



HAL
open science

Asymptotic properties of the small data solutions of the Vlasov-Maxwell system

Léo Bigorgne

► **To cite this version:**

Léo Bigorgne. Asymptotic properties of the small data solutions of the Vlasov-Maxwell system. Analysis of PDEs [math.AP]. Université Paris Saclay (COmUE), 2019. English. NNT : 2019SACLS164 . tel-02435592

HAL Id: tel-02435592

<https://theses.hal.science/tel-02435592>

Submitted on 11 Jan 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THÈSE DE DOCTORAT

de

L'UNIVERSITÉ PARIS-SACLAY

École doctorale de mathématiques Hadamard (EDMH, ED 574)

Établissement d'inscription : Université Paris-Sud

Laboratoire d'accueil : Laboratoire de mathématiques d'Orsay, UMR 8628 CNRS

Spécialité de doctorat : Mathématiques fondamentales

Léo BIGORGNE

Propriétés asymptotiques des solutions à données petites du système de
Vlasov-Maxwell

Date de soutenance : 25 JUIN 2019

Après avis des rapporteurs : JONATHAN LUK (Stanford University)
JÉRÉMIE SZEFTTEL (Sorbonne Université)

Jury de soutenance :

CÉCILE HUNEAU	(École Polytechnique) Examineur
PHILIPPE G. LEFLOCH	(Sorbonne Université) Président du jury
CHRISTOPHE PALLARD	(Université Paris-Sud) Examineur
FRÉDÉRIC ROUSSET	(Université Paris-Sud) Examineur
JACQUES SMULEVICI	(Sorbonne Université) Directeur de thèse
JÉRÉMIE SZEFTTEL	(Sorbonne Université) Rapporteur

Propriétés asymptotiques des solutions à données petites du système de Vlasov-Maxwell

Résumé. L'objectif de cette thèse est de décrire le comportement asymptotique des solutions à données petites du système de Vlasov-Maxwell. En particulier, on s'attachera à étudier tant le champ électromagnétique que le champ de Vlasov par des méthodes de champs de vecteurs, nous permettant ainsi d'éviter toute contrainte de support sur les données initiales. La structure isotrope du système de Vlasov-Maxwell est d'une importance capitale pour compenser le phénomène de résonance causé par les particules approchant la vitesse de propagation du champ électromagnétique. De ce fait, plusieurs parties de ce manuscrit sont dédiées à sa description. Ajoutons également que les méthodes de champs de vecteurs sont connues pour être robustes et s'adapter relativement bien à d'autres situations telles que l'étude des solutions de l'équation des ondes sur un espace-temps courbé. Cette souplesse nous a notamment permis, contrairement aux travaux précédents sur ce sujet, de considérer des plasmas avec des particules sans masse.

Notre étude débute par le cas des grandes dimensions $d \geq 4$ où les effets dispersifs sont plus importants et permettent ainsi d'obtenir de meilleurs taux de décroissance sur les solutions du système et leurs dérivées. Une nouvelle inégalité de décroissance pour les solutions d'une équation de transport relativiste constitue d'ailleurs un élément central de la démonstration. Afin d'établir un résultat analogue dans le cas où les particules sont sans masse, nous avons dû imposer que le champ de Vlasov s'annule initialement pour les petites vitesses puis nous avons ensuite montré que cette hypothèse était nécessaire. Dans un second temps, nous nous intéressons au cas tridimensionnel avec des particules sans masse, où une étude plus poussée de la structure des équations sera nécessaire afin d'obtenir les taux de décroissance optimaux pour les composantes isotropes du champ électromagnétique, les moyennes en vitesse de la fonction de distribution et leurs dérivées. Nous nous concentrons ensuite sur l'étude du comportement asymptotique des solutions à données petites du système de Vlasov-Maxwell massif en dimension 3. Des difficultés spécifiques nous forcent à modifier les champs de vecteurs utilisés précédemment pour l'équation de transport dans le but de compenser les pires termes d'erreurs des équations commutées. Enfin, on considère le même problème en se restreignant à l'étude des solutions à l'extérieur d'un cône de lumière. Les fortes propriétés de décroissance vérifiées par la moyenne en vitesse de la densité de particules dans cette région nous permettent d'affaiblir les hypothèses sur les données initiales et d'avoir une démonstration considérablement plus simple.

Mots-clés. EDP hyperboliques, système de Vlasov-Maxwell, équations non linéaires, équations d'ondes et de transport, méthode des champs de vecteurs, structure isotrope.

Asymptotic properties of the small data solutions of the Vlasov-Maxwell system

Abstract. The purpose of this thesis is to study the asymptotic properties of the small data solutions of the Vlasov-Maxwell system using vector field methods for both the electromagnetic field and the particle density. No compact support assumption is required on the initial data. Instead, we make crucial use of the null structure of the equations in order to deal with a resonant phenomenon caused by the particles approaching the speed of propagation of the Maxwell equations. Due to the robustness of vector field methods and contrary to previous works on this topic, we also study plasmas with massless particles.

We start by investigating the high dimensional cases ($d \geq 4$) where dispersive effects allow us to derive strong decay rate on the solutions of the system and their derivatives. For that purpose, we proved a new decay estimate for solutions to massive relativistic transport equations. In order to obtain an analogous result for massless particles, we required the velocity support of the distribution function to be initially bounded away from 0 and we then proved that this assumption is actually necessary. The second part of this thesis is devoted to the three dimensional massless case, where a stronger understanding of the null structure of the Vlasov-Maxwell system is essential in order to derive the optimal decay rate of the null components of the electromagnetic field, the velocity average of the particle density and their derivatives. We then focus on the asymptotic behavior of the small data solutions of the massive Vlasov-Maxwell system in $3d$. Specific problems force us to modify the vector fields used previously to study the Vlasov field in order to compensate the worst error terms in the commuted transport equations. Finally, still for the massive system in $3d$, we restrict our study of the solutions to the exterior of a light cone. The strong decay properties satisfied by the velocity average of the particle density in such a region permit us to relax the hypothesis on the initial data and lead to a much simpler proof.

Key-words. Hyperbolic PDE, Vlasov-Maxwell system, non linear equations, wave and transport equations, vector field methods, null structure.

