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Dimbihery Rabenoro. Asymptotic distribution of independent not identically distributed random vectors given their sum. Functional limit laws for the increments of Lévy processes.. Mathematics [math]. Sorbonne Université, Université Pierre et Marie Curie, Paris 6; Laboratoire de Probabilités, Statistique et Modélisation (LPSM), 2018. English. NNT: . tel-02428701

**HAL Id: tel-02428701**

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Submitted on 9 Jan 2020

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# THÈSE

présentée pour obtenir

## LE GRADE DE DOCTEUR EN SCIENCES DE SORBONNE UNIVERSITÉ

*Discipline : Mathématiques*

par

Dimbihery RABENORO

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Asymptotic distribution of independent not identically distributed  
random vectors given their sum. Functional limit laws for the  
increments of Levy Processes.

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dirigée par les Professeurs Paul DEHEUVELS et Michel BRONIATOWSKI.

Soutenue le 11 décembre 2018 à Sorbonne Université, devant le jury composé de

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# Chapter 1

## Introduction

This document contains two independent Parts. The first one is composed by Chapters 2 and 3, while the second one is Chapter 4.

### 1.1 Introduction to Chapters 2 and 3

In these chapters, we study the asymptotic distribution of independent not necessarily identically distributed random variables or random vectors, which is closely linked to some Statistical Mechanics issue.

#### 1.1.1 Statistical Mechanics Framework

Following (Khinchin, 1949), we present hereafter the Statistical Mechanics framework within which this question is natural.

#### The Phase Space

Let  $G$  be a mechanical system with  $s$  degrees of freedom. The state of  $G$  is described by values of its  $2s$  dynamical variables denoted by  $q_1, \dots, q_s; p_1, \dots, p_s$ . In other words, there is a one-to-one correspondence which associates to each possible state of  $G$ , a point of an Euclidian space  $\Gamma$ , whose coordinates are the values of  $(q_i, p_i)_{1 \leq i \leq s}$ .  $\Gamma$  is called the phase space of  $G$ . During any interval of time  $\Delta t$ , each point  $P \in \Gamma$  describes a curve corresponding to some successive changes of states of  $G$  during  $\Delta t$ . Thus, the whole space  $\Gamma$  is transformed into itself during  $\Delta t$ . This motion of  $\Gamma$  is called its natural motion. A subset  $M$  of  $\Gamma$  which is stable under the natural motion is called an invariant part of  $\Gamma$ . From point of view of physics, the most important function on  $\Gamma$  is the total energy of  $G$ , denoted by

$$E = E(q_i; p_i), \quad 1 \leq i \leq s. \quad (1.1.1)$$

Assume that  $G$  is an isolated system. Then, by the law of conservation of energy, the function  $E$  has a constant value. Consequently, for any constant  $a$ , the set

$$\Sigma_a := \{E = a\} \subset \Gamma \quad (1.1.2)$$

is an invariant part of  $\Gamma$  and is called a surface of constant energy. We can assume that  $E$  is positive over  $\Gamma$ . Set

$$V_x := \{E < x\} \subset \Gamma \quad \text{and} \quad V(x) := \text{Volume of } V_x. \quad (1.1.3)$$

$V(\cdot)$  is a monotone function which increases from 0 to  $\infty$  as  $x$  varies between the same limits. Then, we have the following lemma.

**Lemma 1.** *Let  $f(\cdot)$  be a function defined on  $\Gamma$ , integrable over  $V_x$ . Then,*

$$\frac{d}{dx} \int_{V_x} f(P) dV = \int_{\Sigma_x} f(P) \frac{d\Sigma}{\|\mathbf{Grad}(\mathbf{E})\|}, \quad (1.1.4)$$

where  $dV$  and  $d\Sigma$  are the volume elements of  $\Gamma$  and of  $\Sigma_x$ .

Let  $M$  be a measurable subset of  $\Sigma_x$ . Then in the natural motion of  $\Gamma$ ,  $M$  is transformed into a set  $M' \subset \Sigma_x$ . However, if we define the measure of  $M$  by  $\mu(M) := \int_M d\Sigma$ , then in general,  $\mu(M) \neq \mu(M')$ . We are deprived of important mathematical tools without this invariance. Therefore, we consider another measure of any set  $M$  contained in  $\Sigma_x$  as follows. At each point of  $M$ , draw the outward normal to  $\Sigma_x$  to its intersection with the infinitely near surface  $\Sigma_{x+dx}$ . Denote by  $D$  the bounded part of  $\Gamma$  which is filled by these normal vectors. Then set

$$\mathcal{M}(M) := \int_D dV = \int_{x < E < x+dx} \mathbf{1}_D(\mathbf{P}) d\mathbf{V}. \quad (1.1.5)$$

This volume is clearly invariant with respect to the natural motion. Its ratio to  $\Delta x$  and the limit of this ratio as  $\Delta x \rightarrow 0$  are also invariant. Now, by Lemma 1, this limit is

$$\int_{\Sigma_x} \mathbf{1}_D(\mathbf{P}) \frac{d\Sigma}{\|\mathbf{Grad}(\mathbf{E})\|} = \int_M \frac{d\Sigma}{\|\mathbf{Grad}(\mathbf{E})\|}. \quad (1.1.6)$$

Therefore, we obtain an invariant measure on subsets of  $\Sigma_x$  by considering the measure  $\mathcal{M}$  defined by

$$\mathcal{M}(M) = \int_M \frac{d\Sigma}{\|\mathbf{Grad}(\mathbf{E})\|} \quad (1.1.7)$$

**Definition 1.** *The measure  $\Omega(x)$  of the whole surface  $\Sigma_x$  is*

$$\Omega(x) = \int_{\Sigma_x} \frac{d\Sigma}{\|\mathbf{Grad}(\mathbf{E})\|} = \mathcal{M}(\Sigma_x). \quad (1.1.8)$$

Assume that for all  $P$ ,  $f(P) = 1$  in Lemma 1. Then we obtain that

$$\Omega(x) = V'(x). \quad (1.1.9)$$

The function  $\Omega(\cdot)$  determines the most important features of the mechanical structure of  $G$  and is therefore called the structure function of  $G$ .

**Definition 2.** *We denote by  $x_1, \dots, x_{2s}$  the dynamical coordinates of a point of  $\Gamma$ , where the order of numeration is irrelevant. Assume that the energy  $E = E(x_1, \dots, x_{2s})$  can be written as*

$$E(x_1, \dots, x_{2s}) = E_1(x_1, \dots, x_r) + E_2(x_{r+1}, \dots, x_{2s}) \quad (1.1.10)$$

We say that the set  $\{x_1, \dots, x_{2s}\}$  is decomposed in two components, that is

$$\{x_1, \dots, x_{2s}\} = \{x_1, \dots, x_r\} \bigsqcup \{x_{r+1}, \dots, x_{2s}\}, \quad (1.1.11)$$

which we write

$$G = G_1 \bigsqcup G_2. \quad (1.1.12)$$

A component defined in this sense does not necessarily coincide with a separate physical subsystem of  $G$ . The isolated character of such components is of a purely energy nature.

Each component, being a subset of dynamical coordinates, has its own phase space. With obvious notations, if  $G = G_1 \bigsqcup G_2$  then

$$\Gamma = \Gamma_1 \times \Gamma_2 \quad \text{and} \quad dV = dV_1 dV_2. \quad (1.1.13)$$

Let  $\Omega, \Omega_1$  and  $\Omega_2$  be the respective structure functions of  $G, G_1$  and  $G_2$ . Then we prove that

$$\Omega(x) = \int_0^\infty \Omega_1(y)\Omega_2(x-y)dy. \quad (1.1.14)$$

We deduce readily that if  $G = G_1 \sqcup G_2 \sqcup \dots G_n$ , then

$$\Omega(x) = \int \left\{ \prod_{i=1}^{n-1} \Omega_i(u_i) du_i \right\} \Omega_n \left( u - \sum_{i=1}^{n-1} u_i \right). \quad (1.1.15)$$

In order to be able to split  $G = G_1 \sqcup G_2$  in two components in this sense, we need to neglect the mixed terms of energy interactions which would involve variables from both  $G_1$  and  $G_2$ .

### Reduction to Probability Theory

We shall now consider the dynamical variables  $(x_1, \dots, x_{2s})$  as a random vector  $X = (X_1, \dots, X_{2s})$ . We still assume that  $G$  is an isolated system, so that the natural motion of  $\Gamma$  is limited within  $\Sigma_{na}$  and the support of  $X$  is contained in  $\Sigma_{na}$ . We assume that the distribution law of  $X$  is given by

$$P(X \in M) = \frac{\mathcal{M}(M)}{\mathcal{M}(\Sigma_{na})} = \frac{1}{\Omega(na)} \int_M \frac{d\Sigma}{\|\mathbf{Grad}(\mathbf{E})\|}, \quad \text{for any set } M \subset \Sigma_{na}. \quad (1.1.16)$$

Assume that  $G$  is divided into two components  $G^{(1)}$  and  $G^{(2)}$ . Therefore, we can write  $X = (X^{(1)}; X^{(2)})$  with  $X^{(1)} = (X_1, \dots, X_r)$  and  $X^{(2)} = (X_{r+1}, \dots, X_{2s})$ . Then, we can prove that for any subset  $M_1$  contained in  $\Gamma_1$ ,

$$P(X^{(1)} \in M_1) = \frac{1}{\Omega(na)} \int_{M_1} \Omega^{(2)}(na - E_1) dV_1. \quad (1.1.17)$$

Consequently, the distribution law of  $X^{(1)}$  is absolutely continuous w.r.t the Lebesgue measure with density given by

$$p_{X^{(1)}}(x^1) = \frac{\Omega^{(2)}(na - E_1(x^1))}{\Omega(na)}, \quad \text{for any } x^1 \in \Gamma_1. \quad (1.1.18)$$

We can then deduce that the random variable  $E_1$  is absolutely continuous w.r.t the Lebesgue measure with density given by

$$p_{E_1}(x) = \frac{\Omega^{(1)}(x)\Omega^{(2)}(na - x)}{\Omega(na)}. \quad (1.1.19)$$

Let  $\Psi(\cdot)$  be the Laplace transform of the function  $\Omega(\cdot)$ , called the partition function of  $G$ . We assume that for any  $\alpha > 0$ ,

$$\Psi(\alpha) := \int \exp(-\alpha x)\Omega(x)dx < \infty \quad (1.1.20)$$

Then, we have the following facts.

**Fact 1.** For any constant  $c > 0$ , there exist a unique solution  $\beta_n^c > 0$  to the equation of unknown  $\alpha$

$$-\frac{d}{d\alpha} \log \Psi(\alpha) = c. \quad (1.1.21)$$

**Fact 2.** The partition function of a system  $G$  is equal to the product of the partition functions of its components.

We introduce now the family  $(U^\alpha)_{\alpha>0}$  of distribution laws conjugate with the system  $G$ , defined by

$$U^\alpha(x) = \frac{1}{\Psi(\alpha)} \exp(-\alpha x)\Omega(x) \quad \text{if } x \geq 0, \quad (1.1.22)$$

and

$$U^\alpha(x) = 0 \quad \text{if } x < 0. \quad (1.1.23)$$

For any  $\alpha > 0$ ,  $U^\alpha(x)$  is the probability density of a random variable  $\tilde{X}^\alpha$ , since

$$U^\alpha(x) \geq 0 \quad \text{and} \quad \int U^\alpha(x) dx = 1, \quad (1.1.24)$$

Furthermore, we have the following

**Fact 3.** For any  $\alpha > 0$ ,

$$\mathbb{E}[\tilde{X}^\alpha] = \int x U^\alpha(x) dx = -\frac{d}{d\alpha} \log \Psi(\alpha). \quad (1.1.25)$$

### Gibbs Measure

We intend to evaluate the energy  $E_1$  of a given component  $G^{(1)}$  of  $G$ . However, we can not approximate directly the structure functions which appear in (1.1.19). Instead, we will be able to approximate the  $U^\alpha$ 's, since they are densities. In that purpose, we assume that  $G$  is divided into a large number  $n$  of components and that  $G^{(1)}$  is a collection of some of them, that is

$$G = \bigsqcup_{j=1}^n g_j = G^{(1)} \bigsqcup G^{(2)}, \quad \text{where } G^{(1)} = \bigsqcup_{j=1}^k g_j \quad \text{and } k < n. \quad (1.1.26)$$

We still assume that  $G$  is an isolated system, so that its energy has some constant value denoted by  $na$ , where  $a$  is the average energy of  $g_1, \dots, g_n$ .

Let  $(U_1^\alpha)_{\alpha>0}$  (resp.  $(U_2^\alpha)_{\alpha>0}$ ) be the family of distribution laws conjugate with  $G^{(1)}$  (resp.  $G^{(2)}$ ). Using that  $\Omega(x) = \Psi(\alpha) \exp(\alpha x) U^\alpha(x)$ , we readily get that for any  $\alpha > 0$ ,

$$p_{E_1}(x) = U_1^\alpha(x) \frac{U_2^\alpha(na - x)}{U^\alpha(na)}. \quad (1.1.27)$$

The objective is now to evaluate  $U_2^\alpha(na - x)$  and  $U^\alpha(na)$ . We can prove the following fact.

**Fact 4.** Assume that  $G = \bigsqcup_{j=1}^n g_j$ . Then, for any  $\alpha > 0$ ,

$$U^\alpha(x) = \int \left\{ \prod_{j=1}^{n-1} u_j^\alpha(y_j) dy_j \right\} u_n^\alpha \left( x - \sum_{j=1}^{n-1} y_j \right), \quad (1.1.28)$$

where for all  $1 \leq j \leq n$ , the  $(u_j^\alpha)_{\alpha>0}$  are the distribution laws conjugate with  $g_j$ .

In other words, for any  $\alpha > 0$ , one can interpret  $U^\alpha(\cdot)$  as the density of a sum of independent random variables  $\tilde{X}_j^\alpha$ , which are not necessarily identically distributed.

The Theory of Probability provides then an asymptotic approximation of  $U^\alpha(\cdot)$ . More precisely, we may apply the following Central Limit Theorem.

**Theorem 1.** Consider a sequence of independent random variables  $(X_j)_{j \geq 1}$  with probability densities  $(u_j)_{j \geq 1}$  and characteristic functions  $(g_j)_{j \geq 1}$ , that is  $g_j(t) = \int \exp(itx) u_j(x) dx$ .

Let  $(a_j)_{j \geq 1}$  be the sequence of expectations of the  $X_j$ 's and for  $2 \leq \ell \leq 5$ , let  $(a_j^\ell)_{j \geq 1}$  be the sequence of their centered absolute moments of order  $\ell$ . Assume that

(1) For any  $j \geq 1$ ,  $u_j$  is differentiable and there exists  $L > 0$  such that  $\sup_{j \geq 1} \int |u_j'(x)| dx < L$ .

(2) There exist  $0 < \alpha < \beta_n^a$  such that  $\inf_{j \geq 1} a_j^2 > \alpha$  and  $\sup_{j \geq 1} \max_{2 \leq \ell \leq 5} a_j^\ell \leq \beta_n^a$ .

(3) There exist positive constants  $\lambda$  and  $\tau$  such that in the region  $|t| \leq \tau$ ,  $\sup_{j \geq 1} |g_j(t)| > \lambda$ .

(4) For any  $0 < c_1 < c_2$ , there exists  $\rho = \rho(c_1, c_2) < 1$  such that for any  $t \in (c_1, c_2)$ ,  $\sup_{j \geq 1} |g_j(t)| < \rho$ .

Set  $A_n = \sum_{j=1}^n a_j$  and  $B_n = \sum_{j=1}^n a_j^2$ . Let  $U_n(x)$  be the density of  $\sum_{j=1}^n X_j$ . Then,

$$U_n(x) = \frac{1}{(2\pi B_n)^{1/2}} \exp\left[-\frac{(x - A_n)^2}{2B_n}\right] + v_n, \quad (1.1.29)$$

where

$$v_n = o\left(\frac{1 + |x - A_n|}{n^{3/2}}\right) \text{ for } |x - A_n| < 2 \log^2 n \quad (1.1.30)$$

and

$$v_n = o\left(\frac{1}{n}\right) \text{ for all } x. \quad (1.1.31)$$

Recall that  $(U^\alpha)_{\alpha > 0}$  is the family of distribution laws *conjugate* with  $G$ , which is composed of  $n$  components. We will write  $U^\alpha$ , the number  $n$  being omitted. We assume that for any  $\alpha > 0$ , for very large  $n$ , the densities  $(U_i^\alpha)_{1 \leq i \leq n}$  satisfy the assumptions (1), (2), (3), (4). This essentially means that the components  $(g_i)$  are of a small number of different kinds, which is a reasonable assumption.

Applying Theorem 1, we obtain that for any  $\alpha > 0$ , (1.1.29) holds for  $U^\alpha$ , with  $A_n = \mathbb{E}[\tilde{X}^\alpha]$  and  $B_n = \text{Var}(\tilde{X}^\alpha)$ . We get from Fact 3 that  $A_n = -\frac{d}{d\alpha} \log \Phi(\alpha)$ . Then, (1.1.21) implies that there exists a unique  $\beta_n^a > 0$  such that

$$A_n = \left(-\frac{d}{d\alpha} \log \Psi(\alpha)\right)_{\alpha=\beta_n^a} = na. \quad (1.1.32)$$

We deduce that

$$U^{\beta_n^a}(na) = \frac{1}{(2\pi B_n)^{1/2}} + o(n^{-3/2}). \quad (1.1.33)$$

We assume that the number  $k$  of components of  $G^{(1)}$  satisfies that  $k = o(n)$ . Therefore,  $n - k \sim n$  and we may apply Theorem 1 to  $U_2^{\beta_n^a}$  to obtain that

$$U_2^{\beta_n^a}(na - x) = \frac{\exp\left[-\frac{(x - A_{1,k})^2}{2B_{k+1,n}}\right]}{(2\pi B_{k+1,n})^{1/2}} + o\left(\frac{1}{n}\right), \quad (1.1.34)$$

where  $A_{1,k} = \sum_{j=1}^k \mathbb{E}[\tilde{X}_j^{\beta_n^a}]$  and  $B_{k+1,n} = \sum_{j=k+1}^n \text{Var}(\tilde{X}_j^{\beta_n^a})$ . The assumptions of Theorem 1 imply that  $B_n$  and  $B_{k+1,n}$  are respectively of order  $n$  and  $n - k$ , and are therefore of the same order since  $k = o(n)$ . Consequently, for any  $x > 0$ ,

$$\frac{U_2^{\beta_n^a}(na - x)}{U^{\beta_n^a}(na)} = \left\{ \exp\left[-\frac{(x - A_{1,k})^2}{2B_{k+1,n}}\right] \right\} \{1 + o(1)\} \quad (1.1.35)$$

However, if we only consider those  $x$  such that  $x - A_{1,k} = o(n^{1/2})$ , we obtain that

$$\exp\left[-\frac{(x - A_{1,k})^2}{2B_{k+1,n}}\right] = \{1 + o(1)\}. \quad (1.1.36)$$



Therefore, writing (1.1.27) for  $\alpha = \beta_n^a$ , we get that for  $x$  satisfying  $x - A_{1,k} = o(n^{1/2})$ ,

$$p_{E_1}(x) = U_1^{\beta_n^a}(x) \{1 + o(1)\}. \quad (1.1.37)$$

Thus, when  $x$  belongs to an interval of wide radius (equal to  $n^{1/2}$ ), the density of  $E_1$  is approximated by  $U_1^{\beta_n^a}(x)$ , which is the density of a Gibbs measure. One can interpret (1.1.37) as follows.  $G$  is an isolated system divided in two components : a small one,  $G^{(1)}$ , immersed in a large heat bath  $G^{(2)}$ .  $G^{(1)}$  and  $G^{(2)}$  interact only by exchanges of energy and their temperatures are equal to the same value  $T$  when thermal equilibrium is reached. Then, the distribution of energy in  $G^{(1)}$  and in any small component of  $G$  is given by (1.1.37), and the parameter  $\beta_n^a$ , usually called an inverse temperature, is equal to  $\frac{1}{k_B T}$ , where  $k_B$  is Boltzmann's constant.

We will explain in chapter 2 why these Statistical Mechanics considerations are linked to the issues of Chapters 2 and 3.

### 1.1.2 Presentation of Chapters 2 and 3

In both Chapters 2 and 3, the essential technique is the commonly called Saddlepoint Approximation (see (Jensen, 1995)), which is an asymptotic local approximation of the density of a sum of independent random variables. It is composed of two steps. Firstly, one performs an exponential change of measure, as in Large Deviations Theory, in order to localize around a given value of the sum. Here, we call it the tilting operation. Secondly, one performs an Edgeworth expansion of the density of the resulting sum. This expansion is a central Limit Theorem, as the Theorem stated hereabove in (Khinchin, 1949), but at higher orders.

In Chapters 2 and 3, we consider a sequence  $(X_j)_{j \geq 1}$  of independent random vectors, valued in  $\mathbb{R}^d$ ,  $d \geq 1$ . Let  $(k_n)_{n \geq 1}$  be a sequence of integers with  $1 \leq k_n < n$ , for all  $n \geq 1$ . We write  $k$  instead of  $k_n$ . We assume that the  $(X_j)$  have a common support  $\mathcal{S}_X$  and that their moment generating functions have a common domain of finiteness, denoted by  $\Theta$ . For  $a \in \mathcal{S}_X$  and  $n \geq 1$ , let  $Q_{nak}$  be a regular version of the conditional distribution of  $X_1^k := (X_1, \dots, X_k)$  given  $\{S_{1,n} = na\}$ , where  $S_{1,n} := \sum_{j=1}^n X_j$ . We study the asymptotic behaviour (as  $n \rightarrow \infty$ ) of  $Q_{nak}$ , under various assumptions on  $k$ .

Our results are given in total variation distance. We denote by  $\|P - Q\|_{TV}$  the total variation distance between probability measures  $P$  and  $Q$ .

The tilting operation is described hereunder.

**Definition 3.** Let  $X$  be a r.v. valued in  $\mathbb{R}^d$ ,  $d \geq 1$ . Denote by  $\Phi_X$  its mgf. Let  $\Theta_X := \{\theta \in \mathbb{R}^d : \Phi_X(\theta) < \infty\}$ . For any  $\theta \in \Theta_X$ , denote by  $\tilde{X}^\theta$  a random vector having the tilted density, defined by

$$p_{\tilde{X}^\theta}(x) := \frac{\exp\langle \theta, x \rangle p_X(x)}{\Phi_X(\theta)} \quad (1.1.38)$$

## 1.2 Summary of Chapter 2

We present here our strategy to obtain the asymptotic behaviour of  $Q_{nak}$  when  $k = o(n)$ .

Since the conditioning event is  $\{S_{1,n} = na\}$ , we prove that for any  $n \geq 1$ , there exists a unique  $\theta_n^a \in \Theta$  such that

$$\mathbb{E} \left[ \widetilde{S_{1,n}^{\theta_n^a}} \right] = na. \quad (1.2.1)$$

The main result of Chapter 2 is the following. In the sequel, all the tilted densities pertain to  $\theta = \theta_n^a$ .

**Theorem 2.** *Under suitable assumptions, if  $k = o(n)$ , then,*

$$\left\| Q_{nak} - \tilde{P}_1^k \right\|_{TV} = \mathcal{O} \left( \frac{k}{n} \right), \quad (1.2.2)$$

where  $\tilde{P}_1^k$  is the joint distribution of independent r.v.'s  $(\tilde{X}_j)_{1 \leq j \leq k}$ .

*Proof.* We give a sketch of the proof. Let  $R_{nak}$  be the distribution of  $S_{1,k} := \sum_{j=1}^k X_j$  given  $S_{1,n} = na$ .

Let  $\tilde{R}_{1,k}$  be the distribution of  $\tilde{S}_{1,k} := \sum_{j=1}^k \tilde{X}_j$ . Then, we obtain from Sufficiency Theory that

$$\left\| Q_{nak} - \tilde{P}_1^k \right\|_{TV} = \left\| R_{nak} - \tilde{R}_{1,k} \right\|_{TV}. \quad (1.2.3)$$

Now, by Scheffe's theorem, we deduce that

$$\left\| Q_{nak} - \tilde{P}_{1,k} \right\|_{TV} = \int \left| p(S_{1,k} = t | S_{1,n} = na) - p_{\tilde{S}_{1,k}}(t) \right| dt, \quad (1.2.4)$$

where  $p(S_{1,k} = \cdot | S_{1,n} = na)$  is the density of  $S_{1,k}$  given  $S_{1,n} = na$  and  $p_{\tilde{S}_{1,k}}$  is the density of  $\tilde{S}_{1,k}$ . Then, we can check readily the following invariance of the conditional density : for any  $t \in \mathbb{R}^d$ ,

$$p(S_{1,k} = t | S_{1,n} = na) = p(\tilde{S}_{1,k} = t | \tilde{S}_{1,n} = na) = p_{\tilde{S}_{1,k}}(t) \left( \frac{p_{\tilde{S}_{k+1,n}}(na - t)}{p_{\tilde{S}_{1,n}}(na)} \right). \quad (1.2.5)$$

For any integers  $\ell, m$  with  $1 \leq \ell \leq m$ , we denote by  $f_{\ell,m}$  the density of  $\tilde{S}_{\ell,m} := \sum_{j=\ell}^m \tilde{X}_j$ . Therefore, we deduce readily from (2.5.6), (2.5.7) and (2.5.8) that

$$\left\| Q_{nak} - \tilde{P}_{1,k} \right\|_{TV} = \int \left| \frac{f_{k+1,n}(na - t)}{f_{1,n}(na)} - 1 \right| f_{1,k}(t) dt. \quad (1.2.6)$$

Finally, we perform Edgeworth expansions for  $f_{k+1,n}$  and  $f_{1,n}$  and we get the desired result.  $\square$

### 1.3 Summary of Chapter 3

This Chapter contains a generalisation of the preceding one. Indeed, we obtain the asymptotic behaviour of  $Q_{nak}$  when  $k$  is not necessarily a  $o(n)$ , and even when  $\frac{k}{n}$  converges to 1.

We need to consider some quantities inspired from an Importance Sampling setting, which allow to use a criterion for convergence in total variation distance. We perform an Adaptive Scheme to estimate the density of  $Q_{nak}$ , and still perform a Saddlepoint Approximation to conclude the proof.

### 1.4 Introduction to Chapter 4

Let  $(X_i)_{i \geq 1}$  be an i.i.d. sequence of random variables. Let  $F_X$  be their common distribution function. For  $n \geq 1$ , set  $S_0 = 0$  and  $S_n = \sum_{i=1}^n X_i$ .

Assume first that  $(X_i)_{i \geq 1}$  is a sequence of Bernoulli of parameter  $p$ . For  $n \geq 1$ , let  $L_n$  be the longest chain of consecutive 1 among  $(X_i)_{1 \leq i \leq n}$ .

**Theorem 3.** *For any  $p \in (0, 1)$ , we have that almost surely,*

$$\frac{L_n}{\log n} \longrightarrow -\frac{1}{\log p} \quad \text{as } n \rightarrow \infty. \quad (1.4.1)$$

The study of  $L_n$  is of interest in insurance, finance and even molecular biology. For any  $n \geq 1$ , and any integer  $k$  such that  $1 \leq k \leq n$ , set

$$\begin{aligned} M_n(k) &:= \max_{0 \leq i \leq n-k} \{S_{i+k} - S_i\} \\ &= \max_{0 \leq i \leq n-k} \{X_{i+1} + \dots + X_{i+k}\}. \end{aligned}$$

Then, in a Bernoulli model,

$$L_n = \max \{k \in \{1, \dots, n\} : M_n(k) = k\}. \quad (1.4.2)$$

From now, we consider any sequence  $(X_i)_{i \geq 1}$  and we focus on

$$I_n(k) := \frac{M_n(k)}{k}, \quad \text{for } n \geq 1 \quad \text{and } 1 \leq k \leq n. \quad (1.4.3)$$

Then,  $I_n(k)$  is called an Erdős-Rényi increment. For example, if  $(X_i)$  represent daily values of a financial asset, then  $I_n(k)$  is the maximal average gain over a period of  $k$  days. Notice that

$$I_n(1) = \max_{1 \leq i \leq n} X_i \quad \text{while} \quad I_n(n) = \frac{S_n}{n}. \quad (1.4.4)$$

Therefore,  $I_n(k)$  can be viewed as an intermediate object between Extreme Value Theory and the classical Theory of mean of variables. Notice also that, if  $\mathbb{E}[X_1] = 0$  then by the law of large numbers,  $I_n(n) \rightarrow 0$  as  $n \rightarrow \infty$ . On the other hand, if  $F_X(t) < 1$  for any real  $t$ , then  $I_n(1) \rightarrow \infty$  as  $n \rightarrow \infty$ . The following result asserts that somewhere between these two extremes, the limit is positive and finite. It is the classical Erdős and Rényi theorem for the partial sum process.

**Theorem 4.** *Assume that the distribution of  $X_1$  is nondegenerate,  $\mathbb{E}[X_1] = 0$  and*

$$\inf \{s : \psi(s) < \infty\} < 0 < \sup \{s : \psi(s) < \infty\}, \quad \text{where } \psi(s) := \mathbb{E}[\exp(sX_1)]. \quad (1.4.5)$$

Fix  $c > 0$ . Let  $k_n$  be the integer part of  $c \log n$ . Then,

$$M_n(k_n) \rightarrow \alpha_c, \quad (1.4.6)$$

where

$$\alpha_c := \inf \{\alpha \geq u : \Psi(\alpha) \geq 1/c\} \quad \text{and} \quad \Psi(\alpha) = \sup_{t: \psi(t) < \infty} \{t\alpha - \log \psi(t)\}. \quad (1.4.7)$$

*Proof.* The proof makes use of classical large deviations results for  $\frac{S(n)}{n}$  and on the Borel-Cantelli lemma. See (Deheuvels, 2007) for details.  $\square$

This result has given rise to many developments and extensions to processes related to the partial sum one, among which a functional version, established in Deheuvels (1991). In Chapter 3, we extend it to Lévy processes.

## 1.5 Summary of Chapter 4

This chapter is devoted to functional Erdős-Rényi theorems for Lévy processes. Let  $Z$  be a Lévy process. For  $x \geq 0$ , and  $\ell > 0$ , define the standardized increment functions of  $Z(\cdot)$  by setting

$$\eta_{x,\ell}(s) := \frac{Z(x + \ell s) - Z(x)}{\ell} \quad \text{for } s \in [0, 1]. \quad (1.5.1)$$

For any  $c > 0$ , consider the following random sets  $\mathcal{G}_T$  of increment functions.

$$\mathcal{G}_T := \{\eta_{x,a_T} : 0 \leq x \leq T - a_T\}, \quad \text{where } a_T = c \log T. \quad (1.5.2)$$

and

$$\mathcal{M}_{n,a_n} := \{\eta_{m,a_n} : m \in \{0, \dots, n - a_n\}\}, \quad \text{where } a_n \text{ is the integer part of } c \log n. \quad (1.5.3)$$

We have established that, under suitable assumptions, the random sets  $\mathcal{G}_T$  and  $\mathcal{M}_{n,a_n}$  converge (in the sense of the Hausdorff distance defined below) almost surely (a.s.) to deterministic sets of functions.

Now, we define the Hausdorff distance. Let  $\mathcal{E}$  be a set of functions on  $[0, 1]$  such that for all  $T > 0$ ,  $\mathcal{G}_T \subseteq \mathcal{E}$ . We endow  $\mathcal{E}$  with a metric topology  $\mathcal{T}$ , defined by a distance  $d_{\mathcal{T}}$ . For any subset  $A \subseteq \mathcal{E}$ , and  $\epsilon > 0$ , consider an enlargement of  $A$  defined by

$$A^\epsilon = A^{\epsilon;\mathcal{T}} := \left\{ g \in \mathcal{E} : d_{\mathcal{T}}(f, g) < \epsilon \text{ for some } f \in A \right\}. \quad (1.5.4)$$

The Hausdorff distance between the subsets  $A, B \subseteq \mathcal{E}$  is defined by

$$\Delta_{\mathcal{T}}(A, B) := \inf \left\{ \epsilon > 0 : A \subseteq B^\epsilon \text{ and } B \subseteq A^\epsilon \right\}. \quad (1.5.5)$$

Let  $K(c)$  be a fixed set. Then,  $\lim_{T \rightarrow \infty} \Delta_{\mathcal{T}}(\mathcal{G}_T, K(c)) = 0$  a.s. if and only if, for any  $\epsilon > 0$ , there exists a.s.  $T(\epsilon) < \infty$  such that for all  $T \geq T(\epsilon)$ ,

$$\mathcal{G}_T \subset (K(c))^\epsilon \text{ and } K(c) \subset (\mathcal{G}_T)^\epsilon. \quad (1.5.6)$$

Let  $\psi$  be the moment generating function of  $Z(1)$ . Introduce the following assumptions.

(C) :  $\psi(t) < \infty$  for all  $t \in \mathbb{R}$ .

(A) :  $\inf \{t : \psi(t) < \infty\} < 0 < \sup \{t : \psi(t) < \infty\}$  and  $Z(1)$  has no Gaussian component.

(E) : There exists a constant  $\mu$  such that for all  $t \geq 0$ ,  $\mathbb{E}[Z(t)] = \mu t$ .

We have obtained the following theorems, called functional Erdős-Rényi laws.

**Theorem 5.** *Assume that (C) and (E) hold. Then, for any  $c > 0$ , there exists a fixed set  $K(c)$  such that*

$$\lim_{T \rightarrow \infty} \Delta_{\mathcal{U}}(\mathcal{G}_T, K(c)) = 0 \quad \text{a.s.} \quad (1.5.7)$$

where  $\Delta_{\mathcal{U}}$  is associated to the distance  $d_{\mathcal{U}}$  defined on the set of bounded functions on  $[0, 1]$  by

$$d_{\mathcal{U}}(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|. \quad (1.5.8)$$

**Theorem 6.** *Assume that (A) and (E) hold. Let  $BV_0(0, 1)$  be the set of functions on  $[0, 1]$  which are right-continuous, with bounded variations and vanish at the origin. Then, for all  $c$  large enough, there exists a fixed set  $L(c)$  such that*

$$\lim_{n \rightarrow \infty} \Delta_{\mathcal{W}}(\mathcal{M}_{n,a_n}, L(c)) = 0 \quad \text{a.s.} \quad (1.5.9)$$

where  $\Delta_{\mathcal{W}}$  is associated to the distance  $d_{\mathcal{W}}$  defined on  $BV_0(0, 1)$  by

$$d_{\mathcal{W}}(f, g) = \int_0^1 |f(u) - g(u)| du + |f(1) - g(1)|. \quad (1.5.10)$$

The proofs of these results rely heavily on functional large deviations theorems for processes with stationary and independent increments.



# Chapter 2

## Asymptotic distribution of independent random vectors given their sum

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Remark Proposition Corollary Lemma Exercise Example Property Theorem Definition Fact

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### 2.1 Introduction

#### 2.1.1 Context and scope

Let  $(X_j)_{j \geq 1}$  be a sequence of independent, not necessarily identically distributed (i.d.), random vectors (r.v.) valued in  $\mathbb{R}^d$ ,  $d \geq 1$ . Let  $(k_n)_{n \geq 1}$  be a sequence of integers with  $1 \leq k_n < n$ , for all  $n \geq 1$ . We write  $k$  instead of  $k_n$ . We assume that the  $(X_j)$  have a common support  $\mathcal{S}_X$ . For  $a \in \mathcal{S}_X$  and  $n \geq 1$ , let  $Q_{nak}$  be a regular version of the conditional distribution of  $X_1^k := (X_1, \dots, X_k)$  given  $\{S_{1,n} = na\}$ , where  $S_{1,n} := \sum_{j=1}^n X_j$ . Such a version exists since  $\mathbb{R}^d$  is a Polish space (see (Stroock, 1994)). In this paper, we study the asymptotic behaviour (as  $n \rightarrow \infty$ ) of  $Q_{nak}$ .

This question is closely related to the well-known Gibbs Conditioning Principle (GCP) (see (Stroock and Zeitouni, 1991)), which states that when the r.v.'s are independent and identically distributed (i.i.d.) and valued in any Polish space, the distribution of  $X_1^k$  given  $\left\{ \frac{1}{n} \sum_{j=1}^n f(X_j) = a \right\}$ , where  $f$  is a measurable real function, converges weakly to some limit distribution. Let  $P_X$  be the common law of the  $(X_j)$ . Denote by  $\mathcal{B}(\mathbb{R}^d)$  the Borel  $\sigma$ -algebra of  $\mathbb{R}^d$ . Then, under suitable conditions, the GCP asserts that for fixed  $k$ , we have for any  $B \in (\mathcal{B}(\mathbb{R}^d))^k$  and  $a \neq \mathbb{E}_{P_X}[f]$ ,

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} P \left( X_1^k \in B \mid A(a, \delta) := \left\{ \frac{1}{n} \sum_{j=1}^n f(X_j) \in [a - \delta, a + \delta] \right\} \right) = (\gamma^a)^k(B), \quad (2.1.1)$$

where the measure  $\gamma^a$ , called a Gibbs measure, minimizes the relative entropy  $H(\cdot | P_X)$  under an

energy constraint. Let  $\theta^a \in \mathbb{R}$  be a solution of the equation

$$\frac{d \log(\Phi^f)}{d\theta}(\theta) = a, \quad \text{where } \Phi^f(\theta) := \int_{\mathbb{R}^d} \exp(\theta f(x)) dP_X(x). \quad (2.1.2)$$

Then  $\gamma_a$  is absolutely continuous (a.c.) with respect to (w.r.t.)  $P_X$ , with

$$\frac{d\gamma^a}{dP_X}(x) = \frac{\exp(\theta^a f(x))}{\Phi^f(\theta^a)}. \quad (2.1.3)$$

The GCP extends to the case where  $k \rightarrow \infty$  as  $n \rightarrow \infty$ , provided that  $k = o(n)$ . (See (Dembo and Zeitouni, 1996)). It has an interpretation in Statistical Mechanics, since it describes the distribution of a typical small subset in a system composed of a very large number  $n$  of particles, under a constraint of averaged energy. The classical approach to obtain statements of the form (2.1.1) is to interpret the event  $A(a, \delta)$  in terms of the empirical distribution and to use Sanov's large deviations theorem (see Section 7.3. in (Dembo and Zeitouni, 1993)). *However, this method uses the exchangeability of the  $(X_j)$  under the conditioning event, which does not hold anymore when the r.v.'s are not i.i.d..*

In this paper, we consider the conditioning point approach of (Diaconis and Freedman, 1988). Instead of enlarging the conditioning event as in (2.1.1), this approach uses that, when all the  $X_j$ 's are a.c. w.r.t the Lebesgue measure on  $\mathbb{R}^d$ ,  $Q_{nak}$  may be defined by a conditional density (see Fact 16 below). We prove that this method can be applied to r.v.'s which are not i.i.d. More precisely, we generalize Theorem 1.6 in (Diaconis and Freedman, 1988), which holds, when  $k = o(n)$ , for a sequence of i.i.d. r.v.'s valued in  $\mathbb{R}$  ( $d = 1$ ). We extend it to a sequence of independent non i.i.d. r.v.'s. valued in  $\mathbb{R}^d$  with  $d \geq 1$ . We obtain that  $Q_{nak}$  is asymptotically approximated in total variation distance, by the product of  $k$  probability measures  $(\gamma_{j,n}^a)_{1 \leq j \leq k}$  described as follows. For any  $j \geq 1$ , let  $\Phi_j(\cdot) := \int_{\mathbb{R}^d} \exp(\langle \cdot, x \rangle) dP_{X_j}(x)$  be the moment generating function (mgf) of  $X_j$ . Then, for any  $n \geq 1$  and  $1 \leq j \leq k$ ,  $\gamma_{j,n}^a$  is a.c. w.r.t.  $P_j := P_{X_j}$ , with

$$\frac{d\gamma_{j,n}^a}{dP_j}(x) = \frac{\exp(\langle \theta_n^a, x \rangle)}{\Phi_j(\theta_n^a)}, \quad \text{for } x \in \mathbb{R}^d, \quad (2.1.4)$$

where for any  $n \geq 1$ ,  $\theta_n^a \in \mathbb{R}^d$  is a solution of the equation

$$\frac{1}{n} \sum_{j=1}^n \nabla \log \Phi_j(\theta) = a. \quad (2.1.5)$$

*Although our conditioning event is less general than in the GCP, our result still has a Statistical Mechanics interpretation, as explained in Section 2. After some preliminary results in Section 3, we precise our assumptions in Section 4. Then, we state and prove our main theorem in Section 5, while some technical lemmas are deferred to the Appendix.*

### 2.1.2 Notations and elementary Facts

All the r.v.'s considered are a.c. w.r.t. the Lebesgue measure on  $\mathbb{R}^d$ . For any r.v.  $X$ , let  $P_X$  be its distribution and  $p_X$  its density. For any  $j \geq 1$ , set

$$P_j := P_{X_j} \quad \text{and} \quad p_j := p_{X_j}.$$

### Conditional density

Let  $U$  and  $V$  be r.v.'s having a joint density denoted by  $p_{(U,V)}$ . Then, there exists a conditional density of  $U$  given  $V$ , denoted as follows.

$$p(U = u | V = v) = \frac{p_{(U,V)}(u, v)}{p_V(v)}.$$

**Fact 5.** Let  $(X_j)_{j \geq 1}$  be a sequence of independent r.v.'s valued in  $\mathbb{R}^d$ ,  $d \geq 1$ . For any  $n \geq 1$ , let  $J_n$  be a subset of  $\{1, \dots, n\}$  s.t.  $\alpha_n := |J_n| < n$ . Let  $L_n$  be the complement of  $J_n$  in  $\{1, \dots, n\}$ . Set  $S_{L_n} := \sum_{j \in L_n} X_j$ . Then, there exists a conditional density of  $(X_j)_{j \in J_n}$  given  $S_{1,n}$ , defined by

$$p((X_j)_{j \in J_n} = (x_j) | S_{1,n} = s) = \frac{\left\{ \prod_{j \in J_n} p_j(x_j) \right\} p_{S_{L_n}}\left(s - \sum_{j \in J_n} x_j\right)}{p_{S_{1,n}}(s)}, \quad (2.1.6)$$

*Proof.* For any measurable function  $\phi : (\mathbb{R}^d)^{\alpha_n} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ , we calculate

$$\mathbb{E}[\phi((X_j)_{j \in J_n}; S_{1,n})] = \int \phi((x_j); s_{1,n}) \left\{ \prod_{j=1}^n p_j(x_j) \right\} dx_1 \dots dx_1, \quad \text{where } s_{1,n} = \sum_{j=1}^n x_j. \quad (2.1.7)$$

Then, we apply the change of variables formula with the diffeomorphism of class  $\mathcal{C}^1$  defined by

$$(x_1, \dots, x_{n-1}, x_n) \mapsto (x_1, \dots, x_{n-1}, s_{1,n}). \quad (2.1.8)$$

We obtain thus that the joint density of  $(X_j)_{j \in J_n}$  and  $S_{1,n}$  is the numerator of (3.2.2).  $\square$

### The Tilted Density

**Definition 4.** Let  $X$  be a r.v. valued in  $\mathbb{R}^d$ ,  $d \geq 1$ . Denote by  $\Phi_X$  its mgf. Let  $\Theta_X := \{\theta \in \mathbb{R}^d : \Phi_X(\theta) < \infty\}$ . For any  $\theta \in \Theta_X$ , denote by  $\tilde{X}^\theta$  a random vector having the tilted density, defined by

$$p_{\tilde{X}^\theta}(x) := \frac{\exp\langle \theta, x \rangle p_X(x)}{\Phi_X(\theta)} \quad (2.1.9)$$

**Fact 6.** For any  $\theta \in \Theta_X$ , the mean of the r.v.  $\tilde{X}^\theta$  is equal to the gradient of  $\kappa$  at  $\theta$ . Thus,

$$\mathbb{E}[\tilde{X}^\theta] = \nabla \kappa(\theta). \quad (2.1.10)$$

The covariance matrix of  $\tilde{X}^\theta$  is equal to the Hessian matrix of  $\kappa$  at  $\theta$ . Thus, for for any  $1 \leq i, j \leq d$ ,

$$[\text{Cov}(\tilde{X}^\theta)]_{i,j} = \left[ \frac{\partial^2 \kappa}{\partial \theta_i \partial \theta_j}(\theta) \right]_{i,j}. \quad (2.1.11)$$

For any  $j \geq 1$ , set  $\Phi_j := \Phi_{X_j}$ . We suppose throughout the text that the functions  $(\Phi_j)_{j \geq 1}$  have the same domain of finiteness denoted by  $\Theta$ , which is assumed to be of non void interior. We write, for any  $j \geq 1$ ,

$$\Theta := \left\{ \theta \in \mathbb{R}^d : \Phi_j(\theta) < \infty \right\}.$$



**Fact 7.** For any  $j \geq 1$ , there exists a probability space  $(\Omega^\theta, \mathcal{A}^\theta, \mathcal{P}^\theta)$  such that for all finite subset  $J \subset \mathbb{N}$  and for all  $(B_j)_{j \in J} \in \mathcal{B}(\mathbb{R}^d)^{|J|}$ ,

$$\mathcal{P}^\theta \left( \left( \tilde{X}_j^\theta \right)_{j \in J} \in (B_j)_{j \in J} \right) = \prod_{j \in J} \tilde{P}_j^\theta(B_j) = \prod_{j \in J} \int_{B_j} \tilde{p}_j^\theta(x) dx, \quad (2.1.12)$$

where  $\tilde{P}_j^\theta := P_{\tilde{X}_j^\theta}$  and  $\tilde{p}_j^\theta := p_{\tilde{X}_j^\theta}$ . In other words,  $\left( \tilde{X}_j^\theta \right)_{j \geq 1}$  is a sequence of independent r.v.'s defined on  $(\Omega^\theta, \mathcal{A}^\theta, \mathcal{P}^\theta)$ .

