently of time. This is the content of the following result:

Proposition IV.8 (Uniform bound on (q^*, V^*)).
There exists a constant K^* such that

$$\max\left(\|V^*\|_*, \|q^*\|_{L^\infty(\mathbb{R}_+)}, \|\partial_t V^*\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})}, \|\partial_t V^*\|_* \right) \leqslant K^*.$$

,

Proof of Proposition IV.8.
For the estimates upon V^* and $\partial_t V^*$, it is a direct consequence of the definition of $\mathcal{E}^*$ and the explicit formula (IV.1.8). The technique to bound the sums is to distinguish between the small and large indices, it was detailed in Calvez et al. [2019].

For q^*, one must look to (IV.1.16): The boundedness of q^* is a straightforward consequence of the convexity of m at $z_*(t)$ for large times, see (IV.1.13). □

Therefore, we have established that there exists a ball B^* of $\mathbb{R} \times \mathcal{E}^*$ such that $(q^*, V^*) \in B^*$.

IV.3.2 *Estimates of I_ε^* and its derivatives*

We next define a notational shortcut for the functional I_ε introduced in equation (IV.1.3), when it is evaluated at the reference (q^*, V^*) :

$$I_\varepsilon^* := \mathcal{I}_\varepsilon(q^*, V^*). \tag{IV.3.3}$$

This section is devoted to get precise estimates of this function of time and space. This will be crucial when for the linearization of $I_\varepsilon(q^* + \varepsilon^2 \kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)$ as can be seen on the full equation (IV.2.2).

Proposition IV.9 (Estimation of I_ε^*).

$$I_\varepsilon^*(t, z) = 1 + O^*(\varepsilon^2),$$

where the constants of $O^*(\varepsilon^2)$ depend only on K^*, introduced in Proposition IV.8, as defined by definition IV.5.

The proof consists in exact Taylor expansion in ε. Very similar expansions were performed in [Calvez et al. 2019, Lemma 3.1], we adapt the method of proof here, since it will be used extensively throughout this article.

Proof of Proposition IV.9.
We recall that by Proposition IV.8, $\max(|q^*|, \|V^*\|_*) \leqslant K^*$, and, by definition

$$I_\varepsilon^*(t,z) = \frac{\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} \exp\Big(-\varepsilon q^*(t)(y_1+y_2) + 2V^*(t,\overline{z}) - V^*(t,\overline{z}+\varepsilon y_1) - V^*(t,\overline{z}+\varepsilon y_2)\Big) dy_1 dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2} \exp\Big(-\varepsilon q^*(t) y + V^*(t,z_*) - V^*(t,z_*+\varepsilon y)\Big) dy} := \frac{N(t,z)}{D(t)}.$$

We denoted Q the quadratic form appearing after the rescaling of the infinitesimal operator in equation (IV.1.3) :

$$Q(y_1,y_2) := \frac{1}{2}y_1y_2 + \frac{3}{4}(y_1^2+y_2^2).$$

We start the estimates with the more complicated term, the numerator N. With an exact Taylor expansion inside the exponential, there exists generic $\widetilde{y_i}$ which verifies $|\widetilde{y_i}| \leqslant |y_i|$, such that

$$N(t,z) = \iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} \exp\Big[-\varepsilon q^*(t)(y_1+y_2) - \varepsilon(y_1+y_2)\partial_z V^*(t,\overline{z}) - \frac{\varepsilon^2}{2}\Big(y_1^2\partial_z^2 V^*(t,\overline{z}+\varepsilon\widetilde{y_1}) + y_2^2\partial_z^2 V^*(t,\overline{z}+\varepsilon\widetilde{y_2})\Big)\Big] dy_1 dy_2$$

Moreover we can write for some $\theta = \theta(y_1,y_2) \in (0,1)$,

$$\exp(-\varepsilon P) = 1 - \varepsilon P + \frac{\varepsilon^2 P^2}{2}\exp(-\theta\varepsilon P),$$
$$P = (y_1+y_2)\Big(q^*(t) + \partial_z V^*(t,\overline{z})\Big) + \frac{\varepsilon}{2}\Big(y_1^2\partial_z^2 V^*(t,\overline{z}+\varepsilon\widetilde{y_1}) + y_2^2\partial_z^2 V^*(t,\overline{z}+\varepsilon\widetilde{y_2})\Big).$$

such that

$$|P| \leqslant K^*\left(|y_1| + |y_2| + \frac{\varepsilon(y_1^2+y_2^2)}{2}\right). \tag{IV.3.4}$$

Combining the expansions, we find:

$$\begin{aligned} N(t,z) &= \frac{1}{\sqrt{\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}\Big(1 - \varepsilon P + \frac{\varepsilon^2 P^2}{2}\exp(-\theta\varepsilon P)\Big) dy_1 dy_2, \\ &= 1 - \varepsilon\frac{1}{\sqrt{\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} P dy_1 dy_2 + \frac{\varepsilon^2}{2\sqrt{\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} P^2 \exp(-\theta\varepsilon P) dy_1 dy_2 \end{aligned} \tag{IV.3.5}$$

The key part is the cancellation of the terms $O(\varepsilon)$ due to the symmetry of Q :

$$\frac{1}{\sqrt{\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(y_1+y_2) dy_1 dy_2 = 0.$$

Therefore :

$$\frac{\varepsilon}{\sqrt{\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}P dy_1 dy_2 = \frac{\varepsilon^2}{2}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}\Big(y_1^2\partial_z^2 V^*(t,\overline{z}+\varepsilon\widetilde{y_1}) + y_2^2\partial_z^2 V^*(t,\overline{z}) + \varepsilon\widetilde{y_2}\Big)dy_1 dy_2.$$

And we get the estimate

$$\begin{aligned}\left|\frac{\varepsilon}{\sqrt{\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}P dy_1 dy_2\right| &\leqslant \frac{\varepsilon^2}{2}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}\Big(y_1^2+y_2^2\Big)K^* dy_1 dy_2 \\ &\leqslant O^*(\varepsilon^2).\end{aligned}$$

Thanks to equation (IV.3.4) it is easy to verify that the last term of equation (IV.3.5) behaves similarly:

$$\frac{\varepsilon^2}{2\sqrt{\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}P^2\exp(-\theta\varepsilon P)dy_1 dy_2 = O^*(\varepsilon^2).$$

Indeed, it states that the term P is at most quadratic with respect to y_i so $Q+\theta\varepsilon P$ is uniformly bounded below by a positive quadratic form for ε small enough. This shows that

$$N(t,z) = 1 + O^*(\varepsilon^2).$$

The denominator is easier, with the same arguments, using the Gaussian density :

$$D(t) = 1 + O^*(\varepsilon^2).$$

Combining the estimates of N and D, we get the desired result. □

There exists a link between q^* and $\partial_z I_\varepsilon^*(t,\mathrm{z}_*)$, which is in fact the motivation behind the choice of q^*.

Proposition IV.10 (Link between q^* and $\partial_z I_\varepsilon^*(t,\mathrm{z}_*)$)**.**

$$\partial_z I_\varepsilon^*(t,\mathrm{z}_*(t)) = \varepsilon^2\left(m''(\mathrm{z}_*(t))q^*(t) - \frac{m^{(3)}(\mathrm{z}_*(t))}{2}\right) + O^*(\varepsilon^4),$$

where the constants of $O^*(\varepsilon^4)$ only depend on K^*.

The proof of this result was the content of [Calvez et al. 2019, Lemma 3.1] and only requires that (q^*, V^*) is uniformly bounded, as stated in Proposition IV.8. Its proof follows the same procedure of exact Taylor expansions as in the one of Proposition IV.9.

It will be useful to dispose of estimates of $\partial_z I_\varepsilon^*$ not only at the point z_*. They are less precise, as stated in the following proposition:

Proposition IV.11 (Estimates of the decay of the derivatives of I_ε^*)**.**
There exists a constant ε_* that depend only on K^* such that for all $\varepsilon \leqslant \varepsilon*$, for $j = 1, 2, 3$ or 4 :

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}}\Big(1+|z-\mathrm{z}_*(t)|\Big)\left|\partial_z^{(j)} I_\varepsilon^*(t,z)\right| = O^*(\varepsilon^2).$$

To shortcut notations, we introduce the following difference operator that appear in the integral I_ε see equation (IV.1.3) :

$$\mathcal{D}_\varepsilon(V)(Y,t,z) := V(t,\overline{z}) - \frac{1}{2}V(t,\overline{z}+\varepsilon y_1) - \frac{1}{2}V(t,\overline{z}+\varepsilon y_2), \quad Y=(y_1,y_2), \tag{IV.3.6}$$

$$\mathcal{D}^*_\varepsilon(V)(y,t) := V(t,z_*) - V(t,z_*+\varepsilon y). \tag{IV.3.7}$$

We will use the following technical lemma giving an estimate of the weight function against the derivatives of a given function.

Lemma IV.12 (Influence of the weight function.)**.**
There exists a constant C such that for each ball B of $\mathcal{E}^*$ or $\mathcal{F}$, there exists ε_B such that for every $W \in B$, for every $y \in \mathbb{R}$ and $\varepsilon \leqslant \varepsilon_B$, for $j = 1, 2$ or 3:

$$\Big(1+|z-z_*(t)|\Big)\left|\partial_z^{(j)} W(t,\overline{z}(t)+\varepsilon y)\right| \leqslant C\,\|W\| \qquad \text{if } |y| \leqslant |z-z_*(t)|,$$
$$\leqslant (1+|y|)\,\|W\|. \qquad \text{otherwise,}$$

with $\|W\| = \|W\|_*$ or $\|W\|_{\mathcal{F}}$ depending on the case.

The Proposition IV.11 is a prototypical result. It will be followed by a series of similar statements. Therefore, we propose two different proofs. In the first one, we write exact Taylor expansions. However the formalism is heavy, which is why we propose next a formal argument, where the Taylor expansions are written without exact rests.

In the rest of this paper more complicated estimates will be proved, in the spirit of Proposition IV.11, see Proposition IV.13 and Lemma IV.20 for instance. The notations and formulas will be very heavy, so we shall only write the "formal" parts of the argument. However it can all be made rigorous, as below.

Proof of Proposition IV.11.
First, write the expression for the derivative, using our notation $\mathcal{D}_\varepsilon$ introduced in (IV.3.6) :

$$\partial_z I^*_\varepsilon(t,z) = \frac{\displaystyle\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} \exp\Big(-\varepsilon q^*(y_1+y_2) + 2\mathcal{D}_\varepsilon(V^*)(Y,t,\overline{z})\Big)\mathcal{D}_\varepsilon(\partial_z V^*)(Y,t,\overline{z})dy_1dy_2}{\sqrt{\pi}\displaystyle\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon q^* y + \mathcal{D}^*_\varepsilon(V^*)(y,t)\Big)dy}, \tag{IV.3.8}$$

$$:= \frac{N(t,z)}{D(t)}.$$

We only focus on the numerator. The denominator D can be handled similarly as in the proof of Proposition IV.9, where we show that it is essentially $1+O^*(\varepsilon^2)$. We perform two Taylor expansions in the numerator N, namely:

$$\begin{cases} 2\mathcal{D}_\varepsilon(V^*)(Y,t,\overline{z}) = -\varepsilon(y_1+y_2)\partial_z V^*(t,\overline{z}) - \dfrac{\varepsilon^2}{2}\Big(y_1^2\partial_z^2 V^*(t,\overline{z}+\varepsilon\widetilde{y_1}) + y_2^2\partial_z^2 V^*(t,\overline{z}+\varepsilon\widetilde{y_2})\Big), \\ \mathcal{D}_\varepsilon(\partial_z V^*)(Y,t,\overline{z}) = -\dfrac{\varepsilon(y_1+y_2)}{2}\partial_z^2 V^*(t,\overline{z}) - \dfrac{\varepsilon^2}{4}\Big(y_1^2\partial_z^3 V^*(t,\overline{z}+\varepsilon\widetilde{y_1}) + y_2^2\partial_z^3 V^*(t,\overline{z}+\varepsilon\widetilde{y_2})\Big), \end{cases} \tag{IV.3.9}$$

where $\widetilde{y_i}$ denote some generic number such that $|\widetilde{y_i}| \leqslant |y_i|$ for $i = 1, 2$. Moreover, we can write

$$\exp(-\varepsilon P) = 1 - \varepsilon P \exp(-\theta \varepsilon P) \quad \text{with}$$
$$P := (y_1 + y_2)\Big(\partial_z V^*(t, \bar{z}) + q^*\Big) + \frac{1}{2}\Big(y_1^2 \partial_z^2 V^*(t, \bar{z} + \varepsilon \widetilde{y_1}) + y_2^2 \partial_z^2 V^*(t, \bar{z} + \varepsilon \widetilde{y_2})\Big) \quad \text{(IV.3.10)}$$

for some $\theta = \theta(y_1, y_2) \in (0, 1)$. Combining the expansions, we find:

$$\Big(1 + |z - z_*(t)|\Big)\partial_z I_\varepsilon^*(t, z) = \frac{1 + |z - z_*(t)|}{\sqrt{\pi}} \iint_{\mathbb{R}^2} e^{-Q(y_1, y_2)} \left(1 - \varepsilon P \exp(-\theta \varepsilon P)\right)$$
$$\times \left(-\frac{\varepsilon(y_1 + y_2)}{2} \partial_z^2 V^*(t, \bar{z}) - \frac{\varepsilon^2}{4}(y_1^2 \partial_z^3 V^*(t, \bar{z} + \varepsilon \widetilde{y_1}) + y_2^2 \partial_z^3 V^*(t, \bar{z} + \varepsilon \widetilde{y_2}))\right) dy_1 dy_2 \,.$$

The crucial point is the cancellation of the $O(\varepsilon)$ contribution due to the symmetry of Q:

$$\iint_{\mathbb{R}^2} e^{-Q(y_1, y_2)}(y_1 + y_2) dy_1 dy_2 = 0 \,. \quad \text{(IV.3.11)}$$

So, it remains

$$\Big(1 + |z - z_*(t)|\Big)N(t, z) =$$
$$-\varepsilon^2 \frac{1 + |z - z_*(t)|}{4\sqrt{\pi}} \iint_{\mathbb{R}^2} e^{-Q(y_1, y_2)} \left[y_1^2 \partial_z^3 V^*(t, \bar{z} + \varepsilon \widetilde{y_1}) + y_2^2 \partial_z^3 V^*(t, \bar{z} + \varepsilon \widetilde{y_2}))\right] dy_1 dy_2$$
$$+ \varepsilon^2 \frac{1 + |z - z_*(t)|}{2\sqrt{\pi}} \iint_{\mathbb{R}^2} e^{-Q(y_1, y_2)} P \exp(-\theta \varepsilon P)(y_1 + y_2) \partial_z^2 V^*(t, \bar{z}) dy_1 dy_2 +$$
$$\varepsilon^3 \frac{1 + |z - z_*(t)|}{4\sqrt{\pi}} \iint_{\mathbb{R}^2} e^{-Q(y_1, y_2)} P \exp(-\theta \varepsilon P) \left(y_1^2 \partial_z^3 V^*(t, \bar{z} + \varepsilon \widetilde{y_1}) + y_2^2 \partial_z^3 V^*(t, \bar{z} + \varepsilon \widetilde{y_2})\right) dy_1 dy_2.$$

If we forget about the weight in front of each term, clearly the last two contributions are uniform $\mathcal{O}^*(\varepsilon)$ since V^* and q^* are uniformly bounded by K^*, see Proposition IV.8 and $\varepsilon \leqslant \varepsilon_*$ small enough. The term P is at most quadratic with respect to y_i, see (IV.3.10), so $Q + \theta \varepsilon P$ is uniformly bounded below by a positive quadratic form for ε small enough.

The difficulty is to add the weight to those estimates. To do so, we use the Lemma IV.12, for every integral terms appearing in the previous formula, since every time appears a term of the form :

$$\Big(1 + |z - z_*(t)|\Big)\partial_z^{(j)} V^*(t, \bar{z} + \varepsilon \widetilde{y_i}).$$

Since every $\widetilde{y_i}$ verifies $|\widetilde{y_i}| \leqslant |y_i|$ in each of the integrals, the bounds given by Lemma IV.12 ensure that those integrals are bounded by moments of the bivariate Gaussian defined by Q, as if there were no weight function. This concludes the proof of the first estimate Proposition IV.11.

Bounding the quantity $\Big(1 + |z - z_*(t)|\Big)\Big|\partial_z^{(j)} I_\varepsilon^*(t, z)\Big|$ for $j = 2, 3$ follows the same steps, as it can be seen on the explicit formulas :

$$\partial_z^2 I_\varepsilon^*(t, z) =$$
$$\frac{\displaystyle\iint_{\mathbb{R}^2} \exp\Big(-Q(y_1, y_2) - \varepsilon g(y_1 + y_2) + 2\mathcal{D}_\varepsilon(V^*)(Y, t, z)\Big)\Big[\mathcal{D}_\varepsilon(\partial_z V^*)^2 + \frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V^*)\Big](Y, t, z) dy_1 y_2}{\sqrt{\pi} \displaystyle\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2} \exp\Big(-\varepsilon q^* y + \mathcal{D}_\varepsilon^*(V^*)(y, t)\Big) dy}, \quad \text{(IV.3.12)}$$

$$\partial_z^3 I_\varepsilon^*(t,z) = \iint_{\mathbb{R}^2} \exp\Big(- Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V^*)(Y,t,z)\Big)\times \frac{\Big[\mathcal{D}_\varepsilon(\partial_z V^*)^3 + \frac{3}{2}\mathcal{D}_\varepsilon(\partial_z V^*)\mathcal{D}_\varepsilon(\partial_z^2 V^*) + \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V^*)\Big](Y,t,z)}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon q^* y + \mathcal{D}_\varepsilon^*(V^*)(y,t)\Big)dy} dy_1 y_2. \quad \text{(IV.3.13)}$$

□

We now propose a formal argument, much simpler to read.

Formal proof of Proposition IV.11.
We use the same notations as previously, see equation (IV.3.8), and again focus on the numerator N. Formally,

$$N(t,z) = \iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}\exp\Big[-\varepsilon(y_1+y_2)\Big(q^* + \partial_z V^*(t,\bar z)\Big) - \frac{\varepsilon^2}{2}(y_1^2+y_2^2)\partial_z^2 V^*(t,\bar z) + O^*(\varepsilon^2)\Big] \times\Big[-\varepsilon(y_1+y_2)\partial_z^2 V^*(t,\bar z) - \frac{\varepsilon^2(y_1^2+y_2^2)}{2}\partial_z^3 V^*(t,\bar z) + O^*(\varepsilon^2)\Big] dy_1 dy_2.$$

This can be slightly simplified studying the orders in ε :

$$N(t,z) = -\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}\times \Big[1-\varepsilon(y_1+y_2)\Big(q^* + \partial_z V^*(t,\bar z)\Big)\Big]\Big[\varepsilon(y_1+y_2)\partial_z^2 V^*(t,\bar z) + \frac{\varepsilon^2(y_1^2+y_2^2)}{2}\partial_z^3 V^*(t,\bar z)\Big] dy_1 dy_2 + O^*(\varepsilon^2). \quad \text{(IV.3.14)}$$

This can be rewritten

$$N(t,z) = \varepsilon I_1 + \varepsilon^2 I_2 + O^*(\varepsilon^3).$$

By symmetry :

$$I_1 := -\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}\Big[\varepsilon(y_1+y_2)\partial_z^2 V^*(t,\bar z)\Big] dy_1 dy_2 = 0.$$

Thanks uniform bounds on (q^*, V^*), :

$$I_2 = O^*(1). \quad \text{(IV.3.15)}$$

To conclude the proof, we must look at $\Big(1+|z-z_*(t)|\Big)N(t,z)$. To keep the estimate of equation (IV.3.15) we use that :

$$\Big(1+|z-z_*(t)|\Big)\max\Big(\partial_z^2 V^*(t,\bar z(t)), \partial_z^3 V^*(t,\bar z(t))\Big) = O^*(1). \quad \text{(IV.3.16)}$$

This is the alternative, in this special case, of Lemma IV.12. The proof of this inequality relies only on the following property of the weight function :

$$\frac{1+|z-z_*(t)|}{1+|\bar z(t)|} \leqslant 1 + \frac{|z+z_*(t)|}{2+|z+z_*(t)|} \leqslant 2.$$

As a consequence, we recover the bound of equation (IV.3.16) :

$$\Big(1+|z-z_*(t)|\Big)\,|\partial_z^2 V^*(t,\bar z(t))| \leqslant 2\Big(1+|\bar z(t)|\Big)\,|\partial_z^2 V^*(t,\bar z(t))| \leqslant O^*(1),$$

by the Proposition IV.8. Thanks to equation (IV.3.16), we have in fact ,

$$\Big(1+|z-z_*(t)|\Big)I_2 = O^*(1)$$

which proves that

$$\Big(1+|z-z_*(t)|\Big)N(t,z) = O^*(\varepsilon^2).$$

We recover the estimate of Proposition IV.11. □

We know detail the proof of the Lemma IV.12.

Proof of Lemma IV.12.
If $|z-z_*| < 1$, then $1+|z-z_*| \leqslant 2$ and the result is immediate by the definitions IV.3 and IV.6 of the adequate functional spaces. Therefore, one can suppose that $|z-z_*| > 1$. We first look at the regime $|y| < |z-z_*|$. Then, by definition of the norms,

$$\begin{aligned}\Big(1+|z-z_*|\Big)\left|\partial_z^{(j)} W(t,\bar z+\varepsilon y)\right| &\leqslant 2\frac{|z-z_*|}{|\bar z+\varepsilon y-z_*|}\Big(|\bar z+\varepsilon y-z_*|\Big)\left|\partial_z^{(j)} W(t,\bar z+\varepsilon y)\right| \\ &\leqslant 2\frac{|z-z_*|}{|\bar z+\varepsilon y-z_*|}\,\|W\|\,. \end{aligned} \tag{IV.3.17}$$

To bound the last quotient, we use the following inequality, that holds true because we are in the regime $|y| \leqslant |z-z_*|$:

$$|\bar z+\varepsilon Y-z_*| \geqslant -|\varepsilon y|+|\bar z-z_*| \geqslant \frac{1}{2}|z-z_*| - \varepsilon|z-z_*|\,.$$

This yields

$$2\frac{|z-z_*|}{|\bar z+\varepsilon y-z_*|} \leqslant \frac{2}{\frac{1}{2}-\varepsilon}. \tag{IV.3.18}$$

Bridging together equations (IV.3.17) and (IV.3.18), one gets the Lemma IV.12 in the regime $|y| \leqslant |z-z_*|$; on the condition that $\varepsilon < \frac{1}{2}$.

On the contrary, when $|z-z_*| \leqslant |y|$, we have immediately:

$$\Big(1+|z-z_*|\Big)\left|\partial_z^{(j)} W(t,\bar z+\varepsilon y)\right| \leqslant (1+|y|)\,\|W\|\,.$$

□

IV.4 LINEARIZATION OF I_ε AND ITS DERIVATIVES

The first step to obtain a linearized equation on W_ε is to study the non linear terms of equation (IV.2.2). A key point is the study of the functional $\mathcal{I}_\varepsilon$ defined in equation (IV.1.3), which plays a major role in our study. We will show that it converges uniformly to 1 and that its derivatives are uniformly small, with some decay, similarly to the function I_ε^*.

IV.4.1 *Linearization of I_ε*

We first bound uniformly all the terms that appear during the linearization of $\mathcal{I}_\varepsilon$. One starts by measuring the first order directional derivatives.

Proposition IV.13 (Bounds on the directional derivatives of $\mathcal{I}_\varepsilon$)**.**
For any ball B of $\mathbb{R}\times\mathcal{E}$, there exist a constant ε_B that depends only on B such that for all $\varepsilon\leqslant\varepsilon_B$ we have for all $(g,W)\in B$, and $H\in\mathcal{E}$:

$$\sup_{(t,z)\in\mathbb{R}^+\times\mathbb{R}}\left|\partial_g\mathcal{I}_\varepsilon(g,W)(t,z)\right|\leqslant\|(g,W)\|\,O(\varepsilon^2), \tag{IV.4.1}$$

$$\sup_{(t,z)\in\mathbb{R}^+\times\mathbb{R}}\left|\partial_V\mathcal{I}_\varepsilon(g,W)\cdot H(t,z)\right|\leqslant\|(g,W)\|\,\|H\|_{\mathcal{E}}\,O(\varepsilon^2). \tag{IV.4.2}$$

Proof of Proposition IV.13.
As in the estimates of I_ε^* and its derivatives in the previous section, the argument to obtain the result will be to perform exact Taylor expansion. As before we will not pay attentions to the exact rests that can be handled exactly as before, and we refer to the proof of Propositions IV.9 and IV.11 to see the details. However our computations will make clear the order ε^2 of equations (IV.4.1) and (IV.4.2). First, thanks to the derivation with respect to g an order of ε is gained:

$$\begin{aligned}\partial_g\mathcal{I}_\varepsilon(g,W)(t,z)=&\\ -\varepsilon\Bigg(&\frac{\iint_{\mathbb{R}^2}\exp\Big[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big](y_1+y_2)dy_1dy_2}{\sqrt{\pi}\int_{\mathbb{R}}e^{-\frac12|y|^2}\exp\Big(-\varepsilon gy+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy}\\ &-\mathcal{I}_\varepsilon(g,W)(t,z)\frac{\int_{\mathbb{R}}e^{-\frac12|y|^2}\exp\Big(-\varepsilon gy+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)y\,dy}{\sqrt{\pi}\int_{\mathbb{R}}e^{-\frac12|y|^2}\exp\Big(-\varepsilon gy+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy}\Bigg).\end{aligned} \tag{IV.4.3}$$

The common denominator is bounded :

$$\int_{\mathbb{R}}e^{-\frac12|y|^2}\exp\Big(-\varepsilon gy+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy\geqslant\int_{\mathbb{R}}\exp\left[-\frac12|y|^2-2\varepsilon\,|y|\,\|(g,W)\|\right]dy.$$

For the numerators, a supplementary order in ε is gained by symmetry of Q, as in other estimates, Proposition IV.11 for instance. For the single integral :

$$\begin{aligned}\int_{\mathbb{R}}e^{-\frac12|y|^2}y\exp\Big(-\varepsilon qy+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy&=\int_{\mathbb{R}}e^{-\frac12|y|^2}y\exp\Big(-\varepsilon gy+O(\varepsilon^2)\,\|W\|_{\mathcal{E}}\Big)dy\\ &=\int_{\mathbb{R}}e^{-\frac12|y|^2}y\Big[1-\varepsilon gy-O(\varepsilon^2)\,\|W\|_{\mathcal{E}}\Big]dy.\end{aligned}$$

Finally

$$\int_{\mathbb{R}}e^{-\frac12|y|^2}y\exp\Big(-\varepsilon qy+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy\leqslant\|(q,W)\|\,O(\varepsilon).$$

For the first numerator of (IV.4.3), computations work the same way :

$$\begin{aligned}&\iint_{\mathbb{R}^2}\exp\Big[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big](y_1+y_2)dy_1dy_2\\ &=\iint_{\mathbb{R}^2}\exp\Big(-Q(y_1,y_2)\Big)\Big[1-\varepsilon(y_1+y_2)\Big(g+\partial_zW(t,\bar z)\Big)+O(\varepsilon)\,\|W\|_{\mathcal{E}}\Big](y_1+y_2)dy_1dy_2,\end{aligned} \tag{IV.4.4}$$

$$= \iint_{\mathbb{R}^2} \exp\Big(-Q(y_1,y_2)\Big)(y_1+y_2)dy_1dy_2 + \varepsilon\Big(g+\partial_z W(t,\bar{z})\Big)\times$$
$$\iint_{\mathbb{R}^2} \exp\Big(-Q(y_1,y_2)\Big)(y_1+y_2)^2 dy_1dy_2 + O(\varepsilon^2)\,\|W\|_{\mathcal{E}} \leqslant \|(g,W)\|\,O(\varepsilon). \quad \text{(IV.4.5)}$$

Therefore, combining equations (IV.4.3) to (IV.4.5) we have proven the bound upon the first derivative of $\mathcal{I}_\varepsilon$ in (IV.4.1).

Concerning equation (IV.4.2), one starts by writing the following formula for the Frechet Derivative:

$$\partial_V \mathcal{I}_\varepsilon(g,W)\cdot H(t,z) =$$
$$\frac{\iint_{\mathbb{R}^2} \exp\Big[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big]2\mathcal{D}_\varepsilon(H)(Y,t,z)dy_1dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y + \mathcal{D}^*_\varepsilon(W)(y,t)\Big)dy}$$
$$- \mathcal{I}_\varepsilon(g,W)(t,z)\frac{\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y + \mathcal{D}^*_\varepsilon(W)(y,t)\Big)\mathcal{D}^*_\varepsilon(H)(y,t)dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y + \mathcal{D}^*_\varepsilon(W)(y,t)\Big)dy}. \quad \text{(IV.4.6)}$$

The claimed order ε^2 holds true, by similar symmetry arguments. For instance, when we do the Taylor expansions on the numerator of the first term of (IV.4.6):

$$\iint_{\mathbb{R}^2} \exp\Big[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big]2\mathcal{D}_\varepsilon(H)(Y,t,z)dy_1dy_2$$
$$= 2\iint_{\mathbb{R}^2}\exp\Big(-Q(y_1,y_2)\Big)\Big[1-\varepsilon(y_1+y_2)\Big(g+\partial_z W(t,\bar{z})\Big) + O(\varepsilon^2)\,\|W\|_{\mathcal{E}}\Big]$$
$$\times\Big[-\varepsilon(y_1+y_2)\partial_z H(t,\bar{z}) + O(\varepsilon^2)(y_1^2+y_2^2)\,\|H\|_{\mathcal{E}}\Big]dy_1dy_2,$$

$$= 2\varepsilon\partial_z H(t,\bar{z})\iint_{\mathbb{R}^2}\exp\Big(-Q(y_1,y_2)\Big)(y_1+y_2)dy_1dy_2 + \varepsilon^2\partial_z H(t,\bar{z}\Big(g+\partial_z W(t,\bar{z})\Big)\times$$
$$\left(\iint_{\mathbb{R}^2}\exp\Big(-Q(y_1,y_2)\Big)(y_1+y_2)^2dy_1dy_2\right) + O(\varepsilon^2)\,\|H\|_{\mathcal{E}}\,\|(g,W)\| \leqslant \|(g,W)\|\,\|H\|_{\mathcal{E}}\,O(\varepsilon^2). \quad \text{(IV.4.7)}$$

For the second term of equation (IV.4.6), we also gain an order ε^2, for the same symmetry reasons, when making Taylor expansions of $\mathcal{D}^*_\varepsilon(W)$:

$$\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon q y + \mathcal{D}^*_\varepsilon(W)(t,y)\Big)\mathcal{D}^*_\varepsilon(H)(y,t)dy$$
$$= -\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y - O(\varepsilon^2)\,\|W\|_{\mathcal{E}}\Big)y^2 O(\varepsilon^2)\,\|H\|_{\mathcal{E}}\,dy,$$
$$= -\int_{\mathbb{R}} y e^{-\frac{1}{2}|y|^2}\Big[1-\varepsilon g y + O(\varepsilon^2)\,\|W\|_{\mathcal{E}}\Big]y^2 O(\varepsilon^2)\,\|H\|_{\mathcal{E}}\,dy$$
$$\leqslant \|(g,W)\|\,\|H\|_{\mathcal{E}}\,O(\varepsilon^2). \quad \text{(IV.4.8)}$$

As before the denominator of equation (IV.4.6) has an universal lower bound, therefore combining equations (IV.4.6) to (IV.4.8) concludes the proof. □

We have proven all the tools to linearize $\mathcal{I}_\varepsilon$ as follows, thanks to the previous estimates on the derivatives of $\mathcal{I}_\varepsilon$.

Proposition IV.14 (Linearization of $\mathcal{I}_\varepsilon$)**.**
For any ball B of $\mathbb{R}\times\mathcal{E}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon\leqslant\varepsilon_B$ we have for all $(g,W)\in B$:

$$\mathcal{I}_\varepsilon(q^*+\varepsilon^2 g, V^*+\varepsilon^2 W) = I_\varepsilon^* + O(\varepsilon^3)\,\|(g,W)\|\,, \tag{IV.4.9}$$
$$= 1 + O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(g,W)\|\,, \tag{IV.4.10}$$

where $O(\varepsilon^3)$ only depends on the ball B.

Proof of proposition (IV.14)*.*
We write an exact Taylor expansion:

$$\mathcal{I}_\varepsilon(q^*+\varepsilon^2 g, V^*+\varepsilon^2 W) = \mathcal{I}_\varepsilon^* + \varepsilon^2\Big[\partial_g\mathcal{I}_\varepsilon(q^*+\varepsilon^2\xi g, V^*+\varepsilon^2\xi W) + \partial_V\mathcal{I}_\varepsilon(q^*+\varepsilon^2\xi g, V^*+\varepsilon^2\xi W)\cdot W\Big].$$

for some $0<\xi<1$. Therefore (IV.4.9) is a direct application of Proposition IV.13 to $g'=q^*+\varepsilon^2\xi g$, $W'=V^*+\varepsilon^2\xi W$ and $H=W$. One deduces the estimation of equation (IV.4.10) from Proposition IV.9. □

IV.4.2 *Linearization of $\partial_z I_\varepsilon$ and decay estimates*

To prove Theorem IV.4, we need to bound uniformly $\|W\|_{\mathcal{F}}$, and this implies L^∞ bounds of the derivatives. To obtain those, we must compute the linearized equations of the derivatives of W_ε. Therefore, linearizing $\mathcal{I}_\varepsilon$ is not enough, we need to linearize $\partial_z^{(j)}\mathcal{I}_\varepsilon$ as well, for $j=1,2$ and 3. For that purpose we need more details than previously upon the nature of the negligible terms. More precisely we need to know how it behaves relatively to the weight function of the space $\mathcal{F}$. The objective of this section is to linearize $\partial_z^{(j)}\mathcal{I}_\varepsilon$ to obtain similar results to Proposition IV.14. We first prove the following estimates on the derivatives of $\mathcal{I}_\varepsilon$:

Proposition IV.15 (Decay estimate of $\partial_z\mathcal{I}_\varepsilon$)**.**
For any ball B of $\mathbb{R}\times\mathcal{E}$, there exists a constant ε_B that depends only on B such that for any pair (g,W) in B , for all $\varepsilon\leqslant\varepsilon_B$:

$$\sup_{(t,z)\in\mathbb{R}\times\mathbb{R}}\Big(\big(1+|z-z_*(t)|\big)\,|\partial_z\mathcal{I}_\varepsilon(g,W)(t,z)|\Big)\leqslant\|(g,W)\|\,O(\varepsilon^2),$$
$$\sup_{(t,z)\in\mathbb{R}\times\mathbb{R}}\Big(\big(1+|z-z_*(t)|\big)\,|\partial_z^2\mathcal{I}_\varepsilon(g,W)(t,z)|\Big)\leqslant\|(g,W)\|\,O(\varepsilon^2),$$
$$\sup_{(t,z)\in\mathbb{R}\times\mathbb{R}}\Big(\big(1+|z-z_*(t)|\big)\,|\partial_z^3\mathcal{I}_\varepsilon(g,W)(t,z)|\Big)\leqslant\|(g,W)\|\,O(\varepsilon)+\frac{1}{2}\|W\|_{\mathcal{E}}\,.$$

where all $O(\varepsilon)$ depend only on the ball B.

This proposition has to be put in parallel with [Calvez et al. 2019, Propositon 4.6]. We are not able to propagate an order ε for all derivatives, but the factor $\frac{1}{2}\|W\|_{\mathcal{E}}$ that we gain will be involved in a contraction argument, just as in Calvez et al. [2019]. This is the core of the perturbative analysis strategy we use.

Proof of Proposition IV.15.
We focus on the first derivative, the proof for the second can be straightforwardly adapted.

$$\Big(1+|z-z_*(t)|\Big)\partial_z\mathcal{I}_\varepsilon(g,W)(t,z) = \Big(1+|z-z_*(t)|\Big)\times \\ \frac{\iint_{\mathbb{R}^2}\exp\Big[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big]\mathcal{D}_\varepsilon\left(\partial_z W\right)(Y,t,z)dy_1dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}^*_\varepsilon(W)(y,t)\Big)dy}. \quad \text{(IV.4.11)}$$

But, as before the following formal Taylor expansions hold true for the numerator:

$$\iint_{\mathbb{R}^2}\exp\Big[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big]\mathcal{D}_\varepsilon(\partial_z W)(y_i,t,z)dy_1dy_2 \\ = \iint_{\mathbb{R}^2}\exp\Big[-Q(y_1,y_2)\Big]\Big[1-\varepsilon(y_1+y_2)\Big(g+\partial_z W(t,\bar{z})\Big)+O(\varepsilon^2)\|W\|_{\mathcal{E}}\Big]\times \\ \Big[-\varepsilon(y_1+y_2)\partial_z^2 W(t,\bar{z}(t))+O(\varepsilon^2)\|W\|_{\mathcal{E}}\Big]dy_1dy_2,$$

$$= -\varepsilon\partial_z^2 W(t,\bar{z})\iint_{\mathbb{R}^2}\exp\Big(-Q(y_1,y_2)\Big)(y_1+y_2)dy_1dy_2+O(\varepsilon^2)\|W\|_{\mathcal{E}} \\ +\varepsilon^2\partial_z^2 W(t,\bar{z})\Big(g+\partial_z W(t,\bar{z})\Big)\iint_{\mathbb{R}^2}\exp\Big(-Q(y_1,y_2)\Big)(y_1+y_2)^2dy_1dy_2 \leqslant O(\varepsilon^2)\|W\|_{\mathcal{E}}. \quad \text{(IV.4.12)}$$

Meanwhile the denominator has a uniform lower bound :

$$\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}^*_\varepsilon(W)(y,t)\Big)dy \geqslant \int_{\mathbb{R}}\exp\left[-\frac{1}{2}|y|^2-2\varepsilon|y|\,\|(g,W)\|\right]dy.$$

The estimate of equation (IV.4.12) can be made rigorous as in the proof of Proposition IV.11 for instance. Moreover, one can add the weight to bound equation (IV.4.11) thanks to Lemma IV.12, as explained in the proof of Proposition IV.11. Therefore, the proof of the first estimate of Proposition IV.15 is achieved.