Remerciements

Je souhaite naturellement exprimer toute ma gratitude à mon directeur de thèse Jacques Smulevici. Je te suis très reconnaissant pour m'avoir accordé ta confiance et pour t'être autant investi durant ces trois années. Ta vision des mathématiques et les intuitions que tu as pu me transmettre m'ont été très utiles et continueront assurément à me servir. Je voudrais aussi te remercier pour m'avoir montré que les échanges étaient un point central de la recherche. Cela m'a permis entre autres de voyager afin d'assister à des conférences, d'exposer mes travaux et surtout de rencontrer d'autres personnes passionnées par la relativité. Outre tes qualités d'encadrant et de scientifique, ta bienveillance et ton dévouement m'ont rendu particulièrement chanceux d'avoir pu préparer ma thèse sous ta direction.

Je remercie vivement Jérémie Szeftel et Jonathan Luk pour avoir aimablement accepté d'être mes rapporteurs de thèse ainsi que Philippe LeFloch, Cécile Huneau, Christophe Pallard et Frédéric Rousset pour avoir accepté de faire partie de mon jury de soutenance. Merci de l'intérêt que vous avez porté à mes travaux.

Ces trois années de thèse se sont déroulées dans un environnement de travail idéal et furent riches en rencontres. Je voudrais donc remercier de nouveau Jérémie, Jacques et tout particulièrement Philippe pour l'organisation des séminaires de relativité. Je suis également très reconnaissant envers Olivier et Cécile pour avoir organisé de nombreuses sessions du groupe de lecture en relativité générale ainsi qu'envers tous les participants, à savoir Allen, Annalaura, Duan, Fatima, Grigorios, Shijie, Siyuan, Volker et Zoey, pour toutes les discussions que nous avons pu avoir. Je remercie également chaleureusement David, pour ses invitations à Vienne, Jérémie et Maximilian pour les échanges que nous avons pu avoir et pour la collaboration que nous avons entamée. Je tiens évidemment à remercier l'équipe administrative du LMO, tout particulièrement Estelle, Ophélie et Stéphane, et les doctorants que j'ai pu côtoyer, en particulier Camille, que je fréquente presque quotidiennement depuis 7 ans. A ce titre, je voudrais également saluer mes autres camarades de l'ENS Rennes, notamment Kévin, mais surtout Joackim et Simon, avec qui j'ai passé autant de temps à courir qu'à parler de mathématiques.

Enfin, je ne serais arrivé jusque là si ma mère ne m'avait pas autant poussé durant ma scolarité et sans Jacques Bozec, mon professeur de mathématiques de terminale qui m'a redonné goût à cette discipline. Diane a également joué un rôle prépondérant ces dernières années en me motivant et soutenant sans relâche. Je salue aussi ma famille, tout particulièrement mon père, Pauline, Enora, Gene, Simon et bien sûr Jade, qui sont venus assister à ma soutenance, ainsi que mes fidèles amis Quentin et Mickaël, sur qui j'ai toujours pu compter. Merci à eux.

Contents

1	Introduction	1
1.1	Présentation du système et le problème de l'existence globale des solutions	1
1.2	Méthodes de champs de vecteurs	4
1.2.1	Grands principes et premières applications	4
1.2.2	La méthode des champs de vecteurs pour les équations de Maxwell	6
1.2.3	Méthode de champs de vecteurs pour les équations de transports cinétiques	7
1.2.4	Résultat : inégalité de décroissance pour une solution de $\mathbf{T}_1(f) = G$	9
1.3	Étude des solutions à données petites pour des systèmes de Vlasov	10
1.3.1	Système de Vlasov-Nordström	10
1.3.2	Le système d'Einstein-Vlasov	11
1.4	Derniers prérequis et présentation de la structure isotrope du système de Vlasov-Maxwell	12
1.4.1	Quelques notations	12
1.4.2	Aspect général des normes utilisées	13
1.4.3	Commutation des équations et conservation de la structure	13
1.4.4	La charge totale du plasma	14
1.4.5	Contrôler les grandes vitesses en exploitant la structure isotrope du système	15
1.5	Résultats : système de Vlasov-Maxwell massif	18
1.5.1	Le cas des grandes dimensions	18
1.5.2	Étude des solutions à l'extérieur d'un cône de lumière en dimension 3	20
1.5.3	Étude des solutions à données petites dans l'espace-temps de Minkowski en dimension 3	22
1.6	Résultats : système de Vlasov-Maxwell sans masse	24
1.6.1	Le problème causé par les petites vitesses	25
1.6.2	Le cas des grandes dimensions	25
1.6.3	Le cas $3d$	26
1.7	Perspectives	27
2	Asymptotic properties of small data solutions of the Vlasov-Maxwell system in high dimensions	31
2.1	Introduction	31
2.1.1	Global in time solutions for the Vlasov-Maxwell system	32
2.1.2	Previous work on small data solutions of the Vlasov-Maxwell system	32
2.1.3	Optimal gradient estimates for Vlasov systems	32
2.1.4	The vector field method for Vlasov fields	33
2.1.5	The Lorenz gauge	33
2.1.6	Results for the massive Vlasov-Maxwell system	34
2.1.7	Results for the massless Vlasov-Maxwell system	35
2.1.8	The main difficulties and ingredients of our proof	35
2.1.9	Structure of the paper	37
2.1.10	Acknowledgements	37
2.2	Notations and preliminaries	38
2.2.1	Basic notations	38
2.2.2	The relativistic transport operator	39
2.2.3	Vector fields	39
2.2.4	Weights preserved by the flow	41
2.2.5	Decay estimates	43
2.2.6	The Vlasov-Maxwell system	44
2.3	Energy estimates for the Vlasov-Maxwell system	52