**Fact 8.** For any  $\theta \in \Theta$ ,  $j \geq 1$  and  $j' \geq 1$ ,

$$\mathbb{E} \left[ \widetilde{X_j + X_{j'}}^\theta \right] = \mathbb{E} \left[ \tilde{X}_j^\theta + \tilde{X}_{j'}^\theta \right]. \quad (2.1.13)$$

**Corollary 1.** For any  $n \geq 1$ , for any  $\theta \in \Theta$ ,

$$\mathbb{E} \left[ \widetilde{S_{1,n}}^\theta \right] = \sum_{j=1}^n m_j(\theta). \quad (2.1.14)$$

We will prove in Section 2.1 that, for a suitable choice of  $a$ , the equation (3.2.24) has a unique solution denoted by  $\theta_n^a$ . Throughout the text, when we write  $\tilde{X}_j$  without any subscript  $\theta$ , this means that we refer implicitly to  $\tilde{X}_j^{\theta_n^a}$ .

## 2.2 Analogies with Statistical Mechanics

We have the following analogies between the mathematical point of view and the statistical mechanics one developed in the Chapter of Introduction.

$$\begin{aligned} S_{1,k} &\longleftrightarrow \text{Energy of } g_1 \sqcup \dots \sqcup g_k \\ \text{Density of } S_{1,k} &\longleftrightarrow \text{Structure function of } g_1 \sqcup \dots \sqcup g_k \\ \text{Moment generating function of } S_{1,k} &\longleftrightarrow \text{Partition function of } g_1 \sqcup \dots \sqcup g_k \\ \theta_n^a &\longleftrightarrow \beta_n^a \end{aligned}$$

Notice that, although the energies ( $e_i$ ) of the components ( $g_i$ ) are the analogues of the ( $X_i$ ), the ( $e_i$ ) are not stochastically independent. However, splitting  $G$  in components ( $g_i$ ) in this sense, gives raise to some ( $U_i^\alpha$ ) such that ( $U^\alpha$ ) is the density of a sum of *independent* random variables ( $\tilde{X}_i^\alpha$ ). The assumptions on the ( $\tilde{X}_i^\alpha$ ) of Theorem 1 are actually analytical conditions of uniformity on their densities ( $U_i^\alpha$ ). They mean that the components ( $g_i$ ) have rather similar characteristics, although they are not identical. Now, we have from (1.1.37) that

$$p(S_{1,k} = x | S_{1,n} = na) \longleftrightarrow \frac{\Omega^{(1)}(x) \Omega^{(2)}(na-x)}{\Omega(na)} = U_1^\alpha(x) \frac{U_2^\alpha(na-x)}{U^\alpha(na)} \approx \frac{\Omega^{(1)}(x) \exp(-\beta_n^a x)}{\Psi^{(1)}(\beta_n^a)}.$$

Therefore, we expect that  $p(S_{1,k} = x | S_{1,n} = na)$  should be approximated by  $\frac{p(S_{1,k}=x) \exp(\theta_n^a x)}{\Phi_{1,k}(\theta_n^a)}$ , where  $\Phi_{1,k}$  is the mgf of  $S_{1,k}$ . This approximation is a consequence of our general result, which is therefore natural.

## 2.3 Preliminary Results

### 2.3.1 Existence of the tilted density

For any set  $E \subset \mathbb{R}^d$ , we denote respectively by  $\text{int}(E)$ ,  $\text{cl}(E)$  and  $\text{conv}(E)$  the interior, the closure and the convex hull of  $E$ . Let  $\mathcal{S}_X$  be the common support of the  $(X_j)_{j \geq 1}$ . Set

$$C_X := \text{cl}(\text{conv}(\mathcal{S}_X)).$$

**Definition 5.** Let  $f$  be a convex function on  $\mathbb{R}^d$ . Set  $\text{dom}(f) := \{x \in \mathbb{R}^d : f(x) < \infty\}$ . Assume that  $\text{int}(\text{dom}(f)) \neq \emptyset$  and  $f$  is differentiable throughout  $\text{int}(\text{dom}(f))$ . Then, for any boundary point  $x$  of  $\text{dom}(f)$ , we say that  $f$  is steep at  $x$  if

$$\|\nabla f(x_i)\| \rightarrow \infty \tag{2.3.1}$$

whenever  $(x_i)$  is a sequence of points in  $\text{int}(\text{dom}(f))$  converging to  $x$ . Furthermore,  $f$  is called steep if it is steep at all boundary point of  $\text{dom}(f)$ .

We have the following characterization of steepness, which is Theorem 5.27 in (Barndorff-Nielsen Ole, 2014).

**Theorem 7.** Let  $f$  be a convex function on  $\mathbb{R}^d$ . Assume that  $\text{int}(\text{dom}(f)) \neq \emptyset$  and that  $f$  is differentiable throughout  $\text{int}(\text{dom}(f))$ . Then  $f$  is steep if and only if for any  $z \in \text{int}(\text{dom}(f))$  and any boundary point  $x \in \text{dom}(f)$ ,

$$\frac{df}{d\lambda}(x + \lambda(z - x)) \downarrow -\infty, \quad \text{as } \lambda \downarrow 0. \tag{2.3.2}$$

**Fact 9.** Assume that for all  $j \geq 1$ ,  $\kappa_j$  is steep. For all  $n \geq 1$ , set

$$\bar{\kappa}_n := \frac{1}{n} \sum_{j=1}^n \kappa_j. \tag{2.3.3}$$

Then, for all  $n \geq 1$ ,  $\bar{\kappa}_n$  is steep.

*Proof.* For all  $n \geq 1$ ,  $\bar{\kappa}_n$  clearly satisfies the assumptions of Theorem 7. Now, for all  $j \geq 1$ ,  $\kappa_j$  being steep,  $\kappa_j$  satisfies (2.3.2). We deduce readily that  $\bar{\kappa}_n$  satisfies (2.3.2), which implies that  $\bar{\kappa}_n$  is steep.  $\square$

**Definition 6.** Let  $C$  be an open convex subset of  $\mathbb{R}^d$ . Let  $f$  be a strictly convex and differentiable function on  $C$ . Assume that  $f$  is steep. Then the pair  $(C, f)$  is said to be of Legendre type.

**Definition 7.** Let  $f$  be a convex function on  $\mathbb{R}^d$ . Its conjugate function is defined on  $\mathbb{R}^d$  by

$$f^*(a) = \sup_{x \in \mathbb{R}^d} \{\langle x, a \rangle - f(x)\} \tag{2.3.4}$$

We have the following result, which is Theorem 5.33. in (Barndorff-Nielsen Ole, 2014).

**Theorem 8.** Let  $f$  be a convex and lower semi-continuous function. Let  $C = \text{int}(\text{dom}(f))$  and  $C^* = \text{int}(\text{dom}(f^*))$ . If the pair  $(C, f)$  is of Legendre type, then the gradient mapping  $\nabla f$  is a homeomorphism from the open convex set  $C$  onto the open convex set  $C^*$ , and  $\nabla(f^*) = (\nabla f)^{-1}$ .

Then, we can state and prove the main result of this Section.

**Theorem 9.** *Assume that for all  $j \geq 1$ ,  $\kappa_j := \log \Phi_j$  is strictly convex and steep. Then, for all  $n \geq 1$  and any  $a \in \text{int}(C_X)$ , there exists a unique  $\theta_n^a \in \text{int}(\Theta)$  such that*

$$\nabla \bar{\kappa}_n(\theta_n^a) = a. \quad (2.3.5)$$

Namely, for any  $n \geq 1$  and  $a \in \text{int}(C_X)$ ,

$$\theta_n^a = \nabla(\bar{\kappa}_n)^*(a). \quad (2.3.6)$$

*Proof.* For all  $n \geq 1$ ,  $\text{dom}(\bar{\kappa}_n) = \Theta$  is an open convex set and  $\bar{\kappa}_n$  is strictly convex and differentiable on  $\text{int}(\Theta)$ , since by assumption, the  $\kappa_j$ 's are. Now, we get from Fact 9 that  $\bar{\kappa}_n$  is steep. Therefore, the pair  $(\Theta, \bar{\kappa}_n)$  is of Legendre type. Furthermore,  $\bar{\kappa}_n$  is lower semi-continuous. Therefore, we obtain from Theorem 8 that the gradient mapping  $\nabla \bar{\kappa}_n : \Theta \rightarrow \text{int}(\text{dom}((\bar{\kappa}_n)^*))$  is a homeomorphism. We conclude the proof by Lemma 2 below.  $\square$

**Lemma 2.** *For any  $n \geq 1$ , we have that  $\text{int}(\text{dom}((\bar{\kappa}_n)^*)) = \text{int}(C_X)$ .*

*Proof.* The proof is given in Appendix.  $\square$

### 2.3.2 Sufficiency Theory

**Definition 8.** *Let  $(\mathcal{E}, \mathcal{A})$  be a measurable space. Let  $\Sigma$  be a sub  $\sigma$ -algebra of  $\mathcal{A}$ . Let  $P$  and  $Q$  be probability measures on  $(\mathcal{E}, \mathcal{A})$ . We say that  $\Sigma$  is sufficient w.r.t.  $P$  and  $Q$  if for all  $A \in \mathcal{A}$ ,*

$$P(A|\Sigma) = Q(A|\Sigma) \text{ almost everywhere (a.e.) } P \text{ and a.e. } Q. \quad (2.3.7)$$

**Lemma 3.** *For any sub  $\sigma$ -algebra  $\mathcal{G}$  of  $\mathcal{A}$ , set*

$$\|P - Q\|_{\mathcal{G}} := 2 \sup_{A \in \mathcal{G}} |P(A) - Q(A)|.$$

*Assume that  $\Sigma \subset \mathcal{A}$  is sufficient w.r.t.  $P$  and  $Q$ . Then*

$$\|P - Q\|_{\Sigma} = \|P - Q\|_{\mathcal{A}}. \quad (2.3.8)$$

*Proof.* The proof is elementary. See Lemma (2.4) in (Diaconis and Freedman, 1987) for details.  $\square$

**Lemma 4.** *Let  $\mathcal{P}$  be a probability measure on  $((\mathbb{R}^d)^k, \mathcal{B}((\mathbb{R}^d)^k))$  with density  $p$  w.r.t the Lebesgue measure. Let  $T$  be the map defined on  $(\mathbb{R}^d)^k$  by  $T(x) = \sum_{i=1}^k x_i$ , for  $x = (x_i)_{1 \leq i \leq k} \in (\mathbb{R}^d)^k$ . For any  $t \in \mathbb{R}^d$ , let*

$$L_t := \left\{ x \in (\mathbb{R}^d)^k : T(x) = t \right\}.$$

*and let  $\sigma^t$  be the natural measure on  $L_t$ . (The definition is recalled in Appendix). Then, the map  $\nu_{\mathcal{P}}$  defined on  $\mathbb{R}^d \times \mathcal{B}((\mathbb{R}^d)^k)$  by*

$$\nu_{\mathcal{P}}(t, A) = \frac{\int_{L_t \cap A} p(x) d\sigma^t(x)}{\int_{L_t} p(x) d\sigma^t(x)} \text{ if } L_t \cap A \neq \emptyset, \text{ and } \nu_{\mathcal{P}}(t, A) = 0 \text{ if } L_t \cap A = \emptyset \quad (2.3.9)$$

*is a regular conditional  $\mathcal{P}$ -distribution for  $I_d$  given  $T$ , where  $I_d$  is the identity map on  $(\mathbb{R}^d)^k$ .*

*Proof.* The proof, which uses some elementary differential geometry, is given in Appendix.  $\square$

**Lemma 5.** *Let  $T$  be the map defined on  $(\mathbb{R}^d)^k$  by  $T(x) = \sum_{i=1}^k x_i$ , for  $x = (x_i)_{1 \leq i \leq k} \in (\mathbb{R}^d)^k$ . Let  $\Sigma$  the sub  $\sigma$ -algebra of  $\mathcal{B}((\mathbb{R}^d)^k)$  generated by  $T$ . Then, for any  $\theta \in \Theta$ ,*

$$\|Q_{nak} - \tilde{P}_{1,k}^\theta\|_\Sigma = \|Q_{nak} - \tilde{P}_{1,k}^\theta\|_{\mathcal{A}}, \quad \text{where} \quad \tilde{P}_{1,k}^\theta := \prod_{j=1}^k \tilde{P}_j^\theta. \quad (2.3.10)$$

*Proof.* Let  $\theta \in \Theta$ . Recall that  $Q_{nak}$  and  $\tilde{P}_{1,k}^\theta$  are a.c. w.r.t. the Lebesgue measure with respective densities  $q_{nak}$  and  $\tilde{p}_{1,k}^\theta$  given by

$$q_{nak}(x_1^k) = \frac{p_1^k(x_1^k) p_{S_{k+1},n}(na - T(x_1^k))}{p_{S_{1,n}}(na)}, \quad \text{where} \quad p_1^k(x_1^k) := \prod_{j=1}^k p_j(x_j), \quad (2.3.11)$$

and

$$\tilde{p}_{1,k}^\theta(x_1^k) = \frac{p_1^k(x_1^k) \exp\langle \theta, T(x_1^k) \rangle}{\Phi_1^k(\theta)}, \quad \text{where} \quad \Phi_1^k := \prod_{j=1}^k \Phi_j. \quad (2.3.12)$$

Since on  $L_t$ , we have that  $T(x_1^k) = t$ , we deduce readily that for any  $t \in \mathbb{R}^d$  and  $A \in \mathcal{B}((\mathbb{R}^d)^k)$ ,

$$\nu_{Q_{nak}}(t, A) = \nu_{\tilde{P}_{1,k}^\theta}(t, A) = \frac{\int_{L_t \cap A} p_1^k(x_1^k) d\sigma^t(x)}{\int_{L_t} p_1^k(x_1^k) d\sigma^t(x)} \quad \text{if } L_t \cap A \neq \emptyset, \quad (2.3.13)$$

and

$$\nu_{Q_{nak}}(t, A) = \nu_{\tilde{P}_{1,k}^\theta}(t, A) = 0 \quad \text{if } L_t \cap A = \emptyset. \quad (2.3.14)$$

Consequently,  $\Sigma$  is sufficient w.r.t  $Q_{nak}$  and  $\tilde{P}_{1,k}^\theta$ , which concludes the proof.  $\square$

### 2.3.3 Edgeworth expansion

We obtain from the following theorem (theorem 19.3 in (Bhattacharya and Rao, 1976)) an Edgeworth expansion for a sequence of independent random vectors.

**Theorem 10.** *Let  $\{X_n : n \geq 1\}$  be a sequence of independent random vectors with values in  $\mathbb{R}^d$ , having zero means and average positive-definite covariance matrices  $V_n$  for any  $n$  large enough. Set*

$$B_n := (V_n)^{-1/2}, \quad \text{where} \quad V_n := \frac{1}{n} \sum_{j=1}^n \text{Cov}(X_j). \quad (2.3.15)$$

Assume that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \mathbb{E} \left[ \|B_n X_j\|^4 \right] < \infty. \quad (2.3.16)$$

Assume also the existence of an integer  $p > 0$  such that for  $n \geq p + 1$  and  $0 \leq m \leq n - p$ , the functions

$$g_{m,n}(t) := \prod_{j=m+1}^{j=m+p} |\mathbb{E}[\exp\{i\langle t, B_n X_j \rangle\}]| \quad (2.3.17)$$

satisfy

$$\gamma := \sup_{n \geq p+1} \sup_{0 \leq m \leq n-p} \int g_{m,n}(t) dt < \infty \quad (2.3.18)$$

and, for all  $b > 0$ ,

$$\delta(b) := \sup_{n \geq p+1} \sup_{0 \leq m \leq n-p} \sup_{\|t\| > b} g_{m,n}(t) < 1. \quad (2.3.19)$$

Let  $\phi$  be the density of the standard normal distribution on  $\mathbb{R}^d$ . Then, the distribution  $Q_n$  of  $n^{-1/2}B_n S_n$  has a density  $q_n$  for all  $n$  large enough, and

$$\sup_{x \in \mathbb{R}^d} (1 + \|x\|^4) \left| q_n(x) - \left[ \phi(x) + n^{-1/2} P_1(-\phi : \{\bar{\chi}_{\nu,n}\})(x) \right] \right| = \mathcal{O}\left(\frac{1}{n}\right), \quad (2.3.20)$$

where  $P_1(-\phi : \{\bar{\chi}_{\nu,n}\})(x) = \phi(x) P_1^\#(x)$  and

$$P_1^\#(x) = \sum_{|\nu|=3} \bar{\chi}_{\nu,n} H_3^{(\nu)}(x), \quad (2.3.21)$$

where  $H_3^{(\nu)}$  is a polynomial function of degree 3 which vanish at 0 and  $\bar{\chi}_{\nu,n}$  is the average of the  $\nu$ th cumulants of  $B_n X_j$  with  $1 \leq j \leq n$ , for  $|\nu| = 3$ . See (7.20) in (Bhattacharya and Rao, 1976) for the precise expressions.

*Proof.* We write hereafter a sketch of the proof. For a given nonnegative integral vector  $\alpha$  with  $|\alpha| \leq 4$ , set

$$h_n(x) = x^\alpha \left( q_n(x) - \left[ \phi(x) + n^{-1/2} P_1(-\phi : \{\bar{\chi}_{3,n}\})(x) \right] \right) \quad (2.3.22)$$

Let  $\hat{h}_n$  be the Fourier transform of  $h_n$ . Then, the Fourier inversion theorem implies that

$$\sup_{x \in \mathbb{R}^d} |h_n(x)| \leq (2\pi)^{-d} \int |\hat{h}_n(t)| dt \quad (2.3.23)$$

The aim is then to bound  $\int |\hat{h}_n(t)| dt$ , by splitting it into a sum of three integrals which are bounded by some  $\mathcal{O}(\frac{1}{n})$ . The key point is that these controls are made *at fixed*  $n$ .  $\square$

We recall that all the notations  $\tilde{\sim}$  considered in the sequel pertain to  $\theta = \theta_n^\alpha$ .

**Corollary 2.** For  $n \geq 1$ , let  $J_n$  be a subset of  $\{1, \dots, n\}$  and  $L_n$  be its complement in  $\{1, \dots, n\}$ . Set  $\alpha_n := |J_n|$  and assume that

$$\lim_{n \rightarrow \infty} |L_n| = \lim_{n \rightarrow \infty} n - \alpha_n = \infty. \quad (2.3.24)$$

Set

$$\tilde{V}_{L_n} := \frac{1}{n - \alpha_n} \sum_{j \in L_n} \text{Cov}(\tilde{X}_j). \quad (2.3.25)$$

Assume that

$$\varliminf_{n \rightarrow \infty} \lambda_{\min}(\tilde{V}_{L_n}) > 0, \quad (2.3.26)$$

which implies in particular that for all  $n$  large enough,  $\tilde{V}_{L_n}$  is positive-definite, so that we may set

$$\tilde{B}_{L_n} := \left( \tilde{V}_{L_n} \right)^{-1/2}. \quad (2.3.27)$$

Suppose that

$$\varlimsup_{n \rightarrow \infty} \frac{1}{n - \alpha_n} \sum_{j \in L_n} \mathbb{E} \left[ \left\| \tilde{B}_{L_n} \left( \tilde{X}_j - m_j(\theta_n^a) \right) \right\|^4 \right] < \infty. \quad (2.3.28)$$

Suppose also that there exists an integer  $p > 0$  such that for all  $n$  larger than some  $N_p$ , to insure that  $\alpha_n \geq p + 1$ , the functions

$$\tilde{g}_{m,n}(t) := \prod_{j=m+1}^{m+p} \left| \mathbb{E} \left[ \exp \left\{ i \left\langle t, \tilde{B}_{L_n} \tilde{X}_{j_\ell} \right\rangle \right\} \right] \right| \quad (0 \leq m \leq \alpha_n - p) \quad (2.3.29)$$

satisfy

$$\tilde{\gamma} := \sup_{n \geq N_p} \sup_{0 \leq m \leq \alpha_n - p} \int \tilde{g}_{m,n}(t) dt < \infty, \quad (2.3.30)$$

and, for all  $b > 0$ ,

$$\tilde{\delta}(b) := \sup_{n \geq N_p} \sup_{0 \leq m \leq \alpha_n - p} \sup_{\|t\| > b} \tilde{g}_{m,n}(t) < 1. \quad (2.3.31)$$

Then the density  $\bar{q}_{L_n}$  of  $\bar{S}_{L_n} = \alpha_n^{-1/2} \tilde{B}_{L_n} \left( \tilde{S}_{L_n} - \sum_{j_\ell \in L_n} m_{j_\ell}(\theta_n^a) \right)$  satisfies

$$\sup_{x \in \mathbb{R}^d} (1 + \|x\|^4) \left| \bar{q}_{L_n}(x) - \left[ \phi(x) + \alpha_n^{-1/2} \tilde{P}_1(-\phi : \{\bar{\chi}_{\nu, L_n}\})(x) \right] \right| = \mathcal{O} \left( \frac{1}{\alpha_n} \right), \quad (2.3.32)$$

where  $\bar{\chi}_{\nu, L_n}$  is the average of the  $\nu$ th cumulants of  $\tilde{B}_{L_n} \left( \tilde{X}_j - m_j(\theta_n^a) \right)$  with  $j \in L_n$ , for  $|\nu| = 3$ .

*Proof.* We need to perform an Edgeworth expansion when, instead of a sequence  $\{X_n : n \geq 1\}$  of independent random vectors, we consider a triangular array whose row of index  $n$  is composed of the  $\alpha_n$  independent random vectors

$$\left( \tilde{X}_j^{\theta_n^a} - \mathbb{E} \left[ \tilde{X}_j^{\theta_n^a} \right] \right)_{j \in L_n}, \quad \text{where we recall that } \mathbb{E} \left[ \tilde{X}_j^{\theta_n^a} \right] = m_j(\theta_n^a). \quad (2.3.33)$$

Therefore, in the framework of triangular arrays, we can write analogously these controls, for a fixed row of the array. So, we consider the row of index  $n$  of the triangular array defined by (2.3.33). A careful study of the preceding proof implies that (2.3.32) holds if the assumptions of this corollary hold.  $\square$

## 2.4 Assumptions and Examples

### 2.4.1 Assumptions

The following assumptions are essentially those of our main Theorem, since they imply those of the Preliminary Results.

(*Supp*) : The  $(X_j)$ ,  $j \geq 1$  have a common support  $\mathcal{S}_X$ , and they have positive densisties  $p_j$ .

(*Mgf*) : The mgf's  $(\Phi_j)_{j \geq 1}$  have the same domain of finiteness  $\Theta$ , and  $\text{int}(\Theta) \neq \emptyset$ .

(*Stp*) : For all  $j \geq 1$ ,  $\kappa_j := \log \Phi_j$  is a strictly convex and steep function.

(*Bdθ*) : For any  $a \in \text{int}(C_X)$ , there exists a compact set  $K_a$  of  $\mathbb{R}^d$  such that

$$\{\theta_n^a : n \geq 1\} \subset K_a \subset \text{int}(\Theta). \quad (2.4.1)$$

(*Cv*) : For all  $j \geq 1$  and  $\theta \in \Theta$ ,  $C_j^\theta := \text{Cov}(\tilde{X}_j^\theta)$  is a positive definite matrix and for any compact  $K \subset \text{int}(\Theta)$ ,

$$0 < \inf_{j \geq 1} \inf_{\theta \in K} \lambda_{\min}(C_j^\theta) \leq \sup_{j \geq 1} \sup_{\theta \in K} \lambda_{\max}(C_j^\theta) < \infty, \quad (2.4.2)$$

where  $\lambda_{\min}(C_j^\theta)$  (resp.  $\lambda_{\max}(C_j^\theta)$ ) is the smallest (resp. largest) eigenvalue of  $C_j^\theta$ .

(*AM4*) : For any compact  $K \subset \text{int}(\Theta)$ ,

$$\sup_{j \geq 1} \sup_{\theta \in K} \mathbb{E} \left[ \left\| \tilde{X}_j^\theta - m_j(\theta) \right\|^4 \right] < \infty. \quad (2.4.3)$$

For any  $j \geq 1$ , let  $\xi_j$  be the characteristic function of  $X_j$  and for any  $\theta \in \Theta$ , denote respectively by  $\tilde{p}_j^\theta$  and  $\tilde{\xi}_j^\theta$  the density and the characteristic function of  $\tilde{X}_j^\theta$ .

(*Cf1*) : For any compact  $K \subset \text{int}(\Theta)$ , there exist positive constants  $\delta_K, C_K, R_K$  such that

$$\forall j \geq 1, \forall \|t\| \geq R_K, \sup_{\theta \in K} |\tilde{\xi}_j^\theta(t)| \leq \frac{C_K}{\|t\|^{\delta_K}}. \quad (2.4.4)$$

(*Cf2*) : For any  $j \geq 1$ ,  $p_j$  is a function of class  $\mathcal{C}^1$  and for any compact  $K \subset \text{int}(\Theta)$ ,

$$\max_{\ell \in \{1, \dots, d\}} \sup_{j \geq 1} \sup_{\theta \in K} \left\| \frac{\partial \tilde{p}_j^\theta}{\partial x_\ell} \right\|_{L^1} < \infty. \quad (2.4.5)$$

(*Cf3*) : For any compact  $K \subset \text{int}(\Theta)$ , for all  $\beta > 0$ ,

$$\sup_{j \geq 1} \sup_{\|t\| > \beta} \sup_{\theta \in K} |\tilde{\xi}_j^\theta(t)| =: \epsilon_{K, \beta} < 1. \quad (2.4.6)$$

**Remark 1.**  $(\mathcal{B}d\theta)$  is reasonable, since  $\nabla \bar{\kappa}_n$  is a mean of functions. We will see that, when  $d = 1$ , it can be replaced by a natural uniformity assumption, denoted by  $(\mathcal{U}f)$ .

Denote by  $[Ad]$  the set of assumptions  $(\mathcal{B}d\theta)$ ,  $(Cv)$ ,  $(\mathcal{AM}4)$  and  $(Cf1)$ ,  $(Cf2)$ ,  $(Cf3)$ .

**Remark 2.**  $[Ad]$  is natural since it concerns each individual r.v.  $X_j$ ,  $j \geq 1$ . Thereby, the order of the r.v.'s is irrelevant (as in Statistical Mechanics), which makes sense since we intend to study the distribution of any small subset of r.v.'s among those defining the global constraint  $\{S_{1,n} = na\}$ .

**Remark 3.** Most of the assumptions in  $[Ad]$  are of the form  $\sup_{j \geq 1} \sup_{\theta \in K} F_j(\theta)$ , where for any  $j \geq 1$ ,  $F_j$  is a continuous function. Therefore, for fixed  $j \geq 1$ ,  $\sup_{\theta \in K} F_j(\theta) < \infty$ , since  $K$  is compact. So  $[Ad]$  is a convenient to check set of uniformity assumptions.

We prove hereunder that  $[Ad]$  implies the assumptions of Corollary 2. We also prove that  $(\mathcal{B}d\theta)$  and  $(Cf2)$  imply  $(Cf1)$ .

### Covariance

**Fact 10.** Assume that  $(\mathcal{B}d\theta)$  holds and that for any compact  $K \subset \text{int}(\Theta)$ ,

$$\lambda_{\min}^K := \inf_{j \geq 1} \inf_{\theta \in K} \lambda_{\min}(C_j^\theta) > 0. \quad (2.4.7)$$

Then,

$$\liminf_{n \rightarrow \infty} \lambda_{\min}(\tilde{V}_{L_n}) > 0. \quad (2.4.8)$$

*Proof.* Recall from the Courant-Fischer min-max theorem that for any Hermitian matrix  $M$ ,

$$\lambda_{\min}(M) = \inf_{\{x \in \mathbb{R}^d : x \neq 0\}} \frac{x^t M x}{x^t x}. \quad (2.4.9)$$

Let  $K_a$  be a compact subset of  $\text{int}(\Theta)$  such that  $(\theta_n^a)_{n \geq 1} \subset K_a$ . Then, for any  $\theta \in K_a$ , any  $x \in \mathbb{R}^d$  ( $x \neq 0$ ), and any  $j \in L_n$ ,

$$\frac{x^t C_j^\theta x}{x^t x} \geq \lambda_{\min}(C_j^\theta) \geq \lambda_{\min}^{K_a}. \quad (2.4.10)$$

Therefore, for all  $n \geq 1$ ,

$$\inf_{\theta \in K_a} \lambda_{\min}(\tilde{V}_{L_n}^\theta) = \inf_{\theta \in K_a} \inf_{\{x \in \mathbb{R}^d : x \neq 0\}} \frac{x^t \tilde{V}_{L_n}^\theta x}{x^t x} \geq \lambda_{\min}^{K_a} > 0. \quad (2.4.11)$$

□

### Absolute Moments of order 4

**Fact 11.**  $(\mathcal{AM}4)$ ,  $(\mathcal{B}d\theta)$  and  $(Cv)$  imply that (2.3.28) holds.

*Proof.* For any  $j \in L_n$ ,

$$\mathbb{E} \left[ \left\| \tilde{B}_{L_n} \left( \tilde{X}_j - m_j(\theta_n^a) \right) \right\|^4 \right] \leq \lambda_{\min}(\tilde{V}_{L_n})^{-2} \mathbb{E} \left[ \left\| \tilde{X}_j - m_j(\theta_n^a) \right\|^4 \right]. \quad (2.4.12)$$

Therefore, (2.3.28) holds if  $\liminf_{n \rightarrow \infty} \lambda_{\min}(\tilde{V}_{L_n}) > 0$  and  $(\mathcal{B}d\theta)$  together with  $(\mathcal{AM}4)$  hold. □



### Characteristic function

**Lemma 6.** *Assume that  $(\mathcal{B}d\theta)$  and  $(Cf1)$  hold. Then, (2.3.30) holds for any  $p > \frac{1}{\delta_{K_A}}$ .*

*Proof.* The proof is given in Appendix.  $\square$

**Corollary 3.** *Assume that  $(\mathcal{B}d\theta)$  and  $(Cf2)$  hold. Then,  $(Cf1)$  holds for all  $t \in \mathbb{R}^d$ ,  $t \neq 0$ , with  $\delta_K = 1$ .*

*Proof.* The proof is given in Appendix.  $\square$

**Lemma 7.** *Assume that  $(\mathcal{B}d\theta)$  and  $(Cf3)$  hold. Then, (2.3.31) holds for any  $p > 0$ .*

*Proof.* Let  $p > 0$ ,  $n \geq N_p$  and  $0 \leq m \leq \alpha_n - p$ . For any  $b > 0$  and  $t \in \mathbb{R}^d$  such that  $\|t\| > b$ , we have that

$$\tilde{g}_{m,n}(t) := \prod_{j=m+1}^{m+p} \left| \tilde{\xi}_j \left( \tilde{B}_{L_n} t \right) \right| \leq (\epsilon_{K_a, \lambda_{\min} b})^p < 1. \quad (2.4.13)$$

$\square$

### 2.4.2 The one-dimensional case

Assume here that  $d = 1$ . For any r.v.  $X$  or for a sequence  $(X_j)_{j \geq 1}$  of i.i.d. r.v.'s, set

$$\kappa := \log(\Phi_X) \quad ; \quad m := \frac{d\kappa}{d\theta} \quad \text{and} \quad s^2 := \frac{d^2\kappa}{d\theta^2}.$$

If  $(X_j)_{j \geq 1}$  is not an identically distributed sequence of r.v.'s, then for any  $j \geq 1$ , set

$$\kappa_j := \log(\Phi_{X_j}) \quad ; \quad m_j := \frac{d\kappa_j}{d\theta} \quad \text{and} \quad s_j^2 := \frac{d^2\kappa_j}{d\theta^2}.$$

**Fact 12.** *For any  $\theta \in \Theta_X$ ,*

$$\mathbb{E}[\tilde{X}^\theta] = m(\theta) \quad \text{and} \quad \text{Var}(\tilde{X}^\theta) = s^2(\theta). \quad (2.4.14)$$

In the sequel,  $\Theta$  and  $\mathcal{S}_X$  pertain to r.v.'s  $X_j$ ,  $j \geq 1$ , with common support and common domain of finiteness of their mgf's. Since  $\Theta$  and  $\text{conv}(\mathcal{S}_X)$  are convex,  $\text{int}(\Theta)$  and  $\text{int}(C_X)$  are open convex subsets of  $\mathbb{R}$ , which are open intervals. Therefore, we can write  $\text{int}(\Theta) = (\alpha, \beta)$  and  $\text{int}(C_X) = (A, B)$ , where  $\alpha, \beta, A, B$  may be finite or not.

**Definition 9.** *Let  $f : (\alpha, \beta) \rightarrow (A, B)$  be a differentiable function. Consider the following property.*

$(\mathcal{H})$  : *For all  $\theta \in \text{int}(\Theta)$ ,  $\frac{df}{d\theta}(\theta) > 0$  and  $\lim_{\theta \rightarrow \alpha} f(\theta) = A$  ;  $\lim_{\theta \rightarrow \beta} f(\theta) = B$ .*

**Fact 13.** *If  $f$  satisfies  $(\mathcal{H})$ , then  $f$  is a homeomorphism from  $\text{int}(\Theta) = (\alpha, \beta)$  to  $\text{int}(C_X) = (A, B)$ .*

If  $d > 1$ , then Theorem 8 requires that  $\bar{\kappa}_n$  is steep, in the sense of Definition 5, while when  $d = 1$ , this notion of steepness is not necessary to get the conclusion of Theorem 8. Indeed, for all  $n \geq 1$ ,  $\frac{d\bar{\kappa}_n}{d\theta}$  is a homeomorphism from  $\text{int}(\Theta)$  to  $\text{int}(C_X)$ , provided that  $\frac{d\bar{\kappa}_n}{d\theta}$  satisfies  $(\mathcal{H})$ . Consider the following assumptions.

$(\mathcal{H}\kappa)$  : For all  $j \geq 1$ ,  $m_j := \frac{d\kappa_j}{d\theta}$  satisfies  $(\mathcal{H})$ .

$(\mathcal{U}f)$  : There exist functions  $f_+$  and  $f_-$  which satisfy  $(\mathcal{H})$  and such that

$$\forall j \geq 1, \forall \theta \in \Theta, f_-(\theta) \leq m_j(\theta) \leq f_+(\theta). \quad (2.4.15)$$

**Fact 14.**  $(\mathcal{H}\kappa)$  implies that  $\frac{d\bar{\kappa}_n}{d\theta}$  is a homeomorphism from  $\text{int}(\Theta)$  to  $\text{int}(C_X)$  and in particular that for any  $a \in \text{int}(C_X)$ , for any  $n \geq 1$ , there exists a unique  $\theta_n^a$  such that

$$\frac{d\bar{\kappa}_n}{d\theta}(\theta_n^a) = \frac{1}{n} \sum_{j=1}^n m_j(\theta_n^a) = a. \quad (2.4.16)$$

**Fact 15.** The uniformity assumption  $(\mathcal{U}f)$  implies that  $(\mathcal{B}d\theta)$  holds.

*Proof.* For any  $j \geq 1$  and  $n \geq 1$ , we have that

$$f_-(\theta_n^a) \leq m_j(\theta_n^a) \leq f_+(\theta_n^a). \quad (2.4.17)$$

Therefore, for all  $n \geq 1$ ,

$$f_-(\theta_n^a) \leq \bar{m}_n(\theta_n^a) = a \leq f_+(\theta_n^a), \quad (2.4.18)$$

which implies that

$$(f_+)^{-1}(a) \leq \theta_n^a \leq (f_-)^{-1}(a). \quad (2.4.19)$$

□

We deduce from these considerations that, when  $d = 1$ , we can replace  $(\mathcal{S}tp)$  and  $(\mathcal{B}d\theta)$  by respectively  $(\mathcal{H}\kappa)$  and  $(\mathcal{U}f)$ .

### 2.4.3 Examples

#### Normal distribution

For any  $j \geq 1$ ,  $X_j$  is a r.v. with normal distribution. Set  $\mu_j := \mathbb{E}[X_j]$  and  $\Gamma_j := \text{Cov}(X_j)$ . Assume that

$$\sup_{j \geq 1} \|\mu_j\| < \infty \quad \text{and} \quad 0 < \inf_{j \geq 1} \lambda_{\min}(\Gamma_j) \leq \sup_{j \geq 1} \lambda_{\max}(\Gamma_j) < \infty. \quad (2.4.20)$$

We recall that, for any  $j \geq 1$ , for all  $\theta \in \Theta = \mathbb{R}^d$ ,

$$\kappa_j(\theta) = \mu_j' \theta + \frac{1}{2} \theta' \Gamma_j \theta \quad \text{and} \quad \nabla \kappa_j(\theta) = \left( (\mu_j)_\ell + \sum_{\ell'=1}^d \theta_{\ell'} (\Gamma_j)_{\ell, \ell'} \right)_{1 \leq \ell \leq d}. \quad (2.4.21)$$

So, for all  $\theta \in \mathbb{R}^d$ , the Hessian matrix of  $\kappa_j$  at  $\theta$  is equal to  $\Gamma_j$ . Since for all  $\theta \in \mathbb{R}^d$ , this matrix is equal to  $C_j^\theta$ , we get that  $(\mathcal{C}v)$  holds. Since for any  $j \geq 1$ ,  $\Gamma_j$  is positive definite, we deduce also that  $\kappa_j$  is strictly convex. Clearly,  $\nabla \kappa_j$  satisfies (2.3.1), so that  $\kappa_j$  is steep and  $(\mathcal{S}tp)$  holds.

Set  $\bar{\mu}_n := \frac{1}{n} \sum_{j=1}^n \mu_j$  and  $\bar{\Gamma}_n := \frac{1}{n} \sum_{j=1}^n \Gamma_j$ . We get after some elementary calculations that for any  $a \in \text{int}(C_X)$  and  $n \geq 1$ , the equation  $\nabla \bar{\kappa}_n(\theta) = a$  is equivalent to

$$(\bar{\Gamma}_n) \theta = a - \bar{\mu}_n. \quad (2.4.22)$$

Then, (2.4.20) implies readily that (2.4.22) defines a unique  $\theta_n^a$  and that the sequence  $(\theta_n^a)_{n \geq 1}$  is bounded, so that (Bd $\theta$ ) holds. Finally, it is straightforward to get from the expression of  $p_j$  and the boundedness conditions, that (AM4) and (Cf2) hold.

