



Mesures de Gibbs p -adiques sur les arbres de Cayley et systèmes dynamiques sur le corps des nombres p -adiques

Mohd Ali Khameini Bin Ahmad

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T H È S E

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par

Mohd Ali Khameini AHMAD

Mesures de Gibbs p -adiques sur les arbres de
Cayley et systèmes dynamiques sur le corps des
nombres p -adiques

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RÉSUMÉ

Cette thèse est consacrée à l'étude du modèle de Potts p -adique à q -états sur les arbres de Cayley. Plus précisément, nous étudions les mesures de Gibbs p -adiques du modèle de Potts sur les arbres de Cayley d'ordres 3 et 4 et leurs systèmes dynamiques p -adiques associés. Dans la première partie, nous décrivons les mesures de Gibbs p -adiques invariantes par translation pour le modèle de Potts sur l'arbre de Cayley d'ordre 4. L'existence de mesures de Gibbs p -adiques invariantes par translation est équivalente à l'existence de points fixes d'une application rationnelle appelée application de Potts–Bethe. Cette application de Potts–Bethe est obtenue à partir de l'équation récurrente d'une application à valeur dans \mathbb{Q}_p^q rencontrée lors de la construction des mesures de Gibbs p -adiques du modèle de Potts sur les arbres de Cayley. Afin de décrire ces mesures de Gibbs p -adiques invariantes par translation, nous trouvons les solutions d'une équation quartique dans certains domaines \mathcal{E}_p de \mathbb{Q}_p . En général, nous trouvons aussi des conditions de solvabilité pour les équations quartiques dépréciées sur \mathbb{Q}_p . Dans la deuxième partie, nous étudions la dynamique des applications de Potts–Bethe dans le cas d'arbres de Cayley d'ordres 3 et 4. Premièrement, nous décrivons l'application de Potts–Bethe ayant une bonne réduction. Pour une application de Potts–Bethe ayant une bonne réduction, la droite projective $\mathbb{P}^1(\mathbb{Q}_p)$ peut être décomposée en composants minimaux et leur bassins attractifs. Cependant, les applications de Potts–Bethe associées au modèle de Potts p -adiques sur les arbres de Cayley d'ordres 3 et 4 ont une mauvaise réduction: pour de nombreux nombres premiers p , ces applications correspondantes sont en effet chaotiques. En fait, pour ces nombres premiers p , nous prouvons que, restreintes à leurs ensembles de Julia, les applications de Potts–Bethe sont topologiquement conjuguées à une dynamique de décalage. Pour des autres nombres premiers p , restreintes à leurs ensembles de Julia, les applications de Potts–Bethe ne sont pas topologiquement conjuguées à une dynamique de décalage. La propriété chaotique de l'application de Potts–Bethe implique l'immensité de l'ensemble des mesures de Gibbs p -adiques et une transition de phase. Comme application, nous obtenons que pour de nombreux nombres premiers p , les modèles de Potts p -adiques sur les arbres de Cayley d'ordres 3 et 4 ont une transition de phase. Nous remarquons également que l'affirmation que la transition de phase implique le chaos n'est pas vraie.

ABSTRACT

This thesis is devoted to the study of the q -state p -adic Potts model on Cayley trees. Specifically, we investigate the p -adic Gibbs measures of the Potts model on the Cayley trees of orders 3 and 4 and their related p -adic dynamical systems. In the first part, we describe the existence of the translation-invariant p -adic Gibbs measures of the Potts model on the Cayley tree of order 4. The existence of translation-invariant p -adic Gibbs measures is equivalent to the existence of fixed points of a rational map called Potts–Bethe mapping. The Potts–Bethe mapping is derived from the recurrent equation of a \mathbb{Q}_p^q -valued function in the construction of the p -adic Gibbs measures of the Potts model on Cayley trees. In order to describe the existence of these translation-invariant p -adic Gibbs measures, we find the solutions of some quartic equation in some domains $\mathcal{E}_p \subset \mathbb{Q}_p$. In general, we also provide some solvability conditions for the depressed quartic equation over \mathbb{Q}_p . In the second part, we study the dynamics of the Potts–Bethe mapping of degrees 3 and 4. First, we describe the Potts–Bethe mapping having good reduction. For a Potts–Bethe mapping with good reduction, the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ can be decomposed into minimal components and their attracting basins. However, the Potts–Bethe mapping associated to the Potts model over \mathbb{Q}_p on the Cayley trees of orders 3 and 4 have bad reduction. For many prime numbers p , such Potts–Bethe mappings are chaotic. In fact, for these primes p , we prove that restricted to their Julia sets, the Potts–Bethe mappings are topologically conjugate to the full shift dynamics. For other primes p , the corresponding dynamics restricted to Julia sets are not topologically conjugate to the full shift dynamics or subshift of finite type. The chaotic property of the Potts–Bethe mapping implies the vastness of the set of the p -adic Gibbs measures, and hence implies a phase transition. As an application, for many prime numbers p , the Potts models over \mathbb{Q}_p on the Cayley trees of orders 3 and 4 have phase transition. We also remark the statement that phase transition implies chaos is not true.

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To my beloved parents

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CHAPTER 1

APERÇU DES TRAVAUX

1.1 FONDEMENTS THÉORIQUES

La théorie des probabilités à valeurs p -adiques – un modèle ne satisfaisant pas les axiomes de Kolmogorov dans lequel les probabilités prennent des valeurs dans le corps \mathbb{Q}_p des nombres p -adiques – a été proposée dans une série de papiers (Khrennikov, 1990a,b, 1991, 1992a,b, 1993, 1996a,b; Khrennikov et al., 1999) dans le but de résoudre la question de l'interprétation statistique de fonctions d'ondes p -adiques en physique quantique non-Archimédienne (Beltrametti & Cassinelli, 1972; Volovich, 1987; Vladimirov et al., 1994; Khrennikov, 1994, 2009; Albeverio et al., 2010). Par ailleurs, afin de formaliser une approche de théorie de la mesure pour les probabilités p -adiques, Ilic-Stepic et al. (2016) ont développé de nombreuses propriétés logiques des probabilités p -adiques, constituant d'importantes extensions décidables et complètes de propositions logiques classiques.

Plus précisément, une probabilité p -adique est définie comme étant la limite des fréquences relatives pour la topologie p -adique. Dès lors les aspects de théorie de la mesure de la théorie probabiliste p -adique ont pu être développés. Ensuite, des théories des processus stochastiques à valeurs p -adiques, ou plus généralement celle de champs admettant des distributions probabilistes (non-Kolmogorov) à valeurs non-Archimédiennes ont été intensément développées et divers théorèmes limites pour des processus à valeurs p -adiques ont été obtenus. D'autres parts, des applications issues d'analyses fonctionnelles et harmoniques p -adiques ont également été établies en physique théorique et mécanique quantique (Albeverio et al., 1997a,b,c, 2009, 2010; Dragovich et al., 2009). Un résumé de tous les récents développements en physique mathématique p -adique peut être trouvé dans un article de Dragovich et al. (2017).

Les mesures de Gibbs, qui jouent un rôle central en mécanique statistique, prennent leurs origines à Boltzmann et Gibbs qui introduisent une approche statistique en thermodynamique afin de déduire des comportements collectifs macroscopiques à partir d'informations individuelles microscopiques. Les mesures de Gibbs associées à l'hamiltonien d'un système physique (un modèle) formalisent la notion d'ensemble canonique. Dans le cas classique où le modèle mathématique est basé sur le corps des nombres réels, le phénomène de transition de phase est alors illustré par la multiplicité des mesures de Gibbs. Ainsi, un problème central dans l'étude des mesures de Gibbs est de décrire la taille de l'ensemble des mesures de Gibbs.

En raison de la structure convexe de l'ensemble des mesures de Gibbs sur le

corps des nombres réels, pour déterminer la taille de l'ensemble des mesures de Gibbs, il suffit alors d'étudier le nombre de ses éléments extrémaux. Dès lors, dans le cas classique, pour prédire une transition de phase, l'attention principale a été de rechercher toutes les mesures extrémales possibles (Georgii, 2011). Cependant, il apparut que rechercher toutes celles-ci pour des modèles de spins pouvait s'avérer un problème difficile et non complètement soluble (Gandolfo, Maes, et al., 2017; Gandolfo et al., 2012). L'étude des mesures de Gibbs pour le modèle de Potts sur des arbres de Cayley ont attiré une grande attention (Kulske et al., 2013; Rozikov & Khakimov, 2013b; Gandolfo, Rahmatullaev, & Rozikov, 2017). La méthode classique pour caractériser les mesures de Gibbs sur des arbres de Cayley, inventée par Preston (1974) et Spitzer (1975), est la théorie des champs de Markov à base d'équations de récurrence. Cependant, la théorie moderne des mesures de Gibbs sur des arbres de Cayley se base également sur la théorie des groupes, les flots d'informations, les marches aléatoires à noeuds pondérés et l'analyse non-linéaire.

Les développements récents de la théorie des mesures de Gibbs sur les arbres de Cayley peuvent être trouvés dans le livre par Rozikov (2013). Mezard et al. (1987) ont ensuite découvert que la structure des fonctions de corrélation était liée à une ultramétrie. Ce résultat nous amena alors à une mécanique statistique basée sur les nombres p -adiques. Cette analogie p -adiques de la théorie des mesures de Gibbs sur les arbres de Cayley a été activement étudiée ces dernières décennies, présentant des exemples concrets naturels de processus à valeurs p -adiques (Mukhamedov, 2013; Ludkovsky & Khrennikov, 2003). L'étude des mesures de Gibbs sur les arbres de Cayley a été initiée dans Ganikhodjaev et al. (1998) and Mukhamedov & Rozikov (2004, 2005). L'existence de mesures de Gibbs p -adiques ainsi qu'une transition de phase pour les modèles sur réseaux a été établi dans Mukhamedov (2013); Mukhamedov & Akin (2013); Mukhamedov et al. (2015) et Rozikov & Khakimov (2013a). Plus récemment, toutes les mesures de Gibbs p -adiques invariantes par translation pour le modèle d'Ising et le modèle de Potts à q -états sur les arbres de Cayley d'ordre deux (Rozikov & Khakimov, 2015) et trois (Saburov & Ahmad, 2015b) ont été décrites. Soulignons que dans le cas p -adique, en raison de l'absence de structure convexe de l'ensemble des mesures de Gibbs, il est assez difficile de prédire transitions de phases ou autres propriétés de cet ensemble.

L'étude de l'ensemble des mesures de Gibbs p -adiques de modèles de Potts sur des arbres de Cayley est fortement liée à un problème diophantien (Rozikov & Khakimov, 2015; Saburov & Ahmad, 2015b) consistant à déterminer le nombre de solutions d'un système d'équations polynômes, ou à donner une borne sur le nombre de solutions sur le corps des nombres p -adiques. En général, le même problème diophantien peut avoir différentes solutions sur le corps des nombres p -adiques ou sur celui des nombres réels en raison des structures topologiques différentes. Pour des arbres d'ordres

supérieurs, le problème diphotantien correspondant peut s'avérer difficile d'accès sur le corps des nombres p -adiques. Récemment, ce problème a été totalement résolu pour des équations monomiales (Mukhamedov & Saburov, 2013), des équations quadratiques (Saburov & Ahmad, 2015c), et des équations cubiques dépréciées pour des entiers $p > 3$ (Mukhamedov et al., 2013, 2014).

1.2 OBJECTIFS

Les objectifs de cette thèse sont :

1. Décrire l'existence des mesures de Gibbs p -adiques invariantes par translation sur l'arbre de Cayley d'ordre 3 et d'ordre 4.
2. Étudier la dynamique de l'application de Potts-Bethe de degré 3 et de degré 4 sur \mathbb{Q}_p .
3. Établir la relation entre les propriétés chaotique de l'application de Potts-Bethe sur \mathbb{Q}_p and la transition de phase du modèle de Potts sur \mathbb{Q}_p sur les arbres de Cayley.

1.3 PANORAMA DE LA THÈSE

Cette thèse contient cinq chapitres. Dans le Chapitre 1, nous présentons l'état des recherches et un aperçu de la thèse. Dans le Chapitre 2, nous dressons les préliminaires des notions de nombres p -adiques, polynômes, d'arbres de Cayley, de mesures de probabilités p -adiques du modèle de Potts sur l'arbre de Cayley et système dynamique. Dans le Chapitre 3, pour montrer l'existence de mesures de Gibbs p -adiques invariantes par translation du modèle de Potts sur l'arbre de Cayley de degré 4, nous résolvons le système suivant d'équations

$$z_i = \left(\frac{(\theta - 1)z_i + \sum_{\ell=1}^{q-1} z_\ell + 1}{\theta + \sum_{\ell=1}^{q-1} z_\ell} \right)^4 \quad \text{for } 1 \leq i \leq q-1$$

où $\theta = \exp_p(J)$ et J une constante de couplage. Nous trouvons la solution de ce système d'équations de la forme $\mathbf{z} = (z_1, \dots, z_{q-1})$ tel que $z_i = 1$ or $z_i = z^* \in \mathcal{E}_p \subset \mathbb{Q}_p$ pour $i = 1, q-1$, avec z^* un point fixe, *i.e.* $z^* = f(z^*)$ pour l'application f :

$$f(z) = \left(\frac{(\theta - 1 + m)z + q - m}{mz + \theta - 1 + q - m} \right)^4$$

appelée "application de Potts-Bethe". Notons que f satisfait $f(1) = 1$ et a la forme $\left(\frac{ax+b}{cx+d}\right)^k$. Pour trouver les points fixes de f autres que 1, nous décrivons des équations quartiques ayant une solution $z \in \mathcal{E}_p$ (Propositions 3.2.1 et 3.2.2). Cette description nous permet de prouver l'existence et la non-unicité des mesures p -adiques invariantes par translation (Théorème 3.2.3). En général, nous produisons aussi des conditions pour que l'équation quartique dépréciée

$$x^4 + ax^2 + bx + c = 0$$

où $a, b, c \in \mathbb{Q}_p$, admette une solution dans \mathbb{Q}_p en termes de ses coefficients (Théorème 3.3.8).

Dans le Chapitre 4, nous étudions la dynamique des applications de Potts-Bethe sur \mathbb{Q}_p . Tout d'abord, nous décrivons les applications de Potts-Bethe ayant une bonne réduction (Proposition 4.2.1) et montrons que la dynamique est alors décomposée en systèmes minimaux avec leurs bassins d'attraction (Théorème 4.2.2). Ensuite, nous étudions séparément les dynamiques des applications de Potts-Bethe de degré 3 et de degré 4 ayant mauvaise réduction. Nous considérons l'application de Potts-Bethe de degré 3 suivante :

$$f_{\theta,q,3}(x) = \left(\frac{\theta x + q - 1}{x + \theta + q - 2} \right)^3.$$

Nous distinguons deux cas: $0 < |\theta - 1|_p < |q|_p < 1$ et $0 < |q|_p < |\theta - 1|_p < 1$.

Cas $0 < |\theta - 1|_p < |q|_p < 1$: Lorsque $p \equiv 5 \pmod{6}$, nous trouvons un bassin d'attractions $\mathfrak{B}(\mathbf{x}^{(0)})$ du point fixe $x^{(0)} = 1$ (Théorème 4.3.8) de la forme

$$\mathfrak{B}(\mathbf{x}^{(0)}) = \mathbb{Q}_p \setminus \left(\{\mathbf{x}^{(1)}\} \cup \bigcup_{n=0}^{\infty} f_{\theta,q,3}^{-n} \{\mathbf{x}^{(\infty)}\} \right)$$

où $x^{(\infty)} = 2 - \theta - q$ et $x^{(1)}$ est le point fixe répulsif. Quand $p \equiv 1 \pmod{6}$, l'ensemble de Julia \mathcal{J} de l'application de Potts-Bethe est non-vide. En outre, nous divisons en deux nouveaux cas : $0 < |\theta - 1|_p < |q|_p^2 < 1$ et $0 < |q|_p^2 \leq |\theta - 1|_p < |q|_p < 1$. Pour $0 < |\theta - 1|_p < |q|_p^2 < 1$, il existe un système $(\mathcal{J}, f_{\theta,q,3})$ qui est isométriquement conjugué à une dynamique du décalage complet sur 3 symboles (Théorème 4.3.13). Pour $|q|_p^2 < |\theta - 1|_p < |q|_p < 1$, il existe un système $(\mathcal{J}, f_{\theta,q,3})$ qui est isométriquement conjugué à une dynamique du décalage de type fini sur r symboles, $r \geq 4$. Cependant, ces sous-décalage sur r symboles sont tous topologiquement conjugué au décalage complet sur 3 symboles (Théorème 4.3.19). Dans tous les cas, nous obtenons la décomposition suivante:

$$\mathbb{Q}_p = \mathfrak{B}(\mathbf{x}^{(0)}) \cup \mathcal{J} \cup \bigcup_{n=0}^{+\infty} f_{\theta,q,3}^{-n} \{\mathbf{x}^{(\infty)}\}.$$

où $x^{(\infty)} = 2 - \theta - q$ et $\mathfrak{B}(\mathbf{x}^{(0)})$ est le bassin d'attraction du point fixe $x^{(0)}$.

Cas $0 < |q|_p < |\theta - 1|_p < 1$: Dans ce cas, nous trouvons les disques de Siegel des points fixes neutres $\mathbf{x}^{(0)}$ et $\mathbf{x}^{(1)}$ (Théorème 4.3.27). Lorsque $p \equiv 5 \pmod{6}$, dans certains cas, nous trouvons des orbites périodiques attractives ou des disques de Siegel des orbites périodiques neutres avec leurs bassins. Quand $p \equiv 1 \pmod{6}$, l'ensemble de Julia \mathcal{J} de l'application de Potts-Bethe est non-vide et il existe un système $(\mathcal{J}, f_{\theta,q,3})$ qui est isométriquement conjugué à la dynamique du décalage complet sur 2 symboles (Théorème 4.3.29).

Ensuite nous considérons l'application de Potts-Bethe de degré 4 suivante :

$$f_{\theta,q,4}(x) = \left(\frac{\theta x + q - 1}{x + \theta + q - 2} \right)^4.$$

De même que pour l'application de Potts-Bethe de degré 3, nous considérons 2 cas puis retrouvons la même décomposition. Nous distinguons deux cas: $0 < |\theta - 1|_p < |q|_p < 1$ et $0 < |q|_p < |\theta - 1|_p < 1$.

Cas $0 < |\theta - 1|_p < |q|_p < 1$: Lorsque $p \equiv 3 \pmod{4}$, l'ensemble de Julia \mathcal{J} de l'application de Potts-Bethe est non-vide. Il existe un système $(\mathcal{J}, f_{\theta,q,4})$ qui est topologiquement conjugué à la dynamique du décalage complet sur 2 symboles (Théorème 4.4.9). Lorsque $p \equiv 1 \pmod{4}$, l'ensemble de Julia \mathcal{J} de l'application de Potts-Bethe est non-vide. Il existe un système $(\mathcal{J}, f_{\theta,q,4})$ topologiquement conjugué à la dynamique du décalage complet sur 4 symboles (Théorème 4.4.9). Dans les deux cas

$$\mathbb{Q}_p = \mathfrak{B}(\mathbf{x}^{(0)}) \cup \mathcal{J} \cup \bigcup_{n=0}^{+\infty} f_{\theta,q,4}^{-n} \{\mathbf{x}^{(\infty)}\}.$$

Cas $0 < |q|_p < |\theta - 1|_p < 1$: Dans ce cas, nous trouvons les disques de Siegel des points fixes neutres $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ (Théorème 4.4.17). Lorsque $p \equiv 3 \pmod{4}$, dans certains cas, nous trouvons des orbites périodiques attractives ou des disques Siegel d'orbites périodiques neutres avec leurs bassins. Lorsque $p \equiv 1 \pmod{4}$, l'ensemble de Julia \mathcal{J} de l'application de Potts-Bethe est non-vide et il existe un système $(\mathcal{J}, f_{\theta,q,4})$ qui est isométriquement conjuguées à la dynamique du décalage complet sur 3 lettres (Théorème 4.4.17).

Nous montrons également pour de nombreux nombres premiers p que l'application de Potts-Bethe possède des propriétés de chaotité. Cette propriété chaotique implique l'immensité de l'ensemble des mesures de Gibbs p -adiques (Théorème 4.6.1). Comme application, pour de nombreux nombres premiers p , la transition de phase a lieu pour le modèle de Potts p -adique sur les arbres de Cayley de degrés 3 et degrés 4. Nous remarquons également que l'affirmation qu'une transition de phase implique le chaos est fausse en général, dans la dernière partie de la Section 4.6.

Dans le Chapitre 5, nous résumons cette thèse et donnons quelques suggestions

pour la recherche future.

1.4 OVERVIEW OF THE THESIS

This thesis contains five chapters. In Chapter 1, we give the research background and the overview of the thesis. In Chapter 2, we provide the preliminaries on p -adic numbers, polynomials, Cayley trees, p -adic probability measures (Potts model over the field \mathbb{Q}_p of p -adic numbers) and the related dynamical systems.

In Chapter 3, we study the existence translation-invariant p -adic Gibbs measures of the q -state Potts model on the Cayley tree of order 4. To show the existence of translation-invariant p -adic Gibbs measures, we solve the following system of equations

$$z_i = \left(\frac{(\theta - 1)z_i + \sum_{\ell=1}^{q-1} z_\ell + 1}{\theta + \sum_{\ell=1}^{q-1} z_\ell} \right)^4 \quad \text{for } 1 \leq i \leq q-1$$

where $\theta = \exp_p(J)$ and J is a coupling constant. We find the solution of this system of equations of the form $\mathbf{z} = (z_1, \dots, z_{q-1})$ such that $z_i = 1$ or $z_i = z^* \in \mathcal{E}_p \subset \mathbb{Q}_p$ for $i = \overline{1, q-1}$ with z^* being the fixed points, i.e. $z^* = f(z^*)$ of the following map f

$$f(z) = \left(\frac{(\theta - 1 + m)z + q - m}{mz + \theta - 1 + q - m} \right)^4$$

called ‘‘Potts–Bethe mapping’’. We note that f satisfies $f(1) = 1$ and has the form $\left(\frac{ax+b}{cx+d} \right)^k$. To find the fixed points of f other than 1, we describe some quartic equation having a solution $z \in \mathcal{E}_p$ (Propositions 3.2.1 and 3.2.2). This descriptions allows us to prove the non-uniqueness of the translation-invariant p -adic Gibbs measures (Theorem 3.2.3). In general, we also provide some conditions for the depressed quartic equation

$$x^4 + ax^2 + bx + c = 0$$

where $a, b, c \in \mathbb{Q}_p$ to have a solution in \mathbb{Q}_p in terms of its coefficients (Theorem 3.3.8).

In Chapter 4, we study the dynamics of the Potts–Bethe mapping on \mathbb{Q}_p . First, we describe the Potts–Bethe mapping having good reduction (Proposition 4.2.1) and show that its dynamics is decomposed into minimal subsystems and their attracting basins (Theorem 4.2.2). Then we study separately the dynamics of the Potts–Bethe mapping of degree 3 and of degree 4 which has bad reduction. We consider the following Potts–Bethe mapping of degree 3

$$f_{\theta,q,3}(x) = \left(\frac{\theta x + q - 1}{x + \theta + q - 2} \right)^3.$$

We distinguish two cases: $0 < |\theta - 1|_p < |q|_p < 1$ and $0 < |q|_p < |\theta - 1|_p < 1$.

Case $0 < |\theta - 1|_p < |q|_p < 1$: When $p \equiv 5 \pmod{6}$, we find the attracting basin $\mathfrak{B}(\mathbf{x}^{(0)})$ of the fixed point $\mathbf{x}^{(0)} = 1$ (Theorem 4.3.8) is as follows

$$\mathfrak{B}(\mathbf{x}^{(0)}) = \mathbb{Q}_p \setminus \left(\{\mathbf{x}^{(1)}\} \cup \bigcup_{n=0}^{\infty} f_{\theta,q,3}^{-n} \{\mathbf{x}^{(\infty)}\} \right)$$

where $\mathbf{x}^{(\infty)} = 2 - \theta - q$ and $\mathbf{x}^{(1)}$ is the repelling fixed point. When $p \equiv 1 \pmod{6}$, the Julia set \mathcal{J} of the Potts–Bethe mapping is non-empty. We divide into two subcases: $0 < |\theta - 1|_p < |q|_p^2 < 1$ and $0 < |q|_p^2 \leq |\theta - 1|_p < |q|_p < 1$. For $0 < |\theta - 1|_p < |q|_p^2 < 1$, there exists a subsystem $(\mathcal{J}, f_{\theta,q,3})$ that is isometrically conjugate to the full shift dynamics on 3 symbols (Theorem 4.3.13). For $0 < |q|_p^2 \leq |\theta - 1|_p < |q|_p < 1$, there exists a subsystem $(\mathcal{J}, f_{\theta,q,3})$ that is isometrically conjugate to a subshift of finite type on r symbols where $r \geq 4$. However, these subshifts on r symbols are all topologically conjugate to the full shift on 3 symbols (Theorem 4.3.19). In both case, we have the following decomposition:

$$\mathbb{Q}_p = \mathfrak{B}(\mathbf{x}^{(0)}) \cup \mathcal{J} \cup \bigcup_{n=0}^{+\infty} f_{\theta,q,3}^{-n} \{\mathbf{x}^{(\infty)}\}$$

where $\mathbf{x}^{(\infty)} = 2 - \theta - q$ and $\mathfrak{B}(\mathbf{x}^{(0)})$ is the attracting basin of the attracting fixed point $\mathbf{x}^{(0)}$.

Case $0 < |q|_p < |\theta - 1|_p < 1$: In this case we find the Siegel disks of the neutral fixed points $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ (Theorem 4.3.27). When $p \equiv 1 \pmod{6}$, the Julia set \mathcal{J} of the Potts–Bethe mapping is non-empty. There exists a subsystem $(\mathcal{J}, f_{\theta,q,3})$ that is isometrically conjugate to the full shift dynamics on 2 symbols (Theorem 4.3.29).

Next we consider the following Potts–Bethe mapping of degree 4

$$f_{\theta,q,4}(x) = \left(\frac{\theta x + q - 1}{x + \theta + q - 2} \right)^4.$$

Similar to the Potts–Bethe mapping of degree 3, we consider two case: $0 < |\theta - 1|_p < |q|_p < 1$ and $0 < |q|_p < |\theta - 1|_p < 1$.

Case $0 < |\theta - 1|_p < |q|_p < 1$: In this case the Julia set \mathcal{J} is non-empty. When $p \equiv 3 \pmod{4}$, there exists a subsystem $(\mathcal{J}, f_{\theta,q,4})$ that is topologically conjugate to the full shift dynamics on 2 symbols (Theorem 4.4.9). Whereas when $p \equiv 1 \pmod{4}$, there exists a subsystem $(\mathcal{J}, f_{\theta,q,4})$ that is topologically conjugate to the full shift dynamics on 4 symbols (Theorem 4.4.9). In both case, we have the following decomposition:

$$\mathbb{Q}_p = \mathfrak{B}(\mathbf{x}^{(0)}) \cup \mathcal{J} \cup \bigcup_{n=0}^{+\infty} f_{\theta,q,4}^{-n} \{\mathbf{x}^{(\infty)}\}$$

where $\mathbf{x}^{(\infty)} = 2 - \theta - q$ and $\mathfrak{B}(\mathbf{x}^{(0)})$ is the attracting basin of the attracting fixed point $\mathbf{x}^{(0)}$.

Case $0 < |q|_p < |\theta - 1|_p < 1$: We calculate the Siegel disks of neutral fixed points $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ (Theorem 4.4.17). When $p \equiv 1 \pmod{4}$, the corresponding Julia set \mathcal{J} is non-empty. We obtain a subsystem $(\mathcal{J}, f_{\theta,q,4})$ that is isometrically conjugate to the full shift dynamics on 3 symbols (Theorem 4.4.17).

We show for many prime numbers p , the Potts–Bethe mapping has chaotic properties. These chaotic properties of the Potts–Bethe mapping imply the vastness of the set of the p -adic Gibbs measures (Theorems 4.6.1 and 4.6.2). As application, for many prime numbers p , the phase transition occurs for the Potts models over \mathbb{Q}_p on the Cayley trees of orders 3 and 4. We also remark that the statement phase transition implies chaos is not true, in the last part of Section 4.6.

In Chapter 5, we give the summary of this thesis and suggestion for the future research.

CHAPTER 2

PRELIMINARIES

2.1 p -ADIC NUMBERS

For an introduction to p -adic numbers and p -adic analysis, we recommend the books of Schikhof (1984), Koblitz (1984) and S. Katok (2007).

2.1.1 p -Adic Integers and p -Adic Numbers

Most of the materials here are taken from Caruso (2017). Recall that each positive integer n can be written in base p

$$n = a_0 + a_1p + a_2p^2 + \cdots + a_\ell p^\ell.$$

Definition 2.1.1. A p -adic integer a is a formal series

$$a = a_0 + a_1p + a_2p^2 + \cdots$$

where $0 \leq a_i \leq p-1$. The set of all p -adic integers is denoted by \mathbb{Z}_p .

For any $a \in \mathbb{Z}_p$, we define

$$\alpha_n = a_0 + a_1p + a_2p^2 + \cdots + a_{n-1}p^{n-1} \in \mathbb{Z}/p^n\mathbb{Z}.$$

Then we set the following functions

$$\pi_n : \mathbb{Z}_p \rightarrow \mathbb{Z}/p^n\mathbb{Z}, \quad \pi_n(a) = \alpha_n.$$

We have for any $a \in \mathbb{Z}_p$, $\pi_{n+1}(a) \equiv \pi_n(a) \pmod{p^n}$. More generally, $\pi_m(a) \equiv \pi_n(a) \pmod{p^n}$ for $m \geq n$. Then putting π_n 's together we get

$$\pi : \mathbb{Z}_p \rightarrow \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}, \quad a \mapsto (\pi_1(a), \pi_2(a), \dots).$$

where

$$\varprojlim_n \mathbb{Z}/p^n\mathbb{Z} := \left\{ a = (a_n) \in \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z} : \pi_{n+1}(a) \equiv \pi_n(a) \pmod{p^n} \forall n \right\}$$

is the projective limit of $\mathbb{Z}/p^n\mathbb{Z}$.

Consider a sequence $\alpha = (\alpha_1, \alpha_2, \dots) \in \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$. Write

$$\alpha_n = a_{n,0} + a_{n,1}p + \dots + a_{n,n-1}p^{n-1}.$$

The condition $\alpha_{n+1} \equiv \alpha_n \pmod{p^n}$ implies $a_{n+1,i} = a_{n,i}$ for $0 \leq i \leq n-1$. This means $(a_{n,i})_{n \geq i}$ is constant and converges to some a_i . Set

$$\psi(\alpha) = a = (a_0, a_1, \dots) \in \mathbb{Z}_p.$$

We can check that ψ is the inverse function of π . Thus we have the following

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}.$$

The descriptions of \mathbb{Z}_p as a limit of $\mathbb{Z}/p^n\mathbb{Z}$ allows us to endow \mathbb{Z}_p with a commutative ring structure, that is for $a, b \in \mathbb{Z}_p$ we look at their sequences $\alpha_n, \beta_n \in \mathbb{Z}/p^n\mathbb{Z}$. Then $\alpha_n + \beta_n \in \mathbb{Z}/p^n\mathbb{Z}$ yields a well defined element $a + b \in \mathbb{Z}_p$. Note that $\mathbb{Z} \subset \mathbb{Z}_p$. One can see $\frac{1}{p} \notin \mathbb{Z}_p$. We have \mathbb{Q}_p as a fraction field of \mathbb{Z}_p .

Definition 2.1.2. A *p*-adic number a is a series

$$a = \sum_{i \geq m} a_i p^i$$

where $m \in \mathbb{Z}$ and $0 \leq a_i \leq p-1$ with $a_m \neq 0$. The set of all *p*-adic numbers is denoted by \mathbb{Q}_p .

2.1.2 *p*-Adic Norm and Completeness

According to Ostrowski's Theorem (see Koblitz (1984)), there are only two kinds of completions of the field \mathbb{Q} of rational numbers. These two kinds of completions give the field \mathbb{R} of real numbers or the field \mathbb{Q}_p of *p*-adic numbers. For a fixed prime p , we introduce the notion of *p*-adic valuation and *p*-adic norm (absolute value). Let $x = p^k \frac{m}{n} \in \mathbb{Q}$ with $k, m \in \mathbb{Z}, n \in \mathbb{N}, p \nmid m$ and $p \nmid n$. The number k is called the *p*-adic valuation of x and is denoted by $\text{ord}_p(x)$. By convention, we define $\text{ord}_p(0) = \infty$. Thus

$$\text{ord}_p(x) = \begin{cases} k & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}$$

Then we can define the p -adic norm (absolute value) as follows:

$$|x|_p = \begin{cases} p^{-k} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This norm is non-Archimedean because it satisfies the strong triangle inequality: $|x + y|_p \leq \max(|x|_p, |y|_p)$. The metric induced by this norm, $d(x, y) = |x - y|_p$, satisfies the ultrametric property: for all $x, y, z \in \mathbb{Q}$, $d(x, y) \leq \max(d(x, z), d(z, y))$. We notice that d takes values from the proper subset $\{0\} \cup \{p^n : n \in \mathbb{Z}\}$ of \mathbb{R} . Recall the following definitions.

Definition 2.1.3. A sequence (x_n) in a field K is called Cauchy if for every $\epsilon > 0$ there exists N such that $|x_n - x_m| < \epsilon$ whenever $n, m > N$.

Definition 2.1.4. A field K is complete with respect to the norm (absolute value) $|\cdot|$ if every Cauchy sequence of K converges in K .

By these definitions, we have the following proposition.

Proposition 2.1.5 (see S. Katok (2007)). The field \mathbb{Q}_p of p -adic numbers is the completion of the field \mathbb{Q} of rational numbers with respect to the p -adic norm (absolute value) $|\cdot|_p$.

Denote by $\mathbb{B}_r(a) := \{x \in \mathbb{Q}_p : |x - a|_p < r\}$, $\overline{\mathbb{B}_r(a)} := \{x \in \mathbb{Q}_p : |x - a|_p \leq r\}$ and $\mathbb{S}_r(a) := \{x \in \mathbb{Q}_p : |x - a|_p = r\}$ the open ball, closed ball and the sphere in \mathbb{Q}_p with center a and radius r . Remark that the open ball is also closed. Then the set of all p -adic integers and p -adic units of \mathbb{Q}_p are denoted by $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ and $\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\}$ respectively. Any p -adic unit $x \in \mathbb{Z}_p^*$ has the unique canonical form

$$x = x_0 + x_1 \cdot p + x_2 \cdot p^2 + \dots$$

where $x_0 \in \{1, 2, \dots, p-1\}$ and $x_i \in \{0, 1, 2, \dots, p-1\}$ for $i \in \mathbb{N}$. Furthermore, any p -adic number $x \in \mathbb{Q}_p$ has the following unique canonical form

$$x = p^{\text{ord}_p(x)} (x_0 + x_1 \cdot p + x_2 \cdot p^2 + \dots)$$

where $x_0 \in \{1, 2, \dots, p-1\}$ and $x_i \in \{0, 1, 2, \dots, p-1\}$ for $i \in \mathbb{N}$. Therefore,

$$x = \frac{x^*}{|x|_p}$$

such that $x^* \in \mathbb{Z}_p^*$.

As parallel to the construction of the field \mathbb{C} of complex numbers, we can also construct an analogue for p -adic numbers, see for example Koblitz (1984) and Schikhof (1984). We consider the algebraic extension of \mathbb{Q}_p . The p -adic absolute value will be

uniquely extended. However, any extension of finite order of \mathbb{Q}_p is not algebraically closed. Hence, the algebraic closure \mathbb{Q}_p^a of \mathbb{Q}_p is an infinite extension. This algebraic closure is not topologically complete. Fortunately, the topological completion of \mathbb{Q}_p^a is algebraically closed. This final field denoted by \mathbb{C}_p is called the field of complex p -adic numbers.

Next, denote by $\mathbb{P}^1(\mathbb{C}_p)$ the projective line over \mathbb{C}_p . The field \mathbb{C}_p may be identified with the subset of $\mathbb{P}^1(\mathbb{C}_p)$ given by

$$\{[x : 1] \in \mathbb{P}^1(\mathbb{C}_p) : x \in \mathbb{C}_p\}.$$

A point in the projective line $\mathbb{P}^1(\mathbb{C}_p)$ is given in homogeneous coordinates by a pair $[x_1 : x_2]$ of points in \mathbb{C}_p which are not both zero. Two such pairs are equal if they differ by an overall (nonzero) factor $\lambda \in \mathbb{C}_p^*$:

$$[x_1 : x_2] = [\lambda x_1 : \lambda x_2].$$

This subset contains all points in $\mathbb{P}^1(\mathbb{C}_p)$ except one: the point of infinity, which may be given as $\infty = [1 : 0]$.

The spherical metric defined on $\mathbb{P}^1(\mathbb{C}_p)$ is analogous to the standard spherical metric on Reimann sphere. For $P = [x_1 : x_2]$ and $Q = [y_1 : y_2]$ in $\mathbb{P}^1(\mathbb{C}_p)$, we define

$$\rho(P, Q) = \frac{|x_1 y_2 - y_1 x_2|_p}{\max\{|x_1|_p, |x_2|_p\} \max\{|y_1|_p, |y_2|_p\}}$$

or, viewing $\mathbb{P}^1(\mathbb{C}_p)$ as $\mathbb{C}_p \cup \{\infty\}$, for $z_1, z_2 \in \mathbb{C}_p \cup \{\infty\}$, we define

$$\rho(z_1, z_2) = \frac{|z_1 - z_2|_p}{\max\{|z_1|_p, 1\} \max\{|z_2|_p, 1\}}$$

and

$$\rho(z, \infty) = \begin{cases} 1, & \text{if } |z|_p \leq 1 \\ \frac{1}{|z|_p}, & \text{if } |z|_p > 1. \end{cases}$$

Remark that the restriction of the spherical metric on the ring $\mathcal{O} = \{z \in \mathbb{C}_p : |z|_p \leq 1\}$ is same as the metric induced by the absolute value $|\cdot|_p$. One can also think the same for the case of projective line $\mathbb{P}^1(\mathbb{Q}_p)$ over \mathbb{Q}_p .

2.2 POLYNOMIAL

2.2.1 Resultant and Discriminant

We refer the book of Gelfand et al. (1994) and the article of Dilcher & Stolarsky (2005) for the resultant and dicriminant of the polynomials.

We write a polynomial over a field K as $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ where $a_n \neq 0$ and $a_i \in K$ for $i = 0, \dots, n$. We denote $n = \deg(a_n x^n + a_{n-1} x^{n-1} + \dots + a_0)$ the degree of the polynomial $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$. The examples of the field K that can be considered are \mathbb{F}_p , \mathbb{Q} , \mathbb{R} , \mathbb{C} and etc. Later, we will concentrate on the polynomial equations over \mathbb{Q}_p . Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0$ be polynomials of degrees $n \geq 1$ and $m \geq 1$ (a_n and a_m do not vanish) with coefficient in an arbitrary field K respectively. Denote by $R_{m,n}(f, g)$ of their resultant. Sometimes we denote it as $R(f, g)$.

Definition 2.2.1. Let $a_n \neq 0$ and $b_m \neq 0$.

1)

$$R(f, g) = a_n^m b_m^n \prod_{i,j} (x_i - y_j)$$

where x_1, \dots, x_n and y_1, \dots, y_m are roots of f and g respectively.

2) $R(f, g)$ is equal to the determinant of the following $(n + m)$ by $(n + m)$ Sylvester matrix,

$$R(f, g) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n & 0 & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} & a_n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & a_1 & a_2 & a_3 & \cdots & a_n \\ b_0 & b_1 & b_2 & \cdots & b_m & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_{m-1} & b_m & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_0 & b_1 & b_2 & \cdots & b_m \end{vmatrix}.$$

The following is one of the result on resultant of two polynomials f and g .

Theorem 2.2.2 (see Gelfand et al. (1994)). For two concrete polynomials f and g , $R(f, g) = 0$ is equivalent to the fact that f and g have a common root. This also means that $R(f, g) \neq 0$ is equivalent to the fact that f and g have no common root.

Resultant and discriminant of polynomial has a relation. We denote by $\Delta_n(f)$ the discriminant of polynomial f . Frequently, we write it as $\Delta(f)$.

Definition 2.2.3. Let $a_n \neq 0$. Then

$$\Delta(f) = (-1)^{\frac{n(n-1)}{2}} a_n^{2n-2} \prod_{i < j} (x_i - x_j)^2$$

where x_1, \dots, x_n are roots of f .

The following is the relation between discriminant and resultant.

Theorem 2.2.4 (Dilcher & Stolarsky (2005)). Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ be a polynomial of degrees $n \geq 1$ with coefficient in an arbitrary field K . Then the discriminant of f is given by

$$\Delta(f) = (-1)^{\frac{n(n-1)}{2}} a_n^{-1} R(f, f').$$

This relation allows us to find the discriminant of a polynomial. For example, the quadratic polynomial has

$$\Delta(ax^2 + bx + c) = b^2 - 4ac,$$

the cubic polynomial has

$$\Delta(ax^3 + bx^2 + cx + d) = b^2 c^2 - 4ac^3 - 4b^3 d - 27a^2 d^2 + 18abcd,$$

and the quartic polynomial has

$$\begin{aligned} \Delta(ax^4 + bx^3 + cx^2 + dx + e) = & 256a^3 e^3 - 192a^2 b d e^2 - 128a^2 c^2 e^2 + 144a^2 c d^2 e \\ & - 27a^2 d^4 + 144ab^2 c e^2 - 6ab^2 d^2 e - 80abc^2 d e \\ & + 18abcd^3 + 16ac^4 e - 4ac^3 d^2 - 27b^4 e^2 \\ & + 18b^3 c d e - 4b^3 d^3 - 4b^2 c^3 e + b^2 c^2 d^2. \end{aligned}$$

2.2.2 Polynomial Congruences

For linear and quadratic congruences, we refer to the book of Rosen (2011). The simplest congruence is linear

$$ax \equiv b \pmod{m} \tag{2.1}$$

where $a, b, m > 0$ are integers. The following theorem tells us when the linear congruence (2.1) has solutions.

Theorem 2.2.5 (see Rosen (2011)). Let $a, b, m > 0$ be integers and $\gcd(a, m) = d$.

(i) If $d \nmid b$, then the congruence (2.1) has no solution.

(ii) If $d \mid b$, then the congruence (2.1) has exactly d incongruent solutions modulo m .

The quadratic congruence

$$ax^2 + bx + c \equiv 0 \pmod{n} \quad (2.2)$$

where $a, b, c, n > 0$ are integers is the simplest non-linear congruence. For simplicity, a prime number p is considered instead of an integer $n > 0$. So we have

$$ax^2 + bx + c \equiv 0 \pmod{p} \quad (2.3)$$

where a, b, c are integers and p a prime number. Denote $\left(\frac{a}{p}\right)$ the Legendre symbol. Then we have the following theorem about the quadratic congruence.

Theorem 2.2.6 (see Rosen (2011)). *Let p be a prime number and a, b, c are integers with $p \nmid a$.*

(a) *Let $p = 2$. We have $a \equiv 1 \pmod{p}$. Then the quadratic congruence (2.3) has solutions if and only if not both $b, c \equiv 1 \pmod{p}$.*

(b) *Let p be an odd prime number. Let $D_2 = b^2 - 4ac$.*

(i) *If $D_2 \equiv 0 \pmod{p}$, then the quadratic congruence (2.3) has exactly one solution modulo p .*

(ii) *If $\left(\frac{D_2}{p}\right) = 1$, i.e. D_2 is a quadratic residue modulo p , then the quadratic congruence (2.3) has two incongruent solutions modulo p .*

(iii) *If $\left(\frac{D_2}{p}\right) = -1$, i.e. D_2 is a quadratic nonresidue modulo p , then the quadratic congruence (2.3) has no solution.*

Now we consider the following cubic congruence

$$x^3 + qx^2 + rx + t \equiv 0 \pmod{p} \quad (2.4)$$

such that q, r, t are integers and p a prime number. Let

$$D_3 = q^2r^2 - 4r^3 - 4q^3t - 27t^2 + 18qrt$$

be the discriminant of the cubic polynomial. Denote $N_{\mathbf{K}}(a_nx^n + a_{n-1}x^{n-1} + \dots + a_0)$ be the number of solutions of polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \dots + a_0$ in the field \mathbf{K} . Then we have the following result.

Theorem 2.2.7 (see Sun (2003)). *Let $p \geq 5$ be a prime number and $p \nmid D_3$. Then*

$$N_{\mathbb{F}_p}(x^3 + qx^2 + rx + t) = \begin{cases} 0 & \text{or } 3 \text{ if } \left(\frac{D_3}{p}\right) = 1, \\ 1 & \text{if } \left(\frac{D_3}{p}\right) = -1. \end{cases}$$

Most results on cubic congruences are due to Sun (2003). Let $\{u_n\}$ and $\{s_n\}$ be the third-order recurrence sequences for $q, r, t \in \mathbb{Z}$ defined by

$$u_{-2} = u_{-1} = 0, u_0 = 1, u_{n+3} + qu_{n+2} + ru_{n+1} + tu_n = 0, \quad (n \geq -2)$$

and

$$s_0 = 3, s_1 = -q, s_2 = q^2 - 2r, s_{n+3} + qs_{n+2} + rs_{n+1} + ts_n = 0, \quad (n \geq 0).$$

Lemma 2.2.8 (Sun (2003)). *Let $p \geq 5$ be a prime number and $p \mid D_3$. Then*

$$N_{\mathbb{F}_p}(x^3 + qx^2 + rx + t) = 3.$$

Theorem 2.2.9 (Sun (2003)). *Let $p \geq 5$ be a prime number $\alpha = (q^2 - 3r)^3$, $\beta = -2q^3 + 9qr - 27t$ and $D_3 = -\frac{1}{27}(\beta^2 - 4\alpha)$. Let $p \nmid ab$. Then*

$$N_{\mathbb{F}_p}(x^3 + qx^2 + rx + t) = \begin{cases} 3 & \text{if } D_3 u_{p-2}^2 \equiv 0 \pmod{p}, \\ 0 & \text{if } D_3 u_{p-2}^2 \equiv (q^2 - 3r)^2 \pmod{p}, \\ 1 & \text{if } D_3 u_{p-2}^2 \not\equiv 0, (q^2 - 3r)^2 \pmod{p}. \end{cases}$$

Theorem 2.2.10 (Sun (2003)). *Let $p \geq 5$ be a prime number. Let $p \nmid (q^2 - 3r)$. Then*

$$N_{\mathbb{F}_p}(x^3 + qx^2 + rx + t) = \begin{cases} 3 & \text{if } s_{p+1} \equiv q^2 - 2r \pmod{p}, \\ 0 & \text{if } s_{p+1} \equiv r \pmod{p}, \\ 1 & \text{if } s_{p+1} \not\equiv q^2 - 2r, r \pmod{p}. \end{cases}$$

The following is a result on specific case of cubic congruence, i.e. depressed cubic polynomial.

Proposition 2.2.11 (Mukhamedov et al. (2014)). *Let $p \geq 5$ be a prime number and $q \equiv 0 \pmod{p}$ and $rt \not\equiv 0 \pmod{p}$. We have $D_3 = -4r^3 - 27t^2$.*

(i) *Let $D_3 u_{p-2}^2 \equiv 0 \pmod{p}$. Then*

- 1) *The cubic congruence (2.4) has 3 incongruent solutions if and only if $D_3 \not\equiv 0 \pmod{p}$. In this case, we have $3\bar{x} + r \not\equiv 0 \pmod{p}$ for any solution \bar{x} .*

2) The cubic congruence (2.4) has 2 incongruent solutions while one of them has multiplicity 2 if and only if $D_3 \equiv 0 \pmod{p}$. If \bar{x}_1 and \bar{x}_2 are 2 incongruent solution while \bar{x}_1 has multiplicity-2, then $\bar{x}_1 \equiv \frac{3t}{2r} \pmod{p}$ and $\bar{x}_2 \equiv -\frac{3t}{r} \pmod{p}$. In this case, we have $3\bar{x}_2 + r \not\equiv 0 \pmod{p}$.

3) The cubic congruence (2.4) does not have any solution of multiplicity 3.

(ii) Let $D_3 u_{p-2}^2 \not\equiv 0 \pmod{p}$. Then $3\bar{x} + r \not\equiv 0 \pmod{p}$ for any solution \bar{x} .

Remark that Sun (2003) also give the explicit form of the solution for the cubic congruence (2.4). Lastly, let us consider the quartic congruence

$$x^4 + ax^2 + bx + c \equiv 0 \pmod{p} \quad (2.5)$$

such that a, b, c are integers and p a prime number. It is also due to Sun (2003), we have the results on quartic congruence (2.5). Denote

$$D_4 = -b^2(4a^3 + 27b^2) + 16c(a^4 + 9ab^2 - 8a^2c + 16c^2)$$

the discriminant of the depressed quartic polynomial $x^4 + ax^2 + bx + c$. It could be easily check that D_4 is also the discriminant of the following cubic polynomial

$$y^3 + 3ay^2 + (a^2 - 4c)y - b^2.$$

Lemma 2.2.12 (Sun (2003)). *Let p be an odd prime number and $a, b, c \in \mathbb{Z}$ with $p \nmid b$. Then the quartic congruence (2.5) has one multiple solution if and only if $p \mid D_4$.*

Theorem 2.2.13 (Sun (2003)). *Let $p \geq 5$ be a prime number and $a, b, c \in \mathbb{Z}$ with $p \nmid bD_4$. Then $N_{\mathbb{F}_p}(x^4 + ax^2 + bx + c) = 0$ if and only if there exists an integer y such that*

$$y^3 + 3ay^2 + (a^2 - 4c)y - b^2 \equiv 0 \pmod{p}$$

and $\left(\frac{y}{p}\right) = -1$. Furthermore, when $N_{\mathbb{F}_p}(x^4 + ax^2 + bx + c) > 0$, we have

$$N_{\mathbb{F}_p}(x^4 + ax^2 + bx + c) = N_{\mathbb{F}_p}(y^3 + 3ay^2 + (a^2 - 4c)y - b^2) + 1.$$

Theorem 2.2.14 (Sun (2003)). *Let $p \geq 5$ be a prime number and $a, b, c \in \mathbb{Z}$ with $p \nmid bD_4$. Then*

$$N_{\mathbb{F}_p}(x^4 + ax^2 + bx + c) = 1 + \sum_y \left(\frac{y}{p}\right)$$

where y runs over the solutions of the cubic congruence

$$y^3 + 3ay^2 + (a^2 - 4c)y - b^2 \equiv 0 \pmod{p}.$$

Notice that all previous results on quartic congruence deal when $p \nmid b$. Now we will discuss the results for the quartic congruence (2.5) when $p \mid b$. For simplicity, suppose $b = 0$. Then the quartic congruence (2.5) takes the form

$$x^4 + ax^2 + c \equiv 0 \pmod{p}. \quad (2.6)$$

The discriminant of quartic congruence (2.6) is

$$D_4 = 16c(a^4 - 8a^3c + 16c^2) = 16c(a^2 - 4c)^2.$$

Suppose $p \nmid ac$. Then to discuss the quartic congruence (2.6), we divide into two cases: $p \mid D_4$ and $p \nmid D_4$.

Lemma 2.2.15. *Let p be an odd prime number and $p \mid D_4$. Then the quartic congruence (2.6) has solutions if and only if $\left(\frac{-2a}{p}\right) = 1$.*

Proof. When $p \mid D_4$, we have $4c \equiv a^2 \pmod{p}$. The quartic congruence (2.6) can be written as

$$x^4 + ax^2 + \frac{a^2}{4} \equiv \left(x^2 + \frac{a}{2}\right)^2 \equiv 0 \pmod{p}.$$

So $-2a$ is a quadratic residue modulo p . □

The following are the old results by Carlitz (1956). For p an odd prime number, we may assume $a \equiv -2m \pmod{p}$ for some $m \in \mathbb{Z}$. So the quartic congruence (2.6) is equivalent to

$$x^4 - 2mx^2 + c \equiv 0 \pmod{p}.$$

Theorem 2.2.16 (Carlitz (1956)). *Let p be an odd prime number and $p \nmid D_4$.*

(i) *Let $c \equiv n^2 \pmod{p}$. Consider the numbers*

$$2(m-n), \quad 2(m+n), \quad m^2 - n^2. \quad (2.7)$$

- 1) *If all numbers in (2.7) are quadratic residues, then the quartic polynomial $x^4 - 2mx^2 + n^2$ is congruent to the product of four linear factors.*
- 2) *If only one number in (2.7) is a quadratic residue, then the quartic polynomial $x^4 - 2mx^2 + n^2$ is congruent to the product of two irreducible quadratics.*

(ii) *Let c be quadratic non-residue modulo p .*

- 1) *If $m^2 - c$ is a quadratic residue modulo p , then the quartic polynomial $x^4 - 2mx^2 + c$ is congruent to an irreducible quadratic and two distinct linear factors.*

- 2) If $m^2 - c$ is a quadratic non-residue modulo p , then the quartic polynomial $x^4 - 2mx^2 + c$ is irreducible.

From the theorem above, we obtain the following corollary.

Corollary 2.2.17. *Let p be an odd prime number and $p \nmid D_4$.*

(i) *Let $c \equiv n^2 \pmod{p}$.*

- 1) *If all of $2(m - n)$, $2(m + n)$ and $m^2 - n^2$ are quadratic residues, then the quartic congruence (2.6) has four solutions.*
- 2) *If only one of $2(m - n)$, $2(m + n)$ and $m^2 - n^2$ is a quadratic residue, then the quartic congruence (2.6) has no solution.*

(ii) *Let c be a quadratic non-residue modulo p .*

- 1) *If $m^2 - c$ is a quadratic residue modulo p , then the quartic congruence (2.6) has two solutions.*
- 2) *If $m^2 - c$ is a quadratic non-residue modulo p , then the quartic congruence (2.6) has no solution.*

Proof. By Theorem 2.2.16, the linear factor means that the quartic congruence (2.6) has a solution. While irreducible quartic and quadratic polynomials imply that the quartic congruence (2.6) is not solvable. Then we get the statements of the corollary. \square

2.2.3 Polynomial Equation Over \mathbb{Q}_p

As in the field \mathbb{R} of real numbers, the field \mathbb{Q}_p of p -adic numbers is not algebraically closed. For example, the simplest monomial equation $x^k = p$ for $k \geq 2$ does not have solution in \mathbb{Q}_p . Meanwhile, the quadratic equation $x^2 = -1$ has solution in \mathbb{Q}_p when $p \equiv 1 \pmod{4}$.

Let us discuss first some techniques for finding roots of polynomials in \mathbb{Q}_p : Newton polygon and Hensel's lemma. For more detailed informations, we refer to the books of Silverman (2007); Borevich & Shafarevich (1966) and S. Katok (2007).

Definition 2.2.18 (Newton polygon). *Let $g(x) = \sum_{i=1}^n a_i x^i$ be a polynomial with $a_i \in \mathbb{C}_p$. The Newton polygon of g is the convex hull of the set of points $\{(i, -\log_p |a_i|_p) : 0 \leq i \leq n\}$. By convention, we set $-\log_p |0|_p = \infty$. To construct Newton polygon, we take a vertical ray starting at the point $(0, -\log_p |a_0|_p)$ and aiming down the y -axis. Then rotate the ray counterclockwise, keeping the point $(0, -\log_p |a_0|_p)$ fixed, until it bends around all of the points $(i, -\log_p |a_i|_p)$.*

The following lemma is one of the results on Newton polygon.

Lemma 2.2.19 (see Silverman (2007), Theorem 5.11). *Let $g(x) = \sum_{i=1}^n a_i x^i$ be a polynomial with $a_i \in \mathbb{C}_p$. Suppose that the Newton polygon of g includes a line segment of slope m whose horizontal length is L , i.e., the Newton polygon has a line segment from*

$$(i, -\log_p |a_i|_p) \quad \text{to} \quad (i + L, -\log_p |a_{i+L}|_p)$$

whose slope is

$$m = \frac{-\log_p |a_{i+L}|_p + \log_p |a_i|_p}{L}.$$

Then g has exactly L roots \bar{x} (counting multiplicity), satisfying $|\bar{x}|_p = p^m$.

Next is the statement of Hensel's lemma.

Lemma 2.2.20 (see Borevich & Shafarevich (1966), Theorem 3). *Let g be a polynomial whose coefficients are p -adic integers. Let θ be a p -adic integer such that for some $i \geq 0$,*

$$g(\theta) \equiv 0 \pmod{p^{2i+1}}, \quad g'(\theta) \equiv 0 \pmod{p^i}, \quad g'(\theta) \not\equiv 0 \pmod{p^{i+1}}.$$

Then g has a unique p -adic integer root \bar{x} which satisfies $\bar{x} \equiv \theta \pmod{p^{i+1}}$.

Now we state some results on lower degree (quadratic and cubic) polynomial equations. The following results are the solvability conditions of the quadratic equation over \mathbb{Q}_p .

Theorem 2.2.21 (see Mukhamedov et al. (2014)). *Let $d = p^{\text{ord}_p(d)} d^*$ such that $d^* = d_0 + d_1 p + \dots$. Then the quadratic equation $x^2 = d$ has solution in \mathbb{Q}_p if and only if $2 \mid \text{ord}_p(d)$ and*

$$(i) \quad \left(\frac{d_0}{p}\right) = 1 \text{ when } p \neq 2.$$

$$(ii) \quad d^* \equiv 1 \pmod{8} \text{ when } p = 2.$$

Theorem 2.2.22 (Saburov & Ahmad (2015c)). *The quadratic equation $x^2 + ax + b = 0$ has solutions in \mathbb{Q}_p if and only if either one of the following conditions holds:*

1. $|a|_p^2 < |b|_p, \quad \sqrt{b} - \exists;$
2. $|a|_p^2 = |b|_p, \quad \sqrt{D} - \exists;$
3. $|a|_p^2 > |b|_p.$

Next we mention some results on cubic equations over \mathbb{Q}_p . Let $a, b \in \mathbb{Q}_p$ be two nonzero p -adic numbers with $a = \frac{a^*}{|a|_p}, b = \frac{b^*}{|b|_p}$ where $a^*, b^* \in \mathbb{Z}_p^*$ with $a^* = a_0 + a_1 \cdot p + a_2 \cdot p^2 + \dots$ and $b^* = b_0 + b_1 \cdot p + b_2 \cdot p^2 + \dots$.

We set $D_0 = -4a_0^3 - 27b_0^2$ and $u_{n+3} = b_0u_n - a_0u_{n+1}$ with $u_1 = 0$, $u_2 = -a_0$, and $u_3 = b_0$ for $n = \overline{1, p-3}$. Then we have the following result on the depressed cubic equation over \mathbb{Q}_p .

Theorem 2.2.23 (Mukhamedov et al. (2014)). *Let $p > 3$ be a prime. Then the depressed cubic equation $x^3 + ax + b = 0$ has solutions in \mathbb{Q}_p if and only if one of the following conditions holds.*

1. $|a|_p^3 < |b|_p^2$, $3 \mid \log_p |b|_p$ and $b_0^{\frac{p-1}{(3, p-1)}} \equiv 1 \pmod{p}$;
2. $|a|_p^3 = |b|_p^2$ and $D_0u_{p-2}^2 \not\equiv 9a_0^2 \pmod{p}$;
3. $|a|_p^3 > |b|_p^2$.

Moreover, Saburov & Ahmad (2018) provide the local descriptions of the roots of the cubic equation over \mathbb{Q}_p for $p \geq 5$. By their study, we could determine the location (p -adic norm) of all roots of a given cubic equation. On the other hand, the cubic equation over \mathbb{Q}_3 and over \mathbb{Q}_2 are study by Saburov & Ahmad (2015a,d) and Saburov & Ahmad (2014) respectively. Recall from Mukhamedov & Saburov (2013) that there exists \sqrt{B} (resp. $\sqrt[3]{B}$) if and only if $B_0^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ (resp. $B_0^{\frac{p-1}{(3, p-1)}} \equiv 1 \pmod{p}$) and $\log_p |B|_p$ is divisible by 2 (resp. by 3). We shall use the notation $\sqrt{B} - \exists$ (resp. $\sqrt[3]{B} - \exists$) whenever there exists \sqrt{B} (resp. $\sqrt[3]{B}$). We set $D_0 \equiv -4A_0^3 - 27B_0^2 \pmod{p}$ and $u_{n+3} = B_0u_n - A_0u_{n+1}$ with $u_1 = 0$, $u_2 = -A_0$, and $u_3 = B_0$ for $n \geq 1$.