2.3.1	Energy estimates for the transport equation	52
2.3.2	Energy estimates for the wave equation	54
2.3.3	Energy estimates for the Maxwell equations	57
2.4	Some technical results	62
2.4.1	An integral estimate	62
2.4.2	The null coordinates of $\nabla_v f$	63
2.4.3	Some Sobolev inequalities	65
2.4.4	Pointwise decay estimate for the null decomposition of the electromagnetic field	66
2.4.5	A Grönwall inequality	71
2.5	Decay estimate for the massive case	72
2.5.1	Sobolev inequalities	73
2.5.2	An energy estimate	74
2.5.3	Proof of Theorem 2.5.1	75
2.5.4	Improved decay for the derivatives of the velocity averages	76
2.6	The massive Vlasov-Maxwell equations	77
2.6.1	Global existence for small data	77
2.6.2	Structure and beginning of the proof	78
2.6.3	Step 1: Decay estimates	79
2.6.4	Step 2: Improving the energy estimates for the transport equation	79
2.6.5	Step 3: Improved decay estimates for velocity averages	84
2.6.6	Step 4: L^2 estimates for the velocity averages	84
2.6.7	Step 5: Improvement of the electromagnetic field energy estimates	88
2.7	The massless Vlasov-Maxwell equations	90
2.7.1	Global existence for small data	90
2.7.2	Structure and beginning of the proof	91
2.7.3	Step 1: Decay estimates	92
2.7.4	Step 2: the Vlasov fields vanishes for small velocities	93
2.7.5	Step 3: Improving the Energy estimates for the transport equations	94
2.7.6	Step 4: L^2 estimates for the velocity averages	97
2.7.7	Step 5: Improvement of the electromagnetic field estimates	103
2.8	Non existence	106
2.8.1	A symmetry property for the Vlasov-Maxwell system	107
2.8.2	Proof of Proposition 2.8.1	109
3 Sharp asymptotics for the solutions of the three-dimensional massless Vlasov-Maxwell system with small data		115
3.1	Introduction	115
3.1.1	Previous results on small data solutions for the massive Vlasov-Maxwell system	116
3.1.2	Previous works on Vlasov systems using vector field methods	117
3.1.3	Statement of the main result	117
3.1.4	Strategy of the proof and main difficulties	118
3.1.5	Structure of the paper	119
3.1.6	Acknowledgements	119
3.2	Notations and preliminaries	119
3.2.1	Basic notations	119
3.2.2	The problem of the small velocities	120
3.2.3	Basic tools for the study of the electromagnetic field	120
3.2.4	The vector fields of the Poincaré group and their complete lifts	121
3.2.5	Commutation of the Vlasov-Maxwell system	122
3.2.6	Weights preserved by the flow and null components of the velocity vector	124
3.2.7	Various subsets of the Minkowski spacetime	125
3.3	Energy and pointwise decay estimates	125
3.3.1	Energy estimates	125
3.3.2	Decay estimates	127
3.4	The null structure of the non linearity $\mathcal{L}_{Z^\gamma}(F) \left(v, \nabla_v \widehat{Z}^\beta f \right)$	133
3.5	Bootstrap assumptions and strategy of the proof	134
3.6	Improvement of the energy bound on the particle density	135
3.6.1	Proof of Proposition 3.6.1	136
3.6.2	Proof of Proposition 3.6.2	136

3.7	L^2 estimates on the velocity averages of the Vlasov field	137
3.7.1	The homogeneous system	138
3.7.2	The inhomogenous system	139
3.7.3	The L^2 estimates	141
3.8	The energy bounds of the electromagnetic field	141
3.A	The Vlasov field vanishes for small velocities	142
3.B	Bounding the initial norms	146
3.C	All derivatives of F are chargeless	147
3.D	Proof of Lemmas 3.3.8, 3.3.9 and 3.3.10	148
4	Sharp asymptotic behavior of solutions of the 3d Vlasov-Maxwell system with small data	151
4.1	Introduction	151
4.1.1	Small data results for the Vlasov-Maxwell system	152
4.1.2	Vector fields and modified vector fields for the Vlasov equations	152
4.1.3	Charged electromagnetic field	153
4.1.4	Statement of the main result	153
4.1.5	Key elements of the proof	154
4.1.6	Structure of the paper	156
4.2	Notations and preliminaries	157
4.2.1	Basic notations	157
4.2.2	Basic tools for the study of the electromagnetic field	157
4.2.3	Weights preserved by the flow and null components of the velocity vector	158
4.2.4	Various subsets of the Minkowski spacetime	159
4.2.5	An integral estimate	160
4.3	Vector fields and modified vector fields	160
4.3.1	The vector fields of the Poincaré group and their complete lift	160
4.3.2	Modified vector field and the first order commutation formula	163
4.3.3	Higher order commutation formula	166
4.3.4	The null structure of $G(v, \nabla_v g)$	171
4.3.5	Source term of $T_F(z^j P_\xi(\Phi) Y^\beta f)$	172
4.3.6	Commutation of the Maxwell equations	177
4.4	Energy and pointwise decay estimates	179
4.4.1	Energy estimates	180
4.4.2	Pointwise decay estimates	182
4.5	The pure charge part of the electromagnetic field	187
4.6	Bootstrap assumptions and strategy of the proof	188
4.7	Immediate consequences of the bootstrap assumptions	190
4.8	Improvement of the bootstrap assumptions (4.49), (4.50) and (4.51)	196
4.8.1	The terms of (category 0)	196
4.8.2	Bounds on several spacetime integrals	197
4.8.3	Completion of the bounds on the spacetime integrals	199
4.8.4	Estimates for $\mathbb{E}_{N-1}^X[f]$, $\mathbb{E}_N^X[f]$ and obtention of optimal decay near the lightcone for velocity averages	203
4.8.5	The critical terms	204
4.9	L^2 decay estimates for the velocity averages of the Vlasov field	205
4.9.1	The homogeneous part	207
4.9.2	The inhomogeneous part	208
4.9.3	L^2 estimates on the velocity averages of f	212
4.10	Improvement of the energy estimates of the electromagnetic field	214
4.10.1	For $\mathcal{E}_N^0[F]$	215
4.10.2	The weighted norm for the exterior region	215
4.10.3	The weighted norms for the interior region	216
5	Asymptotic properties of the solutions to the Vlasov-Maxwell system in the exterior of a light cone	219
5.1	Introduction	219
5.1.1	Small data results for the VM system	220
5.1.2	Previous works on Vlasov systems using vector field methods	220
5.1.3	Statement of the main result	221

5.1.4	Main ingredients of the proof	222
5.1.5	Structure of the paper	223
5.1.6	Acknowledgements	223
5.2	Preliminaries	223
5.2.1	Basic notations	223
5.2.2	A null foliation	224
5.2.3	The commutation vector fields	224
5.2.4	The null components of the velocity vector and the weights preserved by \mathbf{T}	226
5.2.5	The null decomposition of the electromagnetic field	227
5.3	Energy and pointwise decay estimates	227
5.3.1	Estimates for velocity averages	228
5.3.2	Estimates for the electromagnetic field	229
5.4	Null properties of the Vlasov equation	231
5.5	Bootstrap assumptions and strategy of the proof	232
5.6	Improvement of the energy bound on the particle density	233
5.6.1	Proof of inequality (5.22)	233
5.6.2	Proof of Proposition 5.6.1	234
5.6.3	The remaining energy norm	236
5.6.4	L^2 estimates on velocity averages	236
5.7	The energy bound on the electromagnetic field	237
5.8	L^2 estimates for the higher order derivatives of the Vlasov field	237
5.8.1	The homogeneous system	238
5.8.2	The inhomogenous system	239
5.8.3	End of the proof of Proposition 5.6.3	243

Chapter 1

Introduction

L'objectif de cette introduction est de replacer dans leur contexte mathématique les résultats obtenus durant cette thèse, qui sont présentés dans la sous-section 1.2.4 ainsi que dans les sections 1.5 et 1.6.