### Gamma distribution

Fix  $t > 0$ . For any  $j \geq 1$ ,  $X_j$  is a random variable ( $d = 1$ ) with distribution  $\Gamma(k_j, t)$ , such that

$$2 < k_- := \inf_{j \geq 1} k_j \leq k_+ := \sup_{j \geq 1} k_j < \infty. \quad (2.4.23)$$

For any  $j \geq 1$  and  $x \geq 0$ ,

$$p_j(x) = \frac{x^{k_j-1} \exp(-\frac{x}{t})}{\Gamma(k_j)t^{k_j}} \quad (2.4.24)$$

Recall that for any  $j \geq 1$ ,

$$S_X = C_X = (0; \infty) ; \Phi_j(\theta) = (1 - t\theta)^{-k_j} ; \Theta = (-\infty, \frac{1}{t}). \quad (2.4.25)$$

We check readily that (Mgf), (Stp) and (Cv) hold, since, for any  $j \geq 1$  and  $\theta \in \Theta$ ,

$$\kappa_j(\theta) = -k_j \log(1 - t\theta) ; m_j(\theta) = k_j t (1 - t\theta)^{-1} ; s_j^2(\theta) = k_j (1 - t\theta)^{-1} [1 + t\theta(1 - t\theta)^{-1}]. \quad (2.4.26)$$

Furthermore, (Uf) holds, since for any  $j \geq 1$  and  $\theta \in \Theta$ ,

$$f_-(\theta) := \frac{(k_-)t}{1 - t\theta} \leq m_j(\theta) \leq f_+(\theta) := \frac{(k_+)t}{1 - t\theta}. \quad (2.4.27)$$

Now, we have that, for any  $j \geq 1$  and  $\theta \in \Theta$ ,

$$\tilde{p}_j^\theta(x) = \frac{x^{k_j-1} \exp[x(\theta - \frac{1}{t})]}{\Phi_j(\theta)\Gamma(k_j)t^{k_j}}. \quad (2.4.28)$$

For  $\theta \in \Theta$ , we have that  $\theta - \frac{1}{t} < 0$ . Thereby, we deduce readily that (AM4) holds. We also get (Cf2), since  $\frac{d\tilde{p}_j^\theta}{dx}(x)$  is of the form  $P(x) \exp[x(\theta - \frac{1}{t})]$ , where  $P$  is a polynomial function.

## 2.5 Main Result

In the sequel, for any probability measures  $P$  and  $Q$  on  $\mathbb{R}^k$ , we denote the total variation distance between  $P$  and  $Q$  by

$$\|P - Q\|_{TV} := \sup_{B \in \mathcal{B}(\mathbb{R}^k)} |P(B) - Q(B)|.$$

### 2.5.1 Theorem of Diaconis and Freedman

**Theorem 11.** *Let  $(X_j)_{j \geq 1}$  be a sequence of i.i.d. random variables ( $d = 1$ ). Assume that  $\Theta = (\alpha, \beta)$  and  $\mathcal{S}_X = (A, B)$ , where  $\alpha, \beta, A, B$  may be finite or not. This implies that  $\text{int}(\Theta) = (\alpha, \beta)$  and  $\text{int}(C_X) = (A, B)$ . Assume that the function*

$$m := \frac{d(\log(\Phi_X))}{d\theta} \text{ satisfies } (\mathcal{H})$$

and that for any  $\theta \in \Theta$ ,

$$\mathbb{E} \left[ \left( \frac{\tilde{X}^\theta - m(\theta)}{s(\theta)} \right)^4 \right] < \infty. \quad (2.5.1)$$

Suppose that there exists  $\nu \geq 1$  such that for any  $\theta \in \Theta$ ,

$$\int \left| \mathbb{E} \left[ \exp \left( it \frac{\tilde{X}^\theta}{s(\theta)} \right) \right] \right|^\nu dt < \infty, \quad (2.5.2)$$

and that for any  $\theta \in \Theta$ , for all  $b > 0$ ,

$$\sup_{|t| > b} \left| \mathbb{E} \left[ \exp \left( it \frac{\tilde{X}^\theta}{s(\theta)} \right) \right] \right| < 1. \quad (2.5.3)$$

Assume that  $\frac{k}{n} \rightarrow 0$  and  $k \rightarrow \infty$ , as  $n \rightarrow \infty$ . Set  $\gamma := \frac{1}{2} \mathbb{E} [|1 - Z^2|]$ , where  $Z$  is of standard normal distribution. Then, for any  $a \in \mathcal{S}_X$ ,

$$\left\| Q_{nak} - \tilde{P}_1^k \right\|_{TV} = \gamma \frac{k}{n} + o\left(\frac{k}{n}\right), \quad (2.5.4)$$

where  $\tilde{P}_1^k$  is the joint distribution of independent r.v.'s  $(\tilde{X}_j)_{1 \leq j \leq k}$ , having the tilted density defined by  $\theta^a$  such that  $m(\theta^a) = a$ .

### 2.5.2 Main Theorem and Proof

**Theorem 12.** *When  $d > 1$ , assume that  $(Mgf)$ ,  $(Stp)$ ,  $(Bd\theta)$ ,  $(Cv)$ ,  $(AM4)$ ,  $(Cf2)$ ,  $(Cf3)$  hold. (See Section 4 for weaker assumptions). When  $d = 1$ , we can replace  $(Stp)$  and  $(Bd\theta)$  by respectively  $(\mathcal{H}\kappa)$  and  $(Uf)$ . If  $k = o(n)$ , then for any  $a \in \text{int}(C_X)$ ,*

$$\left\| Q_{nak} - \tilde{P}_1^k \right\|_{TV} = \mathcal{O}\left(\frac{k}{n}\right), \quad (2.5.5)$$

where  $\tilde{P}_1^k$  is the joint distribution of independent r.v.'s  $(\tilde{X}_j)_{1 \leq j \leq k}$ .

*Proof.* Let  $n \geq 1$ . Let  $R_{nak}$  be the distribution of  $S_{1,k} := \sum_{j=1}^k X_j$  given  $S_{1,n} = na$ . Let  $\tilde{R}_{1,k}$  be the distribution of  $\tilde{S}_{1,k} := \sum_{j=1}^k \tilde{X}_j$ . Then, we obtain from Sufficiency Theory (Section 3.2) that

$$\left\| Q_{nak} - \tilde{P}_1^k \right\|_{TV} = \left\| R_{nak} - \tilde{R}_{1,k} \right\|_{TV}. \quad (2.5.6)$$

Now, by Scheffe's theorem, we deduce that

$$\left\| Q_{nak} - \tilde{P}_{1,k} \right\|_{TV} = \int \left| p(S_{1,k} = t | S_{1,n} = na) - p_{\tilde{S}_{1,k}}(t) \right| dt. \quad (2.5.7)$$

Then, we can check readily the following invariance of the conditional density : for any  $t \in \mathbb{R}^d$ ,

$$p(S_{1,k} = t | S_{1,n} = na) = p(\tilde{S}_{1,k} = t | \tilde{S}_{1,n} = na) = p_{\tilde{S}_{1,k}}(t) \left( \frac{p_{\tilde{S}_{k+1,n}}(na - t)}{p_{\tilde{S}_{1,n}}(na)} \right). \quad (2.5.8)$$

For any integers  $\ell, m$  with  $1 \leq \ell \leq m$ , we denote by  $f_{\ell,m}$  the density of  $\tilde{S}_{\ell,m} := \sum_{j=\ell}^m \tilde{X}_j$ . Therefore, we deduce readily from (2.5.6), (2.5.7) and (2.5.8) that

$$\left\| Q_{nak} - \tilde{P}_{1,k} \right\|_{TV} = \int \left| \frac{f_{k+1,n}(na - t)}{f_{1,n}(na)} - 1 \right| f_{1,k}(t) dt. \quad (2.5.9)$$

First, we need to normalize in order to perform Edgeworth expansions. Recall that if  $X$  is a random vector with density  $p_X$ , then the normalized random vector  $\bar{X}$  has a density given by

$$p_X(x) = \det \left[ Cov(X)^{-1/2} \right] p_{\bar{X}} \left( Cov(X)^{-1/2} (x - \mathbb{E}[X]) \right). \quad (2.5.10)$$

Set

$$\tilde{t} := Cov(\tilde{S}_{1,k})^{-1/2} (t - \mathbb{E}[\tilde{S}_{1,k}]) = k^{-1/2} \tilde{B}_{1,k} \left[ t - \sum_{j=1}^k m_j(\theta_n^a) \right] \quad (2.5.11)$$

and

$$t^\# := Cov(\tilde{S}_{k+1,n})^{-1/2} (na - t - \mathbb{E}[\tilde{S}_{k+1,n}]) = (n - k)^{-1/2} \tilde{B}_{k+1,n} \left[ \sum_{j=1}^k m_j(\theta_n^a) - t \right] \quad (2.5.12)$$

Therefore,  $\tilde{t}$  and  $t^\#$  are linked by

$$t^\# = - \left[ \frac{k}{n - k} \right]^{1/2} \tilde{B}_{k+1,n} (\tilde{B}_{1,k})^{-1} \tilde{t}. \quad (2.5.13)$$

□

**Lemma 8.** *Let  $0 < \theta_1 < 1$ . Then,*

$$\frac{f_{k+1,n}(na - t)}{f_{1,n}(na)} = \left[ 1 + \mathcal{O} \left( \frac{k}{n} \right) \right] \exp \left( - \frac{\|t^\#\|^2}{2} \right) + \mathcal{O} \left( \frac{\sqrt{k}}{n} \right) \|\tilde{t}\| + \mathcal{O} \left( \frac{1}{n} \right). \quad (2.5.14)$$

(2.5.14) holds uniformly in  $n, k, a, t$  with  $k < \theta_1 n$ .

*Proof.* For any integers  $\ell, m$  with  $1 \leq \ell \leq m$ , we denote by  $g_{\ell, m}$  the density of the *normalized* r.v. associated to  $\tilde{S}_{\ell, m}$ . So, we have that

$$\frac{f_{k+1, n}(na - t)}{f_{1, n}(na)} = \frac{\det \left( \text{Cov}(\tilde{S}_{1, n}) \right)^{1/2}}{\det \left( \text{Cov}(\tilde{S}_{k+1, n}) \right)^{1/2}} \frac{g_{k+1, n}(t^\#)}{g_{1, n}(0)} \quad (2.5.15)$$

The assumptions allow us to perform Edgeworth expansions to obtain that

$$g_{1, n}(0) = \phi(0) + \mathcal{O} \left( \frac{1}{n} \right), \quad \text{since } \tilde{P}_1(0) = 0. \quad (2.5.16)$$

and

$$g_{k+1, n}(t^\#) = \phi(t^\#) + \frac{1}{(n-k)^{1/2}} \tilde{P}_1(-\phi : \{\bar{\chi}_{\nu, L_n}\})(t^\#) + \mathcal{O} \left( \frac{1}{n-k} \right) \quad (2.5.17)$$

where  $L_n = \{k+1, \dots, n\}$  and  $\tilde{P}_1(-\phi : \{\bar{\chi}_{\nu, L_n}\})(t^\#) = \phi(t^\#) \tilde{P}_1^\#(t^\#)$ , with

$$\tilde{P}_1^\#(t^\#) = \sum_{|\nu|=3} \bar{\chi}_{\nu, L_n} H_3^{(\nu)}(t^\#). \quad (2.5.18)$$

Now, for  $|\nu| = 3$ , the  $\nu$ -cumulant of a centered random vector is equal to its  $\nu$ -moment. See (6.21) in (Bhattacharya and Rao, 1976) for details. Furthermore, the cumulants are invariant by any translation. Therefore,

$$\bar{\chi}_{\nu, L_n} = \frac{1}{n-k} \sum_{k+1 \leq j \leq n} \mathbb{E}[(\tilde{B}_{L_n} \tilde{X}_j)^\nu]. \quad (2.5.19)$$

Then, we have that

$$|\bar{\chi}_{\nu, L_n}| \leq \frac{1}{n-k} \sum_{k+1 \leq j \leq n} \mathbb{E} \left[ |(\tilde{B}_{L_n} \tilde{X}_j)^\nu| \right] \leq \frac{1}{n-k} \sum_{k+1 \leq j \leq n} \mathbb{E} \left[ \left\| \tilde{B}_{L_n} \tilde{X}_j \right\|^{|\nu|} \right]. \quad (2.5.20)$$

Now, we have that

$$\left\| \tilde{B}_{L_n} \tilde{X}_j \right\|_\infty^{|\nu|} = \left\| \tilde{B}_{L_n} \tilde{X}_j \right\|_\infty^3 \leq A \left\| \tilde{B}_{L_n} \right\|^3 \cdot \left\| \tilde{X}_j \right\|^3, \quad (2.5.21)$$

where  $A$  is an absolute constant which appears by equivalence of the norms. Now, the assumptions on the covariance matrices and on the absolute moments of order 4 imply that  $\mathbb{E} \left[ \left\| \tilde{B}_{L_n} \tilde{X}_j \right\|_\infty^{|\nu|} \right] = O(1)$ , so that  $\bar{\chi}_{\nu, L_n} = O(1)$  and

$$g_{k+1, n}(t^\#) = \phi(t^\#) \left[ 1 + \mathcal{O} \left( \frac{1}{(n-k)^{1/2}} \right) \sum_{|\nu|=3} H_3^{(\nu)}(t^\#) \right] + \mathcal{O} \left( \frac{1}{n-k} \right) \quad (2.5.22)$$

Now, since  $H_3^{(\nu)}(0) = 0$ , we can factorize by  $t^\#$  in  $\sum_{|\nu|=3} H_3^{(\nu)}(t^\#)$  and get that

$$\phi(t^\#) \cdot \mathcal{O} \left( \frac{1}{(n-k)^{1/2}} \right) \cdot \left| \sum_{|\nu|=3} H_3^{(\nu)}(t^\#) \right| = \mathcal{O} \left( \frac{1}{(n-k)^{1/2}} \right) \cdot \mathcal{O} \left( \|t^\#\| \right) = \mathcal{O} \left( \frac{\sqrt{k}}{n-k} \right) \cdot \|\tilde{t}\| \quad (2.5.23)$$

We deduce readily, after some elementary calculations, that

$$\frac{g_{k+1,n}(t^\#)}{g_{1,n}(0)} = \exp\left(-\frac{\|t\|^2}{2}\right) + \mathcal{O}\left(\frac{\sqrt{k}}{n}\right) \|\tilde{t}\| + \mathcal{O}\left(\frac{1}{n}\right) \quad (2.5.24)$$

Now,

$$\frac{\det\left(\text{Cov}(\tilde{S}_{1,n})\right)}{\det\left(\text{Cov}(\tilde{S}_{k+1,n})\right)} = \det\left[I_d + \left(\text{Cov}(\tilde{S}_{k+1,n})\right)^{-1} \text{Cov}(\tilde{S}_{1,k})\right] \quad (2.5.25)$$

Furthermore, we have that

$$\left\|\left(\text{Cov}(\tilde{S}_{k+1,n})\right)^{-1}\right\| \leq \frac{1}{n-k} \left\|\left(\tilde{B}_{k+1,n}\right)^2\right\| \leq \frac{1}{(n-k)\left(\lambda_{\min}^{K_a}\right)^2} \quad \text{and} \quad \left\|\text{Cov}(\tilde{S}_{1,k})\right\| \leq k\left(\lambda_{\max}^{K_a}\right)^2 \quad (2.5.26)$$

Therefore,

$$\left\|\left(\text{Cov}(\tilde{S}_{k+1,n})\right)^{-1} \text{Cov}(\tilde{S}_{1,k})\right\| = \mathcal{O}\left(\frac{k}{n-k}\right). \quad (2.5.27)$$

Consequently, performing a Taylor expansion of  $\det$  at  $I_d$ , we obtain that

$$\det\left[I_d + \left(\text{Cov}(\tilde{S}_{k+1,n})\right)^{-1} \text{Cov}(\tilde{S}_{1,k})\right] = 1 + \text{Tr}\left[\left(\text{Cov}(\tilde{S}_{k+1,n})\right)^{-1} \text{Cov}(\tilde{S}_{1,k})\right] + o\left(\frac{k}{n-k}\right). \quad (2.5.28)$$

Now, we have that  $\text{Tr}\left[\left(\text{Cov}(\tilde{S}_{k+1,n})\right)^{-1} \text{Cov}(\tilde{S}_{1,k})\right] = \mathcal{O}\left(\frac{k}{n-k}\right)$ , since  $\text{Tr}(\cdot) = \text{Trace}(\cdot)$  is a linear and continuous mapping. Therefore,

$$\frac{\det\left(\text{Cov}(\tilde{S}_{1,n})\right)^{1/2}}{\det\left(\text{Cov}(\tilde{S}_{k+1,n})\right)^{1/2}} = \left[1 + \mathcal{O}\left(\frac{k}{n-k}\right)\right]^{1/2} = 1 + \mathcal{O}\left(\frac{k}{n}\right) \quad (2.5.29)$$

**Lemma 9.** *If  $k = o(n)$ , and  $\|t^\#\| < \theta_2 < \infty$ , then uniformly in  $a$  and  $t$ , we have that*

$$\frac{f_{k+1,n}(na-t)}{f_{1,n}(na)} = 1 + \mathcal{O}\left(\frac{k}{n}\right) + \mathcal{O}\left(\frac{k}{n}\right) \|\tilde{t}\|^2 + \mathcal{O}\left(\frac{\sqrt{k}}{n}\right) \|\tilde{t}\| + \mathcal{O}\left(\frac{k^2}{n^2}\right) \|\tilde{t}\|^4 + \mathcal{O}\left(\frac{1}{n}\right) \quad (2.5.30)$$

*Proof.* Since  $\|t^\#\|$  is bounded, we get from the Taylor-Lagrange inequality that

$$\exp\left(-\frac{\|t^\#\|^2}{2}\right) = 1 - \frac{\|t^\#\|^2}{2} + \mathcal{O}\left(\|t^\#\|^4\right) = 1 + \mathcal{O}\left(\frac{k}{n-k}\right) \|\tilde{t}\|^2 + \mathcal{O}\left(\frac{k^2}{(n-k)^2}\right) \|\tilde{t}\|^4 \quad (2.5.31)$$

Therefore,

$$\left[1 + \mathcal{O}\left(\frac{k}{n}\right)\right] \exp\left(-\frac{\|t^\#\|^2}{2}\right) = 1 + \mathcal{O}\left(\frac{k}{n}\right) + \mathcal{O}\left(\frac{k}{n}\right) \|\tilde{t}\|^2 + \mathcal{O}\left(\frac{k^2}{n^2}\right) \|\tilde{t}\|^4 \quad (2.5.32)$$

□

**Lemma 10.** For  $\nu = 1, 2, 3, 4$  we have that

$$\int \|\tilde{t}\|^\nu f_k(t) dt = \mathcal{O}(1) \quad (2.5.33)$$

*Proof.* We only need to prove the case  $\nu = 4$ . Setting  $I_4 := \int \|\tilde{t}\|^4 f_k(t) dt$ , we readily obtain that

$$I_4 = \int \left\| k^{-1/2} \tilde{B}_{1,k} \left( \sum_{j=1}^k \tilde{X}_j - m_j(\theta_n^a) \right) \right\|^4 dP \leq k^{-2} \|\tilde{B}_{1,k}\|^4 \int \left( \sum_{j=1}^k \|\tilde{X}_j - m_j(\theta_n^a)\| \right)^4 dP. \quad (2.5.34)$$

Since the  $(\tilde{X}_j - m_j(\theta_n^a))$  are centered and mutually independent, we obtain that

$$I_4 \leq k^{-2} \|\tilde{B}_{1,k}\|^4 \left[ \sum_{j=1}^k \int \|\tilde{X}_j - m_j(\theta_n^a)\|^4 dP + \sum_{j_1 \neq j_2} \int \|\tilde{X}_{j_1} - m_{j_1}(\theta_n^a)\|^2 \|\tilde{X}_{j_2} - m_{j_2}(\theta_n^a)\|^2 dP \right] \quad (2.5.35)$$

The assumption on the absolute moments of order 4 and the inequality of Cauchy-Schwartz imply that  $\sum_{j=1}^k \int \|\tilde{X}_j - m_j(\theta_n^a)\|^4 dP = \mathcal{O}(k)$  and  $\sum_{j_1 \neq j_2} \int \|\tilde{X}_{j_1} - m_{j_1}(\theta_n^a)\|^2 \|\tilde{X}_{j_2} - m_{j_2}(\theta_n^a)\|^2 dP = \mathcal{O}(k^2)$ .

Then, since  $\|\tilde{B}_{1,k}\|^4 = \mathcal{O}(1)$ , we conclude from (2.5.35) that  $I_4 = \mathcal{O}(1)$ .  $\square$

We are now able to prove (2.5.5). Setting  $\kappa(t) := \left| \frac{f_{k+1,n}(na-t)}{f_{1,n}(na)} - 1 \right| f_{1,k}(t)$ , we have that

$$\|Q_{nak} - \tilde{P}_{1,k}\|_{TV} = \int_{\|t^\# \|\leq \theta_2} \kappa(t) dt + \int_{\|t^\# \|\>\theta_2} \kappa(t) dt. \quad (2.5.36)$$

Now, Lemma 9 and Lemma 10 imply that

$$\int_{\|t^\# \|\leq \theta_2} \kappa(t) dt = \mathcal{O}\left(\frac{k}{n}\right) \quad (2.5.37)$$

On the other hand, we get from Lemma 8 and Lemma 10 that

$$\int_{\|t^\# \|\>\theta_2} \kappa(t) dt = \int_{\|t^\# \|\>\theta_2} \left[ \left[ 1 + \mathcal{O}\left(\frac{k}{n}\right) \right] \exp\left(-\frac{\|t^\#\|^2}{2}\right) - 1 \right] f_k(t) dt + \mathcal{O}\left(\frac{k}{n}\right) \quad (2.5.38)$$

Recall that  $\|t^\#\| = \mathcal{O}\left(\sqrt{\frac{k}{n}}\right) \|\tilde{t}\|$ . Therefore,  $\|t^\#\| > \theta_2$  implies that there exists an absolute constant  $A$ , with  $0 < A < \infty$ , such that  $A\sqrt{\frac{k}{n}} \|\tilde{t}\| > \theta_2$ . This is equivalent to  $\|\tilde{t}\|^4 > A^{-4} \theta_2^4 \left(\frac{n}{k}\right)^2$ .



Then, since  $\left| \left[ 1 + \mathcal{O}\left(\frac{k}{n}\right) \right] \exp\left(-\frac{\|t^\#\|^2}{2}\right) - 1 \right|$  is uniformly bounded, we get from Markov's inequality and Lemma 10 that

$$\int_{\|t^\#\| > \theta_2} \left| \left[ 1 + \mathcal{O}\left(\frac{k}{n}\right) \right] \exp\left(-\frac{\|t^\#\|^2}{2}\right) - 1 \right| f_k(t) dt = \mathcal{O}\left(\frac{k^2}{n^2}\right). \quad (2.5.39)$$

□

## 2.6 Appendix

### 2.6.1 Proof of Lemma 2

*Proof.* We adapt the proof of Theorem 9.1. (ii)\* in (Barndorff-Nielsen Ole, 2014). So, it is enough to prove that

$$\text{int}(C_X) \subset \text{dom}((\bar{\kappa}_n)^*) \subset C_X. \quad (2.6.1)$$

Let  $t \notin C_X$ . Let  $H$  be a hyperplane separating  $C_X$  and  $t$  strongly, and let  $e$  be the unit vector in  $\mathbb{R}^d$  which is normal to  $H$  and such that  $C_X$  lies in the negative halfspace determined by  $H$  and  $e$ . For any  $r > 0$ , we have that

$$\ell_n(re; t) := \langle re, t \rangle - \bar{\kappa}_n(re) = \frac{1}{n} \left[ \sum_{j=1}^n (rd - \kappa_j(re)) \right], \quad \text{where } d := \langle e, t \rangle. \quad (2.6.2)$$

Since  $t \notin C_X$ , we obtain from (5) of Section 7.1 in (Barndorff-Nielsen Ole, 2014) that for all  $1 \leq j \leq n$ ,  $rd - \kappa_j(re) \rightarrow \infty$  as  $r \rightarrow \infty$ . Therefore,  $\ell_n(re; t) \rightarrow \infty$  as  $r \rightarrow \infty$ . So  $(\bar{\kappa}_n)^*(t) = \sup_{\theta \in \Theta} \{\langle \theta, t \rangle - \bar{\kappa}_n(\theta)\} = \infty$ , which means that  $t \notin \text{dom}((\bar{\kappa}_n)^*)$ . Consequently,  $\text{dom}((\bar{\kappa}_n)^*) \subset C_X$ .

Conversely, let  $t \in \text{int}(C_X)$ . Applying Jensen's inequality, we have that for any  $\theta \in \mathbb{R}^d$ ,

$$\bar{\kappa}_n(\theta) \geq \log \mathbb{E}[\exp\langle \theta, S_{1,n}/n \rangle]. \quad (2.6.3)$$

Now, we apply Lemma 9.1. in (Barndorff-Nielsen Ole, 2014) (which follows readily from Markov's inequality) to the random vector  $S_n/n$  to get that for any  $\theta, \tau \in \mathbb{R}^d$ ,

$$\langle \theta, \tau \rangle - \log \mathbb{E}[\exp\langle \theta, S_{1,n}/n \rangle] \leq -\log \rho_n(\tau) \quad (2.6.4)$$

where

$$\rho_n(\tau) = \inf_e P(\langle e, S_{1,n}/n \rangle \geq \langle e, \tau \rangle), \quad (2.6.5)$$

the infimum being taken over all unit vectors in  $\mathbb{R}^d$ . Then, Lemma 9.2. in (Barndorff-Nielsen Ole, 2014) implies that, since  $t \in \text{int}(C_X)$ , we have that  $\rho_n(t) > 0$ . Consequently, we have that  $t \in \text{dom}((\bar{\kappa}_n)^*)$ , since for any  $\theta \in \mathbb{R}^d$ ,

$$\langle \theta, t \rangle - \bar{\kappa}_n(\theta) \leq \langle \theta, t \rangle - \log \mathbb{E}[\exp\langle \theta, S_{1,n}/n \rangle] \leq -\log \rho_n(t) < \infty, \quad (2.6.6)$$

and  $\rho_n(t)$  is independent of  $\theta$ . □

### 2.6.2 Proof of Lemma 4

#### A Preliminary result

Let  $\Omega$  be an open subset of  $\mathbb{R}^{m+q} = \mathbb{R}^m \times \mathbb{R}^q$ . Let  $T$  be a function of class  $\mathcal{C}^1$  from  $\Omega$  to  $\mathbb{R}^q$  such that for any  $a \in \Omega$ , the differential at  $a$  of  $T$  in the second direction (of  $\mathbb{R}^q$ ) is invertible. Define the map  $h : \mathbb{R}^{m+q} \rightarrow \mathbb{R}^{m+q}$  by

$$h : (x_1, \dots, x_m; x_{m+1}, \dots, x_{m+q}) \mapsto (x_1, \dots, x_m; T(x_1, \dots, x_{m+q})) \quad (2.6.7)$$

The local inversion theorem implies that for any  $a \in \Omega$ , there exist an open neighborhood  $\omega_a$  of  $a$  and open sets  $U_a \subset \mathbb{R}^m$  and  $\mathcal{T}_a \subset \mathbb{R}^q$  such that  $h$  induces a diffeomorphism of class  $\mathcal{C}^1$  from  $\omega_a$  to  $U_a \times \mathcal{T}_a$ . Denote by  $\xi_a$  the inverse of the restriction of  $h$  to  $\omega_a$ .

**Lemma 11.** *Assume that for any  $a \in \Omega$ , and any  $(u, t) \in U_a \times \mathcal{T}_a$ ,*

$$|J_{\xi_a}(u, t)| = 1, \quad (2.6.8)$$

where  $J_{\xi_a}(u, t)$  is the determinant of the jacobian matrix of  $\xi_a$  at  $(u, t)$ . For any fixed  $t \in \mathcal{T}_a$ , let  $\xi_a^t$  be the map from  $U_a$  to  $\omega_a$  defined by  $\xi_a^t(u) = \xi_a(u, t)$ . Then,  $\xi_a^t$  is a diffeomorphism of class  $\mathcal{C}^1$  and clearly, we have that

$$\xi_a^t(U_a) = \{T = t\} \cap \omega_a. \quad (2.6.9)$$

For any  $u \in U_a$ , let  $g_{\xi_a^t}(u)$  be the Gram determinant of the partial derivatives of  $\xi_a^t$  at  $u$ . Assume that  $g_{\xi_a^t}(u)$  is independent of  $u, t$  and  $a$ , that is

$$g_{\xi_a^t}(u) = g, \quad (2.6.10)$$

for some constant  $g > 0$ . For any  $t \in \mathbb{R}^q$ , set  $L_t := \{T = t\}$ . Then, for any measurable non negative function  $f$  on  $\Omega$ , we have that

$$\int_{\Omega} f(x) dx = \frac{1}{\sqrt{g}} \int_{\{t: L_t \cap \Omega \neq \emptyset\}} \left( \int_{L_t \cap \Omega} f(x) d\sigma^t(x) \right) dt, \quad (2.6.11)$$

where  $\sigma^t$  is the natural measure on the submanifold  $L_t \cap \Omega$ .

*Proof.* We recall that  $\sigma^t$  is a Borel measure on  $L_t$ , defined as follows for any submanifold  $V$  of dimension  $p$ . Let  $\omega$  be a neighborhood of a point of  $V$  such that there exists a local parametrization  $(U, \xi)$  of  $V$ , where  $U$  is an open subset of  $\mathbb{R}^p$  with  $\xi(U) = V \cap \omega$ . Then, we define a measure  $\sigma_{\omega}$  on  $V \cap \omega$  by

$$\sigma_{\omega} = \xi(\sqrt{g_{\xi}} \lambda_U), \quad (2.6.12)$$

where  $\lambda_U$  is the Lebesgue measure on  $U$  and  $g_{\xi}$  is the Gram determinant of the partial derivatives of  $\xi$ . Then,  $\sigma$  is a Borel measure on  $V$ , satisfying that for any such  $\omega$ , the restriction of  $\sigma$  to  $V \cap \omega$  is  $\sigma_{\omega}$ .

Now, we have  $\Omega = \bigcup_{a \in \Omega} \omega_a$ , from which we can extract a countable subcover, that is  $\Omega = \bigcup_{n \geq 1} \omega_{a_n}$ .

Without loss of generality, we can assume that the  $(\omega_{a_n})_{n \geq 1}$  are non-overlapping and that

$$\Omega = \left( \bigcup_{n \geq 1} \omega_{a_n} \right) \cup \mathcal{N},$$

for some negligible set  $\mathcal{N}$ . Therefore,  $f = \sum_{n \geq 1} f \mathbf{1}_{\omega_{a_n}}$  a.e., so

$$\int_{\{t: L_t \cap \Omega \neq \emptyset\}} \left( \int_{L_t \cap \Omega} f(x) d\sigma^t(x) \right) dt = \sum_{n \geq 1} \int_{\{t: L_t \cap \omega_{a_n} \neq \emptyset\}} \left( \int_{L_t \cap \omega_{a_n}} f(x) d\sigma_{a_n}^t(x) \right) dt, \quad (2.6.13)$$

where  $\sigma_{a_n}^t$  is the natural measure on the submanifold  $L_t \cap \omega_{a_n}$ . Now, (2.6.9) implies that the couple  $(U_{a_n}, \xi_{a_n}^t)$  is a local parametrization of the submanifold  $L_t$ . Furthermore, we clearly have that  $\{t : L_t \cap \omega_{a_n} \neq \emptyset\} = \mathcal{T}_{a_n}$ . Therefore,

$$\int_{\{t: L_t \cap \Omega \neq \emptyset\}} \left( \int_{L_t \cap \Omega} f(x) d\sigma^t(x) \right) dt = \sum_{n \geq 1} \int_{\mathcal{T}_{a_n}} \left( \int_{U_{a_n}} f(\xi_{a_n}^t(u)) \sqrt{g_{\xi_{a_n}^t}(u)} du \right) dt. \quad (2.6.14)$$

Now, we obtain from (2.6.10) and the definition of  $\xi_{a_n}^t$  that

$$\int_{\{t: L_t \cap \Omega \neq \emptyset\}} \left( \int_{L_t \cap \Omega} f(x) d\sigma^t(x) \right) dt = \sqrt{g} \sum_{n \geq 1} \int_{\mathcal{T}_{a_n}} \left( \int_{U_{a_n}} f(\xi_{a_n}(u, t)) du \right) dt. \quad (2.6.15)$$

We deduce from Fubini's theorem and the change of variables formula that, under (2.6.8),

$$\int_{\{t: L_t \cap \Omega \neq \emptyset\}} \left( \int_{L_t \cap \Omega} f(x) d\sigma^t(x) \right) dt = \sqrt{g} \sum_{n \geq 1} \int_{\omega_{a_n}} f(x) dx = \sqrt{g} \int_{\Omega} f(x) dx. \quad (2.6.16)$$

□

#### Proof of Lemma 4

*Proof.* For any open set  $A \subset (\mathbb{R}^d)^k$  and any measurable set  $B \subset \mathbb{R}^d$ ,

$$\mathcal{P}(A \cap \{T \in B\}) = \int_A \mathbf{1}_B(\mathbf{T}(\mathbf{x})) \mathbf{p}(\mathbf{x}) d\mathbf{x}. \quad (2.6.17)$$

The map  $h : (\mathbb{R}^d)^k \rightarrow (\mathbb{R}^d)^k$  is defined by

$$h : x = (x_1, \dots, x_{k-1}; x_k) \mapsto (x_1, \dots, x_{k-1}; T(x)) \quad (2.6.18)$$

We readily get from the local inversion theorem that  $h$  is a local diffeomorphism of class  $\mathcal{C}^1$ . Furthermore, for any  $a \in (\mathbb{R}^d)^k$ , the maps  $\xi_a$  and  $\xi_a^t$  are defined by

$$\xi_a : (u, t) = (u_1, \dots, u_{k-1}; t) \mapsto (u_1, \dots, u_{k-1}; t - s_{1, k-1}), \quad (2.6.19)$$

where  $s_{1, k-1} = \sum_{i=1}^{k-1} u_i$ , and

$$\xi_a^t : u \mapsto (u; t - s_{1, k-1}) \quad (2.6.20)$$

We readily check that (2.6.8) and (2.6.10) hold here. Therefore, we get from the preceding Lemma that

$$\mathcal{P}(A \cap \{T \in B\}) = \frac{1}{\sqrt{g}} \int_{\{t: L_t \cap A \neq \emptyset\}} \mathbf{1}_B(\mathbf{t}) \left( \int_{L_t \cap A} \mathbf{p}(\mathbf{x}) d\sigma^{\mathbf{t}}(\mathbf{x}) \right) d\mathbf{t}. \quad (2.6.21)$$

In particular, applying (2.6.21) with  $A = (\mathbb{R}^d)^k$ , we get that for any  $B \in \mathcal{B}(\mathbb{R}^d)$ ,

$$(\mathcal{P}T^{-1})(B) = \frac{1}{\sqrt{g}} \int_{\mathbb{R}^d} \mathbf{1}_B(\mathbf{t}) \left( \int_{L_t} \mathbf{p}(\mathbf{x}) d\sigma^{\mathbf{t}}(\mathbf{x}) \right) d\mathbf{t}. \quad (2.6.22)$$

Therefore, the probability measure  $\mathcal{P}T^{-1}$  is a.c. w.r.t. the Lebesgue measure, with

$$\frac{d(\mathcal{P}T^{-1})}{dx} = \frac{1}{\sqrt{g}} \int_{L_t} p(x) d\sigma^t(x) \quad (2.6.23)$$

So, we deduce from (2.6.21) that for any open set  $A$ ,

$$\mathcal{P}(A \cap \{T \in B\}) = \int_B \nu_{\mathcal{P}}(t, A) (\mathcal{P}T^{-1})(dt) \quad (2.6.24)$$

First, we clearly have that for any fixed  $t \in \mathbb{R}^d$ , the map  $A \mapsto \nu_{\mathcal{P}}(t, A)$  is a probability measure. We deduce from this fact and the monotone class theorem that (2.6.24) holds for any Borel set  $A$ .

Finally, we need to prove that for any fixed Borel set  $A$ , the map  $t \mapsto \nu_{\mathcal{P}}(t, A)$  is measurable. Notice that  $L_t \cap A = \emptyset$  if and only if  $t \in T(A)$ . Therefore, it is enough to prove that  $T(A)$  is a Borel set and that the map  $t \mapsto \int_{L_t \cap A} p(x) d\sigma^t(x)$  is measurable.

For the first point, write  $A = F \cup (A \cap F^c)$ , for some  $F \in \mathcal{F}_\sigma$  included in  $A$  (which means that  $F$  is a countable union of closed sets). The key point is then that  $A \cap F^c$  is negligible w.r.t. the Lebesgue measure, and so is  $T(A \cap F^c)$ , which is obtained using that  $T$  is Lipschitz. We conclude by the completeness of the Lebesgue measure.

For the second point, it is enough to prove it when  $A = \omega_a$ , for some  $a \in A$ . Then, we have that

$$\int_{L_t \cap \omega_a} p(x) d\sigma^t(x) = \sqrt{g} \int_{U_a} p(u; t - s_{1,k-1}) du, \quad (2.6.25)$$

which is clearly measurable w.r.t  $t$ . □

### 2.6.3 Proof of Lemma 6

*Proof.* Let  $a \in \text{int}(C_X)$ . Consequently, we may apply (2.4.4) to  $K_a$ . We set  $\delta := \delta_{K_a}$ ,  $C := C_{K_a}$  and  $R := R_{K_a}$ . Now, for any  $p > 0$ , any  $n$  large enough to insure that  $\alpha_n \geq p + 1$ , and any  $0 \leq m \leq \alpha_n - p$ ,

$$\int \widetilde{g}_{m,n}(t) dt = \int_{\{t: \|\widetilde{B}_{L_n} t\| \geq R\}} \widetilde{g}_{m,n}(t) dt + \int_{\{t: \|\widetilde{B}_{L_n} t\| < R\}} \widetilde{g}_{m,n}(t) dt. \quad (2.6.26)$$

Then, for  $\|\tilde{B}_{L_n} t\| \geq R$ , we get from (2.4.4) that

$$|\tilde{\xi}_j(\tilde{B}_{L_n} t)| \leq \frac{C}{\|\tilde{B}_{L_n} t\|^\delta}. \quad (2.6.27)$$

Furthermore, setting  $\lambda_{\min} := \lambda_{\min}^{K_a}$ , we have that  $\|\tilde{B}_{L_n} t\| \geq \lambda_{\min} \|t\|$ . Therefore,

$$\int_{\|\tilde{B}_{L_n} t\| \geq R} \widetilde{g}_{m,n}(t) dt = \int_{\|\tilde{B}_{L_n} t\| \geq R} \prod_{j=m+1}^{m+p} |\tilde{\xi}_j(\tilde{B}_{L_n} t)| dt \leq C^p \int_{\|\tilde{B}_{L_n} t\| \geq R} \frac{1}{(\lambda_{\min} \|t\|)^\delta} dt. \quad (2.6.28)$$

So, if  $p > \frac{1}{\delta}$ , then  $\int_{\|\tilde{B}_{L_n} t\| \geq R} \widetilde{g}_{m,n}(t) dt \leq D_{K_a} < \infty$ , for some constant  $D_{K_a}$  depending only on  $a$ .

Notice that, without loss of generality, we can assume that  $R > 2C^{\frac{1}{\delta}}$ . Therefore, (2.4.4) implies that for all  $t$  satisfying  $\|\tilde{B}_{L_n} t\| \geq R$ , for all  $j \geq 1$ ,

$$|\tilde{\xi}_j(\tilde{B}_{L_n} t)| \leq \frac{C}{R^\delta} < 1. \quad (2.6.29)$$

Therefore, we obtain from Theorem 1, Chapter 1 in (Petrov, 1975) that for all  $t$  satisfying  $\|\tilde{B}_{L_n} t\| < R$ , for all  $j \geq 1$ ,

$$|\tilde{\xi}_j(\tilde{B}_{L_n} t)| \leq 1 - \frac{1 - (\frac{C}{R^\delta})^2}{8R^2} \|\tilde{B}_{L_n} t\|^2. \quad (2.6.30)$$

Setting  $\Gamma := \frac{1 - (\frac{C}{R^\delta})^2}{8R^2}$ , we deduce that for all  $t$  satisfying  $\|\tilde{B}_{L_n} t\| < R$ , for all  $j \geq 1$ ,

$$|\tilde{\xi}_j(\tilde{B}_{L_n} t)| \leq \exp(-\Gamma \|\tilde{B}_{L_n} t\|^2) \leq \exp(-\Gamma \lambda_{\min}^2 \|t\|^2) \quad (2.6.31)$$

Consequently,

$$\int_{\|\tilde{B}_{L_n} t\| < R} \tilde{g}_{m,n}(t) dt \leq \int_{\|\tilde{B}_{L_n} t\| \geq R} \exp(-p\Gamma \lambda_{\min}^2 \|t\|^2) dt. \quad (2.6.32)$$

Therefore,  $\int_{\|\tilde{B}_{L_n} t\| < R} \tilde{g}_{m,n}(t) dt \leq E_{K_a} < \infty$ , for some constant  $E_{K_a}$  depending only on  $a$ . So, we obtain that

$$\sup_{n \geq N_p} \sup_{0 \leq m \leq \alpha_n - p} \int \tilde{g}_{m,n}(t) dt \leq D_{K_a} + E_{K_a} < \infty \quad (2.6.33)$$

□

**Lemma 12.** Let  $p \in L^1(\mathbb{R}^d)$ ,  $d \geq 1$ . Let  $\hat{p}$  be the characteristic function of  $p$ , defined for all  $t \in \mathbb{R}^d$  by  $\hat{p}(t) := \int \exp\langle it, x \rangle p(x) dx$ . Assume that  $p \in C^1(\mathbb{R}^d)$  and that for all  $\ell \in \{1, \dots, d\}$ ,  $\frac{\partial p}{\partial x_\ell} \in L^1(\mathbb{R}^d)$ . Then, there exists an absolute constant  $C$  such that for all  $t \in \mathbb{R}^d$ ,

$$|\hat{p}(t)| \leq \frac{C}{\|t\|} \quad (2.6.34)$$

*Proof.* Let  $t = (t_\ell)_{1 \leq \ell \leq d} \in \mathbb{R}^d$ . For any  $\ell \in \{1, \dots, d\}$ , we have that

$$t_\ell \widehat{p}(t) = i \widehat{\left( \frac{\partial p}{\partial x_\ell} \right)} \quad (2.6.35)$$

The preceding equality is obtained by applying a multidimensional version of integration by parts, which holds when one of the involved functions has compact support. Then, notice that  $p$  can be approximated in  $L^1$  - norm by a sequence of functions of compact support. We deduce that

$$\|t_\ell \widehat{p}(t)\|_\infty = \left\| \widehat{\left( \frac{\partial p}{\partial x_\ell} \right)} \right\|_\infty \leq \left\| \frac{\partial p}{\partial x_\ell} \right\|_{L^1} < \infty. \quad (2.6.36)$$

Setting  $C := \max_{1 \leq \ell \leq d} \left\| \frac{\partial p}{\partial x_\ell} \right\|_{L^1}$ , we deduce that for all  $t \in \mathbb{R}^d$ ,

$$|\widehat{p}(t)| \leq \frac{C}{\|t\|}. \quad (2.6.37)$$

□



# Chapter 3

## A conditional limit theorem for independent random variables

### 3.1 Introduction

#### 3.1.1 Context and Scope, Importance Sampling Framework

Let  $(X_j)_{j \geq 1}$  be a sequence of independent, not necessarily identically distributed (i.d.), random variables (r.v.) valued in  $\mathbb{R}$ , such that (s.t.) the  $(X_j)$  have a common support  $\mathcal{S}_X$ . In this chapter, we restrict ourselves to the one-dimensional case, for technical reasons. Indeed, the proof of the Edgeworth expansion theorem which we use here (see (Petrov, 1975)) is specific to the case  $d = 1$  and can be extended to our framework (see Section 3.3.1 below). We keep the notations of the preceding chapter. For  $a \in \mathcal{S}_X$  and  $n \geq 1$ , we denote by  $Q_{nak}$  a regular version of the conditional distribution of  $X_1^k := (X_1, \dots, X_k)$  given  $\{S_{1,n} = na\}$ .

