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THÈSE DE DOCTORAT DE

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ÉCOLE DOCTORALE N° 601
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de l'Information et de la Communication*
Spécialité : *Mathématiques et leurs Interactions*

Par

Duc Tho NGUYEN

Classical and semi-classical analysis of magnetic fields in two dimensions

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To my parents

*“It is said that there are no mistakes,
only lessons, and our biggest mistakes
are our greatest source of learning.”*

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Résumé
Analyse classique et semi-classique des champs magnétiques
en deux dimensions

Ma thèse porte sur l'analyse classique et semi-classique, notamment en présence d'un champ magnétique. En mécanique classique, nous utilisons la dynamique hamiltonienne pour décrire le mouvement d'une particule chargée dans un domaine soumis à un champ magnétique. Nous nous intéressons à deux problèmes classiques de la physique : le problème de confinement et le problème de scattering. Dans le cas quantique, nous étudions le problème spectral du Laplacien magnétique au niveau semi-classique, dans des domaines bidimensionnels : sur une variété Riemannienne compacte à bord et sur \mathbb{R}^2 .

Le premier chapitre est l'introduction aux problèmes étudiés dans ce manuscrit. Ce chapitre commence par donner un aperçu de la dynamique hamiltonienne ainsi que de la méthode WKB pour les potentiels électriques. Ensuite, nous soulignons les différences entre le cas électrique et le cas purement magnétique. Ensuite, nous mentionnons les principales motivations de cette thèse. La première est liée à la dynamique classique : nous considérons une particule chargée dans une région magnétisée Ω , nous nous intéressons à la question du confinement :

“La particule atteindra-t-elle le bord du domaine en temps fini ?”

Nous apportons une réponse précise à cette question en fonction du comportement du champ magnétique à la frontière et des conditions initiales. Nos résultats améliorent ceux, récents, de Martins [42]. En particulier, nous verrons que, même si le champ magnétique est infini à la frontière, certaines trajectoires peuvent s'échapper de Ω . Ce type de problèmes (ouverts) est mentionné dans [10, Section 1.4].

Du côté classique également, nous nous intéressons au problème du scattering. Nous considérons une particule chargée en dehors de la région magnétisée Ω . Avant d'atteindre la région Ω , sa trajectoire est en ligne droite. Si la particule entre dans la région Ω , s'en échappe-t-elle en temps fini ? Et si c'est le cas, quel est l'angle de déviation entre les directions entrantes et sortantes ? Nous répondrons explicitement à ces questions dans le cas de champs magnétiques radiaux et lorsque Ω est un disque. Dans ce cas, la quantité de mouvement angulaire commute avec l'hamiltonien et permet une réduction à un système avec un degré de liberté.

Ces questions ont des motivations physiques intrinsèques. Leurs réponses permettent une meilleure compréhension de la dynamique classique des particules chargées dans

les champs magnétiques. La description des trajectoires classiques a également de nombreuses applications. En particulier, le confinement de particules chargées a des applications pratiques dans le fonctionnement d’un tokamak, un dispositif utilisant un champ magnétique pour piéger un plasma sous une forme torique. Nos résultats peuvent également s’étendre à la dimension trois. De plus, au niveau quantique, les trajectoires piégées peuvent être liées au caractère essentiellement auto-adjoint du Laplacien magnétique (voir [10, 44, 45, 52]). Nous voulons savoir s’il existe ou non des relations entre eux. Le confinement d’une particule chargée est également un point clé pour décrire le spectre/résonances des Laplaciens magnétiques. À notre connaissance, alors que la description de la dynamique magnétique a permis d’estimer le spectre des Laplaciens magnétiques (voir [26, 51]), aucun résultat ne semble exister pour estimer leur résonances près de l’axe réel. Etudier les trajectoires piégées est une étape nécessaire dans cette direction.

La deuxième motivation vient du côté quantique : l’étude du Laplacien magnétique. L’étude de la théorie spectrale du Laplacien magnétique est attachée à la théorie de la supraconductivité, voir [17]. En se concentrant sur la théorie spectrale semi-classique du Laplacien magnétique lorsque le champ magnétique, noté \mathbf{B} , a un minimum unique et non dégénéré, Helffer et Kordyukov ont fourni les développements asymptotiques pour les valeurs propres

$$\forall \ell \in \mathbb{N}, \quad \lambda_\ell(\mathcal{L}_{h,\mathbf{A}}) = \mathbf{B}(p_0)h + \left(2\ell \frac{\sqrt{\det H}}{\mathbf{B}(p_0)} + \frac{(\text{Tr } H^{\frac{1}{2}})^2}{2\mathbf{B}(p_0)} \right) h^2 + o(h^2), \quad (1)$$

où p_0 est le point minimum de \mathbf{B} et $H = \frac{1}{2}\text{Hess } \mathbf{B}(p_0)$. Dans [28], ces auteurs ont également considéré le cas général d’une surface équipée d’une métrique Riemannienne et ont également obtenu les développements asymptotiques des valeurs propres. À l’aide de la géométrie symplectique et des techniques pseudo-différentielles, Raymond et Vũ Ngọc ont récupéré les développements asymptotiques des valeurs propres via une forme normale de Birkhoff et les ont liés à la dynamique magnétique classique dans [51]. Nous rappelons que la méthode WKB a été appliquée pour résoudre le problème spectral du Laplacien électrique. Une question naturelle concerne la description des fonctions propres magnétiques : existe-t-il des constructions WKB proches d’un minimum magnétique, comme pour le cas électrique? Cette question apparaît par exemple dans [24, Section 6.1]. Elle a ensuite été attaquée dans [5] par l’idée de Born-Oppenheimer dans un cadre multi-échelles. De plus, on pourrait aussi poser la question suivante

“Pouvons-nous récupérer les développements asymptotiques (1)
par la méthode WKB?”

Dans [20], Bonthonneau et Raymond ont obtenu une réponse positive lorsque le champ magnétique est analytique. Dans cette thèse, nous étendrons leur travail en considérant les surfaces générales et en assouplissant également les hypothèses d'analyticité.

Nous consacrons la dernière section du premier chapitre au résumé de tous les résultats obtenus dans cette thèse. Comme indiqué dans la partie motivations, nous énonçons respectivement les principaux théorèmes de l'analyse classique à l'analyse semi-classique. Dans le cas classique, nous énonçons le théorème de confinement dans le cas général et sa version quantitative. Nous obtenons de meilleurs résultats lorsque nous limitons le problème de confinement au cas radial sur un disque unitaire. De nombreux exemples, remarques et images sont fournis pour comparer les travaux de cette thèse avec les travaux précédents. Ensuite, nous énonçons le théorème du scattering dans le cas radial pour donner une formule explicite pour l'angle du scattering. Dans le cas semi-classique, nous introduisons brièvement le laplacien magnétique sur une variété Riemannienne. En supposant que le champ magnétique a un minimum unique et non dégénéré, nous décrivons les valeurs propres et les fonctions propres par la méthode WKB. Ensuite, en utilisant un argument spectral, nous pouvons estimer les véritables fonctions propres et leurs approximations WKB. En particulier, lorsque le champ magnétique est radial, nous développons les résultats obtenus via la méthode WKB en une dimension. La fonction de phase donnée par la méthode WKB a maintenant une formule précise. Les résultats les plus intéressants apparaissant dans ce cas sont la décroissance exponentielle des fonctions propres avec l'exposant lié à la fonction de phase et l'estimation entre les vraies fonctions propres et leurs approximations WKB dans un espace pondéré de façon exponentielle.

Le chapitre 2 est consacré à la dynamique d'une particule chargée soumise au champ magnétique. Dans ce chapitre, nous expliquons brièvement comment écrire la célèbre équation de Newton en présence de la force de Lorentz sous une forme hamiltonienne. Pour ce faire, nous rappelons les bases de la géométrie symplectique dans \mathbb{R}^{2n} . En particulier, nous rappelons comment passer d'un changement de variables (d'espace) à une transformation symplectique. Cela sera nécessaire pour décrire le mouvement d'une particule chargée s'approchant de la frontière (dont un voisinage est décrit par des coordonnées tubulaires). À la fin de ce chapitre, nous considérons un exemple de base lorsque le champ magnétique est constant. Nous étudions le mouvement de la particule via la mécanique de Newton et la mécanique Hamiltonienne, puis nous comparons les deux approches.

Le chapitre 3 est consacré à prouver les résultats en mécanique classique. Tout d'abord,

nous écrivons l'hamiltonien magnétique en coordonnées tubulaires. Ensuite, nous utilisons la mécanique hamiltonienne pour attaquer le problème de confinement et du scattering. C'est le travail de l'article [46] en collaboration avec Nicolas Raymond et San Vũ Ngọc.

Dans le chapitre 4, nous utilisons la méthode WKB pour étudier le spectre du Laplacien magnétique sur une variété Riemannienne bidimensionnelle. Nous observerons que les développements asymptotiques des valeurs propres et des fonctions propres peuvent être obtenues via une analyse WKB. Premièrement, nous introduisons clairement et complètement la définition du Laplacien magnétique sur une variété Riemannienne. Il est défini comme l'extension de Friedrichs de l'opérateur

$$\begin{cases} H_{h,\mathbf{A}} = (D_{h,\mathbf{A}})^* D_{h,\mathbf{A}} \\ \text{Dom}(H_{h,\mathbf{A}}) = C_c^\infty(M), \end{cases} \quad (2)$$

où $D_{h,\mathbf{A}} := -(ihd + \mathbf{A})$ et $(D_{h,\mathbf{A}})^*$ est le $L^2(M)$ -adjoint opérateur de $D_{h,\mathbf{A}}$.

Lorsque la variété est compacte, le domaine de l'opérateur est fourni avec précision et la compacité de ses résolvantes est également prouvée. Ensuite, nous réintroduisons le Laplacien magnétique sur \mathbb{R}^2 dont le spectre sera étudié au chapitre 5. La section suivante est utilisée pour la construction d'une paire de coordonnées isothermes locales et pour la description du Laplacien magnétique dans ces coordonnées. Ces coordonnées isothermes jouent un rôle important dans cette partie, car elles nous aident à considérer le "pull-back" de l'opérateur de la variété au plan \mathbb{R}^2 . Ensuite, nous nous préparons pour le processus WKB en écrivant le problème propre du Laplacien magnétique dans l'équation eikonale et les équations de transport. L'analyse WKB qui est effectuée en détail comprend la résolution de l'équation eikonale et des équations de transport grâce à des séries formelles. Enfin, nous faisons la comparaison entre les vraies fonctions propres et leurs quasi-modes dans la dernière section.

Le chapitre 5 se concentre sur l'étude du spectre du Laplacien magnétique

$$\mathcal{L}_{h,\mathbf{A}} = (-ih\nabla - \mathbf{A})^2.$$

Nous commençons le chapitre en écrivant le laplacien magnétique dans les coordonnées radiales. Avec l'hypothèse supplémentaire que le champ magnétique est radialement symétrique, nous pouvons décomposer en Fourier le Laplacien magnétique en somme directe d'opérateurs électriques $(\mathcal{L}_{m,h})_{m \in \mathbb{Z}}$. En étudiant le spectre de chaque opérateur $\mathcal{L}_{m,h}$, nous avons trouvé des relations entre ces spectres et le spectre de $\mathcal{L}_{h,\mathbf{A}}$. Pour plus de détails, nous pouvons montrer que pour $m \in \mathbb{N}$, la $m^{\text{ième}}$ valeur propre de

$\mathcal{L}_{h,\mathbf{A}}$ est exactement la première valeur propre de $\mathcal{L}_{m,h}$. Ensuite, nous avons appliqué la méthode WKB pour le problème spectral des opérateurs fibrés $(\mathcal{L}_{m,h})_{m \in \mathbb{N}}$ et obtenu les valeurs propres du laplacien magnétique $\mathcal{L}_{h,\mathbf{A}}$ sous la forme (1). De plus, avec l'aide des estimations d'Agmon, nous avons obtenu deux résultats importants. Tout d'abord, nous pouvons montrer que les fonctions propres du laplacien magnétique se désintègrent de façon exponentielle à l'infini et à une vitesse contrôlée par la fonction de phase créée dans la procédure WKB. Deuxièmement, les fonctions propres sont très bien approchées dans un espace à pondération exponentielle.

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Chapter 1

Introduction

The goal isn't to live forever, the goal is to create something that will.

Chuck Palahniuk, Diary

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This thesis is devoted to some aspects of classical and quantum mechanics with magnetic fields. The deep relations between classical and quantum mechanics will only be evoked in this thesis, and, as written by Isaac Newton in his *Philosophiae Naturalis Principia Mathematica*, we will “stand on the shoulders of the Giants”. The reader interested in the history of modern physics and its relations with mathematics might want to consider the following books [11, 18, 21].

1.1 A long time ago

In the 17th century, Isaac Newton (1642-1727) was one of those who founded classical mechanics. The motions of objects were described by means of laws involving the concept of force. For instance, his second law is expressed as an equation allowing to describe the movement of objects, from the trajectory of a bullet to the orbits of the planets. It allowed mathematics to enter many physical fields. Newton’s work was then developed by Lagrange (1736-1813) and Hamilton (1805-1865). The increasing mathematization of physics did not only make the description of physical problems easier, but also opened wide perspectives in mathematics (see [58, 60]).

1.1.1 A question from classical mechanics

We consider a particle moving in a domain Ω in \mathbb{R}^n in the presence of the force $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (we assume that the force depends only on the position of the particle). With Newton’s approach, the particle evolves according to the differential equation

$$m \ddot{q}(t) = F(q(t)), \tag{1.1}$$

where q represents the position of the particle and m is its mass.

Consider the following natural question

“In the presence of the force F , will the particle reach the boundary $\partial\Omega$ in finite time?”

We assume that F is conservative, i.e. F can be written as

$$F = -\nabla V,$$

for some smooth function V . We can find the answer for some special F , through the Hamiltonian equations

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q}, \end{cases}$$

where variable $p \in \mathbb{R}^n$ is the generalized momentum of the particle. The function H is called ***Hamiltonian function*** and has the form

$$H(q, p) = \frac{1}{2m} \|p\|^2 + V(q), \quad (1.2)$$

where $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^n . In the classical sense, the Hamiltonian plays the role of the energy which is the sum of the kinetic energy $\frac{1}{2}m\|\dot{q}\|^2$ and the potential energy V . It is easy to check that the energy is constant along the Hamiltonian trajectories

$$\frac{1}{2m} \|p(t)\|^2 + V(q(t)) = H(q(0), p(0)) =: H_0.$$

Let us come back to the above question. If $V(q) \rightarrow \pm\infty$ as q comes close to the boundary $\partial\Omega$, then no particle can reach the boundary $\partial\Omega$ in finite time. Indeed, if a particle reaches the boundary, it implies that there is some finite time T for which the quantity $V(q(T))$ is very large, even larger than H_0 . This contradicts the conservation of the energy. This tells us that if the scalar potential blows up at the boundary, then the particles never get out of the domain Ω , or even touch the boundary $\partial\Omega$. In other words, the behaviour of the force in a neighborhood of the boundary can keep the particles inside the domain. This thesis will tackle the case when the electric field is replaced by a magnetic field. We will see that the explosion of the magnetic field at the boundary does not necessarily imply the confinement.

This kind of confinement question has somehow quantum analogs. Let us first recall that quantum theory was developed in the early 20th century by numerous physicists. Quantum theory was not created by one individual, but it is the result from experiments and observations. Planck (1858-1947) is considered one of the first persons to have opened the door to the quantum universe when he discovered that the energy of an electromagnetic wave is radiated and absorbed in discrete amounts, in terms of a constant \hbar , the Planck constant. Einstein (1879-1955) went further by giving the idea that the electro-magnetic radiation itself consists of particles, which nowadays are called photons. The quantum mechanics was developed through mathematical models from the matrix theory of Born (1882-1970) and Heisenberg (1901-1976) to the wave mechanics of de Broglie (1892-1987) and Schrödinger (1887-1961).

The equation describing a quantum particle, given by Schrödinger in 1925, is

$$i\hbar \frac{\partial \psi}{\partial t} = \mathcal{H}\psi, \quad (1.3)$$

where

$$\mathcal{H} := -\frac{\hbar^2}{2m} \Delta + V.$$

Here Δ is the Laplacian operator according to the spatial variable. \mathcal{H} is called the Schrödinger operator. The solution of the Schrödinger equation is called the wave function. It does not give us the position of the particle like Newton's equation (1.1). Instead of that, it tells us the probability to find a particle at some place and at some time. Therefore, when a solution is given, we can not say where the particle is. Precisely, for each t , $|\psi(q, t)|^2$ gives the probability law for finding the particle at the point q and at the time t . Thus, we need the condition that

$$\int_{\mathbb{R}^n} |\psi(q, t)|^2 dq = 1.$$

It means that the particle must be somewhere in the whole space. In order to find solutions to the Schrödinger equation (1.3), since \mathcal{H} does not depend on t , we can try to look for solutions in the form

$$\psi(q, t) = \varphi(q)e^{-iE/\hbar t}, \quad (1.4)$$

where $E \in \mathbb{R}$ represents the energy of the particle. Insert this form into (1.3) and then remove $e^{-iE/\hbar t}$ on both sides. We get the eigenvalue equation:

$$\mathcal{H}(\varphi) = E\varphi. \quad (1.5)$$

Therefore, the more information we know about the Schrödinger operator \mathcal{H} , the easier we can solve the equation. One of the properties of the operator \mathcal{H} receiving much attention of mathematicians is its essentially self-adjointness (see Appendix A). Consider the domain Ω . Finding a criterium ensuring essentially self-adjointness shares common features with the classical confinement problem (see [44, 52]). By looking back to the classical case, we could be tempted to think that the condition $V(q) \rightarrow \infty$ as $q \rightarrow \partial\Omega$ is enough to decide the essentially self-adjointness of the Schrödinger operator \mathcal{H} , because no classical particle has the ability to get closed the boundary $\partial\Omega$. Nevertheless, it is not exactly the case: the quantum particle can leak through the infinite barrier of the potential V . This effect is called ‘‘Tunneling effect’’ (see [18]). It turns out that the potential should not only blow up at the boundary, but also blow up at an appropriate rate to make sure that the probability of finding a particle at the boundary is zero. For example, in [44], the essentially self-adjointness of \mathcal{H} is ensured by the behaviour of V near the boundary $\partial\Omega$

$$V(q) \geq \frac{\hbar^2}{2m} \frac{3}{4d(q)^2},$$

with $d(q) = \text{dist}(q, \partial\Omega)$, see [8, 52] for the references.

1.1.2 The WKB method with the electric potential

In this thesis, we will focus on another quantum question. Going back to the Schrödinger equation, after the restriction of looking for the solution in the form (1.4), we obtain an equation

$$-\frac{\hbar^2}{2m}\Delta\varphi + (V - E)\varphi = 0.$$

In order to "solve" this equation, we shall introduce briefly a fundamental method named after Wentzel (1898-1978), Kramers (1894-1952) and Brillouin (1889-1969) which is often used to find an approximate solution for a linear differential equation. This tool is called the WKB method. The reader can consult the books [3, 12, 23] for the following part. The method starts by setting up the solution in the form

$$\varphi(q) = e^{iS(q)/\hbar}a(q, \hbar),$$

for some real-valued function S known as a phase function. By replacing $\phi(q)$ into the time-independent Schrödinger equation, we get the equation

$$\left(\frac{\|\nabla S\|^2}{2m} + V - E\right)e^{iS/\hbar}a - \frac{i\hbar}{2m}(2\nabla S \cdot \nabla a + (\Delta S)a)e^{iS/\hbar} - \frac{\hbar^2}{2m}(\Delta a)e^{iS/\hbar} = 0.$$

Removing $e^{iS/\hbar}$ from this equation, we obtain

$$\left(\frac{\|\nabla S\|^2}{2m} + V - E\right)a - \frac{i\hbar}{2m}(2\nabla S \cdot \nabla a + (\Delta S)a) - \frac{\hbar^2}{2m}\Delta a = 0.$$

Since \hbar is considered small, the first WKB idea is trying to kill the coefficient associated with \hbar^0 , that is

$$\frac{\|\nabla S\|^2}{2m} + V - E = 0. \tag{1.6}$$

This equation is called the Eikonal equation. Assuming that a solution S of (1.6) can be found, we will have

$$(\mathcal{H} - E)\varphi = \mathcal{O}(\hbar).$$

This tells us that φ satisfies the eigenfunction equation up to order \hbar . If a smooth function a solves the equation

$$2\nabla S \cdot \nabla a + (\Delta S)a = 0, \tag{1.7}$$

then we will have a better approximate solution with order \hbar^2 . The equation (1.7) is called the transport equation. This idea leads us to a better approximation by considering the solution a of the equation

$$i(2\nabla S \cdot \nabla a + (\Delta S)a) + \hbar\Delta a = 0, \quad (1.8)$$

in the asymptotic form

$$a(q, \hbar) = \sum_{k=0}^{\infty} a_k(q) \hbar^k. \quad (1.9)$$

Here $(a_k)_{k \in \mathbb{N}}$ is a sequence of smooth complex-valued functions and we say that a function $f : \mathbb{R}^n \times (0, 1) \rightarrow \mathbb{C}$ (\hbar is considered as a parameter ranging in $(0, 1)$) is an asymptotic series $\sum_{k=0}^{\infty} a_k(q) \hbar^k$ if for all $N \in \mathbb{N}$ and for all $\alpha \in \mathbb{N}^n$,

$$\partial_q^\alpha \left(f(q, \hbar) - \sum_{k=0}^N a_k(q) \hbar^k \right) = \mathcal{O}(\hbar^{N+1}) \quad \text{locally uniformly in } q, \quad (1.10)$$

i.e. for all K compact contained in \mathbb{R}^n , there exists a constant $C_{N,\alpha,K}$ such that

$$\left| \partial_q^\alpha \left(f(q, \hbar) - \sum_{k=0}^N a_k(q) \hbar^k \right) \right| \leq C_{N,\alpha,K} \hbar^{N+1} \quad \text{for all } q \in K.$$

Substitute the series form (1.9) of $a(q, \hbar)$ into (1.8) and assign the condition that every terms associated with the orders of \hbar vanish, we have the transport equations

$$\begin{aligned} \hbar^0 : & \quad 2\nabla S \cdot \nabla a_0 + (\Delta S)a_0 = 0, \\ \hbar^1 : & \quad 2\nabla S \cdot \nabla a_1 + (\Delta S)a_1 = i\Delta a_0, \\ \hbar^2 : & \quad 2\nabla S \cdot \nabla a_2 + (\Delta S)a_2 = i\Delta a_1, \\ & \quad \dots\dots\dots \\ \hbar^n : & \quad 2\nabla S \cdot \nabla a_n + (\Delta S)a_n = i\Delta a_{n-1}, \\ & \quad \dots\dots\dots \end{aligned}$$

Since all the above equations have the same structure, hence if we can find smooth solutions a_0 and a_1 to the equations associated with \hbar^1 and \hbar^2 , the other equations will be easy to solve by induction. After we obtain all the a_k , the smooth solution $a(q, \hbar)$ satisfying (1.10) is given by Borel's lemma (see [37, Theorem 1.2.6]). Indeed, let $N \in \mathbb{N}$, Borel's lemma tells us that the polynomial $\sum_{k=1}^N a_k(q) \hbar^k$ coincides with the Taylor series of the function $a(q, \hbar)$ up to order N according to variable \hbar .

Looking at the eikonal equation (1.6), the above WKB construction with a real-valued

phase S is performed in the classically allowed region: $E - V(q) \geq 0$. To deal with the classically forbidden region: $E - V(q) < 0$, we can approximate the solution by the form

$$e^{-\widehat{S}/\hbar} a(q, \hbar),$$

where \widehat{S} is a real-valued smooth function. Essentially, we just replace S by $i\widehat{S}$ in the above WKB construction and we obtain the eikonal equation

$$-\frac{\|\nabla\widehat{S}\|^2}{2m} + V - E = 0, \quad (1.11)$$

and the transport equations

$$\begin{aligned} \hbar^0 : & & 2\nabla\widehat{S} \cdot \nabla a_0 + (\Delta\widehat{S})a_0 &= 0, \\ \hbar^1 : & & 2\nabla\widehat{S} \cdot \nabla a_1 + (\Delta\widehat{S})a_1 &= \Delta a_0, \\ \hbar^2 : & & 2\nabla\widehat{S} \cdot \nabla a_2 + (\Delta\widehat{S})a_2 &= \Delta a_1, \\ & & \dots\dots\dots & \\ \hbar^n : & & 2\nabla\widehat{S} \cdot \nabla a_n + (\Delta\widehat{S})a_n &= \Delta a_{n-1}, \\ & & \dots\dots\dots & \end{aligned}$$

Later, in the magnetic case, we will see that we consider a phase function which is neither purely real nor purely imaginary.

Remark 1.1. In [23, Chapter 2] and [12, Chapter 3], the authors explain how to construct the local WKB approximation for the eigenfunctions of the Schrödinger operator when the potential V has a zero minimum and this minimum is non-degenerate, it means that

$$V(0) = 0, \quad \nabla V(0) = 0, \quad \text{Hess } V(0) > 0.$$

1.1.3 When the magnetic field appears

Above is the story of classical and quantum particles submitted to an electric field. Let us now consider the case of a pure magnetic field. A charged particle submitted to a magnetic field feels the Lorentz force $F = e\dot{q} \times \mathbf{B}$. As we will recall later on, the corresponding Newton equation may be written in a Hamiltonian form. The associated Hamiltonian is

$$H(q, p) = \frac{\|p - \mathbf{A}(q)\|^2}{2},$$

where \mathbf{A} is such that $\text{rot } \mathbf{A} = \mathbf{B}$ (and \mathbf{B} is assumed to be divergence free, according to the Maxwell equations). The aim of this thesis is to explore pure magnetic effects (in two dimensions) on the classical and quantum sides: classical confinement properties

and WKB constructions. In particular, we will see that, contrary to the electric case, an infinite magnetic field at the boundary does not imply the confinement.

On the quantum side, the magnetic Schrödinger operator has the form

$$\mathcal{L}_{h,\mathbf{A}} = (-ih\nabla - \mathbf{A})^2 . \quad (1.12)$$

Here the parameter h will play the role of the Plank constant \hbar and will be assumed to be small. One will see that the WKB analysis of the eigenvalue equation is more subtle than in the electric case. The structure of the system of PDEs will turn out to be more "complex". In particular, the eikonal equation will not be enough to determine the phase S and the phase will be a complex function partially determined by the transport equations.

1.2 Motivations

In this section, we explain the main two "magnetic" motivations of this thesis. The first one is related to classical dynamics. We tackle two well-known physical problems: confinement and scattering. The second one concerns the quantum side. We exhibit general magnetic WKB constructions on a $2D$ Riemannian manifold. In the special case of radial magnetic fields on the Euclidean plane, we prove that our constructions are very good approximations of the true eigenfunctions.

1.2.1 On the classical side

Here are the two problems under consideration:

- (Confinement) Consider a charged particle in the magnetized region Ω . We are curious about the same question which appears in subsection 1.1.1 :

"Will the particle reach the boundary in finite time?"

We will provide a precise answer to this question, depending on the behaviour of the magnetic field at the boundary and on the initial conditions. Our results will improve recent results by Martins in [42]. In particular, we will see that, even if the magnetic field is infinite at the boundary, some trajectories can escape from Ω . This kind of (open) problems is mentioned in [10, Section 1.4].

- (Scattering) Consider a charged particle outside the magnetized region Ω . Before it reaches the region Ω , the trajectory is a straight line. If it enters the region Ω , does the particle escape from it in finite time? And, if it does so, what is the deviation

angle between the ingoing and outgoing directions? We will explicitly answer these questions in the case of radial magnetic fields and when Ω is a disc. In this case, the angular momentum commutes with the Hamiltonian and allows a reduction to a one degree of freedom system.

These questions have intrinsic physical motivations. Their answers allow a better understanding of the classical dynamics of charged particles in the magnetic fields. The description of the classical trajectories has also many applications. In particular, the confinement of charged particles has practical applications in operation of a tokamak, a device using a magnetic field to trap a plasma in a toral shape. Our results also have the potential to expand on three dimensions. In the regime of large magnetic field and small energy, a special treatment of the confinement problem can be done and takes advantage of the near-integrable structure of the Hamiltonian dynamics, either via Birkhoff normal form [51], or KAM theorems [9]. On the contrary, our results here will give more explicit initial conditions and allow regimes where the guiding center motion is not necessarily meaningful.

Furthermore, in the quantum level, the trapped trajectories can be related to the essentially self-adjoint character of the magnetic Laplacian (1.12) (see [10, 44, 45, 52]). We want to know if there are some relations between them or not. The confinement of a charged particle is also a key point to describe the spectrum/resonances of magnetic Laplacians. As far as we know, whereas the description of the magnetic dynamics has allowed to estimate the spectrum of magnetic Laplacians (see [26, 51]), no result seems to exist to estimate their resonances near the real axis. Investigating the trapped trajectories is a necessary step in this direction.

1.2.2 On the quantum side

The investigation of the magnetic Laplacian has now a long story. One of the initial motivations to study the spectral theory of the magnetic Schrödinger operator was the mathematical study of superconductivity, see [17]. The ground-energy is indeed related to the third critical field in the Ginzburg-Landau theory. From this motivation, there are a series of papers devoted to study the spectrum of the magnetic Schrödinger operator at the semi-classical level [26–35, 43, 47, 51]. Among them, the case of a magnetic field having a unique and non-degenerate minimum was investigated. Namely, in [31, Theorem 1.7], Helffer and Kordyukov provide the following asymptotic expansions

$$\forall \ell \in \mathbb{N}, \quad \lambda_\ell(\mathcal{L}_{h,\mathbf{A}}) = \mathbf{B}(p_0)h + \left(2\ell \frac{\sqrt{\det H}}{\mathbf{B}(p_0)} + \frac{(\mathrm{Tr} H^{\frac{1}{2}})^2}{2\mathbf{B}(p_0)} \right) h^2 + o(h^2), \quad (1.13)$$

where p_0 is the minimum point of \mathbf{B} and $H = \frac{1}{2}\text{Hess } \mathbf{B}(p_0)$. In [28], these authors also considered the general case of a surface equipped with a Riemannian metric and also obtain the eigenvalues expansions. With the help of symplectic geometry and pseudo-differential techniques, Raymond and Vũ Ngọc recovered the eigenvalues expansions through a Birkhoff normal form and related them to the magnetic classical dynamics in [51]. As we recall, the WKB method was applied to solve the spectral problem of the electric Laplacian. A natural question concerns the description of the magnetic eigenfunctions: Are there WKB constructions near a magnetic minimum, as for the electric case? This question appears for instance in [24, Section 6.1]. It was then attacked in [5] by Born-Oppenheimer’s idea in a multi-scale framework. Moreover, one could also ask the following question

“Can we recover the spectrum expansion (1.13) by the WKB method?”

In [20], Bonthonneau and Raymond obtained a positive answer when the magnetic field is analytic. In this thesis, we will extend their work by considering general surfaces and also relaxing the analyticity assumptions.

1.3 Results of the thesis

1.3.1 In classical mechanics

We study the dynamics of a charged particle in a smooth bounded domain $\Omega \subset \mathbb{R}^2$ in the presence of a non homogeneous magnetic field \mathbf{B} . We suppose that \mathbf{B} is perpendicular to the plane \mathbb{R}^2 , i.e. $\mathbf{B}(q) = (0, 0, b(q))$ for $q \in \mathbb{R}^2$. This assumption forces particles lying in the \mathbb{R}^2 plane and whose initial velocities are in the plane to stay in this same plane for all time. Since a vector field in \mathbb{R}^3 can be identified with a 2-form via the Euclidean structure, we write the magnetic field as $\mathbf{B} = b(q)dq_1 \wedge dq_2$. Then, if there is a 1-form $\mathbf{A} = A_1dq_1 + A_2dq_2$ such that $d\mathbf{A} = \mathbf{B}$, we can describe the motion of the charged particle through the magnetic Hamiltonian

$$\mathcal{H}(q, p) = \frac{\|p - \mathbf{A}(q)\|^2}{2}, \quad (1.14)$$

where $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^2 . The dynamics is defined by the system

$$\begin{cases} \dot{q} = \partial_p \mathcal{H}(q, p) \\ \dot{p} = -\partial_q \mathcal{H}(q, p). \end{cases} \quad (1.15)$$

We shall always assume that $q \mapsto b(q)$ is locally Lipschitz-continuous, ensuring that the system (1.15) has a unique local maximal solution, thanks to Cauchy-Lipschitz theorem. Then, the vector potential \mathbf{A} will always be chosen to be C^1 -smooth.

We assume that the connected components of $\partial\Omega$ are C^2 -smooth closed curves without self-intersections. Let \mathcal{C} be a connected component of $\partial\Omega$. It can be parametrized by its arc length $\gamma : \mathbb{R}/L\mathbb{Z} \rightarrow \mathcal{C}$ where L is the length of \mathcal{C} .

There exists $\delta > 0$ such that

$$\psi : \begin{cases} (0, \delta) \times \mathbb{R}/L\mathbb{Z} \rightarrow \Omega_{\mathcal{C}}(\delta) \\ (n, s) \mapsto \gamma(s) + nN(s) = q \end{cases}$$

is a smooth diffeomorphism. $N(s)$ denotes the inward pointing normal at $\gamma(s)$ and

$$\Omega_{\mathcal{C}}(\delta) = \{q \in \Omega : d(q, \mathcal{C}) < \delta\}.$$

Note that

$$\mathbf{B} = b(q)dq_1 \wedge dq_2 = b(\psi(n, s))(1 - n\kappa(s))ds \wedge dn, \quad (1.16)$$

where $\kappa(s)$ is the signed curvature of \mathcal{C} at $\gamma(s)$.

After denoting

$$B(n, s) := -b(\psi(n, s))(1 - n\kappa(s)),$$

we can now state our confinement results.

1.3.1.1 General confinement theorems

Our first theorem provides a sufficient condition on \mathbf{B} so that no trajectory can escape from Ω .

Theorem 1.2 (Theorem 2.1 in [46]). *For every connected component \mathcal{C} of $\partial\Omega$, we assume that*

$$\lim_{n \rightarrow 0} \left| \int_n^{\delta_{\mathcal{C}}} \int_0^{L_{\mathcal{C}}} B(\eta, \xi) d\xi d\eta \right| = +\infty, \quad (1.17)$$

and that there exists $M_{\mathcal{C}} \geq 0$ such that, for all $(n, s) \in (0, \delta_{\mathcal{C}}) \times \mathbb{R}/L_{\mathcal{C}}\mathbb{Z}$,

$$\left| B(n, s) - \frac{1}{L_{\mathcal{C}}} \int_0^{L_{\mathcal{C}}} B(n, \xi) d\xi \right| \leq M_{\mathcal{C}}. \quad (1.18)$$

Then the magnetic Hamiltonian dynamics is complete (i.e. no solution of (1.15), starting in Ω , reaches $\partial\Omega$ in finite time).

Of course, given a starting point $q \in \Omega$, only the components \mathcal{C} that bound the connected component of q in Ω need to be taken into account. Actually, there is a more quantitative version of the previous theorem.

Theorem 1.3 (Theorem 2.2 in [46]). *Consider a connected component \mathcal{C} of $\partial\Omega$. Let*

$$K = \sup_{s \in \mathbb{R}/L\mathbb{Z}} |\kappa(s)|, \quad K' = \sup_{s \in \mathbb{R}/L\mathbb{Z}} |\kappa'(s)|.$$

We assume that, for some $\varepsilon \in (0, 1)$, δ satisfies $0 < \delta \leq \varepsilon/K$. We assume that there exists $M \geq 0$ such that, for all $(n, s) \in (0, \delta) \times \mathbb{R}/L\mathbb{Z}$,

$$\left| B(n, s) - \frac{1}{L} \int_0^L B(n, \xi) d\xi \right| \leq M. \quad (1.19)$$

Consider $T > 0$ and $q(t) = \psi(n(t), s(t))$ a trajectory contained in $\Omega_{\mathcal{C}}(\delta)$ for $t \in [0, T]$ with energy H_0 . Let

$$f(n) = -\frac{1}{L} \int_n^{\delta} \int_0^L B(\eta, \xi) d\xi d\eta, \quad (1.20)$$

and assume that

$$\liminf_{n \rightarrow 0} |f(n)| > C(T, q(0), \dot{q}(0)), \quad (1.21)$$

where

$$\begin{aligned} C(T, q(0), \dot{q}(0)) = & \left| \dot{s}(0)[1 - \kappa(s(0))n(0)] + \int_{n(0)}^{\delta} \int_0^L B(\eta, \xi) d\xi d\eta \right| \\ & + \sqrt{2H_0}(1 + \varepsilon) + \left(M\sqrt{2H_0} + \frac{2H_0K'\delta}{1 - \varepsilon} \right) T. \end{aligned}$$

Let g^1 be a continuous and strictly decreasing function such that

$$\lim_{n \rightarrow 0} g(n) = \liminf_{n \rightarrow 0} |f(n)|, \quad g \leq |f| \quad \text{on } [0, \delta].$$

Then, g takes the value $C(T, q(0), \dot{q}(0))$ and, for all $t \in [0, T)$,

$$n(t) > g^{-1}(C(T, q(0), \dot{q}(0))). \quad (1.22)$$

Remark 1.4. Theorems 1.2 and 1.3 are improvements of [42, Theorems 1&2]. They tell us that a particle in Ω never reaches the boundary of Ω . In [42], it is assumed that $\partial_s B$ is integrable:

$$\sup_{s \in \mathcal{C}} \int_0^N |\partial_s B(m, s)| dm < +\infty, \quad (1.23)$$

¹such a function g always exists.

and the question of removing this assumption was explicitly mentioned as important (*op. cit.*, section 3). Our theorems give a partially positive answer to this question, thus allowing for magnetic fields having wilder tangential behaviors.

- Theorem 1.2 generalizes [42, Theorem 1] by replacing the integrability assumption by (1.18). This allows in particular to consider a magnetic field (on the unit disc) of the form

$$B(n, s) = \frac{1}{n} + \sin\left(\frac{\chi(s)}{n}\right),$$

where χ is a smooth function supported in $(-\pi, \pi)$ such that $\chi'(0) \neq 0$ and $\chi(0) = 0$. For this magnetic field, it is easy to check that (1.23) is not satisfied. In fact, the C^∞ smoothness is actually not required; in order to draw Figure 1.1, we took, for simplicity, a small perturbation of $\chi(s) = \arcsin(\sin(s))$.

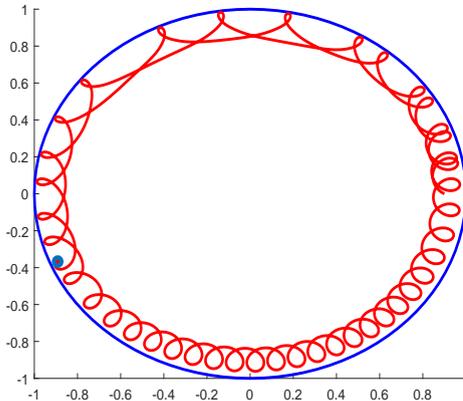


FIGURE 1.1: A trajectory obtained with a magnetic field on the unit disc that is strong near the boundary with a non-integrable tangential derivative:

$$B(q) = \frac{1}{1 - \sqrt{q_1^2 + q_2^2}} + \sin\left(\frac{\arcsin(q_2)}{1 - \sqrt{q_1^2 + q_2^2}}\right) + 5q_1^3 - 7q_2.$$

- An explicit lower bound for the escaping time of a magnetized region is given in [42, Theorem 2] in the case when

$$B(n, s) = \frac{M}{n^\alpha} + h(n, s), \quad \alpha \geq 1, \quad (1.24)$$

where $M \neq 0$ and h is bounded and smooth in $\Omega_C(\delta)$, and so that (1.23) holds. In Figure 1.3, we provide some examples of magnetic fields under this form. Theorem 1.3 not only release the condition integrability s -partial of the function h used in [42, Theorem 2], but also provides an explicit lower bound for magnetic fields that are not in the form (1.24), see Figure 1.2 where the magnetic field changes sign infinitely many times.

- Note that, at the quantum level, a magnetic field (on the unit disc D) like

$$B(n, s) = \frac{2 + \sin s}{n^2}, \quad n = 1 - \sqrt{q_1^2 + q_2^2}, \quad s \in \mathbb{R}/2\pi\mathbb{Z} \quad (1.25)$$

is confining (i.e. the magnetic Laplacian acting on $C_0^\infty(D)$ is essentially self-adjoint), see [10]. Nevertheless, this magnetic field does not satisfy our assumption (1.18) and thus we can not establish the classical confinement with our method.

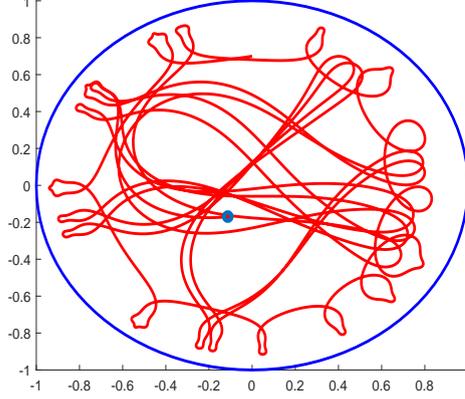


FIGURE 1.2: A trajectory obtained with a magnetic field on the unit disc that strongly oscillates near the boundary:

$$B(q) = \frac{\frac{1}{2} - \sin\left(\frac{1}{1 - \sqrt{q_1^2 + q_2^2}}\right)}{(1 - \sqrt{q_1^2 + q_2^2})^2} + 10q_1 - 2q_1^2 - 10q_2^2.$$

1.3.1.2 Confinement results in the radial case

When $\Omega = D(0, 1)$ and when B is radial, i.e. $B(q) = b(\sqrt{q_1^2 + q_2^2})$, the dynamics is completely integrable, and hence can be entirely described by a one degree of freedom Hamiltonian; concerning the confinement problem, this of course leads to stronger results.

Proposition 1.5 (Proposition 2.3 in [46]). *Let $q(t) = (q_1(t), q_2(t))$ be a solution to (1.15) starting at $t = 0$ from inside the unit disc. If the initial data $(q(0), \dot{q}(0))$ satisfies either **H1** or **H2** below:*

H1:

$$\liminf_{r \rightarrow 1^-} \left| \frac{1}{2\pi} \int_{\|q(0)\| \leq \|q\| \leq r} B(q) dq - \det(q(0), \dot{q}(0)) \right| > \|\dot{q}(0)\|, \quad (1.26)$$

H2:

$$\liminf_{r \rightarrow 1^-} \left| \frac{1}{2\pi} \int_{\|q(0)\| \leq \|q\| \leq r} B(q) dq - \det(q(0), \dot{q}(0)) \right| = \|\dot{q}(0)\|, \quad (1.27)$$

and

$$\limsup_{r \rightarrow 1^-} \frac{\left| \frac{1}{2\pi} \int_{\|q(0)\| \leq \|q\| \leq r} B(q) dq - \det(q(0), \dot{q}(0)) \right| - \|\dot{q}(0)\|}{r-1} < 0, \quad (1.28)$$

then the solution exists for all $t \geq 0$, and there exists $\eta \in [0, 1)$ such that

$$\forall t \geq 0, \quad \|q(t)\| < \eta. \quad (1.29)$$

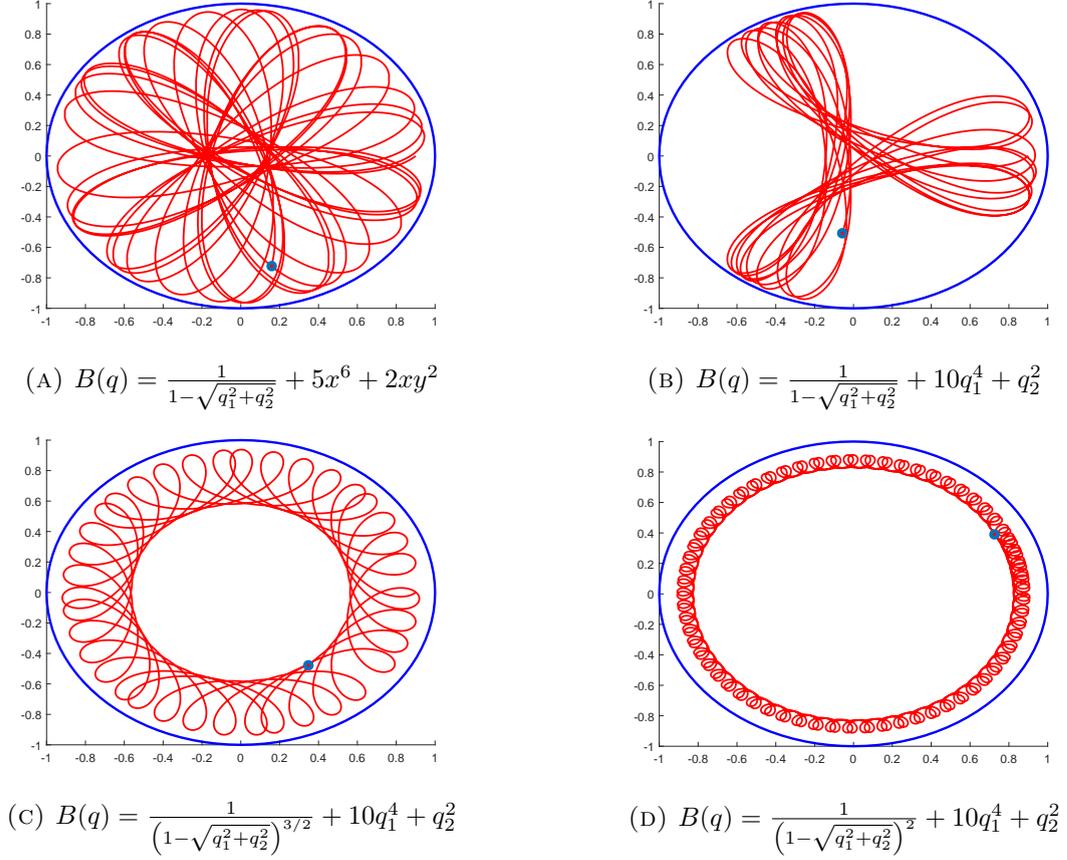


FIGURE 1.3: Some examples of confined charged particles which start at the point $(0.9, 0)$ in the general case.

Example 1.1. *If the given magnetic field satisfies*

$$\lim_{r \rightarrow 1} \left| \int_{D(0,r)} B(q) dq \right| = +\infty,$$

then for arbitrary the particle which start to move inside the unit disk will be kept inside some smaller disk forever, see the Figure 1.4 for magnetic fields whose flux on $D(0, 1)$ is infinite.

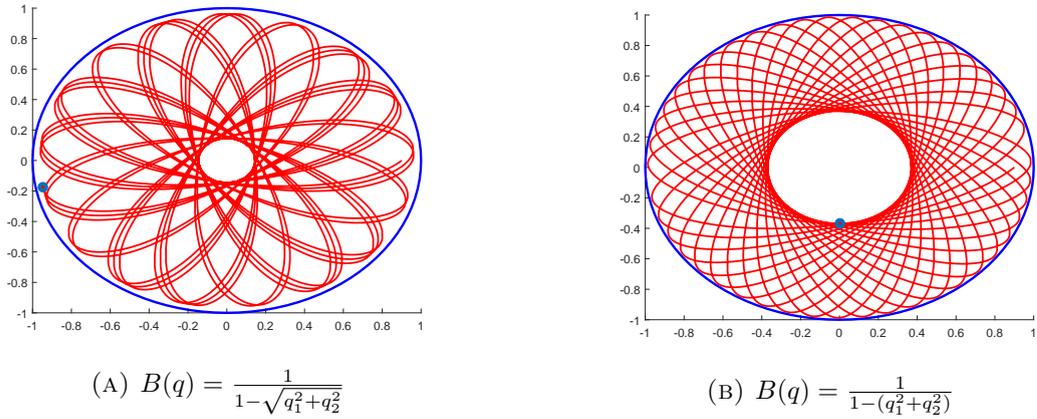


FIGURE 1.4: Some examples of confined charged particles which start at the point $(0.9, 0)$ in the radial case.

One can find situations where none of the hypothesis of Proposition 1.5 hold and the trajectory can be arbitrarily close to the boundary. (see Figure 1.5: this unusual behaviour can be explained by a critical point of the radial Hamiltonian at $r = 1$, see (1.33)).

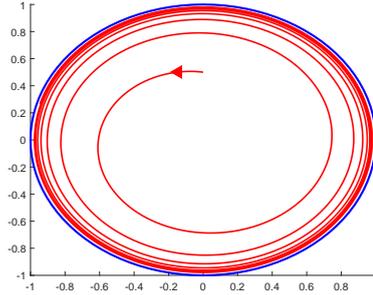


FIGURE 1.5: $B(r) = e^{-r} - \frac{2}{r}$.

If the magnetic field is L^1 -integrable near the boundary of Ω , we can prove that there exist trajectories escaping from Ω in finite time. In particular, even if the magnetic field is infinite at the boundary, the confinement is not ensured.

Proposition 1.6 (Proposition 2.4 in [46]). *When*

$$\limsup_{r \rightarrow 1^-} \left| \int_{D(0,r)} B(q) dq \right| < +\infty, \quad (1.30)$$

there exists a trajectory starting in Ω and reaching the boundary in finite time.

Of course, even under assumption (1.30), some trajectory may be confined, depending on initial conditions (see Figure 1.6 where the simulations are performed with $B(r) = \ln^2(1 - r)$).

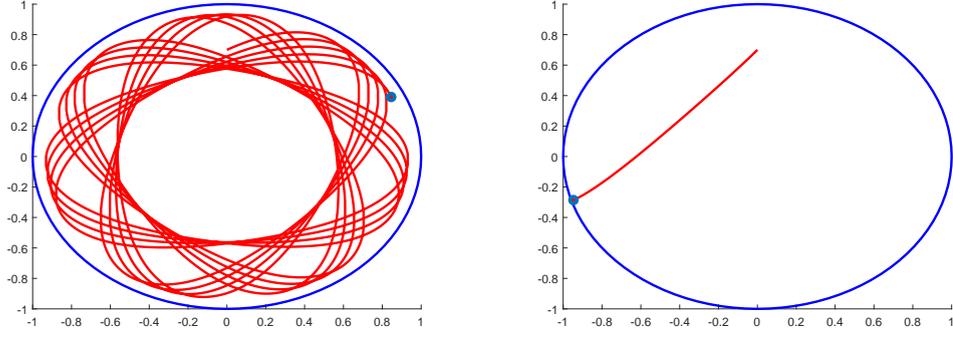


FIGURE 1.6: $B(r) = \ln^2(1 - r)$: the particle is confined or not.

1.3.1.3 Scattering in the radial case

Let us now describe our scattering result in the radial case. We assume that $\mathbf{B}|_{\Omega}$ admits a locally Lipschitz extension in a neighborhood of Ω .

In polar coordinates, we have

$$\mathbf{B} = B(r)rdr \wedge d\theta = d(G(r)d\theta) ,$$

where

$$G(r) = \int_0^r \tau B(\tau) d\tau .$$

Via the symplectic change of coordinates

$$\begin{aligned} \mathbb{R}_+^* \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^2 &\rightarrow (D \setminus \{0\}) \times \mathbb{R}^2 \\ (r, \theta, p_r, p_\theta) &\mapsto \left(r \cos \theta, r \sin \theta, \cos \theta p_r - \frac{\sin \theta}{r} p_\theta, \sin \theta p_r + \frac{\cos \theta}{r} p_\theta \right) = (q, p) , \end{aligned} \quad (1.31)$$

the Hamiltonian becomes

$$\tilde{H}(r, \theta, p_r, p_\theta) = \frac{p_r^2}{2} + \frac{(p_\theta - G(r))^2}{2r^2} . \quad (1.32)$$

In particular, the angular momentum p_θ is constant along the flow and we consider the reduced one dimensional Hamiltonian on $T^*\mathbb{R}_+^*$

$$H(r, p_r) := \frac{p_r^2}{2} + V(r) , \quad V(r) := \frac{(p_\theta - G(r))^2}{2r^2} , \quad (1.33)$$

where $V \in C^1(\mathbb{R}_+^*, \mathbb{R})$. We notice that (see, for example, Lemma 3.1)

$$v_r = p_r , \quad v_\theta = r^{-1}(p_\theta - G(r)) ,$$

where v_r and v_θ are the classical radial and tangential components of the velocity v .

We consider a charged particle with energy H_0 arriving into the disk with velocity v_1 . In particular, $H_0 = \frac{1}{2}\|v_1\|^2$. If the particle escapes from the disk with velocity v_2 (see Figure 1.7), we have $\|v_2\| = \|v_1\|$, and a natural question is to compute the (scattering) angle between these two vectors. Let $\omega \in (-\pi, \pi]$ be the oriented angle between v_1 and v_2 .