1.1 Présentation du système et le problème de l'existence globale des solutions

Le système de Vlasov-Maxwell est utilisé afin de modéliser des plasmas où aucun choc ne se produit entre les particules. Ces dernières ne sont ainsi soumises qu'à la force électromagnétique de Lorentz $\vec{E} + \hat{v} \times \vec{B}$ et la fonction de distribution f_k de la $k^{\text{ème}}$ famille de particules, ayant pour masse m_k et pour charge e_k , est alors solution de l'équation de Vlasov

$$\partial_t f_k + \sum_{i=1}^3 \frac{v^i}{\sqrt{m_k^2 + |v|^2}} \partial_i f_k + e_k \left(\vec{E} + \frac{v}{\sqrt{m_k^2 + |v|^2}} \times \vec{B} \right) \cdot \nabla_v f_k = 0, \quad 1 \leq k \leq K.$$

Le champ électromagnétique satisfait pour sa part les équations de Maxwell avec un terme source dépendant des fonctions de distribution f_k ,

$$\begin{aligned} \nabla \cdot \vec{E} &= \sum_{k=1}^K e_k \int_{v \in \mathbb{R}^3} f_k dv, & \nabla \times \vec{E} &= -\partial_t \vec{B} \\ \nabla \cdot \vec{B} &= 0, & \nabla \times \vec{B} &= \partial_t \vec{E} + \sum_{k=1}^K \int_{v \in \mathbb{R}^3} \frac{v}{\sqrt{m_k^2 + |v|^2}} f_k dv. \end{aligned}$$

La densité de charge ρ et le vecteur densité de courant \vec{j} sont donc donnés par

$$\rho = \sum_{k=1}^K e_k \int_{v \in \mathbb{R}^3} f_k dv \quad \text{et} \quad \vec{j} = \sum_{k=1}^K \int_{v \in \mathbb{R}^3} \frac{v}{\sqrt{m_k^2 + |v|^2}} f_k dv.$$

Une des questions naturelles que l'on peut se poser est celle de l'existence globale en temps des solutions. Étant donné des données initiales $(f_1^0, \dots, f_K^0, \vec{E}^0, \vec{E}^1, \vec{B}^0, \vec{B}^1)$ régulières, satisfaisant les équations de contraintes

$$\nabla \cdot \vec{E}^0 = \sum_{k=1}^K e_k \int_{v \in \mathbb{R}^3} f_k^0 dv \quad \text{et} \quad \nabla \cdot \vec{B}^0 = 0,$$

existe-t-il une solution classique $(f_1, \dots, f_K, \vec{E}, \vec{B})$ du système de Vlasov-Maxwell vérifiant

$$f_k(t=0) = f_k^0, \quad \vec{E}(t=0) = \vec{E}^0, \quad \partial_t \vec{E}(t=0) = \vec{E}^1, \quad \vec{B}(t=0) = \vec{B}^0, \quad \partial_t \vec{B}(t=0) = \vec{B}^1$$

et qui soit définie sur $\mathbb{R}_+ \times \mathbb{R}^3$? Cela constitue encore aujourd'hui un problème ouvert et on ne sait y répondre que sous des hypothèses de symétries ou dans des régimes perturbatifs¹.

¹Le problème de l'existence globale dans le cadre des solutions faibles a par contre été résolu dans [15] puis revisité dans [42].

- Lorsque la solution est invariante par translation dans une variable spatiale (il est commun de dire que l'on travaille avec deux dimensions et demie). Dans ce cadre et sous une hypothèse de support compact en vitesse sur les données initiales, Glassey et Schaeffer ont démontré dans [20] que les solutions étaient globales. Plus récemment, Luk et Strain sont parvenus à généraliser le résultat aux plasmas composés de particules pouvant avoir des vitesses arbitrairement proches de la vitesse de la lumière (voir [37]).
- Rein a prouvé un résultat d'existence globale dans [41] pour des perturbations de certaines solutions du système de Vlasov-Maxwell à symétrie sphérique. Ces dernières peuvent être étudiées plus facilement car elles sont aussi solutions du système de Vlasov-Poisson relativiste.
- L'existence globale pour les solutions à données petites fut d'abord obtenue par Glassey-Strauss dans [24] sous une hypothèse de support compact (en espace et en vitesse). Un résultat similaire fut ensuite prouvé par [22] pour des plasmas presque électriquement neutres, i.e. $\sum_k e_k m_k^3 f_{0k}(x, m_k v)$ est petit (individuellement, les densités de particules ne sont pas nécessairement petites). Schaeffer est ensuite parvenu à montrer un résultat analogue sans restriction sur le support en vitesse des données initiales (voir [44]). Enfin, mentionnons également que les récents résultats [50] de Wang prouvent en particulier que les solutions à données petites du système de Vlasov-Maxwell sont globales sans aucune restriction sur le support des données initiales².

Pour le cas général, à défaut d'être en mesure de prouver des résultats d'existence globale, plusieurs critères de prolongement sont connus. Le premier, obtenu par Glassey-Strauss dans [23] puis retrouvé par d'autres méthodes dans [8] et [32], nous dit que les solutions à régularité C^1 du système de Vlasov-Maxwell, dont les données initiales sont à supports compacts, ne développent pas de singularités tant que les supports en vitesse des fonctions f_k restent bornés. D'autres critères permettant de prolonger la solution au-delà d'un temps $T^* > 0$ existent et requièrent que

$$\left\| \sqrt{1 + |v|^2}^\theta f_k \right\|_{L^\infty([0, T^*[, L_x^q L_v^1])} < +\infty, \quad (1.1)$$

pour un certain q et un certain θ . Les cas $6 \leq q \leq \infty$ et $\theta > \frac{4}{q}$ furent couverts par Pallard dans [38]. Les cas $q = +\infty$ et $\theta = 0$ ainsi que $q = 6$ et $\theta = 0$ furent traités respectivement par [48] et [39]. Citons également les récents travaux de Kunze [33] et Patel [40], permettant de couvrir $q \geq 1$ et $\theta > \frac{4}{q} - 1$ ainsi que $1 \leq q \leq 2$ et $\theta > \frac{18}{5q} - 1$. De plus anciens résultats de Glassey-Strauss traitent le cas $q = \infty$ et $\theta = 1$ pour des données initiales non nécessairement à support compact en v (voir [25]). Récemment, Luk et Strain dans [37] (respectivement Patel dans [40]) ont étendu ce critère pour tout $2 < q \leq +\infty$ et $\theta > \frac{2}{q}$ (respectivement $q = 1$ et $\theta > 3$). Notons en particulier que ces derniers résultats ne requièrent aucune restriction sur les supports des données initiales, tant en espace qu'en vitesse.