We have obtained in the preceding chapter an approximation of  $Q_{nak}$  when  $k = o(n)$ . A natural question arises : What can be said about the distribution of the  $n - k$  other r.v.'s, that is of  $(X_j)_{k+1 \leq j \leq n}$ , given  $\{S_{1,n} = na\}$ . In terms of Statistical Mechanics, the question would be : What can be said about the distribution of energy for the large component ? Set

$$k' := n - k, \quad \text{so that } \frac{k'}{n} \rightarrow 1 \text{ as } n \rightarrow \infty. \quad (3.1.1)$$

Therefore, we study the distribution of  $Q_{nak}$  when  $\frac{k}{n}$  is allowed to converge to 1 as  $n \rightarrow \infty$ . In (Dembo and Zeitouni, 1996), it is explained that the condition  $k = o(n)$  is necessary to get a Gibbs Conditioning Principle. In this paper, as expected we do not obtain a Gibbs type measure as an approximation of  $Q_{nak}$ , if  $\frac{k}{n}$  does not converge to 0.

Now, we describe an Importance Sampling (IS) framework within which it is natural to consider  $Q_{nak}$  for large  $k$ . Consider a sequence  $(X_j)_{j \geq 1}$  of r.v.'s. For large  $n$  but *fixed*, we intend to estimate

$$\Pi_n := P(X_1^n \in \mathcal{E}_n), \quad \text{for some event } \mathcal{E}_n. \quad (3.1.2)$$

A classical IS estimator of  $\Pi_n$  is the following.

$$\hat{\Pi}_n(N) := \frac{1}{N} \sum_{i=1}^N \frac{p_1^n(Y_1^n(i))}{q_1^n(Y_1^n(i))} 1_{\mathcal{E}_n}(Y_1^n(i)), \quad (3.1.3)$$



where  $p_1^n$  is the density of  $X_1^n$  and the  $(Y_1^n(i))$  are i.i.d copies of a random vector  $Y_1^n$  with density  $q_1^n$ . Then, the law of large numbers insures that  $\widehat{\Pi}_n(N)$  converges almost surely to  $\Pi_n$ , as  $N \rightarrow \infty$ . The interest of this resampling procedure is to reduce the variance of the resulting estimator, compared to the usual Monte Carlo method. It is well known that the optimal density from the point of view of the variance is the conditional density  $p(X_1^n | \mathcal{E}_n)$ . Therefore, it is natural to search an approximation of  $p(X_1^n | \mathcal{E}_n)$ . This approach has been developed in ?, for an i.i.d. sequence  $(X_j)_{j \geq 1}$  of centered r.v.'s, with

$$\mathcal{E}_n = \left\{ (x_i)_{1 \leq i \leq n} \in \mathbb{R}^n : \sum_{i=1}^n x_i \geq na_n \right\}, \quad (3.1.4)$$

for some sequence  $(a_n)$  converging slowly to 0. Therefore,  $\widehat{\Pi}_n(N)$  estimates the moderate deviation probability of  $S_{1,n}/n$ . In ?, they get an approximation of  $p(X_1^k | \mathcal{E}_n)$ , which should be close to  $p(X_1^n | \mathcal{E}_n)$  if  $k$  is large. For a r.v.  $X$ , denote by  $\mathcal{L}(X)$  its probability distribution. They obtain that, for some density  $g_k$  on  $\mathbb{R}^k$ ,

$$p\left(X_1^k = Y_1^k \mid S_{1,n} \geq na_n\right) \approx g_k(Y_1^k), \quad \text{where } Y_1^k \sim \mathcal{L}\left(X_1^k \mid S_{1,n} \geq na_n\right). \quad (3.1.5)$$

The precise sense of  $\approx$  is given in Section 3.2.3 below. They deduce from an elementary lemma that

$$g_k(Z_1^k) \approx p\left(X_1^k = Z_1^k \mid S_{1,n} \geq na_n\right), \quad \text{where } Z_1^k \text{ has density } g_k. \quad (3.1.6)$$

Then, the approximation density  $g_k$  has a computable expression, which allows to simulate  $Z_1^k$ . A density  $\bar{g}_n$  on  $\mathbb{R}^n$  is constructed from  $g_k$ . In (3.1.3),  $q_1^n$  and  $(Y_1^n(i))$  are replaced respectively by  $\bar{g}_n$  and copies of a r.v. with density  $\bar{g}_n$ . The IS estimator obtained has better performances than the existing ones which estimate  $\Pi_n$ .

Now, it is reasonable to expect that (3.1.5) implies that the distribution of  $X_1^k$  given  $\{S_{1,n} \geq na_n\}$  is close to the distribution associated to  $g_k$ . We can use this idea to get an approximation of  $Q_{nak}$  for some  $k$  such that  $\frac{k}{n} \rightarrow 1$  (see Theorem 18), but also for a class of  $k$  which are some  $o(n)$  (see Theorem 17). However, in both cases, the condition  $n - k \rightarrow \infty$  is required for the Edgeworth expansions.

We consider a sequence  $(X_j)_{j \geq 1}$  of independent r.v.'s. For any  $a \in \mathcal{S}_X$ , let  $p(X_1^k = \cdot | S_{1,n} = na)$  be the density of  $X_1^k$  given  $\{S_{1,n} = na\}$ . In this paper, we obtain that, for some density  $g_k$  on  $\mathbb{R}^k$ ,

$$p\left(X_1^k = Y_1^k \mid S_{1,n} = na\right) \approx g_k(Y_1^k), \quad \text{where } Y_1^k \sim \mathcal{L}\left(X_1^k \mid S_{1,n} = na\right). \quad (3.1.7)$$

We deduce (see Section 2.4) that

$$\|Q_{nak} - G_k\|_{TV} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (3.1.8)$$

where  $G_k$  is the distribution associated to  $g_k$ . More precisely, when  $k$  is small ( $k = o(n^\rho)$  with  $0 < \rho < 1/2$ ),  $G_k$  is the same Gibbs type measure as in the preceding chapter, while for large  $k$  (see the assumptions of Theorem 18),  $G_k$  is a slight modification of this measure.

Kolmogorov's extension theorem does not apply to the sequence  $(Q_{nan})_{n \geq 1}$  of probability measures. Therefore, we need to consider a sequence  $((\Omega_n, \mathcal{A}_n, \mathcal{P}_n))_{n \geq 1}$  of probability spaces s.t. for any  $n \geq 1$ ,

$Y_1^n$  is a random vector defined on  $(\Omega_n, \mathcal{A}_n, \mathcal{P}_n)$  and the distribution of  $Y_1^n$  is  $Q_{nan}$ . Then, for  $k \leq n$ ,  $Q_{nak}$  is the distribution of  $Y_1^k$ . The properties of  $(Y_1^n)_{n \geq 1}$  are studied in Section 3, after some elementary results and statement of the Assumptions in Section 2, while Section 4 is devoted to our main Results and their proofs.

## 3.2 Assumptions and elementary results

All the r.v.'s considered are a.c. w.r.t. the Lebesgue measure on  $\mathbb{R}$ . For any r.v.  $X$ , let  $P_X$  be its distribution,  $p_X$  its density and  $\Phi_X$  its moment generating function (mgf). For any  $j \geq 1$ , set

$$P_j := P_{X_j} \quad ; \quad p_j := p_{X_j} \quad ; \quad \Phi_j := \Phi_{X_j}. \quad (3.2.1)$$

### 3.2.1 Conditional density

Let  $U$  and  $V$  be r.v.'s having respective densities  $p_U$  and  $p_V$  and a joint density denoted by  $p_{(U,V)}$ . Then, there exists a conditional density of  $U$  given  $V$ , denoted as follows.

$$p(U = u | V = v) = \frac{p_{(U,V)}(u, v)}{p_V(v)}.$$

**Fact 16.** Let  $(X_j)_{j \geq 1}$  be a sequence of independent r.v.'s. For any  $n \geq 1$  and  $1 \leq i \leq n$ , let  $J_n$  be a subset of  $\{i, \dots, n\}$  s.t.  $\alpha_n := |J_n| < n - i + 1$ . Let  $L_n$  be the complement of  $J_n$  in  $\{i, \dots, n\}$ . Set  $S_{L_n} := \sum_{j \in L_n} X_j$ . Then, there exists a conditional density of  $(X_j)_{j \in J_n}$  given  $S_{i,n}$ , defined by

$$p((X_j)_{j \in J_n} = (x_j) | S_{i,n} = s) = \frac{\left\{ \prod_{j \in J_n} p_j(x_j) \right\} p_{S_{L_n}} \left( s - \sum_{j \in J_n} x_j \right)}{p_{S_{i,n}}(s)}, \quad (3.2.2)$$

### 3.2.2 The tilted density

**Definition 10.** For a r.v.  $X$ , let  $\Phi_X$  be its mgf and let  $\Theta_X := \{\theta \in \mathbb{R} : \Phi_X(\theta) < \infty\}$ . For any  $\theta \in \Theta_X$ , denote by  $\tilde{X}^\theta$  a random vector having the tilted density, defined by

$$p_{\tilde{X}^\theta}(x) := \frac{(\exp \theta x) p_X(x)}{\Phi_X(\theta)}. \quad (3.2.3)$$

For any  $j \geq 1$ , set  $\Phi_j := \Phi_{X_j}$ . We suppose throughout the text that the functions  $(\Phi_j)_{j \geq 1}$  have the same domain of finiteness denoted by  $\Theta$ , which is assumed to be of non void interior. We write, for any  $j \geq 1$ ,

$$\Theta := \left\{ \theta \in \mathbb{R}^d : \Phi_j(\theta) < \infty \right\}.$$

**Fact 17.** For any  $j \geq 1$ , there exists a probability space  $(\Omega^\theta, \mathcal{A}^\theta, \mathcal{P}^\theta)$  such that for all finite subset  $J \subset \mathbb{N}$  and for all  $(B_j)_{j \in J} \in \mathcal{B}(\mathbb{R})^{|J|}$ ,

$$\mathcal{P}^\theta \left( \left( \tilde{X}_j^\theta \right)_{j \in J} \in (B_j)_{j \in J} \right) = \prod_{j \in J} \tilde{P}_j^\theta(B_j) = \prod_{j \in J} \int_{B_j} \tilde{p}_j^\theta(x) dx, \quad (3.2.4)$$

where  $\tilde{P}_j^\theta := P_{\tilde{X}_j^\theta}$  and  $\tilde{p}_j^\theta := p_{\tilde{X}_j^\theta}$ . In other words,  $(\tilde{X}_j^\theta)_{j \geq 1}$  is a sequence of independent r.v.'s defined on  $(\Omega^\theta, \mathcal{A}^\theta, \mathcal{P}^\theta)$ .

**Fact 18.** For any  $j \geq 1$ , and  $\theta \in \Theta$ , we have that

$$\mathbb{E} \left[ \tilde{X}_j^\theta \right] = m_j(\theta) \quad \text{where} \quad m_j(\theta) := \frac{d\kappa_j}{d\theta}(\theta) \quad \text{and} \quad \kappa_j(\theta) := \log \Phi_j(\theta). \quad (3.2.5)$$

**Fact 19.** For any  $\theta \in \Theta$ ,  $j \geq 1$  and  $j' \geq 1$ ,

$$\mathbb{E} \left[ \widetilde{X_j + X_{j'}}^\theta \right] = \mathbb{E} \left[ \tilde{X}_j^\theta + \tilde{X}_{j'}^\theta \right]. \quad (3.2.6)$$

**Corollary 4.** For any  $n \geq 1$  and  $1 \leq \ell \leq n$ , for any  $\theta \in \Theta$ ,

$$\mathbb{E} \left[ \widetilde{S_{\ell,n}}^\theta \right] = \sum_{j=\ell}^n m_j(\theta). \quad (3.2.7)$$

**Fact 20.**

For any  $j \geq 1$  and  $\theta \in \Theta$ , set

$$\bar{X}_j^\theta := \tilde{X}_j^\theta - \mathbb{E}[\tilde{X}_j^\theta] = \tilde{X}_j^\theta - m_j(\theta)$$

and for any  $\ell \geq 3$ ,

$$s_j^2(\theta) := \text{Var} \left( \tilde{X}_j^\theta \right) \quad ; \quad \sigma_j(\theta) := \sqrt{s_j^2(\theta)} \quad ; \quad \mu_j^\ell(\theta) := \mathbb{E} \left[ \left( \bar{X}_j^\theta \right)^\ell \right] \quad ; \quad |\mu_j^\ell(\theta)| := \mathbb{E} \left[ \left| \bar{X}_j^\theta \right|^\ell \right].$$

Then,

$$s_j^2(\theta) = \frac{d^2 \kappa_j}{d\theta^2}(\theta) \quad \text{and} \quad \mu_j^\ell(\theta) = \frac{d^\ell \kappa_j}{d\theta^\ell}(\theta). \quad (3.2.8)$$

### 3.2.3 Landau Notations

**Definition 11.** Let  $(X_n)_{n \geq 1}$  be a sequence of r.v.'s such that for any  $n \geq 1$ ,  $X_n$  is defined on a probability space  $(\Omega_n, \mathcal{A}_n, \mathcal{P}_n)$ . Let  $(u_n)$  be a sequence of real numbers. We say that

$(X_n)_{n \geq 1}$  is a  $\mathcal{O}_{\mathcal{P}_n}(u_n)$  if for all  $\epsilon > 0$ , there exists  $A \geq 0$  and  $N_\epsilon \in \mathbb{N}$ , s.t. for all  $n \geq N_\epsilon$ ,

$$\mathcal{P}_n \left( \left| \frac{X_n}{u_n} \right| \leq A \right) \geq 1 - \epsilon. \quad (3.2.9)$$

$(X_n)_{n \geq 1}$  is a  $o_{\mathcal{P}_n}(u_n)$  if for all  $\epsilon > 0$  and  $\delta > 0$ , there exists  $N_{\epsilon,\delta} \in \mathbb{N}$  s.t. for all  $n \geq N_{\epsilon,\delta}$ ,

$$\mathcal{P}_n \left( \left| \frac{X_n}{u_n} \right| \leq \delta \right) \geq 1 - \epsilon. \quad (3.2.10)$$

$(X_n)_{n \geq 1}$  converges to  $\ell \in \mathbb{R}$  in  $\mathcal{P}_n$ -probability and we note  $X_n \xrightarrow{\mathcal{P}_n} \ell$  if

$$X_n = \ell + o_{\mathcal{P}_n}(1). \quad (3.2.11)$$

**Remark 4.** *These notations differ from the classical Landau notations in probability by the fact that here, the rv's  $(X_n)$  are not defined on the same probability space. However, they satisfy similar properties, which we will use implicitly in the proofs.*

### 3.2.4 A criterion for convergence in Total Variation Distance

**Definition 12.** *Set*

$$\mathcal{A}_{\rightarrow 1} := \left\{ (B_n)_{n \geq 1} \in \prod_{n \geq 1} \mathcal{A}_n : \mathcal{P}_n(B_n) \xrightarrow{n \rightarrow \infty} 1 \right\}.$$

**Lemma 13.** *For all integer  $n \geq 1$ , let  $Y_1^n : (\Omega_n, \mathcal{A}_n, \mathcal{P}_n) \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$  be a random vector. For any  $1 \leq k \leq n$ , the distribution of  $Y_1^k$  is denoted by  $P_k$ . Let  $G_k$  be a probability measure on  $\mathbb{R}^k$ . Assume that  $P_k$  and  $G_k$  have positive densities  $p_k$  and  $g_k$ , and that  $k \rightarrow \infty$  as  $n \rightarrow \infty$ . If there exists  $(B_n)_{n \geq 1} \in \mathcal{A}_{\rightarrow 1}$  s.t. for any  $n \geq 1$ , we have on  $B_n$  that*

$$p_k(Y_1^k) = g_k(Y_1^k) [1 + T_n] \quad \text{where } T_n = o_{\mathcal{P}_n}(1), \quad (3.2.12)$$

then,

$$\|P_k - G_k\|_{TV} \xrightarrow{n \rightarrow \infty} 0. \quad (3.2.13)$$

*Proof.* For any  $\delta > 0$ , set

$$E(n, \delta) := \left\{ (y_1^k) \in \mathbb{R}^k : \left| \frac{p_k(y_1^k)}{g_k(y_1^k)} - 1 \right| \leq \delta \right\}. \quad (3.2.14)$$

Then,

$$\begin{aligned} \mathcal{P}_n(\{|T_n| \leq \delta\} \cap B_n) &\leq \mathcal{P}_n\left(\left|\frac{p_k(Y_1^k)}{g_k(Y_1^k)} - 1\right| \leq \delta\right) \\ &= P_k(E(n, \delta)) \\ &= \int_{E(n, \delta)} \frac{p_k(y_1^k)}{g_k(y_1^k)} g_k(y_1^k) dy_1^k \\ &\leq (1 + \delta) G_k(E(n, \delta)). \end{aligned}$$

By (3.2.12), for all  $n$  large enough,

$$\begin{aligned} \mathcal{P}_n(\{|T_n| \leq \delta\} \cap B_n) &\geq 1 - \mathcal{P}_n(\{|T_n| > \delta\}) - \mathcal{P}_n(B_n^c) \\ &\geq 1 - 2\delta. \end{aligned}$$

Combining the preceding inequalities, we obtain that for all  $n$  large enough,

$$1 - 2\delta \leq P_k(E(n, \delta)) \leq (1 + \delta) G_k(E(n, \delta)). \quad (3.2.15)$$

Therefore,

$$\sup_{C \in \mathcal{B}(\mathbb{R}^k)} |P_k(C) - P_k(C \cap E(n, \delta))| \leq P_k(E(n, \delta)^c) \leq 2\delta \quad (3.2.16)$$

and

$$\begin{aligned} \sup_{C \in \mathcal{B}(\mathbb{R}^k)} |G_k(C) - G_k(C \cap E(n, \delta))| &\leq 1 - G_k(E(n, \delta)) \\ &\leq 1 - \frac{1 - 2\delta}{1 + \delta} \\ &= \frac{3\delta}{1 + \delta}. \end{aligned}$$

Now, we have that

$$\sup_{C \in \mathcal{B}(\mathbb{R}^k)} |P_k(C \cap E(n, \delta)) - G_k(C \cap E(n, \delta))| \leq \sup_{C \in \mathcal{B}(\mathbb{R}^k)} \int_{C \cap E(n, \delta)} |p_k(y_1^k) - g_k(y_1^k)| dy_1^k \quad (3.2.17)$$

From the definition of  $E(n, \delta)$ , we deduce that

$$\begin{aligned} \sup_{C \in \mathcal{B}(\mathbb{R}^k)} |P_k(C \cap E(n, \delta)) - G_k(C \cap E(n, \delta))| &\leq \delta \sup_{C \in \mathcal{B}(\mathbb{R}^k)} \int_{C \cap E(n, \delta)} g_k(y_1^k) dy_1^k \\ &\leq \delta. \end{aligned}$$

Finally, applying the triangle inequality, we have that for all  $n$  large enough,

$$\begin{aligned} \sup_{C \in \mathcal{B}(\mathbb{R}^k)} |P_k(C) - G_k(C)| &\leq 2\delta + \delta + \frac{3\delta}{1 + \delta} \\ &= 3\delta \left( \frac{2 + \delta}{1 + \delta} \right), \end{aligned}$$

which converges to 0 as  $\delta \rightarrow 0$ . □

**Remark 5.** *A rate of convergence is not obtainable by this method.*

### 3.2.5 A first calculus

Set

$$p_k(Y_1^k) := p(X_1^k = Y_1^k | S_{1,n} = na). \quad (3.2.18)$$

First, we have that

$$p(X_1^k = Y_1^k | S_{1,n} = na) = p(X_k = Y_k | X_1^{k-1} = Y_1^{k-1}; S_{1,n} = na) p(X_1^{k-1} = Y_1^{k-1} | S_{1,n} = na) \quad (3.2.19)$$

Set  $p_k(Y_1^k) := p(X_1^k = Y_1^k | S_{1,n} = ns)$ , then we deduce by induction on  $k$  that

$$p_k(Y_1^k) = \left\{ \prod_{i=1}^{k-1} p(X_{i+1} = Y_{i+1} | X_i = Y_i; S_{1,n} = na) \right\} p(X_1 = Y_1 | S_{1,n} = na). \quad (3.2.20)$$

For  $1 \leq i_1 \leq i_2 \leq n$ , set  $\Sigma_{i_1, i_2} := \sum_{j=i_1}^{i_2} Y_j$ . We deduce from (3.2.20) that

$$p_k \left( Y_1^k \right) = \left\{ \prod_{i=1}^{k-1} p \left( X_{i+1} = Y_{i+1} \mid S_{i+1, n} = na - \Sigma_{1, i} \right) \right\} p \left( X_1 = Y_1 \mid S_{1, n} = na \right). \quad (3.2.21)$$

Let  $\Sigma_{1, 0} = 0$ . Then,

$$p_k \left( Y_1^k \right) = \prod_{i=0}^{k-1} \pi_i, \quad \text{where } \pi_i := p \left( X_{i+1} = Y_{i+1} \mid S_{i+1, n} = na - \Sigma_{1, i} \right). \quad (3.2.22)$$

The conditioning event being  $\{S_{i+1, n} = na - \Sigma_{1, i}\}$ , we search  $\theta$  s.t.

$$E \left[ \widetilde{S_{i+1, n}}^\theta \right] = \sum_{j=i+1}^n m_j(\theta) = na - \Sigma_{1, i}. \quad (3.2.23)$$

Since  $\mathcal{P}_n$ -a.s.,  $\Sigma_{1, i} + \Sigma_{i+1, n} = na$ , this is equivalent to solve the following equation, where  $\theta$  is unknown.

$$\bar{m}_{i+1, n}(\theta) := \frac{\sum_{j=i+1}^n m_j(\theta)}{n - i} = \frac{\Sigma_{i+1, n}}{n - i}. \quad (3.2.24)$$

We will see below (see Definition 15) that, under suitable assumptions, equation (3.2.24) has a unique solution  $t_{i, n}$ . In the following lines, the tilted densities pertain to  $\theta = t_{i, n}$ .

For  $e = 1, 2$ , let  $\bar{q}_{i+e, n}$  be the density of  $\widetilde{S_{i+e, n}}$ , where

$$\bar{S}_{i+e, n} := \frac{\widetilde{S_{i+e, n}} - \mathbb{E} \left[ \widetilde{S_{i+e, n}} \right]}{\sqrt{\text{Var} \left( \widetilde{S_{i+e, n}} \right)}} = \frac{\widetilde{S_{i+e, n}} - \sum_{j=i+e}^n m_j(t_{i, n})}{\sqrt{\sum_{j=i+e}^n s_j^2(t_{i, n})}}. \quad (3.2.25)$$

Using the invariance of the conditional density under the tilting operation, Fact 16 and then renormalizing, we obtain that

$$\pi_i = p \left( \widetilde{X_{i+1}} = Y_{i+1} \mid \widetilde{S_{i+1, n}} = na - \Sigma_{1, i} \right) = \tilde{p}_{i+1}(Y_{i+1}) \frac{\sigma_{i+1, n} \bar{q}_{i+2, n}(Z_{i+1})}{\sigma_{i+2, n} \bar{q}_{i+1, n}(0)}, \quad (3.2.26)$$

where

$$Z_{i+1} := \frac{m_{i+1} - Y_{i+1}}{\sigma_{i+2, n}}.$$

### 3.2.6 Assumptions

**Definition 13.** Let  $f : (\alpha, \beta) \rightarrow (A, B)$  be a function, where  $\alpha, \beta, A$  and  $B$  may be finite or not. Consider the following condition  $(\mathcal{H})$ .

$(\mathcal{H}) : f$  is strictly increasing and  $\lim_{\theta \rightarrow \alpha} f(\theta) = A$  ;  $\lim_{\theta \rightarrow \beta} f(\theta) = B$ .

**Statements**

We suppose *throughout the text* that the following assumptions hold. So in the statements of the results, we will not always precise which among them are required.

(*Supp*) : The  $(X_j)$ ,  $j \geq 1$  have a common support  $\mathcal{S}_X = (A, B)$ , where  $A$  and  $B$  may be finite or not.

(*Mgf*) : The mgf's  $(\Phi_j)_{j \geq 1}$  have the same domain of finiteness  $\Theta = (\alpha, \beta)$ , where  $\alpha$  and  $\beta$  may be finite or not.

(*H $\kappa$* ) : For all  $j \geq 1$ ,  $m_j := \frac{d\kappa_j}{d\theta}$  satisfies (*H*).

(*Uf*) : There exist functions  $f_+$  and  $f_-$  which satisfy (*H*) and such that

$$\forall j \geq 1, \forall \theta \in \Theta, f_-(\theta) \leq m_j(\theta) \leq f_+(\theta). \tag{3.2.27}$$

(*Cv*) : For any compact  $K \subset \Theta$ ,

$$0 < \inf_{j \geq 1} \inf_{\theta \in K} s_j^2(\theta) \leq \sup_{j \geq 1} \sup_{\theta \in K} s_j^2(\theta) < \infty, \tag{3.2.28}$$

(*AM6*) : For any compact  $K \subset \Theta$ ,

$$\sup_{j \geq 1} \sup_{\theta \in K} |\mu_j^6(\theta)| < \infty. \tag{3.2.29}$$

(*Cf*) : For any  $j \geq 1$ ,  $p_j$  is a function of class  $\mathcal{C}^1$  and for any compact  $K \subset \Theta$ ,

$$\sup_{j \geq 1} \sup_{\theta \in K} \left\| \frac{d\tilde{p}_j^\theta}{dx} \right\|_{L^1} < \infty. \tag{3.2.30}$$

**Elementary Facts**

**Fact 21.** *If a function  $f$  satisfies (*H*), then  $f$  is a homeomorphism from  $(\alpha, \beta)$  to  $(A, B)$ .*

**Fact 22.** *If a function  $f$  is defined as the mean of functions satisfying (*H*), then  $f$  satisfies (*H*). In particular,  $f$  is a homeomorphism from  $(\alpha, \beta)$  to  $(A, B)$ .*

**Corollary 5.** *Let  $\ell, n$  be integers with  $1 \leq \ell \leq n$ . Set*

$$\bar{m}_{\ell, n} := \frac{1}{n - \ell + 1} \sum_{j=\ell}^n m_j.$$

*Then, we deduce from (*H $\kappa$* ) and Fact 22 that  $\bar{m}_{\ell, n}$  is a homeomorphism from  $(\alpha, \beta)$  to  $(A, B)$ . Consequently, for any  $s \in \mathcal{S}_X$ , the equation*

$$\bar{m}_{\ell, n}(\theta) = s \tag{3.2.31}$$

*has a unique solution in  $\Theta = (\alpha, \beta)$ .*

**Definition 14.** We deduce from Corollary 5 that for any  $a \in \mathcal{S}_X$ , for any  $n \geq 1$ , there exists a unique  $\theta_n^a \in \Theta$  s.t.

$$\bar{m}_{1,n}(\theta_n^a) = a.$$

**Fact 23.** We deduce from  $(\mathcal{H}\kappa)$  that for any  $a \in \mathcal{S}_X$ , there exists a compact set  $K_a$  of  $\mathbb{R}$  s.t.

$$\{\theta_n^a : n \geq 1\} \subset K_a \subset \Theta. \quad (3.2.32)$$

**Corollary 6.** We deduce from the preceding Fact and the Assumptions that, for any  $a \in \mathcal{S}_X$ ,

$$\sup_{n \geq 1} \sup_{j \geq 1} |m_j(\theta_n^a)| < \infty, \quad (3.2.33)$$

$$0 < \inf_{n \geq 1} \inf_{j \geq 1} \Phi_j(\theta_n^a) \leq \sup_{n \geq 1} \sup_{j \geq 1} \Phi_j(\theta_n^a) < \infty, \quad (3.2.34)$$

$$0 < \inf_{n \geq 1} \inf_{j \geq 1} s_j^2(\theta_n^a) \leq \sup_{n \geq 1} \sup_{j \geq 1} s_j^2(\theta_n^a) < \infty, \quad (3.2.35)$$

and for any  $3 \leq \ell \leq 6$ ,

$$\sup_{n \geq 1} \sup_{j \geq 1} |\mu_j^\ell(\theta_n^a)| \leq \sup_{n \geq 1} \sup_{j \geq 1} |\mu_j^\ell(\theta_n^a)| < \infty. \quad (3.2.36)$$

**Definition 15.** We deduce from Corollary 5 that for any  $n \geq 1$  and  $0 \leq i \leq k-1$ , there exists a unique  $t_{i,n} \in \Theta$  s.t.

$$\bar{m}_{i+1,n}(t_{i,n}) = \frac{\sum_{j=i+1}^n Y_j}{n-i}. \quad (3.2.37)$$

Since  $\bar{m}_{i+1,n}$  is a homeomorphism from  $\mathcal{S}_X$  to  $\Theta$ ,  $t_{i,n}$  is a r.v. defined on  $(\Omega_n, \mathcal{A}_n)$ .

**Fact 24.** Assume that

$$\max_{0 \leq i \leq k-1} |t_{i,n}| = \mathcal{O}_{\mathcal{P}_n}(1) \quad (3.2.38)$$

Then, under the Assumptions, we have that

$$\max_{0 \leq i \leq k-1} \sup_{j \geq 1} |m_j(t_{i,n})| = \mathcal{O}_{\mathcal{P}_n}(1), \quad (3.2.39)$$

$$\max_{0 \leq i \leq k-1} \sup_{j \geq 1} \max \left\{ \frac{1}{\Phi_j(t_{i,n})}; \Phi_j(t_{i,n}) \right\} = \mathcal{O}_{\mathcal{P}_n}(1), \quad (3.2.40)$$

$$\max_{0 \leq i \leq k-1} \sup_{j \geq 1} \max \left\{ \frac{1}{s_j^2(t_{i,n})}; s_j^2(t_{i,n}) \right\} = \mathcal{O}_{\mathcal{P}_n}(1), \quad (3.2.41)$$

and for any  $3 \leq \ell \leq 6$ ,

$$\max_{0 \leq i \leq k-1} \sup_{j \geq 1} |\mu_j^\ell(t_{i,n})| \leq \max_{0 \leq i \leq k-1} \sup_{j \geq 1} |\mu_j^\ell(t_{i,n})| = \mathcal{O}_{\mathcal{P}_n}(1). \quad (3.2.42)$$



*Proof.* We prove only (3.2.41), the other proofs being similar. Let  $\epsilon > 0$ . Then, (3.2.38) implies that there exists  $A_\epsilon > 0$  s.t. for all  $n$  large enough,

$$\mathcal{P}_n \left( \max_{0 \leq i \leq k-1} |t_{i,n}| \leq A_\epsilon \right) \geq 1 - \epsilon. \quad (3.2.43)$$

Now, (Cv) implies that

$$s_{A_\epsilon}^2 := \sup_{j \geq 1} \sup_{\theta \in [-A_\epsilon, A_\epsilon]} s_j^2(\theta) < \infty. \quad (3.2.44)$$

Therefore,

$$\mathcal{P}_n \left( \max_{0 \leq i \leq k-1} \sup_{j \geq 1} s_j^2(t_{i,n}) \leq s_{A_\epsilon}^2 \right) \geq 1 - \epsilon. \quad (3.2.45)$$

□

**Remark 6.** We will prove in Section 3.4. that, under the Assumptions, (3.2.38) holds.

### 3.3 Properties of $(Y_1^n)_{n \geq 1}$

#### 3.3.1 Edgeworth expansion

Let  $(X_j)_{j \geq 1}$  be a sequence of independent r.v.'s with zero means and finite variances. For any  $j \geq 1$  and  $\ell \geq 3$ , set

$$s_j^2 := \mathbb{E}[X_j^2] = \text{Var}(X_j) \quad ; \quad \sigma_j := \sqrt{s_j^2} \quad ; \quad \mu_j^\ell := \mathbb{E}[X_j^\ell] \quad ; \quad |\mu|_j^\ell := \mathbb{E} \left[ |X_j|^\ell \right].$$

For any  $p, q$  with  $1 \leq p \leq q$  and  $\ell > 2$ , set

$$s_{p,q}^2 := \sum_{j=p}^q s_j^2 \quad ; \quad \sigma_{p,q} := \sqrt{s_{p,q}^2} \quad ; \quad \mu_{p,q}^\ell := \sum_{j=p}^q \mu_j^\ell.$$

For any  $j \geq 1$ , if  $p_j$  is of class  $\mathcal{C}^1$ , set

$$d_j := \left\| \frac{dp_j}{dx} \right\|_{L^1}.$$

For  $\nu \geq 3$ , let  $H_\nu$  be the Hermite polynomial of degree  $\nu$ . For example,

$$H_3(x) = x^3 - 3x \quad ; \quad H_4(x) = x^4 - 6x^2 + 3 \quad ; \quad H_5(x) = x^5 - 10x^3 + 15x.$$

**Theorem 13.** Let  $m$  be an integer with  $m \geq 3$ . Assume that

$$\sup_{j \geq 1} \frac{1}{s_j^2} < \infty, \quad (3.3.1)$$

$$\sup_{j \geq 1} |\mu|_j^{m+1} < \infty, \quad (3.3.2)$$

$$\sup_{j \geq 1} d_j < \infty. \quad (3.3.3)$$

Let  $\mathbf{n}$  be the density of the standard normal distribution. For any  $n \geq 1$ , let  $q_n$  be the density of  $(s_{1,n}^2)^{-1/2} S_{1,n}$ . Then, for all  $n$  large enough, we have that

$$\sup_{x \in \mathbb{R}} \left| q_n(x) - \mathbf{n}(x) \left( 1 + \sum_{\nu=3}^m P_{\nu,n}(x) \right) \right| = \frac{o(1)}{n^{(m-2)/2}}, \quad (3.3.4)$$

where, for example,

$$P_{3,n}(x) = \frac{\mu_{1,n}^3}{6(s_{1,n}^2)^{3/2}} H_3(x)$$

$$P_{4,n}(x) = \frac{(\mu_{1,n}^3)^2}{72(s_{1,n}^2)^3} H_6(x) + \frac{\mu_{1,n}^4 - 3 \sum_{j=1}^n (s_j^2)^2}{24(s_{1,n}^2)^2} H_4(x)$$

$$P_{5,n}(x) = \frac{(\mu_{1,n}^3)^3}{1296(s_{1,n}^2)^{9/2}} H_9(x) + \frac{\mu_{1,n}^3 \left( \mu_{1,n}^4 - 3 \sum_{j=1}^n (s_j^2)^2 \right)}{144(s_{1,n}^2)^{7/2}} H_7(x) + \frac{\mu_{1,n}^5 - 10 \sum_{j=1}^n \mu_j^3 s_j^2}{120(s_{1,n}^2)^{5/2}} H_5(x)$$

**Remark 7.** We obtain from (3.3.1) and (3.3.2) that

$$P_{3,n}(x) = \mathcal{O} \left( \frac{1}{n^{1/2}} \right) H_3(x) \quad (3.3.5)$$

$$P_{4,n}(x) = \mathcal{O} \left( \frac{1}{n} \right) H_6(x) + \mathcal{O} \left( \frac{1}{n} \right) H_4(x) \quad (3.3.6)$$

$$P_{5,n}(x) = \mathcal{O} \left( \frac{1}{n^{3/2}} \right) H_9(x) + \mathcal{O} \left( \frac{1}{n^{3/2}} \right) H_7(x) + \mathcal{O} \left( \frac{1}{n^{3/2}} \right) H_5(x) \quad (3.3.7)$$

### 3.3.2 Extensions of the Edgeworth expansion

For any integers  $p, q$  with  $1 \leq p \leq q$  and  $\theta \in \Theta$ , set

$$s_{p,q}^2(\theta) := \sum_{j=p}^q s_j^2(\theta) \quad ; \quad \sigma_{p,q}(\theta) := \sqrt{s_{p,q}^2(\theta)} \quad ; \quad \mu_{p,q}^\ell(\theta) := \sum_{j=p}^q \mu_j^\ell(\theta).$$

For any  $j \geq 1$  and  $\theta \in \Theta$ , set

$$d_j(\theta) := \left\| \frac{d\tilde{p}_j^\theta}{dx} \right\|_{L^1}.$$

**First Extension**

For any  $n \geq 1$ , let  $J_n$  be a subset of  $\{1, \dots, n\}$  s.t.  $\alpha_n := |J_n| < n$ . Let  $L_n$  be the complement of  $J_n$  in  $\{1, \dots, n\}$ . Set

$$\bar{S}_{L_n} := \sum_{j \in L_n} \tilde{X}_j^{\theta_n^a} - \mathbb{E} \left[ \tilde{X}_j^{\theta_n^a} \right] = \tilde{X}_j^{\theta_n^a} - m_j(\theta_n^a).$$

For any  $\theta \in \Theta$  and  $\ell \geq 3$ , set

$$s_{L_n}^2(\theta) := \sum_{j \in L_n} s_j^2(\theta) \quad ; \quad \sigma_{L_n}(\theta) := \sqrt{s_{L_n}^2(\theta)} \quad ; \quad \mu_{L_n}^\ell(\theta) := \sum_{j \in L_n} \mu_j^\ell(\theta).$$

**Theorem 14.** *Let  $m$  be an integer with  $m \geq 3$ . Assume that*

$$\sup_{j \geq 1} \frac{1}{s_j^2(\theta_n^a)} = \mathcal{O}(1), \tag{3.3.8}$$

$$\sup_{j \geq 1} |\mu_j^{m+1}(\theta_n^a)| = \mathcal{O}(1), \tag{3.3.9}$$

$$\sup_{j \geq 1} d_j(\theta_n^a) = \mathcal{O}(1). \tag{3.3.10}$$

For any  $n \geq 1$ , let  $\bar{q}_{L_n}$  be the density of  $(s_{L_n}^2)^{-1/2} \bar{S}_{L_n}$ . Then, for all  $n$  large enough, we have that

$$\sup_{x \in \mathbb{R}} \left| \bar{q}_{L_n}(x) - \mathbf{n}(x) \left( 1 + \sum_{\nu=3}^m \bar{P}_{\nu, L_n}(x) \right) \right| = \frac{o(1)}{(n - \alpha_n)^{(m-2)/2}}, \tag{3.3.11}$$

where the  $\bar{P}_{\nu, L_n}$  are defined as the  $P_{\nu, n}$ , except that the  $s_{1, n}^2$  and the  $\mu_{1, n}^\ell$  are replaced respectively by  $s_{L_n}^2(\theta_n^a)$  and  $\mu_{L_n}^\ell(\theta_n^a)$ .

**Corollary 7.** *Assume that  $(C\nu)$ ,  $(AM(m+1))$ ,  $(Cf)$  and  $(Uf)$  hold. Then, (3.3.11) holds.*

**Remark 8.** *By Remark 7, for  $\nu = 3, 4, 5$ , some  $\mathcal{O}\left(\frac{1}{n^{(\nu-2)/2}}\right)$  appear in  $P_{\nu, n}$ . They are replaced by some  $\frac{\mathcal{O}(1)}{(n - \alpha_n)^{(\nu-2)/2}}$  in  $\bar{P}_{\nu, L_n}$ .*

**Second Extension**

**Theorem 15.** *Let  $m$  be an integer with  $m \geq 3$ . Assume that*

$$\max_{0 \leq i \leq k-1} \sup_{j \geq 1} \frac{1}{s_j^2(t_{i, n})} = \mathcal{O}_{\mathcal{P}_n}(1), \tag{3.3.12}$$

$$\max_{0 \leq i \leq k-1} \sup_{j \geq 1} |\mu_j^{m+1}(t_{i, n})| = \mathcal{O}_{\mathcal{P}_n}(1), \tag{3.3.13}$$

$$\max_{0 \leq i \leq k-1} \sup_{j \geq 1} d_j(t_{i, n}) = \mathcal{O}_{\mathcal{P}_n}(1). \tag{3.3.14}$$

Let  $e \in \{1, 2\}$ . We recall that  $\bar{q}_{i+e,n}$  is the density of  $\bar{S}_{i+e,n}$ , defined by (3.2.25). Then,

$$\sup_{x \in \mathbb{R}} \left| \bar{q}_{i+e,n}(x) - \mathbf{n}(x) \left( 1 + \sum_{\nu=3}^m \bar{P}_{\nu,n}^{(i,e)}(x) \right) \right| = \frac{o_{\mathcal{P}_n}(1)}{(n-i-e+1)^{(m-2)/2}}, \quad (3.3.15)$$

where the  $\bar{P}_{\nu,n}^{(i,e)}$  are defined as the  $P_{\nu,n}$ , except that the  $s_{1,n}^2$  and the  $\mu_{1,n}^\ell$  are replaced respectively by  $s_{i+e,n}^2(t_{i,n})$  and  $\mu_{i+e,n}^\ell(t_{i,n})$ .