We define the set $\Phi = \Phi_1 \cup \Phi_2 \cup \Phi_3$ (the set Φ is the solvability region of the cubic equation over \mathbb{Q}_p) where

$$\begin{aligned}\Phi_1 &= \{(A, B) \in \mathbb{Q}_p \times \mathbb{Q}_p : |A|_p^3 < |B|_p^2, \sqrt[3]{B} - \exists\} \\ \Phi_2 &= \{(A, B) \in \mathbb{Q}_p \times \mathbb{Q}_p : |A|_p^3 = |B|_p^2, D_0u_{p-2}^2 \not\equiv 9A_0^2 \pmod{p}\} \\ \Phi_3 &= \{(A, B) \in \mathbb{Q}_p \times \mathbb{Q}_p : |A|_p^3 > |B|_p^2\}\end{aligned}$$

We introduce some notations. Let $\delta_1 = b^2 - 4ac$, $\delta_2 = a^2 - 4b$, $\delta_3 = -2a^3 - 27c$, $A = \frac{3b-a^2}{3}$, $B = \frac{-2a^3+9ab-27c}{27}$, and $\Delta = a^2b^2 - 4a^3c - 4b^3 - 27c^2 + 18abc = -4A^3 - 27B^2$. Let also $D = -4(A|A|_p)^3 - 27(B|B|_p)^2$. Then the following theorem extends the Theorem 2.2.23.

Theorem 2.2.24 (Ahmad (2016); Saburov & Ahmad (2017b)). *Let $p > 3$ be a prime. Then the cubic equation $x^3 + ax^2 + bx + c = 0$ has solutions in \mathbb{Q}_p if and only if one of the following conditions holds:*

- A. 1. $\sqrt{|b|_p} < |a|_p$ and $\sqrt[3]{|c|_p} < |a|_p$;
2. $|a|_p < \sqrt{|b|_p}$ and $\sqrt[3]{|c|_p} < \sqrt{|b|_p}$;
3. $|a|_p < \sqrt[3]{|c|_p}$, $\sqrt{|b|_p} < \sqrt[3]{|c|_p}$ and $\sqrt[3]{-c} - \exists$.

- B. 4. $|a|_p < \sqrt{|b|_p} = \sqrt[3]{|c|_p}$ and $D_0 u_{p-2}^2 \not\equiv 9b_0^2 \pmod{p}$;
 5. $\sqrt{|b|_p} < |a|_p = \sqrt[3]{|c|_p}$ and
 i. $|\delta_3|_p < |a|_p^3 = |c|_p$ or
 ii. $|\delta_3|_p = |a|_p^3 = |c|_p$ and $D_0 u_{p-2}^2 \not\equiv a_0^4 \pmod{p}$;
 6. $\sqrt[3]{|c|_p} < |a|_p = \sqrt{|b|_p}$.
- C. 7. $|a|_p = \sqrt{|b|_p} = \sqrt[3]{|c|_p}$ and $(A, B) \in \Phi$.

2.3 DYNAMICAL SYSTEMS

For introduction to dynamical systems, we recommend the books of A. Katok & Hasselblatt (1995) and Walters (1982). For more informations on p -adic dynamical systems and its applications, we refer to the books of Khrennikov (1997); Anashin & Khrennikov (2009) and Silverman (2007). Let X be a space, G be a semigroup and $T_t : X \rightarrow X$ ($t \in G$) be a family of transformation such that $T_{s+t} = T_s \circ T_t$. Then $\Gamma := \{T_t\}_{t \in G}$ defines a semigroup action of G on X . Then (X, Γ) is called a dynamical system. We will focus on discrete dynamical system (X, T) where the semigroup G is generated by a single transformation $T : X \rightarrow X$.

2.3.1 p -Adic Dynamics

We will use the standard terminology of dynamical system, see Khrennikov (1997). Let us consider the dynamical system (B, f) in \mathbb{Q}_p , where $f : B \rightarrow B$ is an analytic function and $B = \mathbb{B}_r(a)$ or \mathbb{Q}_p . Denote $x_n = f^n(x_0)$, where $x_0 \in B$ and $f^n(x)$ is the n th composition of f . A point $x^{(0)}$ is called fixed point if $f(x^{(0)}) = x^{(0)}$. If $f^n(x^{(0)}) = x^{(0)}$, then $x^{(0)}$ is called a periodic point. If n is the smallest natural number with this property then n is said to be the period of $x^{(0)}$. We denote the corresponding cycle by $\gamma = (x^{(0)}, x^{(1)}, \dots, x^{(n-1)})$. A fixed point $x^{(0)}$ is called an attractor if there exists a neighborhood $N_{x^{(0)}}$ of $x^{(0)}$ such that all points $x \in N_{x^{(0)}}$ are attracted to $x^{(0)}$. Its attracting basin is the set $A(x^{(0)}) = \{x \in \mathbb{Q}_p : \lim_{n \rightarrow \infty} f^n(x) = x^{(0)}\}$. A fixed point $x^{(0)}$ is called an repeller if there exists a neighborhood $N_{x^{(0)}}$ of $x^{(0)}$ such that $|f(x) - x^{(0)}|_p > |x - x^{(0)}|_p$ for $x^{(0)} \neq x \in N_{x^{(0)}}$. A cycle $\gamma = (x^{(0)}, x^{(1)}, \dots, x^{(n-1)})$ is said to be an attractor (repeller) if $x^{(0)}$ is attractor (repeller) of the map f^n .

Next is the analogue version of non-Archimedean Siegel disk. The ball $\mathbb{B}_r(x^{(0)})$ is said to be a Siegel disk where $x^{(0)}$ is a fixed point if each sphere $\mathbb{S}_\rho(x^{(0)})$ ($\rho < r$) is an invariant sphere of f . The union of all Siegel disks with center in $x^{(0)}$ is said to be a maximal Siegel disk. In the same way we define a Siegel disk with center at a periodic point $x^{(0)}$ with the corresponding cycle $\gamma = (x^{(0)}, x^{(1)}, \dots, x^{(n-1)})$ of the period n . Here

the spheres $\mathbb{S}_\rho(x^{(0)})$ ($\rho < r$), are invariant spheres of the map f^n .

Since the function $f : B \rightarrow \mathbb{Q}_p$ is analytic, then we can find attractors, repellers, and Siegel disks using properties of the derivative of f . Let $x^{(0)}$ be a periodic point of f . Set $\lambda = (f^n)'(x^{(0)})$. The point $x^{(0)}$ is called attracting if $|\lambda|_p < 1$, repelling if $|\lambda|_p > 1$ and indifferent if $|\lambda|_p = 1$.

Lemma 2.3.1 (Khrennikov (1997), page 288, Lemma 2.1). *Let $f \in \mathbb{Q}_p(x)$ and $a \in \mathbb{Q}_p$. If there exists $r > 0$ such that for all $k \geq 2$*

$$\left| \frac{f^{(k)}(a)}{k!} \right|_p r^{k-1} < |f'(a)|_p,$$

then for all $x, y \in \overline{\mathbb{B}_r(a)} := \{x : |x - a|_p \leq r\}$,

$$|f(x) - f(y)|_p = |f'(a)|_p |x - y|_p.$$

Note that a compact open set in \mathbb{Q}_p is a finite union of balls. The following is the definition of local scaling on a compact open set $X \subset \mathbb{Q}_p$ (Kingsbery et al., 2009).

Definition 2.3.2. *Let $X = \bigcup_{i=1}^n \mathbb{B}_{r_i}(a_i) \subset \mathbb{Q}_p$ be a compact open set. We say that a mapping $f : X \rightarrow X$ is locally scaling for $r_i \in \{p^n, n \in \mathbb{Z}\}$ if there exists a function $S : X \rightarrow \mathbb{R}_{\geq 0}$ such that for any $x, y \in \mathbb{B}_{r_i}(a_i)$, $S(x) = S(y) = S(a_i)$ and*

$$|f(x) - f(y)|_p = S(a_i) |x - y|_p.$$

The function S is called a scaling function. The following lemma will be useful.

Lemma 2.3.3 (Kingsbery et al. (2009), Theorem 5.1). *Let $X = \bigcup_{i=1}^n \mathbb{B}_{r_i}(a_i) \subset \mathbb{Q}_p$ be a compact open and $f : X \rightarrow X$ be a scaling for $r_i \in \{p^n, n \in \mathbb{Z}\}$. Let $S : X \rightarrow \mathbb{R}_{\geq 0}$ be the correspondence scaling function. Then for all $r' \leq r_i$, the restricted map*

$$f : \mathbb{B}_{r'}(a_i) \rightarrow \mathbb{B}_{r'S(a_i)}(f(a_i))$$

is a bijection.

The following result is a direct consequence of Lemma 2.3.3.

Corollary 2.3.4. *Keep the same assumption as in Lemma 2.3.3, we have for all $r' \leq r$, the restricted map*

$$f : \mathbb{S}_{r'}(a_i) \rightarrow \mathbb{S}_{r'S(a_i)}(f(a_i))$$

is a bijection.

Proof. Recall $\mathbb{S}_{r'}(a_i) = \{x \in \mathbb{Q}_p : |x - a_i|_p = r'\}$. Then $\mathbb{S}_{r'}(a_i) = \mathbb{B}_{pr'}(a_i) \setminus \mathbb{B}_{r'}(a_i)$ is a difference of consecutive balls. By Lemma 2.3.3, f are bijections from balls to balls. Thus when restricted to the spheres, f is also bijective. \square

The two most fundamental objects in the complex dynamics are the Fatou and Julia sets associated to a rational map ϕ . This is also the case for p -adic dynamics. As refer to (Benedetto, 2002), let X and Y be metric spaces and F be a family of maps from X to Y . Then F is equicontinuous if there exists a positive constant C such that for any $f \in F$ and $x, y \in X$,

$$d_Y(f(x), f(y)) \leq C d_X(x, y).$$

Then for $\phi \in \mathbb{C}_p(z)$ a rational function, the Fatou set $\mathcal{F} = \mathcal{F}_\phi$ of ϕ is defined as the set of all $x \in \mathbb{P}^1(\mathbb{C}_p)$ having a neighbourhood on which the family of iterates $\{\phi^n\}_{n \geq 0}$ is equicontinuous. Meanwhile, the Julia set $\mathcal{J} = \mathcal{J}_\phi$ is defined as $\mathbb{P}^1(\mathbb{C}_p) \setminus \mathcal{F}$. The notion of neighbourhood and equicontinuity are defined by the spherical metric on $\mathbb{P}^1(\mathbb{C}_p)$.

The Fatou set is open and the Julia set is closed under this spherical metric. They are invariant under the iteration of ϕ , that is $\phi(\mathcal{F}) = \phi^{-1}(\mathcal{F}) = \mathcal{F}$ and $\phi(\mathcal{J}) = \phi^{-1}(\mathcal{J}) = \mathcal{J}$. Moreover, if $f \in \text{PGL}_2(\mathbb{C}_p)$ and $\omega = f^{-1} \circ \phi \circ f$, then $\mathcal{F}_\omega = f^{-1}(\mathcal{F}_\phi)$ and $\mathcal{J}_\omega = f^{-1}(\mathcal{J}_\phi)$. For more background on Fatou and Julia sets in complex theory, one could consult to Wilnor (2006). The following result mention about a periodic point and the Fatou-Julia set.

Proposition 2.3.5 (Benedetto (2001)). *Let $\phi \in \mathbb{C}_p(z)$ be a rational function with Fatou set \mathcal{F} and Julia set \mathcal{J} . Let $a \in \mathbb{P}^1(\mathbb{C}_p)$ be a periodic point of ϕ . Then $a \in \mathcal{J}$ if and only if a is repelling.*

Proposition 2.3.6 (Benedetto (2001)). *Let $\phi \in \mathbb{C}_p(z)$ be a rational function. Then the Fatou set \mathcal{F} is non-empty.*

The following is the analogue components of the Fatou set in non-Archimedean (p -adic) dynamics.

Definition 2.3.7 (Benedetto (2000)). *Let X be a topological space with a set \mathcal{D} of distinguished subsets. Let $U \subset X$ be an open subset. For any $x \in U$ we define the D -component of U containing x to be the set of all $y \in U$ with the following property: there exists a finite sequence of distinguished subsets*

$$D_1, \dots, D_n \in \mathcal{D}$$

with $x \in D_1$ and $y \in D_n$ such that for any $i = 1, \dots, n-1$,

$$D_i \cap D_{i+1} \neq \emptyset.$$

Definition 2.3.8 (Benedetto (2000)). A closed $\mathbb{P}^1(\mathbb{C}_p)$ -disk is a closed ball $\overline{\mathbb{B}_r(a)}$, or the complement of an open ball $\mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{B}_r(a)$ for some $a \in \mathbb{C}_p$ and $r > 0$. Similarly, an open $\mathbb{P}^1(\mathbb{C}_p)$ -disk is an open ball $\mathbb{B}_r(a)$, or the complement of a closed ball $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{\mathbb{B}_r(a)}$ for some $a \in \mathbb{C}_p$ and $r > 0$.

If X is a metric space, we set \mathcal{D} to be the set of all disks with positive radius in X . By this definition, one could verify that the \mathcal{D} -component of an open subset U of the field \mathbb{C} of complex numbers are precisely the connected components of U . If $X = \mathbb{C}_p$, then the \mathcal{D} -component of an open subset U containing a given point $x \in U$ is the largest ball $\mathbb{B}_r(x)$ or $\overline{\mathbb{B}_r(x)}$ centered at x and subset of U . Meanwhile, for the case $X = \mathbb{P}^1(\mathbb{C}_p)$, we choose \mathcal{D} to be the set of all $\mathbb{P}^1(\mathbb{C}_p)$ -disk and we have the following result.

Proposition 2.3.9 (Benedetto (2001)). Let U be an open subset $\mathbb{P}^1(\mathbb{C}_p)$ and $x \in U$. If $U = \mathbb{P}^1(\mathbb{C}_p) \setminus \{x\}$ is the complement of a single point of $\mathbb{P}^1(\mathbb{C}_p)$, then the \mathcal{D} -component of U containing x is U . Otherwise, the \mathcal{D} -component is the largest $\mathbb{P}^1(\mathbb{C}_p)$ -disk containing x and contained in U .

Let X be a metric space and $\phi : X \rightarrow X$. A point $x \in X$ is recurrent if x is contained in the closure of its orbit $\{\phi^n(x) : n \geq 1\}$.

Next, we consider a rational function $\phi(z) = \frac{f(z)}{g(z)}$. Its derivative

$$\phi'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}.$$

is defined for all a such that $a \neq \infty$ and $g(a) \neq 0$. The Taylor's expansion around a is given as follows

$$\phi(z) = \phi(a) + \phi'(a)(z-a) + O((z-a)^2).$$

The function $\phi : \mathbb{C}_p \rightarrow \mathbb{C}_p$ is ramified at a if $\phi'(a) = 0$ where a is called a critical point or ramification point. For $a \neq \infty$ and $g(a) \neq 0$, the ramification index of ϕ at a is denoted by $\deg_\phi(a)$ such that there is constant A satisfies

$$\phi(z) = \phi(a) + A(z-a)^{\deg_\phi(a)} + O((z-a)^{\deg_\phi(a)+1}).$$

The number $\deg_\phi(a)$ is greater than 2 when a is a critical point, i.e. the function ϕ is ramified at a .

For $a = \infty$ and $g(a) = 0$, in order to find their ramification index, we need to make a linear change of variables: Let $f \in \text{PGL}_2(\mathbb{C}_p)$ be a linear fractional transformation, $\phi^f = f^{-1} \circ \phi \circ f$ and $b = f^{-1}(a)$. Then if $b \neq \infty$ and $\phi^f(b) \neq \infty$ we define the ramification index of ϕ at a to be

$$\deg_\phi(a) = \deg_{\phi^f}(b).$$

Let $\phi \in \mathbb{C}_p(z)$ be a rational function. A critical point $a \in \mathbb{P}^1(\mathbb{C}_p)$ of ϕ is wild if $p \mid \deg_\phi(a)$. Then we have the following “No Wandering Domains” results.

Theorem 2.3.10 (Benedetto (2000)). *Let K be a finite extension of \mathbb{Q}_p and $\phi \in K(z)$ have no recurrent wild critical points in its Julia set. Then the Fatou set of ϕ has no wandering D -component and has only finitely many periodic D -components which contain points in K .*

Now, we consider one-parameter family (ϕ, c) of rational maps with marked critical point. We say that (ϕ, c) has a Misiurewicz bifurcation at $t = t_0$ if the following conditions are satisfied (see Rivera-Letelier (2005)).

(M1) There is an integer $\ell \geq 1$ such that $\phi_{t_0}^\ell(c_{t_0})$ is a repelling fixed point of ϕ_{t_0} .

(M2) c_{t_0} is unramified for ϕ_{t_0} .

(M3) $\phi_t^\ell(c_t)$ is not a fixed point of ϕ_t for some t .

(M4) $\deg_{\phi_t^\ell}(c_t) = \deg_{\phi_{t_0}^\ell}(c_{t_0})$ for every t close to t_0 .

The following theorem tells us on the existence of a rational map with the (wild) recurrent critical point.

Theorem 2.3.11 (Rivera-Letelier (2005)). *In every one-parameter family of rational maps that is defined over a finite extension of \mathbb{Q}_p and has a Misiurewicz bifurcation, there is a rational map whose coefficients are algebraic over \mathbb{Q}_p and has non-periodic recurrent critical point.*

2.3.2 p -Adic Weak Repeller

We keep the same notation as in (A. H. Fan et al., 2007) and (S. L. Fan & Liao, 2018). Let $f : X \rightarrow \mathbb{Q}_p$ be a mapping from a compact open set $X \subset \mathbb{Q}_p$ into \mathbb{Q}_p . We assume that (i) $f^{-1}(X) \subset X$ and (ii) $X = \bigcup_{j \in I} \mathbb{B}_{p^{-\tau}}(a_j)$ can be written as a finite disjoint union of balls of centers a_j and of the same radius $p^{-\tau}$, $\tau \in \mathbb{Z}$ such that for each $j \in I$ there is an integer $\tau_j \in \mathbb{Z}$ such that for any $x, y \in \mathbb{B}_{p^{-\tau}}(a_j)$

$$|f(x) - f(y)|_p = p^{\tau_j} |x - y|_p. \quad (2.8)$$

For such a map f , we define its filled Julia set by

$$K_f = \bigcap_{n=0}^{\infty} f^{-n}(X).$$

It is clear that $f^{-1}(K_f) = K_f$ and $f(K_f) \subset K_f$.

The triple (X, K_f, f) is called a *p-adic weak repeller* if all τ_j in (2.8) are non-negative, but at least one is positive. It is called a *p-adic repeller* if all τ_j in (2.8) are positive. For any $i \in I$, let

$$I_i := \{j \in I : \mathbb{B}_r(a_j) \cap f(\mathbb{B}_r(a_i)) \neq \emptyset\} = \{j \in I : \mathbb{B}_r(a_j) \subset f(\mathbb{B}_r(a_i))\}$$

We then define a so-called *incidence matrix* $A = (a_{ij})_{I \times I}$ as follows

$$a_{ij} = \begin{cases} 1 & \text{if } j \in I_i, \\ 0 & \text{if } j \notin I_i. \end{cases}$$

If A is irreducible, we say that (X, K_f, f) is *transitive*. For an index set I and an irreducible incidence matrix A given above, we denote by

$$\Sigma_A = \{(x_k)_{k \geq 0} : x_k \in I, A_{x_k, x_{k+1}} = 1, k \geq 0\}$$

the subshift space. We equip Σ_A with a metric d_f depending on the dynamics which is defined as follows. For $i, j \in I$ and $i \neq j$, let $\kappa(i, j)$ be the integer such that $|a_i - a_j|_p = p^{-\kappa(i, j)}$. It is clear that $\kappa(i, j) < \tau$. By the ultrametric inequality, we have

$$|x - y|_p = |a_i - a_j|_p, i \neq j, \forall x \in \mathbb{B}_r(a_i), \forall y \in \mathbb{B}_r(a_j).$$

For $x = (x_0, x_1, \dots, x_n, \dots) \in \Sigma_A$ and $y = (y_0, y_1, \dots, y_n, \dots) \in \Sigma_A$, we define

$$d_f(x, y) = \begin{cases} p^{-\tau_{x_0} - \tau_{x_1} - \dots - \tau_{x_{n-1}} - \kappa(x_n, y_n)} & \text{if } n \neq 0, \\ p^{-\kappa(x_0, y_0)} & \text{if } n = 0 \end{cases}$$

where $n = n(x, y) = \min\{k \geq 0 : x_k \neq y_k\}$. It is clear that d_f defines the same topology as the classical metric defined by $d(x, y) = p^{-n(x, y)}$. Let σ be the (left) shift transformation on Σ_A . Then (Σ_A, σ, d_f) becomes a dynamical system.

Theorem 2.3.12 (A. H. Fan et al. (2007)). *Let (X, K_f, f) be a transitive p-adic weak repeller with incidence matrix A . Then the dynamics $(K_f, f, |\cdot|_p)$ is isometrically conjugate to the shift dynamics (Σ_A, σ, d_f) .*

Next, we generalized the concept of *p-adic repeller*. Let $f : X \rightarrow \mathbb{Q}_p$ be a mapping from a compact open set $X = \bigcup_{j \in I} \mathbb{B}_{p^{-\tau_j}}(a_j) \subset \mathbb{Q}_p$ into \mathbb{Q}_p such that $X \subset f(X)$ and for each $j \in I$ there is an integer $\alpha_j \in \mathbb{Z}$ such that for any $x, y \in \mathbb{B}_{p^{-\tau_j}}(a_j)$

$$|f(x) - f(y)|_p = p^{\alpha_j} |x - y|_p.$$

We simply denote the ball $\mathbb{B}_{p^{-\tau_j}}(a_j)$ as \mathbb{B}_j . The triple (X, K_f, f) is called a

generalized p -adic weak repeller if (i) for each $i \in I$, $f(\mathbb{B}_i)$ contains at least some \mathbb{B}_j and (ii) there exists at least one $i \in I$ such that $\mathbb{B}_j \subsetneq f(\mathbb{B}_i)$. We define the incidence matrix A similar as before. The generalized p -adic repeller is transitive if A is transitive.

Theorem 2.3.13 (S. L. Fan & Liao (2018)). *Let (X, K_f, f) be a transitive generalized p -adic weak repeller with incidence matrix A . Then the dynamics $(K_f, f, |\cdot|_p)$ is topologically conjugate to the shift dynamics (Σ_A, σ, d_f) .*

The shift dynamics (Σ_A, σ) is called a subshift of finite type determined by matrix A . When A is a $n \times n$ matrix whose coefficients are all equal to 1, (Σ_A, σ) is called a full shift over an alphabet of n symbols, and is usually denoted by (Σ_n, σ) . For the detailed on shift dynamics, we recommend the book of Lind & Marcus (1995).

We use the following definition of chaos by Devaney (1989). Let us introduce some terminologies. *Transitive*. For all non-empty open subsets U and V of X there exists a natural number k such that $f^k(U) \cap V$ is non-empty. *Sensitive dependence on initial conditions*. If there is a positive real number δ (a sensitivity constant) such that for every point $x \in X$ and every neighborhood $N_\epsilon(x)$ of x there exists a point $y \in N_\epsilon(x)$ and a nonnegative integer n such that the distance $d(f^n(x), f^n(y)) \geq \delta$.

Definition 2.3.14. *Let X be a metric space. A continuous map $f : X \rightarrow X$ is said to be chaotic on X if*

- (i) f is transitive,
- (ii) the periodic points of f are dense in X ,
- (iii) f has sensitive dependence on initial conditions.

By this definition, it is well-known that the shift dynamics (subshift of finite type) is chaotic.

2.3.3 Reduction Modulo p of a Rational Map

Most of the materials here are taken from (A. H. Fan et al., 2017). Let $\bar{\cdot}$ denote the reduction modulo p from \mathbb{Z}_p to $\mathbb{Z}/p\mathbb{Z}$ such that $a \mapsto \bar{a}$ with $a \equiv \bar{a} \pmod{p}$. For a polynomial $f \in \mathbb{Z}_p[x]$, the reduction of f modulo p is defined as

$$\bar{f}(x) = \sum_{i=0}^n \bar{a}_i x^i.$$

Notice that a rational map ϕ can be written as quotient of two polynomials $f, g \in \mathbb{Z}_p[x]$ having no common factors such that at least one coefficient of f or g has p -adic absolute value 1. Denote by $\deg(\phi)$ the degree of the rational map, the maximum degree of

its denominator and numerator without common factors. The reduction of ϕ modulo p is the rational map (of degree at most $\deg(\phi)$)

$$\bar{\phi}(x) = \frac{\bar{f}(x)}{\bar{g}(x)}$$

obtain by cancelling common factors in the reductions $\bar{f}(x)$ and $\bar{g}(x)$. If $\deg \bar{\phi} = \deg(\phi)$, we say ϕ has good reduction. If $\deg \bar{\phi} < \deg(\phi)$, we say ϕ has bad reduction.

Recall that a dynamical system (X, T) is minimal if for all $x \in X$, the orbit set $\mathcal{O}_T(x) := \{T^n(x) : n \in \mathbb{N}\}$ is dense in X . The set X is then called a minimal set.

Theorem 2.3.15 (A. H. Fan et al. (2017)). *Let $\phi \in \mathbb{Q}_p(x)$ be a rational map of $\deg(\phi) \geq 2$ with good reduction. Then we have*

$$\mathbb{P}^1(\mathbb{Q}_p) = A \cup B \cup C$$

where A is the finite set consisting of all periodic points of ϕ , $B = \bigcup_j B_j$ is the union of all (at most countably many) clopen invariant sets such that each B_j is a finite union of balls and each subsystem $\phi : B_j \rightarrow B_j$ is minimal and points in C lie in the attracting basin of a periodic orbit or of a minimal subsystem. Moreover, the length of a periodic orbit has one of the following forms:

$$k \quad \text{or} \quad k\ell \quad \text{if} \quad p \geq 5$$

$$k \quad \text{or} \quad k\ell \quad \text{or} \quad kp \quad \text{if} \quad p = 2 \quad \text{or} \quad 3$$

where $1 \leq k \leq p+1$ and $\ell \mid (p-1)$.

2.4 CAYLEY TREES

We remind that for a concise analysis between Cayley trees and Bethe lattices, we consult the article of Ostilli (2012). But in our terminology, both of them are the same and we prefer to use the term of Cayley tree.

Let $\Gamma_k = (V, L)$ be a Cayley tree of order k where $k \geq 1$ with the root x^0 , i.e. each vertex has exactly $k+1$ edges. Let $\Gamma_k^+ = (V, L)$ be a semi-infinite Cayley tree of order k where $k \geq 1$ with the root x^0 , i.e. each vertex has exactly $k+1$ edges except for the root x^0 which has k edges. Here V is the set of vertices and L is the set of edges. The vertices x and y are called nearest neighbors if there exists an edge $l \in L$ connecting them. This edge is also denoted by $l = \langle x, y \rangle$. A collection of the pairs $\langle x, x_1 \rangle, \dots, \langle x_{d-1}, y \rangle$ is called a path between vertices x and y . The distance $d(x, y)$ between $x, y \in V$ on the Cayley tree Γ_k or semi-infinite Cayley tree Γ_k^+ is the length of

the shortest path between x and y . Let

$$W_n = \{x \in V : d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^n W_m \quad \text{and} \quad L_n = \{\langle x, y \rangle \in L : x, y \in V_n\}.$$

The set of direct successors of x is defined by

$$\forall x \in W_n, \quad S(x) = \{y \in W_{n+1} : d(x, y) = 1\}.$$

We now introduce a coordinate structure in the Cayley tree Γ_k and semi-infinite Cayley tree Γ_k^+ . We begin with the Cayley tree Γ_k . Let G_k be a free product of $k + 1$ cyclic group of the order 2 with generator $a_1, a_2, \dots, a_k, a_{k+1}$.

Proposition 2.4.1 (see Rozikov (2013)). *There exist a one-to-one correspondence between the set of vertices V of the Cayley tree Γ and the group G_k .*

Proof. Fix an arbitrary element $x^0 \in V$. Correspond it to the identity element e of the group G_k . Without loss of generality, we assume the Cayley tree is a planar graph. Label the neighbours of e by a_1, \dots, a_{k+1} counter-clockwise. Now we label the neighbours of each a_i for $i = 1, \dots, k + 1$. All a_i have the common neighbour e . So we have $a_i a_i = a_i^2 = e$ and the other neighbours are labelled $a_i a_j$ for $j = 1, \dots, k + 1$ starting $a_i a_i = e$ counter-clockwise. Next we label the neighbours of each $a_i a_j$ by $a_i a_j a_\ell$ for $\ell = 1, \dots, k + 1$ starting by $a_i a_j a_i = a_i$. Iterating this argument, we get a one-to-one correspondence between the set V of the Cayley tree Γ_k and the group G_k . \square

The group representation above is called right representation. In the same manner, one could define the left representation. Now we define the (left) group translation $T_g : V \rightarrow V$ for $g \in V$ on the Cayley tree Γ_k as follows:

$$T_g(x) = gx, \quad \forall x \in V.$$

Next we consider the semi-infinite Cayley tree Γ_k^+ . Every vertex $x \neq x^0$ has the coordinate (i_1, \dots, i_n) where $i_m \in \{1, \dots, k\}$, $1 \leq m \leq n$ and the vertex x^0 has the coordinate (\emptyset) . More precisely, the symbol (\emptyset) constitutes level 0 and the coordinates (i_1, \dots, i_n) form level n of V (from the root x^0). In this case, for any $x \in V$, $x = (i_1, \dots, i_n)$, we have

$$S(x) = \{(x, i) : 1 \leq i \leq k\}$$

where (x, i) means (i_1, \dots, i_n, i) . Let us define a binary operation $\circ : V \times V \rightarrow V$ as follows: for any two elements $x = (i_1, \dots, i_n)$ and $y = (j_1, \dots, j_m)$, we define

$$x \circ y = (i_1, \dots, i_n) \circ (j_1, \dots, j_m) = (i_1, \dots, i_n, j_1, \dots, j_m)$$

and

$$y \circ x = (j_1, \dots, j_m) \circ (i_1, \dots, i_n) = (j_1, \dots, j_m, i_1, \dots, i_n).$$

Then (V, \circ) is a noncommutative semigroup with the unit $x^0 = (\emptyset)$. In a similar manner to the Cayley tree case, we define a translation $\tau_g : V \rightarrow V$ for $g \in V$ as follows:

$$\tau_g(x) = g \circ x \quad \forall x \in V.$$

On a semi-infinite Cayley tree Γ_k^+ , we are interested in periodic functions.

Definition 2.4.2. *Let $G \subset V$ be a sub-semigroup of V and $h : V \rightarrow Y$ be a Y -valued function. A function h is said to be G -periodic if $h(\tau_g(x)) = h(x)$ for all $g \in G$ and $x \in V$. A V -periodic function is called translation-invariant.*

Remark 2.4.3. *The periodicity defined for semi-infinite Cayley tree is different to that of Rozikov (2013) where the Cayley tree Γ^k was considered. In fact, for Cayley tree, there is a group representation and the periodicity was defined by using normal subgroup of this representation. It is known that a tree has a group representation if and only if it is a Cayley tree Rozikov (2002). Thus, we could not have a group representation of the semi-infinite Cayley tree. Moreover, we would like to emphasize that the multiplication of the group of the Cayley tree and the binary operation of the corresponding semi-infinite Cayley tree are different. The former involves cancellation while the latter does not.*

2.5 P -ADIC PROBABILITY MEASURES

For more detailed information about p -adic measures, we recommend the books of Khrennikov et al. (1999) and Khrennikov (2009).

Let (X, \mathcal{B}) be a measurable space where \mathcal{B} is an algebra of subsets of X . A function $\mu : \mathcal{B} \rightarrow \mathbb{Q}_p$ is said to be a p -adic measure if for any $A_1, \dots, A_n \in \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ ($i \neq j$) the following equality holds:

$$\mu\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu(A_j).$$

A p -adic measure is called a probability measure if $\mu(X) = 1$. One of the important feature of p -adic probability, invented by Monna & Springer (1963), is boundedness. A p -adic probability is bounded if $\sup\{|\mu(A)|_p < \infty : A \in \mathcal{B}\}$. We are interested in this important class of p -adic measures in which the boundedness condition itself provides a fruitful integration theory, see for example Khrennikov (2007). In general, a p -adic probability measure need not be bounded (Khrennikov, 1996b; Ludkovsky &

Khrennikov, 2003).

2.5.1 Potts Model Over \mathbb{Q}_p

The study of Potts model over \mathbb{Q}_p was initiated in several papers (Mukhamedov & Rozikov, 2004, 2005; Mukhamedov & Akin, 2013; Rozikov & Khakimov, 2015; Mukhamedov & Khakimov, 2016). Let $\Phi = \{1, 2, \dots, q\}$ be a finite set and V be the set of vertices of a semi-infinite Cayley tree Γ_k^+ . A configuration (resp. a finite-volume configuration, a boundary configuration) is a function $\sigma : V \rightarrow \Phi$ (resp. $\sigma_n : V_n \rightarrow \Phi$, $\sigma^{(n)} : W_n \rightarrow \Phi$). We denote by Ω (resp. Ω_{V_n} , Ω_{W_n}) the set of all configurations (resp. all finite-volume configurations, all boundary configurations). For given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\sigma^{(n)} \in \Omega_{W_n}$, we define their concatenation to be a finite-volume configuration $\sigma_{n-1} \vee \sigma^{(n)} \in \Omega_{V_n}$ such that

$$(\sigma_{n-1} \vee \sigma^{(n)})(v) = \begin{cases} \sigma_{n-1}(v) & \text{if } v \in V_{n-1}, \\ \sigma^{(n)}(v) & \text{if } v \in W_n. \end{cases}$$

The Hamiltonian of the q -state Potts model over \mathbb{Q}_p on the finite-volume configurations is defined for all $\sigma_n \in \Omega_{V_n}$ and $n \in \mathbb{N}$ as follows:

$$H_n(\sigma_n) = J \sum_{\langle x, y \rangle \in L_n} \delta_{\sigma_n(x)\sigma_n(y)}$$

where $J \in \mathbb{B}_{p^{-1/(p-1)}}(0)$ is a coupling constant, $\langle x, y \rangle$ stands for nearest neighbor vertices and δ is Kronecker's delta symbol.

2.5.2 p -Adic Gibbs Measures

Now we present the construction of the p -adic Gibbs measures of the q -state Potts model on the Cayley tree. Let $\tilde{\mathbf{h}} : V \rightarrow \mathbb{Q}_p^q$, be a \mathbb{Q}_p^q -valued function such that $\tilde{\mathbf{h}}(x) = (\tilde{h}_x^{(1)}, \dots, \tilde{h}_x^{(q)})$ for $x \in V$. For $n \in \mathbb{N}$, we define a p -adic measure $\mu_{\tilde{\mathbf{h}}}^{(n)} : \Omega_{V_n} \rightarrow \mathbb{Q}_p$ on Ω_{V_n} , associated to $\tilde{\mathbf{h}}$, as follows: $\forall \sigma_n \in \Omega_{V_n}$

$$\mu_{\tilde{\mathbf{h}}}^{(n)}(\sigma_n) = \frac{1}{\mathcal{Z}_{\tilde{\mathbf{h}}}^{(n)}} \exp_p \left\{ H_n(\sigma_n) + \sum_{x \in W_n} \tilde{h}_x^{(\sigma_n(x))} \right\}, \quad (2.9)$$

where $\exp_p(\cdot) : \mathbb{B}_{p^{-1/(p-1)}}(0) \rightarrow \mathbb{B}_1(1)$ is the p -adic exponential function and $\mathcal{Z}_{\tilde{\mathbf{h}}}^{(n)}$ is the partition function defined by

$$\mathcal{Z}_{\tilde{\mathbf{h}}}^{(n)} = \sum_{\sigma_n \in \Omega_{V_n}} \exp_p \left\{ H_n(\sigma_n) + \sum_{x \in W_n} \tilde{h}_x^{(\sigma_n(x))} \right\}.$$

Here precisely, for any $x \in \mathbb{B}_{p^{-1/(p-1)}}(0)$ we have $\exp_p(x) \in \mathcal{E}_p := \{x \in \mathbb{Q}_p : |x-1|_p < p^{-\frac{1}{p-1}}\}$. Throughout, we consider the function $\tilde{\mathbf{h}}$ satisfying $\tilde{\mathbf{h}}(x) \in (\mathbb{B}_{p^{-1/(p-1)}}(0))^q$ for any $x \in V$. The p -adic measures (2.9) are called *compatible* if for all $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $n \in \mathbb{N}$, one has

$$\sum_{\sigma^{(n)} \in \Omega_{W_n}} \mu_{\tilde{\mathbf{h}}}^{(n)}(\sigma_{n-1} \vee \sigma^{(n)}) = \mu_{\tilde{\mathbf{h}}}^{(n-1)}(\sigma_{n-1}). \quad (2.10)$$

Due to Kolmogorov's extension theorem for the p -adic measures (2.9) (see Ganikhodjaev et al. (1998); Ludkovsky & Khrennikov (2003)), there exists a unique p -adic measure $\mu_{\tilde{\mathbf{h}}} : \Omega \rightarrow \mathbb{Q}_p$ such that $\forall \sigma_n \in \Omega_{V_n}, \forall n \in \mathbb{N}$

$$\mu_{\tilde{\mathbf{h}}}(\{\sigma | \{V_n\} = \sigma_n\}) = \mu_{\tilde{\mathbf{h}}}^{(n)}(\sigma_n).$$

This uniquely extended measure $\mu_{\tilde{\mathbf{h}}} : \Omega \rightarrow \mathbb{Q}_p$ is called a p -adic Gibbs measure.

The problem of finding the p -adic Gibbs measure is then reduced to the problem of finding the function $\tilde{\mathbf{h}}$ that satisfy the compatibility condition (2.10). The following theorem describes the condition on the function $\tilde{\mathbf{h}} : V \rightarrow \mathbb{Q}_p^q$ for which the compatibility condition (2.10) is satisfied.

Theorem 2.5.1 (Mukhamedov & Rozikov (2005, 2004)). *Let $\tilde{\mathbf{h}} : V \rightarrow \mathbb{Q}_p^q$, $\tilde{\mathbf{h}}(x) = (\tilde{h}_x^{(1)}, \dots, \tilde{h}_x^{(q)})$ be a given function and $\mathbf{h} : V \rightarrow \mathbb{Q}_p^{q-1}$, $\mathbf{h}(x) = (h_x^{(1)}, \dots, h_x^{(q-1)})$ be a function defined as $h_x^{(i)} = \tilde{h}_x^{(i)} - \tilde{h}_x^{(q)}$ for all $i = 1, \dots, q-1$. Then the p -adic probability distributions $\{\mu_{\tilde{\mathbf{h}}}^{(n)}\}_{n \in \mathbb{N}}$ are compatible if and only if*

$$\mathbf{h}(x) = \sum_{y \in S(x)} \mathbf{F}(\mathbf{h}(y)), \quad \forall x \in V \setminus \{x^0\} \quad (2.11)$$

where $S(x)$ is the set of direct successors of x and the function $\mathbf{F} : \mathbb{Q}_p^{q-1} \rightarrow \mathbb{Q}_p^{q-1}$, $\mathbf{F}(\mathbf{h}) = (F_1, \dots, F_{q-1})$ for $\mathbf{h} = (h_1, \dots, h_{q-1})$ is defined as follows:

$$F_i = \ln_p \left(\frac{(\theta - 1) \exp_p(h_i) + \sum_{j=1}^{q-1} \exp_p(h_j) + 1}{\theta + \sum_{j=1}^{q-1} \exp_p(h_j)} \right), \quad \theta = \exp_p(J).$$

Definition 2.5.2. *Let (V, \circ) be the semigroup of the semi-infinite Cayley tree $\Gamma_+^k = (V, L)$. Let $G \subset V$ be a sub-semigroup. A p -adic Gibbs measure determined by a G -periodic function $\tilde{\mathbf{h}}$ is called G -periodic. A V -periodic p -adic Gibbs measure is called translation-invariant.*

Let us describe or find the translation-invariant p -adic Gibbs measures. Note that a p -adic Gibbs measure of the Potts model on the Cayley tree is translation-invariant if and only if the function $\tilde{\mathbf{h}} : \Gamma_k^+ \rightarrow \mathbb{Q}_p^q$ in (2.9) is constant, i.e. $\tilde{\mathbf{h}}(x) = \tilde{\mathbf{h}}$ for $x \in \Gamma_k^+$. Recall k is the order of the Cayley tree. Then the condition (2.11) of the Theorem 2.5.1

takes the form $\mathbf{h} = k\mathbf{F}(\mathbf{h})$ or equivalently

$$h_i = \ln_p \left(\frac{(\theta - 1) \exp_p(h_i) + \sum_{j=1}^{q-1} \exp_p(h_j) + 1}{\theta + \sum_{j=1}^{q-1} \exp_p(h_j)} \right)^k \quad \text{for } 1 \leq i \leq q-1.$$

Let $\mathbf{z} = (z_1, \dots, z_{q-1}) \in \mathbb{Q}_p^{q-1}$ such that $z_i = \exp_p(h_i)$ for any $1 \leq i \leq q-1$. Then the following theorem tells us how to find the translation-invariant p -adic Gibbs measures of the Potts model on the Cayley tree.

Theorem 2.5.3 (Mukhamedov & Rozikov (2005)). *Let $\tilde{\mathbf{h}} : V \rightarrow \mathbb{Q}_p^q$ be a function, $\tilde{\mathbf{h}}(x) = \tilde{\mathbf{h}} = (\tilde{h}_1, \dots, \tilde{h}_q)$ and $\mathbf{h} = (h_1, \dots, h_{q-1})$ such that $h_i = \tilde{h}_i - \tilde{h}_q$ for $1 \leq i \leq q-1$. Then there exists a translation-invariant p -adic Gibbs measure $\mu_{\tilde{\mathbf{h}}} : \Omega \rightarrow \mathbb{Q}_p$ associated with a function $\tilde{\mathbf{h}}$ if and only if $\mathbf{z} = (z_1, \dots, z_{q-1}) \in \mathcal{E}_p^{q-1}$ such that $z_i = \exp_p(h_i)$ is a solution of the following system of equations*

$$z_i = \left(\frac{(\theta - 1)z_i + \sum_{j=1}^{q-1} z_j + 1}{\theta + \sum_{j=1}^{q-1} z_j} \right)^k \quad \text{for } 1 \leq i \leq q-1. \quad (2.12)$$

By this theorem, in order to describe the translation-invariant p -adic Gibbs measures, we describe the possible form of the solution to the system of equations (2.12).

In what follows in which it can be found in (Saburov & Ahmad, 2015b), we explain the steps to describe the possible solution of the system of equations (2.12). Let $\mathbf{I}_{q-1} = \{1, \dots, q-1\}$ and δ_{ij} be Kronecker's delta symbol. For $j \in \mathbf{I}_{q-1}$, suppose $\mathbf{e}_j = (\delta_{1j}, \delta_{2j}, \dots, \delta_{q-1j}) \in \mathbb{Q}_p^{q-1}$ and for $\alpha \subset \mathbf{I}_{q-1}$, suppose $\mathbf{e}_\alpha = \sum_{j \in \alpha} \mathbf{e}_j$. For any $\mathbf{z} = (z_1, \dots, z_{q-1}) \in \mathbb{Q}_p^{q-1}$, assume $\{\alpha_j(\mathbf{z})\}_{j=1}^d$ be a disjoint partition of the index set \mathbf{I}_{q-1} , i.e. $\bigcup_{j=1}^d \alpha_j(\mathbf{z}) = \mathbf{I}_{q-1}$, $\alpha_{j_1}(\mathbf{z}) \cap \alpha_{j_2}(\mathbf{z}) = \emptyset$ for $j_1 \neq j_2$ such that $z_{i_1} = z_{i_2}$ for all $i_1, i_2 \in \alpha_j(\mathbf{z})$ and $z_{i_1} \neq z_{i_2}$ for all $i_1 \in \alpha_{j_1}(\mathbf{z})$, $i_2 \in \alpha_{j_2}(\mathbf{z})$. Then for any $i \in \alpha_j(\mathbf{z})$ and $1 \leq j \leq d$ such that $z_i = z_j^\circ$ (where z_j° is a value that corresponds to $\alpha_j(\mathbf{z})$), we have

$$\mathbf{z} = \sum_{j=1}^d z_j^\circ \mathbf{e}_{\alpha_j(\mathbf{z})}. \quad (2.13)$$

Theorem 2.5.4 (Saburov & Ahmad (2015b)). *Let $\mathbf{z} \in \mathcal{E}_p^{q-1}$ is a solution of the system (2.12). Then $1 \leq d \leq k$.*

Proof. Let $\mathbf{z} \in \mathcal{E}_p^{q-1}$ be a solution of the system of equations (2.12). The case $q-1 \leq k$ is trivial. We suppose that $q > k+1$. Then we can write \mathbf{z} in the form (2.13). Since \mathbf{z} is a fixed solution, $S(\mathbf{z}) = \sum_{j=1}^{q-1} z_j = \sum_{j=1}^d z_j^\circ |\alpha_j(\mathbf{z})|$ is also a fixed number where $z_i = z_j^\circ$ for any $i \in \alpha_j(\mathbf{z})$ and $j = 1, \dots, d$.

Let us consider the following polynomial equation of the variable x

$$x(\theta + S(\mathbf{z}))^k = [(\theta - 1)x + S(\mathbf{z}) + 1]^k. \quad (2.14)$$

Since $\mathbf{z} = (z_1, \dots, z_{q-1})$ is a solution of the system of equation (2.12), for each $j = \overline{1, d}$, the number z_j° is a distinct root of the polynomial equation (2.14). On the other hand, the polynomial equation (2.14) of degree k may have at most k number of roots in \mathbb{Q}_p . Thus $d \leq k$. \square

This theorem tells us that for a set of solution of the system of equations (2.12), the distinct values should not exceed the order k of the Cayley tree. This result is useful when describing the translation-invariant p -adic Gibbs measures.

Next we find other kind of periodic p -adic Gibbs measures. We would like to find other function $\tilde{\mathbf{h}}$ that satisfy the compatibility condition (2.10). Let m be some fixed natural number and $H_m = \{x \in V : d(x, x^0) \equiv 0 \pmod{m}\}$. Observe that the subset H_m is a sub-semigroup of V and the function $\tilde{\mathbf{h}} : V \rightarrow \mathbb{Q}_p^q$ is H_m -periodic if $\tilde{\mathbf{h}}(x) = \tilde{\mathbf{h}}^{(j)}$ whenever $d(x, x^{(0)}) \equiv j \pmod{m}$ for $0 \leq j < m$. The semigroup H_1 is equal to the set of vertices V of the semi-infinite Cayley tree. Thus, H_1 is associated to the translation-invariant p -adic Gibbs measures. There exists an H_m -periodic p -adic Gibbs measure associated with this H_m -periodic function $\tilde{\mathbf{h}}$. In this case, the condition (2.11) takes the form $\mathbf{h}^{(j)} = k\mathbf{F}(\mathbf{h}^{(j+1)})$ for all $0 \leq j < m$ with $\mathbf{h}^{(m)} := \mathbf{h}^{(0)}$, i.e.,

$$h_i^{(j)} = \ln_p \left(\frac{(\theta - 1) \exp_p(h_i^{(j+1)}) + \sum_{l=1}^{q-1} \exp_p(h_l^{(j+1)}) + 1}{\theta + \sum_{l=1}^{q-1} \exp_p(h_l^{(j+1)})} \right)^k, \quad 1 \leq i \leq q-1.$$

Remark 2.5.5. It is proven in Theorem 2 of Rozikov & Khakimov (2013b) that the Potts model (over the field \mathbb{R} of the real numbers) on the Cayley trees admits only translation-invariant Gibbs measures and periodic Gibbs measures of period two which correspond to G_k -periodic and $G_k^{(2)}$ -periodic functions respectively. Here G_k is the free group representation of the Cayley tree Γ^k and $G_k^{(2)}$ is the subgroup of G_k consisting of all words of even length. As we have argued in (2.4.3), the definition of the periodic Gibbs measure on the semi-infinite Cayley trees is different to that on the Cayley trees. The former is based on a semigroup structure and the later is based on a group structure.

Suppose $\mathbf{z}^{(j)} = (z_1^{(j)}, \dots, z_{q-1}^{(j)}) \in \mathcal{E}_p^{q-1}$ such that $z_i^{(j)} = \exp_p(h_i^{(j)})$ for any $1 \leq i \leq q-1$. In what follows, we write $\mathbf{z}^{(j)} = \exp_p(\mathbf{h}^{(j)})$.

Proposition 2.5.6. There exists an H_m -periodic p -adic Gibbs measure if and only if the following system of equations

$$z_i^{(j)} = \left(\frac{(\theta - 1)z_i^{(j+1)} + \sum_{l=1}^{q-1} z_l^{(j+1)} + 1}{\theta + \sum_{l=1}^{q-1} z_l^{(j+1)}} \right)^k, \quad 1 \leq i \leq q-1, \quad 0 \leq j \leq m-1 \quad (2.15)$$

has a solution $\{\mathbf{z}^{(j)}\}_{j=0}^{m-1} \subset \mathcal{E}_p^{q-1}$ in which $\mathbf{z}^{(m)} := \mathbf{z}^{(0)}$.

Proof. It follows directly from the definition of semi-group H_m and Theorem 2.5.3. \square

Let us defined the physical phenomenon of the phase transition. It can be found in (Mukhamedov & Akin, 2013). We denote by \mathcal{G}_p the set of all p -adic Gibbs measures of the Potts model on the Cayley tree associated to the functions $\tilde{\mathbf{h}} = \{\tilde{\mathbf{h}}(x), x \in V\}$. The notation $|A|$ means the cardinality of the set A .

Definition 2.5.7 (Mukhamedov & Akin (2013)). *Let the p -adic Gibbs measures for the associated model are non-unique, i.e. $|\mathcal{G}_p| > 1$ and at least one of them unbounded. Then the phase transition occurs.*

CHAPTER 3

TRANSLATION-INVARIANT p -ADIC GIBBS MEASURES

3.1 INTRODUCTION

We will study a class of p -adic Gibbs measures so-called the class of translation-invariant p -adic Gibbs measures of the q -state Potts model on the semi-infinite Cayley trees. In the classical case, Blekher & Ganikhodjaev (1990) show that there is a relation between the Ising on the Cayley trees and the semi-infinite Cayley trees. Later without any confusion, we will drop the term semi-infinite. We recall that a p -adic Gibbs measure of the Potts model on the Cayley tree is translation-invariant if and only if the function $\tilde{\mathbf{h}} : V \rightarrow \mathbb{Q}_p^q$ in (2.9) is constant, i.e. $\tilde{\mathbf{h}}(x) = \tilde{\mathbf{h}}$ for any $x \in \Gamma_k^+$. We also recall the following theorem which inform us about the existence of the translation-invariant p -adic Gibbs measures of the Potts model on the Cayley tree.

Theorem 3.1.1 (Mukhamedov & Rozikov (2005)). *Let $\tilde{\mathbf{h}} : V \rightarrow \mathbb{Q}_p^q$ be a function, $\tilde{\mathbf{h}}(x) = \tilde{\mathbf{h}} = (\tilde{h}_1, \dots, \tilde{h}_q)$ and $\mathbf{h} = (h_1, \dots, h_{q-1})$ such that $h_i = \tilde{h}_i - \tilde{h}_q$ for $1 \leq i \leq q-1$. Then there exists a translation-invariant p -adic Gibbs measures $\mu_{\tilde{\mathbf{h}}} : \Omega \rightarrow \mathbb{Q}_p$ associated with a function $\tilde{\mathbf{h}}$ if and only if $\mathbf{z} = (z_1, \dots, z_{q-1}) \in \mathcal{E}_p^{q-1}$ such that $z_i = \exp_p(h_i)$ is a solution of the following system of equations*

$$z_i = \left(\frac{(\theta - 1)z_i + \sum_{j=1}^{q-1} z_j + 1}{\theta + \sum_{j=1}^{q-1} z_j} \right)^k \quad \text{for } 1 \leq i \leq q-1. \quad (3.1)$$

The following theorem describes the existence of all translation-invariant p -adic Gibbs measures of the Potts model with q -spin values on the Cayley tree of order three.

Theorem 3.1.2 (Saburov & Ahmad (2015b)). *There exists a translation-invariant p -adic Gibbs measure $\mu_{\tilde{\mathbf{h}}} : \Omega \rightarrow \mathbb{Q}_p$ associated with a function $\tilde{\mathbf{h}} = (\tilde{h}_1, \dots, \tilde{h}_q)$ if and only if $\tilde{h}_j = \log_p(hz_j)$ for all $j = \overline{1, q-1}$ and $\tilde{h}_q = \log_p(h)$ where $h \in \mathcal{E}_p$ is any p -adic number and $\mathbf{z} = (z_1, \dots, z_{q-1}) \in \mathcal{E}_p^{q-1}$ is defined either one of the following form*

(A) $\mathbf{z} = (1, \dots, 1);$

(B) $\mathbf{z} = (z, \dots, z)$ where $z \in \mathcal{E}_p \setminus \{1\}$ is a root of the following cubic equation

$$(q-1)^3 z^3 + (3(q-1)^2 - (\theta-1)^2(\theta + 3(q-1) - 1)) z^2 + (3(q-1) - (\theta-1)^2(\theta + 2)) z + 1 = 0;$$

(C) $\mathbf{z} = \mathbf{e}_{\alpha_1} + z\mathbf{e}_{\alpha_2}$ with $|\alpha_i| = m_i$, $m_1 + m_2 = q - 1$ such that $z \in \mathcal{E}_p \setminus \{1\}$ is a root of the following cubic equation

$$m_2^3 z^3 + [3m_2^2(m_1 + 1) - (\theta - 1)^2(\theta + 3m_2 - 1)]z^2 + [3m_2(m_1 + 1)^2 - (\theta - 1)^2(\theta + 3(m_1 + 1) - 1)]z + (m_1 + 1)^3 = 0;$$

(D) $\mathbf{z} = z_1\mathbf{e}_{\alpha_1} + z_2\mathbf{e}_{\alpha_2}$ with $|\alpha_i| = m_i$, $m_1 + m_2 = q - 1$ such that

(i) if $m_1 \neq \frac{1-\theta}{3}$ then $z_1 = -\frac{(\theta-1+3m_2)z_2+\theta+2}{(\theta-1+3m_1)} \in \mathcal{E}_p \setminus \{1\}$ and $z_2 \in \mathcal{E}_p \setminus \{1\}$ is a root of the following cubic equation

$$[(m_1 - m_2)z + (m_1 - 1)]^3 + (\theta - 1 + 3m_1)^2 [(\theta - 1 + 3m_2)z^2 + (\theta + 2)z] = 0;$$

(ii) if $m_2 \neq \frac{1-\theta}{3}$ then $z_2 = -\frac{(\theta-1+3m_1)z_1+\theta+2}{(\theta-1+3m_2)} \in \mathcal{E}_p \setminus \{1\}$ and $z_1 \in \mathcal{E}_p \setminus \{1\}$ is a root of the following cubic equation

$$[(m_2 - m_1)z + (m_2 - 1)]^3 + (\theta - 1 + 3m_2)^2 [(\theta - 1 + 3m_1)z^2 + (\theta + 2)z] = 0;$$

(E) $\mathbf{z} = z_1\mathbf{e}_{\alpha_1} + z_2\mathbf{e}_{\alpha_2} + \mathbf{e}_{\alpha_3}$ with $|\alpha_i| = m_i$, $m_1 + m_2 + m_3 = q - 1$ such that

(i) if $m_1 \neq \frac{1-\theta}{3}$ then $z_1 = -\frac{(\theta-1+3m_2)z_2+3m_3+\theta+2}{(\theta-1+3m_1)} \in \mathcal{E}_p \setminus \{1\}$ and $z_2 \in \mathcal{E}_p \setminus \{1\}$ is a root of the following cubic equation

$$[(m_1 - m_2)z + (m_1 - m_3 - 1)]^3 + (\theta - 1 + 3m_1)^2 (\theta - 1 + 3m_2)z^2 + (\theta - 1 + 3m_1)^2 (3m_3 + \theta + 2)z = 0;$$

(ii) if $m_2 \neq \frac{1-\theta}{3}$ then $z_2 = -\frac{(\theta-1+3m_1)z_1+3m_3+\theta+2}{(\theta-1+3m_2)} \in \mathcal{E}_p \setminus \{1\}$ and $z_1 \in \mathcal{E}_p \setminus \{1\}$ is a root of the following cubic equation

$$[(m_2 - m_1)z + (m_2 - m_3 - 1)]^3 + (\theta - 1 + 3m_2)^2 (\theta - 1 + 3m_1)z^2 + (\theta - 1 + 3m_2)^2 (3m_3 + \theta + 2)z = 0;$$

(iii) if $\theta = 1 - q$ and $m_1 = m_2 = m_3 + 1$ then either $z_1 \in \mathcal{E}_p \setminus \{1\}$ or $z_2 \in \mathcal{E}_p \setminus \{1\}$ is any p -adic number so that the second one is a root of the cubic equation $(z_1 + z_2 + 1)^3 = 27z_1z_2$.

By these theorems, one can see that the existence of the translation-invariant p -adic Gibbs measures of the Potts model on the Cayley tree of order 3 is related to the existence of the roots of cubic equations over \mathbb{Q}_p . In Chapter 2, we gave the solvability conditions of the cubic equation over \mathbb{Q}_p which can be use to describe the existence of the translation-invariant p -adic Gibbs measures of the Potts model on the Cayley tree

of order 3. In next section, we will describe the existence of the translation-invariant p -adic Gibbs measures of the Potts model on the Cayley tree of order 4.

3.2 TRANSLATION-INVARIANT P -ADIC GIBBS MEASURES OF THE POTTS MODEL ON THE CAYLEY TREE OF ORDER 4

We shall describe the existence of the translation-invariant p -adic Gibbs measures of the Potts model with q -state values on the Cayley tree of order 4. By Theorem 3.1.1, we will analyze the following system of equations:

$$z_i = \left(\frac{(\theta - 1)z_i + \sum_{\ell=1}^{q-1} z_\ell + 1}{\theta + \sum_{\ell=1}^{q-1} z_\ell} \right)^4 \quad \text{for } 1 \leq i \leq q-1. \quad (3.2)$$

Due to Theorem 2.5.4, we are interested in the solution of (3.2) of the form $\mathbf{z} = \mathbf{e}_\beta + z\mathbf{e}_\alpha$ with $|\beta| = q - 1 - |\alpha|$. Here $|\alpha|$ means the cardinality of the set α . In this case, the distinct values of the set of the solution to the system of equations (3.2) is $2 < 4 = k$. If we substitute $\mathbf{z} = \mathbf{e}_\beta + z\mathbf{e}_\alpha$ into (3.2) we obtain

$$z = f(z) \quad (3.3)$$

where

$$f(z) = \left(\frac{(\theta - 1 + |\alpha|)z + q - |\alpha|}{|\alpha|z + \theta - 1 + q - |\alpha|} \right)^4 \quad (3.4)$$

$$= \left(\frac{\Theta z + Q - 1}{z + \Theta + Q - 2} \right)^4 \quad (3.5)$$

with $\Theta = \frac{\theta-1}{|\alpha|} + 1$ and $Q = \frac{q}{|\alpha|}$. Recall

$$\mathcal{E}_p = \left\{ x \in \mathbb{Q}_p : |x - 1|_p < p^{-\frac{1}{p-1}} \right\}.$$

Our aim is to find the solutions of the equation (3.3) that belong to \mathcal{E}_p so that we can have the translation-invariant p -adic Gibbs measures. One could check that $1 \in \mathcal{E}_p$ is the solution of the equation (3.3). So from $z - 1 = f(z) - 1$ and the equality (3.4), we get

$$\begin{aligned} & \frac{(z-1)(A-|\alpha|)}{(|\alpha|z + A + B - |\alpha|)^4} \left((Az + B)^3 + (Az + B)^2(|\alpha|z + A + B - |\alpha|) \right. \\ & \quad \left. + (Az + B)(|\alpha|z + A + B - |\alpha|)^2 + (|\alpha|z + A + B - |\alpha|)^3 \right) = (z-1) \end{aligned}$$

where $A = \theta - 1 + |\alpha|$ and $B = q - |\alpha|$. This means that any other solution $1 \neq z \in \mathcal{E}_p$ is the solution of the following quartic equation

$$(A - |\alpha|) \left((Az + B)^3 + (Az + B)^2(|\alpha|z + A + B - |\alpha|) + (Az + B)(|\alpha|z + A + B - |\alpha|)^2 + (|\alpha|z + A + B - |\alpha|)^3 \right) = (|\alpha|z + A + B - |\alpha|)^4.$$

Expanding the terms, we have

$$\begin{aligned} & |\alpha|^4 z^4 + (-A^4 + 4A|\alpha|^3 + 4B|\alpha|^3 - 3|\alpha|^4)z^3 + (-A^4 - 4A^3B \\ & + 6A^2|\alpha|^2 + 12AB|\alpha|^2 - 8A|\alpha|^3 + 6B^2|\alpha|^2 - 8B|\alpha|^3 + 3|\alpha|^4)z^2 \\ & + (-A^4 - 4A^3B + 4A^3|\alpha| - 6A^2B^2 + 12A^2B|\alpha| - 6A^2|\alpha|^2 + 12AB^2|\alpha| \\ & - 12AB|\alpha|^2 + 4A|\alpha|^3 + 4B^3|\alpha| - 6B^2|\alpha|^2 + 4B|\alpha|^3 - |\alpha|^4)z + B^4 = 0 \end{aligned} \quad (3.6)$$

On the other hand, from $z - 1 = f(z) - 1$ and the equality (3.5), we obtain

$$\begin{aligned} & \frac{(z-1)(\Theta-1)}{(z+\Theta+Q-2)^4} \left((\Theta z + Q - 1)^3 + (\Theta z + Q - 1)^2(z + \Theta + Q - 2) \right. \\ & \left. + (\Theta z + Q - 1)(z + \Theta + Q - 2)^2 + (z + \Theta + Q - 2)^3 \right) = (z-1). \end{aligned}$$

This implies that any other solution $1 \neq z \in \mathcal{E}_p$ is the solution of the following quartic equation:

$$\begin{aligned} & (\Theta - 1) \left((\Theta z + Q - 1)^3 + (\Theta z + Q - 1)^2(z + \Theta + Q - 2) \right. \\ & \left. + (\Theta z + Q - 1)(z + \Theta + Q - 2)^2 + (z + \Theta + Q - 2)^3 \right) = (z + \Theta + Q - 2)^4. \end{aligned}$$

Let $z = 1 - Q - (\Theta - 1) + (\Theta - 1)y$. Then the quartic equation above can be written as follows

$$y^4 - (1 + \Theta + \Theta^2 + \Theta^3)y^3 - (3\Theta^2 + 2\Theta + 1)(1 - \Theta - Q)y^2 - (3\Theta + 1)(1 - \Theta - Q)^2y - (1 - \Theta - Q)^3 = 0. \quad (3.7)$$

So, in order to find the solution of the equation (3.3), we could look for the solutions of the quartic equations (3.6) or (3.7). In what follows, we assume $|\alpha| = m$ and $\theta \neq 1, 1 - q$. Recall from Proposition 2.5.1 that $|\theta - 1| < 1$ where $\theta = \exp_p(J) \in \mathcal{E}_p$.