Remark 1.1.1. *Pour l'ensemble des travaux cités ici, il est essentiel que les masses des particules constituant le plasma soient strictement positives, i.e. $m_k > 0$ pour tout $1 \leq k \leq K$.*

Une autre question que l'on peut se poser est celle du comportement asymptotique des solutions du système de Vlasov-Maxwell. Une solution suffisamment régulière de l'équation de transport relativiste

$$\partial_t f + \sum_{i=1}^3 \frac{v^i}{\sqrt{1 + |v|^2}} \partial_i f = 0$$

vérifie les estimées suivantes :

$$\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad \left| \partial_{t,x}^\beta \int_{v \in \mathbb{R}^3} f(t, x, v) dv \right| \leq \frac{C_f^\beta}{(1 + t + |x|)^{3+|\beta|}},$$

où C_f^β est une constante positive dépendant de l'ordre de dérivation $|\beta|$ ainsi que de $f(t = 0)$. On peut donc se demander si l'on peut retrouver de tels taux de décroissance pour les solutions du système de Vlasov-Maxwell.

- Dans leurs travaux sur les solutions à données petites [24], Glassey-Strauss ont obtenu la décroissance optimale sur les moyennes en vitesses des densités de particules, i.e.³

$$\int_{v \in \mathbb{R}^3} f_k dv \lesssim \frac{\epsilon}{(1 + t)^3}.$$

²Les résultats de Wang apportent également de nouvelles informations sur le comportement asymptotique des solutions. Nous reviendrons sur cela ci-dessous.

³On utilise ici la notation $A \lesssim B$ pour désigner une inégalité de la forme $A \leq CB$, où C est une constante absolue.

Leur démonstration ne fournit par contre pas d'estimations sur les dérivées de $\int_v f_k dv$. Pour le champ électromagnétique, ils ont prouvé les inégalités

$$\begin{aligned} |\vec{E}|(t, x) + |\vec{B}|(t, x) &\lesssim \frac{\epsilon}{(1+t)(1+|t-|x|)}, \\ |\nabla_{t,x}\vec{E}|(t, x) + |\nabla_{t,x}\vec{B}|(t, x) &\lesssim \frac{\epsilon \log(3+t+|x|)}{(1+t)(1+|t-|x|)^2} \end{aligned}$$

mais n'ont pas contrôlé les dérivées d'ordre supérieur.

- La méthode utilisée par Schaeffer afin de traiter les grandes vitesses ne lui a pas permis d'obtenir la décroissance optimale sur $\int_v f_k dv$.
- De son côté, Wang est parvenu à retrouver les taux de décroissance du cas linéaire⁴ sur les densités de particules et il a également réussi à contrôler les dérivées d'ordres supérieurs du champ électromagnétique. Plus précisément il a montré que, pour tout $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$\begin{aligned} \left| \partial_{t,x}^\alpha \int_{v \in \mathbb{R}^3} f(t, x, v) dv \right| &\lesssim \frac{\epsilon}{(1+t+|x|)^{3+|\alpha|}} \\ |\nabla_{t,x}^\alpha \vec{E}|(t, x) + |\nabla_{t,x}^\alpha \vec{B}|(t, x) &\lesssim \frac{\epsilon}{(1+t)(1+|t-|x|)^{1+|\alpha|}}. \end{aligned}$$

Soulignons que ces résultats ont été obtenus indépendamment de ceux de cette thèse et requièrent de fortes hypothèses de décroissance polynomiales en x et en v sur les données initiales.

Compte tenu des méthodes utilisées dans cette thèse, nous allons représenter le champ électromagnétique (\vec{E}, \vec{B}) sous forme géométrique par une 2-forme F . Elle est définie (en coordonnées cartésiennes), par

$$F_{0i} = \vec{E}^i \quad \text{et} \quad F_{jk} = -\varepsilon_{ijk} \vec{B}^i,$$

où ε_{ijk} est le symbole de Levi-Cevita. Afin d'alléger les notations et puisque le nombre de familles de particules ne complique pas les démonstrations des résultats que nous allons établir, on va supposer que $K = 1$. Pour les mêmes raisons, on suppose également que la charge des particules est égale à 1 et que leur masse est égale à 1 ou 0. Enfin, dans le but de pouvoir supposer, si besoin, que le plasma est électriquement neutre, nous ne conservons pas la restriction sur la positivité de la densité de particules f . Le système de Vlasov-Maxwell peut alors se récrire sous la forme⁵

$$v^0 \partial_t f + v^i \partial_i f + v^\mu F_\mu^j \partial_{v_j} f = 0, \quad (1.2)$$

$$\nabla^\mu F_{\mu\nu} = J(f)_\nu := \int_{v \in \mathbb{R}^3} \frac{v_\nu}{v^0} f dv, \quad (1.3)$$

$$\nabla^\mu {}^* F_{\mu\nu} = 0, \quad (1.4)$$

où

- ${}^*F(t, x)$ est le dual de Hodge de F . Il est défini par⁶

$${}^*F_{\mu\nu} = \frac{1}{2} F^{\lambda\sigma} \varepsilon_{\lambda\sigma\mu\nu}.$$

- $v^0 = \sqrt{1+|v|^2}$ si les particules sont massives et $v^0 = |v|$ si elles sont sans masse.

Bien que les résultats que nous allons prouver sur ces deux systèmes (massif et sans masse) soient analogues, nous verrons que leurs démonstrations présentent des difficultés spécifiques à chaque cas. Le but de cette thèse est de généraliser les résultats de Glassey-Strauss sur les solutions à données petites de (1.2) – (1.4) afin de

1. supprimer toutes les hypothèses de supports compacts sur les données initiales.

⁴C'est à dire que $\partial^\beta \int_v f_k dv$ se comporte comme $\partial^\beta \int_v g dv$, où $\sqrt{m_k^2 + |v|^2} \partial_t g + \sum_{i=1}^3 v^i \partial_i g = 0$.