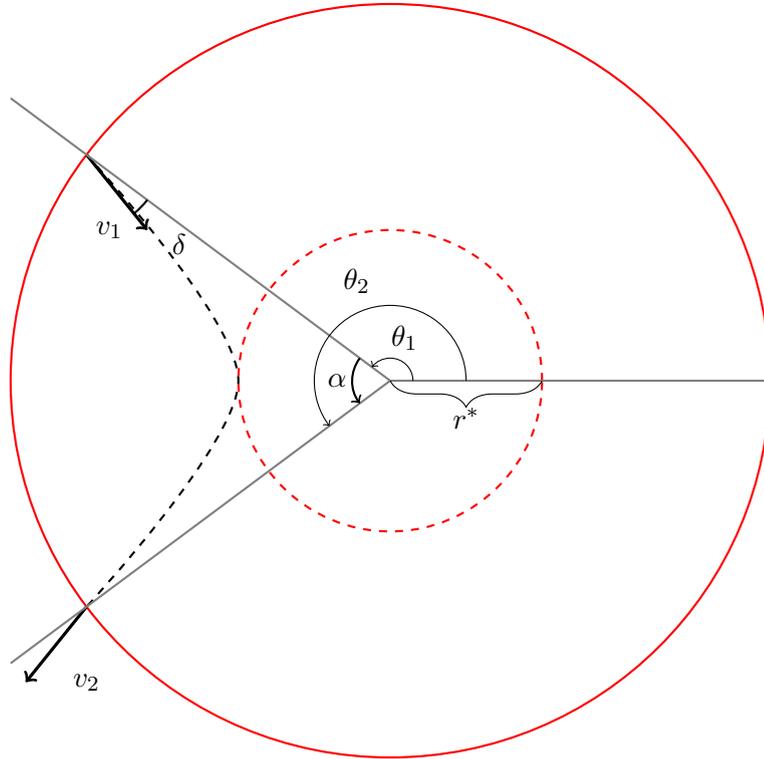


FIGURE 1.7: The scattering arrows.

Theorem 1.7 (Theorem 2.5 in [46]). *Consider a trajectory starting on $\partial\Omega$, with velocity $v_1 \neq 0$ and entering Ω . This means that either $v_r < 0$, or $v_r = 0$ and $\frac{B(1)}{v_\theta} < -1$. We define δ as the angle between the inward pointing normal and v_1 .*

We also assume

- i. *either that the equation $V(r) = H_0$ has a solution for $r \in (0, 1)$ and that the closest solution to 1, denoted by r^* , satisfies $V'(r^*) < 0$.*
- ii. *or, only when $p_\theta = 0$, that the equation $V(r) = H_0$ has no solution.*

Then the trajectory escapes from Ω in finite time with velocity v_2 , and we can compute the scattering angle $\omega \bmod 2\pi$:

i. either the trajectory does not pass through the origin and

$$\omega = \alpha - \pi + 2\delta,$$

where

$$\alpha = 2 \int_{r^*}^1 \frac{p_\theta - G(r)}{r \sqrt{2H_0 r^2 - (p_\theta - G(r))^2}} dr, \quad (1.34)$$

ii. or the trajectory passes through the origin (in this case $p_\theta = 0$) and

$$\omega = \alpha + 2\delta,$$

where

$$\alpha = 2 \int_0^1 \frac{-G(r)}{r \sqrt{2H_0 r^2 - G(r)^2}} dr. \quad (1.35)$$

1.3.2 In semi-classical spectral theory

1.3.2.1 On a compact manifold

Let (M, g) be a two-dimensional compact connected oriented Riemannian manifold with boundary equipped with a metric g . Let \mathbf{A} be a smooth real-valued 1-form defined on M . Since M is two-dimensional, there exists a smooth real-valued function B such that

$$d\mathbf{A} = B d\mathcal{V}_g,$$

in which $d\mathcal{V}_g$ is the Riemannian volume form on M . We call B the magnetic field.

The magnetic Laplacian, denoted by $\mathcal{L}_{h,\mathbf{A}}$, is defined as the Friedrichs extension (the Dirichlet realization) of the operator

$$\begin{cases} H_{h,\mathbf{A}} = (D_{h,\mathbf{A}})^* D_{h,\mathbf{A}} \\ \text{Dom}(H_{h,\mathbf{A}}) = C_c^\infty(M), \end{cases} \quad (1.36)$$

where $D_{h,\mathbf{A}} := -(i\hbar d + \mathbf{A})$ and $(D_{h,\mathbf{A}})^*$ is the $L^2(M)$ -adjoint operator of $D_{h,\mathbf{A}}$. The reader can find the exact definition of these operators in Chapter 4.

From the compactness of M , we will prove that (in Section 4.1.3):

Theorem 1.8. *For each $h > 0$, the magnetic Laplacian $\mathcal{L}_{h,\mathbf{A}}$ has compact resolvent.*

We also give an explicit formula for the operator's domain (in Section 4.1.4):

Theorem 1.9. For each $h > 0$, the domain of the magnetic Laplacian is

$$\text{Dom}(\mathcal{L}_{h,\mathbf{A}}) = H_0^1(M) \cap H^2(M). \quad (1.37)$$

Since $\mathcal{L}_{h,\mathbf{A}}$ is self-adjoint and has compact resolvent, its spectrum is discrete and described by an unbounded increasing sequence:

$$\lambda_0(\mathcal{L}_{h,\mathbf{A}}) \leq \lambda_1(\mathcal{L}_{h,\mathbf{A}}) \leq \dots$$

In order to announce our assumption and main results, we need to recall the definition of the Hessian of a function defined on a manifold at a critical point. In the Riemannian case, we also define its trace and its determinant.

Definition 1.10 (Hessian of a function at a critical point.). The Hessian, denoted by d^2f , of a smooth function $f : M \rightarrow \mathbb{R}$ at a critical point $p \in M$, i.e. $(df)_p = 0$, is defined via

$$\begin{aligned} (d^2f)_p &: T_pM \times T_pM \rightarrow \mathbb{R}, \\ (d^2f)_p(X_0, Y_0) &= X_p(Yf) \quad \text{for all } X_0, Y_0 \in T_pM, \end{aligned}$$

where X, Y are vector fields on M such that $X_p = X_0$ and $Y_p = Y_0$.

Since $(df)_p = 0$, the definition of the Hessian is independent of the choice of the vector fields X, Y extending X_0, Y_0 . Furthermore, it directly implies that $(d^2f)_p$ is a symmetric bilinear form on the vector space T_pM .

Definition 1.11. Let $p \in M$ be the critical point of f , there exists a unique endomorphism \mathcal{H} of T_pM such that

$$(d^2f)_p(U, V) = g_p(\mathcal{H}U, V) \quad \text{for all } U, V \in T_pM.$$

The trace and the determinant of the Hessian of the function f are defined by, respectively,

$$\text{Tr} (d^2f)_p = \text{Tr} \mathcal{H} \quad \text{and} \quad \det (d^2f)_p = \det \mathcal{H}. \quad (1.38)$$

Note that the definition of the trace and the determinant of the Hessian is invariant under diffeomorphisms preserving the Riemannian metric g .

Remark 1.12. Since the Hessian of a function f at a critical point $p \in M$ is symmetric, the linear map \mathcal{H} is also symmetric on T_pM with respect to the inner product g , i.e.

$$g_p(\mathcal{H}U, V) = g_p(U, \mathcal{H}V) \quad \text{for all } U, V \in T_pM.$$

If the Hessian of f at p is positive, we can define the unique positive square-root $\mathcal{H}^{1/2}$ of \mathcal{H} .

Now we state the assumption on the magnetic field B :

Assumption 1.1. *Let $p_0 \in M$, we assume that*

(1) *The magnetic field $B \in C^\infty(M, \mathbb{R})$ has a positive minimum at p_0 , i.e.*

$$B(p_0) = \min_{p \in M} B(p) > 0.$$

(2) *The Hessian of function B at p_0 is positive non-degenerate, i.e.*

$$(d^2 B)_{p_0}(V, V) > 0 \quad \text{for all } V \in T_{p_0}M \setminus \{0\}.$$

Under the same assumptions, in [28, Theorem 1.2], Helffer and Korodyukov provided the expansion of the eigenvalues

$$\forall \ell \in \mathbb{N}, \quad \lambda_\ell(\mathcal{L}_{h, \mathbf{A}}) = b_0 h + \left(2\ell \frac{\sqrt{\det H}}{b_0} + \frac{(\text{Tr } H^{\frac{1}{2}})^2}{2b_0} \right) h^2 + o(h^2), \quad (1.39)$$

where $b_0 = B(p_0)$ and $H = \frac{1}{2}\mathcal{H}$ with \mathcal{H} is the endomorphism given by

$$(d^2 B)_{p_0}(U, V) = g_{p_0}(\mathcal{H}U, V) \quad \text{for all } U, V \in T_{p_0}M.$$

From the expansion (1.39), with the notice that $\det H > 0$, we can state that: for arbitrary large $k \in \mathbb{N}$, there exists $h_0 > 0$ such that for all $h \in (0, h_0)$,

$$\lambda_0(\mathcal{L}_{h, \mathbf{A}}) < \lambda_1(\mathcal{L}_{h, \mathbf{A}}) < \dots < \lambda_k(\mathcal{L}_{h, \mathbf{A}}).$$

In other words, we can choose h small enough such that $(\lambda_j(\mathcal{L}_{h, \mathbf{A}}))_{0 \leq j \leq k}$ are simple eigenvalues. The expansion (1.39) is a strong improvement of the [33, Theorem 7.2] because it obtained the result on a manifold and more asymptotic terms were provided. Our result will show that we can recover this asymptotic expansions by the WKB method and also provide the approximate eigenfunctions. In order to state our main theorems, let us introduce the coordinates on which our works is established:

Definition 1.13 (Isothermal coordinates). Let (M, g) be a Riemannian manifold of two dimensions, a local chart

$$(\Omega, \phi : \Omega \rightarrow \phi(\Omega) \subset \mathbb{R}^2)$$

is called an isothermal coordinate if there exist a function $\eta \in C^\infty(\phi(\Omega))$ such that

$$\phi^* (e^{2\eta} g_0) = g, \quad (1.40)$$

where g_0 is the Euclidean metric on \mathbb{R}^2 and the pull-back metric $\phi^* (e^{2\eta} g_0)$ is a metric on M defined by

$$(\phi^* (e^{2\eta} g_0))_p (U, V) = e^{2\eta} g_0 (d\phi_p U, d\phi_p V),$$

for all $p \in M$ and $U, V \in T_p M$.

The local existence of the isothermal coordinates is also recalled in this thesis (precisely in Theorem 4.6). Our first result is an application of the WKB method:

Theorem 1.14. *Let $p^* \in M$ and assume that the magnetic field B has a local positive minimum at p^* and its Hessian at p^* is positive non-degenerate. Then, there exists an isothermal local chart $(\Omega, \phi : \Omega \rightarrow U \subset \mathbb{R}^2)$ centered at p^* in which the magnetic field has the form*

$$(B \circ \phi^{-1})(q) = b_0 + \alpha q_1^2 + \gamma q_2^2 + \mathcal{O}(\|q\|^3),$$

where $b_0 > 0$, $0 < \alpha \leq \gamma$ and for all $\ell \in \mathbb{N}$, there exist

i) a smooth complex-valued function P defined on Ω satisfying

$$\operatorname{Re}(P \circ \phi^{-1})(q) = \frac{e^{2\eta(0)} b_0}{2} \left(\frac{\sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\gamma}} q_1^2 + \frac{\sqrt{\gamma}}{\sqrt{\alpha} + \sqrt{\gamma}} q_2^2 \right) + \mathcal{O}(\|q\|^3), \quad (1.41)$$

on U , where η is given in Definition 1.13 associated with the isothermal coordinates (U, ϕ) ,

ii) a sequence of smooth complex-valued functions $(U_{\ell,j})_{j \in \mathbb{N}}$ defined on Ω ,

iii) a sequence of real numbers $(\mu_{\ell,j})_{j \in \mathbb{N}}$ with

$$\mu_{\ell,0} = b_0, \quad \mu_{\ell,1} = 2\ell \frac{\sqrt{\det H}}{b_0} + \frac{(\operatorname{Tr} H^{\frac{1}{2}})^2}{2b_0},$$

iv) a sequence of smooth functions $(F_{\ell,j})_{j \in \mathbb{N}}$ defined on Ω which are flat at p^* ,

such that, for all $J \in \mathbb{N}$,

$$e^{P/h} \left(\mathcal{L}_{h,\mathbf{A}} - h \sum_{j=0}^J \mu_{\ell,j} h^j \right) \left(e^{-P/h} \sum_{j=0}^J U_{\ell,j} h^j \right) = \sum_{j=0}^{J+1} h^j F_{\ell,j} + \mathcal{O}(h^{J+2}),$$

locally uniformly on Ω .

The flatness of a function f on the manifold in the above theorem is understood in the meaning

$$\frac{\partial^{m+n} f}{(\partial q_1)^m (\partial q_2)^n}(p^*) = 0, \quad (1.42)$$

in any particular (and hence all) coordinate system, for all $m, n \in \mathbb{N}$.

The road to the proof of this theorem is a long process of applying the WKB method. For details, this road will have to go through Section 4.2, Section 4.3 and Section 4.4. Here is the plan to prove this theorem. Firstly, we use the isothermal coordinates to pull back the magnetic Laplacian to an operator which is defined on a neighborhood of 0 in \mathbb{R}^2 . We use the WKB method to approach the spectral problem of the magnetic Laplacian by means of the formal series. But it is in contrast to the electric case in which the real-valued phase is determined completely by the eikonal equation, the magnetic phase will only be partially determined by the eikonal equation. Up to solving the first transport equation, it will be completely determined. We also provide some tools to solve the transport equations in the formal series analysis. The conclusion is given by Borel's lemma.

Theorem 1.15. *Let $p^* \in M$ and assume that the magnetic field satisfies the assumptions of Theorem 1.14. For any $\ell \in \mathbb{N}$, there exist*

- i) a non-negative function $\widehat{P} \in C_c^\infty(M)$,
- ii) a sequence of functions $(\widehat{U}_{\ell,j})_{j \in \mathbb{N}} \subset C_c^\infty(M)$,

and for any $(\varepsilon, J) \in (0, 1) \times \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$,

$$\|e^{\varepsilon \widehat{P}/h} (\mathcal{L}_{h,\mathbf{A}} - \lambda_{h,\ell}^J) \Upsilon_{h,\ell}^J\|_{L^2(M)} \leq Ch^{J+2}, \quad (1.43)$$

where

$$\lambda_{h,\ell}^J = h \sum_{j=0}^J \mu_{\ell,j} h^j \quad \text{and} \quad \Upsilon_{h,\ell}^J = \sum_{j=0}^J \widehat{U}_{\ell,j} h^j.$$

Hence, it follows that

$$\|(\mathcal{L}_{h,\mathbf{A}} - \lambda_{h,\ell}^J) \Upsilon_{h,\ell}^J\|_{L^2(M)} \leq Ch^{J+2}. \quad (1.44)$$

Theorem 1.15 can be used to prove that there is no odd powers of $h^{\frac{1}{2}}$ in the expansion given by [28, Theorem 1.2]. Furthermore, Theorem 1.15 is an extension of [28, Theorem 2.1] in the case $k = 0$.

For each $\ell \in \mathbb{N}$, let $\Upsilon_{h,\ell}$ be the eigenfunction associated with $\lambda_\ell(\mathcal{L}_{h,\mathbf{A}})$. We introduce the projection into the eigenspace of $\lambda_\ell(\mathcal{L}_{h,\mathbf{A}})$:

$$\begin{aligned} \Pi_\ell : L^2(M) &\rightarrow \text{Dom}(\mathcal{L}_{h,\mathbf{A}}) \\ u &\mapsto \Pi_\ell u = \langle u, \Upsilon_{h,\ell} \rangle_{L^2(M)} \Upsilon_{h,\ell}. \end{aligned}$$

Next theorem will be an application of the Theorem 1.15. We assume that $p^* = p_0$ and let $\Upsilon_{h,\ell}^J$ be the functions which are given in Theorem 1.15. Using spectral arguments, the approximation for the eigenfunctions $\Pi_\ell \Upsilon_{h,\ell}^J$ are provided by the following theorem:

Theorem 1.16. *Assume that the magnetic field satisfies the assumption (1.1). For all $(J, \ell) \in \mathbb{N} \times \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$,*

$$\|\Upsilon_{h,\ell}^J - \Pi_\ell \Upsilon_{h,\ell}^J\|_{L^2(M)} \leq Ch^{J+1}. \quad (1.45)$$

1.3.2.2 On \mathbb{R}^2

We consider the operator

$$H_{h,\mathbf{A}} = (-ih\partial_{q_1} - A_1)^2 + (-ih\partial_{q_1} - A_2)^2,$$

with $\text{Dom}(H_{h,\mathbf{A}}) = C_c^\infty(\mathbb{R}^2)$. Here $\mathbf{A} = A_1 dq_1 + A_2 dq_2$ is the magnetic potential associated with the magnetic field B through

$$d\mathbf{A} = B dq_1 \wedge dq_2.$$

When $\mathbf{A} \in C^1(\mathbb{R}^2)$, this operator is essentially self-adjoint (see [17, Theorem 1.2.2]). There exists a unique self-adjoint extension of $H_{h,\mathbf{A}}$. We call this operator the magnetic Laplacian on \mathbb{R}^2 , and denote it by $\mathcal{L}_{h,\mathbf{A}}$. The magnetic field is assumed radial.

Assumption 1.2. *We assume that the magnetic field B has the form*

$$B(q_1, q_2) = \beta \left(\frac{q_1^2 + q_2^2}{2} \right),$$

where $\beta : \mathbb{R} \rightarrow \mathbb{R}^+$ is a smooth function such that

$$i) \quad \beta(r) > \beta(0) \quad \text{for all } r > 0. \quad (1.46)$$

$$ii) \quad \beta'(0) > 0. \quad (1.47)$$

The assumption (1.46) makes sure that the magnetic field has a unique minimum on \mathbb{R}^2 and the condition (1.47) makes sure that the minimum point is non-degenerate, i.e. the Hessian of B at 0 is a positive definite matrix. Furthermore, we will assume that the magnetic Laplacian has some discrete spectrum which can be described as a sequence

$$\lambda_0(\mathcal{L}_{h,\mathbf{A}}) \leq \lambda_1(\mathcal{L}_{h,\mathbf{A}}) \leq \dots$$

This assumption can be assured when the magnetic field goes to infinity or converges to some number larger than $B(0)$ at infinity (see Theorem 4.3 and 4.4). We now state the main theorems in this part. The first theorem is devoted to the WKB construction for the eigenvalues and the eigenfunctions:

Theorem 1.17. *Assume that the magnetic field satisfies the assumption 1.2. For all $m \in \mathbb{N}$, there exist*

i) a smooth positive function φ defined on $[0, \infty)$ having the formula

$$\varphi(\rho) := \frac{1}{2} \int_0^\rho \int_0^1 \beta(\xi\tau) \, d\xi d\tau, \quad (1.48)$$

ii) a sequence of smooth real-valued functions $(a_{m,j})_{j \in \mathbb{N}}$ defined on $[0, \infty)$ in which $a_{m,0}$ is positive,

iii) a sequence of real numbers $(\mu_{m,j})_{j \in \mathbb{N}}$ with

$$\mu_{m,0} = b_0, \quad \mu_{m,1} = 2m \frac{\sqrt{\det H}}{b_0} + \frac{(\operatorname{Tr} H^{\frac{1}{2}})^2}{2b_0},$$

where $b_0 = B(0)$ and $H = \frac{1}{2} \operatorname{Hess} B(0)$.

We define

$$P(q) := \varphi \left(\frac{\|q\|^2}{2} \right),$$

$$U_{m,j}(q) := a_{m,j} \left(\frac{\|q\|^2}{2} \right),$$

$\theta(q)$: the argument of the complex number $q = q_1 + iq_2$.

Then, for all $J \in \mathbb{N}$,

$$\begin{aligned} e^{P/h} \left(\frac{\|q\|^2}{2} \right)^{\frac{-m}{2}} \left(\mathcal{L}_{h,\mathbf{A}} - h \sum_{j=0}^J \mu_{m,j} h^j \right) & \left(e^{im\theta(q)} \left(\frac{\|q\|^2}{2} \right)^{\frac{m}{2}} e^{-P/h} \sum_{j=0}^J U_{m,j} h^j \right) \\ & = \mathcal{O}(h^{J+2}), \end{aligned}$$

locally uniformly in \mathbb{R}^2 .

Let $K > 0$ be a given number (K is assumed to be large). We define the smooth cut-off function on $[0, \infty)$

$$\chi(\rho) = \begin{cases} 1 & \text{on } [0, K] \\ 0 & \text{on } [K+1, +\infty). \end{cases} \quad (1.49)$$

By multiplying the WKB Ansatz in Theorem 1.17 with this cut-off function, we have the following estimate:

Theorem 1.18. *Assume that the magnetic field satisfies the assumption 1.2. For all $(\varepsilon, m, J) \in (0, 1) \times \mathbb{N} \times \mathbb{N}$, there exist a constant $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$,*

$$\|e^{\varepsilon P/h} (\mathcal{L}_{h,\mathbf{A}} - \lambda_{h,m}^J) \Upsilon_{h,m}^J\|_{L^2(\mathbb{R}^2)} \leq Ch^{J+2}, \quad (1.50)$$

where

$$\begin{aligned} \lambda_{h,m}^J &:= h \sum_{j=0}^J \mu_{m,j} h^j, \\ \Upsilon_{h,m}^J &= \chi\left(\frac{\|\cdot\|^2}{2}\right) e^{im\theta(q)} \left(\frac{\|q\|^2}{2}\right)^{\frac{m}{2}} e^{-P/h} \sum_{j=0}^J U_{m,j} h^j. \end{aligned}$$

Then, it follows that

$$\|(\mathcal{L}_{h,\mathbf{A}} - \lambda_{h,m}^J) \Upsilon_{h,m}^J\|_{L^2(\mathbb{R}^2)} \leq Ch^{J+2}. \quad (1.51)$$

By Agmon estimate, the next result shows us that the eigenfunctions of the magnetic Laplacian decay exponentially as $\|q\| \rightarrow \infty$ at a rate controlled by the function P defined in the above theorem.

Theorem 1.19. *Let $U_{h,m}$ be an eigenfunction associated with $\lambda_m(\mathcal{L}_{h,\mathbf{A}})$. Then, for all $\varepsilon > 0$, there exist $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$,*

$$\|e^{\varepsilon P/h} U_{h,m}\|_{L^2(\mathbb{R}^2)} \leq C \|U_{h,m}\|_{L^2(\mathbb{R}^2)},$$

where P is given in Theorem 1.17.

Let $\Upsilon_{h,m}$ be an eigenfunction associated with $\lambda_m(\mathcal{L}_{h,\mathbf{A}})$. We introduce the projection into the eigenspace of $\lambda_m(\mathcal{L}_{h,\mathbf{A}})$

$$\begin{aligned} \Pi_m &: L^2(\mathbb{R}^2) \rightarrow \text{Dom}(\mathcal{L}_{h,\mathbf{A}}) \\ u &\mapsto \Pi_m u = \langle u, \Upsilon_{h,m} \rangle_{L^2(\mathbb{R}^2)} \Upsilon_{h,m}. \end{aligned}$$

The L^2 -norm estimation for the true eigenfunctions and their Ansatz is given by the following theorem:

Theorem 1.20. *Assume that the magnetic field satisfies assumption 1.2. For all $(J, m) \in \mathbb{N} \times \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$,*

$$\|\Upsilon_{h,m}^J - \Pi_m \Upsilon_{h,m}^J\|_{L^2(\mathbb{R}^2)} \leq Ch^{J+1}. \quad (1.52)$$

With the Agmon estimate, we even have better approximation in an exponentially weighted space.

Theorem 1.21. *Assume that the magnetic field satisfies assumption 1.2. For all $(J, m) \in \mathbb{N} \times \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$,*

$$\left\| e^{\varepsilon P/h} (\Upsilon_{h,m}^J - \Pi_m \Upsilon_{h,m}^J) \right\|_{L^2(\mathbb{R}^2)} \leq Ch^{J+1}. \quad (1.53)$$

Let us sketch the plan to obtain the above results. We first write the magnetic Laplacian $\mathcal{L}_{h,\mathbf{A}}$ in the radial coordinates. It has the form

$$\mathcal{K}_h = -h^2 r^{-2} (r\partial_r)^2 + r^{-2} (-ih\partial_\theta + G(r))^2,$$

where

$$G(r) := \int_0^r \tau \beta \left(\frac{\tau^2}{2} \right) d\tau.$$

By the Fourier decomposition, we can write \mathcal{K}_h in the direct sum of the fibered operators:

$$\mathcal{K}_h = \bigoplus_{m \in \mathbb{Z}} \mathfrak{L}_{h,m}, \quad (1.54)$$

where

$$\mathfrak{L}_{h,m} := -h^2 r^{-2} (r\partial_r)^2 + r^{-2} (hm - G(r))^2. \quad (1.55)$$

Under the assumption 1.46, we can prove that these fibered operators $\mathfrak{L}_{h,m}$ have compact resolvents (see Theorem 5.5). Therefore, the spectrum of each operator $\mathfrak{L}_{h,m}$ is discrete. By changing variable $\rho = \frac{r^2}{2}$, we obtain the equivalent operator

$$\mathcal{N}_{h,m} = -2h^2 \partial_\rho (\rho \partial_\rho) + \frac{(hm - a(\rho))^2}{2\rho}.$$

By considering the rescaled operator of $\mathcal{N}_{h,m}$:

$$\mathcal{M}_{h,m} = -2h \partial_t t \partial_t + \frac{(hm - a(ht))^2}{2ht},$$

and studying its ground-state energy (Subsection 5.2.2), we can show that

$$\lambda_0(\mathcal{M}_{h,m}) = hb_0 + h^2 \left(2m \frac{\sqrt{\det H}}{b_0} + \frac{(\operatorname{Tr} H^{\frac{1}{2}})^2}{2b_0} \right) + o(h^2),$$

where $H = \frac{1}{2}\operatorname{Hess}B(0)$. This tells us that the m -th eigenvalue of the magnetic Laplacian is exactly the first eigenvalue of the m -th fibered operator (see Theorem 5.14). At this stage, we only need to apply WKB method for the operator $\mathcal{N}_{h,m}$ to obtain the eigenvalue expansion and the WKB Ansatz. Then they are transferred to the original magnetic Laplacian by the unitary transformations.

1.4 Organization of the thesis

Chapter 2 is devoted to lead the reader into the classical mechanics with magnetic field. It opens with the introduction to the theory of symplectic geometry and Hamiltonian dynamics. It allows us to define the magnetic Hamiltonian dynamics and provides us the tools to solve the classical problems in the Chapter 3. This chapter ends with the relation between Newton's mechanics and Hamiltonian's mechanics in the presence of the magnetic field.

Chapter 3 is devoted to prove the results in classical mechanics. This is the work of the article [46] in collaboration with Nicolas Raymond and San Vũ Ngọc.

Chapter 4 starts with the introduction of the magnetic Laplacian on a Riemannian manifold. The domain of the operator and compact resolvent property are studied on the compact manifold. The picture of the operator and its spectrum on \mathbb{R}^2 are also considered. The main object of this chapter is performing WKB analysis for investigating the spectral problem of the magnetic Laplacian. Thanks to the spectral theorem, we estimated efficiently the true eigenfunctions and the approximate eigenfunctions locally near the minimum point.

Chapter 5 demonstrates the results in Subsection 1.3.2.2. We start the chapter by writing the magnetic Laplacian in the radial coordinates. After the Fourier decomposition, we investigate the spectrum of the fibered operators. In a next step, we describe the relation between spectrum of the fibered operators and the original magnetic Laplacian. It allows us to construct the WKB eigenvalues and eigenfunctions of the magnetic Laplacian on \mathbb{R}^2 . Finally, we obtain the approximation of the eigenfunctions in a convenient non-exponentially and exponentially weighted space.

Chapter 2

About classical mechanics with magnetic field

Free curiosity has greater power to
stimulate learning than rigorous
coercion.

St. Augustine, Confessions

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This chapter is devoted to the dynamics of a charged particle submitted to the magnetic field. This story was first told by Störmer, who was attracted by the aurora phenomenon and tried to bring this phenomenon into mathematics (see [14]). With the motivation for interpreting the trajectory of the charged particles in the earth’s magnetic field, he was a pioneer person attempted to approach the trapping trajectory problems. His and other’s understanding of the trapped orbits within the earth’s field played essential role in application of the Van Allen radiation [7, 13]. In [41], Littlejohn drew the attention to the importance of Hamiltonian methods in such problems. Symplectic methods (and normal forms) have shown their power in [51] where the authors study the long time dynamics at low energy. There is a huge amount of literature using the Hamiltonian method to investigate the magnetic dynamics, for example [7, 9, 41, 42, 51, 62].

In this chapter, we briefly explain how to write the famous Newton equation in the presence of the Lorentz under a Hamiltonian form. To do so, we recall the basics of symplectic geometry in \mathbb{R}^{2n} . In particular, we recall how to lift a (space) change of variables to a symplectic transformation. This will be needed when describing the motion of a charged particle approaching the boundary (a neighborhood of which being described through tubular coordinates).

The reader can consult the books [16, 36] for the following section and an account of symplectic techniques can be found in [64, Chapter 2]. After that, we will travel to the world of magnetic field with Hamiltonian tools in our hands. Therefore, the reader can skip this section at the first reading.

2.1 Reminder of symplectic geometry

First of all, we introduce the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$ equipped with a bilinear form ω_0 defined as

$$\omega_0(u, v) = \langle Ju, v \rangle \quad \text{for all } u, v \in \mathbb{R}^{2n}, \quad (2.1)$$

where the bracket denotes the Euclidean inner product in \mathbb{R}^{2n} , and J is a $2n \times 2n$ matrix defined by

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

with I_n is the unit matrix in linear algebra.

Since $J^{-1} = J^T = -J$, it can be seen that the bilinear form ω_0 satisfies the following properties:

i) **Anti-symmetric** : $\omega_0(u, v) = -\omega_0(v, u)$ for all $u, v \in \mathbb{R}^{2n}$.

ii) **Non-degenerate**: Let $u \in \mathbb{R}^{2n}$, if

$$\omega_0(u, v) = 0 \quad \text{for all } v \in \mathbb{R}^{2n},$$

then $u = 0$.

Furthermore, ω_0 is usually considered as a 2-form on \mathbb{R}^{2n} in practice, see [1, Chapter 7] for basic knowledge of differential forms. Let us recall that, in the standard coordinates (x_1, \dots, x_{2n}) , for $i, j \in \{1, \dots, 2n\}$ the 1-form dx_i is defined on \mathbb{R}^{2n} as

$$dx_i(u) = u_i \quad \text{for all } u \in \mathbb{R}^{2n},$$

and the 2-form $dx_i \wedge dx_j$ is the exterior product of dx_i and dx_j defined as

$$dx_i \wedge dx_j(u, v) = \begin{vmatrix} dx_i(u) & dx_j(u) \\ dx_i(v) & dx_j(v) \end{vmatrix} = u_i v_j - u_j v_i \quad \text{for all } u, v \in \mathbb{R}^{2n}.$$

By rewriting the standard coordinates as $(q_1, \dots, q_n, p_1, \dots, p_n)$, we can represent ω_0 as a 2-form $\omega_0 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$:

$$\omega_0 = \sum_{k=1}^n dp_k \wedge dq_k.$$

Furthermore, we can observe that

$$\omega_0 = d\sigma \quad \text{for } \sigma = \sum_{k=1}^n p_k dq_k.$$

Since $d^2 = 0$, it implies that

$$d\omega_0 = 0. \tag{2.2}$$

Thus, we also say that ω_0 is **closed**. Since ω_0 is bilinear form, we can naturally define a linear mapping

$$\begin{aligned} T : \quad \mathbb{R}^{2n} &\rightarrow (\mathbb{R}^{2n})^* \\ u &\mapsto \omega_0(u, \cdot). \end{aligned}$$

The non-degeneracy of ω_0 is equivalent to the fact that T is an injective. Since \mathbb{R}^{2n} and $(\mathbb{R}^{2n})^*$ has the same dimensions, T must be an isomorphism.

2.1.1 Hamiltonian vector field on \mathbb{R}^{2n}

Let U be an open set in \mathbb{R}^{2n} and $H : U \rightarrow \mathbb{R}$ be a smooth function. Thanks to the non-degeneracy of ω_0 , for a smooth function, we can define a unique vector field X_H on U such that, for each $x \in U$

$$\omega_0(X_H(x), Y) = -dH(x)Y \quad \text{for all } Y \in \mathbb{R}^{2n}, \quad (2.3)$$

where $dH(x) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is differential map of H defined by

$$dH(x)Y = \langle \nabla H(x), Y \rangle \quad \text{for all } Y \in \mathbb{R}^{2n}.$$

We call X_H is a **Hamiltonian vector field** associated to the function H on \mathbb{R}^{2n} , and we also often call H **Hamiltonian**. Now, we can define the **flow of the Hamiltonian vector field** X_H , we denote it by $\varphi_t(x)$, that is the solution of the following ordinary differential equations (ODE)

$$\begin{cases} \frac{d}{dt}\varphi_t(x) = X_H(\varphi_t(x)), \\ \varphi_0(x) = x, \quad x \in \mathbb{R}^{2n}. \end{cases} \quad (2.4)$$

It means that for each $x \in \mathbb{R}^{2n}$, the curve $c(t) := \varphi_t(x)$ solves the Cauchy initial value problem for the initial condition $c(0) = x$. Since the Hamiltonian H belongs to the class $C^\infty(\mathbb{R}^{2n}, \mathbb{R})$, the below theorem of existence and uniqueness of $\varphi_t(x)$ is ensured by the Theorem of Cauchy-Lipschitz.

Theorem 2.1. *For all $x \in \mathbb{R}^{2n}$, the (ODE) (2.4) has a unique smooth solution*

$$\varphi_t(x) : J(x) \rightarrow \mathbb{R}^{2n},$$

where $J(x)$ is the maximal interval containing 0 on which $\varphi_x(t)$ is defined.

Furthermore, $\varphi_t(x)$ possesses following properties

- i) $\varphi_0(x) = x$ for all $x \in \mathbb{R}^{2n}$.
- ii) $\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$ whenever both sides are defined.
- iii) For each time $t \in \mathbb{R}$, the mapping $\varphi_t : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a diffeomorphism.

To see the connection between Hamiltonian defined above on symplectic space and well-known Hamiltonian equations of a system of n degree, we write condition (2.3) as

$$\langle JX_H(x), Y \rangle = -\langle \nabla H(x), Y \rangle.$$

Therefore, thanks to the equality $J^2 = -J$, it results that

$$X_H(x) = J\nabla H(x) \quad \text{for all } x \in \mathbb{R}^{2n}. \quad (2.5)$$

If we demonstrate $c(t)$ as $(q(t), p(t))$, we can rewrite the equation (2.4) explicitly in the form

$$\begin{cases} \dot{q}_i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t)) \\ \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}(q(t), p(t)) \end{cases} \quad \text{for } i = 1, \dots, n. \quad (2.6)$$

These are called ***Hamilton's equations***. They are equations of motion of the system expressed as $2n$ first-order differential equations. Their nice property is that the derivatives with respect to time are isolated on the left-hand sides of the equation. The following theorem tells us that H is a conserved quantity on its flows, i.e. $H(q(t), p(t))$ is independent of t for each solution $(q(t), p(t))$ of Hamilton's equations.

Theorem 2.2. *Let $(q(t), p(t))$ be the solution of equations (2.6) with initial condition $(q_0, p_0) = (q(0), p(0))$, then*

$$H(q(t), p(t)) = H(q_0, p_0), \quad (2.7)$$

for all $t \in I$, where I is some interval around 0.

Proof. We consider the derivative of $H(q(t), p(t))$ according to variable t :

$$\begin{aligned} \frac{dH}{dt}(q(t), p(t)) &= \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i} \dot{q}_i(t) + \frac{\partial H}{\partial p_i} \dot{p}_i(t) \right) \\ &= \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \\ &= 0. \end{aligned}$$

Thus, the function $H(q(t), p(t))$ is constant and satisfies (2.7). □

Conserved quantities are very useful in that they keep the solutions $(q(t), p(t))$ staying in the level surfaces of any conserved quantity. For example, suppose that we are working with a particle moving in \mathbb{R}^2 and we use Hamilton's equations to investigate its motion. We need to consider the problem with four unknowns. But, if we can find one other conserved quantity F instead of H , then rather than looking for the solutions in four dimensional phase space, we look for them inside the joint level sets of H and F :

$$H(q, p) = H_0, \quad F(q, p) = F_0,$$

for some constant H_0 and F_0 . These joint level sets are two-dimensional instead of four-dimensional, so using conserved quantities greatly simplifies the problem.

The thing happens here is that if we change the coordinates from (q, p) to (\tilde{q}, \tilde{p}) , we wonder that the system (2.6) is still true in the new coordinates or not? Next section will introduce the change of coordinate which preserves the Hamiltonian property.

2.1.2 Symplectic transformation

Let us recall the definition of pull-back of two-form first, and then introduce the definition of a symplectic transformation. Let U, V be open sets in \mathbb{R}^{2n} and ω be the 2-form on U , the pull-back of ω by a smooth function $\varphi : U \rightarrow V$ is defined by

$$(\varphi^*\omega)_x(u, v) = \omega_{\varphi(x)}((d\varphi)_x u, (d\varphi)_x v),$$

for $x \in U$ and for all $u, v \in \mathbb{R}^{2n}$, where $(d\varphi)_x : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is the differential of map φ at x represented by the Jacobian matrix of φ at x .

Definition 2.3. The diffeomorphism $\varphi : U \rightarrow V$ is called *symplectic* if

$$\varphi^*\omega_0 = \omega_0. \quad (2.8)$$

In view of the definition of ω_0 , a symplectic property is equivalent to

$$\langle J(d\varphi)_x u, (d\varphi)_x v \rangle = \langle Ju, v \rangle \quad \text{for all } u, v \in \mathbb{R}^{2n},$$

or in the matrix language, a diffeomorphism φ is symplectic if and only if

$$(d\varphi)_x^T J (d\varphi)_x = J \quad \text{for all } x \in U. \quad (2.9)$$

Example 2.1. On \mathbb{R}^{2n} , the linear map φ defined by

$$\varphi(x) = Jx \quad \text{for } x \in \mathbb{R}^{2n}$$

is symplectic, because

$$(d\varphi)_x^T J (d\varphi)_x = J^T J J = J.$$

Example 2.2. On \mathbb{R}^2 , the linear map φ defined by

$$\varphi(x_1, x_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

with $a, b, c, d \in \mathbb{R}$ satisfy $ad - bc = 1$. Then φ is symplectic, because

$$(\mathrm{d}\varphi)_x^T J (\mathrm{d}\varphi)_x = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & ad - bc \\ bc - ad & 0 \end{pmatrix} = J.$$

The above examples can be generalized by the following theorem

Theorem 2.4 (Linear symplectic mapping). *Let $\varphi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be a linear mapping*

$$\varphi(q, p) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{for } (q, p) \in \mathbb{R}^n \times \mathbb{R}^n, \quad (2.10)$$

where A, B, C, D are $n \times n$ matrices.

Then, φ is symplectic if and only if $A^T C$ and $B^T D$ are symmetric and $A^T D - C^T B = I_n$.

Proof. We just need check that φ is symplectic if and only if

$$\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \quad (2.11)$$

This is equivalent to the equality

$$\begin{pmatrix} A^T C - C^T A & A^T D - C^T B \\ B^T C - D^T A & B^T D - D^T B \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

This is the argument stating in the theorem. Note that the diffeomorphism of mapping φ in the equation (2.11) when compute the determinant of two sides. \square

In general, to check that a mapping is symplectic, we often use next theorem.

Theorem 2.5. *Let U_1, V_1 be open sets in \mathbb{R}^n . Assume that a mapping φ defined by*

$$\begin{aligned} \varphi : U_1 \times \mathbb{R}^n &\rightarrow V_1 \times \mathbb{R}^n \\ (q, p) &\mapsto \varphi(q, p) = (Q(q, p), P(q, p)). \end{aligned}$$

We write the Jacobian matrix of φ in the form of block matrices

$$(\mathrm{d}\varphi)_{(q,p)} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}, \quad (2.12)$$

where $\frac{\partial Q}{\partial q}, \frac{\partial Q}{\partial p}, \frac{\partial P}{\partial q}, \frac{\partial P}{\partial p}$ are the $n \times n$ matrices.
Then, φ is symplectic if and only if

$$\left(\frac{\partial Q}{\partial q}\right)^T \frac{\partial P}{\partial q} \text{ and } \left(\frac{\partial Q}{\partial p}\right)^T \frac{\partial P}{\partial p} \text{ are symmetric,} \quad (2.13)$$

and

$$\left(\frac{\partial Q}{\partial q}\right)^T \frac{\partial P}{\partial p} - \left(\frac{\partial P}{\partial q}\right)^T \frac{\partial Q}{\partial p} = I_n \quad (2.14)$$

at every point $(q, p) \in U_1 \times \mathbb{R}^n$.

Proof. The proof is the same as the last Theorem Linear symplectic mapping, we just replace the matrices A, B, C, D with, respectively, $\frac{\partial Q}{\partial q}, \frac{\partial Q}{\partial p}, \frac{\partial P}{\partial q}, \frac{\partial P}{\partial p}$. Note that, if we have

$$(\mathrm{d}\varphi)_x^T J(\mathrm{d}\varphi)_x = J \quad \text{for } x = (q, p),$$

it implies that $\det((\mathrm{d}\varphi)_x) \neq 0$ at every point $x \in U_1 \times \mathbb{R}^n$, the Inverse Theorem help to confirm the diffeomorphism of φ . \square

In next theorem, we represent the construction of a symplectic transformation on \mathbb{R}^{2n} from a non-linear diffeomorphism on \mathbb{R}^n . We call this technique as **symplectic lifting** (see [64, Chapter 2]) and it will be used in the thesis to construct the Hamiltonian in tubular coordinate and in radial coordinate later.

Theorem 2.6. *Let U_1, V_1 be open subsets in \mathbb{R}^n . Assume that*

$$\psi : U_1 \rightarrow V_1,$$

be a diffeomorphism.

Then, the mapping Ψ defined by

$$\Psi : U_1 \times \mathbb{R}^n \rightarrow V_1 \times \mathbb{R}^n$$

$$(q, p) \mapsto (\psi(q), \eta(q, p)) = \left(\psi(q), \left[\left(\frac{\partial \psi}{\partial q} \right)^{-1} \right]^T p \right)$$

is symplectic, where $\frac{\partial \psi}{\partial q}$ indicates the Jacobian matrix of ψ .

Proof. Apply the Theorem 2.5, we need to verify that

$$\left(\frac{\partial \psi}{\partial q}\right)^T \frac{\partial \eta}{\partial p} = I_n \quad \text{and} \quad \left(\frac{\partial \psi}{\partial q}\right)^T \frac{\partial \eta}{\partial q} = \left(\frac{\partial \eta}{\partial q}\right)^T \frac{\partial \psi}{\partial q}.$$

The first of them is easily satisfied by the definition η . To check the other, we need to look at each element of product matrix of each side.

We denote $n \times n$ matrix $M = (m_{ij})_{1 \leq i, j \leq n} := \left[\left(\frac{\partial \psi}{\partial q} \right)^{-1} \right]^T$. Notice that matrix M depends only on q and $\eta(q, p) = Mp$. For $i, j \in \{1, \dots, n\}$, we have

$$\begin{aligned} \left[\left(\frac{\partial \psi}{\partial q} \right)^T \frac{\partial \eta}{\partial q} \right]_{ij} &= \sum_{k=1}^n \left(\frac{\partial \psi}{\partial q} \right)_{ki} \left(\frac{\partial \eta}{\partial q} \right)_{kj} \\ &= \sum_{k=1}^n \frac{\partial \psi_k}{\partial q_i} \left(\sum_{\ell=1}^n \frac{\partial m_{k\ell}}{\partial q_j} p_\ell \right) \\ &= \sum_{\ell=1}^n \left(\sum_{k=1}^n \frac{\partial \psi_k}{\partial q_i} \frac{\partial m_{k\ell}}{\partial q_j} \right) p_\ell. \end{aligned}$$

Since $M^T \frac{\partial \psi}{\partial q} = I_n$, it implies that

$$\sum_{k=1}^n m_{k\ell} \frac{\partial \psi_k}{\partial q_i} = \delta_{\ell i} \quad \text{for all } \ell, i \in \{1, \dots, n\}.$$

Take $\frac{\partial}{\partial q_j}$ two sides, we get

$$\sum_{k=1}^n \frac{\partial \psi_k}{\partial q_i} \frac{\partial m_{k\ell}}{\partial q_j} = - \sum_{k=1}^n \frac{\partial^2 \psi_k}{\partial q_j \partial q_i} m_{k\ell}.$$

Thus,

$$\left[\left(\frac{\partial \psi}{\partial q} \right)^T \frac{\partial \eta}{\partial q} \right]_{ij} = - \sum_{\ell=1}^n \left(\sum_{k=1}^n \frac{\partial^2 \psi_k}{\partial q_j \partial q_i} m_{k\ell} \right) p_\ell.$$

With the same steps, we also have

$$\left[\left(\frac{\partial \eta}{\partial q} \right)^T \frac{\partial \psi}{\partial q} \right]_{ij} = - \sum_{\ell=1}^n \left(\sum_{k=1}^n \frac{\partial^2 \psi_k}{\partial q_i \partial q_j} m_{k\ell} \right) p_\ell.$$

Then clearly $\left(\frac{\partial \psi}{\partial q} \right)^T \frac{\partial \eta}{\partial q}$ is symmetric. □

2.1.3 Hamiltonian and symplectic transformation

After travelling through procedures for creating symplectomorphism linear and non-linear, we explain the relation between Hamiltonian and symplectic transformation in the following lines. Next theorem again provides us a way to get a symplectic mapping from the flow of Hamiltonian vector field:

Theorem 2.7. *A flow of the Hamiltonian vector field X_H generated by H , we call φ_t , preserves the symplectic structure:*

$$(\varphi_t)^* \omega_0 = \omega_0,$$

and is, therefore, a symplectic transformation.

Proof. From the Theorem 2.1, φ^t is a diffeomorphism for each t . We only need to check the symplectic formula. Since

$$\begin{aligned} \frac{d}{dt}(\varphi_t)^* \omega_0 &= \frac{d}{ds}(\varphi^{t+s})^*|_{s=0} \omega_0 \\ &= \frac{d}{ds}(\varphi^t)^*(\varphi^s)^*|_{s=0} \omega_0 \\ &= (\varphi^t)^* \mathfrak{L}_{X_H} \omega_0. \end{aligned}$$

Here $\mathfrak{L}_{X_H} \omega_0$ is the Lie derivative of ω_0 . From the Cartan's formula and (2.2), we have

$$\begin{aligned} \mathfrak{L}_{X_H} \omega_0 &= (i_{X_H} d + di_{X_H}) \omega_0 \\ &= i_{X_H} d\omega_0 + di_{X_H} \omega_0 \\ &= -d(dH) = 0. \end{aligned}$$

It tells us that $(\varphi_t)^* \omega_0$ does not depend on t , thus

$$(\varphi_t)^* \omega_0 = (\varphi_t)^*|_{t=0} \omega_0 = \omega_0.$$

□

As discussed above, with a smooth function H on an open set U of \mathbb{R}^{2n} , we can define naturally the Hamiltonian vector field X_H on U and then establish its flow through the ODE

$$\dot{u}_t = J\nabla H(u_t) \quad \text{on } U.$$

Let $V \subset \mathbb{R}^{2n}$ be an open set and $\Psi : U \rightarrow V$ be a symplectic transformation. We perform a change of variable

$$K = H \circ \Psi^{-1} : V \rightarrow \mathbb{R}^{2n},$$

and define the curve $v_t := \Psi(u_t)$ in V . By computation, we will see that symplectic transformations can keep the Hamilton form. Indeed, we reckon

$$\nabla H(u_t) = (d\Psi)_{u_t}^T \nabla K(v_t),$$

thus, since Ψ is symplectic and (2.9), we have

$$\dot{v}_t = (d\Psi)_{u_t} \dot{u}_t = (d\Psi)_{u_t} J \nabla H(u_t) = (d\Psi)_{u_t} J (d\Psi)_{u_t}^T \nabla K(v_t) = J \nabla K(v_t).$$

Therefore, we have the following result:

Theorem 2.8. *Any symplectic transformation preserves Hamilton's equations.*

2.2 Magnetic Hamiltonian mechanics

2.2.1 Newton's law with magnetic field

2.2.1.1 What is the magnetic field?

It seems that the word "magnetic" comes to our mind through the image of a magnet. In our real life, we can see magnets everywhere such as in a compass, in a refrigerator door with magnetic stickers and our earth is also a huge magnet. While at school, we use iron chips to make magnetic field lines emerge for observation. In mathematics, we describe these quantities by the vectors. The length of a vector at one point tells us the magnitude of the magnetic field at that point, and the direction of this vector tells us the direction of the magnetic field at that point. Similarly, the magnetic field around the magnet will be represented by a vector field. We will denote this vector field as $\vec{\mathbf{B}}$.

2.2.1.2 Magnetic Newton's equation

Now, we consider a particle of charge e and mass m put in the domain affected by the magnetic field $\vec{\mathbf{B}}$. We want to know the motion of this particle in the space. The second Newton's law gives us the equation of motion.

Let $q(t) := (q_1(t), q_2(t), q_3(t)) \in \mathbb{R}^3$ denote the particle's position at time t . The particle's velocity and acceleration are, respectively,

$$v(t) := \dot{q}(t), \quad a(t) := \ddot{q}(t) = \dot{v}(t).$$

We assume that the net force applied on the particle is only the magnetic force, called Lorentz force, and it has form

$$F = ev \times \vec{\mathbf{B}}.$$

Here e is a constant which indicates the electric charge of the particle. This Lorentz force depends on the velocity of the particle and perpendicular to v and \mathbf{B} . Under the action of the Lorentz force, the Newton's second law ($F = ma$) becomes

$$m\ddot{q} = e\dot{q} \times \vec{\mathbf{B}}. \quad (2.15)$$

To simplify our discussion, we assume that $e = 1$ and $m = 1$. Assume that $\vec{\mathbf{B}} =$

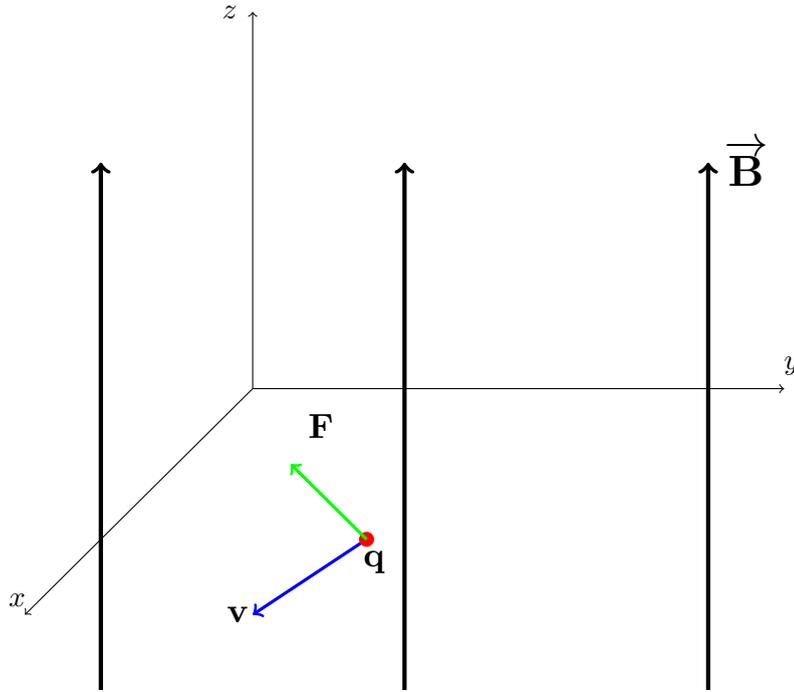


FIGURE 2.1: A charged particle moving in a magnetic field that points in the z direction.