Proposition 3.2.1. *Let $p \geq 3$ be a prime number and $q \neq 3(\theta - 1)$. Suppose $|m|_p = 1$.*

(i) *Let $|q|_p = 1$. Then the solutions of the equation (3.3) are not in \mathcal{E}_p except $z = 1$.*

(ii) *Let $|q|_p < 1$. Then all the solutions of the equation (3.3) are in \mathcal{E}_p .*

Moreover, if $|q|_p < 1$, then the equation (3.7) always has the solutions.

Proof. We recall $|\theta - 1|_p < 1$ and $\theta \neq 1, 1 - q$. Suppose $p \geq 3$ be a prime number, $q \neq 3(\theta - 1)$ and $|m|_p = 1$. In this case we will consider the equation (3.7) in order

to find the solutions $\neq 1$ of the equation (3.3). Since $q \neq 3(\theta - 1)$, one could check that the solutions of the equation (3.7) are not equal to $\frac{\theta-1+q}{\theta-1}$. For $|m|_p = 1$, we have $|Q|_p \leq 1$ and $|\Theta - 1|_p < 1$. If \bar{y} is the solution of the equation (3.7), then

$$|\bar{y}|_p^4 \leq \max\{|1 - \Theta - Q|_p^3, |1 - \Theta - Q|_p^2 |3\Theta^2 + 2\Theta + 1|_p |\bar{y}|_p, |1 - \Theta - Q|_p |3\Theta + 1|_p |\bar{y}|_p^2, |1 + \Theta + \Theta^2 + \Theta^3|_p |\bar{y}|_p^3\}.$$

This implies $|\bar{y}|_p \leq 1$. Thus if $|q|_p = 1$, then $|\bar{z} - 1|_p = |(\Theta - 1)(\bar{y} - 1) - Q|_p = |Q|_p = 1$ where \bar{z} is the solution of the equation (3.3). This means $\bar{z} \notin \mathcal{E}_p$ and the statement (i) holds.

Now suppose $|q|_p < 1$. Then one can see $|\bar{z} - 1|_p = |(\Theta - 1)(\bar{y} - 1) - Q|_p = |Q|_p < 1$ where \bar{z} is any solution of the equation (3.3). So $\bar{z} \in \mathcal{E}_p$ and the statement (ii) holds. Next we will show that the equation (3.7) always has a solution. Without loss of generality, let $m = 1$. Then we have

$$y^4 - (1 + \theta + \theta^2 + \theta^3)y^3 - (3\theta^2 + 2\theta + 1)(1 - \theta - q)y^2 - (3\theta + 1)(1 - \theta - q)^2y - (1 - \theta - q)^3 = 0. \quad (3.8)$$

Let $\bar{y} \equiv 4 \pmod{p}$. Consider the polynomial $P(y) = y^4 - (1 + \theta + \theta^2 + \theta^3)y^3 - (3\theta^2 + 2\theta + 1)(1 - \theta - q)y^2 - (3\theta + 1)(1 - \theta - q)^2y - (1 - \theta - q)^3$ and its derivative $P'(y) = 4y^3 - 3(1 + \theta + \theta^2 + \theta^3)y^2 - 2(3\theta^2 + 2\theta + 1)(1 - \theta - q)y - (3\theta + 1)(1 - \theta - q)^2$. Then

$$P(\bar{y}) \equiv 0 \pmod{p} \quad \text{and} \quad P'(\bar{y}) \equiv 4\bar{y}^3 - 12\bar{y}^2 \equiv \bar{y}^3 \not\equiv 0 \pmod{p}.$$

By Hensel's lemma, there exists a root $y \in \mathbb{Z}_p$ of the quartic equation (3.8) such that $y \equiv \bar{y} \pmod{p}$. This means the equation (3.7) always has a solution. This completes the proof. \square

Proposition 3.2.2. *Let $p \geq 5$ be a prime number, $|m|_p < 1$ and $q \neq 3(\theta - 1)$. Then there exists a solution ($\neq 1$) of (3.3) in \mathcal{E}_p if and only if we have one of the following conditions:*

- (i) $|\theta - 1|_p < |m|_p$ and $|q|_p < |m|_p$;
- (ii) $|q|_p < |\theta - 1|_p = |m|_p$ and exists a solution y of the quartic equation (3.7) such that $|y - 1|_p < 1$;
- (iii) $|\theta - 1|_p = |q|_p = |m|_p$ or $|m|_p < |\theta - 1|_p \leq |q|_p$ and exists a solution y of the quartic equation (3.7) such that $|(\theta - 1)(y - 1) - q|_p < |m|_p$.

Proof. Recall $|\theta - 1|_p < 1$ and $\theta \neq 1, 1 - q$. Let $p \geq 3$ be a prime number, $q \neq 3(\theta - 1)$ and $|m|_p = 1$. The condition $q \neq 3(\theta - 1)$ means that the solutions of the equation (3.7) are not equal to $\frac{\theta-1+q}{\theta-1}$. To find the solution $\neq 1$ of the equation (3.3), we will consider the following cases.

1. $|\theta - 1|_p < |m|_p, |q|_p < |m|_p$;

2. $|\theta - 1|_p < |m|_p = |q|_p$;
3. $|\theta - 1|_p < |m|_p < |q|_p$;
4. $|q|_p < |\theta - 1|_p = |m|_p$;
5. $|\theta - 1|_p = |m|_p = |q|_p$;
6. $|\theta - 1|_p = |m|_p < |q|_p$;
7. $|q|_p < |m|_p < |\theta - 1|_p$;
8. $|m|_p = |q|_p < |\theta - 1|_p$;
9. $|m|_p < |\theta - 1|_p, |m|_p < |q|_p$.

Consider the quartic equation

$$y^4 + A_3y^3 + A_2y^2 + A_1y + A_0 = 0$$

where $A_0 = -(1 - \Theta - Q)^3$, $A_1 = -(3\Theta + 1)(1 - \Theta - Q)^2$, $A_2 = -(3\Theta^2 + 2\Theta + 1)(1 - \Theta - Q)$ and $A_3 = -(\Theta^3 + \Theta^2 + \Theta + 1)$.

CASE 1: Let $|\theta - 1|_p < |m|_p, |q|_p < |m|_p$. Then we have $|\Theta - 1|_p < 1$ and $|Q|_p < 1$. This case is similar to the Proposition 3.2.1.(ii). So there exists a solution ($\neq 1$) of (3.3).

CASE 2: Let $|\theta - 1|_p < |m|_p = |q|_p$. Then we have $|\Theta - 1|_p < 1$ and $|Q|_p = 1$. It is similar to the Proposition 3.2.1.(i). So there is no solution ($\neq 1$) of (3.3) in \mathcal{E}_p .

CASE 3: Let $|\theta - 1|_p < |m|_p < |q|_p$. Then we have $|\Theta - 1|_p < 1$ and $|Q|_p > 1$. If \bar{y} is the solution of the equation (3.7), then

$$|\bar{y}|_p^4 \leq \max\{|A_0|_p^3, |A_1|_p|\bar{y}|_p, |A_2|_p|\bar{y}|_p^2, |A_3|_p|\bar{y}|_p^3\}.$$

This implies $|\bar{y}|_p < |Q|_p$. Thus $|\bar{z} - 1|_p = |(\Theta - 1)(\bar{y} - 1) - Q|_p = |Q|_p > 1$ where \bar{z} is the solution of the equation (3.3) and there is no solution ($\neq 1$) of (3.3) in \mathcal{E}_p .

CASE 4: Let $|q|_p < |\theta - 1|_p = |m|_p$. Then we have $|\Theta|_p \leq 1$, $|\Theta - 1|_p = 1$ and $|Q|_p < 1$. We also have $\text{ord}_p(A_0) = 0$, $\text{ord}_p(A_1) \geq 0$, $\text{ord}_p(A_2) \geq 0$ and $\text{ord}_p(A_3) \geq 0$. We construct its Newton polygon and we find $|\bar{y}|_p = 1$ where \bar{y} is the solution of the equation (3.7). If there is $\bar{y} \in \mathbb{Q}_p$ and $|\bar{y} - 1|_p < 1$, then $|\bar{z} - 1|_p = |(\Theta - 1)(\bar{y} - 1) - Q|_p < 1$ where \bar{z} is the solution of the equation (3.3). This means $\bar{z} \in \mathcal{E}_p$. Otherwise, there is no solution ($\neq 1$) of the equation (3.3) in \mathcal{E}_p .

CASE 5: Let $|\theta - 1|_p = |m|_p = |q|_p$. Then we have $|\Theta|_p \leq 1$, $|\Theta - 1|_p = |Q|_p = 1$. We also have $\text{ord}_p(A_0) \leq 0$, $\text{ord}_p(A_1) \geq 0$, $\text{ord}_p(A_2) \geq 0$ and $\text{ord}_p(A_3) \geq 0$. We construct its Newton polygon and we find $|\bar{y}|_p \leq 1$ where \bar{y} is the solution of the equation (3.7). If there is $\bar{y} \in \mathbb{Q}_p$ and $|(\theta - 1)(\bar{y} - 1) - q|_p < |m|_p$, then $|\bar{z} - 1|_p =$

$|(\Theta - 1)(\bar{y} - 1) - Q|_p < 1$ where \bar{z} is the solution of the equation (3.3). This means $\bar{z} \in \mathcal{E}_p$. Otherwise, there is no solution ($\neq 1$) of the equation (3.3) in \mathcal{E}_p .

CASE 6: Let $|\theta - 1|_p = |m|_p < |q|_p$. Then we have $|\Theta - 1|_p = 1$ and $|Q|_p > 1$. If \bar{y} is the solution of the equation (3.7), then

$$|\bar{y}|_p^4 \leq \max\{|A_0|_p^3, |A_1|_p|\bar{y}|_p, |A_2|_p|\bar{y}|_p^2, |A_3|_p|\bar{y}|_p^3\}.$$

This implies $|\bar{y}|_p < |Q|_p$. Thus $|\bar{z} - 1|_p = |(\Theta - 1)(\bar{y} - 1) - Q|_p = |Q|_p > 1$ where \bar{z} is the solution of the equation (3.3) and there is no solution ($\neq 1$) of (3.3) in \mathcal{E}_p .

CASE 7: Let $|q|_p < |m|_p < |\theta - 1|_p$. Then we have $|\Theta|_p = |\Theta - 1|_p > 1$ and $|Q|_p < 1$. Let $w = y - 1$. Then the equation (3.7) is written as follows

$$w^4 + (3 - \Theta - \Theta^2 - \Theta^3)w^3 + (3Q\Theta^2 + 2Q\Theta - 4\Theta^2 + Q - 4\Theta + 2)w^2 + (-3Q^2\Theta - Q^2 + 8Q\Theta + 4Q - 6\Theta - 2)w + Q^3 - 4Q^2 + 6Q - 3 = 0 \quad (3.9)$$

Let $B_0 = Q^3 - 4Q^2 + 6Q - 3$, $B_1 = -3Q^2\Theta - Q^2 + 8Q\Theta + 4Q - 6\Theta - 2$, $B_2 = 3Q\Theta^2 + 2Q\Theta - 4\Theta^2 + Q - 4\Theta + 2$ and $B_3 = 3 - \Theta - \Theta^2 - \Theta^3$. We also have $\text{ord}_p(A_0) = 0$, $\text{ord}_p(A_1) = \text{ord}_p(\Theta)$, $\text{ord}_p(A_2) = 2\text{ord}_p(\Theta)$ and $\text{ord}_p(A_3) = 3\text{ord}_p(\Theta)$. We construct its Newton polygon and we find $|\bar{w}|_p = |\Theta|^3$ or $|\bar{w}|_p = \frac{1}{|\Theta|_p}$ where \bar{w} is the solution of the equation (3.9). This implies $|\bar{z} - 1|_p = |(\Theta - 1)\bar{w} - Q|_p \geq 1$ where \bar{z} is the solution of the equation (3.3). Thus there is no solution ($\neq 1$) of the equation (3.3) in \mathcal{E}_p .

CASE 8: Let $|\theta - 1|_p > |m|_p = |q|_p$. In this case, we consider the quartic equation (3.6). Let $C_0 = B^4$, $C_1 = -A^4 - 4A^3B + 4A^3m - 6A^2B^2 + 12A^2Bm - 6A^2m^2 + 12AB^2m - 12ABm^2 + 4Am^3 + 4B^3m - 6B^2m^2 + 4Bm^3 - m^4$, $C_2 = -A^4 - 4A^3B + 6A^2m^2 + 12ABm^2 - 8Am^3 + 6B^2m^2 - 8Bm^3 + 3m^4$, $C_3 = -A^4 + 4Am^3 + 4Bm^3 - 3m^4$ and $C_4 = m^4$. Then we obtain $\text{ord}_p(C_0) \geq 4\text{ord}_p(m)$, $\text{ord}_p(C_1) = 4\text{ord}_p(\theta - 1)$, $\text{ord}_p(C_2) = 4\text{ord}_p(\theta - 1)$, $\text{ord}_p(C_3) = 4\text{ord}_p(\theta - 1)$ and $\text{ord}_p(C_4) = 4\text{ord}_p(m)$. We construct its Newton polygon and then we find there may exist \bar{z} the solution of the equation (3.8) such that $|\bar{z}|_p = 1$. When we substitute \bar{z} into equation (3.6), we find $\bar{z} \not\equiv 1 \pmod{p}$ whenever $p \geq 5$. Hence, there is no solution ($\neq 1$) of the equation (3.3) in \mathcal{E}_p .

CASE 9: Let $|\theta - 1|_p > |m|_p$, $|q|_p > |m|_p$. We divide into three subcases: $|m|_p < |q|_p < |\theta - 1|_p$, $|m|_p < |\theta - 1|_p < |q|_p$ and $|m|_p < |\theta - 1|_p = |q|_p$.

CASE 9(I): Suppose $|m|_p < |q|_p < |\theta - 1|_p$. Similar to the Case 8, we will consider the quartic equation (3.6). We have $\text{ord}_p(C_0) \geq 4\text{ord}_p(q)$, $\text{ord}_p(C_1) = 4\text{ord}_p(\theta - 1)$, $\text{ord}_p(C_2) = 4\text{ord}_p(\theta - 1)$, $\text{ord}_p(C_3) = 4\text{ord}_p(\theta - 1)$ and $\text{ord}_p(C_4) = 4\text{ord}_p(m)$. We construct its Newton polygon and we find there may exist \bar{z} the solution of the equation (3.6) such that $|\bar{z}|_p = 1$. If we substitute \bar{z} into quartic equation (3.6), we find $\bar{z} \not\equiv 1 \pmod{p}$ whenever $p \geq 5$. Hence, there is no solution ($\neq 1$) of the equation (3.3) in \mathcal{E}_p .

CASE 9(II): Suppose $|m|_p < |\theta - 1|_p < |q|_p$. Then we have $1 < |\Theta|_p = |\Theta - 1|_p \leq$

$|Q|_p$. We consider the equation quartic (3.9). We have $\text{ord}_p(B_0) = 3\text{ord}_p(Q)$, $\text{ord}_p(B_1) = 2\text{ord}_p(Q) + \text{ord}_p(\Theta)$, $\text{ord}_p(B_2) = \text{ord}_p(Q) + 2\text{ord}_p(\Theta)$ and $\text{ord}_p(B_3) = 3\text{ord}_p(\Theta)$. We construct its Newton polygon and we find there may exist \bar{w} the solution of the quartic equation (3.9) such that $|\bar{w}|_p = \frac{|Q|_p}{|\Theta|_p}$. So if there is $\bar{w} \in \mathbb{Q}_p$ and $|(\theta - 1)(\bar{w} - 1) - q|_p < |m|_p$, then $|\bar{z} - 1|_p = |(\Theta - 1)\bar{w} - Q|_p < 1$ where \bar{z} is the solution of the equation (3.3). This means $\bar{z} \in \mathcal{E}_p$. Otherwise, there is no solution ($\neq 1$) of the equation (3.3) in \mathcal{E}_p .

CASE 9(III): Suppose $|m|_p < |\theta - 1|_p = |q|_p$. Then we have $1 < |\Theta|_p = |\Theta - 1|_p = |Q|_p$. We consider the quartic equation (3.9). We have $\text{ord}_p(B_0) = \text{ord}_p(B_1) = \text{ord}_p(B_2) = \text{ord}_p(B_3) = 3\text{ord}_p(Q) = 3\text{ord}_p(\Theta)$. We construct its Newton polygon and we find there may exist \bar{w} the solution of the quartic equation (3.9) such that $|\bar{w}|_p = 1$. If there is $\bar{w} \in \mathbb{Q}_p$ and $|(\theta - 1)(\bar{w} - 1) - q|_p < |m|_p$, then $|\bar{z} - 1|_p = |(\Theta - 1)\bar{w} - Q|_p < 1$ where \bar{z} is the solution of the equation (3.3). This means $\bar{z} \in \mathcal{E}_p$. Otherwise, there is no solution ($\neq 1$) of the equation (3.3) in \mathcal{E}_p .

There exists a solution ($\neq 1$) of the equation (3.3) in \mathcal{E}_p if the Case 1,4,5,9(II) and 9(III) hold. Otherwise, the solution is not in \mathcal{E}_p . \square

By these two propositions, Proposition 3.2.1 and 3.2.2, we prove the following theorem which state the existence of the translation-invariant p -adic Gibbs measure $\mu_{\mathbf{h}}$ associated to the function \mathbf{h} where $\exp_p(\mathbf{h}) = \mathbf{z} = \mathbf{e}_{\alpha_1} + z\mathbf{e}_{\alpha_2} \neq (1, \dots, 1)$ such that $z = 1 - Q - (\Theta - 1) + (\Theta - 1)y$. Here y is the solution of the quartic equation (3.7).

Theorem 3.2.3. *Let $p \geq 5$ be a prime number, $|\theta - 1|_p < 1$ and $q \neq 3(\theta - 1)$. Then there exists a translation-invariant p -adic Gibbs measure $\mu_{\mathbf{h}}$ associated to the function \mathbf{h} where $\exp_p(\mathbf{h}) = \mathbf{z} = \mathbf{e}_{\alpha_1} + z\mathbf{e}_{\alpha_2} \neq (1, \dots, 1)$ when one of the following conditions holds:*

- (i) $|m|_p = 1$ with $|\theta - 1|_p, |q|_p < 1$;
- (ii) $|\theta - 1|_p, |q|_p < |m|_p < 1$;
- (iii) $|q|_p < |\theta - 1|_p = |m|_p < 1$ and exists a solution y of the quartic equation (3.7) such that $|y - 1|_p < 1$;
- (iv) $|\theta - 1|_p = |q|_p = |m|_p < 1$ or $|m|_p < |\theta - 1|_p \leq |q|_p \leq 1$ and exists a solution y of the quartic equation (3.7) such that $|(\theta - 1)(y - 1) - q|_p < |m|_p$.

This theorem also tells us when the translation-invariant p -adic Gibbs measures is non-unique. The non-uniqueness of translation-invariant p -adic Gibbs measures is essential especially in the study of the set of p -adic Gibbs measures and the phase transition. This theorem also ask us to find a solution to the quartic equation over \mathbb{Q}_p . In next section, we will describe the solvability condition of the quartic equation over \mathbb{Q}_p . Notice that finding the translation-invariant p -adic Gibbs measures of the Potts model on the Cayley

tree is reduced to finding the fixed points of the function f . Later, in Chapter 4, we will study the dynamics of the function f called Potts–Bethe mapping on \mathbb{Q}_p .

3.3 QUARTIC EQUATIONS OVER \mathbb{Q}_p

In the previous section, we notice that the study of translation-invariant p -adic Gibbs measures of Potts model on Cayley tree of order 4 is related to the searching of the roots of the quartic equation over \mathbb{Q}_p . A quartic equation may have or may not have a root in \mathbb{Q}_p . For instance, the quartic equation $x^4 = p$ does not have a root in \mathbb{Q}_p . So in this section we will study the solvability conditions of the quartic equation over \mathbb{Q}_p . It is useful later for describing the existence of the translation-invariant p -adic Gibbs measures of Potts model on the Cayley tree of order 4.

Let us consider the following general quartic equation

$$y^4 + a_3y^3 + a_2y^2 + a_1y + a_0 = 0. \quad (3.10)$$

Replacing the variable y by $x - \frac{a_3}{4}$ we obtain the following depressed quartic equation

$$x^4 + ax^2 + bx + c = 0 \quad (3.11)$$

where $a = \frac{8a_2 - 3a_3^2}{8}$, $b = \frac{a_3^3 - 4a_2a_3 + 8a_1}{8}$ and $c = \frac{-3a_3^4 + 256a_0 - 64a_1a_3 + 16a_2a_3^2}{256}$. Throughout, we will only consider the depressed quartic equation (3.11).

3.3.1 Solvability Over \mathbb{Z}_p^*

Let $\mathbb{A} \subset \mathbb{Z}$ be any subset. We introduce the following set

$$p^{\mathbb{A}}\mathbb{Z}_p^* := \{x \in \mathbb{Q}_p : \log_p |x|_p \in \mathbb{A}\}.$$

Proposition 3.3.1. *Let p be any prime, $a, b, c \in \mathbb{Q}_p$ and $\mathbb{A} \subset \mathbb{Z}$ be any subset. The depressed quartic equation (3.11) is solvable in $p^{\mathbb{A}}\mathbb{Z}_p^*$ if and only if there exists a pair $(y^*, k) \in \mathbb{Z}_p^* \times \mathbb{A}$ such that y^* is a solution of the following depressed quartic equation*

$$y^4 + A_k y^2 + B_k y + C_k = 0 \quad (3.12)$$

where $A_k = ap^{-2k}$, $B_k = bp^{-3k}$ and $C_k = cp^{-4k}$. Moreover, in this case, a solution of the depressed quartic equation (3.11) has the form $x = p^k y^*$.

Proof. Let $x \in \mathbb{Q}_p$ and $|x|_p = p^{-k}$. Then $x \in p^{\mathbb{A}}\mathbb{Z}_p^*$ is a solution of the depressed quartic equation (3.11) if and only if $y^* = x|x|_p \in \mathbb{Z}_p^*$ is a solution of the depressed

quartic equation (3.12). □

This shows that it is enough to study the depressed quartic equation (3.11) over \mathbb{Z}_p^* for $a, b, c \in \mathbb{Q}_p$.

Proposition 3.3.2. *Let p be any prime number. Suppose the depressed quartic equation (3.11) has solutions in \mathbb{Z}_p^* for $a, b, c \in \mathbb{Q}_p$. Then either one of the following statements holds*

- (i) $|b|_p, |c|_p < |a|_p = 1$;
- (ii) $|a|_p, |c|_p < |b|_p = 1$;
- (iii) $|a|_p, |b|_p < |c|_p = 1$;
- (iv) $|a|_p < |b|_p = |c|_p, |b|_p = |c|_p \geq 1$;
- (v) $|b|_p < |a|_p = |c|_p, |a|_p = |c|_p \geq 1$;
- (vi) $|c|_p < |a|_p = |b|_p, |a|_p = |b|_p \geq 1$;
- (v) $|a|_p = |b|_p = |c|_p \geq 1$.

Proof. Let the depressed quartic equation (3.11) has solutions in \mathbb{Z}_p^* . Then one obtain

$$\begin{aligned} |a|_p &= |ax^2|_p = |x^4 + bx + c| \leq \max\{1, |b|_p, |c|_p\}, \\ |b|_p &= |bx|_p = |x^4 + ax^2 + c| \leq \max\{1, |a|_p, |c|_p\}, \\ |c|_p &= |x^4 + ax^2 + bx| \leq \max\{1, |a|_p, |b|_p\}, \\ 1 &= |x^4|_p = |ax^2 + bx + c| \leq \max\{|a|_p, |b|_p, |c|_p\}. \end{aligned}$$

By analyzing and comparing these inequalities, we get the conditions (i)–(vii). □

Next we give the conditions for the depressed equation (3.11) to have solution in \mathbb{Z}_p^* . Recall

$$\Delta = -b^2(4a^3 - 27b^2) + 16c(a^4 + 9ab^2 - 8a^2c + 16c^2)$$

is the discriminant of the depressed quartic equation (3.11). Recall also from Chapter 2 about the quartic congruence

$$x^4 + ax^2 + bx + c \equiv 0 \pmod{p}. \quad (3.13)$$

We recall the Theorems 2.2.13 and 2.2.17. Let $p \geq 5$ be a prime number, $a, b, c \in \mathbb{Z}$ and $\Delta \not\equiv 0 \pmod{p}$. Let $a \equiv -2m \pmod{p}$.

i) Let $p \nmid b$. Then the quartic congruence (3.13) has solution if and only if there is no an integer y such that

$$y^3 + 3ay^2 + (a^2 - 4c)y - b^2 \equiv 0 \pmod{p}$$

$$\text{and } \left(\frac{y}{p}\right) = -1.$$

ii) Let $p \mid b$ and $p \nmid ac$. Then the quartic congruence (3.13) has solution if and only if

$$a) \ c \equiv n^2 \pmod{p} \text{ and } \left(\frac{2(m-n)}{p}\right) = \left(\frac{2(m+n)}{p}\right) = \left(\frac{m^2-n^2}{p}\right) = 1 \text{ or}$$

$$b) \ \left(\frac{c}{p}\right) = -1 \text{ and } \left(\frac{m^2-c}{p}\right) = 1.$$

Then we have the following statements.

$$R_1 : \text{ there is no an integer } y \text{ such that } y^3 + 3ay^2 + (a^2 - 4c)y - b^2 \equiv 0 \pmod{p} \\ \text{and } \left(\frac{y}{p}\right) = -1.$$

$$R_2 : \ c \equiv n^2 \pmod{p} \text{ and } \left(\frac{2(m-n)}{p}\right) = \left(\frac{2(m+n)}{p}\right) = \left(\frac{m^2-n^2}{p}\right) = 1.$$

$$R_3 : \ \left(\frac{c}{p}\right) = -1 \text{ and } \left(\frac{m^2-c}{p}\right) = 1.$$

Let $a = p^{\text{ord}_p(a)}a^*$, $b = p^{\text{ord}_p(b)}b^*$ and $c = p^{\text{ord}_p(c)}c^*$ such that $a^* = a_0 + a_1p + a_2p^2 + \dots$, $b^* = b_0 + b_1p + b_2p^2 + \dots$ and $c^* = c_0 + c_1p + c_2p^2 + \dots$. Here $\sqrt[n]{a} - \exists$ means $\text{ord}_p(a)$ is divisible by n and a_0 is the n th residue modulo p . By Proposition 3.3.2, if the depressed equation (3.11) has solutions in \mathbb{Z}_p^* then $\max\{|a|_p, |b|_p, |c|_p\} \geq 1$.

Proposition 3.3.3. *Let $p \geq 5$ be a prime number. Suppose $\max\{|a|_p, |b|_p, |c|_p\} = 1$ and $|\Delta|_p = 1$. Let either one of the following conditions holds:*

- (i) $|a|_p, |c|_p < |b|_p$ and $\sqrt[3]{-b} - \exists$;
- (ii) $|a|_p, |b|_p < |c|_p$ and $\sqrt[4]{-c} - \exists$;
- (iii) $|a|_p < |b|_p = |c|_p$ and R_1 ;
- (iv) $|b|_p < |a|_p = |c|_p$ and R_2 or R_3 ;
- (v) $|a|_p = |b|_p = |c|_p$ and R_1 .

Then the depressed quartic equation (3.11) has a solution in \mathbb{Z}_p^* .

Proof. Let $p \geq 5$ be a prime number. Let $\max\{|a|_p, |b|_p, |c|_p\} = 1$ and $|\Delta|_p = 1$ where $\Delta = -b^2(4a^3 - 27b^2) + 16c(a^4 + 9ab^2 - 8a^2c + 16c^2)$. Then by Proposition 3.3.2, let one of the following conditions holds:

- (1) $|a|_p, |c|_p < |b|_p = 1$;
- (2) $|a|_p, |b|_p < |c|_p = 1$;
- (3) $|a|_p < |b|_p = |c|_p = 1$;
- (4) $|b|_p < |a|_p = |c|_p = 1$;
- (5) $|a|_p = |b|_p = |c|_p = 1$.

We shall analyze each cases. Let $f_{a,b,c}(x) = x^4 + ax^2 + bx + c$. Then $f'_{a,b,c}(x) = 4x^3 + 2ax + b$. The condition $|\Delta|_p = 1$ means $\Delta \not\equiv 0 \pmod{p}$ and $R(f_{a,b,c}, f'_{a,b,c}) \not\equiv 0 \pmod{p}$. By Theorem 2.2.2, this means that if $f_{a,b,c}(\bar{x}) \equiv 0 \pmod{p}$ for some $\bar{x} \in \mathbb{Z}$, then $f'_{a,b,c}(\bar{x}) \not\equiv 0 \pmod{p}$.

(i) : Case (1). We want to show that the depressed quartic equation (3.11) has solution in \mathbb{Z}_p^* if and only if $\sqrt[3]{-b} - \exists$. IF PART: Let $x \in \mathbb{Z}_p^*$ be a solution of the depressed quartic equation (3.11). Then we get $x_0^4 + b_0x_0 \equiv x_0(x_0^3 + b_0) \equiv x_0^3 + b_0 \equiv 0 \pmod{p}$. It means $(-b_0)^{\frac{p-1}{3}} \equiv 1 \pmod{p}$ or there exists $\sqrt[3]{-b}$. ONLY IF PART: Let $\sqrt[3]{-b} - \exists$. Then there is $\bar{x} \in \mathbb{Z}$ such that $\bar{x}^3 + b_0 \equiv 0 \pmod{p}$. We have $f_{a,b,c}(\bar{x}) \equiv \bar{x}(\bar{x}^3 + b_0) \equiv 0 \pmod{p}$ and $f'_{a,b,c}(\bar{x}) \equiv 4\bar{x}^3 + b_0 \not\equiv 0 \pmod{p}$. Due to Hensel's lemma, there exists $x \in \mathbb{Z}_p$ such that $f_{a,b,c}(x) = 0$ and $x \equiv \bar{x} \pmod{p}$. Since $\bar{x} \not\equiv 0 \pmod{p}$, we have $x \in \mathbb{Z}_p^*$.

(ii) : Case (2). We want to show that the depressed quartic equation (3.11) has solution in \mathbb{Z}_p^* if and only if $\sqrt[4]{-c} - \exists$. IF PART: Let $x \in \mathbb{Z}_p^*$ be a solution of the depressed quartic equation (3.11). Then we have $x_0^4 + c_0 \equiv 0 \pmod{p}$. It means $(-c_0)^{\frac{p-1}{4}}$ or there exists $\sqrt[4]{-c}$. ONLY IF PART: Let $\sqrt[4]{-c} - \exists$. Then there is $\bar{x} \in \mathbb{Z}$ such that $\bar{x}^4 + c_0 \equiv 0 \pmod{p}$. We get $f_{a,b,c}(\bar{x}) \equiv \bar{x}^3 + c_0 \equiv 0 \pmod{p}$ and $f'_{a,b,c}(\bar{x}) \equiv 3\bar{x}^2 \not\equiv 0 \pmod{p}$. Due to Hensel's lemma, there exists $x \in \mathbb{Z}_p$ such that $f_{a,b,c}(x) = 0$ and $x \equiv \bar{x} \pmod{p}$. Since $\bar{x} \not\equiv 0 \pmod{p}$, we have $x \in \mathbb{Z}_p^*$.

(iii) : Case (3). We want to show that the depressed quartic equation (3.11) is solvable in \mathbb{Z}_p^* if and only if R_1 is satisfied. IF PART: Let $x \in \mathbb{Z}_p^*$ be a solution of the depressed quartic equation (3.11). Then we have $x_0^4 + b_0x_0 + c_0 \equiv 0 \pmod{p}$. We know that this congruence has solutions when condition R_1 is holds. ONLY IF PART: We assume that R_1 is satisfied. Then there exists \bar{x} such that $\bar{x}^4 + b_0\bar{x} + c_0 \equiv 0 \pmod{p}$. Since $|\Delta|_p = 1$, we get $f_{a,b,c}(\bar{x}) \equiv \bar{x}^4 + b_0\bar{x} + c_0 \equiv 0 \pmod{p}$ and $f'_{a,b,c}(\bar{x}) \equiv 4\bar{x}^3 + b_0 \not\equiv 0 \pmod{p}$. By Hensel's lemma, there exists $x \in \mathbb{Z}_p$ such that $f_{a,b,c}(x) = 0$ and $x \equiv \bar{x} \not\equiv 0 \pmod{p}$. Hence, $x \in \mathbb{Z}_p^*$.

(iv) : Case (4). We want to show that the depressed quartic equation (3.11) is solvable in \mathbb{Z}_p^* if and only if R_2 or R_3 is satisfied. IF PART: Let $x \in \mathbb{Z}_p^*$ be a solution of the depressed quartic equation (3.11). Then we have $x_0^4 + a_0x_0^2 + c_0 \equiv 0 \pmod{p}$. We know that this congruence has solutions when R_2 or R_3 is holds. ONLY IF PART: Let R_2 or R_3 is satisfied. Then there exists \bar{x} such that $\bar{x}^4 + a_0\bar{x}^2 + c_0 \equiv 0 \pmod{p}$. Since $|\Delta|_p = 1$, we get $f_{a,b,c}(\bar{x}) \equiv 0 \pmod{p}$ and $f'_{a,b,c}(\bar{x}) \equiv 4\bar{x}^3 + 2a_0\bar{x} \not\equiv 0 \pmod{p}$. By

Hensel's lemma, there exists $x \in \mathbb{Z}_p$ such that $f_{a,b,c}(x) = 0$ and $x \equiv \bar{x} \not\equiv 0 \pmod{p}$. Thus, $x \in \mathbb{Z}_p^*$.

(v) : Case (5). We want to show that the depressed quartic equation (3.11) is solvable in \mathbb{Z}_p^* if and only if R_1 is satisfied. IF PART: Let $x \in \mathbb{Z}_p^*$ be a solution of the depressed quartic equation (3.11). Then we have $x_0^4 + a_0x_0^2 + b_0x_0 + c_0 \equiv 0 \pmod{p}$. We know that this congruence has solutions when R_1 holds. ONLY IF PART: We assume R_1 is satisfied. Then there exists \bar{x} such that $\bar{x}^4 + a_0\bar{x}^2 + b_0\bar{x} + c_0 \equiv 0 \pmod{p}$. Since $|\Delta|_p = 1$, we obtain $f_{a,b,c}(\bar{x}) \equiv 0 \pmod{p}$ and $f'_{a,b,c}(\bar{x}) \equiv 4\bar{x}^3 + 2a_0\bar{x} + b_0 \not\equiv 0 \pmod{p}$. By Hensel's lemma, there exists $x \in \mathbb{Z}_p$ such that $f_{a,b,c}(x) = 0$ and $x \equiv \bar{x} \not\equiv 0 \pmod{p}$. Therefore, $x \in \mathbb{Z}_p^*$. \square

Proposition 3.3.4. Let $p \geq 5$ be a prime number. Let $D_0 = -4a_0^3 - 27b_0^2$ and $u_{n+3} + a_0u_{n+1} + b_0u_n = 0$ with $u_1 = 0, u_2 = -a_0$ and $u_3 = -b_0$ for $n = 1, p-3$.

- (i) Let $|b|_p, |c|_p = |a|_p < 1$. Then the depressed quartic equation (3.11) has a solution in \mathbb{Z}_p^* if and only if $\sqrt{-a} - \exists$.
- (ii) Let $|c|_p < |a|_p = |b|_p$. Then the depressed quartic equation (3.11) has a solution in \mathbb{Z}_p^* if and only if $D_0u_{p-2}^2 \not\equiv 9a_0^2 \pmod{p}$.

Proof. (i) Let $|b|_p, |c|_p < |a|_p < 1$. We want to show that the depressed quartic equation (3.11) has solution in \mathbb{Z}_p^* if and only if $\sqrt{-a} - \exists$. IF PART: Let $x \in \mathbb{Z}_p^*$ be a solution of the depressed quartic equation (3.11). Then we get $x_0^4 + b_0x_0 \equiv x_0^2(x_0^2 + a_0) \equiv x_0^2 + a_0 \equiv 0 \pmod{p}$. It means $(-a_0)^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ or there exists $\sqrt{-a}$. ONLY IF PART: Let $\sqrt{-a} - \exists$. Let $\bar{x} \in \mathbb{Z}$ such that $\bar{x}^2 + a_0 \equiv 0 \pmod{p}$. Then we have $f_{a,b,c}(\bar{x}) \equiv \bar{x}^2(\bar{x}^2 + a_0) \equiv 0 \pmod{p}$ and $f'_{a,b,c}(\bar{x}) \equiv 4\bar{x}^3 + 2a_0\bar{x} \equiv 2\bar{x}^3 \not\equiv 0 \pmod{p}$. Due to Hensel's lemma, there exists $x \in \mathbb{Z}_p$ such that $f_{a,b,c}(x) = 0$ and $x \equiv \bar{x} \pmod{p}$. Since $\bar{x} \not\equiv 0 \pmod{p}$, we have $x \in \mathbb{Z}_p^*$.

(ii) We want to show that the depressed quartic equation (3.11) is solvable in \mathbb{Z}_p^* if and only if $D_0u_{p-2}^2 \not\equiv 9a_0^2 \pmod{p}$. IF PART: Let $x \in \mathbb{Z}_p^*$ be a solution of the depressed quartic equation (3.11). Then we have $x_0^4 + a_0x_0^2 + b_0x_0 \equiv 0 \pmod{p}$ or $x_0^3 + a_0x_0 + b_0 \equiv 0 \pmod{p}$. We know that this congruence has solutions when $D_0u_{p-2}^2 \not\equiv 9a_0^2 \pmod{p}$. ONLY IF PART: We assume $D_0u_{p-2}^2 \not\equiv 9a_0^2 \pmod{p}$. Then there exists \bar{x} such that $\bar{x}^4 + a_0\bar{x}^2 + b_0\bar{x} \equiv 0 \pmod{p}$. By Proposition 2.2.11, we get $f_{a,b,c}(\bar{x}) \equiv 0 \pmod{p}$ and $f'_{a,b,c}(\bar{x}) \equiv 4\bar{x}^3 + 2a_0\bar{x} + b_0 \not\equiv 0 \pmod{p}$. By Hensel's lemma, there exists $x \in \mathbb{Z}_p$ such that $f_{a,b,c}(x) = 0$ and $x \equiv \bar{x} \pmod{p}$. Since $\bar{x} \not\equiv 0 \pmod{p}$, we get $x \in \mathbb{Z}_p^*$. \square

Claim A. Let $p \geq 5$ be a prime number and $|\Delta|_p < 1$.

- (i) Let $|a|_p < |b|_p = |c|_p = 1$, then the depressed quartic equation (3.11) have a solution in \mathbb{Z}_p^* if and only if some conditions S_1 hold;

- (ii) Let $|b|_p < |a|_p = |c|_p = 1$ then the depressed quartic equation (3.11) have a solution in \mathbb{Z}_p^* if and only if some conditions S_2 hold;
- (iii) Let $|a|_p = |b|_p = |c|_p = 1$, then the depressed quartic equation (3.11) have a solution in \mathbb{Z}_p^* if and only if some conditions S_3 hold.

Proposition 3.3.5. *Let $p \geq 5$ be a prime number.*

- (i) *Let $a = 0$, $|b|_p = |c|_p = 1$ and $\Delta = 0$. Then the depressed quartic equation (3.11) has a solution in \mathbb{Z}_p^* .*
- (ii) *Let $b = 0$, $|a|_p = |c|_p = 1$ and $\Delta = 0$. Then the depressed quartic equation (3.11) has a solution in \mathbb{Z}_p^* if and only if $\sqrt{-2a} - \exists$.*

Proof. (i) Let $a = 0$, $|b|_p = |c|_p = 1$ and $\Delta = 0$. So

$$\Delta = 27b^4 + 256c^3 = 0.$$

One could verify $\bar{x} = -\frac{4c}{3b} \in \mathbb{Z}_p^*$ is a solution of the depressed quartic equation (3.11).

(ii) Let $b = 0$, $|a|_p = |c|_p = 1$ and $\Delta = 0$. So

$$\Delta = 16c(a^4 - 8a^2c + 16c^2) = 16c(a^2 - 4c)^2 = 0.$$

This means $c = \frac{a^2}{4}$. Then the depressed quartic equation (3.11) is written as follows:

$$x^4 + ax^2 + \frac{a^2}{4} = \left(x^2 + \frac{a}{2}\right)^2 = 0.$$

It clearly shows that the depressed quartic equation (3.11) has a solution if and only if $\sqrt{-\frac{a}{2}}$ exists, equivalently $\sqrt{-2a}$ exists and this solution is in \mathbb{Z}_p^* . □

Proposition 3.3.6. *Let $p \geq 5$ be a prime number. Suppose $\max\{|a|_p, |b|_p, |c|_p\} > 1$ and two of $|a|_p, |b|_p, |c|_p$ are the maximum. Then the depressed quartic equation (3.11) has a solution in \mathbb{Z}_p^* if and only if either one of the following conditions holds:*

- (i) $|a|_p < |b|_p = |c|_p$;
- (ii) $|b|_p < |a|_p = |c|_p$ and $\sqrt{-ac} - \exists$;
- (iii) $|c|_p < |a|_p = |b|_p$.

Proof. Let $p \geq 5$ be a prime number. Let $\max\{|a|_p, |b|_p, |c|_p\} > 1$. Then by Proposition 3.3.2, if the depressed quartic equation (3.11) has solutions in \mathbb{Z}_p^* , then one could check either one of the following conditions holds:

- (1) $|a|_p < |b|_p = |c|_p$;
- (2) $|b|_p < |a|_p = |c|_p$;
- (3) $|c|_p < |a|_p = |b|_p$.

We shall analyze each cases. Let $f_{a,b,c}(x) = x^4 + ax^2 + bx + c$. Then $f'_{a,b,c}(x) = 4x^3 + 2ax + b$.

(i) : Case (1). We want to show that the depressed quartic equation (3.11) is solvable in \mathbb{Z}_p^* . Since $|b|_p = |c|_p = p^k$ for some $k \geq 1$, it is clear that the solvability of the following two depressed quartic equations $x^4 + ax^2 + bx + c = 0$ and $p^k x^4 + p^k ax^2 + b^*x + c^* = 0$ are equivalent. Moreover, any solution of the first depressed quartic equation is a solution of the second one and vice versa. On the other hand, the second depressed quartic equation is suitable for applying the Hensel's lemma. Let us choose \bar{x} such that $b_0\bar{x} + c_0 \equiv 0 \pmod{p}$. Suppose that $g_{b,c}(x) = p^k x^4 + p^k ax^2 + b^*x + c^*$. We have $g_{b,c}(\bar{x}) \equiv b_0\bar{x} + c_0 \equiv 0 \pmod{p}$ and $g'_{b,c}(\bar{x}) \equiv b_0 \not\equiv 0 \pmod{p}$. Due to Hensel's lemma, there exists $x \in \mathbb{Z}_p$ such that $g_{b,c}(x) = 0$ and $x \equiv \bar{x} \pmod{p}$. Since $\bar{x} \not\equiv 0 \pmod{p}$, we have $x \in \mathbb{Z}_p^*$.

(ii) : Case (2). We want to show that the depressed quartic equation (3.11) is solvable in \mathbb{Z}_p^* if and only if there exists $\sqrt{-ac}$. Since $|a|_p = |c|_p = p^k$ for some $k \geq 1$, it is clear that the solvability of the following two depressed quartic equations $x^4 + ax^2 + bx + c = 0$ and $p^k x^4 + a^*x^2 + p^k bx + c^* = 0$ are equivalent and moreover, any solution of the first depressed quartic equation is a solution of the second one and vice versa. On the other hand, the second depressed quartic equation is suitable to apply the Hensel's lemma. IF PART: Let $x \in \mathbb{Z}_p^*$ be a solution of the depressed quartic equation (3.11). Then we have $a_0x_0^2 + c_0 \equiv 0 \pmod{p}$. It means $(-a_0c_0)^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ or equivalently there exists $\sqrt{-ac}$. ONLY IF PART: Assume $\sqrt{-ac}$ exists. Then there exists \bar{x} such that $a_0\bar{x}^2 + c_0 \equiv 0 \pmod{p}$. Suppose $g_{a,c}(x) = p^k x^4 + a^*x^2 + p^k bx + c^*$. We then obtain $g_{a,c}(\bar{x}) \equiv a_0\bar{x}^2 + c_0 \equiv 0 \pmod{p}$ and $g'_{a,c}(\bar{x}) \equiv 2a_0\bar{x} \not\equiv 0 \pmod{p}$. According to Hensel's lemma, there exists $x \in \mathbb{Z}_p$ such that $g_{a,c}(x) = 0$ and $x \equiv \bar{x} \pmod{p}$. Since $\bar{x} \not\equiv 0 \pmod{p}$, we have $x \in \mathbb{Z}_p^*$.

(iii) : Case (3). We want to show that the depressed quartic equation (3.11) is solvable in \mathbb{Z}_p^* . Since $|a|_p = |b|_p = p^k$ for some $k \geq 1$, it is clear that the solvability of the following two depressed quartic equations $x^4 + ax^2 + bx + c = 0$ and $p^k x^4 + a^*x^2 + b^*x + p^k c = 0$ are equivalent and moreover, any solution of the first depressed quartic equation is a solution of the second one and vice versa. On the other hand, the second depressed quartic equation is suitable to apply Hensel's lemma. Let $a_0\bar{x} + b_0 \equiv 0 \pmod{p}$. Suppose that $g_{a,b}(x) = p^k x^4 + a^*x^2 + b^*x + p^k c$. We then have $g_{a,b}(\bar{x}) \equiv \bar{x}(a_0\bar{x} + b_0) \equiv 0 \pmod{p}$ and $g'_{a,b}(\bar{x}) \equiv 2a_0\bar{x} + b_0 \equiv a_0\bar{x} \not\equiv 0 \pmod{p}$. Due to Hensel's lemma, there exists $x \in \mathbb{Z}_p$ such that $g_{a,b}(x) = 0$ and $x \equiv \bar{x} \not\equiv 0 \pmod{p}$. Thus, $x \in \mathbb{Z}_p^*$. \square

Proposition 3.3.7. Let $p \geq 5$ be a prime number. Let $|a|_p = |b|_p = |c|_p > 1$ and $|a|_p < |\bar{\Delta}|_p \leq |a|_p^2$ where $\bar{\Delta} = b^2 - 4ac$. If $\sqrt{\bar{\Delta}} - \exists$, then the depressed quartic equation (3.11) has a solution in \mathbb{Z}_p^* .

Proof. Let $|a|_p = |b|_p = |c|_p > 1$ with $|a|_p = |b|_p = |c|_p = p^\ell$ or $a = p^\ell a^*$, $b = p^\ell b^*$, $c = p^\ell c^*$ where $\ell \geq 1$. Let $\bar{\Delta} = b^2 - 4ac = p^{-2\ell} \delta$ where $\delta = b^{*2} - 4a^*c^*$. We can write the depressed quartic equation (3.11) as follows

$$p^\ell x^4 + a^* x^2 + b^* x + c^* = 0.$$

We get from the last equation that

$$4a^* p^\ell x^4 + (2a^* x + b^*)^2 - \delta = 0, \quad (3.14)$$

$$p^\ell (2a^* x)^4 + 4(a^*)^3 [(2a^* x + b^*)^2 - \delta] = 0. \quad (3.15)$$

Assume that $|a|_p < |\bar{\Delta}|_p \leq |a|_p^2$. It means $\frac{1}{|a|_p} < |\delta|_p \leq 1$. We want to show that the depressed quartic equation (3.11) is solvable in \mathbb{Z}_p^* if and only if there exists $\sqrt{\bar{\Delta}}$. IF PART: Let $x \in \mathbb{Z}_p^*$ be a solution of the depressed quartic equation (3.11). We get from (3.14) that $(2a^* x + b^*)^2 \equiv \delta \pmod{p^\ell}$. It means that there exists $\sqrt{\delta}$ or equivalently $\sqrt{\bar{\Delta}}$. ONLY IF PART: Assume that there exists $\sqrt{\bar{\Delta}}$. We choose $\bar{x} \in \mathbb{Z}$ such that $(2a^* \bar{x} + b^*)^2 \equiv \delta \pmod{p^m}$. Suppose that $\bar{f}_{a,b,c}(x) = p^k x^3 + a^* x^2 + b^* x + c^*$. We then have

$$(2a^*)^4 \bar{f}_{a,b,c}(\bar{x}) = p^\ell (2a^* \bar{x})^4 + 4(a^*)^3 [(2a^* \bar{x} + b^*)^2 - \delta] \equiv 0 \pmod{p^{m+1}} \quad \text{and}$$

$$\begin{aligned} (2a^*)^4 \bar{f}'_{a,b,c}(\bar{x}) &= 4p^\ell (2a^* \bar{x})^3 + (2a^*)^4 (2a^* \bar{x} + b^*) \\ &\equiv (2a^*)^4 (2a^* \bar{x} + b^*) \pmod{p^{\frac{m}{2}}} \equiv 0 \pmod{p^{\frac{m}{2}}} \\ &\not\equiv 0 \pmod{p^{\frac{m}{2}+1}}. \end{aligned}$$

Due to Hensel's lemma, there exists $x \in \mathbb{Z}_p$ such that $\bar{f}_{a,b,c}(x) = 0$ and $x \equiv \bar{x} \pmod{p}$. Since $\bar{x} \not\equiv 0 \pmod{p}$, we have $x \in \mathbb{Z}_p^*$. \square

Claim B. Let $p \geq 5$ be a prime number. Let $|a|_p = |b|_p = |c|_p > 1$. Then the depressed quartic equation (3.11) has a solution in \mathbb{Z}_p^* if and only if some conditions S_4 hold.

We then obtain the following conclusion. Let $p \geq 5$ be a prime number. Then the depressed quartic equation (3.11) has solutions in \mathbb{Z}_p^* if and only if one of the following conditions holds:

- (i) $|a|_p, |c|_p < |b|_p = 1$ and $\sqrt[3]{-b} - \exists$;
- (ii) $|a|_p, |b|_p < |c|_p = 1$ and $\sqrt[4]{-c} - \exists$;
- (iii) $|b|_p, |c|_p < |a|_p = 1$ and $\sqrt{-a} - \exists$;

- (iv) $|a|_p < |b|_p = |c|_p = 1$ with
 - (a) $|\Delta|_p = 1$ and R_1 ;
 - (b) $|\Delta|_p < 1$ and S_1 ;
- (v) $|b|_p < |a|_p = |c|_p = 1$ with
 - (a) $|\Delta|_p = 1$ and R_2 or R_3 ;
 - (b) $|\Delta|_p < 1$ and S_2 ;
- (vi) $|c|_p < |a|_p = |b|_p = 1$ and $D_0 u_{p-2}^2 \not\equiv 9a_0^2 \pmod{p}$;
- (vii) $|a|_p < |b|_p = |c|_p$ and $|b|_p = |c|_p > 1$;
- (viii) $|b|_p < |a|_p = |c|_p$, $|a|_p = |c|_p > 1$ and $\sqrt{-ac} - \exists$;
- (ix) $|c|_p < |a|_p = |b|_p$ and $|a|_p = |b|_p > 1$;
- (x) $|a|_p = |b|_p = |c|_p = 1$ with
 - (a) $|\Delta|_p = 1$ and R_1 ;
 - (b) $|\Delta|_p < 1$ and S_3 ;
- (xi) $|a|_p = |b|_p = |c|_p > 1$ and S_4 .

3.3.2 Solvability Over \mathbb{Q}_p

In this subsection, we provide the solvability condition of the depressed quartic equation (3.11) over \mathbb{Q}_p .

Theorem 3.3.8. *Suppose $p \geq 5$ be a prime number and*

$$\Delta = -b^2(4a^3 - 27b^2) + 16c(a^4 + 9ab^2 - 8a^2c + 16c^2).$$

Let $D_0 = -4a_0^3 - 27b_0^2$, $\overline{\Delta}$ and $u_{n+3} + a_0u_{n+1} + b_0u_n = 0$ with $u_1 = 0, u_2 = -a_0$ and $u_3 = -b_0$ for $n = \overline{1, p-3}$. Let one of the following conditions holds:

- (i) $|b|_p^4, |c|_p^3 < |a|_p^6, \quad \sqrt{-a} - \exists$.
- (ii) $|a|_p^6, |c|_p^3 < |b|_p^4, \quad \sqrt[3]{-b} - \exists$;
- (iii) $|a|_p^6, |b|_p^4 < |c|_p^3, \quad \sqrt[4]{-c} - \exists$;
- (iv) $|a|_p^6 < |b|_p^4 = |c|_p^3, \quad |\Delta|_p = |b|_p^4 = |c|_p^3, \quad R_1$;
- (v) $|b|_p^4 < |a|_p^6 = |c|_p^3, \quad |\Delta|_p = |a|_p^6 = |c|_p^3, \quad R_2 \text{ or } R_3$;

- (vi) $|c|_p^3 < |a|_p^6 = |b|_p^4, \quad D_0 u_{p-2}^2 \not\equiv 9a_0^2 \pmod{p};$
- (vii) $|c|_p^3 < |b|_p^4, \quad |a|_p |c|_p < |b|_p^2;$
- (viii) $|c|_p^3 < |a|_p^6, \quad |b|_p^2 < |a|_p |c|_p, \quad \sqrt{-ac} - \exists;$
- (ix) $|b|_p^4 < |a|_p^6, \quad |a|_p |c|_p < |b|_p^2;$
- (x) $|a|_p^6 = |b|_p^4 = |c|_p^3, \quad |\Delta|_p = |a|_p^6 = |b|_p^4 = |c|_p^3, \quad R_1;$
- (xi) $|c|_p^3 < |b|_p^4 < |a|_p^6, \quad |b|_p^2 = |a|_p |c|_p, \quad \frac{|c|_p^2}{|a|_p} < |\bar{\Delta}|_p \leq |a|_p |c|_p, \quad \sqrt{\bar{\Delta}} - \exists.$

Then the depressed quartic equation (3.11) has solutions in \mathbb{Q}_p .

Proof. Let $p \geq 5$ be a prime number, $x \in \mathbb{Q}_p$ be a nonzero p -adic number and $|x|_p = p^{-k}$ where $k \in \mathbb{Z}$. By Proposition 3.3.1, x is a solution of the depressed quartic equation (3.11) in \mathbb{Q}_p if and only if $y = p^k x$ is a solution of the depressed quartic equation (3.12) in \mathbb{Z}_p^* . Let $A_k = p^{-2k} a$, $B_k = p^{-3k} b$ and $C_k = p^{-4k} c$.

Due to Propositions 3.3.3, 3.3.4, 3.3.6 and 3.3.7, if either one of the following conditions holds true:

1. I. $|A_k|_p = 1, |B_k|_p < 1, |C_k|_p < 1, \sqrt{-A_k} - \exists;$
 II. $|B_k|_p = 1, |A_k|_p < 1, |C_k|_p < 1, \sqrt[3]{-B_k} - \exists;$
 III. $|C_k|_p = 1, |A_k|_p < 1, |B_k|_p < 1, \sqrt[4]{-C_k} - \exists;$
2. IV. $|A_k|_p < |B_k|_p = |C_k|_p = 1, \quad |D_k|_p = 1, \quad R_1;$
 V. $|B_k|_p < |A_k|_p = |C_k|_p = 1, \quad |D_k|_p = 1, \quad R_2 \quad \text{or} \quad R_3;$
 VI. $|C_k|_p < |A_k|_p = |B_k|_p = 1, \quad D_0 u_{p-2}^2 \not\equiv 9a_0^2 \pmod{p};$
 VII. $|A_k|_p < |B_k|_p = |C_k|_p, \quad |B_k|_p = |C_k|_p > 1;$
 VIII. $|B_k|_p < |A_k|_p = |C_k|_p, \quad |A_k|_p = |C_k|_p > 1, \quad \sqrt{-A_k C_k} - \exists;$
 IX. $|C_k|_p < |A_k|_p = |B_k|_p, \quad |A_k|_p = |B_k|_p > 1;$
3. X. $|A_k|_p = |B_k|_p = |C_k|_p = 1, \quad |D_k|_p = 1, \quad R_1.$
 XI. $|A_k|_p = |B_k|_p = |C_k|_p > 1, \quad |A_k|_p < |\bar{\Delta}_k|_p \leq |A_k|_p^2, \quad \sqrt{\bar{\Delta}_k} - \exists;$

then the depressed quartic equation (3.12) has a solution in \mathbb{Z}_p^* where $\bar{\Delta}_k = B_k^2 - 4A_k C_k$, $A_k = p^{-2k} \frac{a^*}{|a|_p}$, $B_k = p^{-3k} \frac{b^*}{|b|_p}$, $C_k = p^{-4k} \frac{c^*}{|c|_p}$ and $a^*, b^*, c^* \in \mathbb{Z}_p^*$ with

$$a^* = a_0 + a_1 \cdot p + \dots, \quad b^* = b_0 + b_1 \cdot p + \dots, \quad c^* = c_0 + c_1 \cdot p + \dots$$

It is clear $|A_k|_p = p^{2k} |a|_p$, $|B_k|_p = p^{3k} |b|_p$ and $|C_k|_p = p^{4k} |c|_p$. Now, in every case (I)-(XI), we want to describe all p -adic numbers $a, b, c \in \mathbb{Q}_p$ for which all the conditions should be satisfied for some $k \in \mathbb{Z}$.