⁵On utilise ici la convention de sommation d'Einstein. Par exemple $v^i \partial_i = \sum_{i=1}^3 v^i \partial_i$ et $v^\mu \partial_\mu = \sum_{\mu=0}^3 v^\mu \partial_\mu$. La variable muette de la somme décrira [1, 3] si c'est une lettre latine et [0, 3] si c'est une lettre grecque.

⁶On montera et descendra les indices des tenseurs en utilisant la métrique de Minkowski $\eta = \text{diag}(-, 1, 1, 1)$. Ainsi $F^{\lambda\sigma} = F_{\mu\nu} \eta^{\lambda\mu} \eta^{\sigma\nu}$.

2. Obtenir les taux de décroissance optimaux sur $\int_v f dv$ et ses dérivées.
3. Obtenir le comportement précis de F et de ses dérivées.
4. Utiliser des méthodes dites "robustes".

Concernant le premier objectif, ajoutons que nous nous attacherons à affaiblir, et ce de manière optimale, les hypothèses de décroissance sur la variable v . Plus précisément, nous supposons seulement que $f(0, x, \cdot)$ est intégrable en v , ce qui est une condition nécessaire pour que le terme source des équations de Maxwell soit bien défini. En particulier, aucune contrainte du type

$$\int_{\mathbb{R}_v^3} (1 + |v|)^\delta f(0, x, v) dv < +\infty, \quad \delta > 0,$$

ne sera imposée. Pour le troisième point, il est connu que pour une solution suffisamment régulière des équations de Maxwell dans le vide, certaines composantes du champ électromagnétique, telles que les composantes radiales $\frac{x_i}{r} \vec{E}^i$ et $\frac{x_i}{r} \vec{B}^i$ du champ électrique et du champ magnétique, décroissent plus rapidement que F lui-même. Notre but est donc de montrer que cela reste vrai pour les champs électromagnétiques d'un plasma suffisamment dilué. Pour le dernier point, rappelons tout d'abord les outils utilisés dans les preuves des résultats cités.

- Les démonstrations des résultats de Glassey-Strauss et Schaeffer sont basées sur des formules de représentations pour l'équation des ondes ainsi qu'un contrôle précis des caractéristiques de l'équation de Vlasov.
- Wang a quant à lui utilisé une méthode de champs de vecteurs ainsi que de l'analyse de Fourier.

Notre objectif est d'utiliser uniquement des méthodes qui s'adaptent bien à des problèmes de relativité générale, où l'espace-temps ambiant est courbé, tel que l'étude de plasmas en astrophysique. Il serait notamment intéressant d'étudier les propriétés de décroissance ponctuelle des solutions de l'équation de Vlasov dans un espace-temps de type trou noir. Nous excluons donc les méthodes basées sur des formules de représentations ou de l'analyse de Fourier et nous allons ainsi principalement utiliser des méthodes de champs de vecteurs, tant pour étudier la densité de particules f et l'équation de Vlasov que pour le champ électromagnétique F et les équations de Maxwell⁷. Le but de la section suivante est donc de présenter les principes de telles méthodes ainsi que certains résultats qu'elles ont permis d'établir.

1.2 Méthodes de champs de vecteurs

1.2.1 Grands principes et premières applications

Au cours du temps, différentes méthodes ont été développées afin de prouver qu'une solution de l'équation des ondes

$$\square u := -\partial_t^2 u + \sum_{i=1}^3 \partial_i^2 u = 0, \quad (1.5)$$

sous des hypothèses raisonnables sur les données initiales $u(0, \cdot)$ et $\partial_t u(0, \cdot)$, décroît. On peut notamment utiliser l'expression exacte de la solution ou de l'analyse de Fourier. Dans [29], Klainerman a utilisé un tout autre processus, aujourd'hui appelé *méthode des champs de vecteurs*, et qui est basé sur

1. des commutateurs, qui sont des champs de vecteurs reflétant les symétries de l'équation étudiée.
 - Dans le cas de l'équation des ondes (1.5), ces commutateurs sont les champs de Killing de l'espace-temps de Minkowski ainsi que le champ de vecteurs de changement d'échelle, qui est quant à lui Killing conforme. Plus précisément, on commute l'équation avec les éléments de \mathbb{K} , qui sont

les translations	$\partial_\mu, \quad 0 \leq \mu \leq 3,$
les rotations	$\Omega_{ij} = x^i \partial_j - x^j \partial_i, \quad 1 \leq i < j \leq 3,$
les rotations hyperboliques	$\Omega_{0k} = t \partial_k + x^k \partial_t, \quad 1 \leq k \leq 3.$
le champ de vecteurs de changement d'échelle	$S = t \partial_t + r \partial_r.$

Pour tout $Z \in \mathbb{K}$, on a $\square Z(u) = 0$.

⁷Les démonstrations des résultats de décroissance pour des solutions de l'équation des ondes sur les espace-temps de Schwarzschild ou de Kerr reposent essentiellement sur des méthodes de champs de vecteurs.

2. Des inégalités d'énergies permettant de propager des normes L^p à poids des solutions et de leurs dérivées.

- Pour notre exemple, on utilise tout simplement la conservation de la norme L^2 du gradient de u

$$\|\nabla_{t,x}u\|_{L^2(\mathbb{R}^3)}(t) = \|\nabla_{t,x}u\|_{L^2(\mathbb{R}^3)}(0).$$

3. Des inégalités de Sobolev à poids afin d'obtenir de la décroissance ponctuelle.

- Pour l'étude de l'équation des ondes, on utilise généralement l'inégalité fonctionnelle suivante, dite de Klainerman-Sobolev, qui est

$$|\nabla_{t,x}v|(t,x) \lesssim \frac{1}{(1+t+r)(1+|t-r|)^{\frac{1}{2}}} \sum_{\substack{Z^\beta \in \mathbb{K}^{|\beta|} \\ |\beta| \leq 2}} \|\nabla_{t,x}Z^\beta v\|_{L^2(\mathbb{R}^3)}(t), \quad (1.6)$$

où Z^β parcourt l'ensemble des dérivées d'ordre au plus 2 formées par les commutateurs de \mathbb{K} .