(B_1, B_2, B_3) , we can rewrite (2.15) as

$$\begin{cases} \ddot{q}_1 = B_3(q)\dot{q}_2 - B_2(q)\dot{q}_3 \\ \ddot{q}_2 = B_1(q)\dot{q}_3 - B_3(q)\dot{q}_1 \\ \ddot{q}_3 = B_2(q)\dot{q}_1 - B_1(q)\dot{q}_2 \end{cases} \quad (2.16)$$

Suppose that $\vec{\mathbf{B}}$ is a C^1 vector field, the Cauchy-Lipschitz theorem tells us that the system (2.16) has a unique local maximal solution for each pair of initial conditions at $t = 0$:

$$\begin{cases} q(0) = q_0 \\ \dot{q}(0) = v_0 \end{cases}.$$

Definition 2.9. A solution $q(t)$ of Newton's equation (2.15) is called a *trajectory*.

From the Newton's equation (2.15), we get a result that: the energy function

$$E(q, v) := \frac{1}{2} \|v\|^2, \quad (2.17)$$

where $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^3 , is conserved. It means that the value of the energy function along any trajectory is constant. Indeed, verifying this by differentiation:

$$\begin{aligned} \frac{d}{dt} E(q(t), v(t)) &= \frac{d}{dt} \left(\frac{1}{2} \|\dot{q}(t)\|^2 \right) \\ &= \dot{q}(t) \cdot \ddot{q}(t) \\ &= \dot{q}(t) \cdot (\mathbf{B} \times \dot{q}(t)) \\ &= 0. \end{aligned}$$

Thus, $E(q(t), v(t))$ is independent of the time t .

In particular, it tells us that when the charged particle travels in the pure magnetic field (means that there is no electric field), it always moves with constant speed (the speed is the magnitude of the velocity). No matter how large the magnetic field is and no matter which direction the particle goes, its speed never changes. Its speed only depends on the initial condition. Later, we will see that this energy function is our Hamiltonian. Before going to define the Hamiltonian formulation in this chapter, we need to define an important ingredient which connects the magnetic field and the Hamiltonian function: the magnetic potential.

2.2.1.3 A magnetic potential

So, what is a magnetic potential? Before we go to the definition, we need to pay our attention to one observation in the real life. Imagine that we are holding a bar magnet in our hand, we divide it into two pieces. Then, two small magnets are automatically created with their own south and north poles. If we continue this splitting process down to the atomic level, we find that even elementary particles behave as magnetic dipoles, each with a North and South pole. It appears that nature does not allow us to create magnetic monopoles in this way. This watching is explained by the Gauss' Law for Magnetism, it states that the magnetic field $\vec{\mathbf{B}}$ has free divergence, i.e.

$$\nabla \cdot \vec{\mathbf{B}} = 0. \quad (2.18)$$

By the divergence theorem, this Law implies that for a magnetic dipole, any closed surface the magnetic flux directed inward toward the south pole will equal the flux outward from the north pole. Thus, if there was a magnetic monopole source, the net flux of magnetic field \mathbf{B} out of some closed surface is non-zero, and this breaks the Law. We can see [15, Chapter 7] for details.

But the Law does not stop here, it also produces one vector field which is a bridge between Newton's mechanic and Hamiltonian formulation by Poincaré lemma below.

Theorem 2.10 (Poincaré's Lemma). *Let S be open star-shaped subset of \mathbb{R}^n and let $k \in \mathbb{N}^*$. Assume that σ is a k -form defined on S and if σ is closed, i.e.*

$$d\sigma = 0, \quad (2.19)$$

then there exists a $(k-1)$ -form ω on S such that

$$d\omega = \sigma. \quad (2.20)$$

Proof. The proof can be easily found in textbooks, for example [49, Chapter 1], [64, Appendix B]. \square

Now, let $\vec{\mathbf{B}} = (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$ in \mathbb{R}^3 , we set up the compatible 2-form in \mathbb{R}^3

$$\sigma_{\mathbf{B}} = \mathbf{B}_1 dq_2 \wedge dq_3 - \mathbf{B}_2 dq_1 \wedge dq_3 + \mathbf{B}_3 dq_1 \wedge dq_2.$$

We check that $\sigma_{\mathbf{B}}$ is closed,

$$d\sigma_{\mathbf{B}} = \left(\frac{\partial \mathbf{B}_1}{\partial q_1} + \frac{\partial \mathbf{B}_2}{\partial q_2} + \frac{\partial \mathbf{B}_3}{\partial q_3} \right) dq_1 \wedge dq_2 \wedge dq_3 = (\nabla \cdot \vec{\mathbf{B}}) dq_1 \wedge dq_2 \wedge dq_3 = 0.$$

Applying Poincaré's Lemma, there exists a 1-form $\omega = \mathbf{A}_1 dq_1 + \mathbf{A}_2 dq_2 + \mathbf{A}_3 dq_3$ such that

$$\begin{aligned} d\omega &= \left(\frac{\partial \mathbf{A}_2}{\partial q_1} - \frac{\partial \mathbf{A}_1}{\partial q_2} \right) dq_1 \wedge dq_2 + \left(\frac{\partial \mathbf{A}_3}{\partial q_1} - \frac{\partial \mathbf{A}_1}{\partial q_3} \right) dq_1 \wedge dq_3 + \left(\frac{\partial \mathbf{A}_3}{\partial q_2} - \frac{\partial \mathbf{A}_2}{\partial q_3} \right) dq_2 \wedge dq_3 \\ &= \sigma_{\mathbf{B}}. \end{aligned}$$

So, if we let $\vec{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)$, then it implies that

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}}. \quad (2.21)$$

We will call $\sigma_{\vec{\mathbf{B}}}$ the magnetic 2-form and its primitives ω the magnetic potential 1-form. We call any vector field $\vec{\mathbf{A}}$ satisfied (2.21) the *magnetic potential* of magnetic field $\vec{\mathbf{B}}$.

We can see that there are many possible choices of vector potential $\vec{\mathbf{A}}$ satisfying (2.21), we just add $\vec{\mathbf{A}}$ with some free-curl vector field V , i.e. $\nabla \times V = 0$, we obtain a new vector field which satisfies (2.21) too. How can we relate two choices of vector potential? Poincaré Lemma gives the answer.

Lemma 2.11. *Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{A}}'$ be two magnetic potential vector fields of $\vec{\mathbf{B}}$. Then, there exists a function $\Psi \in C^1(\mathbb{R}^3, \mathbb{R})$ such that*

$$\vec{\mathbf{A}}' = \vec{\mathbf{A}} + \nabla \Psi. \quad (2.22)$$

Conversely, if $\vec{\mathbf{A}}$ and $\vec{\mathbf{A}}'$ satisfied (2.22), then they are the magnetic potential of the same magnetic field.

Proof. Let $\vec{U} = (U_1, U_2, U_3) := \vec{\mathbf{A}}' - \vec{\mathbf{A}}$ be the difference vector field of $\vec{\mathbf{A}}$ and $\vec{\mathbf{A}}'$. Since $\nabla \times \vec{U} = \nabla \times (\vec{\mathbf{A}}' - \vec{\mathbf{A}}) = 0$, it deduces that the 1-form $\sigma_{\vec{U}} = U_1 dq_1 + U_2 dq_2 + U_3 dq_3$ satisfies

$$d\sigma_{\vec{U}} = 0.$$

By applying Poincaré's Lemma, there exists a differentiable function Ψ such that

$$U = \nabla \Psi.$$

The final conclusion of the Lemma is easy to check by noting that the curl of the gradient of a function is zero. \square

In this section, we have used a vector field to define the magnetic field in dimension 3. For higher dimension, it is more convenient to use differential forms to define these objects. Namely, on the domain Ω in \mathbb{R}^n , the magnetic potential is a smooth real 1-form \mathbf{A} on Ω , given by

$$\mathbf{A} = \sum_{i=1}^n A_i dx_i.$$

The associated magnetic field is the 2-form $\mathbf{B} = d\mathbf{A}$, more explicitly, we have

$$\mathbf{B} = \sum_{1 \leq j < k \leq n} B_{jk}(x) dx_j \wedge dx_k,$$

with $B_{jk} = \frac{\partial A_k}{\partial j} - \frac{\partial A_j}{\partial k}$.

Since $d^2 = 0$, we have $d\mathbf{B} = 0$ and this is equivalent to the condition (2.18) in \mathbb{R}^3 . The relation $d\mathbf{A} = \mathbf{B}$ is similar to the equation $\nabla \times \vec{\mathbf{A}} = \vec{\mathbf{B}}$ in \mathbb{R}^3 .

2.2.2 From magnetic Newton's mechanics to magnetic Hamiltonian

Let us get back to our journey by considering a charged particle that is submitted to a pure magnetic field \mathbf{B} in a domain Ω in \mathbb{R}^3 . Let $\mathbf{A} = A_1 dq_1 + A_2 dq_2 + A_3 dq_3$ be the associated magnetic potential of \mathbf{B} .

The matrix representing the right cross product with \mathbf{B} in the canonical basis is

$$M_{\mathbf{B}} = J_{\mathbf{A}}^T - J_{\mathbf{A}},$$

where $J_{\mathbf{A}}$ is the Jacobian matrix of \mathbf{A} . Hence Newton's equation (2.16) becomes

$$\ddot{q} = M_{\mathbf{B}} \dot{q},$$

so that

$$\frac{d}{dt} (\dot{q} + \mathbf{A}(q)) = J_{\mathbf{A}}^T \dot{q}.$$

By introducing the generalized momentum variable $p = \dot{q} + \mathbf{A}(q)$, and define the Hamiltonian function

$$\mathcal{H}(q, p) = \frac{\|p - \mathbf{A}(q)\|^2}{2}, \quad (2.23)$$

where $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^3 . We obtain the system of Hamiltonian's equations

$$\begin{cases} \dot{q} = \partial_p \mathcal{H}(q, p) \\ \dot{p} = -\partial_q \mathcal{H}(q, p). \end{cases} \quad (2.24)$$

Conversely, if we start from Hamiltonian equations (2.24) of Hamiltonian \mathcal{H} , we get

$$\begin{cases} \dot{q} = p - A(q) \\ \dot{p} = J_A^T (p - A(q)). \end{cases} \quad (2.25)$$

By reversing the above process, we take the derivative of \dot{q} , we also obtain the Newton's equation (2.15). Therefore, Newton's equation (2.15) and Hamiltonian's equation system (2.24) are equivalent to describe the dynamic motion of the charged particle. Furthermore, from the equation of \dot{q} , we can express the Hamiltonian as the energy function $E(q, \dot{q})$ mentioned at the end of Subsection 2.2.1.2,

$$\mathcal{H}(q, p) = \frac{1}{2} \|\dot{q}\|^2.$$

2.2.2.1 Gauge invariance

As discussed earlier, we see that the magnetic potential is not defined in the unique way, but up to a gradient of a smooth function. That is, if \mathbf{A} and $\tilde{\mathbf{A}}$ produce the same magnetic field, then there exists a function $\Psi \in C^\infty(\Omega)$ such that

$$\tilde{\mathbf{A}} = \mathbf{A} + \nabla\Psi.$$

We obtain two Hamiltonians

$$\mathcal{H}(q, p) = \frac{\|p - \mathbf{A}(q)\|^2}{2} \quad \text{and} \quad \tilde{\mathcal{H}}(\tilde{q}, \tilde{p}) = \frac{\|\tilde{p} - \tilde{\mathbf{A}}(\tilde{q})\|^2}{2}.$$

It leads to two corresponding Hamiltonian dynamics. The natural question is if these Hamiltonian systems describe a same movement of the particle. To have the answer for this question, we need to find the link between two Hamiltonian systems associated with two arbitrary magnetic potentials \mathbf{A} and $\tilde{\mathbf{A}}$. We consider the mapping φ defined by

$$\varphi : \quad \Omega \times \mathbb{R}^3 \rightarrow \Omega \times \mathbb{R}^3 \tag{2.26}$$

$$(q, p) \mapsto \varphi(q, p) = (q, p + \nabla\Psi(q)). \tag{2.27}$$

We can easily verify that φ is a diffeomorphism from $\Omega \times \mathbb{R}^3$ to itself, and its inverse is

$$\varphi^{-1}(q, p) = (q, p - \nabla\Psi(q)).$$

Furthermore, the Jacobian matrix of φ at the point (q, p) has the form

$$(\mathrm{d}\varphi)_{(q,p)} = \begin{pmatrix} I_3 & 0 \\ (\mathrm{d}^2\Psi)_q & I_3 \end{pmatrix},$$

where $(\mathrm{d}^2\Psi)_q$ is the Hessian matrix of Ψ at the point q . By applying the Theorem 2.5, we can confirm that φ is symplectic. From the definition of \mathcal{H} and $\tilde{\mathcal{H}}$, we see that

$$\tilde{\mathcal{H}} = \mathcal{H} \circ \varphi^{-1}.$$

Therefore, the dynamics defined by \mathcal{H} and $\tilde{\mathcal{H}}$ are equivalent via a symplectic transformation and does not depends on \mathbf{A} , we call this property Gauge invariance of magnetic Hamiltonian.

2.2.2.2 With a constant magnetic field

Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be the standard basis of \mathbb{R}^3 . We restrict ourselves to the case the magnetic field pointing in the direction perpendicular to the plane $\mathbb{R}^2 := \{q_1 \vec{e}_1 + q_2 \vec{e}_2 \mid q_1, q_2 \in \mathbb{R}\}$ and does not depend on time, that is

$$\vec{\mathbf{B}}(q) = \mathbf{B}(q_1, q_2) \vec{e}_3.$$

We assume that $\mathbf{B} = \mathbf{B}_0$, where $B_0 \in \mathbb{R}$, is the constant magnetic field. We will use Newton's mechanics and Hamiltonian's system to investigate the motion of the particle.

Newton's approach: We rewrite the system (2.16) as

$$\begin{cases} \ddot{q}_1 = \mathbf{B}_0 \dot{q}_2, \\ \ddot{q}_2 = -\mathbf{B}_0 \dot{q}_1, \\ \ddot{q}_3 = 0. \end{cases} \quad (2.28)$$

The last of these is easy to solve

$$q_3(t) = q_3(0) + \dot{q}_3(0)t. \quad (2.29)$$

It results that we just need to focus on the system of q_1 and q_2 :

$$\begin{cases} \ddot{q}_1 = \mathbf{B}_0 \dot{q}_2, \\ \ddot{q}_2 = -\mathbf{B}_0 \dot{q}_1. \end{cases} \quad (2.30)$$

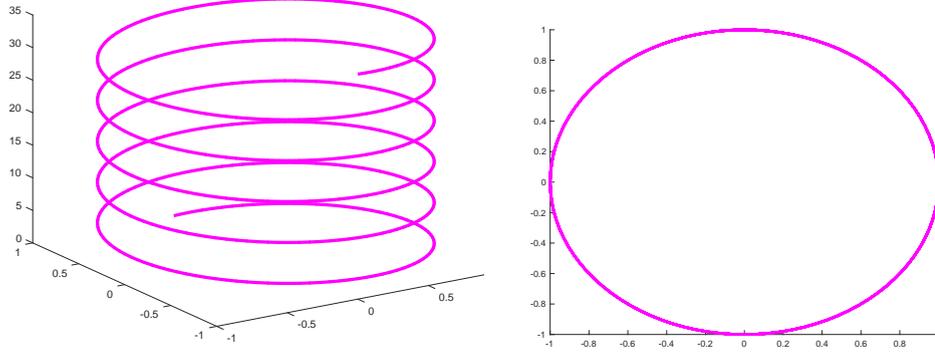
Let $v = \dot{q}_1 + i\dot{q}_2$, use (2.30), we find that

$$\dot{v} = -i\mathbf{B}_0 v.$$

It follows that $v(t) = v(0)e^{-i\mathbf{B}_0 t}$. Then we have the trajectory

$$q(t) := q_1(t) + iq_2(t) = q(0) - \frac{iv(0)}{\mathbf{B}_0} + \frac{iv(0)}{\mathbf{B}_0} e^{-i\mathbf{B}_0 t}. \quad (2.31)$$

This formula describes that the charged particle moves along the path whose projection on the plane \mathbb{R}^2 is a circle whose center is $q(0) - \frac{iv(0)}{\mathbf{B}_0}$ with radius $\frac{\|v(0)\|}{|\mathbf{B}_0|}$. Meanwhile q_3 given by (2.29) increases steadily, so the particle actually describes a uniform helix curve whose axis is parallel to the magnetic field. When the particle starts from \mathbb{R}^2 with initial velocity lines in \mathbb{R}^2 , it means that $q_3(0) = 0$ and $\dot{q}_3(0) = 0$, then the spiral reduces to a circle (see Figure 2.2). It results that the particle is trapped in a cyclotron motion forever in a constant magnetic field.



(A) A Helix path.

(B) A Cyclotron path.

FIGURE 2.2: The movement of the particle when \mathbf{B} is constant.

Hamiltonian's approach: To deal with magnetic Hamiltonian, first of all we find a magnetic potential \mathbf{A} associated with magnetic field

$$\mathbf{B} = \mathbf{B}_0 dq_1 \wedge dq_2 + 0dq_1 \wedge dq_3 + 0dq_2 \wedge dq_3 = \mathbf{B}_0 dq_1 \wedge dq_2.$$

We choose $\mathbf{A} = \left(-\frac{\mathbf{B}_0}{2}q_2\right) dq_1 + \left(\frac{\mathbf{B}_0}{2}q_1\right) dq_2 + 0dq_3 = \left(-\frac{\mathbf{B}_0}{2}q_2\right) dq_1 + \left(\frac{\mathbf{B}_0}{2}q_1\right) dq_2$, thus

$$d\mathbf{A} = \mathbf{B}.$$

Then, the Hamiltonian is

$$\mathcal{H}(q, p) = \frac{\left(p_1 + \frac{\mathbf{B}_0}{2}q_2\right)^2}{2} + \frac{\left(p_2 - \frac{\mathbf{B}_0}{2}q_1\right)^2}{2} + \frac{p_3^2}{2}.$$

We look at the third variable first, we obtain

$$\begin{cases} \dot{q}_3 = p_3, \\ \dot{p}_3 = 0. \end{cases} \quad (2.32)$$

It implies that $q_3(t)$ satisfies (2.29). Also from Hamiltonian's equation, we have

$$\begin{cases} \dot{q}_1 = p_1 + \frac{\mathbf{B}_0}{2}q_2, \\ \dot{q}_2 = p_2 - \frac{\mathbf{B}_0}{2}q_1, \\ \dot{p}_1 = \frac{\mathbf{B}_0}{2}\left(p_2 - \frac{\mathbf{B}_0}{2}q_1\right), \\ \dot{p}_2 = \frac{\mathbf{B}_0}{2}\left(p_1 + \frac{\mathbf{B}_0}{2}q_2\right). \end{cases} \quad (2.33)$$

A solution of this linear system has the form

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = C_1 \begin{pmatrix} \frac{2}{\mathbf{B}_0} \\ 0 \\ 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -\frac{2}{\mathbf{B}_0} \\ 1 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} -\frac{2}{\mathbf{B}_0} \\ \frac{2i}{\mathbf{B}_0} \\ i \\ 1 \end{pmatrix} e^{-i\mathbf{B}_0 t} + C_4 \begin{pmatrix} -\frac{2}{\mathbf{B}_0} \\ -\frac{2i}{\mathbf{B}_0} \\ -i \\ 1 \end{pmatrix} e^{i\mathbf{B}_0 t}, \quad (2.34)$$

where C_1, C_2, C_3, C_4 are constant depending on the initial data $(q(0), p(0))$. To compare the solutions given by the two approaches, we just need to write q and \dot{q} in complex form as above. We have

$$q(t) = q_1(t) + iq_2(t) = \frac{2}{\mathbf{B}_0} C_1 - \frac{2i}{\mathbf{B}_0} C_2 - \frac{4}{\mathbf{B}_0} C_3 e^{-i\mathbf{B}_0 t},$$

and thus

$$\dot{q}(t) = \dot{q}_1(t) + i\dot{q}_2(t) = 4iC_3 e^{-i\mathbf{B}_0 t}.$$

Then, we get the constant $C_3 = \frac{-i}{4} \dot{q}(0)$.

Let $t = 0$ in the equation of $q(t)$, we obtain

$$\frac{2}{\mathbf{B}_0} C_1 - \frac{2i}{\mathbf{B}_0} C_2 = q(0) - \frac{i\dot{q}(0)}{\mathbf{B}_0}.$$

Replace this term in $q(t)$, we recover the solution same as (2.31)

$$q(t) = q(0) - \frac{i\dot{q}(0)}{\mathbf{B}_0} + \frac{i\dot{q}(0)}{\mathbf{B}_0} e^{-i\mathbf{B}_0 t}.$$

Chapter 3

Study of the magnetic Hamiltonian dynamics

Not all of us can do great things. But
we can do small things with great
love.

Mother Teresa

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The present chapter is devoted to produce the proof for the results obtained in classical mechanics. The chapter is organized as follow. In Section 3.1, we write the magnetic Hamiltonian in tubular coordinates. In Section 3.2, we provide the proofs for results in confinement. Section 3.3 is devoted for the proof of scattering Theorem 1.7.

3.1 The magnetic Hamiltonian in tubular coordinates

In Chapter 2, we considered the magnetic domain in three dimensions which appears naturally in physics. We also saw that the Hamiltonian is a function of 6 variables defined by

$$\mathcal{H}(q, p) = \frac{\|p - \mathbf{A}(q)\|^2}{2} \quad \text{for } (q, p) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

with $\mathbf{A} = A_1 dq_1 + A_2 dq_2 + A_3 dq_3$ is a 1-form satisfies

$$d\mathbf{A} = \mathbf{B}.$$

In our problem, we study the motion of a charged particle in $\Omega \subset \mathbb{R}_{(q_1, q_2)}^2$ submitted to the magnetic field $\mathbf{B} = b(q_1, q_2) dq_1 \wedge dq_2$. We can set up the magnetic potential in the form $\mathbf{A} = A_1(q_1, q_2) dq_1 + A_2(q_1, q_2) dq_2$ with A_1, A_2 satisfies

$$\frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2} = b.$$

Since the charged particles which we are studying have their initial positions and velocities in the domain $\Omega \subset \mathbb{R}_{(q_1, q_2)}^2$, via Hamiltonian's equations of q_3 and p_3 , we see that q_3 and p_3 are always zero. For that reason, we can ignore the variable q_3 and p_3 in energy function and we get

$$\mathcal{H}(q, p) = \frac{\|p - \mathbf{A}(q)\|^2}{2} \quad \text{for } (q, p) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Let us recall again the definition of the tubular coordinate which we mentioned briefly in the statement of results in Chapter 1.

We are working on the bounded domain Ω in \mathbb{R}^2 . We assume that any connected components of $\partial\Omega$ are C^2 -smooth closed curves without self-intersections. Let \mathcal{C} be a connected component of $\partial\Omega$, since \mathcal{C} is regular, we may parametrize it by arc length $\gamma : \mathbb{R}/L\mathbb{Z} \rightarrow \mathcal{C}$ where L is the length of \mathcal{C} .

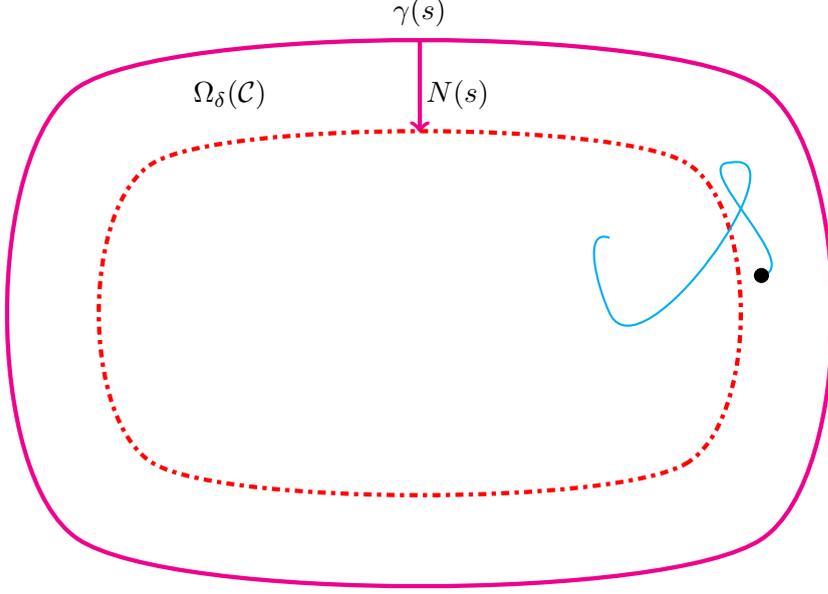


FIGURE 3.1: Tubular Coordinates.

There exists $\delta > 0$ such that

$$\psi : \begin{cases} (0, \delta) \times \mathbb{R}/L\mathbb{Z} \rightarrow \Omega_{\mathcal{C}}(\delta) \\ (n, s) \mapsto \gamma(s) + nN(s) = q \end{cases} \quad (3.1)$$

is a smooth diffeomorphism. $N(s)$ denotes the inward pointing normal at $\gamma(s)$ and

$$\Omega_{\mathcal{C}}(\delta) = \{q \in \Omega : d(q, \mathcal{C}) < \delta\}.$$

Via the Theorem 2.6, we can lift ψ to a symplectic change of coordinates Ψ defined by

$$\Psi : \begin{cases} (0, \delta) \times \mathbb{R}/L\mathbb{Z} \times \mathbb{R}^2 \rightarrow \Omega_{\mathcal{C}}(\delta) \times \mathbb{R}^2, \\ (n, s, p_n, p_s) \mapsto (\psi(n, s), ((d\psi)_{(n,s)}^{-1})^T(p_n, p_s)) = (q, p), \end{cases}$$

where we have explicitly $p = (1 - n\kappa(s))^{-1}p_s\gamma'(s) + p_nN(s)$ with $\kappa(s)$ is the signed curvature of \mathcal{C} at $\gamma(s)$.

Note that 2-form $dq_1 \wedge dq_2$ in tubular coordinates has the form

$$dq_1 \wedge dq_2 = (1 - n\kappa(s))ds \wedge dn.$$

Therefore, we can write magnetic field \mathbf{B} as

$$\mathbf{B} = b(q)dq_1 \wedge dq_2 = b(\psi(n, s))(1 - n\kappa(s))ds \wedge dn,$$

and write the magnetic potential as

$$\mathbf{A} = A_n(n, s)dn + A_s(n, s)ds,$$

with A_n, A_s defined on $(0, \delta) \times \mathbb{R}/L\mathbb{Z}$ such that

$$\frac{\partial A_s}{\partial n} - \frac{\partial A_n}{\partial s} = B(n, s) := -b(\psi(n, s))(1 - n\kappa(s)).$$

The lemma below shows us that the Hamiltonian takes the form

$$H(n, s, p_n, p_s) = \frac{1}{2}(p_n - A_n(n, s))^2 + \frac{(p_s - A_s(n, s))^2}{2(1 - \kappa(s)n)^2}.$$

Lemma 3.1. *We write $\mathbf{A} = A_1dq_1 + A_2dq_2$. With (3.1), we have*

$$\mathbf{A} = A_n dn + A_s ds, \quad \tilde{A} = (A_n, A_s)^T = (d\psi)^T(A_1, A_2)^T.$$

We have

$$H(n, s, p_n, p_s) = \mathcal{H} \circ \Psi(n, s, p_n, p_s) = \frac{(p_n - A_n(n, s))^2}{2} + \frac{(p_s - A_s(n, s))^2}{2(1 - \kappa(s)n)^2}. \quad (3.2)$$

Moreover, $v_n = p_n - A_n(n, s)$ and $v_s = (1 - n\kappa(s))^{-1}(p_s - A_s)$ are the normal and tangential component of v .

Proof. We write

$$2H(q, p) = \|p - A\|^2 = \|(d\psi^{-1})^T(\tilde{p} - \tilde{A})\|^2 = \langle (d\psi^{-1})(d\psi^{-1})^T(\tilde{p} - \tilde{A}), \tilde{p} - \tilde{A} \rangle,$$

with $\tilde{p} = (p_n, p_s)^T$. Note that

$$(d\psi^{-1})^T = [N(s), (1 - n\kappa(s))\gamma'(s)]. \quad (3.3)$$

We get

$$(d\psi^{-1})(d\psi^{-1})^T = \begin{pmatrix} 1 & 0 \\ 0 & (1 - n\kappa(s))^{-2} \end{pmatrix}.$$

Concerning the velocity v , since $q = \gamma(s) + nN(s)$ and thanks to the Frenet-Serret formula $N'(s) = -\kappa(s)\gamma'(s)$, we have

$$v = \dot{s}(1 - n\kappa(s))\gamma'(s) + \dot{n}N(s) =: v_s\gamma' + v_nN,$$

and thus we get the result by using the Hamilton equations $\dot{s} = (1 - \kappa(s)n)^{-2}(p_s - A_s)$ and $\dot{n} = p_n - A_n$. \square

3.2 On the confinement problem

3.2.1 In the general case

We will prove Theorems 1.2 and 1.3 in this subsection. To reach the boundary, the particle has to be close to a connected component \mathcal{C} of $\partial\Omega$ (see Figure 3.1). Thus, we can assume that, for all $t \in [0, T)$,

$$q(t) \in \Omega_{\mathcal{C}}(\delta).$$

Modifying the vector potential corresponds to a symplectic transformation of the form $(q, p) \mapsto (q, p + dS(q))$, for some smooth function S , and hence does not modify the trajectory of the particle. Thus, we consider the function

$$\alpha(n, s) = \frac{s}{L} \int_0^L B(n, \xi) d\xi - \int_0^s B(n, \xi) d\xi.$$

Notice that $\alpha(n, \cdot)$ is L -periodic. Recalling (1.20) and letting $\mathbf{A} = \alpha(n, s)dn + f(n)ds$, we have $\mathbf{B} = d\mathbf{A}$.

By (3.2), the corresponding Hamiltonian is

$$H(n, s, p_n, p_s) = \frac{(p_n - \alpha(n, s))^2}{2} + \frac{(p_s - f(n))^2}{2(1 - \kappa(s)n)^2}.$$

Concerning Hamilton's equations, we have in particular

$$\dot{n} = p_n - \alpha(n, s), \quad \dot{p}_s = \tilde{B}(n, s)\dot{n} - \frac{(p_s - f(n))^2}{(1 - \kappa(s)n)^3} \kappa'(s)n,$$

where

$$\tilde{B}(n, s) = \frac{1}{L} \int_0^L B(n, \xi) d\xi - B(n, s).$$

We recall that, for all $t \in [0, T)$, $H(n(t), s(t), p_n(t), p_s(t)) = H_0$. We get

$$\begin{aligned} |\dot{n}| &\leq \sqrt{2H_0} \\ |p_s - f(n)| &\leq \sqrt{2H_0}(1 + \varepsilon) \\ \left| \frac{(p_s - f(n))^2}{(1 - \kappa(s)n)^3} \kappa'(s)n \right| &\leq \frac{2H_0 K' \delta}{1 - \varepsilon}, \end{aligned} \tag{3.4}$$

where in the last estimates we have used the notation of Theorem 1.3 and in particular $|\kappa|n \leq K\delta \leq \varepsilon$. With our assumption (1.19) on $\tilde{B}(n, s)$, we find, for all $t \in [0, T)$,

$$|p_s(t)| \leq |p_s(0)| + \left(M\sqrt{2H_0} + \frac{2H_0 K' \delta}{1 - \varepsilon} \right) T,$$

and thus

$$|f(n(t))| \leq |p_s(t)| + |p_s(t) - f(n(t))| \leq C(T, q(0), \dot{q}(0)), \quad (3.5)$$

with

$$C(T, q(0), \dot{q}(0)) = |p_s(0)| + \sqrt{2H_0}(1 + \varepsilon) + \left(M\sqrt{2H_0} + \frac{2H_0K'\delta}{1 - \varepsilon} \right) T.$$

If the trajectory reaches the boundary at $t = T$, then

$$\lim_{t \rightarrow T} n(t) = 0.$$

This, with (3.5) and (1.17), gives a contradiction. This proves Theorem 1.2.

Now, consider a function g as in Theorem 1.3. We have, for all $t \in [0, T)$,

$$g(n(t)) \leq |f(n(t))| \leq C(T, q(0), \dot{q}(0)).$$

From (1.21), we have $\lim_{n \rightarrow 0} g(n) > C(T, q(0), \dot{q}(0))$; hence g must take the value $C(T, q(0), \dot{q}(0))$ and the conclusion follows.

3.2.2 In the radial case

Let us recall (1.33). When Ω is the unit disc and when \mathbf{B} is radial, we can use radial coordinates to approach the confinement problem. As in the tubular coordinates, from the diffeomorphism

$$\phi : \begin{cases} \mathbb{R}_+^* \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \Omega \setminus \{0\} \\ (r, \theta) \mapsto (r \cos \theta, r \sin \theta) = q, \end{cases}$$

we can lift up to a symplectic transformation

$$\Phi : \begin{cases} \mathbb{R}_+^* \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^2 \rightarrow (D \setminus \{0\}) \times \mathbb{R}^2 \\ (r, \theta, p_r, p_\theta) \mapsto (\phi(r, \theta), ((d\phi)_{(r, \theta)}^{-1})^\top(p_r, p_\theta)) = (q, p). \end{cases}$$

The explicit formula of Φ is given in (1.31). Combining this with the form of magnetic potential

$$\mathbf{A} = G(r)d\theta \quad \text{where } G(r) = \int_0^r \tau B(\tau) d\tau,$$

the Hamiltonian in this coordinate becomes

$$\tilde{H}(r, \theta, p_r, p_\theta) = \frac{p_r^2}{2} + \frac{(p_\theta - G(r))^2}{2r^2}.$$

From the formula of the Hamiltonian, we can immediately see that the angular momentum p_θ is constant. Therefore, there are two constants of motion- the total energy and the component p_θ , this fact is used to reduce the Hamiltonian to a function of r and p_r

$$H(r, p_r) = \frac{p_r^2}{2} + V(r), \quad \text{where } V(r) = \frac{(p_\theta - G(r))^2}{2r^2}.$$

3.2.2.1 Proof of Proposition 1.5

The assumptions of Proposition 1.5 can be written in terms of V .

(H1) If

$$\liminf_{r \rightarrow 1^-} V(r) > H_0, \quad (3.6)$$

we consider $\eta = \sup\{x \in (0, 1) : V(x) = H_0\} \in (0, 1)$. Consider a trajectory $(q(t), p(t))$ with $q(0) \in D(0, 1)$. We can assume that $q(0) \neq 0$. Let T be the maximal time of existence in $D(0, 1)$. By energy conservation, we have, for all $t \in [0, T)$,

$$V(r(t)) \leq H_0,$$

so that $r(t) \leq \eta$.

Note that (3.6) means

$$\liminf_{r \rightarrow 1^-} |G(r) - p_\theta| > \sqrt{2H_0}.$$

Using the usual complex coordinate in the plane \mathbb{R}^2 , we can write $\dot{q} = (\dot{r} + i\dot{\theta}r) e^{i\theta}$ and thus

$$\det(q(t), \dot{q}(t)) = r^2(t)\dot{\theta}(t) = p_\theta - G(r(t)).$$

Finally, we notice that $\|\dot{q}(0)\| = \sqrt{2H_0}$ and write

$$G(r) - p_\theta = G(r) - G(r(0)) - [p_\theta - G(r(0))],$$

which gives (1.26).

(H2) If

$$\liminf_{r \rightarrow 1^-} V(r) = H_0, \quad (3.7)$$

and

$$\limsup_{r \rightarrow 1^-} \frac{V(r) - H_0}{r - 1} < 0,$$

then we must again have

$$\sup\{x \in (0, 1) : V(x) = H_0\} < 1,$$

and we can proceed as above.

3.2.2.2 Proof of Proposition 1.6

Consider $p_\theta = 0$. Let $|V|_\infty := \sup_{r \in (0,1)} |V(r)|$. By assumption, $|V|_\infty < +\infty$.

Let $r(0) \in (0, 1)$ and choose $p_r(0) > 0$ such that $p_r^2(0) = 2(|V|_\infty - V(r(0))) + v^2$, with $v > 0$. Since, for all $t \in [0, T)$,

$$\frac{p_r^2(t)}{2} + V(r(t)) = \frac{p_r^2(0)}{2} + V(r(0)),$$

we get $\dot{r}(t) = p_r(t) \geq v$ so that

$$r(t) \geq vt + r(0).$$

The escape time is at most $t = \frac{1-r(0)}{v}$.

3.3 On the scattering problem

We distinguish between the cases $p_\theta = 0$ and $p_\theta \neq 0$.

3.3.1 Case when $p_\theta \neq 0$

In this case, $\lim_{r \rightarrow 0} V(r) = +\infty$; hence, due to energy conservation, the trajectory does not approach the origin.

- i. Assume that $p_r(0) < 0$. We have $V(1) < H_0$ and we can consider the right most turning point $r^* \in (0, 1)$. By definition $V(r^*) = H_0$, and necessarily $V'(r^*) \leq 0$.

If $V'(r^*) < 0$, it is easy to check that r reaches r^* in finite time, say $t = t^*$. This time is given by

$$t^* = \int_{r^*}^1 \frac{dr}{\sqrt{2(H_0 - V(r))}}.$$

By symmetry, the escape time is $2t^*$. Since $\dot{\theta} = \frac{p_\theta - G(r)}{r^2}$, we have

$$\theta(t^*) - \theta(0) = \int_0^{t^*} \frac{p_\theta - G(r)}{r^2} dt = \int_0^{t^*} \frac{(p_\theta - G(r))\dot{r}}{r^2 p_r} dt = \int_0^{t^*} -\frac{(p_\theta - G(r))\dot{r}}{r^2 \sqrt{2(H_0 - V(r))}} dt,$$

so that

$$\theta(t^*) - \theta(0) = \int_{r^*}^1 \frac{p_\theta - G(r)}{r^2 \sqrt{2(H_0 - V(r))}} dr.$$

By symmetry, we have

$$\theta(2t^*) - \theta(0) = 2 \int_{r^*}^1 \frac{p_\theta - G(r)}{r^2 \sqrt{2(H_0 - V(r))}} dr.$$

If $V'(r^*) = 0$, $(r^*, 0)$ is a critical point of the Hamiltonian and we get that r reaches r^* in infinite time (see Figure 3.2).

- ii. Assume that $p_r(0) = 0$. Then $V(1) = H_0$. By assumption (the trajectory enters $D(0, 1)$), we have $V'(1) \geq 0$, *i.e.*, $(p_\theta - G(1))B(1) + (p_\theta - G(1))^2 \leq 0$. If $V'(1) = 0$, the particle sits at a fixed point of the Hamiltonian system, and hence $r(t) \equiv 1$ is constant. If $V'(1) > 0$, it enters $D(0, 1)$ and the discussion is the same as previously.

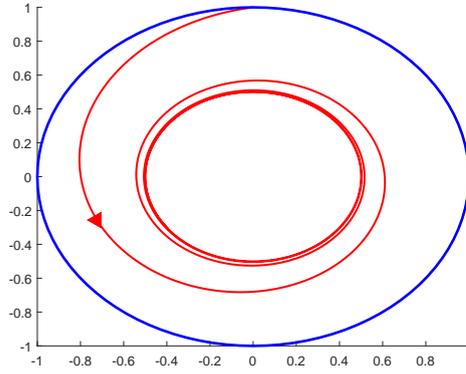


FIGURE 3.2: $B(r) = e^{-r} - \frac{2}{r}$.

3.3.2 Case when $p_\theta = 0$

In this case, since $G(0) = 0$, $V(r) = \frac{1}{2r^2}G(r)^2$ admits a continuous extension at $r = 0$.

- i. Assume that $p_r(0) < 0$. We have $V(1) < H_0$. The existence of r^* such that $V(r^*) = H_0$ is not ensured. If $V(r) < H_0$ on $[0, 1]$, the particle reaches $r = 0$ in finite time $t = t^*$:

$$t^* = \int_0^1 \frac{dr}{\sqrt{2(H_0 - V(r))}}.$$

We get, by symmetry,

$$\theta(2t^*) - \theta(0) = 2 \int_0^1 \frac{-G(r)}{r^2 \sqrt{2(H_0 - V(r))}} dr + \pi.$$

If there exists $r^* \in (0, 1)$ such that $V(r^*) = H_0$, the trajectory does not reach the origin and the discussion is the same as in the case $p_\theta \neq 0$.

ii. Assume that $p_r(0) = 0$. The discussion is the same as when $p_\theta \neq 0$.

3.3.3 Scattering angle

We can now end the proof of Theorem 1.7. In terms of complex numbers, we can write

$$v_1 = (v_r(0) + iv_\theta(0))e^{i\theta_1}, \quad v_2 = (-v_r(0) + iv_\theta(0))e^{i\theta_2}.$$

The scattering angle is

$$\theta_2 - \theta_1 + \text{Arg} \left(\frac{-v_r(0) + iv_\theta(0)}{v_r(0) + iv_\theta(0)} \right).$$

Since δ is the argument of $-v_r(0) + iv_\theta(0)$, the scattering angle is

$$\theta_2 - \theta_1 - \pi + 2\delta.$$

Chapter 4

Semi-classical magnetic Laplacian on a two-dimensional Riemannian manifold

I am not sure exactly what heaven
will be like, but I know that when we
die and it comes time for God to
judge us, he will not ask, 'How many
good things have you done in your
life?' rather he will ask, 'How much
love did you put into what you did?'

Mother Teresa

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In this chapter, we use the WKB method to investigate the spectrum of the magnetic Laplacian on a two-dimensional Riemannian manifold. We will observe that the asymptotic expansions of the eigenvalues and the eigenfunctions can be obtained via a WKB analysis. Firstly, we introduce the definition of the magnetic Laplacian on a Riemannian manifold in Section 4.1. Section 4.2 is utilized for the construction of a pair of local isothermal coordinates and for the description of the magnetic Laplacian in these coordinates. Then, we prepare for the WKB process by writing the eigen-problem of the magnetic Laplacian into the eikonal equation and the transport equations in Section 4.3. The WKB analysis which is performed in Section 4.4 includes solving the eikonal equation and the transport equations thanks to formal series. Finally, we make the comparison between the true eigenfunctions and their quasi-modes in Section 4.5.

4.1 The magnetic Laplacian on a Riemannian manifold

The aim of this section is giving the basic definition of the magnetic Laplacian. We will show that upon assuming the compactness of the manifold (with or without boundary), the operator will have compact resolvent, then its spectrum will be described by a real, discrete sequence tending to $+\infty$. The special case when the manifold is \mathbb{R}^2 is also considered.

4.1.1 Some facts about Riemannian manifolds

Let us recall some properties of Riemannian manifolds which can be found in Riemannian textbooks such as [38, 39].

Let (M, g) be an oriented Riemannian manifold of dimension $n \geq 2$, i.e. M is a smooth oriented manifold equipped with a Riemannian metric g . This metric g is defined on M such that, for each $p \in M$,

$$g_p : T_p M \times T_p M \rightarrow \mathbb{R}$$

is a symmetric inner product on the tangent space $T_p M$ which varies smoothly from point to point in the sense that if X and Y are differentiable vector fields on M , then $p \mapsto g_p(X(p), Y(p))$ is a smooth function on M .

Notation 4.1. *Let $(\mathcal{U}, \varphi : \mathcal{U} \rightarrow \varphi(\mathcal{U}) \subset \mathbb{R}^n)$ be a local chart with coordinates denoted by (x^1, \dots, x^n) . At each point $p \in \mathcal{U}$, we denote*

1. $[V]_p, [\omega]_p$, respectively, the coordinates of a tangent vector $V = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \Big|_p$ and a covector $\omega = \sum_{i=1}^n \omega^i dx^i \Big|_p$, i.e.

$$[V]_p = (v^1, \dots, v^n)^T \quad \text{and} \quad [\omega]_p = (\omega^1, \dots, \omega^n)^T.$$

2. G_p the matrix whose components are

$$G_{ij}(p) = g_p \left(\frac{\partial}{\partial x^i} \Big|_p, \frac{\partial}{\partial x^j} \Big|_p \right) \quad \text{for all } 1 \leq i, j \leq n.$$

Thus, G_p is symmetric and

$$g_p(V, W) = [V]_p^T G_p [W]_p \quad \text{for all } V, W \in T_p M.$$

3. $G^{ij}(p)$ the components of the matrix G_p^{-1} and $|G_p|$ the determinant of the matrix G_p .
4. $d\mathcal{V}_g$ the classical Riemannian volume form on M , which is the unique smooth orientation form that satisfies

$$d\mathcal{V}_g(E_1, \dots, E_n) = 1,$$

for every local oriented orthonormal frame (E_1, \dots, E_n) on M , c.f. [39, Chapter 15]. In local coordinates,

$$d\mathcal{V}_g = \sqrt{|G|} dx^1 \wedge dx^2 \wedge \dots \wedge dx^n.$$

In terms of the Riemannian metric g , we can define a canonical isomorphism \hat{g} between the tangent bundle TM and the cotangent bundle T^*M point-wise as follows. For each $p \in M$ and $V \in T_p M$, $\hat{g}_p(V)$ is a co-vector which is defined by

$$\hat{g}_p(V)(W) = g_p(V, W) \quad \text{for all } W \in T_p M. \quad (4.1)$$

Furthermore, it also induces a dual metric g^* on the set of 1-forms on M by setting

$$g_p^*(w_1, w_2) = g_p(\hat{g}_p^{-1}(w_1), \hat{g}_p^{-1}(w_2)) \quad \text{for all } w_1, w_2 \in T_p^* M.$$

In local coordinates, the mapping \hat{g}_p is considered as multiplication with the matrix G_p , i.e.

$$[\hat{g}_p(V)] = G_p[V].$$

The integration of a measurable function f on M is defined as follows. If f has a compact support contained in a local chart (\mathcal{U}, φ) , then

$$\int_M f \, d\mathcal{V}_g = \int_{\varphi(\mathcal{U})} (\varphi^{-1})^* (f \, d\mathcal{V}_g) = \int_{\varphi(\mathcal{U})} (f \circ \varphi^{-1}) \sqrt{|G|} \circ \varphi^{-1} \, dx.$$

The definition is independent of choosing the local charts. We use a partition of unity on M to define the integration in the general case. Let $(\mathcal{U}_i, \varphi_i)$ be a finite family of local charts which covers M and satisfies

$$0 \leq \varphi_i \leq 1, \quad \text{supp}(\varphi_i) \subset \mathcal{U}_i,$$

and

$$\sum_i \varphi_i = 1.$$

The integration of f on M is defined by

$$\int_M f \, d\mathcal{V}_g = \sum_i \int_M f \varphi_i \, d\mathcal{V}_g.$$

Note that this definition does not depend on the choice of partitions.

Next, we define $L^2(M)$ to be the completion of $C_c^\infty(M)$, the space of smooth functions having compact supports on M , with the inner product

$$\langle u, v \rangle_{L^2(M)} = \int_M u \bar{v} \, d\mathcal{V}_g \quad \text{for } u, v \in C_c^\infty(M),$$

where the bar is the complex conjugation. The explicit expression of the Hilbert space $L^2(M)$ is given by

$$L^2(M) = \left\{ f : M \rightarrow \mathbb{C} \text{ is measurable and } \int_M |f|^2 \, d\mathcal{V}_g < \infty \right\}.$$

Notation 4.2. We make the notation that $C^m(\Lambda^k(M))$ is a space of all C^m -smooth complex-valued k -form on M , with $m \in \mathbb{N} \cup \{\infty\}$ and $k \in \mathbb{N}$.

We define $L^2(\Lambda^1(M))$ to be the completion of the space of smooth 1-forms having compact supports in $C^\infty(\Lambda^1(M))$ with the inner product

$$\langle \alpha, \beta \rangle_{L^2(\Lambda^1(M))} = \int_M g^*(\alpha, \bar{\beta}) \, d\mathcal{V}_g,$$

where α, β are 1-forms with compact supports and $\bar{\beta} := \sum_{i=1}^n \bar{\beta}_i dx^i$ if $\beta = \sum_{i=1}^n \beta_i dx^i$.

4.1.2 Definition of the magnetic Laplacian

Since the main subject of this chapter is studying the spectrum of the magnetic Laplacian on a two-dimensional Riemannian manifold, we will restrict ourselves to the case $n = 2$. The definition of this operator can be found in the introduction parts of [28, 32, 57].

Let \mathbf{A} be a real-valued 1-form in $C^1(\Lambda^1(M))$. Since M is the two-dimensional manifold, there exists a real-valued function $B \in C(M)$ such that

$$d\mathbf{A} = B d\mathcal{V}_g.$$

We call B the *magnetic field* and \mathbf{A} the *magnetic potential*. In the sequel, we define the magnetic Laplacian on a manifold through \mathbf{A} as follows. Firstly, associated to the 1-form \mathbf{A} , we introduce a natural differential

$$\begin{aligned} D_{h,\mathbf{A}} : C_c^\infty(M) &\rightarrow C^1(\Lambda^1(M)) \\ u &\mapsto D_{h,\mathbf{A}}(u) := -(ihd + \mathbf{A})u, \end{aligned}$$

where $h > 0$ is a semi-classical parameter which is assumed to be small.

Both $L^2(M)$ and $L^2(\Lambda^1(M))$ are Hilbert spaces with their corresponding inner products, that allows to determine the formal adjoint of $D_{h,\mathbf{A}}$ as

$$(D_{h,\mathbf{A}})^* : C^1(\Lambda^1(M)) \rightarrow C(M)$$

which is defined by the identity

$$\langle D_{h,\mathbf{A}}u, \omega \rangle_{L^2(\Lambda^1(M))} = \langle u, (D_{h,\mathbf{A}})^*(\omega) \rangle_{L^2(M)}, \quad \text{for all } u \in C_c^\infty(M), \omega \in C^1(\Lambda^1(M)).$$

We consider the operator

$$\text{Dom}(H_{h,\mathbf{A}}) = C_c^\infty(M), \quad H_{h,\mathbf{A}} = (D_{h,\mathbf{A}})^* D_{h,\mathbf{A}},$$

which has the following properties:

1. $H_{h,\mathbf{A}}$ is symmetric because

$$\langle H_{h,\mathbf{A}}u, v \rangle_{L^2(M)} = \langle D_{h,\mathbf{A}}u, D_{h,\mathbf{A}}v \rangle_{L^2(\Lambda^1(M))} = \langle u, H_{h,\mathbf{A}}v \rangle,$$

for all $u, v \in C_c^\infty(M)$.

2. $H_{h,\mathbf{A}}$ is positive since

$$\langle H_{h,\mathbf{A}}u, u \rangle_{L^2(M)} = \int_M g^* ((-ihd - \mathbf{A})u, (-ihd - \mathbf{A})u) d\mathcal{V}_g \geq 0.$$

Thus, $H_{h,\mathbf{A}}$ has a Friedrichs extension which is self-adjoint, see [25, Theorem 4.4]. We denote this extension as $\mathcal{L}_{h,\mathbf{A}}$, and we call it the *magnetic Laplacian*.

4.1.3 Compact resolvent property

From now, M is assumed to be a compact manifold (possibly with boundary). In the next lines, we will explain the way to get the magnetic Laplacian through Friedrichs procedure. Through that, we obtain the form domain of the operator naturally and then we prove the resolvent compactness thanks to this form domain. In other words, we are on the way proving Theorem 1.8 which is stated in Chapter 1. Recall that the Sobolev space $H^1(M)$ on a Riemannian manifold is the Hilbert space

$$H^1(M) = \{u \in L^2(M) : du \in L^2(\Lambda^1(M))\},$$

with the inner product

$$\langle u, v \rangle_{H^1(M)} = \langle u, v \rangle_{L^2(M)} + \langle du, dv \rangle_{L^2(\Lambda^1(M))}.$$

$H_0^1(M)$ is defined as the closure of $C_c^\infty(M)$ in $H^1(M)$ with respect to the associated norm

$$\|u\|_{H^1(M)} = \sqrt{\|u\|_{L^2(M)}^2 + \|du\|_{L^2(\Lambda^1(M))}^2}.$$

We denote by $Q_{h,\mathbf{A}}$ the sesquilinear form induced by the operator $H_{h,\mathbf{A}}$,

$$Q_{h,\mathbf{A}}(u, v) = \langle H_{h,\mathbf{A}}u, v \rangle_{L^2(M)} = \langle (-ihd - \mathbf{A})u, (-ihd - \mathbf{A})v \rangle_{L^2(\Lambda^1(M))},$$

for all $u, v \in C_c^\infty(M)$.