For the second term of (IV.15), involving the second order derivative, the arguments and decomposition of the space are the same, with the formula :

$$\partial_z^2\mathcal{I}_\varepsilon(g,W)(t,z) = \\ \frac{\iint_{\mathbb{R}^2}\exp\Big(-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big)\Big[\mathcal{D}_\varepsilon(\partial_z W)^2+\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 W)\Big](Y,t,z)dy_1y_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}^*_\varepsilon(W)(y,t)\Big)dy}.$$

Things are a little bit different for the third derivative, as can be seen on the following explicit formula:

$$\partial_z^3 \mathcal{I}_\varepsilon(t,z) = \frac{\iint_{\mathbb{R}^2} \exp\Big(-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big)}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2} \exp\Big(-\varepsilon g y + \mathcal{D}_\varepsilon^*(W)(y,t)\Big) dy} \times \\ \left[\mathcal{D}_\varepsilon(\partial_z W)^3 + \frac{3}{2}\mathcal{D}_\varepsilon(\partial_z W)\mathcal{D}_\varepsilon(\partial_z^2 W) + \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 W)\right](Y,t,z) dy_1 y_2. \quad \text{(IV.4.13)}$$

All terms in this formula will provide an order ε exactly as before, except for the term $\mathcal{D}_\varepsilon(\partial_z^3 W)$ since we lack a priori controls of the fourth derivative of W in $\mathcal{F}$. Therefore, for this term we proceed as follows :

$$\Big(1+|z-z_*(t)|\Big)\left|\mathcal{D}_\varepsilon(\partial_z^3 W)(Y,t,z)\right| = \\ \Big(1+|z-z_*(t)|\Big)\left|\partial_z^3 W(t,\bar{z}) - \frac{1}{2}\partial_z^3 W(t,\bar{z}+\varepsilon y_1) - \frac{1}{2}\partial_z^3 W(t,\bar{z}+\varepsilon y_2)\right| \leqslant 2\|W\|_{\mathcal{E}}\Big(1+\varepsilon(|y_1|+|y_2|)\Big).$$

For this computation, we used the following property of the weight function, which was also of huge importance in Calvez et al. [2019] :

$$\Big(1+|\bar{z}-z_*|\Big)\frac{1}{2}\left|\partial_z^3 W(\bar{z}+\varepsilon y_1)\right| \leqslant \frac{1+|\bar{z}-z_*|}{2\Big(1+|\bar{z}+\varepsilon y_1 - z_*|\Big)}\|W\|_{\mathcal{E}} \\ \leqslant \frac{\|W\|_{\mathcal{E}}}{2}\left(1+\frac{|\varepsilon y_1|}{1+|\bar{z}+\varepsilon y_1 - z_*|}\right) \leqslant (1+|\varepsilon y_1|)\frac{\|W\|_{\mathcal{E}}}{2}.$$

As a consequence, we deduce that

$$\Big(1+|z-z_*(t)|\Big)\frac{\iint_{\mathbb{R}^2} \exp\Big(-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big)}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2} \exp\Big(-\varepsilon g y + \mathcal{D}_\varepsilon^*(W)(y,t)\Big) dy} \\ \times \left[\frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 W)(Y,t,z)\right] dy_1 y_2 \leqslant \left(\frac{1}{2}+O(\varepsilon)\right)\|W\|_{\mathcal{E}}.$$

Once added to other estimates of the terms of equation (IV.4.13), obtained by Taylor expansions of $\mathcal{D}_\varepsilon$ as before we get the desired estimate. □

We now detail the control upon the directional derivatives of $\mathcal{I}_\varepsilon$.

Proposition IV.16 (Bound of the directional derivatives of $\mathcal{I}_\varepsilon$, with weight)**.**
For any ball B of $\mathbb{R}\times\mathcal{E}$, there exists a constant ε_B that depends only on B such that for any pair (g,W) in B and any function $H\in\mathcal{E}$, for $j=1,2$, for $\varepsilon\leqslant\varepsilon_B$:

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}}\left(\left(1+|z-z_*(t)|\right)\left|\partial_g\partial_z^{(j)}\mathcal{I}_\varepsilon(g,W)(t,z)\right|\right)\leqslant\|(g,W)\|\,O(\varepsilon), \tag{IV.4.14}$$

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}}\left(\left(1+|z-z_*(t)|\right)\left|\partial_V\partial_z^{(j)}\mathcal{I}_\varepsilon(g,W)\cdot H(t,z)\right|\right)\leqslant\|(g,W)\|\,\|H\|_{\mathcal{E}}\,O(\varepsilon), \tag{IV.4.15}$$

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}}\left(\left(1+|z-z_*(t)|\right)\left|\partial_g\partial_z^{3}\mathcal{I}_\varepsilon(g,W)(t,z)\right|\right)\leqslant\|(g,W)\|\,O(\varepsilon), \tag{IV.4.16}$$

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}}\left(1+|z-z_*(t)|\right)\left|\partial_V\partial_z^{3}\mathcal{I}_\varepsilon(g,W)\cdot H(t,z)\right|\leqslant\|(g,W)\|\,\|H\|_{\mathcal{E}}\,O(\varepsilon)+\frac{1}{2}\|H\|_{\mathcal{E}}\,. \tag{IV.4.17}$$

where the $O(\varepsilon)$ depend only on the ball B.

In those estimates, the order of precisions $O(\varepsilon)$ is not optimal and we could improve it, but it is not necessary to our purposes. We will not give the full proof for each estimate of this Proposition. However, we see that it follows the same pattern than in Proposition IV.15, and we will even use those results for the proof. In particular for the third derivative, it is not possible to completely recover an order ε, because of the term $\frac{1}{2}\|H\|_{\mathcal{E}}$. It comes from the *linear* part $\mathcal{D}_\varepsilon(\partial_z^3 W)$ that appears in $\partial_z^3\mathcal{I}_\varepsilon$, see equation (IV.4.13). However, it does not prevent us from carrying our analysis since the factor $\frac{1}{2}$ will be absorbed by a contraction argument, see section IV.8.

Proof of Proposition IV.16.
We detail the proof of estimates (IV.4.14), because derivative in g are somehow easier. The formula for the first derivative is:

$$\begin{aligned}&\partial_g\partial_z\mathcal{I}_\varepsilon(g,W)(t,z)=\\ &-\varepsilon\left(\frac{\iint_{\mathbb{R}^2}\exp\Big[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big](y_1+y_2)\mathcal{D}_\varepsilon(\partial_z W)(Y,t,z)dy_1dy_2}{\sqrt{\pi}\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon gy+\mathcal{D}_\varepsilon^*(y,t)\Big)dy}\right.\\ &\qquad\left.-\partial_z\mathcal{I}_\varepsilon(g,W)(t,z)\frac{\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon gy+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)ydy}{\sqrt{\pi}\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon gy+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy}\right).\end{aligned} \tag{IV.4.18}$$

The first term of this formula closely resembles the one for $\partial_z I_\varepsilon(g,W)$, with an additional factor $(\varepsilon(y_1+y_2)$. We de not detail how to bound it, as it follows the same steps, see the work done following equation (IV.4.11). For the second term we first use the following bound :

$$\frac{\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon gy+D^*(W)(y,t)\Big)ydy}{\sqrt{\pi}\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon gy+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy}\leqslant\frac{\int_{\mathbb{R}}\exp\left[-\frac{1}{2}|y|^2+2\varepsilon|y|\,\|(g,W)\|\right]ydy}{\int_{\mathbb{R}}\exp\left[-\frac{1}{2}|y|^2-2\varepsilon|y|\,\|(g,W)\|\right]dy}. \tag{IV.4.19}$$

For ε sufficiently small that depends only on $\|(g,W)\|$ we deduce an uniform bound with moments of the Gaussian distribution. We then use the estimate from Proposition IV.15 on $\partial_z I_\varepsilon(g,W)$, which absorbs the weight, to conclude.

Every other estimate of Proposition IV.16 works along the same lines. We illustrate this with the second derivative in g :

$$\begin{aligned}\partial_g\partial_z^2\mathcal{I}_\varepsilon(g,W)(t,z) = -\varepsilon\Bigg(&\frac{\iint_{\mathbb{R}^2}\exp\Big(-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big)}{\sqrt{\pi}\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy}\times\\ &\Big[\mathcal{D}_\varepsilon(\partial_z W)^2+2\mathcal{D}_\varepsilon(\partial_z W)\mathcal{D}_\varepsilon(\partial_z^2 W)\Big](Y,t,z)dy_1y_2-\\ &-\partial_z^2\mathcal{I}_\varepsilon(g,W)\frac{\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}y\exp\Big(-\varepsilon g y+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)\,dy}{\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy}\Bigg). \quad \text{(IV.4.20)}\end{aligned}$$

This is very close to $\partial_z^2 I_\varepsilon$ that has already been estimated in Proposition IV.15, and therefore the same arguments as before hold.

The structure is different for the derivatives in V, as can be seen for $\partial_V\partial_z\mathcal{I}_\varepsilon(g,W)\cdot H$:

$$\begin{aligned}&\partial_V\partial_z\mathcal{I}_\varepsilon(g,W)\cdot H(t,z) =\\ &\frac{\iint_{\mathbb{R}^2}\exp\Big(-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big)\Big[2\mathcal{D}_\varepsilon(\partial_z W)\mathcal{D}_\varepsilon(H)+\mathcal{D}_\varepsilon(\partial_z H)\Big](Y,t,z)dy_1dy_2}{\sqrt{\pi}\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy}\\ &-\partial_z\mathcal{I}_\varepsilon(g,W)(t,z)\frac{\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)\mathcal{D}_\varepsilon^*(H)(y,t)dy}{\sqrt{\pi}\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy}. \quad \text{(IV.4.21)}\end{aligned}$$

The second term can still be bounded using Proposition IV.15 and the estimate (IV.4.19), and the following immediate result :

$$|\mathcal{D}_\varepsilon^*(W)(t,y)| \leqslant \varepsilon\,|y|\,\|W\|_{\mathcal{E}}\,.$$

For the first term, we must do Taylor expansions of $2\mathcal{D}_\varepsilon(\partial_z W)\mathcal{D}_\varepsilon(H)+\mathcal{D}_\varepsilon(\partial_z H)$ to control them with the weight. One ends up with moments of the multidimensional Gaussian distribution just as in all the previous proofs. The same method holds for the second derivative in V.

For the terms with the third derivative, the derivative in g is similar to the previous computations:

$$\begin{aligned}\partial_g\partial_z^3\mathcal{I}_\varepsilon(t,z) = -\varepsilon\Bigg(&\frac{\iint_{\mathbb{R}^2}\exp\Big(-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big)}{\sqrt{\pi}\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy}\times\\ &(y_1+y_2)\left[\mathcal{D}_\varepsilon(\partial_z W)^3+\frac{3}{2}\mathcal{D}_\varepsilon(\partial_z W)\mathcal{D}_\varepsilon(\partial_z^2 W)+\frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 W)\right](Y,t,z)dy_1y_2\\ &+\partial_z^3\mathcal{I}_\varepsilon(t,z)\,\frac{\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}y\exp\Big(-\varepsilon g y+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)\,dy}{\int_{\mathbb{R}}e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy}\Bigg). \quad \text{(IV.4.22)}\end{aligned}$$

To get the bound (IV.4.17) things are a little bit different, because of the term $\mathcal{D}_\varepsilon(\partial_z^3 W_\varepsilon)$:

$$\partial_V \partial_z^3 \mathcal{I}_\varepsilon(g,W)\cdot H(t,z) = \frac{\iint_{\mathbb{R}^2} \exp\Big(-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(W)(Y,t,z)\Big)}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy}\times$$
$$\Big[\mathcal{D}_\varepsilon(H)\Big(2\mathcal{D}_\varepsilon(\partial_z W)^3+3\mathcal{D}_\varepsilon(\partial_z W)\mathcal{D}_\varepsilon(\partial_z^2 W)+\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^3 W)\Big)+$$
$$\Big(3\mathcal{D}_\varepsilon(\partial_z H)\mathcal{D}_\varepsilon(\partial_z W)^2+\frac{3}{2}\Big(\mathcal{D}_\varepsilon(\partial_z W)\mathcal{D}_\varepsilon(\partial_z^2 H)+\mathcal{D}_\varepsilon(\partial_z H)\mathcal{D}_\varepsilon(\partial_z^2 W)\Big)+\frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 H)\Big)\Big](Y,t,z)dy_1 y_2$$
$$+\partial_z^3\mathcal{I}_\varepsilon(t,z)\frac{\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)\mathcal{D}_\varepsilon^*(H)(y,t)dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon g y+\mathcal{D}_\varepsilon^*(W)(y,t)\Big)dy}.$$

We do not get an order ε from the linear part $\mathcal{D}_\varepsilon(\partial_z^3 H)$, since we do not control the fourth derivative in $\mathcal{E}$. We then proceed with arguments following equation (IV.4.13) in the proof of Proposition IV.15. □

Thanks to those estimates we are able to write our main result for this part, which is a precise control of the linearization of the derivatives of $\mathcal{I}_\varepsilon$:

Proposition IV.17 (Decay linearization)**.**
For any ball B of $\mathbb{R}\times\mathcal{E}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon\leqslant\varepsilon_B$ we have for all $(g,W)\in B$:

$$\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2 g, V^*+\varepsilon^2 W)(t,z) = \partial_z I_\varepsilon^*(t,z)+\frac{\|(g,W)\|}{1+|z-z_*(t)|}O(\varepsilon^3), \tag{IV.4.23}$$
$$\partial_z^2\mathcal{I}_\varepsilon(q^*+\varepsilon^2 g, V^*+\varepsilon^2 W)(t,z) = \partial_z^2 I_\varepsilon^*(t,z)+\frac{\|(g,W)\|}{1+|z-z_*(t)|}O(\varepsilon^3), \tag{IV.4.24}$$
$$\partial_z^3\mathcal{I}_\varepsilon(q^*+\varepsilon^2 g, V^*+\varepsilon^2 W)(t,z) = \partial_z^3 I_\varepsilon^*(t,z)+\frac{\|(g,W)\|}{1+|z-z_*(t)|}O(\varepsilon^3)+\frac{\varepsilon^2}{2}\|W\|_{\mathcal{E}}. \tag{IV.4.25}$$

where $O(\varepsilon^3)$ only depends on the ball B.

Proof of Proposition IV.17.
The methodology for equations (IV.4.23) to (IV.4.25) is the same. We detail for instance how to prove equation (IV.4.23). One begins by writing the following exact Taylor expansion up to the second order:

$$\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2 g, V^*+\varepsilon^2 W)(t,z) =$$
$$\partial_z\mathcal{I}_\varepsilon^*(t,z)+\varepsilon^2\Big[\partial_g\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\xi g, V^*+\varepsilon^2\xi W)(t,z)+\partial_V\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\xi g, V^*+\varepsilon^2\xi W)\cdot W(t,z)\Big].$$

with $0<\xi<1$. The result for equation (IV.4.23) is then given by the directional decay estimates of Proposition IV.16 applied to to $g'=q^*+\varepsilon^2\xi g$, $W'=V^*+\varepsilon^2\xi W$, and $H=W$. □

IV.4.3 *Uniform controls of the second derivative of I_ε^**

To conclude this section dedicated to estimates of I_ε, we now show that our estimates above can be

made much more precise when looking at the particular case of the function I_ε^* evaluated at the point z_*. In particular we will gain information upon the sign of the derivatives, that will prove crucial regarding the stability of κ_ε. This additional precision is similar to what was needed in the stationary case, [Calvez et al. 2019, Lemma 3.1], where detailed expansions of $\mathcal{I}_\varepsilon$ were needed for the study of the affine part, hereby named γ_ε. We will find convenient to use the following notations, as in Calvez et al. [2019] :

Definition IV.18 (Measures definition)**.**
We introduce the following measures :

$$dG_\varepsilon^*(Y,z,t) := \frac{G_\varepsilon^*(Y,t,z)}{\iint_{\mathbb{R}^2} G_\varepsilon^*(Y,t,z)dy_1dy_2}, \quad \text{with } Y=(y_1,y_2),$$
$$= \frac{\exp\Big[-Q(y_1,y_2)+2V^*(t,\bar z)-V^*(t,\bar z+\varepsilon y_1)-V^*(\bar z+\varepsilon y_2)-\varepsilon q^*(y_1+y_2)\Big]}{\iint_{\mathbb{R}^2}\exp\Big[-Q(y_1,y_2)+2V^*(t,\bar z)-V^*(t,\bar z+\varepsilon y_1)-V^*(\bar z+\varepsilon y_2)-\varepsilon q^*(y_1+y_2)\Big]dy_1dy_2}. \tag{IV.4.26}$$

And :

$$dN_\varepsilon^*(y,t) := \frac{N_\varepsilon^*(y,t)}{\int_{\mathbb{R}^2} N_\varepsilon^*(\cdot,t)} := \frac{\exp\left[-\frac{1}{2}|y|^2+V^*(t,z_*)-V^*(t,z_*+\varepsilon y)-\varepsilon q^* y\right]}{\int_{\mathbb{R}}\exp\left[-\frac{1}{2}|y|^2+V^*(t,z_*)-V^*(t,z_*+\varepsilon y)-\varepsilon q^* y\right]dy}. \tag{IV.4.27}$$

Proposition IV.19 (Uniform control of the directional derivatives of $\partial_z\mathcal{I}_\varepsilon^*$)**.**
There exist a positive function of time R^*, such that for any ball B of $\mathcal{E}$, there exists a constant ε_B that depends only on B , that verifies for all $\varepsilon \leqslant \varepsilon_B$, $H \in B$:

$$\partial_g\partial_z\mathcal{I}_\varepsilon^*(t,z_*) = \varepsilon^2 R^*(t)+O^*(\varepsilon^3). \tag{IV.4.28}$$
$$\partial_V\partial_z\mathcal{I}_\varepsilon^*\cdot H(t,z_*) = O(\varepsilon^3)\,\|H\|_{\mathcal{E}}+\varepsilon^2 S_\varepsilon(H), \tag{IV.4.29}$$

where $O(\varepsilon^2)$ depends only on B. Moreover, R^* is uniformly bounded and there exists a constant R_0 such that $R^* > R_0 > 0$ for all $t > t_0$. Finally, S_ε is given by the following formula :

$$S_\varepsilon(H) := \frac{1}{4}\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)\Big[y_1^2\partial_z^3H(t,z_*+\varepsilon\xi y_1)+y_2^2\partial_z^3H(t,z_*+\varepsilon\xi y_2)\Big]dy_1dy_2, \tag{IV.4.30}$$

for some fixed number $0<\xi<1$.

The sign of R^* is directly connected to the behavior on z_* we assumed in the introduction, see (IV.1.13). We are not able to completely show the order ε^3 in (IV.4.29) since we lack fourth order derivatives. This is the same issue we faced for the estimate of the third derivative in Proposition IV.15 and in (IV.4.17). Once the complete control of every derivative is achieved, we will be able to bound S_ε uniformly, see section IV.8.

Proof of proposition (IV.19).
First we focus on the bound of equation (IV.4.28). Similarly to equation (IV.4.18), the explicit formula for the derivative is :

$$\partial_g \partial_z I_\varepsilon^*(t,z_*) := -\varepsilon(I_1+I_2) =$$
$$-\varepsilon\left(\frac{\iint_{\mathbb{R}^2}\exp\left[-Q(y_1,y_2)-\varepsilon q^*(y_1+y_2)+2\mathcal{D}_\varepsilon(V^*)(Y,t,z_*)\right](y_1+y_2)\mathcal{D}_\varepsilon(\partial_z V^*)(Y,t,z_*)dy_1dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon q^* y+V^*(t,z_*)-V^*(t,z_*+\varepsilon y)\Big)dy}\right.$$
$$\left.-\partial_z I_\varepsilon^*(t,z_*)\frac{\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2} y\exp\Big(-\varepsilon q^* y+V^*(t,z_*)-V^*(t,z_*+\varepsilon y)\Big)dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon q^* y+V^*(t,z_*)-V^*(t,z_*+\varepsilon y)\Big)dy}\right). \quad \text{(IV.4.31)}$$

Thanks to the Proposition IV.16, we already know that $|\partial_z I_\varepsilon^*(t,z_*)| = O^*(\varepsilon^2)$. Moreover we bound uniformly the second term:

$$\left|\frac{\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2} y\exp\Big(-\varepsilon q^* y+V^*(t,z_*)-V^*(t,z_*+\varepsilon y)\Big)dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-\varepsilon q^* y+V^*(t,z_*)-V^*(t,z_*+\varepsilon y)\Big)dy}\right| \leqslant \frac{\int_{\mathbb{R}}\exp\left(-\frac{1}{2}|y|^2+2\varepsilon K^*|y|\right)|y|\,dy}{\sqrt{\pi}\int_{\mathbb{R}}\exp\left(-\frac{1}{2}|y|^2-2\varepsilon K^*|y|\right)dy}$$
$$\leqslant O^*(1),$$

where K^* was defined in Proposition IV.8. This shows that $I_2 = O^*(\varepsilon^2)$. Therefore one can focus on I_1. In order to gather information upon the sign of this quantity and not only get a bound in absolute value, we perform exact Taylor expansions of $\mathcal{D}_\varepsilon(\partial_z V^*)$. We divide I_1 by $I_\varepsilon^*(t,z_*)$, the formula is more aesthetic that we are accustomed to, thanks to the definitions of equations (IV.4.26) and (IV.4.27) :

$$\frac{I_1}{\mathcal{I}_\varepsilon^*(t,z_*)} = \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(y_1+y_2)\mathcal{D}_\varepsilon(\partial_z V^*)(Y,t,z)dy_1dy_2.$$

As usual, we make Taylor expansions : there exists $0<\xi_1,\xi_2<1$ such that

$$\frac{I_1}{\mathcal{I}_\varepsilon^*(t,z_*)} = \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)\left[-\varepsilon\frac{(y_1+y_2)^2}{2}\partial_z^2 V^*(t,z_*)-\frac{\varepsilon^2 y_1^2(y_1+y_2)}{4}\partial_z^3 V^*(t,z_*+\varepsilon\xi_1 y_1)\right.$$
$$\left.-\frac{\varepsilon^2 y_2^2(y_1+y_2)}{4}\partial_z^3 V^*(t,z_*+\varepsilon\xi_2 y_2)\right]dy_1dy_2. \quad \text{(IV.4.32)}$$

The first term has a sign for $t>t_0$, see equations (IV.1.13) and (IV.3.2). We then define R^* as :

$$\varepsilon\partial_z^2 V^*(t,z_*(t))\left\langle dG_\varepsilon^*(y_i,z_*(t),t),\frac{(y_1+y_2)^2}{2}\right\rangle =: \varepsilon R^*(t),$$

with the following uniform bounds, that come from moments of a Gaussian distribution :

$$0<R_0<R^*(t)\leqslant R_\infty \quad \forall t>t_0.$$

The next terms of equation (IV.4.32) are of order superior to ε^2 , and can be bounded uniformly by :

$$\frac{\varepsilon^2}{4}\left|\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)\left[y_1^2(y_1+y_2)+y_2^2(y_1+y_2)\right]\|V^*\|_*\,dy_1dy_2\right| \leqslant O^*(\varepsilon^2).$$

Therefore one can rewrite (IV.4.32) as

$$\frac{I_1}{\mathcal{I}_\varepsilon^*(t,\mathrm{z}_*)} = -\varepsilon R^*(t) + O^*(\varepsilon^2).$$

Thanks to Proposition IV.9 we recover a similar estimate for I_1 :

$$I_1 = -\varepsilon R^*(t) + O^*(\varepsilon^2).$$

Finally coming back to equation (IV.4.31), we have shown that

$$\partial_g \partial_z I_\varepsilon^*(t,\mathrm{z}_*) = \varepsilon^2 R^*(t) + O^*(\varepsilon^3).$$

This concludes the proof of the estimate (IV.4.28). Next we tackle the proof of the estimate upon the Frechet derivative (IV.4.29), where, again, we first divide by $I_\varepsilon^*(t,\mathrm{z}_*)$:

$$\frac{\partial_V \partial_z \mathcal{I}_\varepsilon^* \cdot H(t,\mathrm{z}_*)}{\mathcal{I}_\varepsilon^*(t,\mathrm{z}_*)} = \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,\mathrm{z}_*)\Big[\mathcal{D}_\varepsilon(\partial_z V^*)2\mathcal{D}_\varepsilon(H) + \mathcal{D}_\varepsilon(\partial_z H)\Big](Y,t,\mathrm{z}_*)dy_1dy_2 - \frac{\partial_z I_\varepsilon^*(t,\mathrm{z}_*)}{I_\varepsilon^*(t,\mathrm{z}_*)}\int_{\mathbb{R}} dN_\varepsilon^*(y,t)\mathcal{D}_\varepsilon^*(H)(y,t)dy. \quad \text{(IV.4.33)}$$

Thanks to Propositions IV.9 and IV.11, and an uniform bound on $\mathcal{D}_\varepsilon^*(W)$:

$$\left|\frac{\partial_z I_\varepsilon^*(t,\mathrm{z}_*)}{I_\varepsilon^*(t,\mathrm{z}_*)}\int_{\mathbb{R}} dN_\varepsilon^*(y,t)\mathcal{D}_\varepsilon^*(H)(y,t)dy.\right| \leqslant O(\varepsilon^3)\,\|W\|_{\mathcal{E}}\,. \quad \text{(IV.4.34)}$$

For the first term of equation (IV.4.33), we first make a brutal bound based on Taylor expansions of $\mathcal{D}_\varepsilon(H)$:

$$|\mathcal{D}_\varepsilon(H)(Y,t,\mathrm{z}_*)\,| \leqslant \frac{\varepsilon^2}{2}(|y_1|^2 + |y_2|^2)\,\|H\|_{\mathcal{E}}\,.$$

The key element here is that since $\mathcal{D}_\varepsilon$ is evaluated at z_* one gains an order in ε because $\partial_z H(t,\mathrm{z}_*) = 0$, by definition of $\mathcal{E}$. Therefore, one gets

$$\left|\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,\mathrm{z}_*)\Big[\mathcal{D}_\varepsilon(\partial_z V^*)2\mathcal{D}_\varepsilon(H)\Big](Y,t,\mathrm{z}_*)dy_1dy_2\right| \leqslant O(\varepsilon^3)\,\|H\|_{\mathcal{E}}\,, \quad \text{(IV.4.35)}$$

where the additional order in ε is gained through a Taylor expansion of $\mathcal{D}_\varepsilon(\partial_z V^*)$. We finally tackle the last term of (IV.4.33) we did not yet estimate, involving $\mathcal{D}_\varepsilon(\partial_z H)$. Based only on Taylor expansions in $\mathcal{E}$, we do not gain an order ε^3 as in the previous terms. Rather, we obtain, for some $0 < \xi < 1$:

$$\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,\mathrm{z}_*)\mathcal{D}_\varepsilon(\partial_z H)(Y,t,\mathrm{z}_*)dy_1dy_2 = \varepsilon\frac{\partial_z^2 H(t,\mathrm{z}_*)}{2}\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,\mathrm{z}_*)(y_1+y_2)dy_1dy_2 + \frac{\varepsilon^2}{4}\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,\mathrm{z}_*)\Big[y_1^2\partial_z^3 H(t,\mathrm{z}_* + \varepsilon\xi y_1) + y_2^2\partial_z^3 H(t,\mathrm{z}_* + \varepsilon\xi y_2)\Big]dy_1dy_2 \quad \text{(IV.4.36)}$$

It is straightforward based on multiple similar computations to deduce that the first moment of dG_ε^* is zero. Therefore,

$$\varepsilon\frac{\partial_z^2 H(t,\mathrm{z}_*)}{2}\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,\mathrm{z}_*)(y_1+y_2)dy_1dy_2 = \varepsilon\frac{\partial_z^2 H(t,\mathrm{z}_*)}{2}O(\varepsilon^2)\,\|H\|_{\mathcal{E}} = O(\varepsilon^3)\,\|H\|_{\mathcal{E}}\,. \quad \text{(IV.4.37)}$$

In the second term of (IV.4.36) we recognize the expression of S_ε.

Finally, by putting together (IV.4.34), (IV.4.35) and equation (IV.4.36), the estimate (IV.4.29) is proven. □

The order ε^2 of equation (IV.4.29) will be crucial in our analysis, especially when working around κ_ε. We will also need this order when we linearize, which is required to get an order of precision further from what we have done so far. This is the objective of the following lemma.

Lemma IV.20 (Uniform control of the second Fréchet derivative of $\partial_z \mathcal{I}_\varepsilon$)**.**
For any ball B of $\mathbb{R} \times \mathcal{E}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \leqslant \varepsilon_B$ we have for all $(g, W) \in B$:

$$g^2 \partial_g^2 \partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t,z) + 2g \partial_V \partial_g \partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W) \cdot W(t,z) \\ + \partial_V^2 \partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t,z) \cdot (W,W)(t,z) \ \leqslant O(\varepsilon^2) \, \|(g,W)\| \, . \quad \text{(IV.4.38)}$$

Proof of Lemma IV.20.
We will denote $f(p) := \partial_z \mathcal{I}_\varepsilon(q^* + pg, V^* + pW)(t,z)$. Then we can rewrite the left hand side of (IV.4.38) as $f''(1)$. This follows from the motivation behind Lemma IV.20, that is to be able to linearize $\partial_z \mathcal{I}_\varepsilon$. The formula for f'' is very long , so for clarity we will denote respectively $A_\varepsilon(p)$ for the numerator and $B_\varepsilon(p)$ the denominator of $f(p)$, so that when we differentiate we have the structure :

$$f''(p) = \frac{A_\varepsilon''(p)}{B_\varepsilon(p)} - 2\frac{A_\varepsilon'(p)B_\varepsilon'(p)}{B_\varepsilon(p)^2} - \frac{A_\varepsilon(p)B_\varepsilon''(p)}{B_\varepsilon(p)^2} + 2\frac{A_\varepsilon(p)B_\varepsilon'(p)^2 B_\varepsilon(p)}{B_\varepsilon(p)^4}. \quad \text{(IV.4.39)}$$

The numerator is defined as :

$$A_\varepsilon(p) := \iint_{\mathbb{R}^2} \exp\Big[-Q(y_1,y_2) + 2\mathcal{D}_\varepsilon(V^* + pW)(y_i,t,z) - \varepsilon(q^* + pg)(y_1+y_2)\Big] \\ \Big(\partial_z(V^* + pW)(t, z_*(t)) - \frac{1}{2}\partial_z(V^* + pW)(z_*(t) + \varepsilon y_1) - \frac{1}{2}\partial_z(V^* + pW)(z_*(t) + \varepsilon y_2)\Big)\, dy_1 dy_2,$$

while the denominator reads :

$$B_\varepsilon(p) := \int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2} \exp\Big(-\varepsilon(q^* + pg)y + \mathcal{D}_\varepsilon^*(V^* + pW)(y,t)\Big)\, dy.$$

Therefore we will divide each term by I_ε^* to simplify the notations, this will make appear the measures $dG_\varepsilon^*, dN_\varepsilon^*$ introduced in equations (IV.4.26) and (IV.4.27). For instance :

$$\frac{A_\varepsilon(p)}{\mathcal{I}_\varepsilon^*(t,z_*)B_\varepsilon(p)} := \\ \frac{\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*))\Big[\exp\Big(2p\mathcal{D}_\varepsilon(W) - \varepsilon g p(y_1+y_2)\Big)\mathcal{D}_\varepsilon(\partial_z V^* + p\partial_z W)(Y,t,z_*)\Big] dy_1 dy_2}{\int_{\mathbb{R}} dN_\varepsilon^*(y,t)\exp\Big(2\mathcal{D}_\varepsilon^*(W)(y,t) - \varepsilon g y\Big) dy}.$$

We notice that any factor of the sum in (IV.4.39) (divided by $\mathcal{I}_\varepsilon^*$) is a sum (and a product) of terms of the form

$$\frac{A_\varepsilon^{(j)}(p)B_\varepsilon^{(k)}(p)}{B_\varepsilon(p)\mathcal{I}_\varepsilon^*(t,z)} = \frac{A_\varepsilon^{(j)}(p)}{\mathcal{I}_\varepsilon^*(t,z)B_\varepsilon(p)}\frac{B_\varepsilon^{(k)}(p)}{B_\varepsilon(p)},$$

with $j = 0, 1, 2$, $k = 1, 2$ and the constraint $j + k = 2$. It is rather convenient to bound separately each of those terms. For instance we deal with the second :

$$\frac{A'_\varepsilon(p)B'_\varepsilon(p)}{B_\varepsilon(p)^2\mathcal{I}^*_\varepsilon(t,z)} = \frac{A'_\varepsilon(p)}{\mathcal{I}^*_\varepsilon(t,z)B_\varepsilon(p)}\frac{B'_\varepsilon(p)}{B_\varepsilon(p)}, \tag{IV.4.40}$$

The first term of this product is

$$\frac{A'_\varepsilon(p)}{\mathcal{I}^*_\varepsilon(t,z_*)B_\varepsilon(p)} := \frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y,t,z_*)\Big[\exp\Big(2p\mathcal{D}_\varepsilon(W) - \varepsilon g p(y_1+y_2)\Big)\mathcal{D}_\varepsilon(\partial_z W)(Y,t,z_*)\Big]\, dy_1dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y,t)\exp\Big(2\mathcal{D}^*_\varepsilon(W)(y,t) - \varepsilon g y\Big)dy} + \frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y,t,z_*)\exp\Big(2p\mathcal{D}_\varepsilon(W) - \varepsilon g p(y_1+y_2)\Big)2\mathcal{D}_\varepsilon(\partial_z V^* + p\partial_z W)\Big(\mathcal{D}_\varepsilon(W) - \varepsilon g(y_1+y_2)\Big)dy_1dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y,t)\exp\Big(2\mathcal{D}^*_\varepsilon(W)(y,t) - \varepsilon g y\Big)dy}.$$

The numerator and denominator can be bounded by estimating naively $\mathcal{D}_\varepsilon$:

$$\left|\frac{A'_\varepsilon(p)}{B_\varepsilon(p)\mathcal{I}^*_\varepsilon(t,z)}\right| \leqslant \frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y,t,z_*)\exp\Big(3\varepsilon\,\|(g,W)\|\,(|y_1|+|y_2|)\Big)\varepsilon(|y_1|+|y_2|)\,\|(g,W)\|\,dy_1dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y,t)\exp\Big(-3\varepsilon\,\|(g,W)\|\,|y|\Big)dy} + \frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y,t,z_*)\exp\Big(3\varepsilon\|(g,W)\|(|y_1|+|y_2|)\Big)\varepsilon^2(|y_1|+|y_2|)^2(3\,\|(g,W)\|+2\,\|V^*\|_*)3\|(g,W)\|dy_1dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y,t)\exp\Big(-3\varepsilon\,\|(g,W)\|\,|y|\Big)dy}. \tag{IV.4.41}$$

Therefore, we only get moments of a Gaussian distribution, so the previous bound is in fact

$$\left|\frac{A'_\varepsilon(p)}{B_\varepsilon(p)\mathcal{I}^*_\varepsilon(t,z)}\right| \leqslant O(\varepsilon)\,\|(g,W)\|\,.$$

With the exact same arguments but more convoluted formulas, one shows that

$$\left|\frac{A''_\varepsilon(p)}{B_\varepsilon(p)\mathcal{I}^*_\varepsilon(t,z)}\right| \leqslant O(\varepsilon)\,\|(g,W)\|\,. \tag{IV.4.42}$$

For the quotients of B in equation (IV.4.39), we lose the structure of the measures dG_ε^* and dN_ε^*, but they are replaced by an actual Gaussian measure $\exp(-y^2/2)$. Therefore, with the same arguments as before, we bound the quotient by the moments of a Gaussian distribution :

$$\begin{aligned}\left|\frac{B_\varepsilon^{(k)}(p)}{B_\varepsilon(p)}\right| &= \left|\frac{\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(2\mathcal{D}_\varepsilon^*(V^*+pW)-\varepsilon(q^*+gp)y\Big)\Big(2\mathcal{D}_\varepsilon^*(W)-\varepsilon g y\Big)^k dy}{\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(2\mathcal{D}_\varepsilon^*(V^*+pW)-\varepsilon(q^*+gp)y\Big)dy}\right| \\ &\leqslant \frac{\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(2\varepsilon\,|y|\,\|V^*\|_\alpha+\varepsilon G^*\,|y|+3\varepsilon\,\|(g,W)\|\,|y|\Big)\Big(3\varepsilon\,\|(g,W)\|\,|y|\Big)^k dy}{\int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2}\exp\Big(-3\varepsilon\,\|(g,W)\|\,|y|\Big)dy}, \\ &\leqslant O(\varepsilon^k)\,\|(g,W)\|\,. \end{aligned} \tag{IV.4.43}$$

Multiplying each term of (IV.4.42) by (IV.4.43) and then combining them yields the desired estimate result, given $j+k=2$, and the separation of terms made in equation (IV.4.39) :

$$\left|\frac{f''(p)}{I_\varepsilon^*(t,z)}\right| \leqslant O(\varepsilon^2)\,\|(g,W)\|\,.$$

Thanks to Proposition IV.9 , Lemma IV.20 is proven. □

IV.5 RESULTS FOR κ_ε AND p_ε

IV.5.1 *Result on κ_ε*

Thanks to the estimates of the previous sections, every useful tool to look at the perturbation κ_ε is made available. We recall that our final goal is to show that κ_ε is bounded as it is the perturbation from q^*, see equation (IV.1.20). We show in this section that one gets an ordinary differential equation on κ_ε when linearizing, see Proposition IV.21. It is obtained by differentiating the equation (IV.2.2) and evaluating at $z=\mathrm{z}_*$. This is exactly what suggested the spectral analysis of the formal linearized operator in equation (IV.2.1). Now, thanks to our previous set of estimates of section IV.4, we are able to carefully justify our linearization. Finally, the limit ODE we introduced for q^* out of the blue in equation (IV.1.16) will appear clearly when we do our analysis to counterbalance contributions of higher order. To shortcut expressions, we introduce the following alternative notations for all $t,z\in\mathbb{R}_+\times\mathbb{R}$:

$$M_\varepsilon(t,z) := W_\varepsilon(t,z)-2W_\varepsilon(t,\bar{z}(t)). \tag{IV.5.1}$$

Compared to previous sections, we will work in the space $\mathcal{F}$ that is well suited to show the boundedness of W_ε.

Proposition IV.21 (Equation on κ_ε)**.**
For any ball B of $\mathbb{R}\times\mathcal{F}$ there exists a constant ε_B that depends only on B such that if $(\kappa_\varepsilon,W_\varepsilon)\in B$ is a solution of (IV.2.2)*, then for all $\varepsilon\leqslant\varepsilon_B$, κ_ε is a solution of the following ordinary equation :*

$$-\dot{\kappa}_\varepsilon(t) = R^*(t)\kappa_\varepsilon + S_\varepsilon(W_\varepsilon)+O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|+m'(\mathrm{z}_*)\partial_z^2 W_\varepsilon(t,\mathrm{z}_*). \tag{IV.5.2}$$

where the $O(\varepsilon)$ depends only on B, and R^*, S_ε are defined in Proposition IV.19.