- (i) : We consider the condition 1.I. From $|A_k|_p = 1$, we obtain $k = \frac{1}{2} \log_p |a|_p$. It follows from $|B_k|_p < 1$ and $|C_k|_p < 1$ that $|b|_p^2 < |a|_p^3$ and $|c|_p < |a|_p^2$ respectively. The condition $\sqrt{-A_k} - \exists$ is equivalent to $\sqrt{-a} - \exists$. Therefore, if $|b|_p^4, |c|_p^3 < |a|_p^6$ and $\sqrt{-a} - \exists$, then the condition 1.I. is satisfied with $k = \frac{1}{2} \log_p |a|_p \in \mathbb{Z}$.
- (ii) : Let us consider the condition 1.II. From $|B_k|_p = 1$, we have that $k = \frac{1}{3} \log_p |b|_p$. It follows from $|A_k|_p < 1$ and $|C_k|_p < 1$ that $|a|_p^3 < |b|_p^2$ and $|c|_p^3 < |b|_p^4$ respectively. The existence of $\sqrt[3]{-B_k}$ is equivalent to the existence of $\sqrt[3]{-b}$. Hence, if $|a|_p^6, |c|_p^3 < |b|_p^4$ and $\sqrt[3]{-b} - \exists$, then the condition 1.II. is satisfied with $k = \frac{1}{3} \log_p |b|_p \in \mathbb{Z}$.
- (iii) : We consider the condition 1.III. From $|C_k|_p = 1$, we get that $k = \frac{1}{4} \log_p |c|_p$. It follows from $|A_k|_p < 1$ and $|B_k|_p < 1$ that $|a|_p^2 < |c|_p$ and $|b|_p^4 < |c|_p^3$ respectively. One could check that $\sqrt[4]{-C_k} - \exists$ is equivalent to $\sqrt[4]{-c} - \exists$. Thus if $|a|_p^6, |b|_p^4 < |c|_p^3$ and $\sqrt[4]{-c} - \exists$ then the condition 1.III. is satisfied with $k = \frac{1}{4} \log_p |c|_p \in \mathbb{Z}$.
- (iv) : Let us consider the condition 2.IV. From $|B_k|_p = 1$ and $|C_k|_p = 1$, we obtain that $k = \frac{1}{3} \log_p |b|_p$ and $k = \frac{1}{4} \log_p |c|_p$. It means that $|b|_p^4 = |c|_p^3$. Moreover, from $|A_k|_p < 1$, $|A_k|_p < |B_k|_p$ and $|A_k|_p < |C_k|_p$ we have $|a|_p^3 < |b|_p^2$ and $|a|_p^2 < |c|_p$. One could check $|D_k|_p = 1$ is equivalent to $|\Delta|_p = |b|_p^4 = |c|_p^3$. Therefore, if $|a|_p^6 < |b|_p^4 = |c|_p^3$, $|\Delta|_p = |b|_p^4 = |c|_p^3$ and R_1 , then the condition 2.IV is satisfied with $k = \frac{1}{3} \log_p |b|_p = \frac{1}{4} \log_p |c|_p \in \mathbb{Z}$.
- (v) : Let us consider the condition 2.V. From $|A_k|_p = 1$ and $|C_k|_p = 1$, we obtain $k = \frac{1}{2} \log_p |a|_p$ and $k = \frac{1}{4} \log_p |c|_p$. This means $|a|_p^2 = |c|_p$. Moreover, from $|B_k|_p < 1$, $|B_k|_p < |A_k|_p$ and $|B_k|_p < |C_k|_p$ we get $|b|_p^2 < |a|_p^3$ and $|b|_p^3 < |c|_p^2$. The condition $|D_k|_p = 1$ is equivalent to $|\Delta|_p = |b|_p^4 = |c|_p^3$. Therefore, if $|b|_p^4 < |a|_p^6 = |c|_p^3$ and $|\Delta|_p = |a|_p^6 = |c|_p^3$ with R_2 or R_3 , then the condition 2.V is satisfied with $k = \frac{1}{2} \log_p |a|_p = \frac{1}{4} \log_p |c|_p \in \mathbb{Z}$.
- (vi) : We consider the condition 2.VI. From $|A_k|_p = 1$ and $|B_k|_p = 1$, we obtain $k = \frac{1}{2} \log_p |a|_p$ and $k = \frac{1}{3} \log_p |b|_p$. It means that $|a|_p^3 = |b|_p^2$. Meanwhile, from $|C_k|_p < 1$, $|C_k|_p < |A_k|_p$ and $|C_k|_p < |B_k|_p$, we have $|c|_p < |a|_p^2$ and $|c|_p^3 < |b|_p^4$. Therefore, if $|c|_p^3 < |a|_p^6 = |b|_p^4$ and $D_0 u_{p-2}^2 \not\equiv 9a_0^2 \pmod{p}$, then the condition 2.VI is satisfied with $k = \frac{1}{2} \log_p |a|_p = \frac{1}{3} \log_p |b|_p \in \mathbb{Z}$.
- (vii) : Let us consider the condition 2.VII. From $|B_k|_p = |C_k|_p$, we obtain that $k = \log_p \frac{|b|_p}{|c|_p}$. It follows from $|B_k|_p > 1$ and $|C_k|_p > 1$ that $|c|_p^3 < |b|_p^4$. Moreover, from $|A_k|_p < |B_k|_p$ and $|A_k|_p < |C_k|_p$ we have $|a|_p |c|_p < |b|_p^2$. Therefore, if $|c|_p^3 < |b|_p^4$ and $|a|_p |c|_p < |b|_p^2$, then the condition 2.VII. is satisfied with $k = \log_p \frac{|b|_p}{|c|_p} \in \mathbb{Z}$.
- (viii) : Let us consider the condition 2.VIII. From $|A_k|_p = |C_k|_p$, we obtain $k = \frac{1}{2} \log_p \frac{|a|_p}{|c|_p}$. It follows from $|A_k|_p > 1$ and $|C_k|_p > 1$ that $|c|_p < |a|_p^2$. Moreover, the conditions $|B_k|_p < |A_k|_p$ and $|B_k|_p < |C_k|_p$ imply $|b|_p^2 < |a|_p |c|_p$. It is clear that the existence of $\sqrt{-A_k C_k}$ is equivalent to the existence of $\sqrt{-ac}$. Thus if $|c|_p^3 < |a|_p^6, |b|_p^2 < |a|_p |c|_p$ and $\sqrt{-ac} - \exists$, then the condition 2.VIII. is satisfied with $k = \frac{1}{2} \log_p \frac{|a|_p}{|c|_p} \in \mathbb{Z}$.
- (ix) : We consider the condition 2.IX. From $|A_k|_p = |B_k|_p$, we obtain $k = \log_p \frac{|a|_p}{|b|_p}$.

and from $|A_k|_p > 1$ and $|B_k|_p > 1$, we get $|b|_p^2 < |a|_p^3$. Moreover, the conditions $|C_k|_p < |A_k|_p$ and $|C_k|_p < |B_k|_p$ imply $|a|_p|c|_p < |b|_p^2$. Therefore, if $|b|_p^2 < |a|_p^3$ and $|a|_p|c|_p < |b|_p^2$, then the condition 2.IX. is satisfied with $k = \log_p \frac{|b|_p}{|a|_p} \in \mathbb{Z}$.

(x) : Let us consider the condition 3.X. From $|A_k|_p = 1$, $|B_k|_p = 1$ and $|C_k|_p = 1$, we obtain $2k = \log_p |a|_p$, $3k = \log_p |b|_p$ and $4k = \log_p |c|_p$ respectively. This implies $|a|_p^6 = |b|_p^4 = |c|_p^3$. One could check $|D_k|_p = 1$ is equivalent to $|\Delta|_p = |a|_p^6 = |b|_p^4 = |c|_p^3$. Therefore, if $|a|_p^6 = |b|_p^4 = |c|_p^3$, $|\Delta|_p = |a|_p^6 = |b|_p^4 = |c|_p^3$ and R_1 , then the condition 3.II is satisfied with $k = \frac{1}{2} \log_p |a|_p = \frac{1}{3} \log_p |b|_p = \frac{1}{4} \log_p |c|_p \in \mathbb{Z}$.

(xi) : Let us consider the condition 3.XI. From $|A_k|_p = |B_k|_p$, $|A_k|_p = |C_k|_p$ and $|B_k|_p = |C_k|_p$, we obtain that $k = \log_p \frac{|a|_p}{|b|_p}$, $2k = \log_p \frac{|a|_p}{|c|_p}$ and $k = \log_p \frac{|b|_p}{|c|_p}$ respectively. This means $|b|_p^2 = |a|_p|c|_p$. Moreover, from $|A_k|_p > 1$, $|B_k|_p > 1$ and $|C_k|_p > 1$, we have $|b|_p^2 < |a|_p^3$, $|c|_p < |a|_p^2$ and $|c|_p^3 < |b|_p^4$. This means $|c|_p^3 < |b|_p^4 < |a|_p^6$. We can check $|A_k|_p < |\bar{\Delta}|_p \leq |A_k|_p^2$ implies $\frac{|c|_p^2}{|a|_p} < |\bar{\Delta}|_p \leq |a|_p|c|_p$ and the existence of $\sqrt{\bar{\Delta}}$ is equivalent to the existence of $\sqrt{\bar{\Delta}}$. Therefore, if $|c|_p^3 < |b|_p^4 < |a|_p^6$, $|b|_p^2 = |a|_p|c|_p$, $\frac{|c|_p^2}{|a|_p} < |\bar{\Delta}|_p \leq |a|_p|c|_p$ and $\sqrt{\bar{\Delta}} - \exists$, then the condition 3.X is satisfied with $k = \log_p \frac{|a|_p}{|b|_p} = \frac{1}{2} \log_p \frac{|a|_p}{|c|_p} = \log_p \frac{|b|_p}{|c|_p} \in \mathbb{Z}$. \square

Claim C. Let $p \geq 5$ be a prime number.

- (i) Let $|a|_p^6 < |b|_p^4 = |c|_p^3$, $|\Delta|_p < |b|_p^4 = |c|_p^3$. Then the depressed quartic equation (3.11) has a solution in \mathbb{Q}_p if and only if some conditions S_1 hold;
- (ii) Let $|b|_p^4 < |a|_p^6 = |c|_p^3$, $|\Delta|_p < |a|_p^6 = |c|_p^3$. Then the depressed quartic equation (3.11) has a solution in \mathbb{Q}_p if and only if some conditions S_2 hold;
- (iii) Let $|a|_p^6 = |b|_p^4 = |c|_p^3$, $|\Delta|_p < |a|_p^6 = |b|_p^4 = |c|_p^3$. Then the depressed quartic equation (3.11) has a solution in \mathbb{Q}_p if and only if some conditions S_3 hold;
- (iv) Let $|c|_p^3 < |b|_p^4 < |a|_p^6$, $|b|_p^2 = |a|_p|c|_p$. Then the depressed quartic equation (3.11) has a solution in \mathbb{Q}_p if and only if some conditions S_4 hold.

Our concluding remark on the depressed quartic equation (3.11) on \mathbb{Q}_p is the following: Let $p \geq 5$ be a prime number. Then the depressed quartic equation (3.11) has solution in \mathbb{Q}_p if and only if one of the following conditions holds

- A. 1. $|b|_p^4, |c|_p^3 < |a|_p^6$,
 - i. $|c|_p^3 < |b|_p^4$
 - a. $|b|_p^2 < |a|_p|c|_p$, $\sqrt{-ac} - \exists$ or $\sqrt{-a} - \exists$;
 - b. $|b|_p^2 = |a|_p|c|_p$, $\sqrt{-a} - \exists$ or S_4 ;
 - c. $|b|_p^2 > |a|_p|c|_p$;
 - ii. $|b|_p^4 \leq |c|_p^3$

- a. $|b|_p^2 < |a|_p |c|_p, \quad \sqrt{-ac} - \exists \quad \text{or} \quad \sqrt{-a} - \exists;$
- b. $|b|_p^2 \geq |a|_p |c|_p, \quad \sqrt{-a} - \exists;$
- 2. $|a|_p^6, |c|_p^3 < |b|_p^4;$
- 3. $|a|_p^6, |b|_p^4 < |c|_p^3, \quad \sqrt[3]{-c} - \exists;$
- B. 4. $|a|_p^6 < |b|_p^4 = |c|_p^3$
 - i. $|\Delta|_p = |b|_p^4, \quad R_1;$
 - ii. $|\Delta|_p < |b|_p^4, \quad S_1;$
- 5. $|b|_p^4 < |a|_p^6 = |c|_p^3$
 - i. $|\Delta|_p = |a|_p^6, \quad R_2 \quad \text{or} \quad R_3;$
 - ii. $|\Delta|_p < |a|_p^6, \quad S_2;$
- 6. $|c|_p^3 < |a|_p^6 = |b|_p^4.$
- C. 7. $|a|_p^6 = |b|_p^4 = |c|_p^3$
 - i. $|\Delta|_p = |a|_p^6, \quad R_1;$
 - ii. $|\Delta|_p < |a|_p^6, \quad S_3;$

Demonstration. We will consider the results in Theorem 3.3.8 and Remark 3.3.2. Let

$$\begin{aligned}
\mathbf{H}_1 &= \{(a, b, c) \in \mathbb{Q}_p^3 : |b|_p^4, |c|_p^3 < |a|_p^6, \quad \sqrt{-a} - \exists\}; \\
\mathbf{H}_2 &= \{(a, b, c) \in \mathbb{Q}_p^3 : |a|_p^6, |c|_p^3 < |b|_p^4, \quad \sqrt[3]{-b} - \exists\}; \\
\mathbf{H}_3 &= \{(a, b, c) \in \mathbb{Q}_p^3 : |a|_p^6, |b|_p^4 < |c|_p^3, \quad \sqrt[4]{-c} - \exists\}; \\
\mathbf{H}_{4,1} &= \{(a, b, c) \in \mathbb{Q}_p^3 : |a|_p^6 < |b|_p^4 = |c|_p^3, \quad |\Delta|_p = |b|_p^4 = |c|_p^3, \quad (I)\}; \\
\mathbf{H}_{4,2} &= \{(a, b, c) \in \mathbb{Q}_p^3 : |a|_p^6 < |b|_p^4 = |c|_p^3, \quad |\Delta|_p < |b|_p^4 = |c|_p^3, \quad S_1\}; \\
\mathbf{H}_{5,1} &= \{(a, b, c) \in \mathbb{Q}_p^3 : |b|_p^4 < |a|_p^6 = |c|_p^3, \quad |\Delta|_p = |a|_p^6 = |c|_p^3, \quad (II) \quad \text{or} \quad (IV)\}; \\
\mathbf{H}_{5,2} &= \{(a, b, c) \in \mathbb{Q}_p^3 : |b|_p^4 < |a|_p^6 = |c|_p^3, \quad |\Delta|_p < |a|_p^6 = |c|_p^3, \quad S_2\}; \\
\mathbf{H}_{6,1} &= \{(a, b, c) \in \mathbb{Q}_p^3 : |c|_p^3 < |a|_p^6 = |b|_p^4, \quad |\Delta|_p = |a|_p^6 = |b|_p^4, \quad D_0 u_{p-2}^2 \not\equiv 9a_0^2 \pmod{p}\}; \\
\mathbf{H}_7 &= \{(a, b, c) \in \mathbb{Q}_p^3 : |c|_p^3 < |b|_p^4, \quad |a|_p |c|_p < |b|_p^2\}; \\
\mathbf{H}_8 &= \{(a, b, c) \in \mathbb{Q}_p^3 : |c|_p^3 < |a|_p^6, \quad |b|_p^2 < |a|_p |c|_p, \quad \sqrt{-ac} - \exists\}; \\
\mathbf{H}_9 &= \{(a, b, c) \in \mathbb{Q}_p^3 : |b|_p^4 < |a|_p^6, \quad |a|_p |c|_p < |b|_p^2\}; \\
\mathbf{H}_{10,1} &= \{(a, b, c) \in \mathbb{Q}_p^3 : |a|_p^6 = |b|_p^4 = |c|_p^3, \quad |\Delta|_p = |a|_p^6 = |b|_p^4 = |c|_p^3, \quad (I)\}; \\
\mathbf{H}_{10,2} &= \{(a, b, c) \in \mathbb{Q}_p^3 : |a|_p^6 = |b|_p^4 = |c|_p^3, \quad |\Delta|_p < |a|_p^6 = |b|_p^4 = |c|_p^3, \quad S_3\}; \\
\mathbf{H}_{11} &= \{(a, b, c) \in \mathbb{Q}_p^3 : |c|_p^3 < |b|_p^4 < |a|_p^6, \quad |b|_p^2 = |a|_p |c|_p, \quad S_4\}.
\end{aligned}$$

All these sets are the solvability regions of the depressed quartic equation (3.11) on \mathbb{Q}_p . We will check all these sets so we could simplify the overlapping regions.

One could see $\mathbf{H}_7 = \mathbf{H}_{7,1} \cup \mathbf{H}_{7,2} \cup \mathbf{H}_{7,3}$ where

$$\begin{aligned}
\mathbf{H}_{7,1} &= \{(a, b, c) \in \mathbb{Q}_p^3 : |c|_p^3 < |b|_p^4, \quad |a|_p |c|_p < |b|_p^2, \quad |b|_p^4 > |a|_p^6\} \\
&= \{(a, b, c) \in \mathbb{Q}_p^3 : |c|_p^3, |a|_p^6 < |b|_p^4\}; \\
\mathbf{H}_{7,2} &= \{(a, b, c) \in \mathbb{Q}_p^3 : |c|_p^3 < |b|_p^4, \quad |a|_p |c|_p < |b|_p^2, \quad |b|_p^4 = |a|_p^6\} \\
&= \{(a, b, c) \in \mathbb{Q}_p^3 : |c|_p^3 < |b|_p^4 = |a|_p^6\}; \\
\mathbf{H}_{7,3} &= \{(a, b, c) \in \mathbb{Q}_p^3 : |c|_p^3 < |b|_p^4, \quad |a|_p |c|_p < |b|_p^2, \quad |b|_p^4 < |a|_p^6\} \\
&= \{(a, b, c) \in \mathbb{Q}_p^3 : |c|_p^3 < |b|_p^4 < |a|_p^6, \quad |a|_p |c|_p < |b|_p^2\}.
\end{aligned}$$

A.1: We consider the sets \mathbf{H}_1 , $\mathbf{H}_{7,3}$, \mathbf{H}_8 , \mathbf{H}_9 and \mathbf{H}_{11} .

We will show that $\max\{|a|_p^6, |b|_p^4, |c|_p^3\} = |a|_p^6$. It is true for \mathbf{H}_1 , $\mathbf{H}_{7,3}$ and \mathbf{H}_{11} . We are left to show for \mathbf{H}_8 and \mathbf{H}_9 . For \mathbf{H}_8 : We have $|c|_p^3 < |a|_p^6$ and $|b|_p^2 < |a|_p |c|_p$. This implies $|b|_p^2 < |a|_p |c|_p < |a|_p^3$. Hence $|c|_p^3, |b|_p^4 < |a|_p^6$. For \mathbf{H}_9 : We have $|b|_p^4 < |a|_p^6$ and $|a|_p |c|_p < |b|_p^2$. Then we obtain $|a|_p |c|_p < |b|_p^2 < |a|_p^3$. Thus $|c|_p^3, |b|_p^4 < |a|_p^6$. So we have shown $\max\{|a|_p^6, |b|_p^4, |c|_p^3\} = |a|_p^6$. Next we will show $\mathbf{H}_9 = \mathbf{H}_{7,3}$. We now have

$$\mathbf{H}_9 = \{(a, b, c) \in \mathbb{Q}_p^3 : |b|_p^4 < |a|_p^6, \quad |a|_p |c|_p < |b|_p^2, \quad |c|_p^3 < |a|_p^6\}.$$

From $|c|_p^3 < |a|_p^6$ and $|a|_p |c|_p < |b|_p^2$, we get $|c|_p^{\frac{3}{2}} < |a|_p |c|_p < |b|_p^2$. This means $|c|_p^3 < |b|_p^4$. So we have $\mathbf{H}_9 = \mathbf{H}_{7,3}$ since

$$\mathbf{H}_9 = \{(a, b, c) \in \mathbb{Q}_p^3 : |c|_p^3 < |b|_p^4 < |a|_p^6, \quad |a|_p |c|_p < |b|_p^2\}.$$

A.2: We check for \mathbf{H}_2 and $\mathbf{H}_{7,1}$. One can see $\mathbf{H}_2 \subset \mathbf{H}_{7,1}$.

A.3: It follows directly from \mathbf{H}_3 .

B.4: It follows directly from $\mathbf{H}_{4,1}$ and $\mathbf{H}_{4,2}$.

B.5: It follows directly from $\mathbf{H}_{5,1}$ and $\mathbf{H}_{5,2}$.

B.6: We look at \mathbf{H}_6 and $\mathbf{H}_{7,2}$. One can see easily $\mathbf{H}_6 \subset \mathbf{H}_{7,2}$.

C.7: It follows directly from $\mathbf{H}_{10,1}$ and $\mathbf{H}_{10,2}$.

In this section, we have given some solvability conditions of the quartic equation over \mathbb{Q}_p . These conditions are useful in order to show the existence of p -adic Gibbs measures of the Potts model on the Cayley tree of order 4 as stated in Theorem 3.2.3.

CHAPTER 4

THE POTTS–BETHE MAPPING OVER \mathbb{Q}_p

Some results in this chapter were published by Saburov & Ahmad (2017a) and Ahmad et al. (2018) in “Journal of Physics: Conferences Series” and “Journal of Statistical Physics” respectively.

4.1 INTRODUCTION

We introduce the Potts–Bethe mapping of degree k for $k \in \mathbb{N}$ as follows

$$f_{\theta,q,k}(x) = \left(\frac{\theta x + q - 1}{x + q + \theta - 2} \right)^k \quad (4.1)$$

where $(\theta - 1)(\theta - 1 + q) \neq 0$. Let $\mathbf{x}^{(\infty)} = 2 - \theta - q$ be the singular point of the Potts–Bethe mapping (4.1).

A point $x \in \mathbb{Q}_p$ is called a critical point of a function h if $h'(x) = 0$. Notice that the Potts–Bethe mapping (4.1) has a critical point $\mathbf{x}^{(c)} = \frac{1-q}{\theta}$. Let $g(x) = \frac{1}{x}$ and $(f_{\theta,q,k})_g = g^{-1}(x) \circ f_{\theta,q,k} \circ g(x)$. Then we can easily verify that the singular point $\mathbf{x}^{(\infty)}$ is another critical point of the Potts–Bethe mapping (4.1). If we consider $f_{\theta,q,k} : \mathbb{P}^1(\mathbb{C}_p) \rightarrow \mathbb{P}^1(\mathbb{C}_p)$, then one could check that the point ∞ is not the fixed point of $f_{\theta,q,k}$.

This Potts–Bethe mapping (4.1) appear in the study of statistical mechanics, i.e. the Potts model on the Cayley tree of order k . The Potts–Bethe mapping was given a name by Monroe (1996a,b) where he studied the dynamics of this Potts–Bethe mapping (4.1) over \mathbb{C} . In this chapter, we shall study the Potts–Bethe mapping (4.1) on \mathbb{Q}_p . Recall from Chapter 2 that $\theta = \exp_p(J)$ for some coupling constant J and $q \in \mathbb{N}$. This means $|\theta - 1|_p < 1$ and $|q|_p \leq 1$. The study of the Potts–Bethe mapping (4.1) on \mathbb{Q}_p was first studied by Mukhamedov & Khakimov (2016).

4.2 THE POTTS–BETHE MAPPING WITH GOOD REDUCTION

For generality, we suppose $\theta, q \in \mathbb{Q}_p$. The p -adic rational map with good reduction decomposed the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ into minimal subsystems. It is fully described by Theorem 2.3.15. Now let us find the conditions of the Potts–Bethe mapping (4.1) over \mathbb{Q}_p with good reduction.

Proposition 4.2.1. *Let $p \geq 3$ be a prime number.*

(I) Let $\max\{|\theta|_p, |q|_p\} < 1$. Then the Potts–Bethe mapping (4.1) has a good reduction.

(II) Let $\max\{|\theta|_p, |q|_p\} = 1$.

(i) If $|\theta - 1|_p |\theta - 1 + q|_p = 1$, then the Potts–Bethe mapping (4.1) has a good reduction.

(ii) If $|\theta - 1|_p |\theta - 1 + q|_p < 1$, then the Potts–Bethe mapping (4.1) has a bad reduction.

(III) Let $\max\{|\theta|_p, |q|_p\} > 1$.

(i) Let $|q|_p > |\theta|_p$. Then the Potts–Bethe mapping (4.1) has a bad reduction.

(ii) Let $|q|_p < |\theta|_p$. Then the Potts–Bethe mapping (4.1) has a good reduction.

(iii) Let $|q|_p = |\theta|_p$.

(a) If $|\theta + q|_p = |\theta|_p$, then the Potts–Bethe mapping (4.1) has a good reduction.

(b) If $|\theta + q|_p < |\theta|_p$, then the Potts–Bethe mapping (4.1) has a bad reduction.

Proof. Let $\tilde{f}_{\theta,q,k} = f_{\theta,q,k} \pmod{p}$.

(I) Suppose $\max\{|\theta|_p, |q|_p\} < 1$. Then we get

$$\tilde{f}_{\theta,q,k} = \left(\frac{-1}{x-2} \right)^k.$$

This implies $\deg(\tilde{f}_{\theta,q,k}) = k = \deg(f_{\theta,q,k})$. Thus, $f_{\theta,q,k}$ has a good reduction.

(II)-(i) Suppose $\max\{|\theta|_p, |q|_p\} = 1$ and $|\theta - 1|_p |\theta - 1 + q|_p = 1$. Then

$$\tilde{f}_{\theta,q,k}(x) = \left(\frac{\theta x + q - 1}{x + q + \theta - 2} \right)^k \pmod{p}.$$

Since $(\theta - 1)(\theta - 1 + q) \not\equiv 0 \pmod{p}$, the $\deg(\tilde{f}_{\theta,q,k}) = k = \deg(f_{\theta,q,k})$. Hence, $f_{\theta,q,k}$ has a good reduction.

(II)-(ii) Suppose $\max\{|\theta|_p, |q|_p\} = 1$ and $|\theta - 1|_p |\theta - 1 + q|_p < 1$. Then

$$\tilde{f}_{\theta,q,k}(x) = \left(\frac{\theta x + q - 1}{x + q + \theta - 2} \right)^k \pmod{p}.$$

We have $\deg(\tilde{f}_{\theta,q,k}) < k = \deg(f_{\theta,q,k})$ because $(\theta - 1)(\theta - 1 + q) \equiv 0 \pmod{p}$. Thus, $f_{\theta,q,k}$ has a bad reduction.

(III) Let $\max\{|\theta|_p, |q|_p\} > 1$. Then $\theta = p^m \theta^*$ and $q = p^n q^*$ for some $\theta^*, q^* \in \mathbb{Z}_p^*$ and integers $m, n < 0$. We have

$$f_{\theta,q,k}(x) = \begin{cases} \left(\frac{\theta^* x + p^{n-m} q^* - p^{-m}}{p^{-m} x + p^{n-m} q^* + \theta^* - 2p^{-m}} \right)^k & \text{if } \max\{|\theta|_p, |q|_p\} = |\theta|_p \\ \left(\frac{p^{m-n} \theta^* x + q^* - p^{-n}}{p^{-n} x + q^* + p^{m-n} \theta^* - 2p^{-n}} \right)^k & \text{if } \max\{|\theta|_p, |q|_p\} = |q|_p \\ \left(\frac{\theta^* x + q^* - p^{-m}}{p^{-m} x + q^* + \theta^* - 2p^{-m}} \right)^k & \text{if } |\theta|_p = |q|_p \end{cases}$$

This means

$$\tilde{f}_{\theta,q,k}(x) = \begin{cases} x^k & \text{if } \max\{|\theta|_p, |q|_p\} = |\theta|_p \\ 1 & \text{if } \max\{|\theta|_p, |q|_p\} = |q|_p \\ \left(\frac{\theta^* x + q^*}{\theta^* + q^*} \right)^k & \text{if } |\theta|_p = |q|_p \end{cases}$$

and

$$\deg(\tilde{f}_{\theta,q,k}(x)) = \begin{cases} = k & \text{if } \max\{|\theta|_p, |q|_p\} = |\theta|_p \\ = 0 < k & \text{if } \max\{|\theta|_p, |q|_p\} = |q|_p \\ = k & \text{if } |\theta|_p = |q|_p = |\theta + q|_p \\ \neq k & \text{if } |\theta|_p = |q|_p > |\theta + q|_p. \end{cases}$$

Therefore, the statement (III) holds. □

Due to this proposition, we obtain the following result that tell us about the dynamics of the Potts–Bethe mapping (4.1) with good reduction.

Theorem 4.2.2. *Let $p \geq 3$ be a prime number. Suppose one of the following conditions holds*

- (i) $\max\{|\theta|_p, |q|_p\} < 1$;
- (ii) $\max\{|\theta|_p, |q|_p\} = 1$ and $|\theta - 1|_p |\theta - 1 + q|_p = 1$;
- (iii) $|\theta|_p > 1$ and $|\theta|_p > |q|_p$;
- (iv) $|\theta|_p = |q|_p = |\theta + q|_p > 1$.

Then the Potts–Bethe mapping (4.1) decompose $\mathbb{P}^1(\mathbb{Q}_p)$ as follows

$$\mathbb{P}^1(\mathbb{Q}_p) = A \cup B \cup C$$

where A is the finite set consisting of all periodic points of $f_{\theta,q,k}$, $B = \bigcup_j B_j$ is the union of all (at most countably many) clopen invariant sets such that each B_j is a finite union of balls and each subsystem $f_{\theta,q,k} : B_j \rightarrow B_j$ is minimal and points in C lie in the attracting basin of a periodic orbit or of a minimal subsystem.

Proof. By Proposition 4.2.1, the conditions $\max\{|\theta|_p, |q|_p\} < 1$ and $\max\{|\theta|_p, |q|_p\} \geq 1$ with $|\theta - 1|_p |\theta - 1 + q|_p = |\theta|_p \max\{|\theta|_p, |q|_p\}$ imply that the Potts–Bethe mapping (4.1) has a good reduction. Then the decomposition of $\mathbb{P}^1(\mathbb{Q}_p)$ follows directly from the Theorem 2.3.15. \square

Note that $|\theta - 1|_p < 1$ and $|q|_p \leq 1$ when we associate the Potts–Bethe mapping (4.1) with the Potts model on Cayley trees. By this condition one could check by Proposition 4.2.1 that the Potts–Bethe mapping (4.1) has a bad reduction. There is no global result to characterize completely the dynamics of p -adic rational map with bad reduction. In what follows we will study the dynamics the Potts–Bethe mapping (4.1) of degree 3 and degree 4 with bad reduction.

4.3 THE POTTS–BETHE MAPPING OF DEGREE 3

In this section, we will investigate the dynamics of the Potts–Bethe mapping of degree 3, $f_{\theta,q,3} : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ such that

$$f_{\theta,q,3}(x) = \left(\frac{\theta x + q - 1}{x + \theta + q - 2} \right)^3. \quad (4.2)$$

We first find all the fixed points of the Potts–Bethe mapping (4.2). Let $\mathbf{Fix}\{f\} := \{x \in \mathbb{Q}_p : f(x) = x\}$ be the set of all fixed points of the mapping f . We notice $\mathbf{x}^{(0)} = 1 \in \mathbf{Fix}\{f_{\theta,q,3}\}$. Then it follows from $f_{\theta,q,3}(x) - 1 = x - 1$ that

$$(x - 1)(\theta - 1) \frac{(\theta x + q - 1)^2 + (\theta x + q - 1)(x - \mathbf{x}^{(\infty)}) + (x - \mathbf{x}^{(\infty)})^2}{(x - \mathbf{x}^{(\infty)})^3} = (x - 1).$$

Thus, any other fixed point $x \neq \mathbf{x}^{(0)}$ is a root of the following cubic equation,

$$(\theta - 1)(\theta x + q - 1)((\theta + 1)x + \theta + 2q - 3) = (x + q - 1)(x - \mathbf{x}^{(\infty)})^2. \quad (4.3)$$

Now suppose $y := \frac{x-1+q}{\theta-1} + 1$. Then the cubic equation (4.3) can be written with respect to y as

$$y^3 - (1 + \theta + \theta^2)y^2 - (2\theta + 1)(1 - \theta - q)y - (1 - \theta - q)^2 = 0. \quad (4.4)$$

Let us find all roots of the cubic equation (4.4) whenever $\theta \neq 1, 1 - q$.

Proposition 4.3.1. *Let $p \geq 5$ be a prime numbers and $0 < |\theta - 1|_p, |q|_p < 1$. Then the cubic equation (4.4) has a root $\mathbf{y}^{(1)}$ such that $|\mathbf{y}^{(1)}|_p = 1$ and $|\mathbf{y}^{(1)} - 3|_p < 1$. Furthermore, if $p \equiv 1 \pmod{6}$, then the cubic equation (4.4) has roots $\mathbf{y}^{(2)}$ and $\mathbf{y}^{(3)}$ such*

that

$$\mathbf{y}^{(2)} = \frac{1}{|1-\theta-q|_p} \left(y_0^{(2)} + y_1^{(2)} p + \dots \right) \text{ and } \mathbf{y}^{(3)} = \frac{1}{|1-\theta-q|_p} \left(y_0^{(3)} + y_1^{(3)} p + \dots \right)$$

in which $y_0^{(2)}$ and $y_0^{(3)}$ are roots of the following congruence

$$3t^2 + 3(1-\theta-q)^* t + ((1-\theta-q)^*)^2 \equiv 0 \pmod{p}. \quad (4.5)$$

Remark 4.3.2. The congruence (4.5) has a discriminant $D = -3((1-\theta-q)^*)^2$. It is solvable if and only if D (or equivalently -3) is a quadratic residue modulo p (see Rosen (2011)). Note -3 is a quadratic residue modulo p when $p \equiv 1 \pmod{6}$.

Proof. Let $|\theta-1|_p, |q|_p < 1$. Suppose $a = -(1+\theta+\theta^2)$, $b = -(2\theta+1)(1-\theta-q)$ and $c = -(1-\theta-q)^2$. Then $|b|_p^2 = |c|_p = |1-\theta-q|_p^2 < 1 = |a|_p$ and $|b|_p < |a|_p^2$, $|c|_p < |a|_p^3$. By Theorem 5.1 of Saburov & Ahmad (2015b), the cubic equation (4.4) has a root $\mathbf{y}^{(1)}$ for which $|\mathbf{y}^{(1)}|_p = |a|_p = 1$. Since $|\theta-1|_p, |q|_p < 1$ and $\mathbf{y}^{(1)} = y_0^{(1)} + y_1^{(1)} p + \dots$, then we obtain $y_0^{(1)} - 3 \equiv 0 \pmod{p}$. This implies $|\mathbf{y}^{(1)} - 3|_p < 1$.

Now let $\delta_1 = b^2 - 4ac$. We have $\delta_1 = -3(1-\theta-q)^2$ and $|b|_p^2 = |a|_p |c|_p = |1-\theta-q|_p^2 = |\delta_1|_p$. Since $p \equiv 1 \pmod{3}$, we have $\sqrt{\delta_1} \in \mathbb{Q}_p$ (or equivalently $\sqrt{-3} \in \mathbb{Q}_p$). Therefore, by Theorem 5.1 of Saburov & Ahmad (2015b), if $p \equiv 1 \pmod{6}$, then the cubic equation (4.4) has two more roots $\mathbf{y}^{(2)}$ and $\mathbf{y}^{(3)}$ such that $|\mathbf{y}^{(2)}|_p = |\mathbf{y}^{(3)}|_p = \frac{|b|_p}{|a|_p} = |1-\theta-q|_p < 1$. Assume $|1-\theta-q|_p = p^{-r}$ for some $r \in \mathbb{N}$. Then we get $\mathbf{y}^{(i)} = p^r (\mathbf{y}^{(i)})^*$ for $i = 2, 3$ and $1-\theta-q = p^r (1-\theta-q)^*$. By plugging $\mathbf{y}^{(i)}$ into the cubic equation (4.4), we obtain

$$p^r ((\mathbf{y}^{(i)})^*)^3 - (1+\theta+\theta^2)((\mathbf{y}^{(i)})^*)^2 - (2\theta+1)(1-\theta-q)^* (\mathbf{y}^{(i)})^* - ((1-\theta-q)^*)^2 = 0$$

and for $i = 2, 3$ that

$$3(y_0^{(i)})^2 + 3(1-\theta-q)^* y_0^{(i)} + ((1-\theta-q)^*)^2 \equiv 0 \pmod{p}.$$

This means $y_0^{(2)}$ and $y_0^{(3)}$ are the roots of the congruence

$$3t^2 + 3(1-\theta-q)^* t + ((1-\theta-q)^*)^2 \equiv 0 \pmod{p}$$

and this completes the proof. □

We then obtain the following result about the fixed points of the Potts–Bethe mapping (4.2).

Theorem 4.3.3. Suppose $0 < |\theta - 1|_p, |q|_p < 1$. Let $\mathbf{y}^{(i)}$ be the roots of the cubic equation (4.4) and $\mathbf{x}^{(0)} := 1$, $\mathbf{x}^{(i)} := \mathbf{x}^{(\infty)} + (a - 1)\mathbf{y}^{(i)}$ for $1 \leq i \leq 3$. Then

$$\mathbf{Fix}\{f_{\theta,q,3}\} = \begin{cases} \{\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}\} & \text{if } p \equiv 1 \pmod{6}, \\ \{\mathbf{x}^{(0)}, \mathbf{x}^{(1)}\} & \text{if } p \equiv 5 \pmod{6}. \end{cases}$$

Proof. It is the consequence of Proposition 4.3.1. \square

In the sequel, we will consider the cases $0 < |\theta - 1|_p < |q|_p < 1$ and $0 < |q|_p < |\theta - 1|_p < 1$.

4.3.1 The Case $0 < |\theta - 1|_p < |q|_p < 1$

Proposition 4.3.4. Let $p \geq 5$ be a prime number. Then

- (i) $\mathbf{x}^{(0)} = 1$ and $\mathbf{x}^{(\infty)} = \mathbf{x}^{(0)} - q - (\theta - 1)$;
- (ii) $\mathbf{x}^{(i)} = \mathbf{x}^{(\infty)} + (\theta - 1)\mathbf{y}^{(i)}$ for $i = 1, 2, 3$;
- (iii) $|\mathbf{x}^{(\infty)}|_p = |\mathbf{x}^{(i)}|_p = 1$ and $\mathbf{x}^{(\infty)}, \mathbf{x}^{(i)} \in \mathcal{E}_p$ for $i = 0, 1, 2, 3$;
- (iv) $|\mathbf{y}^{(2)}|_p = |\mathbf{y}^{(1)} - 3|_p = |\mathbf{y}^{(2)} - \mathbf{y}^{(3)}|_p = |q|_p < 1 = |\mathbf{y}^{(1)}|_p$;
- (v) $|\mathbf{x}^{(0)} - \mathbf{x}^{(\infty)}|_p = |q|_p$, $|\mathbf{x}^{(1)} - \mathbf{x}^{(\infty)}|_p = |\theta - 1|_p$;
- (vi) $|\mathbf{x}^{(2)} - \mathbf{x}^{(\infty)}|_p = |\mathbf{x}^{(3)} - \mathbf{x}^{(\infty)}|_p = |q|_p|\theta - 1|_p$;
- (vii) $|\mathbf{x}^{(i)} - \mathbf{x}^{(0)}|_p = |q|_p$ for $i = 1, 2, 3$ and $|\mathbf{x}^{(i)} - \mathbf{x}^{(1)}|_p = |\theta - 1|_p$ for $i = 2, 3$;
- (viii) $|\mathbf{x}^{(2)} - \mathbf{x}^{(3)}|_p = |q|_p|\theta - 1|_p$.

Proof. We have $\mathbf{x}^{(\infty)} = 2 - \theta - q$, $\mathbf{x}^{(0)} = 1$ and $\mathbf{x}^{(i)} = \mathbf{x}^{(i)} + (\theta - 1)\mathbf{y}^{(i)}$ for $i = 1, 2, 3$. Then one could easily calculate all the properties. \square

Then we analyze the local behaviour of each fixed points.

Proposition 4.3.5. Let $p \geq 5$ be a prime number. Then the fixed point $\mathbf{x}^{(0)}$ is attracting and the fixed point $\mathbf{x}^{(1)}$ is repelling. Moreover, if $p \equiv 1 \pmod{6}$, then the fixed points $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ exist and they are repelling.

Proof. We will show $\left|f'_{\theta,q,3}(\mathbf{x}^{(0)})\right|_p < 1$ and $\left|f'_{\theta,q,3}(\mathbf{x}^{(i)})\right|_p > 1$ for $1 \leq i \leq 3$. One could check for $0 \leq i \leq 3$, we have

$$\begin{aligned} f'_{\theta,q,3}(\mathbf{x}^{(i)}) &= \frac{3(\theta - 1)(\theta - 1 + q)(\theta\mathbf{x}^{(i)} + q - 1)^2}{(\mathbf{x}^{(i)} - \mathbf{x}^{(\infty)})^4} \\ &= \frac{3(\theta - 1)(\theta - 1 + q)\mathbf{x}^{(i)}}{(\theta\mathbf{x}^{(i)} + q - 1)(\mathbf{x}^{(i)} - \mathbf{x}^{(\infty)})}. \end{aligned} \quad (4.6)$$

Hence $\left| f'_{\theta,q,3}(\mathbf{x}^{(0)}) \right|_p = \frac{|\theta-1|_p}{|q|_p} < 1$. Recalling $\mathbf{x}^{(i)} = \mathbf{x}^{(\infty)} + (\theta-1)\mathbf{y}^{(i)}$ for $1 \leq i \leq 3$, we deduce from the equality (4.6) that

$$f'_{\theta,q,3}(\mathbf{x}^{(i)}) = \frac{3(\theta-1+q)\mathbf{x}^{(i)}}{(\theta-1)\mathbf{y}^{(i)}(\theta\mathbf{y}^{(i)} + 1 - \theta - q)}. \quad (4.7)$$

Due to Proposition 4.3.1, we have $|\mathbf{y}^{(2)}|_p = |\mathbf{y}^{(3)}|_p = |1 - \theta - q|_p < 1 = |\mathbf{y}^{(1)}|_p$ and $|\mathbf{x}^{(i)}|_p = 1$ for $1 \leq i \leq 3$. Thus $\left| f'_{\theta,q,3}(\mathbf{x}^{(1)}) \right|_p = \frac{|q|_p}{|\theta-1|_p} > 1$.

Next suppose $p \equiv 1 \pmod{6}$. Then, for the fixed points $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$, follows from the equality (4.7) for $i = 2, 3$, we have

$$f'_{\theta,q,3}(\mathbf{x}^{(i)}) = \frac{3(\theta-1+q)|1-\theta-q|_p^2 \mathbf{x}^{(i)}}{(\theta-1)(\mathbf{y}^{(i)})^* (\theta(\mathbf{y}^{(i)})^* + (1-\theta-q)^*)}.$$

By means of the congruence (4.5), we can check $\theta(\mathbf{y}^{(i)})^* \not\equiv -(1-\theta-q)^* \pmod{p}$. Consequently, we obtain

$$\left| f'_{\theta,q,3}(\mathbf{x}^{(i)}) \right|_p = \frac{|\theta-1+q|_p}{|1-\theta-q|_p^2 |\theta-1|_p} = \frac{1}{|q|_p |\theta-1|_p} > 1.$$

□

We will now consider two cases: $p \equiv 5 \pmod{6}$ and $p \equiv 1 \pmod{6}$.

CASE $p \equiv 5 \pmod{6}$. We introduce the following sets

$$\begin{aligned} \mathcal{A}_0 &:= \left\{ x \in \mathbb{Q}_p : |x - \mathbf{x}^{(0)}|_p < |q|_p \right\}, \\ \mathcal{A}_1 &:= \left\{ x \in \mathbb{Q}_p : |x - \mathbf{x}^{(0)}|_p > |q|_p \right\}, \\ \mathcal{A}_{0,\infty} &:= \left\{ x \in \mathbb{Q}_p : |x - \mathbf{x}^{(\infty)}|_p = |x - \mathbf{x}^{(0)}|_p = |q|_p \right\}, \\ \mathcal{A}_2 &:= \left\{ x \in \mathbb{Q}_p : |\theta-1|_p < |x - \mathbf{x}^{(\infty)}|_p < |q|_p \right\}, \\ \mathcal{A}_{1,\infty}^{(1)} &:= \left\{ x \in \mathbb{Q}_p : |x - \mathbf{x}^{(1)}|_p = |x - \mathbf{x}^{(\infty)}|_p = |\theta-1|_p \right\}, \\ \mathcal{A}_{1,\infty}^{(2)} &:= \left\{ x \in \mathbb{Q}_p : |x - \mathbf{x}^{(1)}|_p < |x - \mathbf{x}^{(\infty)}|_p = |\theta-1|_p \right\}, \\ \mathcal{A}_\infty &:= \left\{ x \in \mathbb{Q}_p : 0 < |x - \mathbf{x}^{(\infty)}|_p < |\theta-1|_p \right\}. \end{aligned}$$

Proposition 4.3.6. *The following inclusions hold*

- (i) $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_{0,\infty} \subset f_{\theta,q,3}^{-1}(\mathcal{A}_0)$;
- (ii) $\mathcal{A}_\infty \subset f_{\theta,q,3}^{-1}(\mathcal{A}_1) \subset f_{\theta,q,3}^{-2}(\mathcal{A}_0)$;
- (iii) $\mathcal{A}_{1,\infty}^{(1)} \subset f_{\theta,q,3}^{-1}(\mathcal{A}_{0,\infty}) \subset f_{\theta,q,3}^{-2}(\mathcal{A}_0)$.

Proof. We have

$$f_{\theta,q,3}(x) - \mathbf{x}^{(0)} = \frac{(\theta-1)(x - \mathbf{x}^{(0)})}{(x - \mathbf{x}^{(\infty)})^3} g(x) \quad (4.8)$$

where $g(x) = (\theta x + q - 1)^2 + (\theta x + q - 1)(x - \mathbf{x}^{(\infty)}) + (x - \mathbf{x}^{(\infty)})^2$. We also notice that the Potts–Bethe mapping (4.2) can be written as follows

$$f_{\theta,q,3}(x) = \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right)^3. \quad (4.9)$$

Then we have

$$f_{\theta,q,3}(x) - \mathbf{x}^{(0)} = \frac{(\theta-1)(x - \mathbf{x}^{(0)})}{x - \mathbf{x}^{(\infty)}} g_1(x) \quad (4.10)$$

where $g_1(x) = \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right)^2 + \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right) + 1$.

(i) We will show for any $x \in \mathcal{A}_0 \cup \mathcal{A}_1$ that $f_{\theta,q,3}(x) \in \mathcal{A}_0$. In fact, we have

$$\left| x - \mathbf{x}^{(\infty)} \right|_p = \left| x - \mathbf{x}^{(0)} + q + (\theta - 1) \right|_p = \begin{cases} |q|_p, & x \in \mathcal{A}_0, \\ |x - \mathbf{x}^{(0)}|_p, & x \in \mathcal{A}_1 \end{cases} \text{ and}$$

$$|\theta x + q - 1|_p = \left| \theta(x - \mathbf{x}^{(0)}) + q + (\theta - 1) \right|_p = \begin{cases} |q|_p, & x \in \mathcal{A}_0, \\ |x - \mathbf{x}^{(0)}|_p, & x \in \mathcal{A}_1. \end{cases}$$

These imply

$$|g(x)|_p \leq \begin{cases} |q|_p^2, & x \in \mathcal{A}_0, \\ |x - \mathbf{x}^{(0)}|_p^2, & x \in \mathcal{A}_1. \end{cases}$$

Then from the equality (4.8), we deduce that

$$|f_{\theta,q,3}(x) - \mathbf{x}^{(0)}|_p \leq \begin{cases} \frac{|x - \mathbf{x}^{(0)}|_p}{|q|_p} |\theta - 1|_p, & x \in \mathcal{A}_0 \\ |\theta - 1|_p, & x \in \mathcal{A}_1 \end{cases} \leq |\theta - 1|_p < |q|_p.$$

So we get $f_{\theta,q,3}(x) \in \mathcal{A}_0$ for any $x \in \mathcal{A}_0 \cup \mathcal{A}_1$ which means $f_{\theta,q,3}(\mathcal{A}_0) \subset \mathcal{A}_0$ and $f_{\theta,q,3}(\mathcal{A}_1) \subset \mathcal{A}_0$.

Next, we will show that $f_{\theta,q,3}(x) \in \mathcal{A}_0$ for any $x \in \mathcal{A}_{0,\infty}$. We have $|x - \mathbf{x}^{(0)}|_p = |x - \mathbf{x}^{(\infty)}|_p = |q|_p$ and $|\theta x + q - 1|_p = |\theta(x - \mathbf{x}^{(\infty)}) + (\theta - 1)(1 - \theta - q)|_p = |q|_p$. These imply $|g(x)|_p \leq |q|_p^2$. By the equality (4.8), we get $|f_{\theta,q,3}(x) - \mathbf{x}^{(0)}|_p \leq |\theta - 1|_p < |q|_p$. Thus $f_{\theta,q,3}(x) \in \mathcal{A}_0$ for any $x \in \mathcal{A}_{0,\infty}$ which implies $f_{\theta,q,3}(\mathcal{A}_{0,\infty}) \subset \mathcal{A}_0$.

Lastly, we will show $f_{\theta,q,3}(x) \in \mathcal{A}_0$ for any $x \in \mathcal{A}_2$. Indeed, for any $x \in \mathcal{A}_2$, we have

$$\left| \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right|_p = \frac{|q|_p |\theta - 1|_p}{|x - \mathbf{x}^{(\infty)}|_p} < 1$$

which implies $|g_1(x)|_p = 1$ and $|f_{\theta,q,3}(x) - \mathbf{x}^{(0)}|_p = \frac{|\theta-1|_p}{|x-\mathbf{x}^{(\infty)}|_p} |x - \mathbf{x}^{(0)}|_p < |q|_p$. Hence, we get $f_{\theta,q,3}(x) \in \mathcal{A}_0$ for any $x \in \mathcal{A}_2$ which implies $f_{\theta,q,3}(\mathcal{A}_2) \subset \mathcal{A}_0$ the inclusion (i) holds.

(ii) Now we want show $f_{\theta,q,3}(x) \in \mathcal{A}_1$ for any $x \in \mathcal{A}_\infty$. We consider the following sets

$$\begin{aligned}\mathcal{A}_\infty^{(1)} &:= \left\{ x \in \mathbb{Q}_p : 0 < |x - \mathbf{x}^{(\infty)}|_p < |q|_p |\theta - 1|_p \right\} \\ \mathcal{A}_\infty^{(2)} &:= \left\{ x \in \mathbb{Q}_p : |x - \mathbf{x}^{(\infty)}|_p = |q|_p |\theta - 1|_p \right\} \\ \mathcal{A}_\infty^{(3)} &:= \left\{ x \in \mathbb{Q}_p : |q|_p |\theta - 1|_p < |x - \mathbf{x}^{(\infty)}|_p < |\theta - 1|_p \right\}\end{aligned}$$

where $\mathcal{A}_\infty = \mathcal{A}_\infty^{(1)} \cup \mathcal{A}_\infty^{(2)} \cup \mathcal{A}_\infty^{(3)}$.

Let $x \in \mathcal{A}_\infty^{(1)}$. Then

$$\begin{aligned}\left| \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right|_p &= \frac{|q|_p |\theta-1|_p}{|x - \mathbf{x}^{(\infty)}|_p} > 1, \\ |f_{\theta,q,3}(x)|_p &= \left| \theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right|_p^3 > 1 \quad \text{and} \\ |f_{\theta,q,3}(x) - \mathbf{x}^{(0)}|_p &= |f_{\theta,q,3}(x)|_p > 1 > |q|_p.\end{aligned}$$

Thus $f_{\theta,q,3}(\mathcal{A}_\infty^{(1)}) \subset \mathcal{A}_1$.

Next let $x \in \mathcal{A}_\infty^{(3)}$. Then $|x - \mathbf{x}^{(0)}|_p = |q|_p$ and $\left| \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right|_p = \frac{|\theta-1|_p |q|_p}{|x - \mathbf{x}^{(\infty)}|_p} < 1$.

This implies

$$|g_1(x)|_p = 1 \quad \text{and} \quad |q|_p < |f_{\theta,q,3}(x) - \mathbf{x}^{(0)}|_p = \frac{|\theta-1|_p |x - \mathbf{x}^{(0)}|_p}{|x - \mathbf{x}^{(\infty)}|_p} < 1.$$

It yields $f_{\theta,q,3}(\mathcal{A}_\infty^{(3)}) \subset \mathcal{A}_1$.

Lastly, we suppose $x \in \mathcal{A}_\infty^{(2)}$ such that $y := \frac{x - \mathbf{x}^{(\infty)}}{\theta - 1}$. Since $|y|_p = |q|_p$, we have $y = \frac{y^*}{|q|_p}$ where $y^* \in \mathbb{Z}_p^*$. Then we substitute into $g(x)$ of the equality (4.8) and obtain

$$\begin{aligned}g(x) &= (\theta x + q - 1)^2 + (\theta x + q - 1)(x - \mathbf{x}^{(\infty)}) + (x - \mathbf{x}^{(\infty)})^2 \\ &= (\theta - 1)^2 [(\theta^2 + \theta + 1)y^2 + (2\theta + 1)(1 - \theta - q)y + (1 - \theta - q)^2] \\ &= \frac{(\theta - 1)^2}{|q|_p^2} [(\theta^2 + \theta + 1)(y^*)^2 + (2\theta + 1)(1 - \theta - q)^*(y^*) + ((1 - \theta - q)^*)^2]\end{aligned}$$

Now we will show $|(\theta^2 + \theta + 1)(y^*)^2 + (2\theta + 1)(1 - \theta - q)^*(y^*) + ((1 - \theta - q)^*)^2|_p = 1$ for any $y^* \in \mathbb{Z}_p^*$ when $p \equiv 5 \pmod{6}$. To do so, it is enough to show the following congruent equation

$$3t^2 + 3(1 - \theta - q)^*t + ((1 - \theta - q)^*)^2 \equiv 0 \pmod{p}$$

does not have any root in \mathbb{Z} . Indeed, the discriminant $D = -3((1 - \theta - q)^*)^2$ or equivalently -3 of the last quadratic congruence is a quadratic non-residue modulo p whenever $p \equiv 5 \pmod{6}$. So it does not have any root in \mathbb{Z} and we get $|g(x)|_p = |q|_p^2 |\theta - 1|_p^2$. Since $|x - \mathbf{x}^{(0)}|_p = |q|_p$, by the equality (4.8), we have $|f_{\theta,q,3}(x) - \mathbf{x}^{(0)}|_p = 1 > |q|_p$ for any $x \in \mathcal{A}_{\infty}^{(2)}$. Thus $f_{\theta,q,3}(\mathcal{A}_{\infty}^{(2)}) \subset \mathcal{A}_1$ and we have the inclusion (ii).

(iii) We will show $f_{\theta,q,3}(x) \in \mathcal{A}_{0,\infty}$ for any $x \in \mathcal{A}_{1,\infty}^{(1)}$. We first show $|f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)}|_p = |q|_p$. We have $\left| \frac{(\theta-1)(1-\theta-q)}{x-\mathbf{x}^{(\infty)}} \right|_p = \frac{|\theta-1|_p |q|_p}{|x-\mathbf{x}^{(\infty)}|_p} < 1$ and $|g_1(x) - 3|_p < |g_1(x)|_p = 1$. Then, by the equality we get (4.10),

$$\left| f_{\theta,q,3}(x) - \mathbf{x}^{(0)} \right|_p = \frac{|\theta-1|_p |x - \mathbf{x}^{(0)}|_p}{|x - \mathbf{x}^{(\infty)}|_p} = |q|_p.$$

Next we will show $|f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)}|_p = |q|_p$. Since $|\theta-1|_p < |q|_p$ and $|f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)}|_p = |f_{\theta,q,3}(x) - \mathbf{x}^{(0)} + q + (\theta-1)|_p$, it is enough to show that $|f_{\theta,q,3}(x) - \mathbf{x}^{(0)} + q|_p = |q|_p$. It is easy to check

$$\begin{aligned} f_{\theta,q,3}(x) - \mathbf{x}^{(0)} + q &= \frac{(\theta-1)(x - \mathbf{x}^{(0)})}{x - \mathbf{x}^{(\infty)}} (g_1(x) - 3) \\ &\quad + 3 \left((\theta-1) - \frac{(\theta-1)^2}{x - \mathbf{x}^{(\infty)}} \right) + \frac{q(\theta-1)(\mathbf{y}^{(1)} - 3)}{x - \mathbf{x}^{(\infty)}} + q \frac{x - \mathbf{x}^{(1)}}{x - \mathbf{x}^{(\infty)}}. \end{aligned}$$

From the following assertions, $|\mathbf{y}^{(1)} - 3|_p = |x - \mathbf{x}^{(0)}|_p = |q|_p$, $|x - \mathbf{x}^{(\infty)}|_p = |x - \mathbf{x}^{(1)}|_p = |\theta-1|_p$ and $|g_1(x) - 3|_p < 1$, we obtain from the above equality $|f_{\theta,q,3}(x) - \mathbf{x}^{(0)} + q|_p = |q|_p$. This means $|f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)}|_p = |q|_p$. Consequently, $f_{\theta,q,3}(\mathcal{A}_{1,\infty}^{(1)}) \subset \mathcal{A}_{0,\infty}$ and the inclusion (iii) follows. \square

Proposition 4.3.7. *The following statements are true.*

- (i) For any $\bar{x}, \bar{\bar{x}} \in \mathcal{A}_{1,\infty}^{(2)}$, one has $|f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}})|_p = \frac{|q|_p}{|\theta-1|_p} |\bar{x} - \bar{\bar{x}}|_p$;
- (ii) For any $x \in \mathcal{A}_{1,\infty}^{(2)} \setminus \{\mathbf{x}^{(1)}\}$, there exists $n_0 \in \mathbb{N}$ such that $f_{\theta,q,3}^{n_0}(x) \notin \mathcal{A}_{1,\infty}^{(2)}$.

Proof. We need the following equality

$$f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}}) = \frac{(\theta-1)(1-\theta-q)(\bar{x} - \bar{\bar{x}})}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} F_{\theta,q,3}(\bar{x}, \bar{\bar{x}})$$

where

$$\begin{aligned} F_{\theta,q,3}(\bar{x}, \bar{\bar{x}}) &= 3\theta^2 + 3\theta(1-\theta-q) \left(\frac{\theta-1}{\bar{x} - \mathbf{x}^{(\infty)}} + \frac{\theta-1}{\bar{\bar{x}} - \mathbf{x}^{(\infty)}} \right) \\ &\quad + (1-\theta-q)^2 \left(\frac{(\theta-1)^2}{(\bar{x} - \mathbf{x}^{(\infty)})^2} + \frac{(\theta-1)^2}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} + \frac{(\theta-1)^2}{(\bar{\bar{x}} - \mathbf{x}^{(\infty)})^2} \right). \end{aligned}$$

(i) Let $\bar{x}, \bar{\bar{x}} \in \mathcal{A}_{1,\infty}^{(2)}$. Then we have

$$\left| 3\theta(1-\theta-q) \left(\frac{\theta-1}{\bar{x}-\mathbf{x}^{(\infty)}} + \frac{\theta-1}{\bar{\bar{x}}-\mathbf{x}^{(\infty)}} \right) \right|_p \leq |q|_p \quad \text{and,}$$

$$\left| (1-\theta-q)^2 \left(\frac{(\theta-1)^2}{(\bar{x}-\mathbf{x}^{(\infty)})^2} + \frac{(\theta-1)^2}{(\bar{x}-\mathbf{x}^{(\infty)})(\bar{\bar{x}}-\mathbf{x}^{(\infty)})} + \frac{(\theta-1)^2}{(\bar{\bar{x}}-\mathbf{x}^{(\infty)})^2} \right) \right|_p \leq |q|_p^2.$$

This means $|F_{\theta,q,3}(\bar{x}, \bar{\bar{x}})|_p = 1$. Therefore, $|f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}})|_p = \frac{|q|_p}{|\theta-1|_p} |\bar{x} - \bar{\bar{x}}|_p$.

(ii) Let $x \in \mathcal{A}_{1,\infty}^{(2)} \setminus \{\mathbf{x}^{(1)}\}$ and $r = |x - \mathbf{x}^{(1)}|_p > 0$. Since $\frac{|\theta-1|_p}{|q|_p} < 1$, there exists n_0 such that $\left(\frac{|\theta-1|_p}{|q|_p}\right)^{n_0} \leq \frac{r}{|\theta-1|_p} < \left(\frac{|\theta-1|_p}{|q|_p}\right)^{n_0-1}$ or $\left(\frac{|q|_p}{|\theta-1|_p}\right)^{n_0-1} r < |\theta-1|_p \leq \left(\frac{|q|_p}{|\theta-1|_p}\right)^{n_0} r$. Thus

$$\left| f_{\theta,q,3}^{n_0-1}(x) - \mathbf{x}^{(1)} \right|_p = \frac{|q|_p^{n_0-1} r}{|\theta-1|_p^{n_0-1}} < |\theta-1|_p \leq \frac{|q|_p^{n_0} r}{|\theta-1|_p^{n_0}} = \left| f_{\theta,q,3}^{n_0}(x) - \mathbf{x}^{(1)} \right|_p$$

or equivalently

$$f_{\theta,q,3}^{n_0-1}(x) \in \mathcal{A}_{1,\infty}^{(2)} \quad \text{and} \quad f_{\theta,q,3}^{n_0}(x) \notin \mathcal{A}_{1,\infty}^{(2)}$$

which completes the proof. \square

Now we will describe the attracting basin

$$\mathfrak{B}(\mathbf{x}^{(0)}) := \left\{ x \in \mathbb{Q}_p : \lim_{n \rightarrow \infty} f_{\theta,q,3}^n(x) = \mathbf{x}^{(0)} \right\}$$

of the attracting fixed point $\mathbf{x}^{(0)}$.