Let $h > 0$ and $u \in C_c^\infty(M)$. Notice that $\mathbf{A} \in C^1(\Lambda^1(M))$, then for every $\varepsilon > 0$

$$\begin{aligned} Q_{h,\mathbf{A}}(u, u) &= h^2 \langle du, du \rangle_{L^2(\Lambda^1(M))} + 2h \operatorname{Im} \langle \mathbf{A}u, du \rangle_{L^2(\Lambda^1(M))} + \langle \mathbf{A}u, \mathbf{A}u \rangle_{L^2(\Lambda^1(M))} \\ &\geq h^2 \|du\|_{L^2(\Lambda^1(M))}^2 - 2h\varepsilon \|du\|_{L^2(\Lambda^1(M))}^2 - \frac{2h}{\varepsilon} \|\mathbf{A}u\|_{L^2(\Lambda^1(M))}^2 \\ &= (h^2 - 2h\varepsilon) \|du\|_{L^2(\Lambda^1(M))}^2 - \frac{2h}{\varepsilon} \|\mathbf{A}\|_\infty^2 \|u\|_{L^2(M)}^2, \end{aligned}$$

where $\|\mathbf{A}\|_\infty = \max_{p \in M} \sqrt{g_p^*(A(p), A(p))}$.

By choosing ε small enough, we proved that there exist $c_0 > 0$ and $C_1 > 0$ (depending on h and \mathbf{A}) such that

$$Q_{h,\mathbf{A}}(u, u) + c_0 \|u\|_{L^2(M)}^2 \geq C_1 \|u\|_{H^1(M)}^2 \quad \text{for every } u \in C_c^\infty(M). \quad (4.2)$$

From the boundedness of \mathbf{A} on M , it is easy to see that there exists $C_2 > 0$ (depending on h and \mathbf{A}) such that

$$Q_{h,\mathbf{A}}(u, u) + c_0 \|u\|_{L^2(M)}^2 \leq C_2 \|u\|_{H^1(M)}^2 \quad \text{for every } u \in C_c^\infty(M). \quad (4.3)$$

Following the procedure of constructing Friedrichs extension, we define V as a completion in $L^2(M)$ for the norm

$$p_0(u) = \sqrt{Q_{h,\mathbf{A}}(u, u) + c_0 \|u\|_{L^2(M)}^2}.$$

Precisely, we have

$$V = \left\{ \begin{array}{l} u \in L^2(M) : \text{there exists } (u_n)_{n \in \mathbb{N}} \subset C_c^\infty(M) \text{ such that } u_n \rightarrow u \text{ in } L^2(M) \\ \text{and } (u_n)_{n \in \mathbb{N}} \text{ is a Cauchy sequence for the norm } p_0 \end{array} \right\}. \quad (4.4)$$

From (4.2) and (4.3), it implies that

$$V = H_0^1(M).$$

By the density of $C_c^\infty(M)$ in $H_0^1(M)$, we can define a sesquilinear form on $H_0^1(M)$:

$$\mathcal{B}(u, v) = Q_{h,\mathbf{A}}(u, v) + c_0 \langle u, v \rangle_{L^2(M)}.$$

This sesquilinear form satisfies all the conditions of Lax-Milgram Theorem A.6: the coercivity of \mathcal{B} deduced from the extension of (4.2) on $H_0^1(M)$,

$$\mathcal{B}(u, u) \geq C_1 \|u\|_{H^1(M)}^2 \quad \text{for all } u \in H_0^1(M), \quad (4.5)$$

and the continuity of \mathcal{B} comes from

$$\begin{aligned} & |\mathcal{B}(u, v)| \\ &= \left| h^2 \langle du, dv \rangle_{L^2(\Lambda^1(M))} - ih \langle \mathbf{A}u, dv \rangle_{L^2(\Lambda^1(M))} + ih \langle du, \mathbf{A}v \rangle_{L^2(\Lambda^1(M))} + c_0 \langle u, v \rangle_{L^2(M)} \right| \\ &\leq h^2 \|u\|_{H^1(M)} \|v\|_{H^1(M)} + 2h \|\mathbf{A}\|_\infty \|u\|_{H^1(M)} \|v\|_{H^1(M)} + c_0 \|u\|_{H^1(M)} \|v\|_{H^1(M)} \\ &\leq (h^2 + 2h \|\mathbf{A}\|_\infty + c_0) \|u\|_{H^1(M)} \|v\|_{H^1(M)}. \end{aligned}$$

It leads to the existence of a self-adjoint operator $\mathcal{H}_{h,\mathbf{A}}$ that satisfies

$$\langle \mathcal{H}_{h,\mathbf{A}}u, v \rangle_{L^2(M)} = \mathcal{B}_{h,\mathbf{A}}(u, v) \quad \forall u \in \text{Dom}(\mathcal{H}_{h,\mathbf{A}}), \forall v \in H_0^1(M), \quad (4.6)$$

and its domain is defined as a subspace of $H_0^1(M)$

$$\text{Dom}(\mathcal{H}_{h,\mathbf{A}}) = \left\{ u \in H_0^1(M) : \begin{array}{l} \text{The linear mapping } v \mapsto \mathcal{B}(u, v) \text{ is continuous} \\ \text{on } H_0^1(M) \text{ with the norm } \|\cdot\|_{L^2(M)} \end{array} \right\}. \quad (4.7)$$

The magnetic Laplacian is then defined by

$$\text{Dom}(\mathcal{L}_{h,\mathbf{A}}) = \text{Dom}(\mathcal{H}_{h,\mathbf{A}}), \quad \mathcal{L}_{h,\mathbf{A}} = \mathcal{H}_{h,\mathbf{A}} - c_0 \text{Id}.$$

The domain of the magnetic Laplacian will be discussed in the next section. At this stage, we aim at showing that the magnetic Laplacian has compact resolvent. To do that, we will prove this property for the self-adjoint operator $\mathcal{H}_{h,\mathbf{A}}$. Indeed, following Proposition A.9, we set the norm induced by the sesquilinear \mathcal{B} :

$$\|\cdot\|_{\mathcal{B}} = \sqrt{\mathcal{B}(\cdot, \cdot)}.$$

Notice that the injection $(H_0^1(M), \|\cdot\|_{\mathcal{B}}) \hookrightarrow L^2(M)$ can be seen as a composition of two injections:

$$(H_0^1(M), \|\cdot\|_{\mathcal{B}}) \xrightarrow{i_1} (H_0^1(M), \|\cdot\|_{H^1(M)}) \xrightarrow{i_2} L^2(M).$$

From the coercivity (4.5), it implies directly that the first injection i_1 is continuous. By the Rellich theorem (see for example [22]), the compactness of the manifold M implies that the embedding $(H_0^1(M), \|\cdot\|_{H^1(M)}) \xrightarrow{i_2} L^2(M)$ is compact. Thus, the injection $(H_0^1(M), \|\cdot\|_{\mathcal{B}}) \hookrightarrow L^2(M)$ is compact too. Therefore, the operator $\mathcal{H}_{h,\mathbf{A}}$ has compact resolvent.

Since $\mathcal{H}_{h,\mathbf{A}}$ is self-adjoint, the number $(c_0 + i)$ belongs to the resolvent set of $\mathcal{H}_{h,\mathbf{A}}$. Then, the resolvent

$$(\mathcal{L}_{h,\mathbf{A}} - i \text{Id})^{-1} = (\mathcal{H}_{h,\mathbf{A}} - (c_0 + i) \text{Id})^{-1}$$

is compact.

Finally, we can conclude that $\mathcal{L}_{h,\mathbf{A}}$ has compact resolvent. From this property, $\mathcal{L}_{h,\mathbf{A}}$ only has discrete spectrum and its spectrum is described by an unbounded increasing sequence.

4.1.4 Domain of the magnetic Laplacian

This subsection is devoted to the proof of Theorem 1.9 which is stated in Chapter 1. When we use the Friedrichs method to construct a self-adjoint extension from an initial operator, it seems that it is not easy to provide an explicit formula for the domain of that self-adjoint operator. In our case, when \mathbf{A} belongs to $C^1(\Lambda^1(M))$ we can give an expression for this abstract domain clearly. Namely, we will see that the initial operator $H_{h,\mathbf{A}} = (D_{h,\mathbf{A}})^*(D_{h,\mathbf{A}})$ is actually a differential operator. Then, the theory of distributions give an explicit description for the Magnetic Laplacian operator. Finally, it ends by regularity arguments.

Let us recall some elementary operators defined on a Riemannian manifold. First, we consider the Hodge operator

$$d^* : C^1(\Lambda^1 M) \rightarrow C(M),$$

which is a formal adjoint to d ,

$$\langle du, \omega \rangle_{L^2(\Lambda^1 M)} = \langle u, d^* \omega \rangle_{L^2(M)} \quad \text{for all } u \in C^1(M), \omega \in C^1(\Lambda^1 M).$$

In local coordinates, we have

$$d^* \omega = - \sum_{i,j=1}^2 \frac{1}{\sqrt{|G|}} \frac{\partial}{\partial x^j} \left(\sqrt{|G|} G^{ij} \omega_i \right), \quad \text{for } \omega = \sum_{i=1}^2 \omega_i dx^i. \quad (4.8)$$

Indeed, let $\varphi : M \rightarrow \mathbb{R}$ be a smooth function that has a compact support, then

$$\begin{aligned} \int_M (d^* \omega) \varphi d\mathcal{V}_g &= \int_M g^*(\omega, d\varphi) d\mathcal{V}_g \\ &= \int_M g^*(\omega, d\varphi) d\mathcal{V}_g \\ &= \int_M \sum_{i,j=1}^2 G^{ij} \omega_i \frac{\partial \varphi}{\partial x^j} \sqrt{|G|} dx^1 dx^2 \\ &= - \int_M \sum_{i,j=1}^2 \frac{1}{\sqrt{|G|}} \frac{\partial}{\partial x^j} \left(\sqrt{|G|} G^{ij} \omega_i \right) \varphi d\mathcal{V}_g, \end{aligned}$$

and since it is true for all $\varphi \in C_0^\infty(M, \mathbb{R})$, (4.8) is established. Note that d^* satisfies the following Leibniz rule

$$d^*(f\omega) = f d^* \omega - g^*(df, \omega) \quad \text{for all } f \in C^1(M), \omega \in C^1(\Lambda^1 M).$$

From d and d^* , we define the Laplacian (acting on functions) as

$$\Delta = -d^*d.$$

Besides, we identify the magnetic potential \mathbf{A} with the multiplication operator

$$\begin{aligned} \mathbf{A} : C^1(M) &\rightarrow C^1(\Lambda^1 M) \\ u &\mapsto uA. \end{aligned}$$

We also define the formal adjoint of \mathbf{A} by

$$\begin{aligned} \mathbf{A}^* : C^1(\Lambda^1 M) &\rightarrow C(M) \\ \mathbf{A}^* \omega &= g^*(\mathbf{A}, \omega). \end{aligned}$$

In local coordinates, \mathbf{A}^* is computed by

$$\mathbf{A}^* \omega = \sum_{i,j=1}^2 G^{ij} \omega_i A_j, \quad \text{for } \omega = \sum_{i=1}^2 \omega_i dx^i, \quad \mathbf{A} = \sum_{j=1}^2 A_j dx^j. \quad (4.9)$$

It is easy to see that

$$(\mathbf{D}_{h,\mathbf{A}})^* = ihd^* - \mathbf{A}^*.$$

Here the explicit formula for the magnetic Laplacian $(\mathbf{D}_{h,\mathbf{A}})^* \mathbf{D}_{h,\mathbf{A}}$ can be written as follows,

$$\begin{aligned} (\mathbf{D}_{h,\mathbf{A}})^* \mathbf{D}_{h,\mathbf{A}} u &= (ihd^* - \mathbf{A}^*) (-ihd - \mathbf{A}) u \\ &= h^2 d^* d u + ih \mathbf{A}^* d u - ih d^* (\mathbf{A} u) + \mathbf{A}^* (\mathbf{A} u) \\ &= -h^2 \Delta u + ih g^*(\mathbf{A}, du) - ih (d^* \mathbf{A}) u + ih g^*(du, \mathbf{A}) + g^*(\mathbf{A}, \mathbf{A}) u \\ &= -h^2 \Delta u + 2ih g^*(\mathbf{A}, du) + (-ih d^* \mathbf{A} + g^*(\mathbf{A}, \mathbf{A})) u. \end{aligned}$$

Moreover, using (4.8) and (4.9), the operator can be expressed as

$$(\mathbf{D}_{h,\mathbf{A}})^* \mathbf{D}_{h,\mathbf{A}} = \frac{1}{\sqrt{|G|}} \sum_{k,\ell=1}^2 (hD_k - A_k) \left[\sqrt{|G|} G^{k\ell} (hD_\ell - A_\ell) \right],$$

where $D_j := -i \frac{\partial}{\partial x^j}$.

It tells us that the operator $(D_{h,\mathbf{A}})^*D_{h,\mathbf{A}}$ is a differential operator. From (4.7), let us call to mind the definition of the domain of the magnetic operator

$$\text{Dom}(\mathcal{L}_{h,\mathbf{A}}) = \left\{ u \in H_0^1(M) : \text{there exists } f \in L^2(M) \text{ such that} \right. \\ \left. Q_{h,\mathbf{A}}(u, v) = \langle f, v \rangle_{L^2(M)} \quad \text{for all } v \in H_0^1(M) \right\}.$$

Using the arguments from distribution theory, it implies that

$$\text{Dom}(\mathcal{L}_{h,\mathbf{A}}) = \{u \in H_0^1(M) : (D_{h,\mathbf{A}})^*D_{h,\mathbf{A}}u \in L^2(M)\}.$$

Let $u \in \text{Dom}(\mathcal{L}_{h,\mathbf{A}})$, thus $u \in H_0^1(M)$ and

$$-h^2\Delta u + 2ihg^*(\mathbf{A}, du) + (-ihd^*\mathbf{A} + g^*(\mathbf{A}, \mathbf{A}))u \in L^2(M).$$

Since $\mathbf{A} \in C^1(\Lambda^1 M)$, we have

$$(-ihd^*\mathbf{A} + g^*(\mathbf{A}, \mathbf{A}))u \in L^2(M),$$

and

$$g^*(\mathbf{A}, du) \in H^{-1}(M).$$

By applying the elliptic regularity theorem [61, Chapter 5, Theorem 1.3] for $-\Delta u$, we have $u \in H^1(M)$. It follows that $g^*(\mathbf{A}, du) \in L^2(M)$. Once again, we obtain $u \in H^2(M)$ by the elliptic regularity theorem. That yields

$$\text{Dom}(\mathcal{L}_{h,\mathbf{A}}) = H_0^1(M) \cap H^2(M).$$

Remark 4.1. When M has no boundary, it is geodesically complete. We apply [57, Theorem 1.1] which tells that $H_{h,\mathbf{A}}$ is essentially self-adjoint, i.e. there exists a unique self-adjoint extension of $H_{h,\mathbf{A}}$, that is the magnetic Laplacian which is defined above. Moreover, since $H_0^1(M) = H^1(M)$, it follows that

$$\text{Dom}(\mathcal{L}_{h,\mathbf{A}}) = H^2(M). \tag{4.10}$$

4.1.5 In the case of \mathbb{R}^2

On the manifold, we defined the magnetic Laplacian which depends on \mathbf{A} and thus its spectrum also depends on \mathbf{A} . When the Riemannian manifold is now \mathbb{R}^2 with the Euclidean metric $g = (dq_1)^2 + (dq_2)^2$, we can observe that although the magnetic Laplacian is defined by a formula of \mathbf{A} , its spectrum depends only on the magnetic field B .

4.1.5.1 Redefinition of the magnetic Laplacian

Let us reintroduce the magnetic Laplacian on \mathbb{R}^2 whose spectrum investigation will be mentioned in Chapter 5. Let $\mathbf{A} = A_1 dq_1 + A_2 dq_2$ be the magnetic potential and B be the associated magnetic field which is given by the exterior derivative of \mathbf{A} :

$$d\mathbf{A} = B dq_1 \wedge dq_2,$$

or B is given by the formula

$$B = \frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2}.$$

Let us consider the space

$$\mathbf{H}_{h,\mathbf{A}}^1(\mathbb{R}^2) = \{u \in L^2(\mathbb{R}^2) : (-ih\nabla - \mathbf{A})u \in L^2(\mathbb{R}^2)\},$$

which is equipped with a sesquilinear form

$$\langle u, v \rangle_{\mathbf{H}_{h,\mathbf{A}}^1} := \int_{\mathbb{R}^2} (-ih\nabla - \mathbf{A})u \cdot \overline{(-ih\nabla - \mathbf{A})v} dq + \int_{\mathbb{R}^2} u \bar{v} dq, \quad \forall u, v \in \mathbf{H}_{h,\mathbf{A}}^1(\mathbb{R}^2). \quad (4.11)$$

The above expression $(-ih\nabla - \mathbf{A})u \in L^2(\mathbb{R}^2)$ is understood in the sense that

$$\int_{\mathbb{R}^2} |(-ih\partial_{q_1}u - A_1u)|^2 + |(-ih\partial_{q_2}u - A_2u)|^2 dq < \infty.$$

The dot product in the integral which involves with the 1-form \mathbf{A} in the sesquilinear (4.11) is understood in the sense that

$$\begin{aligned} (-ih\nabla - \mathbf{A})u \cdot \overline{(-ih\nabla - \mathbf{A})v} &= (-ih\partial_{q_1}u - A_1u) \overline{(-ih\partial_{q_1}v - A_1v)} \\ &\quad + (-ih\partial_{q_2}u - A_2u) \overline{(-ih\partial_{q_2}v - A_2v)}. \end{aligned}$$

We call $\mathbf{H}_{h,\mathbf{A}}^1(\mathbb{R}^2)$ the *magnetic Sobolev space*, this space has the two following useful properties:

i) The space $(\mathbf{H}_{h,\mathbf{A}}^1(\mathbb{R}^2), \langle u, v \rangle_{\mathbf{H}_{h,\mathbf{A}}^1})$ is a Hilbert space.

It is easy to see that the sesquilinear $\langle u, v \rangle_{\mathbf{H}_{h,\mathbf{A}}^1}$ define an inner product. We just need to show that every Cauchy sequence in $\mathbf{H}_{h,\mathbf{A}}^1(\mathbb{R}^2)$ converges in $\mathbf{H}_{h,\mathbf{A}}^1(\mathbb{R}^2)$ with the norm

$$\|u\|_{\mathbf{H}_{h,\mathbf{A}}^1} = \sqrt{\|(-ih\nabla - \mathbf{A})u\|_{L^2(\mathbb{R}^2)}^2 + \|u\|_{L^2(\mathbb{R}^2)}^2}, \quad \text{for } u \in \mathbf{H}_{h,\mathbf{A}}^1(\mathbb{R}^2).$$

Let $(u_n)_{n \in \mathbb{N}}$ be a Cauchy sequence with the norm $\|\cdot\|_{H_{h,\mathbf{A}}^1}$. Since $L^2(\mathbb{R}^2)$ is a Banach space, there exist u, v_1, v_2 in $L^2(\mathbb{R}^2)$ such that

$$\begin{cases} u_n \rightarrow u, \\ (-ih\partial_{q_1} - A_1)u_n \rightarrow v_1, \\ (-ih\partial_{q_2} - A_2)u_n \rightarrow v_2, \end{cases} \quad (4.12)$$

in $L^2(\mathbb{R}^2)$. We obtain, by distribution theory, the result that

$$\begin{cases} (-ih\partial_{q_1} - A_1)u = v_1 \in L^2(\mathbb{R}^2), \\ (-ih\partial_{q_2} - A_2)u = v_2 \in L^2(\mathbb{R}^2), \end{cases} \quad (4.13)$$

In other words, u_n converges to u in $H_{h,\mathbf{A}}^1(\mathbb{R}^2)$.

ii) $C_c^\infty(\mathbb{R}^2)$ is dense in $H_{h,\mathbf{A}}^1(\mathbb{R}^2)$.

A smooth compact support function after being applied the operation $(-ih\nabla - A)$ is also smooth and has compact support, thus $C_c^\infty(\mathbb{R}^2)$ is obvious contained in $H_{h,\mathbf{A}}^1(\mathbb{R}^2)$. To prove the density of $C_c^\infty(\mathbb{R}^2)$, we first consider its density in a smaller subspace of $H_{h,\mathbf{A}}^1(\mathbb{R}^2)$:

$$H_{c,h,\mathbf{A}}^1(\mathbb{R}^2) = \{f \in H_{h,\mathbf{A}}^1(\mathbb{R}^2) : f \text{ has compact support}\}.$$

Let $f \in H_{c,h,\mathbf{A}}^1(\mathbb{R}^2)$, since $A_i f \in L^2(\mathbb{R}^2)$, thus $\partial_{q_i} f \in L^2(\mathbb{R}^2)$ for $i = 1, 2$. Using the mollifiers ρ_n and setting up $f_n = \rho_n * f$, we have $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^2)$. By the property of the convolution, we obtain that

$$\begin{cases} f_n \rightarrow f \\ \nabla f_n \rightarrow \nabla f, \end{cases} \quad (4.14)$$

in $L^2(\mathbb{R}^2)$. Furthermore, since the supports of $(f_n - f)$ are contained in a fixed compact set, we have $\|A(f_n - f)\|_{L^2(\mathbb{R}^2)} \rightarrow 0$. We conclude that $C_c^\infty(\mathbb{R}^2)$ is dense in $H_{c,\mathbf{A}}^1(\mathbb{R}^2)$. Therefore, the proof is complete when we can demonstrate that $H_{c,h,\mathbf{A}}^1(\mathbb{R}^2)$ is dense in the Hilbert space $H_{h,\mathbf{A}}^1(\mathbb{R}^2)$. To do that, we consider a function $u \in H_{h,\mathbf{A}}^1(\mathbb{R}^2)$ satisfying

$$\int_{\mathbb{R}^2} (-ih\nabla - \mathbf{A})u \cdot \overline{(-ih\nabla - \mathbf{A})\varphi} dq + \int_{\mathbb{R}^2} u \bar{\varphi} dq = 0 \quad \forall \varphi \in H_{c,\mathbf{A}}^1(\mathbb{R}^2),$$

and we will show that $u = 0$.

Consider a smooth function χ with compact support and $0 \leq \chi \leq 1$ which is equal to 1 in the neighborhood of 0. We define $\chi_n(\cdot) = \chi(\frac{\cdot}{n})$. Then we have

$\chi_n u \in H_{c,h,\mathbf{A}}^1(\mathbb{R}^2)$ and

$$\frac{1}{n} \int_{\mathbb{R}^2} (-ih\nabla - \mathbf{A})u \cdot \overline{(-ihu\nabla\chi\left(\frac{\cdot}{n}\right))} dq + \int_{\mathbb{R}^2} \chi_n |(-ih\nabla - \mathbf{A})u|^2 dq + \int_{\mathbb{R}^2} \chi_n |u|^2 dq = 0,$$

for all $n \in \mathbb{N}$.

Since $u \in H_{h,\mathbf{A}}^1(\mathbb{R}^2)$ and $|\nabla\chi|$ is bounded above by a constant on \mathbb{R}^2 , with the application of Hölder inequality, we obtain

$$\left| \int_{\mathbb{R}^2} (-ih\nabla - \mathbf{A})u \cdot \overline{(-ihu\nabla\chi\left(\frac{\cdot}{n}\right))} dq \right| \leq C \|(-ih\nabla - \mathbf{A})u\|_{L^2(\mathbb{R}^2)} \|u\|_{L^2(\mathbb{R}^2)}.$$

Applying Fatou's lemma, it results that

$$\begin{aligned} \int_{\mathbb{R}^2} |(-ih\nabla - \mathbf{A})u|^2 dq + \int_{\mathbb{R}^2} |u|^2 dq &\leq \liminf_{n \rightarrow +\infty} \left(\int_{\mathbb{R}^2} \chi_n |(-ih\nabla - \mathbf{A})u|^2 dq + \int_{\mathbb{R}^2} \chi_n |u|^2 dq \right) \\ &\leq \liminf_{n \rightarrow +\infty} \frac{1}{n} \left| \int_{\mathbb{R}^2} (-ih\nabla - \mathbf{A})u \cdot \overline{(-ihu\nabla\chi\left(\frac{\cdot}{n}\right))} dq \right| \\ &= 0. \end{aligned}$$

Therefore, $u = 0$ and the statement of the density of $C_c^\infty(\mathbb{R}^2)$ is proved.

Applying Theorem A.6, we replace $H = L^2(\mathbb{R}^2)$ and $V = H_{h,\mathbf{A}}^1(\mathbb{R}^2)$ and the sesquilinear $Q(\cdot, \cdot) = \langle \cdot, \cdot \rangle_{H_{h,\mathbf{A}}^1}$. Obviously, Q is continuous, V -elliptic and Hermitian on V . Thus, Theorem A.6 provides the existence of a self-adjoint operator $\mathcal{S}_{h,\mathbf{A}}$ which has the domain

$$\begin{aligned} \text{Dom}(\mathcal{S}_{h,\mathbf{A}}) &= \{u \in H_{h,\mathbf{A}}^1(\mathbb{R}^2) : \exists w \in L^2(\mathbb{R}^2), Q(u, v) = \langle w, v \rangle_{L^2} \quad \forall v \in H_{h,\mathbf{A}}^1(\mathbb{R}^2)\} \\ &= \{u \in H_{h,\mathbf{A}}^1(\mathbb{R}^2) : \exists w \in L^2(\mathbb{R}^2), Q(u, v) = \langle w, v \rangle_{L^2} \quad \forall v \in C_c^\infty(\mathbb{R}^2)\} \\ &= \{u \in H_{h,\mathbf{A}}^1(\mathbb{R}^2) : (-ih\nabla - \mathbf{A})^2 u \in L^2(\mathbb{R}^2)\}. \end{aligned}$$

The second equality is given by the density of $C_c^\infty(\mathbb{R}^2)$ in $H_{h,\mathbf{A}}^1(\mathbb{R}^2)$ and we understand the term $(-ih\nabla + \mathbf{A})^2 u$ in the sense of distribution. The self-adjoint operator $\mathcal{S}_{h,\mathbf{A}}$ satisfies, for all $u \in \text{Dom}(\mathcal{S}_{h,\mathbf{A}})$, $v \in H_{h,\mathbf{A}}^1(\mathbb{R}^2)$,

$$\langle \mathcal{S}_{h,\mathbf{A}} u, v \rangle_{L^2(\mathbb{R}^2)} = \langle (-ih\nabla - \mathbf{A})u, (-ih\nabla - \mathbf{A})v \rangle_{L^2(\mathbb{R}^2)} + \langle u, v \rangle_{L^2(\mathbb{R}^2)}.$$

We define $\mathcal{L}_{h,\mathbf{A}}$ the magnetic Laplacian on \mathbb{R}^2 by

$$\mathcal{L}_{h,\mathbf{A}} = \mathcal{S}_{h,\mathbf{A}} - \text{Id},$$

where Id is the identity operator on $\text{Dom}(\mathcal{S}_{h,\mathbf{A}})$. Therefore, the magnetic Laplacian $\mathcal{L}_{h,\mathbf{A}}$ has the domain

$$\text{Dom}(\mathcal{L}_{h,\mathbf{A}}) = \{u \in H_{h,\mathbf{A}}^1(\mathbb{R}^2) : (-ih\nabla - \mathbf{A})^2 u \in L^2(\mathbb{R}^2)\},$$

and for all $u \in \text{Dom}(\mathcal{L}_{h,\mathbf{A}})$, $v \in H_{h,\mathbf{A}}^1(\mathbb{R}^2)$

$$\langle \mathcal{L}_{h,\mathbf{A}} u, v \rangle_{L^2(\mathbb{R}^2)} = \langle (-ih\nabla - \mathbf{A})u, (-ih\nabla - \mathbf{A})v \rangle_{L^2(\mathbb{R}^2)}. \quad (4.15)$$

Furthermore, we can make the domain of $\mathcal{L}_{h,\mathbf{A}}$ simpler by considering the operator

$$H_{h,\mathbf{A}} = (-ih\nabla - \mathbf{A})^2 \text{ with } D(H_{h,\mathbf{A}}) = C_c^\infty(\mathbb{R}^2). \quad (4.16)$$

This operator is symmetric, hence it is closable. Let $\hat{H}_{h,\mathbf{A}}$ denote the closure operator of $H_{h,\mathbf{A}}$, we have

$$\text{Dom}(\hat{H}_{h,\mathbf{A}}) = \{u \in L^2(\mathbb{R}^2) : (-ih\nabla - \mathbf{A})^2 u \in L^2(\mathbb{R}^2)\} \text{ and } \hat{H}_{h,\mathbf{A}} = (-ih\nabla - \mathbf{A})^2. \quad (4.17)$$

The equality (4.15) leads to

$$\langle \mathcal{L}_{h,\mathbf{A}} u, v \rangle_{L^2(\mathbb{R}^2)} = \langle (-ih\nabla - \mathbf{A})u, (-ih\nabla - \mathbf{A})v \rangle_{L^2(\mathbb{R}^2)} = \langle H_{h,\mathbf{A}} u, v \rangle_{L^2(\mathbb{R}^2)},$$

for all $u, v \in C_c^\infty(\mathbb{R}^2)$. It implies that $\mathcal{L}_{h,\mathbf{A}}$ is a self-adjoint extension of $H_{h,\mathbf{A}}$. Since $\mathbf{A} \in C^1(\mathbb{R}^2)$, the operator $H_{h,\mathbf{A}}$ is essentially self-adjoint (see [17]), then its closure $\hat{H}_{h,\mathbf{A}}$ is self-adjoint. By the uniqueness of the self-adjoint extension of an essentially self-adjoint operator, we conclude that

$$\mathcal{L}_{h,\mathbf{A}} = \hat{H}_{h,\mathbf{A}}.$$

Therefore, we can rewrite the domain of the magnetic Laplacian $\mathcal{L}_{h,\mathbf{A}}$ in an easier way

$$\text{Dom}(\mathcal{L}_{h,\mathbf{A}}) = \{u \in L^2(\mathbb{R}^2) : (-ih\nabla - \mathbf{A})^2 u \in L^2(\mathbb{R}^2)\}, \quad (4.18)$$

and the explicit formula for $\mathcal{L}_{h,\mathbf{A}}$

$$\mathcal{L}_{h,\mathbf{A}} = (-ih\nabla - \mathbf{A})^2.$$

4.1.5.2 Spectrum of the magnetic Laplacian

First of all, let us say a few words about the gauge invariance of the spectrum of the magnetic Laplacian. Let \mathbf{A} and $\tilde{\mathbf{A}}$ be the magnetic potentials associated with B ,

$$d\mathbf{A} = d\tilde{\mathbf{A}} = Bdq_1 \wedge dq_2.$$

By Lemma 2.10, there exists a smooth function ϕ on \mathbb{R}^2 such that

$$\tilde{\mathbf{A}} = \mathbf{A} + d\phi.$$

Notice that

$$e^{i\phi/h}(-ih\nabla - \mathbf{A})^2 e^{-i\phi/h} = (-ih\nabla - \mathbf{A} - d\phi)^2 = (-ih\nabla - \tilde{\mathbf{A}})^2.$$

Thus, two operators $\mathcal{L}_{h,\mathbf{A}}$ and $\mathcal{L}_{h,\tilde{\mathbf{A}}}$ are unitary equivalent. In other words, we showed that the spectrum of magnetic Laplacian is independent from the choice of magnetic potential:

$$\mathrm{Sp}(\mathcal{L}_{h,\mathbf{A}}) = \mathrm{Sp}(\mathcal{L}_{h,\tilde{\mathbf{A}}}). \quad (4.19)$$

From (4.15), for every $u \in \mathrm{Dom}(\mathcal{L}_{h,\mathbf{A}})$, we have the inequality

$$\langle \mathcal{L}_{h,\mathbf{A}} u, u \rangle_{L^2(\mathbb{R}^2)} \geq 0.$$

It shows that $\mathcal{L}_{h,\mathbf{A}}$ is positive, thus $\mathrm{Sp}(\mathcal{L}_{h,\mathbf{A}})$ is contained in $[0, +\infty)$ (see [25, Proposition 11.2]). According to Lax-Milgram (see Theorem A.6), the magnetic Laplacian is bijective from its domain onto $L^2(\mathbb{R}^2)$, it tells us that $0 \in \rho(\mathcal{L}_{h,\mathbf{A}})$. Therefore, we have:

$$\mathrm{Sp}(\mathcal{L}_{h,\mathbf{A}}) \subset (0, +\infty).$$

Recall that $\mathbf{B} = B(q)dq_1 \wedge dq_2$, when B is non-negative, we have the following result:

Theorem 4.2. *For all $u \in H_{h,\mathbf{A}}^1(\mathbb{R}^2)$,*

$$Q_{h,\mathbf{A}}(u) := \int_{\mathbb{R}^2} |(-ih\nabla - \mathbf{A})u|^2 dq \geq h \int_{\mathbb{R}^2} B|u|^2 dq. \quad (4.20)$$

Proof. Since \mathbf{B} is non-negative, the integral in the right-hand side of (4.20) is well defined. We will start the proof with the function u in $C_c^\infty(\mathbb{R}^2)$ first and then we use the density of $C_c^\infty(\mathbb{R}^2)$ in $H_{h,\mathbf{A}}^1(\mathbb{R}^2)$ to finish the proof. Notice that

$$[-ih\partial_{q_1} - A_1, -ih\partial_{q_2} - A_2] = -ihB.$$

Here, the bracket $[\cdot, \cdot]$ is the commutator of two operators which is defined by, for two operators T_1, T_2 ,

$$[T_1, T_2] = T_1 T_2 - T_2 T_1.$$

Thus, we have

$$\langle [-ih\partial_{q_1} - A_1, -ih\partial_{q_2} - A_2]u, u \rangle = -ih \int_{\mathbb{R}^2} B|u|^2 dq.$$

Integrating by part and applying the Cauchy-Schwarz inequality, it leads to

$$\begin{aligned} |\langle [-ih\partial_{q_1} - A_1, -ih\partial_{q_2} - A_2]u, u \rangle| &\leq 2 \|(-ih\partial_{q_1} - A_1)u\|_{L^2(\mathbb{R}^2)} \|(-ih\partial_{q_2} - A_2)u\|_{L^2(\mathbb{R}^2)} \\ &\leq \|(-ih\partial_{q_1} - A_1)u\|_{L^2(\mathbb{R}^2)}^2 + \|(-ih\partial_{q_2} - A_2)u\|_{L^2(\mathbb{R}^2)}^2. \end{aligned}$$

Therefore, we can conclude that

$$\int_{\mathbb{R}^2} |(-ih\nabla - \mathbf{A})u|^2 dq \geq h \int_{\mathbb{R}^2} B|u|^2 dq \quad \text{for all } u \in C_c^\infty(\mathbb{R}^2).$$

Now, let $u \in H_{h,\mathbf{A}}^1(\mathbb{R}^2)$. Since $C_c^\infty(\mathbb{R}^2)$ is dense in $H_{h,\mathbf{A}}(\mathbb{R}^2)$, there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R}^2)$ such that u_n converges to u . We have

$$\int_{\mathbb{R}^2} |(-ih\nabla - \mathbf{A})u_n|^2 dx \geq h \int_{\mathbb{R}^2} B|u_n|^2 dx. \quad (4.21)$$

Let n go to $+\infty$ and apply Fatou's lemma, we get

$$\begin{aligned} \int_{\mathbb{R}^2} |(-ih\nabla - \mathbf{A})u|^2 dx &= \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} |(-ih\nabla - \mathbf{A})u_n|^2 dx \\ &\geq h \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} B|u_n|^2 dx \\ &\geq h \int_{\mathbb{R}^2} B|u|^2 dx. \end{aligned}$$

□

Let us discuss two applications of Theorem 4.2. Its first application is to prove the compact resolvent property of the magnetic Laplacian when B blows up at infinity. The second one is to allow us to estimate the lowest spectrum. In details, we have the following theorem:

Theorem 4.3. *The magnetic Laplacian $\mathcal{L}_{h,\mathbf{A}}$ has compact resolvent if*

$$\lim_{\|q\| \rightarrow \infty} B(q) = +\infty.$$

Proof. Applying Proposition A.9, the plan is to show that the injection

$$(H_{h,\mathbf{A}}^1(\mathbb{R}^2), \|\cdot\|_{H_{h,\mathbf{A}}^1}) \hookrightarrow (L^2(\mathbb{R}^2), \|\cdot\|_{L^2})$$

is compact. Let $B(0;1) = \left\{u \in H_{h,\mathbf{A}}^1(\mathbb{R}^2) : \|u\|_{H_{h,\mathbf{A}}^1} < 1\right\}$ be the unit ball in $H_{h,\mathbf{A}}^1(\mathbb{R}^2)$. We use Riesz-Fréchet-Kolmogorov criterion (see [49, Page 34]) to demonstrate that $B(0;1)$ is relatively compact in $L^2(\mathbb{R}^2)$:

i) $B(0;1)$ is a bounded subset of $L^2(\mathbb{R}^2)$ since

$$\|u\|_{L^2} \leq \|u\|_{H_{h,\mathbf{A}}^1} < 1,$$

for all $u \in B(0;1)$.

ii) Let $\varepsilon > 0$. For all $u \in B(0;1)$, using Theorem 4.2, we obtain

$$h \int_{\mathbb{R}^2} B|u|^2 dq \leq 1.$$

Under the condition of the magnetic field B at infinity, there exists a ball

$$\Omega_R := \{q \in \mathbb{R}^2 : \|q\| < R\}$$

in \mathbb{R}^2 such that

$$B(q) \geq \frac{1}{h\varepsilon^2} \quad \text{for all } q \in \mathbb{R}^2 \setminus \Omega_R.$$

Thus, it is easy to see that

$$\|u\|_{L^2(\mathbb{R}^2 \setminus \Omega_R)} \leq \varepsilon.$$

iii) Let $\varepsilon > 0$ and $\omega \subset\subset \mathbb{R}^2$. Since $C_c^\infty(\mathbb{R}^2)$ is dense in $H_{h,\mathbf{A}}^1(\mathbb{R}^2)$, for each $u \in B(0;1)$, there exists $w_u \in C_c^\infty(\mathbb{R}^2)$ such that

$$\|u - w_u\|_{H_{h,\mathbf{A}}^1} \leq \frac{\varepsilon}{3}.$$

It implies directly that $\|w_u\|_{H_{h,\mathbf{A}}^1} \leq 2$ for all $u \in B(0;1)$. Let $\Omega \supset \omega$ is a compact set such that $\text{dist}(\omega, \partial\Omega) \geq 1$. Because of the continuity of \mathbf{A} on Ω , there exists a constant C_1 (independent of u) such that

$$\|Aw_u\|_{L^2(\Omega)} \leq C_1.$$

As a consequence, there exists a constant C_2 (independent of u) such that

$$\|\nabla w_u\|_{L^2(\Omega)} \leq C_2.$$

Let $k \in \mathbb{R}^2$ such that $\|k\| \leq 1$, we have

$$\|\tau_k w_u - w_u\|_{L^2(\omega)} \leq \|\nabla w_u\|_{L^2(\Omega)} \|k\| \leq C_2 \|k\|,$$

where $\tau_k v(q) := v(q + k)$. It deduces that

$$\|\tau_k u - u\|_{L^2(\omega)} \leq \frac{2\varepsilon}{3} + C_2 \|k\|.$$

Therefore, there exists $\delta > 0$ (independent of u) such that

$$\|\tau_k u - u\|_{L^2(\omega)} \leq \varepsilon \quad \text{for all } k \in \mathbb{R}^2 \text{ such that } \|k\| \leq \delta.$$

□

Combine (4.20) with the min-max principle for the self-adjoint operator $\mathcal{L}_{h,\mathbf{A}}$, we obtain the estimation for the infimum of the spectrum

$$\inf \text{Sp}(\mathcal{L}_{h,\mathbf{A}}) = \inf_{\substack{u \in H_{h,\mathbf{A}}^1(\mathbb{R}^2) \\ u \neq 0}} \frac{Q_{h,\mathbf{A}}(u)}{\|u\|_{L^2(\mathbb{R}^2)}} \geq hb_0,$$

where $b_0 := \inf_{q \in \mathbb{R}^2} B(q)$. Thus,

$$\text{Sp}(\mathcal{L}_{h,\mathbf{A}}) \subset [hb_0, +\infty). \quad (4.22)$$

Now, we study the essence of spectrum of the magnetic Laplacian, is it discrete or continuous? The answer will depend on the behaviour of the magnetic field at infinity. Explicitly, we have the below result which appears in [31, 51]:

Theorem 4.4. *Assume that $b_0 > 0$ and*

$$\liminf_{\|q\| \rightarrow +\infty} B(q) = b_0 + \eta_0, \quad (4.23)$$

with some $\eta_0 > 0$.

Then, for any $\eta_1 \in (0, \eta_0)$, the spectrum lying in the interval $[hb_0, h(b_0 + \eta_1))$, if exists, is discrete.

Proof. This theorem is a direct consequence of the characterization of the bottom of the essential spectrum of the magnetic Laplacian (see [17, Appendix B]):

$$\inf \operatorname{Sp}_{\text{ess}}(\mathcal{L}_{h,\mathbf{A}}) = \sup_{K \subset \mathbb{R}^2} \left(\inf_{\substack{u \in C_c^\infty(\mathbb{R}^2 \setminus K) \\ \|u\|_{L^2(\mathbb{R}^2)} \leq 1}} \langle \mathcal{L}_{h,\mathbf{A}} u, u \rangle \right),$$

where the supremum is taken over all compact subsets $K \subset \mathbb{R}^2$. From (4.23), there exists $M > 0$ such that

$$B(q) \geq b_0 + \eta_1 \quad \text{for all } q \in B(0; M) := \{q \in \mathbb{R}^2 : \|q\| \leq M\}.$$

Combining with 4.2, we deduce that

$$\langle \mathcal{L}_{h,\mathbf{A}} u, u \rangle \geq h(b_0 + \eta_1),$$

for all $u \in C_c^\infty(\mathbb{R}^2 \setminus B(0; M))$ such that $\|u\|_{L^2(\mathbb{R}^2)} \leq 1$. Therefore, we must have

$$\inf \operatorname{Sp}_{\text{ess}}(\mathcal{L}_{h,\mathbf{A}}) \geq h(b_0 + \eta_1),$$

and the conclusion follows. □

4.2 Isothermal coordinates

4.2.1 Construction of the isothermal coordinates

In this section, we would like to show that the isothermal coordinates always exist locally in the two-dimensional Riemannian manifolds. To do that, let us start by recalling the definition of a gradient of a smooth function on a Riemannian manifold.

Definition 4.5. Let $f : M \rightarrow \mathbb{R}$ be a smooth function, we define the gradient of f as $\nabla^g f := \hat{g}^{-1}(df)$, where \hat{g} is defined in (4.1). In other words, it is a unique vector field such that, for all $p \in \Omega$ and for all $X_p \in T_p \Omega$,

$$g((\nabla^g f)_p, X_p) = df_p(X_p) = X_p f.$$

Theorem 4.6. Let $p \in M$, local isothermal coordinates exist around p .

Remark 4.7. The reader is invited to compare this following proof with the proof in [61, Page 438] which used the Hodge star operator to show the existence of the coordinates.

Proof. Let us fix a local chart $(W, \Psi : W \rightarrow \Psi(W) \subset \mathbb{R}^2)$ around p with coordinates x^1, x^2 . We need to look for a diffeomorphism ϕ created from two real-valued smooth functions defined on some small neighborhood of p :

$$\phi(y) = (u(y), v(y)),$$

such that there exists a smooth function η defined on $\phi(\Omega) \subset \mathbb{R}^2$ satisfying

$$g(V, V) = e^{2\eta} g_0(d\phi_y V, d\phi_y V) \quad \text{for all } V \in T_y M. \quad (4.24)$$

Let us sketch the proof. The proof is divided into two main steps: first, we assume that there exists a small neighborhood Ω centered at p such that if we have u and v satisfying the equation (with a condition):

$$\begin{cases} (\nabla^{\perp g} v)_y = (\nabla^g u)_y, \\ (\nabla^g v)_y \neq 0, \end{cases} \quad (4.25)$$

for all $y \in \Omega$, then (4.24) is established. Here R is the rotation for the angle $\frac{\pi}{2}$ represented by the matrix

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and $\nabla^{\perp g} v$ is a vector field which has the coordinates

$$[\nabla^{\perp g} v] := G^{-\frac{1}{2}} R G^{\frac{1}{2}} [\nabla^g v].$$

Second, the existence of Ω and u, v which satisfy (4.25) is showed by the theory of the linear elliptic equations. We start with the first step:

Step 1: We assume that u, v satisfy (4.25) on Ω .

In this theorem, we denote $\nabla^e w := \begin{pmatrix} \frac{\partial w}{\partial x^1} \\ \frac{\partial w}{\partial x^2} \end{pmatrix}$. Since $[\nabla^g v] = G^{-1} \nabla^e v$, we can rewrite $[\nabla^{\perp g} v] = |G|^{-\frac{1}{2}} (\nabla^e v)^\perp$. Two vector fields $\nabla^g v, \nabla^{\perp g} v$ satisfy

$$\begin{aligned} g(\nabla^g v, \nabla^{\perp g} v) &= [\nabla^g v]^T G [\nabla^{\perp g} v] \\ &= |G|^{-\frac{1}{2}} (\nabla^e v)^T (G^{-1})^T G (\nabla^e v)^\perp \\ &= |G|^{-\frac{1}{2}} (\nabla^e v)^T (\nabla^e v)^\perp \\ &= 0, \end{aligned}$$

and

$$\begin{aligned}
g(\nabla^{\perp g}v, \nabla^{\perp g}v) &= [\nabla^{\perp g}v]^T G [\nabla^{\perp g}v] \\
&= \left(G^{-\frac{1}{2}} R G^{\frac{1}{2}} [\nabla^g v]\right)^T G \left(G^{-\frac{1}{2}} R G^{\frac{1}{2}} [\nabla^g v]\right) \\
&= [\nabla^g v]^T G [\nabla^g v] \\
&= g(\nabla^g v, \nabla^g v).
\end{aligned}$$

Since $\nabla^g v \neq 0$, the set $\{\nabla^g v, \nabla^{\perp g}v\}$ become the orthogonal frame on Ω . Let V be a tangent vector in $T_y M$. We can represent $V = \alpha(\nabla^g v)_y + \beta(\nabla^{\perp g}v)_y$ and compute the push-forward tangent vector $d\phi_y V$ in \mathbb{R}^2 as

$$d\phi_y V = \left(\alpha(\nabla^g v)_y u + \beta(\nabla^{\perp g}v)_y u\right) \frac{\partial}{\partial x^1} + \left(\alpha(\nabla^g v)_y v + \beta(\nabla^{\perp g}v)_y v\right) \frac{\partial}{\partial x^2}.$$

Now we use the equalities

$$\begin{aligned}
dv_y(\cdot) &= g((\nabla^g v)_y, \cdot), \\
du_y(\cdot) &= g((\nabla^{\perp g}v)_y, \cdot),
\end{aligned}$$

to calculate two sides of (4.24). That is

$$\begin{aligned}
g_0(d\phi_y V, d\phi_y V) &= \left(\alpha(\nabla^g v)_y u + \beta(\nabla^{\perp g}v)_y u\right)^2 + \left(\alpha(\nabla^g v)_y v + \beta(\nabla^{\perp g}v)_y v\right)^2 \\
&= \left(\alpha du_y(\nabla^g v)_y + \beta du_y(\nabla^{\perp g}v)_y\right)^2 + \left(\alpha dv_y(\nabla^g v)_y + \beta dv_y(\nabla^{\perp g}v)_y\right)^2 \\
&= (\alpha^2 + \beta^2)g((\nabla^g v)_y, (\nabla^g v)_y)^2.
\end{aligned}$$

The left hand side of (4.24) is

$$\begin{aligned}
g(V, V) &= g(\alpha(\nabla^g v)_y + \beta\nabla^{\perp g}v)_y, \alpha(\nabla^g v)_y + \beta\nabla^{\perp g}v)_y) \\
&= (\alpha^2 + \beta^2)g((\nabla^g v)_y, (\nabla^g v)_y).
\end{aligned}$$

Therefore, we get the result (4.24) by setting up $\eta = \frac{1}{2} \log [g(\nabla^g v, \nabla^g v)]^{-1}$, which is a smooth function defined on $\phi(\Omega) \subset \mathbb{R}^2$. The diffeomorphism property of ϕ follows from the linear independence of two non-zero gradient vectors $\nabla^e u$ and $\nabla^e v$.

Step 2: Let us now consider the equation (4.25). Since we are looking for local coordinates, (4.25) can be reformulated as

$$\begin{cases} (\nabla^e v)^\perp = |G|^{\frac{1}{2}} G^{-1} \nabla^e u, \\ (\nabla^g v)_p \neq 0, \end{cases} \quad (4.26)$$

Here p is the point under consideration in the statement of the theorem. By the classical Poincaré lemma, the existence of v such that $(\nabla^g v)_p \neq 0$ is given if there exists u such that $L_g u = 0$ and $(\nabla^g u)_p \neq 0$ where

$$L_g := \nabla^e \cdot \left(|G|^{\frac{1}{2}} G^{-1} \right) \nabla^e.$$

Let us clarify this clause. Indeed, we assume that there exists u such that

$$\begin{cases} \nabla^e \cdot \left(|G|^{\frac{1}{2}} G^{-1} \right) \nabla^e u = 0 \\ (\nabla^g u)_p \neq 0. \end{cases} \quad (4.27)$$

Let us denote $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} := \left(|G|^{\frac{1}{2}} G^{-1} \right) \nabla^e u$ and $\alpha := U_2 dx^1 - U_1 dx^2$ be a 1-form on W , we have

$$d\alpha = - \left(\frac{\partial U_1}{\partial x^1} + \frac{\partial U_2}{\partial x^2} \right) dx^1 \wedge dx^2 = 0.$$

By Poincaré lemma, there exists a smooth function v such that $\alpha = dv$ (here we assume that the local chart allows the exactness of a closed form, see [39, Corollary 17.15]). In other words, we have

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} -\frac{\partial v}{\partial x^2} \\ \frac{\partial v}{\partial x^1} \end{pmatrix}.$$

Therefore, v satisfies the equation (4.26) and $\nabla^e v \neq 0$ (from the relation of u and v in (4.26)).

Now, we focus on finding a function u which satisfies (4.27). This equation and the condition of the solution are rewritten as

$$\begin{cases} \sum_{i,j=1}^2 \partial_{x^i} \left(|G|^{\frac{1}{2}} G^{ij} \partial_{x^j} u \right) = 0 \\ (\nabla^e u)_p \neq 0. \end{cases} \quad (4.28)$$

Without loss of generality, we can assume that $\Psi(p) = 0$ and that the preimage of the ball $\overline{B(0,1)} = \{q \in \mathbb{R}^2 : \|q\| \leq 1\}$ contained in W . Let (q^1, q^2) be the coordinates in \mathbb{R}^2 and let

$$A^{ij} := |G(\Psi^{-1})|^{\frac{1}{2}} G^{ij}(\Psi^{-1}),$$

for all $i, j \in \{1, 2\}$.

Let $\epsilon > 0$, we consider a problem on \mathbb{R}^2 : Finding a smooth solution $\tilde{u} : B(0; \epsilon) \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} \sum_{i,j=1}^2 \partial_{q^i} (A^{ij}(q) \partial_{q^j} \tilde{u}_\epsilon) = 0 & \text{on } B(0; \epsilon), \\ \tilde{u}_\epsilon(q) = \frac{q_1}{\epsilon} & \text{on } \partial B(0; \epsilon). \end{cases} \quad (4.29)$$

We will show that for $\epsilon >$ small enough, $\partial_{q^1} \tilde{u}(0) \neq 0$ and the proof is complete as we put $u = \tilde{u} \circ \Psi$. To do that, let us consider the Dirichlet problem on the $B(0; \epsilon)$, by setting $\tilde{w}_\epsilon(q) = \tilde{u}_\epsilon(q) - \frac{q_1}{\epsilon}$. We have a PDE:

$$\begin{cases} \sum_{i,j=1}^2 \partial_{q^i} (A^{ij}(q) \partial_{q^j} \tilde{w}_\epsilon) = \frac{1}{\epsilon} f & \text{on } B(0; \epsilon), \\ \tilde{w}_\epsilon(q) = 0 & \text{on } \partial B(0; \epsilon), \end{cases} \quad (4.30)$$

where $f(q) = -\left(\frac{\partial A^{11}}{\partial q_1} + \frac{\partial A^{21}}{\partial q_1}\right)$.

By changing variable $q = \epsilon t$ and setting up $w_\epsilon(t) = \tilde{w}_\epsilon(q)$, we obtain an equivalent PDE:

$$\begin{cases} \sum_{i,j=1}^2 \partial_{t^i} (A^{ij}(\epsilon t) \partial_{t^j} w_\epsilon) = \epsilon f(\epsilon t) & \text{on } B(0; 1), \\ w_\epsilon(t) = 0 & \text{on } \partial B(0; 1). \end{cases} \quad (4.31)$$

Since $G(y)^{-1}$ is a positive definite matrix at each point $y \in W$ and its minimum eigenvalue $\lambda_{\min}(y)$ is a positive continuous function on W according to variable y . Then, there exists a constant c_0 such that

$$\sum_{i,j=1}^2 G^{ij}(y) \xi^i \xi^j = \langle G(y)^{-1} \xi, \xi \rangle \geq \lambda_{\min}(y) \|\xi\|^2 \geq c_0 \|\xi\|^2. \quad (4.32)$$

for all $\xi \in \mathbb{R}^2$.