*Proof.* We follow the lines of the proof of Theorem 13, given in (Petrov, 1975). For  $j \geq 1$ , let  $\tilde{\xi}_j$  be the characteristic function of  $\tilde{X}_j^{t_{i,n}}$ . Then, for any  $\tau \in \mathbb{R}$ ,

$$\tilde{\xi}_j(\tau) = \int \exp(i\tau x) \frac{\exp(t_{i,n}x) p_j(x)}{\Phi_j(t_{i,n})} dx \quad (3.3.16)$$

is a r.v. defined on  $(\Omega_n, \mathcal{A}_n)$ . Performing a Taylor expansion of  $\exp(i\tau x)$ , we obtain that

$$\tilde{\xi}_j(\tau) = 1 + \frac{s_j^2(t_{i,n})}{2} (i\tau)^2 + \sum_{\nu=3}^m \frac{\mu_j^\nu(t_{i,n})}{\nu!} (i\tau)^\nu + r_j(\tau). \quad (3.3.17)$$

Then, we deduce from Fact 24 that

$$\sum_{j=i+e}^n r_j \left( \frac{\tau}{\sigma_{i+e,n}} \right) \leq \frac{\delta_{i,n}}{(n-i-e+1)^{(m-2)/2}} |\tau|^m, \quad \text{where } \max_{0 \leq i \leq k-1} |\delta_{i,n}| = o_{\mathcal{P}_n}(1). \quad (3.3.18)$$

For any  $n \geq 1$ , and  $\omega \in \Omega_n$ , we consider a triangular array whose row of index  $n$  is composed of the  $n-i-e+1$  independent r.v.'s

$$\left( \bar{X}_j^{t_{i,n}(\omega)} \right)_{i+e \leq j \leq n}$$

Let  $\bar{\xi}_{i+e,n}$  be the characteristic function of  $\bar{S}_{i+e,n}^{t_{i,n}}$ , given by  $\bar{\xi}_{i+e,n}(\tau) = \int \exp(i\tau x) \bar{q}_{i+e,n}(x) dx$ . By independence of the  $\left( \bar{X}_j^{t_{i,n}(\omega)} \right)_{i+e \leq j \leq n}$  and (3.3.17) combined with (3.3.18), we obtain that for suitable some constant  $\rho > 0$ , for  $|\tau| \leq n^\rho$ ,

$$|\bar{\xi}_{i+e,n}(\tau) - u_{m,n}(\tau)| \leq \frac{\delta_{i,n}}{(n-i-e+1)^{(m-2)/2}} \left( |\tau|^m + |\tau|^{3(m-1)} \right) \exp\left(-\frac{\tau^2}{2}\right), \quad (3.3.19)$$

where  $u_{m,n}$  is the Fourier transform of  $\mathbf{n}(x) \left( 1 + \sum_{\nu=3}^m \bar{P}_{\nu,n}^{(i,e)}(x) \right)$  and  $\max_{0 \leq i \leq k-1} |\delta_{i,n}| = o_{\mathcal{P}_n}(1)$ .

Now, we have that

$$I := \int_{-\infty}^{\infty} |\bar{\xi}_{i+e,n}(\tau) - u_{m,n}(\tau)| d\tau \quad (3.3.20)$$

$$\leq \int_{|\tau| \leq n^\rho} |\bar{\xi}_{i+e,n}(\tau) - u_{m,n}(\tau)| d\tau + \int_{|\tau| > n^\rho} |u_{m,n}(\tau)| d\tau + \int_{|\tau| > n^\rho} |\bar{\xi}_{i+e,n}(\tau)| d\tau. \quad (3.3.21)$$

Then, we obtain from (3.3.19) that

$$\int_{|\tau| \leq n^\rho} |\bar{\xi}_{i+e,n}(\tau) - u_{m,n}(\tau)| d\tau = \frac{o_{\mathcal{P}_n}(1)}{(n-i-e+1)^{(m-2)/2}}.$$

Then, using general results on characteristic functions (see Lemma 12 in (Petrov, 1975)), we prove that

$$\int_{|\tau| > n^\rho} |u_{m,n}(\tau)| d\tau = \frac{o_{\mathcal{P}_n}(1)}{(n-i-e+1)^{(m-2)/2}}. \quad (3.3.22)$$

Now, (3.3.14) implies that for any  $\alpha > 0$  and  $\eta > 0$ ,

$$\max_{0 \leq i \leq k-1} (n-i-e+1)^\alpha \int_{|\tau| > \eta} \prod_{j=i+e}^n |\tilde{\xi}_j(\tau)| d\tau = o_{\mathcal{P}_n}(1), \quad (3.3.23)$$

which implies in turn that

$$\int_{|\tau| > n^\rho} |\bar{\xi}_{i+e,n}(\tau)| d\tau = \frac{o_{\mathcal{P}_n}(1)}{(n-i-e+1)^{(m-2)/2}}. \quad (3.3.24)$$

Considering (3.3.21), we deduce that

$$I = \frac{o_{\mathcal{P}_n}(1)}{(n-i-e+1)^{(m-2)/2}}.$$

Then, Fourier inversion yields that

$$\bar{q}_{i+e,n}(x) - \mathbf{n}(x) \left( 1 + \sum_{\nu=3}^m \bar{P}_{\nu,n}^{(i,e)}(x) \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\tau x) (\bar{\xi}_{i+e,n}(\tau) - u_{m,n}(\tau)) d\tau. \quad (3.3.25)$$

Therefore,

$$\sup_{x \in \mathbb{R}} \left| \bar{q}_{i+e,n}(x) - \mathbf{n}(x) \left( 1 + \sum_{\nu=3}^m \bar{P}_{\nu,n}^{(i,e)}(x) \right) \right| \leq \frac{I}{2\pi} = \frac{o_{\mathcal{P}_n}(1)}{(n-i-e+1)^{(m-2)/2}}. \quad (3.3.26)$$

□

**Corollary 8.** Assume that  $(\mathcal{C}v)$ ,  $(\mathcal{A}\mathcal{M}(m+1))$ ,  $(\mathcal{C}f)$  hold, and that

$$\max_{0 \leq i \leq k-1} |t_{i,n}| = \mathcal{O}_{\mathcal{P}_n}(1) \quad (3.3.27)$$

Then, (3.3.15) holds.

**Remark 9.** By Remark 7, for  $\nu = 3, 4, 5$ , some  $\mathcal{O}\left(\frac{1}{n^{(\nu-2)/2}}\right)$  appear in  $P_{\nu,n}$ . They are replaced by some  $\frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^{(\nu-2)/2}}$  in  $P_{\nu,n}^{(i,e)}$ .

### 3.3.3 Moments of $Y_j$

Throughout this Section 3.3, all the tilted densities considered pertain to  $\theta = \theta_n^a$ , defined by

$$\bar{m}_{1,n}(\theta_n^a) = a. \quad (3.3.28)$$

The moments of the  $Y_j$ 's are obtained by integration of the conditional density. As expected, their first order approximations are the moments of  $\widetilde{X}_j$ .

**Lemma 14.**

$$\max_{1 \leq j \leq n} |\mathbb{E}_{\mathcal{P}_n}[Y_j] - m_j(\theta_n^a)| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \quad (3.3.29)$$

*Proof.* For any  $n \geq 1$  and  $1 \leq j \leq n$ , we have that

$$\mathbb{E}_{\mathcal{P}_n}[Y_j] = \int xp(X_j = x | S_{1,n} = na) dx = \int xp(\widetilde{X}_j = x | \widetilde{S}_{1,n} = na) dx. \quad (3.3.30)$$

Let  $L_n = \{1, \dots, n\} \setminus \{j\}$ . Normalizing, we obtain that

$$p(\widetilde{X}_j = x | \widetilde{S}_{1,n} = na) = \tilde{p}_j(x) \left( \frac{\sigma_{1,n}(\theta_n^a)}{\sigma_{L_n}(\theta_n^a)} \right) \frac{p_{\bar{S}_{L_n}}(\gamma_n^j(x))}{p_{\bar{S}_{1,n}}(0)}, \quad \text{where } \gamma_n^j(x) := \frac{m_j(\theta_n^a) - x}{\sigma_{L_n}(\theta_n^a)}. \quad (3.3.31)$$

Since  $(\mathcal{AM}6)$  implies  $(\mathcal{AM}4)$ , we get from Corollary 7 with  $m = 3$  that

$$p_{\bar{S}_{L_n}}(\gamma_n^j(x)) = \mathbf{n}(\gamma_n^j(x)) \left[ 1 + \frac{\mu_{L_n}^3(\theta_n^a)}{6(s_{L_n}^2(\theta_n^a))^{3/2}} H_3(\gamma_n^j(x)) \right] + \frac{o(1)}{\sqrt{n-1}} \quad (3.3.32)$$

and

$$p_{\bar{S}_{1,n}}(0) = \mathbf{n}(0) + \frac{o(1)}{\sqrt{n}}. \quad (3.3.33)$$

Now,  $(\mathcal{C}v)$ ,  $(\mathcal{AM}6)$  and the boundedness of the sequence  $(\theta_n^a)_{n \geq 1}$  imply readily that

$$\frac{\sigma_{1,n}(\theta_n^a)}{\sigma_{L_n}(\theta_n^a)} = 1 + \mathcal{O}\left(\frac{1}{n}\right) \quad \text{and} \quad \frac{\mu_{L_n}^3(\theta_n^a)}{6(s_{L_n}^2(\theta_n^a))^{3/2}} = \mathcal{O}\left(\frac{1}{\sqrt{n-1}}\right). \quad (3.3.34)$$

Since the functions  $\theta \mapsto \mathbf{n}(\theta)$  and  $\theta \mapsto \mathbf{n}(\theta)H_3(\theta)$  are bounded, we deduce that

$$\frac{p_{\bar{S}_{L_n}}(\gamma_n^j(x))}{p_{\bar{S}_{1,n}}(0)} = \left\{ \mathbf{n}(\gamma_n^j(x)) \left( 1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) H_3(\gamma_n^j(x)) \right) + \frac{o(1)}{\sqrt{n-1}} \right\} \left\{ \frac{1}{\mathbf{n}(0)} + \frac{o(1)}{\sqrt{n}} \right\} \quad (3.3.35)$$

$$= \frac{\mathbf{n}(\gamma_n^j(x))}{\mathbf{n}(0)} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) = \exp\left(-\frac{\gamma_n^j(x)^2}{2}\right) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \quad (3.3.36)$$

Consequently,

$$p(\widetilde{X}_j = x | \widetilde{S}_{1,n} = na) = \tilde{p}_j(x) \left( 1 + \mathcal{O}\left(\frac{1}{n}\right) \right) \left\{ \exp\left(-\frac{\gamma_n^j(x)^2}{2}\right) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \right\}. \quad (3.3.37)$$

Recalling that  $\int x\tilde{p}_j(x)dx = m_j(\theta_n^a)$ , we deduce from (3.3.30) and (3.3.37) that

$$\mathbb{E}_{\mathcal{P}_n}[Y_j] = \left\{ \int x\tilde{p}_j(x) \exp\left(-\frac{\gamma_n^j(x)^2}{2}\right) dx + m_j(\theta_n^a) \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \right\} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right). \quad (3.3.38)$$

Therefore, it is enough to prove that

$$\int x\tilde{p}_j(x) \exp\left(-\frac{\gamma_n^j(x)^2}{2}\right) dx = m_j(\theta_n^a) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \quad (3.3.39)$$

Now, for any  $u \in \mathbb{R}$ ,

$$1 - u^2/2 \leq \exp(-u^2/2) \leq 1, \quad (3.3.40)$$

from which we deduce that

$$\int_0^\infty x\tilde{p}_j(x)dx - \frac{1}{2} \int_0^\infty x\tilde{p}_j(x)\gamma_n^j(x)^2 dx \leq \int_0^\infty x\tilde{p}_j(x) \exp\left(-\frac{\gamma_n^j(x)^2}{2}\right) dx \leq \int_0^\infty x\tilde{p}_j(x)dx \quad (3.3.41)$$

and

$$\int_{-\infty}^0 x\tilde{p}_j(x)dx \leq \int_{-\infty}^0 x\tilde{p}_j(x) \exp\left(-\frac{\gamma_n^j(x)^2}{2}\right) dx \leq \int_{-\infty}^0 x\tilde{p}_j(x)dx - \frac{1}{2} \int_{-\infty}^0 x\tilde{p}_j(x)\gamma_n^j(x)^2 dx. \quad (3.3.42)$$

Adding (3.3.41) and (3.3.42), we obtain that

$$m_j(\theta_n^a) - \frac{1}{2} \int_0^\infty x\tilde{p}_j(x)\gamma_n^j(x)^2 dx \leq \int x\tilde{p}_j(x) \exp\left(-\frac{\gamma_n^j(x)^2}{2}\right) dx \leq m_j(\theta_n^a) - \frac{1}{2} \int_{-\infty}^0 x\tilde{p}_j(x)\gamma_n^j(x)^2 dx. \quad (3.3.43)$$

For any  $B \in \mathcal{B}(\mathbb{R})$ , we have that

$$\int_B x\tilde{p}_j(x)\gamma_n^j(x)^2 dx = \frac{1}{s_{L_n}^2(\theta_n^a)} \left\{ \int_B x\tilde{p}_j(x) (m_j(\theta_n^a) - x)^2 dx \right\} \quad (3.3.44)$$

$$= \frac{1}{s_{L_n}^2(\theta_n^a)} \sum_{i=0}^2 \binom{2}{i} m_j(\theta_n^a)^{2-i} (-1)^i \int_B x^{1+i} \tilde{p}_j(x) dx. \quad (3.3.45)$$

Let  $i \in \{0, 1, 2\}$ . Recalling that  $L_n = \{1, \dots, n\} \setminus \{j\}$ , we get from (Cv) and (Uf) that

$$\max_{1 \leq j \leq n} \frac{1}{s_{L_n}^2(\theta_n^a)} = \mathcal{O}\left(\frac{1}{n}\right) \quad \text{and} \quad \max_{1 \leq j \leq n} |m_j(\theta_n^a)|^{2-i} = \mathcal{O}(1). \quad (3.3.46)$$

Then, (AM6) implies that for all  $n \geq 1$ ,

$$\max_{1 \leq j \leq n} \left| \int_B x^{1+i} \tilde{p}_j(x) dx \right| \leq \max_{1 \leq j \leq n} \int_{\mathbb{R}} |x|^{1+i} \tilde{p}_j(x) dx \leq \sup_{j \geq 1} \left\{ 1 + \sup_{\theta \in K_a} \mathbb{E} \left[ \left| \tilde{X}_j^\theta \right|^6 \right] \right\} < \infty. \quad (3.3.47)$$

So we deduce from (3.3.45) that

$$\max_{1 \leq j \leq n} \int_B x\tilde{p}_j(x)\gamma_n^j(x)^2 dx = \mathcal{O}\left(\frac{1}{n}\right). \quad (3.3.48)$$

Taking  $B = (-\infty, 0)$  and  $B = (0, \infty)$  in (3.3.48), we conclude the proof by (3.3.43).  $\square$

**Lemma 15.** *We have that*

$$\max_{1 \leq j < j' \leq n} |\mathbb{E}_{\mathcal{P}_n}[Y_j Y_{j'}] - m_j(\theta_n^a) m_{j'}(\theta_n^a)| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \quad (3.3.49)$$

and

$$\max_{1 \leq j \leq n} |\mathbb{E}_{\mathcal{P}_n}[Y_j^2] - (s_j^2(\theta_n^a) + m_j(\theta_n^a)^2)| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \quad (3.3.50)$$

*Proof.* For any  $1 \leq j < j' \leq n$ , we have that

$$\mathbb{E}_{\mathcal{P}_n}[Y_j Y_{j'}] = \int x x' p\left(\widetilde{X}_j = x; \widetilde{X}_{j'} = x' \mid \widetilde{S}_{1,n} = na\right) dx dx'.$$

Let  $L_n = \{1, \dots, n\} \setminus \{j, j'\}$ . Normalizing, we obtain that

$$p\left(\widetilde{X}_j = x; \widetilde{X}_{j'} = x' \mid \widetilde{S}_{1,n} = na\right) = \tilde{p}_j(x) \tilde{p}_{j'}(x') \left(\frac{\sigma_{1,n}(\theta_n^a)}{\sigma_{L_n}(\theta_n^a)}\right) \frac{p_{\widetilde{S}_{L_n}}\left(\Gamma_n^{j,j'}(x)\right)}{p_{\widetilde{S}_{1,n}}(0)},$$

where

$$\Gamma_n^j(x) := \frac{m_j(\theta_n^a) + m_{j'}(\theta_n^a) - x - x'}{\sigma_{L_n}(\theta_n^a)}.$$

Since  $(\mathcal{AM}4)$  holds, we get from Corollary 7 with  $m = 3$  that

$$p\left(\widetilde{X}_j = x; \widetilde{X}_{j'} = x' \mid \widetilde{S}_{1,n} = na\right) = \tilde{p}_j(x) \tilde{p}_{j'}(x') \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \left\{ \exp\left(-\frac{\Gamma_n^j(x)^2}{2}\right) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \right\}. \quad (3.3.51)$$

As in the preceding proof, we get from (3.3.40) (applied to  $\exp\left(-\frac{\Gamma_n^j(x)^2}{2}\right)$ ) that, uniformly in  $j$ ,

$$\begin{aligned} \mathbb{E}_{\mathcal{P}_n}[Y_j Y_{j'}] &= \int x x' \tilde{p}_j(x) \tilde{p}_{j'}(x') dx dx' + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \\ &= m_j(\theta_n^a) m_{j'}(\theta_n^a) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

The proof of (3.3.50) is quite similar. □

**Corollary 9.** *We have that*

$$\max_{1 \leq j < j' \leq n} \text{Cov}_{\mathcal{P}_n}(Y_j, Y_{j'}) = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \quad (3.3.52)$$

and

$$\max_{1 \leq j \leq n} |\text{Var}_{\mathcal{P}_n}(Y_j) - (s_j^2(\theta_n^a))| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \quad (3.3.53)$$



*Proof.* We deduce from the preceding Lemmas that for any  $1 \leq j < j' \leq n$ ,

$$\begin{aligned} \text{Cov}_{\mathcal{P}_n}(Y_j, Y_{j'}) &= \mathbb{E}_{\mathcal{P}_n}[Y_j Y_{j'}] - \mathbb{E}_{\mathcal{P}_n}[Y_j] \mathbb{E}_{\mathcal{P}_n}[Y_{j'}] \\ &= \left( m_j(\theta_n^a) m_{j'}(\theta_n^a) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \right) - \left( m_j(\theta_n^a) m_{j'}(\theta_n^a) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \right) \\ &= \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

□

### 3.3.4 Proof of $\max_{0 \leq i \leq k-1} |t_{i,n}| = \mathcal{O}_{\mathcal{P}_n}(1)$

For any  $n \geq 1$  and  $i = 0, \dots, k-1$ , set

$$V_{i+1,n} := \frac{1}{n-i} \sum_{j=i+1}^n Z_j \quad \text{where} \quad Z_j := Y_j - \mathbb{E}[Y_j]. \quad (3.3.54)$$

**Lemma 16.** *We have that*

$$\mathbb{E}_{\mathcal{P}_n}[V_{1,n}^2] = o(1). \quad (3.3.55)$$

*Proof.* We have that

$$\mathbb{E}_{\mathcal{P}_n}[V_{1,n}^2] = \frac{1}{n^2} \left\{ \sum_{j=1}^n \text{Var}_{\mathcal{P}_n}(Y_j) + 2 \sum_{1 \leq j < j' \leq n} \text{Cov}_{\mathcal{P}_n}(Y_j, Y_{j'}) \right\}. \quad (3.3.56)$$

Then, we get from Corollary 9 that

$$\mathbb{E}_{\mathcal{P}_n}[V_{1,n}^2] = \frac{1}{n^2} \left\{ \sum_{j=1}^n \left[ s_j^2(\theta_n^a) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \right] + n(n-1) \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \right\}. \quad (3.3.57)$$

We conclude the proof by Corollary 6 which implies that

$$\frac{1}{n^2} \sum_{j=1}^n \left[ s_j^2(\theta_n^a) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \right] = o(1). \quad (3.3.58)$$

□

**Lemma 17.** *We have that*

$$\max_{0 \leq i \leq k-1} |V_{i+1,n}| = o_{\mathcal{P}_n}(1). \quad (3.3.59)$$

*Proof.* We follow the lines of Kolmogorov's maximal inequality proof. Let  $n \geq 1$  and  $i \in \{0, \dots, k-1\}$ . For any  $\delta > 0$ , set

$$A_{i,n} := \{|V_{i+1,n}| \geq \delta\} \cap \left( \bigcap_{j=0}^{i-1} \{|V_{j+1,n}| < \delta\} \right), \quad (3.3.60)$$

and

$$A_n := \left\{ \max_{0 \leq i \leq k-1} |V_{i+1,n}| \geq \delta \right\} = \bigcup_{i=0}^{k-1} A_{i,n}. \quad (3.3.61)$$

Since the  $(A_{i,n})_{0 \leq i \leq k-1}$  are non-overlapping, we have that

$$\mathbb{E}_{\mathcal{P}_n}[V_{1,n}^2] \geq \sum_{i=0}^{k-1} \int_{A_{i,n}} V_{1,n}^2 d\mathcal{P}_n \quad (3.3.62)$$

$$= \sum_{i=0}^{k-1} \int_{A_{i,n}} \{(V_{1,n} - V_{i+1,n}) + V_{i+1,n}\}^2 d\mathcal{P}_n \quad (3.3.63)$$

$$\geq 2 \sum_{i=0}^{k-1} \int_{A_{i,n}} (V_{1,n} - V_{i+1,n})V_{i+1,n} d\mathcal{P}_n + \sum_{i=0}^{k-1} \int_{A_{i,n}} V_{i+1,n}^2 d\mathcal{P}_n \quad (3.3.64)$$

$$\geq 2 \sum_{i=0}^{k-1} \int_{A_{i,n}} (V_{1,n} - V_{i+1,n})V_{i+1,n} d\mathcal{P}_n + \delta^2 \mathcal{P}_n(A_n). \quad (3.3.65)$$

By Lemma 16, it is enough to prove that

$$\sum_{i=0}^{k-1} \int_{A_{i,n}} (V_{1,n} - V_{i+1,n})V_{i+1,n} d\mathcal{P}_n = o(1). \quad (3.3.66)$$

In the proof of Kolmogorov, the corresponding term is equal to 0, by independence of the involved random variables. Similarly (3.3.66) will follow from Corollary 15, which states that the  $(Z_j)$  are asymptotically uncorrelated. Indeed, we have that

$$\sum_{i=0}^{k-1} \int_{A_{i,n}} (V_{1,n} - V_{i+1,n})V_{i+1,n} d\mathcal{P}_n = \sum_{i=0}^{k-1} \mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} V_{1,n} V_{i+1,n}] - \sum_{i=0}^{k-1} \mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} V_{1,n}^2]. \quad (3.3.67)$$

Then, it is enough to prove that each sum in the right-hand side of (3.3.67) is a  $o(1)$ . We get readily that

$$\mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} V_{1,n} V_{i+1,n}] = \frac{1}{n(n-i)} \left\{ \sum_{j=i+1}^n \mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} Z_j^2] + \sum_{\substack{1 \leq j \leq n \\ i+1 \leq j' \leq n \\ j \neq j'}} \mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} Z_j Z_{j'}] \right\} \quad (3.3.68)$$

and

$$\mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} V_{i+1,n}^2] = \frac{1}{(n-i)^2} \left\{ \sum_{j=i+1}^n \mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} Z_j^2] + \sum_{\substack{i+1 \leq j, j' \leq n \\ j \neq j'}} \mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} Z_j Z_{j'}] \right\}. \quad (3.3.69)$$

Now, the Cauchy-Schwarz inequality applied twice, first in  $\mathcal{L}^2$  and then in  $\mathbb{R}^k$ , implies that

$$\sum_{i=0}^{k-1} \frac{1}{n(n-i)} \sum_{j=i+1}^n \mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} Z_j^2] \leq \frac{1}{n} \sum_{i=0}^{k-1} \mathcal{P}_n(A_{i,n})^{1/2} \left( \frac{\sum_{j=i+1}^n \mathbb{E}_{\mathcal{P}_n} [Z_j^4]^{1/2}}{n-i} \right) \quad (3.3.70)$$

$$\leq \frac{1}{n} \left\{ \sum_{i=0}^{k-1} \mathcal{P}_n(A_{i,n}) \right\}^{1/2} \left\{ \sum_{i=0}^{k-1} \left( \frac{\sum_{j=i+1}^n \mathbb{E}_{\mathcal{P}_n} [Z_j^4]^{1/2}}{n-i} \right)^2 \right\}^{1/2}. \quad (3.3.71)$$

Then,  $\left[ \sum_{i=0}^{k-1} \mathcal{P}_n(A_{i,n}) \right]^{1/2} = \mathcal{P}_n(A_n)^{1/2} \leq 1$  and we obtain from Corollary 15 and Fact 6 that, for all  $i \in \{0, \dots, k-1\}$ ,

$$\left( \frac{\sum_{j=i+1}^n \mathbb{E}_{\mathcal{P}_n} [Z_j^4]^{1/2}}{n-i} \right)^2 = \left( \frac{\sum_{j=i+1}^n \left\{ \mu_j^4(\theta_n^a) + \mathcal{O}\left(\frac{1}{n}\right) \right\}^{1/2}}{n-i} \right)^2 = \mathcal{O}(1). \quad (3.3.72)$$

Finally, we deduce from (3.3.71) and (3.3.72) that

$$\sum_{i=0}^{k-1} \frac{1}{n(n-i)} \sum_{j=i+1}^n \mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} Z_j^2] = \frac{1}{n} \{k\mathcal{O}(1)\}^{1/2} = o(1). \quad (3.3.73)$$

We obtain similarly that

$$\sum_{i=0}^{k-1} \frac{1}{(n-i)^2} \sum_{j=i+1}^n \mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} Z_j^2] \leq \mathcal{P}_n(A_n)^{1/2} \left\{ \sum_{i=0}^{k-1} \frac{1}{(n-i)^2} \left( \frac{\sum_{j=i+1}^n \mathbb{E}_{\mathcal{P}_n} [Z_j^4]^{1/2}}{n-i} \right)^2 \right\}^{1/2} \quad (3.3.74)$$

$$= \mathcal{O}(1) \left\{ \sum_{i=0}^{k-1} \frac{1}{(n-i)^2} \right\}^{1/2} = o(1). \quad (3.3.75)$$

To conclude, we consider the sums involving  $\mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} Z_j Z_{j'}]$ , for  $j \neq j'$ , in (3.3.68) and (3.3.69). The Cauchy-Schwarz inequality brings terms of the form  $\mathbb{E}_{\mathcal{P}_n} [Z_j^2 Z_{j'}^2]$ . Clearly,  $Z_j^2$  and  $Z_{j'}^2$  are similarly asymptotically uncorrelated and thereby, we obtain analogously that

$$\min \left\{ \sum_{i=0}^{k-1} \frac{1}{n(n-i)} \sum_{\substack{1 \leq j \leq n \\ i+1 \leq j' \leq n \\ j \neq j'}} \mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} Z_j Z_{j'}] \ ; \ \sum_{i=0}^{k-1} \frac{1}{(n-i)^2} \sum_{\substack{i+1 \leq j, j' \leq n \\ j \neq j'}} \mathbb{E}_{\mathcal{P}_n} [\mathbf{1}_{A_{i,n}} Z_j Z_{j'}] \right\} = o(1),$$

which ends the proof.  $\square$

**Theorem 16.** *We have that*

$$\max_{0 \leq i \leq k-1} |t_{i,n}| = \mathcal{O}_{\mathcal{P}_n}(1). \quad (3.3.76)$$

*Proof.* The triangle inequality implies that for any  $n \geq 1$ ,

$$\max_{0 \leq i \leq k-1} |\bar{m}_{i+1,n}(t_{i,n})| \leq \max_{0 \leq i \leq k-1} |V_{i+1,n}| + \max_{0 \leq i \leq k-1} \left| \left( \frac{1}{n-i} \sum_{j=i+1}^n \mathbb{E}[Y_j] \right) - \bar{m}_{i+1,n}(\theta_n^a) \right| + \max_{0 \leq i \leq k-1} |\bar{m}_{i+1,n}(\theta_n^a)|. \quad (3.3.77)$$

We get from Lemma 17 and assumption (E) that

$$\max_{0 \leq i \leq k-1} |V_{i+1,n}| = o_{\mathcal{P}_n}(1). \quad (3.3.78)$$

Then, Lemma 14 implies that

$$\max_{0 \leq i \leq k-1} \left| \left( \frac{1}{n-i} \sum_{j=i+1}^n \mathbb{E}[Y_j] \right) - \bar{m}_{i+1,n}(\theta_n^a) \right| \leq \max_{0 \leq i \leq k-1} \left\{ \frac{1}{n-i} \sum_{j=i+1}^n |\mathbb{E}[Y_j] - m_j(\theta_n^a)| \right\} = \mathcal{O}\left(\frac{1}{n}\right). \quad (3.3.79)$$

Now, Fact 6 implies that

$$\max_{0 \leq i \leq k-1} |\bar{m}_{i+1,n}(\theta_n^a)| = \mathcal{O}(1). \quad (3.3.80)$$

Combining (3.3.77), (3.3.78), (3.3.79), and (3.3.80), we obtain that

$$\max_{0 \leq i \leq k-1} |\bar{m}_{i+1,n}(t_{i,n})| = \mathcal{O}_{\mathcal{P}_n}(1). \quad (3.3.81)$$

Now,  $(\mathcal{H}\kappa)$  implies that for all  $i = 0, \dots, k-1$ ,  $\bar{m}_{i+1,n}$  is a homeomorphism from  $\Theta$  to  $\mathcal{S}_X$ . Then, we get from  $(\mathcal{U}f)$  that for all  $s \in \mathcal{S}_X$ ,

$$(f_+)^{-1}(s) \leq (\bar{m}_{i+1,n})^{-1}(s) \leq (f_-)^{-1}(s). \quad (3.3.82)$$

We deduce that  $\mathcal{P}_n$  - a.s.,

$$(f_+)^{-1}(\bar{m}_{i+1,n}(t_{i,n})) \leq t_{i,n} \leq (f_-)^{-1}(\bar{m}_{i+1,n}(t_{i,n})), \quad (3.3.83)$$

which combined to (3.3.81) concludes the proof.  $\square$

### 3.3.5 The max of the trajectories

Throughout this Section 3.5, all the tilted densities considered pertain to  $\theta = \theta_n^a$ , defined by

$$\bar{m}_{1,n}(\theta_n^a) = a. \quad (3.3.84)$$

**Lemma 18.** *We have that*

$$\max_{1 \leq j \leq n} |Y_j| = \mathcal{O}_{\mathcal{P}_n}(\log n). \quad (3.3.85)$$

*Proof.* For any  $n \geq 1$ , set  $M_n := \max_{1 \leq j \leq n} |Y_j|$ . For all  $s > 0$ , we have that

$$\mathcal{P}_n(M_n \geq s) \leq \sum_{j=1}^n \mathcal{P}_n(Y_j \leq -s) + \mathcal{P}_n(Y_j \geq s) \quad (3.3.86)$$

$$= \sum_{j=1}^n \int_{-\infty}^{-s} P(\tilde{X}_j = x | \tilde{S}_{1,n} = na) dx + \int_s^{\infty} P(\tilde{X}_j = x | \tilde{S}_{1,n} = na) dx. \quad (3.3.87)$$

Now, we recall from (3.3.37) that

$$p(\tilde{X}_j = x | \tilde{S}_{1,n} = na) = \tilde{p}_j(x) \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \left\{ \exp\left(-\frac{\gamma_n^j(x)^2}{2}\right) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \right\} = \tilde{p}_j(x) \mathcal{O}(1). \quad (3.3.88)$$

Consequently, there exists an absolute constant  $C > 0$  s.t. for all  $n \geq 1$ ,

$$\mathcal{P}_n(M_n \geq s) \leq C \left\{ \sum_{j=1}^n P(\tilde{X}_j \leq -s) + P(\tilde{X}_j \geq s) \right\}. \quad (3.3.89)$$

We get from Markov's inequality that for any  $\lambda > 0$ ,

$$P(\tilde{X}_j \leq -s) = P(\exp(-\lambda \tilde{X}_j) \geq \exp(\lambda s)) \leq \mathbb{E}[\exp(-\lambda \tilde{X}_j)] \exp(-\lambda s) \quad (3.3.90)$$

and

$$P(\tilde{X}_j \geq s) \leq \mathbb{E}[\exp(\lambda \tilde{X}_j)] \exp(-\lambda s). \quad (3.3.91)$$

Then, for any  $\lambda \neq 0$ ,

$$\mathbb{E}[\exp(\lambda \tilde{X}_j)] = \int \exp(\lambda x) \left[ \frac{\exp(\theta_n^a x) p_j(x)}{\Phi_j(\theta_n^a)} dx \right] = \frac{\Phi_j(\theta_n^a + \lambda)}{\Phi_j(\theta_n^a)}. \quad (3.3.92)$$

Therefore,

$$\mathcal{P}_n(M_n \geq s) \leq C \left\{ \sum_{j=1}^n \frac{\Phi_j(\theta_n^a - \lambda)}{\Phi_j(\theta_n^a)} + \frac{\Phi_j(\theta_n^a + \lambda)}{\Phi_j(\theta_n^a)} \right\} \exp(-\lambda s). \quad (3.3.93)$$

Since the sequence  $(\theta_n^a)_{n \geq 1}$  is bounded, we can find  $\lambda > 0$  s.t. each of the sequences  $(\theta_n^a - \lambda)_{n \geq 1}$  and  $(\theta_n^a + \lambda)_{n \geq 1}$  is included in a compact subset of  $\Theta$ . Therefore, we deduce that there exists an absolute constant  $D$  s.t.

$$\sup_{n \geq 1} \sup_{j \geq 1} \max \left\{ \frac{\Phi_j(\theta_n^a - \lambda)}{\Phi_j(\theta_n^a)} ; \frac{\Phi_j(\theta_n^a + \lambda)}{\Phi_j(\theta_n^a)} \right\} \leq D. \quad (3.3.94)$$

Therefore,

$$\mathcal{P}_n(M_n \geq s) \leq CDn \exp(-\lambda s) = CD \exp(\log n - \lambda s). \quad (3.3.95)$$

Consequently, for all sequence  $(s_n)_{n \geq 1}$  s.t.  $\frac{s_n}{\log n} \rightarrow \infty$  as  $n \rightarrow \infty$ , we have that

$$\mathcal{P}_n(M_n \geq s_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.3.96)$$

Set  $Z_n := \frac{M_n}{\log n}$ . For any sequence  $(a_n)_{n \geq 1}$  s.t.  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$ , we have that

$$\mathcal{P}_n(Z_n \geq a_n) = \mathcal{P}_n(M_n \geq s_n) \quad \text{where } s_n := a_n \log n, \quad \text{so that } \frac{s_n}{\log n} \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (3.3.97)$$

Finally, we conclude the proof by applying the following Fact, since we get from (3.3.96) that

$$\mathcal{P}_n(Z_n \geq a_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.3.98)$$

□

**Fact 25.** For all  $n \geq 1$ , let  $Z_n : (\Omega_n, \mathcal{A}_n, \mathcal{P}_n) \rightarrow \mathbb{R}$  be a r.v. Assume that for any sequence  $(a_n)_{n \geq 1}$  s.t.  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$ , we have that  $\mathcal{P}_n(|Z_n| \geq a_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Then,

$$Z_n = \mathcal{O}_{\mathcal{P}_n}(1). \quad (3.3.99)$$

*Proof.* Suppose that the sequence  $(Z_n)$  is not a  $\mathcal{O}_{\mathcal{P}_n}(1)$ . This means that there exists  $\epsilon > 0$  s.t. for all  $k \in \mathbb{N}$ , there exists  $n(k) \in \mathbb{N}$  s.t.

$$\mathcal{P}_{n(k)}(|Z_{n(k)}| \geq k) > \epsilon. \quad (3.3.100)$$

If the sequence  $(n(k))_k$  is bounded, then there exists a fixed  $n_0 \in \mathbb{N}$  and a subsequence  $(n(k_j))_{j \geq 1}$  such that for all  $j \geq 1$ ,  $n(k_j) = n_0$ . We can clearly assume that  $k_j \rightarrow \infty$  as  $j \rightarrow \infty$ , which implies that

$$\lim_{j \rightarrow \infty} \mathcal{P}_{n(k_j)}(|Z_{n(k_j)}| \geq k_j) = \lim_{j \rightarrow \infty} \mathcal{P}_{n_0}(|Z_{n_0}| \geq k_j) = 0, \quad (3.3.101)$$

which contradicts (3.3.100).

If the sequence  $(n(k))_k$  is not bounded, then there exists a strictly increasing subsequence  $(n(k_j))_j$  s.t.  $n(k_j) \rightarrow \infty$  as  $j \rightarrow \infty$ . Now, we can define a sequence  $(a_n)$  s.t. for all  $j \geq 1$ ,  $a_{n(k_j)} = k_j$ . We still can assume that  $k_j \rightarrow \infty$  as  $j \rightarrow \infty$ . Therefore, we can assume that  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$ , which implies that

$$\lim_{j \rightarrow \infty} \mathcal{P}_{n(k_j)}(|Z_{n(k_j)}| \geq k_j) = \lim_{j \rightarrow \infty} \mathcal{P}_{n(k_j)}(|Z_{n(k_j)}| \geq a_{n(k_j)}) = 0, \quad (3.3.102)$$

which contradicts (3.3.100).

□

### 3.3.6 Taylor expansion

**Lemma 19.** Let  $I$  be an interval of  $\mathbb{R}$  containing 0, of non void interior, and  $f : I \rightarrow \mathbb{R}$  a function of class  $C^2$ . Let  $(U_n)$  be a sequence of random variables  $U_n : (\Omega_n, \mathcal{A}_n) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  s.t.

$$U_n = o_{\mathcal{P}_n}(1). \quad (3.3.103)$$

Then, there exists  $(B_n)_{n \geq 1} \in \mathcal{A}_{\rightarrow 1}$  s.t. for any  $n \geq 1$ ,

$$f(U_n) = f(0) + U_n f'(0) + U_n^2 \mathcal{O}_{\mathcal{P}_n}(1) \quad \text{on } B_n. \quad (3.3.104)$$

Furthermore, if  $U_n = o_{\mathcal{P}_n}(u_n)$ , with  $u_n \xrightarrow[n \rightarrow \infty]{} 0$ , then

$$f(U_n) = f(0) + o_{\mathcal{P}_n}(u_n) \quad \text{on } B_n. \quad (3.3.105)$$

*Proof.* Let  $\epsilon > 0$ . Let  $\delta > 0$  s.t.  $(-\delta, \delta) \subset I$ . Set

$$B_n := \{|U_n| < \delta\}.$$

Since  $U_n = o_{\mathcal{P}_n}(1)$ , we have that  $(B_n)_{n \geq 1} \in \mathcal{A}_{\rightarrow 1}$ . For any  $n \geq 1$ ,  $f(U_n)$  is well defined on  $B_n$ , and the Taylor-Lagrange formula provides a  $C_n$  with  $|C_n| \leq |U_n|$ , s.t.

$$f(U_n) = f(0) + U_n f'(0) + \frac{U_n^2}{2} f''(C_n). \quad (3.3.106)$$

Now,  $C_n$  can be obtained from a dichotomy process, initialized with  $U_n$ . This implies that for all  $n$ ,  $C_n$  is a measurable mapping from  $(\Omega_n, \mathcal{A}_n)$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , for  $C_n$  is the limit of such mappings. Then, as  $|C_n| \leq |U_n|$  and  $f''$  is continuous, we have that

$$C_n \xrightarrow{\mathcal{P}_n} 0 \implies f''(C_n) \xrightarrow{\mathcal{P}_n} f''(0) \implies f''(C_n) = \mathcal{O}_{\mathcal{P}_n}(1). \quad (3.3.107)$$

Furthermore, if  $U_n = o_{\mathcal{P}_n}(u_n)$  with  $u_n \xrightarrow[n \rightarrow \infty]{} 0$ , then  $\frac{U_n^2}{2} f''(C_n)$  is also a  $o_{\mathcal{P}_n}(u_n)$ .  $\square$

## 3.4 Main Results

### 3.4.1 Theorem with small $k$

**Theorem 17.** *Suppose that the Assumptions stated in Section 2.6 hold. Assume that*

$$k \longrightarrow \infty \text{ as } n \longrightarrow \infty \quad \text{and that} \quad k = o(n^\rho), \quad \text{with } 0 < \rho < 1/2. \quad (3.4.1)$$

Then,

$$\left\| Q_{nak} - \tilde{P}_1^k \right\|_{TV} \xrightarrow[n \rightarrow \infty]{} 0, \quad (3.4.2)$$

where  $\tilde{P}_1^k$  is the joint distribution of independent r.v.'s  $(\tilde{X}_j^{\theta_n^a})_{1 \leq j \leq k}$ .