Proof of Proposition IV.21.
As announced, one starts by differentiating equation (IV.2.2). This yields, with the notation M_ε introduced in (IV.5.1) :

$$\begin{aligned}\partial_z\Gamma(t,z) - \varepsilon^2\dot{q}^*(t) - \varepsilon^2\partial_z\partial_t V^*(t,z) - \varepsilon^4\dot{\kappa}_\varepsilon(t) - \varepsilon^4\partial_z\partial_t W_\varepsilon(t,z) = \\ \Gamma(t,z)\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp(\varepsilon^2 M_\varepsilon(t,z)) \\ + \partial_z\Gamma(t,z)\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp(\varepsilon^2 M_\varepsilon(t,z)) \\ + \Gamma(t,z)\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp(\varepsilon^2 M_\varepsilon(t,z))\partial_z M_\varepsilon(t,z).\end{aligned}$$

When we evaluate the expression in $z = z_*$, the last two terms vanish, since $\partial_z\Gamma(t,z_*) = \partial_z M_\varepsilon(t,z_*) = 0$. Therefore, the equation becomes, since $M_\varepsilon(t,z_*) = 0$ and $\Gamma(t,z_*) = 1$,

$$-\varepsilon^2\dot{q}^*(t) - \varepsilon^2\partial_z\partial_t V^*(t,z_*) - \varepsilon^4\dot{\kappa}_\varepsilon(t) - \varepsilon^4\partial_z\partial_t W_\varepsilon(t,z_*) = \partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z_*). \tag{IV.5.3}$$

We then expand the right hand side with respect to ε, *i.e.* we expand $\mathcal{I}_\varepsilon$. There exists $0<\xi<1$ such that

$$\begin{aligned}\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z_*) = \partial_z\mathcal{I}^*_\varepsilon(t,z_*) + \varepsilon^2\left[\partial_g\partial_z\mathcal{I}^*_\varepsilon(t,z_*)\kappa_\varepsilon + \partial_V\partial_z\mathcal{I}^*_\varepsilon\cdot W_\varepsilon(t,z_*)\right] \\ + \frac{\varepsilon^4}{2}\Big[\kappa_\varepsilon^2\partial_g^2\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\xi\kappa_\varepsilon, V^*+\varepsilon^2\xi W_\varepsilon)(t,z) + 2\kappa_\varepsilon\partial_V\partial_g\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\xi\kappa_\varepsilon, V^*+\varepsilon^2\xi W_\varepsilon)\cdot W_\varepsilon(t,z) \\ + \partial_V^2\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\xi\kappa_\varepsilon, V^*+\varepsilon^2\xi W_\varepsilon)(t,z)\cdot(W_\varepsilon,W_\varepsilon)(t,z)\Big].\end{aligned} \tag{IV.5.4}$$

We see that for most of the terms, we provided a careful estimate in the previous section IV.4. First, by Proposition IV.10,

$$\partial_z I^*_\varepsilon(t,z_*) = \varepsilon^2\left(m''(z_*)q^*(t) - \frac{m^{(3)}(z_*)}{2}\right) + O^*(\varepsilon^4).$$

Plugging this in the asymptotic development of (IV.5.4), and adding the estimate of Lemma IV.20 for the exact rests of order 2, we get the following :

$$\begin{aligned}\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z_*) = \varepsilon^2\left(m''(z_*)q^*(t) - \frac{m^{(3)}(z_*)}{2}\right) + \\ \varepsilon^2\left[\partial_g\partial_z\mathcal{I}^*_\varepsilon(t,z_*)\kappa_\varepsilon + \partial_V\partial_z\mathcal{I}^*_\varepsilon\cdot W_\varepsilon(t,z_*)\right] + O^*(\varepsilon^4) + O(\varepsilon^6)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|.\end{aligned}$$

Combining this with the Proposition IV.19 where we got precise expansions at the point z_*, we complete the expansion of $\partial_z\mathcal{I}_\varepsilon$:

$$\begin{aligned}\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z_*) = \\ \varepsilon^2\left(m''(z_*)q^*(t) - \frac{m^{(3)}(z_*)}{2}\right) + \varepsilon^4 R^*(t)\kappa_\varepsilon + \varepsilon^4 S_\varepsilon(W_\varepsilon) + O^*(\varepsilon^4) + O(\varepsilon^6)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|.\end{aligned}$$

When we turn back to equation (IV.5.3), we have shown at this point the following relationship:

$$\begin{aligned}-\varepsilon^2\dot{q}^*(t) - \varepsilon^2\partial_z\partial_t V^*(t,z_*) - \varepsilon^4\dot{\kappa}_\varepsilon(t) - \varepsilon^4\partial_z\partial_t W_\varepsilon(t,z_*) = \\ \varepsilon^2\left(m''(z_*)q^*(t) - \frac{m^{(3)}(z_*)}{2}\right) + \varepsilon^4 R^*(t)\kappa_\varepsilon + \varepsilon^4 S_\varepsilon(W_\varepsilon) + O^*(\varepsilon^4) + O(\varepsilon^6)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|.\end{aligned} \tag{IV.5.5}$$

To get a stable equation on κ_ε, the terms of order ε^2 must vanish. This is precisely where the role of the dynamics of q^* defined in equation (IV.1.16) is highlighted. To see it, we just rewrite a term of equation (IV.5.5) using that $\partial_z W_\varepsilon(t, \mathrm{z}_*) = 0$, and Lemma IV.7:

$$\partial_z \partial_t V^*(t, \mathrm{z}_*) = m'(\mathrm{z}_*)\partial_z^2 V^*(t, \mathrm{z}_*) = 2m'(\mathrm{z}_*)m''(\mathrm{z}_*).$$

Therefore, we recognize that by definition of q^* in (IV.1.16), the following terms cancel:

$$\varepsilon^2 \left(\dot{q}^*(t) + m''(\mathrm{z}_*)q^*(t) - \frac{\partial_z^3 m(\mathrm{z}_*)}{2} + 2m''(\mathrm{z}_*)m'(\mathrm{z}_*) \right) = 0.$$

We then rewrite the second term of equation (IV.5.5) of order ε^4 :

$$\partial_z \partial_t W_\varepsilon(t, \mathrm{z}_*) = m'(\mathrm{z}_*)\partial_z^2 W_\varepsilon(t, \mathrm{z}_*).$$

Finally, equation (IV.5.5) can be rewritten as

$$-\dot{\kappa}_\varepsilon(t) = R^*(t)\kappa_\varepsilon + S_\varepsilon(W_\varepsilon) + O^*(1) + O(\varepsilon^2)\,\|(\kappa_\varepsilon, W_\varepsilon)\| + m'(\mathrm{z}_*(t))\partial_z^2 W_\varepsilon(t, \mathrm{z}_*).$$

We have proven equation (IV.5.2). □

The remaining term $m'(\mathrm{z}_*)\partial_z^2 W_\varepsilon(t, \mathrm{z}_*)$ is not meaningful. At this point, we cannot put it into either $O^*(1)$ or $O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\|$. This will be possible when we have better estimates of $\partial_z^2 W(t, \mathrm{z}_*)$ that we state in Proposition IV.32. The same thins holds true for the number $S_\varepsilon(W_\varepsilon)$.

IV.5.2 *Equation on p_ε*

We chose no to perturb the number p_ε as we did for $(q_\varepsilon, V_\varepsilon)$ since it can be straightforwardly computed from our reference equation (IV.2.2). Given the spectral decomposition of heuristics section IV.2, it is consistent to evaluate equation (IV.2.2) at $z = \mathrm{z}_*(t)$ to gain necessary informations upon p_ε :

$$1 - \varepsilon^2 \Big(\dot{p}_\varepsilon(t) + m'(\mathrm{z}_*)q^*(t) \Big) - \varepsilon^4 m'(\mathrm{z}_*)\kappa_\varepsilon(t) = \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, \mathrm{z}_*). \tag{IV.5.6}$$

Thanks to Propositions IV.8 and IV.14, and as long as κ_ε is bounded, which we will show in section IV.8,

$$\varepsilon^2 \Big(\dot{p}_\varepsilon(t) + \partial_z m(\mathrm{z}_*(t))q^*(t) \Big) = O(\varepsilon^2) \tag{IV.5.7}$$

By letting formally $\varepsilon \to 0$, we find that p^* solves

$$\dot{p}^*(t) + m'(\mathrm{z}_*)q^*(t) = O(1).$$

This is compatible with the what is stated in equation (IV.1.17). More precisely, we can define p_ε as the solution of equation (IV.5.6) associated to the initial condition $p_\varepsilon(0)$ that verifies the assumption of the Theorem IV.4. Then it is easy to see that $p_\varepsilon \to p^*$ when $\varepsilon \to 0$ with Propositions IV.8 and IV.14.

We finally tackle the complete linearization of the equation (IV.2.2). A foretaste was given when we studied the equation on κ_ε, however it was local since we had beforehand evaluated at $z_*(t)$. Here, we will provide global (in space) results. The weight function introduced in the definition of $\mathcal{E}$ is meant to be able to control the behavior at infinity.

IV.6.1 *Linearization for W_ε*

A first step is to control the function M_ε.

Lemma IV.22 (Control of M_ε)**.**
For any ball B of $\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \leqslant \varepsilon_B$, if $W_\varepsilon \in B$, M_ε defined in (IV.5.1) verifies

$$\exp(\varepsilon^2 M_\varepsilon(t,z)) = 1 + \varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|W_\varepsilon\|_{\mathcal{F}}\,.$$

where $O(\varepsilon^4)$ depends only on the ball B.

Proof of Lemma IV.22.
By the choice of the norm in $\mathcal{F}$, and in the setting of $W_\varepsilon \in B$ we have the uniform control for all t, z :

$$|M_\varepsilon(t,z)| \leqslant \|W_\varepsilon\|_{\mathcal{F}}\,.$$

Then, by performing an exact Taylor expansion, there exists $0 \leqslant \xi \leqslant 1$ such that

$$\exp(\varepsilon^2 M_\varepsilon(t,z)) = 1 + \varepsilon^2 M_\varepsilon(t,z) + \frac{\varepsilon^4}{2} M_\varepsilon(t,z)^2 \exp\Big(\varepsilon^2 \xi M_\varepsilon(t,z)\Big).$$

To conclude we uniformly bound the rest for $\varepsilon^2 \leqslant 1/\|W_\varepsilon\|_{\mathcal{F}}$:

$$\left|\frac{\varepsilon^4}{2} M_\varepsilon(t,z)^2 \exp\Big(\varepsilon^2 \xi M_\varepsilon(t,z)\Big)\right| \leqslant \varepsilon^4 \frac{\|W_\varepsilon\|_{\mathcal{F}}^2}{2}.$$

□

This first result gives us will be important to linearize the problem (IV.2.2) solved by $(\kappa_\varepsilon, W_\varepsilon)$. We see that the boundedness of M_ε is necessary to linearize the term $\exp\Big(\varepsilon^2 M(t,z)\Big)$ appearing in equation (IV.2.2). We now write the linearized problem for W_ε.

Proposition IV.23 (Linearization for W_ε)**.**
For any ball B of $\mathbb{R} \times \mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \leqslant \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$ solution of equation (IV.2.2) verifies the following estimate :

$$\varepsilon^2 \partial_t W_\varepsilon(t,z) = \Gamma(t,z)\Big(M_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\Big), \tag{IV.6.1}$$

where $O(\varepsilon)$ depends only on B.

Proof of Proposition IV.23.
One starts from the equation (IV.2.2),

$$\Gamma(t,z) - \varepsilon^2\Big(\dot{p}_\varepsilon(t) + m'(\mathrm{z}_*)q^*(t) + \dot{q}^*(t)(z-\mathrm{z}_*) + \partial_t V^*(t,z)\Big) \\ - \varepsilon^4\Big(\dot{\kappa}_\varepsilon(t)(z-\mathrm{z}_*) + m'(\mathrm{z}_*)\kappa_\varepsilon(t) + \partial_t W_\varepsilon(t,z)\Big) \\ = \Gamma(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp\left(\varepsilon^2 M_\varepsilon(t,z)\right) \quad \text{(IV.6.2)}$$

Thanks to Lemma IV.22 and Proposition IV.14 where we linearized $\mathcal{I}_\varepsilon$ and the term in M_ε, one can expand the right hand side :

$$\Gamma(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp\left(\varepsilon^2 M_\varepsilon(t,z)\right) = \\ \Gamma(t,z)\Big(1 + O^*(\varepsilon^2) + O(\varepsilon^3)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big)\Big(1 + \varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big) \\ = \Gamma(t,z) + \varepsilon^2\Gamma(t,z)M_\varepsilon(t,z) + \Gamma(t,z)\Big(O^*(\varepsilon^2) + O(\varepsilon^3)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big). \quad \text{(IV.6.3)}$$

The left hand side of equation (IV.6.2) is a little bit more involved, we will use our previous work on $(p_\varepsilon, \kappa_\varepsilon)$. First, thanks to equation (IV.5.6) that states the relationship verified by p_ε, we have

$$-\varepsilon^2\Big(\dot{p}_\varepsilon(t) + m'(\mathrm{z}_*)q^*(t)\Big) - \varepsilon^4\kappa_\varepsilon m'(\mathrm{z}_*) = 1 - \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,\mathrm{z}_*).$$

We then use Proposition IV.14 about the linearization of I_ε to get that

$$-\varepsilon^2\Big(\dot{p}_\varepsilon(t) + m'(\mathrm{z}_*)q^*(t)\Big) - \varepsilon^4\kappa_\varepsilon m'(\mathrm{z}_*) = O^*(\varepsilon^2) + O(\varepsilon^3)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|. \quad \text{(IV.6.4)}$$

From Proposition IV.8, we have the following uniform bound :

$$|\partial_t V^*(t,z)| \leqslant K^*. \quad \text{(IV.6.5)}$$

Thanks to our preliminary work on κ_ε, and more precisely the equation (IV.5.5) we know that

$$\dot{q}^*(t) + \varepsilon^2\dot{\kappa}_\varepsilon(t) = O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|.$$

Therefore, the affine terms are comparable to Γ, since Γ admits an uniform lower bound by hypothesis, see equation (IV.1.11) :

$$\left|\frac{\Big(\dot{q}^*(t) + \varepsilon^2\dot{\kappa}_\varepsilon(t)\Big)(z-\mathrm{z}_*)}{\Gamma(t,z)}\right| = O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|. \quad \text{(IV.6.6)}$$

When adding up the estimates of (IV.6.5) and (IV.6.6), we have shown :

$$-\varepsilon^2\Big(\dot{p}_\varepsilon(t) + m'(\mathrm{z}_*)q^*(t) + \dot{q}^*(t)(z-\mathrm{z}_*) + \partial_t V^*(t,z)\Big) - \\ \varepsilon^4\Big(\dot{\kappa}_\varepsilon(t)(z-\mathrm{z}_*) + \partial_z m(\mathrm{z}_*(t))\kappa_\varepsilon(t) + \partial_t W_\varepsilon(t,z)\Big) = \Gamma(t,z)\Big(O^*(\varepsilon^2) + O(\varepsilon^3)\Big) - \varepsilon^4\partial_t W_\varepsilon(t,z). \quad \text{(IV.6.7)}$$

We have divided by Γ the relationships (IV.6.4) and (IV.6.5), which is possible thanks to the uniform lower bound of Γ.

Finally, when putting together (IV.6.6) and (IV.6.3) in equation (IV.6.2), the terms Γ cancel each other, and we find equation (IV.6.1) factoring out ε^2. □

One can notice the similarity between what we just proved rigorously and the heuristics made in equation (IV.2.1). From this result one can straightforwardly deduce a linear approximated equation verified by M_ε.

Corollary IV.24 (Linearization in $M_\varepsilon(t,z)$)**.**
For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon\leqslant\varepsilon_B$, any pair $(\kappa_\varepsilon,W_\varepsilon)\in B$ verifies the following estimate

$$\varepsilon^2\partial_t M_\varepsilon(t,z)=\Gamma(t,z)\left(2\frac{\Gamma(t,\bar z(t))}{\Gamma(t,z)}M_\varepsilon(t,\bar z(t))-M_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|_B\right) \tag{IV.6.8}$$

where the $O(\varepsilon)$ depends only on B.

IV.6.2 *Linearization for $\partial_z W_\varepsilon$*

The computations for $\partial_z W_\varepsilon$ are slightly more complex because of the differentiation of the triple product in the right-hand side equation (IV.2.2). However, the key point is that when we linearize $\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2\kappa_\varepsilon)$ the derivatives of $\mathcal{I}_\varepsilon$ are negligible in ε. Therefore the intuitive linearized problem for $\partial_z W_\varepsilon$, given by the derivation of the linearized equation for W_ε, actually holds true. This is the content of the following proposition :

Proposition IV.25 (Linearization in $\partial_z W_\varepsilon$)**.**
For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon\leqslant\varepsilon_B$, any pair $(\kappa_\varepsilon,W_\varepsilon)\in B$ solution of equation (IV.2.2) verifies the following estimate :

$$\begin{aligned}\varepsilon^2\partial_t\partial_z W_\varepsilon(t,z)=&\\ \Gamma(t,z)\Big(1+O^*(\varepsilon)+O(\varepsilon^2)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)&\left(\partial_z M_\varepsilon(t,z)+\frac{O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{1+|z-z_*(t)|}\right)\\ &+\partial_z\Gamma(t,z)\Big(M_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big),\end{aligned} \tag{IV.6.9}$$

where $O(\varepsilon)$ depends only on B.

Remark IV.26. *Under the same conditions, W_ε also is solution of the following linearized problem*

$$\varepsilon^2\partial_t\partial_z W_\varepsilon(t,z)=\Gamma(t,z)\Big(\partial_z M_\varepsilon(t,z)+O(1)\Big)+\partial_z\Gamma(t,z)\Big(M_\varepsilon(t,z)+O(\varepsilon)\Big). \tag{IV.6.10}$$

However in the next sections we will see that one need precise estimates about the structure of the nonlinear negligible terms, which explains the more detailed equation (IV.6.9), that may seem surprising at first.

Proof of Proposition IV.25.
One starts by differentiating equation (IV.2.2) as in the proof of Proposition IV.21 to highlight κ_ε. This yields :

$$\begin{aligned}\partial_z\Gamma(t,z)-\varepsilon^2\dot q^*(t)-\varepsilon^2\partial_z\partial_t V^*(t,z)-&\varepsilon^4\dot\kappa_\varepsilon(t)-\varepsilon^4\partial_z\partial_t W_\varepsilon(t,z)=\\ &\Gamma(t,z)\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2W_\varepsilon)(t,z)\exp(\varepsilon^2M_\varepsilon(t,z))\\ &+\partial_z\Gamma(t,z)\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2W_\varepsilon)(t,z)\exp(\varepsilon^2M_\varepsilon(t,z))\\ &+\Gamma(t,z)\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2W_\varepsilon)(t,z)\exp(\varepsilon^2M_\varepsilon(t,z))\partial_z M_\varepsilon(t,z).\end{aligned}$$

However contrary to the case where we were studying $\dot{\kappa}_\varepsilon$, we will not evaluate in z_*. We introduce the notations R_i corresponding to each of the three terms of the right hand side of the previous equation. We will linearize each R_i starting with R_1 where we linearize thanks to Proposition IV.17 and Lemma IV.22, paired with the estimate of Proposition IV.11 :

$$\begin{aligned}
R_1 :=& \partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\Gamma(t,z)\exp(\varepsilon^2 M_\varepsilon(t,z)) \\
=& \Gamma(t,z)\left(\partial_z I_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\left(1+\varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right) \\
=& \Gamma(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\left(1+\varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right).
\end{aligned}$$

Therefore, the final contribution of R_1 is:

$$R_1 = \Gamma(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right). \tag{IV.6.11}$$

Next, one looks at R_2. Thanks to Proposition IV.14,

$$\begin{aligned}
R_2 :=& \partial_z\Gamma(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp(\varepsilon^2 M_\varepsilon(t,z)) \\
=& \partial_z\Gamma(t,z)\Big(1 + O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\Big)\Big(1+\varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\Big), \\
=& \partial_z\Gamma(t,z) + \varepsilon^2\partial_z\Gamma(t,z)M_\varepsilon(t,z) + \partial_z\Gamma(t,z)\Big(O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\Big).
\end{aligned} \tag{IV.6.12}$$

We finally tackle R_3 with the same techniques, using Proposition IV.14 and Lemma IV.22 :

$$\begin{aligned}
R_3 :=& \varepsilon^2\Gamma(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp(\varepsilon^2 M_\varepsilon(t,z))\partial_z M_\varepsilon(t,z), \\
=& \varepsilon^2\Gamma(t,z)\partial_z M_\varepsilon(t,z)\Big(1 + O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\Big)\Big(1+\varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\Big), \\
=& \varepsilon^2\Gamma(t,z)\Big(1 + O^*(\varepsilon) + O(\varepsilon^2)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\Big)\partial_z M_\varepsilon(t,z).
\end{aligned} \tag{IV.6.13}$$

At this point, we have deliberately "lost" an order of precision in ε in the term $O^*(\varepsilon)$. The motivation is to always keep the precision in O^* lower than the rest of the terms, just for clarity and consistency reasons throughout this paper. Finally, if we come back to our initial problem, when we assemble equations (IV.6.11) to (IV.6.13).

$$\begin{aligned}
&\partial_z\Gamma(t,z) - \varepsilon^2\dot{q}^*(t) - \varepsilon^2\partial_z\partial_t V^*(t,z) - \varepsilon^4\dot{\kappa}_\varepsilon(t) - \varepsilon^4\partial_z\partial_t W_\varepsilon(t,z) \\
&\qquad = \partial_z\Gamma(t,z) + \varepsilon^2\partial_z\Gamma(t,z)\Big(M_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\Big) + \\
&\varepsilon^2\Gamma(t,z)\Big(1 + O^*(\varepsilon^2) + O(\varepsilon^2)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\Big)\left(\partial_z M_\varepsilon(t,z) + \frac{O^*(\varepsilon) + O(\varepsilon^2)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right),
\end{aligned} \tag{IV.6.14}$$

where, for the last term, we used that

$$\frac{O^*(\varepsilon) + O(\varepsilon^2)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1 + O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|} = O^*(\varepsilon) + O(\varepsilon^2)\,\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

We now deal with the left hand side of the equation (IV.6.14). First, the terms $\partial_z\Gamma(t,z)$ on each side cancel. Moreover, as in the proof of Proposition IV.23, thanks to equation (IV.5.5), we have

$$\begin{aligned}
&-\varepsilon^2\Big(\dot{q}^*(t) + \partial_z\partial_t V^*(t,z) + \varepsilon^2\dot{\kappa}_\varepsilon(t)\Big) - \varepsilon^4\partial_z\partial_t W_\varepsilon(t,z) = \\
&-\varepsilon^4\partial_z\partial_t W_\varepsilon(t,z) - \varepsilon^2\Big(\partial_z\partial_t V^*(t,z) - \partial_z\partial_t V^*(t,z_*)\Big) + O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|.
\end{aligned} \tag{IV.6.15}$$

But, with Proposition IV.8

$$\varepsilon^2 \left|\partial_z \partial_t V^*(t,z)\right| \leqslant O^*(\varepsilon^2).$$

Finally, if we divide by Γ, the following estimate holds true.

$$\left|\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|}{\Gamma(t,z)}\right| \leqslant \frac{O^*(\varepsilon^2) + O(\varepsilon^3)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|}{1+|z-z_*(t)|}$$

Plugging this into equation (IV.6.14), and dividing each side by ε^2, we therefore recover the relationship we wanted to prove :

$$\varepsilon^2 \partial_t \partial_z W_\varepsilon(t,z) = \Gamma(t,z)\Big(1 + O^*(\varepsilon^2) + O(\varepsilon^3)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big) \times \left(\partial_z M_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|}{1+|z-z_*(t)|}\right) + \partial_z \Gamma(t,z)\Big(M_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big).$$

□

We deduce straightforwardly a linearization result upon the quantity $\partial_z M_\varepsilon$.

Corollary IV.27 (Linearization for $\partial_z M_\varepsilon(t,z)$)**.**
For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \leqslant \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$ solution of equation (IV.2.2) verifies the following estimate :

$$\varepsilon^2 \partial_t \partial_z M_\varepsilon(t,z) = \Gamma(t,z)\Big(1 + O^*(\varepsilon) + O(\varepsilon^2)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big)\left[\frac{\Gamma(t,\bar z)}{\Gamma(t,z)}\partial_z M_\varepsilon(t,\bar z) - \partial_z M_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|}{1+|z-z_*(t)|}\right] + \partial_z\Gamma(t,z)\left[\frac{\partial_z\Gamma(t,\bar z)}{\partial_z\Gamma(t,z)} M_\varepsilon(t,\bar z) - M_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\right].$$

where the $O(\varepsilon)$ depends only on B.

IV.6.3 *Linearization for $\partial_z^2 W_\varepsilon(t,z)$*

We now tackle the linearized equation for $\partial_z^2 W_\varepsilon$. Every term is controlled as in the linearization of $\partial_z \mathcal{I}_\varepsilon$.

Proposition IV.28 (Linearization for $\partial_z^2 W_\varepsilon$)**.**
For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \leqslant \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$ solution of equation (IV.2.2) verifies the following estimate:

$$-\varepsilon^2 \partial_t \partial_z^2 W_\varepsilon(t,z) = \partial_z^2\Gamma(t,z)\Big(M_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big) + 2\partial_z\Gamma(t,z)\Big(\partial_z M_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big) + \Gamma(t,z)\Big(1 + O^*(\varepsilon) + O(\varepsilon^2)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big) \times \left(\partial_z^2 W_\varepsilon(t,z) - \frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z) + \frac{O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|}{1+|z-z_*(t)|}\right), \quad \text{(IV.6.16)}$$

where the $O(\varepsilon)$ depends only on B.

We choose to write the second derivative $\partial_z^2 M_\varepsilon(t,z)$ in full : $\partial_z^2 W_\varepsilon(t,z) - \frac{1}{2}\,\partial_z^2 W_\varepsilon(t,\overline{z})$ as the factor $\frac{1}{2}$ will be the key to ensure the boundedness of $\partial_z^2 W_\varepsilon$, see section IV.7.

Proof of Proposition IV.28.
We start by differentiating twice equation (IV.2.2). This yields :

$$\partial_z^2\Gamma(t,z) - \varepsilon^2\partial_z^2\partial_t V^*(t,z) - \varepsilon^4\partial_z^2\partial_t W_\varepsilon(t,z) = R_1 + R_2 + R_3 + R_4 + R_5 + R_6,$$

with the following notations :

$$\begin{aligned}
R_1 &:= \partial_z^2\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\Gamma(t,z)\exp(\varepsilon^2 M_\varepsilon(t,z)),\\
R_2 &:= 2\partial_z\Gamma(t,z)\partial_z\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp(\varepsilon^2 M_\varepsilon(t,z)),\\
R_3 &:= 2\Gamma(t,z)\varepsilon^2\partial_z\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp(\varepsilon^2 M_\varepsilon(t,z))\partial_z M_\varepsilon(t,z),\\
R_4 &:= \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\partial_z^2\Gamma(t,z) \times \exp(\varepsilon^2 M_\varepsilon(t,z)),\\
R_5 &:= 2\varepsilon^2\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\partial_z\Gamma(t,z)\exp(\varepsilon^2 M_\varepsilon(t,z))\partial_z M_\varepsilon(t,z),
\end{aligned}$$

and finally :

$$\begin{aligned}
R_6 := \varepsilon^2\Gamma(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp(\varepsilon^2 M_\varepsilon(t,z))\\
\times\left(\varepsilon^2\partial_z M_\varepsilon(t,z)^2 + \partial_z^2 W_\varepsilon(t,z) - \frac{1}{2}\partial_z^2 W_\varepsilon(t,\overline{z})\right).
\end{aligned}$$

We will estimate each term separately, starting with R_1, for which we apply the Proposition IV.17 and Lemma IV.22 :

$$\begin{aligned}
R_1 =&\Gamma(t,z)\left(\partial_z^2 I_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\left(1 + \varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right)\\
=&\Gamma(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\left(1 + \varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right).
\end{aligned}$$

Therefore, the final estimate of R_1 is :

$$R_1 = \Gamma(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right). \tag{IV.6.17}$$

Next, for the other term R_2 we use Propositions IV.11 and IV.17 :

$$\begin{aligned}
R_2 =&2\left(\partial_z I_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\partial_z\Gamma(t,z)\left(1 + \varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right),\\
=&2\partial_z\Gamma(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\left(1 + \varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right).
\end{aligned}$$

We can simplify this expression :

$$R_2 = 2\partial_z\Gamma(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right). \tag{IV.6.18}$$

The term, R_3 will not contribute at the order ε^2, because of Proposition IV.11, and $|\partial_z M_\varepsilon(t,z)| \leqslant \|W_\varepsilon\|_{\mathcal{F}}$:

$$\begin{aligned}
R_3 =&2\varepsilon^2\Gamma(t,z)\partial_z M_\varepsilon(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\left(1 + \varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right)\\
=&\frac{O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\Gamma(t,z).
\end{aligned} \tag{IV.6.19}$$

For R_4, zeroth order terms are more entangled. With Proposition IV.14 and Lemma IV.22 :

$$\begin{aligned} R_4 &= \partial_z^2\Gamma(t,z)\Big(1+O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\Big(1+\varepsilon^2 M_\varepsilon(t,z)+O(\varepsilon^4)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big), \\ &= \partial_z^2\Gamma(t,z)+\varepsilon^2\partial_z^2\Gamma(t,z)\Big(M_\varepsilon(t,z)+O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big). \end{aligned} \tag{IV.6.20}$$

We see in R_4 the appearance of the term $\varepsilon^2\partial_z^2\Gamma(t,z)M_\varepsilon(t,z)$ that is also in equation (IV.6.16), and so it is a good opportunity to do a first a summary of the computations when adding equations (IV.6.17) to (IV.6.20) :

$$\begin{aligned} R_1+R_2+R_3+R_4 = \partial_z^2\Gamma(t,z)+\varepsilon^2\partial_z^2\Gamma(t,z)\Big(M_\varepsilon(t,z)+O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big) \\ +\Big(\Gamma(t,z)+\partial_z\Gamma(t,z)\Big)\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}. \end{aligned} \tag{IV.6.21}$$

We continue the estimations by looking at R_5, thanks to Proposition IV.14 :

$$\begin{aligned} R_5 &= 2\varepsilon^2\partial_z\Gamma(t,z)\partial_z M_\varepsilon(t,z)\Big(1+O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\Big[1+\varepsilon^2 M_\varepsilon(t,z)+O(\varepsilon^4)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big], \\ &= 2\varepsilon^2\partial_z\Gamma(t,z)\partial_z M_\varepsilon(t,z)+\partial_z\Gamma(t,z)\Big(O^*(\varepsilon^3)+O(\varepsilon^4)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big). \end{aligned} \tag{IV.6.22}$$

Finally, we tackle the last term, R_6, with Proposition IV.14

$$\begin{aligned} R_6 = \varepsilon^2\Gamma(t,z)\Big(1+O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\Big(1+\varepsilon^2 M_\varepsilon(t,z)+O(\varepsilon^4)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big) \\ \times\left(\frac{O(\varepsilon^2)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}+\partial_z^2 W_\varepsilon(t,z)-\frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z)\right). \end{aligned}$$

$$R_6 = \varepsilon^2\Gamma(t,z)\Big(1+O^*(\varepsilon^2)+O(\varepsilon^2)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\left(\partial_z^2 W_\varepsilon(t,z)-\frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z(t))+\frac{O(\varepsilon^2)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}\right). \tag{IV.6.23}$$

Thanks to those last two estimates (IV.6.22) and (IV.6.23), that we add with the previous result of equation (IV.6.21), we obtain for the full equation :

$$\begin{aligned} &\partial_z^2\Gamma(t,z)-\varepsilon^2\partial_z^2\partial_t V^*(t,z)-\varepsilon^4\partial_z^2\partial_t W_\varepsilon(t,z) = \partial_z^2\Gamma(t,z)+ \\ &\varepsilon^2\partial_z^2\Gamma(t,z)\Big(M_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)+2\varepsilon^2\partial_z\Gamma(t,z)\Big(\partial_z M_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big) \\ &+\varepsilon^2\Gamma(t,z)\Big(1+O^*(\varepsilon)+O(\varepsilon^2)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\left(\partial_z^2 W_\varepsilon(t,z)-\frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z)+\frac{O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}\right). \end{aligned}$$

Thanks to Proposition IV.8 we know that $\left\|\varepsilon^2\partial_z^2\partial_t V^*(t,z)\right\|_\infty \leqslant O^*(\varepsilon^2)$. Then,

$$\begin{aligned} -\varepsilon^4\partial_z^2\partial_t W_\varepsilon(t,t) = \varepsilon^2\partial_z^2\Gamma(t,z)\Big(M_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)+ \\ 2\varepsilon^2\partial_z\Gamma(t,z)\Big(\partial_z M_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)+ \\ \varepsilon^2\Gamma(t,z)\Big(1+O^*(\varepsilon)+O(\varepsilon^2)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\left(\partial_z^2 W_\varepsilon(t,z)-\frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z)+\frac{O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}\right). \end{aligned}$$

which proves equation (IV.6.16) after dividing by ε^2. □

IV.6.4 *Linearization of $\partial_z^3 W_\varepsilon(t,z)$*

Our last linearized equation is the one for $\partial_z^3 W_\varepsilon$ and we proceed with the same technique.

Proposition IV.29 (Linearization in $\partial_z^3 W_\varepsilon$)**.**
For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon\leqslant\varepsilon_B$, any pair $(\kappa_\varepsilon,W_\varepsilon)\in B$ solution of equation (IV.2.2) verifies the following estimate:

$$\begin{aligned}-\varepsilon^2\partial_t\partial_z^3 W_\varepsilon(t,z) = \partial_z^3\Gamma(t,z)\Big(M_\varepsilon(t,z)+O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)+\ 3\partial_z^2\Gamma(t,z)\times\\ \Big(\partial_z M_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)+\ 3\partial_z\Gamma(t,z)\times\\ \Big(\partial_z^2 W_\varepsilon(t,z)-\frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z)+O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)+\Gamma(t,z)\Big(1+O^*(\varepsilon)+O(\varepsilon^2)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\\ \times\left(\partial_z^3 W_\varepsilon(t,z)-\frac{1}{4}\partial_z^3 W_\varepsilon(t,\bar z)+\frac{1}{2}\|W_\varepsilon\|_{\mathcal{F}}+\ \frac{O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}\right).\end{aligned}\tag{IV.6.24}$$

where the $O(\varepsilon)$ depends only on B.

Proof of Proposition IV.28.
We start, as ever, by differentiating equation (IV.2.2), but now three times. This yields for the right hand side ten terms :

$$\partial_z^3\Gamma(t,z)-\varepsilon^2\partial_z^3\partial_t V^*(t,z)-\varepsilon^4\partial_z^3\partial_t W_\varepsilon(t,t)=R_1+R_2+R_3+R_4+R_5+R_6+R_7+R_8+R_9+R_{10},\tag{IV.6.25}$$

with the following notations :

$$\begin{aligned}R_1&:=\partial_z^3\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2W_\varepsilon)(t,z)\Gamma(t,z)\exp(\varepsilon^2M_\varepsilon(t,z)),\\ R_2&:=3\partial_z^2\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2W_\varepsilon)(t,z)\partial_z\Gamma(t,z)\exp(\varepsilon^2M_\varepsilon(t,z)),\\ R_3&:=3\varepsilon^2\partial_z^2\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2W_\varepsilon)(t,z)\Gamma(t,z)\exp(\varepsilon^2M_\varepsilon(t,z))\partial_z M_\varepsilon(t,z),\\ R_4&:=6\varepsilon^2\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2W_\varepsilon)(t,z)\partial_z\Gamma(t,z)\exp(\varepsilon^2M_\varepsilon(t,z))\partial_z M_\varepsilon(t,z),\\ R_5&:=3\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2W_\varepsilon)(t,z)\partial_z^2\Gamma(t,z)\exp(\varepsilon^2M_\varepsilon(t,z)),\end{aligned}$$

and moreover :

$$R_6:=3\varepsilon^2\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2W_\varepsilon)(t,z)\Gamma(t,z)\exp(\varepsilon^2M_\varepsilon(t,z))\Big(\varepsilon^2\partial_z M_\varepsilon(t,z)^2+\partial_z^2 M_\varepsilon(t,z)\Big),$$

$$R_7:=3\varepsilon^2\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2W_\varepsilon)(t,z)\partial_z\Gamma(t,z)\exp(\varepsilon^2M_\varepsilon(t,z))\Big(\varepsilon^2\partial_z M_\varepsilon(t,z)^2+\partial_z^2 M_\varepsilon(t,z)\Big),$$

$$\begin{aligned}R_8&:=3\varepsilon^2\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2W_\varepsilon)(t,z)\partial_z^2\Gamma(t,z)\exp(\varepsilon^2M_\varepsilon(t,z))\partial_z M_\varepsilon(t,z),\\ R_9&:=\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon,V^*+\varepsilon^2W_\varepsilon)(t,z)\partial_z^3\Gamma(t,z)\exp(\varepsilon^2M_\varepsilon(t,z)).\end{aligned}$$

The last term corresponds to the third derivative of the exponential term $\exp(\varepsilon^2 M_\varepsilon)$.

$$R_{10} := \varepsilon^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\Gamma(t,z)\exp(\varepsilon^2 M_\varepsilon(t,z)) \\ \times \left(\varepsilon^4 \partial_z M_\varepsilon(t,z)^3 + 3\varepsilon^2 \partial_z M_\varepsilon(t,z)\partial_z^2 M_\varepsilon(t,z) + \partial_z^3 W_\varepsilon(t,z) - \frac{1}{4}\partial_z^2 W_\varepsilon(t,\overline{z})\right).$$

We first tackle R_1. We use the linearization of the third derivative of $\mathcal{I}_\varepsilon$ in Proposition IV.17.

$$\begin{aligned} R_1 =& \Gamma(t,z)\left(\partial_z^3 I_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|} + \frac{\varepsilon^2}{2}\|W_\varepsilon\|_{\mathcal{E}}\right)\left(1+\varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right) \\ =& \varepsilon^2\Gamma(t,z)\left(\frac{O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|} + \frac{1}{2}\|W_\varepsilon\|_{\mathcal{E}}\right)\left(1+\varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right). \end{aligned}$$

We end up with the following estimate

$$R_1 = \varepsilon^2\Gamma(t,z)\left(\frac{O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|} + \frac{1}{2}\|W_\varepsilon\|_{\mathcal{F}}\right). \tag{IV.6.26}$$

For R_2, with Proposition IV.17 we have

$$\begin{aligned} R_2 =& 3\partial_z\Gamma(t,z)\left(\partial_z^2 I_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\left(1+\varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right), \\ =& 3\partial_z\Gamma(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\left(1+\varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right). \end{aligned}$$

We can simplify this expression to

$$R_2 = \varepsilon^2\partial_z\Gamma(t,z)\left(\frac{O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right). \tag{IV.6.27}$$

For R_3 we get

$$\begin{aligned} R_3 =& 3\varepsilon^2\Gamma(t,z)\partial_z M_\varepsilon(t,z)\left(\partial_z^2 I_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\left(1+\varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right), \\ =& 3\varepsilon^2\Gamma(t,z)\partial_z M_\varepsilon(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\left(1+\varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right). \end{aligned}$$

We can simplify this expression to

$$R_3 = \frac{O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\Gamma(t,z). \tag{IV.6.28}$$

For R_4 one has very similarly

$$\begin{aligned} R_4 =& 6\varepsilon^2\partial_z\Gamma(t,z)\partial_z M_\varepsilon(t,z)\left(\partial_z I_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\left(1+\varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right), \\ =& 6\varepsilon^2\partial_z\Gamma(t,z)\partial_z M_\varepsilon(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\left(1+\varepsilon^2 M_\varepsilon(t,z) + O(\varepsilon^4)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right). \end{aligned}$$

We can simplify this expression to

$$R_4 = \frac{O(\varepsilon^3)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}\partial_z\Gamma(t,z). \tag{IV.6.29}$$

The expression for R_5 still follows the same road

$$\begin{aligned} R_5 =& 3\partial_z^2\Gamma(t,z)\left(\partial_z I_\varepsilon^*(t,z)+\frac{O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\Big(1+\varepsilon^2 M_\varepsilon(t,z)+O(\varepsilon^4)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big),\\ =& 3\partial_z^2\Gamma(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\Big(1+\varepsilon^2 M_\varepsilon(t,z)+O(\varepsilon^4)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big). \end{aligned}$$

The last expression can be shortened in

$$R_5 = 3\varepsilon^2\partial_z^2\Gamma(t,z)\frac{O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}. \tag{IV.6.30}$$

For R_6, the expression is a little more involved due to the second derivative of the exponential

$$\begin{aligned} R_6 = 3\varepsilon^2\Gamma(t,z)&\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}\right)\Big(1+\varepsilon^2 M_\varepsilon(t,z)+O(\varepsilon^4)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\\ &\times\left(\frac{O(\varepsilon^2)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}+\partial_z^2 W_\varepsilon(t,z)-\frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z)\right). \end{aligned}$$

We eventually shorten R_6 as

$$R_6 = 3\Gamma(t,z)\frac{O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}. \tag{IV.6.31}$$

If we bridge together all of our previous estimates in (IV.6.26), (IV.6.27), (IV.6.28), (IV.6.29) and (IV.6.30), (IV.6.31) we obtain that

$$\begin{aligned} &R_1+R_2+R_3+R_4+R_5+R_6 =\\ &\varepsilon^2\Big(\Gamma(t,z)+\partial_z\Gamma(t,z)+\partial_z^2\Gamma(t,z)\Big)\left(\frac{O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}\right)+\frac{\varepsilon^2}{2}\Gamma(t,z)\,\|W_\varepsilon\|_{\mathcal F}. \end{aligned} \tag{IV.6.32}$$

In that first round of estimates, we have shown that all the contributions of the terms with the derivatives of $\mathcal{I}_\varepsilon$ do not appear when linearizing because they are of high order in ε. Therefore, the most meaningful contribution will now appear, because $\mathcal{I}_\varepsilon$ now contributes mainly as 1 and no longer vanishes.