En mettant tout ceci bout à bout, on obtient des informations sur le comportement asymptotique de u . En effet, si $Z^\beta \in \mathbb{K}^{|\beta|}$ on a par 1) que

$$\square Z^\beta(u) = 0 \quad \text{et donc, par 2), que} \quad \|\nabla_{t,x}Z^\beta u\|_{L^2(\mathbb{R}^3)}(t) = \|\nabla_{t,x}Z^\beta u\|_{L^2(\mathbb{R}^3)}(0).$$

Par conséquent, en appliquant l'inégalité de Klainerman-Sobolev à u , on obtient

$$\begin{aligned} |\nabla_{t,x}u|(t,x) &\lesssim \frac{1}{(1+t+r)(1+|t-r|)^{\frac{1}{2}}} \sum_{\substack{Z^\beta \in \mathbb{K}^{|\beta|} \\ |\beta| \leq 2}} \|\nabla_{t,x}Z^\beta u\|_{L^2(\mathbb{R}^3)}(t) \\ &= \frac{1}{(1+t+r)(1+|t-r|)^{\frac{1}{2}}} \sum_{\substack{Z^\beta \in \mathbb{K}^{|\beta|} \\ |\beta| \leq 2}} \|\nabla_{t,x}Z^\beta u\|_{L^2(\mathbb{R}^3)}(0). \end{aligned}$$

Ce type de méthode, basé sur des lois de conservations approchées et des commutateurs, est particulièrement robustes et a permis de traiter nombre de problèmes non linéaires comme la stabilité de l'espace-temps de Minkowski pour les équations d'Einstein (voir [12] et [35]). Des méthodes des champs de vecteurs s'avèrent aussi être efficace pour étudier les solutions de l'équation des ondes sur une variété lorentzienne courbe telle que Schwarzschild (voir [13]) ou Kerr (voir [14]).

Nous allons maintenant énoncer quelques résultats sur le problème de l'existence globale pour certaines équations d'ondes semi-linéaires.

Theorem 1.2.1. *Les solutions à données petites de*

$$\square u = |\partial_t u|^2, \quad u(0, \cdot), \partial_t u(0, \cdot) \in C_c^\infty(\mathbb{R}^d) \quad (1.7)$$

sont globales en dimensions $d \geq 4$ mais peuvent exploser en temps fini en dimension 3.

Le résultat d'explosion en temps fini est dû à Fritz John (voir [28]) tandis que le résultat d'existence globale, concernant les grandes dimensions, fut prouvé en utilisant la méthode des champs de vecteurs. Il est pertinent de comparer ce théorème avec le suivant.

Theorem 1.2.2. *Les solutions à données petites de*

$$\square u = |\partial_t u|^2 - \sum_{i=1}^3 |\partial_i u|^2, \quad u(0, \cdot), \partial_t u(0, \cdot) \in C_c^\infty(\mathbb{R}^3) \quad (1.8)$$

sont globales en dimension 3.

On dit qu'une telle non-linéarité satisfait la condition isotrope⁸. Klainerman dans son article [30] a remarqué que le terme source de (1.8) se comportait mieux que, par exemple, celui de (1.7). Cela peut s'expliquer assez simplement et de manière géométrique si l'on introduit une base isotrope

$$(L, \underline{L}, e_1, e_2), \quad \text{où} \quad L = \partial_t + \partial_r, \quad \underline{L} = \partial_t - \partial_r$$

⁸Ou plus communément, *the null condition*.

et (e_1, e_2) est une base orthonormale sur les sphères $(t, r) = \text{constante}$. Les dérivées tangentées au cône de lumière $t = r$ d'une solution d'une équation d'onde se comportent mieux que la dérivée transverse. Plus précisément, on a les inégalités suivantes, valables pour toute fonction u suffisamment régulière,

$$|\underline{L}u| \leq \frac{1}{1+|t-r|} \sum_{Z \in \mathbb{K}} |Zu| \quad \text{et} \quad |Lu| + |e_1u| + |e_2u| \leq \frac{1}{1+t+r} \sum_{Z \in \mathbb{K}} |Zu|. \quad (1.9)$$

Le fait que les dérivées classiques $\partial_{t,x}$ nous donnent un gain de décroissance par rapport aux rotations, aux rotations hyperboliques et à S n'est pas étonnant compte-tenu des poids en x^i et t présents dans ces derniers champs de vecteurs. Il est important de remarquer que ce gain dégénère près du cône de lumière pour $\partial_{t,x}u$ et $\underline{L}u$ mais pas pour Lu notamment. En exprimant nos deux non-linéarités dans la base isotrope, on peut voir que leur comportement asymptotique est donné par

$$\begin{aligned} |\partial_t u|^2 &\sim |\underline{L}u|^2, \\ |\partial_t u|^2 - \sum_{i=1}^3 |\partial_i u|^2 &\sim |\underline{L}u||Lu|, \end{aligned}$$

ce qui explique pourquoi les solutions de (1.8) se comportent mieux que celles de (1.7). Pour de tels problèmes de stabilité, la dimension 3 constitue souvent un seuil critique dans le sens où il est plus simple de traiter les grandes dimensions (le taux de décroissance des solutions étant meilleur dû à la dispersion) et que la stabilité ne peut être prouvée en $3d$ que si les équations présentent une bonne structure à l'image de (1.8). Généralement, les équations provenant de la physique possèdent une forme de structure isotrope. C'est notamment le cas pour les équations d'Einstein (voir [12], [35]), de Maxwell [11], de Yang-Mills [51], d'Einstein-Klein-Gordon [34] ou d'Einstein-Vlasov (voir [17] et [26]) par exemple.

Dans [35] est introduite la condition isotrope faible que l'on peut illustrer par l'exemple suivant.

Proposition 1.2.3. *Les solutions à données petites de*

$$\square u = 0, \quad u(0, \cdot), \partial_t u(0, \cdot) \in C_c^\infty(\mathbb{R}^3), \quad (1.10)$$

$$\square v = |\partial_t u|^2, \quad v(0, \cdot), \partial_t v(0, \cdot) \in C_c^\infty(\mathbb{R}^3) \quad (1.11)$$

sont globales en temps.

Pour le montrer, on résout tout d'abord la première équation, qui est une équation d'onde linéaire et on peut ensuite déterminer v (le terme source de (1.11) est défini sur $\mathbb{R}_+ \times \mathbb{R}^3$ et a un bon comportement). On aurait pu considérer un exemple un peu plus compliqué en remplaçant (1.10) par

$$\square u = \partial_t u \partial_t v - \sum_{i=1}^3 \partial_i u \partial_i v$$

et obtenir un résultat analogue. Cet exemple trivial permet d'illustrer que l'on peut avoir existence globale en temps pour les solutions à données petites d'un système d'équations d'ondes semi-linéaires qui ne satisfait pas la condition isotrope (la non-linéarité dans (1.11) est $|\partial_t u|^2$). La stabilité de la solution triviale pour un tel système provient de l'existence d'une hiérarchie dans les équations. On peut dire, de façon schématique, qu'il est triangulaire.