*Proof.* We have that

$$\pi_k(Y_1^k) := p\left(X_1^k = Y_1^k | S_{1,n} = na\right) = \frac{p_{\tilde{X}_1^k}(Y_1^k) p_{\tilde{S}_{k+1,n}}(na - \Sigma_{1,k})}{p_{\tilde{S}_{1,n}}(na)}. \quad (3.4.3)$$

Then we normalize, so that

$$\pi_k(Y_1^k) = p_{\tilde{X}_1^k}(Y_1^k) \frac{\sigma_{1,n}(\theta_n^a)}{\sigma_{k+1,n}(\theta_n^a)} \frac{p_{\tilde{S}_{k+1,n}}(Z_k)}{p_{\tilde{S}_{1,n}}(0)} \quad \text{where} \quad Z_k := \frac{\sum_{j=1}^k m_j(\theta_n^a) - Y_j}{\sigma_{k+1,n}(\theta_n^a)}. \quad (3.4.4)$$

Since  $(\mathcal{AM}4)$  holds, we get from Corollary 7 with  $m = 3$  that

$$\pi_k(Y_1^k) = p_{\tilde{X}_1^k}(Y_1^k) \frac{\sigma_{1,n}(\theta_n^a)}{\sigma_{k+1,n}(\theta_n^a)} \frac{\mathfrak{n}(Z_k) \left(1 + \frac{\mu_{k+1,n}^3(\theta_n^a)}{6(s_{k+1,n}^2(\theta_n^a))^{3/2}} H_3(Z_k)\right) + \frac{o(1)}{(n-k)^{3/2}}}{\mathfrak{n}(0) + \frac{o(1)}{n^{3/2}}} \quad (3.4.5)$$

First, we get from Corollary 6 that

$$\frac{\sigma_{1,n}(\theta_n^a)}{\sigma_{k+1,n}(\theta_n^a)} = \left(1 + \frac{s_{1,k}^2(\theta_n^a)}{s_{k+1,n}^2(\theta_n^a)}\right)^{1/2} = \left(1 + \frac{k}{n-k} \mathcal{O}(1)\right)^{1/2} \quad \text{and} \quad \frac{\mu_{k+1,n}^3(\theta_n^a)}{6(s_{k+1,n}^2(\theta_n^a))^{3/2}} = \frac{\mathcal{O}(1)}{(n-k)^{1/2}}.$$

Then, (3.4.1) implies that

$$\frac{\sigma_{1,n}(\theta_n^a)}{\sigma_{k+1,n}(\theta_n^a)} = 1 + o(1) \quad \text{and} \quad \frac{\mu_{k+1,n}^3(\theta_n^a)}{6(s_{k+1,n}^2(\theta_n^a))^{3/2}} = o(1). \quad (3.4.6)$$

Now, we get from Corollary 6 and Lemma 18 that

$$Z_k = \frac{k \log n}{\sqrt{n-k}} \mathcal{O}_{\mathcal{P}_n}(1). \quad (3.4.7)$$

Then, (3.4.1) implies that

$$Z_k = o_{\mathcal{P}_n}(1), \quad \text{so that} \quad \mathfrak{n}(Z_k) \xrightarrow{\mathcal{P}_n} \mathfrak{n}(0) \quad \text{and} \quad H_3(Z_k) \xrightarrow{\mathcal{P}_n} H_3(0) = 0. \quad (3.4.8)$$

We obtain from the preceding lines that

$$\pi_k(Y_1^k) = p_{\tilde{X}_1^k} \left( Y_1^k \right) (1 + o_{\mathcal{P}_n}(1)). \quad (3.4.9)$$

Finally, we apply Lemma 13 to conclude the proof.  $\square$

### 3.4.2 Theorem with large $k$

#### Statement of the Theorem

Let  $y_1^n \in (\mathcal{S}_X)^n$ . Then, for any  $0 \leq i \leq k-1$ , there exists a unique  $\tau_i(y_1^n)$  s.t.

$$\bar{m}_{i+1,n}(\tau_i(y_1^n)) = \frac{\sum_{j=i+1}^n y_j}{n-i}. \quad (3.4.10)$$

For  $0 \leq i \leq k-1$ , define a density  $g(y_{i+1}|y_1^i)$  by

$$g(y_{i+1}|y_1^i) := C_i^{-1} \tilde{p}_{i+1}(y_{i+1}) \exp\left(-\frac{(y_{i+1} - m_{i+1}(\tau_i(y_1^n)))^2}{2s_{i+2,n}^2(\tau_i(y_1^n))}\right) \exp\left(\frac{3\alpha_{i+2,n}^{(3)}(\tau_i(y_1^n))}{\sigma_{i+2,n}(\tau_i(y_1^n))} y_{i+1}\right),$$

where  $C_i$  is a normalizing constant which insures that  $\int g(y_{i+1}|y_1^i) dy_{i+1} = 1$  and

$$\alpha_{i+e,n}^{(3)}(\tau_i(y_1^n)) := \frac{\mu_{i+e,n}^3(\tau_i(y_1^n))}{6(s_{i+e,n}^2(\tau_i(y_1^n)))^{3/2}}.$$

Then, we define the limiting density on  $\mathbb{R}^k$  by

$$g_k(y_1^k) := \prod_{i=0}^{k-1} g(y_{i+1}|y_1^i). \quad (3.4.11)$$



**Theorem 18.** *Suppose that the Assumptions stated in Section 2.6 hold. Assume that  $k$  is of order  $n - (\log n)^\tau$  with  $\tau > 6$ .*

$$\|Q_{nak} - G_k\|_{TV} \xrightarrow{n \rightarrow \infty} 0, \quad (3.4.12)$$

where  $G_k$  is the distribution associated to the density  $g_k$ .

*Proof.* We get from the criterion for convergence in total variation distance stated in Section 2.4. that it is enough to prove the following Theorem.  $\square$

**Theorem 19.** *Suppose that the Assumptions stated in Section 2.6 hold. Assume that  $k$  is of order  $n - (\log n)^\tau$  with  $\tau > 6$ . Then, there exists  $(B_n)_{n \geq 1} \in \mathcal{A}_{\rightarrow 1}$  s.t. for any  $n \geq 1$ ,*

$$p_k(Y_1^k) := p(X_1^k = Y_1^k | S_{1,n} = na) = g_k(Y_1^k)[1 + o_{\mathcal{P}_n}(1)] \quad \text{on } B_n. \quad (3.4.13)$$

The proof is given hereafter, in three steps. Throughout the proof, all the tilted densities considered pertain to  $\theta = t_{i,n}$ . We write  $s_j^2, \mu_j^\ell$  instead of  $s_j^2(t_{i,n}), \mu_j^\ell(t_{i,n})$ .

**Identifying  $g(Y_{i+1}|Y_1^i)$**

When  $y_1^n = Y_1^n$ , we have that

$$\sum_{j=i+1}^n y_j = \sum_{j=i+1}^n Y_j = na - \sum_{j=1}^i Y_j \quad \mathcal{P}_n \text{ a.s.} \quad (3.4.14)$$

and

$$\tau_i(Y_1^n) = t_{i,n}. \quad (3.4.15)$$

We recall from the first calculus of Section 3.2.5 that

$$\pi_i = \tilde{p}_{i+1}(Y_{i+1}) \frac{\sigma_{i+1,n} \bar{q}_{i+2,n}(Z_{i+1})}{\sigma_{i+2,n} \bar{q}_{i+1,n}(0)}, \quad \text{where } Z_{i+1} := \frac{m_{i+1} - Y_{i+1}}{\sigma_{i+2,n}}. \quad (3.4.16)$$

Since  $(\mathcal{AM}6)$  holds, we get from Corollary 8 with  $m = 5$  that

$$\pi_i = \tilde{p}_{i+1}(Y_{i+1}) \frac{\sigma_{i+1,n}}{\sigma_{i+2,n}} \left\{ \frac{\mathfrak{n}(Z_{i+1}) \left[ 1 + \sum_{\nu=3}^5 \bar{P}_\nu^{i+2,n}(Z_{i+1}) \right] + \frac{o_{\mathcal{P}_n}(1)}{(n-i-1)^{3/2}}}{\mathfrak{n}(0) \left[ 1 + \sum_{\nu=3}^5 \bar{P}_\nu^{i+1,n}(0) \right] + \frac{o_{\mathcal{P}_n}(1)}{(n-i)^{3/2}}} \right\}. \quad (3.4.17)$$

For  $e \in \{1, 2\}$ , set

$$\alpha_{i+e,n}^{(3)} := \frac{\mu_{i+e,n}^3}{6(s_{i+e,n}^2)^{3/2}} = \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-e+1)^{1/2}},$$

$$\beta_{i+e,n}^{(6)} := \frac{(\mu_{i+e,n}^3)^2}{72(s_{i+e,n}^2)^3} = \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{n-i-e+1} \quad ; \quad \beta_{i+e,n}^{(4)} := \frac{\mu_{i+e,n}^4 - 3 \sum_{j=i+e}^n (s_j^2)^2}{24(s_{i+e,n}^2)^2} = \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{n-i-e+1},$$

$$\gamma_{i+e,n}^{(9)} := \frac{(\mu_{i+e,n}^3)^3}{1296(s_{i+e,n}^2)^{9/2}} \quad ; \quad \gamma_{i+e,n}^{(7)} := \frac{\mu_{i+e,n}^3 \left( \mu_{i+e,n}^4 - 3 \sum_{j=i+e}^n (s_j^2)^2 \right)}{144(s_{i+e,n}^2)^{7/2}} \quad ; \quad \gamma_{i+e,n}^{(5)} := \frac{\mu_{i+e,n}^5 - 10 \sum_{j=i+e}^n \mu_j^3 s_j^2}{120(s_{i+e,n}^2)^{5/2}},$$

where, for  $\ell \in \{5, 7, 9\}$ ,

$$\gamma_{i+e,n}^{(\ell)} = \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-e+1)^{3/2}}. \quad (3.4.18)$$

For  $m \in \{3, \dots, 9\}$ , replacing  $H_m(Z_{i+1})$  by its expression, we have that

$$\overline{P}_3^{i+e,n}(Z_{i+1}) = \alpha_{i+e,n}^{(3)} [Z_{i+1}^3 - 3Z_{i+1}], \quad (3.4.19)$$

$$\overline{P}_4^{i+e,n}(Z_{i+1}) = \beta_{i+e,n}^{(6)} [Z_{i+1}^6 - 15Z_{i+1}^4 + 45Z_{i+1}^2 - 15] + \beta_{i+e,n}^{(4)} [Z_{i+1}^4 - 6Z_{i+1}^2 + 3], \quad (3.4.20)$$

$$\overline{P}_5^{i+e,n}(Z_{i+1}) = \gamma_{i+e,n}^{(9)} [Z_{i+1}^9 + \dots + 945Z_{i+1}] + \gamma_{i+e,n}^{(7)} [Z_{i+1}^7 + \dots - 105Z_{i+1}] + \gamma_{i+e,n}^{(5)} [Z_{i+1}^5 + \dots + 15Z_{i+1}]. \quad (3.4.21)$$

Therefore,

$$\sum_{\nu=3}^5 \overline{P}_\nu^{i+2,n}(Z_{i+1}) = -3\alpha_{i+2,n}^{(3)} Z_{i+1} - 15\beta_{i+2,n}^{(6)} + 3\beta_{i+2,n}^{(4)} + \mathcal{O}_{\mathcal{P}_n}(1) \frac{(\log n)^3}{(n-i-1)^2}. \quad (3.4.22)$$

and

$$\sum_{\nu=3}^5 \overline{P}_{\nu,n}^{i+1}(0) = -15\beta_{i+1,n}^{(6)} + 3\beta_{i+1,n}^{(4)}.$$

Since  $\mathbf{n}(Z_{i+1}) = \mathcal{O}_{\mathcal{P}_n}(1)$ , we can factorize  $\mathbf{n}(Z_{i+1})$  in the numerator of the bracket of (3.4.17), so that

$$\pi_i = \tilde{p}_{i+1}(Y_{i+1}) \frac{\sigma_{i+1,n}}{\sigma_{i+2,n}} \frac{\mathbf{n}(Z_{i+1}) \left[ 1 - 3\alpha_{i+2,n}^{(3)} Z_{i+1} - 15\beta_{i+2,n}^{(6)} + 3\beta_{i+2,n}^{(4)} + \mathcal{O}_{\mathcal{P}_n}(1) \frac{(\log n)^3}{(n-i-1)^2} + \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^{3/2}} \right]}{\mathbf{n}(0) \left[ 1 - 15\beta_{i+1,n}^{(6)} + 3\beta_{i+1,n}^{(4)} + \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i)^{3/2}} \right]}.$$

Since  $n-k$  is of order  $(\log n)^\tau$  with  $\tau > 6$ , we have for all  $n > 1$ , and  $i = 0, \dots, k-1$ ,

$$0 \leq \frac{(\log n)^3}{(n-i-1)^2} (n-i-1)^{3/2} \leq \frac{(\log n)^3}{(n-k)^{1/2}} \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore,

$$\frac{(\log n)^3}{(n-i-1)^2} = \frac{o(1)}{(n-i-1)^{3/2}}, \text{ so that } \mathcal{O}_{\mathcal{P}_n}(1) \frac{(\log n)^3}{(n-i-1)^2} = \frac{o_{\mathcal{P}_n}(1)}{(n-i-1)^{3/2}}$$

Consequently,

$$\pi_i = \tilde{p}_{i+1}(Y_{i+1}) \frac{\sigma_{i+1,n}}{\sigma_{i+2,n}} \exp\left(-\frac{Z_{i+1}^2}{2}\right) \left\{ \frac{1 + \frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} Y_{i+1} - \frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} m_{i+1} - 15\beta_{i+2,n}^{(6)} + 3\beta_{i+2,n}^{(4)} + \frac{o_{\mathcal{P}_n}(1)}{(n-i-1)^{3/2}}}{1 - 15\beta_{i+1,n}^{(6)} + 3\beta_{i+1,n}^{(4)} + \frac{o_{\mathcal{P}_n}(1)}{(n-i)^{3/2}}} \right\}$$

Now, we need to extract  $Y_{i+1}$  from the numerator of the bracket hereabove. In that purpose, set

$$U_{i,n} := \frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} Y_{i+1} + U'_{i,n} \quad \text{where} \quad U'_{i,n} := -\frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} m_{i+1} - 15\beta_{i+2,n}^{(6)} + 3\beta_{i+2,n}^{(4)} + \frac{o_{\mathcal{P}_n}(1)}{(n-i-1)^{3/2}},$$

and

$$V_{i,n} := -15\beta_{i+1,n}^{(6)} + 3\beta_{i+1,n}^{(4)} + \frac{o_{\mathcal{P}_n}(1)}{(n-i)^{3/2}}.$$

**Fact 26.** For any  $n \geq 1$ , let  $(W_{i,n})_{0 \leq i \leq k-1}$  be r.v.'s defined on  $(\Omega_n, \mathcal{A}_n)$  s.t.  $\max_{0 \leq i \leq k-1} |W_{i,n}| = o_{\mathcal{P}_n}(1)$ .

Then, there exists  $(B_n)_{n \geq 1} \in \mathcal{A}_{\rightarrow 1}$  s.t. for all  $n \geq 1$ , we have on  $B_n$  that for all  $i = 0, \dots, k-1$ ,

$$1 + W_{i,n} = \exp(W_{i,n} + W_{i,n}^2 A_{i,n}) \quad \text{where} \quad \max_{0 \leq i \leq k-1} |A_{i,n}| = \mathcal{O}_{\mathcal{P}_n}(1). \quad (3.4.23)$$

*Proof.* Let  $\epsilon > 0$ . For any  $n \geq 1$ , set

$$B_n := \left\{ \max_{0 \leq i \leq k-1} |W_{i,n}| < 1/2 \right\}.$$

Since  $\max_{0 \leq i \leq k-1} |W_{i,n}| = o_{\mathcal{P}_n}(1)$ , we have that  $(B_n)_{n \geq 1} \in \mathcal{A}_{\rightarrow 1}$ . Now, set

$$f(x) := \log(1+x).$$

Then  $f$  satisfies the conditions of Lemma 19. Therefore, for all  $i = 0, \dots, k-1$ , there exists  $C_{i,n}$  with  $\max_{0 \leq i \leq k-1} |C_{i,n}| \leq \max_{0 \leq i \leq k-1} |W_{i,n}|$  s.t.

$$f(W_{i,n}) = f(0) + W_{i,n} f'(0) + \frac{W_{i,n}^2}{2} f''(C_{i,n}) \quad (3.4.24)$$

For  $n \geq 1$  and  $0 \leq i \leq k-1$ , set  $A_{i,n} := \frac{1}{2} f''(C_{i,n})$ . Now,  $f''(x) = -\frac{1}{(1+x)^2}$ . Clearly, for all  $x \in (0, \frac{1}{2})$ ,  $|f''(x)| \leq \frac{1}{(1-x)^2}$ . Therefore, for any  $n \geq 1$ , we have on  $B_n$  that

$$\max_{0 \leq i \leq k-1} |A_{i,n}| \leq \frac{1}{\left(1 - \max_{0 \leq i \leq k-1} |C_{i,n}|\right)^2} \leq \frac{1}{\left(1 - \max_{0 \leq i \leq k-1} |W_{i,n}|\right)^2}, \quad (3.4.25)$$

which implies that  $\max_{0 \leq i \leq k-1} |A_{i,n}| = \mathcal{O}_{\mathcal{P}_n}(1)$ . □

Since

$$\max_{0 \leq i \leq k-1} |U_{i,n}| = o_{\mathcal{P}_n}(1) \quad \text{and} \quad \max_{0 \leq i \leq k-1} |V_{i,n}| = o_{\mathcal{P}_n}(1), \quad (3.4.26)$$

we have that

$$\frac{1 + U_{i,n}}{1 + V_{i,n}} = \exp(U_{i,n} + U_{i,n}^2 A_{i,n} - V_{i,n} - V_{i,n}^2 B_{i,n}), \quad (3.4.27)$$

where

$$\max_{0 \leq i \leq k-1} |A_{i,n}| = \mathcal{O}_{\mathcal{P}_n}(1) \quad \text{and} \quad \max_{0 \leq i \leq k-1} |B_{i,n}| = \mathcal{O}_{\mathcal{P}_n}(1).$$

Consequently, the preceding Fact implies that there exists  $(B_n)_{n \geq 1} \in \mathcal{A}_{\rightarrow 1}$  s.t. for any  $n \geq 1$  and  $0 \leq i \leq k-1$ ,

$$\pi_i = \Gamma_i \quad \text{on } B_n,$$

where

$$\Gamma_i = \tilde{p}_{i+1}(Y_{i+1}) \exp\left(-\frac{(Y_{i+1} - m_{i+1})^2}{2s_{i+2,n}^2}\right) \exp\left(\frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} Y_{i+1}\right) \frac{\sigma_{i+1,n}}{\sigma_{i+2,n}} \exp\{U'_{i,n} + U_{i,n}^2 A_{i,n} - V_{i,n} - V_{i,n}^2 B_{i,n}\}.$$

In order to identify  $g(Y_{i+1}|Y_1^i)$ , we have grouped the factors containing  $Y_{i+1}$ . Thereby, we obtain a function of  $Y_{i+1}$ , which we normalize to get a density. Thus, set

$$g(Y_{i+1}|Y_1^i) := C_i^{-1} \tilde{p}_{i+1}(Y_{i+1}) \exp\left(-\frac{(Y_{i+1} - m_{i+1})^2}{2s_{i+2,n}^2}\right) \exp\left(\frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} Y_{i+1}\right),$$

where  $C_i$  satisfies that

$$C_i = \int \exp\left(-\frac{(y - m_{i+1})^2}{2s_{i+2,n}^2}\right) \exp\left(\frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} y\right) \tilde{p}_{i+1}(y) dy.$$

Therefore,

$$\Gamma_i = g(Y_{i+1}|Y_1^i) \left\{ C_i \frac{\sigma_{i+1,n}}{\sigma_{i+2,n}} \exp(U'_{i,n} - V_{i,n} + U_{i,n}^2 A_{i,n} - V_{i,n}^2 B_{i,n}) \right\}. \quad (3.4.28)$$

Our objective is now to prove that

$$\prod_{i=0}^{k-1} C_i \frac{\sigma_{i+1,n}}{\sigma_{i+2,n}} \exp(U'_{i,n} - V_{i,n} + U_{i,n}^2 A_{i,n} - V_{i,n}^2 B_{i,n}) = 1 + o_{\mathcal{P}_n}(1). \quad (3.4.29)$$

In that purpose, we consider firstly the following result.

**Lemma 20.** For  $n \geq 1$ , let  $(Z_{i,n})_{0 \leq i \leq k-1}$  be r.v.'s defined on  $(\Omega_n, \mathcal{A}_n)$  and  $(u_{i,n})_{0 \leq i \leq k-1}$  be a sequence of reals. Assume that  $\max_{0 \leq i \leq k-1} |Z_{i,n}| = \mathcal{O}_{\mathcal{P}_n}(1)$  and  $\sum_{i=0}^{k-1} u_{i,n} \rightarrow 0$  as  $n \rightarrow \infty$ . Then,

$$\prod_{i=0}^{k-1} \exp(u_{i,n} Z_{i,n}) = 1 + o_{\mathcal{P}_n}(1).$$

Consequently, for any  $\alpha \geq 0$  and  $\beta > 1$ ,

$$\prod_{i=0}^{k-1} \exp\left(\frac{(\log n)^\alpha}{(n-i-1)^\beta} Z_{i,n}\right) = 1 + o_{\mathcal{P}_n}(1).$$

*Proof.* It is enough to prove that

$$\sum_{i=0}^{k-1} u_{i,n} Z_{i,n} = o_{\mathcal{P}_n}(1). \quad (3.4.30)$$

Let  $\epsilon > 0$  and  $\delta > 0$ . There exists  $A_\epsilon > 0$  and  $N_\epsilon > 0$  s.t. for all  $n \geq N_\epsilon$ ,

$$\mathcal{P}_n\left(\max_{0 \leq i \leq k-1} |Z_{i,n}| \leq A_\epsilon\right) \geq 1 - \epsilon.$$

Now, there exists  $N_{\epsilon,\delta} > 0$  s.t. for all  $n \geq N_{\epsilon,\delta}$ ,

$$\sum_{i=0}^{k-1} |u_{i,n}| < \frac{\delta}{A_\epsilon}.$$

Then, for all  $n \geq \max\{N_\epsilon; N_{\epsilon,\delta}\}$ ,

$$\begin{aligned} \mathcal{P}_n\left(\left|\sum_{i=0}^{k-1} u_{i,n} Z_{i,n}\right| < \delta\right) &\geq \mathcal{P}_n\left(\left\{\sum_{i=0}^{k-1} |u_{i,n}| |Z_{i,n}| < \delta\right\} \cap \left\{\max_{0 \leq i \leq k-1} |Z_{i,n}| \leq A_\epsilon\right\}\right) \\ &\geq \mathcal{P}_n\left(\left\{\sum_{i=0}^{k-1} |u_{i,n}| < \frac{\delta}{A_\epsilon}\right\} \cap \left\{\max_{0 \leq i \leq k-1} |Z_{i,n}| \leq A_\epsilon\right\}\right) \\ &= \mathcal{P}_n\left(\max_{0 \leq i \leq k-1} |Z_{i,n}| \leq A_\epsilon\right) \\ &\geq 1 - \epsilon. \end{aligned}$$

□

**The factors estimated by applying Lemma 20**

**Corollary 10.** *We have that*

$$\prod_{i=0}^{k-1} \{\exp(U_{i,n}^2 A_{i,n} - V_{i,n}^2 B_{i,n})\} = 1 + o_{\mathcal{P}_n}(1). \quad (3.4.31)$$

*Proof.* We may apply Lemma 20, since

$$\max_{0 \leq i \leq k-1} |U_{i,n}| = \frac{\log n}{n-i-1} \mathcal{O}_{\mathcal{P}_n}(1) \quad \text{and} \quad \max_{0 \leq i \leq k-1} |V_{i,n}| = \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{n-i-1}. \quad (3.4.32)$$

□

Unfortunately, (3.4.32) implies that we can not apply Lemma 20 to  $U'_{i,n}$  and  $V_{i,n}$ . However, we have that

$$U'_{i,n} - V_{i,n} = -\frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}}m_{i+1} + 3\left(\beta_{i+2,n}^{(4)} - \beta_{i+1,n}^{(4)}\right) - 15\left(\beta_{i+2,n}^{(6)} - \beta_{i+1,n}^{(6)}\right) + \frac{o_{\mathcal{P}_n}(1)}{(n-i-1)^{3/2}}. \quad (3.4.33)$$

Now,

$$\beta_{i+2,n}^{(4)} - \beta_{i+1,n}^{(4)} = \frac{\lambda_{i+2,n}(s_{i+1,n}^2)^2 - \lambda_{i+1,n}(s_{i+2,n}^2)^2}{24(s_{i+1,n}^2)^2(s_{i+2,n}^2)^2}, \quad \text{where } \lambda_{i+e,n} = \sum_{j=i+e}^n \lambda_j \text{ and } \lambda_j = \mu_j^4 - 3(s_j^2)^2 \quad (3.4.34)$$

$$= \frac{\lambda_{i+2,n} \left[ (s_{i+2,n}^2)^2 + 2s_{i+2,n}^2 s_{i+1,n} + (s_{i+1,n}^2)^2 \right] - (\lambda_{i+2,n} + \lambda_{i+1,n})(s_{i+2,n}^2)^2}{24(s_{i+1,n}^2)^2(s_{i+2,n}^2)^2} \quad (3.4.35)$$

$$= \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2}, \quad (3.4.36)$$

since in the numerator of (3.4.35), the terms of order  $(n-i-1)^3$ , that is the terms  $\lambda_{i+2,n}(s_{i+2,n}^2)^2$ , vanish.

Similarly, we obtain that

$$\left(\beta_{i+2,n}^{(6)} - \beta_{i+1,n}^{(6)}\right) = \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2}. \quad (3.4.37)$$

Combining (3.4.33), (3.4.36), (3.4.37), we obtain that

$$\prod_{i=0}^{k-1} \exp(U'_{i,n} - V_{i,n}) = \left\{ \prod_{i=0}^{k-1} \exp\left(-\frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}}m_{i+1}\right) \right\} \left\{ \prod_{i=0}^{k-1} \exp\left(\frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2} + \frac{o_{\mathcal{P}_n}(1)}{(n-i-1)^{3/2}}\right) \right\} \quad (3.4.38)$$

$$= \left\{ \prod_{i=0}^{k-1} \exp\left(-\frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}}m_{i+1}\right) \right\} \{1 + o_{\mathcal{P}_n}(1)\}, \quad (3.4.39)$$

where the last equality follows from Lemma 20. Notice that  $\frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} = \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{n-i-1}$ , so that the corresponding factor is not in the range of Lemma 20. Finally, (3.4.31) and (3.4.39) imply that

$$\prod_{i=0}^{k-1} C_i \frac{\sigma_{i+1,n}}{\sigma_{i+2,n}} \exp(U'_{i,n} - V_{i,n} + U_{i,n}^2 A_{i,n} - V_{i,n}^2 B_{i,n}) = \left\{ \prod_{i=0}^{k-1} C_i \frac{\sigma_{i+1,n}}{\sigma_{i+2,n}} \exp\left(-\frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}}m_{i+1}\right) \right\} \{1 + o_{\mathcal{P}_n}(1)\}.$$

**The other factors**

Therefore, in order to conclude, it is enough to prove that

$$\prod_{i=0}^{k-1} L_{i,n} = 1 + o_{\mathcal{P}_n}(1), \quad \text{where } L_{i,n} := C_i \frac{\sigma_{i+1,n}}{\sigma_{i+2,n}} \exp\left(-\frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} m_{i+1}\right). \quad (3.4.40)$$

**Fact 27.** *We have that*

$$\frac{\sigma_{i+1,n}}{\sigma_{i+2,n}} = 1 + \frac{s_{i+1}^2}{2s_{i+2,n}^2} + \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2}, \quad (3.4.41)$$

and

$$\exp\left(-\frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} m_{i+1}\right) = 1 - \frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} m_{i+1} + \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2} \quad (3.4.42)$$

*Proof.* We have that

$$\frac{\sigma_{i+1,n}}{\sigma_{i+2,n}} = \left(1 + \frac{s_{i+1}^2}{s_{i+2,n}^2}\right)^{1/2} \quad \text{and} \quad \frac{s_{i+1}^2}{s_{i+2,n}^2} = \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2}. \quad (3.4.43)$$

Therefore, (3.4.41) follows readily from Lemma 19, applied with the function  $f : x \mapsto (1+x)^{1/2}$ . Similarly, we get (3.4.42) by applying Lemma 19 with the function  $f : x \mapsto \exp(x)$ .  $\square$

**Lemma 21.** *We have that*

$$C_i = 1 + \frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} m_{i+1} - \frac{s_{i+1}^2}{2s_{i+2,n}^2} + \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2}. \quad (3.4.44)$$

*Proof.* Recall that

$$C_i = \int \exp(v_i(y)) \tilde{p}_{i+1}(y) dy \quad \text{where} \quad v_i(y) := -\frac{(y-m_{i+1})^2}{2s_{i+2,n}^2} + \frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} y.$$

A Taylor expansion implies the existence of  $w_i(y)$  with  $|w_i(y)| \leq |v_i(y)|$  s.t.

$$\exp(v_i(y)) = 1 + v_i(y) + \frac{v_i(y)^2}{2} \exp(w_i(y)). \quad (3.4.45)$$

Now,

$$\begin{aligned} \int (1 + v_i(y)) \tilde{p}_{i+1}(y) dy &= \int \left[1 - \frac{(y-m_{i+1})^2}{2s_{i+2,n}^2} + \frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} y\right] \tilde{p}_{i+1}(y) dy \\ &= \int \tilde{p}_{i+1}(y) dy - \frac{1}{2s_{i+2,n}^2} \int (y-m_{i+1})^2 \tilde{p}_{i+1}(y) dy + \frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} \int y \tilde{p}_{i+1}(y) dy \\ &= 1 - \frac{s_{i+1}^2}{2s_{i+2,n}^2} + \frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} m_{i+1}. \end{aligned}$$

Consequently, it is enough to prove the following Fact.  $\square$

**Fact 28.** *We have that*

$$J_i := \int \frac{v_i(y)^2}{2} \exp(w_i(y)) \tilde{p}_{i+1}(y) dy = \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2}. \quad (3.4.46)$$

*Proof.* We have that  $|w_i(y)| \leq |v_i(y)|$ . Moreover,  $w_i(y)$  and  $v_i(y)$  are actually of the same sign, so that  $\exp(w_i(y)) \leq 1 + \exp(v_i(y))$ . Therefore,

$$J_i \leq J_i^{(1)} + J_i^{(2)} \quad \text{where } J_i^{(1)} := \int \frac{v_i(y)^2}{2} \tilde{p}_{i+1}(y) dy \quad \text{and } J_i^{(2)} := \int \frac{v_i(y)^2}{2} \exp(v_i(y)) \tilde{p}_{i+1}(y) dy. \quad (3.4.47)$$

Now, expanding  $v_i(y)$ , we get readily that

$$J_i^{(1)} = \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2}. \quad (3.4.48)$$

Fix  $\epsilon > 0$ .

Then there exist  $\alpha_\epsilon, \beta_\epsilon, \gamma_\epsilon$  positive and a compact  $K_\epsilon$  s.t., for all  $n$  large enough,

$$\mathcal{P}_n \left( B_n^\epsilon := \bigcap_{i=0}^{k-1} \left\{ t_i \in K_\epsilon ; |m_{i+1}| \leq \alpha_\epsilon ; \frac{1}{2s_{i+2,n}^2} \leq \frac{\beta_\epsilon}{n-i-1} ; \left| \frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} \right| \leq \frac{\gamma_\epsilon}{n-i-1} \right\} \right) \geq 1 - \epsilon.$$

The following lines hold on  $B_n^\epsilon$ .

For all real  $y$ , we have that

$$|v_i(y)| \leq \frac{\beta_\epsilon(|y| + \alpha_\epsilon)^2}{n-i-1} + \frac{\gamma_\epsilon|y|}{n-i-1} \quad (3.4.49)$$

For  $|y| \geq \alpha_\epsilon$ , we have that  $|y - m_{i+1}| \geq |y - \alpha_\epsilon|$ , so that

$$v_i(y) \leq -\frac{\beta_\epsilon(y - \alpha_\epsilon)^2}{n-i-1} + \frac{\gamma_\epsilon|y|}{n-i-1}. \quad (3.4.50)$$

Therefore,

$$J_i^{(2)} \leq \frac{1}{2(n-i-1)^2} \int_{|y| \leq \alpha_\epsilon} [\beta_\epsilon(|y| + \alpha_\epsilon)^2 + \gamma_\epsilon|y|]^2 \exp(v_i(y)) \tilde{p}_{i+1}(y) dy \quad (3.4.51)$$

$$+ \frac{1}{2(n-i-1)^2} \int_{|y| \geq \alpha_\epsilon} [\beta_\epsilon(|y| + \alpha_\epsilon)^2 + \gamma_\epsilon|y|]^2 \exp\left(-\frac{\beta_\epsilon(y - \alpha_\epsilon)^2}{n-i-1} + \frac{\gamma_\epsilon|y|}{n-i-1}\right) \tilde{p}_{i+1}(y) dy. \quad (3.4.52)$$

Clearly, on  $B_n^\epsilon$ , the first integral hereabove is bounded by a constant  $I_\epsilon$ . For the second integral, an integration by parts and Assumption (Cf) imply that, on  $B_n^\epsilon$ , it is also bounded by a constant  $L_\epsilon$ . So,

$$J_i^{(2)} = \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2}, \quad (3.4.53)$$

which concludes the proof. □



Combining (3.4.44), (3.4.41) and (3.4.42), we obtain that

$$\begin{aligned}
L_{i,n} &:= C_i \frac{\sigma_{i+1,n}}{\sigma_{i+2,n}} \exp(-\kappa_{i,n} m_{i+1}) \quad \text{where} \quad \kappa_{i,n} := \frac{3\alpha_{i+2,n}^{(3)}}{\sigma_{i+2,n}} \\
&= \left[ 1 + \kappa_{i,n} m_{i+1} - \frac{s_{i+1}^2}{2s_{i+2,n}^2} + \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2} \right] \left[ 1 + \frac{s_{i+1}^2}{2s_{i+2,n}^2} + \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2} \right] \left[ 1 - \kappa_{i,n} m_{i+1} + \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2} \right] \\
&= \left[ 1 + \kappa_{i,n} m_{i+1} - \frac{s_{i+1}^2}{2s_{i+2,n}^2} + \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2} \right] \left[ 1 - \kappa_{i,n} m_{i+1} + \frac{s_{i+1}^2}{2s_{i+2,n}^2} + \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2} \right] \\
&= 1 + \frac{\mathcal{O}_{\mathcal{P}_n}(1)}{(n-i-1)^2}.
\end{aligned}$$

Therefore, we may write  $L_{i,n} = 1 + \frac{W_{i,n}}{(n-i-1)^2}$ , where  $\max_{0 \leq i \leq k-1} |W_{i,n}| = \mathcal{O}_{\mathcal{P}_n}(1)$ . Then, we get from Lemma 19 applied with  $f : x \mapsto \log(1+x)$  that

$$\log(L_{i,n}) = \log \left( 1 + \frac{W_{i,n}}{(n-i-1)^2} \right) = \frac{W_{i,n}}{(n-i-1)^2} + \left( \frac{W_{i,n}}{(n-i-1)^2} \right)^2 \mathcal{O}_{\mathcal{P}_n}(1)$$

Therefore,

$$\log \left( \prod_{i=0}^{k-1} L_{i,n} \right) = \sum_{i=0}^{k-1} \log(L_{i,n}) = \mathcal{O}_{\mathcal{P}_n}(1) \sum_{i=0}^{k-1} \frac{1}{(n-i-1)^2} = o_{\mathcal{P}_n}(1). \quad (3.4.54)$$

Consequently,

$$\prod_{i=0}^{k-1} L_{i,n} = 1 + o_{\mathcal{P}_n}(1). \quad (3.4.55)$$

Finally, we have proved that there exists  $(B_n)_{n \geq 1} \in \mathcal{A}_{\rightarrow 1}$  s.t. for any  $n \geq 1$ ,

$$p_k(Y_1^k) = \prod_{i=0}^{k-1} \Gamma_i \quad \text{on } B_n$$

and

$$\prod_{i=0}^{k-1} \Gamma_i = g_k(Y_1^k) [1 + o_{\mathcal{P}_n}(1)].$$

# Chapter 4

## Functional Limit Laws for the increments of Lévy processes

### 4.1 Introduction

Let  $\{Z(t) : t \geq 0\}$  be a right-continuous with left-hand limits Lévy process (see, e.g., [Bertoin \(1998\)](#)). We assume that  $Z(0) = 0$ , so that the distribution of  $\{Z(t) : t \geq 0\}$  is fully characterized by the law of  $Z := Z(1)$ . Denote the increment functions on  $Z(\cdot)$  by  $\gamma_{x,a}^Z(s) := Z(x + as) - Z(x)$ , for  $x \geq 0$ ,  $a > 0$  and  $s \in [0, 1]$ . The purpose of the present paper is to establish functional limit laws for sets of rescaled increments of  $Z(\cdot)$ , of the form

$$\mathcal{H}_T = \mathcal{H}_T^Z := \{b_T^{-1} \gamma_{x,a_T}^Z(\cdot) : 0 \leq x \leq T - a_T\} \quad \text{for } T > 0. \quad (4.1.1)$$

Limit laws for increment functions such that we have in mind have been established by [Révész \(1979\)](#), [Borovkov \(1990\)](#), [Deheuvels \(1991\)](#), [Deheuvels and Mason \(1993\)](#), [Sanchis \(1994a\)](#) and [Sanchis \(1994b\)](#), among others, for variants of  $Z(\cdot)$ , and under various assumptions on  $a_T$  and  $b_T$ . Here,  $0 < a_T \leq T$  and  $b_T > 0$  are functions of  $T > 0$  which will be specified later on. Our aim is to show that, under appropriate conditions, we have almost surely (a.s.)

$$\lim_{T \rightarrow \infty} \Delta(\mathcal{H}_T^Z, K) = 0, \quad (4.1.2)$$

where  $K$  is a deterministic limit set of functions, and  $\Delta(\cdot, \cdot)$ , a Hausdorff-type set-distance (see, e.g., [\(4.1.4\)](#) below). Our main results will be stated in [Theorems 25](#) and [26](#), in the sequel.

We will make use of the following notation. We denote by  $\mathcal{E}$  be a set of right-continuous functions on  $[0, 1]$  fulfilling  $\mathcal{H}_T^Z \subseteq \mathcal{E}$  for all  $T > 0$ . We endow  $\mathcal{E}$  with a metric topology  $\mathcal{T}$ , defined by a suitable distance  $d_{\mathcal{T}}$ . Mostly, we shall limit ourselves to the cases where  $d_{\mathcal{T}}$  is either the uniform distance (denoted by  $d_{\mathcal{U}}$ ), or the Högnäs distance (denoted by  $d_{\mathcal{W}}$ ), whose definition is postponed until [\(4.1.14\)](#) below. For each  $f \in \mathcal{E}$  and  $\epsilon > 0$ , we denote by  $B_{\mathcal{T}}(f, \epsilon) := \{g \in \mathcal{E} : d_{\mathcal{T}}(f, g) < \epsilon\}$ , the open ball of center  $f$  and radius  $\epsilon$ , pertaining to  $(\mathcal{E}, \mathcal{T})$ . For each non-empty subset  $A \subseteq \mathcal{E}$ , and each  $\epsilon > 0$ , we set

$$A^\epsilon = A^{\epsilon; \mathcal{T}} := \left\{ g \in \mathcal{E} : d_{\mathcal{T}}(f, g) < \epsilon \text{ for some } f \in A \right\} = \bigcup_{f \in A} B_{\mathcal{T}}(f, \epsilon). \quad (4.1.3)$$

We extend the definition (4.1.3) to  $A = \emptyset$ , by setting  $\emptyset^\epsilon = \emptyset$  for all  $\epsilon > 0$ . The Hausdorff distance between the subsets  $A, B \subseteq \mathcal{E}$  of  $\mathcal{E}$  is defined, in turn, by

$$\Delta_{\mathcal{T}}(A, B) := \inf \left\{ \epsilon > 0 : A \subseteq B^\epsilon \text{ and } B \subseteq A^\epsilon \right\}. \quad (4.1.4)$$

Let  $X$  denote a random variable [rv], with moment-generating function [mgf] defined by  $\psi_X(t) = \mathbb{E}(\exp(tX)) \in (0, \infty]$  for  $t \in \mathbb{R}$ . The Legendre transform  $\Psi_X$  of  $\psi_X$  is denoted by

$$\Psi_X(\alpha) := \sup_{\{t: \psi_X(t) < \infty\}} \{\alpha t - \log \psi_X(t)\} \text{ for } \alpha \in \mathbb{R}. \quad (4.1.5)$$

Set  $t_1 := \inf \{t : \psi_X(t) < \infty\}$  and  $t_0 := \sup \{t : \psi_X(t) < \infty\}$ . We refer to [Deheuvels \(1991\)](#) for a discussion of the properties of the mgf  $\psi_X(\cdot)$ , and its Legendre transform  $\Psi_X(\cdot)$ . The observation that  $\psi_X(0) = 1$  entails that  $-\infty \leq t_0 \leq 0 \leq t_1 \leq \infty$ . We shall make use, at times, of the following assumptions on  $\psi_X$ .

$$(\mathcal{C}_X) \quad \psi_X(t) < \infty \text{ for all } t \in \mathbb{R} \quad \Leftrightarrow \quad t_1 = -\infty \text{ and } t_0 = \infty.$$

$$(\mathcal{A}_X) \quad \psi_X(t) < \infty \text{ in a neighborhood of } 0 \quad \Leftrightarrow \quad t_1 < 0 < t_0.$$

Also, at times, we shall impose the following conditions upon  $\{a_T : T > 0\}$ .

(A1)  $a_T$  and  $Ta_T^{-1}$  are ultimately non-decreasing functions of  $T > 0$ ;

(A2)  $\frac{a_T}{\log T} \rightarrow d \in (0, \infty]$  and  $\frac{\log Ta_T^{-1}}{\log \log T} \rightarrow \infty$  as  $T \rightarrow \infty$ .