We start with R_7 :

$$\begin{aligned} R_7 = 3\varepsilon^2\partial_z\Gamma(t,z)&\Big(1+O^*(\varepsilon)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\Big(1+\varepsilon^2 M_\varepsilon(t,z)+O(\varepsilon^4)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\\ &\times\left(\frac{O(\varepsilon^2)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}+\partial_z^2 W_\varepsilon(t,z)-\frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z)\right), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &R_7 =\\ &3\varepsilon^2\partial_z\Gamma(t,z)\Big(1+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|+O^*(\varepsilon^2)\Big)\left(\partial_z^2 W_\varepsilon(t,z)-\frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z)+\frac{O(\varepsilon^2)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}\right). \end{aligned}$$

Finally, for R_7 :

$$R_7 = 3\varepsilon^2\partial_z\Gamma(t,z)\left(\partial_z^2 W_\varepsilon(t,z)-\frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z)\right)+\partial_z\Gamma(t,z)\Big(O^*(\varepsilon^3)+O(\varepsilon^4)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big). \tag{IV.6.33}$$

For R_8, the following estimates hold true,

$$R_8 = 3\varepsilon^2\partial_z^2\Gamma(t,z)\partial_z M_\varepsilon(t,z)\Big(1+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|+O^*(\varepsilon^2)\Big)\partial_z M_\varepsilon(t,z).$$

Therefore

$$R_8 = 3\varepsilon^2\partial_z^2\Gamma(t,z)\partial_z M_\varepsilon(t,z)+\partial_z^2\Gamma(t,z)\Big(O^*(\varepsilon^3)+O(\varepsilon^4)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big). \tag{IV.6.34}$$

For the last two terms, the derivatives up to the third order appear. The simplest is given by R_9 :

$$\begin{aligned} R_9 &= \partial_z^3\Gamma(t,z)\Big(1+O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\Big(1+\varepsilon^2M_\varepsilon(t,z)+O(\varepsilon^4)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big),\\ &= \partial_z^3\Gamma(t,z)+\varepsilon^2\partial_z^3\Gamma(t,z)\Big(M_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big). \end{aligned} \tag{IV.6.35}$$

At last, for the term R_{10},

$$\begin{aligned} R_{10} = \varepsilon^2\Gamma(t,z)\Big(1+O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\Big(1+\varepsilon^2M_\varepsilon(t,z)+O(\varepsilon^4)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)\\ \times\left(\frac{O(\varepsilon^2)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}+\partial_z^3W_\varepsilon(t,z)-\frac14\partial_z^3W_\varepsilon(t,\bar z)\right). \end{aligned} \tag{IV.6.36}$$

It is shortened to

$$R_{10} = \varepsilon^2\Gamma(t,z)\Big(1+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|+O^*(\varepsilon^2)\Big)\left(\partial_z^3W_\varepsilon(t,z)-\frac14\partial_z^3W_\varepsilon(t,\bar z)+\frac{O(\varepsilon^2)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}\right). \tag{IV.6.37}$$

We now add every estimate, starting from (IV.6.32) and with (IV.6.33), (IV.6.34), (IV.6.35) and (IV.6.37) to obtain

$$\begin{aligned} \sum_{j=1}^{10}R_j = \varepsilon^2\Gamma(t,z)\Big(1+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|+O^*(\varepsilon^2)\Big)\left(\partial_z^3W_\varepsilon(t,z)-\frac14\partial_z^3W_\varepsilon(t,\bar z)+\right.\\ \left.\frac{O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|}{1+|z-z_*(t)|}+\frac12\,\|W_\varepsilon\|_{\mathcal F}\right)+3\varepsilon^2\partial_z\Gamma(t,z)\times\\ \left(\partial_z^2W_\varepsilon(t,z)-\frac12\partial_z^2W_\varepsilon(t,\bar z)+O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\right)+3\varepsilon^2\partial_z^2\Gamma(t,z)\Big(\partial_zM_\varepsilon(t,z)\\ +O^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big)+\partial_z^3\Gamma(t,z)+\varepsilon^2\partial_z^3\Gamma(t,z)\Big(M_\varepsilon(t,z)+O^*(\varepsilon^2)+O(\varepsilon^3)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\Big). \end{aligned} \tag{IV.6.38}$$

To conclude the proof, we deal with the left hand side of equation (IV.6.25) as in the linearization of the second derivative, noticing that the terms $\partial_z^3\Gamma$ cancel on each side. □

IV.7 STABILITY OF THE LINEARIZED EQUATIONS

Building upon the series of linear approximations, we can study the stability of W_ε in the space $\mathcal{F}$. The first result is to control the different terms of $\mathcal{F}$.

Theorem IV.30 (Stability analysis)**.**
For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B, a ball B_0 of $\mathbb{R}$ that depends only on V^* such that for all $\varepsilon\leqslant\varepsilon_B$, a pair $(\kappa_\varepsilon,W_\varepsilon)\in B$ solution of equation (IV.2.2) verifies the following bounds :

$$\begin{aligned}
\|M_\varepsilon\|_\infty^0 &\leqslant O_0^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\,,\\
\|\partial_z M_\varepsilon\|_\infty^0 &\leqslant O_0^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\,,\\
\left\|\Big(1+|\cdot-z_*|\Big)\partial_z^2 W_\varepsilon\right\|_\infty^0 &\leqslant O_0^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|\,,\\
\left\|\Big(1+|\cdot-z_*|\Big)\partial_z^3 W_\varepsilon\right\|_\infty^0 &\leqslant O_0^*(1)+O(\varepsilon)\,\|(\kappa_\varepsilon,W_\varepsilon)\|+\frac{2}{3}\,\|W_\varepsilon\|_{\mathcal{F}}\,.
\end{aligned}$$

where $\|W\|_\infty^0:=\sup_{(t,z)\in\ \mathbb{R}_+\times B_0}|W(t,z)|$ *and* $O_0^*(1)=\max\Big(O^*(1),\|W_\varepsilon(0,\cdot)\|_{\mathcal{F}}\Big)$.

The proof of this theorem is quite intricate and will be divided in several subsections. The plan is a follows :

- First, we focus on a small ball around $z_*(t)$.
- Second, we propagate this bound on the whole space by dividing it in successive dyadic rings centered around z_*.

The main arguments will be the maximum principle and an ad hoc division of the space that accounts for the non local nature of the infinitesimal operator. The purpose of this dyadic decomposition in rings is to obtain a decay of the norm with respect to the radius of the rings.

IV.7.1 *Division of the space in a ball surrounded by dyadic rings*

Let us first consider a time T_*. Then for all times such that $0\leqslant t,s\leqslant T_*$, the inequality

$$|z_*(t)-z_*(s)|\leqslant\sup_{s\geqslant 0}\left|m'(z_*(s))\right|T_*:=R_*$$

holds true, and the supremum is finite because z_* lives in a bounded domain uniquely determined by m and $z_*(0)$, see equation (IV.1.5).

We slightly expand this ball by a constant R_0 to be defined later and define

$$B_0:=\{z\text{ such that }\ |z-z_0|\leqslant R_0+R_*\}\,.$$

Our intention behind the choice is that the ball B_0 verifies the following property :

$$\forall t\leqslant T_*,\ z\in B_0,|z-\bar z(t)|=\frac{|z-z_*(t)|}{2}=\frac{|z-z_*(0)+z_*(0)-z_*(t)|}{2}\leqslant\frac{R_0}{2}+R_*.\quad\text{(IV.7.1)}$$

We recall that $\bar z(t):=\dfrac{z+z_*(t)}{2}$. We will split the rest of the space around B_0 with successive dyadic rings. The first ring is defined as $D_1=\{\ z:R_0+R_*\leqslant|z-z_*(0)|\leqslant 2R_0+R_*\}$ It verifies for every $t\leqslant T_*$ the following identity on the middle point :

$$\begin{aligned}
\left|\frac{z+z_*(t)}{2}-z_0\right|&\leqslant\left|\frac{z-z_*(0)}{2}\right|+\left|\frac{z_*(0)-z_*(t)}{2}\right|,\\
&\leqslant R_0+R_*.
\end{aligned}$$

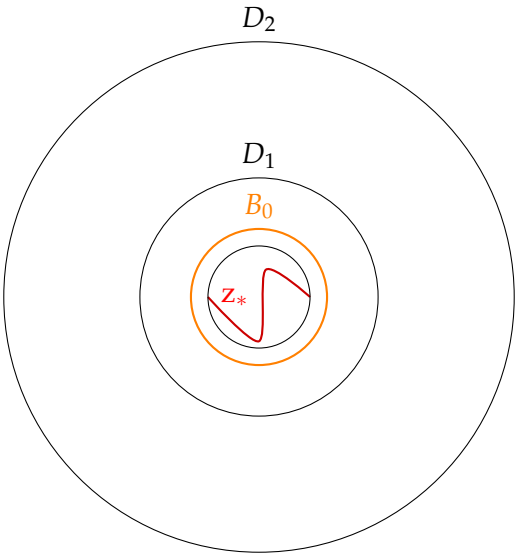


Figure IV.2: Illustration of the division of the space in successive dyadic rings.

This shows that for any $z \in D_1$, and time $t \leqslant T_*$, any middle point $\bar{z}(t)$ lies in B_0. More generally, the following lemma holds true if we define for $n \geqslant 2$: $D_n := \left\{ 2^{n-1}R_0 + R_* \leqslant |z - z_0| \leqslant 2^n R_0 + R_* \right\}$,

Lemma IV.31 (Middle point property)**.**
For every time $0 \leqslant t \leqslant T_*$ *:*

$$\forall n \geqslant 1 \ \forall z \in D_n, \quad \bar{z}(t) \ \in D_{n-1},$$

with the convention $D_0 = B_0$.

The following inequalities are a direct consequence of the definition of D_n :

$$\forall t \leqslant T_*, \forall z \in D_n, \quad 2^{n-1}R_0 \leqslant |z - z_*(t)| \leqslant 2^n R_0 + 2R_*. \tag{IV.7.2}$$

notations for this section : We will denote $\|\cdot\|_\infty^n$ the L^∞ norm on the $\mathbb{R}_+ \times D_n$.

IV.7.2 *Local bounds on* B_0

Our first step consists in getting bounds locally on $\mathbb{R} \times B_0$:

Proposition IV.32 (Local bound)**.**
There exists a constant ε_B *that depend only on* B*, such that upon the conditions of Theorem IV.30,* W_ε *verifies for* $\varepsilon \leqslant \varepsilon_B$

$$\begin{aligned}
\|M_\varepsilon\|_\infty^0 &\leqslant O_0^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|, \\
\|\partial_z M_\varepsilon\|_\infty^0 &\leqslant O_0^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|, \\
\left\|\partial_z^2 W_\varepsilon\right\|_\infty^0 &\leqslant O_0^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|.
\end{aligned}$$

Proof of Proposition IV.32.
By the derivation of the linearized equation in Proposition IV.28, W_ε verifies

$$\varepsilon^2\partial_t\partial_z^2 W_\varepsilon(t,z) = \partial_z^2\Gamma(t,z)\Big(- W_\varepsilon(t,z) + 2W_\varepsilon\left(t,\bar z\right) + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big) +$$
$$2\partial_z\Gamma(t,z)\Big[\partial_z W_\varepsilon(t,\bar z) - \partial_z W_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big] +$$
$$\Gamma(t,z)\Big(1 + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big) \times \left(\frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z) - \partial_z^2 W_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\right).$$

We will use the maximum principle on the ball B_0. The key point is that all factors are small compared to $\left\|\partial_z^2 W_\varepsilon\right\|_\infty^0$. First, we write that for any $z \in B_0$, thanks to equation (IV.7.1),

$$\partial_z W_\varepsilon(t,\bar z) - \partial_z W_\varepsilon(t,z) \leqslant \left(\frac{R_0}{2} + R_*\right)\left\|\partial_z^2 W_\varepsilon\right\|_{L^\infty([0,T^*]\times B_0)}.$$

Similarly, there exists $\xi \in (z,\bar z)$ and $\xi' \in (z_*,\bar z)$ such that

$$2W_\varepsilon(t,\bar z) - W_\varepsilon(t,z) - W_\varepsilon(t,z_*) = \left(\frac{z_* - z}{2}\right)\partial_z W(\bar z) - \frac{1}{2}\frac{(z-z_*)^2}{4}\partial_z^2 W(\xi)$$
$$+ \left(\frac{z - z_*}{2}\right)\partial_z W(\bar z) - \frac{1}{2}\frac{(z-z_*)^2}{4}\partial_z^2 W(\xi'), \leqslant \frac{1}{4}\left(\frac{R_0}{2} + R_*\right)^2\left\|\partial_z^2 W_\varepsilon\right\|_{L^\infty([0,T^*]\times B_0)}. \quad \text{(IV.7.3)}$$

Moreover by the hypothesis of equation (IV.1.12) on Γ, for $j = 1, 2$

$$\sup_{(t,z)\in\mathbb{R}_+\times B_0}\left|\frac{\partial_z^{(j)}\Gamma(t,z)}{\Gamma(t,z)}\right| \leqslant O^*(1).$$

Thanks to those a priori bounds, when we evaluate equation (IV.6.16) at the point of maximum of $\partial_z^2 W_\varepsilon$ on $[0,T_*]\times B_0$ we get

$$\varepsilon^2\partial_t\left(\left\|\partial_z^2 W_\varepsilon\right\|_{L^\infty([0,T^*]\times B_0)}\right) \leqslant \Gamma(t,z)\Big(1 + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big)\Bigg[\frac{1}{2}\left\|\partial_z^2 W_\varepsilon\right\|_{L^\infty([0,T^*]\times B_0)}$$
$$- \left\|\partial_z^2 W_\varepsilon\right\|_{L^\infty([0,T^*]\times B_0)} + O^*(1)\left(\frac{1}{4}\left(\frac{R_0}{2} + R_*\right)^2 + \frac{R_0}{2} + R_*\right)\left\|\partial_z^2 W_\varepsilon\right\|_{L^\infty([0,T^*]\times B_0)}$$
$$+ O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Bigg].$$

The *crucial* step is that we choose T^* and R_0 so small that

$$O^*(1)\left(\frac{1}{2}\left(\frac{R_0}{2} + R_*\right)^2 + \frac{R_0}{2} + R_*\right) \leqslant \frac{1}{4}. \quad \text{(IV.7.4)}$$

The consequence is that

$$\varepsilon^2\partial_t\left(\left\|\partial_z^2 W_\varepsilon\right\|_{L^\infty([0,T^*]\times B_0)}\right) \leqslant$$
$$\Gamma(t,z)\Big(1 + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big)\left[-\frac{1}{4}\left\|\partial_z^2 W_\varepsilon\right\|_{L^\infty([0,T^*]\times B_0)} + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\right].$$

For ε small enough, that depends only on K^* and B, we have a positive lower bound of $\Gamma(t,z)\Big(1+O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)$. Therefore, we can apply the maximum principle, on the ball B_0 :

$$\left\|\partial_z^2 W_\varepsilon\right\|_\infty^0 \leqslant \max\left(O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|, \left\|\partial_z^2 W_\varepsilon(0,\cdot)\right\|_{L^\infty([0,T^*]\times B_0)}\right).$$

We take advantage that both W_ε and $\partial_z W_\varepsilon$ vanish at z^*, and the mid point property of equation (IV.7.1) to write

$$|\partial_z W_\varepsilon(t,z)| = \left|\int_{z_*(t)}^{z} \partial_z^2 W_\varepsilon(t,z')dz'\right| \leqslant (R_0+2R_*)\left\|\partial_z^2 W_\varepsilon\right\|_{L^\infty([0,T^*]\times B_0)}.$$

As a consequence, using again the expansion of (IV.7.3),

$$|M_\varepsilon(t,z)| = |2W_\varepsilon(t,\bar{z}(t)) - W_\varepsilon(t,z)| \leqslant \frac{1}{4}\left(\frac{R_0}{2}+R_*\right)^2\left\|\partial_z W_\varepsilon\right\|_{L^\infty([0,T^*]\times B_0)}.$$

We can use the same arguments to prove the bound upon $\partial_z M_\varepsilon$. At this point we have proven:

$$\begin{aligned}
\|M_\varepsilon\|_{L^\infty([0,T^*]\times B_0)} &\leqslant \max\left(O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|, \left\|\partial_z^2 W_\varepsilon(0,\cdot)\right\|_{L^\infty(B_0)}\right),\\
\|\partial_z M_\varepsilon\|_{L^\infty([0,T^*]\times B_0)} &\leqslant \max\left(O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|, \left\|\partial_z^2 W_\varepsilon(0,\cdot)\right\|_{L^\infty(B_0)}\right),\\
\left\|\partial_z^2 W_\varepsilon\right\|_{L^\infty([0,T^*]\times B_0)} &\leqslant \max\left(O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|, \left\|\partial_z^2 W_\varepsilon(0,\cdot)\right\|_{L^\infty(B_0)}\right).
\end{aligned}$$

The next step is to adapt the previous method to all positive times. We renew every previous estimate on each interval $I_k := [kT_*,(k+1)T_*]$. By going over the same steps, we notice that the only argument that changes for different k is the ball B_0 around z_*, but interestingly not its radius see equation (IV.7.4). Every other estimate is the same and is independent of the ball B_0. Therefore, once the radius is chosen small enough depending only on K^*, see equation (IV.7.4), we can repeat recursively the estimates on each interval I_k. Considering all $k\in\mathbb{N}$, we have therefore proven the Proposition IV.32. □

We explain how to get an estimate on the ring thanks to the estimate on the ball, first on W_ε.

IV.7.3 *Bound on the rings, M_ε*

We will know propagate those bounds beyond the small ball. It is very important to keep the level of precision of $O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|$, to which we will add some decay property due to the increasing size of the rings.

Proposition IV.33 (In the rings M_ε)**.**
There exists a constant ε_B that depend only on B such that upon the conditions of Theorem IV.30, W_ε verifies for $\varepsilon\leqslant\varepsilon_B$

$$\|M_\varepsilon\|_\infty^n \leqslant O_0^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|. \tag{IV.7.5}$$

for all $n\geqslant 1$.

Proof of Proposition IV.33.
For any $n \geqslant 1$, take z in the ring D_{n+1} defined previously. Then, $\bar{z} \in D_n$ by Lemma IV.31. Next, we use the linearized equation given by corollary IV.24. For $t \in \mathbb{R}$ and $z \in D_{n+1}$ the following inequality holds true

$$\varepsilon^2 \partial_t M_\varepsilon(t,z) \leqslant \Gamma(t,z) \left(2\frac{\Gamma(t,\bar{z})}{\Gamma(t,z)} \|M_\varepsilon\|_\infty^n - M_\varepsilon(t,z) + O^*(1) + O(\varepsilon) \|(\kappa_\varepsilon, W_\varepsilon)\| \right)$$

But the quotient of Γ verifies :

$$\sup_{(t,z)\in\mathbb{R}_+\times D_n} \left| \frac{\Gamma(t,\bar{z})}{\Gamma(t,z)} \right| := a_n,$$

where the sequence a_n is bounded and verifies $a_n \to a < \frac{1}{2}$ as $n \to \infty$ by the hypothesis of equation (IV.1.14).

Moreover since by hypothesis Γ admits an uniform lower bound, (IV.1.11), we can apply the maximum principle :

$$\|M_\varepsilon\|_\infty^n \leqslant \max \left(2a_n \|M_\varepsilon\|_\infty^{n-1} + O^*(1) + O(\varepsilon) \|(\kappa_\varepsilon, W_\varepsilon)\|, \|M_\varepsilon(0,\cdot)\|_{L^\infty(D_n)} \right). \tag{IV.7.6}$$

By recursion, this imposes :

$$\|M_\varepsilon\|_\infty^n \leqslant \max \left(\left(\prod_{k=1}^n 2a_k \right) \|M_\varepsilon\|_\infty^0 + \left(O^*(1) + O(\varepsilon) \|(\kappa_\varepsilon, W_\varepsilon)\| \right) \sum_{k=1}^n \left(\prod_{j=k}^n 2a_j \right)^k, \|M_\varepsilon(0,\cdot)\|_{L^\infty(D_n)} \right).$$

However, since $a < \frac{1}{2}$, the sum is finite, and there exists a generic constant C^* that depend only on B^* and m such that

$$\begin{aligned} \|M_\varepsilon\|_\infty^n &\leqslant \max \left(C^* \|M_\varepsilon\|_\infty^0 + \left(O^*(1) + O(\varepsilon) \|(\kappa_\varepsilon, W_\varepsilon)\| \right), \|M_\varepsilon(0,\cdot)\|_{L^\infty(D_n)} \right) \\ &\leqslant \max \left(O^*(1) + O(\varepsilon) \|(\kappa_\varepsilon, W_\varepsilon)\|, \|W_\varepsilon(0,\cdot)\|_{\mathcal{F}} \right). \end{aligned}$$

We uniformly bounded the product $\prod_{k=1}^n 2a_k$, and made use of our previous estimate on the ball to initiate the recursion : Proposition IV.32. □

IV.7.4 *Bound on the rings : $\partial_z M_\varepsilon$*

We now state a similar result for $\partial_z M_\varepsilon$. We see the appearance of a decay of the norm with respect to the index n. This is crucial to later obtain the uniform bound of Theorem IV.30.

Proposition IV.34 (In the rings $\partial_z M_\varepsilon$)**.**
There exists a constant ε_B that depends only on B such that upon the condition of Theorem IV.30, W_ε verifies for $\varepsilon \leqslant \varepsilon_B$

$$\|\partial_z M_\varepsilon\|_\infty^n \leqslant \frac{O_0^*(1) + O(\varepsilon) \|(\kappa_\varepsilon, W_\varepsilon)\|}{2^n}.$$

for $n \geqslant 1$.

Proof of Proposition IV.34.
The proof is very similar to the bound on M_ε, the linearized problem for $\partial_z M_\varepsilon$ is given in corollary IV.27. The following inequalities on Γ are true by our hypothesis of (IV.1.14) :

$$\sup_{(t,z)\in\mathbb{R}_+\times D_{n+1}}\left|\frac{\Gamma(t,\bar z(t))}{\Gamma(t,z)}\right|\leqslant a_n,\qquad \sup_{(t,z)\in\mathbb{R}_+\times D_{n+1}}\left|\frac{\partial_z\Gamma(t,\bar z(t))}{\partial_z\Gamma(t,z)}\right|\leqslant b_n, \tag{IV.7.7}$$

where the sequence b_n is uniformly bounded. Moreover we also know from equation (IV.1.12) that

$$\sup_{(t,z)\in\mathbb{R}^2}\left(\Big(1+|z-\mathrm{z}_*(t)|\Big)\left|\frac{\partial_z\Gamma(t,z)}{\Gamma(t,z)}\right|\right)\leqslant O^*(1). \tag{IV.7.8}$$

As a consequence, for $z\in D_n$ and $t\in\mathbb{R}$,

$$\varepsilon^2\partial_t\partial_z M_\varepsilon(t,z)\leqslant\Gamma(t,z)\Big(1+O^*(\varepsilon)+O(\varepsilon^2)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)\Big[a_n\left\|\partial_z M_\varepsilon\right\|_\infty^n-\partial_z M_\varepsilon(t,z)+$$
$$\frac{O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{1+|z-\mathrm{z}_*(t)|}+\frac{O^*(1)}{1+|z-\mathrm{z}_*(t)|}\Big(b_n\left\|M_\varepsilon\right\|_\infty^n+\left\|M_\varepsilon\right\|_\infty^{n+1}+O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)\Big]. \tag{IV.7.9}$$

Moreover, there exists C^* such that $\frac{1}{1+|z-\mathrm{z}_*(t)|}\leqslant\frac{C^*}{2^n}$. Therefore, since for ε small enough, that depends only on K^* and B, we have a positive lower bound of $\Gamma(t,z)\Big(1+O^*(\varepsilon)+O(\varepsilon^2)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)$. We apply the maximum principle, and thanks to Proposition IV.33 :

$$\left\|\partial_z M_\varepsilon\right\|_\infty^n\leqslant\max\left(a_n\left\|\partial_z M_\varepsilon\right\|_\infty^{n-1}+\frac{O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^n},\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\right).$$

By recursion,

$$\left\|\partial_z M_\varepsilon\right\|_\infty^n\leqslant$$
$$\max\left(\left(\prod_{k=1}^n a_k\right)\left\|\partial_z M_\varepsilon\right\|_\infty^0+\Big(O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)\left(\sum_{k=1}^n\frac{1}{2^k}\prod_{j=k}^n a_j\right),\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\right).$$

Because a_n is eventually smaller than $\frac{1}{2}$, and by the initial bound upon $\partial_z M_\varepsilon$ on the ball B_0 from Proposition IV.32, we know that

$$\prod_{k=1}^n a_k\leqslant\frac{O^*(1)}{2^n}.$$

while

$$\sum_{k=0}^{n-1}\frac{1}{2^{n-k}}\prod_{j=0}^k a_{k-j}\lesssim\frac{1}{2^n}\sum_k\prod_{j=0}^k(2a_j),\qquad\text{when } n\to\infty$$

which is convergent series. Therefore, one gets

$$\left\|\partial_z M_\varepsilon\right\|_\infty^n\leqslant\left(\frac{O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^n},\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\right).$$

□

We now make a similar statement upon the second derivative. We introduce a weight function φ_α to make use of its scaling properties, as first introduced in Calvez et al. [2019]. However its role is completely transparent in the result, it only appears throughout the proof.

Proposition IV.35 (In the rings $\partial_z^2 W_\varepsilon$)**.**
There exists a constant ε_B that depends only on B such that upon the condition of Theorem IV.30, W_ε verifies for $\varepsilon \leqslant \varepsilon_B$

$$\left\|\partial_z^2 W_\varepsilon\right\|_\infty^n \leqslant \frac{O_0^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|}{2^n}.$$

for $n \geqslant 1$.

Proof of Proposition IV.35.
We define $\varphi_\alpha(t,z) = (1+|z-z_*(t)|)^\alpha$, for some arbitrary $0<\alpha<1$. Before estimating on the rings, one should notice that the weight φ_α is uniformly bounded on B_0. Therefore, on that ball, Proposition IV.32 implies :

$$\left\|\varphi_\alpha \partial_z^2 W_\varepsilon\right\|_\infty^0 \leqslant O_0^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|.$$

Thanks to the linearization of equation (IV.6.16), $\varphi_\alpha \partial_z^2 W_\varepsilon$ solves

$$\begin{aligned}\varepsilon^2 \partial_t\left[\varphi_\alpha(t,z)\partial_z^2 W_\varepsilon(t,z)\right] &= -\partial_z^2\Gamma(t,z)\varphi_\alpha(t,z)\Big(M_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big)\\ &\quad - 2\partial_z\Gamma(t,z)\varphi_\alpha(t,z)\Big[\partial_z M_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big] + \Gamma(t,z)\varphi_\alpha(t,z)\times\\ &\Big(1 + O^*(\varepsilon^2) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big)\left(\frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z) - \partial_z^2 W_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|}{1+|z-z_*(t)|}\right)\\ &\quad + \partial_t\varphi_\alpha(t,z) O(\varepsilon^2)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|. \quad \text{(IV.7.10)}\end{aligned}$$

By our hypothesis upon the decay of Γ, (IV.1.14), for any $n \geqslant 1$, and $z \in D_{n+1}$

$$\left|\frac{\partial_t\varphi_\alpha(t,z)}{\Gamma(t,z)}\right| \leqslant \frac{O^*(1)}{2^{n(2-\alpha)}}, \quad \frac{\varphi_\alpha(t,z)}{1+|z-z_*(t)|} \leqslant \frac{O^*(1)}{2^{n(1-\alpha)}},$$

and the weight function was chosen precisely to satisfy the following scaling es timate

$$\sup_{\mathbb{R}^+\times\mathbb{R}}\left|\frac{\varphi_\alpha(t,z)}{\varphi_\alpha(t,\bar z(t))}\right| \leqslant 2^\alpha.$$

Therefore, we can bound the right hand side of equation (IV.7.10) after factorizing by Γ, for $t \in \mathbb{R}$ and $z \in D_{n+1}$

$$\begin{aligned}\varepsilon^2\partial_t\left[\varphi_\alpha(t,z)\partial_z^2 W_\varepsilon(t,z)\right] &\leqslant \Gamma(t,z)\Big(1 + O^*(\varepsilon^2) + O(\varepsilon^2)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big)\Big[-\varphi_\alpha(t,z)\partial_z^2 W_\varepsilon(t,z)\\ &\quad + \frac{1}{2^{1-\alpha}}\left\|\varphi_\alpha\partial_z^2 W_\varepsilon\right\|_\infty^n + \frac{O^*(1)+O(\varepsilon)}{2^{n(1-\alpha)}} + \frac{O(\varepsilon^2)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|}{2^{n(2-\alpha)}}\\ &\quad + \left|\frac{\partial_z^2\Gamma(t,z)\varphi_\alpha(t,z)}{\Gamma(t,z)}\right|\Big(\left\|M_\varepsilon\right\|_\infty^{n+1} + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big)\\ &\quad + 2\left|\frac{\partial_z\Gamma(t,z)\varphi_\alpha(t,z)}{\Gamma(t,z)}\right|\Big(\left\|\partial_z M_\varepsilon\right\|_\infty^{n+1} + O^*(1) + O(\varepsilon)\left\|(\kappa_\varepsilon, W_\varepsilon)\right\|\Big)\Big].\end{aligned}$$

Moreover we dispose of the following uniform controls on the ring by equation (IV.1.12):

$$\text{For } j=1,2: \sup_{z\in D_n} \varphi_\alpha(z)\left|\frac{\partial_z^j\Gamma(t,z)}{\Gamma(t,z)}\right| \leqslant \frac{O^*(1)}{2^{n(1-\alpha)}}.$$

We also control M_ε and $\partial_z M_\varepsilon$ thanks to Proposition IV.34. Therefore the following bound holds true :

$$\begin{aligned}\varepsilon^2\partial_t\Big[\varphi_\alpha(t,z)\partial_z^2 W_\varepsilon(t,z)\Big] \leqslant \Gamma(t,z)\Big(1+O^*(\varepsilon^2)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)\Big[-\varphi_\alpha(t,z)\partial_z^2 W_\varepsilon(t,z)\\ +\frac{1}{2^{1-\alpha}}\left\|\varphi_\alpha\partial_z^2 W_\varepsilon\right\|_\infty^n+\frac{O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^{n(1-\alpha)}}\\ +\frac{O^*(1)}{2^{n(2-\alpha)}}\Big(\max\Big(O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big),\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\Big)\Big].\end{aligned}$$

Since $n\geqslant 1$, we can simplify the right hand side , keeping only the relevant terms :

$$\begin{aligned}\varepsilon^2\partial_t\Big[\varphi_\alpha(t,z)\partial_z^2 W_\varepsilon(t,z)\Big] \leqslant \Gamma(t,z)\Big(1+O^*(\varepsilon^2)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)\Big[-\varphi_\alpha(t,z)\partial_z^2 W_\varepsilon(t,z)\\ +\frac{1}{2^{1-\alpha}}\left\|\varphi_\alpha\partial_z^2 W_\varepsilon\right\|_\infty^n+\max\left(\frac{O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^{n(1-\alpha)}},\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\right)\Big].\end{aligned}$$

Therefore, since for ε small enough, that depends only on K^* and $\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|$, we have a positive lower bound of $\Gamma(t,z)\Big(1+O^*(\varepsilon^2)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)$, we can apply the maximum principle,

$$\left\|\varphi_\alpha\partial_z^2 W_\varepsilon\right\|_\infty^{n+1}\leqslant\max\left(\frac{1}{2^{1-\alpha}}\left\|\varphi_\alpha\partial_z^2 W_\varepsilon\right\|_\infty^n+\frac{O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^{n(1-\alpha)}},\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\right).$$

By recursion, this implies that

$$\begin{aligned}\left\|\varphi_\alpha\partial_z^2 W_\varepsilon\right\|_\infty^n &\leqslant \max\left(\frac{1}{2^{(1-\alpha)n}}\left\|\varphi_\alpha\partial_z^2 W_\varepsilon\right\|_\infty^0+\frac{O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^n}\sum_{k=0}^{n-1}2^{\alpha k},\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\right)\\ &\leqslant \max\left(\frac{1}{2^{(1-\alpha)n}}\left\|\varphi_\alpha\partial_z^2 W_\varepsilon\right\|_\infty^0+\frac{O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^{n(1-\alpha)}},\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\right) \quad \text{(IV.7.11)}\end{aligned}$$

Thanks to Proposition IV.32, we conclude with

$$\left\|\varphi_\alpha\partial_z^2 W_\varepsilon\right\|_\infty^n\leqslant\max\left(\frac{O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^{n(1-\alpha)}},\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\right).$$

Therefore we get Proposition IV.35 by dividing each side of the previous equation by $\varphi_\alpha(t,z)$, since by definition of the rings D_n, we have $|z-z_*|\sim 2^n$, see equation (IV.7.2). □

IV.7.6 *Local and on the rings bound for $\partial_z^3 W_\varepsilon$*

We dedicate this section to the study of $\partial_z^3 W_\varepsilon$ since it does not exactly fits the mold of the previous estimates due to the additional factor $\frac{1}{2}\left\|W\right\|_{\mathcal{F}}$ in the linearized equation in Proposition IV.29.

We highlight the difference by proving the initial bound on the small initial B_0. First, we recall that from the computation of the linearized equation in Proposition IV.29, W_ε verifies

$$
\begin{aligned}
-\varepsilon^2\partial_t\partial_z^3 W_\varepsilon(t,z) &= \partial_z^3\Gamma(t,z)\Big(M_\varepsilon(t,z)+O^*(\varepsilon^2)+O(\varepsilon^3)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)+\\
3\partial_z^2\Gamma(t,z)\Big(\partial_z M_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)&+3\partial_z\Gamma(t,z)\left(\partial_z^2 W_\varepsilon(t,z)-\frac{1}{2}\partial_z^2 W_\varepsilon(t,\bar z)\right.\\
&\left.+O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\right)+\Gamma(t,z)\Big(1+O^*(\varepsilon)+O(\varepsilon^2)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)\times\\
&\left(\partial_z^3 W_\varepsilon(t,z)-\frac{1}{4}\partial_z^3 W_\varepsilon(t,\bar z)+\ \frac{O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{1+|z-\mathrm{z}_*(t)|}+\frac{1}{2}\left\|W_\varepsilon\right\|_{\mathcal{F}}\right). \quad \text{(IV.7.12)}
\end{aligned}
$$

The main ingredient is that all factors will be small compared to $\left\|\partial_z^3 W_\varepsilon\right\|_\infty^0$. First, we recall that $M_\varepsilon, \partial_z M_\varepsilon$ and $\partial_z^2 M_\varepsilon$ are all uniformly bounded on B_0 by Proposition IV.32. Therefore,

$$
\partial_z^2 M_\varepsilon(t,z)-\partial_z^2 M_\varepsilon(t,\bar z)\leqslant\frac{3}{2}\left\|\partial_z^2 W_\varepsilon\right\|_\infty^0\leqslant O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|.
$$

Moreover, from equation (IV.1.11), for $j=1,2,3$

$$
\sup_{(t,z)\in\mathbb{R}+\times B_0}\left|\frac{\partial_z^{(j)}\Gamma(t,z)}{\Gamma(t,z)}\right|\leqslant O^*(1).
$$

When plugging all of this into equation (IV.7.12), we obtain

$$
\begin{aligned}
\varepsilon^2\partial_t\left\|\partial_z^3 W_\varepsilon\right\|_\infty^0\leqslant\Gamma(t,z)&\Big(1+O^*(\varepsilon)+O(\varepsilon^2)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)\times\\
&\left(-\left\|\partial_z^3 W_\varepsilon\right\|_\infty^0+\frac{1}{4}\left\|\partial_z^3 W_\varepsilon\right\|_\infty^0+O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|+\frac{1}{2}\left\|W_\varepsilon\right\|_{\mathcal{F}}\right).
\end{aligned}
$$

For ε small enough, that depends only on K^* and $\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|$, we have a positive lower bound of $\Gamma(t,z)\Big(1+O^*(\varepsilon)+O(\varepsilon^2)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)$. Therefore, we can apply the maximum principle, on the ball B_0 :

$$
\begin{aligned}
\left\|\partial_z^3 W_\varepsilon\right\|_\infty^0&\leqslant\max\ \left(O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|+\frac{\varepsilon^2}{2}\left\|W_\varepsilon\right\|_{\mathcal{F}},\left\|\partial_z^3 W_\varepsilon(0,\cdot)\right\|_\infty^0\right),\\
&\leqslant O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|+\frac{1}{2}\left\|W_\varepsilon\right\|_{\mathcal{F}}. \quad \text{(IV.7.13)}
\end{aligned}
$$

As explained before, we can now repeat the procedure on each interval of time $I_k:=[kT_*,(k+1)T_*]$. We now proceed to propagate this bound on the rings, starting again from equation (IV.7.12) and using the maximum principle. For any $t\in\mathbb{R}$ and $z\in D_{n+1}$, we have

$$
\begin{aligned}
\varepsilon^2\partial_t\partial_z^3 W_\varepsilon(t,z)\leqslant\Gamma(t,z)&\Big(1+O^*(\varepsilon)+O(\varepsilon^2)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)\left[-\partial_z^3 W_\varepsilon(t,z)+\frac{1}{4}\left\|\partial_z^3 W_\varepsilon\right\|_\infty^n+\right.\\
&\frac{O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{1+|z-\mathrm{z}_*(t)|}+\frac{1}{2}\left\|W_\varepsilon\right\|_{\mathcal{F}}+\left|\frac{\partial_z^3\Gamma(t,z)}{\Gamma(t,z)}\right|\Big(\left\|M_\varepsilon\right\|_\infty^{n+1}+O^*(\varepsilon^2)+O(\varepsilon^3)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)+\\
&3\left|\frac{\partial_z^2\Gamma(t,z)}{\Gamma(t,z)}\right|\Big(\left\|\partial_z M_\varepsilon\right\|_\infty^{n+1}+O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)+3\left|\frac{\partial_z\Gamma(t,z)}{\Gamma(t,z)}\right|\times\\
&\left.\left(\frac{3}{2}\left\|\partial_z^2 W_\varepsilon\right\|_\infty^{n+1}+O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\right)\right].
\end{aligned}
$$

We dispose of the following uniform controls on the ring by equation (IV.1.12):

$$\text{For } j=1,2,3: \sup_{z\in D_n}\left|\frac{\partial_z^j\Gamma(t,z)}{\Gamma(t,z)}\right|\leqslant\frac{O^*(1)}{2^{jn}}.$$

Therefore, using all of our previous estimates on the rings, Propositions IV.33 to IV.35, we obtain

$$\begin{aligned}\varepsilon^2\partial_t\partial_z^3W_\varepsilon(t,z)\leqslant\Gamma(t,z)\Big(1+O(\varepsilon^2)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|+O^*(\varepsilon^2)\Big)\Bigg[-\partial_z^3W_\varepsilon(t,z)+\frac{1}{4}\left\|\partial_z^3W_\varepsilon\right\|_\infty^n+\\ \frac{O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{1+|z-z_*(t)|}+\frac{1}{2}\left\|W_\varepsilon\right\|_{\mathcal{F}}+\frac{O^*(1)}{2^{3n}}\Big(O_0^*(1)+\ O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)\\ +\ \frac{O^*(1)}{2^{2n}}\Big(O_0^*(1)+\ O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)\ +\frac{O^*(1)}{2^n}\Big(O_0^*(1)+\ O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|\Big)\Bigg].\end{aligned}$$

We keep only the relevant terms :

$$\begin{aligned}\varepsilon^2\partial_t\partial_z^3W_\varepsilon(t,z)\leqslant\Gamma(t,z)\Big(1+O(\varepsilon^2)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|+O^*(\varepsilon^2)\Big)\Bigg[-\partial_z^3W_\varepsilon(t,z)+\frac{1}{4}\left\|\partial_z^3W_\varepsilon\right\|_\infty^n+\\ \frac{O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^n}+\frac{1}{2}\left\|W_\varepsilon\right\|_{\mathcal{F}}+\frac{O_0^*(1)+\ O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^n}\Bigg].\end{aligned}$$

Thanks to the maximum principle, one gets

$$\left\|\partial_z^3W_\varepsilon\right\|_\infty^{n+1}\leqslant\max\left(\frac{1}{4}\left\|\partial_z^3W_\varepsilon\right\|_\infty^n+\frac{1}{2}\left\|W_\varepsilon\right\|_{\mathcal{F}}+\frac{O_0^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^n},\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\right).$$

By recursion, this implies that

$$\begin{aligned}\left\|\partial_z^3W_\varepsilon\right\|_\infty^n&\leqslant\left(\frac{1}{4}\left\|\partial_z^3W_\varepsilon\right\|_\infty^0+\sum_{k=0}^{n-1}4^{-k}\left(\frac{1}{2}\left\|W_\varepsilon\right\|_{\mathcal{F}}+\frac{O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^n}\right),\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\right)\\ &\leqslant\left(\frac{1}{4}\left\|\partial_z^3W_\varepsilon\right\|_\infty^0+\frac{O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{4^n}+\frac{1}{2}\sum_{k=0}^{\infty}4^{-k}\left\|W_\varepsilon\right\|_{\mathcal{F}},\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\right)\end{aligned}$$

Moreover, $\frac{1}{2}\sum_{k=0}^{\infty}4^{-k}=\frac{2}{3}$. Thanks to the estimate on B_0 of equation (IV.7.13), we conclude with

$$\left\|\partial_z^3W_\varepsilon\right\|_\infty^n\leqslant\left(\frac{O^*(1)+O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^n}+\frac{2}{3}\left\|W_\varepsilon\right\|_{\mathcal{F}},\left\|W_\varepsilon(0,\cdot)\right\|_{\mathcal{F}}\right).$$

We have therefore proven the following result :

Proposition IV.36 (In the rings $\partial_z^3W_\varepsilon$)**.**
There exists a constant ε_B that depends only on B such that upon the condition of Theorem IV.30, W_ε verifies for $\varepsilon\leqslant\varepsilon_B$

$$\left\|\partial_z^3W_\varepsilon\right\|_\infty^n\leqslant\frac{O_0^*(1)+\ O(\varepsilon)\left\|(\kappa_\varepsilon,W_\varepsilon)\right\|}{2^n}+\frac{2}{3}\left\|W_\varepsilon\right\|_{\mathcal{F}}.$$

for $n\geqslant1$.