Theorem 4.3.8. *The following is the attracting basin of the attracting fixed point $\mathbf{x}^{(0)}$.*

$$\mathfrak{B}(\mathbf{x}^{(0)}) = \mathbb{Q}_p \setminus \left(\{\mathbf{x}^{(1)}\} \cup \bigcup_{n=0}^{\infty} f_{\theta,q,3}^{-n} \{\mathbf{x}^{(\infty)}\} \right).$$

Proof. Let $\bigcup_{n=0}^{\infty} f_{\theta,q,3}^{-n} \{\mathbf{x}^{(\infty)}\}$ be a set of all eventually singular points of the Potts–Bethe mapping (4.2). Let $x \in \mathcal{A}_0$. It follows from Proposition 4.3.6 (i) that $|f_{\theta,q,3}(x) - \mathbf{x}^{(0)}|_p \leq \frac{|\theta-1|_p}{|q|_p} |x - \mathbf{x}^{(0)}|_p$ where $\frac{|\theta-1|_p}{|q|_p} < 1$. This means $f_{\theta,q,3}(\mathcal{A}_0) \subset \mathcal{A}_0$. Let $x \in \mathfrak{B}(\mathbf{x}^{(0)})$ be any point. This means $\lim_{n \rightarrow \infty} f_{\theta,q,3}^n(x) = \mathbf{x}^{(0)}$. Then there exist some $n_0 \in \mathbb{N}$ such that $f_{\theta,q,3}^{n_0}(x) \in \mathcal{A}_0$. This implies $x \in f_{\theta,q,3}^{-n_0}(\mathcal{A}_0)$ or equivalently

$$\mathfrak{B}(\mathbf{x}^{(0)}) = \bigcup_{n=0}^{\infty} f_{\theta,q,3}^{-n}(\mathcal{A}_0).$$

Due to Proposition 4.3.6 (i)-(iii), we have

$$\mathbb{Q}_p \setminus \mathcal{A}_{1,\infty}^{(2)} \subset f_{\theta,q,3}^{-1}(\mathcal{A}_0) \cup f_{\theta,q,3}^{-2}(\mathcal{A}_0) \subset \mathfrak{B}(\mathbf{x}^{(0)}).$$

Moreover, due to Proposition 4.3.7 (ii), for any $x \in \mathcal{A}_{1,\infty}^{(2)} \setminus \{\mathbf{x}^{(1)}\}$, there exists n_0 such that $f_{\theta,q,3}^{(n_0)}(x) \in \mathbb{Q}_p \setminus \mathcal{A}_{1,\infty}^{(2)} \subset \mathfrak{B}(\mathbf{x}^{(0)}) \cup \{\mathbf{x}^{(\infty)}\}$. \square

CASE 2: $p \equiv 1 \pmod{6}$. We introduce the following sets.

$$\begin{aligned}
\mathcal{A}_0 &:= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(0)} \right|_p < |q|_p \right\}, \\
\mathcal{A}_1 &:= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(0)} \right|_p > |q|_p \right\}, \\
\mathcal{A}_{0,\infty} &:= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(\infty)} \right|_p = \left| x - \mathbf{x}^{(0)} \right|_p = |q|_p \right\}, \\
\mathcal{A}_2 &:= \left\{ x \in \mathbb{Q}_p : |\theta - 1|_p < \left| x - \mathbf{x}^{(\infty)} \right|_p < |q|_p \right\}, \\
\mathcal{A}_{1,\infty} &:= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(1)} \right|_p = \left| x - \mathbf{x}^{(\infty)} \right|_p = |\theta - 1|_p \right\}, \\
\mathcal{C}_1 &:= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(1)} \right|_p < |\theta - 1|_p \right\}, \\
\mathcal{A}_3 &:= \left\{ x \in \mathbb{Q}_p : |q|_p |\theta - 1|_p < \left| x - \mathbf{x}^{(\infty)} \right|_p < |\theta - 1|_p \right\}, \\
\mathcal{A}_{2,3,\infty} &:= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(\infty)} \right|_p = \left| x - \mathbf{x}^{(2)} \right|_p = \left| x - \mathbf{x}^{(3)} \right|_p = |q|_p |\theta - 1|_p \right\}, \\
\mathcal{C}_2 &:= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(2)} \right|_p < |q|_p |\theta - 1|_p \right\}, \\
\mathcal{C}_3 &:= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(3)} \right|_p < |q|_p |\theta - 1|_p \right\}, \\
\mathcal{A}_\infty &:= \left\{ x \in \mathbb{Q}_p : 0 < \left| x - \mathbf{x}^{(\infty)} \right|_p < |q|_p |\theta - 1|_p \right\}.
\end{aligned}$$

Proposition 4.3.9. *The following inclusions hold:*

- (i) $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_{0,\infty} \subset f_{\theta,q,3}^{-1}(\mathcal{A}_0)$;
- (ii) $\mathcal{A}_\infty \cup \mathcal{A}_3 \cup \mathcal{A}_{2,3,\infty} \subset f_{\theta,q,3}^{-1}(\mathcal{A}_1) \subset f_{\theta,q,3}^{-2}(\mathcal{A}_0)$;
- (iii) $\mathcal{A}_{1,\infty} \subset f_{\theta,q,3}^{-1}(\mathcal{A}_{0,\infty}) \subset f_{\theta,q,3}^{-2}(\mathcal{A}_0)$.

Proof. The proof of the inclusions (i) and (iii) follow from the inclusions (i) and (iii) of Proposition 4.3.6 respectively. We are left to proof the assertions (ii). Recall

$$f_{\theta,q,3}(x) - \mathbf{x}^{(0)} = \frac{(\theta - 1)(x - \mathbf{x}^{(0)})}{(x - \mathbf{x}^{(\infty)})^3} g(x) \quad (4.11)$$

$$= \frac{(\theta - 1)(x - \mathbf{x}^{(0)})}{x - \mathbf{x}^{(\infty)}} g_1(x) \quad (4.12)$$

where $g(x) = (\theta x + q - 1)^2 + (\theta x + q - 1)(x - \mathbf{x}^{(\infty)}) + (x - \mathbf{x}^{(\infty)})^2$ and

$$g_1(x) = \left(\theta + \frac{(\theta - 1)(1 - \theta - q)}{x - \mathbf{x}^{(\infty)}} \right)^2 + \left(\theta + \frac{(\theta - 1)(1 - \theta - q)}{x - \mathbf{x}^{(\infty)}} \right) + 1.$$

- (ii) We will show $f_{\theta,q,3}(x) \in \mathcal{A}_1$ for any $x \in \mathcal{A}_\infty \cup \mathcal{A}_3 \cup \mathcal{A}_{2,3,\infty}$.

Let $x \in \mathcal{A}_\infty$. We have $\left| \frac{(\theta-1)(1-\theta-q)}{x-\mathbf{x}^{(\infty)}} \right|_p = \frac{|q|_p |\theta-1|_p}{|x-\mathbf{x}^{(\infty)}|_p} > 1$ which implies

$$|f_{\theta,q,3}(x)|_p = \left| \theta + \frac{(\theta-1)(1-\theta-q)}{x-\mathbf{x}^{(\infty)}} \right|_p^3 > 1 > |q|_p$$

and $|f_{\theta,q,3}(x) - \mathbf{x}^{(0)}|_p = |f_{\theta,q,3}(x)|_p > |q|_p$. It means $f_{\theta,q,3}(x) \in \mathcal{A}_1$ for any $x \in \mathcal{A}_\infty$.

Let $x \in \mathcal{A}_3$. We have $\left| \frac{(\theta-1)(1-\theta-q)}{x-\mathbf{x}^{(\infty)}} \right|_p = \frac{|\theta-1|_p |q|_p}{|x-\mathbf{x}^{(\infty)}|_p} < 1$ which implies $|g_1(x)|_p = 1$. By the equality (4.12), we get

$$|q|_p < |f_{\theta,q,3}(x) - \mathbf{x}^{(0)}|_p = \frac{|\theta-1|_p |x - \mathbf{x}^{(0)}|_p}{|x - \mathbf{x}^{(\infty)}|_p} < 1.$$

Thus $f_{\theta,q,3}(x) \in \mathcal{A}_1$ for any $x \in \mathcal{A}_3$.

Let $x \in \mathcal{A}_{2,3,\infty}$. Write $y = \frac{x-\mathbf{x}^{(\infty)}}{\theta-1}$. Then $|y|_p = |q|_p$ with $y = \frac{y^*}{|q|_p}$ and

$$\begin{aligned} g(x) &= (\theta x + q - 1)^2 + (\theta x + q - 1) \left(x - \mathbf{x}^{(\infty)} \right) + \left(x - \mathbf{x}^{(\infty)} \right)^2 \\ &= (\theta - 1)^2 \left((\theta^2 + \theta + 1)y^2 + (2\theta + 1)(1 - \theta - q)y + (1 - \theta - q)^2 \right) \\ &= \frac{(\theta - 1)^2}{|q|_p^2} \left((\theta^2 + \theta + 1)(y^*)^2 + (2\theta + 1)(1 - \theta - q)^*(y^*) + ((1 - \theta - q)^*)^2 \right). \end{aligned}$$

By the properties of the set $\mathcal{A}_{2,3,\infty}$, we get $|g(x)|_p = |q|_p^2 |\theta - 1|_p^2$. It follows directly from the equality (4.11) that $|f_{\theta,q,3}(x) - \mathbf{x}^{(0)}|_p = 1 > |q|_p$. Hence $f_{\theta,q,3}(x) \in \mathcal{A}_1$ for any $x \in \mathcal{A}_{2,3,\infty}$ and the inclusion (ii) follows. \square

By this Proposition 4.3.9, we can now describe the attracting basin $\mathfrak{B}(\mathbf{x}^{(0)})$ of the attracting fixed point $\mathbf{x}^{(0)} = 1$.

Theorem 4.3.10. *We have*

$$\mathfrak{B}(\mathbf{x}^{(0)}) = \bigcup_{n=0}^{+\infty} f_{\theta,q,3}^{-n}(\mathcal{A}_0)$$

and $\text{Dom}\{f_{\theta,q,3}\} \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3) \subset \mathfrak{B}(\mathbf{x}^{(0)})$.

Proof. The proof of the first part is similar to the proof of Theorem 4.3.8. For the second part, due to Proposition 4.3.9 (i)-(iii), we have

$$\text{Dom}\{f_{\theta,q,3}\} \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3) \subset f_{\theta,q,3}^{-1}(\mathcal{A}_0) \cup f_{\theta,q,3}^{-2}(\mathcal{A}_0) \subset \mathfrak{B}(\mathbf{x}^{(0)}).$$

\square

Proposition 4.3.11. *The following equality holds*

$$|f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}})|_p = \begin{cases} \frac{|q|_p}{|\theta-1|_p} |\bar{x} - \bar{\bar{x}}|_p & \text{if } \bar{x}, \bar{\bar{x}} \in \mathcal{C}_1; \\ \frac{1}{|q|_p |\theta-1|_p} |\bar{x} - \bar{\bar{x}}|_p & \text{if } \bar{x}, \bar{\bar{x}} \in \mathcal{C}_2; \\ \frac{1}{|q|_p |\theta-1|_p} |\bar{x} - \bar{\bar{x}}|_p & \text{if } \bar{x}, \bar{\bar{x}} \in \mathcal{C}_3. \end{cases} \quad (4.13)$$

Proof. We need the following equality

$$f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}}) = \frac{(\theta-1)(1-\theta-q)(\bar{\bar{x}} - \bar{x})}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} F_{\theta,q,3}(\bar{x}, \bar{\bar{x}}) \quad (4.14)$$

such that

$$\begin{aligned} F_{\theta,q,3}(\bar{x}, \bar{\bar{x}}) &= \\ & 3\theta^2 + 3\theta(1-\theta-q) \left(\frac{\theta-1}{\bar{x} - \mathbf{x}^{(\infty)}} + \frac{\theta-1}{\bar{\bar{x}} - \mathbf{x}^{(\infty)}} \right) \\ & + (1-\theta-q)^2 \left(\frac{(\theta-1)^2}{(\bar{x} - \mathbf{x}^{(\infty)})^2} + \frac{(\theta-1)^2}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} + \frac{(\theta-1)^2}{(\bar{\bar{x}} - \mathbf{x}^{(\infty)})^2} \right). \end{aligned}$$

Let $\bar{x}, \bar{\bar{x}} \in \mathcal{C}_1$. Then $\left| 3\theta(1-\theta-q) \left(\frac{\theta-1}{\bar{x} - \mathbf{x}^{(\infty)}} + \frac{\theta-1}{\bar{\bar{x}} - \mathbf{x}^{(\infty)}} \right) \right|_p \leq |q|_p$ and

$$\left| (1-\theta-q)^2 \left(\frac{(\theta-1)^2}{(\bar{x} - \mathbf{x}^{(\infty)})^2} + \frac{(\theta-1)^2}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} + \frac{(\theta-1)^2}{(\bar{\bar{x}} - \mathbf{x}^{(\infty)})^2} \right) \right|_p \leq |q|_p^2.$$

These imply $|F_{\theta,q,3}(\bar{x}, \bar{\bar{x}})|_p = 1$. Thus, $|f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}})|_p = \frac{|q|_p}{|\theta-1|_p} |\bar{x} - \bar{\bar{x}}|_p$.

Let $\bar{x}, \bar{\bar{x}} \in \mathcal{C}_i$ for $i = 2, 3$. Then

$$\frac{\mathbf{x}^{(i)} - \mathbf{x}^{(\infty)}}{\theta-1} = \mathbf{y}^{(i)} = \frac{y_0^{(i)} + y_1^{(i)} p + \dots}{|q|_p}$$

where $y_0^{(2)}$ and $y_0^{(3)}$ are roots of the following congruent equation

$$3t^2 + 3(1-\theta-q)^* t + ((1-\theta-q)^*)^2 \equiv 0 \pmod{p}.$$

We also have

$$\left| \frac{\bar{x} - \mathbf{x}^{(\infty)}}{\theta-1} \right|_p = \left| \frac{\bar{\bar{x}} - \mathbf{x}^{(\infty)}}{\theta-1} \right|_p = |q|_p \text{ and } \left(\frac{\bar{x} - \mathbf{x}^{(\infty)}}{\theta-1} \right)^* = \left(\frac{\bar{\bar{x}} - \mathbf{x}^{(\infty)}}{\theta-1} \right)^* \equiv y_0^{(i)} \pmod{p}.$$

By all these properties, we have

$$\begin{aligned} F_{\theta,q,3}(\bar{x}, \bar{\bar{x}}) &= \\ 3\theta^2 &+ 3\theta(1-\theta-q)^* \left(\left(\frac{\theta-1}{\bar{x}-\mathbf{x}^{(\infty)}} \right)^* + \left(\frac{\theta-1}{\bar{\bar{x}}-\mathbf{x}^{(\infty)}} \right)^* \right) \\ &+ ((1-\theta-q)^*)^2 \left(\left(\left(\frac{\theta-1}{\bar{x}-\mathbf{x}^{(\infty)}} \right)^* \right)^2 + \left(\frac{\theta-1}{\bar{x}-\mathbf{x}^{(\infty)}} \right)^* \left(\frac{\theta-1}{\bar{\bar{x}}-\mathbf{x}^{(\infty)}} \right)^* + \left(\left(\frac{\theta-1}{\bar{\bar{x}}-\mathbf{x}^{(\infty)}} \right)^* \right)^2 \right) \end{aligned}$$

and

$$\begin{aligned} \left(y_0^{(i)} \right)^2 F_{\theta,q,3}(\bar{x}, \bar{\bar{x}}) &\equiv 3 \left(y_0^{(i)} \right)^2 + 6(1-\theta-q)^* \left(y_0^{(i)} \right) + 3((1-\theta-q)^*)^2 \\ &\equiv 3 \left(\left(y_0^{(i)} \right) + (1-\theta-q)^* \right)^2 \not\equiv 0 \pmod{p}. \end{aligned}$$

This yields $|F_{\theta,q,3}(\bar{x}, \bar{\bar{x}})|_p = 1$. Therefore, $|f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}})|_p = \frac{|\bar{x} - \bar{\bar{x}}|_p}{|q|_p |\theta - 1|_p}$. \square

To continue we will distinguish two subcases: $0 < |\theta - 1|_p < |q|_p^2 < 1$ and $0 < |q|_p^2 \leq |\theta - 1|_p < |q|_p < 1$. Suppose

$$r = |q|_p |\theta - 1|_p.$$

SUBCASE $0 < |\theta - 1|_p < |q|_p^2 < 1$. We consider the disjoint open balls $\mathbb{B}_r(\mathbf{x}^{(i)})$ for $i = 1, 2, 3$. It is clear that $\mathbb{B}_r(\mathbf{x}^{(1)}) \subsetneq \mathcal{C}_1$ and $\mathbb{B}_r(\mathbf{x}^{(i)}) = \mathcal{C}_i$ for $i = 2, 3$. We define

$$\mathbb{X} = \mathbb{B}_r(\mathbf{x}^{(1)}) \cup \mathbb{B}_r(\mathbf{x}^{(2)}) \cup \mathbb{B}_r(\mathbf{x}^{(3)}).$$

Then $(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3) \setminus \mathbb{X} = \mathcal{C}_1 \setminus \mathbb{B}_r(\mathbf{x}^{(1)})$.

Proposition 4.3.12. (i) For $i = 1, 2, 3$, $\mathbb{X} \subset \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \subset f_{\theta,q,3}(\mathbb{B}_r(\mathbf{x}^{(i)}))$;

(ii) $f_{\theta,q,3}(\mathcal{C}_1 \setminus \mathbb{B}_r(\mathbf{x}^{(1)})) \subset \mathbb{Q}_p \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$;

(iii) $f_{\theta,q,3}^{-1}(\mathbb{X}) \subset \mathbb{X}$.

Proof. (i) It is clear $\mathbb{B}_r(\mathbf{x}^{(i)}) \subseteq \mathcal{C}_i$ for $i = 1, 2, 3$. This means $\mathbb{X} \subset \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. Hence, it is enough to show $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \subset f_{\theta,q,3}(\mathbb{B}_r(\mathbf{x}^{(i)}))$.

Let $x \in \mathbb{B}_r(\mathbf{x}^{(1)})$. By the equality (4.13), we have $|f_{\theta,q,3}(x) - \mathbf{x}^{(1)}|_p = \frac{|q|_p}{|\theta - 1|_p} |x - \mathbf{x}^{(1)}|_p$. Thus $f_{\theta,q,3}(\mathbb{B}_r(\mathbf{x}^{(1)})) = \mathbb{B}_{|q|_p^2}(\mathbf{x}^{(1)})$. Since $|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|_p = |\mathbf{x}^{(1)} - \mathbf{x}^{(3)}|_p = |\theta - 1|_p < |q|_p^2$, we have $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)} \in \mathbb{B}_{|q|_p^2}(\mathbf{x}^{(1)})$. It implies $|x - \mathbf{x}^{(1)}|_p \leq |\theta - 1|_p < |q|_p^2$ for any $x \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. Hence $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \subset f_{\theta,q,3}(\mathbb{B}_r(\mathbf{x}^{(1)}))$.

Let $x \in \mathbb{B}_r(\mathbf{x}^{(i)})$ for $i = 2, 3$. By the equality (4.13), we have $|f_{\theta,q,3}(x) - \mathbf{x}^{(i)}|_p = \frac{|x - \mathbf{x}^{(i)}|_p}{|q|_p |\theta - 1|_p}$. This means $f_{\theta,q,3}(\mathbb{B}_r(\mathbf{x}^{(i)})) = \mathbb{B}_1(\mathbf{x}^{(i)})$ for $i = 2, 3$. Since $|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|_p = |\mathbf{x}^{(1)} - \mathbf{x}^{(3)}|_p = |\theta - 1|_p < 1$ and $|\mathbf{x}^{(2)} - \mathbf{x}^{(3)}|_p = |q|_p |\theta - 1|_p < 1$, we deduce $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)} \in \mathbb{B}_1(\mathbf{x}^{(i)})$. So $|x - \mathbf{x}^{(i)}|_p \leq |\theta - 1|_p < 1$ for any $x \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. Therefore $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \subset f_{\theta,q,3}(\mathbb{B}_r(\mathbf{x}^{(i)}))$ for $i = 2, 3$.

(ii) Let $x \in \mathcal{C}_1 \setminus \mathbb{B}_r(\mathbf{x}^{(1)})$. Note

$$\mathcal{C}_1 \setminus \mathbb{B}_r(\mathbf{x}^{(1)}) = \left\{ x \in \mathbb{Q}_p : |q|_p |\theta - 1|_p \leq |x - \mathbf{x}^{(1)}|_p < |x - \mathbf{x}^{(\infty)}|_p = |\theta - 1|_p \right\}.$$

By the equality (4.13) and the fact $|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|_p = |\mathbf{x}^{(1)} - \mathbf{x}^{(3)}|_p = |\theta - 1|_p$, we have

$$|\theta - 1|_p < |q|_p^2 \leq |f_{\theta,q,3}(x) - \mathbf{x}^{(1)}|_p = \frac{|q|_p |x - \mathbf{x}^{(1)}|_p}{|\theta - 1|_p} < |q|_p$$

and $|q|_p^2 \leq |f_{\theta,q,3}(x) - \mathbf{x}^{(i)}|_p = |f_{\theta,q,3}(x) - \mathbf{x}^{(1)}|_p < |q|_p$ for $i = 2, 3$. Hence, $f_{\theta,q,3}(x) \notin \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. Thus, the assertion (ii) follows.

(iii) The inclusion $f_{\theta,q,3}^{-1}(\mathbb{X}) \subset \mathbb{X}$ which is equivalent to $f_{\theta,q,3}(\text{Dom}\{f_{\theta,q,3}\} \setminus \mathbb{X}) \subset \text{Dom}\{f_{\theta,q,3}\} \setminus \mathbb{X}$ follows directly from (ii) and Proposition 4.3.9. \square

Now we are going to describe of the dynamics of the Potts–Bethe mapping (4.2) over \mathbb{Q}_p .

Theorem 4.3.13. *Let $0 < |\theta - 1|_p < |q|_p^2 < 1$. Then*

(i) *the dynamical system $(\mathcal{J}_0, f_{\theta,q,3}, |\cdot|_p)$ is isometrically conjugate to the full shift $(\Sigma_3, \sigma, d_{f_{\theta,q,3}})$ over an alphabet of three symbols where*

$$\mathcal{J}_0 = \bigcap_{n=0}^{\infty} f_{\theta,q,3}^{-n}(\mathbb{X}).$$

where the incidence matrix is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix};$$

(ii) $\mathbb{Q}_p = \mathfrak{B}(\mathbf{x}^{(0)}) \cup \mathcal{J}_0 \cup \bigcup_{n=0}^{+\infty} f_{\theta,q,3}^{-n}\{\mathbf{x}^{(\infty)}\}$.

Proof. (i) By the equality (4.13), Proposition 4.3.12 and Theorem 2.3.13, the dynamical system $f_{\theta,q,3} : \mathcal{J}_0 \rightarrow \mathcal{J}_0$ is isometrically conjugate to the full shift dynamics $(\Sigma_3, \sigma, d_{f_{\theta,q,3}})$.

(ii) Due to Theorem 4.3.10, the following inclusion holds,

$$\text{Dom}\{f_{\theta,q,3}\} \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3) \subset \mathfrak{B}(\mathbf{x}^{(0)}).$$

Let $x \in (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3) \setminus \mathcal{J}_0$. There exists n_0 such that $x \notin f_{\theta,q,3}^{-n_0}(\mathbb{X})$ or equivalently $f_{\theta,q,3}^{n_0}(x) \notin \mathbb{X}$. Then by Proposition 4.3.12 (iii), we get $f_{\theta,q,3}^{n_0+1}(x) \in \mathfrak{B}(\mathbf{x}^{(0)}) \cup \{\mathbf{x}^{(\infty)}\}$.

Consequently, we obtain the following decomposition

$$\mathbb{Q}_p = \mathfrak{B}(\mathbf{x}^{(0)}) \cup \mathcal{I}_0 \cup \bigcup_{n=0}^{+\infty} f_{\theta,q,3}^{-n} \{\mathbf{x}^{(\infty)}\}.$$

□

SUBCASE $0 < |q|_p^2 \leq |\theta - 1|_p < |q|_p < 1$. Let $m \geq 1$ be an integer such that

$$|q|_p^{\frac{m+1}{m}} \leq |\theta - 1|_p < |q|_p^{\frac{m+2}{m+1}}.$$

Consider for $\ell = 0, 1, \dots, m$, the spheres

$$\mathcal{G}_\ell := \left\{ x : |x - \mathbf{x}^{(1)}| = \frac{|\theta - 1|_p^{\ell+1}}{|q|_p^\ell} \right\}.$$

Let $r = |q|_p |\theta - 1|_p$. Then \mathcal{G}_ℓ can be written as a union of disjoint balls of radius r , i.e. $\mathcal{G}_\ell = \bigcup_{i=1}^{m_\ell} \mathbb{B}_r(a_{i,\ell})$ where $a_{i,\ell} \in \mathcal{G}_\ell$ and $\mathbb{B}_r(a_{i,\ell}) \cap \mathbb{B}_r(a_{j,\ell}) = \emptyset$ for $1 \leq i \neq j \leq m_\ell$.

Since the sphere \mathcal{G}_ℓ is a union of $p - 1$ open balls with radius $\frac{|\theta - 1|_p^{\ell+1}}{|q|_p^\ell}$, by the following Remark 4.3.14, we have $m_\ell = (p - 1) \frac{|\theta - 1|_p^\ell}{|q|_p^{\ell+1}}$.

Remark 4.3.14. Let $\gamma \leq \delta$ be two integers. Any open ball $\mathbb{B}_{p^\delta}(a)$ is decomposed into $p^{\delta-\gamma}$ disjoint open balls $\mathbb{B}_{p^\gamma}(a_i)$, i.e., $\mathbb{B}_{p^\delta}(a) = \bigcup_{i=1}^{p^{\delta-\gamma}} \mathbb{B}_{p^\gamma}(a_i)$ where $a_i \in \mathbb{B}_{p^\delta}(a)$, and $|a_i - a_j|_p = p^\gamma$ if $a_i \neq a_j$.

Proposition 4.3.15. The map $f_{\theta,q,3} : \mathcal{C}_1 \rightarrow \mathbb{B}_{|q|_p}(\mathbf{x}^{(1)})$ is a one-to-one correspondence. In particular, $f_{\theta,q,3} : \mathcal{G}_\ell \rightarrow \mathcal{G}_{\ell-1}$ is a one-to-one correspondence for $1 \leq \ell \leq m$.

Proof. Recall $\mathcal{C}_1 = \mathbb{B}_{|\theta-1|_p}(\mathbf{x}^{(1)})$. By equality (4.13), we have a scaling on \mathcal{C}_1 such that for any $x, y \in \mathcal{C}_1$,

$$|f_{\theta,q,3}(x) - f_{\theta,q,3}(y)|_p = \frac{|q|_p}{|\theta - 1|_p} |x - y|_p.$$

Since $\mathbf{x}^{(1)}$ is the fixed point of $f_{\theta,q,3}$, we get $f_{\theta,q,3}(\mathcal{C}_1) = \mathbb{B}_{|q|_p}(\mathbf{x}^{(1)})$. By Lemma 2.3.3, $f_{\theta,q,3}$ is a one-to-one correspondence between \mathcal{C}_1 and $\mathbb{B}_{|q|_p}(\mathbf{x}^{(1)})$. Furthermore, we have for $1 \leq l \leq m$, $\mathcal{G}_\ell \subset \mathcal{C}_1$ and $f_{\theta,q,3}(\mathcal{G}_\ell) \subseteq \mathcal{G}_{\ell-1}$. Since \mathcal{G}_ℓ are spheres, by the Corollary 2.3.4, $f_{\theta,q,3}$ is one-to-one between \mathcal{G}_ℓ and $\mathcal{G}_{\ell-1}$. □

We have $\mathcal{G}_\ell \subset \mathcal{C}_1$ for all $\ell = 1, 2, \dots, m$ and $(\mathcal{C}_2 \cup \mathcal{C}_3) \subset \mathcal{G}_0$. For all $\ell = 1, 2, \dots, m$, suppose $\mathcal{H}_\ell = \bigcup_{i=1}^{n_\ell} \mathbb{B}_r(a_{i,\ell}) \subset \mathcal{G}_\ell$ with $a_{i,\ell} \in \mathcal{G}_\ell$ such that for $1 \leq i \neq j \leq n_\ell \leq m_\ell$, $\mathbb{B}_r(a_{i,\ell}) \cap \mathbb{B}_r(a_{j,\ell}) = \emptyset$ and $(\mathcal{C}_2 \cup \mathcal{C}_3) \subset f_{\theta,q,3}^\ell(\mathbb{B}_r(a_{i,\ell}))$. Then we have the following crucial result.

Proposition 4.3.16. For $\ell = 1, 2, \dots, m$, we have $n_\ell = 1$.

Proof. We need only to show $n_1 = 1$ and the rest follows from the Proposition 4.3.15. By the fact $|\mathbf{x}^{(2)} - \mathbf{x}^{(3)}| = |q|_p |\theta - 1|_p$, we have $|x - y| = |q|_p |\theta - 1|_p$ for any $x \in \mathcal{C}_2$, $y \in \mathcal{C}_3$. Since $|q|_p |\theta - 1|_p < |q|_p^2$, we have that \mathcal{C}_2 and \mathcal{C}_3 are in the image of a same ball $\mathbb{B}_r(a_{i,1}) \subset \mathcal{G}_1$ for some $1 \leq i \leq m_1$, i.e., $\mathcal{C}_2 \cup \mathcal{C}_3 \subset f_{\theta,q,3}(\mathbb{B}_r(a_{i,1}))$. By the Proposition 4.3.15, this ball is unique. Hence $n_1 = 1$. \square

Without loss of generality, we may assume $\mathcal{H}_\ell = \mathbb{B}_r(a_{1,\ell})$ with $a_{1,\ell} \in \mathcal{G}_\ell$. Let $\mathcal{D}_1 = \mathbb{B}_r(\mathbf{x}^{(1)})$ and for $m \geq 1$,

$$\mathbb{X}_m = \mathcal{D}_1 \cup \mathcal{H}_m \cup \dots \cup \mathcal{H}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3.$$

Then we have the following result on the pre-image of \mathbb{X}_m under $f_{\theta,q,3}$.

Proposition 4.3.17. Suppose $0 < |q|_p^{\frac{m+1}{m}} \leq |\theta - 1|_p < |q|_p^{\frac{m+2}{m+1}} < 1$ for some $m \geq 1$. Then $f_{\theta,q,3}^{-1}(\mathbb{X}_m) \subset \mathbb{X}_m$.

Proof. It suffices to show $f_{\theta,q,3}(\text{Dom}\{f_{\theta,q,3}\} \setminus \mathbb{X}_m) \cap \mathbb{X}_m = \emptyset$. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. By the Proposition 4.3.9, we have $f_{\theta,q,3}(\text{Dom}\{f_{\theta,q,3}\} \setminus \mathcal{C}) \cap \mathcal{C} = \emptyset$. Since $\mathbb{X}_m \subset \mathcal{C}$, we are left to show $f_{\theta,q,3}(\mathcal{C} \setminus \mathbb{X}_m) \cap \mathbb{X}_m = \emptyset$. It is clear that $\mathcal{C} \setminus \mathbb{X}_m = \mathcal{C}_1 \setminus (\mathcal{D}_1 \cup \mathcal{H}_1 \cup \dots \cup \mathcal{H}_m)$. By the one-to-one property in the Proposition 4.3.15, the uniqueness in the Proposition 4.3.16 and the definition of \mathcal{H}_ℓ for $1 \leq \ell \leq m$, we have $f_{\theta,q,3}(\mathcal{C}_1 \setminus (\mathcal{D}_1 \cup \mathcal{H}_1 \cup \dots \cup \mathcal{H}_m)) \cap \mathbb{X}_m = \emptyset$. \square

Next we describe the image of \mathbb{X}_m .

Proposition 4.3.18. Suppose $0 < |q|_p^{\frac{m+1}{m}} \leq |\theta - 1|_p < |q|_p^{\frac{m+2}{m+1}}$ for some $m \geq 1$. Then

- (i) For $i = 2, 3$, $(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3) \subset f_{\theta,q,3}(\mathcal{C}_i)$;
- (ii) $(\mathcal{C}_2 \cup \mathcal{C}_3) \cap f_{\theta,q,3}(\mathcal{D}_1) = \emptyset$;
- (iii) $(\mathcal{C}_2 \cup \mathcal{C}_3) \subset f_{\theta,q,3}(\mathcal{H}_1)$;
- (iv) for $m = 1$, we have $\mathcal{H}_1 \subset f_{\theta,q,3}(\mathcal{D}_1)$ and $f_{\theta,q,3}(\mathcal{H}_1) \cap \mathcal{D}_1 = \emptyset$;
- (v) for $m \geq 2$, we have

(a) $\mathcal{H}_m \subset f_{\theta,q,3}(\mathcal{D}_1)$ and $f_{\theta,q,3}(\mathcal{D}_1) \cap (\mathcal{H}_{m-1} \cup \dots \cup \mathcal{H}_1) = \emptyset$;

(b) for any \mathcal{H}_ℓ , $2 \leq \ell \leq m$,

$$\mathcal{H}_{\ell-1} \subset f_{\theta,q,3}(\mathcal{H}_\ell) \text{ and } f_{\theta,q,3}(\mathcal{H}_\ell) \cap \left(\mathcal{D}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \bigcup_{i=1, i \neq \ell-1}^m \mathcal{H}_i \right) = \emptyset;$$

(c) $f_{\theta,q,3}(\mathcal{H}_1) \cap \left(\mathcal{D}_1 \cup \bigcup_{\ell=2}^m \mathcal{H}_\ell \right) = \emptyset$.

Proof. Recall for $1 \leq l \leq m$ that $\mathcal{H}_l = \mathbb{B}_r(a_{1,l})$, $|a_{1,l} - \mathbf{x}^{(1)}|_p = \frac{|\theta-1|_p^{l+1}}{|q|_p^l}$ and $(\mathcal{C}_2 \cup \mathcal{C}_3) \subset f_{\theta,q,3}^\ell(\mathbb{B}_r(a_{1,l}))$.

(i) By the equality (4.13), we have for $i = 2, 3$ that $f_{\theta,q,3}(\mathcal{C}_i) = \mathbb{B}_1(\mathbf{x}^{(i)})$. Since $|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|_p = |\mathbf{x}^{(1)} - \mathbf{x}^{(3)}|_p = |\theta - 1|_p < 1$ and $|\mathbf{x}^{(2)} - \mathbf{x}^{(3)}|_p = |q|_p |\theta - 1|_p < 1$, we have $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)} \in \mathbb{B}_1(\mathbf{x}^{(i)})$ for $i = 2, 3$. Moreover, since $|x - \mathbf{x}^{(i)}|_p \leq |\theta - 1|_p < 1$ for any $x \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, we get the inclusion $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \subset f_{\theta,q,3}(\mathcal{C}_i)$.

(ii) By the equality (4.13), we obtain $f_{\theta,q,3}(\mathcal{D}_1) = \mathbb{B}_{|q|_p^2}(\mathbf{x}^{(1)})$. Let $x \in \mathcal{C}_2$ or $x \in \mathcal{C}_3$. Then $|x - \mathbf{x}^{(1)}|_p = |\mathbf{x}^{(i)} - \mathbf{x}^{(1)}|_p = |\theta - 1|_p \geq |q|_p^2$. Thus, $f_{\theta,q,3}(\mathcal{D}_1) \cap (\mathcal{C}_2 \cup \mathcal{C}_3) = \emptyset$.

(iii) By the definition of \mathcal{H}_1 , it is clear $(\mathcal{C}_2 \cup \mathcal{C}_3) \subset f_{\theta,q,3}(\mathcal{H}_1)$.

(iv) Let $m = 1$. Recall from the argument in the proof of (ii) that $f_{\theta,q,3}(\mathcal{D}_1) = \mathbb{B}_{|q|_p^2}(\mathbf{x}^{(1)})$. Since $|q|_p^2 \leq |\theta - 1|_p < |q|_p^{\frac{3}{2}}$, we obtain $\frac{|\theta-1|_p^2}{|q|_p} < |q|_p^2 \leq |\theta - 1|_p$. Let $x \in \mathcal{H}_1$. Then $|x - \mathbf{x}^{(1)}|_p = |a_{1,1} - \mathbf{x}^{(1)}|_p = \frac{|\theta-1|_p^2}{|q|_p} < |q|_p^2$. Thus $\mathcal{H}_1 \subset f_{\theta,q,3}(\mathcal{D}_1)$.

Next, by the equality (4.13), we have $f_{\theta,q,3}(\mathcal{H}_1) = \mathbb{B}_{|q|_p^2}(f_{\theta,q,3}(a_{1,1}))$ and $|f_{\theta,q,3}(a_{1,1}) - \mathbf{x}^{(1)}|_p = |\theta - 1|_p$. Let $x \in \mathcal{D}_1$. Then $|x - f_{\theta,q,3}(a_{1,1})|_p = |\mathbf{x}^{(1)} - f_{\theta,q,3}(a_{1,1})|_p = |\theta - 1|_p \geq |q|_p^2$. Hence $f_{\theta,q,3}(\mathcal{H}_1) \cap \mathcal{D}_1 = \emptyset$.

(v) Let $m \geq 2$. Recall that $f_{\theta,q,3}(\mathcal{D}_1) = \mathbb{B}_{|q|_p^2}(\mathbf{x}^{(1)})$. From $0 < |q|_p^{\frac{m+1}{m}} \leq |\theta - 1|_p < |q|_p^{\frac{m+2}{m+1}}$, we have $\frac{|\theta-1|_p^{m+1}}{|q|_p^m} < |q|_p^2 \leq \frac{|\theta-1|_p^m}{|q|_p^{m-1}}$. Let $x \in \mathcal{H}_m$. Then $|x - \mathbf{x}^{(1)}|_p = |a_{1,m} - \mathbf{x}^{(1)}|_p = \frac{|\theta-1|_p^{m+1}}{|q|_p^m} < |q|_p^2$. Thus $\mathcal{H}_m \subset f_{\theta,q,3}(\mathcal{D}_1)$.

By the Proposition 4.3.15, we deduce $f_{\theta,q,3} : \mathcal{C}_1 \rightarrow \mathbb{B}_{|q|_p}(\mathbf{x}^{(1)})$ and $f_{\theta,q,3} : \mathcal{G}_\ell \rightarrow \mathcal{G}_{\ell-1}$ for $1 \leq \ell \leq m$ are all one-to-one. By the definition of \mathcal{H}_ℓ , we have $\mathcal{H}_{\ell-1} \subset f_{\theta,q,3}(\mathcal{H}_1) = \mathbb{B}_{\bar{r}}(f_{\theta,q,3}(a_{1,\ell}))$, $2 \leq \ell \leq m$ where $\bar{r} = |q|_p^2$. Therefore, the rests, (a)–(c), are true. \square

We define the the following set

$$\mathcal{J}_m = \bigcap_{n=0}^{\infty} f_{\theta,q,3}^{-n}(\mathbb{X}_m).$$

Theorem 4.3.19. *Let $0 < |q|_p^{\frac{m+1}{m}} \leq |\theta - 1|_p < |q|_p^{\frac{m+2}{m+1}} < 1$ for some $m \geq 1$. Then*

(i) *$(\mathbb{X}_m, \mathcal{J}_m, f_{\theta,q,3})$ is transitive and the dynamical system $(\mathcal{J}_m, f_{\theta,q,3}, |\cdot|_p)$ is isometrically conjugate to the subshift dynamics $(\Sigma_{A_m}, \sigma_{A_m}, d_{f_{\theta,q,3}})$ on $(m+3)$ symbols with*

$$A_m = \begin{pmatrix} \mathbf{e}_1^T & \mathbf{I}_m & \mathbf{0}^T & \mathbf{0}^T \\ 0 & \mathbf{0} & 1 & 1 \\ 1 & \mathbf{1} & 1 & 1 \\ 1 & \mathbf{1} & 1 & 1 \end{pmatrix}$$

where \mathbf{I}_m is an $m \times m$ identity matrix, $\mathbf{e}_1 = (1, 0, \dots, 0)$, $\mathbf{1} = (1, 1, \dots, 1)$ and $\mathbf{0} = (0, 0, \dots, 0)$. Here \mathbf{v}^T is the transpose of the vector \mathbf{v} . Moreover, this subshift is topologically conjugate to the full shift on three symbols;

$$(ii) \mathbb{Q}_p = \mathfrak{B}(\mathbf{x}^{(0)}) \cup \mathcal{J}_m \cup \bigcup_{n=0}^{+\infty} f_{\theta,q,3}^{-n} \{\mathbf{x}^{(\infty)}\}.$$

Proof. Let us consider the coding $\mathcal{D}_1 \rightarrow 0$, $\mathcal{H}_\ell \rightarrow \ell$ where $\ell = 1, \dots, m$, $\mathcal{C}_2 \rightarrow m+1$ and $\mathcal{C}_3 \rightarrow m+2$. Due to the Theorem 2.3.13 and Propositions 4.3.16–4.3.18, the dynamical system $f_{\theta,q,3} : \mathcal{J}_m \rightarrow \mathcal{J}_m$ is isometrically conjugate to the subshift dynamics $(\Sigma_{A_m}, \sigma_{A_m}, d_{f_{\theta,q,3}})$ associated with the $(m+3) \times (m+3)$ matrix A_m for $m \geq 1$.

Next, we want show that the subshift $\Sigma_{A_m} \subset \Sigma_{m+3}$ is topologically conjugate to the full shift Σ_3 on three symbols. For details, one can refer to Sections 1.4–1.5 by Lind & Marcus (1995). We will construct a bijection $\pi : \Sigma_3 \rightarrow \Sigma_{A_m}$. First, we let $\pi_m : \mathcal{B}_{m+1}(\Sigma_3) \rightarrow \mathcal{B}_1(\Sigma_{A_m})$, where $\mathcal{B}_N(\Sigma)$ is the set of all N -block on a shift space Σ over $\{1, 2, \dots, s\}$ i.e., the N -block $(x_j x_{j+1} \dots x_{j+N})$, $j \leq n \leq j+N$, $x_n \in \{1, 2, \dots, s\}$ and $j \geq 1$. We define for $x_i \in \{0, 1, 2\}$ and $l, i = 1, \dots, m$, π_m :

$$\begin{aligned} 0^{m+1} &\mapsto 0 \\ 0^l 1 x_1 \dots x_{m-l} &\mapsto l \\ 0^l 2 x_1 \dots x_{m-l} &\mapsto l \\ 1 x_1 \dots x_m &\mapsto m+1 \\ 2 x_1 \dots x_m &\mapsto m+2. \end{aligned}$$

Now we are ready to construct the map π . We have for any $x \in \Sigma_3$,

$$\pi(x) := \pi_m(x_0 \dots x_m) \pi_m(x_1 \dots x_{m+1}) \pi_m(x_2 \dots x_{m+2}) \dots \pi_m(x_i \dots x_{m+i}) \dots$$

Then $\pi \circ \sigma = \sigma_{A_m} \circ \pi$. Moreover, π is bijective. Therefore, we have shown a conjugacy between Σ_A and Σ_{A_m} .

(ii) The proof is similar to the proof of (ii) of Theorem 4.3.13. □

4.3.2 The Case $0 < |q|_p < |\theta - 1|_p < 1$

Let us first check the behaviour of the fixed points of the Potts–Bethe mapping (4.2).

Proposition 4.3.20. *Let $p \geq 5$ be a prime number. Then the fixed points $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ are indifferent. Moreover, if $p \equiv 1 \pmod{6}$, then the fixed points $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ exist and they are repelling.*

Proof. The scheme of the proof is similar to Proposition 4.3.5. We need to show $\left| f'_{\theta,q,3}(\mathbf{x}^{(i)}) \right|_p = 1$ for $i = 0, 1$ and $\left| f'_{\theta,q,3}(\mathbf{x}^{(i)}) \right|_p > 1$ for $i = 2, 3$. We have

$$\begin{aligned} f'_{\theta,q,3}(\mathbf{x}^{(i)}) &= 3(\theta - 1)(\theta - 1 + q) \frac{(\theta \mathbf{x}^{(i)} + q - 1)^2}{(\mathbf{x}^{(i)} - \mathbf{x}^{(\infty)})^4} \\ &= \frac{3(\theta - 1)(\theta - 1 + q) \mathbf{x}^{(i)}}{(\theta \mathbf{x}^{(i)} + q - 1)(\mathbf{x}^{(i)} - \mathbf{x}^{(\infty)})}. \end{aligned} \tag{4.15}$$

Then $\left| f'_{\theta,q,3}(\mathbf{x}^{(0)}) \right|_p = 1$. It follows from the equality (4.15) and $\mathbf{x}^{(i)} = 1 - q + (\theta - 1)(\mathbf{y}^{(i)} - 1)$ for $1 \leq i \leq 3$ that

$$f'_{\theta,q,3}(\mathbf{x}^{(i)}) = \frac{3(\theta - 1 + q)\mathbf{x}^{(i)}}{(\theta - 1)\mathbf{y}^{(i)}(\theta\mathbf{y}^{(i)} + 1 - \theta - q)}. \quad (4.16)$$

By Proposition 4.3.1, we have $|\mathbf{y}^{(2)}|_p = |\mathbf{y}^{(3)}|_p = |\theta - 1|_p < 1 = |\mathbf{y}^{(1)}|_p$ and $|\mathbf{x}^{(i)}|_p = 1$ for $1 \leq i \leq 3$. Then we obtain $\left| f'_{\theta,q,3}(\mathbf{x}^{(1)}) \right|_p = 1$.

Now suppose $p \equiv 1 \pmod{6}$. This implies the existence of the fixed points $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$. It follows from the equality (4.16) that for $i = 2, 3$

$$f'_{\theta,q,3}(\mathbf{x}^{(i)}) = \frac{3(\theta - 1 + q)|1 - \theta - q|_p^2 \mathbf{x}^{(i)}}{(\theta - 1)(\mathbf{y}^{(i)})^* (\theta (\mathbf{y}^{(i)})^* + (1 - \theta - q)^*)}. \quad (4.17)$$

By the congruence (4.5), we have $\theta (\mathbf{y}^{(i)})^* \not\equiv -(1 - \theta - q)^* \pmod{p}$ for $i = 2, 3$. Consequently,

$$\left| f'_{\theta,q,3}(\mathbf{x}^{(i)}) \right|_p = \frac{|\theta - 1 + q|_p}{|1 - \theta - q|_p^2 |\theta - 1|_p} = \frac{1}{|\theta - 1|_p^2} > 1.$$

□

Recall that the Potts–Bethe mapping (4.2) has a critical point $\mathbf{x}^{(c)}$. One could verify $\mathbf{x}^{(c)} = \mathbf{x}^{(\infty)} + (\theta - 1)\mathbf{y}^{(c)} = \frac{1-q}{\theta}$ where $\mathbf{y}^{(c)} = \frac{\theta-1+q}{\theta}$.

Proposition 4.3.21. *Let $p \geq 5$ be a prime number. Then*

- (i) $|\mathbf{y}^{(c)}|_p = |\mathbf{y}^{(2)}|_p = |\mathbf{y}^{(3)}|_p = |\theta - 1|_p < 1 = |\mathbf{y}^{(1)}|_p$ and $|\mathbf{y}^{(1)} - 3|_p = |\theta - 1|_p$,
- (ii) $|\mathbf{y}^{(2)} - \mathbf{y}^{(3)}|_p = |\mathbf{y}^{(2)} - \mathbf{y}^{(c)}|_p = |\mathbf{y}^{(3)} - \mathbf{y}^{(c)}|_p = |\theta - 1|_p$,
- (iii) $|\mathbf{x}^{(c)}|_p = |\mathbf{x}^{(\infty)}|_p = |\mathbf{x}^{(i)}|_p = 1$ for $i = 0, 1, 2, 3$,
- (iv) $|\mathbf{x}^{(0)} - \mathbf{x}^{(\infty)}|_p = |\mathbf{x}^{(1)} - \mathbf{x}^{(\infty)}|_p = |\theta - 1|_p$,
- (v) $|\mathbf{x}^{(2)} - \mathbf{x}^{(\infty)}|_p = |\mathbf{x}^{(3)} - \mathbf{x}^{(\infty)}|_p = |\mathbf{x}^{(c)} - \mathbf{x}^{(\infty)}|_p = |\mathbf{x}^{(2)} - \mathbf{x}^{(3)}|_p = |\mathbf{x}^{(2)} - \mathbf{x}^{(c)}|_p = |\mathbf{x}^{(3)} - \mathbf{x}^{(c)}|_p = |\theta - 1|_p^2$,
- (vi) $|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}|_p = |\theta - 1|_p$ and $|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}|_p = |\mathbf{x}^{(i)} - \mathbf{x}^{(c)}|_p = |\theta - 1|_p$ for $i = 0, 1$, and $j = 2, 3$.

Proof. Recall $\mathbf{x}^{(\infty)} = 2 - \theta - q$, $\mathbf{x}^{(c)} = \mathbf{x}^{(\infty)} + (\theta - 1)\mathbf{y}^{(c)}$, $\mathbf{x}^{(0)} = 1$ and $\mathbf{x}^{(i)} = \mathbf{x}^{(\infty)} + (\theta - 1)\mathbf{y}^{(i)}$ for $i = 1, 2, 3$. Then one could easily checked all the properties. □

We introduce the following sets

$$\begin{aligned}\mathcal{A} &= \left\{x \in \mathbb{Q}_p : 0 < |x - \mathbf{x}^{(\infty)}|_p < |\theta - 1|_p\right\}, \\ \mathcal{B} &= \left\{x \in \mathbb{Q}_p : |x - \mathbf{x}^{(\infty)}|_p > |\theta - 1|_p\right\}, \\ \mathcal{C} &= \left\{x \in \mathbb{Q}_p : |x - \mathbf{x}^{(\infty)}|_p = |\theta - 1|_p\right\}\end{aligned}$$

with $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{D}$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{E}$ such that

$$\begin{aligned}\mathcal{A}_1 &= \left\{x \in \mathbb{Q}_p : |\theta - 1|_p^2 < |x - \mathbf{x}^{(\infty)}|_p < |\theta - 1|_p\right\}, \\ \mathcal{D} &= \left\{x \in \mathbb{Q}_p : 0 < |x - \mathbf{x}^{(\infty)}|_p \leq |\theta - 1|_p^2\right\}, \\ \mathcal{B}_1 &= \left\{x \in \mathbb{Q}_p : |\theta - 1|_p < |x - \mathbf{x}^{(\infty)}|_p < 1\right\}, \\ \mathcal{E} &= \left\{x \in \mathbb{Q}_p : |x - \mathbf{x}^{(\infty)}|_p \geq 1\right\}.\end{aligned}$$

In order to study the dynamics of the Potts–Bethe mapping (4.2), we will calculate the image of the sets \mathcal{A} , \mathcal{B} and \mathcal{C} .

Now we begin with the set \mathcal{C} . Suppose $\bar{x}^{(i)} = \mathbf{x}^{(\infty)} + (\theta - 1)\bar{y}^{(i)} \in \mathcal{C}$ where $|\bar{y}^{(i)} - i| < 1$ and $\mathcal{C}_i := \mathbb{B}_{|\theta-1|_p}(\bar{x}^{(i)})$. Then

$$\mathcal{C} := \bigcup_{i=1}^{p-1} \mathcal{C}_i.$$

Lemma 4.3.22. *Let $p \geq 5$ be a prime number.*

(i) *Let $x \in \mathcal{C}$. Then $|f'_{\theta,q,3}(x)|_p = 1$.*

(ii) *Let $\bar{x}, \bar{\bar{x}} \in \mathcal{C}_i$, $1 \leq i \leq p-1$. Then $|f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}})|_p = |\bar{x} - \bar{\bar{x}}|_p$.*

Proof. We calculate the k -th derivative of $f_{\theta,q,3}$ for $k \geq 1$ as follows

$$\begin{aligned}f_{\theta,q,3}^{(k)}(x) &= \frac{3k!(-1)^{k+1}(\theta-1)(\theta-1+q)}{(x - \mathbf{x}^{(\infty)})^{k+1}} \left(\theta^2 + \frac{(k+1)\theta(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right. \\ &\quad \left. + \frac{(k+1)(k+2)(\theta-1)^2(1-\theta-1-q)^2}{3!(x - \mathbf{x}^{(\infty)})^2} \right).\end{aligned}\tag{4.18}$$

(i) Suppose $x \in \mathcal{C}$. Then we have $|x - \mathbf{x}^{(\infty)}|_p = |\theta - 1|_p = |\theta(x - \mathbf{x}^{(\infty)}) + (\theta - 1)(1 - \theta - q)|_p$. By the equality (4.18), we obtain

$$|f'_{\theta,q,3}(x)|_p = \left| \frac{3(\theta-1)(\theta-1+q)(\theta(x - \mathbf{x}^{(\infty)}) + (\theta-1)(1-\theta-q))^2}{(x - \mathbf{x}^{(\infty)})^4} \right|_p = 1$$

(ii) Suppose $\bar{r} = \frac{|\theta-1|_p}{p}$ and $\bar{x}^{(i)} \in \mathcal{C}_i$. By the equality (4.18), we have for $k \geq 2$

$$\left| \frac{(k+1)\theta(\theta-1)(1-\theta-q)}{\bar{x}^{(i)} - \mathbf{x}^{(\infty)}} \right|_p = |k+1|_p |\theta-1|_p \leq |\theta-1|_p < 1 \quad \text{and}$$

$$\left| \frac{(k+1)(k+2)(\theta-1)^2(1-\theta-1-q)^2}{3!(\bar{x}^{(i)} - \mathbf{x}^{(\infty)})^2} \right|_p = |k+1|_p |k+2|_p |\theta-1|_p^2 \leq |\theta-1|_p^2 < 1.$$

These imply

$$\left| f_{\theta,q,3}^{(k)}(\bar{x}^{(i)}) \right|_p \bar{r}^{k-1} = \frac{1}{|\theta-1|_p^{k-1}} \cdot \frac{|\theta-1|_p^{k-1}}{p^{k-1}} = \frac{1}{p^{k-1}} < 1 = \left| f'_{\theta,q,3}(\bar{x}^{(i)}) \right|_p.$$

We also notice for $1 \leq i \leq p-1$ that $\overline{\mathbb{B}_{\bar{r}}(\bar{x}^{(i)})} = \mathcal{C}_i$. Therefore, by (i) and Lemma 2.3.1, we have for $\bar{x}, \bar{\bar{x}} \in \mathcal{C}_i$ that $|f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}})|_p = |\bar{x} - \bar{\bar{x}}|_p$. \square

Recall $\mathbb{P}^1(\mathbb{F}_p) = \{0, 1, \dots, p-1, \infty\}$. By definition, $\mathcal{C}_0 = \mathcal{A}$ and $\mathcal{C}_\infty = \mathcal{B}$. Then we have the following proposition.

Proposition 4.3.23. *Let $p \geq 5$ be a prime number. Suppose $m : \mathbb{P}^1(\mathbb{F}_p) \rightarrow \mathbb{P}^1(\mathbb{F}_p)$ and $m(x) = 4 - \frac{3}{x}$. Then for $1 \leq i \leq p-1$ we have $f_{\theta,q,3}(\mathcal{C}_i) = \mathcal{C}_{m(i)}$. Moreover, we have*

- (i) $f_{\theta,q,3}(\mathcal{C}_i) = \mathcal{C}_i$ for $i = 1, 3$;
- (ii) $f_{\theta,q,3}^k(\mathcal{C}_4) = \mathcal{A}$ for some $k \in \{1, \dots, p-3\}$.

Proof. Recall that the Potts–Bethe mapping (4.2) can be written as follows

$$f_{\theta,q,3}(x) = \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right)^3.$$

Then we get

$$\begin{aligned} f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)} &= f_{\theta,q,3}(x) - \mathbf{x}^{(0)} + q + (\theta-1) \\ &= \frac{(\theta-1)(x - \mathbf{x}^{(0)})}{x - \mathbf{x}^{(\infty)}} g_1(x) + q + (\theta-1) \end{aligned} \quad (4.19)$$

where $g_1(x) = \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right)^2 + \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right) + 1$. By modifying the equality (4.19), we obtain

$$\begin{aligned} f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)} &= \frac{(\theta-1)(x - \mathbf{x}^{(0)})(g_1(x) - 3)}{x - \mathbf{x}^{(\infty)}} + \\ &\quad q \left(\frac{(\theta-1)(\mathbf{y}^{(1)} - 3) + (x - \mathbf{x}^{(1)})}{x - \mathbf{x}^{(\infty)}} \right) + \\ &\quad (\theta-1) \left(4 - 3 \frac{(\theta-1)}{x - \mathbf{x}^{(\infty)}} \right). \end{aligned} \quad (4.20)$$

Suppose $\bar{x}^{(i)} = \mathbf{x}^{(\infty)} + (\theta - 1)\bar{y}^{(i)} \in \mathcal{C}_i$ with $|\bar{y}^{(i)} - i|_p < 1$. Then we obtain by the equality (4.20) that

$$\begin{aligned} f_{\theta,q,3}(\bar{x}^{(i)}) - \mathbf{x}^{(\infty)} &\equiv (\theta - 1) \left(4 - 3 \frac{(\theta - 1)}{\bar{x}^{(i)} - \mathbf{x}^{(\infty)}} \right) \pmod{p^{\text{ord}_p(\theta-1)+1}} \\ &\equiv (\theta - 1) \left(4 - \frac{3}{\bar{y}^{(i)}} \right) \pmod{p^{\text{ord}_p(\theta-1)+1}} \\ \frac{f_{\theta,q,3}(\bar{x}^{(i)}) - \mathbf{x}^{(\infty)}}{\theta - 1} &\equiv 4 - \frac{3}{i} \equiv m(i) \pmod{p}. \end{aligned}$$

So $f_{\theta,q,3}(\bar{x}^{(i)}) = \mathbf{x}^{(\infty)} + (\theta - 1)w^{(i)}$ such that $w^{(i)} \equiv m(i) \pmod{p}$. It implies $f_{\theta,q,3}(\bar{x}^{(i)}) \in \mathcal{C}_{m(i)}$. By Lemmas 4.3.22 and 2.3.3, we have $f_{\theta,q,3}(\mathcal{C}_i) = \mathcal{C}_{m(i)}$.

(i) It is clear $f_{\theta,q,3}(\mathcal{C}_1) = \mathcal{C}_1$ and $f_{\theta,q,3}(\mathcal{C}_3) = \mathcal{C}_3$ since 1 and 3 are the fixed points of function m . (ii) We have for $1 \leq i \leq p - 1$ that $f_{\theta,q,3}(\mathcal{C}_i) = \mathcal{C}_{m(i)}$. One also could verify that $m(i) = 0$ for some $i = 1, \dots, p - 1$ and $m(i) \neq \infty$ for any $i = 1, \dots, p - 1$. In order to check whether $f_{\theta,q,3}^k(\mathcal{C}_4) = \mathcal{C}_0$ for some k or not, it is enough to check whether 4 and 0 are in the same orbit or not under iteration of m . By simple calculation, we can see that $m^2(0) = 4$. It shows 4 and 0 are in the same orbit. By the bijectivity of m , there exists $k \in \mathbb{N}$ such that $m^k(4) = 0$ and the possible least number $k \in \{1, \dots, p - 3\}$ since 1 and 3 are the fixed points of m . Thus, $f_{\theta,q,3}^k(\mathcal{C}_4) = \mathcal{C}_0$ for some $k \in \{1, \dots, p - 3\}$. \square

Next we calculate the image of the sets \mathcal{B} and \mathcal{A}_1 .

Proposition 4.3.24. *Let $p \geq 5$ be a prime number. Then $f_{\theta,q,3}(\mathcal{B}) \subseteq \mathcal{C}_4$ and $f_{\theta,q,3}(\mathcal{A}_1) \subseteq \mathcal{B}_1$. Moreover, the restricted maps $f_{\theta,q,3} : \mathcal{B} \rightarrow \mathcal{C}_4 \setminus \theta^3$ and $f_{\theta,q,3} : \mathcal{A}_1 \rightarrow \mathcal{B}_1$ are bijective.*

Proof. Let $x \in \mathcal{B}$. Then $|x - \mathbf{x}^{(\infty)}|_p > |\theta - 1|_p$. By the equality (4.19), we have

$$\begin{aligned} f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)} &\equiv (\theta - 1) \left(4 - 3 \frac{(\theta - 1)}{x - \mathbf{x}^{(\infty)}} \right) \pmod{p^{\text{ord}_p(\theta-1)+1}} \\ &\equiv 4(\theta - 1) \not\equiv 0 \pmod{p^{\text{ord}_p(\theta-1)+1}}. \end{aligned} \quad (4.21)$$

It means $|f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)}|_p = |\theta - 1|_p$ and $f_{\theta,q,3}(x) = \mathbf{x}^{(\infty)} + (\theta - 1)\bar{w}$ with $|\bar{w}|_p = 1$ and $|\bar{w} - 4|_p < 1$. Hence $f_{\theta,q,3}(x) \in \mathcal{C}_4$ for any $x \in \mathcal{B}$ which implies $f_{\theta,q,3}(\mathcal{B}) \subseteq \mathcal{C}_4$.