Therefore, our PDE (according to variable t) is also uniformly elliptic:

$$\sum_{i,j=1}^2 A^{ij}(\epsilon t) \xi^i \xi^j \geq c_0 \|\xi\|^2 \quad \text{for all } t \in B(0; 1), \xi \in \mathbb{R}^2.$$

By Lax-Milgram theorem, there exists a unique weak solution $w_\epsilon \in H_0^1(B(0; 1))$ satisfying the PDE (4.31), i.e.

$$Q(w_\epsilon, b) = \epsilon \langle f(\epsilon t), b \rangle_{L^2(B(0;1))} \quad \text{for all } b \in H_0^1(B(0; 1)),$$

where Q is the bilinear form defined by

$$Q(a, b) = \int_{B(0;1)} \sum_{i,j=1}^2 A^{ij}(\varepsilon t) (\partial_{t^i} a) (\partial_{t^j} b) \, dq.$$

By Poincaré inequality, we have

$$\|w_\varepsilon\|_{L^2(B(0;1))}^2 \leq C \|\nabla w_\varepsilon\|_{L^2(B(0;1))}^2 \leq CQ(w_\varepsilon, w_\varepsilon) \leq C\varepsilon \|f(\varepsilon t)\|_{L^2(B(0;1))} \|w_\varepsilon\|_{L^2(B(0;1))}$$

From the boundedness of f , there exists $M > 0$ (not depending on ε) such that

$$\|w_\varepsilon\|_{L^2(B(0;1))} \leq M\varepsilon.$$

From the regularity arguments, we obtain, for all $m \in \mathbb{N}$,

$$\|w_\varepsilon\|_{H^{m+2}(B(0;1))} \leq \varepsilon \|f(\varepsilon t)\|_{H^m(B(0;1))} + \|w_\varepsilon\|_{L^2(B(0;1))} \leq C\varepsilon.$$

By choosing m large enough such that $H^m(B(0, 1)) \hookrightarrow C^1(\overline{B(0; 1)})$, then we have

$$\left| \frac{\partial w_\varepsilon}{\partial t^1}(0) \right| \leq \|w_\varepsilon\|_{C^1} \leq C\varepsilon.$$

Since $\frac{\partial w_\varepsilon}{\partial t^1}(0) = \varepsilon \frac{\partial \tilde{w}_\varepsilon}{\partial q^1}(0)$, it implies that $\left| \frac{\partial \tilde{w}_\varepsilon}{\partial q^1}(0) \right|$ is bounded above. By choosing ε small enough we have the conclusion. □

4.2.2 The magnetic Laplacian in isothermal coordinates.

We can start the journey to prove Theorem 1.14 now. Let $p^* \in M$ be the point in Theorem 1.14. By Theorem 4.6, there exists an isothermal local chart $(\Omega, \phi : \Omega \rightarrow \phi(\Omega))$ centered at p^* . Regarding Definition 1.13, we set up $U := \phi(\Omega)$ which is an open set in \mathbb{R}^2 and $\tilde{g} := e^{2\eta} g_0$ be the metric on U in \mathbb{R}^2 . We use the conformal map

$$\varphi := \phi^{-1} : U \rightarrow \Omega,$$

to pull-back the sesquilinear from manifold to \mathbb{R}^2 .

Then, for all $u, v \in C_c^\infty(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} g^*((-ihd - \mathbf{A})u, (-ihd - \mathbf{A})v) \, d\mathcal{V}_g &= \int_U \varphi^*(g^*((-ihd - \mathbf{A})u, (-ihd - \mathbf{A})v)) \, d\mathcal{V}_g \\ &= \int_U \tilde{g}^*((-ihd - \varphi^* \mathbf{A})\tilde{u}, (-ihd - \varphi^* \mathbf{A})\tilde{v}) |\tilde{G}|^{\frac{1}{2}} \, dq, \end{aligned}$$

where $\tilde{u} := \varphi^*u$ and $\tilde{v} := \varphi^*v$.

Let $\mathcal{M} = \varphi^*\mathbf{A}$ be the pull-back of 1-form \mathbf{A} . By considering \mathcal{M} as a function vector $(\mathcal{M}_1, \mathcal{M}_2)^T$ and note that $\tilde{G} = \begin{pmatrix} e^{-2\eta} & 0 \\ 0 & e^{-2\eta} \end{pmatrix}$, we have

$$\begin{aligned} & \int_{\Omega} g^*((-i\hbar d - \mathbf{A})u, (-i\hbar d - \mathbf{A})v) d\mathcal{V}_g \\ &= \int_U \left\langle \tilde{G}^{-1}(-i\hbar\nabla_q - \mathcal{M})\tilde{u}, (-i\hbar\nabla_q - \mathcal{M})\tilde{v} \right\rangle_{\mathbb{C}^2} |\tilde{G}|^{\frac{1}{2}} dq \\ &= \int_U e^{-2\eta} \left\langle (-i\hbar\nabla_q - \mathcal{M})\tilde{u}, (-i\hbar\nabla_q - \mathcal{M})\tilde{v} \right\rangle_{\mathbb{C}^2} e^{2\eta} dq \\ &= \int_U \left[(-i\hbar\partial_{q_1} - \mathcal{M}_1)^2 + (-i\hbar\partial_{q_2} - \mathcal{M}_2)^2 \right] \tilde{u}\tilde{v} dq, \end{aligned}$$

where the last equality is obtained by the integration by part.

We label $\mathcal{L}_{h,\mathcal{M}}$ be an operator acting on $L^2(U, e^{2\eta}dq)$ defined by

$$\begin{cases} \text{Dom}(\mathcal{L}_{h,\mathcal{M}}) = C_c^\infty(U), \\ \mathcal{L}_{h,\mathcal{M}} = e^{-2\eta} \left[(-i\hbar\partial_{q_1} - \mathcal{M}_1)^2 + (-i\hbar\partial_{q_2} - \mathcal{M}_2)^2 \right]. \end{cases}$$

We deduce that, for all $u, v \in C_c^\infty(\Omega)$,

$$\langle \mathcal{L}_{h,\mathbf{A}}u, v \rangle_{L^2(\Omega)} = \langle \mathcal{L}_{h,\mathcal{M}}\varphi^*u, \varphi^*v \rangle_{L^2(U, e^{2\eta}dq)} = \langle (\varphi^{-1})^*\mathcal{L}_{h,\mathcal{M}}\varphi^*u, v \rangle_{L^2(\Omega)}. \quad (4.33)$$

Therefore, the relation between $\mathcal{L}_{h,\mathbf{A}}$ and $\mathcal{L}_{h,\mathcal{M}}$ is shown through

$$\mathcal{L}_{h,\mathbf{A}} = (\varphi^{-1})^*\mathcal{L}_{h,\mathcal{M}}(\varphi)^* \quad \text{on } C_c^\infty(\Omega). \quad (4.34)$$

Take the exterior derivative of both sides of $\mathcal{M} = \varphi^*\mathbf{A}$, we have

$$\left(\frac{\partial\mathcal{M}_2}{\partial q_1} - \frac{\partial\mathcal{M}_1}{\partial q_2} \right) dq_1 \wedge dq_2 = d\varphi^*\mathbf{A} = \varphi^*d\mathbf{A} = \varphi^*(Bd\mathcal{V}_g) = (B \circ \varphi)\sqrt{|\tilde{G}|}dq_1 \wedge dq_2.$$

Thus, the formula $d\mathbf{A} = \mathbf{B}d\mathcal{V}_g$ on Ω is equivalent to the formula

$$\frac{\partial\mathcal{M}_2}{\partial q_1} - \frac{\partial\mathcal{M}_1}{\partial q_2} = (B \circ \varphi)e^{2\eta} \quad \text{on } U. \quad (4.35)$$

4.3 Spectral analysis with the WKB method

Notation 4.3. In Section 4.3 and Section 4.4, we will denote \hat{f} be the Taylor formal series of function $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ at zero. It means that

$$\hat{f}(q_1, q_2) = \sum_{m, n \geq 0} \frac{1}{m!n!} \frac{\partial^{m+n} f(0)}{\partial q_1^m \partial q_2^n} q_1^m q_2^n.$$

We denote the \tilde{f} be the formal series after changing variable $(q_1, q_2) = \left(\frac{z+w}{2}, \frac{z-w}{2i}\right)$ with $(z, w) \in \mathbb{C}^2$. We designate $\mathbb{C}[[z]]$ for the ring of formal series in the variable z with coefficients in \mathbb{C} and $\mathbb{C}[[z, w]]$ for the ring of formal series in the variable (z, w) with coefficients in \mathbb{C} .

4.3.1 A choice of the magnetic potential

We continue on the way to prove Theorem 1.14. We recall that p^* is the local minimum point of the magnetic field B on Ω and the Hessian of B at p^* is positive non-degenerate. We assume that $\varphi(0) = p^*$ and denote $\mathcal{B}(q) = B(\varphi(q))$ as the magnetic field on U in \mathbb{R}^2 . The relation between the Hessian of B on manifold and the Hessian matrix of \mathcal{B} is given by

$$(d^2 B)_{p^*}(U, V) = \langle \text{Hess } \mathcal{B}(0)(d\varphi^{-1})_{p^*} U, (d\varphi^{-1})_{p^*} V \rangle_{\mathbb{R}^2} \quad \text{for all } U, V \in T_{p^*} U.$$

If the Hessian of the function B at p^* is positive non-degenerate, then the Hessian of \mathcal{B} at 0 is also positive non-degenerate. Therefore, we deduce that \mathcal{B} has local minimum at 0 and the Hessian matrix of \mathcal{B} at 0 is positive definite. Furthermore, Taylor expansion of \mathcal{B} at 0 is

$$\mathcal{B}(q) = \mathcal{B}(0) + \frac{1}{2} \langle \text{Hess } \mathcal{B}(0)q, q \rangle_{\mathbb{R}^2} + \mathcal{O}(\|q\|^3).$$

Since $\text{Hess } \mathcal{B}(0)$ is positive definite, there exists an orthogonal matrix P and two positive numbers α, γ such that

$$P^{-1} \text{Hess } \mathcal{B}(0) P = \begin{pmatrix} 2\alpha & 0 \\ 0 & 2\gamma \end{pmatrix}.$$

Thus, through a linear change of variable in \mathbb{R}^2 , without loss of generality, we can write

$$\mathcal{B}(q_1, q_2) = b_0 + \alpha q_1^2 + \gamma q_2^2 + \mathcal{O}(\|q\|^3) \quad \text{with } b_0 > 0 \text{ and } 0 < \alpha \leq \gamma.$$

The following lemma will be useful to define a special vector potential.

Lemma 4.8. *There exists a smooth solution of equation*

$$\Delta \Psi = e^{2\eta} \mathcal{B}, \quad (4.36)$$

in a neighborhood of U such that

$$\Psi(q_1, q_2) = \frac{e^{2\eta(0)} \mathcal{B}(0)}{4} (q_1^2 + q_2^2) + \mathcal{O}(\|q\|^3).$$

Proof. It is well-known that the Poisson equation (4.36) always has smooth solutions modulo by some Harmonic function. It means that if u is a particular solution of (4.36), then so is $\Psi = u + \varphi$, where φ is the solution of the equation

$$\Delta \varphi = 0.$$

In our case, by Taylor's theorem, we have

$$u(q_1, q_2) = u(0) + \frac{\partial u(0)}{\partial q_1} q_1 + \frac{\partial u(0)}{\partial q_2} q_2 + \frac{1}{2} \frac{\partial^2 u(0)}{\partial q_1^2} q_1^2 + \frac{\partial^2 u(0)}{\partial q_1 q_2} q_1 q_2 + \frac{1}{2} \frac{\partial^2 u(0)}{\partial q_2^2} q_2^2 + \mathcal{O}(\|x\|^3).$$

We choose Harmonic polynomial φ as

$$\varphi(q) = - \left[u(0) + \frac{\partial u(0)}{\partial q_1} q_1 + \frac{\partial u(0)}{\partial q_2} q_2 + \frac{\partial^2 u(0)}{\partial q_1 q_2} q_1 q_2 \right] + \frac{1}{4} \left(\frac{\partial^2 u(0)}{\partial q_2^2} - \frac{\partial^2 u(0)}{\partial q_1^2} \right) (q_1^2 - q_2^2).$$

Then the solution $\Psi = u + \varphi$ has the form

$$\Psi(q) = a(q_1^2 + q_2^2) + \mathcal{O}(\|q\|^3).$$

The Taylor expansion implies that

$$a = \frac{\partial^2 \Psi(0)}{\partial q_1^2} = \frac{\partial^2 \Psi(0)}{\partial q_2^2}.$$

Therefore, from equation (4.36) at 0, we see that

$$a = \frac{e^{2\eta(0)} \mathcal{B}(0)}{4}.$$

□

Let Ψ be the function given by Lemma 4.8, we choose $A = (-\partial_{q_2} \Psi, \partial_{q_1} \Psi)$. Then A will satisfy (4.35) and

$$\frac{\partial \mathcal{M}_2}{\partial q_1} - \frac{\partial \mathcal{M}_1}{\partial q_2} = \frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2}.$$

By setting 1-form $\omega = (A_1 - \mathcal{M}_1)dq_1 + (A_2 - \mathcal{M}_2)dq_2$ on U , ω will be closed. We can assume that the local neighborhood U allows the exactness of a closed form (see [39, Corollary 17.15]), it leads to the existence of a function $\theta \in C^\infty(U)$ such that $\omega = d\theta$ or

$$A = \mathcal{M} + \nabla\theta.$$

We notice that

$$e^{i\theta/h}(-ih\nabla - \mathcal{M})^2 e^{-i\theta/h} = (-ih\nabla - \mathcal{M} - \nabla\theta)^2 = (-ih\nabla - A)^2,$$

so that

$$\mathcal{L}_{h,A} = e^{i\theta/h} \mathcal{L}_{h,\mathcal{M}} e^{-i\theta/h}. \quad (4.37)$$

Furthermore, with the choice of the magnetic potential $A = (-\partial_{q_2}\Psi, \partial_{q_1}\Psi)$, we have the divergence of A is zero:

$$\nabla \cdot A = 0. \quad (4.38)$$

We will focus on performing WKB method for the eigen-problem for the operator $\mathcal{L}_{h,A}$ in the next subsection.

4.3.2 Asymptotic expansion of the eigen-problem of the magnetic Laplacian

Let us consider the eigen-problem of the magnetic Laplacian

$$\mathcal{L}_{h,A}u(q, h) = \lambda(h)u(q, h). \quad (4.39)$$

We start the WKB method by looking for solution $u(q, h)$ in the form

$$u(q, h) = e^{-S(q)/h} a(q, h),$$

where a and S are complex-valued functions. We emphasize that we will search for a complex phase S which is neither purely real nor purely imaginary as working in subsection 1.1.2 for the electric potential. Now, we focus on constructing an approximate solution of the equation

$$e^{S/h} (\mathcal{L}_{h,A} - \lambda(h)) e^{-S/h} a(q, h) = 0, \quad (4.40)$$

in the semi-classical limit: $\hbar \rightarrow 0$.

Let denote $\mathcal{L}_{\hbar,A}^S = e^{S/\hbar} \mathcal{L}_{\hbar,A} e^{-S/\hbar}$, we calculate it as

$$\begin{aligned} \mathcal{L}_{\hbar,A}^S &= e^{-2\eta} e^{S/\hbar} (-i\hbar \nabla - A)^2 e^{-S/\hbar} \\ &= e^{-2\eta} \left[(-A_1 + i\partial_{q_1} S)^2 + (-A_2 + i\partial_{q_2} S)^2 \right. \\ &\quad \left. + i\hbar \nabla \cdot A + \hbar \Delta S + 2\hbar (\nabla S + iA) \cdot \nabla - \hbar^2 \Delta \right]. \end{aligned}$$

Since $\nabla \cdot A = 0$, gathering the terms according to order of \hbar , we can write $\mathcal{L}_{\hbar,A}^S$ as

$$\mathcal{L}_{\hbar,A}^S = E_0^S + \hbar E_1^S - \hbar^2 \Delta, \quad (4.41)$$

where E_0^S is the multiplication operator

$$E_0^S a = e^{-2\eta} \left[(-A_1 + i\partial_{q_1} S)^2 + (-A_2 + i\partial_{q_2} S)^2 \right] a,$$

and E_1^S is the differential operator

$$E_1^S a = e^{-2\eta} (\Delta S + 2(\nabla S + iA) \cdot \nabla) a.$$

We now look for $a(q, \hbar)$ in the form

$$a(q, \hbar) = \sum_{j=0}^{\infty} a_j(q) \hbar^j, \quad (4.42)$$

where $(a_j)_{j \geq 0}$ are smooth complex-valued functions, and

$$\lambda(\hbar) = \hbar \sum_{j=0}^{\infty} \mu_j \hbar^j. \quad (4.43)$$

The equalities in (4.42) and (4.43) are in the sense of (1.10). Let us substitute (4.42) and (4.43) into (4.40), and require that each term associate with \hbar^j for $j \in \mathbb{N}$ vanishes,

we get the sequence of equations

$$\begin{aligned}
h^0 : & \quad E_0^S a_0 = 0 \\
h^1 : & \quad E_0^S a_1 + (E_1^S - \mu_0) a_0 = 0, \\
h^2 : & \quad E_0^S a_2 + (E_1^S - \mu_0) a_1 = (\mu_1 + e^{-2\eta} \Delta) a_0, \\
& \quad \dots\dots\dots \\
h^n (n \geq 3) : & \quad E_0^S a_n + (E_1^S - \mu_0) a_{n-1} = (\mu_1 + e^{-2\eta} \Delta) a_{n-2} + \sum_{j=2}^{n-1} \mu_j a_{n-1-j}, \\
& \quad \dots\dots\dots
\end{aligned}$$

These equations will be solved in formal series. As a starting point, we determine partially phase function S through the eikonal equation.

4.4 WKB construction

4.4.1 The eikonal equation

Let us find \hat{S} in $\mathbb{C}[[q_1, q_2]]$ such that

$$\left(-\hat{A}_1 + i\partial_{q_1} \hat{S}\right)^2 + \left(-\hat{A}_2 + i\partial_{q_2} \hat{S}\right)^2 = 0,$$

and thus

$$\left(-\hat{A}_1 + i\partial_{q_1} \hat{S} + i(-\hat{A}_2 + i\partial_{q_2} \hat{S})\right) \left(-\hat{A}_1 + i\partial_{q_1} \hat{S} - i(-\hat{A}_2 + i\partial_{q_2} \hat{S})\right) = 0.$$

Let us consider an \hat{S} such that

$$-\hat{A}_1 + i\partial_{q_1} \hat{S} + i(-\hat{A}_2 + i\partial_{q_2} \hat{S}) = 0.$$

It satisfies

$$2\partial_{\bar{z}} \hat{S} = -i\hat{A}_1 + \hat{A}_2, \quad \text{with } \partial_{\bar{z}} := \frac{1}{2}(\partial_{q_1} + i\partial_{q_2}).$$

Notice that we also have $\partial_{\bar{z}} \hat{\Psi} = -i\hat{A}_1 + \hat{A}_2$ (by choosing the magnetic potential A which is mentioned in the above section). It implies that

$$\partial_{\bar{z}} \hat{S} = \partial_{\bar{z}} \hat{\Psi}. \tag{4.44}$$

After changing the variable $q_1 = \frac{z+w}{2}$ and $q_2 = \frac{z-w}{2i}$ in the formal series \hat{S} and $\hat{\Psi}$, we have $\partial_w = \partial_{\bar{z}}$. Thus, \hat{S} satisfies the equation (4.44) if and only if \tilde{S} satisfies

$$\partial_w \tilde{S}(z, w) = \partial_w \tilde{\Psi}(z, w),$$

or \tilde{S} has the form

$$\tilde{S}(z, w) = \tilde{\Psi}(z, w) + f(z), \quad (4.45)$$

where $f(z) = \sum_{m \geq 0} f_m z^m$ be a formal series in $\mathbb{C}[[z]]$ which is determined later.

Next, we will write the transport equations (the equations associated with h^k for $k \geq 1$) in the formal series form. In order to simplify the notation, we denote $\mathcal{E} = e^{-2\eta}$. Under the linear transformation variables in $\mathbb{C}[[z, w]]$,

$$\begin{cases} q_1 = \frac{z+w}{2} \\ q_2 = \frac{z-w}{2i} \end{cases}, \quad (4.46)$$

the relation between the derivatives in $\mathbb{C}[[q_1, q_2]]$ and in $\mathbb{C}[[z, w]]$ are shown through

$$\begin{cases} \partial_z = \frac{1}{2}(\partial_{q_1} - i\partial_{q_2}) \\ \partial_w = \frac{1}{2}(\partial_{q_1} + i\partial_{q_2}) \end{cases}, \quad (4.47)$$

Let us represent the operator E_1^S and Δ in the formal series form. It is obvious that

$$\Delta = \partial_{q_1}^2 + \partial_{q_2}^2 = 4\partial_z \partial_w.$$

With the choice of \tilde{S} in (4.45), we have

$$\Delta \tilde{S} = \Delta \tilde{\Psi} = \tilde{\mathcal{E}}^{-1} \tilde{\mathcal{B}}. \quad (4.48)$$

Looking at the term $(\nabla S + iA) \cdot \nabla$ in the operator E_1^S , we write it, in the formal series, as

$$\begin{aligned} & (\partial_{q_1} \hat{S} + iA_1) \partial_{q_1} + (\partial_{q_2} \hat{S} + iA_2) \partial_{q_2} \\ &= (\partial_{q_1} \hat{S} - i\partial_{q_2} \hat{\Psi}) \partial_{q_1} + (\partial_{q_2} \hat{S} + i\partial_{q_1} \hat{\Psi}) \partial_{q_2} \\ &= \left[(\partial_z + \partial_w) \tilde{S} + (\partial_z - \partial_w) \tilde{\Psi} \right] (\partial_z + \partial_w) + i \left[i(\partial_z - \partial_w) \tilde{S} + i(\partial_z + \partial_w) \tilde{\Psi} \right] (\partial_z - \partial_w) \\ &= \left(2\partial_z \tilde{\Psi} + f'(z) \right) (\partial_z + \partial_w) - \left(2\partial_z \tilde{\Psi} + f'(z) \right) (\partial_z - \partial_w) \\ &= 2 \left(2\partial_z \tilde{\Psi} + f'(z) \right) \partial_w. \end{aligned}$$

The operator E_1^S will become

$$\begin{aligned} E_1^S &= \tilde{\mathcal{E}} \left(\tilde{\mathcal{E}}^{-1} \tilde{\mathcal{B}} + 4 \left(2\partial_z \tilde{\Psi} + f'(z) \right) \right) \\ &= 4\tilde{\mathcal{E}} \left(2\partial_z \tilde{\Psi} + f'(z) \right) \partial_w + \tilde{\mathcal{B}}. \end{aligned}$$

Finally, we obtain the system of the transport equations in the formal series form:

$$\begin{aligned} h^1 : & \quad \left[4\tilde{\mathcal{E}} \left(2\partial_z \tilde{\Psi} + f'(z) \right) \partial_w + \tilde{\mathcal{B}} - \mu_0 \right] A^{(0)} = 0, \\ h^2 : & \quad \left[4\tilde{\mathcal{E}} \left(2\partial_z \tilde{\Psi} + f'(z) \right) \partial_w + \tilde{\mathcal{B}} - \mu_0 \right] A^{(1)} = \left(\mu_1 + 4\tilde{\mathcal{E}} \partial_z \partial_w \right) A^{(0)}, \\ & \dots\dots\dots \\ h^n : & \quad \left[4\tilde{\mathcal{E}} \left(2\partial_z \tilde{\Psi} + f'(z) \right) \partial_w + \tilde{\mathcal{B}} - \mu_0 \right] A^{(n-1)} = \left(\mu_1 + 4\tilde{\mathcal{E}} \partial_z \partial_w \right) A^{(n-1)} + \sum_{j=2}^{n-1} \mu_j A^{(n-1-j)}, \\ & \dots\dots\dots \end{aligned}$$

4.4.2 Some tools to solve the transport equations

In this subsection, we prove some useful lemma for solving the transport equations.

Lemma 4.9. *There exists a formal series $w(z) = \sum_{k \geq 1} w_k z^k$ in $\mathbb{C}[[z]]$ satisfying*

$$\tilde{\mathcal{B}}(z, w(z)) = b_0, \tag{4.49}$$

and such that $w_1 = \frac{\sqrt{\gamma} - \sqrt{\alpha}}{\sqrt{\gamma} + \sqrt{\alpha}}$.

Proof. We have

$$\begin{aligned} \tilde{\mathcal{B}}(z, w) &= \hat{\mathcal{B}} \left(\frac{z+w}{2}, \frac{z-w}{2i} \right) \\ &= b_0 + \alpha \frac{(z+w)^2}{4} - \gamma \frac{(z-w)^2}{4} + \dots \\ &= b_0 + \frac{1}{4}(\alpha - \gamma)z^2 + \frac{1}{2}(\alpha + \gamma)zw + \frac{1}{4}(\alpha - \gamma)w^2 + \dots \end{aligned}$$

If we write $\tilde{\mathcal{B}}(z, w) = \sum_{m, n \geq 0} \tilde{b}_{mn} z^m w^n$, notice that $\tilde{b}_{01} = 0$, it helps us to find the term w_k by recursion. Namely, write out the equation (4.49), we have

$$\sum_{m, n \geq 0} \tilde{b}_{mn} z^m \left(\sum_{k \geq 1} w_k z^k \right)^n = b_0.$$

We will look for w_k for $k \geq 1$ by induction. Collect term by term, we have

- Term z^0 : $\tilde{b}_{00} = b_0$ (it is true).
- Term z^1 : $\tilde{b}_{10} + \tilde{b}_{01}w_1 = 0$ (it is true).
- Term z^2 : To create order 2 of z , m, n has to be smaller or equal to 2, we have some cases $(m, n) = (0, 2)$, $(m, n) = (1, 1)$, $(m, n) = (2, 0)$. Then we have the equation to find w_1 :

$$\tilde{b}_{02}w_1^2 + \tilde{b}_{11}w_1 + \tilde{b}_{20} = 0.$$

There are two solutions for w_1 which are

$$\frac{\sqrt{\gamma} + \sqrt{\alpha}}{\sqrt{\gamma} - \sqrt{\alpha}}, \quad \text{and} \quad \frac{\sqrt{\gamma} - \sqrt{\alpha}}{\sqrt{\gamma} + \sqrt{\alpha}}.$$

In order to adapt to even the case $\gamma = \alpha$, we choose $w_1 = \frac{\sqrt{\gamma} - \sqrt{\alpha}}{\sqrt{\gamma} + \sqrt{\alpha}}$.

- Term z^3 : notice that the equation obtained by collecting the coefficients of the term z^3 does not contain w_k for $k \geq 3$ because $\tilde{b}_{01} = 0$. So, we got the equation containing w_2 and w_1 , \tilde{b}_{mn} for $0 \leq m, n \leq 3$. Furthermore, this equation is linear according to w_2 , if otherwise, the order of z is larger than 3. We just need to care about the coefficients attached to w_2 , they are

$$\tilde{b}_{11} + 2\tilde{b}_{02}w_1 = \frac{1}{2}(\alpha + \gamma) + \frac{1}{2}(\alpha - \gamma)\frac{\sqrt{\gamma} - \sqrt{\alpha}}{\sqrt{\gamma} + \sqrt{\alpha}} = \sqrt{\alpha\gamma},$$

which is non-zero, then w_2 is easy to computed.

By induction, let $p \in \mathbb{N} \setminus \{0\}$, we assume that $(w_k)_{1 \leq k \leq p-1}$ are determined and we need to look for w_p . We collect all coefficients of z^{p+1} , and since $b_{01} = 0$, we get an equation containing only finite $(w_k)_{1 \leq k \leq p}$ and $(\tilde{b}_{mn})_{0 \leq m, n \leq p+1}$. The equation is linear according to variable w_p which has attached coefficient $((m, n) = (1, 1)$ and $(m, n) = (0, 2))$:

$$\tilde{b}_{11} + 2\tilde{b}_{02}w_1 = \sqrt{\alpha\gamma} \neq 0.$$

So, w_p is determined. □

Lemma 4.10. *Let $V(s, t)$ and $F(s, t)$ be formal series in $\mathbb{C}[[s, t]]$. We write $V(s, t)$ and $F(s, t)$ in the form*

$$V(s, t) := \sum_{m \geq 0} v_m(s)t^m,$$

and

$$F(s, t) := \sum_{m \geq 0} f_m(s)t^m,$$

where $(v_n(s))_{n \in \mathbb{N}}$ and $(f_n(s))_{n \in \mathbb{N}}$ are the sequences in $\mathbb{C}[[s]]$. We assume that $v_0(s) = 0$, $v_1(s) = v_1$, $f_0(s) = f_0$ with $v_1 \in \mathbb{R} \setminus \{0\}$ and $f_0 \in \mathbb{R}$. Let $\ell \in \mathbb{N}$, then

i. the homogeneous equation

$$(V(s, t)\partial_t + F(s, t))u(s, t) = 0, \quad (4.50)$$

has solutions in the set

$$W(\ell) = \left\{ \sum_{m \geq 0} w_m(s)t^m \in \mathbb{C}[[s, t]] : w_k(s) = 0 \text{ for } k \in \{0, \dots, (\ell-1)\} \text{ and } w_\ell(s) \neq 0 \right\}$$

if and only if $f_0 + \ell v_1 = 0$.

ii. Under the previous condition $f_0 + \ell v_1 = 0$, there exist a family $(c_k(s))_{k=0 \dots \ell} \subset \mathbb{C}[[s]]$ such that the inhomogeneous equation

$$(V(s, t)\partial_z + F(s, t))u(s, t) = G(s, t), \quad (4.51)$$

for $G(s, t) = \sum_{m \geq 0} g_m(s)t^m$, has formal series solutions in the form

$$u(s, t) = \sum_{m \geq 0} u_m(s)t^m, \quad (4.52)$$

if and only if

$$c_\ell(s)g_0(s) + c_{\ell-1}(s)g_1(s) + \dots + c_0(s)g_\ell(s) = 0. \quad (4.53)$$

Here, the coefficients $(c_k(s))_{k=0 \dots \ell} \subset \mathbb{C}[[s]]$ are determined by $(v_j(s))_{1 \leq j \leq (\ell+1)}$ and $(f_j(s))_{1 \leq j \leq \ell}$, and $c_0(s) = 1$. Furthermore, assume that the condition (4.53) is satisfied, if $u_\ell(s)$ is given, the formal series solution u will be determined uniquely by the recursion formula

$$u_m(s) = \frac{g_m(s) - \sum_{j=0}^{m-1} (jv_{m-j+1}(s) + f_{m-j}(s))u_j(s)}{(m-\ell)v_1},$$

for all $m \in \mathbb{N} \setminus \{\ell\}$.

Proof. Let us start with the homogeneous case, we look for a solution $u(s, t)$ in the form

$$u(s, t) = \sum_{m \geq 0} u_m(s)t^m,$$

of the equation

$$\left(\sum_{m \geq 1} v_m(s) t^m \right) \left(\sum_{m \geq 1} m u_m(s) t^{m-1} \right) + \left(\sum_{m \geq 0} f_m(s) t^m \right) \left(\sum_{m \geq 0} u_m(s) t^m \right) = 0.$$

For arbitrary $k \in \mathbb{N}$, we get the equations corresponding to t^k

$$\begin{aligned} \sum_{j=1}^k j v_{k-j+1}(s) u_j(s) + \sum_{j=0}^k f_{k-j}(s) u_j(s) &= 0 \\ \Leftrightarrow (k v_1 + f_0) u_k(s) + \sum_{j=0}^{k-1} (j v_{k-j+1}(s) + f_{k-j}(s)) u_j(s) &= 0. \end{aligned}$$

For the sake of convenience, we write here some equations

$$\begin{aligned} t^0 : \quad & f_0 u_0(s) = 0, \\ t^1 : \quad & [v_1 + f_0] u_1(s) + f_1(s) u_0(s) = 0, \\ t^2 : \quad & [2v_1 + f_0] u_2(s) + f_2(s) u_0(s) + (v_2(s) + f_1(s)) u_1(s) = 0. \end{aligned}$$

For $\ell \in \mathbb{N}$, consider a non-zero solution u in $W(\ell)$. Then $u_k(s) = 0$ for $0 \leq k \leq \ell - 1$ and $u_\ell(s) \neq 0$. It implies directly that $\ell v_1 + f_0 = 0$. Now, if $\ell v_1 + f_0 = 0$, then for all $k \neq \ell$, we get the recursion formula

$$u_k(s) = - \frac{\sum_{j=0}^{k-1} (j v_{k-j+1}(s) + f_{k-j}(s)) u_j(s)}{(k - \ell) v_1}.$$

We consider two cases:

- Case 1 : $\ell = 0$. It leads to $f_0 = 0$. Since the first equation is

$$f_0 u_0(s) = 0,$$

we can choose any $u_0(s) \neq 0$, and compute next coefficients by the following recursion formula

$$u_k(s) = - \frac{\sum_{j=0}^{k-1} (j v_{k-j+1}(s) + f_{k-j}(s)) u_j(s)}{k v_1},$$

for all $k \geq 1$.

- Case 2: $\ell \neq 0$. Then f_0 has to be non-zero. From the first equation, it leads to $u_0(s) = 0$. From the recursion formula, it implies that $u_k(s) = 0$ for all $k < \ell$. For

$k = \ell$, we get the equation

$$(k - \ell)v_1(s)u_\ell(s) = 0,$$

we can choose any $u_\ell(s) \neq 0$ and keep going with other u_k by the recursion formula.

In any case, we always get non-trivial solutions u in the set $W(\ell)$.

Now we consider the inhomogeneous case with solution u in the form (4.52):

$$\left(\sum_{m \geq 1} v_m(s)t^m \right) \left(\sum_{m \geq 1} mu_m(s)t^{m-1} \right) + \left(\sum_{m \geq 0} f_m(s)t^m \right) \left(\sum_{m \geq 0} u_m(s)t^m \right) = \sum_{m \geq 0} g_m(s)t^m.$$

Like the homogeneous case, for arbitrary $k \in \mathbb{N}$, the equations corresponding to t^k are

$$(kv_1 + f_0)u_k(s) + \sum_{j=0}^{k-1} (jv_{k-j+1}(s) + f_{k-j}(s))u_j(s) = g_k(s).$$

Assume that there exists $\ell \in \mathbb{N}$ such that $\ell v_1 + f_0 = 0$, these equations becomes

$$(k - \ell)v_1u_k(s) + \sum_{j=0}^{k-1} (jv_{k-j+1}(s) + f_{k-j}(s))u_j(s) = g_k(s). \quad (4.54)$$

The equation corresponding to t^ℓ is

$$(\ell - \ell)u_\ell(s) + \sum_{j=0}^{\ell-1} (jv_{\ell-j+1}(s) + f_{\ell-j}(s))u_j(s) = g_\ell(s).$$

The inhomogeneous equation has solutions in the form (4.52) if and only if

$$g_\ell(s) - \sum_{j=0}^{\ell-1} (jv_{\ell-j+1}(s) + f_{\ell-j}(s))u_j(s) = 0.$$

This relation is in the form (4.53) after computing $(u_k(s))$ according to $g_k(s)$, $f_k(s)$ and $v_k(s)$ for $k = 0 \dots (\ell - 1)$. For example, we can compute $c_1(s)$ by collecting all coefficients connecting with $g_{\ell-1}(s)$. Notice that $g_{\ell-1}(s)$ only appears in the formula of $u_{\ell-1}$ and its coefficient in $u_{\ell-1}$ is $\frac{-1}{v_1}$, then we can compute

$$c_1(s) = -[(\ell - 1)v_2(s) + f_1(s)] \frac{-1}{v_1} = \frac{(\ell - 1)v_2(s) + f_1(s)}{v_1}.$$

The statement at the end of the lemma is easy obtained from (4.54). \square

4.4.3 The first transport equation

We consider the first transport equation

$$v(z, w)\partial_w A^{(0)}(z, w) + \left(\tilde{\mathcal{B}}(z, w) - \mu_0\right) A^{(0)}(z, w) = 0, \quad (4.55)$$

where $v(z, w) = 4\tilde{\mathcal{E}}(z, w) \left(f'(z) + 2\partial_z \tilde{\Psi}(z, w)\right)$.

Let $w(z)$ be the formal series defined in Lemma 4.9. Applying the change of variables $(z, w) = (z, y + w(z))$, which is licit in $\mathbb{C}[[z, w]]$ because $w_0 = 0$, we get the equation

$$V(z, y)\partial_y A^{(0)}(z, y + w(z)) + F(z, y)A^{(0)}(z, y + w(z)) = 0, \quad (4.56)$$

where

$$V(z, y) = v(z, y + w(z)) = 4\tilde{\mathcal{E}}(z, y + w(z)) \left(f'(z) + 2\partial_z \tilde{\Psi}(z, y + w(z))\right),$$

and

$$F(z, y) = \tilde{\mathcal{B}}(z, y + w(z)) - \mu_0.$$

4.4.3.1 Choosing formal series f and determining \tilde{S} completely

We recall that \tilde{S} given in (4.45) is expressed by the sum of the formal series $\tilde{\Psi}(z, w)$ and $f(z)$. The formal series $\tilde{\Psi}(z, w)$ is known by Lemma 4.8 and the formal series $f(z) \in \mathbb{C}[[z]]$ is waiting to be determined. We will choose $f(z)$ such that we can apply Lemma 4.10 to solve the first transport equation (4.55). To do that, the formal series $f(z)$ will be chosen such that

$$f'(z) + 2\partial_z \tilde{\Psi}(z, w(z)) = 0. \quad (4.57)$$

This statement will be clearer in the next subsection. At this stage, we focus on finding the formal series $f(z)$ satisfying (4.57). Since $w(z)$ is a formal series which has $w_0 = 0$, then the composition $\partial_z \tilde{\Psi}(z, w(z))$ is well-defined, it means that each coefficient of $\partial_z \tilde{\Psi}(z, w(z))$ can be computed by a finite number of coefficients of $w(z)$ and $\partial_z \tilde{\Psi}(z, w)$. According to Lemma 4.8, the Taylor series $\tilde{\Psi}(z, w)$ has the expression

$$\tilde{\Psi}(z, w) = \frac{e^{2\eta(0)} b_0}{4} zw + \sum_{m+n \geq 3} \psi_{mn} z^m w^n.$$

We denote the formal series $\partial_z \tilde{\Psi}(z, w(z))$ by $\sum_{k \geq 0} \nu_k z^k$, we have

$$\sum_{k \geq 0} \nu_k z^k = \frac{e^{2\eta(0)} b_0}{4} w(z) + \sum_{\substack{m+n \geq 3 \\ m \geq 1}} m \psi_{mn} z^{m-1} (w(z))^n.$$

Let $f(z) = \sum_{k \geq 0} \hat{f}_k z^k$, and we can solve (4.57)

$$\sum_{k \geq 1} k \hat{f}_k z^{k-1} = -2 \sum_{k \geq 1} \nu_{k-1} z^{k-1},$$

to get

$$\hat{f}_k = -\frac{2\nu_{k-1}}{k} \quad \text{for all } k \geq 1.$$

Since there is no restriction for $f(0)$, we can choose $\hat{f}_0 = 0$. Furthermore, we can compute some initial coefficients of $f(z)$

$$\hat{f}_1 = -2\nu_0 = 0, \quad \hat{f}_2 = -\nu_1 = \frac{e^{2\eta(0)} b_0}{4} \frac{\sqrt{\alpha} - \sqrt{\gamma}}{\sqrt{\alpha} + \sqrt{\gamma}}.$$

Now, $\tilde{S}(z, w)$ is totally determined and

$$\tilde{S}(z, w) = \frac{e^{2\eta(0)} b_0}{4} zw + \frac{e^{2\eta(0)} b_0}{4} \frac{\sqrt{\alpha} - \sqrt{\gamma}}{\sqrt{\alpha} + \sqrt{\gamma}} z^2 + \sum_{m+n \geq 3} [\tilde{S}]_{mn} z^m w^n. \quad (4.58)$$

4.4.3.2 Solving the first transport equation

Let us come back to the transport equation (4.55). We write $V(z, y)$ and $F(z, y)$ in the form

$$V(z, y) = \sum_{m \geq 0} v_m(z) y^m \quad \text{and} \quad F(z, y) = \sum_{m \geq 0} f_m(z) y^m.$$

We now check the assumptions of the Lemma 4.10. Firstly, from choosing the formal series $f(z)$ satisfying (4.57), we have

$$v_0(z) = V(z, 0) = 4\tilde{\mathcal{E}}(z, w(z)) \left(f'(z) + 2\partial_z \tilde{\Psi}(z, w(z)) \right) = 0.$$

Secondly, from (4.57) and Lemma 4.8, we obtain

$$\begin{aligned}
v_1(z) &= \partial_y V(z, 0) \\
&= 4\partial_w \tilde{\mathcal{E}}(z, y + w(z)) \left(f'(z) + 2\partial_z \tilde{\Psi}(z, y + w(z)) \right) \Big|_{y=0} \\
&\quad + 8\tilde{\mathcal{E}}(z, y + w(z)) \partial_w \partial_z \tilde{\Psi}(z, y + w(z)) \Big|_{y=0} \\
&= 8\tilde{\mathcal{E}}(z, w(z)) \partial_w \partial_z \tilde{\Psi}(z, w(z)) \\
&= 2\tilde{\mathcal{E}}(z, w(z)) \tilde{\mathcal{E}}^{-1}(z, w(z)) \tilde{\mathcal{B}}(z, w(z)) \\
&= 2\tilde{\mathcal{B}}(z, w(z)) \\
&= 2b_0.
\end{aligned}$$

Finally, from (4.9), we get

$$f_0(z) = F(z, 0) = \tilde{\mathcal{B}}(z, w(z)) - \mu_0 = b_0 - \mu_0.$$

Thanks to Lemma 4.10 for the homogeneous case, the equation (4.56) has solutions in the form $\sum_{m \geq 0} A_m^{(0)}(z) y^m$ such that $A_0^{(0)}(z) \neq 0$ if and only if

$$f_0(z) = 0 \Leftrightarrow \mu_0 = b_0.$$

In this case, the solution of the first transport equation (4.55) which obtained by the change of variables $(z, y) = (z, z - w(z))$ in the solution of the equation (4.56) has the form

$$A^{(0)}(z, w) = \sum_{m \geq 0} A_m^{(0)}(z) (w - w(z))^m,$$

where $A_m^{(0)}(z)$ can be computed by the recursion formula

$$A_m^{(0)}(z) = - \frac{\sum_{j=0}^{m-1} (j v_{m-j+1}(z) + f_{m-j}(z)) A_j^{(0)}(z)}{2mb_0}.$$

The series $A_0^{(0)}(z)$ will be determined later in the process of solving the second transport equation.

4.4.4 The second transport equation

We consider the second transport equation

$$\left(v(z, w) \partial_w + \tilde{B}(z, w) - \mu_0 \right) A^{(1)} = (\mu_1 + 4\tilde{\mathcal{E}}(z, w) \partial_z \partial_w) A^{(0)}. \quad (4.59)$$

We deal with this equation as same as we did for the first transport equation (4.55). By changing variables $(z, w) = (z, y + w(z))$ in the second transport equation (4.59), we obtain

$$V(z, y)\partial_y A^{(1)}(z, y + w(z)) + F(z, y)A^{(1)}(z, y + w(z)) = G^{(1)}(z, y), \quad (4.60)$$

where $G^{(1)}(z, y) = \left(\mu_1 + 4\tilde{\mathcal{E}}(z, y + w(z))\partial_z\partial_w\right)A^{(0)}(z, y + w(z))$.

We write $G^{(1)}(z, y)$ in the form $G^{(1)}(z, y) = \sum_{m \geq 0} g_m^{(1)}(z)y^m$. Since $v_0(z) = 0$, $v_1(z) = 2b_0 \neq 0$ and $f_0(z) = 0$, applying Lemma 4.10 in the inhomogeneous case ($\ell = 0$ in this case), the equation (4.60) has solutions if and only if $g_0^{(1)}(z) = 0$ or

$$G^{(1)}(z, 0) = (\mu_1 + 4\tilde{\mathcal{E}}(z, w(z))\partial_z\partial_w)A^{(0)}(z, w(z)) = 0.$$

This is equivalent to

$$\mu_1 A_0^{(0)}(z) + 4\tilde{\mathcal{E}}(z, w(z)) \left(\partial_z A_1^{(0)}(z) - 2A_2^{(0)}(z)w'(z) \right) = 0.$$

Since

$$A_1^{(0)}(z) = -\frac{f_1(z)}{2b_0}A_0^{(0)}(z),$$

and

$$\begin{aligned} A_2^{(0)}(z) &= -\frac{(v_2(z) + f_1(z))A_1^{(0)}(z) + f_2(z)A_0^{(0)}(z)}{4b_0} \\ &= \frac{1}{8b_0^2}f_1(z)(v_2(z) + f_1(z))A_0^{(0)}(z) - \frac{1}{4b_0}f_2(z)A_0^{(0)}(z), \end{aligned}$$

the equation related to μ_1 can be rewritten as

$$V(z)\partial_z A_0^{(0)}(z) + F(z)A_0^{(0)}(z) = 0, \quad (4.61)$$

where

$$V(z) := \frac{2}{b_0}\tilde{\mathcal{E}}(z, w(z))f_1(z),$$

and

$$F(z) := \tilde{\mathcal{E}}(z, w(z)) \left(\frac{2}{b_0}f_1'(z) + \frac{1}{b_0^2}f_1(z)(f_1(z) + v_2(z))w'(z) + \frac{2}{b_0}f_2(z)w'(z) \right) - \mu_1.$$

Notation 4.4. Below, with a given formal series $X(z) = \sum_{k \geq 0} x_k z^k \in C[[z]]$, we use the notation $[X(z)]_k$ to extract the coefficient of z^k , so that

$$[X(z)]_k = x_k.$$

We have

$$f_1(z) = \partial_2 \tilde{\mathcal{B}}(z, w(z)) = \sum_{\substack{m \geq 0 \\ n \geq 1}} n \tilde{b}_{mn} z^m (w(z))^{n-1},$$

$$f_2(z) = \frac{1}{2} \partial_2^2 \tilde{\mathcal{B}}(z, w(z)) = \sum_{\substack{m \geq 0 \\ n \geq 2}} \frac{n(n-1)}{2} \tilde{b}_{mn} z^m (w(z))^{n-2}.$$

It is easy to check that

$$[V(z)]_0 = \frac{2}{b_0} [\tilde{\mathcal{E}}(z, w(z))]_0 [f_1(z)]_0 = 0 \quad (\text{since } \tilde{b}_{01} = 0),$$

and

$$\begin{aligned} [V(z)]_1 &= \frac{2}{b_0} [\tilde{\mathcal{E}}(z, w(z))]_0 [f_1(z)]_1 + \frac{2}{b_0} [\tilde{\mathcal{E}}(z, w(z))]_1 [f_1(z)]_0 \\ &= \frac{2e^{-2\eta(0)}}{b_0} (2\tilde{b}_{02}w_1 + \tilde{b}_{11}) \\ &= \frac{2e^{-2\eta(0)}}{b_0} \left(\frac{1}{2}(\alpha - \gamma) \frac{\sqrt{\gamma} - \sqrt{\alpha}}{\sqrt{\gamma} + \sqrt{\alpha}} + \frac{1}{2}(\alpha + \gamma) \right) \\ &= \frac{2e^{-2\eta(0)} \sqrt{\alpha\gamma}}{b_0}. \end{aligned}$$

Furthermore, we can compute

$$\begin{aligned} [F(z)]_0 &= \frac{2e^{-2\eta(0)}}{b_0} (\tilde{b}_{11} + \tilde{b}_{02}w_1) - \mu_1 \\ &= \frac{e^{-2\eta(0)} (\sqrt{\gamma} + \sqrt{\alpha})^2}{2b_0} - \mu_1. \end{aligned}$$

Applying Lemma 4.10, the equation (4.61) has solutions if and only if there exists $\ell \in \mathbb{N}$ such that

$$\mu_1 = e^{-2\eta(0)} \left(2\ell \frac{\sqrt{\alpha\gamma}}{b_0} + \frac{(\sqrt{\gamma} + \sqrt{\alpha})^2}{2b_0} \right).$$

Then, $A_0^{(0)}$ can be determined by the formal series which has ℓ first terms vanishing and the other coefficients gained by the recursion formula starting from certain $[A_0^{(0)}(z)]_\ell \neq 0$. In detail, $[A_0^{(0)}(z)]_k$ (for $k > \ell$) is determined by the recursion formula

$$\begin{aligned} [A_0^{(0)}(z)]_k &= - \frac{\sum_{j=0}^{k-1} (j[V(z)]_{k-j+1} + [F(z)]_{k-j}) [A_0^{(0)}(z)]_j}{(k-\ell)[V(z)]_1} \\ &= - \frac{b_0 e^{2\eta(0)}}{2\sqrt{\alpha\gamma}(k-\ell)} \left(\sum_{j=0}^{k-1} (j[V(z)]_{k-j+1} + [F(z)]_{k-j}) [A_0^{(0)}(z)]_j \right). \end{aligned}$$

and we choose $[A_0^{(0)}(z)]_\ell = 1$.

After the right-hand side formal series of equation (4.59) is known, we can find a particular solution of equation (4.59). That is

$$\sum_{m \geq 0} \alpha_m^{(1)}(z)(w - w(z))^m.$$

The formal series $\alpha_m^{(1)}(z)$ is determined by recursion formula

$$\alpha_m^{(1)}(z) = \frac{g_m^{(1)}(z) - \sum_{j=0}^{m-1} (jv_{m-j+1}(z) + f_{m-j}(z))\alpha_j^{(1)}(z)}{2mb_0},$$

starting with $\alpha_0^{(1)}(z) = 0$.

The formal series solutions of equation (4.59) take the form

$$A^{(1)}(z, w) = \sum_{m \geq 0} \alpha_m^{(1)}(z)(w - w(z))^m + \sum_{m \geq 0} A_m^{(1)}(z)(w - w(z))^m, \quad (4.62)$$

where $\sum_{m \geq 0} A_m^{(1)}(z)(w - w(z))^m$ is the solution of the first transport equation (4.55), that is, $A_m^{(1)}(z)$ is determined by

$$A_m^{(1)}(z) = -\frac{\sum_{j=0}^{m-1} (jv_{m-j+1}(z) + f_{m-j}(z))A_j^{(1)}(z)}{2mb_0},$$

and $A_0^{(1)}(z)$ remains to be determined.

4.4.5 Induction

Let $p \in \mathbb{N} \setminus \{0\}$. We assume that the sequences $(\mu_j)_{0 \leq j \leq p}$ and $(A^{(j)})_{0 \leq j \leq p-1}$ are determined from the first $(p+1)$ transport equations:

$$\begin{aligned} & \left(v(z, w)\partial_w + \tilde{B}(z, w) - \mu_0 \right) A^{(0)} = 0 \\ & \left(v(z, w)\partial_w + \tilde{B}(z, w) - \mu_0 \right) A^{(1)} = (\mu_1 + 4\tilde{\mathcal{E}}(z, w)\partial_z\partial_w)A^{(0)} \\ & \dots\dots\dots \\ & \left(v(z, w)\partial_w + \tilde{B}(z, w) - \mu_0 \right) A^{(p)} = (\mu_1 + 4\tilde{\mathcal{E}}(z, w)\partial_z\partial_w)A^{(p-1)} + \sum_{j=2}^p \mu_j A^{(p-j)} \end{aligned}$$

Let us also assume that the $A^{(j)}$'s, for $j \in \{1, \dots, p\}$, are in the form

$$A^{(j)}(z, w) = \sum_{m \geq 0} \alpha_m^{(j)}(z)(w - w(z))^m + \sum_{m \geq 0} A_m^{(j)}(z)(w - w(z))^m,$$

where

- i) $\sum_{m \geq 0} \alpha_m^{(j)}(z)(w - w(z))^m$, which is a particular solution for the j -th transport equation, is a determined formal series with $\alpha_0^{(j)}(z) = 0$ in $\mathbb{C}[[z]]$ for $j \in \{1, \dots, p\}$.
- ii) $\sum_{m \geq 0} A_m^{(j)}(z)(w - w(z))^m$, which is a solution of the first transport equation (4.55), is also determined and satisfies $[A_0^{(j)}(z)]_\ell = 0$ in \mathbb{C} for $j \in \{1, \dots, p-1\}$.