The Proposition 1 below, which is an easy consequence of the results of [Deheuvels and Mason \(1993\)](#), gives some motivation to our work. Denote by  $B(0, 1)$  (resp.  $C(0, 1)$ , resp.  $AC(0, 1)$ ) the set of bounded left-continuous (resp., continuous, resp., absolutely continuous) functions on  $[0, 1]$ . We endow, at first,  $B(0, 1) \supseteq C[0, 1] \supseteq AC(0, 1)$  with the uniform distance defined by  $d_{\mathcal{U}}(f, g) := \|f - g\|$ , where  $\|f\| := \sup_{0 \leq s \leq 1} |f(s)|$ . Introduce a functional  $I_W$ , defined on  $B(0, 1)$  by

$$I_W(f) = \begin{cases} \int_0^1 \dot{f}(s)^2 ds & \text{if } f \in AC(0, 1) \text{ and } f(0) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Set  $\log_+ t = \log(t \vee e)$  and  $\log_2(t) = \log_+ \log_+ t$  for  $t \in \mathbb{R}$ . Consider the Strassen-type set (refer to [Strassen \(1964\)](#))

$$\mathbb{S} := \left\{ f \in C[0, 1] : I_W(f) \leq 1 \right\}. \quad (4.1.6)$$

**Proposition 1.** *Let (A1) and (A2) hold with  $d = \infty$ . Assume that, for all  $t \geq 0$ ,  $E[Z(t)] = 0$  and  $\text{Var}[Z(t)] = t$ . Set  $b_T := [2a_T (\log_+(T/a_T) + \log_2 T)]^{1/2}$  for  $T > 0$ . Then, under  $(\mathcal{A}_Z)$ , we have*

$$\lim_{T \rightarrow \infty} \Delta_{\mathcal{U}}(\mathcal{H}_T^Z, \mathbb{S}) = 0 \text{ a.s.} \quad (4.1.7)$$

*Proof.* Let  $\{W(t) : t \geq 0\}$  be a standard Wiener process. In view of the notation (4.1.1) and under the assumptions above, we infer from the results of [Révész \(1979\)](#) that

$$\lim_{T \rightarrow \infty} \Delta_{\mathcal{U}}(\mathcal{H}_T^W, \mathbb{S}) = 0 \text{ a.s.} \quad (4.1.8)$$

Under the assumptions of the proposition, the following strong invariance principle holds (see, e.g., [Deheuvels and Mason \(1993\)](#)). There exists a probability space on which we can define a standard Wiener process  $\{W(t) : t \geq 0\}$  jointly with  $\{Z(t) : t \geq 0\}$ , in such a way that, as  $T \rightarrow \infty$ ,

$$\sup_{0 \leq t \leq T} |Z(t) - W(t)| = \mathcal{O}(\log T) \text{ a.s.} \quad (4.1.9)$$

By combining (4.1.8) with (4.1.9), we conclude readily (4.1.7).  $\square$

In the remainder of our paper, we investigate the case of "intermediate increments"  $a_T$ , namely, when (A2) holds for some  $0 < d < \infty$  (we leave aside "small increments", for which  $d = 0$ ). Intermediate increments correspond to the increment sizes investigated by Erdős and Rényi (see, e.g., [Erdős and Rényi, \(1970\)](#)) for increments of partial sums. In this case, invariance principles of the form given in (4.1.9) cannot be used to obtain variants of Proposition 1. This follows from the fact that the rate in (4.1.9) cannot be reduced to  $o(\log T)$  (see, e.g., [Deheuvels and Mason \(1993\)](#)). For such "intermediate increments", functional limit laws for rescaled increments of the form (4.1.2), when  $Z(\cdot)$  is formally replaced in the definition (4.1.1) by a partial sum process  $S(\cdot)$ , are well known. For their statement, we need the following notation. Letting  $\{X_i : i \geq 1\}$ , with  $X := X_1$ , denote a sequence of independent and identically distributed [iid] rv's, the corresponding partial sum process is defined by

$$S(t) := \sum_{i=1}^{\lfloor t \rfloor} X_i \quad \text{for } t \geq 0, \quad (4.1.10)$$

where  $\lfloor t \rfloor \leq t < \lfloor t \rfloor + 1$  denotes the integer part of  $t$ . In this setup, functional limit laws may be obtained through the formal replacement of  $Z(\cdot)$  by  $S(\cdot)$ . The appropriate choice of the scaling factor is here  $b_T = a_T$ , in relation with Cramér type large deviations principles (see, e.g., ([Cramér, 1937](#))). We refer to [Borovkov \(1990\)](#), [Deheuvels \(1991\)](#), [Sanchis \(1994a\)](#) and [Sanchis \(1994b\)](#), for the corresponding results. The following Theorem 20 is a consequence of their results, for sets of increment functions of  $S(\cdot)$  of the form

$$\mathcal{M}_{n,k} := \{s_{i,k} : 0 \leq i \leq n - k\},$$

where

$$s_{i,k}(t) := \frac{S(i + \lfloor kt \rfloor) - S(i) + (kt - \lfloor kt \rfloor)X_{i+\lfloor kt \rfloor+1}}{k} \quad \text{for } 0 \leq t \leq 1.$$

For each pair of integers  $n$  and  $k$  such that  $1 \leq k \leq n$ ,  $\mathcal{M}_{n,k} \subseteq C_0(0,1) := \{f \in C(0,1) : f(0) = 0\}$ . Define a functional  $I_X$  on  $C_0(0,1)$  by setting, for each  $f \in C_0(0,1)$ ,

$$I_X(f) = \begin{cases} \int_0^1 \Psi_X(\dot{f}(s)) ds & \text{if } f \in AC(0,1), \\ \infty & \text{otherwise.} \end{cases}$$

Now, for any  $\alpha > 0$ , introduce the set

$$K_\alpha := \{f \in C_0(0,1) : I_X(f) \leq \alpha\}. \quad (4.1.11)$$

**Theorem 20.** *Under  $(\mathcal{C}_X)$ , for any  $c > 0$  and  $k = \lfloor c \log n \rfloor$ , we have that*

$$\lim_{T \rightarrow \infty} \Delta_{\mathcal{U}}(\mathcal{M}_{n,k}, K_{1/c}) = 0 \quad \text{a.s.} \quad (4.1.12)$$

Theorem 21 below is a variant of Theorem 20, obtained under more general assumptions. For  $c > 0$  and  $a_T = c \log_+ T$ , let  $\mathcal{F}_T$  be the set of Erdős-Rényi-type increment functions of  $\{S(t) : t \geq 0\}$ , defined by

$$\mathcal{F}_T := \{c \beta_{x,T} : 0 \leq x \leq T - a_T\} \quad \text{where} \quad \beta_{x,T}(s) := \frac{S(x + sa_T) - S(x)}{a_T} \quad \text{for} \quad 0 \leq s \leq 1. \quad (4.1.13)$$

We see here that  $\mathcal{F}_T \subseteq BV_0(0, 1)$ , where  $BV_0(0, 1)$  denotes the set of all right-continuous functions  $f$  on  $[0, 1]$  with bounded variation, and such that  $f(0) = 0$ . In words,  $BV_0(0, 1)$  is the space of all distribution functions of totally bounded signed Radon measures on  $[0, 1]$ . In the sequel, we will identify  $f \in BV_0(0, 1)$  with the signed measure  $\mu_f$  such that  $df = \mu_f$ . We define below on  $BV_0(0, 1)$  a distance  $d_{\mathcal{W}}$  (see, e.g., (Högnäs, 1977)), which metricizes the weak convergence of signed measures. We denote by  $\Delta_{\mathcal{W}}$  the corresponding Hausdorff distance. Set, for  $f, g \in BV_0(0, 1)$ ,

$$d_{\mathcal{W}}(f, g) = \int_0^1 |f(u) - g(u)| du + |f(1) - g(1)|. \quad (4.1.14)$$

For  $f \in BV_0(0, 1)$ , write  $f = f_+ - f_-$ , where  $df = df_+ - df_-$  is the Hahn-Jordan decomposition of  $df$ . For any  $g \in BV_0(0, 1)$ , denote by  $g = g^A + g^S$ , where  $dg = dg^A + dg^S$  stands for the Lebesgue decomposition of  $dg$  into an absolutely continuous and a singular component. For  $c > 0$ , we define functionals  $J_X$  and  $J_{X,c}$  on  $BV_0(0, 1)$  by setting, for each  $f \in BV_0(0, 1)$

$$J_X(f) = \int_0^1 \Psi_X \left( \frac{d}{ds} f^A(s) \right) ds + t_0 f_+^S(1) - t_1 f_-^S(1) \quad (4.1.15)$$

and

$$J_{X,c}(f) := c J_X \left( \frac{f}{c} \right). \quad (4.1.16)$$

We set, further,

$$D_c := \{f \in BV_0(0, 1) : J_{X,c}(f) \leq 1\}. \quad (4.1.17)$$

**Theorem 21.** *Let  $X$  be centered, with finite variance, and nondegenerate, meaning that  $P(X = x) < 1$  for all  $x$ . Assume further that (A2) holds with  $0 < d < \infty$ . Then, under  $(\mathcal{A}_{X_1})$ , we have*

$$\lim_{T \rightarrow \infty} \Delta_{\mathcal{W}}(\mathcal{F}_{T,c}, D_c) = 0 \quad a.s. \quad (4.1.18)$$

In addition, under  $(\mathcal{C}_{X_1})$ , we have

$$\lim_{T \rightarrow \infty} \Delta_{\mathcal{U}}(\mathcal{F}_{T,c}, D_c) = 0 \quad a.s. \quad (4.1.19)$$

The limiting behaviour of rescaled increments of Lévy processes mimicks closely the limiting behavior of rescaled increments of partial sums processes given in Theorems 20 and 21. As expected, we obtain results analogous to Theorem 21 when we replace partial sum processes by Lévy processes. The following additional notation will be useful. For  $x \geq 0$ , and  $\ell > 0$ , define standardized increment functions of  $Z(\cdot)$  by setting

$$\eta_{x,\ell}(s) := \frac{Z(x + \ell s) - Z(x)}{\ell} \quad \text{for} \quad s \in [0, 1]. \quad (4.1.20)$$

In this paper, we consider intermediate Erdős-Rényi-type increments of the form

$$a_T = c \log T, \text{ for some } c > 0. \quad (4.1.21)$$

Our aim is to characterize the limiting behavior of the random set of increment functions

$$\mathcal{G}_T := \{\eta_{x, a_T} : 0 \leq x \leq T - a_T\}. \quad (4.1.22)$$

The forthcoming Fact 32 in Section 3.1 will show that the Theorem 22, stated below, due to (Frolov, 2008), is a consequence of a functional limit theorem for  $\mathcal{G}_T$ .

**Theorem 22.** *Set  $\Psi := \Psi_{Z(1)}$ . Define  $c_0$  by  $1/c_0 = \sup\{\Psi(z) : \Psi(z) < \infty\}$ . For any  $c > c_0$ , set  $a_T = c \log(T)$ . For any  $u > 0$ , set  $\gamma(u) := \sup\{z \geq 0 : \Psi(z) \leq u\}$ . Then, under  $(\mathcal{A}_{Z(1)})$ , we have*

$$\lim_{T \rightarrow \infty} \left\{ \sup_{f \in \mathcal{G}_T} f(1) \right\} = \lim_{T \rightarrow \infty} \left\{ \sup_{0 \leq x \leq T - a_T} \frac{Z(x + a_T) - Z(x)}{a_T} \right\} = \gamma(1/c) \quad a.s. \quad (4.1.23)$$

This remainder of the present paper is organized as follows. In Section 2, we present general results on Lévy processes which will be needed in our proofs. Our main results are stated in Section 3, with proofs detailed in Section 4. Some technical results are deferred to the Appendix.

## 4.2 General results

### 4.2.1 Lévy Processes

Let  $\{Z(t) : t \geq 0\}$  be a Lévy process. The similarities between the structure of  $\{Z(t) : t \geq 0\}$  and that of  $\{S(t) : t \geq 0\}$  are essentially due to the infinitely divisibility of the distribution of  $Z(1)$ . This implies that, for each  $\lambda > 0$ , the discretized version  $\{Z(n\lambda) : n \geq 0\}$  of  $Z(\cdot)$  is a sequence of partial sums of iid random variables. The law of the random variable  $Z(1)$  (and hence, the distribution of  $\{Z(t) : t \geq 0\}$ , see, e.g., Bertoin (1998)) is characterized by a unique triple of constants,  $(a, \sigma^2, \pi)$ , with  $a \in \mathbb{R}$ ,  $\sigma^2 \geq 0$ , and where  $\pi$  denotes a measure on  $\mathbb{R} - \{0\}$  such  $\int_{\mathbb{R} - \{0\}} \min\{x^2, 1\} \pi(dx) < \infty$ . This relies on the Lévy-Itô decomposition, implying that the process  $\{Z(t) : t \geq 0\}$  may be decomposed into

$$Z(t) = Z^{(1)}(t) + Z^{(2)}(t) + Z^{(3)}(t), \quad (4.2.1)$$

where  $Z^{(1)}$ ,  $Z^{(2)}$  and  $Z^{(3)}$  are three independent stochastic processes described hereafter.  $Z^{(1)}(\cdot)$  is a Wiener process with linear drift, namely such that  $Z^{(1)}(t) = \sigma W(t) - at$ , where  $\{W(t) : t \geq 0\}$  is a standard Wiener process;  $Z^{(2)}(\cdot)$  is a compound Poisson process and  $Z^{(3)}(\cdot)$  is a square integrable martingale, both defined in terms of  $\pi$ . Consequently, the distribution of  $\{Z(t) : t \geq 0\}$  is fully determined by that of  $Z(1)$ , which is itself characterized by its mgf, denoted by  $\psi(t) = \mathbb{E}(\exp(tZ(1)))$ . Denote by  $\Psi$  the Legendre transform of  $\psi$ . Introduce the following assumptions.

(C) :  $\psi(t) < \infty$  for all  $t \in \mathbb{R}$ .

(A) :  $t_1 := \inf\{t : \psi(t) < \infty\} < 0 < t_0 := \sup\{t : \psi(t) < \infty\}$  and  $Z(1)$  has no Gaussian component.

### 4.2.2 Functional spaces

Our results rely heavily on the large deviations principles (LDP) for the distributions  $(P_\lambda)_{\lambda>0}$  of the processes  $\{Z_\lambda(t) : 0 \leq t \leq 1\}$ , defined by

$$Z_\lambda(t) := \frac{1}{\lambda} Z(\lambda t). \quad (4.2.2)$$

Below we present some useful results concerning the spaces of functions which contain the sample paths of  $\{Z_\lambda(t) : 0 \leq t \leq 1\}$ .

#### The Skorohod space

By definition of a Lévy process, for any  $\lambda > 0$ , the sample paths of  $\{Z_\lambda(t) : 0 \leq t \leq 1\}$  belong to the space  $D(0, 1)$  of right-continuous with left-hand limits functions on  $[0, 1]$ . We endow  $D(0, 1)$  either with the uniform topology  $\mathcal{U}$ , or with the Skorohod topology  $\mathcal{S}$ . We recall from (Billingsley, 1999) that the Skorohod topology  $\mathcal{S}$  on  $D(0, 1)$  is induced by the distance  $d_{\mathcal{S}}$  defined by

$$d_{\mathcal{S}}(f, g) = \inf_{\nu \in \Lambda} \left\{ \max(\|\nu - I\|; \|f - g \circ \nu\|) \right\}, \quad (4.2.3)$$

where  $\Lambda$  is the class of strictly increasing, continuous mappings of  $[0, 1]$  onto itself.

Let  $C(0, 1)$  denote the set of continuous functions on  $[0, 1]$ . For any  $f \in C(0, 1)$ , let  $\omega_f$  denote the modulus of continuity of  $f$ , defined by

$$\omega_f(\delta) = \sup_{|s-t| \leq \delta} |f(s) - f(t)|. \quad (4.2.4)$$

**Fact 29.** *Let  $K$  be a compact subset of  $(C(0, 1), \mathcal{U})$ . Then, for any  $\epsilon > 0$ , there exists a  $\zeta > 0$  such that for all  $g \in K$ ,*

$$B_{\mathcal{U}}(g, \epsilon) \supseteq B_{\mathcal{S}}(g, \zeta). \quad (4.2.5)$$

*Proof.* Let  $\epsilon > 0$ . By the Arzelà-Ascoli theorem, there exists a  $\delta > 0$  such that

$$\sup_{g \in K} \omega_g(\delta) < \frac{\epsilon}{2}. \quad (4.2.6)$$

Set  $\zeta := \min\{\delta; \frac{\epsilon}{2}\}$ . We infer from (4.2.3) that, for all  $g \in K$  and  $h \in B_{\mathcal{S}}(g, \zeta)$ , there exists a  $\nu_h \in \Lambda$  satisfying

$$\|\nu_h - I\| < \zeta \leq \delta \text{ and } \|h - g \circ \nu_h\| < \zeta \leq \frac{\epsilon}{2}. \quad (4.2.7)$$

Therefore,

$$\|g \circ \nu_h - g\| \leq w_g(\|\nu_h - I\|) < \frac{\epsilon}{2}$$

and

$$\|h - g\| \leq \|h - g \circ \nu_h\| + \|g \circ \nu_h - g\| < \epsilon.$$

□

**Fact 30.** *Let  $x \geq 0$ . Then, for any  $B \in \mathcal{B}_{\mathcal{S}}$ , we have*

$$\mathbb{P}((Z(x + \cdot) - Z(x)) \in B) = \mathbb{P}(Z(\cdot) \in B). \quad (4.2.8)$$

*Proof.* Let  $k \in \mathbb{N}$  and  $(t_i)_{1 \leq i \leq k}$  with  $0 \leq t_1 < \dots < t_k \leq 1$ . Let  $\pi_{(t_i)}$  be the natural projection from  $D(0, 1)$  to  $\mathbb{R}^k$  defined by  $\pi_{(t_i)}(f) = (f(t_i))$ . Since  $Z$  has independent and stationary increments, for all Borel subset  $H$  of  $\mathbb{R}^k$ , we have that

$$P\left((Z(x + \cdot) - Z(x)) \in \pi_{(t_i)}^{-1}(H)\right) = P\left(Z(\cdot) \in \pi_{(t_i)}^{-1}(H)\right). \quad (4.2.9)$$

Finally, we conclude our proof by an application of Theorem 12.5 of (Billingsley, 1999), which asserts that if the laws of two processes valued in  $D(0, 1)$  agree on all sets of the form  $\pi_{(t_i)}^{-1}(H)$ , then they agree on the whole family of sets  $\mathcal{B}_S$ .  $\square$

**Fact 31.** *For all  $x \geq 0$  and  $u > 0$ , we have*

$$P\left(\sup_{0 \leq s \leq 1} |Z(x + s) - Z(x)| > u\right) = P\left(\sup_{0 \leq s \leq 1} |Z(s)| > u\right). \quad (4.2.10)$$

*Proof.* Let  $(s_k)_{k \in \mathbb{N}}$  be an enumeration of  $\mathbb{Q} \cap [0, 1]$ . Since any  $f \in D(0, 1)$  is right-continuous, we have that

$$P\left(\sup_{0 \leq s \leq 1} |Z(x + s) - Z(x)| > u\right) = P\left((Z(x + \cdot) - Z(x)) \in \bigcup_{k \in \mathbb{N}} \{f \in D(0, 1) : |f(s_k)| > u\}\right). \quad (4.2.11)$$

We recall from Section 12 of (Billingsley, 1999) that for any  $s \in [0, 1]$ , the map  $f \in D(0, 1) \mapsto f(s)$  is measurable with respect to  $\mathcal{B}_S$ . Therefore,  $\{\bigcup_{k \in \mathbb{N}} \{f \in D(0, 1) : |f(s_k)| > u\}\} \in \mathcal{B}_S$ , so we conclude by an application of Fact 30.  $\square$

### The space $BV_{0,M}(0, 1)$

Whenever  $(\mathcal{A})$  is in force, it follows from general properties of infinitely divisible distributions (see, e.g., the discussion, Section 5 in (Lynch and Sethuraman, 1987)), that for each  $\lambda > 0$ , the sample paths of  $\{Z_\lambda(t) : 0 \leq t \leq 1\}$  belong to  $BV_0(0, 1)$ . We endow this set with the topology  $\mathcal{W}$  of weak convergence of the underlying signed measures. Next, for each  $A \subset BV_0(0, 1)$  and  $\epsilon > 0$ , we set

$$A^\epsilon = \bigcup_{f \in A} B_{\mathcal{W}}(f, \epsilon),$$

where  $B_{\mathcal{W}}(f, \epsilon) := \{g \in BV_0(0, 1) : d_{\mathcal{W}}(f, g) < \epsilon\}$ . We observe that the  $A^\epsilon$  is not necessarily open with respect to  $\mathcal{W}$ , since  $d_{\mathcal{W}}$  does not define the weak topology on the whole set  $BV_0(0, 1)$ . Thus, we are not allowed to apply a LDP to some  $A^\epsilon$  or to its complement. Therefore, we need to restrict the weak topology as follows. For  $f \in BV_0(0, 1)$ , let  $|f|_v(1) := f_+(1) + f_-(1)$  be the total variation of  $f$  in the interval  $[0, 1]$ . For any  $M > 0$ , set

$$BV_{0,M}(0, 1) := \{f \in BV_0(0, 1) : |f|_v(1) \leq M\}. \quad (4.2.12)$$

Then, for  $M > 0$ , the restriction of the weak topology to  $BV_{0,M}(0, 1)$  is metricized by the distance  $d_{\mathcal{W}}$ . In the sequel, we endow  $BV_{0,M}(0, 1)$  with the weak topology.

**Lemma 22.** *For any  $M > 0$ ,  $BV_{0,M}(0, 1)$  is a compact metric space.*

*Proof.* see, e.g., Proposition 1.4. in (Deheuvels, 2007).  $\square$



### 4.2.3 Functional large deviations

We recall some definitions and results on large deviations theory.

**Definition 16.** Let  $\mathcal{E}$  be a topological space, endowed with a topology  $\mathcal{T}$  and its Borel  $\sigma$ -algebra, denoted by  $\mathcal{B}_{\mathcal{T}}$ . A function  $\mathcal{I} : \mathcal{E} \rightarrow [0, \infty)$  is a rate function if  $\mathcal{I}$  is lower semicontinuous. Furthermore, we say that  $\mathcal{I}$  is a good rate function if for all  $\alpha < \infty$ , the level set  $K_{\mathcal{I}}(\alpha) := \{f \in \mathcal{E} : \mathcal{I}(f) \leq \alpha\}$  is compact. We say that a family of probability measures  $(P_{\lambda})_{\lambda \geq 0}$  on  $(\mathcal{E}, \mathcal{B}_{\mathcal{T}})$  satisfies a Large Deviation Principle (LDP) with rate function  $\mathcal{I}$  if for any closed (resp. open) subset  $F$  (resp.  $G$ ) of  $\mathcal{T}$ , we have :

$$\overline{\lim}_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log P_{\lambda}(F) \leq -\mathcal{I}(F) \quad (\text{upper bound}) \quad (4.2.13)$$

$$\text{and } \underline{\lim}_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log P_{\lambda}(G) \geq -\mathcal{I}(G) \quad (\text{lower bound}). \quad (4.2.14)$$

where for any non-empty subset  $A$  of  $\mathcal{E}$ ,  $\mathcal{I}(A) := \inf_{f \in A} \mathcal{I}(f)$ .

First, we have the following general result.

**Lemma 23.** Let  $(\mathcal{E}, d)$  be a metric space. Let  $\mathcal{I}$  be a good rate function on  $(\mathcal{E}, d)$ . Then, for all positive  $\alpha$  and  $\epsilon$ , we have that

$$\inf_{f \notin (K_{\alpha})^{\epsilon}} \mathcal{I}(f) > \alpha. \quad (4.2.15)$$

*Proof.* Suppose that  $\mathcal{I}((K_{\alpha})^c) = \alpha$ . Then there's a sequence  $(x_n)$  with  $x_n \notin (K_{\alpha})^{\epsilon}$  for all  $n$ , such that  $\mathcal{I}(x_n) \searrow \alpha$ . For some  $N$  and all  $n \geq N$ , we have  $\mathcal{I}(x_n) \leq \alpha + 1$ , so that  $x_n \in \{x \in \mathcal{E} : \mathcal{I}(x) \leq \alpha + 1\}$ , which is a compact set. Hence,  $(x_n)_{n \geq N}$  has a convergent subsequence  $x_{n_k} \rightarrow x$ , as  $k \rightarrow \infty$ . Since  $\mathcal{I}$  is lower semicontinuous, we have that  $\mathcal{I}(x) \leq \lim_{k \rightarrow \infty} \mathcal{I}(x_{n_k}) = \alpha$ . Therefore,  $x \in K_{\alpha}$ . Now, for all  $n$ ,  $x_n \notin (K_{\alpha})^{\epsilon}$ , so that  $d(x_n, x) \geq \epsilon$ , which leads to a contradiction.  $\square$

Now, we state the functional LDP, under  $(\mathcal{C})$  and then under  $(\mathcal{A})$  on which our proofs rely. In the sequel,  $\Psi_{Z(1)}$  is denoted by  $\Psi$ .

Let  $I$  be the functional defined on  $D[0, 1]$ , by

$$I(f) = \begin{cases} \int_0^1 \Psi(\dot{f}(s)) ds & \text{if } f \in AC(0, 1) \text{ and } f(0) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

**Lemma 24.** Under  $(\mathcal{C})$ ,  $I$  is a good rate function.

*Proof.* Since  $I(f) = \infty$  whenever  $f \notin AC(0, 1)$ , we obtain that for any  $\alpha > 0$ ,  $K_I(\alpha) := \{f \in D(0, 1) : I(f) \leq \alpha\} = \{f \in AC(0, 1) : I(f) \leq \alpha\}$ . Now, since  $(\mathcal{C})$  holds, Theorem 1.1. in (Deheuvels, 2007) implies that the set  $\{f \in AC(0, 1) : I(f) \leq \alpha\}$  is a compact subset of  $(C(0, 1), \mathcal{U})$ . We conclude by recalling that the restriction of  $\mathcal{S}$  to  $C(0, 1)$  coincides there with  $\mathcal{U}$ , which implies that a compact subset of  $(C(0, 1), \mathcal{U})$  is also a compact subset of  $(D(0, 1), \mathcal{S})$ .  $\square$

**Theorem 23.** *Under (C), the distributions  $(P_\lambda)_{\lambda>0}$  satisfy a LDP in  $(D(0,1), \mathcal{S})$ , with rate function  $I$ .*

*Proof.* See, e.g., (Varadhan, 1966). □

Let  $J$  be the functional defined on  $BV_0(0,1)$  by

$$J(f) = \int_0^1 \Psi \left( \frac{d}{ds} f^A(s) \right) ds + t_0 f_+^S(1) - t_1 f_-^S(1). \quad (4.2.16)$$

**Theorem 24.** *Under (A), the distributions  $(P_\lambda)_{\lambda>0}$  satisfy a LDP in  $(BV_0(0,1), \mathcal{W})$ , with good rate function  $J$ .*

*Proof.* See, e.g., (Lynch and Sethuraman, 1987). □

### 4.3 Functional Erdős-Rényi theorems

#### 4.3.1 Main results

We consider the following assumption.

( $\mathcal{E}$ ) : There exists a constant  $\mu$  such that for all  $t \geq 0$ ,  $\mathbb{E}[Z(t)] = \mu t$ .

The next two theorems, called Erdős-Rényi functional laws (ERFL), are the main results of this paper.

**Theorem 25.** *Assume that (C) and ( $\mathcal{E}$ ) hold. For  $c > 0$ , set*

$$K_{1/c} := \{f \in D(0,1) : I(f) \leq 1/c\}, \quad (4.3.1)$$

where  $I$  is the rate function defined in Section 2. Then, for any  $c > 0$  and  $a_T = c \log(T)$ , we have

$$\lim_{T \rightarrow \infty} \Delta_{\mathcal{U}}(\mathcal{G}_T, K_{1/c}) = 0 \quad a.s. \quad (4.3.2)$$

**Theorem 26.** *Assume that (A) and ( $\mathcal{E}$ ) hold. For any integers  $n > 0$  and  $q < n$ , set*

$$\mathcal{M}_{n,q} := \{\eta_{m,q} : m \in \{0, \dots, n - q\}\}. \quad (4.3.3)$$

For any  $c > 0$ , set  $A_n := \lfloor c \log n \rfloor$ . Assume that for all  $c$  large enough, there exists a constant  $M > 0$  such that almost surely for all  $n$  large enough, we have that

$$\mathcal{M}_{n,A_n} \subset BV_{0,M}(0,1). \quad (4.3.4)$$

For  $c > 0$ , set

$$L_{1/c} := \{f \in BV_0(0,1) : J(f) \leq 1/c\}, \quad (4.3.5)$$

where  $J$  is the rate function defined in Section 2. Then, for all  $c$  large enough, we have that

$$\lim_{n \rightarrow \infty} \Delta_{\mathcal{W}}(\mathcal{M}_{n,A_n}, L_{1/c}) = 0 \quad a.s. \quad (4.3.6)$$

The following Fact usually provides applications when functional limit theorems are established.

**Fact 32.** *Let  $(\mathcal{E}, d)$  be a metric space of functions defined on  $[0, 1]$ . Let  $\Theta : (\mathcal{E}, d) \rightarrow \mathbb{R}$  be a continuous map. Assume that for some compact subset  $K$  of  $\mathcal{E}$ ,  $\lim_{T \rightarrow \infty} \Delta(\mathcal{H}_T, K) = 0$  a.s. Then, we have that*

$$\lim_{T \rightarrow \infty} \left\{ \sup_{f \in \mathcal{H}_T} \Theta(f) \right\} = \sup_{f \in K} \Theta(f) \quad \text{a.s.} \quad (4.3.7)$$

Corollary 1 below follows from Theorems 25 and 26 (and their proofs) combined to Fact 32 applied to the functional  $f \mapsto f(1)$ , which is continuous with respect to the uniform topology and the weak topology.

**Corollary 11.** *Under the assumptions of Theorem 25, we have that for any  $c > 0$ ,*

$$\lim_{T \rightarrow \infty} \sup_{0 \leq x \leq T - a_T} \frac{Z(x + a_T) - Z(x)}{a_T} = \gamma(1/c) \quad \text{a.s.} \quad (4.3.8)$$

$$\text{and } \lim_{n \rightarrow \infty} \sup_{0 \leq m \leq n - A_n} \frac{Z(m + A_n) - Z(m)}{A_n} = \gamma(1/c) \quad \text{a.s.} \quad (4.3.9)$$

*Under the assumptions of Theorem 26, we have that for all  $c$  large enough, (4.3.9) holds.*

## 4.3.2 Examples

### Continuous paths

Let  $\{Z(t) : t \geq 0\}$  be a Lévy process with continuous paths. We recall that in this case,  $\{Z(t) : t \geq 0\}$  is a brownian motion with drift. Therefore, Theorem 25 yields an ERFLL for  $\{Z(t) : t \geq 0\}$ , since it satisfies (E) and (C).

### Subordinators

Let  $\{Z(t) : t \geq 0\}$  be a subordinator, that is a Lévy process with almost surely increasing paths. Then, for any  $n > 0$  and  $m \in \{0, \dots, n - A_n\}$ , we have that

$$|\eta_{m, A_n}|_v(1) = \frac{Z(m + A_n) - Z(m)}{A_n}. \quad (4.3.10)$$

Then, we deduce easily from Theorem 22 that for any  $c > 0$  large enough, there exists a constant  $M_c < \infty$  such that

$$\overline{\lim}_{n \rightarrow \infty} \sup_{0 \leq m \leq n - A_n} |\eta_{m, A_n}|_v(1) \leq M_c \quad \text{a.s.} \quad (4.3.11)$$

So if we assume in addition that (E) and (A) hold, then  $\{Z(t) : t \geq 0\}$  satisfies the assumptions of Theorem 26.

### Compound Poisson process

Recall that given a sequence  $\{Y_i : i \geq 1\}$  of i.i.d. r.v.'s. and a homogeneous, right continuous Poisson process  $\{N(t) : t \geq 0\}$  which we assume to be independent of  $\{Y_i : i \geq 1\}$ , the compound Poisson process  $\{S_N(t) : t \geq 0\}$  is defined by

$$S_N(t) = \sum_{i=1}^{N(t)} Y_i. \quad (4.3.12)$$

Denote by  $\lambda$  the parameter of  $\{N(t) : t \geq 0\}$ . Recall that for any  $t \geq 0$ , we have that

$$E[S_N(t)] = \lambda t E[Y_1] \quad ; \quad \psi_{S_N(1)}(u) = \exp[\lambda(\psi_{Y_1}(u) - 1)]. \quad (4.3.13)$$

Consequently,  $(\mathcal{E})$  holds. If we assume that  $(\mathcal{C}_{Y_1})$  holds, then  $\{S_N(t) : t \geq 0\}$  satisfies  $(\mathcal{C})$ , and therefore the assumptions of Theorem 25.

Now, assume that  $(\mathcal{A}_{Y_1})$  holds. Notice that for any  $x \geq 0$  and  $\ell > 0$ , we have that

$$|\eta_{x,\ell}|_v(1) = \ell^{-1} \sum_{j=N(x)+1}^{N(x+\ell)} |Y_j| = \ell^{-1} \{|S_N|(x+\ell) - |S_N|(x)\}, \quad (4.3.14)$$

where  $|S_N|(t) := \sum_{j=1}^{N(t)} |Y_j|$ . Then,  $\{|S_N|(t) : t \geq 0\}$  is also a compound Poisson process with  $\psi_{|S_N|(1)}(u) = \exp[\lambda(\psi_{|Y_1|}(u) - 1)]$ . So for  $|u| < \beta := \min\{|t_1|, |t_0|\}$ , we have that

$$\psi_{|Y_1|}(u) = \int_{-\infty}^0 \exp(-ux) dP(Y_1 \leq x) + \int_0^{\infty} \exp(ux) dP(Y_1 \leq x) \leq \psi_{Y_1}(-u) + \psi_{Y_1}(u) < \infty. \quad (4.3.15)$$

Therefore,  $(\mathcal{A}_{|S_N|(1)})$  holds and we deduce from (4.3.14) combined with Theorem 22 that  $\{S_N(t) : t \geq 0\}$  satisfies the assumptions of Theorem 26.

## 4.4 Proofs

The statement that  $\lim_{T \rightarrow \infty} \Delta_{\mathcal{T}}(\mathcal{H}_T, K) = 0$  a.s. is equivalent to the statement that, for any  $\epsilon > 0$ , there exists a.s.  $T(\epsilon) < \infty$  such that for all  $T \geq T(\epsilon)$ ,

$$\mathcal{H}_T \subset K^\epsilon \text{ and } K \subset (\mathcal{H}_T)^\epsilon. \quad (4.4.1)$$

We give several preliminary lemmas which lead to the proofs of Theorems 25 and 26 in Section 4.3. We call the first statement in (4.4.1) the upper bound, while the second is called the lower bound. The reason is that for the first we use an upper bound in a functional LDP while we use a lower bound for the second one. For any  $c > 0$ , set

$$\mathcal{L}_n := \{\eta_{x, A_n} : 0 \leq x \leq n - A_n\}. \quad (4.4.2)$$

For any integer  $j$  large enough, set  $n_j := \max\{n : A_n = j\}$ , so that  $\exp(\frac{j}{c}) \leq n_j < \exp(\frac{j+1}{c})$ .

### 4.4.1 The upper bound

**Lemma 25.** *Assume that (C) holds. Then, for any  $\epsilon > 0$ , the series  $\sum_j P(\mathcal{M}_{n_j, j} \not\subset (K_{1/c})^{\epsilon; \mathcal{U}})$  is convergent.*

*Proof.* Fix  $\epsilon > 0$ . Since for all  $c > 0$ ,  $K_{1/c}$  is a compact subset of  $(C(0, 1), \mathcal{U})$ , Fact 29 implies that there exists  $\zeta > 0$  such that

$$(K_{1/c})^{\epsilon; \mathcal{U}} = \bigcup_{g \in K_{1/c}} B_{\mathcal{U}}(g, \epsilon) \supset \bigcup_{g \in K_{1/c}} B_{\mathcal{S}}(g, \zeta) = (K_{1/c})^{\zeta; \mathcal{S}}. \quad (4.4.3)$$

Let  $F$  be the complement in  $D(0, 1)$  of  $(K_{1/c})^{\zeta; \mathcal{S}}$ . Therefore, for all  $n > 0$ , we have that

$$P(\mathcal{M}_{n, A_n} \not\subset (K_{1/c})^{\epsilon; \mathcal{U}}) \leq \sum_{m=0}^{n-A_n} P(\eta_{m, A_n} \in F) \leq nP\left(\frac{Z(A_n \cdot)}{A_n} \in F\right). \quad (4.4.4)$$

The last inequality is justified by Fact 30. Since (C) holds, we can apply Theorem 23 :  $F$  being a closed subset of  $(D(0, 1), \mathcal{S})$ , for any  $\theta > 0$ , we have for all  $n$  large enough,

$$P(\mathcal{M}_{n, A_n} \not\subset (K_{1/c})^{\epsilon; \mathcal{U}}) \leq n \exp[A_n(-I(F) + \theta)]. \quad (4.4.5)$$

Then, by Lemma 23,  $I$  is a good rate function. Therefore, applying Lemma 22 with  $(\mathcal{E}, d) = (D(0, 1), d_{\mathcal{S}})$ , we can write  $I(F) = \frac{1}{c} + \delta$  with  $\delta > 0$ . So applying (4.4.5) with  $\theta = \frac{\delta}{4}$ , we have for all  $n$  large enough,

$$P(\mathcal{M}_{n, A_n} \not\subset (K_{1/c})^{\epsilon; \mathcal{U}}) \leq n \exp\left(-A_n \left(\frac{1}{c} + \frac{3\delta}{4}\right)\right). \quad (4.4.6)$$

Applying this inequality with  $n = n_j$ , so that  $A_n = j$ , we obtain that

$$\begin{aligned} P(\mathcal{M}_{n_j, j} \not\subset (K_{1/c})^{\epsilon; \mathcal{U}}) &\leq n_j \exp\left(-j \left(\frac{1}{c} + \frac{3\delta}{4}\right)\right) \\ &< \exp\left(\frac{j+1}{c}\right) \exp\left(-j \left(\frac{1}{c} + \frac{3\delta}{4}\right)\right) \\ &= \exp\left(\frac{1}{c} - j \frac{3\delta}{4}\right). \end{aligned}$$

□

**Lemma 26.** *Suppose that the assumptions of Theorem 26 hold. Then, for any  $\epsilon > 0$ , the series  $\sum_j P(\mathcal{M}_{n_j, j} \not\subset (L_{1/c})^{\epsilon; \mathcal{W}})$  is convergent.*

*Proof.* Set  $\tilde{L}_{1/c}^{\epsilon} := (L_{1/c})^{\epsilon; \mathcal{W}} \cap BV_{0, M}(0, 1)$ . Then, it is enough to prove that the series  $\sum_j P(\mathcal{M}_{n_j, j} \not\subset \tilde{L}_{1/c}^{\epsilon})$

is convergent. Denote by  $\mathcal{F}$  the complement of  $\tilde{L}_{1/c}^{\epsilon}$  in  $BV_{0, M}(0, 1)$ . Then  $\mathcal{F}$  is a closed subset of  $(BV_{0, M}(0, 1), \mathcal{W})$ , and therefore a closed subset of  $(BV_0(0, 1), \mathcal{W})$ . Since (A) holds, we can apply Theorem 24. As in (3.5) in Deheuvels (1991), for any  $\theta > 0$ , we have for all  $n$  large enough,

$$P(\mathcal{M}_{n, a_n} \not\subset \tilde{L}_{1/c}^{\epsilon}) \leq nP\left(\frac{Z(A_n \cdot)}{A_n} \in \mathcal{F}\right) \leq n \exp[A_n(-J(\mathcal{F}) + \theta)]. \quad (4.4.7)$$

Denote by  $J_M$  the restriction of  $J$  to  $BV_{0,M}(0,1)$ . Since  $\mathcal{F} \subset BV_{0,M}(0,1)$ , we have that  $J(\mathcal{F}) = J_M(\mathcal{F})$ . Now, a level set of  $J_M$  is a closed subset of the compact set  $BV_{0,M}(0,1, \mathcal{W})$ . Therefore,  $J_M$  is a good rate function on the metric space  $(BV_{0,M}(0,1), d_{\mathcal{W}})$ . So applying Lemma 23, we get that  $J(\mathcal{F}) > 1/c$ . We conclude as in the proof of Lemma 25.  $\square$

**Lemma 27.** *Assume that, for all  $\epsilon > 0$ , we have that*

$$\sum_j P(\mathcal{M}_{n_j, j} \not\subset K^\epsilon) < \infty \quad (4.4.8)$$

and

$$\sum_j n_j P\left(\sup_{0 \leq \tau \leq 1} |Z(\tau)| \geq j\epsilon\right) < \infty. \quad (4.4.9)$$

Then almost surely, for all  $n$  large enough, we have that

$$\mathcal{L}_n \subset K^\epsilon. \quad (4.4.10)$$

*Proof.* For any  $\epsilon > 0$ , we have for all  $j$  large enough,

$$P(\mathcal{L}_{n_j} \not\subset K^\epsilon) \leq P\left(\{\mathcal{L}_{n_j} \not\subset K^\epsilon\} \cap \{\mathcal{M}_{n_j, j} \subset K^{\epsilon/2}\}\right) + P\left(\{\mathcal{M}_{n_j, j} \not\subset K^{\epsilon/2}\}\right). \quad (4.4.11)$$

Set  $P_j := P\left(\{\mathcal{L}_{n_j} \not\subset K^\epsilon\} \cap \{\mathcal{M}_{n_j, j} \subset K^{\epsilon/2}\}\right)$ . For any real  $x$ , let  $k_x$  be the integer which is nearest to  $x$  (we choose  $k_x = \lfloor x \log n \rfloor$  if  $x \in \frac{1}{2}\mathbb{N}$ ). Assume that the event  $\{\{\mathcal{L}_{n_j} \not\subset K^\epsilon\} \cap \{\mathcal{M}_{n_j, j} \subset K^{\epsilon/2}\}\}$  is realized. So there exists  $\eta_{x_0, j} \in \mathcal{L}_{n_j}$ , with  $x_0 \notin \mathbb{N}$ , such that for all  $g \in K$ ,  $d_{\mathcal{U}}(\eta_{x_0, j}, g) \geq \epsilon$ . Now, there exists  $g_{k_{x_0}} \in K$  such that  $d_{\mathcal{U}}(\eta_{k_{x_0}, j}, g_{k_{x_0}}) < \frac{\epsilon}{2}$ . Then, the triangle inequality implies that necessarily,  $d_{\mathcal{U}}(\eta_{x_0, j}, \eta_{k_{x_0}, j}) \geq \frac{\epsilon}{2}$ . Therefore,

$$P_j \leq P\left(\sup_{0 \leq x \leq n_j - j} \left\| \left\| \frac{Z(x + j \cdot) - Z(x)}{j} - \frac{Z(k_x + j \cdot) - Z(k_x)}{j} \right\| \right\| \geq \frac{\epsilon}{2}\right) \quad (4.4.12)$$

and so

$$P_j \leq P\left(\sup_{0 \leq x \leq n_j - j} \{\|Z(x + j \cdot) - Z(k_x + j \cdot)\| + \|Z(x) - Z(k_x)\|\} \geq j \frac{\epsilon}{2}\right). \quad (4.4.13)$$

Now, for all  $s \in [0, 1]$ , we have that  $|(x + js) - (k_x + js)| \leq \frac{1}{2}$ , so that

$$\sup_{0 \leq x \leq n_j - j} \sup_{0 \leq s \leq 1} |Z(x + js) - Z(k_x + js)| \leq \sup_{0 \leq y \leq n_j} \sup_{0 \leq a \leq \frac{1}{2}} |Z(y + a) - Z(y)|. \quad (4.4.14)$$

Consequently,

$$P_j \leq P\left(\sup_{0 \leq y \leq n_j} \sup_{0 \leq a \leq \frac{1}{2}} |Z(y + a) - Z(y)| \geq j \frac{\epsilon}{4}\right). \quad (4.4.15)$$

Applying Fact 33 below we obtain that there exists a constant  $\beta > 0$  such that for all  $j$  large enough,

$$P_j \leq \beta n_j P\left(\sup_{0 \leq \tau \leq 1} |Z(\tau)| > j \frac{\epsilon}{12}\right). \quad (4.4.16)$$

Therefore, (4.4.9) and (4.4.8) imply that the series  $\sum_j P_j$  and  $\sum_j P(\mathcal{L}_{n_j} \not\subset K^\epsilon)$  are convergent. So, the Borel-Cantelli Lemma implies that  $P(\mathcal{L}_{n_j} \not\subset K^\epsilon \text{ i.o. (in } j)) = 0$ . Finally, we observe that, for all  $n$  satisfying  $n_{j-1} < n \leq n_j$ , we have that  $\mathcal{L}_n \subset \mathcal{L}_{n_j}$ . Consequently, the event  $\{\mathcal{L}_n \not\subset K^\epsilon \text{ i.o. (in } n)\}$  is included in  $\{\mathcal{L}_{n_j} \not\subset K^\epsilon \text{ i.o. (in } j)\}$ , which concludes the proof.  $\square$

**Fact 33.** *Let  $0 < h < 1$ . Then there exists a constant  $\beta > 0$  such that for all  $u > 0$  and  $n \geq 1$ ,*

$$P\left(\sup_{0 \leq x \leq n} \sup_{0 \leq a \leq h} |Z(x+a) - Z(x)| \geq u\right) \leq \beta n P\left(\sup_{0 \leq \tau \leq 1} |Z(\tau)| > \frac{u}{3}\right). \quad (4.4.17)$$

*Proof.* Fix  $n \geq 1$  and let  $R_n := \lceil \frac{n}{h} \rceil + 1$ . Then,  $\delta_n := \frac{n}{R_n} < h$ . For any  $x \in [0, n]$ , let  $i_x$  be the unique integer such that  $i_x \delta_n \leq x < (i_x + 1) \delta_n$ . Then, for  $a \in [0, h]$ , two cases occur.