Recall

$$f_{\theta,q,3}(x) - \mathbf{x}^{(0)} = \frac{(\theta - 1)(x - \mathbf{x}^{(0)})}{x - \mathbf{x}^{(\infty)}} g_1(x) \quad (4.22)$$

where $g_1(x) = \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right)^2 + \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right) + 1$. Let $x \in \mathcal{A}_1$. Since $|\theta -$

$1|_p^2 < |x - \mathbf{x}^{(\infty)}|_p < |\theta - 1|_p$, we have

$$\left| \frac{(\theta - 1)(1 - \theta - q)}{x - \mathbf{x}^{(\infty)}} \right|_p = \frac{|\theta - 1|_p^2}{|x - \mathbf{x}^{(\infty)}|_p} < 1.$$

This implies $|g_1(x)|_p = 1$. We get by the equality (4.22) that

$$|\theta - 1|_p < \left| f_{\theta,q,3}(x) - \mathbf{x}^{(0)} \right|_p = \frac{|\theta - 1|_p |x - \mathbf{x}^{(0)}|_p}{|x - \mathbf{x}^{(\infty)}|_p} < 1.$$

Then we have $|f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)}|_p = |f_{\theta,q,3}(x) - \mathbf{x}^{(0)}|_p$. So for any $x \in \mathcal{A}_1$ We have $f_{\theta,q,3}(x) \in \mathcal{B}_1$. Thus $f_{\theta,q,3}(\mathcal{A}_1) \subseteq \mathcal{B}_1$.

We proceed by showing the bijectiveness of the restricted maps. Recall for $\bar{x}, \bar{\bar{x}} \in \mathbb{Q}_p$ that

$$f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}}) = \frac{(\theta - 1)(1 - \theta - q)(\bar{x} - \bar{\bar{x}})}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} F(\bar{x}, \bar{\bar{x}})$$

where

$$\begin{aligned} F(\bar{x}, \bar{\bar{x}}) &= 3\theta^2 + 3\theta(1 - \theta - q) \left(\frac{\theta - 1}{\bar{x} - \mathbf{x}^{(\infty)}} + \frac{\theta - 1}{\bar{\bar{x}} - \mathbf{x}^{(\infty)}} \right) \\ &+ (1 - \theta - q)^2 \left(\frac{(\theta - 1)^2}{(\bar{x} - \mathbf{x}^{(\infty)})^2} + \frac{(\theta - 1)^2}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} + \frac{(\theta - 1)^2}{(\bar{\bar{x}} - \mathbf{x}^{(\infty)})^2} \right). \end{aligned}$$

One could check $|F(\bar{x}, \bar{\bar{x}})|_p = 1$ for $\bar{x}, \bar{\bar{x}} \in \mathcal{B}$ or \mathcal{A}_1 . So we obtain

$$|f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}})|_p = \frac{|\theta - 1|_p^2}{|\bar{x} - \mathbf{x}^{(\infty)}|_p |\bar{\bar{x}} - \mathbf{x}^{(\infty)}|_p} |\bar{x} - \bar{\bar{x}}|_p. \quad (4.23)$$

Let $\bar{x}, \bar{\bar{x}} \in \mathcal{B}$. We know $\mathcal{B} = \bigcup_{\bar{r}} \mathbb{S}_{\bar{r}}(\mathbf{x}^{(\infty)})$ in which $\bar{r} > |\theta - 1|_p$. For each \bar{r} , we have $f_{\theta,q,3}(\mathbb{S}_{\bar{r}}(\mathbf{x}^{(\infty)})) \subset \mathcal{C}_4$ is a sphere with center θ^3 . We know the restricted map $f_{\theta,q,3} : \mathbb{S}_{\bar{r}}(\mathbf{x}^{(\infty)}) \rightarrow f_{\theta,q,3}(\mathbb{S}_{\bar{r}}(\mathbf{x}^{(\infty)}))$ is a bijection by the Corollary 2.3.4. Furthermore, by the equality (4.23), we get $f_{\theta,q,3}(\mathbb{S}_{\bar{r}}(\mathbf{x}^{(\infty)})) \cap f_{\theta,q,3}(\mathbb{S}_{r'}(\mathbf{x}^{(\infty)})) = \emptyset$ for $\bar{r} \neq r'$. For any $\theta^3 \neq y \in \mathcal{C}_4$, there exists a preimage $x = f^{-1}(y) \in \mathcal{B}$. Thus, the restricted map $f : \mathcal{B} \rightarrow \mathcal{C}_4 \setminus \theta^3$ is a bijection.

Next, assume $\bar{x}, \bar{\bar{x}} \in \mathcal{A}_1$. We know $\mathcal{A}_1 = \bigcup_{\bar{r}} \mathbb{S}_{\bar{r}}(\mathbf{x}^{(\infty)})$ in which $|\theta - 1|_p^2 < \bar{r} < |\theta - 1|_p$. Then we have $f_{\theta,q,3}(\mathbb{S}_{\bar{r}}(\mathbf{x}^{(\infty)})) = \mathbb{S}_{r'}(\mathbf{x}^{(\infty)}) \subset \mathcal{B}_1 = \bigcup_r \mathbb{S}_r(\mathbf{x}^{(\infty)})$ is a sphere for $|\theta - 1|_p < r' < 1$. By the Corollary 2.3.4, restricted map $f_{\theta,q,3} : \mathbb{S}_{\bar{r}}(\mathbf{x}^{(\infty)}) \rightarrow \mathbb{S}_{r'}(\mathbf{x}^{(\infty)})$ is a bijection. By the equality (4.23), we have $f_{\theta,q,3}(\mathbb{S}_{\bar{r}}(\mathbf{x}^{(\infty)})) \cap f_{\theta,q,3}(\mathbb{S}_{\hat{r}}(\mathbf{x}^{(\infty)})) = \emptyset$ for $\bar{r} \neq \hat{r}$. Thus the restricted map $f_{\theta,q,3} : \mathcal{A}_1 \rightarrow \mathcal{B}_1$ is a bijection. \square

Now we discuss the crucial part, the set \mathcal{D} . Suppose $\tilde{x}^{(i)} = \mathbf{x}^{(\infty)} + (\theta - 1)\tilde{y}^{(i)} \in \mathcal{D}$

with $\tilde{y}^{(i)} = \frac{(\tilde{y}^{(i)})^*}{|\theta-1|_p}$ and $\left|(\tilde{y}^{(i)})^* - i\right|_p < 1$. Let $\mathcal{D}_i := \mathbb{B}_{|\theta-1|_p^2}(\tilde{x}^{(i)})$. Then

$$\mathcal{D} = \left(\bigcup_{i=0}^{p-1} \mathcal{D}_i \right) \setminus \{\mathbf{x}^{(\infty)}\}$$

such that $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$ for $i \neq j$. Next we will describe the sets \mathcal{D}_i having the singular point $\mathbf{x}^{(\infty)}$ and the critical point $\mathbf{x}^{(c)}$. We have $\mathbf{x}^{(\infty)} \in \mathcal{D}_0$ and

$$\mathcal{D}_0 = \left(\bigcup_{i=1}^{\infty} \mathbb{S}_i^{(0)} \right) \cup \{\mathbf{x}^{(\infty)}\} \quad (4.24)$$

where $\mathbb{S}_i^{(0)} := \mathbb{S}_{\frac{|\theta-1|_p^2}{p^i}}(\mathbf{x}^{(\infty)})$. Let $\hat{x}_j \in \mathbb{S}_i^{(0)}$ and $\mathcal{D}_{ij}^{(0)} := \mathbb{B}_{\frac{|\theta-1|_p^2}{p^i}}(\hat{x}_j)$. Then

$$\mathbb{S}_i^{(0)} = \bigcup_{j=1}^{p-1} \mathcal{D}_{ij}^{(0)} \quad (4.25)$$

such that $\mathcal{D}_{ik}^{(0)} \cap \mathcal{D}_{i\ell}^{(0)} = \emptyset$ for $k \neq \ell$.

Assume $\gamma \equiv (\theta - 1 + q)^* \pmod{p}$. Then one can check that $\mathbf{x}^{(c)} \in \mathcal{D}_\gamma$ and

$$\mathcal{D}_\gamma = \left(\bigcup_{i=1}^{\infty} \mathbb{S}_i^{(\gamma)} \right) \cup \{\mathbf{x}^{(c)}\} \quad (4.26)$$

where $\mathbb{S}_i^{(\gamma)} := \mathbb{S}_{\frac{|\theta-1|_p^2}{p^i}}(\mathbf{x}^{(c)})$. Now suppose $\check{x}_j \in \mathbb{S}_i^{(\gamma)}$ and $\mathcal{D}_{ij}^{(\gamma)} := \mathbb{B}_{\frac{|\theta-1|_p^2}{p^i}}(\check{x}_j)$. Then

$$\mathbb{S}_i^{(\gamma)} = \bigcup_{j=1}^{p-1} \mathcal{D}_{ij}^{(\gamma)} \quad (4.27)$$

such that $\mathcal{D}_{ik}^{(0)} \cap \mathcal{D}_{i\ell}^{(0)} = \emptyset$ for $k \neq \ell$. We can easily calculate $\left|f'_{\theta,q,3}(\mathbf{x}^{(\infty)})\right|_p$ is unbounded and $\left|f'_{\theta,q,3}(\mathbf{x}^{(c)})\right|_p = 1$. Then the following results inform us about the derivative near the points $\mathbf{x}^{(\infty)}$ and $\mathbf{x}^{(c)}$.

Lemma 4.3.25. *Let $p \geq 5$ be a prime number and $\gamma \equiv (\theta - 1 + q)^* \pmod{p}$.*

(i) *Let $x \in \mathbb{S}_i^{(0)} \subset \mathcal{D}_0$ where integers $i > 0$. Then $\left|f'_{\theta,q,3}(x)\right|_p = \frac{p^{4i}}{|\theta-1|_p^2}$.*

(ii) *Let $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_{ij}^{(0)} \subset \mathbb{S}_i^{(0)}$ for $1 \leq j \leq p-1$. Then we have*

$$\left|f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}})\right|_p = \frac{p^{4i}}{|\theta-1|_p^2} |\bar{x} - \bar{\bar{x}}|_p. \quad (4.28)$$

(iii) Let $x \in \mathbb{S}_i^{(\gamma)} \subset \mathcal{D}_\gamma$ where integers $i > 0$. Then $\left| f'_{\theta,q,3}(x) \right|_p = \frac{1}{p^{2i}|\theta-1|_p^2}$.

(iv) Let $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_{ij}^{(\gamma)} \subset \mathbb{S}_i^{(\gamma)}$ for $1 \leq j \leq p-1$. Then we have

$$\left| f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}}) \right|_p = \frac{1}{p^{2i}|\theta-1|_p^2} |\bar{x} - \bar{\bar{x}}|_p. \quad (4.29)$$

(v) Let $x \in \mathcal{D} \setminus (\mathcal{D}_0 \cup \mathcal{D}_\gamma)$. Then $\left| f'_{\theta,q,3}(x) \right|_p = \frac{1}{|\theta-1|_p^2}$.

(vi) Let $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_i$ for $1 \leq i \leq p-1$ and $i \neq \gamma$. Then we have

$$\left| f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}}) \right|_p = \frac{1}{|\theta-1|_p^2} |\bar{x} - \bar{\bar{x}}|_p. \quad (4.30)$$

Proof. We again need the following equality. For any $\bar{x}, \bar{\bar{x}} \in \mathbb{Q}_p \setminus \{\mathbf{x}^{(\infty)}\}$ we have

$$f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}}) = \frac{(\theta-1)(1-\theta-q)(\bar{\bar{x}} - \bar{x})}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} F(\bar{x}, \bar{\bar{x}}) \quad (4.31)$$

such that

$$\begin{aligned} F(\bar{x}, \bar{\bar{x}}) &= 3\theta^2 + 3\theta(1-\theta-q) \left(\frac{\theta-1}{\bar{x} - \mathbf{x}^{(\infty)}} + \frac{\theta-1}{\bar{\bar{x}} - \mathbf{x}^{(\infty)}} \right) \\ &+ (1-\theta-q)^2 \left(\frac{(\theta-1)^2}{(\bar{x} - \mathbf{x}^{(\infty)})^2} + \frac{(\theta-1)^2}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} + \frac{(\theta-1)^2}{(\bar{\bar{x}} - \mathbf{x}^{(\infty)})^2} \right). \end{aligned}$$

(i) Let $x \in \mathbb{S}_i^{(0)}$. Then we have $|\theta(x - \mathbf{x}^{(\infty)}) + (\theta-1)(1-\theta-q)|_p = |\theta-1|_p^2$ and $|x - \mathbf{x}^{(\infty)}|_p = \frac{|\theta-1|_p^2}{p^i}$. Recall

$$\begin{aligned} f_{\theta,q,3}^{(k)}(x) &= \frac{3k!(-1)^{k+1}(\theta-1)(\theta-1+q)}{(x - \mathbf{x}^{(\infty)})^{k+1}} \left(\theta^2 + \frac{(k+1)\theta(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right. \\ &\quad \left. + \frac{(k+1)(k+2)(\theta-1)^2(1-\theta-1-q)^2}{3!(x - \mathbf{x}^{(\infty)})^2} \right). \end{aligned}$$

Then we get

$$\begin{aligned} \left| f'_{\theta,q,3}(x) \right|_p &= \left| \frac{3(\theta-1)(\theta-1+q)(\theta(x - \mathbf{x}^{(\infty)}) + (\theta-1)(1-\theta-q))^2}{(x - \mathbf{x}^{(\infty)})^4} \right|_p \\ &= \frac{|\theta-1|_p^6}{|x - \mathbf{x}^{(\infty)}|_p^4} = \frac{p^{4i}}{|\theta-1|_p^2}. \end{aligned}$$

(ii) Let $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_{ij}^{(0)}$ for $1 \leq j \leq p-1$. We have $\bar{x} = \mathbf{x}^{(\infty)} + (\theta-1)\bar{y}$ and $\bar{\bar{x}} =$

$\mathbf{x}^{(\infty)} + (\theta - 1)\bar{\bar{y}}$. Then

$$\begin{aligned} F(\bar{x}, \bar{\bar{x}}) &= 3\theta^2 + 3\theta(1 - \theta - q) \left(\frac{1}{\bar{y}} + \frac{1}{\bar{\bar{y}}} \right) + (1 - \theta - q)^2 \left(\left(\frac{1}{\bar{y}} \right)^2 + \frac{1}{\bar{y}\bar{\bar{y}}} + \left(\frac{1}{\bar{\bar{y}}} \right)^2 \right) \\ &= 3\theta^2 + 3 \frac{\theta(1 - \theta - q)^*}{p^i} \left(\frac{1}{\bar{y}^*} + \frac{1}{\bar{\bar{y}}^*} \right) + \\ &\quad + \frac{((1 - \theta - q)^*)^2}{p^{2i}} \left(\frac{1}{(\bar{y}^*)^2} + \frac{1}{\bar{y}^* \bar{\bar{y}}^*} + \frac{1}{(\bar{\bar{y}}^*)^2} \right) \end{aligned}$$

Since $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_{ij}^{(0)}$, it implies $\bar{y}^* \equiv \bar{\bar{y}}^* \equiv w_0 \pmod{p}$. We have

$$\left(\frac{1}{(\bar{y}^*)^2} + \frac{1}{\bar{y}^* \bar{\bar{y}}^*} + \frac{1}{(\bar{\bar{y}}^*)^2} \right) \equiv \frac{3}{w_0^2} \not\equiv 0 \pmod{p}$$

which implies

$$\left| \frac{((1 - \theta - q)^*)^2}{p^{2i}} \left(\frac{1}{(\bar{y}^*)^2} + \frac{1}{\bar{y}^* \bar{\bar{y}}^*} + \frac{1}{(\bar{\bar{y}}^*)^2} \right) \right|_p = p^{2i}.$$

Therefore, $|F(\bar{x}, \bar{\bar{x}})|_p = p^{2i}$ and by the equality (4.14), we get $|f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}})|_p = \frac{p^{4i}}{|\theta-1|_p^2} |\bar{x} - \bar{\bar{x}}|_p$.

(iii) Let $x \in \mathbb{S}_i^{(\gamma)}$ and $x = \mathbf{x}^{(\infty)} + (\theta - 1)y$ such that $|y - \mathbf{y}^{(c)}|_p = \frac{|\theta-1|_p}{p^i}$. Then we have $|\theta(x - \mathbf{x}^{(\infty)}) + (\theta - 1)(1 - \theta - q)|_p = \frac{|\theta-1|_p^2}{p^i}$ and $|x - \mathbf{x}^{(\infty)}|_p = |\theta - 1|_p^2$. Thus we obtain

$$\begin{aligned} |f'_{\theta,q,3}(x)|_p &= \left| \frac{3(\theta - 1)(\theta - 1 + q)(\theta(x - \mathbf{x}^{(\infty)}) + (\theta - 1)(1 - \theta - q))^2}{(x - \mathbf{x}^{(\infty)})^4} \right|_p \\ &= \frac{|\theta(x - \mathbf{x}^{(\infty)}) + (\theta - 1)(1 - \theta - q)|_p^2}{|\theta - 1|_p^6} = \frac{1}{p^{2i} |\theta - 1|_p^2}. \end{aligned}$$

(iv) Let $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_{ij}^{(\gamma)} \subset \mathbb{S}_i^{(1)}$ for $1 \leq j \leq p - 1$. Then $\bar{x} = \mathbf{x}^{(\infty)} + (\theta - 1)\bar{y}$ and $\bar{\bar{x}} = \mathbf{x}^{(\infty)} + (\theta - 1)\bar{\bar{y}}$ such that $\bar{y} = \frac{(\bar{y})^*}{|\theta-1|_p}$ and $\bar{\bar{y}} = \frac{(\bar{\bar{y}})^*}{|\theta-1|_p}$. We also have for $w = w_0 + \dots + w_i p^i$ that $(\bar{y})^* = w + p^{i+1}A$ and $(\bar{\bar{y}})^* = w + p^{i+1}B$ in which $|w - (y^{(c)})^*|_p = \frac{1}{p^i}$ and $|A - B|_p \leq 1$. Then by the equality (4.31) we have

$$\begin{aligned} F(\bar{x}, \bar{\bar{x}}) &= 3\theta^2 + 3\theta(1 - \theta - q)^* \left(\frac{1}{(\bar{y})^*} + \frac{1}{(\bar{\bar{y}})^*} \right) \\ &\quad + ((1 - \theta - q)^*)^2 \left(\frac{1}{(\bar{y}^*)^2} + \frac{1}{(\bar{y}^*)(\bar{\bar{y}}^*)} + \frac{1}{(\bar{\bar{y}}^*)^2} \right). \end{aligned}$$

After replacing $(\bar{y})^*$ and $(\bar{\bar{y}})^*$, we get

$$\begin{aligned}
(\bar{y}^*)^2 (\bar{\bar{y}}^*)^2 F(\bar{x}, \bar{\bar{x}}) &= 3w^2 (\theta w + (1 - \theta - q)^*)^2 + 3w(A + B)p^{i+1} \times \\
&\quad \times \left(2\theta^2 w^2 + 3\theta w(1 - \theta - q)^* + ((1 - \theta - q)^*)^2 \right) \\
&\quad + p^{2(i+1)} \bar{X} \\
&= 3w^2 (\theta w + (1 - \theta - q)^*)^2 + 3w(A + B)p^{i+1} \times \\
&\quad \times \left(\theta w (\theta w + (1 - \theta - q)^*) + (\theta w + (1 - \theta - q)^*)^2 \right) \\
&\quad + p^{2(i+1)} \bar{X}
\end{aligned}$$

such that $|\bar{X}|_p \leq 1$. We also know $|\theta w + (1 - \theta - q)^*|_p = |\theta(w - (y^{(c)})^*)|_p = |w - (y^{(c)})^*|_p$. Then we get $|F(\bar{x}, \bar{\bar{x}})|_p = \frac{1}{p^{2i}}$. Thus we have $|f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}})|_p = \frac{|\bar{x} - \bar{\bar{x}}|_p}{p^{2i} |\theta - 1|_p^2}$.

(v) Let $x \in \mathcal{D} \setminus (\mathcal{D}_0 \cup \mathcal{D}_\gamma)$. We have $x = \mathbf{x}^{(\infty)} + (\theta - 1)y$ such that $|y - \mathbf{y}^{(c)}|_p = |\theta - 1|_p$. We also have $|\theta(x - \mathbf{x}^{(\infty)}) + (\theta - 1)(1 - \theta - q)|_p = |\theta - 1|_p^2$ and $|x - \mathbf{x}^{(\infty)}|_p = |\theta - 1|_p^2$. Hence

$$\begin{aligned}
|f'_{\theta,q,3}(x)|_p &= \left| \frac{3(\theta - 1)(\theta - 1 + q)(\theta(x - \mathbf{x}^{(\infty)}) + (\theta - 1)(1 - \theta - q))^2}{(x - \mathbf{x}^{(\infty)})^4} \right|_p \\
&= \frac{1}{|\theta - 1|_p^2}.
\end{aligned}$$

(vi) Let $\gamma \equiv (\theta - 1 + q)^* \pmod{p}$. Suppose $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_i$ for $1 \leq i \leq p - 1$ and $i \neq \gamma$. We have $\bar{x} = \mathbf{x}^{(\infty)} + (\theta - 1)\bar{y}$ and $\bar{\bar{x}} = \mathbf{x}^{(\infty)} + (\theta - 1)\bar{\bar{y}}$. Then we obtain

$$\begin{aligned}
\left| \frac{\bar{x} - \mathbf{x}^{(\infty)}}{\theta - 1} \right|_p &= \left| \frac{\bar{\bar{x}} - \mathbf{x}^{(\infty)}}{\theta - 1} \right|_p = |\theta - 1|_p \\
\left(\frac{\bar{x} - \mathbf{x}^{(\infty)}}{\theta - 1} \right)^* &= \left(\frac{\bar{\bar{x}} - \mathbf{x}^{(\infty)}}{\theta - 1} \right)^* \equiv w_0^{(i)} \pmod{p}.
\end{aligned}$$

From the equality (4.31), we get

$$\begin{aligned}
F(\bar{x}, \bar{\bar{x}}) &= 3\theta^2 + 3\theta(1 - \theta - q)^* \left(\left(\frac{\theta - 1}{\bar{x} - \mathbf{x}^{(\infty)}} \right)^* + \left(\frac{\theta - 1}{\bar{\bar{x}} - \mathbf{x}^{(\infty)}} \right)^* \right) \\
&\quad + ((1 - \theta - q)^*)^2 \left(\left(\left(\frac{\theta - 1}{\bar{x} - \mathbf{x}^{(\infty)}} \right)^* \right)^2 + \left(\frac{\theta - 1}{\bar{\bar{x}} - \mathbf{x}^{(\infty)}} \right)^* \left(\frac{\theta - 1}{\bar{x} - \mathbf{x}^{(\infty)}} \right)^* + \left(\left(\frac{\theta - 1}{\bar{\bar{x}} - \mathbf{x}^{(\infty)}} \right)^* \right)^2 \right)
\end{aligned}$$

$$\begin{aligned}
\text{and } (w_0^{(i)})^2 F(\bar{x}, \bar{\bar{x}}) &\equiv 3(w_0^{(i)})^2 + 6(1 - \theta - q)^* (w_0^{(i)}) + 3((1 - \theta - q)^*)^2 \\
&\equiv 3 \left((w_0^{(i)}) + (1 - \theta - q)^* \right)^2 \not\equiv 0 \pmod{p}.
\end{aligned}$$

This means $|F(\bar{x}, \bar{\bar{x}})|_p = 1$ because $w_0^{(i)} \not\equiv (\mathbf{y}^{(c)})^* \equiv -(1 - \theta - q)^* \pmod{p}$. Therefore, $|f_{\theta,q,3}(\bar{x}) - f_{\theta,q,3}(\bar{\bar{x}})|_p = \frac{|\bar{x} - \bar{\bar{x}}|_p}{|\theta - 1|_p^2}$. \square

It is clear $\mathcal{E} = \mathbb{Q}_p \setminus \mathbb{B}_1(\mathbf{x}^{(\infty)})$ and $\mathbb{Z}_p^* = \bigcup_{j=1}^{p-1} \mathbb{B}_1(j)$. Remember that the congruence

$$x^3 \equiv b \pmod{p} \quad (4.32)$$

has solution(s) if and only if $b^{\frac{p-1}{\gcd(p-1,3)}} \equiv 1 \pmod{p}$, i.e. b is a cubic residue modulo p . Moreover, if b is a cubic residue modulo p then (4.32) has $\gcd(p-1, 3)$ solution(s). Let

$$(\mathbb{Z}_p^*)^{\frac{1}{3}} = \{x \in \mathbb{Z}_p^* : x_0 \text{ is a cubic residue modulo } p\}.$$

We can easily check that $(\mathbb{Z}_p^*)^{\frac{1}{3}} = \mathbb{Z}_p^*$ when $p \equiv 5 \pmod{6}$ and $(\mathbb{Z}_p^*)^{\frac{1}{3}} \neq \mathbb{Z}_p^*$ when $p \equiv 1 \pmod{6}$. If $p \equiv 5 \pmod{6}$, then there are no fixed points in \mathcal{D} , whereas if $p \equiv 1 \pmod{6}$, then there exist the fixed points $\mathbf{x}^{(2)}, \mathbf{x}^{(3)} \in \mathcal{D}$. Let α and β be the solutions of the equation (4.5). Then we have $\mathbf{x}^{(2)} \in \mathcal{D}_\alpha$ and $\mathbf{x}^{(3)} \in \mathcal{D}_\beta$.

One could easily verify that $f_{\theta,q,3}(\mathbf{x}^{(c)}) = 0$ and $f_{\theta,q,3}(\mathbf{x}^{(\infty)})$ is undefined. We have the following result that tells us the image of the points near the points $\mathbf{x}^{(\infty)}$ and $\mathbf{x}^{(c)}$.

Proposition 4.3.26. *Let $\gamma \equiv (\theta - 1 + q)^* \pmod{p}$. Let α and β be the solutions of the equation (4.5).*

(I) *If $p \equiv 5 \pmod{6}$ then*

- (i) *for integer $i \geq 1$, $\mathbb{S}_i^{(0)} \subset \mathcal{D}_0$ as in (4.24), we have $f_{\theta,q,3}(\mathbb{S}_i^{(0)}) = p^{-3i} \mathbb{Z}_p^*$;*
- (ii) *for integer $i \geq 1$, $\mathbb{S}_i^{(\gamma)} \subset \mathcal{D}_\gamma$ as in (4.26), we have $f_{\theta,q,3}(\mathbb{S}_i^{(\gamma)}) = p^{3i} \mathbb{Z}_p^*$;*
- (iii) $f_{\theta,q,3}\left(\bigcup_{i=1, i \neq \gamma}^{p-1} \mathcal{D}_i\right) = \mathbb{Z}_p^* \setminus \mathbb{B}_1(\mathbf{x}^{(\infty)})$.

(II) *If $p \equiv 1 \pmod{6}$ then*

- (i) *for integer $i \geq 1$, $\mathbb{S}_i^{(0)} \subset \mathcal{D}_0$ as in (4.24), we have $f_{\theta,q,3}(\mathbb{S}_i^{(0)}) = p^{-3i} (\mathbb{Z}_p^*)^{\frac{1}{3}}$;*
- (ii) *for integer $i \geq 1$, $\mathbb{S}_i^{(\gamma)} \subset \mathcal{D}_\gamma$ as in (4.26), we have $f_{\theta,q,3}(\mathbb{S}_i^{(\gamma)}) = p^{3i} (\mathbb{Z}_p^*)^{\frac{1}{3}}$;*
- (iii) $f_{\theta,q,3}(\mathcal{D}_i) = \mathbb{B}_1(\mathbf{x}^{(\infty)})$ where $i = \alpha, \beta$;
- (iv) $f_{\theta,q,3}\left(\bigcup_{\substack{i=1 \\ i \neq \alpha, \beta, \gamma}}^{p-1} \mathcal{D}_i\right) = (\mathbb{Z}_p^*)^{\frac{1}{3}} \setminus \mathbb{B}_1(\mathbf{x}^{(\infty)})$.

Proof. (I) Let $p \equiv 5 \pmod{6}$. Remember that the Potts–Bethe mapping (4.2) can be written as follows

$$f_{\theta,q,3}(x) = \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right)^3. \quad (4.33)$$

(i) Recall $\mathcal{D}_0 = \bigcup_{i=1}^{\infty} \mathbb{S}_i^{(0)}$ and $\mathbb{S}_i^{(0)} = \bigcup_{j=1}^{p-1} \mathcal{D}_{ij}^{(0)}$. Let us calculate the image of the sphere $\mathbb{S}_i^{(0)}$ for integer $i \geq 1$. Suppose $x \in \mathbb{S}_i^{(0)}$. Then $x - \mathbf{x}^{(\infty)} = (\theta - 1)y$ with $y = \frac{p^i y^*}{|\theta - 1|_p}$. By substituting it into the equality (4.33), we obtain

$$f_{\theta,q,3}(x) = \left(\theta + \frac{(1 - \theta - q)^*}{p^i y^*} \right)^3.$$

So we have

$$|f_{\theta,q,3}(x)|_p = \left| \theta + \frac{(1 - \theta - q)^*}{p^i y^*} \right|_p^3 = p^{3i} > 1.$$

This means $f_{\theta,q,3}(x) = \frac{1}{p^{3i}} f_{\theta,q,3}^*(x)$ where $|f_{\theta,q,3}^*(x)|_p = 1$ and the first digit of $f_{\theta,q,3}^*(x)$ is a cubic residue modulo p . This implies $f_{\theta,q,3}(x) \in p^{-3i} (\mathbb{Z}_p^*)^{\frac{1}{3}} \subset \mathcal{E}$. Next we calculate the image of $\mathcal{D}_{ij}^{(0)} \subset \mathbb{S}_i^{(0)}$. Let $\hat{x}_j \in \mathcal{D}_{ij}^{(0)}$ and $f_{\theta,q,3}(\hat{x}) = p^{-3i} f_{\theta,q,3}^*(\hat{x})$. By the equality (4.28) and the Corollary 2.3.3 we get

$$f_{\theta,q,3}(\mathcal{D}_{ij}^{(0)}) = \mathbb{B}_{p^{3i}}(f_{\theta,q,3}(\hat{x}_j)) = p^{-3i} \mathbb{B}_1(f_{\theta,q,3}^*(\hat{x}_j))$$

for $1 \leq j \leq p-1$ and $f_{\theta,q,3}^*(\hat{x}_j) \pmod{p}$ is a cubic residue modulo p . Thus $f_{\theta,q,3}(\mathbb{S}_i^{(0)}) = p^{-3i} (\mathbb{Z}_p^*)^{\frac{1}{3}} = p^{-3i} \mathbb{Z}_p^*$ and we have the statement (i).

(ii) Let $\gamma \equiv (\theta - 1 + q)^* \pmod{p}$. Recall $\mathcal{D}_\gamma = \mathbf{x}^{(c)} \cup \bigcup_{i=1}^{\infty} \mathbb{S}_i^{(\gamma)}$ and $\mathbb{S}_i^{(\gamma)} = \bigcup_{j=1}^{p-1} \mathcal{D}_{ij}^{(\gamma)}$. We calculate the image of each spheres $\mathbb{S}_i^{(\gamma)}$ for integer $i \geq 1$. Let $x = \mathbf{x}^{(\infty)} + (\theta - 1)y \in \mathbb{S}_i^{(\gamma)}$. Then $|y - \mathbf{y}^{(c)}|_p = \frac{|\theta - 1|_p}{p^i}$ and $|y^* - (y^{(c)})^*|_p = \frac{1}{p^i}$. After substituting into (4.33), we get

$$f_{\theta,q,3}(x) = \left(\theta + \frac{(1 - \theta - q)^*}{y^*} \right)^3 = \left(\frac{\theta}{y^*} (y^* - (y^{(c)})^*) \right)^3.$$

We have $|f_{\theta,q,3}(x)|_p = |y - (y^{(c)})^*|_p = \frac{1}{p^{3i}}$. We also have $f_{\theta,q,3}(x) = p^{3i} f_{\theta,q,3}^*(x)$ where $|f_{\theta,q,3}^*(x)|_p = 1$ and the first digit of $f_{\theta,q,3}^*(x)$ is a cubic residue modulo p . These mean $f_{\theta,q,3}(x) \in p^{3i} (\mathbb{Z}_p^*)^{\frac{1}{3}}$. Next, let us calculate the image of $\mathcal{D}_{ij}^{(\gamma)} \subset \mathbb{S}_i^{(\gamma)}$. Let $\check{x}_j \in \mathcal{D}_{ij}^{(\gamma)}$ and $f_{\theta,q,3}(\check{x}) = p^{3i} f_{\theta,q,3}^*(\check{x})$. By the equality (4.28) and the Corollary 2.3.3, we get

$$f_{\theta,q,3}(\mathcal{D}_{ij}^{(\gamma)}) = \mathbb{B}_{p^{-3i}}(f_{\theta,q,3}(\check{x}_j)) = p^{3i} \mathbb{B}_1(f_{\theta,q,3}^*(\check{x}_j))$$

for $1 \leq j \leq p-1$ and $f_{\theta,q,3}^*(\check{x}_j) \pmod{p}$ is a cubic residue modulo p . Thus $f_{\theta,q,3}(\mathbb{S}_i^{(\gamma)}) = p^{3i} (\mathbb{Z}_p^*)^{\frac{1}{3}} = p^{3i} \mathbb{Z}_p^*$ and the statement (ii) holds true.

(iii) Let $x = \mathbf{x}^{(\infty)} + (\theta - 1)y \in \bigcup_{\substack{i=1 \\ i \neq \gamma}}^{p-1} \mathcal{D}_i$. Then $|y - \mathbf{y}^{(c)}|_p = |\theta - 1|_p$ and $|y^* -$

$(y^{(c)})^*|_p = 1$. After substituting into (4.33), we get

$$f_{\theta,q,3}(x) = \left(\frac{\theta}{y^*} \left(y^* - (y^{(c)})^* \right) \right)^3.$$

We see that $|f_{\theta,q,3}(x)|_p = |y^* - (y^{(c)})^*|_p = 1$ and its first digit is a cubic residue modulo p . So $f_{\theta,q,3}(x) \in (\mathbb{Z}_p^*)^{\frac{1}{3}}$. We also have $|f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)}|_p = 1$. It means $f_{\theta,q,3}(x) \notin \mathbb{B}_1(\mathbf{x}^{(\infty)}) = \mathbb{B}_1(1)$. Hence $f_{\theta,q,3}(x) \in (\mathbb{Z}_p^*)^{\frac{1}{3}} \setminus \mathbb{B}_1(1)$.

Next we calculate the image of \mathcal{D}_i where $i = 1, \dots, p-1$, $i \neq \gamma$. By the equality (4.30) and Lemma 2.3.3, we obtain $f_{\theta,q,3}(\mathcal{D}_j) = \mathbb{B}_1(f_{\theta,q,3}(\tilde{x}_j))$ and $f_{\theta,q,3}(\tilde{x}_j) \pmod{p}$ is a cubic residue modulo p . Thus $f_{\theta,q,3} \left(\bigcup_{i=1, i \neq \gamma}^{p-1} \mathcal{D}_i \right) = (\mathbb{Z}_p^*)^{\frac{1}{3}} \setminus \mathbb{B}_1(\mathbf{x}^{(\infty)}) = \mathbb{Z}_p^* \setminus \mathbb{B}_1(\mathbf{x}^{(\infty)})$ and we have the statement (iii).

(II) Let $p \equiv 1 \pmod{6}$. The scheme of the proof is similar to the proof when $p \equiv 5 \pmod{6}$. But first we consider the following. Let $x = \mathbf{x}^{(\infty)} + (\theta - 1)y \in \mathcal{D} \setminus \mathcal{D}_0$. Then $y = \frac{y^*}{|\theta - 1|_p}$ and we write expression of $g_0(x)$ in (4.19) as follows

$$\begin{aligned} g_0(x) &= (\theta x + q - 1)^2 + (\theta x + q - 1)(x - \mathbf{x}^{(\infty)}) + (x - \mathbf{x}^{(\infty)})^2 \\ &= (\theta - 1)^2 ((\theta^2 + \theta + 1)y^2 + (2\theta + 1)(1 - \theta - q)y + (1 - \theta - q)^2) \\ &= \frac{(\theta - 1)^2}{|\theta - 1|_p^2} g_2(x). \end{aligned}$$

where $g_2(x) = (\theta^2 + \theta + 1)(y^*)^2 + (2\theta + 1)(1 - \theta - q)^*(y^*) + ((1 - \theta - q)^*)^2$. The following congruence will be useful later

$$g_2(x) \equiv 3(y^*)^2 + 3(1 - \theta - q)^*(y^*) + ((1 - \theta - q)^*)^2 \pmod{p}. \quad (4.34)$$

The proof for the statements (II)-(i) and (II)-(ii) are the same to the proof of statements (I)-(i) and (I)-(ii). So we are left to prove the statements (II)-(iii) and (II)-(iv).

(iii) Let $x \in \mathcal{D}_i$, $i = \alpha, \beta$ for α and β the only solutions of (4.34). Then $y^* \equiv \tilde{y}^{(i)} \equiv y_0^{(2)}$ or $y_0^{(3)} \pmod{p}$. Then we have $|g_0(x)|_p < |\theta - 1|_p^4$. This implies by the equality (4.19) that $|f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)}|_p < 1$. So $f_{\theta,q,3}(x) \in \mathbb{B}_1(1) = \mathbb{B}_1(\mathbf{x}^{(\infty)})$. By the equality (4.30) and Lemma 2.3.3, we have for $i = \alpha, \beta$ that $f_{\theta,q,3}(\mathcal{D}_i) = \mathbb{B}_1(\mathbf{x}^{(\infty)})$. Hence the statement (iii) holds.

(iv) Let $x = \mathbf{x}^{(\infty)} + (\theta - 1)y \in \bigcup_{\substack{i=1 \\ i \neq \alpha, \beta, \gamma}}^{p-1} \mathcal{D}_i$. Then $|y - \mathbf{y}^{(c)}|_p = |\theta - 1|_p$ and $|y^* - (y^{(c)})^*|_p = 1$. Substitute into (4.33) and we get

$$f_{\theta,q,3}(x) = \left(\frac{\theta}{y^*} \left(y^* - (y^{(c)})^* \right) \right)^3.$$

One could see $|f_{\theta,q,3}(x)|_p = \left| y^* - (y^{(c)})^* \right|_p = 1$ and its first digit is a cubic residue modulo p . These imply that $f_{\theta,q,3}(x) \in (\mathbb{Z}_p^*)^{\frac{1}{3}}$. Now we will show that $|f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)}|_p = 1$. We know $y^* \not\equiv y_0^{(i)}$ for $i = 2, 3$. It means $g_2(x) \not\equiv 0 \pmod{p}$ and $|g_0(x)|_p = |\theta - 1|_p^4$. This implies, by the equality (4.19), the assertion $|f_{\theta,q,3}(x) - \mathbf{x}^{(\infty)}|_p = 1$. Thus $f_{\theta,q,3}(x) \in (\mathbb{Z}_p^*)^{\frac{1}{3}} \setminus \mathbb{B}_1(1) = (\mathbb{Z}_p^*)^{\frac{1}{3}} \setminus \mathbb{B}_1(\mathbf{x}^{(\infty)})$.

Next we calculate the image of \mathcal{D}_i for $i = 1, \dots, p-1$, $i \neq \alpha, \beta, \gamma$. By the equality (4.30) and Lemma 2.3.3, we get $f(\mathcal{D}_j) = \mathbb{B}_1(f(\tilde{x}_j))$ and $f_{\theta,q,3}(\tilde{x}_j) \pmod{p}$ is a cubic residue modulo p . Thus $f_{\theta,q,3} \left(\bigcup_{\substack{i=1 \\ i \neq \alpha, \beta, \gamma}}^{p-1} \mathcal{D}_i \right) = (\mathbb{Z}_p^*)^{\frac{1}{3}} \setminus \mathbb{B}_1(\mathbf{x}^{(\infty)})$ and the statement (iv) is true. \square

We have calculated the image of the sets \mathcal{A} , \mathcal{B} and \mathcal{C} for $p \equiv 1 \pmod{6}$ and $p \equiv 5 \pmod{6}$. It is helpful later when we discuss about the dynamics of the Potts–Bethe mapping.

Now we find the Siegel disks of the neutral fixed points $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$.

Theorem 4.3.27. *Let $p \geq 5$ be a prime number. Then the sets \mathcal{C}_1 and \mathcal{C}_3 are the Siegel disks of the fixed points $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ respectively.*

Proof. Notice $\mathbf{x}^{(0)} \in \mathcal{C}_1$ and $\mathbf{x}^{(1)} \in \mathcal{C}_3$. By the Corollary 4.3.23, we get $f_{\theta,q,3}(\mathcal{C}_1) = \mathcal{C}_1$ and $f_{\theta,q,3}(\mathcal{C}_3) = \mathcal{C}_3$. Also have by Lemma 4.3.22, we have for any $x, y \in \mathcal{C}_1$ or \mathcal{C}_3 that $|f_{\theta,q,3}(x) - f_{\theta,q,3}(y)|_p = |x - y|_p$. This implies that each spheres with center $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ in \mathcal{C}_1 and \mathcal{C}_3 respectively are invariant under $f_{\theta,q,3}$. Hence \mathcal{C}_1 and \mathcal{C}_3 are the Siegel disks of the fixed points $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ respectively. \square

Remark 4.3.28. *We have an isometry in each Siegel disks \mathcal{C}_1 and \mathcal{C}_3 . This defines a 1-Lipschitz dynamical system. By the work of S. L. Fan & Liao (2018), S. L. Fan & Liao (2015) and A. H. Fan et al. (2014), the dynamical behaviour of such a system is completely described by its minimal decompositions.*

Next, we look into the specific examples of the Potts–Bethe mapping (4.2). For simplicity, we may assume $\mathcal{A}_1 = \mathcal{B}_1 = \emptyset$. This means $|\theta - 1|_p = \frac{1}{p}$. We will consider the two cases: $p \equiv 5 \pmod{6}$ and $p \equiv 1 \pmod{6}$.

CASE 1: $p \equiv 5 \pmod{6}$. By the Propositions 4.3.24 and 4.3.23, we have the following

$$\mathcal{B} \longrightarrow \mathcal{C}_4 \setminus \{\theta^3\} \longrightarrow \mathcal{C}_2 \longrightarrow \mathcal{A}$$

and by Proposition 4.3.26, we have $f_{\theta,q,3}(\mathcal{A}) \subset \mathcal{B}$. The arrow means a bijection.

EXAMPLE 1: Suppose $p = 5$, $\theta = 6$ or (i) $q = 25$ and (ii) $q = -25$. Then

$$f_{6,25,3}(x) = \left(\frac{6x + 24}{x + 29} \right)^3 \text{ and } f_{6,-25,3}(x) = \left(\frac{6x - 26}{x - 21} \right)^3.$$

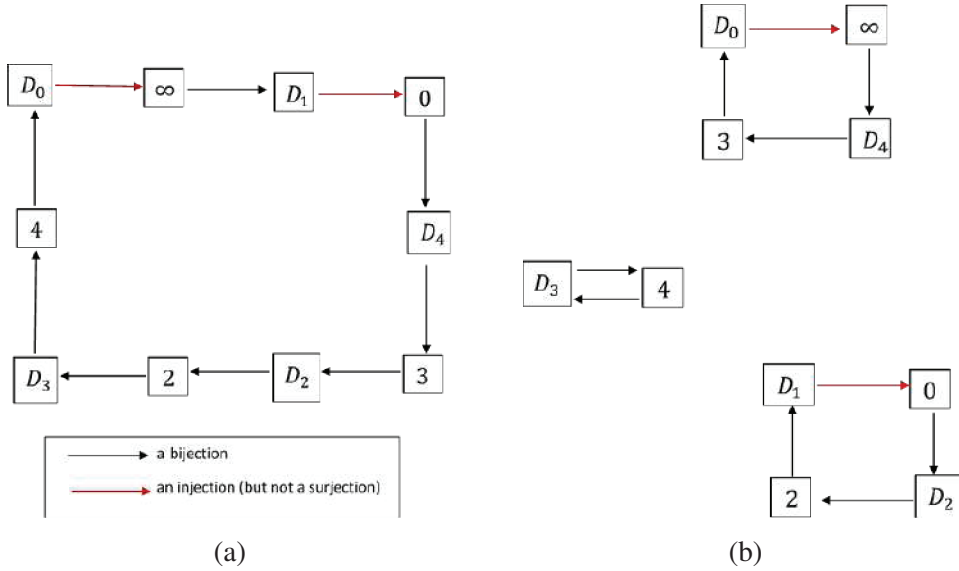


Figure 4.1: (a) $q = 25$, (b) $q = -25$

Since $\mathcal{A}_1 = \mathcal{B}_1 = \emptyset$, we have $\mathcal{A} = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \dots \cup \mathcal{D}_4$ and $\mathcal{B} = \mathbb{Q}_p \setminus \mathbb{B}_1(1) = \mathbb{B}_1(0) \cup \mathbb{B}_1(2) \cup \mathbb{B}_1(3) \cup \mathbb{B}_1(4) \cup \mathbb{Q}_p \setminus \mathbb{Z}_p$. One could verify $\mathbf{x}^{(c)} \in \mathcal{D}_1$ since $\gamma \equiv 1 \pmod{p}$. We denote $\mathbb{B}_1(k)$ as k for $k = 0, 2, 3, 4$ and $\mathbb{Q}_p \setminus \mathbb{Z}_p$ as ∞ . Then the Figures 4.1a and 4.1b show the dynamics of the Potts–Bethe mapping on the sets \mathcal{A} and \mathcal{B} of our examples.

Claim I Let $p \equiv 5 \pmod{6}$. Then the Potts–Bethe mapping (4.2) decompose \mathbb{Q}_p into several minimal sets (including the attracting periodic orbits and the Siegel disks of periodic orbits) and their basins.

$$\mathcal{B} \longrightarrow \mathcal{C}_4 \setminus \{\theta^3\} \longrightarrow \mathcal{A}$$

By Proposition 4.3.26, we have $f(\mathcal{A}) \subset \mathcal{B}$.

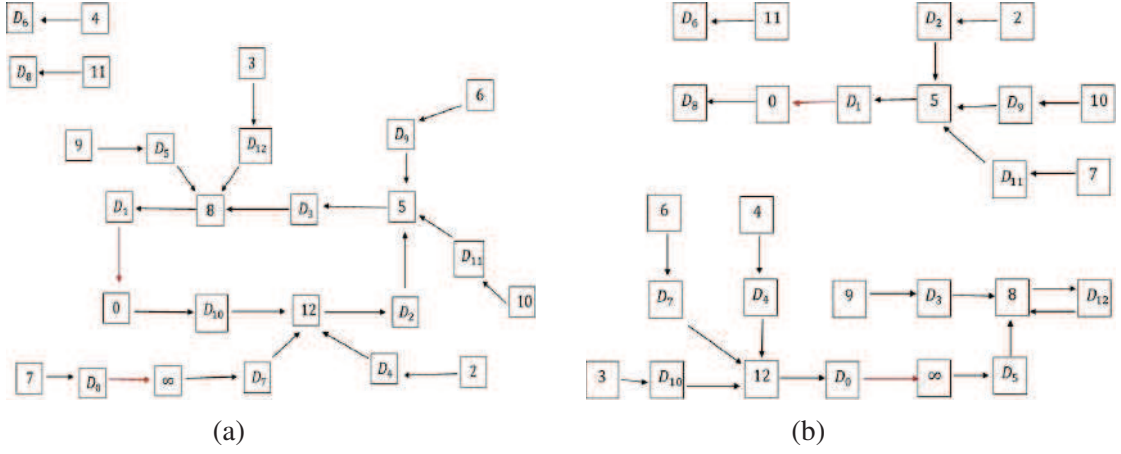


Figure 4.2: (a) $q_0 = 1$, (b) $q_0 = 2$

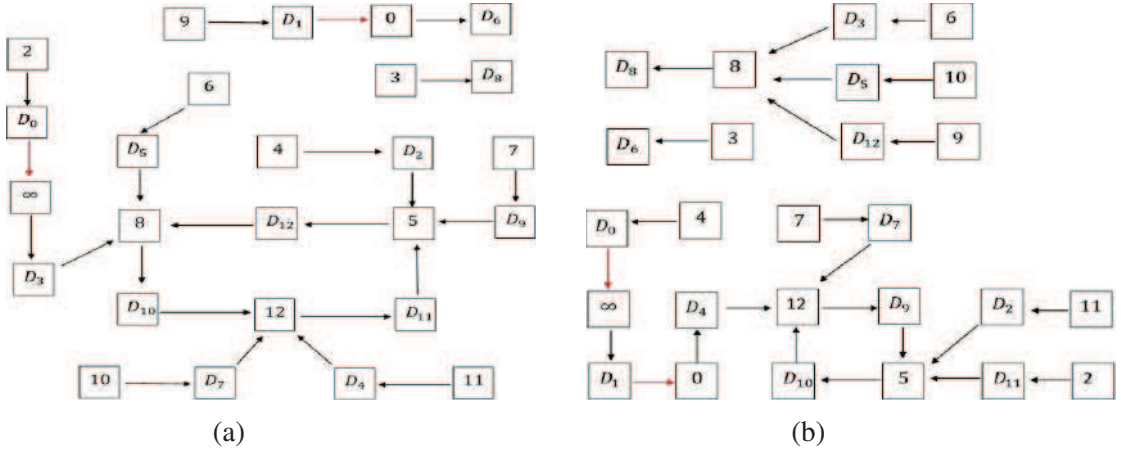


Figure 4.3: (a) $q_0 = 3$, (b) $q_0 = 4$

EXAMPLE 2: Assume $p = 13$, $\theta = 14$ and $q = q_0 \cdot 169$ where $q_0 = 1, \dots, 12$. Then

$$f_{14, q_0 \cdot 169, 3}(x) = \left(\frac{14x + q_0 \cdot 169 - 1}{x + q_0 \cdot 169 + 12} \right)^3.$$

Since $\mathcal{A}_1 = \mathcal{B}_1 = \emptyset$, we have $\mathcal{A} = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \dots \cup \mathcal{D}_{12}$ and $\mathcal{B} = \mathbb{Q}_p \setminus \mathbb{B}_1(1) = \mathbb{B}_1(0) \cup \mathbb{B}_1(2) \cup \dots \cup \mathbb{B}_1(12) \cup \mathbb{Q}_p \setminus \mathbb{Z}_p$. By simple calculation, we can see $\mathbf{x}^{(c)} \in \mathcal{D}_1$ since $\gamma \equiv 1 \pmod{p}$. One can also calculate that the repelling fixed points $\mathbf{x}^{(2)} \in \mathcal{D}_6$ and $\mathbf{x}^{(3)} \in \mathcal{D}_8$ since $\alpha = 6$ and $\beta = 8$ are the solutions of the equation (4.5). We denote $\mathbb{B}_1(k)$ by k , $k = 0, 2, 3, \dots, 12$ and $\mathbb{Q}_p \setminus \mathbb{Z}_p$ by ∞ . Then the Figures 4.2–4.7 show the image between the sets \mathcal{A} and \mathcal{B} under $f_{14, q_0 \cdot 169, 3}$ for $q_0 = 1, \dots, 12$.

let us consider the dynamics of $f_{14, q_0 \cdot 169, 3} : \mathbb{P}^1(\mathbb{C}_p) \rightarrow \mathbb{P}^1(\mathbb{C}_p)$ for $q_0 = 1, \dots, 12$. Then $\mathbf{x}^{(\infty)} = -12 - q_0 \cdot 169$ and $\mathbf{x}^{(c)} = -\frac{1 - q_0 \cdot 169}{14}$ are the critical points of $f_{14, q_0 \cdot 169, 3}$. One could easily check that all ramification indices $\deg_{f_{14, q_0 \cdot 169, 3}}(-12 - q_0 \cdot 169) = \deg_{f_{14, q_0 \cdot 169, 3}}(-\frac{1 - q_0 \cdot 169}{14}) = 3$. If $p \neq 3$, then all the critical points are not wild.

We know from Proposition 4.3.26 that for $i = 6, 8$, we have $f_{14, q_0 \cdot 169, 3}(\mathcal{D}_i) = \mathbb{B}_1(1)$. Noted by these examples $\bigcap_{n=0}^{\infty} \tilde{f}_{14, q_0 \cdot 169, 3}^{-n}(\mathcal{D}_6 \cup \mathcal{D}_8) = \emptyset$ for $x \in \mathcal{D}_6 \cup \mathcal{D}_8$ with

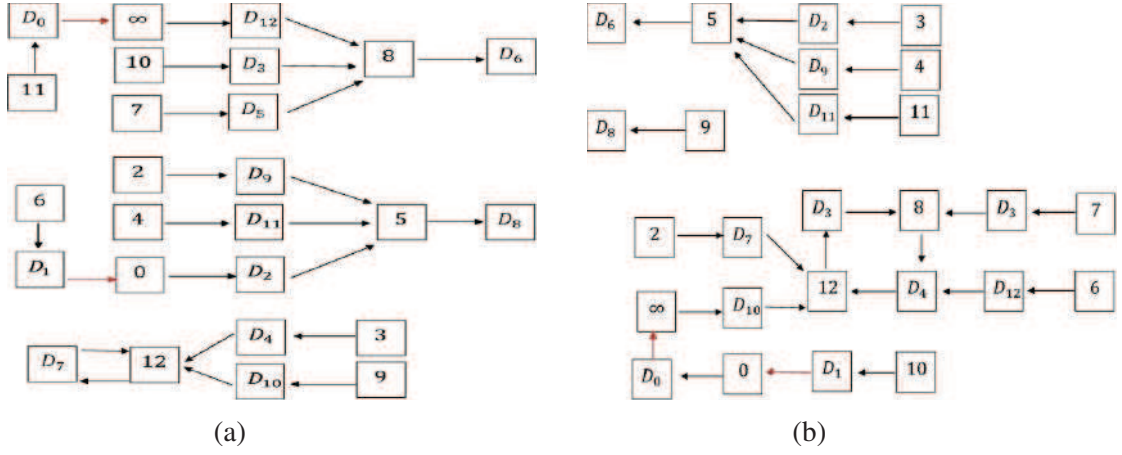


Figure 4.4: (a) $q_0 = 5$, (b) $q_0 = 6$

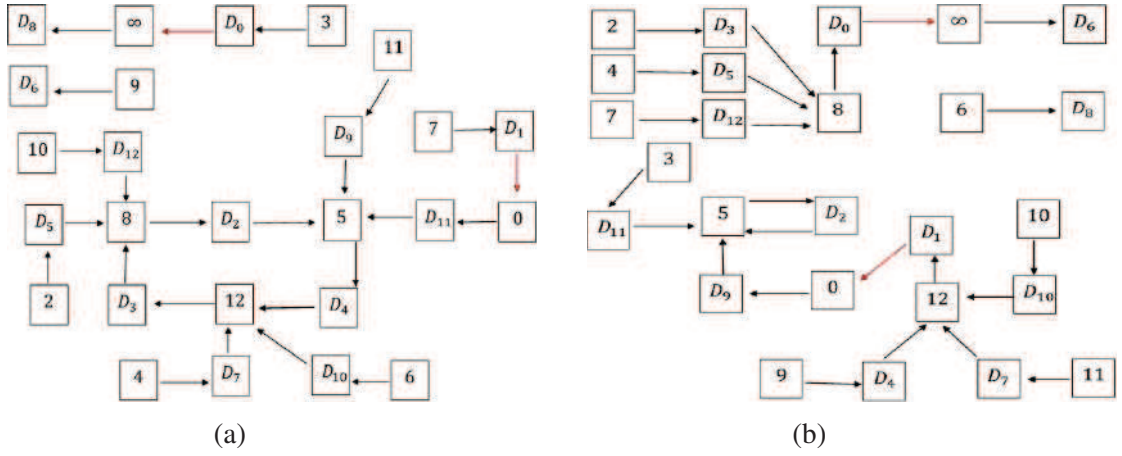


Figure 4.5: (a) $q_0 = 7$, (b) $q_0 = 8$

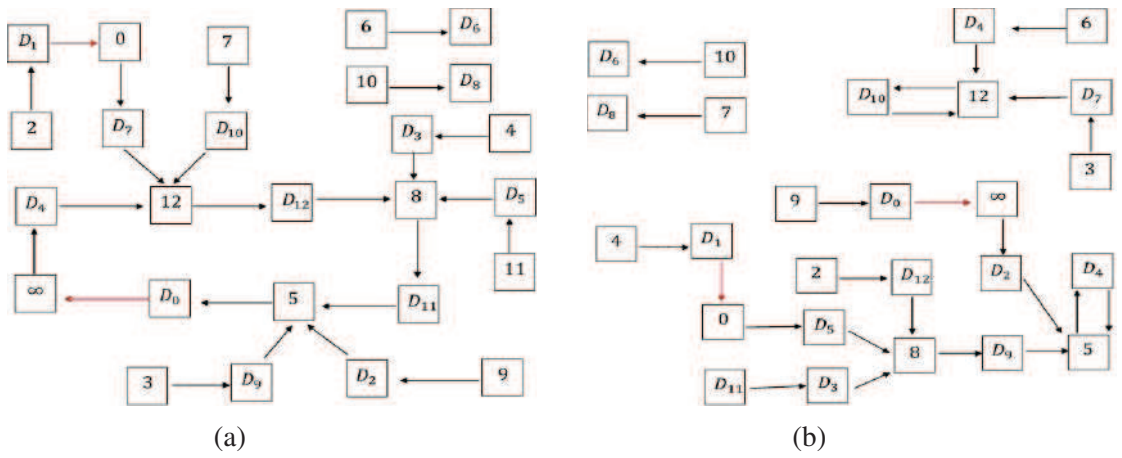


Figure 4.6: (a) $q_0 = 9$, (b) $q_0 = 10$

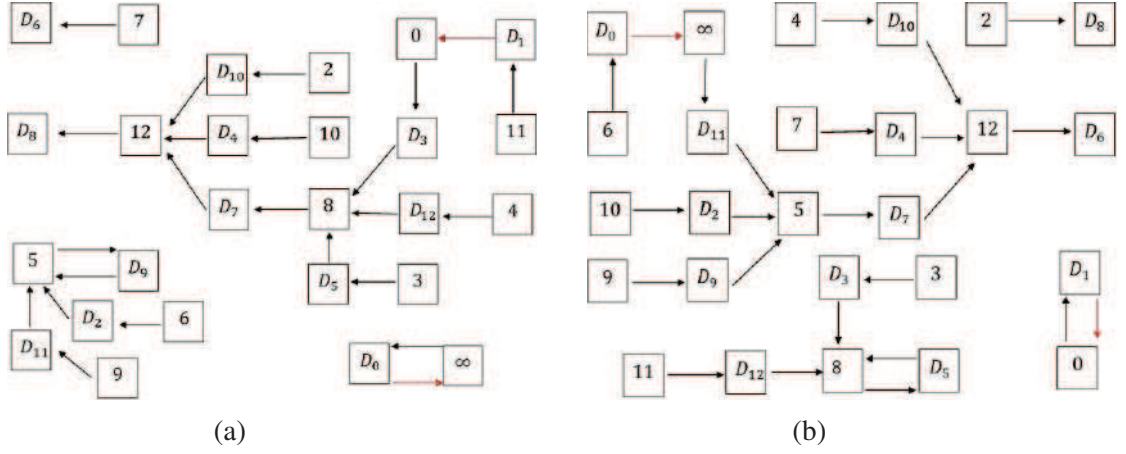


Figure 4.7: (a) $q_0 = 11$, (b) $q_0 = 12$

$\tilde{f}_{14,q_0,169,3}^{-1}(x) \in \mathcal{C}_\ell$ where $4\ell - 3 \equiv 0 \pmod{p}$. Now we could prove the following theorem.

Theorem 4.3.29. *Let $p \equiv 1 \pmod{6}$ be a prime number and $\mathbb{X} = \mathcal{D}_\alpha \cup \mathcal{D}_\beta$ such that α and β are the solutions of the congruence (4.5). Then there exists a subsystem $(\mathcal{J}, f_{\theta,q,3}, |\cdot|_p)$ that is isometrically conjugate to the full shift $(\Sigma_2, \sigma, d_{f_{\theta,q,3}})$ on two symbols.*

Proof. By Propositions 4.3.23, 4.3.24 and 4.3.26, we have $f_{\theta,q,3}^{-1}(\mathbb{X}) \subset \mathcal{C}_\ell \cup \mathcal{D}_\alpha \cup \mathcal{D}_\beta$ where $m(\ell) = 4\ell - 3 \equiv 0 \pmod{p}$. We also have for any $x \in \mathcal{D}_\alpha$ or \mathcal{D}_β that $f_{\theta,q,3}^{-1}(x) := \{x_1 \in \mathcal{D}_\alpha, x_2 \in \mathcal{D}_\beta, x_3 \in \mathcal{C}_\ell\}$. Let $\tilde{f}_{\theta,q,3}^{-1}(x) := \{x_1 \in \mathcal{D}_\alpha, x_2 \in \mathcal{D}_\beta\}$ for $x \in \mathcal{D}_\alpha$ or \mathcal{D}_β . Then $\tilde{f}_{\theta,q,3}^{-1}(\mathbb{X}) \subset \mathcal{D}_\alpha \cup \mathcal{D}_\beta$. Suppose

$$\mathcal{J} = \bigcap_{n=0}^{\infty} \tilde{f}_{\theta,q,3}^{-n}(\mathbb{X}) \subset \mathbb{X}.$$

Then we have the incidence matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

By Theorem 2.3.13, subsystem $f_{\theta,q,3} : \mathcal{J} \rightarrow \mathcal{J}$ is isometrically conjugate to the full shift dynamics $(\Sigma_2, \sigma, d_{f_{\theta,q,3}})$ on two symbols. \square

Claim II Let $p \equiv 1 \pmod{6}$. Then the dynamics of the Potts–Bethe mapping (4.2) consists of a subsystem that is topologically conjugate to the full shift on two symbols, minimal subsystems (including the attracting periodic orbits and the Siegel disks of periodic orbits) and their basins.