Thanks to the previous results we have estimated $\|W\|_{\mathcal{F}}$ upon the rings. To translate to a global result, we simply remark that there exists an universal constant C such that for every $z \in D_n$, $\frac{1}{2^n} \leqslant \frac{C}{1+|z-z_*|}$, see equation (IV.7.2) . Therefore, we have proven in Propositions IV.33 to IV.36 that

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} |M_\varepsilon(t,z)| \leqslant \max\left(O^*(1) + O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\,, \|W_\varepsilon(0,\cdot)\|_{\mathcal{F}}\right), \tag{IV.7.14}$$

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} |\partial_z M_\varepsilon(t,z)| \leqslant \max\left(\left(1+|z-z_*(t)|\right)\left(O^*(1) + O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\right), \|W_\varepsilon(0,\cdot)\|_{\mathcal{F}}\right), \tag{IV.7.15}$$

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \left|\partial_z^2 W_\varepsilon(t,z)\right| \leqslant \max\left(\frac{O^*(1) + O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|}, \|W_\varepsilon(0,\cdot)\|_{\mathcal{F}}\right), \tag{IV.7.16}$$

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \left|\partial_z^3 W_\varepsilon(t,z)\right| \leqslant \max\left(\frac{O^*(1) + O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\|}{1+|z-z_*(t)|} + \frac{2}{3}\|W_\varepsilon\|_{\mathcal{F}}, \|W_\varepsilon(0,\cdot)\|_{\mathcal{F}}\right). \tag{IV.7.17}$$

The estimates equations (IV.7.14), (IV.7.16) and (IV.7.17) are exactly as stated in Theorem IV.30. However, we dot not have the one in $\partial_z M_\varepsilon$ because our estimates on the rings are too rough.

Thanks to our estimate of the second derivative **with the weight**, we manage to bound uniformly $\partial_z M_\varepsilon$:

$$\begin{aligned}
|\partial_z M_\varepsilon(t,z)| &\leqslant |\partial_z W_\varepsilon(t,z) - \partial_z W_\varepsilon(t,\bar{z})|\,,\\
&\leqslant \frac{|z-z_*|}{2}\left\|\partial_z^2 W_\varepsilon\right\|_\infty,\\
&\leqslant O_0^*(1) + O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\,.
\end{aligned}$$

The crucial point is therefore the presence of the weight in the estimate (IV.7.16) that compensate the factor $(z - z_*)$ from the Taylor expansion.

IV.8 Proof of Theorem IV.4

We now prove the main result of this paper, that is the boundedness of $(\kappa_\varepsilon, W_\varepsilon)$ in $\mathbb{R}\times\mathcal{F}$. We look for a constant K_0 such that $\|W_\varepsilon\|_{\mathcal{F}} \leqslant K_0$ and $|\kappa_\varepsilon| \leqslant K_0$. Therefore we first suppose that

$$|\kappa_\varepsilon(0)| \leqslant K \quad \text{and} \quad \|W_\varepsilon(0,\cdot)\|_{\mathcal{F}} \leqslant K. \tag{IV.8.1}$$

with K to be determined by the analysis.

By Theorem IV.30, that we can apply with our assumption (IV.8.1), we have precise bounds of W_ε. More precisely, there exists a constant C such that :

$$\|W_\varepsilon\|_{\mathcal{F}} \leqslant \max\left(O^*(1) + \left(C\varepsilon + \frac{2}{3}\right)K, K\right).$$

Next, we choose ε_0 and K_0 such that

$$K_0 \geqslant C^* + \left(C\varepsilon + \frac{2}{3}\right)K, \quad \varepsilon_0 := \frac{1}{4}C.$$

Then for $\varepsilon \leqslant \varepsilon_0$, starting from an initial data $\|W_\varepsilon(0,\cdot)\| \leqslant K_0$, the bound is propagated in time and

$$\boxed{\|W_\varepsilon\|_{\mathcal{F}} \leqslant K_0.}$$

This is part of Theorem IV.4. Now it only remains to show a similar bound for κ_ε. We go back to Proposition IV.21 since we made suitable assumptions and we get that as long κ_ε is bounded it solves

$$-\dot{\kappa}_\varepsilon(t) = R^*(t)\kappa_\varepsilon + O^*(1) + O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\| + \partial_z m(\mathrm{z}_*(t))\partial_z^2 W(t,\mathrm{z}_*(t)) + S_\varepsilon(W_\varepsilon). \qquad \text{(IV.8.2)}$$

Thanks to Proposition IV.32 we can estimate the term $\partial_z^2 W(t,\mathrm{z}_*)$, because $\mathrm{z}_* \in B_0$, see figure IV.2.

$$\left|\partial_z^2 W(t,\mathrm{z}_*)\right| \leqslant \max\left(O^*(1) + O(\varepsilon)\,\|(\kappa, W_\varepsilon)\|\,, \|W_\varepsilon(0,\cdot)\|_{\mathcal{F}}\right).$$

We now seek a similar bound upon the number $S_\varepsilon(W_\varepsilon)$. We make use of the uniform bounds we just proved for $\partial_z^3 W_\varepsilon$:

$$\begin{aligned} S_\varepsilon(W_\varepsilon) = \frac{1}{4}\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,\mathrm{z}_*)\Big[y_1^2\partial_z^3 W_\varepsilon(t,\mathrm{z}_* + \varepsilon\xi y_1) + y_2^2\partial_z^3 W_\varepsilon(t,\mathrm{z}_* + \varepsilon\xi y_2)\Big]dy_1 dy_2 \\ \leqslant \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,\mathrm{z}_*)\frac{(y_1^2+y_2^2)}{4}\Big(O^*(1)+O(\varepsilon)\Big)dy_1 dy_2 \end{aligned}$$

Therefore, by computing the moments of dG_ε^*, we have shown that

$$S_\varepsilon(W_\varepsilon) = O^*(1) + O(\varepsilon). \qquad \text{(IV.8.3)}$$

Since R^* is a positive function that admits an uniform lower bound R_0, see Proposition IV.21, it is straightforward from (IV.8.2) and our subsequent bounds that there exists a generic constant C such that for all time t

$$|\kappa_\varepsilon(t)| \leqslant C^* + C\varepsilon K + O^*(1)\max\left(O^*(1) + O(\varepsilon)\,\|(\kappa_\varepsilon, W_\varepsilon)\|\,, \|W_\varepsilon(0,\cdot)\|_{\mathcal{F}}\right).$$

As for W_ε this ensures that there exists K_0 such that $\kappa_\varepsilon(0) \leqslant K_0$ and

$$\forall t > 0 \quad |\kappa_\varepsilon(t)| \leqslant K_0.$$

Since $V_\varepsilon = V^* + \varepsilon^2 W_\varepsilon, \quad q_\varepsilon = q^* + \varepsilon^2\kappa_\varepsilon$, Theorem IV.4 is proven.

Cinquième partie

Lignées phénotypiques et mal-adaptation à un environnement changeant

Nous nous intéressons enfin à la réponse d'une population structurée en trait à un changement graduel d'environnement. La population est à l'équilibre dans un repère mobile se déplaceant à la même vitesse que l'environnement. Nous cherchons à comprendre quelle est la dynamique interne au sein de la distribution stationnaire. Une manière d'aborder le problème est de suivre les trajectoires phénoptypiques le long de la généalogie des individus au cours du temps. Nous proposons dans un régime de petite variance plusieurs méthodes pour obtenir les trajectoires de ces lignées au cours du temps : en interprétant les géodésiques d'une équation de Hamilton-Jacobi et en utilisant les "fractions neutres". Les formules données dans chaque cas sont concordantes, et sont illustrées par des simulations numériques stochastiques et déterministes. Nous abordons le mode de reproduction sexué et également le cas de la reprodction asexuée. Ces différents travaux vont être soumis pour publication.

In this work, we are interested in studying the lineages inside a population keeping pace with an environmental change. We will detail later what we mean exactly by the study of the lineages, first we detail the ecological model. The population is structured by a one–dimensional phenotypic trait, denoted by $x \in \mathbb{R}$, the density of individuals with trait x is $f(t,x)$ at any time $t > 0$. We suppose that the population is subject to selection through an intrinsic mortality rate $\mu(x)$, that admits a minimal value at an "optimal trait" : $x = 0$. The function μ can be decomposed as follows :

$$\mu(x) = \mu_0 + m(x),$$

where μ_0 is the basic mortality rate common to all phenotypes, and $m(x)$ an increment of mortality that models the weight of an non-optimal phenotype : m is a convex function that admits a minimum at $x = 0$, $m(0) = 0$. A way to measure the strength of selection is through the parameter

$$\alpha := \partial_z^2 m(0) > 0.$$

The smaller α, the flatter the selection function, and therefore the less selection around the optimal trait acts on the population. The birth rate is considered constant among traits, at a value β. Under the assumption $\beta > \mu_0$, the population at the optimal trait does not go extinct in the absence of environmental changes.

The changing environment is modeled through a linear drift of the optimal trait at the speed c. Mal-adaptation of given individual with trait x is thus defined as the difference between its phenotypic value x and the optimal value for the trait at time t : $z := x - ct$. The proposed model for the density function is the following equation :

$$\partial_t f(t,x) + \Big(\mu(x-ct) + (\beta - \mu_0) \int_{\mathbb{R}} f(t,x')dx'\Big) f(t,x) = \beta \mathcal{B}(f(t,\cdot))(x)\,. \tag{V.0.1}$$

The non linear term $\Big((\beta - \mu_0) \int_{\mathbb{R}} f(t,x')dx\Big) f(t,x)$ accounts for the density dependence of the population. Finally, the operator $\mathcal{B}$ describes the birth of new individuals via reproduction at the rate β. It will always be an integro-differential operator which means that the birth of given individual depends on the whole distribution of population f. With this terminology, two types of reproduction are encapsulated :

- ▷ **Asexual type of reproduction** :

$$\mathcal{B}(F)(x) = \frac{1}{\sigma} \int_{\mathbb{R}} K\left(\frac{x - x'}{\sigma}\right) F(x')\, dx'\,, \tag{V.0.2}$$

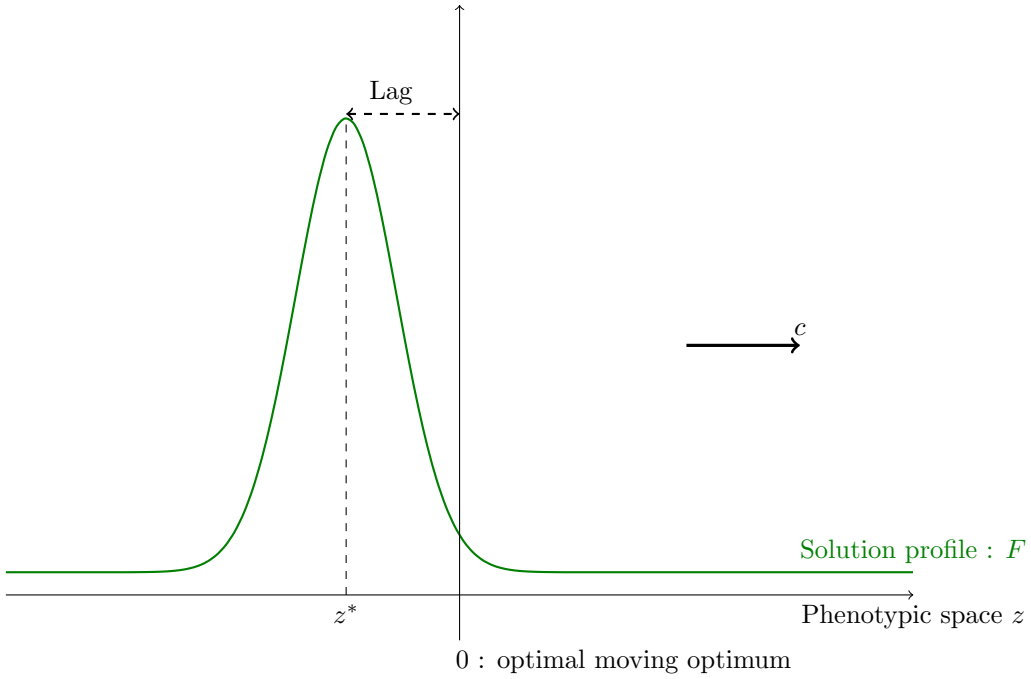
  where σ^2 is the variance of mutations and K is a symmetric normalized probability density function, that encodes the shape of kernel of the deviation due to mutations.

- ▷ **Infinitesimal type of reproduction**

$$\mathcal{B}(F)(x) = \iint_{\mathbb{R}^2} G_{\sigma^2}\left(x - \frac{x_1 + x_2}{2}\right) F(x_1) \left(\frac{F(x_2)}{\int_{\mathbb{R}} F(x_2')\, dx_2'}\right) dx_1 dx_2\,, \tag{V.0.3}$$

  where G_{σ^2} denotes the centred Gaussian distribution with variance $\sigma^2/2$.

In both cases one expects that the combination of selection of a (moving) trait and the creation of diversity creates an equilibrium of the phenotypic distribution. This will be true if the speed of environmental change c is not too fast for the mutations to act. In that scenario, the population always tries to keep up with the moving optimum, but never catches up, creating a "lag", leading to a profile with the following shape :



We suppose the existence of the following :

◁ An equilibrium profile F in the moving frame *i.e.* a solution of equation (V.0.1) of the form $\exp(\lambda t)F(z-ct)$.

◁ A (unique) dominant trait , z^*, among the population.

A straightforward computation implies that a non trivial equilibrium F is the solution to the following eigenvalue problem:

$$\lambda F(z) - c\partial_z F(z) + \mu(z)F(z) = \beta\mathcal{B}(F)(z)\,, \tag{V.0.4}$$

with

$$\lambda = (\beta - \mu_0)\int_{\mathbb{R}} F(z')dz'. \tag{V.0.5}$$

By hypothesis $\lambda > 0$. The backward transport term $-c\partial_z F$ corresponds to the effect of keeping pace with moving optimum for the phenotypic distribution F at equilibrium. A formal integration of equation (V.0.4) shows that

$$\lambda = \int_{\mathbb{R}} (\beta - \mu(z))\frac{F(z)}{\int_{\mathbb{R}} F(z')dz'}dz := \beta - \bar{\mu}. \tag{V.0.6}$$

The eigenvalue λ can thus be interpreted as a measure of the mean "fitness" of the population, or its mean intrinsic rate of increase, where $\beta - \mu(z)$ is the contribution to population growth rate of an individual with maladaptation z. The mean mortality $\overline{\mu}$ caused by selection is defined by

$$\overline{\mu} = \int_{\mathbb{R}} \mu(z) \frac{F(z)}{\int_{\mathbb{R}} F(z')dz'} dz.$$

This eigenvalue delivers information about the size N of the population at equilibrium in presence of a changing environment. It is detailed in Bouin et al. [2019] how to obtain a precise analytical formula for the critical speed of environmental change above which extinction is predicted, that is the case $\lambda < 0$. They precisely monitor a *weak selection regime*, as either a small variance σ^2 or a weak selection compared to birth (α/β small). They explain how, under that regime, the equilibrium F concentrates around a mean value of trait z^*, with variance depending on the product $\sigma^2\alpha/\beta$. They perform formal computations around equation (V.0.4) to deduce approximations of biological macroscopic variables of interest : variance at equilibrium, phenotypic lag, mean fitness..

OBJECTIVES

Our aim is to describe in both scenarios of reproduction the dynamics of individuals intra-equilibrium. More precisely, we will suppose that initially, say $t = 0$, the population of the model of equation (V.0.1) is at equilibrium. Letting the population of density evolve with time, the profile F is unchanged by hypothesis. One wants to understand the dynamics that led the population to stay at equilibrium. For any given $z \in \mathbb{R}$ one asks, what was the trajectory of the phenotype across its ancestors that led to this individual with the phenotype z at time t. This trajectory will be referenced as *lineage* in the following. For an asexual mode of reproduction, it is clear that it is a function of time, whose value at time t is the phenotype of an ancestor living at that time. In the case of sexual reproduction, because each individual has two parents a lineage is *stricto sensu* a sample of a trajectory among the genealogical tree, but we will try to justify our glossary of terms in that case also.

Conjecture 1. *In the asexual model described in equations (V.0.1) and (V.0.2), one expects that for every trait at equilibrium, asymptotically, when $t \to -\infty$, the* typical *trait of the ancestors will be the optimal trait $z = 0$.*

In fact, thanks to the approximation procedure that we will sketch up in the following, we will first be able to suggest an equation for the whole trajectory of ancestors in the regime of small variance, see equation (V.1.12). Next, in section 2 we will also be capable of proposing an equation for the whole evolution in time of the mean trait of the ancestors, through a completely different point of view, see equation (V.3.4).

Conjecture 2. *In the model of reproduction of the infinitesimal operator, described by equations (V.0.1) and (V.0.3), one expects that for every trait at equilibrium, asymptotically when $t \to -\infty$, the* typical *trait of the ancestors will be at the dominant trait $z = z^*$.*

Those two conjectures illustrates a real dichotomy between the two modes of reproduction. One can interpret them by reasoning backwards :

- ▷ In the asexual mode of reproduction, even if they are very few individuals at the optimum trait, they are the one that are responsible for the genetic diversity of the populations going forward in time. Ill-adapted individuals play a role minor in the profile of the population after a long time.

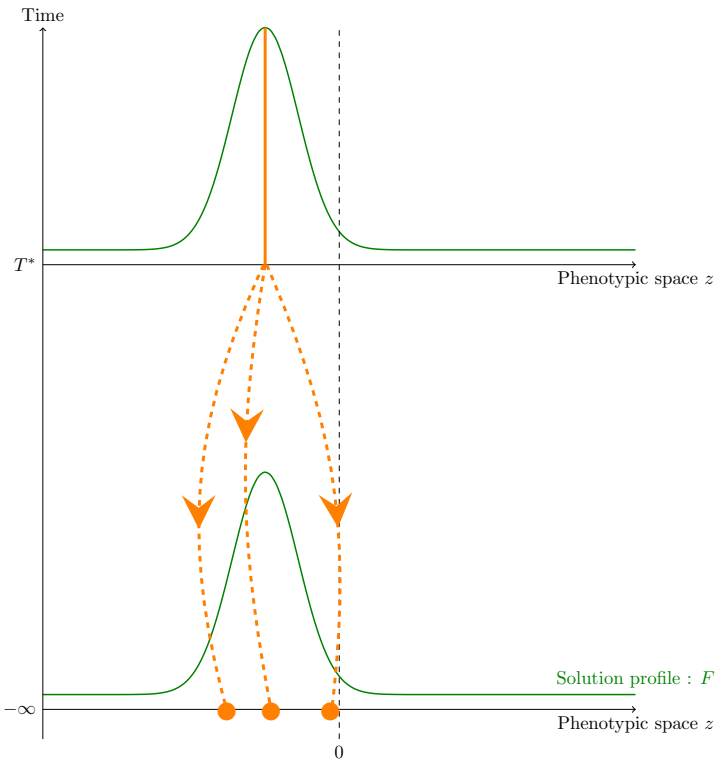


Figure V.1: Potential lineages arriving at the dominant trait z^* at time T^*. Which one of the orange arrows is the typical behavior of lineages?

▷ Conversely in the infinitesimal model, the main contribution to the ongoing phenotypic diversity is assured by the most frequent phenotype.

We will provide heuristics for each of the conjecture in the following sections. The rest of this chapter is organized as follows :

▷ First, we explain some heuristics behind the conjecture 1, we detail and explain the numerics that back up this assumption, after a quick sketch of the approximation procedure for (λ, F) solution of equation (V.0.1).

▷ We then proceed to the same steps for the infinitesimal model.

▷ Next we provide an alternative justification or rather a reversed point of view for both conjectures by means of the neutral phenotypic fractions, that were introduced in Hallatschek and Nelson [2008; 2010], and in Garnier et al. [2012]; Roques et al. [2012] to study the inside dynamic of a traveling wave for Fisher-KPP type reaction diffusion equations.

▷ We finally provide some similar heuristics upon lineages in another sexual model of selection, more linked to the mechanism of genetics, with different alleles for each individual, rather than a single trait.

V.1.1 *Approximation method and corresponding heuristics*

We are interested in the following equation for the equilibrium distribution F :

$$\lambda F(z) - c\partial_z F(z) + \mu(z)F(z) = \frac{\beta}{\sigma}\int_{\mathbb{R}} K\left(\frac{z-z'}{\sigma}\right) F(z')\,dz' \tag{V.1.1}$$

The methodology to read an approximated problem of the form of equation (V.1.1) in the asymptotic "$\sigma \ll 1$", has been introduced in the context of evolutionary biology by Diekmann et al. [2005]; Perthame [2007]; Perthame and Barles [2008]. For their part, Bouin et al. [2019] propose a rescaling of equation (V.1.1) by introducing the small parameter

$$\varepsilon := \sqrt{\sigma^2 \frac{\beta}{\alpha}}.$$

In the limit $\varepsilon \to 0$ they get quantitative results measuring the impact of environmental change on for instance the fitness of individuals or the shape of the distribution, but staying in the framework of Barles et al. [2009].

Here, to proceed with our heuristics, we will keep the biological parameters, and study the small variance limit, that is $\sigma \to 0$. We expect when that happens concentration around specific traits, see Barles et al. [2009]; Lorz et al. [2011], which motivates the following logarithmic transform :

$$F(z) = \exp\left(-\frac{U(z)}{\sigma}\right). \tag{V.1.2}$$

Plugging equation (V.1.2) into equation (V.1.1), yields the following equality :

$$\lambda + \frac{c}{\sigma}\partial_z U(z) + \mu(z) = \frac{\beta}{\sigma}\int_{\mathbb{R}} K\left(\frac{z-z'}{\sigma}\right)\exp\left(\frac{U(z)-U(z')}{\sigma}\right) dz'. \tag{V.1.3}$$

To get a finite limit when $\sigma \to 0$, it seems clear that the speed of environmental change must be scaled as σ, this is why we make the following assumption :

$$c := \sigma c'. \tag{V.1.4}$$

We abuse notation by in the following by dropping the $'$. Moreover, we suppose that K is a standardized, positive, symmetric and thin-tailed probability kernel :

$$\int_{\mathbb{R}} K(y)dy = \int_{\mathbb{R}} y^2 K(y)dy = 1, \qquad \exists \eta > 0 \text{ s.t. } \int_{\mathbb{R}} K(y)e^{\eta|y|}dy < +\infty.$$

We finally make the single approximation of these computations, for the integral term of equation (V.1.3):

$$\forall z' \in \mathbb{R}, \quad \exp\left(\frac{U(z)-U(z')}{\sigma}\right) \approx \exp\left(-\frac{z-z'}{\sigma}\partial_z U(z)\right).$$

Therefore by an affine change of variable, equation (V.1.3) is expressed as :

$$\lambda + c\partial_z U(z) + \mu(z) = \beta\int_{\mathbb{R}} K(y)\exp\left(y\partial_z U(z)\right) dy.$$

The assumption on the exponential decay of K makes possible to define the Hamiltonian for all $p \in \mathbb{R}$:

$$H(p) := \int_{\mathbb{R}} K(y) \exp(yp) dy - 1. \tag{V.1.5}$$

And finally, we have that

$$\lambda + c\partial_z U(z) + \mu(z) = \beta + \beta H(p). \tag{V.1.6}$$

We introduce the Lagrangian function L corresponding of K as the Legendre transform of H :

$$L(v) := \max_{p \in \mathbb{R}} \Big(pv - H(p) \Big).$$

With those notations, Bouin et al. [2019] explain how to get the following first order (in ε) approximation for the growth rate λ :

$$\lambda \approx \beta - \mu_0 - \beta L\left(\frac{c}{\sigma\beta}\right) + O(\varepsilon), \tag{V.1.7}$$

$$\tag{V.1.8}$$

The interest of this formula is for instance to compute a threshold on c so that the population does not go extinct, corresponding to the limit case $\lambda < 0$. The dominant trait is located at z^* such that it solves

$$m(z^*) = \beta L\left(\frac{c}{\sigma\beta}\right), \tag{V.1.9}$$

and the phenotypic variance is :

$$\text{Var}\ (F) \approx -\frac{c}{\partial_z \mu(z^*)} + O(\varepsilon).$$

MOST LIKELY LINEAGE AND LAGRANGIAN STRUCTURE

Thanks to the introduction of the Lagrangian L, there actually may exist a representation function of U solution of equation (V.1.6). Inspired by Barles and Roquejoffre [2006], we expect to have :

$$U(z) = \inf_{\gamma \text{ s.t. } \gamma(0)=z} \int_{-\infty}^{0} \left[L\left(\frac{\dot\gamma(s) + c}{\beta}\right) - \beta + \mu(\gamma(s)) + \lambda \right] ds. \tag{V.1.10}$$

The infimum is taken over all functions γ that reach the phenotype z at time 0, when going backward in time. Since U_0 is linked to the distribution of phenotype by the logarithmic transform of equation (V.1.2), solving the minimization problem equation (V.1.10) amounts to solving the spectral problem of equation (V.1.6). Along the phenotypic path γ , minimizing the cost comes from the combined weight of mutations through L (at speed $\dot\gamma + c$) and selection through m. The birth rate β plays an opposite role of selection, while λ is the term that balances the expression.

Therefore an interpretation of the formula is to see a potential minimizing trajectory Γ as a representation of the typical phenotype inside the equilibrium across times that will eventually lead to the phenotype z at final time. The knowledge of the function Γ for any z is somehow richer than the one of the profile U since it accounts, backwardly in time, of the whole dynamic of the equilibrium

from time $-\infty$ to 0. Given the supposed existence of Γ, or of a trajectory that is arbitrarily close to optimum, we can make the following formal analysis, starting with the identity

$$U_0(z) = \int_{-\infty}^{0} \left[L\left(\frac{\dot{\Gamma}(s) + c}{\beta} \right) - \beta + \mu(\Gamma(s)) + \lambda \right] ds.$$

One should expect $\dot{\Gamma}(s)$ to converge to 0 when $s \to -\infty$. Otherwise, the mortality $\mu(\Gamma)$ would become arbitrarily large and so the trajectory would not be minimizing since $U_0(z)$ could take the value $+\infty$. Formally, this implies that Γ converge when $s \to -\infty$, then necessarily it is towards the value 0, in order to minimize the selection function μ. With this formal argument we expect that :

$$\Gamma(s) \xrightarrow[s \to -\infty]{} 0.$$

This precisely the result motivating the conjecture 1. Assuming it holds true, it prescribes the value of λ such that the integrand vanishes, since otherwise U would be infinite.

$$L\left(\frac{c}{\beta} \right) - \beta + \mu(0) + \lambda = 0. \tag{V.1.11}$$

Since $\mu(0) = 0$, and c was the rescaling of the original speed of adaptation $\frac{c}{\sigma}$ to the variance of mutations, the formula coincides with equation (V.1.7). The term $L\left(\frac{c}{\sigma\beta}\right)$ measures the constraint of having to keep pace with the environment evolving at speed c, while $\mu_0 - \beta$ is the cost of fitness for the optimally fitted asymptotic individuals of the lineage.

To conclude the remarks around the variational equation (V.1.10), one should note that thanks again to the Hamiltonian structure, Γ solves formally an Ordinary Differential Equation, see for instance Hairer et al. [2006] :

$$\begin{aligned} \dot{\Gamma}(t) &= -c + \beta\, \partial_p H\Big(\partial_z U(\Gamma(t))\Big), \\ \Gamma(0) &= z. \end{aligned} \tag{V.1.12}$$

This result comes from the work on the characteristics of the Hamilton Jacobi equation (V.1.6). It is not simpler to solve equation (V.1.12), since U itself depends on Γ, but it gives information on the whole behavior of Γ. It will prove to be nonetheless a useful equation to simulate numerically Γ, if one is externally handed the profile U.

Moreover it is known since the works of Champagnat et al. [2006; 2007] that there exists an underlying individual based Markov process that converge in the weak sense of measures towards the equilibrium F when the size of population goes to infinity. Therefore, to assess that interpretation of Γ, we will run individual based stochastic simulations.

V.1.2 *Numerical methods and results*

The Lagrangian structure shed some light on the backwards behavior of lineages, that is for a given individual the traits of his ancestors. To justify this interpretation, we will let the population evolve inside the equilibrium profile, and stop at a given time t. If we are able to keep a memory of the phenotypic lineage of the individuals alive, we will artificially recreate a backwards trajectory despite the simulation running forward.

We will use the description of the underlying Markov process given by Champagnat et al. [2006; 2007] for this kind of processes. The population is described by a vector of individuals, each labeled

by its trait. In addition to the previous parameters, one must prescribe K the theoretical size of the population around which the actual size of the simulated population $K(t)$ will fluctuate. It must be big enough so that the asymptotic $N \sim +\infty$ is verified, but numerically a larger population leads to a larger time of computation, so N must be set reasonably big.

Let us consider any alive random individual i, alive at time t, $1 \leqslant i \leqslant N(t)$, with trait $z_i \in \mathbb{R}$. The outcomes for this individual at time t is one of the following :

- ▷ **Birth of a descendant :** it happens with an uniform rate among individuals $r_B^i = \beta$.
- ▷ **Death of** i Its probability to die is decomposed in two separate events :
  - (a) **Death by selection** The individual may die because its phenotype is really ill adapted. This is given by the relationship :

$$r_{Ds}^i = \mu(z_i - ct).$$

  - (b) **Death by density dependence** Alternatively, an individual may die because of the density dependence in the population, at a rate that is prescribed by the size of population at time t and the carrying capacity N

$$r_{Ddd} = \sum_{j=1}^{N(t)} \frac{1}{N} = \frac{N(t)}{N}.$$

- ▷ **Incrementation of the time step** The time step is the smallest time for all individuals to go through one of the previous steps. By the Markovian property each event occurs following an exponential law of parameter dictated by its rate r :

$$dt \sim \min_{1 \leqslant i \leqslant N(t)} \mathcal{E}(r_B^i + r_{Ds}^i + r_{Ddd}).$$

  By the property of "absence of memory" of the exponential law, dt also follows the law :

$$\mathcal{E}\left(\sum_i (r_B^i + r_{Ds}^i + r_{Ddd})\right)$$

It would have been more succinct to present the population as the point measure (empirical density)

$$\nu_t^K(dz) := \frac{1}{N} \sum_{j=1}^{N_t^N} \delta_{z_i(t)}(dz),$$

and describe its evolution through its actions on measurable bounded functions by the generator of ν_t^K, since it contains all of the previous informations on the infinitesimal fluctuations of the population. We tried a more pedestrian explanation in the hope to be more concrete. One should note that Méléard et al. [2012] proposed a model for the lineages itself, where the population is no longer characterized by a measure on R^K (corresponding to the trait of individuals), but by a measure on the space of càdlàg functions. Each càdlàg function corresponds to a lineage of the individuals in the population. With the same tools as in Champagnat et al. [2006; 2007], they propose an asymptotic limit for the dynamics of lineages. The limit object is not a Partial Differential Equation in the general case, they used a more theoretical tool, and it seems interesting to study this approach later.

Finally, the implementation of the operator $\mathcal{B}$ in equation (V.0.2), is straightforward using the following interpretation of the convolution term :

$$z_{\text{offspring}} = z_{\text{parent}} + \mathcal{N}(0, \sigma).$$

We get the population at time $t + dt$ by either adding the individual that was born ($K(t + dt) = K(t) + 1$) or subtracting the one that died ($K(t + dt) = K(t) - 1$), and we repeat all the steps until reaching the desired final time of simulation. Therefore, starting from an initial population, and applying the previous recursive steps, one can simulate the equilibrium F with an individual based random walk. However, several difficulties remain. The variance σ must be small enough and K big enough to have precise approximations. Decreasing σ increases automatically the time of computation if the population is not already at equilibrium since there is little diversity created to shift the profile. The time step dt decreases when K rises because of the density dependence, individuals will die more often, and more individuals means more events (birth or death). Moreover the procedure is very avid of computational times because of all the random draws one has to make, and the more individuals, the more one has to make computations, even if the population is at equilibrium.

Consequentially, we performed the simulations using an approximated model, by first fixing dt to a small but deterministic value. Then, for each individual, we draw a time of birth following the law $\mathcal{E}(\beta)$ and a time of death following the law $\mathcal{E}(\mu(z_i) + N(t)/N)$. Then we simply count which individuals led to a reproduction event and which died on the time-window $[t, t + dt]$. This amounts to the supposition that on this interval of time, individuals cannot reproduce more than once. This led to the figure V.2 comparing the profile at different times in the simulation. Actually, for a long time while doing the simulations we witnessed huge fluctuations inside the equilibrium, even if all the explicit parameters –given the approximations formulas of equation (V.1.7)– indicated that we were in a context of positive equilibrium ($\lambda > 0$). The dominant trait would oscillate around its theoretical value, and the size of the populations $K(t)$ would suddenly plummet and then explode without explanations. We figured out eventually since we often tried to minimize the time of computation by decreasing K and toying with the other parameters, we were in fact working very close to the critical regime of the Markov process, which explained those random variations. Those disappeared when working with more reasonable parameters, at the price of longer time of computations.

Finally, let's explain how we follow the lineage of individuals. We create a huge matrix at the beginning which is where we will stock the lineage of every individual along the simulation. Every time an individual appears, since we kept track of its parent lineage we are able to know its lineage. The procedure works as described in figure V.3, where each line corresponds to a generation for each individual, without distinction of time.

This lead to the main numerical result of this section. We are able to simulate the PDE (V.0.1), which gives an approximated profile U, from which, by solving equation (V.1.12) we plot a theoretical lineage (in red). We compare that to the simulated lineages that were obtained by the aforementioned algorithm. The following parameters were used :

$$\alpha = 2, \quad \beta = 2, \quad \sigma = 0,1, \quad K = 20000, \quad c = 0,2. \tag{V.1.13}$$

For the needs of representation, we select z^* as the starting point of the lineages see figure V.1:

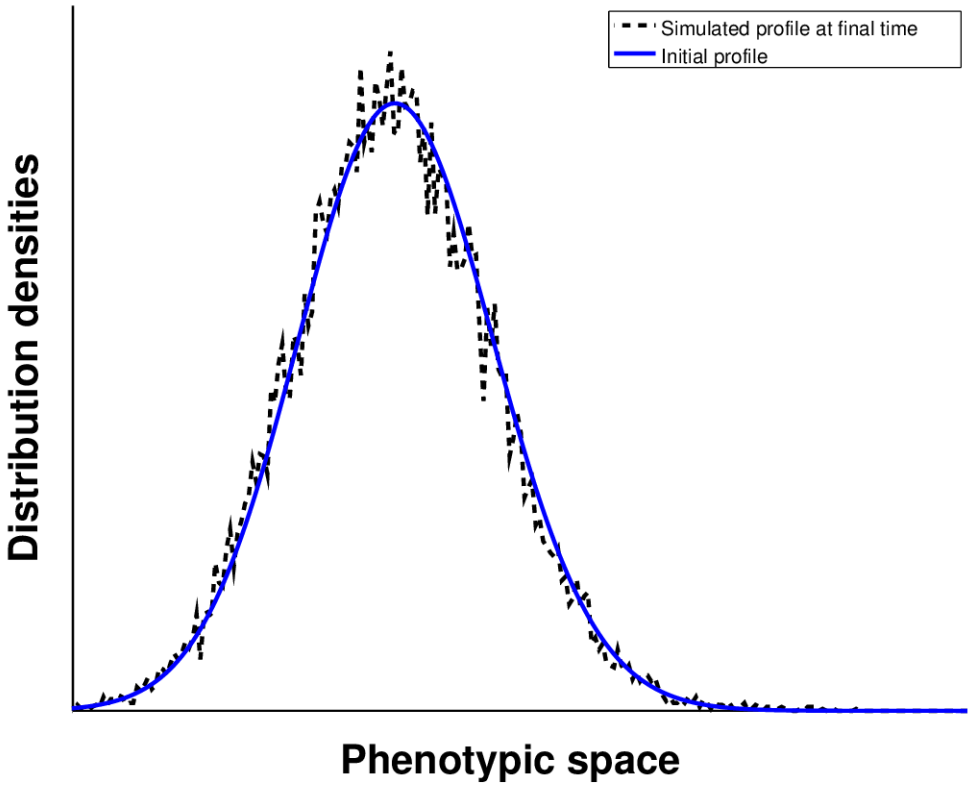


Figure V.2: The grey background is the stationary density of individuals F, from which we initiated the algorithm. The ordinate axis represents the phenotype z in the moving frame ($z = x - ct$).