Only $\sum_{m \geq 0} A_m^{(p)}(z)(w - w(z))^m$ is waiting for being determined. We just need to determine $A_0^{(p)}(z)$, because other terms can be computed by the recursion formula

$$A_m^{(p)}(z) = -\frac{\sum_{j=0}^{m-1} (jv_{m-j+1}(z) + f_{m-j}(z))A_j^{(p)}(z)}{2mb_0},$$

for $m \geq 1$.

Let us now consider the equation satisfied by $A^{(p+1)}$:

$$\left(v \partial_w + \tilde{B} - b_0\right) A^{(p+1)} = (\mu_1 + 4\tilde{\mathcal{E}}\partial_z\partial_w)A^{(p)} + \mu_{p+1}A^{(0)} + \sum_{j=2}^p \mu_j A^{(p+1-j)}. \quad (4.63)$$

As before, the fact that this equation has solutions will fix value of μ_{p+1} and determine $A_0^{(p)}(z)$. Indeed, by Lemma 4.10, the existence of solutions to (4.63) is equivalent to

$$(\mu_1 + 4\tilde{\mathcal{E}}\partial_z\partial_w)A^{(p)}(z, w(z)) = -\mu_{p+1}A^{(0)}(z, w(z)) - \sum_{j=2}^p \mu_j A^{(p+1-j)}(z, w(z)).$$

This can be rewritten as

$$V(z)\partial_z A_0^{(p)}(z) + F(z)A_0^{(p)}(z) = g(z), \quad (4.64)$$

where

$$g(z) = -\mu_{p+1}A_0^{(0)}(z) - \sum_{j=2}^p \mu_j A_0^{(p+1-j)}(z) - 4\partial_z\alpha_1^{(p)}(z) + 8\alpha_2^{(p)}(z).$$

We are in the inhomogeneous case of Equation (4.61). Lemma 4.10 is applied here. There are coefficients c_0, \dots, c_ℓ in \mathbb{C} , with $c_0 = 1$ such that the equation (4.64) has solutions in $\mathbb{C}[[z]]$ if and only if

$$c_0[g(z)]_\ell + \dots + c_{\ell-1}[g(z)]_1 + c_\ell[g(z)]_0 = 0. \quad (4.65)$$

Under the inductive assumption, every term appearing in the formula of $g(z)$ is known except μ_{p+1} . Because $[A_0^{(0)}(z)]_k = 0$ for $k \in \{0, \dots, \ell-1\}$ and $[A_0^{(0)}(z)]_\ell = 1$, this helps

us to determine μ_{p+1} because the coefficient associated to μ_{p+1} appearing in (4.65) is finally 1. Furthermore, there exists unique solution $A_0^{(p)}(z)$ such that $[A_0^{(p)}(z)]_\ell = 0$ and we pick this solution.

Coming back to the equation (4.63) of $A^{(p+1)}$, with this choice of μ_{p+1} and $A_0^{(p)}(z)$, there are solutions and they can be written as

$$A^{(p+1)}(z, w) = \sum_{m \geq 0} \alpha_m^{(p+1)}(z)(w - w(z))^m + \sum_{m \geq 0} A_m^{(p+1)}(z)(w - w(z))^m,$$

where

i) the first formal series $\sum_{m \geq 0} \alpha_m^{(p+1)}(z)(w - w(z))^m$ is defined by the formula

$$\begin{cases} \alpha_0^{(p+1)}(z) = 0 \\ \alpha_m^{(p+1)}(z) = \frac{g_m^{(p+1)}(z) - \sum_{j=0}^{m-1} (jv_{m-j+1}(z) + f_{m-j}(z))\alpha_j^{(p+1)}(z)}{2mb_0} \end{cases} \quad \forall m \geq 1, \quad (4.66)$$

with $g_m^{p+1} = (\mu_1 + 4\partial_z \partial_w)A^{(p)} + \mu_{p+1}A^{(0)} + \sum_{j=2}^p \mu_j A^{(p+1-j)}$,

ii) the second formal series $\sum_{m \geq 0} A_m^{(p+1)}(z)(w - w(z))^m$ is the solution of the first transport equation (4.55), that is, $A_m^{(p+1)}(z)$ determined by

$$A_m^{(p+1)}(z) = -\frac{\sum_{j=0}^{m-1} (jv_{m-j+1}(z) + f_{m-j}(z))A_j^{(p+1)}(z)}{2mb_0},$$

and $A_0^{(p+1)}(z)$ is specified by the next transport equation.

4.4.6 Conclusion

We use Borel's Lemma to finish the WKB process.

Lemma 4.11 (Borel's Lemma). *Let $(u_{mn})_{m,n \in \mathbb{N}}$ be the sequence in \mathbb{R} . There exists a function $f \in C_0^\infty(\mathbb{R}^2)$ such that*

$$\frac{\partial^{m+n} f}{\partial q_1^m \partial q_2^n}(0) = u_{mn}.$$

Proof. Let χ be a cut-off function equal 1 near 0 and have a support compact in $B(0, 1) \subset \mathbb{R}^2$. We set up

$$f_j(x) = u_j \frac{x^j}{j!} \chi\left(\frac{x}{\varepsilon_j}\right),$$

where $j := (j_1, j_2) \in \mathbb{N}^2$, $x^j := q_1^{j_1} q_2^{j_2}$ and $j! := j_1! j_2!$.

Then $f_j \in C_0^\infty(\mathbb{R}^2)$. We can choose ε_j small enough such that, for all $x \in \mathbb{R}^2$,

$$|D^\alpha f_j(x)| \leq \frac{2^{-|j|}}{|j|+1} \quad \text{for all } |\alpha| < |j|. \quad (4.67)$$

Indeed, for $|x| \leq \varepsilon_j$,

$$|D^\alpha f_j(x)| \leq u_j \varepsilon_j^{|j|-|\alpha|}.$$

For small ε_j , we get the estimate (4.67) since $|\alpha| < |j|$. As a result, the function series

$$f(q_1, q_2) = \sum_{|j| \geq 0} f_j(x),$$

is uniformly convergent. The series obtained by differentiation is uniformly convergent too. Therefore, $f \in C_0^\infty(\mathbb{R}^2)$ and satisfies

$$D^j f(0) = u_j.$$

□

Let us recall that U is the neighborhood of 0 in \mathbb{R}^2 , which is defined by $U = \phi(\Omega)$. Here, Ω is the neighborhood of p^* in which the isothermal local chart $(\Omega, \phi : \Omega \rightarrow \phi(\Omega))$ is defined. We have the following theorem:

Theorem 4.12. *For all $\ell \in \mathbb{N}$, there exist*

i) a smooth complex-valued function T on U satisfying

$$\operatorname{Re}(T)(q) = \frac{e^{2\eta(0)} b_0}{2} \left(\frac{\sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\gamma}} q_1^2 + \frac{\sqrt{\gamma}}{\sqrt{\alpha} + \sqrt{\gamma}} q_2^2 \right) + \mathcal{O}(\|q\|^3), \quad (4.68)$$

ii) a sequence of smooth complex-valued function $(a_{\ell,j})_{j \in \mathbb{N}}$ on U ,

iii) a sequence of real numbers $(\mu_{\ell,j})_{j \in \mathbb{N}}$ with

$$\mu_{\ell,0} = b_0, \quad \mu_{\ell,1} = e^{-2\eta(0)} \left(2\ell \frac{\sqrt{\alpha\gamma}}{b_0} + \frac{(\sqrt{\gamma} + \sqrt{\alpha})^2}{2b_0} \right),$$

iv) a sequence of flat functions $(f_j)_{j \in \mathbb{N}}$ on U ,

such that, for all $J \in \mathbb{N}$,

$$e^{T/h} \left(\mathcal{L}_{h,\mathcal{M}} - h \sum_{j=0}^J \mu_{\ell,j} h^j \right) \left(e^{-T/h} \sum_{j=0}^J a_{\ell,j} h^j \right) = \sum_{j=0}^{J+1} h^j f_j + \mathcal{O}(h^{J+2}),$$

locally uniformly in U .

Proof. After solving the Eikonal equation and the transport equations, we collect a set of formal series : $\tilde{S}(z, w), (A^{(j)}(z, w))_{j \in \mathbb{N}}$ in $\mathbb{C}[[z, w]]$ and a series $\mu_{\ell, j}$ depending on ℓ . By applying the change of variable (z, w) into (q_1, q_2) , we obtain the formal series in $\mathbb{C}[[q_1, q_2]]$. Applying Borel's Lemma for real part and imaginary part of each sequence, we get a smooth complex-valued functions S and $(a_j)_{j \in \mathbb{N}}$ so that for each $J \in \mathbb{N}$, there exists flat functions f_0, f_1, \dots, f_{J+1} and a smooth function F on \mathbb{R}^2 (precisely F is a polynomial according to variable h whose coefficients are smooth functions depending on a_j and $\mu_{\ell, j}$) such that

$$\left(\mathcal{L}_{h,A}^S - h \sum_{j=0}^J \mu_{\ell, j} h^j \right) \left(\sum_{j=0}^J a_{\ell, j} h^j \right) = \sum_{j=0}^{J+1} h^j f_j + h^{J+2} F.$$

Note that $\mathcal{L}_{h,A}^S = e^{S/h} \mathcal{L}_{h,A} e^{-S/h}$, we have the expression

$$e^{S/h} \left(\mathcal{L}_{h,A} - h \sum_{j=0}^J \mu_{\ell, j} h^j \right) \left(e^{-S/h} \sum_{j=0}^J a_{\ell, j} h^j \right) = \sum_{j=0}^{J+1} h^j f_j + h^{J+2} F. \quad (4.69)$$

We recall that we applied WKB method for the eigen-problem of the magnetic Laplacian $\mathcal{L}_{h,A}$ with the special magnetic potential A which is mentioned in Subsection 4.3.1. The operator $\mathcal{L}_{h,A}$ and operator $\mathcal{L}_{h,\mathcal{M}}$ related to each other through the smooth real valued function θ :

$$\mathcal{L}_{h,A} = e^{i\theta/h} \mathcal{L}_{h,\mathcal{M}} e^{-i\theta/h}. \quad (4.70)$$

Therefore, we set up the function $T = S + i\theta$ to have

$$e^{T/h} \left(\mathcal{L}_{h,\mathcal{M}} - h \sum_{j=0}^J \mu_{\ell, j} h^j \right) \left(e^{-T/h} \sum_{j=0}^J a_{\ell, j} h^j \right) = \sum_{j=0}^{J+1} h^j f_j + h^{J+2} F \quad (4.71)$$

Restricting these functions on U , we get the conclusion. Since $\text{Re}(T) = \text{Re}(S)$, the formula of $\text{Re}(T)$ is implied directly from (4.58) after the change of variable:

$$\begin{aligned} \hat{S}(q_1, q_2) &= \frac{e^{2\eta(0)} b_0}{4} (q_1^2 + q_2^2) + \frac{e^{2\eta(0)} b_0}{4} \frac{\sqrt{\alpha} - \sqrt{\gamma}}{\sqrt{\alpha} + \sqrt{\gamma}} (q_1^2 - q_2^2 + 2q_1 q_2 i) + \mathcal{O}(\|q\|^3) \\ &= \frac{e^{2\eta(0)} b_0}{2} \left(\frac{\sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\gamma}} q_1^2 + \frac{\sqrt{\gamma}}{\sqrt{\alpha} + \sqrt{\gamma}} q_2^2 \right) + i \frac{e^{2\eta(0)} b_0}{2} \frac{\sqrt{\alpha} - \sqrt{\gamma}}{\sqrt{\alpha} + \sqrt{\gamma}} q_1 q_2 + \mathcal{O}(\|q\|^3). \end{aligned}$$

□

Proof of Theorem 1.14. Let us recall (4.34) for the relation of the magnetic Laplacian on manifold and the magnetic Laplacian on \mathbb{R}^2 . To bring the results in Theorem 4.12 to

manifold, we first multiply all functions appeared in (4.71) with a fixed cut-off function which is equal to 1 in the neighborhood of 0 and has compact support contained in U , we call temporarily χ_1 this cut-off function. This process provides us the functions in $C_c^\infty(U)$. Next, we apply the operator $(\varphi^{-1})^* = \phi^*$ for two sides of (4.71), we obtain the statement of Theorem 1.14 with

- i) $P = \phi^*(\chi_1 T)$.
- ii) $U_{\ell,j} = \phi^*(\chi_1 a_{\ell,j})$ for all $(\ell, j) \in \mathbb{N}^2$.

The formula of $\mu_{\ell,1}$ is given by the following matrix argument. From the definition of isothermal coordinates, we have

$$g_{p^*}(V_1, V_2) = e^{2\eta(0)} g_0(d\phi_{p^*} V_1, d\phi_{p^*} V_2), \quad \text{for all } V_1, V_2 \in T_{p^*} M.$$

In the matrix expression, we have

$$G_{p^*} = e^{2\eta(0)} (D\phi)_{p^*}^T (D\phi)_{p^*}, \quad (4.72)$$

where $(D\phi)_{p^*}$ is the representative matrix of the linear differential of $d\phi_{p^*}$. From Definition 1.10, the relation of the Hessian of B on manifold and the Hessian of $\mathcal{B} = B \circ \phi^{-1}$ is given by

$$d^2 B_{p^*}(V_1, V_2) = \langle \text{Hess} \mathcal{B}(0) d\phi_{p^*} V_1, d\phi_{p^*} V_2 \rangle_{\mathbb{R}^2}. \quad (4.73)$$

In order to compute the trace and determinant of the Hessian at p^* , we need to connect with the endomorphism \mathcal{H} of $T_{p^*} M$ defined by

$$(d^2 B)_{p^*}(V_1, V_2) = g_{p^*}(\mathcal{H} V_1, V_2) \quad \forall V_1, V_2 \in T_{p^*} M. \quad (4.74)$$

Additionally, (4.73) and (4.74) imply that

$$\langle \text{Hess} \mathcal{B}(0) d\phi_{p^*} V_1, d\phi_{p^*} V_2 \rangle_{\mathbb{R}^2} = g_{p^*}(\mathcal{H} V_1, V_2) \quad \forall V_1, V_2 \in T_{p^*} M, \quad (4.75)$$

or in the matrix expression

$$(D\phi)_{p^*}^T \text{Hess} \mathcal{B}(0) (D\phi)_{p^*} = \mathcal{H} G_{p^*}.$$

Combined with (4.72), we get

$$(D\phi)_{p^*}^T \text{Hess} \mathcal{B}(0) [(D\phi)_{p^*}^T]^{-1} = e^{2\eta(0)} \mathcal{H}.$$

Notice that

$$\text{Hess } \mathcal{B}(0) = \begin{pmatrix} 2\alpha & 0 \\ 0 & 2\gamma \end{pmatrix}.$$

Let $H = \frac{1}{2}\mathcal{H}$, then we can easily compute the determinant of H and trace of $H^{\frac{1}{2}}$:

$$\det(H) = \frac{e^{-4\eta(0)}}{4} \det(\text{Hess } \mathcal{B}(0)) = e^{-4\eta(0)} \alpha \gamma,$$

and

$$\begin{aligned} \text{Tr } H^{1/2} &= \text{Tr} \left[\frac{e^{-\eta(0)}}{\sqrt{2}} (D\phi)_{p^*}^T (\text{Hess } \mathcal{B}(0))^{1/2} [(D\phi)_{p^*}^T]^{-1} \right] \\ &= \frac{e^{-\eta(0)}}{\sqrt{2}} \text{Tr} \left[(\text{Hess } \mathcal{B}(0))^{1/2} \right] \\ &= e^{-\eta(0)} (\sqrt{\alpha} + \sqrt{\gamma}). \end{aligned}$$

We can see that

$$\mu_1 = 2\ell \frac{\sqrt{\det H}}{b_0} + \frac{(\text{Tr } H^{\frac{1}{2}})^2}{2b_0},$$

which is consistent with the formula of Helffer and Kordyukov in (1.39). \square

4.5 Comparison of the eigenfunctions and their WKB approximations

Next, in order to prove Theorem 1.15, we need to restrict the supports of functions in Theorem 1.14 in a smaller domain. We will perform this restriction for functions defined on \mathbb{R}^2 first, and then we pull-back these functions to manifold as the above proof. From the formula of $\text{Re}(T)$ in (4.68), there exist $K > 0$ and $\delta > 0$ such that $D(0; K) := \{q \in \mathbb{R}^2 : \|q\| \leq K\} \subset U$ and

$$\text{Re}(T)(q) \geq \delta \|q\|^2 \text{ for all } q \in D(0; K). \quad (4.76)$$

Let $\chi_2 : U \rightarrow \mathbb{R}$ be the cut-off function which is equal to 1 near 0 and has the support compact contained in $D(0; K)$. For brevity, we denote R is the real part of T

$$R := \text{Re}(T).$$

Theorem 4.13. *For all $(\varepsilon, J, \ell) \in (0, 1) \times \mathbb{N} \times \mathbb{N}$, there exist a constant $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$\|e^{\varepsilon T/h} (\mathcal{L}_{h, \mathcal{M}} - \lambda_{h, \ell}^J) \Psi_{h, \ell}^J\|_{L^2(U, e^{2\eta_{\text{dq}}})} \leq Ch^{J+2}, \quad (4.77)$$

with

$$\lambda_{h,\ell}^J = h \sum_{j=0}^J h^j \mu_{\ell,j} \quad \text{and} \quad \Psi_{h,\ell}^J = \chi_2 e^{-T/h} \left(\sum_{j=0}^J h^j a_{\ell,j} \right).$$

In particular,

$$\| (\mathcal{L}_{h,\mathcal{M}} - \lambda_{h,\ell}^J) \Psi_{h,\ell}^J \|_{L^2(U, e^{2\eta} dq)} \leq Ch^{J+2}. \quad (4.78)$$

Proof. Let $(\varepsilon, J, \ell) \in (0, 1) \times \mathbb{N} \times \mathbb{N}$, we check that

$$\begin{aligned} & e^{\varepsilon T/h} (\mathcal{L}_{h,\mathcal{M}} - \lambda_{h,\ell}^J) \Psi_{h,\ell}^J \\ &= e^{\varepsilon T/h} (\mathcal{L}_{h,\mathcal{M}} - \lambda_{h,\ell}^J) \left(\chi_2 e^{-T/h} \left(\sum_{j=0}^J h^j a_{\ell,j} \right) \right) \\ &= e^{\varepsilon T/h} [\mathcal{L}_{h,\mathcal{M}}, \chi_2] e^{-T/h} \left(\sum_{j=0}^J h^j a_{\ell,j} \right) + \chi_2 e^{\varepsilon T/h} (\mathcal{L}_{h,\mathcal{M}} - \lambda_{h,\ell}^J) \left(e^{-T/h} \left(\sum_{j=0}^J h^j a_{\ell,j} \right) \right). \end{aligned} \quad (4.79)$$

We compute

$$\begin{aligned} [\mathcal{L}_{h,\mathcal{M}}, \chi_2] &= e^{-2\eta} [(-ih\partial_1 - \mathcal{M}_1)^2 + (-ih\partial_2 - \mathcal{M}_2)^2, \chi_2] \\ &= e^{-2\eta} [-h^2\Delta + 2ih\mathcal{M} \cdot \nabla, \chi_2] \\ &= e^{-2\eta} (-h^2\Delta\chi_2 - 2h^2\nabla\chi_2 \cdot \nabla + 2ih\mathcal{M} \cdot \nabla\chi_2). \end{aligned}$$

Notice that supports of $\Delta\chi_2$ and $\nabla\chi_2$ are contained in $D(0; R)$ and stay away from zero. On these supports, we have the estimation, for all $N \in \mathbb{N}$,

$$e^{(\varepsilon-1)R(q)/h} \leq e^{(\varepsilon-1)\delta\|q\|^2/h} \leq \frac{N!}{(1-\varepsilon)^N \delta^N \|q\|^{2N}} h^N.$$

Here, we use the simple inequality $e^x \geq \frac{x^n}{n!}$ for $x \geq 0$. Notice that $T, \mathcal{M}, a_{j,\ell}, \eta$ are smooth functions on \bar{U} . Therefore, for all $N \in \mathbb{N}$, there exists a constant C (depending on the triple (ε, J, ℓ)) such that

$$\left\| e^{\varepsilon T/h} [\mathcal{L}_{h,\mathcal{M}}, \chi_2] e^{-T/h} \left(\sum_{j=0}^J h^j a_{j,\ell} \right) \right\|_{L^2(U, e^{2\eta} dq)} \leq Ch^N.$$

Or we can write

$$\left\| e^{\varepsilon T/h} [\mathcal{L}_{h,\mathcal{M}}, \chi_2] e^{-T/h} \left(\sum_{j=0}^J h^j a_{\ell,j} \right) \right\|_{L^2(U, e^{2\eta} dq)} = \mathcal{O}(h^\infty). \quad (4.80)$$

We look at the second term of (4.79), from (4.71), we have

$$\chi_2 e^{\varepsilon T/h} (\mathcal{L}_{h,\mathcal{M}} - \lambda_{h,\ell}^J) \left(e^{-T/h} \left(\sum_{j=0}^J h^j a_{\ell,j} \right) \right) = \chi_2 e^{(\varepsilon-1)T/h} \left(\sum_{j=0}^{J+1} h^j f_j + h^{J+2} F \right).$$

In the following lines, we will prove that

$$\left\| \chi_2 e^{(\varepsilon-1)T/h} \left(\sum_{j=0}^{J+1} h^j f_j \right) \right\|_{L^2(U, e^{2\eta} dq)} = \mathcal{O}(h^\infty). \quad (4.81)$$

Indeed, let $N \in \mathbb{N}$. Since $\sum_{j=0}^{J+1} h^j f_j$ is a flat function, we consider a small disc $D(0; r) \subset D(0; K)$ such that

$$\left| \sum_{j=0}^{J+1} h^j f_j(q) \right| \leq C \|q\|^{2N} \quad \forall q \in D(0; r).$$

On $D(0; r)$, we have the estimation

$$\begin{aligned} \left| \chi_2(q) e^{(\varepsilon-1)T(q)/h} \left(\sum_{j=0}^{J+1} h^j f_j(q) \right) \right| &= \left| \chi_2(q) e^{(\varepsilon-1)R(q)/h} \left(\sum_{j=0}^{J+1} h^j f_j(q) \right) \right| \\ &\leq e^{(\varepsilon-1)\delta \|q\|^2/h} \left| \sum_{j=0}^{J+1} h^j f_j(q) \right| \\ &\leq \frac{h^N}{N! (1-\varepsilon)^N \delta^N}. \end{aligned}$$

On $D(0; K) \setminus D(0; r)$, we have $\|q\| > r$, then

$$\begin{aligned} \left| \chi_2(q) e^{(\varepsilon-1)T(q)/h} \left(\sum_{j=0}^{J+1} h^j f_j(q) \right) \right| &= \left| \chi_2(q) e^{(\varepsilon-1)R(q)/h} \left(\sum_{j=0}^{J+1} h^j f_j(q) \right) \right| \\ &\leq \frac{h^N}{N! (1-\varepsilon)^N \delta^N r^{2N}} \left| \sum_{j=0}^{J+1} h^j f_j(q) \right| \\ &\leq C h^N. \end{aligned}$$

Thus, (4.81) is proved. From (4.79), (4.80) and (4.81), we obtain the following estimate

$$\begin{aligned}
& \left\| e^{\varepsilon T/h} (\mathcal{L}_{h,\mathcal{M}} - \lambda_{h,\ell}^J) \Psi_{h,\ell}^J \right\|_{\mathbb{L}^2(U, e^{2n} dq)} \\
& \leq \left\| e^{\varepsilon T/h} [\mathcal{L}_{h,\mathcal{M}}, \chi_2] e^{-T/h} \left(\sum_{j=0}^J h^j a_{\ell,j} \right) \right\|_{\mathbb{L}^2(U, e^{2n} dq)} \\
& \quad + \left\| \chi_2 e^{\varepsilon T/h} (\mathcal{L}_{h,\mathcal{M}} - \lambda_{h,\ell}^J) \left(e^{-T/h} \left(\sum_{j=0}^J h^j a_{\ell,j} \right) \right) \right\|_{\mathbb{L}^2(U, e^{2n} dq)} \\
& \leq \left\| e^{\varepsilon T/h} [\mathcal{L}_{h,\mathcal{M}}, \chi_2] e^{-T/h} \left(\sum_{j=0}^J h^j a_{\ell,j} \right) \right\|_{\mathbb{L}^2(U, e^{2n} dq)} + \left\| \chi_2 e^{(\varepsilon-1)T/h} \left(\sum_{j=0}^{J+1} h^j f_j \right) \right\|_{\mathbb{L}^2(U, e^{2n} dq)} \\
& \quad + h^{J+2} \left\| \chi_2 e^{(\varepsilon-1)T/h} F \right\|_{\mathbb{L}^2(U, e^{2n} dq)} \\
& \leq Ch^{J+2}.
\end{aligned}$$

□

Proof of Theorem 1.15. This theorem is just the version of Theorem 4.13 on a manifold. Notice that R is not non-negative, we multiply R with χ_2 to get a non-negative function defined on U . From Theorem 4.13, we have

$$\| e^{\varepsilon \chi_2 R/h} (\mathcal{L}_{h,\mathcal{M}} - \lambda_{h,\ell}^J) \Psi_{h,\ell}^J \|_{\mathbb{L}^2(U, e^{2n} dq)} \leq \| e^{\varepsilon T/h} (\mathcal{L}_{h,\mathcal{M}} - \lambda_{h,\ell}^J) \Psi_{h,\ell}^J \|_{\mathbb{L}^2(U, e^{2n} dq)} \leq Ch^{J+2}.$$

We define, for $(\ell, j) \in \mathbb{N}^2$,

i)

$$\widehat{P}(x) = \begin{cases} \phi^*(\chi_2 R)(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in M \setminus \Omega. \end{cases}$$

ii)

$$\widehat{U}_{\ell,j}(x) = \begin{cases} \phi^*(\chi_2 e^{-T/h} a_{\ell,j})(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in M \setminus \Omega. \end{cases}$$

These functions are smooth functions on M and $\widehat{P}(x) \geq 0$. □

Proof of Theorem 1.16. We fix the couple $(J, \ell) \in \mathbb{N} \times \mathbb{N}$. From the estimation (4.78), we apply the spectral theorem to get

$$\| \Upsilon_{h,\ell}^J \|_{\mathbb{L}^2(M)} \text{dist}(\lambda_{h,\ell}^J, \text{Sp}(\mathcal{L}_{h,\mathbf{A}})) \leq Ch^{J+2}.$$

Recall that, from the WKB construction, the expansion of $\lambda_{h,\ell}^J$ is given in the form

$$\lambda_{h,\ell}^J = b_0 h + \left(2\ell \frac{\sqrt{\det H}}{b_0} + \frac{(\operatorname{Tr} H^{\frac{1}{2}})^2}{2b_0} \right) h^2 + \dots + \mu_J h^{J+1}.$$

Comparing this with the eigenvalues $(\lambda_k(\mathcal{L}_{h,\mathbf{A}}))_{k \in \mathbb{N}}$ of the magnetic Laplacian given by Helffer and Kordyukov in (1.39), the nearest eigenvalue with $\lambda_{h,\ell}^J$ has to be $\lambda_\ell(\mathcal{L}_{h,A})$. For convenience, we denote $\lambda_\ell(h)$ instead of $\lambda_\ell(\mathcal{L}_{h,\mathbf{A}})$ for the ℓ -th eigenvalue of the magnetic Laplacian. Therefore,

$$\|\Upsilon_{h,\ell}^J\|_{L^2(M)} |\lambda_\ell(h) - \lambda_{h,\ell}^J| \leq Ch^{J+2}. \quad (4.82)$$

Combine this with (1.44) and notice that $\Pi_0 \Upsilon_{h,\ell}^J$ belongs to the kernel of $\mathcal{L}_{h,\mathbf{A}} - \lambda_\ell(h)$, we obtain

$$\begin{aligned} \|(\mathcal{L}_{h,\mathbf{A}} - \lambda_\ell(h)) (\Upsilon_{h,\ell}^J - \Pi_0 \Upsilon_{h,\ell}^J)\|_{L^2(M)} &\leq \|(\mathcal{L}_{h,\mathbf{A}} - \lambda_{h,\ell}^J) \Upsilon_{h,\ell}^J\|_{L^2(M)} \\ &\quad + |\lambda_\ell(h) - \lambda_{h,\ell}^J| \|\Upsilon_{h,\ell}^J\|_{L^2(M)} \\ &\leq Ch^{J+2}. \end{aligned}$$

By definition, $\Upsilon_{h,\ell}^J - \Pi_0 \Upsilon_{h,\ell}^J \in [\ker(\mathcal{L}_{h,\mathbf{A}} - \lambda_0(h))]^\perp$ and, since the gap between $\lambda_\ell(h)$ and other eigenvalues is of order h^2 , the spectral theorem proves that there exists $c > 0$ such that

$$\begin{aligned} ch^2 \|\Upsilon_{h,\ell}^J - \Pi_0 \Upsilon_{h,\ell}^J\|_{L^2(M)} &\leq \|(\mathcal{L}_{h,\mathbf{A}} - \lambda_\ell(h)) (\Upsilon_{h,\ell}^J - \Pi_0 \Upsilon_{h,\ell}^J)\|_{L^2(M)} \\ &\leq Ch^{J+2}. \end{aligned}$$

It implies the statement of the theorem. \square

Chapter 5

WKB analysis on \mathbb{R}^2 with a radial magnetic field

Nothing is impossible. The word itself says "I'm possible!"

Audrey Hepburn

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In this chapter, we are interested in the magnetic Laplacian

$$\mathcal{L}_{h,\mathbf{A}} = (-ih\nabla - \mathbf{A})^2, \quad (5.1)$$

which was carefully introduced in Subsection 4.1.5 of Chapter 4. We consider the semi-classical spectral problem of the magnetic Laplacian in \mathbb{R}^2 . With the assumption that the magnetic field B is in the radial form and has a unique non-degenerate minimum, we can use the WKB method to describe its spectrum through a family of electric operators which are created by the Fourier decomposition. Moreover, an approximation of the eigenfunction in an exponentially weighted space is also obtained. In other words, the present chapter is devoted to proving all the results in Subsection 1.3.2.2.

5.1 Magnetic Laplacian in the radial coordinates

5.1.1 The magnetic potential

Let \mathbf{A} be the magnetic potential associate with a magnetic field B , we write it as $\mathbf{A} = A_1 dq_1 + A_2 dq_2$. Because of the gauge of invariance (see Chapter 4), we can choose a magnetic potential compatible with the radial symmetry such that our problem can be solved in the convenient way. In our case, we choose

$$A_1(q) = -q_2\alpha(q), \quad A_2(q) = q_1\alpha(q), \quad (5.2)$$

where

$$\alpha(q) := \int_0^1 tB(tq) dt.$$

This potential is indeed associated with B because

$$\begin{aligned} \frac{\partial A_2}{\partial q_1}(q) - \frac{\partial A_1}{\partial q_2}(q) &= 2 \int_0^1 tB(tq) dt + \int_0^1 t^2 (q_1\partial_1 B(tq) + q_2\partial_2 B(tq)) dt \\ &= \int_0^1 \frac{d}{dt} (t^2 B(tq)) dt \\ &= B(q). \end{aligned}$$

5.1.2 Expression of the operator

Since the magnetic field only depends on $r := \sqrt{q_1^2 + q_2^2}$, we will borrow the polar coordinates to perform our analysis. Let us introduce the change of variable

$$\psi : \begin{cases} \mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}^2 \setminus \{0\} \\ (r, \theta) \mapsto (r \cos \theta, r \sin \theta) = (q_1, q_2). \end{cases} \quad (5.3)$$

The magnetic Laplacian in this radial coordinate is characterized by the following theorem

Theorem 5.1. *Under the transformation (5.3), the magnetic potential \mathbf{A} has the form*

$$\mathbf{A} = A_r dr + A_\theta d\theta,$$

where $\tilde{\mathbf{A}} = (A_r, A_\theta)^T := (d\psi)^T (A_1, A_2)^T$, with $d\psi$ denotes the Jacobian matrix of ψ .

The magnetic Laplacian $\mathcal{L}_{h, \mathbf{A}}$ in (5.1) is unitary equivalent to the operator

$$\mathcal{K}_{h, \tilde{\mathbf{A}}} = r^{-2} (r(-ih\partial_r + A_r))^2 + r^{-2} (-ih\partial_\theta + A_\theta)^2, \quad (5.4)$$

whose domain is

$$\text{Dom}(\mathcal{K}_{h, \tilde{\mathbf{A}}}) = \left\{ w \in L^2(\mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}, r dr d\theta) : \mathcal{K}_{h, \tilde{\mathbf{A}}}(w) \in L^2(\mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}, r dr d\theta) \right\}.$$

Proof. We introduce the unitary operator

$$\begin{aligned} T : L^2(\mathbb{R}^2, dq_1 dq_2) &\rightarrow L^2(\mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}, r dr d\theta) \\ u(q_1, q_2) &\mapsto \tilde{u}(r, \theta) = u \circ \psi(r, \theta). \end{aligned} \quad (5.5)$$

We also denote

$$\tilde{\nabla} := \begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix}, \quad \nabla := \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix}.$$

Notice that

$$\tilde{\nabla} \tilde{u}(r, \theta) = (d\psi)^T \nabla u(\psi(r, \theta)).$$

We just need to prove the statement in the restriction domain of the operator, precisely $u, v \in C_0^\infty(\mathbb{R}^2)$ and the rest is obtained by the density of $C_0^\infty(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$. Now, let

$u, v \in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned}
\langle \mathcal{L}_{h,\mathbf{A}} u, v \rangle_{L^2(\mathbb{R}^2)} &= \langle (-ih\nabla - \mathbf{A})u, (-ih\nabla - \mathbf{A})v \rangle_{L^2(\mathbb{R}^2)} \\
&= \int_{\mathbb{R}^2} (-ih\nabla - \mathbf{A})u \cdot \overline{(-ih\nabla - \mathbf{A})v} \, dq \\
&= \int_0^{2\pi} \int_{\mathbb{R}^+} (d\psi^{-1})^T (-ih\tilde{\nabla} - \tilde{\mathbf{A}})\tilde{u} \cdot \overline{(d\psi^{-1})^T (-ih\tilde{\nabla} - \tilde{\mathbf{A}})\tilde{v}} \, r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_{\mathbb{R}^+} (d\psi^{-1})(d\psi^{-1})^T (-ih\tilde{\nabla} - \tilde{\mathbf{A}})\tilde{u} \cdot \overline{(-ih\tilde{\nabla} - \tilde{\mathbf{A}})\tilde{v}} \, r \, dr \, d\theta.
\end{aligned}$$

Note that

$$(d\psi^{-1})(d\psi^{-1})^T = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}.$$

Then, thanks to integrating by parts according to variable r ,

$$\begin{aligned}
\langle \mathcal{L}_{h,\mathbf{A}} u, v \rangle_{L^2(\mathbb{R}^2)} &= \int_0^{2\pi} \int_{\mathbb{R}^+} [r^{-2} (r(-ih\partial_r - A_r))^2 \tilde{u} + r^{-2} (-ih\partial_\theta - A_\theta)^2 \tilde{u}] \tilde{v} \, r \, dr \, d\theta \\
&= \langle \mathcal{K}_{h,\tilde{\mathbf{A}}} \circ T(u), T(v) \rangle_{L^2(r \, dr \, d\theta)}.
\end{aligned}$$

Thus,

$$\mathcal{L}_{h,\mathbf{A}} = T^{-1} \circ \mathcal{K}_{h,\tilde{\mathbf{A}}} \circ T.$$

We recall from Chapter 4 that

$$\text{Dom}(\mathcal{L}_{h,\mathbf{A}}) = \{u \in L^2(\mathbb{R}^2) : \mathcal{L}_{h,\mathbf{A}} u \in L^2(\mathbb{R}^2)\}.$$

It gives us the $\mathcal{K}_{h,\tilde{\mathbf{A}}}$'s domain

$$\text{Dom}(\mathcal{K}_{h,\tilde{\mathbf{A}}}) = T(\text{Dom}(\mathcal{L}_{h,\mathbf{A}})) = \{Tw : w \in \text{Dom}(\mathcal{L}_{h,\mathbf{A}})\}.$$

□

With the choice of the magnetic potential \mathbf{A} in (5.2), we get immediately

$$A_r(r, \theta) = 0, \quad A_\theta(r, \theta) = G(r) := \int_0^r \tau \beta \left(\frac{\tau^2}{2} \right) \, d\tau.$$

The magnetic Laplacian becomes

$$\mathcal{K}_{h,\tilde{\mathbf{A}}} = -h^2 r^{-2} (r\partial_r)^2 + r^{-2} (-ih\partial_\theta - G(r))^2.$$

For the sake of simplification, \mathcal{K}_h is replaced for $\mathcal{K}_{h,\tilde{\mathbf{A}}}$ hereafter.

5.1.3 Fourier decomposition

Next, Fourier series is used in order to decompose the radial magnetic Laplacian into a direct sum of self-adjoint operators. Notice that for each $u(r, \theta) \in L^2(\mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}, r dr d\theta)$, we can express u as a Fourier series

$$u(r, \theta) = \sum_{m \in \mathbb{Z}} u_m(r) \frac{e^{im\theta}}{2\pi},$$

where

$$u_m(r) = \int_0^{2\pi} u(r, \theta) \frac{e^{-im\theta}}{2\pi} d\theta \in L^2(\mathbb{R}^+, r dr),$$

and $\sum_{m \in \mathbb{Z}} \|u_m\|_{L^2(\mathbb{R}^+, r dr)}^2 < +\infty$.

This allows us to write the Hilbert space $L^2(\mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}, r dr d\theta)$ as a Hilbert direct sum of Hilbert spaces $L^2(\mathbb{R}^+, r dr)$,

$$L^2(\mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}, r dr d\theta) \cong \bigoplus_{m \in \mathbb{Z}} L^2(\mathbb{R}^+, r dr). \quad (5.6)$$

Here, we use the isomorphism to associate each function $u \in L^2(\mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}, r dr d\theta)$ with the sequence of functions $(u_m(r))_{m \in \mathbb{Z}} \subset L^2(\mathbb{R}^+, r dr)$. Through that, we write the operator \mathcal{K}_h corresponding to this decomposition as

$$\begin{aligned} \mathcal{K}_h u &= \left[-h^2 r^{-2} (r \partial_r)^2 + r^{-2} (-ih \partial_\theta - G(r))^2 \right] \sum_{m \in \mathbb{Z}} \left(u_m(r) \frac{e^{im\theta}}{2\pi} \right) \\ &= \sum_{m \in \mathbb{Z}} \left[-h^2 r^{-2} (r \partial_r)^2 + r^{-2} (-ih \partial_\theta - G(r))^2 \right] \left(u_m(r) \frac{e^{im\theta}}{2\pi} \right) \\ &= \sum_{m \in \mathbb{Z}} \left[-h^2 r^{-2} (r \partial_r)^2 + r^{-2} (hm - G(r))^2 \right] \left(u_m(r) \frac{e^{im\theta}}{2\pi} \right). \end{aligned}$$

Therefore, the magnetic Laplacian \mathcal{K}_h is described as the direct sum of the radial electric Schrödinger operator

$$\mathcal{K}_h = \bigoplus_{m \in \mathbb{Z}} \mathfrak{L}_{h,m}, \quad (5.7)$$

where

$$\begin{cases} \mathfrak{L}_{h,m} := -h^2 r^{-2} (r \partial_r)^2 + r^{-2} (hm - G(r))^2 \\ \text{Dom}(\mathfrak{L}_{h,m}) = \{u \in L^2(\mathbb{R}^+, r dr) : \mathfrak{L}_{h,m} u \in L^2(\mathbb{R}^+, r dr)\} \end{cases} \quad (5.8)$$

are the self-adjoint operators.

Because of that, we will study the spectrum of the magnetic Laplacian through analysing the spectrum of these electric Schrödinger operators. Namely, we find the asymptotic

expansion for the ground state energy of each $\mathfrak{L}_{h,m}$, $m \in \mathbb{N}$. After that, we figure out the relation between the first eigenpair of $\mathfrak{L}_{h,m}$ and the m -th eigenpair of \mathfrak{K}_h . Next, the WKB method is then used to describe the first eigenfunction for each $\mathfrak{L}_{h,m}$. Finally, we show that these WKB quasi-modes are very good approximations, in L^2 sense, of the corresponding eigenfunctions.

5.2 Spectral analysis of the radial electric Schrödinger operators

5.2.1 Compact resolvent property

Before proving the compact resolvent property, let us prove two basic results:

Lemma 5.2. $L^2(\mathbb{R}^+, r dr)$ is a Banach space.

Proof. Take a Cauchy sequence u_n in $L^2(\mathbb{R}^+, r dr)$, so $v_n(q_1, q_2) := u_n(\sqrt{q_1^2 + q_2^2})$ is also a Cauchy sequence in $L^2(\mathbb{R}^2, dq)$ since

$$\begin{aligned} \|v_n\|_{L^2(\mathbb{R}^2, dq)}^2 &= \int_{\mathbb{R}^2} |v_n(q)|^2 dq \\ &= \int_0^{2\pi} \int_0^{+\infty} |u_n(r)|^2 r dr d\theta \\ &= 2\pi \|u_n\|_{L^2(\mathbb{R}^+, r dr)}^2. \end{aligned}$$

Since $L^2(\mathbb{R}^2, dq)$ is a Banach space, v_n converges to some v in $L^2(\mathbb{R}^2, dq)$. Let ϕ be an arbitrary θ -rotation transformation

$$\begin{aligned} \phi : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ \phi \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \end{aligned}$$

Since $v_n \circ \phi(q) = v_n(q)$ and $|\det \phi| = 1$, it implies that

$$\begin{aligned} \|v - v \circ \phi\|_{L^2(\mathbb{R}^2, dq)} &\leq \|v - v_n \circ \phi\|_{L^2(\mathbb{R}^2, dq)} + \|v_n \circ \phi - v \circ \phi\|_{L^2(\mathbb{R}^2, dq)} \\ &= 2\|v - v_n\|_{L^2(\mathbb{R}^2, dq)} \\ &\rightarrow 0. \end{aligned}$$

It yields that v is independent of θ and if $u(r)$ is chosen as $v(r \cos \theta, r \sin \theta)$ then u_n converges to u in $L^2(\mathbb{R}^+, r dr)$. \square

Lemma 5.3. *Let R be an arbitrary positive number, then the injection $H^1((0, R), r dr)$ is compact embedded in $L^2((0, R), r dr)$.*

Proof. Take a bounded sequence u_n in $H^1((0, R), r dr)$

$$\int_0^R |u_n|^2 r dr + \int_0^R |\partial_r u_n|^2 r dr \leq 1.$$

As above lemma, we set $v_n(q_1, q_2) := u_n(\sqrt{q_1^2 + q_2^2})$ defined on the disc $D(0; R)$. Hence, v_n satisfies

$$\begin{aligned} \|v_n\|_{H^1(D(0; R), dq)}^2 &= \int_{\mathbb{R}^2} |v_n|^2 + \left| \frac{\partial v_n}{\partial x} \right|^2 + \left| \frac{\partial v_n}{\partial y} \right|^2 dq \\ &= \int_0^{2\pi} \int_0^R |u_n(r)|^2 + \left| \frac{\partial u_n}{\partial r} \cos \theta \right|^2 + \left| \frac{\partial u_n}{\partial r} \sin \theta \right|^2 r dr d\theta \\ &= \int_0^{2\pi} \int_0^R |u_n|^2 + |\partial_r u_n|^2 r dr d\theta \\ &= 2\pi \|u_n\|_{H^1((0, R), r dr)}^2. \end{aligned}$$

Because the injection $H^1(D(0; R), dq) \hookrightarrow L^2(D(0; R), dq)$ is compact, there exists a subsequence v_{n_k} converging to some v in $L^2(D(0; R), dq)$. Next, reasoning in the end of the proof of Lemma 5.2, we can prove that v is independent of θ and we obtain the result. \square

Let us now give a criterion to get a compact resolvent.

Theorem 5.4. *Let $\gamma > 0$ and $V \in C^\infty(\mathbb{R}^+, \mathbb{R}^+)$ satisfy*

$$\lim_{r \rightarrow \infty} V(r) = +\infty.$$

We set up a sesquilinear form

$$\mathcal{Q}(u, v) = \gamma \int_0^{+\infty} \partial_r u \overline{\partial_r v} r dr + \int_0^{+\infty} V u \bar{v} r dr + \int_0^{+\infty} u \bar{v} r dr,$$

defined on the domain

$$\text{Dom}(\mathcal{Q}) = \left\{ u \in L^2(\mathbb{R}^+, r dr) : \partial_r u \in L^2(\mathbb{R}^+, r dr), \sqrt{V} u \in L^2(\mathbb{R}^+, r dr) \right\}.$$

Then $\text{Dom}(\mathcal{Q})$ is a Hilbert space with the inner product $\mathcal{Q}(\cdot, \cdot)$. Furthermore, the self-adjoint operator generated from Lax-Milgram (see Theorem A.6) has compact resolvent.

Proof. All properties of the inner product are easy to check. We will check that all the Cauchy sequences in $\text{Dom}(\mathcal{Q})$ converge. Assume that $(u_n)_{n \in \mathbb{N}} \subset \text{Dom}(\mathcal{Q})$ is a Cauchy

sequence with the norm

$$\|u\|_{\mathcal{Q}}^2 = \gamma \|\partial_r u\|_{L^2(\mathbb{R}^+, r dr)}^2 + \|\sqrt{V}u\|_{L^2(\mathbb{R}^+, r dr)}^2 + \|u\|_{L^2(\mathbb{R}^+, r dr)}^2.$$

Hence, u_n , $\partial_r u_n$ and $\sqrt{V}u_n$ are Cauchy sequences in $L^2(\mathbb{R}^+, r dr)$. Since $L^2(\mathbb{R}^+, r dr)$ is a Banach space, with the distribution technique, we imply that there exists $u \in L^2(\mathbb{R}^+, r dr)$ such that

$$\begin{cases} u_n \rightarrow u \\ \partial_r u_n \rightarrow \partial_r u \\ \sqrt{V}u_n \rightarrow \sqrt{V}u \end{cases},$$

or $\|u_n - u\|_{\mathcal{Q}} \rightarrow 0$. Thus, $(\text{Dom}(\mathcal{Q}), \mathcal{Q}(\cdot, \cdot))$ is a Hilbert space. In order to examine the compact resolvent of the self-adjoint induced from Lax-Milgram Theorem, we just need to check that the injection $(\text{Dom}(\mathcal{Q}), \|\cdot\|_{\mathcal{Q}}) \hookrightarrow L^2(\mathbb{R}^+, r dr)$ is compact (see Proposition A.9).

Take a sequence (u_n) in $\text{Dom}(\mathcal{Q})$ such that

$$\gamma \int_0^{+\infty} |\partial_r u_n|^2 r dr + \int_0^{+\infty} V(r) |u_n|^2 r dr + \int_0^{+\infty} |u_n|^2 r dr \leq 1. \quad (5.9)$$

Since V go to infinity when $r \rightarrow +\infty$, then for any $\varepsilon > 0$, there exists a number $R_\varepsilon > 0$ such that

$$\int_{R_\varepsilon}^{+\infty} |u_n|^2 r dr \leq \frac{\varepsilon}{4} \quad \text{for all } n \geq 0.$$

Furthermore, since $H^1((0, R_\varepsilon), r dr)$ is compact embedded in $L^2((0, R_\varepsilon), r dr)$, we have a subsequence $(v_n := u_{n_{|_{(0, R_\varepsilon)}}})$ is Cauchy in $L^2((0, R_\varepsilon), r dr)$. For n, k large enough, we get

$$\begin{aligned} \|u_k - u_n\|_{L^2(\mathbb{R}^+, r dr)}^2 &= \int_0^{R_\varepsilon} |u_k - u_n|^2 r dr + \int_{R_\varepsilon}^{+\infty} |u_k - u_n|^2 r dr \\ &\leq \int_0^{R_\varepsilon} |v_k - v_n|^2 r dr + 2 \left(\int_{R_\varepsilon}^{+\infty} |u_k|^2 r dr + \int_{R_\varepsilon}^{+\infty} |u_n|^2 r dr \right) \\ &\leq \frac{\varepsilon}{2} + 2 \frac{\varepsilon}{4} \\ &= \varepsilon. \end{aligned}$$

Since $L^2(\mathbb{R}^+, r dr)$ is a Banach space (see Lemma 5.2), we get the convergence of u_n in $L^2(\mathbb{R}^+, r dr)$. We finish the proof. \square

Applying the above theorem, we have the result:

Theorem 5.5. *For any $m \in \mathbb{Z}$ and $h > 0$, the operator $\mathfrak{L}_{h,m}$ has compact resolvent.*

Proof. We fix $m \in \mathbb{Z}$ and $h > 0$, we take $\gamma = 2h^2$ and consider the sesquilinear

$$\mathcal{Q}_{h,m}(u, v) = 2h^2 \int_0^{+\infty} \partial_r u \overline{\partial_r v} r dr + \int_0^{+\infty} \frac{(hm - G(r))^2}{r^2} u \bar{v} r dr + \int_0^{+\infty} u \bar{v} r dr,$$

defined on the domain

$$\text{Dom}(\mathcal{Q}_{h,m}) = \left\{ u \in L^2(\mathbb{R}^+, r dr) : \partial_r u \in L^2(\mathbb{R}^+, r dr), \sqrt{V_{h,m}} u \in L^2(\mathbb{R}^+, r dr) \right\},$$

with

$$V_{h,m}(r) := \frac{(hm - G(r))^2}{r^2}.$$

We have $\text{Dom}(\mathcal{Q}_{h,m})$ is a Hilbert space equipped with the inner product $\mathcal{Q}_{h,m}$. From Lax-Milgram (see Theorem A.6), the sesquilinear form $\mathcal{Q}_{h,m}$ produces a self-adjoint operator $\mathcal{S}_{h,m}$. We know that $u \in \text{Dom}(\mathcal{S}_{h,m})$ if there exists an element $w \in L^2(\mathbb{R}^+, r dr)$ such that

$$\mathcal{Q}_{h,m}(u, v) = \langle w, v \rangle_{L^2(\mathbb{R}^+, r dr)} \quad \text{for all } v \in \text{Dom}(\mathcal{Q}_{h,m}).$$

Considering $v \in C_c^\infty(\mathbb{R}^+)$ and using the distribution definition, we deduce that

$$\left[-h^2 r^{-2} (r \partial_r)^2 + V_{h,m} \right] u + u \in L^2(\mathbb{R}^+, r dr).$$

Thus, $u \in \text{Dom}(\mathfrak{L}_{h,m})$ and $\mathfrak{L}_{h,m} + Id$ becomes an extension of $\mathcal{S}_{h,m}$. By the self-adjointness of $\mathfrak{L}_{h,m}$ and $\mathcal{S}_{h,m}$, we can conclude that

$$\mathcal{S}_{h,m} = \mathfrak{L}_{h,m} + Id.$$

Note that $\mathfrak{L}_{h,m}$ has compact resolvent if and only if $\mathcal{S}_{h,m}$ has compact resolvent.

From the condition (1.46) of magnetic field, we imply that

$$\lim_{r \rightarrow +\infty} \frac{(hm - G(r))^2}{r^2} = +\infty. \quad (5.10)$$

The conclusion of the theorem is obtained by applying again Theorem 5.4. \square

5.2.2 Spectrum of rescaled radial electric Schrödinger operators.

By the change of variable $\rho = \frac{r^2}{2}$ in integral, we have

$$G(r) = \int_0^r \tau \beta \left(\frac{\tau^2}{2} \right) d\tau = \int_0^\rho \beta(s) ds.$$

We define

$$a(\rho) = \int_0^\rho \beta(s) ds.$$

Using the unitary transformation

$$\begin{aligned} T_1 : L^2(\mathbb{R}^+, d\rho) &\rightarrow L^2(\mathbb{R}^+, r dr), \\ v(\rho) &\mapsto (T_1 v)(r) = v(r^2/2), \end{aligned} \quad (5.11)$$

we get a new operator which is unitary equivalent to $\mathfrak{L}_{h,m}$ and acting on $L^2(\mathbb{R}^+, d\rho)$:

$$\begin{cases} \text{Dom}(\mathcal{N}_{h,m}) = \{u \in L^2(\mathbb{R}^+, d\rho) : \mathcal{N}_{h,m}u \in L^2(\mathbb{R}^+, d\rho)\} \\ \mathcal{N}_{h,m} := T_1^{-1} \mathfrak{L}_{h,m} T_1 = -2h^2 \partial_\rho \rho \partial_\rho + \frac{(hm - a(\rho))^2}{2\rho} \end{cases}. \quad (5.12)$$

Later, we will use the WKB method to find an asymptotic expansion of eigenfunction corresponding to the first eigenvalue of $\mathcal{N}_{h,m}$. Now, we just want to discover the asymptotic expansion $\mathcal{N}_{h,m}$'s first eigenvalue by means of a rescaled operator.