4.4 THE POTTS–BETHE MAPPING OF DEGREE 4

In this section we consider the Potts–Bethe mapping of degree 4

$$f_{\theta,q,4}(x) = \left(\frac{\theta x + q - 1}{x + \theta + q - 2} \right)^4. \quad (4.35)$$

To study the dynamical behaviour of Potts–Bethe mapping (4.35) over \mathbb{Q}_p , we first find its fixed points, i.e. $f_{\theta,q,4}(x) = x$ for $x \in \mathbb{Q}_p$. It is clear $\mathbf{x}^{(0)} = 1$ is the fixed point of the Potts–Bethe mapping (4.35). Then from $f_{\theta,q,4}(x) - 1 = x - 1$, we obtain

$$\frac{(x-1)(\theta-1)}{(x+\theta+q-2)^4} \left((\theta x + q - 1)^3 + (\theta x + q - 1)^2(x + \theta + q - 2) + (\theta x + q - 1)(x + \theta + q - 2)^2 + (x + \theta + q - 2)^3 \right) = (x-1).$$

This means that any other fixed point $x \neq \mathbf{x}^{(0)}$ is the root of the quartic equation

$$(\theta-1) \left((\theta x + q - 1)^3 + (\theta x + q - 1)^2(x + \theta + q - 2) + (\theta x + q - 1)(x + \theta + q - 2)^2 + (x + \theta + q - 2)^3 \right) = (x + \theta + q - 2)^4.$$

Let $y = \frac{x+q-1}{\theta-1} + 1$. Then the quartic equation above can be written as follows

$$y^4 - (1 + \theta + \theta^2 + \theta^3)y^3 - (3\theta^2 + 2\theta + 1)(1 - \theta - q)y^2 - (3\theta + 1)(1 - \theta - q)^2y - (1 - \theta - q)^3 = 0. \quad (4.36)$$

Instead of finding the roots of the previous quartic equation with respect to x , we will find the roots of the quartic equation (4.36) whenever $\theta \neq 1, 1 - q$.

Proposition 4.4.1. *Let $p \geq 5$ be a prime number and $0 < |\theta - 1|_p, |q|_p, |\theta - 1 + q|_p < 1$. Then the quartic equation (3.7) always has roots $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ such that $|\mathbf{y}^{(1)}|_p = 1$ with $|\mathbf{y}^{(1)} - 4|_p < 1$ and*

$$\mathbf{y}^{(2)} = \frac{1}{|1 - \theta - q|_p} \left(y_0^{(2)} + y_1^{(2)}p + \dots \right)$$

in which

$$2y_0^{(2)} + (1 - \theta - q)^* \equiv 0 \pmod{p}. \quad (4.37)$$

Moreover, if $p \equiv 1 \pmod{4}$ then the quartic equation (3.7) has two more roots $\mathbf{y}^{(3)}$ and $\mathbf{y}^{(4)}$ with

$$\mathbf{y}^{(3)} = \frac{1}{|1 - \theta - q|_p} \left(y_0^{(3)} + y_1^{(3)}p + \dots \right) \quad \text{and} \quad \mathbf{y}^{(4)} = \frac{1}{|1 - \theta - q|_p} \left(y_0^{(4)} + y_1^{(4)}p + \dots \right)$$

such that $y_0^{(3)}$ and $y_0^{(4)}$ are roots of the following congruence

$$2t^2 + 2(1 - \theta - q)^*t + ((1 - \theta - q)^*)^2 \equiv 0 \pmod{p}. \quad (4.38)$$

Proof. We refer to Chapter 3 for the study of the solvability of the quartic equation over \mathbb{Q}_p . Now let us find all the roots of the quartic equation (4.36).

Write the quartic equation (4.36) as follows

$$y^4 + A_3y^3 + A_2y^2 + A_1y + A_0 = 0$$

where $A_0 = -(1 - \theta - q)^3$, $A_1 = -(3\theta + 1)(1 - \theta - q)^2$, $A_2 = -(3\theta^2 + 2\theta + 1)(1 - \theta - q)$ and $A_3 = -(\theta^3 + \theta^2 + \theta + 1)$. Let $|1 - \theta - q|_p = p^{-k}$ for some integer $k > 0$. Then $\text{ord}_p(A_0) = 3k$, $\text{ord}_p(A_1) = 2k$, $\text{ord}_p(A_2) = k$ and $\text{ord}_p(A_3) = 0$. Construct the Newton polygon of the following points

$$(0, 3k), (1, 2k), (2, k), (3, 0) \text{ and } (4, 0).$$

From this Newton polygon, we know that the quartic equation (3.7) has four roots such that $|\mathbf{y}^{(1)}|_p = 1$ and $|\mathbf{y}^{(i)}|_p = |1 - \theta - q|_p$ for $i = 2, 3, 4$.

Next we will show when the roots $\mathbf{y}^{(i)}$ for $i = 1, 2, 3, 4$ belong to \mathbb{Q}_p . First we check for the root $\mathbf{y}^{(1)}$. Since $|\mathbf{y}^{(1)}|_p = 1$, we have $\mathbf{y}^{(1)} = \sum_{j=0}^{\infty} y_j^{(1)} p^j$ with $y_0^{(1)} \in \{1, \dots, p-1\}$ and $y_j^{(1)} \in \{0, 1, \dots, p-1\}$ for integers $j \geq 1$. Then we obtain

$$\begin{aligned} \left(y_0^{(1)}\right)^4 - 4\left(y_0^{(1)}\right)^3 &\equiv 0 \pmod{p} \\ y_0^{(1)} - 4 &\equiv 0 \pmod{p} \end{aligned}$$

This implies $|\mathbf{y}^{(1)} - 4|_p < 1$.

Let $|\bar{y} - 4|_p < 1$. Consider the polynomial $P(y) = y^4 + A_3y^3 + A_2y^2 + A_1y + A_0$ and its derivative $P'(y) = 4y^3 - 3A_3y^2 + 2A_2y + A_1$. Then

$$P(\bar{y}) \equiv 0 \pmod{p} \quad \text{and} \quad P'(\bar{y}) \equiv 4\bar{y}^3 - 12\bar{y}^2 \equiv \bar{y}^3 \not\equiv 0 \pmod{p}.$$

By Hensel's lemma, there exists a root $\mathbf{y}^{(1)} \in \mathbb{Z}_p$ of the quartic equation (3.7) such that $\mathbf{y}^{(1)} \equiv \bar{y} \pmod{p}$. Thus the root $\mathbf{y}^{(1)}$ belongs to \mathbb{Q}_p such that $|\mathbf{y}^{(1)} - 4|_p < 1$.

Now we check for the roots $\mathbf{y}^{(i)}$ in which $i = 2, 3, 4$. Assume $|1 - \theta - q|_p = p^{-k}$ for some integer $k > 0$. Then we have $\mathbf{y}^{(i)} = \frac{(\mathbf{y}^{(i)})^*}{|1 - \theta - q|_p} = p^k (\mathbf{y}^{(i)})^*$ such that $(\mathbf{y}^{(i)})^* = \sum_{j=0}^{\infty} y_j^{(i)} p^j$ with $y_0^{(i)} \in \{1, \dots, p-1\}$ and $y_j^{(i)} \in \{0, 1, \dots, p-1\}$ for integers $j \geq 1$. Substitute into the quartic equation (3.7), then $p^k((\mathbf{y}^{(i)})^*)^4 - (\theta^3 + \theta^2 + \theta + 1)((\mathbf{y}^{(i)})^*)^3 - (3\theta^2 + 2\theta + 1)(1 - \theta - q)^*((\mathbf{y}^{(i)})^*)^2 - (3\theta + 1)((1 - \theta - q)^*)^2(\mathbf{y}^{(i)})^* - ((1 - \theta - q)^*)^3 = 0$ and

$$4\left(y_0^{(i)}\right)^3 + 6(1 - \theta - q)^*\left(y_0^{(i)}\right)^2 + 4((1 - \theta - q)^*)^2 y_0^{(i)} + ((1 - \theta - q)^*)^3 \equiv 0 \pmod{p}.$$

This congruence can be factorized as follows

$$\left(2y_0^{(i)} + (1 - \theta - q)^*\right) \left(2\left(y_0^{(i)}\right)^2 + 2(1 - \theta - q)^* y_0^{(i)} + ((1 - \theta - q)^*)^2\right) \equiv 0 \pmod{p}. \quad (4.39)$$

Hence the congruence (4.39) has always a solution because the congruence $2y_0^{(2)} + (1 - \theta - q)^* \equiv 0 \pmod{p}$ is solvable. Furthermore, the factor

$$2\left(y_0^{(i)}\right)^2 + 2(1 - \theta - q)^* y_0^{(i)} + ((1 - \theta - q)^*)^2$$

has a discriminant $D = -((1 - \theta - q)^*)^2$. This means the congruence (4.39) has two more solutions when $p \equiv 1 \pmod{4}$.

Suppose \bar{y}_2 , \bar{y}_3 and \bar{y}_4 be integers such that $2\bar{y}_2 + (1 - \theta - q)^* \equiv 0 \pmod{p}$ with \bar{y}_3 and \bar{y}_4 satisfy

$$2t^2 + 2(1 - \theta - q)^* t + ((1 - \theta - q)^*)^2 \equiv 0 \pmod{p}.$$

Such \bar{y}_3 and \bar{y}_4 exist if $p \equiv 1 \pmod{4}$. Consider the polynomial $\bar{P}(y) = p^k y^4 - (\theta^3 + \theta^2 + \theta + 1)y^3 - (3\theta^2 + 2\theta + 1)((1 - \theta - q)^*)y^2 - (3\theta + 1)((1 - \theta - q)^*)^2 y - ((1 - \theta - q)^*)^3$ and its derivative $\bar{P}'(y) = 4p^k y^3 - 3(1 + \theta + \theta^2 + \theta^3)y^2 - 2(3\theta^2 + 2\theta + 1)((1 - \theta - q)^*)y - (3\theta + 1)((1 - \theta - q)^*)^2$. Then

$$\begin{aligned} \bar{P}(\bar{y}_i) &\equiv -4\bar{y}_i^3 - 6(1 - \theta - q)^* \bar{y}_i^2 - 4((1 - \theta - q)^*)^2 \bar{y}_i - ((1 - \theta - q)^*)^3 \pmod{p} \\ \bar{P}'(\bar{y}_i) &\equiv -12\bar{y}_i^2 - 12(1 - \theta - q)^* \bar{y}_i - 4((1 - \theta - q)^*)^2 \pmod{p}. \end{aligned}$$

The discriminant of $\bar{P}(y) \pmod{p}$ is non-zero modulo p whenever $p \geq 5$, i.e. $\Delta \equiv -16((1 - \theta - q)^*)^6 \pmod{p}$. This implies the resultant $R(\bar{P}(y) \pmod{p}, \bar{P}'(y) \pmod{p}) \not\equiv 0 \pmod{p}$. By Theorem 2.2.2, the congruences $\bar{P}(y) \equiv 0 \pmod{p}$ and $\bar{P}'(y) \equiv 0 \pmod{p}$ has no common solutions. So we have for $i = 2, 3, 4$ that

$$\bar{P}(\bar{y}_i) \equiv 0 \pmod{p} \quad \text{and} \quad \bar{P}'(\bar{y}_i) \not\equiv 0 \pmod{p}.$$

By Hensel's lemma, there exists a root $(\mathbf{y}^{(2)})^* \in \mathbb{Z}_p$ of the quartic equation (3.7) such that $(\mathbf{y}^{(2)})^* \equiv \bar{y}_2 \pmod{p}$. If $p \equiv 1 \pmod{4}$, there exist two more roots $(\mathbf{y}^{(3)})^*, (\mathbf{y}^{(4)})^* \in \mathbb{Z}_p$ of the quartic equation (3.7) such that $(\mathbf{y}^{(i)})^* \equiv \bar{y}_i \pmod{p}$ for $i = 3, 4$. Hence, all the roots $\mathbf{y}^{(2)}, \mathbf{y}^{(3)}, \mathbf{y}^{(4)}$ belong to \mathbb{Q}_p . \square

As a consequence, we obtain the following result about the fixed points set.

Theorem 4.4.2. *Let $p \geq 5$ be a prime number and $\mathbf{y}^{(i)}$ for $1 \leq i \leq 4$ be roots of the quartic equation (4.4). Suppose $\mathbf{x}^{(0)} := 1$ and $\mathbf{x}^{(i)} := \mathbf{x}^{(\infty)} + (\theta - 1)\mathbf{y}^{(i)}$ for $1 \leq i \leq 4$.*

Then

$$\mathbf{Fix}\{f_{\theta,q,4}\} = \begin{cases} \{\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \mathbf{x}^{(4)}\} & \text{if } p \equiv 1 \pmod{4}, \\ \{\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}\} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Similar to the study of the dynamics of the Potts–Bethe mapping of degree 3, we will consider two cases: $0 < |\theta - 1|_p < |q|_p < 1$ and $0 < |q|_p < |\theta - 1|_p < 1$

4.4.1 The Case $0 < |\theta - 1|_p < |q|_p < 1$

The following properties are useful.

Proposition 4.4.3. *Let $p \geq 5$ be a prime number. Then*

- (i) $|q|_p = |\mathbf{y}^{(2)}|_p = |\mathbf{y}^{(3)}|_p = |\mathbf{y}^{(4)}|_p$, $|\mathbf{y}^{(1)} - 4|_p$, $|\mathbf{y}^{(1)}|_p = 1$;
- (ii) $|\mathbf{y}^{(i)} - \mathbf{y}^{(j)}|_p = |q|_p$ and $|\theta \mathbf{y}^{(i)} + (1 - \theta - q)|_p = |q|_p$, $i \neq j$, $i, j = 2, 3, 4$;
- (iii) $|\mathbf{x}^{(0)} - \mathbf{x}^{(\infty)}|_p = |q|_p$ and $|\mathbf{x}^{(1)} - \mathbf{x}^{(\infty)}|_p = |\theta - 1|_p$;
- (iv) $|\mathbf{x}^{(i)} - \mathbf{x}^{(\infty)}|_p = |q|_p |\theta - 1|_p$ for $i = 2, 3, 4$;
- (v) $|\mathbf{x}^{(i)} - \mathbf{x}^{(0)}|_p = |q|_p$ for $i = 1, 2, 3, 4$ and $|\mathbf{x}^{(i)} - \mathbf{x}^{(1)}|_p = |\theta - 1|_p$ for $i = 2, 3, 4$;
- (vi) $|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}|_p = |q|_p |\theta - 1|_p$ for $i \neq j$, $i, j = 2, 3, 4$.

Theorem 4.4.4. *Let $p \geq 5$ be a prime number. The fixed point $\mathbf{x}^{(0)}$ is attracting and the fixed point $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are repelling. Furthermore, if $p \equiv 1 \pmod{4}$, then the fixed points $\mathbf{x}^{(3)}$ and $\mathbf{x}^{(4)}$ exist and they are repelling.*

Proof. We need to show $\left| f'_{\theta,q,4}(\mathbf{x}^{(0)}) \right|_p = 1$ and $\left| f'_{\theta,q,4}(\mathbf{x}^{(i)}) \right|_p > 1$ for $i = 1, 2, 3, 4$. For $0 \leq i \leq 4$, we have

$$\begin{aligned} f'(\mathbf{x}^{(i)}) &= \frac{4(\theta - 1)(\theta - 1 + q)(\theta \mathbf{x}^{(i)} + q - 1)^3}{(\mathbf{x}^{(i)} + \theta + q - 2)^5} \\ &= \frac{4(\theta - 1)(\theta - 1 + q)\mathbf{x}^{(i)}}{(\theta \mathbf{x}^{(i)} + q - 1)(\mathbf{x}^{(i)} + \theta + q - 2)}. \end{aligned} \quad (4.40)$$

Thus $\left| f'(\mathbf{x}^{(0)}) \right|_p = \frac{|\theta - 1|_p}{|q|_p} < 1$. By the equality (4.40) and $\mathbf{x}^{(i)} = \mathbf{x}^{(\infty)} + (\theta - 1)\mathbf{y}^{(i)}$ for $i = 1, 2, 3, 4$, we get

$$f'(\mathbf{x}^{(i)}) = \frac{4(\theta - 1 + q)\mathbf{x}^{(i)}}{(\theta - 1)\mathbf{y}^{(i)}(\theta \mathbf{y}^{(i)} + (1 - \theta - q))}. \quad (4.41)$$

Hence $\left| f'(\mathbf{x}^{(1)}) \right|_p = \frac{|q|_p}{|\theta - 1|_p} > 1$. For the fixed points $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ and $\mathbf{x}^{(4)}$ (the fixed points $\mathbf{x}^{(3)}$ and $\mathbf{x}^{(4)}$ exist when $p \equiv 1 \pmod{4}$), we obtain from (4.41) that

$$f'(\mathbf{x}^{(i)}) = \frac{4(\theta - 1 + q)|1 - \theta - q|_p^2 \mathbf{x}^{(i)}}{(\theta - 1)(\mathbf{y}^{(i)})^* (\theta (\mathbf{y}^{(i)})^* + (1 - \theta - q)^*)}. \quad (4.42)$$

By means of (4.37) and (4.38), we can verify $\theta (\mathbf{y}^{(i)})^* \not\equiv -(1-\theta-q)^* \pmod{p}$. Consequently, we obtain

$$\left| f'(\mathbf{x}^{(i)}) \right|_p = \frac{|\theta-1+q|_p}{|\theta-1+q|_p^2 |\theta-1|_p} = \frac{1}{|q|_p |\theta-1|_p} > 1.$$

□

We introduce the following sets

$$\begin{aligned} \mathcal{A}_0 &= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(0)} \right|_p < |q|_p \right\}, \\ \mathcal{A}_1 &= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(0)} \right|_p > |q|_p \right\}, \\ \mathcal{A}_{0,\infty} &= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(\infty)} \right|_p = \left| x - \mathbf{x}^{(0)} \right|_p = |q|_p \right\}, \\ \mathcal{A}_2 &= \left\{ x \in \mathbb{Q}_p : |\theta-1|_p < \left| x - \mathbf{x}^{(\infty)} \right|_p < |q|_p \right\}, \\ \mathcal{A}_{1,\infty} &= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(1)} \right|_p = \left| x - \mathbf{x}^{(\infty)} \right|_p = |\theta-1|_p \right\}, \\ \mathcal{C}_1 &= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(1)} \right|_p < |\theta-1|_p \right\}, \\ \mathcal{A}_3 &= \left\{ x \in \mathbb{Q}_p : |q|_p |\theta-1|_p < \left| x - \mathbf{x}^{(\infty)} \right|_p < |\theta-1|_p \right\}, \\ \mathcal{A}_{2,\infty} &= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(2)} \right|_p = \left| x - \mathbf{x}^{(\infty)} \right|_p = |q|_p |\theta-1|_p \right\}, \\ \mathcal{C}_2 &= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(2)} \right|_p < |q|_p |\theta-1|_p \right\}, \\ \mathcal{A}_\infty &= \left\{ x \in \mathbb{Q}_p : 0 < \left| x - \mathbf{x}^{(\infty)} \right|_p < |q|_p |\theta-1|_p \right\}. \end{aligned}$$

If $p \equiv 1 \pmod{4}$, we have $\mathcal{A}_{2,\infty} = \mathcal{A}_{2,,3,4,\infty} \cup \mathcal{C}_3 \cup \mathcal{C}_4$ in which

$$\begin{aligned} \mathcal{A}_{2,,3,4,\infty} &= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(2)} \right|_p = \left| x - \mathbf{x}^{(3)} \right|_p = \left| x - \mathbf{x}^{(4)} \right|_p = \left| x - \mathbf{x}^{(\infty)} \right|_p = |q|_p |\theta-1|_p \right\}, \\ \mathcal{C}_3 &= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(3)} \right|_p < |q|_p |\theta-1|_p \right\}, \\ \mathcal{C}_4 &= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(4)} \right|_p < |q|_p |\theta-1|_p \right\}. \end{aligned}$$

One can easily check $\mathbf{x}^{(0)} \in \mathcal{A}_0$, $\mathbf{x}^{(1)} \in \mathcal{C}_1$, $\mathbf{x}^{(2)} \in \mathcal{C}_2$, $\mathbf{x}^{(3)} \in \mathcal{C}_3$ and $\mathbf{x}^{(4)} \in \mathcal{C}_4$.

Proposition 4.4.5. *Let $p \geq 5$ be a prime number.*

- (i) $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_{0,\infty} \subset f_{\theta,q,4}^{-1}(\mathcal{A}_0)$;
- (ii) $\mathcal{A}_\infty \cup \mathcal{A}_3 \subset f_{\theta,q,4}^{-1}(\mathcal{A}_1)$;
- (iii) $\mathcal{A}_{2,\infty} \subset f_{\theta,q,4}^{-1}(\mathcal{A}_1)$ if $p \equiv 3 \pmod{4}$;

(iv) $\mathcal{A}_{2,3,4,\infty} \subset f_{\theta,q,4}^{-1}(\mathcal{A}_1)$ if $p \equiv 1 \pmod{4}$;

(v) $\mathcal{A}_{1,\infty} \subset f_{\theta,q,4}^{-1}(\mathcal{A}_{0,\infty})$.

Proof. We have

$$f_{\theta,q,4}(x) - \mathbf{x}^{(0)} = \frac{(\theta-1)(x - \mathbf{x}^{(0)})}{(x - \mathbf{x}^{(\infty)})^4} g(x) \quad (4.43)$$

where $g(x) = (\theta x + q - 1)^3 + (\theta x + q - 1)^2(x - \mathbf{x}^{(\infty)}) + (\theta x + q - 1)(x - \mathbf{x}^{(\infty)})^2 + (x - \mathbf{x}^{(\infty)})^3$. We rewrite the Potts–Bethe mapping (4.35) as follows

$$f_{\theta,q,4}(x) = \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right)^4. \quad (4.44)$$

Then we have

$$f_{\theta,q,4}(x) - \mathbf{x}^{(0)} = \frac{(\theta-1)(x - \mathbf{x}^{(0)})}{x - \mathbf{x}^{(\infty)}} g_1(x) \quad (4.45)$$

where $g_1(x) = \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right)^3 + \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right)^2 + \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right) + 1$.

(i) We first show for any $x \in \mathcal{A}_0 \cup \mathcal{A}_1$ that $f_{\theta,q,4}(x) \in \mathcal{A}_0$. In fact, we have

$$|x - \mathbf{x}^{(\infty)}|_p = |x - \mathbf{x}^{(0)} + \theta - 1 + q|_p = \begin{cases} |q|_p, & x \in \mathcal{A}_0, \\ |x - \mathbf{x}^{(0)}|_p, & x \in \mathcal{A}_1 \end{cases} \text{ and}$$

$$|\theta x + q - 1|_p = |\theta(x - \mathbf{x}^{(0)}) + \theta - 1 + q|_p = \begin{cases} |q|_p, & x \in \mathcal{A}_0, \\ |x - \mathbf{x}^{(0)}|_p, & x \in \mathcal{A}_1. \end{cases}$$

These imply

$$|g(x)|_p \leq \begin{cases} |q|_p^3, & x \in \mathcal{A}_0, \\ |x - \mathbf{x}^{(0)}|_p^3, & x \in \mathcal{A}_1. \end{cases}$$

So we have $|f_{\theta,q,4}(x) - \mathbf{x}^{(0)}|_p \leq \begin{cases} \frac{|x - \mathbf{x}^{(0)}|_p}{|q|_p} |\theta - 1|_p, & x \in \mathcal{A}_0 \\ |\theta - 1|_p, & x \in \mathcal{A}_1 \end{cases} \leq |\theta - 1|_p < |q|_p$. Thus for $x \in \mathcal{A}_0 \cup \mathcal{A}_1$, we have $f_{\theta,q,4}(x) \in \mathcal{A}_0$ which means $f_{\theta,q,4}(\mathcal{A}_0) \subset \mathcal{A}_0$ and $f_{\theta,q,4}(\mathcal{A}_1) \subset \mathcal{A}_0$.

Next we show for any $x \in \mathcal{A}_{0,\infty}$ that $f_{\theta,q,4}(x) \in \mathcal{A}_0$. We have $|x - \mathbf{x}^{(0)}|_p = |x - \mathbf{x}^{(\infty)}|_p = |q|_p$ and $|\theta x + q - 1|_p = |\theta(x - \mathbf{x}^{(\infty)}) + (\theta - 1)(1 - \theta - q)|_p = |q|_p$ which imply $|g(x)|_p \leq |q|_p^2$. By the equality (4.43), we obtain $|f_{\theta,q,4}(x) - \mathbf{x}^{(0)}|_p \leq |\theta - 1|_p < |q|_p$. Hence $f_{\theta,q,4}(x) \in \mathcal{A}_0$ for any $x \in \mathcal{A}_{0,\infty}$ which means $f_{\theta,q,4}(\mathcal{A}_{0,\infty}) \subset \mathcal{A}_0$.

We continue by showing for any $x \in \mathcal{A}_2$ that $f_{\theta,q,4}(x) \in \mathcal{A}_0$. For any $x \in \mathcal{A}_2$, we have $\left| \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right|_p = \frac{|q|_p |\theta-1|_p}{|x - \mathbf{x}^{(\infty)}|_p} < 1$. This $|g_1(x)|_p = 1$ and $|f_{\theta,q,4}(x) - \mathbf{x}^{(0)}|_p = \frac{|\theta-1|_p}{|x - \mathbf{x}^{(\infty)}|_p} |x - \mathbf{x}^{(0)}|_p < |x - \mathbf{x}^{(0)}|_p = |q|_p$. Therefore, we get for any $x \in \mathcal{A}_2$ that $f_{\theta,q,4}(x) \in \mathcal{A}_0$.

\mathcal{A}_0 which means $f_{\theta,q,4}(\mathcal{A}_2) \subset \mathcal{A}_0$. This conclude the statement (i).

(ii) Let $x \in \mathcal{A}_\infty$. We have $\left| \frac{(\theta-1)(1-\theta-q)}{x-\mathbf{x}^{(\infty)}} \right|_p = \frac{|q|_p |\theta-1|_p}{|x-\mathbf{x}^{(\infty)}|_p} > 1$. This implies

$$|f_{\theta,q,4}(x)|_p = \left| \theta + \frac{(\theta-1)(1-\theta-q)}{x-\mathbf{x}^{(\infty)}} \right|_p^4 > 1 > |q|_p$$

and $|f_{\theta,q,4}(x) - \mathbf{x}^{(0)}|_p = |f_{\theta,q,4}(x)|_p > |q|_p$. Thus $f_{\theta,q,4}(\mathcal{A}_\infty) \subset \mathcal{A}_1$.

Next let $x \in \mathcal{A}_3$. We have $\left| \frac{(\theta-1)(1-\theta-q)}{x-\mathbf{x}^{(\infty)}} \right|_p = \frac{|\theta-1|_p |q|_p}{|x-\mathbf{x}^{(\infty)}|_p} < 1$. This implies $|g_1(x)|_p =$

1. By the equality (4.45), we have $|q|_p < |f_{\theta,q,4}(x) - \mathbf{x}^{(0)}|_p = \frac{|\theta-1|_p |x-\mathbf{x}^{(0)}|_p}{|x-\mathbf{x}^{(\infty)}|_p} < 1$.

Hence, $f_{\theta,q,4}(\mathcal{A}_3) \subset \mathcal{A}_1$ and we obtain the statement (ii).

The proof of the statements (iii) and (iv) are similar. We will show the proof for the statement (iv). The statement (iii) come directly from (iv). Their differences are the existence of the fixed points $\mathbf{x}^{(3)}$ and $\mathbf{x}^{(4)}$.

(iv) Suppose $p \equiv 1 \pmod{4}$. Let $x \in \mathcal{A}_{2,3,4,\infty}$ with $y := \frac{x-\mathbf{x}^{(\infty)}}{\theta-1}$. In this case, $|y|_p = |q|_p$ and $y = \frac{y^*}{|q|_p}$. Then from $g(x)$ of the equality (4.43), we write

$$\begin{aligned} g(x) &= (\theta x + q - 1)^3 + (\theta x + q - 1)^2(x - \mathbf{x}^{(\infty)}) + (\theta x + q - 1)(x - \mathbf{x}^{(\infty)})^2 + (x - \mathbf{x}^{(\infty)})^3 \\ &= (\theta - 1)^3 ((\theta^3 + \theta^2 + \theta + 1)y^3 + (3\theta^2 + 2\theta + 1)(1 - \theta - q)y^2 \\ &\quad + (3\theta + 1)(1 - \theta - q)^2 y + (1 - \theta - q)^3) \\ &= \frac{(\theta - 1)^3}{|q|_p^3} \bar{g}(x) \end{aligned}$$

such that $\bar{g}(x) = (\theta^3 + \theta^2 + \theta + 1)(y^*)^3 + (3\theta^2 + 2\theta + 1)(1 - \theta - q)^*(y^*)^2 + (3\theta + 1)((1 - \theta - q)^*)^2(y^*) + ((1 - \theta - q)^*)^3$. Let $y^* = \sum_{j=0}^{\infty} y_j p^j$. Then $y_0 \not\equiv y_0^{(i)} \pmod{p}$ for $i = 2, 3, 4$. These mean means

$$\begin{aligned} \bar{g}(x) &\equiv (2y^* + (1 - \theta - q)^*) \left(2(y^*)^2 + 2(1 - \theta - q)^* y^* + ((1 - \theta - q)^*)^2 \right) \pmod{p} \\ &\not\equiv 0 \pmod{p} \end{aligned}$$

and $|g(x)|_p = |q|_p^3 |\theta - 1|_p^3$. By the equality (4.43), we get $|f_{\theta,q,4}(x) - \mathbf{x}^{(0)}|_p = 1 > |q|_p$.

Thus, $f_{\theta,q,4}(\mathcal{A}_{2,3,4,\infty}) \subset \mathcal{A}_1$ and we have the statement (iv).

(v) We will show $f_{\theta,q,4}(x) \in \mathcal{A}_{0,\infty}$ for any $x \in \mathcal{A}_{1,\infty}$. First we show $|f_{\theta,q,4}(x) - \mathbf{x}^{(0)}|_p = |q|_p$. Indeed, we have $\left| \frac{(\theta-1)(1-\theta-q)}{x-\mathbf{x}^{(\infty)}} \right|_p = \frac{|\theta-1|_p |q|_p}{|x-\mathbf{x}^{(\infty)}|_p} < 1$. This implies by $g_1(x)$ of the equality (4.45) that $|g_1(x) - 4|_p < |g_1(x)|_p = 1$ and $|f_{\theta,q,4}(x) - \mathbf{x}^{(0)}|_p = \frac{|\theta-1|_p |x-\mathbf{x}^{(0)}|_p}{|x-\mathbf{x}^{(\infty)}|_p} = |q|_p$. Next we show $|f_{\theta,q,4}(x) - \mathbf{x}^{(\infty)}|_p = |q|_p$. Since $|f_{\theta,q,4}(x) - \mathbf{x}^{(\infty)}|_p = |f_{\theta,q,4}(x) - \mathbf{x}^{(0)} + \theta - 1 - q|_p$, it is enough to show $|f_{\theta,q,4}(x) - \mathbf{x}^{(0)} + q|_p =$

$|q|_p$. We have

$$\begin{aligned} f_{\theta,q,4}(x) - \mathbf{x}^{(0)} + q &= \frac{(\theta-1)(x - \mathbf{x}^{(0)})}{x - \mathbf{x}^{(\infty)}} (g_1(x) - 4) \\ &\quad + 4 \left((\theta-1) - \frac{(\theta-1)^2}{x - \mathbf{x}^{(\infty)}} \right) + \frac{q(\theta-1)(\mathbf{y}^{(1)} - 4)}{x - \mathbf{x}^{(\infty)}} + \frac{q(x - \mathbf{x}^{(1)})}{x - \mathbf{x}^{(\infty)}}. \end{aligned}$$

Since $|x - \mathbf{x}^{(0)}|_p = |q|_p$, $|x - \mathbf{x}^{(\infty)}|_p = |x - \mathbf{x}^{(1)}|_p = |\theta - 1|_p$, $|g_1(x) - 4|_p < 1$ and $|\mathbf{y}^{(1)} - 4|_p < 1$, we obtain from the above equality that $|f_{\theta,q,4}(x) - \mathbf{x}^{(0)} + q|_p = |q|_p$. It means $|f_{\theta,q,4}(x) - \mathbf{x}^{(\infty)}|_p = |q|_p$. Thus $f_{\theta,q,4}(\mathcal{A}_{1,\infty}) \subset \mathcal{A}_{0,\infty}$ and the statement (v) holds. \square

Now we describe the attracting basin $\mathfrak{B}(\mathbf{x}^{(0)})$ of the attracting fixed point $\mathbf{x}^{(0)}$.

Theorem 4.4.6. *Let $p \geq 5$ be a prime number. Then $\mathfrak{B}(\mathbf{x}^{(0)}) = \bigcup_{n=0}^{\infty} f_{\theta,q,4}^{-n}(\mathcal{A}_0)$ and*

- (i) $(\mathbb{Q}_p \setminus \{\mathbf{x}^{(\infty)}\}) \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4) \subset \mathfrak{B}(\mathbf{x}^{(0)}) = \bigcup_{n=0}^{\infty} f_{\theta,q,4}^{-n}(\mathcal{A}_0)$ if $p \equiv 1 \pmod{4}$;
- (ii) $(\mathbb{Q}_p \setminus \{\mathbf{x}^{(\infty)}\}) \setminus (\mathcal{C}_1 \cup \mathcal{C}_2) \subset \mathfrak{B}(\mathbf{x}^{(0)}) = \bigcup_{n=0}^{\infty} f_{\theta,q,4}^{-n}(\mathcal{A}_0)$ if $p \equiv 3 \pmod{4}$.

Proof. Let $x \in \mathcal{A}_0$. It follows from Proposition 4.4.5 (i) that $|f_{\theta,q,4}(x) - \mathbf{x}^{(0)}|_p \leq \frac{|\theta-1|_p}{|q|_p} |x - \mathbf{x}^{(0)}|_p$ where $\frac{|\theta-1|_p}{|q|_p} < 1$. It means $f_{\theta,q,4}(\mathbb{S}_r(\mathbf{x}^{(0)})) \subset \mathbb{B}_{\frac{|\theta-1|_p}{|q|_p} r}(\mathbf{x}^{(0)})$. Particularly, $f_{\theta,q,4}(\mathcal{A}_0) \subset \mathcal{A}_0$. Let $x \in \mathfrak{B}(\mathbf{x}^{(0)})$. Then there exist some $n_0 \in \mathbb{N}$ and sufficiently small $r > 0$ such that $f_{\theta,q,4}^{n_0}(x) \in \mathbb{S}_r(\mathbf{x}^{(0)}) \subset \mathcal{A}_0$. Hence $x \in f_{\theta,q,4}^{n_0}(\mathcal{A}_0)$ or equivalently $\mathfrak{B}(\mathbf{x}^{(0)}) = \bigcup_{n=0}^{\infty} f^{-n}(\mathcal{A}_0)$.

Due to Proposition 4.4.5, we directly obtain the statements (i) and (ii). \square

Proposition 4.4.7. *Let $p \geq 5$ be a prime number. Then*

$$\left| f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}}) \right|_p = \begin{cases} \frac{|q|_p}{|\theta-1|_p} |\bar{x} - \bar{\bar{x}}|_p & \text{if } \bar{x}, \bar{\bar{x}} \in \mathcal{C}_1, \\ \frac{1}{|q|_p |\theta-1|_p} |\bar{x} - \bar{\bar{x}}|_p & \text{if } \bar{x}, \bar{\bar{x}} \in \mathcal{C}_2. \end{cases} \quad (4.46)$$

If $p \equiv 1 \pmod{4}$, then we also have

$$\left| f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}}) \right|_p = \begin{cases} \frac{1}{|q|_p |\theta-1|_p} |\bar{x} - \bar{\bar{x}}|_p & \text{if } \bar{x}, \bar{\bar{x}} \in \mathcal{C}_3, \\ \frac{1}{|q|_p |\theta-1|_p} |\bar{x} - \bar{\bar{x}}|_p & \text{if } \bar{x}, \bar{\bar{x}} \in \mathcal{C}_4. \end{cases} \quad (4.47)$$

Proof. For any $\bar{x}, \bar{\bar{x}} \in \mathbb{Q}_p \setminus \{\mathbf{x}^{(\infty)}\}$, we have

$$f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}}) = \frac{(\theta-1)(1-\theta-q)(\bar{\bar{x}} - \bar{x})}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} F(\bar{x}, \bar{\bar{x}}) \quad (4.48)$$

where

$$\begin{aligned}
F(\bar{x}, \bar{\bar{x}}) &= 4\theta^3 + 6\theta^2(1-\theta-q) \left(\frac{\theta-1}{\bar{x}-\mathbf{x}^{(\infty)}} + \frac{\theta-1}{\bar{\bar{x}}-\mathbf{x}^{(\infty)}} \right) \\
&\quad + 4\theta(1-\theta-q)^2 \left(\frac{(\theta-1)^2}{(\bar{x}-\mathbf{x}^{(\infty)})^2} + \frac{(\theta-1)^2}{(\bar{x}-\mathbf{x}^{(\infty)})(\bar{\bar{x}}-\mathbf{x}^{(\infty)})} + \frac{(\theta-1)^2}{(\bar{\bar{x}}-\mathbf{x}^{(\infty)})^2} \right) \\
&\quad + (1-\theta-q)^3 \left(\frac{(\theta-1)^3}{(\bar{x}-\mathbf{x}^{(\infty)})^3} + \frac{(\theta-1)^3}{(\bar{x}-\mathbf{x}^{(\infty)})^2(\bar{\bar{x}}-\mathbf{x}^{(\infty)})} + \frac{(\theta-1)^3}{(\bar{x}-\mathbf{x}^{(\infty)})(\bar{\bar{x}}-\mathbf{x}^{(\infty)})^2} + \frac{(\theta-1)^3}{(\bar{\bar{x}}-\mathbf{x}^{(\infty)})^3} \right).
\end{aligned}$$

Let $\bar{x}, \bar{\bar{x}} \in \mathcal{C}_1$. Then we have

$$\begin{aligned}
&\left| 6\theta^2(1-\theta-q) \left(\frac{\theta-1}{\bar{x}-\mathbf{x}^{(\infty)}} + \frac{\theta-1}{\bar{\bar{x}}-\mathbf{x}^{(\infty)}} \right) \right|_p \leq |q|_p, \\
&\left| 4\theta(1-\theta-q)^2 \left(\frac{(\theta-1)^2}{(\bar{x}-\mathbf{x}^{(\infty)})^2} + \frac{(\theta-1)^2}{(\bar{x}-\mathbf{x}^{(\infty)})(\bar{\bar{x}}-\mathbf{x}^{(\infty)})} + \frac{(\theta-1)^2}{(\bar{\bar{x}}-\mathbf{x}^{(\infty)})^2} \right) \right|_p \leq |q|_p^2 \quad \text{and} \\
&\left| (1-\theta-q)^3 \left(\frac{(\theta-1)^3}{(\bar{x}-\mathbf{x}^{(\infty)})^3} + \frac{(\theta-1)^3}{(\bar{x}-\mathbf{x}^{(\infty)})^2(\bar{\bar{x}}-\mathbf{x}^{(\infty)})} + \frac{(\theta-1)^3}{(\bar{x}-\mathbf{x}^{(\infty)})(\bar{\bar{x}}-\mathbf{x}^{(\infty)})^2} + \frac{(\theta-1)^3}{(\bar{\bar{x}}-\mathbf{x}^{(\infty)})^3} \right) \right|_p \leq |q|_p^3.
\end{aligned}$$

These imply $|F(\bar{x}, \bar{\bar{x}})|_p = 1$. Therefore $|f(\bar{x}) - f(\bar{\bar{x}})|_p = \frac{|q|_p}{|\theta-1|_p} |\bar{x} - \bar{\bar{x}}|_p$.

Next we suppose $\bar{x}, \bar{\bar{x}} \in \mathcal{C}_i$ for $i = 2, 3, 4$ (the sets \mathcal{C}_3 and \mathcal{C}_4 are considered when $p \equiv 1 \pmod{4}$). We know for $i = 2, 3, 4$ that

$$\frac{\mathbf{x}^{(i)} - \mathbf{x}^{(\infty)}}{\theta-1} = \mathbf{y}^i = \frac{y_0^{(i)} + y_1^{(i)}p + \dots}{|q|_p},$$

where $2y_0^{(2)} + (1-\theta-q)^* \equiv 0 \pmod{p}$ and $y_0^{(3)}$ and $y_0^{(4)}$ are the roots of $2t^2 + 2(1-\theta-q)^*t + ((1-\theta-q)^*)^2 \equiv 0 \pmod{p}$. Since $\bar{x}, \bar{\bar{x}} \in \mathcal{C}_i$ for $i = 2, 3, 4$, we have

$$\left| \frac{\bar{x} - \mathbf{x}^{(\infty)}}{\theta-1} \right|_p = \left| \frac{\bar{\bar{x}} - \mathbf{x}^{(\infty)}}{\theta-1} \right|_p = |q|_p \quad \text{and} \quad \left(\frac{\bar{x} - \mathbf{x}^{(\infty)}}{\theta-1} \right)^* = \left(\frac{\bar{\bar{x}} - \mathbf{x}^{(\infty)}}{\theta-1} \right)^* \equiv y_0^{(i)} \pmod{p}.$$

Then we get

$$\begin{aligned}
F(\bar{x}, \bar{\bar{x}}) &= 4a^3 + 6a^2(1-\theta-q)^* \left(\left(\frac{\theta-1}{\bar{x}-\mathbf{x}^{(\infty)}} \right)^* + \left(\frac{\theta-1}{\bar{\bar{x}}-\mathbf{x}^{(\infty)}} \right)^* \right) \\
&\quad + 4a((1-\theta-q)^*)^2 \left(\left(\left(\frac{\theta-1}{\bar{x}-\mathbf{x}^{(\infty)}} \right)^* \right)^2 + \left(\frac{\theta-1}{\bar{x}-\mathbf{x}^{(\infty)}} \right)^* \left(\frac{\theta-1}{\bar{\bar{x}}-\mathbf{x}^{(\infty)}} \right)^* + \left(\left(\frac{\theta-1}{\bar{\bar{x}}-\mathbf{x}^{(\infty)}} \right)^* \right)^2 \right) \\
&\quad + ((1-\theta-q)^*)^3 \left(\left(\left(\frac{\theta-1}{\bar{x}-\mathbf{x}^{(\infty)}} \right)^* \right)^3 + \left(\left(\frac{\theta-1}{\bar{x}-\mathbf{x}^{(\infty)}} \right)^* \right)^2 \left(\frac{\theta-1}{\bar{\bar{x}}-\mathbf{x}^{(\infty)}} \right)^* \right. \\
&\quad \quad \left. + \left(\frac{\theta-1}{\bar{x}-\mathbf{x}^{(\infty)}} \right)^* \left(\left(\frac{\theta-1}{\bar{\bar{x}}-\mathbf{x}^{(\infty)}} \right)^* \right)^2 + \left(\left(\frac{\theta-1}{\bar{\bar{x}}-\mathbf{x}^{(\infty)}} \right)^* \right)^3 \right)
\end{aligned}$$

and

$$\left(y_0^{(i)} \right)^3 F(\bar{x}, \bar{\bar{x}}) \equiv 4 \left(\left(y_0^{(i)} \right) + (1-\theta-q)^* \right)^3 \not\equiv 0 \pmod{p}.$$

This yields $|F(\bar{x}, \bar{\bar{x}})|_p = 1$. Therefore, $|f(\bar{x}) - f(\bar{\bar{x}})|_p = \frac{|\bar{x} - \bar{\bar{x}}|_p}{|q|_p |\theta - 1|_p}$ for any $\bar{x}, \bar{\bar{x}} \in \mathcal{C}_i$, $i = 2, 3, 4$. \square

Let $r = |q|_p |\theta - 1|_p$ and define the following set

$$\mathbb{X} = \begin{cases} \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 & \text{if } p \equiv 1 \pmod{4}, \\ \mathcal{C}_1 \cup \mathcal{C}_2 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proposition 4.4.8. *Let $p \geq 5$ be a prime number.*

(i) *If $p \equiv 3 \pmod{4}$, then $\mathbb{X} \subset f_{\theta,q,4}(\mathcal{C}_i)$ for $i = 1, 2$.*

(ii) *If $p \equiv 1 \pmod{4}$, then $\mathbb{X} \subset f_{\theta,q,4}(\mathcal{C}_i)$ for $i = 1, 2, 3, 4$.*

Proof. Remember that the fixed points $\mathbf{x}^{(3)}$ and $\mathbf{x}^{(4)}$ exist if $p \equiv 1 \pmod{4}$. For any $x \in \mathcal{C}_1$, by the equality (4.46), we have $|f_{\theta,q,4}(x) - \mathbf{x}^{(1)}|_p = \frac{|q|_p}{|\theta - 1|_p} |x - \mathbf{x}^{(1)}|_p$. So $f_{\theta,q,4}(\mathcal{C}_1) = \mathbb{B}_{|q|_p}(\mathbf{x}^{(1)})$ and $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \mathbf{x}^{(4)} \in \mathbb{B}_{|q|_p}(\mathbf{x}^{(1)})$. For any $x \in \mathbb{X}$ we have $|x - \mathbf{x}^{(1)}|_p \leq |\theta - 1|_p < |q|_p$. Therefore $\mathbb{X} \subset f_{\theta,q,4}(\mathcal{C}_1)$.

Next for any $x \in \mathcal{C}_i$ for $i = 2, 3, 4$, by the equalities (4.46) and (4.47), we have $|f_{\theta,q,4}(x) - \mathbf{x}^{(i)}|_p = \frac{|x - \mathbf{x}^{(i)}|_p}{|q|_p |\theta - 1|_p}$, refer (4.46) and (4.47). This means $f_{\theta,q,4}(\mathcal{C}_i) = \mathbb{B}_1(\mathbf{x}^{(i)})$ and $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \mathbf{x}^{(4)} \in \mathbb{B}_1(\mathbf{x}^{(i)})$ for $i = 2, 3, 4$. For any $x \in \mathbb{X}$, we get $|x - \mathbf{x}^{(i)}|_p \leq |\theta - 1|_p < 1$. Hence $\mathbb{X} \subset f_{\theta,q,4}(\mathcal{C}_i)$ for $i = 2, 3, 4$ and we have the statements (i) and (ii). \square

Now we describe the dynamical behaviour of the Potts–Bethe mapping (4.35) over \mathbb{Q}_p . Let

$$\mathcal{J} = \bigcap_{n=0}^{\infty} f^{-n}(\mathbb{X}).$$

Theorem 4.4.9. *Let $p \geq 5$ be a prime number. Then*

(i) *the subsystem $(\mathcal{J}, f_{\theta,q,4}, |\cdot|_p)$ is topologically conjugate to the*

(a) *full shift (Σ_4, σ, d_f) on four symbols when $p \equiv 1 \pmod{4}$;*

(b) *full shift (Σ_2, σ, d_f) on two symbols when $p \equiv 3 \pmod{4}$.*

(ii) $\mathbb{Q}_p = \mathfrak{B}(\mathbf{x}^{(0)}) \cup \mathcal{J} \cup \bigcup_{n=0}^{+\infty} f^{-n}(\mathbf{x}^{(\infty)})$.

Proof. (i) By Propositions 4.4.5, 4.4.8 and Theorems 2.3.13, 4.4.6, the dynamical system $f_{\theta,q,4} : \mathcal{J} \rightarrow \mathcal{J}$ is topologically conjugate to the full shift dynamics (Σ_4, σ, d_f) if $p \equiv 1 \pmod{4}$ and (Σ_2, σ, d_f) if $p \equiv 3 \pmod{4}$ where the incidence matrix is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

when $p \equiv 1 \pmod{4}$ and

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

when $p \equiv 3 \pmod{4}$.

(ii) Due to Theorem 4.4.5, we have $(\mathbb{Q}_p \setminus \{\mathbf{x}^{(\infty)}\}) \setminus \mathbb{X} \subset \mathfrak{B}(\mathbf{x}^{(0)})$. Let $x \in \mathbb{X} \setminus \mathcal{J}$. Then there exists n_0 such that $x \notin f^{-n_0}(\mathbb{X})$ or equivalently $f_{\theta,q,4}^{n_0}(x) \notin \mathbb{X}$. Due to Proposition 4.4.8 (ii), we get $f_{\theta,q,4}^{n_0+1}(x) \in \mathfrak{B}(\mathbf{x}^{(0)}) \cup \{\mathbf{x}^{(\infty)}\}$. Consequently, we obtain the decomposition $\mathbb{Q}_p = \mathfrak{B}(\mathbf{x}^{(0)}) \cup \mathcal{J} \cup \bigcup_{n=0}^{+\infty} f^{-n} \{\mathbf{x}^{(\infty)}\}$. \square

4.4.2 The Case $0 < |q|_p < |\theta - 1|_p < 1$

Recall the singular point $\mathbf{x}^{(\infty)} = 2 - \theta - q$, the critical point $\mathbf{x}^{(c)} = \mathbf{x}^{(\infty)} + (\theta - 1)\mathbf{y}^{(c)} = \frac{1-q}{\theta}$ and the fixed points $\mathbf{x}^{(0)} = 1$, $\mathbf{x}^{(i)} = \mathbf{x}^{(\infty)} + (\theta - 1)\mathbf{y}^{(i)}$ for $i = 1, 2, 3, 4$. Then one could easily verify the following properties.

Proposition 4.4.10. *Let p be a prime number. Then*

- (i) $|\mathbf{y}^{(c)}|_p = |\mathbf{y}^{(i)}|_p = |\theta - 1|_p < 1 = |\mathbf{y}^{(1)}|_p$ for $i = 2, 3, 4$ and $|\mathbf{y}^{(1)} - 4|_p < 1$,
- (ii) $|\mathbf{y}^{(i)} - \mathbf{y}^{(j)}|_p = |\mathbf{y}^{(i)} - \mathbf{y}^{(c)}|_p = |\theta - 1|_p$ for $i \neq j$, $i, j = 2, 3, 4$,
- (iii) $|\mathbf{x}^{(c)}|_p = |\mathbf{x}^{(\infty)}|_p = |\mathbf{x}^{(i)}|_p = 1$ for $i = 0, 1, 2, 3, 4$
- (iv) $|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}|_p = |\mathbf{x}^{(i)} - \mathbf{x}^{(\infty)}|_p = |\mathbf{x}^{(i)} - \mathbf{x}^{(c)}|_p = |\theta - 1|_p$ for $i = 0, 1$,
- (v) $|\mathbf{x}^{(i)} - \mathbf{x}^{(\infty)}|_p = |\mathbf{x}^{(c)} - \mathbf{x}^{(\infty)}|_p = |\mathbf{x}^{(i)} - \mathbf{x}^{(c)}|_p = |\theta - 1|_p^2$ for $i = 2, 3, 4$
- (vi) $|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}|_p = |\theta - 1|_p$, $|\mathbf{x}^{(j)} - \mathbf{x}^{(k)}|_p = |\theta - 1|_p^2$ for $i = 0, 1$, and $j \neq k$, $j, k = 2, 3, 4$.

The following result is about the local behavior of the fixed points.

Theorem 4.4.11. *Let $p \geq 5$ be a prime number. Then the fixed points $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ are indifferent and the fixed point $\mathbf{x}^{(2)}$ is repelling. Moreover, if $p \equiv 1 \pmod{4}$ then the fixed points $\mathbf{x}^{(3)}$ and $\mathbf{x}^{(4)}$ exist and they are repelling.*

Proof. The proof is similar to Theorem 4.4.4. \square

We consider the following sets

$$\begin{aligned} \mathcal{A} &= \left\{ x \in \mathbb{Q}_p : 0 < |x - \mathbf{x}^{(\infty)}|_p < |\theta - 1|_p \right\}, \\ \mathcal{B} &= \left\{ x \in \mathbb{Q}_p : |x - \mathbf{x}^{(\infty)}|_p > |\theta - 1|_p \right\}, \\ \mathcal{C} &= \left\{ x \in \mathbb{Q}_p : |x - \mathbf{x}^{(\infty)}|_p = |\theta - 1|_p \right\}. \end{aligned}$$

We have $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{D}$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{E}$ in which

$$\begin{aligned}\mathcal{A}_1 &= \left\{ x \in \mathbb{Q}_p : |\theta - 1|_p^2 < \left| x - \mathbf{x}^{(\infty)} \right|_p < |\theta - 1|_p \right\}, \\ \mathcal{D} &= \left\{ x \in \mathbb{Q}_p : 0 < \left| x - \mathbf{x}^{(\infty)} \right|_p \leq |\theta - 1|_p^2 \right\}, \\ \mathcal{B}_1 &= \left\{ x \in \mathbb{Q}_p : |\theta - 1|_p < \left| x - \mathbf{x}^{(\infty)} \right|_p < 1 \right\}, \\ \mathcal{E} &= \left\{ x \in \mathbb{Q}_p : \left| x - \mathbf{x}^{(\infty)} \right|_p \geq 1 \right\}.\end{aligned}$$

Now we will calculate the image of the sets \mathcal{A} , \mathcal{B} and \mathcal{C} . We first calculate the image of \mathcal{C} . Suppose $\bar{x}^{(i)} = \mathbf{x}^{(\infty)} + (\theta - 1)\bar{y}^{(i)} \in \mathcal{C}$ with $|\bar{y}^{(i)} - i| < 1$ and $\mathcal{C}_i = \mathbb{B}_{|\theta-1|_p}(\bar{x}^{(i)})$. Then we have

$$\mathcal{C} = \bigcup_{i=1}^{p-1} \mathcal{C}_i.$$

Notice $\mathbf{x}^{(0)} \in \mathcal{C}_1$ and $\mathbf{x}^{(1)} \in \mathcal{C}_4$.

Lemma 4.4.12. *Let $p \geq 5$ be a prime number.*

- (i) *Let $x \in \mathcal{C}$. Then $\left| f'_{\theta,q,4}(x) \right|_p = 1$.*
- (ii) *Let $\bar{x}, \bar{\bar{x}} \in \mathcal{C}_i$ for $1 \leq i \leq p-1$. Then $\left| f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}}) \right|_p = |\bar{x} - \bar{\bar{x}}|_p$.*

Proof. The following is the k th derivative of $f_{\theta,q,4}$ for $k \geq 1$

$$\begin{aligned}f_{\theta,q,4}^{(k)}(x) &= \frac{4(-1)^{k+1}k!(\theta-1)(\theta-1+q)}{(x-\mathbf{x}^{(\infty)})^{k+1}} \left(\theta^3 + \frac{3(k+1)\theta^2(\theta-1)(1-\theta-q)}{2!(x-\mathbf{x}^{(\infty)})} \right. \\ &\quad \left. + \frac{(k+2)!\theta(\theta-1)^2(1-\theta-q)^2}{2!k!(x-\mathbf{x}^{(\infty)})^2} + \frac{(k+3)!(\theta-1)^3(1-\theta-q)^3}{4!k!(x-\mathbf{x}^{(\infty)})^3} \right). \quad (4.49)\end{aligned}$$

(i) Suppose $x \in \mathcal{C}$. Then we have $|x - \mathbf{x}^{(\infty)}|_p = |\theta(x - \mathbf{x}^{(\infty)}) + (\theta - 1)(1 - \theta - q)|_p = |\theta - 1|_p$. By the equality (4.49), one could check $\left| f'_{\theta,q,4}(x) \right|_p = 1$

(ii) Suppose $r = \frac{|\theta-1|_p}{p}$ and $\bar{x}^{(i)} \in \mathcal{C}_i$. By the equality (4.49), we have for $k \geq 2$

$$\left| \frac{3(k+1)\theta^2(\theta-1)(1-\theta-q)}{2!(\bar{x}^{(i)} - \mathbf{x}^{(\infty)})} \right|_p = |k+1|_p |\theta-1|_p \leq |\theta-1|_p < 1,$$

$$\left| \frac{(k+2)!\theta(\theta-1)^2(1-\theta-q)^2}{2!k!(\bar{x}^{(i)} - \mathbf{x}^{(\infty)})^2} \right|_p = |k+1|_p |k+2|_p |\theta-1|_p^2 \leq |\theta-1|_p^2 < 1$$

and

$$\left| \frac{(k+3)!(\theta-1)^3(1-\theta-q)^3}{4!k!(\bar{x}^{(i)} - \mathbf{x}^{(\infty)})^3} \right|_p = |k+1|_p |k+2|_p |k+3|_p |\theta-1|_p^3 \leq |\theta-1|_p^3 < 1.$$

Then we obtain $\left| f_{\theta,q,4}^{(k)}(\bar{x}^{(i)}) \right|_p r^{k-1} = \frac{1}{|\theta-1|_p^{k-1}} \cdot \frac{|\theta-1|_p^{k-1}}{p^{k-1}} = \frac{1}{p^{k-1}} < 1 = \left| f'_{\theta,q,4}(\bar{x}^{(i)}) \right|_p$. We notice for $1 \leq i \leq p-1$ that $\overline{\mathbb{B}_r(\bar{x}^{(i)})} = C_i$. Thus by (i) and Lemma 2.3.1, we have for all $\bar{x}, \bar{\bar{x}} \in C_i$ that $|f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}})|_p = |\bar{x} - \bar{\bar{x}}|_p$. \square

Let $\mathbb{P}^1(\mathbb{F}_p) = \{0, 1, \dots, p-1, \infty\}$. By definition, $C_0 = \mathcal{A}$ and $C_\infty = \mathcal{B}$.

Proposition 4.4.13. *Let $p \geq 5$ be a prime number. Let $m : \mathbb{P}^1(\mathbb{F}_p) \rightarrow \mathbb{P}^1(\mathbb{F}_p)$ and $m(x) = 5 - \frac{4}{x}$. Then for $1 \leq i \leq p-1$, we have $f(C_i) = C_{m(i)}$. Moreover, we have*

- (i) $f_{\theta,q,4}(C_i) = C_i$ for $i = 1, 4$;
- (ii) $f_{\theta,q,4}^k(C_5) = \mathcal{A}$ for some $k \in \mathbb{Z}$.

Proof. By the equality (4.45), we obtain

$$\begin{aligned} f_{\theta,q,4}(x) - \mathbf{x}^{(\infty)} &= \frac{(\theta-1)(x - \mathbf{x}^{(0)})(g_1(x) - 4)}{x - \mathbf{x}^{(\infty)}} + q \left(\frac{(\theta-1)(y^{(1)} - 4) + (x - \mathbf{x}^{(1)})}{x - \mathbf{x}^{(\infty)}} \right) \\ &+ (\theta-1) \left(5 - 4 \frac{(\theta-1)}{x - \mathbf{x}^{(\infty)}} \right). \end{aligned} \quad (4.50)$$

Let $x^{(i)} = \mathbf{x}^{(\infty)} + (\theta-1)y^{(i)} \in C_i$ with $|\bar{y}^{(i)} - i|_p < 1$. Then, by the equality (4.50), we get

$$\begin{aligned} f_{\theta,q,4}(x^{(i)}) - \mathbf{x}^{(\infty)} &\equiv (\theta-1) \left(5 - \frac{4(\theta-1)}{x^{(i)} - \mathbf{x}^{(\infty)}} \right) \pmod{p^{ord_p(\theta-1)+1}} \\ &\equiv (\theta-1) \left(5 - \frac{4}{y^{(i)}} \right) \pmod{p^{ord_p(\theta-1)+1}} \\ \frac{f_{\theta,q,4}(x^{(i)}) - \mathbf{x}^{(\infty)}}{\theta-1} &\equiv 5 - \frac{4}{i} \equiv m(i) \pmod{p}. \end{aligned}$$

This means $f_{\theta,q,4}(x^{(i)}) = \mathbf{x}^{(\infty)} + (\theta-1)z^{(i)}$ with $z^{(i)} \equiv m(i) \pmod{p}$. So $f_{\theta,q,4}(\bar{x}^{(i)}) \in C_{m(i)}$. By Lemma 4.4.12 and Lemma 2.3.3, we have $f_{\theta,q,4}(C_i) = C_{m(i)}$.