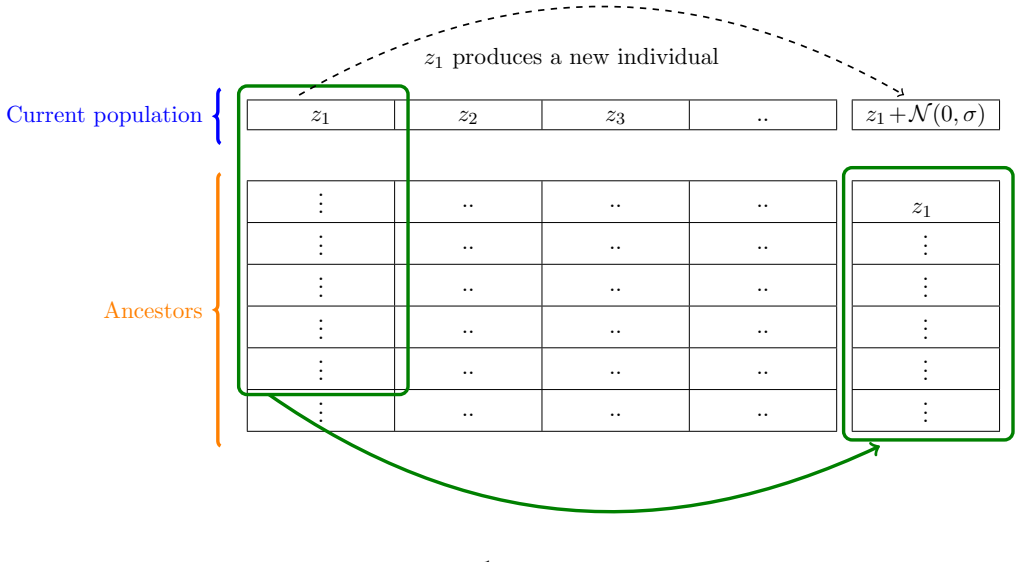


Figure V.3: Keeping track of the lineages : a tentative explicative drawing

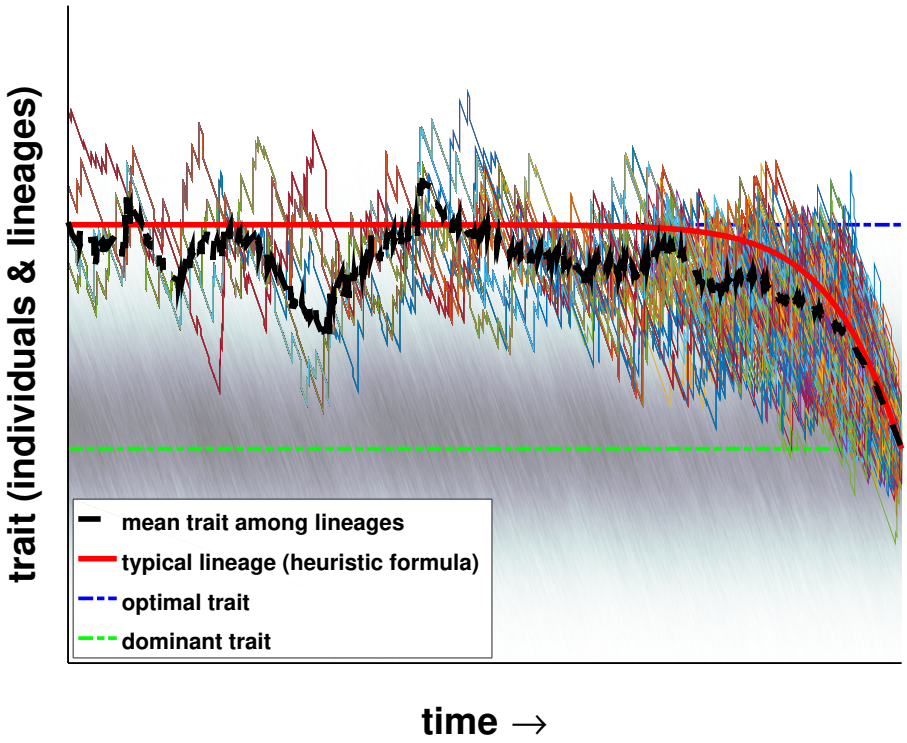


Figure V.4: The grey background is the stationary density of individuals F. The y axis represents the phenotype z in the moving frame ($z = x - ct$).

As it is announced in conjecture 1, one can observe that despite having very little density around the optimum trait, it is where the lineages take their origin. Some sort of coalescent phenomenon can also be observed, since there are approximately 900 lineages at the final time, while one observe less than ten different lineages when $t \to -\infty$, meaning a lot of individuals share a common ancestor. It is already known that a small number of individuals can recreate a whole population. Authors of Allendorf and Luikart [2009] explain that in Isla Guadalupe, Mexico, the 20.000 northern elephant seals alive today are the descendants of less than 20 individuals that lived in the nineteenth-century. Nowadays, the amount of DNA sequence data available has led to an intensive research for coalescent alleles ("common ancestors"). Our new result explains that in our model, individuals responsible for the genetic diversity will be the small number who are optimally fitted.

If we go back to figure V.1, we have plotted the trajectory depicted by figure V.5.

V.2 LINEAGES DYNAMICS OF THE SEXUAL MODEL

We are now interested in the following equation for the equilibrium distribution F with a sexual (infinitesimal) mode of reproduction:

$$\lambda F(z) - c\partial_z F(z) + \mu(z)F(z) = \beta \iint_{\mathbb{R}^2} G_{\sigma^2}\left(z - \frac{y_1+y_2}{2}\right) F(y_1)\left(\frac{F(y_2)}{\int_{\mathbb{R}} F(y_2')\,dy_2'}\right) dy_1 dy_2. \tag{V.2.1}$$

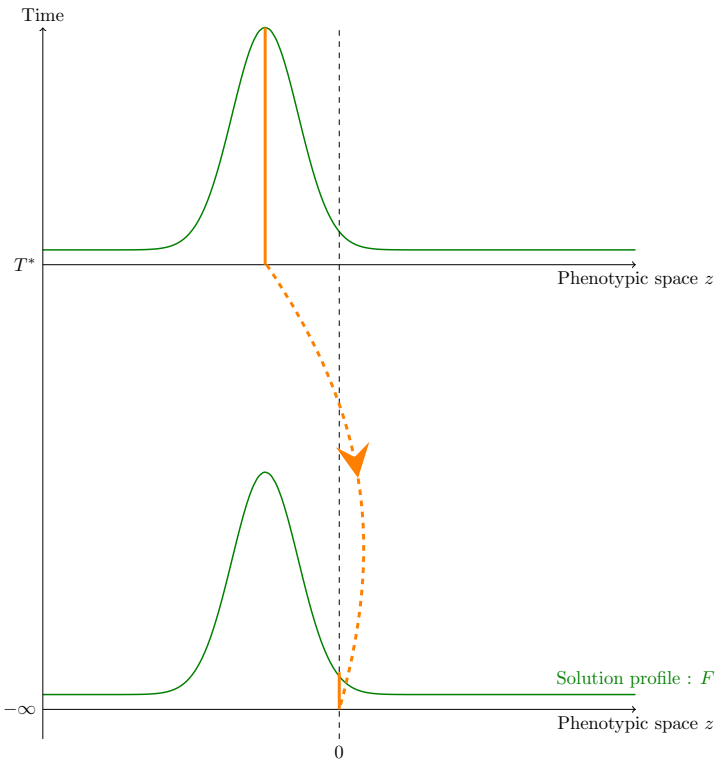


Figure V.5: Lineage arriving at dominant trait mostly come from optimum trait.

We are still interested in understanding the phenotypic dynamics of the ancestors of individuals at equilibrium. Compared to the asexual mode of reproduction, one loses the Hamiltonian structure that hold true in the limit $\sigma \to 0$. Therefore, we must understand how the traits of the offspring are linked with the ones of the parents directly through the operator $\mathcal{B}$ to propose an heuristic behind the conjecture 2. The idea is to look both at the trait distribution of the parents from the point of view of an offspring and the distribution of offspring from the point of view of the parents. The former is quite simple, because it is a reasoning that goes forward in time, and it can be read straightforwardly on the infinitesimal operator. If the two parents have trait z_1 and z_2, then the offspring has the trait

$$z_{offspring} = \frac{z_1 + z_2}{2} + \sigma \mathcal{N}\left(0, \frac{1}{2}\right). \tag{V.2.2}$$

Conversely, it is natural to seek, given the trait of an offspring z, what are the most likely traits of his two parents? A response was given by Bouin et al. [2019] by studying how the operator $\mathcal{B}$ concentrates around a trait when $\sigma \to 0$. The arguments, that we recalled in part IV, show that the double integral concentrates around $(\bar{z}, \bar{z})$ defined as the midpoint

$$\left(z_1 = \frac{z + z^*}{2}, z_2 = \frac{z + z^*}{2}\right), \tag{V.2.3}$$

where z^* is the dominant trait of the population. This can be interpreted as the two most likely parents for the offspring. Therefore, we guess that for any individual with a trait z, given the population at equilibrium, the most likely scenario is that both his parents have a trait that is midway between his and the dominant trait *i.e.* $\bar{z}$. As a consequence, one expects that in the matter of few generations, the individual will have most of his ancestors that have (or are arbitrarily close to it)

the dominant trait z^*. This justifies the conjecture 2. The following figures tries to illustrate an heuristics on why the infinitesimal operator concentrates around that particulars middle points.

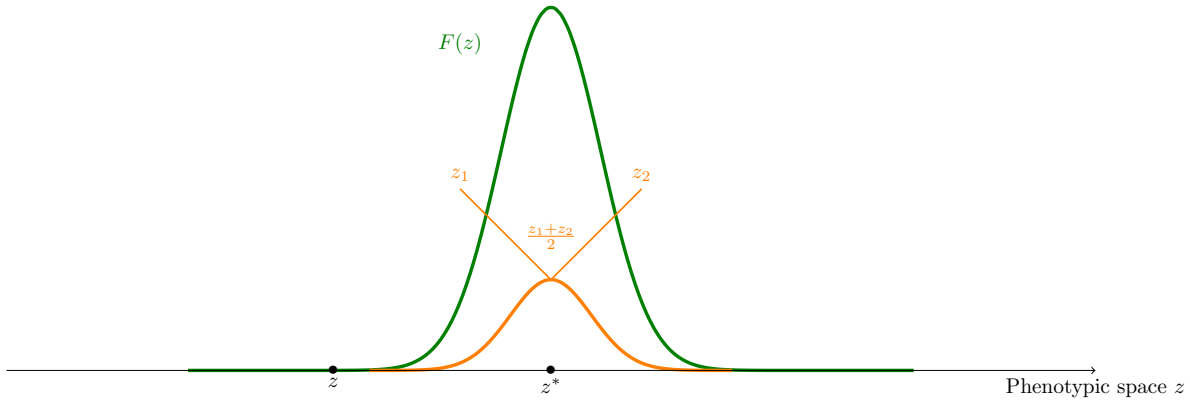


Figure V.6: In orange the distribution law $\sigma\mathcal{N}\left(0, \frac{1}{2}\right)$ of the deviation from the mean trait of parents.

From the point of view of parents, it seems most likely that both of them are at the most represented trait in the population, z^*. However like the previous graphic tries to illustrate, the outcome of the trait of the offspring (distribution in orange) is too narrow, which means that it is not likely for them to have an offspring far from z^*, like the z in the picture. Moreover, given the distribution of the offspring, this is even more the case the closest σ to 0.

Conversely given the phenotype of an individual of trait z, it would seem likely to have any parents of traits (z_1, z_2) that verify $\frac{z_1+z_2}{2} = z$, as pictured below :

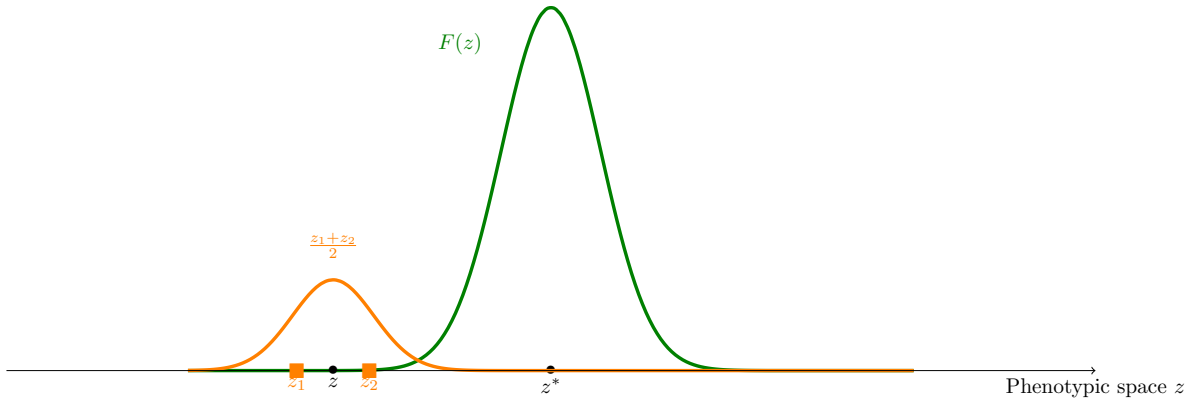


Figure V.7: Both parents have a trait in a very low density zone of F.

However, in the scenario depicted, the traits of the parent are in very low density regions for the population. Therefore we also discard that scenario of reproduction. Interestingly the trade-off between the two scenarios consists in both parents having traits that realizes a compromise between the two situations, see figure V.8.

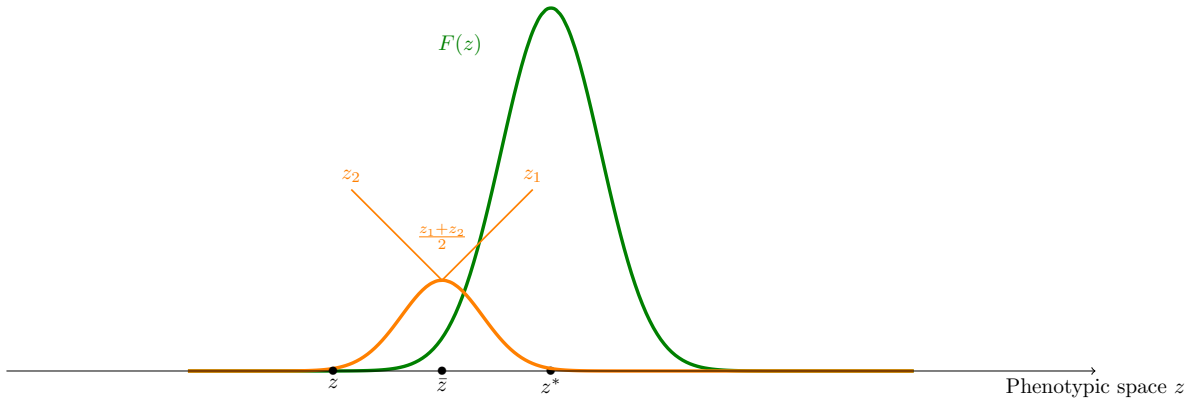


Figure V.8: Most probable scenario : both parents at $\bar{z}$.

Those heuristics were verified by the numerics on the individual based Markov chain process linked to equation (V.2.1). The structure of the algorithm is exactly the same, the main difference comes from when a birth event occur. In that eventuality, the individual that rung the clock for a birth event constitutes the first parent. To choose the other parent, one draws uniformly randomly a second individual among the whole population, faithfully to the interpretation of the infinitesimal model. Interestingly, the profile F solution of equation (V.2.1) is more narrow in the sexual case, since the variance is of order σ, while it is of order $\sqrt{\sigma}$ in the asexual case :

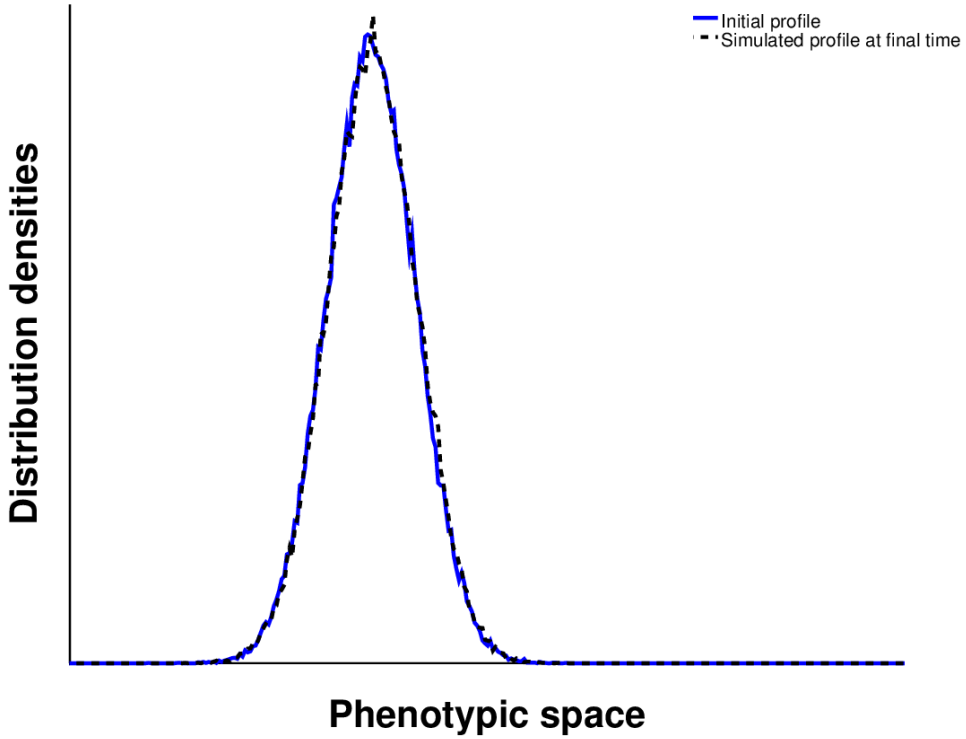


Figure V.9: The x axis represents the phenotype z in the moving frame ($z = x - ct$).

To keep track of the lineages, one faces a memory problem. Because of the nature of sexual reproduction, each individual has 2^n ancestors at the n-th generation. To deal with it -in our simulations we often trade with thousands of generations- we choose to keep the information on only one parent at each birth event, which means that we forget about the trait of one parent at each generation. However, we do not consider this to be a limitation at all. Firstly both parent play a symmetric role in the infinitesimal operator. Second, our heuristics have highlighted that both parents are expected to have most of the time the same trait $(z + z^*)/2$. Therefore, whatever parents one chooses to keep in memory, the overall behavior is not disturbed. This led to the trajectories described by figure V.10.

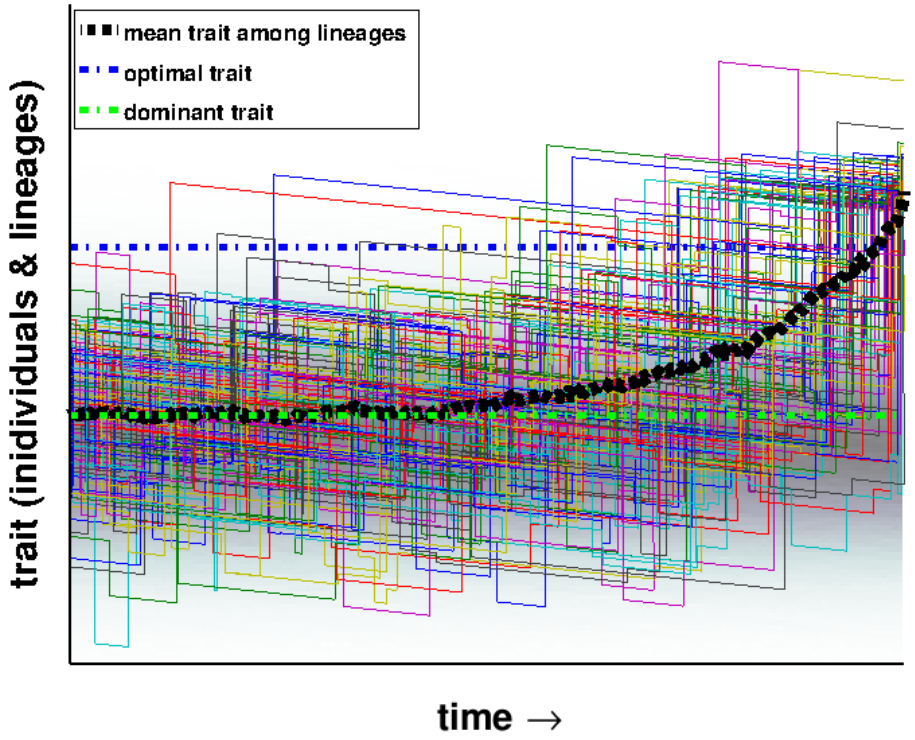


Figure V.10: The grey background is the stationary density of individuals F. The y axis represents the phenotype z in the moving frame ($z = x - ct$).

We sampled the origin of the lineages far from both the optimal and dominant trait so that the trajectory is not trivial. Compared to figure V.4, the time window is in fact much shorter. This is logical, since at each generation one expects the lineage to halve the distance with the dominant trait, so the return towards the dominant trait can be conjectured to be exponential. The figure V.10 confirms the intuition of conjecture 2 concerning the origin of the population. Their mean behavior (in black) clearly indicates the trajectory to originate from the dominant trait, even if it lacks a comparable "theoretical" trajectory. It means that there is much more mixing in the population compared to the asexual mode of reproduction, since an ill-adapted trait is the common trait for the ancestors of the majority of individuals. Somehow the weight of having a parent ill-adapted is less in this sexual mode of reproduction, probably because it can be counterbalanced by having a second

parent much better adapted. In terms of dynamics of the equilibrium, the following scheme proposes an explanation of both conjecture 2 and figure V.10 :

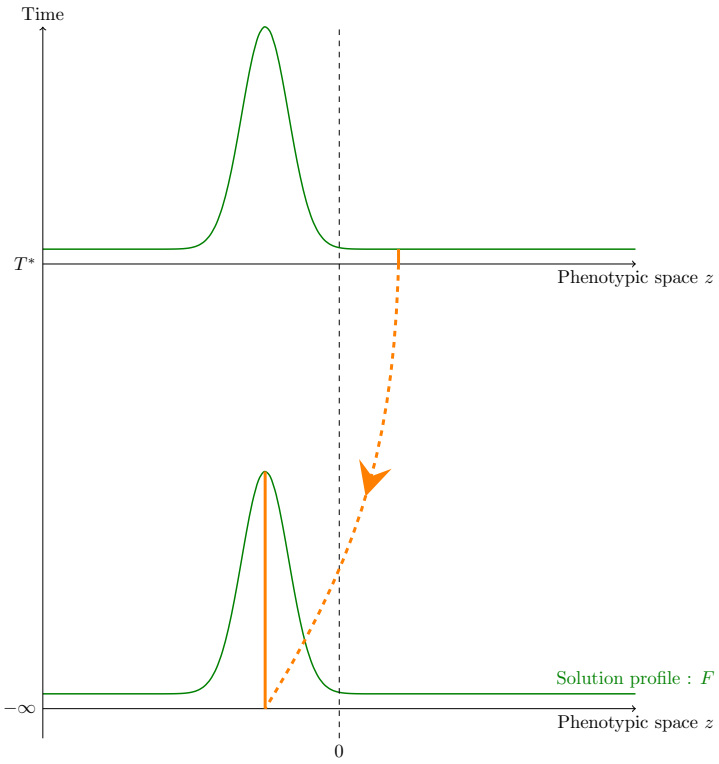


Figure V.11: Lineage arriving at any trait mostly come from dominant trait.

V.3 INSIDE DYNAMICS OF THE EQUILIBRIUM THROUGH NEUTRAL FRACTIONS

In this section we try to provide another heuristics that may explain conjecture 1 and conjecture 2, this time by looking at the descendants created by a group of individuals. Let us assume that the individuals are labeled and they transmit their label to their offspring even if the offspring differs in trait. Somehow, those label can be seen as neutral genes that are promoted or demoted because they are close to gene under selection, creating a phenomenon of genetic hitchhiking, see Barton [2000]. Mathematically, this approach has been introduced in the context of reaction diffusion equations by Hallatschek and Nelson [2008; 2010], and in Garnier et al. [2012]; Roques et al. [2012] to understand the inside dynamic of the traveling wave solution arisen in this context. They were able thanks to this methodology to establish a robust dichotomy between two different possible behaviors for the inside dynamic of the front, "pulled" or "pushed" front, generalizing a concept of Stokes [1976].

Our purpose is very similar to Garnier et al. [2012], since it lies in understanding the dynamics of an equilibrium profile. We assume that the population is made of several *neutral* fractions v^k that can have different traits. In particular, at time $t = 0$:

$$f_0(x) := f(0,x) = \sum_{k\geqslant 1} v_0^k(x), \text{ with } v_0^k \geqslant 0 \text{ for all } k \geqslant 1.$$

The key assumption to these heuristics is to assume that each fraction only differ by their label, while the mutation and selection processes act the same way they do on the entire population f, in the sense that the density of each fraction verifies an equation of the form :

$$\begin{aligned}&\partial_t v^k(t,x) + \Big(\mu(x-ct) + (\beta-\mu_0)\int_{\mathbb{R}} f(t,x')\,dx'\Big)v^k(t,x) = \beta\mathcal{B}_f(v^k(t,\cdot))(x)\,,\ \text{for}\ t>0,\ x\in\mathbb{R}\\ &v^k(0,x) = v_0^k(x),\ x\in\mathbb{R}.\end{aligned} \tag{V.3.1}$$

The concept of "neutral" fractions lies in the choice of the linear operator $\mathcal{B}_f$. In the asexual scenario, $\mathcal{B}$ is already linear, so :

$$\mathcal{B}_f(v)(x) = \frac{1}{\sigma}\int_{\mathbb{R}} K\left(\frac{x-x'}{\sigma}\right) v(x')\,dx'.\ \textbf{(Asexual reproduction)}$$

In the sexual case however, since it is not linear we choose the following

$$\mathcal{B}_F(v)(x) = \iint_{\mathbb{R}^2} G_{\sigma^2}\left(x - \frac{x_1+x_2}{2}\right) v(x_1)\left(\frac{f(x_2)}{\int_{\mathbb{R}} f(x_2')\,dx_2'}\right) dx_1dx_2.\ \textbf{(Sexual)} \tag{V.3.2}$$

This translates the idea of neutrality of each fraction : each parent x_1 can choose a second parent in the whole population, and not only its fraction. Therefore, in both cases, the sum of the fraction densities verifies equation (V.0.1).

Next, we consider an arbitrarily chosen fraction v^k that satisfies equation (V.3.1) –we call it v in the sequel– and we study its evolution. As before, we assume the total population (and sum of all fractions) f to be at an equilibrium of the form $f(t,x) = F(x-c\,t)$. Moreover, it is convenient to work, as previously in the moving frame of speed c, the density $v(t,x+ct) := v(t,z)$. All in all, v satisfies the following equation:

$$\partial_t v(t,z) - c\partial_z v(t,z) + \Big(\mu(z) + (\beta-\mu_0)\int_{\mathbb{R}} F(z')\,dz'\Big)v(t,z) = \beta\mathcal{B}_F(v(t,\cdot))(z)\,, t>0\ z\in\mathbb{R} \tag{V.3.3}$$

By definition of λ in equation (V.0.6), we can write the equation (V.3.3) as follows, in order to shift the main eigenvalue to 0 :

$$\partial_t v(t,z) = \beta\Big(\mathcal{B}_F(v(t,\cdot))(z) - v(t,z)\Big) + c\partial_z v(t,z) - \Big(\mu(z)-\overline{\mu}\Big)v(t,z)\,, t>0\ x\in\mathbb{R} \tag{V.3.4}$$

We recall that we supposed that we were in the regime where $\lambda > 0$.

FORMAL RESOLUTION OF EQUATION (V.3.4).

We will perform a spectral analysis of the previous equation. Let us denote $\mathcal{L}$ the following operator on $L^2(\mathbb{R})$:

$$\mathcal{L}(v) = \beta\Big(\mathcal{B}_F(v) - v\Big) + c\partial_z v - \Big(\mu-\overline{\mu}\Big)v,$$

such that

$$\partial_t v(t,z) = \mathcal{L}(v)(t,z). \tag{V.3.5}$$

Its domain $D(\mathcal{L})$ is $H^1(\mathbb{R})$. Moreover, we can define its adjoint operators $\mathcal{L}^*$ in $L^2(\mathbb{R})$ as follows

$$\mathcal{L}^*(\phi) = \beta\Big(\mathcal{B}_F^*(\phi) - \phi\Big) - c\partial_z\phi - \Big(\mu-\overline{\mu}\Big)\phi \tag{V.3.6}$$

where $\mathcal{B}_F^*(\phi)$ is the adjoint operator to $\mathcal{B}_F$, that differs in each reproduction scenario.

By definition, F is the eigenvector of $\mathcal{L}$ associated to the eigenvalue 0 , see equation (V.3.4). Let us denote φ the eigenvector of the adjoint operator $\mathcal{L}^*$ associated to eigenvalue 0. Then we know that any solution of equation (V.3.3) starting from v_0 satisfies

$$v(t,z) \xrightarrow[t\to\infty]{} p[v_0]F(z), \quad \text{uniformly,} \quad t\to\infty, \quad \text{where} \quad p[v_0] := \frac{\int_{\mathbb{R}} v_0(z)\varphi(z)dz}{\int_{\mathbb{R}} F(z)\varphi(z)dz}. \tag{V.3.7}$$

Let's sketch the proof of this classical result.

- The domain of $\mathcal{L}$ generates a C^0- semi-group on $L^2(\mathbb{R})$ because
  1. $D(L)$ is dense in $L^2(\mathbb{R})$.
  2. For any $\gamma > 0$, the resolvent R verifies $R(\gamma I - \mathcal{L}) = L^2(\mathbb{R})$.
  3. There exists ω such that $L - \omega Id$ is dissipative.
- One can show that the kernel of $\mathcal{L}$ is in fact $Ker(\mathcal{L}) = F\mathbb{R}$, where F is the steady state.
- We decompose $L^2(\mathbb{R}) = F\mathbb{R} + Y$, where Y is the orthogonal of the eigenspace $F\mathbb{R}$, and is defined by $Y = \left\{ v | \int_{\mathbb{R}} v\varphi = 0 \right\}$ where ϕ is the eigenvector associated to the adjoint operator $\mathcal{L}^*$ in equation (V.3.6).
- Therefore, the semi-group $(S(t))$ generated by $\mathcal{L}$ stabilizes Y : $(S(t)Y \subset Y)$ and moreover the restriction of $S(t)$ to Y, $S(t)|Y$ is also a C^0 semi-group generated by the restriction of $\mathcal{L}$ on Y, $L_0 = L|Y$.
- Since 0 is simple and isolated (*which remains to be proven*), the spectrum of L_0 is in $(-\infty, -\eta)$, with η corresponding to the spectral gap.

From a spectral projection Theorem, for instance Engel and Nagel [2001], we can conclude on the asymptotic behavior of the solutions of equation (V.3.7).

LINK WITH THE LINEAGES.

To link the neutral fractions with the phenotypic lineages ones asks: what is the contribution of a fraction of gene with initially the trait y? To answer this question we just need to look at an initial condition of the form

$$v_0(z;y) = \delta(z-y)F(z), \text{ with } \quad y \in \mathbb{R}, \tag{V.3.8}$$

where δ is the dirac function. This means that we look at a sampling of the profile with a single trait. We denote $v(t,z;y)$ the density of population with trait z at time t, given that initially the individuals had the trait y. From the previous result, the distribution of this fraction starting from y is given by

$$v(t,z;y) \xrightarrow[t\to\infty]{} p[y]F(z) \text{ as } t\to\infty, \quad \text{with} \quad p[y] = \frac{F(y)\varphi(y)}{\int_{\mathbb{R}} F(z')\varphi(z')dz'}. \tag{V.3.9}$$

It is meaningful that the function p does not depend on z. This formulas tells us that the number of individuals arriving at z is in fact an independent number of z that is $p(y)$ multiplied by the density at given point z. Therefore $p(y)$ is a proportion of individuals coming from y in the final population, it is constant among the distribution. This holds true as long as the asymptotic spectral result of equation (V.3.9) holds true.

Coming back to the lineages, we were looking for an answer to the question : given a trait z, what are the ancestor of those individuals? With the previous notations, the lineage is the following function :

$$y \mapsto v(t,z;y). \tag{V.3.10}$$

This explicits well the fact that a lineage goes backwards in time while the study of fractions works forwards. Therefore at any time t the mean trait $\overline{y}(t,z)$ and the most frequent trait $y^*(t,z)$ of the ancestors of individuals of trait z is given by

$$\overline{y}(t,z) = \frac{\displaystyle\int_{\mathbb{R}} y\, v(t,z;y)\, dy}{\displaystyle\int_{\mathbb{R}} v(t,z;y)\, dy} \quad \text{and} \quad y^*(t,z) = \underset{y\in\mathbb{R}}{\arg\max}\ v(t,z;y).$$

But since by definition $\displaystyle\int_{\mathbb{R}} v(t,z;y)\, dy = F(z)$, we deduce that

$$\overline{y}(t,z) = \frac{\displaystyle\int_{\mathbb{R}} y\, v(t,z;y)\, dy}{F(z)}.$$

From the above formal equation (V.3.9), those quantities converge, when $t \to +\infty$, to

$$\overline{y}(t,z) \xrightarrow[t\to+\infty]{} \frac{\displaystyle\int_{\mathbb{R}} y\, F(y)\varphi(y)\, dy}{\displaystyle\int_{\mathbb{R}} F(z')\varphi(z')dz'}, \quad \text{and} \tag{V.3.11}$$

$$y^*(t,z) \xrightarrow[t\to+\infty]{} \underset{y\in\mathbb{R}}{\arg\max}\, F(y)\varphi(y). \tag{V.3.12}$$

It should be noted that those expressions do not depend on the point z where the sampling of the lineage occurs. This is a by-product of p not depending on z. It means that in our model there is a single trait shared by the majority of the ancestors of all individuals in the profile. With those quantitative results, we will try to justify conjecture 1 and conjecture 2. As a remark, one should notice that we can have access to the whole trajectory of $\bar{y}$ in time when multiplying equation (V.3.4) by z and integrating with respect to y.

V.3.1 *The asexual reproduction case*

To tackle the asexual reproduction, we first use the *diffusive approximation*, where the computations can be made explicit.

V.3.1.1 *The diffusion approximation case.*

If the mutational variance σ^2 is small, $\mathcal{B}_F$ equation (V.0.2) can be approximated by the following diffusive operator:

$$\mathcal{B}_F(F)(z) \approx \beta F(z) + \beta \frac{\sigma^2}{2} \partial_z^2 F(z).$$

Under this approximation, the operator $\mathcal{L}$ becomes :

$$\mathcal{L}(v) = \beta \frac{\sigma^2}{2} \partial_z^2 v + c \partial_z v - \Big(\mu - \overline{\mu}\Big) v$$

By definition, F satisfies $\mathcal{L}(F) = 0$, and therefore thanks to the quadratic decay of μ, $\widetilde{F} := F e^{cx/(\beta\sigma^2)}$ is in $L^2(\mathbb{R})$. Moreover, it satisfies $\widetilde{\mathcal{L}}(\widetilde{F}) = 0$, where $\widetilde{\mathcal{L}}$ is the following self-adjoint operator in $L^2(\mathbb{R})$:

$$\widetilde{\mathcal{L}}(\widetilde{v}) = \beta \frac{\sigma^2}{2} \partial_z^2 \widetilde{v} - \Big(\mu - \overline{\mu} + \frac{c^2}{2\beta\sigma^2}\Big) \widetilde{v}.$$

For any dual eigenfunction φ such that $\mathcal{L}^*(\varphi) = 0$, the function $\widetilde{\varphi} := \varphi e^{-cz/(\beta\sigma^2)}$ also is in $L^2(\mathbb{R})$, again by decay of μ. A simple computations shows that $\widetilde{\varphi}$ satisfies $\widetilde{\mathcal{L}}(\widetilde{\varphi}) = 0$, and therefore $\widetilde{F} = \widetilde{\varphi}$. As a conclusion we have found an explicit expression for φ in the diffusion approximation,

$$\varphi(z) = F(z) e^{2cz/(\beta\sigma^2)}.$$

In the perspective of the lineages, it means that the asymptotic mean trait and most frequent ancestor can also be computed explicitly for large time:

$$\overline{y}(\infty, z) = \frac{\displaystyle\int_{\mathbb{R}} y F(y)^2 e^{2cy/(\beta\sigma^2)} dy}{\displaystyle\int_{\mathbb{R}} \Big(F(z') e^{cz'/(\beta\sigma^2)}\Big)^2 dz'} \quad \text{and} \quad y^*(\infty, z) = \arg\max_{y \in \mathbb{R}} F(y)^2 e^{2cy/(\beta\sigma^2)}$$

Moreover, the function defined for each z as $\widetilde{F}(z) := F e^{cz/(\beta\sigma^2)}$ is even, since μ is an even function and $\widetilde{F}$ satisfies $\widetilde{\mathcal{L}}(\widetilde{F}) = 0$. Therefore $y \mapsto \Big(F(y) e^{cy/(\beta\sigma^2)}\Big)^2$ is also even, and as a consequence :

$$\overline{y}(\infty, z) = 0 \quad \text{for all} \quad z \in \mathbb{R}.$$

In addition, we deduce that $\widetilde{F}$ has a (local) extrema at $z = 0$. To get more information we investigate the sign of $\partial_z^2 \widetilde{F}(0)$. By definition $\widetilde{\mathcal{L}}(\widetilde{F}) = 0$, but $\mu_0 - \bar{\mu} < 0$. Therefore, one must choose σ sufficiently small so that $\partial_z^2 \widetilde{F}(0) > 0$. Then since F can have at most one maximum, with equation (V.3.12),

$$y^*(\infty, z) = 0 \quad \text{for all} \quad z \in \mathbb{R}.$$

This confirms once again the conjecture 1 in the case of the diffusive approximation. We can also compute the equation satisfies by $\bar{y}$ to obtain the entire trajectory;

$$\partial_t \overline{y}(t, z) = \frac{\beta\sigma^2}{2} \partial_z^2 \overline{y}(t, z) + \left(c + \frac{\beta\sigma^2}{2}\right) \partial_z \overline{y}(t, z),$$

which can be transformed into the following equation

$$\partial_t \overline{y} = \frac{\beta\sigma^2}{2} \Big(\partial_z^2 \overline{y} + \partial_z(\ln(F\varphi)) \partial_z \overline{y}(t, z)\Big).$$

V.3.1.2 *The general asexual case.*

We will try to adapt the key arguments of the diffusive approximation. We recall that we look at the following equation

$$\partial_t v(t,z) = \mathcal{L}(v)(t,z).$$

where

$$\mathcal{L}(v) = \beta\Big(K * v - v\Big) + c\partial_z v - \Big(\mu - \overline{\mu}\Big)v.$$

The notations $*$ stands for the convolution between the probability kernel K, of variance σ and the function v. We defined the adjoint operator $\mathcal{L}^*$ on $L^2(\mathbb{R})$ as follows, since the convolution is a self adjoint operator :

$$\mathcal{L}^*(\phi) = \beta\Big(K * \phi - \phi\Big) - c\partial_z\phi - \Big(\mu - \overline{\mu}\Big)\phi$$

The key point is to notice that $\varphi(z) := F(-z)$ satisfies

$$\mathcal{L}^*(\varphi) = 0,$$

because μ is an even function. Therefore, **the function $F\varphi$ is even.** This parity arguments was a key point of the diffusive approximation. Thanks to equation (V.3.12), this implies

$$\overline{y}(\infty,z) = 0 \quad \text{for all} \quad z \in \mathbb{R}.$$

Moreover, we deduce that $F\varphi$ has an extrema at $z = 0$. As before, for σ small enough this is a maximum and we conclude that

$$y^*(\infty,z) = 0 \quad \text{for all} \quad z \in \mathbb{R}.$$

As a conclusion, we used the formalism of neutral genetics fractions to give heuristics justifying conjecture 1.