5.2.2.1 Rescaling

In order to find the quasi-modes and quasi-eigenvalues in terms of formal power series of h , we use the scaling $\rho = ht$ and expand the resulting operator, called $\mathcal{M}_{h,m}$, into a formal series of h . Precisely, we obtain the operator $\mathcal{M}_{h,m}$ through the unitary transformation

$$\begin{aligned} T_2 : L^2(\mathbb{R}^+, dt) &\rightarrow L^2(\mathbb{R}^+, d\rho), \\ v(t) &\mapsto (T_2 v)(\rho) = h^{-1/2} v(h^{-1} \rho). \end{aligned} \quad (5.13)$$

and

$$\begin{cases} \text{Dom}(\mathcal{M}_{h,m}) = \{u \in L^2(\mathbb{R}^+, dt) : \mathcal{M}_{h,m}u \in L^2(\mathbb{R}^+, dt)\} \\ \mathcal{M}_{h,m} := T_2^{-1} \mathcal{N}_{h,m} T_2 = -2h \partial_t t \partial_t + \frac{(hm - a(ht))^2}{2ht} \end{cases}.$$

Since $\beta \in C^\infty(\mathbb{R})$, so is a . By applying the Taylor's Theorem for the function a at 0, we have

$$a(ht) = \beta(0)ht + \frac{\beta'(0)}{2} h^2 t^2 + \mathcal{O}(h^3 t^3).$$

We compute directly the operator $\mathcal{M}_{h,m}$

$$\begin{aligned}
\mathcal{M}_{h,m} &= -2h\partial_t t \partial_t + \frac{a^2(ht)}{2ht} - \frac{m a(ht)}{t} + h \frac{m^2}{2t} \\
&= -2h\partial_t t \partial_t + \frac{\left(\beta(0)ht + \frac{\beta'(0)}{2}h^2t^2 + \mathcal{O}(h^3t^3)\right)^2}{2ht} \\
&\quad - \frac{m \left(\beta(0)ht + \frac{\beta'(0)}{2}h^2t^2 + \mathcal{O}(h^3t^3)\right)}{t} + h \frac{m^2}{2t} \\
&= h \left(-2\partial_t t \partial_t + \frac{\beta(0)^2t}{2} + \frac{m^2}{2t} - m\beta(0) \right) \\
&\quad + h^2 \left(-\frac{m\beta'(0)}{2}t + \frac{\beta(0)\beta'(0)}{2}t^2 \right) + \frac{\mathcal{O}(h^3t^3)}{t}.
\end{aligned}$$

By setting up

$$\begin{aligned}
\mathfrak{L}_m^{[0]} &:= -2\partial_t t \partial_t + \frac{\beta(0)^2t}{2} + \frac{m^2}{2t} - m\beta(0) \\
\mathfrak{L}_m^{[1]} &:= -\frac{m\beta'(0)}{2}t + \frac{\beta(0)\beta'(0)}{2}t^2,
\end{aligned}$$

we can rewrite $\mathcal{M}_{h,m}$ as

$$\mathcal{M}_{h,m} = h\mathfrak{L}_m^{[0]} + h^2\mathfrak{L}_m^{[1]} + \frac{R(ht)}{t},$$

where R is the reminder satisfying that there exist a constant $C > 0$ and $\delta > 0$ such that

$$|R(s)| \leq Cs^3 \quad \text{for all } s \in [0, \delta]. \quad (5.14)$$

We consider $\mathfrak{L}_m^{[0]}$ as a self-adjoint operator given by Lax-Milgram Theorem through the sesquilinear

$$\mathcal{Q}_m^{[0]}(u, v) = \int_0^{+\infty} 2t\partial_t u \overline{\partial_t v} dt + \int_0^{+\infty} \left(\frac{\beta(0)^2t}{2} + \frac{m^2}{2t} - m\beta(0) \right) u \bar{v} dt \quad (5.15)$$

with the form domain

$$\text{Dom}(\mathcal{Q}_m^{[0]}) = \left\{ u \in L^2(\mathbb{R}^+, dt) : \sqrt{t}\partial_t u \in L^2(\mathbb{R}^+, dt), \sqrt{\frac{\beta(0)^2t}{2} + \frac{m^2}{2t} - m\beta(0)}u \in L^2(\mathbb{R}^+, dt) \right\}.$$

Therefore, we have

$$\mathcal{Q}_m^{[0]}(u, v) = \langle \mathfrak{L}_m^{[0]}u, v \rangle_{L^2(\mathbb{R}^+, dt)} \quad \text{for all } u \in \text{Dom}(\mathfrak{L}_m^{[0]}), v \in \text{Dom}(\mathcal{Q}_m^{[0]}). \quad (5.16)$$

5.2.2.2 Investigating spectrum of the operator $\mathfrak{L}_m^{[0]}$

This subsection is devoted to the spectrum of the operators $\mathfrak{L}_m^{[0]}$ for all $m \in \mathbb{Z}$.

Theorem 5.6. *Let $m \in \mathbb{Z}$, the operator $\mathfrak{L}_m^{[0]}$ has discrete spectrum and its spectrum is given by*

$$\text{Sp}(\mathfrak{L}_m^{[0]}) = \{(2k + 1 + |m| - m)\beta(0) : k \in \mathbb{N}\} .$$

To prove this theorem, we will follow two steps: first, we show that the self-adjoint operator $\mathfrak{L}_m^{[0]}$ has compact resolvent. Second, we relate $\mathfrak{L}_m^{[0]}$ to the Laguerre operator and then deduce its spectrum. We start with their compact resolvent properties.

Theorem 5.7. *For each $m \in \mathbb{Z}$, the operator $\mathfrak{L}_m^{[0]}$ has compact resolvent.*

Proof. Let $m \in \mathbb{Z}$. By changing variable $t = \frac{r^2}{2}$, the plan is to show that the operator $\mathfrak{L}_m^{[0]}$ is unitarily equivalent to an operator which has compact resolvent. Let us introduce an unitary operator

$$\begin{aligned} U : L^2(\mathbb{R}^+, dt) &\rightarrow L^2(\mathbb{R}^+, r dr), \\ v(t) &\mapsto (Uv)(r) = v(r^2/2). \end{aligned}$$

We organize a quadratic form as following:

$$\begin{cases} \text{Dom}(\tilde{\mathcal{Q}}_m^{[0]}) := U(\text{Dom}(\mathcal{Q}_m^{[0]})), \\ \tilde{\mathcal{Q}}_m^{[0]}(a, b) = \mathcal{Q}_m^{[0]}(U^{-1}a, U^{-1}b) + \int_{\mathbb{R}^+} a \bar{b} r dr. \end{cases}$$

From the definition of U , we have $(U^{-1}a)(t) = a(\sqrt{2t})$. By computing straightforwardly and upon observing that $\partial_r = \sqrt{2t}\partial_t$, it turns out that

$$\text{Dom}(\tilde{\mathcal{Q}}_m^{[0]}) = \left\{ a \in L^2(\mathbb{R}^+, r dr) : \partial_r a \in L^2(\mathbb{R}^+, r dr), \sqrt{\frac{\beta(0)^2 r^2}{4} + \frac{m^2}{r^2} - m\beta(0)} a \in L^2(\mathbb{R}^+, r dr) \right\},$$

and

$$\tilde{\mathcal{Q}}_m^{[0]}(a, b) = \int_{\mathbb{R}^+} \partial_r a \overline{\partial_r b} r dr + \int_{\mathbb{R}^+} \left(\frac{\beta(0)^2 r^2}{4} + \frac{m^2}{r^2} - m\beta(0) \right) a \bar{b} r dr + \int_{\mathbb{R}^+} a \bar{b} r dr .$$

Applying Theorem 5.4 and noticing that

$$\lim_{r \rightarrow +\infty} \left(\frac{\beta(0)^2 r^2}{4} + \frac{m^2}{r^2} - m\beta(0) \right) = +\infty ,$$

the operator $S_m^{[0]}$, which is defined from Lax-Milgram (see Theorem A.6) through the sesquilinear $\tilde{Q}_m^{[0]}$, has compact resolvent. As a consequence, the operator $\tilde{\mathfrak{L}}_m^{[0]} := S_m^{[0]} - Id$ also has compact resolvent. According to (5.16), it implies that

$$\tilde{\mathfrak{L}}_m^{[0]} = U \mathfrak{L}_m^{[0]} U^{-1}.$$

Thus, $\mathfrak{L}_m^{[0]}$ is unitary equivalent to $\tilde{\mathfrak{L}}_m^{[0]}$ and also has compact resolvent. \square

In the next step, we bring the operator $\mathfrak{L}_m^{[0]}$ to the Laguerre form. By letting $t = \frac{s}{\beta(0)}$, the operator $\mathfrak{L}_m^{[0]}$ becomes

$$\beta(0) \left(-2\partial_s s \partial_s + \frac{s}{2} + \frac{m^2}{2s} - m \right). \quad (5.17)$$

Consider the self-adjoint operator

$$\mathcal{T}_m = s^{-\frac{|m|}{2}} e^{\frac{s}{2}} \left(-2\partial_s s \partial_s + \frac{s}{2} + \frac{m^2}{2s} - m \right) s^{\frac{|m|}{2}} e^{-\frac{s}{2}}, \quad (5.18)$$

acting in the Hilbert space $L^2(\mathbb{R}^+, s^{-|m|} e^s ds)$ and equivalent to the operator $-2\partial_s s \partial_s + \frac{s}{2} + \frac{m^2}{2s} - m$ by the unitary

$$\begin{aligned} \Lambda : L^2(\mathbb{R}^+, s^{|m|} e^{-s} ds) &\rightarrow L^2(\mathbb{R}^+, ds) \\ v(s) &\mapsto (\Lambda v)(s) = s^{\frac{|m|}{2}} e^{-\frac{s}{2}} v(s). \end{aligned}$$

A straightforward computation gives

$$\begin{aligned} \mathcal{T}_m &= -2 \left(\partial_s + \frac{|m|}{2s} - \frac{1}{2} \right) s \left(\partial_s + \frac{|m|}{2s} - \frac{1}{2} \right) + \frac{s}{2} + \frac{m^2}{2s} - m \\ &= -2 \left(\partial_s + \frac{|m|}{2s} - \frac{1}{2} \right) - 2s \left(\partial_s + \frac{|m|}{2s} - \frac{1}{2} \right)^2 + \frac{s}{2} + \frac{m^2}{2s} - m \\ &= -2 \left(\partial_s + \frac{|m|}{2s} - \frac{1}{2} \right) - 2s \left[\partial_s^2 + \partial_s \left(\frac{|m|}{2s} - \frac{1}{2} \right) + \left(\frac{|m|}{2s} - \frac{1}{2} \right) \partial_s \right] \\ &\quad - 2s \left(\frac{|m|}{2s} - \frac{1}{2} \right)^2 + \frac{s}{2} + \frac{m^2}{2s} - m \\ &= -2 \left(\partial_s + \frac{|m|}{2s} - \frac{1}{2} \right) - 2s \left[\partial_s^2 - \frac{|m|}{2s^2} + \left(\frac{|m|}{s} - 1 \right) \partial_s \right] \\ &\quad - 2s \left(\frac{m^2}{4s^2} - \frac{|m|}{2s} + \frac{1}{4} \right) + \frac{s}{2} + \frac{m^2}{2s} - m \\ &= -2s \partial_s^2 + (2s - 2 - 2|m|) \partial_s + |m| - m + 1. \end{aligned}$$

It is well-known that the generalized Laguerre polynomials $L_n^{(m)}$ are solutions of the differential equation

$$s\partial_s^2 y + (|m| + 1 - s)\partial_s y + ny = 0, \quad (5.19)$$

with $n \in \mathbb{Z}$, see [59]. Furthermore, these polynomials are orthogonal with the inner product of space $L^2(\mathbb{R}^+, s^{|m|}e^{-s}ds)$ and satisfies

$$\int_0^{+\infty} L_k^{(m)}(s)L_n^{(m)}(s)s^{|m|}e^{-s}ds = \frac{\Gamma(n + |m| + 1)}{n!}\delta_{k,n}, \quad (5.20)$$

where $\delta_{k,n}$ denotes Kronecker delta notation.

Then for each $m \in \mathbb{Z}$, we have

$$\mathcal{T}_m(L_n^{(m)}) = (2n + 1 + |m| - m)L_n^{(m)}. \quad (5.21)$$

In particular,

$$\{2n + |m| - m + 1 : n \in \mathbb{N}\} \subset \text{sp}(\mathcal{T}_m). \quad (5.22)$$

Theorem 5.8. *For each $m \in \mathbb{Z}$, the family $(L_n^{(m)})_{n \in \mathbb{N}}$ is total in $L^2(\mathbb{R}^+, s^{|m|}e^{-s}ds)$.*

Proof. Take $f \in L^2(\mathbb{R}^+, s^{|m|}e^{-s}ds)$ such that

$$\int_0^{+\infty} f(s)L_n^{(m)}(s)s^{|m|}e^{-s}ds = 0 \quad \text{for all } n \in \mathbb{N}.$$

It follows that, for all $n \in \mathbb{N}$,

$$\int_0^{+\infty} s^n f(s)s^{|m|}e^{-s}ds = 0.$$

Let \bar{f} be the extension of f on \mathbb{R} by setting that \bar{f} is zero on the negative axis. We put

$$F(\xi) = \int_{\mathbb{R}} e^{-is\xi} \bar{f}(s)s^{|m|}e^{-s}ds.$$

This function is well defined thanks to the Cauchy-Schwarz inequality

$$\begin{aligned} \int_{\mathbb{R}} |e^{-is\xi} \bar{f}(s)s^{|m|}e^{-s}| ds &= \int_0^{+\infty} |\bar{f}(s)s^{|m|}e^{-s}| ds \\ &\leq \left(\int_0^{+\infty} |\bar{f}(s)|^2 s^{|m|}e^{-s} ds \right) \left(\int_0^{+\infty} s^{|m|}e^{-s} ds \right) \\ &< +\infty. \end{aligned}$$

Using power series for an exponential function, we can rewrite

$$F(\xi) = \int_{\mathbb{R}} \sum_{k=0}^{+\infty} \frac{(-is\xi)^k}{k!} \bar{f}(s)s^{|m|}e^{-s}ds.$$

We can apply the Fubini theorem to get

$$F(\xi) = \sum_{k=0}^{+\infty} \xi^k \int_{\mathbb{R}} \frac{(-is)^k}{k!} \bar{f}(s) s^{|m|} e^{-s} ds = 0.$$

Therefore the Fourier transform of $\bar{f} s^{|m|} e^{-s}$ is 0 and of course $f = 0$. \square

Theorem 5.9. *Spectrum of the operator \mathcal{T}_m is*

$$\text{Sp}(\mathcal{T}_m) = \{2k + 1 + |m| - m : k \in \mathbb{N}\}.$$

Proof. From (5.22), we just need to prove the remaining direction.

If we denote by $(f_n)_{n \in \mathbb{N}}$ the L^2 -normalization of the family $L_n^{(m)}$, then $(f_n)_{n \in \mathbb{N}}$ is a Hilbert basis of $L^2(\mathbb{R}^+, s^{|m|} e^{-s} ds)$ such that $\mathcal{T}_m(f_n) = (2n + 1 + |m| - m)f_n$. Finally, since \mathcal{T}_m is self-adjoint with compact resolvent then \mathcal{T}_m has a real, discrete spectrum. For this reason, we search for the eigenvalue $\lambda \in \mathbb{R}$ such that there exists a non-zero $\Psi \in \text{Dom}(\mathcal{T}_m)$ which satisfies

$$\mathcal{T}_m(\Psi) = \lambda \Psi.$$

We write the following decomposition, converging in $L^2(\mathbb{R}^+, s^{|m|} e^{-s} ds)$,

$$\Psi = \sum_{n \in \mathbb{N}} \langle \Psi, f_n \rangle_{L^2} f_n.$$

Here we denote $\langle \cdot, \cdot \rangle_{L^2}$ as an inner product on $L^2(\mathbb{R}^+, s^{|m|} e^{-s} ds)$. For all $\phi \in \text{Dom}(\mathcal{T}_m)$, we have

$$\langle \Psi, (\mathcal{T}_m - \lambda)\phi \rangle_{L^2} = \langle (\mathcal{T}_m - \lambda)\Psi, \phi \rangle_{L^2} = 0.$$

Then by convergence in $L^2(\mathbb{R}^+, s^{|m|} e^{-s} ds)$,

$$\sum_{n \in \mathbb{N}} \langle \Psi, f_n \rangle_{L^2} \langle f_n, (\mathcal{T}_m - \lambda)\phi \rangle_{L^2} = 0.$$

Let $k \in \mathbb{N}$ and choose $\phi = f_k$, we obtain

$$\sum_{n \in \mathbb{N}} \langle \Psi, f_n \rangle_{L^2} \langle f_n, (\mathcal{T}_m - \lambda)f_k \rangle_{L^2} = \langle \Psi, f_k \rangle_{L^2} (2k + |m| - m + 1 - \lambda) = 0.$$

If Ψ is orthogonal to all members of family $(f_k)_{k \in \mathbb{N}}$ then $\Psi = 0$. Therefore, there exists $k \in \mathbb{N}$ such that

$$\lambda = 2k + 1 + |m| - m.$$

In conclusion,

$$\text{Sp}(\mathcal{T}_m) = \{2k + 1 + |m| - m : k \in \mathbb{N}\}.$$

□

Proof of Theorem 5.6. According to the unitary equivalence arguments, the spectrum of $\mathfrak{L}_m^{[0]}$ is easily obtained as stated in the theorem. □

Example 5.1. When $m \geq 0$, the first and the second eigenpairs of $\mathfrak{L}_m^{[0]}$ are respectively

$$\left(\beta(0), t^{\frac{m}{2}} e^{-\frac{\beta(0)t}{2}}\right) \quad \text{and} \quad \left(3\beta(0), (\beta(0)t - m - 1)t^{\frac{m}{2}} e^{-\frac{\beta(0)t}{2}}\right).$$

5.2.2.3 A quasimode for the rescaled operator

Back to the rescaled operator $\mathcal{M}_{h,m}$ and we restrict ourselves to the case $m \geq 0$. In this part, we aim to show that the first eigenvalue of $\mathcal{M}_{h,m}$ has the expansion

$$\lambda_0(\mathcal{M}_{h,m}) = \beta(0)h + \frac{(m+1)\beta'(0)}{\beta(0)}h^2 + o(h^2).$$

To do that, we prove the accurate upper bound for the eigenvalue in Theorem 5.10. Then, the lower bound of the eigenvalue will be showed in Theorem 5.13 with the help of Agmon estimate.

Theorem 5.10. For all $m \in \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,

$$\text{dist}(\mu_{m,0}h + \mu_{m,1}h^2, \text{Sp}(\mathcal{M}_{h,m})) \leq Ch^3,$$

where $\mu_{m,0} = \beta(0)$ and $\mu_{m,1} = \frac{(m+1)\beta'(0)}{\beta(0)}$.

Proof. For each $m \in \mathbb{N}$, we need to find a non-trivial function $\Psi_h \in \text{Dom}(\mathcal{M}_{h,m})$ such that

$$\|(\mathcal{M}_{h,m} - \lambda_h)\Psi_h\|_{L^2(\mathbb{R}^+, dt)} \leq Ch^3.$$

where $\lambda_h = \mu_{m,0}h + \mu_{m,1}h^2$. Then the conclusion of this theorem follows from the spectral theorem. Although Ψ_h depends on m , whenever there is no confusion, we write Ψ_h instead of $\Psi_{h,m}$. Recall that

$$\mathcal{M}_{h,m} = h\mathfrak{L}_m^{[0]} + h^2\mathfrak{L}_m^{[1]} + \frac{\mathcal{O}(h^3t^3)}{t}.$$

This suggests an idea that we need to look for a quasimode Ψ_h in form of

$$\Psi_0 + h\Psi_1.$$

In details,

$$\begin{aligned}
(\mathcal{M}_{h,m} - \lambda)(\Psi_0 + h\Psi_1) &\sim \left[h \left(\mathfrak{L}_m^{[0]} - \mu_{m,0} \right) + h^2 \left(\mathfrak{L}_m^{[1]} - \mu_{m,1} \right) \right] (\Psi_0 + h\Psi_1) \\
&= h \left(\mathfrak{L}_m^{[0]} - \mu_{m,0} \right) \Psi_0 + h^2 \left[\left(\mathfrak{L}_m^{[0]} - \mu_{m,0} \right) \Psi_1 + \left(\mathfrak{L}_m^{[1]} - \mu_{m,1} \right) \Psi_0 \right] \\
&\quad + h^3 \left(\mathfrak{L}_m^{[1]} - \mu_{m,1} \right) \Psi_1.
\end{aligned}$$

Obviously, we need to solve the following equations to find λ and Ψ :

Equation according to order h :

$$\left(\mathfrak{L}_m^{[0]} - \mu_{m,0} \right) \Psi_0 = 0.$$

We choose $\mu_{m,0} = \beta(0)$ and $\Psi_0 = t^{\frac{m}{2}} e^{-\frac{\beta(0)t}{2}}$ the first eigenpair of $\mathfrak{L}_m^{[0]}$.

Equation according to order h^2 :

$$\left(\mathfrak{L}_m^{[0]} - \mu_{m,0} \right) \Psi_1 = \left(\mu_{m,1} - \mathfrak{L}_m^{[1]} \right) \Psi_0. \quad (5.23)$$

Since $\mathfrak{L}_m^{[0]}$ is a self-adjoint operator and μ_0 is a discrete eigenvalue then $\mathfrak{L}_m^{[0]} - \mu_0$ is a Fredholm operator. Therefore, the equation (5.23) has solution Ψ_1 if and only if

$$\left(\mathfrak{L}_m^{[1]} - \mu_{m,1} \right) \Psi_0 \in \text{span}(\Psi_0)^\perp.$$

This is equivalent to

$$\mu_{m,1} \|\Psi_0\|^2 = \langle \mathfrak{L}_m^{[1]} \Psi_0, \Psi_0 \rangle_{L^2},$$

or

$$\begin{aligned}
\mu_1 \int_0^{+\infty} t^m e^{-\beta(0)t} dt &= \beta'(0) \int_0^{+\infty} \left(-\frac{m}{2}t + \frac{\beta(0)}{2}t^2 \right) t^m e^{-\beta(0)t} dt \\
&= \frac{(m+1)\beta'(0)}{\beta(0)} \int_0^{+\infty} t^m e^{-\beta(0)t} dt \quad (\text{since } \beta(0) > 0).
\end{aligned}$$

Thus,

$$\mu_1 = \frac{(m+1)\beta'(0)}{\beta(0)}.$$

Inserting this value into equation (5.23), we have a differential equation

$$\begin{aligned}
&\left(-2\partial_t t \partial_t + \frac{\beta(0)^2 t}{2} + \frac{m^2}{2t} - (m+1)\beta(0) \right) \Psi_1 \\
&= \left(\frac{m+1}{\beta(0)} + \frac{m}{2}t - \frac{\beta(0)}{2}t^2 \right) \beta'(0) t^{\frac{m}{2}} e^{-\frac{\beta(0)t}{2}}.
\end{aligned}$$

We look for Ψ_1 in the form $\Psi_1 = \beta'(0)y(t)t^{\frac{m}{2}}e^{-\frac{\beta(0)t}{2}}$, and we only need to find a function $y(t)$ satisfies

$$-2ty'' - 2(1 + m - \beta(0)t)y' = \frac{m+1}{\beta(0)} + \frac{m}{2}t - \frac{\beta(0)}{2}t^2.$$

By finding $y(t)$ in the polynomial form $a_2t^2 + a_1t + a_0$, we obtain

$$y(t) = -\frac{1}{8}t^2 - \frac{1}{2\beta(0)}t.$$

Therefore

$$\Psi_1(t) = \left(-\frac{1}{8}t^2 - \frac{1}{2\beta(0)}t\right) \beta'(0)t^{\frac{m}{2}}e^{-\frac{\beta(0)t}{2}},$$

satisfies (5.23). To make sure that Ψ belongs to the domain of $\mathcal{M}_{h,m}$, we multiply $\Psi_0 + h\Psi_1$ with a smooth cut-off function, that is

$$\Psi_h(t) = \chi(ht)(\Psi_0 + h\Psi_1),$$

where

$$\chi(t) := \begin{cases} 1 & \text{for } t \in [0, \frac{\delta}{2}] \\ 0 & \text{for } t \in [\delta, \infty) \end{cases}, \quad (5.24)$$

where δ is given by the estimate of Taylor's remainder of the function a , namely in (5.14).

Before estimating the L^2 -norm, we want to analyse the quasimode near $t = 0$. Remind that the operator $\mathcal{M}_{h,m}$ has the term

$$\frac{(hm - a(ht))^2}{2ht},$$

which goes to infinity as $t \rightarrow 0$ and $m \neq 0$. When we apply $\mathcal{M}_{h,m}$ to Ψ_0 in the neighborhood of 0,

$$\begin{aligned} \mathcal{M}_{h,m}\Psi_0 &= h \left(\frac{-m^2}{2t} - \frac{\beta(0)^2t}{2} + m\beta(0) \right) \Psi_0 + \frac{(hm - a(ht))^2}{2ht} \Psi_0 \\ &= \left(-h\frac{\beta(0)^2t}{2} + mh\beta(0) + \frac{a(ht)^2}{2ht} - \frac{ma(ht)}{t} \right) \Psi_0, \end{aligned}$$

Since $a(0) = 0$, $\frac{a(ht)^2}{2ht}$ and $\frac{ma(ht)}{t}$ are smooth at 0. Thus, $\mathcal{M}_{h,m}\Psi_0$ does not blow up at 0.

For simplicity of notation, we denote $S(t) = \Psi_0 + h\Psi_1$. By using Taylor formula and

the definitions of μ_i and Ψ_i , we have

$$\begin{aligned}
(\mathcal{M}_{h,m} - \lambda_h)\Psi_h &= (\mathcal{M}_{h,m} - \lambda_h)\chi(ht)S \\
&= -2h [\partial_t t \partial_t, \chi(ht)] S + \chi(ht)(\mathcal{M}_{h,m} - \lambda_h)S \\
&= -2h^3 t \chi''(ht)S - 2h^2 \chi'(ht)(S + 2tS') \\
&\quad + \chi(ht) \left[h^3 (\mathfrak{L}_m^{[1]} - \mu_{m,1})\Psi_1 + \frac{R(ht)}{t} S \right].
\end{aligned}$$

With support consideration and changing variable $s = th$ in integral with a notice that S contains the exponential term $e^{-\frac{\beta(0)}{t}}$, the L^2 -norm of the term $[\partial_t t \partial_t, \chi(ht)] S$ is $\mathcal{O}(h^\infty)$. We only need to care about estimating the term $\chi(ht) \frac{R(ht)}{t} S$. We have

$$\begin{aligned}
\int_0^{+\infty} \chi^2(ht) \frac{|R(ht)|^2}{t^2} S^2 dt &= \int_0^{\delta/h} \chi^2(ht) \frac{|R(ht)|^2}{t^2} S^2 dt \\
&\leq Ch^6 \int_0^{+\infty} t^4 S^2 dt \\
&\leq \tilde{C}h^6.
\end{aligned}$$

□

The above theorem states that, for each $m \in \mathbb{N}$, one can find an eigenvalue of $\mathcal{M}_{h,m}$ near $\beta(0)h + \frac{(m+1)\beta'(0)}{\beta(0)}h^2$. We will use the Agmon estimate to assert that this eigenvalue is in fact the first one $\lambda_0(\mathcal{M}_{h,m})$. In this section, let us denote

$$\mathcal{V}_{h,m}(t) = \frac{(hm - a(ht))^2}{2ht},$$

and the quadratic form associated with $\mathcal{M}_{h,m}$ as

$$Q_{h,m}(u) = 2h \int_0^\infty t |\partial_t u|^2 dt + \int_0^\infty \mathcal{V}_{h,m}(t) |u(t)|^2 dt.$$

Moreover, since we only deal with the Hilbert space $L^2(\mathbb{R}^+, dt)$ in this section, we denote $\|\cdot\|_{L^2}$ instead of $\|\cdot\|_{L^2(\mathbb{R}^+, dt)}$ and so is the inner product.

Let us recall that $a(\rho) = \int_0^\rho \beta(s) ds$. Under the condition (1.46), we imply

$$a(ht) \geq \beta(0)ht.$$

For $t > 0$ large enough, we have the inequality

$$\mathcal{V}_{h,m}(t) = \frac{(hm - a(ht))^2}{2ht} \geq h \frac{(\beta(0)t - m)^2}{2t}. \quad (5.25)$$

Theorem 5.11. *Let $m \in \mathbb{N}$ and let $\Phi \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$, then for all $u \in \text{Dom}(\mathcal{M}_{h,m})$, we have*

$$Q_{h,m}(e^\Phi u) = \text{Re}\langle \mathcal{M}_{h,m}u, e^{2\Phi}u \rangle_{L^2} + 2h \int_0^{+\infty} t(\Phi'(t))^2 e^{2\Phi(t)} |u(t)|^2 dt. \quad (5.26)$$

Proof. By Lax-Milgram theorem, we have

$$2h\langle \sqrt{t}\partial_t u, \sqrt{t}\partial_t(e^{2\Phi}u) \rangle_{L^2} + \int_0^{+\infty} \mathcal{V}_{h,m}(t)e^{2\Phi(t)}|u(t)|^2 dt = \langle \mathcal{M}_{h,m}u, e^{2\Phi}u \rangle_{L^2}. \quad (5.27)$$

To simplify the notation, we set up $P := \sqrt{t}\partial_t$. Notice that the commutator

$$[P, e^\Phi] = \sqrt{t}\Phi'(t)e^\Phi$$

is a multiplication operator. We perform the following computation

$$\begin{aligned} \langle Pu, Pe^{2\Phi}u \rangle_{L^2} &= \langle Pu, [P, e^\Phi]e^\Phi u \rangle_{L^2} + \langle Pu, e^\Phi Pe^\Phi u \rangle_{L^2} \\ &= \langle e^\Phi Pu, [P, e^\Phi]u \rangle_{L^2} + \langle e^\Phi Pu, Pe^\Phi u \rangle_{L^2} \\ &= \langle e^\Phi Pu, [P, e^\Phi]u \rangle_{L^2} + \langle Pe^\Phi u, Pe^\Phi u \rangle_{L^2} + \langle [e^\Phi, P]u, Pe^\Phi u \rangle_{L^2} \\ &= \|Pe^\Phi u\|_{L^2}^2 - \|[P, e^\Phi]u\|_{L^2}^2 + \langle e^\Phi Pu, [P, e^\Phi]u \rangle_{L^2} - \langle [P, e^\Phi]u, e^\Phi Pu \rangle_{L^2}. \end{aligned}$$

Take the real part of (5.27), we get

$$2h\|Pe^\Phi u\|_{L^2}^2 - 2h\|[P, e^\Phi]u\|_{L^2}^2 + \int_0^{+\infty} \mathcal{V}_{h,m}(t)e^{2\Phi}|u|^2 dt = \text{Re}\langle \mathcal{M}_{h,m}u, e^{2\Phi}u \rangle_{L^2}.$$

This implies the statement of the theorem. \square

Theorem 5.12. *Under the assumption (1.46), for all $m \in \mathbb{N}$, and for all $\varepsilon \in (0, \frac{\beta(0)}{2})$, there exists $M > 0$ such that*

$$\|e^{\varepsilon t}\Psi\|_{L^2}^2 \leq M\|\Psi\|_{L^2}^2 \quad \text{and} \quad Q_{h,m}(e^{\varepsilon t}\Psi) \leq Mh\|\Psi\|_{L^2}^2,$$

for all eigenfunctions Ψ with eigenvalue of order h of the operator $\mathcal{M}_{h,m}$.

Proof. Let us consider a sequence of functions $(\chi_k)_{k \geq 1}$ defined as follows

$$\chi_k(t) = \begin{cases} t & \text{for } 0 \leq t \leq k, \\ 2k - t & \text{for } k \leq t \leq 2k, \\ 0 & \text{for } t \geq 2k. \end{cases} \quad (5.28)$$

Notice that $\chi_k \in W^{1,\infty}(\mathbb{R}^+)$ and $|\chi_k'(t)| \leq 1$ for all $t \in \mathbb{R}^+$.

Let us consider an eigenvalue $\lambda (= \mathcal{O}(h))$ of the operator $\mathcal{M}_{h,m}$ associated with eigenfunction Ψ . This means that there exists $C > 0$ such that

$$\mathcal{M}_{h,m}\Psi = \lambda\Psi \quad \text{and} \quad \lambda < Ch. \quad (5.29)$$

Take $\Phi = \varepsilon\chi_k(t)$ in equation (5.26), we have

$$\begin{aligned} Q_{h,m}(e^{\varepsilon\chi_k(t)}\Psi) &\leq \lambda \|e^{\varepsilon\chi_k(t)}\Psi\|_{L^2}^2 + 2h\varepsilon^2 \int_0^{+\infty} t|\chi_k'(t)|^2 e^{2\varepsilon\chi_k(t)} |\Psi(t)|^2 dt \\ &\leq Ch \|e^{\varepsilon\chi_k(t)}\Psi\|_{L^2}^2 + 2h\varepsilon^2 \int_0^{+\infty} t e^{2\varepsilon\chi_k(t)} |\Psi(t)|^2 dt. \end{aligned}$$

It implies that

$$\int_0^{+\infty} \mathcal{V}_{h,m}(t) e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt \leq Ch \|e^{\varepsilon\chi_k(t)}\Psi\|_{L^2}^2 + 2h\varepsilon^2 \int_0^{+\infty} t e^{2\varepsilon\chi_k(t)} |\Psi(t)|^2 dt.$$

Bring every terms from the right to the left, we have

$$\int_0^{+\infty} (\mathcal{V}_{h,m}(t) - Ch - 2h\varepsilon^2 t) e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt \leq 0.$$

From the estimate (5.25), there exist a number $R > 0$ and a constant $C_1(R, \varepsilon) > 0$ (depending on R and ε) such that for all $t \geq R$, we have

$$\begin{aligned} \mathcal{V}_{h,m}(t) - Ch - 2\varepsilon^2 th &\geq h \left[\frac{(b_0 t - m)^2}{2t} - 2\varepsilon^2 t - C \right] \\ &= h \left[\left(\frac{b_0^2}{2} - 2\varepsilon^2 \right) t + \frac{m^2}{2t} - b_0 m - C \right] \\ &\geq C_1(R, \varepsilon) h. \end{aligned} \quad (5.30)$$

We deduce the existence of $C_2(R, \varepsilon) > 0$ such that, for all $k \geq 1$,

$$\begin{aligned} C_1(R, \varepsilon) h \int_R^{+\infty} e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt &\leq \int_R^{+\infty} (\mathcal{V}_{h,m}(t) - Ch - 2h\varepsilon^2 t) e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt \\ &\leq \int_0^R (2h\varepsilon^2 t + Ch - \mathcal{V}_{h,m}(t)) e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt \\ &\leq C_2(R, \varepsilon) h \|\Psi\|_{L^2}^2. \end{aligned}$$

Then, there exists a constant $C(R, \varepsilon)$ such that, for all $k \geq 1$,

$$\begin{aligned} \int_0^\infty e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt &= \int_0^R e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt + \int_R^\infty e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt \\ &\leq \int_0^R e^{2\varepsilon R} |\Psi|^2 dt + \int_R^\infty e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt \\ &\leq C(R, \varepsilon) \|\Psi\|_{L^2}^2. \end{aligned}$$

Let $k \rightarrow +\infty$ and use Fatou's lemma, we obtain

$$\int_0^\infty e^{2\varepsilon t} |\Psi|^2 dt \leq C(R, \varepsilon) \|\Psi\|_{L^2}^2.$$

To estimate the quadratic form $Q_{h,m}$, we evaluate more strictly the estimate (5.30)

$$\mathcal{V}_{h,m}(t) - Ch - 2\varepsilon^2 th \geq C_1(R, \varepsilon) ht.$$

Follow the same steps as above, we get the control

$$\int_0^{+\infty} t e^{2\varepsilon\chi_k(t)} |\Psi(t)|^2 dt \leq C \|\Psi\|_{L^2}^2.$$

Then, it leads to

$$Q_{h,m}(e^{\varepsilon\chi_k(t)} \Psi) \leq Ch \|\Psi\|_{L^2}^2,$$

and we get the result. □

Theorem 5.13. *For all $m \in \mathbb{N}$,*

$$\lambda_0(\mathcal{M}_{h,m}) = \beta(0)h + \frac{(m+1)\beta'(0)}{\beta(0)} h^2 + o(h^2). \quad (5.31)$$

Proof. Let us fix $m \in \mathbb{N}$. We can choose the first eigenpairs $(\lambda_i(\mathcal{M}_{h,m}), \Psi_{i,h})_{i=1,2}$ such that $\Psi_{0,h}$ and $\Psi_{1,h}$ are orthogonal. We let

$$E(h) = \text{span}(\Psi_{0,h}, \Psi_{1,h}). \quad (5.32)$$

Thus, $E(h)$ has dimension 2. From Theorem 5.10, we can deduce that $(\Psi_{i,h})_{i=1,2}$ are eigenfunctions having eigenvalues of order h and so are the elements of $E(h)$. Let

$\Psi \in E(h)$, we have

$$\begin{aligned}
Q_{h,m}(\Psi) &= 2h \int_0^{+\infty} t |\partial_t(\Psi)|^2 dt + \int_0^{+\infty} \mathcal{V}_{h,m}(t) |\Psi|^2 dt \\
&= h \int_0^{+\infty} 2t |\partial_t(\Psi)|^2 + \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) |\Psi|^2 dt \\
&\quad + \int_0^{+\infty} \left[\mathcal{V}_{h,m}(t) - h \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) \right] |\Psi|^2 dt \\
&\geq h \mathcal{Q}_m^{[0]}(\Psi) - \int_0^{+\infty} \left| \mathcal{V}_{h,m}(t) - h \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) \right| |\Psi|^2 dt,
\end{aligned}$$

where $\mathcal{Q}_m^{[0]}$ is the quadratic form associated with the operator $\mathfrak{L}_m^{[0]}$. By the Taylor's theorem, we have

$$\mathcal{V}_{h,m}(t) - h \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) = \frac{\mathcal{O}((ht)^2)}{t}.$$

There exists $C_1 > 0$ and $\delta > 0$ such that, for all (h, t) which satisfies $ht \leq \delta$,

$$\left| \mathcal{V}_{h,m}(t) - h \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) \right| \leq C_1 t h^2.$$

Using the Agmon estimate, we have

$$\begin{aligned}
\int_0^{\delta/h} \left| \mathcal{V}_{h,m}(t) - h \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) \right| |\Psi|^2 dt &\leq C_1 h^2 \int_0^{\delta/h} t |\Psi|^2 dt \\
&\leq C_2 h^2 \int_0^{+\infty} e^{2\varepsilon t} |\Psi|^2 dt \\
&\leq C h^2 \|\Psi\|_{L^2}^2.
\end{aligned}$$

Consider on the interval $[\frac{\delta}{h}, +\infty)$,

$$\begin{aligned}
\int_{\delta/h}^{+\infty} |\mathcal{V}_{h,m}(t)| |\Psi|^2 dt &= \int_{\delta/h}^{+\infty} e^{-2\varepsilon t} |\mathcal{V}_{h,m}(t)| e^{2\varepsilon t} |\Psi|^2 dt \\
&\leq \max_{t \geq \delta/h} e^{-2\varepsilon t} \int_0^{+\infty} |\mathcal{V}_{h,m}(t)| e^{2\varepsilon t} |\Psi|^2 dt \\
&\leq C h^2 \|\Psi\|_{L^2}^2.
\end{aligned}$$

Similarly, we also have

$$\int_{\delta/h}^{+\infty} h \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) dt \leq C h^2 \|\Psi\|_{L^2}^2.$$

Thus, we have the estimate

$$Q_{h,m}(\Psi) \geq h\mathcal{Q}_m^{[0]}(\Psi) - Ch^2\|\Psi\|_{L^2}^2.$$

Since $\Psi \in E(h)$, there exists $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ such that $\Psi = \alpha_1\Psi_{0,h} + \alpha_2\Psi_{1,h}$. Since $\Psi_{0,h}$ and $\Psi_{1,h}$ are orthogonal, it leads to $\|\Psi\|_{L^2}^2 = \alpha_1^2\|\Psi_{0,h}\|_{L^2}^2 + \alpha_2^2\|\Psi_{1,h}\|_{L^2}^2$. Therefore, we obtain

$$Q_{h,m}(\Psi) = \lambda_0(\mathcal{M}_{h,m})|\alpha_1|^2\|\Psi_{0,h}\|_{L^2}^2 + \lambda_1(\mathcal{M}_{h,m})|\alpha_2|^2\|\Psi_{1,h}\|_{L^2}^2 \leq \lambda_1(\mathcal{M}_{h,m})\|\Psi\|_{L^2}^2,$$

By the min-max principle for the quadratic form $\mathcal{Q}_m^{[0]}$, we have

$$\lambda_1(\mathcal{M}_{h,m}) \geq 3\beta(0)h - Ch^2. \quad (5.33)$$

From Theorem 5.10, we see that there exists an eigenvalue of $\mathcal{M}_{h,m}$ which is near the quantity $\beta(0)h + \frac{(m+1)\beta'(0)}{\beta(0)}h^2$ in order $o(h^2)$. This eigenvalue can not be $\lambda_k(\mathcal{M}_{h,m})$ with $k \geq 1$, because if it were, from the estimate

$$3\beta(0)h - Ch^2 \leq \lambda_k(\mathcal{M}_{h,m}) = \beta(0)h + \frac{(m+1)\beta'(0)}{\beta(0)}h^2 + o(h^2),$$

we divide both sides by h and let h come to zero, this gives us $\beta(0) = 0$. This can not happen. The conclusion follows. \square

5.3 The relation between eigenfunctions of the magnetic Laplacian and eigenfunctions of the fibered operator

We recall the expression of the magnetic Laplacian operator \mathcal{K}_h

$$\mathcal{K}_h = -h^2r^{-2}(r\partial_r)^2 + r^{-2}(-ih\partial_\theta - G(r))^2. \quad (5.34)$$

For each $m \in \mathbb{N}$, let $\lambda_0(\mathcal{N}_{h,m})$ be the first eigenvalue of the operator $\mathcal{N}_{h,m}$ and $\Psi_{h,m}$ be the associated eigenfunction. Since $\mathcal{M}_{h,m}$ and $\mathcal{N}_{h,m}$ are unitary equivalent, from Theorem 5.13, we obtain

$$\lambda_0(\mathcal{N}_{h,m}) = \lambda_0(\mathcal{M}_{h,m}) = \beta(0)h + \frac{(m+1)\beta'(0)}{\beta(0)}h^2 + o(h^2). \quad (5.35)$$

On the other hand, $\mathfrak{L}_{h,m}$ and $\mathcal{N}_{h,m}$ also are equivalent by

$$\mathfrak{L}_{h,m} = T_1\mathcal{N}_{h,m}T_1^{-1},$$

where T_1 are unitary operators defined at (5.11). Therefore, $\lambda_0(\mathcal{N}_{h,m})$ is also the first eigenvalue of $\mathfrak{L}_{h,m}$ and $T_1(\Psi_{h,m})$ is the associated eigenfunction. It results that

$$[\mathcal{K}_h - \lambda_0(\mathcal{N}_{h,m})] e^{im\theta}(T_1\Psi_{h,m}) = [\mathfrak{L}_{h,m} - \lambda_0(\mathcal{M}_{h,m})] T_1(\Psi_{h,m}) e^{im\theta} = 0.$$

Thus $\lambda_0(\mathcal{N}_{h,m})$ belongs to the spectrum of \mathcal{K}_h . Now, the result by Helffer and Kordyukov (see (1.13)) tells us that, for all $k \in \mathbb{N}$, the k -th eigenvalue of \mathcal{K}_h , denoted by $\lambda_k(\mathcal{K}_h)$ satisfies

$$\lambda_k(\mathcal{K}_h) = B(0)h + \left(2k \frac{\sqrt{\det H}}{B(0)} + \frac{(\text{Tr} H^{1/2})^2}{2B(0)} \right) h^2 + o(h^2), \quad (5.36)$$

where $H = \frac{1}{2} \text{Hess} B(0)$.

We recall the definition of the magnetic field $B(q_1, q_2) = \beta \left(\frac{q_1^2 + q_2^2}{2} \right)$, we have

$$H = \begin{pmatrix} \frac{\beta'(0)}{2} & 0 \\ 0 & \frac{\beta'(0)}{2} \end{pmatrix}.$$

By computation and recall that $\beta'(0) > 0$, we have

$$\det H = \frac{\beta'(0)^2}{4} \quad \text{and} \quad \text{Tr}(H^{1/2}) = \sqrt{2} \sqrt{\beta'(0)}.$$

Thus, $\lambda_k(h)$ has the form

$$\lambda_k(\mathcal{K}_h) = \beta(0)h + \frac{(k+1)\beta'(0)}{\beta(0)} h^2 + o(h^2). \quad (5.37)$$

Since $\lambda_0(\mathcal{N}_{h,m})$ is the eigenvalue of \mathcal{K}_h , thus there exists $k \in \mathbb{N}$ such that

$$\lambda_0(\mathcal{M}_{h,m}) = \lambda_k(\mathcal{K}_h).$$

Comparing with (5.35) and taking h small enough, we immediately obtain $k = m$ and

$$\lambda_m(\mathcal{K}_h) = \lambda_0(\mathcal{M}_{h,m}).$$

Since $\lambda_m(\mathcal{K}_h)$ is a simple eigenvalue, we have the following statement

Theorem 5.14. *The m -th eigenvalue of the magnetic Laplacian \mathcal{K}_h is exactly the first eigenvalue of the operator $\mathcal{N}_{h,m}$:*

$$\lambda_m(\mathcal{K}_h) = \lambda_0(\mathcal{N}_{h,m}).$$

The m -th eigenfunction of \mathcal{K}_h is in the form

$$ce^{im\theta}\Psi_{h,m}\left(\frac{r^2}{2}\right), \quad (5.38)$$

where $\Psi_{h,m}$ is a ground-state eigenfunction of the operator $\mathcal{N}_{h,m}$ and $c \in \mathbb{C} \setminus \{0\}$.

5.4 Magnetic WKB construction in the radial coordinates

In this section, we focus on constructing the WKB Ansatz for the eigenfunction of the magnetic Laplacian. As Theorem 5.14 indicates the relation between this eigenfunction with the eigenfunction of the operator $\mathcal{N}_{h,m}$, we just need to do the WKB analysis for the operator $\mathcal{N}_{h,m}$. Since $\mathcal{N}_{h,m}$ is a real electric Schrödinger operator in dimension 1, one can easily find a WKB approximation for $\Psi_{h,m}$. We recall the unscaled operator

$$\mathcal{N}_{h,m} = -2h^2\partial_\rho(\rho\partial_\rho) + \frac{(hm - a(\rho))^2}{2\rho}, \quad (5.39)$$

where

$$a(\rho) = \int_0^\rho \beta(s) ds.$$

To start the WKB method, we consider its conjugated operator with real-valued smooth function φ :

$$\widehat{\mathcal{N}}_{h,m} = e^{\frac{\varphi(\rho)}{h}} \rho^{-\frac{m}{2}} \mathcal{N}_{h,m} \rho^{\frac{m}{2}} e^{-\frac{\varphi(\rho)}{h}}.$$

We have

$$\begin{aligned} \widehat{\mathcal{N}}_{h,m} &= -2h^2 e^{\frac{\varphi(\rho)}{h}} \rho^{-\frac{m}{2}} [\partial_\rho(\rho\partial_\rho)] \rho^{\frac{m}{2}} e^{-\frac{\varphi(\rho)}{h}} + \frac{(hm - a(\rho))^2}{2\rho} \\ &= -2h^2 \left(\partial_\rho + \frac{m}{2\rho} - \frac{\varphi'(\rho)}{h} \right) \rho \left(\partial_\rho + \frac{m}{2\rho} - \frac{\varphi'(\rho)}{h} \right) + \frac{(hm - a(\rho))^2}{2\rho} \\ &= -2h^2 \left(1 + \rho \left(\partial_\rho + \frac{m}{2\rho} - \frac{\varphi'(\rho)}{h} \right) \right) \left(\partial_\rho + \frac{m}{2\rho} - \frac{\varphi'(\rho)}{h} \right) + \frac{(hm - a(\rho))^2}{2\rho} \\ &= -2h^2 \left(\partial_\rho + \frac{m}{2\rho} - \frac{\varphi'(\rho)}{h} \right) - 2h^2 \rho \left(\partial_\rho + \frac{m}{2\rho} - \frac{\varphi'(\rho)}{h} \right)^2 + \frac{(hm - a(\rho))^2}{2\rho} \\ &= -2h^2 \left(\partial_\rho + \frac{m}{2\rho} - \frac{\varphi'(\rho)}{h} \right) - 2h^2 \rho \left(\partial_\rho^2 - \frac{m}{2\rho^2} + \frac{m}{\rho} \partial_\rho - \frac{\varphi''(\rho)}{h} - \frac{2\varphi'(\rho)}{h} \partial_\rho \right) \\ &\quad - 2h^2 \rho \left(\frac{m}{2\rho} - \frac{\varphi'(\rho)}{h} \right)^2 + \frac{(hm - a(\rho))^2}{2\rho}. \end{aligned}$$

So, the expression of the conjugated operator $\widehat{\mathcal{N}}_{h,m}$ can be written in order of h as follows

$$\begin{aligned}\widehat{\mathcal{N}}_{h,m} &= \left(\frac{(a(\rho))^2}{2\rho} - 2\rho(\varphi'(\rho))^2 \right) \\ &\quad + h \left(4\varphi'(\rho)\rho\partial_\rho + 2\varphi'(\rho) + 2\rho\varphi''(\rho) + 2m\varphi'(\rho) - m\frac{a(\rho)}{\rho} \right) \\ &\quad + h^2 \left(-2\rho\partial_\rho^2 - (2m+2)\partial_\rho \right).\end{aligned}$$

5.4.1 The eikonal equation

We find function φ on $[0, +\infty)$ such that the term of order h^0 is zero, that is

$$\begin{aligned}\frac{(a(\rho))^2}{2\rho} - 2\rho(\varphi'(\rho))^2 &= 0, \\ \Leftrightarrow (\varphi'(\rho))^2 &= \frac{(a(\rho))^2}{4\rho^2}.\end{aligned}\tag{5.40}$$

We choose a positive solution

$$\varphi(\rho) = \int_0^\rho \frac{a(\tau)}{2\tau} d\tau,$$

which is a smooth function on $[0, +\infty)$ because it can be rewritten in the form

$$\varphi(\rho) = \int_0^\rho \frac{a(\tau)}{2\tau} d\tau = \int_0^\rho \frac{1}{2\tau} \int_0^\tau \beta(\xi) d\xi d\tau = \frac{1}{2} \int_0^\rho \int_0^1 \beta(\xi\tau) d\xi d\tau.$$

Then the operator $\mathcal{N}_{h,m}$ becomes

$$\widehat{\mathcal{N}}_{h,m} = h\mathcal{N}^1 + h^2\mathcal{N}_m^2,$$

where

$$\mathcal{N}^1 = 4\varphi'(\rho)\rho\partial_\rho + 2\varphi'(\rho) + 2\rho\varphi''(\rho) = 2a(\rho)\partial_\rho + \beta(\rho),$$

and

$$\mathcal{N}_m^2 = -2\rho\partial_\rho^2 - (2m+2)\partial_\rho.$$

We now look for a WKB Ansatz and a quasi-eigenvalue, respectively, in the form

$$\begin{aligned}a(\rho, h) &\sim a_0(\rho) + ha_1(\rho) + h^2a_2(\rho) + \dots, \\ \lambda(h) &\sim h(\mu_0 + h\mu_1 + h^2\mu_2 + \dots).\end{aligned}$$

We substitute these formal series into the equation

$$\left(\widehat{\mathcal{N}}_{h,m} - \lambda(h) \right) a(\rho, h) = 0.$$

Collect all terms according to h^j , we have an infinite system

$$\begin{aligned}
h : & & (\mathcal{N}^1 - \mu_0) a_0 &= 0, \\
h^2 : & & (\mathcal{N}^1 - \mu_0) a_1 &= (\mu_1 - \mathcal{N}_m^2) a_0, \\
h^3 : & & (\mathcal{N}^1 - \mu_0) a_2 &= (\mu_1 - \mathcal{N}_m^2) b_1 + \mu_2 b_0, \\
h^4 : & & (\mathcal{N}^1 - \mu_0) a_3 &= (\mu_1 - \mathcal{N}_m^2) b_2 + \mu_2 a_1 + \mu_3 a_0, \\
& \dots & &
\end{aligned}$$

We will solve these transport equations by elementary ODEs arguments and by induction.