*Case 1 :*  $x + a \leq (i_x + 1) \delta_n$ .

Then,  $0 \leq (x + a) - i_x \delta_n \leq h$  and  $0 \leq x - i_x \delta_n \leq h$ , so that

$$|Z(x+a) - Z(x)| \leq |Z(i_x \delta_n) - Z(x+a)| + |Z(i_x \delta_n) - Z(x)| \leq 2 \max_{0 \leq i \leq R_n} \sup_{0 \leq \tau \leq h} |Z(i \delta_n) - Z(i \delta_n + \tau)|. \quad (4.4.18)$$

*Case 2 :*  $x + a > (i_x + 1) \delta_n$ .

Then,  $0 \leq (x + a) - (i_x + 1) \delta_n \leq h$ ,  $0 \leq (i_x + 1) \delta_n - i_x \delta_n \leq h$  and  $0 \leq x - i_x \delta_n \leq h$ , so that

$$|Z(x+a) - Z(x)| \leq |Z((i_x + 1) \delta_n) - Z(x+a)| + |Z(i_x \delta_n) - Z((i_x + 1) \delta_n)| + |Z(i_x \delta_n) - Z(x)| \quad (4.4.19)$$

and so

$$|Z(x+a) - Z(x)| \leq 3 \max_{0 \leq i \leq R_n} \sup_{0 \leq \tau \leq h} |Z(i \delta_n) - Z(i \delta_n + \tau)|. \quad (4.4.20)$$

Therefore, we obtain in both cases a bound independent of  $x$  and  $a$ , so that

$$\sup_{0 \leq x \leq n} \sup_{0 \leq a \leq h} |Z(x+a) - Z(x)| \leq 3 \max_{0 \leq i \leq R_n} \sup_{0 \leq \tau \leq h} |Z(i \delta_n + \tau) - Z(i \delta_n)|. \quad (4.4.21)$$

Now, for all  $u > 0$ , we get from Fact 31 that

$$P\left(\max_{0 \leq i \leq R_n} \sup_{0 \leq \tau \leq h} |Z(i \delta_n + \tau) - Z(i \delta_n)| > u\right) \leq R_n P\left(\sup_{0 \leq \tau \leq 1} |Z(\tau)| > u\right). \quad (4.4.22)$$

$\square$

**Lemma 28.** *Assume that (C) holds and that  $\{Z(t) : t \geq 0\}$  is centered. Then, for all  $\epsilon > 0$ , we have that*

$$\sum_j n_j P\left(\sup_{0 \leq \tau \leq 1} |Z(\tau)| \geq j \epsilon\right) < \infty. \quad (4.4.23)$$

*Proof.* Assumption (C) implies that for any  $s > 0$ ,  $\mathcal{A}_s := \psi(s) + \psi(-s)$  is finite. Now, for any fixed  $s > 0$ , we have that for all  $u > 0$ ,

$$P\left(\sup_{0 \leq \tau \leq 1} |Z(\tau)| > u\right) \leq P\left(\sup_{0 \leq \tau \leq 1} \exp(sZ(\tau)) > \exp(su)\right) + P\left(\sup_{0 \leq \tau \leq 1} \exp(-sZ(\tau)) > \exp(su)\right). \quad (4.4.24)$$

Then,  $\{Z(t) : t \geq 0\}$  being a centered Lévy process, it is a martingale. Now, the function  $x \mapsto \exp(\pm sx)$  is nonnegative, continuous and convex. Therefore, the process  $(\exp[\pm sZ(t)])_{t \geq 0}$  is a nonnegative submartingale. Then, by Doob's inequality, we obtain that

$$P\left(\sup_{0 \leq \tau \leq 1} |Z(\tau)| > u\right) \leq \mathcal{A}_s \exp(-su). \quad (4.4.25)$$

Therefore, recalling that  $n_j \leq \exp\left(\frac{j+1}{c}\right)$ , we have that

$$n_j P\left(\sup_{0 \leq \tau \leq 1} |Z(\tau)| \geq j\epsilon\right) \leq n_j \mathcal{A}_s \exp(-sj\epsilon) \leq \mathcal{A}_s \exp\left(\frac{1}{c}\right) \exp\left[-j\left(s\epsilon - \frac{1}{c}\right)\right]. \quad (4.4.26)$$

Now, by choosing  $s$  large enough so that  $s\epsilon > \frac{1}{c}$ , we conclude the proof.  $\square$

#### 4.4.2 The lower bound

For any integer  $n > 0$ , set  $R_n := [(n - A_n)/A_n]$  and  $\mathcal{Q}_n := \{\eta_{rA_n, A_n} : 1 \leq r \leq R_n\}$ .

**Lemma 29.** *Assume that (C) holds and that  $\{Z(t) : t \geq 0\}$  is centered. Let  $c > 0$ . For any  $\epsilon > 0$ , we have a.s. that for all  $n$  large enough,*

$$K_{1/c} \subset \mathcal{L}_n^\epsilon. \quad (4.4.27)$$

*Proof.* Fix  $\epsilon > 0$ . Let  $g \in K_{1/c}$  and set  $G := B_{\mathcal{U}}(g, \epsilon/2)$ . From Fact 29, there exists  $\zeta > 0$  such that  $G \supset G' := B_{\mathcal{S}}(g, \zeta)$ . Therefore,

$$P\left(g \notin \mathcal{Q}_n^{\epsilon/2}\right) = P\left(\bigcap_{1 \leq r \leq R_n} \{\eta_{rA_n, A_n} \notin G\}\right) \leq P\left(\bigcap_{1 \leq r \leq R_n} \{\eta_{rA_n, A_n} \notin G'\}\right). \quad (4.4.28)$$

Then, the mutual independence of the  $\eta_{rA_n, A_n}$  for  $1 \leq r \leq R_n$  and Fact 30 imply that

$$P\left(g \notin \mathcal{Q}_n^{\epsilon/2}\right) \leq \prod_{r=1}^{R_n} (1 - P(\eta_{rA_n, A_n} \in G')) = \left(1 - P\left(\frac{Z(A_n \cdot)}{A_n} \in G'\right)\right)^{R_n}. \quad (4.4.29)$$

Now,  $G'$  is  $\mathcal{S}$ -open. Therefore, under (C), Theorem 23 implies that for all  $\theta > 0$ , we have for all  $n$  large enough,

$$P\left(\frac{Z(A_n \cdot)}{A_n} \in G'\right) \geq \exp(A_n(-I(G') - \theta)). \quad (4.4.30)$$

Since  $I(f) = \infty$  for  $f \notin C(0, 1)$ , we have  $I(G') = I(G' \cap C(0, 1))$ . Now, since the Skorohod topology relativized to  $C(0, 1)$  coincides with the uniform topology, we have that  $G'' := G' \cap C(0, 1)$  is an open subset of  $(C(0, 1), \mathcal{U})$  containing  $g$ .

Now, we claim that  $I(G'') < 1/c$ . Indeed, if  $I(g) < 1/c$ , then it is clear. Otherwise, assume that  $I(g) = 1/c$  and for all  $f \in G''$ ,  $I(f) \geq 1/c$ . Therefore,  $I(g)$  would be a local minimum of the restriction of  $I$  to  $C(0, 1)$ , which is a convex function. Since  $(C(0, 1), \mathcal{U})$  is a convex topological vector space,  $I(g)$  would be a global minimum of this function. Now, since  $\mathbb{E}[Z(1)] = 0$ , we have



that  $\Psi(0) = 0$ . So, for any constant function  $g_0$ ,  $I(g_0) = 0$  is a global minimum of  $I$  on  $C(0, 1)$ . Consequently, we would have that  $I(g) = 0$ , which leads to a contradiction.

So, we can write  $I(G'') = \frac{1}{c} - \delta$  with  $\delta > 0$ . Taking  $\theta = \frac{\delta}{4}$  in (4.4.30), we obtain for all  $n$  large enough,

$$P\left(\frac{Z(A_n \cdot)}{A_n} \in G'\right) \geq \exp\left(-A_n \left(\frac{1}{c} - \frac{3\delta}{4}\right)\right) \geq n^{-(1-c\frac{3\delta}{4})}. \quad (4.4.31)$$

Consequently, for all  $n$  large enough,

$$P\left(g \notin \mathcal{Q}_n^{\epsilon/2}\right) \leq \left(1 - n^{-(1-c\frac{3\delta}{4})}\right)^{(n-A_n)/A_n} \leq \left(1 - n^{-(1-c\frac{3\delta}{4})}\right)^{\frac{n}{c \log n}} \quad (4.4.32)$$

and

$$P\left(g \notin \mathcal{Q}_n^{\epsilon/2}\right) \leq \exp\left(-n^{-(1-c\frac{3\delta}{4})} \frac{n}{c \log n}\right) = \exp\left(-\frac{n^{c\frac{3\delta}{4}}}{c \log n}\right). \quad (4.4.33)$$

Therefore, the Borel-Cantelli lemma implies that

$$P\left(g \notin \mathcal{Q}_n^{\epsilon/2} \text{ i.o. (in } n)\right) = 0. \quad (4.4.34)$$

Finally, since  $K_{1/c}$  is a compact subset of  $(C(0, 1), \mathcal{U})$ , we can find  $d \in \mathbb{N}$  and functions  $(g_q)_{q=1, \dots, d}$  in  $K_{1/c}$  such that  $K_{1/c} \subset \bigcup_{q=1}^d B_{\mathcal{U}}(g_q, \epsilon/2)$ . Then, the triangle inequality and (4.4.34) applied to each  $g_q$  imply that there exists almost surely an integer  $N(\epsilon)$  such that for all  $n \geq N(\epsilon)$ ,

$$K_{1/c} \subset \mathcal{Q}_n^\epsilon \subset \mathcal{L}_n^\epsilon. \quad (4.4.35)$$

□

**Lemma 30.** *Assume that (A) holds and that  $\{Z(t) : t \geq 0\}$  is centered. Furthermore, assume that we can choose  $M < \infty$  large enough, so that a.s., for all  $n$  large enough,*

$$\mathcal{Q}_n \subset BV_{0, M}(0, 1). \quad (4.4.36)$$

Then, for any  $\epsilon > 0$ , there exists almost surely  $N(\epsilon) < \infty$  such that for all  $n \geq N(\epsilon)$ ,

$$L_{1/c} \subset \mathcal{Q}_n^\epsilon. \quad (4.4.37)$$

*Proof.* Let  $\epsilon > 0$ .

Let  $\mathcal{P} = \{0 = t_0 < t_1 < \dots < t_{m-1} < t_m = 1\}$  be a partition of  $[0; 1]$ . Set  $d(\mathcal{P}) := \max_{1 \leq i \leq m} \{t_i - t_{i-1}\}$ . For  $f \in BV_0(0, 1)$  define  $f_\pm^\mathcal{P}(u) = \sum_{i=1}^{k-1} (f(t_i) - f(t_{i-1}))^\pm + \frac{u-t_{k-1}}{t_k-t_{k-1}} (f(t_k) - f(t_{k-1}))^\pm$  for  $t_{k-1} \leq u \leq t_k$ ,  $2 \leq k \leq m$ , and  $f_\pm^\mathcal{P}(u) = \frac{u}{t_1} f(t_1)^\pm$  for  $0 \leq u \leq t_1$ . Let  $f^\mathcal{P} := f_+^\mathcal{P} - f_-^\mathcal{P}$ .

Let  $s \in L_{1/c}$ . Then the triangle inequality implies that for any  $1 \leq r \leq R_n$ , we have that

$$d_{\mathcal{W}}(s, \eta_{rA_n, A_n}) \leq d_{\mathcal{W}}(s, s^\mathcal{P}) + d_{\mathcal{W}}(s^\mathcal{P}, \eta_{rA_n, A_n}^\mathcal{P}) + d_{\mathcal{W}}(\eta_{rA_n, A_n}, \eta_{rA_n, A_n}^\mathcal{P}). \quad (4.4.38)$$

By (2.56) in Deheuvels (1991), we have that for any  $f \in BV_0(0, 1)$ ,

$$d_{\mathcal{W}}(f, f^\mathcal{P}) \leq \frac{d(\mathcal{P})}{2} |f|_v(1). \quad (4.4.39)$$

By (3.9) in Deheuvels (1991), we can choose  $M < \infty$  large enough, so that

$$L_{1/c} \subset BV_{0,M}(0, 1). \quad (4.4.40)$$

Now, we fix  $M < \infty$  such that both (4.4.40) and (4.4.36) hold. Then, we choose  $\mathcal{P}$  such that  $d(\mathcal{P}) \leq \frac{2\epsilon}{3M}$ . Define  $(y_i)_{1 \leq i \leq m}$  by  $y_i = s(t_i) - s(t_{i-1})$ . Then for any  $\delta > 0$ , set

$$P_{n,\delta} := P \left( \min_{0 \leq r \leq R_n} \max_{1 \leq i \leq m} \left| \frac{Z(rA_n + t_i A_n) - Z(rA_n + t_{i-1} A_n)}{A_n} - y_i \right| \geq \delta \right). \quad (4.4.41)$$

Since the process  $Z(\cdot)$  has independent and stationary increments, we get that

$$\begin{aligned} P_{n,\delta} &= \left[ 1 - P \left( \max_{1 \leq i \leq m} \left| \frac{Z(t_i A_n) - Z(t_{i-1} A_n)}{A_n} - y_i \right| < \delta \right) \right]^{R_n} \\ &\leq \exp \left[ -R_n \prod_{1 \leq i \leq m} P \left( \left| \frac{Z((t_i - t_{i-1}) A_n)}{A_n} - y_i \right| < \delta \right) \right]. \end{aligned}$$

Then, under  $(\mathcal{A})$ , we obtain from results of (Lynch and Sethuraman, 1987) that for all  $1 \leq i \leq m$ , the sequence  $\left( \frac{Z((t_i - t_{i-1}) A_n)}{A_n} \right)_n$  satisfies a LDP with rate function

$$z \mapsto \psi \left( \frac{z}{t_i - t_{i-1}} \right) (t_i - t_{i-1}). \quad (4.4.42)$$

Setting  $B(y_i; \delta) := \{z : |z - y_i| < \delta\}$ , we deduce that for any  $\theta > 0$ , we have for all  $n$  large enough,

$$P \left( \left| \frac{Z((t_i - t_{i-1}) A_n)}{A_n} - y_i \right| < \delta \right) \geq \exp \left[ A_n \left( - \inf_{z \in B(y_i; \delta)} \psi \left( \frac{z}{t_i - t_{i-1}} \right) (t_i - t_{i-1}) - \frac{\theta}{m} \right) \right]. \quad (4.4.43)$$

Let  $J^{\mathcal{P}}$  be the function defined on  $\mathbb{R}^m$  by

$$J^{\mathcal{P}}((z_i)) = \sum_{i=1}^{i=m} \psi \left( \frac{z_i}{t_i - t_{i-1}} \right) (t_i - t_{i-1}). \quad (4.4.44)$$

Let  $G_\delta := \{(z_i)_i \in \mathbb{R}^m : \max_i |z_i - y_i| < \delta\}$ . Then we obtain that for all  $n$  large enough,

$$P_{n,\delta} \leq \exp \left[ -R_n \exp \left[ A_n \left( -J^{\mathcal{P}}(G_\delta) - \theta \right) \right] \right]. \quad (4.4.45)$$

Now, we claim that  $J^{\mathcal{P}}(G_\delta) < 1$ . The proof is analogous to that of the fact that  $I(G'') < 1/c$  previously, since  $J^{\mathcal{P}}$  is a convex function on  $\mathbb{R}^m$ . We also obtain analogously that for any  $\delta > 0$ ,  $\sum_{n=1}^{\infty} P_{n,\delta} < \infty$ . By the Borel-Cantelli lemma, there exists almost surely  $N = N(\delta, \mathcal{P})$  such that for all  $n \geq N$ , there exists  $1 \leq r \leq R_n$  satisfying

$$\max_{1 \leq i \leq m} |(\eta_{rA_n, A_n}(t_i) - \eta_{rA_n, A_n}(t_{i-1})) - (s(t_i) - s(t_{i-1}))| = \max_{1 \leq i \leq m} \left| \frac{Z(rA_n + t_i A_n) - Z(rA_n + t_{i-1} A_n)}{A_n} - y_i \right| < \delta.$$

Now, by (2.62) in Deheuvels (1991), we have that for any  $f, g \in BV_0(0, 1)$ ,

$$d_{\mathcal{W}}(f^{\mathcal{P}}, g^{\mathcal{P}}) \leq 2m \max_{1 \leq i \leq m} |(f(t_i) - f(t_{i-1})) - (g(t_i) - g(t_{i-1}))|. \quad (4.4.46)$$

Then, we choose  $\delta > 0$  such that  $\delta \leq \frac{\epsilon}{6m}$ . Therefore we obtain from (??) that almost surely, for all  $n$  large enough, there exists  $1 \leq r \leq R_n$  such that  $d_{\mathcal{W}}(\eta_{rA_n, A_n}^{\mathcal{P}}, s^{\mathcal{P}}) \leq \frac{\epsilon}{3}$  and then from (4.4.38) and the choice of  $d(\mathcal{P}) \leq \frac{2\epsilon}{3M}$  that

$$d_{\mathcal{W}}(s, \eta_{rA_n, A_n}) \leq \epsilon. \quad (4.4.47)$$

Finally, since  $J$  is a good rate function,  $L_{1/c}$  is a compact subset of  $(BV_{0,M}(0,1), \mathcal{W})$ . Therefore, analogously to the end of the previous proof, we obtain that almost surely, for all  $n$  large enough,  $L_{1/c} \subset \mathcal{Q}_n^{2\epsilon}$ .  $\square$

### 4.4.3 Proof of main theorems

**Remark 10.** For any real  $\mu$ , let  $\{Z^{(\mu)}(t) : t \geq 0\}$  be the Lévy process defined by

$$Z^{(\mu)}(t) := Z(t) + \mu t, \text{ for } t \geq 0. \quad (4.4.48)$$

We prove in Appendix that, if an ERFL holds for  $\{Z(t) : t \geq 0\}$ , then an ERFL holds for  $\{Z^{(\mu)}(t) : t \geq 0\}$ . Therefore, in order to prove Theorems 25 and 26 under assumption  $(\mathcal{E})$ , it is enough to obtain them for centered Lévy processes.

#### Proof of Theorem 25

*Proof.* First, assume that  $\{Z(t) : t \geq 0\}$  is centered. Then, we combine Lemmas 25, 27, 28 and Lemma 31 in the Appendix to get the upper bound, that is first part of (4.4.1). Then, we combine Lemma 29 and Lemma 32 in the Appendix to get the lower bound, that is the second part of (4.4.1). We conclude by applying Remark 1.  $\square$

#### Proof of Theorem 26

*Proof.* First, assume that  $\{Z(t) : t \geq 0\}$  is centered. Notice that the following statement

$$\lim_{n \rightarrow \infty} \Delta_{\mathcal{W}}(\mathcal{M}_{n, A_n}, L_{1/c}) = 0 \quad \text{a.s.} \quad (4.4.49)$$

is equivalent to the statement that, for any  $\epsilon > 0$ , there exists a.s. an integer  $N(\epsilon) < \infty$  such that for all  $n \geq N(\epsilon)$ ,

$$\mathcal{M}_{n, A_n} \subset L_{1/c}^{\epsilon} \text{ and } L_{1/c} \subset \mathcal{M}_{n, A_n}^{\epsilon}. \quad (4.4.50)$$

Then, Lemma 26 and the Borel-Cantelli Lemma imply that

$$P(\mathcal{M}_{n_j, j} \not\subset (L_{1/c})^{\epsilon; \mathcal{W}} \text{ i.o. (in } j)) = 0, \quad (4.4.51)$$

from which we deduce readily the first statement of (4.4.50). Then the second statement of (4.4.50) follows from Lemma 30, under the assumptions of Theorem 26. We conclude by applying Remark 1.  $\square$

## 4.5 Appendix

### 4.5.1 Proof of Remark 1

We define the set of Erdős-Rényi-type increment functions associated to  $Z^{(\mu)}(\cdot)$  by

$$\mathcal{G}_T^{(\mu)} := \left\{ \eta_{x, a_T}^{(\mu)}(\cdot) := \frac{Z^{(\mu)}(x + a_T \cdot) - Z^{(\mu)}(x)}{a_T} = \eta_{x, a_T}(\cdot) + \mu I_d : 0 \leq x \leq T - a_T \right\}. \quad (4.5.1)$$

Let  $I^{(\mu)}$  be the functional defined on  $D[0, 1]$  by  $I^{(\mu)}(f) = \int_0^1 \Psi^{(\mu)}(\dot{f}(u)) du$  if  $f$  is absolutely continuous and  $f(0) = 0$ , while  $I^{(\mu)}(f) = \infty$  otherwise. For any  $\alpha > 0$ , set  $K_\alpha^{(\mu)} := \{f \in D(0, 1) : I^{(\mu)}(f) \leq \alpha\}$ .

**Fact 34.** *Assume that for some  $\{a_T, T > 0\}$  and  $\alpha > 0$ , we have that*

$$\lim_{T \rightarrow \infty} \Delta(\mathcal{G}_T, K_\alpha) = 0 \quad a.s. \quad (4.5.2)$$

*Then, for all real  $\mu$ , we have that*

$$\lim_{T \rightarrow \infty} \Delta(\mathcal{G}_T^{(\mu)}, K_\alpha^{(\mu)}) = 0 \quad a.s. \quad (4.5.3)$$

*Proof.* Let  $\Psi^{(\mu)}$  be the Legendre transform of the mgf of  $Z^{(\mu)}(1)$ . Then, for any real  $a$ ,  $\Psi^{(\mu)}(a) = \Psi(a - \mu)$ , which implies easily that  $K_\alpha^{(\mu)} = K_\alpha + \mu I_d$ . Now, the assumption and a translation argument imply that for all  $\epsilon > 0$ , a.s., for all  $T$  large enough,

$$\mathcal{G}_T^{(\mu)} = \mathcal{G}_T + \mu I_d \subset (K_\alpha + \mu I_d)^\epsilon = \left(K_\alpha^{(\mu)}\right)^\epsilon \quad (4.5.4)$$

and

$$K_\alpha^{(\mu)} = K_\alpha + \mu I_d \subset (\mathcal{G}_T + \mu I_d)^\epsilon = \left(\mathcal{G}_T^{(\mu)}\right)^\epsilon. \quad (4.5.5)$$

□

### 4.5.2 End of proof of Theorem 25

For any  $c > 0$ , set  $K := K_{1/c}$ . Denote by  $I$  the identity function on  $[0, 1]$ . Fix  $\lambda_0$  with  $0 < \lambda_0 < 1$ .

**Fact 35.** *For any  $f \in K$ , we define the function  $\rho_f$  by*

$$\rho_f(\lambda) = \|\lambda^{-1} f(\lambda I) - f\|. \quad (4.5.6)$$

*Then there exists a function  $\delta(\cdot)$ , independent of  $f$  and satisfying  $\delta(x) \rightarrow 0$  as  $x \rightarrow 0$ , such that for all  $\lambda, \mu \in [\lambda_0, 1]$ ,*

$$|\rho_f(\lambda) - \rho_f(\mu)| \leq \delta(|\lambda - \mu|). \quad (4.5.7)$$

*Proof.* For any  $\lambda, \mu \in [\lambda_0, 1]$ , we have that

$$|\rho_f(\lambda) - \rho_f(\mu)| = \left| \|\lambda^{-1} f(\lambda I) - f\| - \|\mu^{-1} f(\mu I) - f\| \right| \leq \|\lambda^{-1} f(\lambda I) - \mu^{-1} f(\mu I)\| \quad (4.5.8)$$

and

$$\|\lambda^{-1} f(\lambda I) - \mu^{-1} f(\mu I)\| \leq \|\lambda^{-1} f(\lambda I) - \lambda^{-1} f(\mu I)\| + \|\lambda^{-1} f(\mu I) - \mu^{-1} f(\mu I)\|. \quad (4.5.9)$$

Then, we have that

$$\|\lambda^{-1}f(\lambda I) - \lambda^{-1}f(\mu I)\| \leq \lambda_0^{-1} \sup_{f \in K} w_f(|\lambda - \mu|) \quad (4.5.10)$$

and

$$\|\lambda^{-1}f(\mu I) - \mu^{-1}f(\mu I)\| \leq |\lambda^{-1} - \mu^{-1}| \sup_{f \in K} \|f\|. \quad (4.5.11)$$

Now,  $K$  being  $\mathcal{U}$ -compact, we apply the Arzela-Ascoli theorem to conclude.  $\square$

**Fact 36.** For any  $g, h \in D(0, 1)$  and  $\lambda \in [\lambda_0, 1]$ , we have that

$$\|\lambda^{-1}g(\lambda I) - g\| \leq (1 + \lambda_0^{-1}) \|h - g\| + \|\lambda^{-1}h(\lambda I) - h\|. \quad (4.5.12)$$

*Proof.* For any  $\lambda \in [\lambda_0, 1]$ , we have that

$$\|\lambda^{-1}g(\lambda I) - g\| \leq \|\lambda^{-1}g(\lambda I) - \lambda^{-1}h(\lambda I)\| + \|\lambda^{-1}h(\lambda I) - h\| + \|h - g\|. \quad (4.5.13)$$

Since  $\lambda_0 \leq \lambda \leq 1$ , we get readily that

$$\|\lambda^{-1}g(\lambda I) - \lambda^{-1}h(\lambda I)\| \leq \lambda_0^{-1} \|h - g\|. \quad (4.5.14)$$

$\square$

**Fact 37.** For all  $\delta > 0$ , there exists  $\gamma = \gamma(\delta) > 0$  such that  $|\lambda - 1| < \gamma$  implies that

$$\sup_{g \in K} \|\lambda^{-1}g(\lambda I) - g\| < \delta. \quad (4.5.15)$$

*Proof.* Fix  $\delta > 0$ . Then  $K$  being  $\mathcal{U}$ -compact, for any  $\alpha > 0$ , there exists a finite number of functions  $(g_q)_{q=1, \dots, d}$  in  $K$  such that  $K \subset \bigcup_{q=1}^d \mathcal{B}_{\mathcal{U}}(g_q, \alpha)$ . Now, we obtain from Fact 35 that for all  $\kappa > 0$ , there exists  $\gamma > 0$  depending only on  $\kappa$ , such that  $|\lambda - 1| < \gamma$  implies that for all  $q \in \{1, \dots, d\}$ ,  $\|\lambda^{-1}g_q(\lambda I) - g_q\| < \kappa$ . For any  $g \in K$ , there exists  $q \in \{1, \dots, d\}$  such that  $g \in \mathcal{B}_{\mathcal{S}}(g_q, \alpha)$  and we deduce from (4.5.12) applied to  $h = g_q$  that

$$\|\lambda^{-1}g(\lambda I) - g\| \leq \alpha(1 + \lambda_0^{-1}) + \kappa. \quad (4.5.16)$$

Finally, for any  $\delta > 0$ , we can choose  $\gamma$  and  $\alpha$  small enough to get that  $\alpha(1 + \lambda_0^{-1}) + \kappa < \delta$ .  $\square$

**Fact 38.** For all  $\delta > 0$ , there exist  $\theta = \theta(\delta)$ , with  $0 < \theta < \delta$ , and  $\gamma = \gamma(\delta) > 0$  such that

$$|\lambda - 1| < \gamma \implies \sup_{f \in K^\theta} \|\lambda^{-1}f(\lambda I) - f\| < \delta. \quad (4.5.17)$$

*Proof.* Fix  $\delta > 0$ . For any  $\theta > 0$  and  $f \in K^\theta$ , let  $g \in K$  such that  $\|g - f\| < \theta$ . Then (4.5.12) implies that

$$\|\lambda^{-1}f(\lambda I) - f\| \leq \theta(1 + \lambda_0^{-1}) + \sup_{g \in K} \|\lambda^{-1}g(\lambda I) - g\|. \quad (4.5.18)$$

Observing that  $1 + \lambda_0^{-1} > 1$ , we conclude by applying Fact 37.  $\square$

**Lemma 31.** For any  $\epsilon > 0$ , there exists a.s. a real  $T(\epsilon)$  such that for all real  $T \geq T(\epsilon)$ , we have that

$$\mathcal{G}_T \subset K^\epsilon. \quad (4.5.19)$$

*Proof.* For any  $T > 0$ , set

$$n_T := \inf \{n : A_n \geq a_T\}.$$

Notice that necessarily,  $n_T \geq T$ . Now, the Mean value theorem implies that  $A_{n_T} - a_T \leq \frac{c}{T} (n_T - T)$ , so there exists  $\tau > 0$  such that for all  $T \geq \tau$ ,  $T - a_T \leq n_T - A_{n_T}$ . Furthermore, the definition of  $n_T$  implies that  $A_{n+1} < a_T$  and we prove readily that

$$\lim_{T \rightarrow \infty} \frac{a_T}{A_{n_T}} = 1. \quad (4.5.20)$$

Fix  $\epsilon > 0$ .

We get from Lemmas 25, 27 and 28 that for any  $\theta > 0$ , there exists a.s. an integer  $N(\theta)$  such that

$$\forall n \geq N(\theta), \mathcal{L}_n \subset K^\theta. \quad (4.5.21)$$

Now, Fact 38 implies that there exist  $\theta(\epsilon/2) > 0$  and  $\gamma(\epsilon/2) > 0$  such that

$$|\lambda - 1| < \gamma(\epsilon/2) \implies \sup_{f \in K^{\theta(\epsilon/2)}} \|\lambda^{-1} f(\lambda I) - f\| < \epsilon/2. \quad (4.5.22)$$

From (4.5.20), there exists  $T_0(\epsilon) > 0$  such that for all  $T \geq T_0(\epsilon)$ ,  $\left| \frac{a_T}{A_{n_T}} - 1 \right| < \gamma(\epsilon/2)$ .

Then, we can define a.s.  $T(\epsilon)$  by

$$T(\epsilon) := \max \{\tau; T_0(\epsilon); N(\theta(\epsilon/2))\}. \quad (4.5.23)$$

Let  $T \geq T(\epsilon)$ . Since  $T \geq \tau$ , we have that  $[0, T - a_T] \subset [0, n_T - A_{n_T}]$ . Then for all  $x \in [0, T - a_T]$  and all  $s \in [0, 1]$ , we have that

$$\eta_{x, a_T}(s) = \frac{A_{n_T}}{a_T} \eta_{x, A_{n_T}} \left( s \frac{a_T}{A_{n_T}} \right). \quad (4.5.24)$$

Since  $T(\epsilon) \geq N(\theta(\epsilon/2))$  we have that, for all  $x \in [0, n_T - A_{n_T}]$ ,  $\eta_{x, A_{n_T}} \in K^{\theta(\epsilon/2)}$ . Since  $T \geq T_0(\epsilon)$ , we have that  $\left| \frac{a_T}{A_{n_T}} - 1 \right| < \gamma(\epsilon/2)$ . By (4.5.22), we have consequently that, for all  $x \in [0, T - a_T]$ ,

$$\left\| \frac{A_{n_T}}{a_T} \eta_{x, A_{n_T}} \left( \frac{a_T}{A_{n_T}} I \right) - \eta_{x, A_{n_T}} \right\| < \epsilon/2. \quad (4.5.25)$$

Now, since  $\theta(\epsilon/2) < \epsilon/2$ , we have that  $K^{\theta(\epsilon/2)} \subset K^{\epsilon/2}$ . Therefore, for all  $x \in [0, T - a_T]$ ,  $\eta_{x, A_{n_T}} \in K^{\epsilon/2}$ . Then we obtain from (4.5.25) and the triangle inequality that

$$\forall x \in [0, T - a_T], \eta_{x, a_T} = \frac{A_{n_T}}{a_T} \eta_{x, A_{n_T}} \left( \frac{a_T}{A_{n_T}} I \right) \in K^\epsilon. \quad (4.5.26)$$

□

**Lemma 32.** For any  $\epsilon > 0$ , there exists a.s. a real  $T'(\epsilon)$  such that for all  $T \geq T'(\epsilon)$ , we have that

$$K \subset \mathcal{G}_T^\epsilon. \quad (4.5.27)$$

*Proof.* For all  $T \geq c$ , we have that  $1 \leq T - a_T$ . Set

$$m_T := \max \{n : n \leq T \text{ and } n - A_n \leq T - a_T\}.$$

Then, if  $T \geq c$  then  $m_T$  exists. The definition of  $m_T$  implies that  $m_T + 1 - A_{m_T+1} > T - a_T$ . We deduce readily that

$$\lim_{T \rightarrow \infty} \frac{m_T}{T} = \lim_{T \rightarrow \infty} \frac{A_{m_T}}{a_T} = 1. \quad (4.5.28)$$

Fix  $\epsilon > 0$ .

Lemma 29 implies that a.s., for all  $n$  large enough,  $K \subset \mathcal{L}_n^{\epsilon/2}$ . Since  $m_T \rightarrow \infty$  as  $T \rightarrow \infty$ , we can find a.s.  $T_1(\epsilon) > 0$  such that

$$\forall T \geq T_1(\epsilon), K \subset \mathcal{L}_{m_T}^{\epsilon/2}. \quad (4.5.29)$$

Then for all  $x \in [0, m_T - A_{m_T}] \subset [0, T - a_T]$ , and all  $s \in [0, 1]$ , we have that

$$\eta_{x, A_{m_T}}(s) = \frac{a_T}{A_{m_T}} \eta_{x, a_T} \left( s \frac{A_{m_T}}{a_T} \right). \quad (4.5.30)$$

Now, let  $\gamma(\epsilon/2)$  and  $\theta(\epsilon/2)$  be as in (4.5.22). Let  $T_2(\epsilon) > 0$  such that for all  $T \geq T_2(\epsilon)$ ,  $\left| \frac{A_{m_T}}{a_T} - 1 \right| < \gamma(\epsilon/2)$ .

From Lemma 31, there exists a.s.  $T(\theta(\epsilon/2)) > 0$  such that

$$\forall T \geq T(\theta(\epsilon/2)), \mathcal{G}_T \subset K^{\theta(\epsilon/2)}. \quad (4.5.31)$$

Then, we can define a.s.  $T'(\epsilon)$  by

$$T'(\epsilon) := \max \{c, T_1(\epsilon); T_2(\epsilon); T(\theta(\epsilon/2))\}. \quad (4.5.32)$$

Let  $T \geq T'(\epsilon)$ . Since  $T \geq T(\theta(\epsilon/2))$ , we have that

$$\forall x \in [0, T - a_T], \eta_{x, a_T} \in K^{\theta(\epsilon/2)}. \quad (4.5.33)$$

Since, moreover  $T \geq T_2(\epsilon)$ , and noticing that  $\frac{A_{m_T}}{a_T} \leq 1$ , we obtain that for all  $x \in [0, T - a_T]$

$$\left\| \frac{a_T}{A_{m_T}} \eta_{x, a_T} \left( \frac{A_{m_T}}{a_T} I \right) - \eta_{x, a_T} \right\| < \epsilon/2. \quad (4.5.34)$$

Consequently, for all  $x \in [0, m_T - A_{m_T}] \subset [0, T - a_T]$ , we have that  $\left\| \eta_{x, A_{m_T}} - \eta_{x, a_T} \right\| < \epsilon/2$ . Then, since  $T \geq T_1(\epsilon)$ , for all  $f \in K$ , there exists  $x_f \in [0, m_T - A_{m_T}]$  such that  $\left\| \eta_{x_f, A_{m_T}} - f \right\| < \epsilon/2$  and, by the triangle inequality,  $\left\| \eta_{x_f, a_T} - f \right\| < \epsilon$ .  $\square$

# Chapter 5

## Conclusion

### 5.1 Conclusion of chapters 2 and 3

In this section, we compare the methods of Chapter 1 and Chapter 2 and we present some perspectives.

An obvious similarity between them is the use of the commonly called saddlepoint approximation (see (Jensen, 1995)). This technique is the combination of tilting operation and of Edgeworth expansion of the density of the resulting partial sum. This approximation provides usually an accurate approximation of the tail of partial sums, which justifies its use in the Importance Sampling scheme described in the Introduction of Chapter 2. In the framework of conditional limit theorems, the tilting operation allows to obtain an exponential family for which the statistic of sum is sufficient. The idea of sufficiency is also essential in Chapter 1, since the sub  $\sigma$ -algebra  $\sigma(T)$  generated by the sum is sufficient in the sense given in Chapter 1. Heuristically, in both cases, the notion of sufficiency expresses that the knowledge of the value of the sum contains enough information to deduce the distribution of the sampling given this value.

From Chapter 2 to Chapter 3, we have used several ideas to get a result even when  $k$  is not a  $o(n)$  anymore. Firstly, we have performed an adaptative scheme to estimate  $p_k(Y_1^k)$ . Secondly, we have performed a higher order Edgeworth expansion. However, a rate of convergence is not obtainable by the method of Chapter 2. For  $k = o(n)$ , Chapter 2 provides a rate for the convergence of  $\|Q_{nak} - G_k\|_{TV}$  to 0, which is equal to  $\frac{k}{n}$ . So a possible perspective is to get a rate for this convergence when  $k$  is not a  $o(n)$ .

In statistical mechanics, in an isolated system, we obtain the distribution of a small component given the total energy when thermal equilibrium is reached, that is when the entropy is maximal. Thereby, a further development of our work is to establish a conditional limit theorem of this kind by optimisation of some entropy, and without performing Edgeworth expansions.

### 5.2 Conclusion of chapter 4

Under  $\mathcal{A}$ , we have obtained that for any  $c > 0$ ,

$$\lim_{n \rightarrow \infty} \Delta_{\mathcal{W}}(\mathcal{M}_{n,a_n}, L_{1/c}) = 0 \quad \text{a.s.}, \quad (5.2.1)$$



where  $a_n := \lfloor c \log n \rfloor$  and

$$\mathcal{M}_{n,a_n} := \{\eta_{m,a_n} : m \in \{0, \dots, n - a_n\}\}. \quad (5.2.2)$$

In future work, we would like to obtain, under  $(\mathcal{A})$ , the limiting behavior of a random set containing all the  $\eta_{x,a_n}$ , for *all real*  $x$  with  $0 \leq x \leq n - a_n$ . Since the tail of the distribution of  $Z(1)$  is heavier under  $(\mathcal{A})$  than under  $(\mathcal{C})$ , the increment functions have a wider amplitude under  $(\mathcal{A})$ . Therefore, it is more difficult to study their asymptotic inclusion in some fixed set. However, even under  $(\mathcal{A})$ , the distribution of  $Z(1)$  still has a light tail, so that it is reasonable to expect a positive result.

Another further development is to establish a functional Erdős-Rényi theorem (FERT) for renewal or compound renewal processes. We recall that a compound renewal process is of the form

$$Z(t) = \sum_{i=1}^{N(t)} X_i, \quad (5.2.3)$$

where  $N(\cdot)$  is any renewal process, independent of  $(X_i)$ . We have obtained a (FERT) in the particular case of the compound Poisson process, which is a Lévy process. In [?](#), the author obtains an Erdős-Rényi theorem for compound renewal process, which could be a manifestation of a functional version. In [\(Deheuvels and Steinebach, 2016\)](#), a FERT is established for renewal processes, by reducing to the FERT for the partial sum process which defines the renewal process. Therefore, we can expect a FERT for any compound renewal process, but it should require additional functional large deviations results.

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