Now let's show some detailed numerical pictures of our results. We sampled the phenotypic space at a large number y_k of points and considered all fractions v_k starting with the initial data $\delta(z - y_k)F(z)$. Then we numerically simulate the equation equation (V.3.4). The first figure, figure V.12 represents the cumulative densities of the fraction v_k . As announced, when summing all the contributions of each fraction we recover the density F. However it is not clear from this figure how to quantify exactly each contribution $v_k(z)$ to the profile $F(z)$, which is our objective. This information is given by the function $p(y)$, in the figure V.13. Alternatively, thanks to equation (V.3.9), it is the graphic representation of the fraction

$$\frac{F(y)\varphi(y)}{\displaystyle\int_{\mathbb{R}} F(z')\varphi(z')dz'}.$$

The figure V.13 describes a well broader picture than the heuristics of conjecture 1 or equations (V.3.11) and (V.3.12), since it contains much more information. However, we notice that as we wanted to prove,

$$\overline{y}(\infty,z) = 0, \quad y^*(\infty,z) = 0 \quad \text{for all} \quad z \in \mathbb{R}.$$

Also, p is an even function, which was the main ingredient of our formal proof.

Now thanks to our detailed account of what happens inside the equilibrium, we can have in addition to the behavior at $t = +\infty$, the behavior of $v(t,z;y)$ for all times and any z in the distribution.

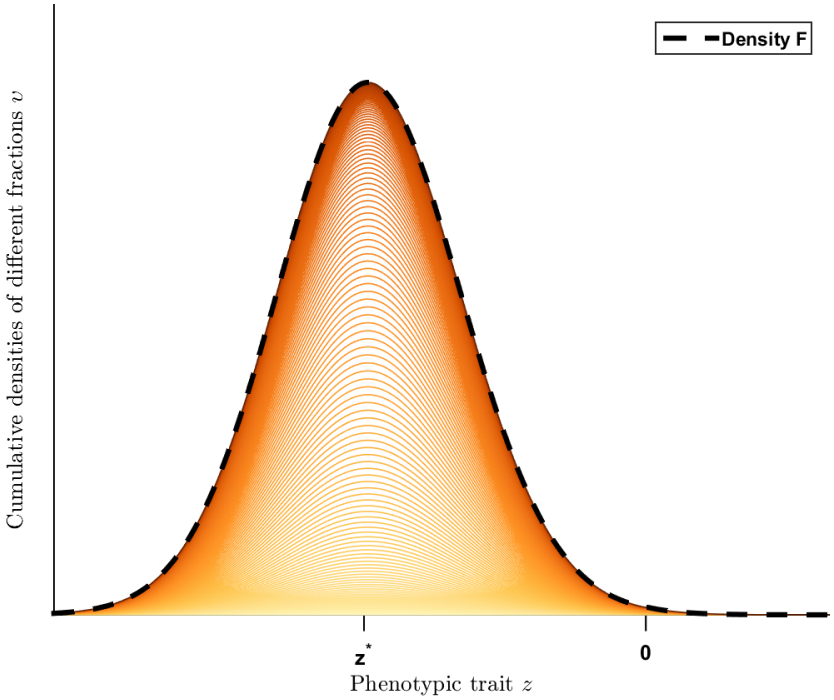


Figure V.12: Cumulative densities of the fractions v_k.

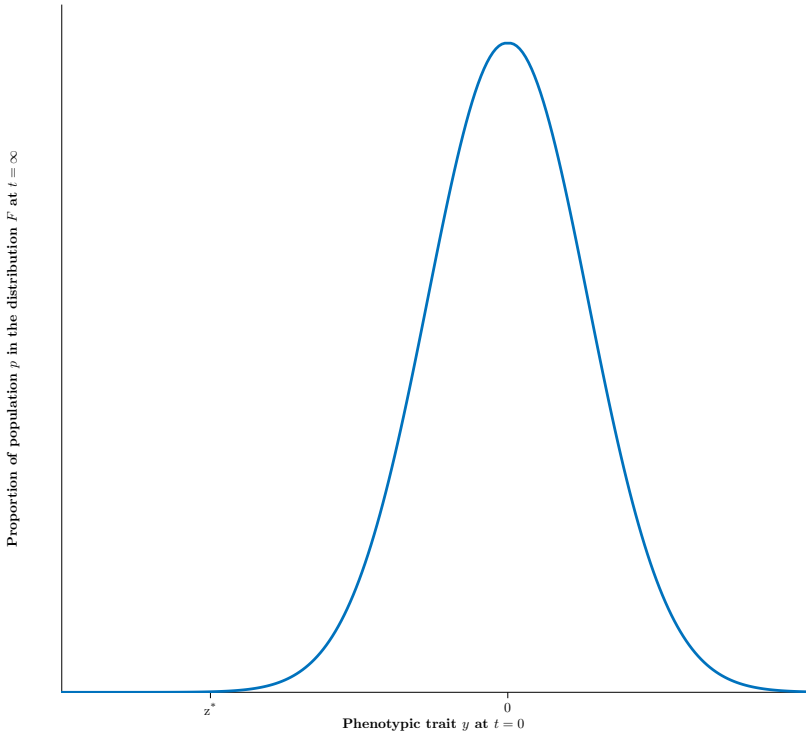


Figure V.13: Proportion $p[y]$ of the population F with ancestors y when $t \to -\infty$.

Therefore we sample the lineages at the dominant trait z^*, and plot the statistics of the function $v(t, z^*; y)$ as time evolves.

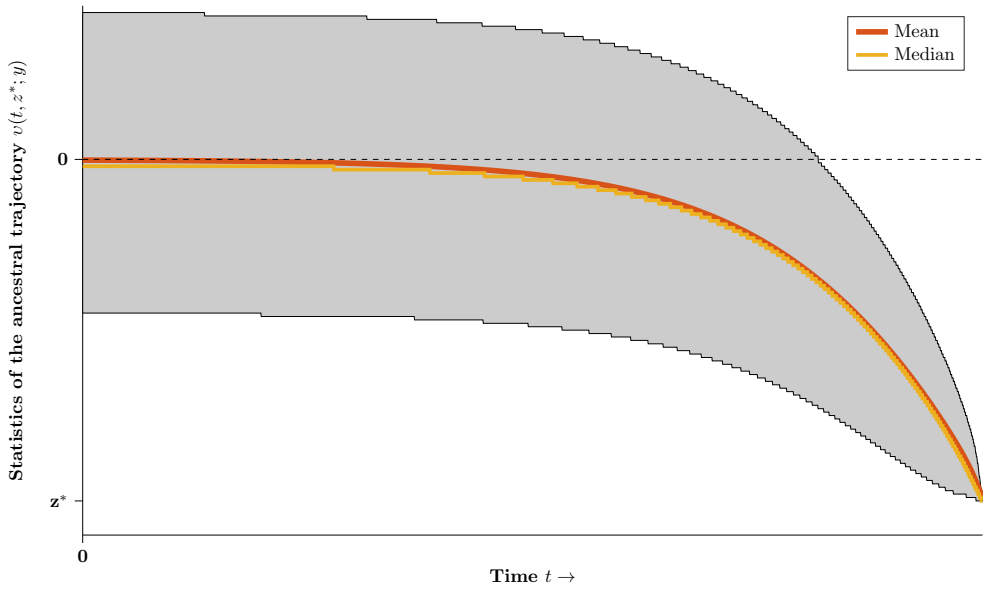


Figure V.14: Evolution of the mean and median of $v(t, z^*; y)$ along time. The grey area corresponds to the region between the 5% and the 95% quantile of the distribution.

One observes on figure V.14 that asymptotically $\bar{y}(\infty, z^*) = 0$, but the relaxation towards 0 is the same than the mean of the individual based lineages of figure V.4. Therefore a strong link is established between the individual based model lineages (that go backwards), and the dynamics of the neutral fractions (that go forward), as guessed in the formula equation (V.3.10). Finally one concludes those simulations by an snapshot of what the evolution of $v(t, z^*; y)$ looks like along time, as summarized by its statistics in figure V.14.

In the figure V.15, the plot that is the more to the left represents the time $t = 0$. It is a Dirac function in z^*, by definition (V.3.8). As time evolves, we see the distribution shift towards 0. Eventually it reaches an asymptotic profile, which is exactly the one pictured in figure V.13.

V.3.2 *The sexual reproduction case*

Contrarily to the asexual model, for the infinitesimal model, it is more involved to estimate the eigenvector of the adjoint operator $\mathcal{L}^*$. In order to bypass this issue, we will work on an approximated model, where one assumes that the phenotypic variance is small, following the methodology of Bouin et al. [2019]. More precisely, we introduce the following parameter ε, defined as

$$\varepsilon = \sqrt{\frac{\sigma^2 \alpha}{2\beta}}$$

Then, we perform a phenotypic scaling

$$z \to z\sqrt{\frac{\alpha}{\beta}}, \quad m(z) \to \frac{m\left(z\sqrt{\frac{\alpha}{\beta}}\right)}{\beta}, \quad \lambda \to \frac{\lambda + \mu_0}{\beta}.$$

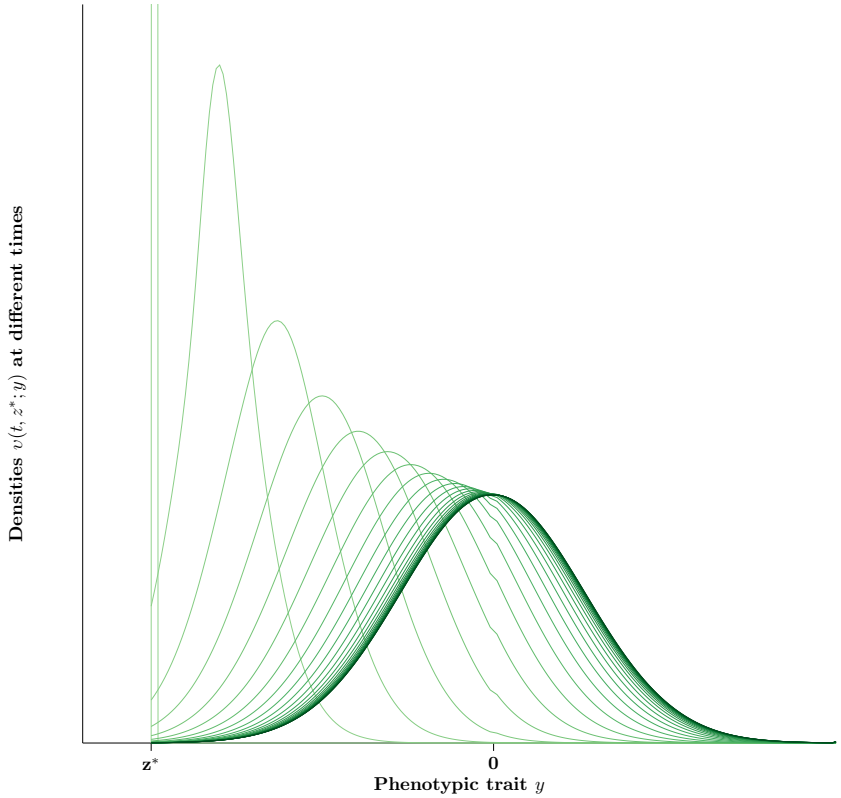


Figure V.15: Snapshots of the whole distribution $v(t,z;y)$ at different times.

and finally as in equation (V.1.4) we scale the speed of change of the environment :

$$c \to \frac{2c}{\sigma^2\sqrt{\alpha\beta}}.$$

Plugging those into equation (V.0.1), we obtain the following dimensionless model in the moving frame,:

$$\lambda F(z) - \varepsilon^2 c\partial_z F(z) + m(z)F(z) = \frac{1}{\varepsilon\sqrt{\pi}} \iint_{\mathbb{R}^2} \exp\left(-\frac{1}{\varepsilon^2}\left(z - \frac{z_1+z_2}{2}\right)^2\right) F(z_1) \frac{F(z_2)}{\int_{\mathbb{R}} F(z_2')\,dz_2'}\, dz_1 dz_2\,. \tag{V.3.13}$$

Under the assumption of small variance that is $\varepsilon \ll 1$, we can describe the function F in both cases using an appropriate Hopf-Cole transformation of the form

$$U = -\varepsilon^2 \log F. \tag{V.3.14}$$

We can formally expand the pair (λ, U), in a power series with respect to ε^2:

$$\begin{cases} U(z) = U_0(z) + \varepsilon^2 U_1(z) + \varepsilon^4 U_2(z) + o(\varepsilon^4) \\ \lambda = \lambda_0 + \varepsilon^2\lambda_1 + \varepsilon^4\lambda_2 + o(\varepsilon^4) \end{cases} \tag{V.3.15}$$

Plugging those into equation (V.0.1), one deduces equations on each U_i, as detailed in Bouin et al. [2019]. One can push the developments as far as desired. Firstly, one can show, that at the first order U is well approximated by a quadratic function :

$$U_0(z) = \frac{(z - z_0^*)^2}{2}\,. \tag{V.3.16}$$

for some z_0^* that is determined by the following equation :

$$\begin{cases} \lambda_0 + m(z_0^*) = 1 \\ c + \partial_z m(z_0^*) = 0\,. \end{cases} \tag{V.3.17}$$

The argument to obtain the form of $_0$ is based on the compensations that must occur inside the operator to guarantee it converges. More precisely U_0 solves the following equation :

$$\inf_{(z_1,z_2)\in\mathbb{R}^2} \left[\left(z - \frac{z_1+z_2}{2} \right)^2 + U_0(z_1) + U_0(z_2) - U_0(z) - \min_{z\in\mathbb{R}} U_0(z) \right].$$

Of course this is in stark contrast with the asexual case, where the first order approximation is solution of an Hamilton Jacobi equation. Moreover, the next order correction U_1 satisfies the following non local equation :

$$\log\left(\lambda_0 + c(z - z_0^*) + m(z)\right) = U_1(z_0^*) - 2U_1\left(\frac{z+z_0^*}{2}\right) + U_1(z)\,. \tag{V.3.18}$$

This was extensively detailed in parts III and IV. In this part, we are focused on the consequences it has to study equation (V.3.4) through the operator $\mathcal{L}$ (see (V.3.5)). Inspired by the previous discussions, it seems natural to seek solutions of

$$\mathcal{L}_\varepsilon(v) = 0,$$

where the operator $\mathcal{L}_\varepsilon$ is defined by

$$\mathcal{L}_\varepsilon(v) = \mathcal{B}_{F,\varepsilon}(v) + \varepsilon^2 c\partial_z v - m\, v - \lambda v.$$

where the operator $\mathcal{B}_{F,\varepsilon}$ is defined by

$$\mathcal{B}_{F,\varepsilon}(v) = \frac{1}{\varepsilon\sqrt{\pi}} \iint_{\mathbb{R}^2} \exp\left(-\frac{1}{\varepsilon^2}\left(z - \frac{z_1+z_2}{2} \right)^2 \right) v(z_1) \frac{F(z_2)}{\int_{\mathbb{R}} F(z_2')\, dz_2'}\, dz_1 dz_2,$$

similarly to equation (V.3.2). As detailed in equations (V.3.11) and (V.3.12), we also need information on the null space of the dual operator, $\mathcal{L}_\varepsilon^*$ which is defined by

$$\mathcal{L}_\varepsilon^*(\varphi) = \mathcal{B}_{F,\varepsilon}^*(\varphi) - \varepsilon^2 c\partial_z \varphi - (\lambda + m)\varphi.$$

The adjoint of $\mathcal{B}_{\varepsilon,F}$ verifies :

$$\mathcal{B}_{F,\varepsilon}^*(\phi)(z) = \frac{1}{\varepsilon\sqrt{\pi}} \iint_{\mathbb{R}^2} \exp\left(-\frac{1}{\varepsilon^2}\left(\frac{z+z_2}{2} - z_1 \right)^2 \right) \phi(z_1) \left(\frac{F(z_2)}{\int_{\mathbb{R}} F(z_2')\, dz_2'} \right) dz_1 dz_2 \tag{V.3.19}$$

Similarly to the study of $\mathcal{B}_\varepsilon$, we will approximate the solutions ψ_ε of

$$\mathcal{L}_\varepsilon^*(\psi_\varepsilon) = 0$$

using the Hop-Cole transformation:

$$\psi_\varepsilon = -\varepsilon^2 \log \phi_\varepsilon,$$

We expand $(\lambda_\varepsilon, \psi_\varepsilon)$, in a power series with respect to ε^2:

$$\begin{cases} \psi_\varepsilon(z) & = \psi_0(z) + \varepsilon^2\psi_1(z) + \varepsilon^4\psi_2(z) + o(\varepsilon^4), \\ \lambda & = \lambda_0 + \varepsilon^2\lambda_1 + \varepsilon^4\lambda_2 + o(\varepsilon^4) \end{cases}$$

and we can show, similarly to equation (V.3.16), that ψ_0 solves

$$\inf_{(z_1,z_2)\in\mathbb{R}^2} \left[\left(\frac{z+z_2}{2} - z_1 \right)^2 + U_0(z_1) + U_0(z_2) - U_0(z) - \min_{z\in\mathbb{R}} U_0(z) \right].$$

One shows that necessarily, ψ_0 takes the form

$$\psi_0(z) = \frac{3(z-z_0^*)^2}{2}. \tag{V.3.20}$$

with as before

$$\begin{cases} \lambda_0 + m(z_0^*) = 1 \\ c + \partial_z m(z_0^*) = 0. \end{cases}$$

Thus the first order approximation of the function $F\varphi$ is

$$F(z)\varphi(z) = \exp\Big(-\frac{(z-z_0^*)^2}{2\varepsilon^2} - \frac{3(z-z_0^*)^2}{2\varepsilon^2} \Big) = \exp\Big(-\frac{2(z-z_0^*)^2}{\varepsilon^2} \Big).$$

With equations (V.3.11) and (V.3.12), we therefore have, for any ε :

$$\begin{aligned} &\overline{y}(\infty, z) = z_0^*, \\ &y^*(\infty, z) = z_0^*, \quad \text{for all} \quad z \in \mathbb{R}. \end{aligned}$$

Therefore, at least very roughly, we once again can justify the heuristics behind conjecture 2.

V.4 MULTIPLE ALLELES MODEL

V.4.1 *Description of the model*

Let us now consider a different model of sexual reproduction, this time for a *diploid* species, which is also subject to a gradual environmental change. This means that each chromosome is paired with another copy of itself. However every copy of a given gene may not contain exactly the same DNA sequence. Those different copies are called alleles. For a diploid species, the two chromosomes are copies of the genetic information given by two different parents. Here we are interested in one peculiar locus on the chromosomes, and we suppose that each allele admits a continuous representation in $\mathbb{R}$. We will denote x (respectively y) the value taken by the allele on the first (respectively second) copy of the gene at the given locus, since in the literature chromosomes are often labeled X and Y according to which parent it belongs to.

We are interested in the density of alleles distribution, in the limit of small variance. Similarly to the previous model of equation (V.1.1), we will derive quantitative results thanks to formal Taylor

expansions leading to Hamilton-Jacobi equations for the logarithmic density of alleles. We consider the following stationary equation:

$$-\sigma C\cdot\nabla F(x,y)+\Big(\lambda+m(x,y)\Big)F(x,y)=\\ \frac{1}{\iint F(x',y')dx'dy'}\left(\iint K_\sigma(x-x')F(x',y^\dagger)\,dx'dy^\dagger\right)\left(\iint K_\sigma(y-y')F(x^\dagger,y')\,dx^\dagger dy'\right). \tag{V.4.1}$$

We denote m the following symmetric quadratic form

$$m(x,y)=m(x,y)=\frac{1}{4}(x+y)^2. \tag{V.4.2}$$

We can interpret the mean of allele $\frac{x+y}{2}$ as the expressed trait of an individual with the pair of alleles (x,y), or directly $m(x,y)$ as the expressed fitness of the individual.

The idea is that selection acts around a fitness optimum 0 reached by the pair $(0,0)$, but the population stands on a moving phenotypic belt, translated on each generation by

$$\sigma C:=(\sigma c,\,\sigma c).$$

This velocity is crucially of order of the standard deviation σ as in equation (V.1.4). K_σ is typically the normal distribution with variance σ^2:

$$K_\sigma(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right),$$

or any other (thin-tailed) centered probability kernel of variance σ. Therefore, the reproduction operator means that an offspring is born with a random normal mutation for each allele. In this model depicted by Equation (V.4.1), each parent gives to the offspring an allele, which is then submitted to mutations according to K_σ. The other allele of the parents, denoted $x^\dagger$ and $y^\dagger$ do not play any role in the genetic transmission, as is seen on equation (V.4.1). However, albeit a sexual model, the generation of alleles in x and y independently follows the pattern of the asexual model equation (V.1.1). In addition to that, as in the infinitesimal model defined in (V.0.2), the equation is non linear and $1-$ homogeneous. Moreover one should note that the mode of reproduction described by equation (V.4.1) is not equivalent to a mutation (x,y) to (x',y') through a multivariate kernel $K(x-x',y-y')$, since here we sample twice the population. We also introduce the following logarithmic transformation

$$U=-\sigma\log F.$$

It yields the following equation :

$$C\cdot\nabla U(x,y)+\lambda+m(x,y)=\frac{\exp\left(\frac{U(x,y)}{\sigma}\right)}{\iint\exp\left(-\frac{U(x',y')}{\sigma}\right)dx'dy'}\times\\ \left(\iint K_\sigma(x-x')\exp\left(-\frac{U(x',y^\dagger)}{\sigma}\right)dx'dy^\dagger\right)\left(\iint K_\sigma(y-y')\exp\left(-\frac{U(x\dagger,y')}{\sigma}\right)dx^\dagger dy'\right).$$

We expand U relatively to σ , as in the sexual model :

$$U = U_0 + \sigma U_1.$$

We look at the asymptotic $\sigma \ll 1$. Then the integrals will concentrate around the points of minimum of U_0, therefore it must satisfy for all $(x, y) \in \mathbb{R}^2$:

$$U_0(x,y) + \min_{x,y} U_0(x,y) = \min_{y^\dagger} U_0(x,y^\dagger) + \min_{x^\dagger} U_0(x^\dagger,y), \tag{V.4.3}$$

where the left hand side comes from the double integral $\iint F$, and the second from the concentration of the other two integrals of the reproduction terms. From equation (V.4.3), we deduce that necessarily U_0 must be a function of separate variables:

$$U_0(x,y) = V_0(x) + V_0(y). \tag{V.4.4}$$

Biologically this corresponds to the so called Hardy-Weinberg equilibrium of Mendelian genetics : an independent pairing of alleles in the absence of selection. Plugging this decomposition into the equation, and denoting K the standardized probability kernel $\left(K_\sigma = \frac{1}{\sigma} K(\cdot/\sigma)\right)$, we get the following identity:

$$\begin{aligned} cV_0'(x) + cV_0'(y) + O(\sigma) + \lambda + m(x,y) = \frac{\exp\left(\frac{U_1(x,y)}{\sigma}\right)}{\iint \exp\left(-\frac{U(x',y')}{\sigma}\right) dx'dy'} \times \\ \left(\iint K(\xi) \exp\left(\frac{V_0(x) - V_0(x-\sigma\xi) - V_0(y^\dagger)}{\sigma}\right) \exp\Big(-U_1(x-\sigma\xi, y^\dagger)\Big)\, d\xi dy^\dagger\right) \times \\ \left(\iint K(\zeta) \exp\left(\frac{V_0(y) - V_0(y-\sigma\zeta) - V_0(x^\dagger)}{\sigma}\right) \exp\Big(-U_1(x^\dagger, y-\sigma\zeta)\Big)\, dx^\dagger d\zeta\right). \end{aligned}$$

In the regime $\sigma \ll 1$, the denominator of the first fraction is approximated with Laplace's method, introducing a point of minimum (x^*, y^*):

$$\iint \exp\left(-\frac{U(x',y')}{\sigma}\right) dx'dy' \approx \left(\frac{2\pi\sigma}{V_0''(x^*)} \frac{2\pi\sigma}{V_0''(y^*)}\right)^{1/2} \exp\left(\frac{-V_0(x^*) - V_0(y^*)}{\sigma}\right) \exp\Big(-U_1(x^*, y^*)\Big).$$

By definition,

$$V_0'(x^*) = V_0'(y^*) = 0.$$

Similarly, for the term :

$$\begin{aligned} &\iint K(\xi) \exp\left(\frac{V_0(x) - V_0(x-\sigma\xi) - V_0(y^\dagger)}{\sigma}\right) \exp\Big(-U_1(x-\sigma\xi, y^\dagger)\Big)\, d\xi dy^\dagger \\ &\approx \left(\int K(\xi) \exp(\xi \cdot V_0'(x))\, d\xi\right) \left(\frac{2\pi\sigma}{V_0''(y^*)}\right)^{1/2} \exp\left(-\frac{V_0(y^*)}{\sigma}\right) \exp\Big(-U_1(x,y^*)\Big). \end{aligned} \tag{V.4.5}$$

The other contribution offers the same kind of formula, with the roles of $(\xi, y^\dagger)$ replaced by $(x^\dagger, \zeta)$. All in all, we get, assembling all the estimations, the following limit equation as $\sigma \ll 1$:

$$cV_0'(x) + cV_0'(y) + \lambda + m(x,y) = \widehat{K}(V_0'(x))\widehat{K}(V_0'(y)) \frac{\exp\Big(U_1(x,y) + U_1(x^*,y^*)\Big)}{\exp\Big(U_1(x,y^*) + U_1(x^*,y)\Big)}, \tag{V.4.6}$$

where $\widehat{K}$ designs the Laplace transform of the distribution K, as it appeared in the Hamiltonian in equation (V.1.5):

$$\widehat{K}(p) := \int_{\mathbb{R}} K(y) \exp(yp) dy.$$

Observe the symmetry of the roles played by x and y in equation (V.4.6), as a consequence necessarily, $x^* = y^*$. It is not fully an equation for V_0 because it also involves U_1, but this part can be discarded by evaluating the former expression at $y = y^*$:

$$cV_0'(x) + \lambda + m(x, y^*) = \widehat{K}(V_0'(x)). \tag{V.4.7}$$

However, the value of $x^* = y^*$ is still unknown. Equation (V.4.7) translates that one can see the problem as an asexual one in each alleles but in an environment imposed by x^* : the term $m(x, x^*)$ is the only (major) difference with equation (V.1.6). By evaluating at x^*, we obtain that necessarily :

$$\lambda = 1 - m(x^*, x^*).$$

And moreover, by considering the minimum over all $x \in \mathbb{R}$, we can recognize the expression of the Lagrangian function L associated to the "Hamiltonian" $\widehat{K}$, see equation (V.1.7) for a similar formula :

$$\lambda = 1 - \min_x m(x, x^*) - L(c). \tag{V.4.8}$$

Observe that this reasoning is completely oblivious of the precise nature of the selection m. In our case this gives

$$\begin{cases} \lambda = 1 - L(c) \\ \lambda = 1 - (x^*)^2 \end{cases} \tag{V.4.9}$$

but in other scenarios, *e.g.* $m(x, y) := \frac{1}{2}(x^2 + y^2)$, one can guess other formulas. The relation

$$(x^*)^2 = L(c)$$

is not without resemblance to the corresponding one in the one allele-model , equation (V.1.9) :

$$m(z^*) = \beta L\left(\frac{c}{\sigma\beta}\right).$$

V.4.2 *Lineages and numerical simulations*

We can derive from those formulas– (V.4.9)– the same kind of heuristics as in section 2 and 3, seeking to answer the question of what is the trajectory of individuals inside the equilibrium. However this leads to a different intuition on the lineages. If we repeat the reasoning made on equation (V.1.10), most of the "trait" comes from the point that is minimizing the mortality rate in equation (V.4.8). The similar representation formula for equation (V.4.7) is

$$V_0(x) = \inf_{\gamma \text{ s.t. } \gamma(0)=x} \int_{-\infty}^{0} \left[L\Big(\dot{\gamma}(s) + c\Big) + m(\gamma(s), x^*) + \lambda \right] ds. \tag{V.4.10}$$

Similarly to equation (V.1.11), this prescribes the value of λ as in equation (V.4.8) :

$$\lambda = 1 - \min_x m(x, x^*) - L(c).$$

The argument is, as in the asexual model that any minimizing trajectory Γ of the problem (V.4.10) must necessarily converge towards the minimum of the mortality rate m as $s \to -\infty$, otherwise it would not minimize the integral. While, in the asexual mode of reproduction this straightforwardly implied that Γ was converging to 0 the argmin of μ. Here the major difference is that one must take into account the two alleles of each parent, even the one that does not play a role in the reproduction, for instance $y^\dagger$. is the most likely to be at $y^* = x^*$. Indeed, we are now looking for the following quantity

$$\operatorname*{argmin}_x m(x, x^*).$$

Computationally this comes from the evaluation of equation (V.4.6) in x^*, but it really tells the story that the allele that is not transmitted during reproduction takes the dominant value in the population. This implies, by the very nature of the selection function m that to minimize m the genetic diversity is created by the allele

$$\Gamma(s) \xrightarrow[s\to-\infty]{} -x^*.$$

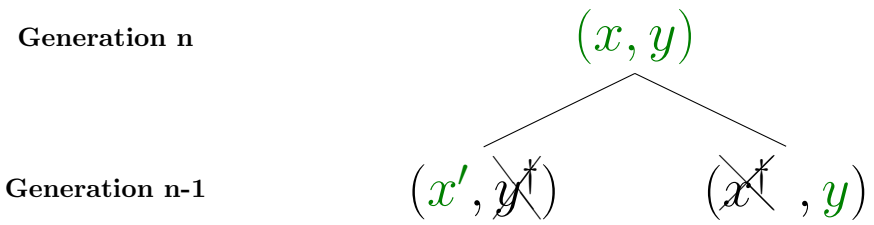
Therefore,

$$m(\Gamma(-\infty), x^*) = \min_x m(x, x^*) = 0. \tag{V.4.11}$$

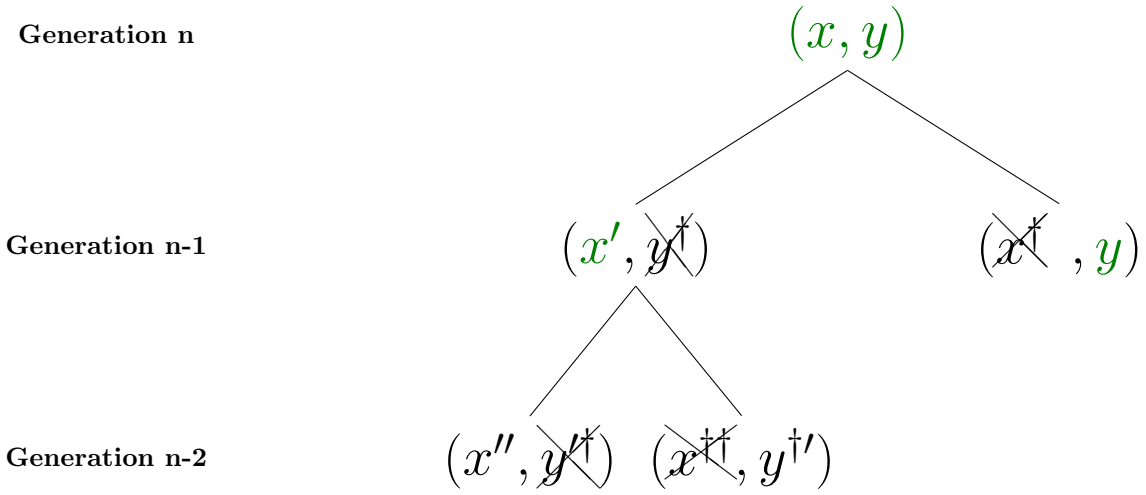
This is encapsulated in the following conjecture:

Conjecture 3. *In the sexual mode of reproduction described by equation (V.4.1), one expects that for every trait at equilibrium, the* typical *allele of the ancestors will be at the value* $-x^*$.

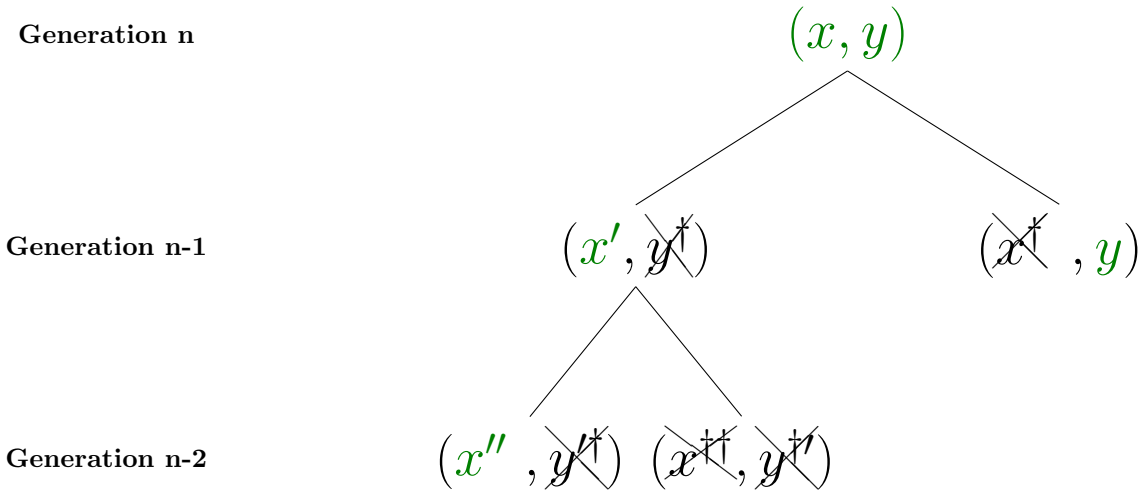
In fact there is some subtlety about this conjecture as it deals with lineages of alleles and not as before with individual based lineages. Let's explain the result on the following schemes : one start with an individual with alleles (x, y) at the n-th generation, and we picture the parents that gave birth to the aforementioned individual :



We erased in blue the parental alleles that are "lost" in the process in the sense that they do not transmit their genetic patrimony, and pictured in green the trajectory we are interested in. Next we do the same for one of the parents, let's say the left one to try to follow the lineage :



But next, since we know that the individual with the pair of alleles $(x^{\dagger\dagger}, y^{\dagger\prime})$ did not contribute to the genetic patrimony of the individual (x, y) *two* generations later, we can also forget about the allele $y^{\dagger\prime}$ although it is kept in the next generation $n-1$. Therefore, the lineage we are interesting in takes the form :



We see that the lineage Γ pictured in the previous figures is completely independent of the notion of individuals. Among the genealogical tree of the 4^n lineages that led to the individual (x, y), we are looking for only two branches (only one pictured above), despite the model being a sexual one. The function Γ is independently one or the other, because of the symmetry between x and y. This means that asymptotically, we expect that both alleles have the value $-x^*$ for the most part, without it corresponding to any particular individual as we see on the figures. The mean and dominant configuration of the alleles for a single individual is $(-x^*, x^*)$, which means that the mean "trait" is 0 when $s \to -\infty$, as in the asexual model. However there is a very low density of alleles at $-x^*$, if any see figure V.16.

To capture numerically the behavior described by conjecture 3, we first adapted the kind of simulations we described in section 2.2. It is a bit more involved since an individual is now characterized by a couple (x, y) of the value of its alleles. Therefore, one must do the procedure explained in figure V.3 twice, one for each allele. The model is built so that an individual inherits an

allele from each parent, so for each one we draw uniformly randomly a number in $\{1,2\}$ to determine the allele inherited by the offspring before mutations. This leads to the following profile :

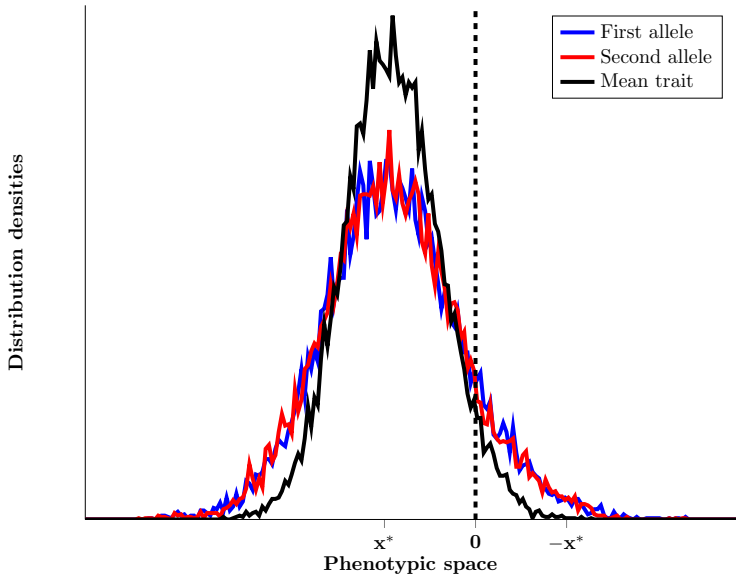


Figure V.16: Profile of the distributions of the first allele and second allele of all individuals in the population, and its mean.

As expected, the distributions of the first and second allele (in red and blue) are indistinguishable. We can notice that as expected, the distribution of the mean of alleles is narrower than the distribution of alleles, since if X and Y are *i.i.d*, Var $\left(\frac{1}{2}(X+Y)\right) = \frac{1}{2}\text{Var}X$. This means that there is a higher probability to have alleles at $-x^*$ than expected if we only considered the mean " trait ".

To keep track of the lineages during the simulation, for each offspring we kept both alleles of a single parent, therefore including one of the allele erased in the previous figures. As in the infinitesimal model, we choose for each reproduction event the fittest of the two parents, *i.e.* the one that minimizes the selection function m. This means that we will not be able to plot the function Γ that is an allele trajectory, but we follow a "trait" trajectory along individuals. This led to the picture of figure V.17.

We did not add the detailed lineages for the clarity of the figure. One sees that the global trend for the mean "trait" in red is to go towards the optimal trait 0. If we plot the largest allele of each individual and take the mean over the whole population, we see that it converges towards a region of phenotype where there is very few density of individuals. Even if it does not reach the value $-x^*$, it is still meaningful and it conveys much of the message of this section : what drives the survival of the population is the presence of very few individuals who are very well adapted. The fact that we do not see the red curve go up to the optimal trait means that we are plotting a lineage Γ' that is not realizing the minimum in equation (V.4.10). It is skewed towards the dominant trait since a lot of individuals have this dominant allele among the non optimal branches of the genealogical tree of the individuals. Moreover, there is an effect of the finite population for the simulation, and σ is also not zero.