5.4.2 The first transport equation

Collecting all terms of order h^1 , we have the first transport equation

$$(2a(\rho)\partial_\rho + \beta(\rho) - \mu_0)a_0 = 0. \quad (5.41)$$

The equation (5.41) has smooth solutions which do not vanish at 0 if and only if

$$\mu_0 = \beta(0).$$

Indeed, since $a(\rho) = \int_0^\rho \beta(s) ds = \rho \int_0^1 \beta(\rho s) ds$ and $\beta(0) > 0$, we can extend the quotient

$$F(\rho) := \frac{\beta(0) - \beta(\rho)}{2a(\rho)},$$

into a smooth function on $[0, +\infty)$. We still denote this extension as F , and the differential equation

$$\partial_\rho a_0 = F a_0,$$

has smooth solutions

$$a_0(\rho) = a_0(0) \exp\left(\int_0^\rho F(s) ds\right),$$

with $a_0(0) \neq 0$. We make a choice $a(0) = 1$ and we obtain a solution

$$a_0(\rho) = \exp\left(\int_0^\rho F(s) ds\right).$$

5.4.3 The second transport equation

Let us gather all terms of order h^2 to get the second transport equation

$$(2a(\rho)\partial_\rho + \beta(\rho) - \mu_0)a_1 = (\mu_1 + (2m + 2)\partial_\rho + 2\rho\partial_\rho^2)a_0, \quad (5.42)$$

we deduce that the necessary condition such that the equation (5.42) has solution which is smooth at 0 is

$$(\mu_1 + (2m + 2)\partial_\rho)a_0(0) = 0,$$

or

$$\mu_1 = -\frac{(2m + 2)\partial_\rho a_0(0)}{a_0(0)}.$$

From the equation (5.41), let ρ go to 0, we get

$$\frac{\partial_\rho a_0(0)}{a_0(0)} = \lim_{\rho \rightarrow 0} \frac{\beta(0) - \beta(\rho)}{2a(\rho)} = \frac{-\beta'(0)}{2\beta(0)}.$$

Thus, we find that

$$\mu_1 = (m + 1)\frac{\beta'(0)}{\beta(0)}. \quad (5.43)$$

we recover the coefficient found in the first part of the analysis. If we take this value of μ_1 , our second transport equation (5.42) can be rewritten as

$$\partial_\rho a_1 - F(\rho)a_1 = g_1(\rho) := \frac{(\mu_1 + (2m + 2)\partial_\rho + 2\rho\partial_\rho^2)a_0}{2a(\rho)}. \quad (5.44)$$

This equation has solutions in the form

$$\begin{aligned} a_1(\rho) &= \exp\left(\int_0^\rho F(s) ds\right) \int_0^\rho \left(\exp\left(-\int_0^\tau F(s) ds\right) g_1(\tau)\right) d\tau \\ &\quad + a_1(0) \exp\left(\int_0^\rho F(s) ds\right). \end{aligned}$$

We prescribe the restriction at $\rho = 0$ that $a_1(0) = 0$ so that the equation (5.42) has the unique solution

$$a_1(\rho) = \exp\left(\int_0^\rho F(s) ds\right) \int_0^\rho \left(\exp\left(-\int_0^\tau F(s) ds\right) g_1(\tau)\right) d\tau.$$

5.4.4 Induction

Let $n \in \mathbb{N}$ and $n \geq 2$. We assume that $(\mu_j)_{0 \leq j \leq n}$ and $(a_j)_{0 \leq j \leq n}$ are determined and $(a_j)_{1 \leq j \leq n}$ are smooth function on $[0, +\infty)$ and vanish at $\rho = 0$. Let us show that we

can determine μ_{n+1} and a_{n+1} by the $(n+1)$ -th transport equation

$$(2a(\rho)\partial_\rho + \beta(\rho) - \mu_0) a_{n+1} = ((2m+2)\partial_\rho + 2\rho\partial_\rho^2) a_n + \sum_{j=1}^{n+1} \mu_j a_{n+1-j}. \quad (5.45)$$

The equation has a smooth solution at 0 if and only if

$$(2m+2)\partial_\rho a_n(0) + \sum_{j=1}^n \mu_j a_{n+1-j}(0) + \mu_{n+1} a_0(0) = 0.$$

Since $a_0(0) = 1$, μ_{n+1} is completely determined by

$$\mu_{n+1} = -(2m+2)\partial_\rho a_n(0).$$

With this value of μ_{n+1} , we can rewrite the equation (5.45) as

$$\partial_\rho a_{n+1} - F(\rho)a_{n+1} = g_n(\rho), \quad (5.46)$$

where g_n is the smooth extension of the function

$$G_n(\rho) = \frac{((2m+2)\partial_\rho + 2\rho\partial_\rho^2) a_n + \sum_{j=1}^{n+1} \mu_j a_{n+1-j}}{2a(\rho)},$$

on $[0, +\infty)$.

There is only one solution a_{n+1} such that $a_{n+1}(0) = 0$, that is

$$a_{n+1}(\rho) = \exp\left(\int_0^\rho F(s) ds\right) \int_0^\rho \left(\exp\left(-\int_0^\tau F(s) ds\right) g_n(\tau)\right) d\tau.$$

Proof of Theorem 1.17. We fix $m \in \mathbb{N}$. The process of performing WKB for the spectral problem of the operator $\widehat{\mathcal{N}}_{h,m}$ provides us the required functions and sequences as follows:

- i) The function $\varphi(\rho)$ is given by the Eikonal equation (5.40):

$$\varphi(\rho) = \frac{1}{2} \int_0^\rho \int_0^1 \beta(\xi\tau) d\xi d\tau. \quad (5.47)$$

- ii) The transport equations give us the existence of a sequence of smooth functions $(a_{m,j})_{j \in \mathbb{N}}$ defined on $[0, +\infty)$ and the sequence $(\mu_{m,j})_{j \in \mathbb{N}}$ which depends on m . Notice that $a_{m,0}$ is positive because

$$a_{m,0}(\rho) = \exp\left(\int_0^\rho F(s) ds\right).$$

For each $J \in \mathbb{N}$, from the WKB construction, there exists a smooth function $f_{m,J}(\rho)$ defined on $[0, +\infty)$ such that

$$e^{\frac{\varphi(\rho)}{h}} \rho^{-\frac{m}{2}} \left(\mathcal{N}_{h,m} - h \sum_{j=0}^J \mu_{m,j} h^j \right) \left(\rho^{\frac{m}{2}} e^{-\frac{\varphi(\rho)}{h}} \sum_{j=0}^J a_{m,j} h^j \right) = f_{m,J}(\rho) h^{J+2}.$$

After changing of variable $\rho = \frac{r^2}{2}$, we obtain

$$\begin{aligned} e^{\frac{\varphi\left(\frac{r^2}{2}\right)}{h}} \left(\frac{r^2}{2}\right)^{-\frac{m}{2}} \left(\mathfrak{L}_{h,m} - h \sum_{j=0}^J \mu_{m,j} h^j \right) \left(\left(\frac{r^2}{2}\right)^{\frac{m}{2}} e^{-\frac{\varphi\left(\frac{r^2}{2}\right)}{h}} \sum_{j=0}^J a_{m,j} \left(\frac{r^2}{2}\right) h^j \right) \\ = f_{m,J} \left(\frac{r^2}{2}\right) h^{J+2}. \end{aligned}$$

By multiplying $\left(\frac{r^2}{2}\right)^{\frac{m}{2}} e^{-\frac{\varphi\left(\frac{r^2}{2}\right)}{h}} \sum_{j=0}^J a_{m,j} \left(\frac{r^2}{2}\right) h^j$ with $e^{im\theta}$ and using the fact that

$$\mathcal{K}_h(e^{im\theta} u) = \mathfrak{L}_{h,m}(e^{im\theta} u),$$

the result of Theorem 1.17 is deduced after changing variables from the radial coordinates to Cartesian coordinates. \square

5.5 The WKB approximation

Let us recall the definition of the cut-off function χ defined at (1.49) associated with the large number $K > 0$. From now, for shortness, we will denote $L^2(\mathbb{R}^+)$ instead $L^2(\mathbb{R}^+, d\rho)$ for the Hilbert space of the operator $\mathcal{N}_{h,m}$.

Theorem 5.15. *For all $(\varepsilon, m, J) \in (0, 1) \times \mathbb{N} \times \mathbb{N}$, there exist a constant $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$\left\| e^{\varepsilon\varphi(\rho)/h} (\mathcal{N}_{h,m} - \lambda_{h,m}^J) \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} \leq Ch^{J+2}, \quad (5.48)$$

where

$$\lambda_{h,m}^J := h \sum_{j=0}^J \mu_{m,j} h^j \quad \text{and} \quad \Psi_{h,m}^J(\rho) := \chi e^{-\varphi(\rho)/h} \rho^{\frac{m}{2}} \left(\sum_{j=0}^J a_{m,j} h^j \right). \quad (5.49)$$

In particular,

$$\left\| (\mathcal{N}_{h,m} - \lambda_{h,m}^J) \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} \leq Ch^{J+2}. \quad (5.50)$$

Proof. Using the definition of the operator $\widehat{\mathcal{N}}_{h,m}$, we check that

$$\begin{aligned} & e^{\varepsilon\varphi(\rho)/h} (\mathcal{N}_{h,m} - \lambda_{h,m}^J) \Psi_{h,m}^J \\ &= e^{\varepsilon\varphi(\rho)/h} [\mathcal{N}_{h,m}, \chi] e^{-\varphi(\rho)/h} \rho^{\frac{m}{2}} S_J + \chi e^{(\varepsilon-1)\varphi(\rho)/h} \rho^{\frac{m}{2}} \left(\widehat{\mathcal{N}}_{h,m} - \lambda_{h,m}^J \right) S_J. \end{aligned} \quad (5.51)$$

where $S_J := \left(\sum_{j=0}^J a_{m,j} h^j \right)$ is a smooth function on $[0, +\infty)$.

Notice that

$$\begin{aligned} [\mathcal{N}_{h,m}, \chi] &= -2h^2 [\partial_\rho \rho \partial_\rho, \chi] \\ &= -2h^2 (\chi' + \rho \chi'') - 4h^2 \chi' \rho \partial_\rho, \end{aligned}$$

and with the support considerations, the first term in the right hand side of (5.51) is $\mathcal{O}(h^\infty)$ with the norm $\|\cdot\|_{L^2(\mathbb{R}^+)}$.

From the WKB construction of λ_J and $\Psi_{h,\eta,J}(\rho)$, we get

$$\left(\widehat{\mathcal{N}}_{h,m} - \lambda_{h,m}^J \right) S_J(\rho) = h^{J+2} f_{m,J}(\rho),$$

where $f_{m,J}(\rho)$ is a smooth function on $[0, +\infty)$, then the second term in the right hand side of (5.51) is estimated by

$$\begin{aligned} \left\| \chi e^{(\varepsilon-1)\varphi(\rho)/h} \rho^{\frac{m}{2}} \left(\widehat{\mathcal{N}}_{h,m} - \lambda_{h,m}^J \right) S_J \right\|_{L^2(\mathbb{R}^+)} &= h^{J+2} \left\| e^{(\varepsilon-1)\varphi(\rho)/h} \rho^{\frac{m}{2}} f_{m,J}(\rho) \right\|_{L^2([0, K+1])} \\ &\leq C_1 h^{J+2}. \end{aligned}$$

This leads to

$$\left\| e^{\varepsilon\varphi(\rho)/h} (\mathcal{N}_{h,m} - \lambda_{h,m}^J) \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} \leq Ch^{J+2},$$

and get (5.48). \square

Proof of Theorem 1.18. Theorem 1.18 is a direct consequence of Theorem 5.15 by unitary equivalence.

$$\| e^{\varepsilon P/h} (\mathcal{L}_{h,\mathbf{A}} - \lambda_{h,m}^J) \Upsilon_{h,m}^J \|_{L^2(\mathbb{R}^2)} = \left\| e^{\varepsilon\varphi(\rho)/h} (\mathcal{N}_{h,m} - \lambda_{h,m}^J) \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} \leq Ch^{J+2}.$$

\square

We may now provide an approximation of the ground-state eigenfunction of the operator $\mathcal{N}_{h,m}$ by the WKB construction $\Psi_{h,m}^J$ defined in (5.49). Let $\Psi_{h,m}$ be an eigenfunction according to $\lambda_0(\mathcal{N}_{h,m})$, we introduce the orthogonal projection of $\Psi_{h,m}^J$ onto eigenspace of $\lambda_0(\mathcal{N}_{h,m})$

$$\Gamma_m \Psi_{h,m}^J = \langle \Psi_{h,m}^J, \Psi_{h,m} \rangle \Psi_{h,m}.$$

Theorem 5.16. For all $(m, J) \in \mathbb{N} \times \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,

$$\|\Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J\|_{L^2(\mathbb{R}^+)} \leq Ch^{J+1}. \quad (5.52)$$

Proof. From the estimate (5.50), we use the spectral theorem to get

$$\|\Psi_{h,m}^J\|_{L^2(\mathbb{R}^+)} \operatorname{dist}(\lambda_{h,m}^J, \operatorname{Sp}(\mathcal{N}_{h,m})) \leq Ch^{J+2}.$$

Recall that, from the WKB construction,

$$\lambda_{h,m}^J = h \sum_{k=0}^J \mu_k h^k$$

with $\mu_0 = \beta(0)$. Note that $\mathcal{N}_{h,m}$ and $\mathcal{M}_{h,m}$ are unitary. We apply Theorem 5.13 and one part of its proof (5.33), we get

$$\lambda_0(\mathcal{N}_{h,m}) = \lambda_0(\mathcal{M}_{h,m}) = \beta(0)h + \frac{(m+1)\beta'(0)}{\beta(0)}h^2 + o(h^2),$$

and

$$3\beta(0)h - Ch^2 \leq \lambda_1(\mathcal{N}_{h,m}) \leq \lambda_2(\mathcal{N}_{h,m}) \leq \dots$$

Since the spectrum of $\mathcal{N}_{h,m}$ is discrete, we can conclude that

$$\operatorname{dist}(\lambda_{h,m}^J, \operatorname{Sp}(\mathcal{N}_{h,m})) = |\lambda_J(h) - \lambda_0(\mathcal{N}_{h,m})|.$$

Thus, we obtain

$$\|\Psi_{h,m}^J\|_{L^2(\mathbb{R}^+)} |\lambda_{h,m}^J - \lambda_0(\mathcal{N}_{h,m})| \leq Ch^{J+2}. \quad (5.53)$$

Combine this with (5.50), and notice that $\Gamma_m \Psi_{h,m}^J$ belongs to the kernel of $\mathcal{N}_{h,m}$, we get

$$\begin{aligned} & \|(\mathcal{N}_{h,m} - \lambda_0(\mathcal{N}_{h,m})) (\Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J)\|_{L^2(\mathbb{R}^+)} \\ &= \|(\mathcal{N}_{h,m} - \lambda_0(\mathcal{N}_{h,m})) \Psi_{h,m}^J\|_{L^2(\mathbb{R}^+)} \\ &\leq \|(\mathcal{N}_{h,m} - \lambda_{h,m}^J) \Psi_{h,m}^J\|_{L^2(\mathbb{R}^+)} + |\lambda_{h,m}^J - \lambda_0(\mathcal{N}_{h,m})| \|\Psi_{h,m}^J\|_{L^2(\mathbb{R}^+)} \\ &\leq Ch^{J+2}. \end{aligned}$$

By definition, $\Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J \in [\ker(\mathcal{N}_{h,m} - \lambda_0(\mathcal{N}_{h,m}))]^\perp$ and, since the gap between the first and the second eigenvalues of $\mathcal{N}_{h,m}$ is larger than $2\beta(0)h + \mathcal{O}(h^2)$, the Spectral

Theorem proves that

$$\begin{aligned}
& (2\beta(0)h - ch^2) \|\Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J\|_{L^2(\mathbb{R}^+)} \\
& \leq (\lambda_1(\mathcal{N}_{h,m}) - \lambda_0(\mathcal{N}_{h,m})) \|\Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J\|_{L^2(\mathbb{R}^+)} \\
& \leq \|(\mathcal{N}_{h,m} - \lambda_0(\mathcal{N}_{h,m})) (\Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J)\|_{L^2(\mathbb{R}^+)} \\
& \leq Ch^{J+2}.
\end{aligned}$$

Dividing by h and take h small enough, the conclusion follows. \square

Moreover, we can show that the eigenfunctions of $\mathcal{N}_{h,m}$ can be approximated better in an exponentially weighted space. In order to do that, we establish the following Agmon estimate and we refer to the reference [6]. Let us recall the expression of the operator $\mathcal{N}_{h,m}$

$$\mathcal{N}_{h,m} = -2h^2 \partial_\rho \rho \partial_\rho + \tilde{V}_{h,m}(\rho),$$

where

$$\tilde{V}_{h,m}(\rho) := \frac{(hm - a(\rho))^2}{2\rho}.$$

Proposition 5.17. *Let $m \in \mathbb{N}$ and let $(\Phi_k)_{k \in \mathbb{N}} \subset W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$. Assume that there exist $M > 0$, $K_1 > 0$, $K_2 > 0$ and $R_0 > 0$ such that for all $h \in (0, 1)$, $k \in \mathbb{N}$*

$$\tilde{V}_{h,m}(\rho) - 2\rho |\Phi_k'(\rho)|^2 \geq Mh \quad \text{for all } \rho \in [R_0h, +\infty), \quad (5.54)$$

$$|\Phi_k'(\rho)| \leq K_1, \quad |\Phi_k(\rho)| \leq K_2h \quad \text{for all } \rho \in [0, R_0h]. \quad (5.55)$$

Then, for all $c_0 \in (0, M)$, there exists a positive constant $C > 0$ such that, for all $h \in (0, 1)$, $k \in \mathbb{N}$, $z \in [0, c_0h]$, and $u \in \text{Dom}(\mathcal{N}_{h,m})$,

$$\|e^{\Phi_k/h} u\|_{L^2(\mathbb{R}^+)} \leq \frac{C}{h} \|e^{\Phi_k/h} (\mathcal{N}_{h,m} - z) u\|_{L^2(\mathbb{R}^+)} + C \|u\|_{L^2(\mathbb{R}^+)}. \quad (5.56)$$

Proof. From Lax-Milgram theorem, we have

$$\begin{aligned}
\left\langle \mathcal{N}_{h,m} u, e^{2\Phi_k/h} u \right\rangle_{L^2(\mathbb{R}^+)} &= 2h^2 \left\langle \sqrt{\rho} \partial_\rho u, \sqrt{\rho} \partial_\rho (e^{2\Phi_k/h} u) \right\rangle_{L^2(\mathbb{R}^+)} \\
&\quad + \int_0^\infty \tilde{V}_{h,m}(\rho) e^{2\Phi_k/h} |u|^2 d\rho.
\end{aligned} \quad (5.57)$$

Setting $P = \sqrt{\rho} \partial_\rho$ and following the same steps as in the proof of Theorem 5.11, then

$$\text{Re} \left(\left\langle Pu, P e^{2\Phi_k/h} u \right\rangle_{L^2(\mathbb{R}^+)} \right) = \|P e^{\Phi_k/h} u\|_{L^2(\mathbb{R}^+)}^2 - \|[P, e^{\Phi_k/h}] u\|_{L^2(\mathbb{R}^+)}^2.$$

Noticing that $[P, e^{\Phi_k/h}] = \frac{\sqrt{\rho}\Phi'_k}{h} e^{\Phi_k/h}$ and take the real part of (5.57), we obtain

$$\begin{aligned} \operatorname{Re} \left\langle \mathcal{N}_{h,m} u, e^{2\Phi_k/h} u \right\rangle &= 2h^2 \int_0^{+\infty} \rho |\partial_\rho (e^{\Phi_k/h} u)|^2 d\rho \\ &\quad + \int_0^{+\infty} \left(\tilde{V}_{h,m} - 2\rho |\Phi'_k(\rho)|^2 \right) e^{2\Phi_k/h} |u|^2 d\rho. \end{aligned}$$

Since $\tilde{V}_{h,m}(\rho) \geq 0$, we get

$$\begin{aligned} \int_{R_0h}^{+\infty} \left(\tilde{V}_{h,m} - 2\rho |\Phi'_k(\rho)|^2 \right) |e^{\Phi_k/h} u|^2 d\rho &\leq \|e^{\Phi_k/h} \mathcal{N}_{h,m} u\|_{L^2(\mathbb{R}^+)} \|e^{\Phi_k/h} u\|_{L^2(\mathbb{R}^+)} \\ &\quad + \int_0^{R_0h} 2\rho |\Phi'_k(\rho)|^2 e^{2\Phi_k/h} |u|^2 d\rho. \end{aligned}$$

Using (5.54), we deduce that

$$\begin{aligned} Mh \int_{R_0h}^{+\infty} |e^{\Phi_k/h} u|^2 d\rho &\leq \|e^{\Phi_k/h} \mathcal{N}_{h,m} u\|_{L^2(\mathbb{R}^+)} \|e^{\Phi_k/h} u\|_{L^2(\mathbb{R}^+)} \\ &\quad + \int_0^{R_0h} 2\rho |\Phi'_k(\rho)|^2 e^{2\Phi_k/h} |u|^2 d\rho. \end{aligned}$$

Thanks to (5.55), Φ_k/h and Φ'_k are uniformly bounded with respect to h and to k on $[0, R_0h)$. Therefore, we deduce that there exists a constant $L > 0$ (independent of h and k) such that

$$Mh \int_0^{+\infty} |e^{\Phi_k/h} u|^2 d\rho \leq \|e^{\Phi_k/h} \mathcal{N}_{h,m} u\|_{L^2(\mathbb{R}^+)} \|e^{\Phi_k/h} u\|_{L^2(\mathbb{R}^+)} + Lh \int_0^{R_0h} |u|^2 d\rho.$$

For $z \in [0, c_0h)$, we get

$$(M - c_0)h \|e^{\Phi_k/h} u\|_{L^2(\mathbb{R}^+)}^2 \leq \|e^{\Phi_k/h} (\mathcal{N}_{h,m} - z) u\|_{L^2(\mathbb{R}^+)} \|e^{\Phi_k/h} u\|_{L^2(\mathbb{R}^+)} + Lh \|u\|_{L^2(\mathbb{R}^+)}^2.$$

Since $M > c_0$, this gives (5.56). □

The first application of the above Agmon estimate is to prove the decay of the eigenfunction of the operator.

Theorem 5.18. *For all $\varepsilon \in (0, 1)$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$ and all eigenfunctions Ψ with eigenvalue of order h of the operator $\mathcal{N}_{h,m}$,*

$$\|e^{\varepsilon\varphi/h} \Psi\|_{L^2(\mathbb{R}^+)} \leq C \|\Psi\|_{L^2(\mathbb{R}^+)}, \quad (5.58)$$

where $\varphi(\rho) = \int_0^\rho \frac{a(\tau)}{2\tau} d\tau$ is given by (5.47).

Proof. Let $(\chi_k)_{k \in \mathbb{N}}$ be a sequence of functions that is defined in the proof of Theorem (5.12). We recall here two remarkable properties of $(\chi_k)_{k \in \mathbb{N}}$.

i) (χ_k) converges pointwise to the function $y(t) = t$ on $[0, +\infty)$, i.e.

$$\lim_{k \rightarrow +\infty} \chi_k(t) = t \quad \text{for all } t \in [0, +\infty).$$

ii) Related to the derivatives of χ_k :

$$|\chi'_k(t)| = 1 \text{ a.e. on } [0, 2k] \quad \text{and} \quad |\chi'_k(t)| = 0 \text{ for } s > 2k.$$

In order to apply Proposition 5.17, we set up the sequence $(\Phi_k)_{k \in \mathbb{N}}$ as follows:

$$\Phi_k(\rho) = \varepsilon \chi_k(\varphi(\rho)).$$

For each $k \in \mathbb{N}$, we need to check that $\Phi_k \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$. Obviously, $\chi_k \circ \varphi \in L^\infty(\mathbb{R}^+, \mathbb{R})$ and for all $u \in C_c^1(\mathbb{R}^+)$,

$$\begin{aligned} \int_{\mathbb{R}^+} \chi_k(\varphi) u' d\rho &= \int_{[0 \leq \varphi \leq k]} \varphi u' d\rho + \int_{[k \leq \varphi \leq 2k]} (2k - \varphi) u' d\rho \\ &= \int_{\varphi^{-1}(0)}^{\varphi^{-1}(k)} \varphi u' d\rho + \int_{\varphi^{-1}(k)}^{\varphi^{-1}(2k)} (2k - \varphi) u' d\rho \\ &= - \int_{\varphi^{-1}(0)}^{\varphi^{-1}(k)} \varphi' u d\rho + \int_{\varphi^{-1}(k)}^{\varphi^{-1}(2k)} \varphi' u d\rho \\ &= - \int_{\mathbb{R}^+} \chi'_k(\varphi) \varphi' u d\rho, \end{aligned}$$

where $[a \leq \varphi \leq b] := \{\rho \in \mathbb{R}^+ : a \leq \varphi(\rho) \leq b\}$, and since φ is strictly increasing $[a \leq \varphi \leq b] = [\varphi^{-1}(a), \varphi^{-1}(b)]$ for all $a, b \in [0, +\infty)$. Since $\chi'_k(\varphi) \varphi' \in L^\infty(\mathbb{R}^+, \mathbb{R})$, we deduce that $\Phi_k \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$. Furthermore, one has

$$|\Phi'_k(\rho)| \leq \varepsilon |\varphi'(\rho)| = \frac{\varepsilon a(\rho)}{2\rho} \quad \text{a.e. on } \mathbb{R}^+.$$

Let us consider an eigenvalue $\lambda = (\mathcal{O}(h))$ and an associated eigenfunction Ψ of the operator $\mathcal{N}_{h,m}$. Then, there exist $c_0 > 0$ and $h_0 > 0$ such that

$$|\lambda| \leq c_0 h \quad \text{for all } h \in (0, h_0).$$

Let M and R_0 be numbers such that

$$\begin{cases} M > c_0, \\ R_0 \geq \frac{2\beta(0)m + 2M}{\beta(0)^2(1 - \varepsilon^2)}. \end{cases}$$

Using the assumption (1.46), we have $a(\rho) \geq b_0\rho$ for all $\rho \in \mathbb{R}^+$. From the definition of R_0 , we have the estimate, for all $h \in (0, h_0)$, $k \in \mathbb{N}$ and $\rho \geq R_0h$,

$$\begin{aligned} \tilde{V}_{h,m}(\rho) - 2\rho|\Phi'_k(\rho)|^2 &\geq \tilde{V}_{h,m}(\rho) - \varepsilon^2 \frac{a^2(\rho)}{2\rho} \\ &\geq \frac{(1 - \varepsilon^2) \left(\beta(0)\rho - \frac{hm}{1 - \varepsilon^2} \right)^2 - \frac{h^2m^2}{1 - \varepsilon^2}}{2\rho} \\ &= \frac{(1 - \varepsilon^2)\beta(0)^2}{2} \rho - hm\beta(0) \\ &\geq \left(\frac{(1 - \varepsilon^2)\beta(0)^2 R_0}{2} - \beta(0)m \right) h \\ &\geq Mh. \end{aligned}$$

On the other hand, there exist $K_1 > 0$ and $K_2 > 0$ such that, for all $h \in (0, h_0)$, $k \in \mathbb{N}$ and $\rho \in [0, R_0h)$,

$$|\Phi'_k(\rho)| \leq \frac{\varepsilon a(\rho)}{2\rho} = \frac{\varepsilon}{2} \int_0^1 \beta(\rho s) ds \leq K_1$$

and

$$|\Phi_k(\rho)| \leq \varepsilon \phi(\rho) = \varepsilon \int_0^\rho \int_0^1 \beta(\tau s) ds d\tau \leq K_2 h.$$

Now, we can apply Proposition 5.17 for $z = \lambda$, there exists a constant $C > 0$ such that, for all eigenfunction Ψ associated with λ ,

$$\int_{\mathbb{R}^+} e^{2\varepsilon\chi_k(\varphi/h)} |\Psi|^2 d\rho \leq C \int_{\mathbb{R}^+} |\Psi|^2 d\rho.$$

By letting $k \rightarrow \infty$ and using Fatou's lemma, we get the result

$$\int_{\mathbb{R}^+} e^{2\varepsilon\varphi/h} |\Psi|^2 d\rho \leq C \int_{\mathbb{R}^+} |\Psi|^2 d\rho.$$

□

Proof of Theorem 1.19. Let $T : L^2(\mathbb{R}^2, dq) \rightarrow L^2(\mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}, r dr d\theta)$ be the unitary operator introduced in (5.5). Then, $T(U_{h,m})$ is the eigenfunction associated with the eigenvalue $\lambda_m(\mathcal{K}_h) = \lambda_m(\mathcal{L}_{h,\mathbf{A}})$ of the operator \mathcal{K}_h . From Theorem 5.14, $T(U_{h,m})$ has the form

$$T(U_{h,m}) = \frac{1}{\sqrt{2\pi}} e^{-im\theta} \Psi_{h,m} \left(\frac{r^2}{2} \right),$$

where $\Psi_{h,m}$ is a eigenfunction associated with the first eigenvalue $\lambda_0(\mathcal{N}_{h,m})$ of the operator $\mathcal{N}_{h,m}$. By computation, we have

$$\begin{aligned} \int_{\mathbb{R}^2} e^{2\varepsilon P/h} |U_{h,m}|^2 dq &= \int_0^{2\pi} \int_{\mathbb{R}^+} e^{2\varepsilon\varphi(r^2/2)/h} |T(U_{h,m})|^2 r dr d\theta \\ &= \int_{\mathbb{R}^+} e^{2\varepsilon\varphi(r^2/2)/h} \left| \Psi_{h,m} \left(\frac{r^2}{2} \right) \right|^2 r dr \\ &= \int_{\mathbb{R}^+} e^{2\varepsilon\varphi(\rho)/h} |\Psi_{h,m}|^2 d\rho. \end{aligned}$$

Applying Theorem 5.18, there exists $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,

$$\int_{\mathbb{R}^+} e^{2\varepsilon\varphi(\rho)/h} |\Psi_{h,m}|^2 d\rho \leq C \int_{\mathbb{R}^+} |\Psi_{h,m}|^2 d\rho = C \int_{\mathbb{R}^2} |U_{h,m}|^2 dq.$$

□

Theorem 5.19. *For all $(\varepsilon, m, J) \in (0, 1) \times \mathbb{N} \times \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$\|e^{\varepsilon\varphi(\rho)/h} (\Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J)\|_{L^2(\mathbb{R}^+)} \leq Ch^{J+1}, \quad (5.59)$$

where $\varphi(\rho) = \int_0^\rho \frac{a(\tau)}{2\tau} d\tau$ is given by (5.47).

Proof. Let us fix $(\varepsilon, m, J) \in (0, 1) \times \mathbb{N} \times \mathbb{N}$. We recall that $\Gamma_m \Psi_{h,m}^J$ is the eigenfunction with the eigenvalue $\lambda_0(\mathcal{N}_{h,m})$ that has order h . From Theorem 5.18, $\Gamma_m \Psi_{h,m}^J$ belongs to $L^2(\mathbb{R}^+)$. In order to prove the estimate in this theorem, we consider the sequence $(\Phi_k)_{k \in \mathbb{N}}$ that is defined in the proof of Theorem 5.18. As well as in that proof, we obtained the positive numbers M, K_1, K_2 and R_0 such that all conditions of Proposition 5.17 are satisfied. Let us apply Proposition 5.17 for the function $\Psi_{h,m}^J \in \text{Dom}(\mathcal{N}_{h,m})$ to get the estimate

$$\begin{aligned} \|e^{\Phi_k/h} \Psi_{h,m}^J\|_{L^2(\mathbb{R}^+)} &\leq \frac{C}{h} \|e^{\Phi_k/h} (\mathcal{N}_{h,m} - \lambda_{h,m}^J) \Psi_{h,m}^J\|_{L^2(\mathbb{R}^+)} + C \|\Psi_{h,m}^J\|_{L^2(\mathbb{R}^+)} \\ &\leq Ch^{J+1} + C \|\Psi_{h,m}^J\|_{L^2(\mathbb{R}^+)}, \end{aligned}$$

where the last inequality is obtained by (5.48).

Applying again Proposition 5.17 for the function $u = \Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J$ which belongs to $\text{Dom}(\mathcal{N}_{h,m})$, we have

$$\|e^{\Phi_k/h} u\| \leq \frac{C}{h} \|e^{\Phi_k/h} (\mathcal{N}_{h,m} - \lambda_0(\mathcal{N}_{h,m})) u\| + C \|u\|. \quad (5.60)$$

Thanks to (5.48) and the inequality (5.53), we have

$$\begin{aligned}
& \left\| e^{\Phi_k/h} (\mathcal{N}_{h,m} - \lambda_0(\mathcal{N}_{h,m})) u \right\|_{L^2(\mathbb{R}^+)} \\
& \leq \left\| e^{\Phi_k/h} (\mathcal{N}_{h,m} - \lambda_{h,m}^J) \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} + |\lambda_0(\mathcal{N}_{h,m}) - \lambda_{h,m}^J| \left\| e^{\Phi_k/h} \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} \\
& \leq Ch^{J+2} + Ch^{J+1} |\lambda_0(\mathcal{N}_{h,m}) - \lambda_{h,m}^J| + C |\lambda_0(\mathcal{N}_{h,m}) - \lambda_{h,m}^J| \left\| \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} \\
& \leq Ch^{J+2}.
\end{aligned}$$

Combining with Theorem 5.16 and 5.60, we get

$$\|e^{\varepsilon \chi_k(\varphi(\rho))/h} u\|_{L^2(\mathbb{R}^+)} \leq Ch^{J+1},$$

for all $k \geq 1$. Then we let $k \rightarrow +\infty$ and use the Fatou's lemma, we get the conclusion. \square

Proof of Theorem 1.20 and Theorem 1.21. These two theorems are consequences of Theorem 5.16 and Theorem 5.19. We fix the couple $(J, m) \in \mathbb{N} \times \mathbb{N}$. Let us recall that $\Upsilon_{h,m}$ be the eigenfunction associated with $\lambda_m(\mathcal{L}_{h,\mathbf{A}})$ and

$$\Pi_m \Upsilon_{h,m}^J = \langle \Upsilon_{h,m}^J, \Upsilon_{h,m} \rangle_{L^2(\mathbb{R}^2, dq)} \Upsilon_{h,m}$$

is the projection of the WKB Ansatz $\Upsilon_{h,m}^J$ into the eigenspace of $\lambda_m(\mathcal{L}_{h,\mathbf{A}})$.

Through the unitary operator $T : L^2(\mathbb{R}^2, dq) \rightarrow L^2(\mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}, r dr d\theta)$ introduced in (5.5), $T(\Upsilon_{h,m})$ is the eigenfunction of the operator \mathcal{H}_h . From Theorem 5.14, $T(\Upsilon_{h,m})$ has the form

$$T(\Upsilon_{h,m}) = \frac{1}{\sqrt{2\pi}} e^{-im\theta} \Psi_{h,m} \left(\frac{r^2}{2} \right),$$

in which $\Psi_{h,m}$ is an eigenfunction associated with the first eigenvalue $\lambda_0(\mathcal{N}_{h,m})$ of the operator $\mathcal{N}_{h,m}$. By computation, we have

$$\begin{aligned}
\langle \Upsilon_{h,m}^J, \Upsilon_{h,m} \rangle_{L^2(\mathbb{R}^2)} &= \langle T \Upsilon_{h,m}^J, T \Upsilon_{h,m} \rangle_{L^2(r dr d\theta)} \\
&= \sqrt{2\pi} \left\langle \Psi_{h,m}^J \left(\frac{r^2}{2} \right), \Psi_{h,m} \left(\frac{r^2}{2} \right) \right\rangle_{L^2(\mathbb{R}^+, r dr)} \\
&= \sqrt{2\pi} \langle \Psi_{h,m}^J, \Psi_{h,m} \rangle_{L^2(\mathbb{R}^+, d\rho)}.
\end{aligned}$$

It yields that

$$T(\Pi_m \Upsilon_{h,m}^J) = \langle \Psi_{h,m}^J, \Psi_{h,m} \rangle_{L^2(\mathbb{R}^+, d\rho)} \Psi_{h,m} \left(\frac{r^2}{2} \right) e^{-im\theta}.$$

Notice that

$$T(\Upsilon_{h,m}^J) = \Psi_{h,m}^J \left(\frac{r^2}{2} \right) e^{-im\theta}.$$

By applying the result (5.52), it follows that

$$\begin{aligned}
& \|\Upsilon_{h,m}^J - \Pi_m \Upsilon_{h,m}^J\|_{L^2(\mathbb{R}^2, dq)} \\
&= \|T(\Upsilon_{h,m}^J) - T(\Pi_m \Upsilon_{h,m}^J)\|_{L^2(rdrd\theta)} \\
&= \sqrt{2\pi} \left\| \Psi_{h,m}^J \left(\frac{r^2}{2} \right) - \langle \Psi_{h,m}^J, \Psi_{h,m} \rangle_{L^2(\mathbb{R}^+)} \Psi_{h,m} \left(\frac{r^2}{2} \right) \right\|_{L^2(\mathbb{R}^+, rdr)} \\
&= \sqrt{2\pi} \left\| \Psi_{h,m}^J - \langle \Psi_{h,m}^J, \Psi_{h,m} \rangle_{L^2(\mathbb{R}^+, d\rho)} \Psi_{h,m} \right\|_{L^2(\mathbb{R}^+, d\rho)} \\
&= \sqrt{2\pi} \left\| \Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+, d\rho)} \\
&\leq Ch^{J+1}.
\end{aligned}$$

In a similar way, by applying the estimate (5.59), we have

$$\begin{aligned}
& \|e^{\varepsilon P/h} (\Upsilon_{h,m}^J - \Pi_m \Upsilon_{h,m}^J)\|_{L^2(\mathbb{R}^2, dq)} \\
&= \sqrt{2\pi} \left\| e^{\varepsilon\varphi/h} \left(\Psi_{h,m}^J - \langle \Psi_{h,m}^J, \Psi_{h,m} \rangle_{L^2(\mathbb{R}^+)} \Psi_{h,m} \right) \right\|_{L^2(\mathbb{R}^+)} \\
&= \sqrt{2\pi} \left\| e^{\varepsilon\varphi/h} (\Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J) \right\|_{L^2(\mathbb{R}^+)} \\
&\leq Ch^{J+1}.
\end{aligned}$$

□

Appendix A

Spectral Theory

In the following, we recall some elementary knowledges about spectral theory which is often used in this thesis. The proofs of the statements in this appendix can be found in the lecture note [50] or in [25]. Below, $(H, \langle \cdot, \cdot \rangle_H)$ denotes the Hilbert space on \mathbb{C} equipped with the inner product $\langle \cdot, \cdot \rangle_H$. We also denote by $\text{Dom}(T)$ the domain of an operator T which is a linear subspace in H .

Definition A.1 (Closed operator). Let $(\text{Dom}(T), T)$ be an operator on H . T is said to be closed if its graph

$$G(T) = \{(x, y) \mid x \in \text{Dom}(T), y = Tx\},$$

is closed in $H \times H$, where the norm on $H \times H$ is defined by

$$\|(x, y)\| = (\|x\|_H^2 + \|y\|_H^2)^{\frac{1}{2}}.$$

Definition A.2 (Closure of an operator). The smallest closed extension of T is called the closure of T , and denoted by \overline{T} .

Definition A.3 (Adjoint of an unbounded operator). Let $T : \text{Dom}(T) \rightarrow H$ be a densely defined linear operator on H (i.e. $\text{Dom}(T)$ is dense in H), we let

$$\text{Dom}(T^*) = \{u \in H : \text{Dom}(T) \ni w \mapsto \langle Tw, u \rangle_H \text{ is continuous for the topology of } H\}.$$

Thus, for all $u \in \text{Dom}(T^*)$, there exists unique $T^*u \in H$ such that

$$\langle Tw, u \rangle_H = \langle w, T^*u \rangle_H \quad \text{for all } u \in \text{Dom}(T^*).$$

We call T^* the adjoint operator of T .

Definition A.4. Let $(\text{Dom}(T), T)$ be a densely defined operator.

1. $(\text{Dom}(T), T)$ is called symmetric if $T \subset T^*$.
2. $(\text{Dom}(T), T)$ is called self-adjoint if $T = T^*$.
3. A symmetric operator is essentially self-adjoint if its closure is self-adjoint.

In the above definition, we used the relation $A \subset B$ between two operators A and B . It means that $\text{Dom}(A) \subset \text{Dom}(B)$ and $Au = Bu$ for all $u \in \text{Dom}(A)$.

Proposition A.5. *If $(\text{Dom}(T), T)$ is essentially self-adjoint operator, it has unique self-adjoint extension.*

We often use the following theorem to produce a self-adjoint operator from a continuous and coercive sesquilinear.

Theorem A.6 (Lax-Milgram). *Let $(V, \langle \cdot, \cdot \rangle_V)$ be a Hilbert space such that V is continuously embedded and dense in H . Let Q be a sesquilinear form define on V which is*

1. *Continuous : There exists a constant $C >$ such that*

$$|Q(u, v)| \leq C \|u\|_V \|v\|_V \quad \text{for all } u, v \in V .$$

2. *V-elliptic (or Coercive): There exists a constant $\alpha >$ such that*

$$|Q(u, u)| \geq \alpha \|u\|_V^2 \quad \text{for all } u, v \in V .$$

3. *Hermitian : $Q(u, v) = \overline{Q(v, u)}$ for all $u, v \in V$.*

Then, there exists a self-adjoint operator \mathcal{S} whose domain is

$$\text{Dom}(\mathcal{S}) = \{u \in V : v \mapsto Q(u, v) \text{ is continuous on } V \text{ for the norm of } H\} .$$

and defined by the way : for each $u \in \text{Dom}(\mathcal{S})$, there exists a unique element in H , we denote it $\mathcal{S}u$ such that

$$Q(u, v) = \langle \mathcal{S}u, v \rangle_H \quad \forall v \in V .$$

Furthermore, \mathcal{L} is bijective from $\text{Dom}(\mathcal{S})$ onto H and $\text{Dom}(\mathcal{S})$ is dense in V and in H .

Definition A.7 (Resolvent and spectrum set). Let $(\text{Dom}(T), T)$ be a self-adjoint operator on H . The resolvent set of T is define by

$$\rho(T) = \{z \in \mathbb{C} : (T - zI) \text{ is a bijective from } \text{Dom}(T) \text{ onto } H\} .$$

and the complement of the resolvent set in \mathbb{C} is called the spectrum of the operator, denoted by $\text{Sp}(T)$,

$$\text{Sp}(T) = \mathbb{C} \setminus \rho(T).$$

And if $\lambda \in \rho(T)$, the operator $(T - \lambda)^{-1}$ is called a resolvent of T .

When $(\text{Dom}(T), T)$ is a self-adjoint operator, its spectrum is classified into discrete spectrum and essential spectrum:

Definition A.8. Let $(\text{Dom}(T), T)$ be a self-adjoint operator.

- i) The discrete spectrum of T , denoted by Sp_{dis} , containing elements which are isolated finite multiplicity values in $\text{Sp}(T)$.
- ii) The essential spectrum of T , denoted by $\text{Sp}_{\text{ess}}(T)$, is the complement of discrete spectrum of T

$$\text{Sp}_{\text{ess}}(T) = \text{Sp}(T) \setminus \text{Sp}_{\text{dis}}(T).$$

The operator T will have a purely discrete spectrum if it has a compact resolvent. There is one useful criterion to check the compactness of the resolvent through the form domain:

Proposition A.9. Consider two Hilbert spaces $(V, \langle \cdot, \cdot \rangle_V)$ and $(H, \langle \cdot, \cdot \rangle_H)$ such that V is continuously embedded and dense in H . Assume that Q is a continuous, coercive and Hermitian sesquilinear form on V and let \mathcal{S} be the self-adjoint operator given by the Theorem A.6. Let us denote by $\|\cdot\|_Q$ the norm induced by Q , i.e. $\|u\|_Q = \sqrt{Q(u, u)}$. If $(V, \|\cdot\|_Q) \hookrightarrow (H, \|\cdot\|_H)$ is compact, then \mathcal{S} has compact resolvent.

Proof. Choose one element $z \in \rho(\mathcal{S})$. We need to show that

$$(\mathcal{S} - z)^{-1} : (H, \|\cdot\|_H) \rightarrow (H, \|\cdot\|_H)$$

is a compact operator. To do that, we consider $(\mathcal{S} - z)^{-1} : (H, \|\cdot\|_H) \rightarrow (H, \|\cdot\|_H)$ as the composition of the following operators:

$$(H, \|\cdot\|_H) \xrightarrow{(\mathcal{S}-z)^{-1}} (\text{Dom}(\mathcal{S}), \|\cdot\|_{\mathcal{S}}) \xrightarrow{i_1} (V, \|\cdot\|_Q) \xrightarrow{i_2} (H, \|\cdot\|_H),$$

where $\|u\|_{\mathcal{S}} := \|u\|_H + \|\mathcal{S}u\|_H$ is the graph norm equipped on $\text{Dom}(\mathcal{S})$. Since \mathcal{S} is closed, $(\text{Dom}(\mathcal{S}), \|\cdot\|_{\mathcal{S}})$ is a Banach space. Applying the closed graph theorem, the mapping $(\mathcal{S} - z)^{-1} : (H, \|\cdot\|_H) \rightarrow (\text{Dom}(\mathcal{S}), \|\cdot\|_{\mathcal{S}})$ is bounded. By the definition of the norm $\|\cdot\|_Q$, we get the estimation, for all $u \in \text{Dom}(\mathcal{S})$

$$\|u\|_Q = \sqrt{\langle \mathcal{S}u, u \rangle_H} \leq \|\mathcal{S}u\|_H^{1/2} \|u\|_H^{1/2} \leq \frac{1}{2} (\|\mathcal{S}u\|_H + \|u\|_H) = \frac{1}{2} \|u\|_{\mathcal{S}}.$$

It means that the injection i_1 is bounded too. Since i_2 is compact, the conclusion of the proposition comes from the fact that the set of compact operators is an ideal of the set of bounded operators. \square

Proposition A.10. *Let $(\text{Dom}(T), T)$ be a self-adjoint operator on H . For all $\lambda \in \rho(T)$, we have*

$$\|(T - \lambda)^{-1}\| := \sup_{u \in H \setminus \{0\}} \frac{\|(T - \lambda)^{-1}u\|_H}{\|u\|_H} \leq \frac{1}{\text{dist}(\lambda, \text{Sp}(T))}.$$

The above proposition leads to the useful result which is usually used to approximate the eigenvalues of the self-adjoint operator.

Corollary A.11. *Let $(\text{Dom}(T), T)$ be a self-adjoint operator on H . For all $\lambda \in \mathbb{C}$ and for all $\psi \in \text{Dom}(T)$, we have*

$$\text{dist}(\lambda, \text{Sp}(T))\|\psi\|_H \leq \|(T - \lambda)\psi\|_H.$$

Lemma A.12. *Let $(\text{Dom}(T), T)$ be a self-adjoint operator on H . Then*

$$\text{Sp}_{\text{dis}}(T) = \{\lambda \in \text{Sp}(T) : (T - \lambda) \text{ is a Fredholm operator}\}.$$

Theorem A.13 (Min-max principle). *Let T be a self-adjoint operator was born from Lax-Milgram theorem A.6 by a sesquilinear Q and $\text{Dom}(Q)$ is the form domain of Q . We define the Rayleigh quotients of the operator T*

$$\mu_n(T) = \sup_{u_1, \dots, u_{n-1} \in \text{Dom}(Q)} \left(\inf_{\substack{u \in \text{span}\{u_1, \dots, u_{n-1}\}^\perp \\ u \in \text{Dom}(T) \setminus \{0\}}} \frac{Q(u, u)}{\|u\|_H^2} \right).$$

Then the Rayleigh quotients $\mu_n(T)$ is a non-decreasing sequence and one of the following holds

- i) $\mu_n(T)$ is the n -th eigenvalue counted with multiplicity of T and the spectrum of T in $(-\infty, \mu_n(T)]$ is discrete.*
- ii) $\mu_n(T)$ is the bottom of the essential spectrum and, for all $j \geq n$, $\mu_j(T) = \mu_n(T)$.*

And the consequence of the Min-Max principle which is often used is the following

Corollary A.14. *Assume that there exists $a \in \mathbb{R}$ with $a < \inf \text{Sp}_{\text{ess}}(T)$ and an n -dimensional linear space $V \subset \text{Dom}(T)$ such that*

$$\langle T\psi, \psi \rangle_H \leq a\|\psi\|_H^2, \quad \text{for all } \psi \in V.$$

Then, the n -eigenvalue $\lambda_n(T)$ exists and satisfies

$$\lambda_n(T) \leq a.$$

Proof. Given arbitrary u_1, \dots, u_{n-1} in H . Since V is a vector space has dimension n , we can find a non-trivial element $\phi \in V \cap \text{span}\{u_1, \dots, u_{n-1}\}^T$. It implies that

$$\inf_{\substack{u \in \text{span}\{u_1, \dots, u_{n-1}\}^T \\ u \in \text{Dom}(T) \setminus \{0\}}} \frac{Q(u, u)}{\|u\|_H^2} \leq \frac{Q(\phi, \phi)}{\|\phi\|_H^2} = \frac{\langle T\phi, \phi \rangle_H}{\|\phi\|_H^2} \leq a.$$

Take supremum on all class containing $(n - 1)$ elements in H , we have

$$\mu_n \leq a.$$

Note that $a < \inf \text{Sp}_{\text{ess}}(T)$, apply the Min-Max principle (see Theorem [A.13](#)), it results that μ_n is the n -th eigenvalue. \square

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Titre : Analyse classique et semi-classique des champs magnétiques en deux dimensions

Mot clés : Hamiltonien magnétique ; Confinement ; Scattering ; Méthodes WKB ; Analyse semi-classique ; Théorie spectrale

Résumé : Ce manuscrit est consacré à l'étude de la mécanique classique et la mécanique quantique en présence d'un champ magnétique. En mécanique classique, nous utilisons un Hamiltonien pour décrire la dynamique d'une particule chargée dans un domaine soumis à un champ magnétique. Nous nous intéressons ici à deux problèmes classiques de physique : le problème de confinement et le problème de scattering. Dans le cas quantique, nous étudions le problème spectral du Laplacien magnétique au niveau semi-classique dans des domaines de dimension deux : sur une variété Riemannienne compacte à bord et dans \mathbb{R}^2 . En supposant que le champ magnétique ait un unique mini-

mum strictement positif et non-dégénéré, nous pouvons décrire les fonctions propres par les méthodes WKB. Grâce au théorème spectral, nous pouvons estimer efficacement les vraies fonctions propres et les fonctions propres approchées localement proche du minimum du champ magnétique. Dans \mathbb{R}^2 , sous l'hypothèse additionnelle d'une symétrie radiale du champ magnétique, nous pouvons montrer que les fonctions propres du Laplacien magnétique décroissent de manière exponentielle à l'infini avec une vitesse contrôlée par la fonction phase de la procédure WKB. De plus, les fonctions propres sont très bien approchées dans un espace à poids exponentiel.

Title: Classical and semi-classical analysis of magnetic fields in two dimensions

Keywords: Magnetic Hamiltonian; Confinement; Scattering; WKB methods; Semiclassical analysis; Spectral theory

Abstract: This manuscript is devoted to classical mechanics and quantum mechanics, especially in the presence of magnetic field. In classical mechanics, we use Hamiltonian dynamics to describe the motion of a charged particle in a domain affected by the magnetic field. We are interested in two classical physical problems: the confinement and the scattering problem. In the quantum case, we study the spectral problem of the magnetic Laplacian at the semi-classical level, in two-dimensional domains: on a compact Riemannian manifold with boundary and on \mathbb{R}^2 . Under the assumption that the magnetic field has a unique posi-

tive and non-degenerate minimum, we can describe the eigenfunctions by WKB methods. Thanks to the spectral theorem, we estimated efficiently the true eigenfunctions and the approximate eigenfunctions locally near the minimum point of the magnetic field. On \mathbb{R}^2 , with the additional assumption that the magnetic field is radially symmetric, we can show that the eigenfunctions of the magnetic Laplacian decay exponentially at infinity and at a rate controlled by the phase function created in WKB procedure. Furthermore, the eigenfunctions are very well approximated in an exponentially weighted space.