(i) Since 1 and 4 are the fixed points of function m , we have $f_{\theta,q,4}(C_1) = C_1$ and $f_{\theta,q,4}(C_4) = C_4$. (ii) For convenience, we suppose that $C_0 = \mathcal{A}$ since \mathcal{A} satisfies the properties of C_0 . We have for $1 \leq i \leq p-1$ that $f_{\theta,q,4}(C_i) = C_{m(i)}$. In order to check whether $f_{\theta,q,4}^k(C_5) = C_0$ for some k or not, it is enough to check whether 4 and 0 are in the same orbit or not under the iteration of m . By simple calculation, we have $m^2(0) = 4$. So indeed 4 and 0 are in the same orbit. By the bijectivity of mapping m , there exists $k \in \mathbb{Z}$ such that $m^k(4) = 0$. The least number k is in $\{1, 2, \dots, p-3\}$ since 1 and 4 are the fixed points of m . Thus $f_{\theta,q,4}^k(C_4) = \mathcal{A}$ for some $k \in \{1, \dots, p-3\}$. \square

Next we calculate the image of the sets \mathcal{A} and \mathcal{B} .

Proposition 4.4.14. *Let $p \geq 5$ be a prime number. Then $f_{\theta,q,4}(\mathcal{A}_1) \subseteq \mathcal{B}_1$ and $f_{\theta,q,4}(\mathcal{B}) \subseteq C_5$ if $p > 5$ or $f_{\theta,q,4}(\mathcal{B}) \subseteq \mathcal{A}$ if $p = 5$. Moreover, the restricted maps $f_{\theta,q,4} : \mathcal{A}_1 \rightarrow \mathcal{B}_1$ and $f_{\theta,q,4} : \mathcal{B} \rightarrow C_5 \setminus \{\theta^4\}$ for $p > 5$ or $f_{\theta,q,4} : \mathcal{B} \rightarrow \mathcal{A} \setminus \{\theta^4\}$ for $p = 5$ are bijective.*

Proof. Recall

$$f_{\theta,q,4}(x) - \mathbf{x}^{(0)} = \frac{(\theta-1)(x - \mathbf{x}^{(0)})}{x - \mathbf{x}^{(\infty)}} g_1(x) \quad (4.51)$$

where $g_1(x) = \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}}\right)^3 + \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}}\right)^2 + \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}}\right) + 1$. First we suppose $x \in \mathcal{A}_1$. Since $|\theta - 1|_p^2 < |x - \mathbf{x}^{(\infty)}|_p < |\theta - 1|_p$, we have

$$\left| \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right|_p = \frac{|\theta - 1|_p^2}{|x - \mathbf{x}^{(\infty)}|_p} < 1.$$

It implies $|g_1(x)|_p = 1$. By the equality (4.51), we get

$$|\theta - 1|_p < \left| f_{\theta,q,4}(x) - \mathbf{x}^{(0)} \right|_p = \frac{|\theta - 1|_p |x - \mathbf{x}^{(0)}|_p}{|x - \mathbf{x}^{(\infty)}|_p} < 1.$$

We have also $|f_{\theta,q,4}(x) - \mathbf{x}^{(\infty)}|_p = |f_{\theta,q,4}(x) - \mathbf{x}^{(0)}|_p$. This means $f_{\theta,q,4}(x) \in \mathcal{B}_1$. Thus $f_{\theta,q,4}(\mathcal{A}_1) \subseteq \mathcal{B}_1$.

Let $\bar{x}, \bar{\bar{x}} \in \mathcal{A}_1$. Then one could calculate

$$|f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}})|_p = \frac{|\theta - 1|_p^2}{|\bar{x} - \mathbf{x}^{(\infty)}|_p |\bar{\bar{x}} - \mathbf{x}^{(\infty)}|_p} |\bar{x} - \bar{\bar{x}}|_p \quad (4.52)$$

We know $\mathcal{A}_1 = \bigcup_{\bar{r}} \mathbb{S}_{\bar{r}}(\mathbf{x}^{(\infty)})$ for $|\theta - 1|_p^2 < \bar{r} < |\theta - 1|_p$. Then $f_{\theta,q,4}(\mathbb{S}_{\bar{r}}(\mathbf{x}^{(\infty)})) = \mathbb{S}_{r'}(\mathbf{x}^{(\infty)}) \subset \mathcal{B}_1 = \bigcup_{r'} \mathbb{S}_{r'}(\mathbf{x}^{(\infty)})$ for $|\theta - 1|_p < r' < 1$. By the equality (4.52), we obtain $f_{\theta,q,4}(\mathbb{S}_{\bar{r}}(\mathbf{x}^{(\infty)})) \cap f_{\theta,q,4}(\mathbb{S}_{\hat{r}}(\mathbf{x}^{(\infty)})) = \emptyset$ for $\bar{r} \neq \hat{r}$ and by Corollary 2.3.4, this restricted map is bijective. One could easily see \bar{r} maps to r' bijectively. Thus the restricted map $f_{\theta,q,4} : \mathcal{A}_1 \rightarrow \mathcal{B}_1$ is a bijection.

Next we assume $x \in \mathcal{B}$. We have $|x - \mathbf{x}^{(\infty)}|_p > |\theta - 1|_p$. By the equality (4.50), we get

$$f_{\theta,q,4}(x) - \mathbf{x}^{(\infty)} \equiv (\theta - 1) \left(5 - \frac{4(\theta - 1)}{x - \mathbf{x}^{(\infty)}} \right) \equiv 5(\theta - 1) \not\equiv 0 \pmod{p^{ord_p(\theta-1)+1}}$$

for $p > 5$. This means $|f_{\theta,q,4}(x) - \mathbf{x}^{(\infty)}|_p = |\theta - 1|_p$ and $f_{\theta,q,4}(x) = \mathbf{x}^{(\infty)} + (\theta - 1)\bar{w}$ with $|\bar{w} - 5|_p < 1$. Therefore, $f_{\theta,q,4}(x) \in \mathcal{C}_5$. For $p = 5$, we have $|f_{\theta,q,4}(x) - \mathbf{x}^{(\infty)}|_p < |\theta - 1|_p$. This implies $f_{\theta,q,4}(x) \in \mathcal{A}$. Hence $f_{\theta,q,4}(\mathcal{B}) \subseteq \mathcal{C}_5$ if $p > 5$ and $f_{\theta,q,4}(\mathcal{B}) \subseteq \mathcal{A}$ if $p = 5$.

Let $\bar{x}, \bar{\bar{x}} \in \mathcal{B}$. Then

$$|f(\bar{x}) - f(\bar{\bar{x}})|_p = \frac{|\theta - 1|_p^2}{|\bar{x} - \mathbf{x}^{(\infty)}|_p |\bar{\bar{x}} - \mathbf{x}^{(\infty)}|_p} |\bar{x} - \bar{\bar{x}}|_p \quad (4.53)$$

We know $\mathcal{B} = \bigcup_r \mathbb{S}_r(\mathbf{x}^{(\infty)})$ for $r > |\theta - 1|_p$. We have $f_{\theta,q,4}(\mathbb{S}_r(\mathbf{x}^{(\infty)})) \subset \mathcal{C}_5$ when $p > 5$ and $f_{\theta,q,4}(\mathbb{S}_r(\mathbf{x}^{(\infty)})) \subset \mathcal{A}$ when $p = 5$ is a sphere center at θ^4 . By the equality (4.53) and $r'' \neq r' > |\theta - 1|_p$, we get $f_{\theta,q,4}(\mathbb{S}_{r''}(\mathbf{x}^{(\infty)})) \cap f(\mathbb{S}_{r'}(\mathbf{x}^{(\infty)})) = \emptyset$. For any $\theta^4 \neq \bar{x} \in \mathcal{C}_5$ if $p > 5$ and $\theta^4 \neq \bar{x} \in \mathcal{A}$ if $p = 5$, there exists a pre-image $x = f^{-1}(\bar{x}) \in \mathcal{B}$. Then by Corollary 2.3.4, the restricted map $f : \mathcal{B} \rightarrow \mathcal{C}_5 \setminus \{\theta^4\}$ for $p > 5$ and $f : \mathcal{B} \rightarrow \mathcal{A} \setminus \{\theta^4\}$ for $p = 5$ is a bijection. \square

Now we discuss the last part, the image of the set $\mathcal{D} \subset \mathcal{A}$. Suppose $\tilde{x}^{(i)} = \mathbf{x}^{(\infty)} + (\theta - 1)\tilde{y}^{(i)} \in \mathcal{D}$ with $\tilde{y}^{(i)} = \frac{(\tilde{y}^{(i)})^*}{|\theta - 1|_p}$ and $|\tilde{y}^{(i)} - i|_p < |\theta - 1|_p$. Let $\mathcal{D}_i := \mathbb{B}_{|\theta - 1|_p^2}(\tilde{x}^{(i)})$. Then

$$\mathcal{D} = \left(\bigcup_{i=0}^{p-1} \mathcal{D}_i \right) \setminus \{\mathbf{x}^{(\infty)}\}$$

such that $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$ for $i \neq j$. Next we describe the sets containing the points $\mathbf{x}^{(\infty)}$ and $\mathbf{x}^{(c)}$. Notice $\mathbf{x}^{(\infty)} \in \mathcal{D}_0$ and

$$\mathcal{D}_0 = \bigcup_{i=1}^{\infty} \mathbb{S}_i^{(0)} \cup \{\mathbf{x}^{(\infty)}\} \quad (4.54)$$

with $\mathbb{S}_i^{(0)} := \mathbb{S}_{\frac{|\theta - 1|_p^2}{p^i}}(\mathbf{x}^{(\infty)})$. Let $\hat{x}_j \in \mathbb{S}_i^{(0)}$ and $\mathcal{D}_{ij}^{(0)} := \mathbb{B}_{\frac{|\theta - 1|_p^2}{p^i}}(\hat{x}_j)$. Then

$$\mathbb{S}_i^{(0)} = \bigcup_{j=1}^{p-1} \mathcal{D}_{ij}^{(0)}$$

such that $\mathcal{D}_{ik}^{(0)} \cap \mathcal{D}_{i\ell}^{(0)} = \emptyset$ for $k \neq \ell$.

Assume $\delta \equiv -(1 - \theta - q)^* \pmod{p}$. Then, one can check $\mathbf{x}^{(c)} \in \mathcal{D}_\delta$ and

$$\mathcal{D}_\delta = \left(\bigcup_{i=1}^{\infty} \mathbb{S}_i^{(\delta)} \right) \cup \{\mathbf{x}^{(c)}\} \quad (4.55)$$

where $\mathbb{S}_i^{(\delta)} := \mathbb{S}_{\frac{|\theta - 1|_p^2}{p^i}}(\mathbf{x}^{(c)})$. Suppose $\check{x}_j \in \mathbb{S}_i^{(\delta)}$ and $\mathcal{D}_{ij}^{(\delta)} := \mathbb{B}_{\frac{|\theta - 1|_p^2}{p^i}}(\check{x}_j)$. Then

$$\mathbb{S}_i^{(\delta)} = \bigcup_{j=1}^{p-1} \mathcal{D}_{ij}^{(\delta)}$$

such that $\mathcal{D}_{ik}^{(\delta)} \cap \mathcal{D}_{i\ell}^{(\delta)} = \emptyset$ for $k \neq \ell$.

Recall $f_{\theta,q,4}(\mathbf{x}^{(\infty)})$ is undefined and $f_{\theta,q,4}(\mathbf{x}^{(c)}) = 0$. We also could verify $\left| f'_{\theta,q,4}(\mathbf{x}^{(\infty)}) \right|_p$ is undefined and $\left| f'_{\theta,q,4}(\mathbf{x}^{(c)}) \right|_p = 0$. The following result informs about the derivative of a point near the points $\mathbf{x}^{(\infty)}$ and $\mathbf{x}^{(c)}$ and its property.

Lemma 4.4.15. *Let $p \geq 5$ be a prime number and $\delta \equiv -(1 - \theta - q)^* \pmod{p}$.*

(A) (i) *Let $x \in \mathbb{S}_i^{(0)} \subset \mathcal{D}_0$ where integers $i > 0$. Then $\left| f'_{\theta,q,4}(x) \right|_p = \frac{p^{5i}}{|\theta - 1|_p^2}$.*

(ii) Let $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_{ij}^{(0)} \subset \mathbb{S}_i^{(0)}$ for $1 \leq j \leq p-1$. Then $|f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}})|_p = \frac{p^{5i}}{|\theta-1|_p^2} |\bar{x} - \bar{\bar{x}}|_p$.

(B) (iii) Let $x \in \mathbb{S}_i^{(1)} \subset \mathcal{D}_\delta$ where integers $i > 0$. Then $|f'_{\theta,q,4}(x)|_p = \frac{1}{p^{3i}|\theta-1|_p^2}$.

(iv) Let $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_{ij}^{(\delta)} \subset \mathbb{S}_i^{(\delta)}$ for $1 \leq j \leq p-1$. Then $|f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}})|_p = \frac{1}{p^{3i}|\theta-1|_p^2} |\bar{x} - \bar{\bar{x}}|_p$.

(C) (v) Let $x \in \mathcal{D} \setminus (\mathcal{D}_0 \cup \mathcal{D}_\delta)$. Then $|f'_{\theta,q,4}(x)|_p = \frac{1}{|\theta-1|_p^2}$.

(vi) Let $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_i$ where $0 \leq i \leq p-1$ and $i \neq 0, \delta$. Then $|f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}})|_p = \frac{1}{|\theta-1|_p^2} |\bar{x} - \bar{\bar{x}}|_p$.

Proof. For any $\bar{x}, \bar{\bar{x}} \in \mathbb{Q}_p \setminus \{\mathbf{x}^{(\infty)}\}$, we have

$$f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}}) = \frac{(\theta-1)(1-\theta-q)(\bar{\bar{x}} - \bar{x})}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} F(\bar{x}, \bar{\bar{x}}) \quad (4.56)$$

where

$$\begin{aligned} F(\bar{x}, \bar{\bar{x}}) &= 4a^3 + 6a^2(1-\theta-q) \left(\frac{\theta-1}{\bar{x} - \mathbf{x}^{(\infty)}} + \frac{\theta-1}{\bar{\bar{x}} - \mathbf{x}^{(\infty)}} \right) \\ &\quad + 4a(1-\theta-q)^2 \left(\frac{(\theta-1)^2}{(\bar{x} - \mathbf{x}^{(\infty)})^2} + \frac{(\theta-1)^2}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} + \frac{(\theta-1)^2}{(\bar{\bar{x}} - \mathbf{x}^{(\infty)})^2} \right) \\ &\quad + (1-\theta-q)^3 \left(\frac{(\theta-1)^3}{(\bar{x} - \mathbf{x}^{(\infty)})^3} + \frac{(\theta-1)^3}{(\bar{x} - \mathbf{x}^{(\infty)})^2(\bar{\bar{x}} - \mathbf{x}^{(\infty)})} + \frac{(\theta-1)^3}{(\bar{x} - \mathbf{x}^{(\infty)})(\bar{\bar{x}} - \mathbf{x}^{(\infty)})^2} + \frac{(\theta-1)^3}{(\bar{\bar{x}} - \mathbf{x}^{(\infty)})^3} \right) \end{aligned}$$

and $f'_{\theta,q,4}(x) = 4(\theta-1)(\theta-1+q) \frac{(\theta(x-\mathbf{x}^{(\infty)})+(\theta-1)(1-\theta-q))^3}{(x-\mathbf{x}^{(\infty)})^5}$

(A)-(i) Let $x \in \mathbb{S}_i^{(0)}$. Then $|\theta(x-\mathbf{x}^{(\infty)})+(\theta-1)(1-\theta-q)|_p = |\theta-1|_p^2$ and $|x-\mathbf{x}^{(\infty)}|_p = \frac{|\theta-1|_p^2}{p^i}$. Thus

$$\begin{aligned} |f'_{\theta,q,4}(x)|_p &= \left| \frac{4(\theta-1)(a+b)(a(x-\mathbf{x}^{(\infty)})+(\theta-1)(1-\theta-q))^3}{|\theta-1|_p^8 (x-\mathbf{x}^{(\infty)})^5} \right|_p \\ &= \frac{p^{5i}}{|x-\mathbf{x}^{(\infty)}|_p^5} = \frac{1}{|\theta-1|_p^2}. \end{aligned}$$

(A)-(ii) Let $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_{ij}^{(0)}$ for $1 \leq j \leq p-1$ such that $\bar{x} = \mathbf{x}^{(\infty)} + (\theta-1)\bar{y}$ and $\bar{\bar{x}} = \mathbf{x}^{(\infty)} + (\theta-1)\bar{\bar{y}}$ where $\bar{y} = \frac{p^i \bar{y}^*}{|\theta-1|_p}$ and $\bar{\bar{y}} = \frac{p^i \bar{\bar{y}}^*}{|\theta-1|_p}$. Substitue them into $F(\bar{x}, \bar{\bar{x}})$, we get

$$\begin{aligned} F(\bar{x}, \bar{\bar{x}}) &= 4\theta^3 + \frac{6\theta^2(1-\theta-q)^*}{p^i} \left(\frac{1}{\bar{y}^*} + \frac{1}{\bar{\bar{y}}^*} \right) + \frac{4\theta((1-\theta-q)^*)^2}{p^{2i}} \left(\frac{1}{(\bar{y}^*)^2} + \frac{1}{\bar{y}^* \bar{\bar{y}}^*} + \frac{1}{(\bar{\bar{y}}^*)^2} \right) \\ &\quad + \frac{((1-\theta-q)^*)^3}{p^{3i}} \left(\frac{1}{(\bar{y}^*)^2} + \frac{1}{(\bar{y}^*)^2 \bar{\bar{y}}^*} + \frac{1}{\bar{y}^* (\bar{\bar{y}}^*)^2} + \frac{1}{(\bar{\bar{y}}^*)^2} \right) \end{aligned}$$

Since $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_{ij}^{(0)}$, it implies $\bar{y}^* \equiv \bar{\bar{y}}^* \equiv w_0 \not\equiv 0 \pmod{p}$ for some integer w_0 . Then

we obtain

$$\frac{1}{(\bar{y}^*)^2} + \frac{1}{(\bar{y}^*)^2 \bar{y}^*} + \frac{1}{\bar{y}^* (\bar{y}^*)^2} + \frac{1}{(\bar{y}^*)^2} \equiv \frac{4}{w_0^3} \not\equiv 0 \pmod{p}.$$

This implies

$$\left| \frac{((1-\theta-q)^*)^3}{p^{3i}} \left(\frac{1}{(\bar{y}^*)^2} + \frac{1}{(\bar{y}^*)^2 \bar{y}^*} + \frac{1}{\bar{y}^* (\bar{y}^*)^2} + \frac{1}{(\bar{y}^*)^2} \right) \right|_p = p^{3i}.$$

Therefore, $|F(\bar{x}, \bar{\bar{x}})|_p = p^{3i}$ and $|f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}})|_p = \frac{p^{5i}}{|\theta-1|_p^2} |\bar{x} - \bar{\bar{x}}|_p$.

(B)-(iii) Let $x \in \mathbb{S}_i^{(\delta)}$ and $x = \mathbf{x}^{(\infty)} + (\theta-1)y$ such that $|y - \mathbf{y}^{(c)}|_p = \frac{|\theta-1|_p}{p^i}$.

Then we have $|x - \mathbf{x}^{(\infty)}|_p = |\theta-1|_p^2$ and $|\theta(x - \mathbf{x}^{(\infty)}) + (\theta-1)(1-\theta-q)|_p = \frac{|\theta-1|_p^2}{p^i}$.

Hence

$$\begin{aligned} |f'_{\theta,q,4}(x)|_p &= \left| \frac{4(\theta-1)(\theta-1+q)(\theta(x - \mathbf{x}^{(\infty)}) + (\theta-1)(1-\theta-q))^3}{(x - \mathbf{x}^{(\infty)})^5} \right|_p \\ &= \frac{|\theta(x - \mathbf{x}^{(\infty)}) + (\theta-1)(1-\theta-q)|_p^3}{|\theta-1|_p^8} \\ &= \frac{1}{p^{3i} |\theta-1|_p^2}. \end{aligned}$$

(B)-(iv) Let $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_{ij}^{(\delta)} \subset \mathbb{S}_i^{(\delta)}$, $1 \leq j \leq p-1$ such that $\bar{x} = \mathbf{x}^{(\infty)} + (\theta-1)\bar{y}$ and $\bar{\bar{x}} = \mathbf{x}^{(\infty)} + (\theta-1)\bar{\bar{y}}$ where $\bar{y} = \frac{(\bar{y})^*}{|\theta-1|_p}$ and $\bar{\bar{y}} = \frac{(\bar{\bar{y}})^*}{|\theta-1|_p}$. We also have for $w = w_0 + \dots + w_i p^i$ such that $(\bar{y})^* = w + p^{i+1}A$ and $(\bar{\bar{y}})^* = w + p^{i+1}B$ in which $|w - (y^{(c)})^*|_p = \frac{1}{p^i}$ and $|A + B|_p \leq 1$. Substitue them into $F(\bar{x}, \bar{\bar{x}})$, we obtain

$$\begin{aligned} F(\bar{x}, \bar{\bar{x}}) &= 4\theta^3 + 6\theta^2(1-\theta-q)^* \left(\frac{1}{(\bar{y})^*} + \frac{1}{(\bar{\bar{y}})^*} \right) \\ &\quad + 4\theta((1-\theta-q)^*)^2 \left(\frac{1}{(\bar{y}^*)^2} + \frac{1}{(\bar{y}^*)(\bar{\bar{y}}^*)} + \frac{1}{(\bar{\bar{y}}^*)^2} \right) \\ &\quad + ((1-\theta-q)^*)^3 \left(\frac{1}{(\bar{y}^*)^3} + \frac{1}{(\bar{y}^*)^2(\bar{\bar{y}}^*)} + \frac{1}{(\bar{y}^*)(\bar{\bar{y}}^*)^2} + \frac{1}{(\bar{\bar{y}}^*)^3} \right). \end{aligned}$$

This yields

$$(\bar{y}^*)^3 (\bar{\bar{y}}^*)^2 F(\bar{x}, \bar{\bar{x}}) = 4w^3 (\theta w + (1-\theta-q)^*)^3 + p^{3i+1} \bar{X}$$

where $|\bar{X}|_p \leq 1$. We know $|\theta w + (1-\theta-q)^*|_p = \left| \theta \left(w - (y^{(c)})^* \right) \right|_p = \left| w - (y^{(c)})^* \right|_p$.

Thus $|F(\bar{x}, \bar{\bar{x}})|_p = \frac{1}{p^{3i}}$ and it implies $|f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}})|_p = \frac{|\bar{x} - \bar{\bar{x}}|_p}{p^{3i} |\theta-1|_p^2}$.

(C)-(v) Let $x \in \mathcal{D} \setminus (\mathcal{D}_0 \cup \mathcal{D}_\delta)$ where $x = \mathbf{x}^{(\infty)} + (\theta - 1)y$ such that $|y|_p = |y - \mathbf{y}^{(c)}|_p = |\theta - 1|_p$. We have $|\theta(x - \mathbf{x}^{(\infty)}) + (\theta - 1)(1 - \theta - q)|_p = \left| \theta(\theta - 1) \left(\frac{y - \mathbf{y}^{(c)}}{\theta} \right) \right|_p = |\theta - 1|_p^2$ and $|x - \mathbf{x}^{(\infty)}|_p = |\theta - 1|_p^2$. By the equality (4.49), we obtain $\left| f'_{\theta,q,4}(x) \right|_p = \frac{1}{|\theta - 1|_p^2}$.

(C)-(vi) Suppose $\bar{x}, \bar{\bar{x}} \in \mathcal{D}_i$ for $0 \leq i \leq p - 1$ and $i \neq 0, \delta$ with $\bar{x} = \mathbf{x}^{(\infty)} + (\theta - 1)\bar{y}$ and $\bar{\bar{x}} = \mathbf{x}^{(\infty)} + (\theta - 1)\bar{\bar{y}}$ such that $\bar{y} = \frac{\bar{y}^*}{|\theta - 1|_p}$ and $\bar{\bar{y}} = \frac{\bar{\bar{y}}^*}{|\theta - 1|_p}$. We have $\bar{y}^* = \bar{\bar{y}}^* \equiv w_0^{(i)} \pmod{p}$. Then we obtain

$$\left(w_0^{(i)} \right)^3 F(\bar{x}, \bar{\bar{x}}) \equiv 4 \left(\left(w_0^{(i)} \right) + (1 - \theta - q)^* \right)^3 \not\equiv 0 \pmod{p}.$$

This implies $|F(\bar{x}, \bar{\bar{x}})|_p = 1$ because $w_0^{(i)} \not\equiv (\mathbf{y}^{(c)})^* \equiv -(1 - \theta - q)^* \pmod{p}$. Therefore, $|f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{\bar{x}})|_p = \frac{|\bar{x} - \bar{\bar{x}}|_p}{|\theta - 1|_p^2}$. \square

One could easily verify $\mathcal{E} = \mathbb{Q}_p \setminus \mathbb{B}_1(\mathbf{x}^{(\infty)})$ and $\mathbb{Z}_p^* = \bigcup_{j=1}^{p-1} \mathbb{B}_1(j)$. Let

$$(\mathbb{Z}_p^*)^{\frac{1}{4}} = \{x \in \mathbb{Z}_p^* : x_0 \text{ is a quartic residue modulo } p\}.$$

The congruence

$$x^4 \equiv b \pmod{p} \tag{4.57}$$

has solution(s) if and only if $b^{\frac{p-1}{\gcd(p-1,4)}} \equiv 1 \pmod{p}$, i.e. b is a quartic residue modulo p . Moreover, if b is a quartic residue modulo p then the congruence (4.57) has $\gcd(p - 1, 4)$ solutions.

We remind that the fixed points $\mathbf{x}^{(3)}$ and $\mathbf{x}^{(4)}$ exist when $p \equiv 1 \pmod{4}$. Let α, β and γ be the solutions of the equation (4.39). Then we suppose $\mathbf{x}^{(2)} \in \mathcal{D}_\alpha$, $\mathbf{x}^{(3)} \in \mathcal{D}_\beta$ and $\mathbf{x}^{(4)} \in \mathcal{D}_\gamma$. Let we calculate the image of the set \mathcal{D}

Proposition 4.4.16. *Let $p \geq 5$ be a prime number. Let $\delta \equiv -(1 - \theta - q)^* \pmod{p}$ and α, β and γ be the solutions of the congruence (4.39).*

- (i) For $i \geq 1$, $\mathbb{S}_i^{(0)} \subset \mathcal{D}_0$ as in (4.54), we have $f_{\theta,q,4}(\mathbb{S}_i^{(0)}) = p^{-4i} (\mathbb{Z}_p^*)^{\frac{1}{4}}$;
- (ii) For $i \geq 1$, $\mathbb{S}_i^{(\delta)} \subset \mathcal{D}_1$ as in (4.55), we have $f_{\theta,q,4}(\mathbb{S}_i^{(\delta)}) = p^{4i} (\mathbb{Z}_p^*)^{\frac{1}{4}}$;
- (iii) (a) If $p \equiv 1 \pmod{4}$ then for $i = \alpha, \beta, \gamma$ we have $f_{\theta,q,4}(\mathcal{D}_i) = \mathbb{B}_1(\mathbf{x}^{(\infty)})$;
 (b) If $p \equiv 3 \pmod{4}$ then $f_{\theta,q,4}(\mathcal{D}_\alpha) = \mathbb{B}_1(\mathbf{x}^{(\infty)})$;
- (iv) (a) If $p \equiv 1 \pmod{4}$ then $f_{\theta,q,4} \left(\bigcup_{\substack{i=1 \\ i \neq \alpha, \beta, \gamma, \delta}}^{p-1} \mathcal{D}_i \right) = (\mathbb{Z}_p^*)^{\frac{1}{4}} \setminus \mathbb{B}_1(\mathbf{x}^{(\infty)})$;
 (b) If $p \equiv 3 \pmod{4}$ $f_{\theta,q,4} \left(\bigcup_{\substack{i=1 \\ i \neq \alpha, \delta}}^{p-1} \mathcal{D}_i \right) = (\mathbb{Z}_p^*)^{\frac{1}{4}} \setminus \mathbb{B}_1(\mathbf{x}^{(\infty)})$.

Proof. Recall

$$f_{\theta,q,4}(x) = \left(\theta + \frac{(\theta-1)(1-\theta-q)}{x - \mathbf{x}^{(\infty)}} \right)^4. \quad (4.58)$$

Let $x = \mathbf{x}^{(\infty)} + (\theta-1)y \in \mathcal{D} \setminus \mathcal{D}_0$. Then, for $y = \frac{y^*}{|\theta-1|_p}$, the expression $g(x)$ in the equality (4.43) can be written as follows

$$\begin{aligned} g(x) &= (\theta x + b)^3 + (\theta x + q - 1)^2(x - \mathbf{x}^{(\infty)}) + (\theta x + q - 1)(x - \mathbf{x}^{(\infty)})^2 + (x - \mathbf{x}^{(\infty)})^3 \\ &= (\theta-1)^2((\theta^3 + \theta^2 + \theta + 1)y^3 + (3\theta^2 + 2\theta + 1)(1-\theta-q)y^2 \\ &\quad + (3\theta + 1)(1-\theta-q)^2y + (1-\theta-q)^3) \\ &= \frac{(\theta-1)^2}{|\theta-1|_p^2} g_2(x). \end{aligned}$$

in which

$$\begin{aligned} g_2(x) &= (\theta^3 + \theta^2 + \theta + 1)(y^*)^3 + (3\theta^2 + 2\theta + 1)(1-\theta-q)^*(y^*)^2 \\ &\quad + (3\theta + 1)((1-\theta-q)^*)^2 y^* + ((1-\theta-q)^*)^3 \\ &\equiv 4(y^*)^3 + 6(1-\theta-q)^*(y^*)^2 + 4((1-\theta-q)^*)^2 y^* \\ &\quad + ((1-\theta-q)^*)^3 \pmod{p}. \end{aligned} \quad (4.59)$$

(i) Consider $\mathcal{D}_0 = \bigcup_{i=1}^{\infty} \mathbb{S}_i^{(0)}$ and $\mathbb{S}_i^{(0)} = \bigcup_{j=1}^{p-1} \mathcal{D}_{ij}^{(0)}$. We will calculate the image of each spheres $\mathbb{S}_i^{(0)}$ for integer $i \geq 1$. Suppose $x \in \mathbb{S}_i^{(0)}$. Then $x - \mathbf{x}^{(\infty)} = (\theta-1)y$ with $y = \frac{p^i y^*}{|\theta-1|_p}$. Substitute into the equality (4.58), we obtain

$$f_{\theta,q,4}(x) = \left(\theta + \frac{(1-\theta-q)^*}{p^i y^*} \right)^4.$$

This means $|f_{\theta,q,4}(x)|_p = \left| \theta + \frac{(1-\theta-q)^*}{p^i y^*} \right|_p^4 = p^{4i} > 1$ and $f_{\theta,q,4}(x) = \frac{1}{p^{4i}} f_{\theta,q,4}^*(x)$ such that $|f_{\theta,q,4}^*(x)|_p = 1$ and the first digit of $f_{\theta,q,4}^*(x)$ is a quartic residue modulo p . So $f_{\theta,q,4}(x) \in p^{-4i} (\mathbb{Z}_p^*)^{\frac{1}{4}}$. Next we calculate the image of each balls $\mathcal{D}_{ij}^{(0)} \subset \mathbb{S}_i^{(0)}$ for $1 \leq j \leq p-1$. Let $\hat{x}_j \in \mathcal{D}_{ij}^{(0)}$. Then by Lemma 4.4.15(A)-(ii) and Corollary 2.3.3, we get

$$f_{\theta,q,4}(\mathcal{D}_{ij}^{(0)}) = \mathbb{B}_{p^{4i}}(f_{\theta,q,4}(\hat{x}_j)) = p^{-4i} \mathbb{B}_1(f_{\theta,q,4}^*(\hat{x}_j))$$

for $1 \leq j \leq p-1$ and $f_{\theta,q,4}^*(\hat{x}_j) \pmod{p}$ is a quartic residue modulo p . Thus $f_{\theta,q,4}(\mathbb{S}_i^{(0)}) = p^{-4i} (\mathbb{Z}_p^*)^{\frac{1}{4}}$ and we have the statement (i).

(ii) Let $\delta \equiv (\theta-1+q)^* \pmod{p}$. Consider $\mathcal{D}_\delta = \mathbf{x}^{(c)} \cup \bigcup_{i=1}^{\infty} \mathbb{S}_i^{(\delta)}$ and $\mathbb{S}_i^{(\delta)} = \bigcup_{j=1}^{p-1} \mathcal{D}_{ij}^{(\delta)}$. Let $x = \mathbf{x}^{(\infty)} + (\theta-1)y \in \mathbb{S}_i^{(\delta)}$. Then $|y - \mathbf{y}^{(c)}|_p = \frac{|\theta-1|_p}{p^i}$ and $|y^* - (y^{(c)})^*|_p =$

$\frac{1}{p^i}$. Substitute into the equality (4.58), we get

$$f_{\theta,q,4}(x) = \left(\frac{\theta}{y^*} \left(y^* - (y^{(c)})^* \right) \right)^4.$$

So $|f_{\theta,q,4}(x)|_p = \left| y - (y^{(c)})^* \right|_p^4 = \frac{1}{p^{4i}}$ and $f_{\theta,q,4}(x) = p^{4i} f_{\theta,q,4}^*(x)$ where $\left| f_{\theta,q,4}^*(x) \right|_p = 1$ and the first digit of $f_{\theta,q,4}^*(x)$ is a quartic residue modulo p . It means $f_{\theta,q,4}(x) \in p^{4i} (\mathbb{Z}_p^*)^4$. Next we calculate the image of each balls $\mathcal{D}_{ij}^{(\delta)}$ for $1 \leq j \leq p-1$. Let $\check{x}_j \in \mathcal{D}_{ij}^{(\delta)}$. Then by Lemma 4.4.15(B)-(iv) and Corollary 2.3.3, we have

$$f_{\theta,q,4}(\mathcal{D}_{ij}^{(\delta)}) = \mathbb{B}_{p^{-4i}}(f_{\theta,q,4}(\check{x}_j)) = p^{4i} \mathbb{B}_1(f_{\theta,q,4}^*(\check{x}_j))$$

for $1 \leq j \leq p-1$ and $f_{\theta,q,4}^*(\check{x}_j) \pmod{p}$ is a quartic residue modulo p . Hence $f_{\theta,q,4}(\mathbb{S}_i^{(\delta)}) = p^{4i} (\mathbb{Z}_p^*)^4$ and statement (ii) holds true.

(iii)-(a) Suppose $p \equiv 1 \pmod{4}$. Let $x \in \mathcal{D}_i$ for $i = 2, 3, 4$. We have $y^* \equiv y_0^{(i)} \pmod{p}$. We know $y_0^{(2)}, y_0^{(3)}$ and $y_0^{(4)}$ are the only solutions of (4.59). So $|g(x)|_p < |\theta - 1|_p^4$ and by the equality (4.43), $|f_{\theta,q,4}(x) - \mathbf{x}^{(\infty)}|_p < 1$. Therefore, $f_{\theta,q,4}(x) \in \mathbb{B}_1(1) = \mathbb{B}_1(\mathbf{x}^{(\infty)})$. By Lemmas 4.4.15(C)-(vi) and 2.3.3, we have for $i = 2, 3, 4$ that $f_{\theta,q,4}(\mathcal{D}_i) = \mathbb{B}_1(\mathbf{x}^{(\infty)})$.

(iii)-(b) The proof is similar to (iii)-(a).

(iv)-(a) Suppose $p \equiv 1 \pmod{4}$. Let $x = \mathbf{x}^{(\infty)} + (\theta - 1)y \in \bigcup_{\substack{i=1 \\ i \neq \alpha, \beta, \gamma, \delta}}^{p-1} \mathcal{D}_i$. Then

$|y - \mathbf{y}^{(c)}|_p = |\theta - 1|_p$ and $\left| y^* - (y^{(c)})^* \right|_p = 1$. Substitute into the equality (4.58), we get

$$f_{\theta,q,4}(x) = \left(\frac{\theta}{y^*} \left(y^* - (y^{(c)})^* \right) \right)^4.$$

This means $|f_{\theta,q,4}(x)|_p = \left| y^* - (y^{(c)})^* \right|_p^4 = 1$ and its first digit is a quartic residue modulo p . This implies $f_{\theta,q,4}(x) \in (\mathbb{Z}_p^*)^{\frac{1}{4}}$. Now we will show that $|f_{\theta,q,4}(x) - \mathbf{x}^{(\infty)}|_p = 1$. We know $y^* \not\equiv y_0^{(i)}, i = 2, 3, 4$. So $g_2(x) \not\equiv 0 \pmod{p}$. This means $|g(x)|_p = |\theta - 1|_p^4$ and by equality (4.43) that $|f_{\theta,q,4}(x) - \mathbf{x}^{(\infty)}|_p = 1$. Thus $f_{\theta,q,4}(x) \in (\mathbb{Z}_p^*)^{\frac{1}{4}} \setminus \mathbb{B}_1(1) = (\mathbb{Z}_p^*)^{\frac{1}{4}} \setminus \mathbb{B}_1(\mathbf{x}^{(\infty)})$. Next we calculate the image of each balls \mathcal{D}_i where $i = 1, \dots, p-1$ and $i \neq \alpha, \beta, \gamma, \delta$. Let $\tilde{x} \in \mathcal{D}_i$. Then by Lemmas 4.4.15(C)-(vi) and 2.3.3, we get

$$f_{\theta,q,4}(\mathcal{D}_j) = \mathbb{B}_1(f(\tilde{x}_j))$$

and $f_{\theta,q,4}(\tilde{x}_j) \pmod{p}$ is a quartic residue modulo p . Thus $f_{\theta,q,4}\left(\bigcup_{i=5}^{p-1} \mathcal{D}_i\right) = (\mathbb{Z}_p^*)^{\frac{1}{4}} \setminus \mathbb{B}_1(\mathbf{x}^{(\infty)})$.

(iv)-(b) The proof is the same as (iv)-(a). □

Then we obtain the following result on the dynamical behaviour of the Potts–Bethe mapping (4.35) on \mathbb{Q}_p .

Theorem 4.4.17. *Let $p \geq 5$ be a prime number and α, β and γ be the solutions of the congruence (4.39).*

- (i) *Then \mathcal{C}_1 and \mathcal{C}_4 are Siegel disks of the fixed points $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ respectively.*
- (ii) *Suppose $p \equiv 1 \pmod{4}$. Let $\mathbb{X} = \mathcal{D}_\alpha \cup \mathcal{D}_\beta \cup \mathcal{D}_\gamma$. Then there is a subsystem $(\mathcal{J}, f_{\theta,q,4}, |\cdot|_p)$ that is isometrically conjugate to the full shift $(\Sigma_3, \sigma, d_{f_{\theta,q,4}})$ on three symbols.*

Proof. (i) We have that the fixed points $\mathbf{x}^{(0)} \in \mathcal{C}_1$ and $\mathbf{x}^{(1)} \in \mathcal{C}_4$. By Corollary 4.4.13, we get $f_{\theta,q,4}(\mathcal{C}_1) = \mathcal{C}_1$ and $f_{\theta,q,4}(\mathcal{C}_4) = \mathcal{C}_4$. We also have by Lemma 4.4.12 that for any $x, y \in \mathcal{C}_1$ or \mathcal{C}_4 that $|f_{\theta,q,4}(\bar{x}) - f_{\theta,q,4}(\bar{y})|_p = |\bar{x} - \bar{y}|_p$. It implies that each spheres with center $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ in \mathcal{C}_1 and \mathcal{C}_4 respectively are invariant under $f_{\theta,q,4}$. Thus we have the statement (i).

(ii) Let $p \equiv 1 \pmod{4}$. By Propositions 4.4.13, 4.4.14 and 4.4.16, we have $f_{\theta,q,4}^{-1}(\mathbb{X}) \subset \mathcal{C}_\ell \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$ where $m(\ell) = 5 - \frac{4}{\ell} \equiv 0 \pmod{p}$. We also have for any $x \in \mathcal{D}_2$ or \mathcal{D}_3 or \mathcal{D}_4 that $f_{\theta,q,4}^{-1}(x) := \{x_1 \in \mathcal{C}_\ell, x_2 \in \mathcal{D}_2, x_3 \in \mathcal{D}_3, x_4 \in \mathcal{D}_4\}$. Let $\bar{f}_{\theta,q,4}^{-1}(x) := \{x_1 \in \mathcal{D}_2, x_2 \in \mathcal{D}_3, x_4 \in \mathcal{D}_4\}$ for $x \in \mathcal{D}_2$ or \mathcal{D}_3 or \mathcal{D}_4 . Then $\bar{f}_{\theta,q,4}^{-1}(\mathbb{X}) \subset \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$. We consider the following (Julia) set

$$\mathcal{J} = \bigcap_{n=0}^{\infty} \bar{f}_{\theta,q,4}^{-n}(\mathbb{X}) \subset \mathbb{X}.$$

Then we have the incidence matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

By Theorem 2.3.13, the dynamical system $(\mathcal{J}, f_{\theta,q,4}, |\cdot|_p)$ is isometrically conjugate to the full shift dynamics $(\Sigma_3, \sigma, d_{f_{\theta,q,4}})$. \square

Claim III

- (i) Let $p \equiv 1 \pmod{4}$. Then the dynamics of the Potts–Bethe mapping (4.35) consists of a subsystem that is topologically conjugate to the full shift on three symbols, minimal subsystems (including the attracting orbits and the Siegel disks of periodic orbits) and their basins.
- (ii) Let $p \equiv 3 \pmod{4}$. Then the dynamics of the Potts–Bethe mapping (4.35) consists of a minimal subsystems (including the attracting orbits and the Siegel disks of periodic orbits) and their basins.

4.5 BRIEF SUMMARY OF THE DYNAMICAL BEHAVIOUR

In this section, we will discuss and give a brief summary for Section 4.2–4.4.

First we have provided the condition when the Potts–Bethe mapping (4.1) having a good reduction, Proposition 4.2.1. By these condition, one could check that the Potts–Bethe mapping derive from the p -adic Potts model on Cayley trees has a bad reduction. We have investigated the dynamics of the Potts–Bethe mapping of degree 3 and degree 4.

Let us discuss first the dynamics of the Potts–Bethe mapping of degree 3. We have considered two cases: $0 < |\theta - 1|_p < |q|_p < 1$ and $0 < |q|_p < |\theta - 1|_p < 1$. We always suppose prime number $p \geq 5$ otherwise stated.

CASE $0 < |\theta - 1|_p < |q|_p < 1$. Let $p \equiv 5 \pmod{6}$. The Potts–Bethe mapping (4.9) has two fixed points, one of them is attracting and another is repelling. Moreover, a trajectory of the Potts–Bethe mapping starting from any eventually non-singular point converges to the attracting point, Theorem 4.3.8. Let $p \equiv 1 \pmod{6}$. Then dynamics on the Julia set \mathcal{J}_m , $m = 0, 1, 2, \dots$ is topologically conjugate to the full shift on three symbols, Theorems 4.3.13 and 4.3.19. We also found the attracting basin of the fixed point $\mathbf{x}^{(0)}$. This enable us to make the following partition:

$$\mathbb{Q}_p = \mathfrak{B}(\mathbf{x}^{(0)}) \cup \mathcal{J}_m \cup \bigcup_{n=0}^{\infty} f_{\theta,q,3}^{-n} \{\mathbf{x}^{(\infty)}\}.$$

CASE $0 < |q|_p < |\theta - 1|_p < 1$. We found the Siegel disks of the neutral fixed points, Theorem 4.3.27. Let $p \equiv 1 \pmod{6}$. Then dynamics on the Julia set is topologically conjugate to the full shift dynamics on two symbols, Theorem 4.3.29. By the examples provided, one can expect there might exist a periodic orbit. Also by these, one can see that the critical points are not wild. But we could not prove whether the critical point is recurrent or not. There are interesting component of Fatou set called "Wandering Domain" as mentioned in Theorem 2.3.10. The Fatou set of the Potts–Bethe mapping has wandering domains if one could show that there exist a wild recurrent critical point. We can use the result in Theorem 2.3.11 in order to find this kind of critical point.

By similar approach, one can show that the case $0 < |\theta - 1 + q|_p < |\theta - 1|_p = |q|_p < 1$ and the case $0 < |\theta - 1 + q|_p = |\theta - 1|_p = |q|_p < 1$ are the same to the case $0 < |\theta - 1|_p < |q|_p < 1$ and the case $0 < |q|_p < |\theta - 1|_p < 1$ respectively. We have a similar conclusion for the Potts–Bethe mapping of degree 4 as the Potts–Bethe mapping of degree 3. Note that our we use only basic calculation here. So the techniques here can be applied to the investigate the dynamics of the Potts–Bethe mapping of any degree $k \in \{1, 2, 3, \dots\}$ over \mathbb{Q}_p .

4.6 CHAOS, PHASE TRANSITION AND THE SET OF P -ADIC GIBBS MEASURES

Now we would like to show the vastness of the set of p -adic Gibbs measures. Again, here $|\alpha|$ means the cardinality of the set α . We are looking for solutions of the form $\mathbf{z}^{(j)} = z^{(j)}\mathbf{e}_\alpha + \mathbf{e}_\beta$ with $|\alpha| + |\beta| = q - 1$ of the system of equations (2.15). In this case, the system of equations (2.15) with $z^{(n)} := z^{(0)}$ takes the following form,

$$z^{(j)} = \left(\frac{(\theta + |\alpha| - 1)z^{(j+1)} + q - |\alpha|}{|\alpha|z^{(j+1)} + \theta + q - |\alpha| - 1} \right)^k, \quad 0 \leq j \leq n-1. \quad (4.60)$$

This means that $\{z^{(j)}\}_{j=0}^{n-1}$ is a periodic orbit of length m of the following Potts–Bethe mapping

$$G_{\theta, \alpha, q, k}(z) = \left(\frac{(\theta + |\alpha| - 1)z + q - |\alpha|}{|\alpha|z + \theta + q - |\alpha| - 1} \right)^k. \quad (4.61)$$

We suppose $0 < |\theta - 1|_p, |q|_p < 1$ and $||\alpha||_p = 1$. This is one of the assumptions to ensure the non-uniqueness of the translation-invariant p -adic Gibbs measures (refer Theorem 3.2.3). We may assume that $1 \leq |\alpha| \leq q - 1$ otherwise if $|\alpha| = 0$ then we obtain a trivial solution $\mathbf{z}^{(j)} = (1, 1, \dots, 1)$ for all $0 \leq j \leq m - 1$ of the system of equations (2.15). Thus, we have

$$G_{\theta, \alpha, q, k}(z) = \left(\frac{\left(\frac{\theta-1}{|\alpha|} + 1 \right) z + \frac{q}{|\alpha|} - 1}{z + \frac{\theta-1+q}{|\alpha|} - 1} \right)^k.$$

If we set $\Theta = \frac{\theta-1}{|\alpha|} + 1$ and $Q = \frac{q}{|\alpha|}$, then we obtain

$$f_{\Theta, Q, k}(z) = \left(\frac{\Theta z + Q - 1}{z + \Theta + Q - 2} \right)^k.$$

Theorem 4.6.1. *There exist H_m -periodic p -adic Gibbs measures associated with the H_m -periodic function $\{\tilde{\mathbf{h}}^{(j)}\}_{j=0}^{m-1}$ where*

$$\tilde{\mathbf{h}}^{(j)} = (\ln_p(hz_1^{(j)}), \dots, \ln_p(hz_{q-1}^{(j)}), \ln_p(h))$$

with $h \in \mathcal{E}_p$ and $\mathbf{z}^{(j)} = (z_1^{(j)}, \dots, z_{q-1}^{(j)}) \in \mathcal{E}_p^{q-1}$ such that $\mathbf{z}^{(j)} = z^{(j)}\mathbf{e}_\alpha + \mathbf{e}_\beta$, $|\alpha| + |\beta| = q - 1$ and $\{z^{(j)}\}_{j=0}^{m-1}$ is a periodic orbit of length m of the Potts–Bethe mapping

$$G_{\theta, \alpha, q, 3}(z) = \left(\frac{(\theta + |\alpha| - 1)z + q - |\alpha|}{|\alpha|z + \theta + q - |\alpha| - 1} \right)^k. \quad (4.62)$$

Proof. Due to Proposition 2.5.6, one could check that there exists an H_m -periodic p -

adic Gibbs measure associated with an H_m -periodic function $\left\{\tilde{\mathbf{h}}^{(j)}\right\}_{j=0}^{m-1}$

$$\tilde{\mathbf{h}}^{(j)} = \left(\ln_p(hz_1^{(j)}), \dots, \ln_p(hz_{q-1}^{(j)}), \ln_p(h)\right)$$

with $h \in \mathcal{E}_p$, $\mathbf{z}^{(j)} = (z_1^{(j)}, \dots, z_{q-1}^{(j)}) \in \mathcal{E}_p^{q-1}$ of the form $\mathbf{z}^{(j)} = z^{(j)}\mathbf{e}_\alpha + \mathbf{e}_\beta$, $|\alpha| + |\beta| = q - 1$ if and only if $\{z^{(j)}\}_{j=0}^{m-1} \subset \mathcal{E}_p$ is a periodic orbit of length m of the Potts–Bethe mapping (4.62). \square

Theorem 4.6.2. *Let prime numbers $p \geq 5$.*

A. Let $k = 3$. Suppose one of the following condition holds

(i) $0 < |\theta - 1|_p < |q|_p < 1$ with $p \equiv 1 \pmod{6}$;

(ii) $0 < |q|_p < |\theta - 1|_p < 1$ with $p \equiv 1 \pmod{6}$;

B. Let $k = 4$. Suppose one of the following condition holds

(i) $0 < |\theta - 1|_p < |q|_p < 1$;

(ii) $0 < |q|_p < |\theta - 1|_p < 1$ with $p \equiv 1 \pmod{4}$.

Then the set of p -adic Gibbs measures \mathcal{G}_p is at least countable.

Proof. The Potts–Bethe mapping (4.62) can be written in the following form

$$f_{\Theta, \mathbf{Q}, 3}(z) = \left(\frac{\Theta z + \mathbf{Q} - 1}{z + \Theta + \mathbf{Q} - 2} \right)^k \quad (4.63)$$

where $\Theta := \frac{\theta-1}{|\alpha|} + 1$ and $\mathbf{Q} := \frac{q}{|\alpha|}$ whenever $|\alpha| \neq 0$. Due to Theorem 4.3.13, 4.3.19, 4.3.29, 4.4.9 and 4.4.17 and $|\alpha| \not\equiv 0 \pmod{p}$, then the Potts–Bethe mapping (4.63) of degree 3 and degree 4 have a subsystem that topologically conjugate to the full shift dynamics. It is well known that the full shift dynamics has periodic orbits of any period m . Thus the Potts–Bethe mapping (4.63) has a periodic orbit of any length m . Moreover, it can be easily verified $\{z^{(j)}\}_{j=0}^{m-1} \subset \mathcal{E}_p$. So there exist many H_m -periodic p -adic Gibbs measures of the Potts model on the Cayley tree of order 3 and order 4 for many prime numbers p . \square

This theorem tells us that the chaotic property of the Potts–Bethe mapping implies the vastness of the set of the p -adic Gibbs measures. Now we would like to compare the chaotic property of the Potts–Bethe mapping and the phase transition of the Potts model over \mathbb{Q}_p on the Cayley trees. We are following the calculation by Mukhamedov & Akin (2013). We shall investigate the measures corresponding to the solutions of equations (4.4) or (4.36). Let $\mathbf{x}^{(i)} = \exp_p(h^{(i)})$ be related the solution of

equations (4.4) or (4.36) and $\mu^{(i)}$ be the corresponding p -adic Gibbs measures. The partition function $\mathcal{Z}_{h^{(i)}}^{(n)}$ corresponding to the measure $\mu^{(i)}$ has the form $\mathcal{Z}_{h^{(i)}}^{(n)} = a^{(i)|V_{n-1}|}$ where $a^{(i)} = (\mathbf{x}^{(i)} - \mathbf{x}^{(\infty)})^k$. For a given configuration $\sigma \in \Omega_{V_n}$, we get by (2.9) that

$$\left| \mu^{(i)}(\sigma) \right|_p = \frac{\left| \exp_p \left\{ H_n(\sigma) + \sum_{x \in W_n} h^{(i)}_x(\sigma(x)) \right\} \right|_p}{\left| (\mathbf{x}^{(i)} - \mathbf{x}^{(\infty)})^{k|V_{n-1}|} \right|_p} = \frac{1}{\left| (\mathbf{x}^{(i)} - \mathbf{x}^{(\infty)})^{k|V_{n-1}|} \right|_p}. \quad (4.64)$$

For $k = 3$ and $k = 4$, one has $\left| \mathbf{x}^{(i)} - \mathbf{x}^{(\infty)} \right|_p \leq \frac{1}{p}$ whenever $|\theta - 1|_p, |q|_p < 1$. This means

$$\left| \mu^{(i)}(\sigma) \right|_p \geq p^{k|V_{n-1}|} \text{ for } n \in \mathbb{N}.$$

Thus $\mu^{(i)}$ is unbounded provided $\mathbf{x}^{(i)}$ exists. Let us give some examples for the occurrence of phase transition of the Potts model over \mathbb{Q}_p on Cayley tree.

EXAMPLE 3:

- (i) Let $p \equiv 5 \pmod{6}$ and $k = 3$. Suppose $0 < |\theta - 1|_p < |q|_p < 1$. Then the translation-invariant p -adic Gibbs measures $\mu^{(0)}$ and $\mu^{(1)}$ exist.
- (ii) Let $p \equiv 1 \pmod{6}$ and $k = 3$. Suppose $0 < |\theta - 1|_p < |q|_p < 1$. Then the translation-invariant p -adic Gibbs measures $\mu^{(0)}$, $\mu^{(1)}$, $\mu^{(2)}$ and $\mu^{(3)}$ exist.

In both cases, all the p -adic Gibbs measures are unbounded. Hence, the phase transition occurs (refer to Definition 2.5.7).

Noted that the translation-invariant p -adic Gibbs measure $\mu^{(i)}$ is associated to the fixed point $x^{(i)}$ of the Potts–Bethe mapping (4.1). In Example 3-(i), the Potts–Bethe mapping is regular (all non-eventually singular point converge to a unique fixed point). Meanwhile, in Example 3-(ii) the Potts–Bethe mapping is chaotic. By these observation, we infer the following implications

$$\text{Chaos (of Potts–Bethe mapping)} \implies \text{Phase Transition}$$

and

$$\text{Phase Transition} \not\Rightarrow \text{Chaos (of Potts–Bethe mapping)}.$$

CHAPTER 5

CONCLUSION

5.1 SUMMARY OF RESEARCH

In summary, we showed that the chaotic property of the Potts–Bethe mapping implies the existence of the phase transition of the Potts model on the Cayley trees. We have studied the Potts model over \mathbb{Q}_p on the Cayley trees of order 3 and order 4. Now we briefly summarize the results obtained from this thesis.

In Chapter 3, we began by stating the result on the existence of the translation-invariant p -adic Gibbs measures of the Potts model on the Cayley tree of order 3. It was related to the existence of roots of some cubic equations over \mathbb{Q}_p . Next, we showed the existence of the translation-invariant p -adic Gibbs measures of the Potts model on the Cayley tree of order 4 associated to some function $\tilde{\mathbf{h}} : \Gamma_k^+ \rightarrow \mathbb{Q}_p^q$. To show the existence of these translation-invariant p -adic Gibbs measures, we solved the following system of equations

$$z_i = \left(\frac{(\theta - 1)z_i + \sum_{\ell=1}^{q-1} z_\ell + 1}{\theta + \sum_{\ell=1}^{q-1} z_\ell} \right)^4 \quad \text{for } 1 \leq i \leq q-1$$

where $\theta = \exp_p(J)$ and J is a coupling constant such that $z_i = \exp_p(h_i)$. We searched the solution of this system of equations of the form $\mathbf{z} = (z_1, \dots, z_{q-1})$ such that $z_i = 1$ or $z_i = z^* \in \mathcal{E}_p \subset \mathbb{Q}_p$ for $i = \overline{1, q-1}$ with z^* being the fixed points, i.e. $z^* = f(z^*)$ of the following map f

$$f(z) = \left(\frac{(\theta - 1 + m)z + q - m}{mz + \theta - 1 + q - m} \right)^4$$

called “Potts–Bethe mapping”. To find the fixed points of f other than 1, we described some quartic equation having a solution $z \in \mathcal{E}_p$ (Propositions 3.2.1 and 3.2.2). By these descriptions, we proved the non-uniqueness of the translation-invariant p -adic Gibbs measures (Theorem 3.2.3). This has motivated us to provide the solvability conditions of the depressed quartic equation

$$x^4 + ax^2 + bx + c = 0$$

with $a, b, c \in \mathbb{Q}_p$ over \mathbb{Q}_p (Theorem 3.3.8).

In Chapter 4, we studied the dynamical behaviour of the Potts–Bethe mapping over \mathbb{Q}_p . We began by describing the Potts–Bethe mapping having good reduction

(Proposition 4.2.1) and showed that its dynamics is decomposed into minimal subsystems and their attracting basins (Theorem 4.2.2). Then we studied separately the dynamics of the Potts–Bethe mapping of degree 3 and of degree 4 which has bad reduction. We recalled that this Potts–Bethe mapping is obtained by studying the p -adic Gibbs measures of the Potts model on the Cayley tree. We considered the following Potts–Bethe mapping of degree 3

$$f_{\theta,q,3}(x) = \left(\frac{\theta x + q - 1}{x + \theta + q - 2} \right)^3$$

and distinguished two cases: $0 < |\theta - 1|_p < |q|_p < 1$ and $0 < |q|_p < |\theta - 1|_p < 1$.

Case $0 < |\theta - 1|_p < |q|_p < 1$: When $p \equiv 5 \pmod{6}$, we found the attracting basin $\mathfrak{B}(\mathbf{x}^{(0)})$ of the fixed point $\mathbf{x}^{(0)} = 1$ (Theorem 4.3.8). When $p \equiv 1 \pmod{6}$, the Julia set of the Potts–Bethe mapping is non-empty. Then we divided into two subcases: $0 < |\theta - 1|_p < |q|_p^2 < 1$ and $0 < |q|_p^2 \leq |\theta - 1|_p < |q|_p < 1$. For $0 < |\theta - 1|_p < |q|_p^2 < 1$, there exists a subsystem $(\mathcal{J}, f_{\theta,q,3})$ that is isometrically conjugate to the full shift dynamics on 3 symbols (Theorem 4.3.13). For $0 < |q|_p^2 \leq |\theta - 1|_p < |q|_p < 1$, there exists a subsystem $(\mathcal{J}, f_{\theta,q,3})$ that is isometrically conjugate to a subshift of finite type on r symbols where $r \geq 4$. However, these subshifts on r symbols are all topologically conjugate to the full shift on 3 symbols (Theorem 4.3.19).

Case $0 < |q|_p < |\theta - 1|_p < 1$: In this case we found the Siegel disks of the neutral fixed points $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ (Theorem 4.3.27). When $p \equiv 5 \pmod{6}$, in certain case, we found the existence of the attracting periodic orbits or Siegel disk of periodic orbits with their basins. When $p \equiv 1 \pmod{6}$, the Julia set of the Potts–Bethe mapping is non-empty. There exists a subsystem $(\mathcal{J}, f_{\theta,q,3})$ that is isometrically conjugate to the full shift dynamics on 2 symbols (Theorem 4.3.29). It is remarked that the case $0 < |\theta - 1 + q|_p < |\theta - 1|_p = |q|_p < 1$ and the case $0 < |\theta - 1 + q|_p = |\theta - 1|_p = |q|_p < 1$ has similar behaviour to the case $0 < |\theta - 1|_p < |q|_p < 1$ and the case $0 < |q|_p < |\theta - 1|_p < 1$ respectively.

The study of the Potts–Bethe mapping of degree 4 was quite similar to the study of the Potts–Bethe mapping of degree 3. The differences were the number of its fixed points and the description on primes p . We noted that the techniques here can be applied for any degree $k \in \{1, 2, 3, \dots\}$. Hence, we showed for many prime numbers p , the Potts–Bethe mapping has chaotic properties. These chaotic properties of the Potts–Bethe mapping imply the vastness of the set of the p -adic Gibbs measures (Theorems 4.6.1 and 4.6.2). As an application, for many prime numbers p , the phase transition occurs for the Potts models over \mathbb{Q}_p on the Cayley trees of orders 3 and 4. We also remarked that the statement phase transition implies chaos is not true, Section 4.6.

5.2 SUGGESTION FOR FUTURE RESEARCH

We saw that the study of the Potts model over \mathbb{Q}_p on Cayley trees, particularly, its p -adic Gibbs measures is associated the the study of a rational maps called Potts–Bethe mapping. In future we could analyze further the properties of the Potts–Bethe mapping, for examples its Fatou–Julia sets and their dimensions, ergodicity, minimality and more.

In addition, we could establish a boundary law argument to Theorem 2.5.3 to a general statistical model over \mathbb{Q}_p on Cayley trees. So it could describe wide class of p -adic Gibbs measures which can be the example of the p -adic Markov chains. It is also for the characterization and development of the set of p -adic Gibbs measures.

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