Another numerical model based on alleles must therefore be considered to build more faithful simulations of Γ. It is still an ongoing work, and it is not completely trivial because one does want to

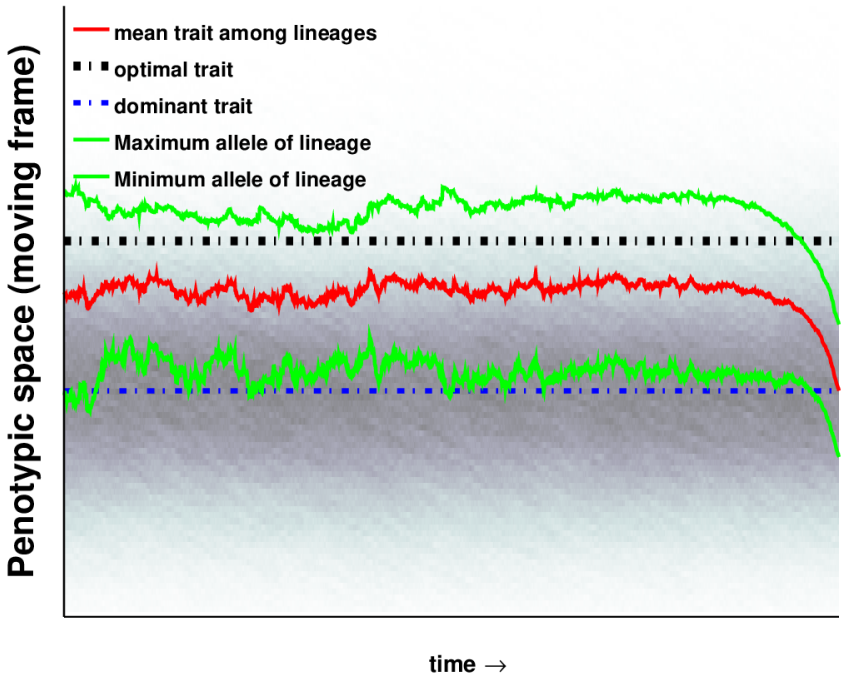


Figure V.17: The green plots represent the mean (over all ancestors) of the maximum and minimum alleles. In gray, the invariant density profile of the mean of alleles.

keep track of too much branches in the genealogical tree. The idea would be to construct the allele lineages of the previous pictures but forwardly in time.

V.4.3 *Time discrete model and Gaussian distributions*

V.4.3.1 *The new model and its explicit solution*

In order to better understand the heuristics that we just presented about the model (V.4.1), we proceed to simplify this model and consider **discrete** time steps. In fact we will show that with that additional simplification, there exist Gaussian explicit solutions. In this part we consider the following model :

$$N_{t+1}(x-c,y-c) = \frac{\exp\Big(-m(x,y)\Big)}{\iint N_t(x',y')\,dx'dy'}\left(\iint K_\sigma(x-x')N_t(x',y^\dagger)\,dx'dy^\dagger\right)\left(\iint K_\sigma(y-y')N_t(x^\dagger,y')\,dx^\dagger dy'\right). \tag{V.4.12}$$

Of course there is a big gap in both modeling and mathematical relevance between this time-discrete model and the stationary one of equation (V.4.1), but the possibility to do explicit computations makes the comparison very sound. In the model of (V.4.12), we changed slightly the point of view, and N_t is now the density of individuals alive at the given time t, structured by both alleles

(x,y). At each generation t, the phenotype profile is shifted by c, which corresponds to a discrete version of the transport term of equation (V.4.1). The new model equation (V.4.12) amounts to the supposition that every individuals in the population reproduced in single generations synchronized and non overlapping. We expect an exponential growth, which explains that we chose to write the mortality as an exponential rate, when the function m is unchanged compared to the definition (V.4.2). As announced, we can prove the following result :

Proposition V.1.
The function $N := \exp(\lambda t)G(x,y)$ where the function G is the following multivariate Gaussian distribution :

$$G(x,y) := \frac{1}{2\pi\sqrt{2\sigma}} \exp\left(-\frac{1}{2}(x-x^*, y-y^*)A^{-1}(x-x^*, y-y^*)^T\right)$$

is an exact solution of equation (V.4.1), with the following covariance matrix :

$$A := \sigma \begin{pmatrix} \sqrt{2+\sigma^2} & -\sigma \\ -\sigma & \sqrt{2+\sigma^2} \end{pmatrix}. \tag{V.4.13}$$

It is centered at the point (x^*, y^*), with the following expression for the dominant allele :

$$x^* = y^* = -\frac{c}{2\sigma}\left(\sigma + \sqrt{2+\sigma^2}\right). \tag{V.4.14}$$

Alternatively, one can say that G is a solution of the stationary problem associated to equation (V.4.12). Moreover, the shape of the matrix A is significant. Once divided by σ as in G, we see that when $\sigma \ll 1$ we get the following approximation from equation (V.4.13) :

$$\begin{pmatrix} \sqrt{2+\sigma^2} & 0 \\ 0 & \sqrt{2+\sigma^2} \end{pmatrix}.$$

The interesting point being that in this regime, both alleles are independent, as was guessed in the continuous case in equation (V.4.4), that was biologically justified by the Hardy-Weinberg equilibrium.

The trait x^* diverges when $\sigma \to 0$, because c is not rescaled of the order of σ, as in equation (V.1.4). However, in this section we do not investigate this asymptotic regime since the computations are explicit for all sigma.

Proof of Proposition V.1.
Let's seek λ a real number and G a Gaussian probability distribution solution of the following stationary state:

$$e^{\lambda}G(x-c,y-c) = \exp\Big(-m(x,y)\Big)\left(\iint K_\sigma(x-x')N_t(x',y^\dagger)\,dx'dy^\dagger\right)\times \left(\iint K_\sigma(y-y')N_t(x^\dagger,y')\,dx^\dagger dy'\right). \tag{V.4.15}$$

It is an elaborated guess to conjecture that the Gaussian solution may have a variance σ^2, and we look for solutions of the form :

$$G(x,y) = \frac{1}{2\pi(\det A)^{1/2}} \exp\left(-\frac{1}{2}(Z-Z^*)^T A^{-1}(Z-Z^*)\right), \quad Z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

Due to invariance by translation, it is sufficient to compute the results of offspring generation with $Z^* = 0$, and then to translate by Z^* back. The computation is done via Fourier transformation: the Fourier transform of $KG(x) := \iint K_\sigma(x - x')G(x', y^\dagger)\, dx'dy^\dagger$ is, for $\xi = (\xi_x, \xi_y)$:

$$\widehat{KG}(\xi) = \left(\widehat{K_\sigma}(\xi_x)\delta_0(\xi_y)\right) \exp\left(-\frac{1}{2}\xi^T A\xi\right) = \exp\left(-\frac{1}{2}\sigma^2\xi_x^2\right) \exp\left(-\frac{1}{2}a\xi_x^2\right) \delta_0(\xi_y).$$

Therefore,

$$KG(x) = \frac{1}{(2\pi\omega^2)^{1/2}} \exp\left(-\frac{x^2}{2\omega^2}\right), \quad \omega^2 := \sigma^2 + a.$$

Finally, re-introducing $Z^* = (x^*, y^*)$ the dominant trait in the population, the equation for the stationary state (V.4.15) can be reformulated as:

$$\frac{1}{2\pi(\det A)^{1/2}} \exp\left(\lambda - \frac{1}{2}(Z - Z^* - C)^T A^{-1}(Z - Z^* - C)\right) = \frac{1}{2\pi\omega^2} \exp\left(-\frac{1}{2}Z^T MZ - \frac{(x - x^*)^2}{2\omega^2} - \frac{(y - y^*)^2}{2\omega^2}\right),$$

with the following notations corresponding to the vector of environmental shift and the matrix of the quadratic form defining m in equation (V.4.2) :

$$C := Z = \frac{1}{2}\begin{pmatrix} c \\ c \end{pmatrix}, \quad M := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

This leads to the following set of necessary equations:

- ▷ The second order terms involve the following matrix equation:

$$A^{-1} = M + \frac{1}{\omega^2}\,\mathrm{Id} \iff \mathrm{Id} = MA + \frac{1}{\omega^2}A.$$

Equivalently,

$$\begin{cases} 1 = a + b + \dfrac{a}{\sigma^2 + a} \\ 0 = b + a + \dfrac{b}{\sigma^2 + a} \end{cases} \iff \begin{cases} \dfrac{\sigma^2}{\sigma^2 + a} = \dfrac{a + b}{2} \\ 0 = \dfrac{b + a}{2} + \dfrac{b}{\sigma^2 + a} \end{cases}$$

This system can be explicitly solved to get the explicit form of the variance-covariance matrix :

$$A := \sigma \begin{pmatrix} \sqrt{2 + \sigma^2} & -\sigma \\ -\sigma & \sqrt{2 + \sigma^2} \end{pmatrix}.$$

- ▷ Secondly, we examine the first order terms. They yield the following equation for the lag Z^*:

$$A^{-1}(Z^* + C) = \frac{1}{\omega^2}Z^* \iff MZ^* = -A^{-1}C \tag{V.4.16}$$

Since all matrices are symmetric and $x^* = y^*$, all relevant information lies in the first line of the previous equality, that yield, inverting the symmetric matrix A:

$$x^* = -\frac{c}{2\sigma}\left(\sigma + \sqrt{2 + \sigma^2}\right).$$

▷ Finally, the zeroth order part contains the following piece of information concerning the lag load λ :

$$\lambda - \log(\sqrt{2}\sigma) - \frac{1}{2}(Z^* + C)^T A^{-1}(Z^* + C) = -\frac{1}{2\omega^2}(Z^*)^T Z^* - \log(\omega^2)$$

Therefore, when plugging equation (V.4.16),

$$\lambda = -\frac{1}{2\omega^2}(Z^*)^T Z^* + \frac{1}{2\omega^2}(Z^* + C)^T Z^* + \log\left(\frac{\sqrt{2}\sigma}{\omega^2}\right).$$

Thanks to the expression of A of equation (V.4.13),

$$\lambda = \frac{1}{2\omega^2} C^T Z^* - \log\left(\frac{\sigma}{\sqrt{2}} + \sqrt{1 + \frac{\sigma^2}{2}}\right).$$

We can conclude with the explicit expression of the lag x^* from equation (V.4.14) :

$$\begin{aligned}
\lambda &= \frac{cx^*}{\sigma^2 + \sqrt{2 + \sigma^2}} - \log\left(\frac{\sigma}{\sqrt{2}} + \sqrt{1 + \frac{\sigma^2}{2}}\right), \\
&= -\frac{c^2}{2\sigma}\frac{\sigma + \sqrt{2 + \sigma^2}}{\sigma^2 + \sqrt{2 + \sigma^2}} - \log\left(\frac{\sigma}{\sqrt{2}} + \sqrt{1 + \frac{\sigma^2}{2}}\right), \\
&= -\frac{c^2}{2\sqrt{2}\sigma'}\frac{\sigma' + \sqrt{1 + \sigma'^2}}{\sigma^2 + \sqrt{1 + \sigma'^2}} - \log\left(\sigma' + \sqrt{1 + \sigma'^2}\right); \quad \sigma' := \frac{\sigma}{\sqrt{2}}.
\end{aligned}$$

□

V.4.3.2 *Explicit lineages*

Thanks to the explicit expression of the profile described in Proposition V.1, it is possible to compute at each time, the most probable value of the alleles that lead to a given offspring. Therefore, we fix an individual with allelic values (x, y). This individual was born with a deviation from the alleles of his parents, and even lost half of the genetic information from his parents. **Our objective is to quantitatively describe what are the most likely parents for this individual.**

Let's focus for instance around the allele x, but both alleles will indulge the same behavior by symmetry. Because of Laplace's principle, that we used previously to derive for instance equation (V.4.5), we know how the integral of equation (V.4.12) (that models mutations) concentrates around

$$\arg\max_{x', y^\dagger} K_\sigma(x - x')G(x', y^\dagger).$$

This is interpreted as looking for the most probable allelic values for a parent of any individual. Therefore, we know thanks to Proposition V.1 that we investigate the following :

$$\operatorname{argmin}_{x', y^\dagger}\left\{\frac{(x - x')^2}{2\sigma^2} + \frac{1}{2}\begin{pmatrix} x' - x^* \\ y^\dagger - y^* \end{pmatrix}^T A^{-1}\begin{pmatrix} x' - x^* \\ y^\dagger - y^* \end{pmatrix}\right\}.$$

The couple $(x', y^\dagger)$ that realizes the extremum necessarily verifies the following relations coming from the first order conditions :

$$\begin{cases} -\dfrac{(x-x')}{\sigma^2} + \dfrac{\sqrt{2+\sigma^2}}{2\sigma}(x'-x^*) + \dfrac{1}{2}(y^\dagger - x^*) = 0 \\ \dfrac{(x'-x^*)}{2} + \dfrac{\sqrt{2+\sigma^2}}{2\sigma}(y^\dagger - x^*) = 0 \end{cases}$$

From it, we deduce successively that x' must solve

$$\frac{x'-x}{\sigma^2} + \frac{x'-x^*}{\sigma\sqrt{2+\sigma^2}} = 0,$$

and so

$$x' = \frac{\dfrac{x}{\sigma} + \dfrac{x^*}{\sqrt{2+\sigma^2}}}{\dfrac{1}{\sigma} + \dfrac{1}{\sqrt{2+\sigma^2}}}.$$

We recognize the expression of a barycenter and therefore we introduce the numbers p, q such that $p+q=1$ and

$$p := \frac{1}{1+\dfrac{\sigma}{\sqrt{2+\sigma^2}}}, \quad q := \frac{1}{1+\dfrac{\sqrt{2+\sigma^2}}{\sigma}}.$$

Moreover, if we want to state that x' is an allele of the parent of the individual (x, y), we must translate back by the allele by c as indicated by equation (V.4.12). All in all this yields the following expression for the most probable parent x' that gives and individual x :

$$x' - c = px + qx^*.$$

This is a very natural formula where the parent in the absence of environmental shift is a convex combination of the offspring and the dominant trait in the population x^*. When one follows the lineage, one must reiterate the procedure and so on to the parent of x'. This is saying that the lineage is a sequence $(x_n)_{n\in\mathbb{N}}$, that verifies the following recursive formula :

$$x_{n+1} = px_n + qx^* + c.$$

If we now look at the allele that is not transmitted during the reproduction, $y^\dagger$, we have

$$y^\dagger = x^* - \frac{\sigma}{\sigma+\sqrt{2+\sigma^2}}(x-x^*).$$

Of course there is not an iterative process on $(y_n)_n$ since this information is not transmitted from a generation to another. As a conclusion, with our computations we have proven the following proposition :

Proposition V.2 (Lineage in time discrete model)**.**
In the model described by equation (V.4.12), for every generation and every individual (x, y), the more probable trajectory in the allelic space for his ancestors $(x_n, y_n^\dagger)$ where only the allele x_n is transmitted at each generation, is given by the following recursive equation :

$$x_{n+1} = px_n + qx^* + c, \qquad x_0 = x$$

with the coefficients p, q such that

$$p := \frac{1}{1 + \frac{\sigma}{\sqrt{2+\sigma^2}}}, \quad q := \frac{1}{1 + \frac{\sqrt{2+\sigma^2}}{\sigma}}.$$

The allele that is not transmitted $y_n^\dagger$ verifies at each generation

$$y_n^\dagger = x^* - \frac{\sigma}{\sigma + \sqrt{2+\sigma^2}}(x_n - x^*). \tag{V.4.17}$$

While we chose to write the proposition for only the allele x, the same recursive equations apply for the allele y, which makes two allelic lineages. A first consequence of Proposition V.2 is that we can have an explicit expression for the allele that corresponds to $n \to +\infty$, that is the "origin" of the lineages. We denote this point x_∞, and it is the solution of the following fixed point equation :

$$x_\infty = p x_\infty + q x^* + c.$$

This means that with equation (V.4.14),

$$\begin{aligned} x_\infty &= x^* + \frac{c}{q}, \\ &= x^* - \frac{2x^*\sigma}{\sigma + \sqrt{2+\sigma^2}} \left(\frac{\sqrt{2+\sigma^2}}{\sigma} + 1 \right), \\ &= -x^*. \end{aligned}$$

In this discrete case the sequence x_n is the equivalent of the minimizing function Γ defined as the solution of equation (V.4.10). **Therefore, the explicit computations we just provided confirm the conjecture 3 concerning the asymptotic behavior of lineages when** $t \to -\infty$, which corresponds here to $n \to +\infty$. Moreover, one may want to know which ancestral value takes the other coupled allele for the individual when the first is x_∞. This corresponds to the value that is lost in the genetic process at each generation, $(y_n^\dagger)$. Thanks to equation (V.4.17),

$$y_\infty^\dagger = x^* + o(\sigma).$$

As we announced earlier, it is at the dominant trait of the population, which corresponds to the fact that it is drawn uniformly in the whole population. Therefore, $x_\infty + y_\infty^\dagger = 0$, which corresponds to a "trait" that originates from the optimal value, as in the asexual model, see conjecture 1. This holds true in the regime of small variance, and it corresponds exactly with the heuristic analysis performed below equation (V.4.10), in particular equation (V.4.11).

V.5 PERSPECTIVES

Since a grand majority of the work presented in this chapter is made of heuristics, there is a tremendous amount of work remaining to be done :

- ▷ With Viet Chi Tran, Vincent Calvez, Benoit Henry and Sylvie Méléard we laid the preliminary groundwork towards a rigorous proof of conjecture 1, in the case of the diffusive approximation. One cannot hope ot recover the information we are looking from the individual point of view of Champagnat et al. [2006]; Méléard et al. [2012], but the idea would be to use Many To One (MTO) formulas for characterizing typical lineages in genealogical trees.

- ▷ With Matthew Osmond and Vincent Calvez, a more in depth work on the models with two alleles is ingoing, where multiple locus with or without recombinations, and more general selection functions can be integrated in the framework of the study of the regime of small variance, and it can still be backed by computations on the time-discrete models, similarly to section 4.3.
- ▷ A complete study of the operator $\mathcal{B}^*_{F,\varepsilon}$ of equation (V.3.19) has not been performed yet. In the spirit of part IV, existence and stability properties of correctors ψ_1 and λ_1 could in theory provide a more detailed understanding of the inside dynamics of equation (V.0.3) and a rigorous proof of conjecture 2.

Alfaro, M. (2017). Slowing Allee effect versus accelerating heavy tails in monostable reaction diffusion equations. *Nonlinearity*, 30(2):687–702.

Alfaro, M. and Coville, J. (2017). Propagation phenomena in monostable integro-differential equations: acceleration or not? *Journal of Differential Equations*, 263(9):5727–5758.

Allendorf, F. W. and Luikart, G. (2009). *Conservation and the genetics of populations*. John Wiley & Sons.

Andersson, M. B. (1994). *Sexual selection*. Princeton University Press.

Aronson, D. G. and Weinberger, H. F. (1975). Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In *Partial differential equations and related topics*, pages 5–49. Springer.

Aronson, D. G. and Weinberger, H. F. (1978). Multidimensional nonlinear diffusion arising in population genetics. *Advances in Mathematics*, 30(1):33–76.

Baeumer, B., Kovács, M., and Meerschaert, M. M. (2007). Fractional reproduction-dispersal equations and heavy tail dispersal kernels. *Bulletin of Mathematical Biology*, 69(7):2281–2297.

Barfield, M., Holt, R. D., and Gomulkiewicz, R. (2011). Evolution in Stage-Structured Populations. *The American naturalist*, 177(4):397–409.

Barles, G. (1994). *Solutions de viscosité des équations de Hamilton-Jacobi*, volume 17. Springer-Verlag, Paris.

Barles, G., Evans, L. C., and Souganidis, P. E. (1990). Wavefront propagation for reaction-diffusion systems of PDE. *Duke Math. J.*, 61(3):835–858.

Barles, G., Mirrahimi, S., and Perthame, B. (2009). Concentration in Lotka-Volterra parabolic or integral equations: a general convergence result. *Methods and Applications of Analysis*, 16(3):321–340.

Barles, G. and Perthame, B. (1988). Exit time problems in optimal control and vanishing viscosity method. *SIAM Journal on Control and Optimization*, 26(5):1133–1148.

Barles, G. and Perthame, B. (1990). Comparison principle for dirichlet-type hamilton-jacobi equations and singular perturbations of degenerated elliptic equations. *Applied Mathematics and Optimization*, 21(1):21–44.

Barles, G. and Perthame, B. (2007). Concentrations and constrained hamilton-jacobi equations arising in adaptive dynamics. *Contemporary Mathematics*, 439:57–68.

Barles, G. and Roquejoffre, J.-M. (2006). Ergodic type problems and large time behaviour of unbounded solutions of hamilton–jacobi equations. *Communications in Partial Differential Equations*, 31(8):1209–1225.

Barles, G. and Souganidis, P. E. (1994). A remark on the asymptotic behavior of the resolution of the kpp equation. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, 319(7):679–684.

Barton, N. (1995). A general model for the evolution of recombination. *Genetics Research*, 65(2):123–144.

Barton, N., Etheridge, A., and Véber, A. (2017). The infinitesimal model: Definition, derivation, and implications. *Theoretical population biology*, 118:50–73.

Barton, N. H. (2000). Genetic hitchhiking. *Philosophical Transactions of the Royal Society of London. Series B: Biological Sciences*, 355(1403):1553–1562.

Berestycki, H. and Larrouturou, B. (1988). *Quelques aspects mathématiques de la propagation des flammes prémélangées*. PhD thesis, INRIA.

Berestycki, H., Nicolaenko, B., and Scheurer, B. (1985). Traveling wave solutions to combustion models and their singular limits. *SIAM Journal on Mathematical Analysis*, 16(6):1207–1242.

Bertin, E., Droz, M., and Grégoire, G. (2006). Boltzmann and hydrodynamic description for self-propelled particles. *Physical Review E*, 74(2):022101.

Bouin, E., Bourgeron, T., Calvez, V., Cotto, O., Garnier, J., Lepoutre, T., and Ronce, O. (2019). Equilibria of quantitative genetics models beyond the gaussian approximation i: Maladaptation to a changing environment. *In preparation*.

Bouin, E., Garnier, J., Henderson, C., and Patout, F. (2018). Thin Front Limit of an Integro-differential Fisher-KPP Equation with Fat-Tailed Kernels. *SIAM Journal on Mathematical Analysis*, 50(3):3365–3394.

Bouin, E. and Mirrahimi, S. (2013). A hamilton-jacobi approach for a model of population structured by space and trait. *arXiv preprint arXiv:1307.8332*.

Bouin, E. and Mirrahimi, S. (2015). A Hamilton–Jacobi approach for a model of population structured by space and trait. *Communications in Mathematical Sciences*, 13(6):1431–1452.

Bourgeron, T., Calvez, V., Garnier, J., and Lepoutre, T. (2017). Existence of recombination-selection equilibria for sexual populations. *arXiv preprint arXiv:1703.09078*.

Bulmer, M. G. (1980). *The mathematical theory of quantitative genetics.* Clarendon Press.

Bürger, R. and Lynch, M. (1995). Evolution and extinction in a changing environment: a quantitative-genetic analysis. *Evolution*, 49(1):151–163.

Burt, A. (2000). Perspective: sex, recombination, and the efficacy of selection—was weismann right? *Evolution*, 54(2):337–351.

Cabré, X. and Roquejoffre, J.-M. (2013). The influence of fractional diffusion in fisher-kpp equations. *Communications in Mathematical Physics*, 320(3):679–722.

Calvez, V., Garnier, J., and Patout, F. (2019). Asymptotic analysis of a quantitative genetics model with nonlinear integral operator. *Journal de l'École polytechnique — Mathématiques*, 6:537–579.

Calvez, V., Henderson, C., Mirrahimi, S., Turanova, O., and Dumont, T. (2018). Non-local competition slows down front acceleration during dispersal evolution. *arXiv:1810.07634 [math]*. arXiv: 1810.07634.

Calvez, V. and Lam, K.-Y. (2018). Uniqueness of the viscosity solution of a constrained Hamilton-Jacobi equation. *arXiv:1809.05317 [math]*. arXiv: 1809.05317.

Carr, A. and Chmaj, J. (2004). Uniqueness of travelling waves for nonlocal monostable equations. *Proc. Amer. Math. Soc.*, 132(8):2433–2439.

Carrillo, J. A., Cuadrado, S., and Perthame, B. (2007). Adaptive dynamics via hamilton–jacobi approach and entropy methods for a juvenile-adult model. *Mathematical Biosciences*, 205(1):137–161.

Champagnat, N., Ferrière, R., and Méléard, S. (2006). Unifying evolutionary dynamics: from individual stochastic processes to macroscopic models. *Theoretical population biology*, 69(3):297–321.

Champagnat, N., Ferrière, R., and Méléard, S. (2007). Individual-based probabilistic models of adaptive evolution and various scaling approximations. In *Seminar on Stochastic Analysis, Random Fields and Applications V*, pages 75–113. Springer.

Chasseigne, E., Chaves, M., and Rossi, J. D. (2006). Asymptotic behavior for nonlocal diffusion equations. *Journal de mathématiques pures et appliquées*, 86(3):271–291.

Clark, J. S. (1998). Why trees migrate so fast: confronting theory with dispersal biology and the paleorecord. *The American Naturalist*, 152(2):204–224.

Clark, J. S., Fastie, C., Hurtt, G., Jackson, S. T., Johnson, C., King, G. A., Lewis, M., Lynch, J., Pacala, S., Prentice, C., et al. (1998). Reid's paradox of rapid plant migration dispersal theory and interpretation of paleoecological records. *BioScience*, 48(1):13–24.

Coron, C., Costa, M., Laroche, F., Leman, H., and Smadi, C. (2019). Emergence of homogamy in a two-loci stochastic population model. *arXiv preprint arXiv:1902.07926*.

Cotto, O. and Ronce, O. (2014). Maladaptation as a source of senescence in habitats variable in space and time. *Evolution*, 68(9):2481–2493.

Coville, J. (2007). *Contribution à l'étude d'équations non locales en dynamique des populations*. habilitation, Aix Marseille Université. HAL:tel-01238013.

Coville, J. and Dupaigne, L. (2005). Propagation speed of travelling fronts in non local reaction–diffusion equations. *Nonlinear Analysis: Theory, Methods & Applications*, 60(5):797–819.

Crandall, M. G., Evans, L. C., and Lions, P.-L. (1984). Some properties of viscosity solutions of hamilton-jacobi equations. *Transactions of the American Mathematical Society*, 282(2):487–502.

Crandall, M. G., Ishii, H., and Lions, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, 27(1):1–67.

Crandall, M. G. and Lions, P.-L. (1983). Viscosity solutions of hamilton-jacobi equations. *Transactions of the American mathematical society*, 277(1):1–42.

Darwin, C. (1859). *On the origin of species, 1859*. Routledge.

Degond, P., Frouvelle, A., and Raoul, G. (2014). Local Stability of Perfect Alignment for a Spatially Homogeneous Kinetic Model. *Journal of Statistical Physics*, 157(1):84–112.

Desvillettes, L., Jabin, P. E., Mischler, S., Raoul, G., et al. (2008). On selection dynamics for continuous structured populations. *Communications in Mathematical Sciences*, 6(3):729–747.

Diekmann, O. (1979). Run for your life. a note on the asymptotic speed of propagation of an epidemic. *Journal of Differential Equations*, 33(1):58–73.

Diekmann, O., Jabin, P.-E., Mischler, S., and Perthame, B. (2005). The dynamics of adaptation: an illuminating example and a Hamilton-Jacobi approach. *Theoretical Population Biology*, 67(4):257–271.

Dimassi, M., Sjostrand, J., et al. (1999). *Spectral asymptotics in the semi-classical limit*. Number 268. Cambridge university press.

Engel, K.-J. and Nagel, R. (2001). One-parameter semigroups for linear evolution equations. In *Semigroup Forum*, volume 63, pages 278–280. Springer.

Evans, L. and Souganidis, P. (1989). A PDE approach to geometric optics for certain semilinear parabolic equations. *Indiana University mathematics journal*, 38(1):141–172.

Evans, L. C. (1989). The perturbed test function method for viscosity solutions of nonlinear pde. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 111(3-4):359–375.

Fife, P. C. (2013). *Mathematical aspects of reacting and diffusing systems*, volume 28. Springer Science & Business Media.

Fife, P. C. and McLeod, J. B. (1977). The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Archive for Rational Mechanics and Analysis*, 65(4):335–361.

Finkelshtein, D., Kondratiev, Y., and Tkachov, P. (2016). Accelerated front propagation for monostable equations with nonlocal diffusion . *ArXiv e-prints*.

Finkelshtein, D. and Tkachov, P. (2017). Accelerated nonlocal nonsymmetric dispersion for monostable equations on the real line. *ArXiv e-prints*.

Fisher, R. A. (1918). The correlation between relatives on the supposition of mendelian inheritance. *Transactions of the Royal Society of Edinburgh*, 52:399–433.

Fisher, R. A. (1919). Xv.—the correlation between relatives on the supposition of mendelian inheritance. *Earth and Environmental Science Transactions of the Royal Society of Edinburgh*, 52(2):399–433.

Fisher, R. A. (1930). *The genetical theory of natural selection: a complete variorum edition*. Oxford University Press.

Fisher, R. A. (1937). The wave of advance of advantageous genes. *Annals of eugenics*, 7(4):355–369.

Fleming, W. H. and Souganidis, P. E. (1986). Pde-viscosity solution approach to some problems of large deviations. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 13(2):171–192.

Freidlin, M. (1985). Limit theorems for large deviations and reaction-diffusion equations. *The Annals of Probability*, pages 639–675.

Gandon, S. and Mirrahimi, S. (2017). A hamilton–jacobi method to describe the evolutionary equilibria in heterogeneous environments and with non-vanishing effects of mutations. *Comptes Rendus Mathematique*, 355(2):155–160.

Gardner, R. (1986). Existence of multidimensional travelling wave solutions of an initial-boundary value problem. *Journal of differential equations*, 61(3):335–379.

Garnier, J. (2011). Accelerating solutions in integro-differential equations. *SIAM Journal on Mathematical Analysis*, 43(4):1955–1974.

Garnier, J., Giletti, T., Hamel, F., and Roques, L. (2012). Inside dynamics of pulled and pushed fronts. *Journal de mathématiques pures et appliquées*, 98(4):428–449.

Gurney, W. and Nisbet, R. (1975). The regulation of inhomogeneous populations. *J. Theor. Biol.*, 52(2):441 – 457.

Hairer, E., Lubich, C., and Wanner, G. (2006). *Geometric numerical integration: structure-preserving algorithms for ordinary differential equations*, volume 31. Springer Science & Business Media.

Hallatschek, O. and Nelson, D. R. (2008). Gene surfing in expanding populations. *Theoretical population biology*, 73(1):158–170.

Hallatschek, O. and Nelson, D. R. (2010). Life at the front of an expanding population. *Evolution: International Journal of Organic Evolution*, 64(1):193–206.

Hamel, F. and Roques, L. (2010). Fast propagation for KPP equations with slowly decaying initial conditions. *Journal of Differential Equations*, 249(7):1726–1745.

Henderson, C. (2016). Propagation of solutions to the Fisher-KPP equation with slowly decaying initial data. *Nonlinearity*, 29(11):3215–3240.

Hofbauer, J. and Sigmund, K. (1998). *Evolutionary games and population dynamics*. Cambridge university press.

Hofbauer, J. and Sigmund, K. (2003). Evolutionary game dynamics. *Bulletin of the American Mathematical Society*, 40(4):479–519.

Hoffmann, A. A. and Sgro, C. M. (2011). Climate change and evolutionary adaptation. *Nature*, 470(7335):479.

Huisman, J. and Tufto, J. (2012). Comparison of non-gaussian quantitative genetic models for migration and stabilizing selection. *Evolution: International Journal of Organic Evolution*, 66(11):3444–3461.

Jabin, P.-E. and Raoul, G. (2011). On selection dynamics for competitive interactions. *Journal of mathematical biology*, 63(3):493–517.

Jin, S. (1999). Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations. *SIAM J. Sci. Comput.*, 21(2):441–454.

Jourdain, B., Méléard, S., and Woyczynski, W. A. (2012). Lévy flights in evolutionary ecology. *Journal of Mathematical Biology*, 65(4):677–707.

Kelly, C. D. and Jennions, M. D. (2011). Sexual selection and sperm quantity: meta-analyses of strategic ejaculation. *Biological Reviews*, 86(4):863–884.

Kirkpatrick, M. and Barton, N. H. (1997). Evolution of a species' range. *The American Naturalist*, 150(1):1–23.

Klar, A. (1998). An asymptotic-induced scheme for nonstationary transport equations in the diffusive limit. *SIAM J. Numer. Anal.*, 35(3):1073–1094.

Kokko, H., Jennions, M. D., and Brooks, R. (2006). Unifying and testing models of sexual selection. *Annu. Rev. Ecol. Evol. Syst.*, 37:43–66.

Kolmogorov, A. N., Petrovsky, I., and Piskunov, N. (1937). Etude de l'équation de la diffusion avec croissance de la quantité de matiere et son applicationa un probleme biologique. *Moscow Univ. Math. Bull*, 1:1–25.

Kopp, M. and Matuszewski, S. (2014). Rapid evolution of quantitative traits: theoretical perspectives. *Evolutionary Applications*, 7(1):169–191.

Kot, M., Lewis, M. A., and van den Driessche, P. (1996). Dispersal data and the spread of invading organisms. *Ecology*, 77(7):2027–2042.

Lam, K.-Y. and Lou, Y. (2017). An integro-PDE model for evolution of random dispersal. *Journal of Functional Analysis*, 272(5):1755–1790.

Lewis, M. and Kareiva, P. (1993). Allee dynamics and the spread of invading organisms. *Theoretical Population Biology*, 43(2):141–158.

Lorz, A., Mirrahimi, S., and Perthame, B. (2011). Dirac mass dynamics in multidimensional nonlocal parabolic equations. *Communications in Partial Differential Equations*, 36(6):1071–1098.

Losos, J. B. (2013). *The Princeton guide to evolution*. Princeton University Press.

Luria, S. E. and Delbrück, M. (1943). Mutations of bacteria from virus sensitivity to virus resistance. *Genetics*, 28(6):491.

Magal, P. and Raoul, G. (2015). Dynamics of a kinetic model describing protein exchanges in a cell population. *arXiv preprint arXiv:1511.02665*.

Mahadevan, R. (2007). A note on a non-linear krein–rutman theorem. *Nonlinear Analysis: Theory, Methods & Applications*, 67(11):3084–3090.

Martin, G. and Roques, L. (2016). The nonstationary dynamics of fitness distributions: asexual model with epistasis and standing variation. *Genetics*, 204(4):1541–1558.

Medlock, J. and Kot, M. (2003). Spreading disease: integro-differential equations old and new. *Mathematical Biosciences*, 184(2):201–222.

Méléard, S. and Mirrahimi, S. (2015). Singular limits for reaction-diffusion equations with fractional Laplacian and local or nonlocal nonlinearity. *Communications in Partial Differential Equations*, 40(5):957–993.

Méléard, S., Tran, V. C., et al. (2012). Nonlinear historical superprocess approximations for population models with past dependence. *Electronic Journal of Probability*, 17.

Mirrahimi, S. (2013). Adaptation and migration of a population between patches. *Discrete and Continuous Dynamical Systems. Series B.*, 18(3):753–768.

Mirrahimi, S. (2017). A Hamilton–Jacobi approach to characterize the evolutionary equilibria in heterogeneous environments. *Mathematical Models and Methods in Applied Sciences*, 27(13):2425–2460.

Mirrahimi, S. (2018). Singular limits for models of selection and mutations with heavy-tailed mutation distribution. *arXiv preprint arXiv:1807.10475*.

Mirrahimi, S. and Gandon, S. (2018). Evolution of specialization in heterogeneous environments: equilibrium between selection, mutation and migration. *bioRxiv preprint*.

Mirrahimi, S. and Perthame, B. (2015). Asymptotic analysis of a selection model with space. *Journal de Mathématiques Pures et Appliquées*, 104(6):1108–1118.

Mirrahimi, S. and Raoul, G. (2013). Dynamics of sexual populations structured by a space variable and a phenotypical trait. *Theoretical population biology*, 84:87–103.

Mirrahimi, S. and Roquejoffre, J.-M. (2015). A class of Hamilton-Jacobi equations with constraint: uniqueness and constructive approach. *arXiv:1505.05994 [math]*. arXiv: 1505.05994.

Méléard, S. and Mirrahimi, S. (2015). Singular Limits for Reaction-Diffusion Equations with Fractional Laplacian and Local or Nonlocal Nonlinearity. *Communications in Partial Differential Equations*, 40(5):957–993.

Otto, S. P. (2009). The evolutionary enigma of sex. *the american naturalist*, 174(S1):S1–S14.

Parmesan, C. (2006). Ecological and evolutionary responses to recent climate change. *Annu. Rev. Ecol. Evol. Syst.*, 37:637–669.

Perthame, B. (2007). *Transport equations in biology*. Frontiers in mathematics. Birkhäuser, Basel.

Perthame, B. and Barles, G. (2008). Dirac concentrations in lotka-volterra parabolic pdes. *Indiana University Mathematics Journal*, pages 3275–3301.

Perthame, B. and Souganidis, P. (2005). Front propagation for a jump process model arising in spatial ecology. *Discrete Contin. Dyn. Syst.*, 13(5):1235–1246.

Raoul, G. (2017). Macroscopic limit from a structured population model to the kirkpatrick-barton model. *arXiv preprint arXiv:1706.04094*.

Roques, L. (2005). Study of the premixed flame model with heat losses the existence of two solutions. *European Journal of Applied Mathematics*, 16(6):741–765.

Roques, L. (2013). *Modèles de réaction-diffusion pour l'écologie spatiale: Avec exercices dirigés*. Editions Quae.

Roques, L., Garnier, J., Hamel, F., and Klein, E. K. (2012). Allee effect promotes diversity in traveling waves of colonization. *Proceedings of the National Academy of Sciences*, 109(23):8828–8833.

Roughgarden, J. (1972). Evolution of niche width. *The American Naturalist*, 106(952):683–718.

Sato, K.-i., Ken-Iti, S., and Katok, A. (1999). *Lévy processes and infinitely divisible distributions*. Cambridge university press.

Schumacher, K. (1980). Travelling-front solutions for integro-differential equations. I. *J. Reine Angew. Math.*, 316:54–70.

Skellam, J. G. (1951). Random dispersal in theoretical populations. *Biometrika*, 38(1/2):196–218.

Slatkin, M. (1970). Selection and polygenic characters. *Proceedings of the National Academy of Sciences*, 66(1):87–93.

Slatkin, M. and Lande, R. (1976). Niche Width in a Fluctuating Environment-Density Independent Model. *The American Naturalist*, 110(971):31–55.

Smith, J. M. (1974). The theory of games and the evolution of animal conflicts. *Journal of theoretical biology*, 47(1):209–221.

Stokes, A. (1976). On two types of moving front in quasilinear diffusion. *Mathematical Biosciences*, 31(3-4):307–315.

Tufto, J. (2000). Quantitative genetic models for the balance between migration and stabilizing selection. *Genetics Research*, 76(3):285–293.

Turchin, P. (1998). *Quantitative analysis of movement*. Sinauer assoc. Sunderland (mass.).

Turelli, M. (2017). Commentary: Fisher's infinitesimal model: A story for the ages. *Theoretical Population Biology*, 118:46–49.

Turelli, M. and Barton, N. H. (1994). Genetic and statistical analyses of strong selection on polygenic traits: what, me normal? *Genetics*, 138(3):913–941.

Weinberger, H. F. (1982). Long-time behavior of a class of biological models. *SIAM J. Math. Anal.*, 13(3):353–396.

Xin, J. (2000). Front propagation in heterogeneous media. *SIAM review*, 42(2):161–230.

Yagisita, H. (2009). Existence and nonexistence of travelling waves for a nonlocal monostable equation. *Publ. RIMS, Kyoto Univ.*