Remark 1.2.4. *Dans notre étude des solutions du système de Vlasov-Maxwell, nous allons également tirer profit de la structure des équations ainsi que de nombreuses hiérarchies entre les équations commutées.*

1.2.2 La méthode des champs de vecteurs pour les équations de Maxwell

Les équations de Maxwell étant philosophiquement des équations d'ondes, il n'est pas étonnant que la méthode des champs de vecteurs ait pu être adaptée assez rapidement à l'étude de leurs solutions. Christodoulou et Klainerman ont ainsi démontré dans [11] le résultat suivant.

Theorem 1.2.5. *Soit F une solution des équations de Maxwell dans le vide*

$$\begin{aligned} \nabla^\mu F_{\mu\nu} &= 0 \\ \nabla^{\mu*} F_{\mu\nu} &= 0, \end{aligned}$$

vérifiant

$$\mathcal{E}[F] := \sum_{|\beta| \leq 2} \int_{\mathbb{R}^3} (1+r)^{2+|\beta|} \left| \nabla_{t,x}^\beta F \right| (0, x) dx < +\infty.$$

Alors, le comportement asymptotique des composantes isotropes de F est donné par

$$\begin{aligned} |\underline{\alpha}_i|(t, x) &:= |F_{e_i \underline{L}}|(t, x) \lesssim \frac{\mathcal{E}[F]}{(1+t+r)(1+|t-r|)^{\frac{3}{2}}} \\ |\alpha_i|(t, x) &:= |F_{e_i L}|(t, x) \lesssim \frac{\mathcal{E}[F]}{(1+t+r)^{\frac{5}{2}}} \\ |\rho|(t, x) &:= \left| \frac{1}{2} F_{LL} \right|(t, x) \lesssim \frac{\mathcal{E}[F]}{(1+t+r)^2(1+|t-r|)^{\frac{1}{2}}} \\ |\sigma|(t, x) &:= |F_{e_1 e_2}|(t, x) \lesssim \frac{\mathcal{E}[F]}{(1+t+r)^2(1+|t-r|)^{\frac{1}{2}}} \end{aligned}$$

On remarque en particulier que,

- en tant que solution d'une équation d'onde, il était attendu que le champ électromagnétique vérifie

$$|F(t, x)| \lesssim \frac{\mathcal{E}[F]}{(1+t+r)(1+|t-r|)^{\frac{3}{2}}}.$$

- Certaines composantes isotropes du champ électromagnétique se comportent mieux que d'autres au niveau du cône de lumière $t = r$. Une base isotrope semble par conséquent être adaptée à l'étude du comportement asymptotique de F . Notons en revanche que ses composantes cartésiennes, autrement dit le champ électrique \vec{E} et magnétique \vec{B} , ne permettent pas d'observer de tels phénomènes car, en général,

$$\left| \vec{E} \right|(t, x) \sim \left| \vec{B} \right|(t, x) \sim |F|(t, x) \sim |\underline{\alpha}|(t, x).$$

- C'est la structure, dite aussi *isotrope*, qui permet d'obtenir de meilleurs taux de décroissance qu'attendu sur certaines composantes du champs électromagnétique.
- La composante ρ (respectivement σ) correspond à la composante radiale du champ électrique $\frac{x_i}{r} \vec{E}^i$ (respectivement à la composante radiale du champ magnétique $\frac{x_i}{r} \vec{B}^i$).

Ce résultat nous sera notamment très utile en vue de contrôler des termes similaires à $v^\mu F_\mu^j \partial_{v^j} f$ présents dans l'équation de transport (1.2) du système de Vlasov-Maxwell. Notons qu'il a aussi permis d'établir la stabilité de la solution triviale du système de Maxwell-Klein-Gordon (voir [36]).

1.2.3 Méthode de champs de vecteurs pour les équations de transports cinétiques

Dans l'optique d'étudier des systèmes couplant une équation de Vlasov avec une (ou plusieurs) équation d'onde, Fajman-Joudioux-Smulevici ont adapté dans [18] la méthode des champs de vecteurs aux équations de transports cinétiques relativistes. Afin de montrer comment, nous allons prendre pour modèle l'étude des solutions (régulières) de

$$\mathbf{T}_m(f) := v^\mu \partial_\mu(f) = 0, \quad (1.12)$$

où $m = 1$ et dans ce cas $v^0 = \sqrt{1 + |v|^2}$ (les particules sont massives) ou $m = 0$ et alors $v^0 = |v|$ (les particules sont sans masse). Pour simplifier la présentation on va supposer que l'on se trouve dans le premier cas $m = 1$. Étant donné la motivation initiale, qui est d'étudier un couplage avec une équation d'onde, il sera important de remarquer que les commutateurs utilisés pour ces deux types d'équations sont compatibles. Remarquons tout d'abord que

$$\begin{aligned} \mathbf{T}_1(\partial_\mu f) &= [\mathbf{T}_1, \partial_\mu](f) + \partial_\mu(\mathbf{T}_1(f)) = 0 + 0 = 0, \\ \mathbf{T}_1(Sf) &= [\mathbf{T}_1, S](f) + S(\mathbf{T}_1(f)) = \mathbf{T}_1(f) + 0 = 0. \end{aligned}$$

Par contre,

$$[\mathbf{T}_1, \Omega_{ij}] = v^i \partial_j - v^j \partial_i \quad \text{et} \quad [\mathbf{T}_1, \Omega_{0k}] = v^0 \partial_k + v^k \partial_t,$$

ce qui nous empêche de commuter (1.12) par les champs de vecteurs de \mathbb{K} . Il est par contre montré dans [43], dans un cadre beaucoup plus général, que si $X = X^\mu \partial_\mu$ est un champ de Killing de l'espace-temps de Minkowski, alors son lift complet⁹

$$\widehat{X} := X^\mu \partial_\mu + v^\mu \partial_\mu X^\lambda \partial_{v^\lambda}$$

commute avec \mathbf{T}_1 . On a

$$\begin{aligned} \widehat{\partial}_\mu &= \partial_\mu, & 0 \leq \mu \leq 3, \\ \widehat{\Omega}_{ij} &= x^i \partial_j - x^j \partial_i + v^i \partial_{v^j} - v^j \partial_{v^i}, & 1 \leq i < j \leq 3 \\ \widehat{\Omega}_{0k} &= x^k \partial_t - t \partial_k + v^0 \partial_{v^k}, & 1 \leq k \leq 3 \\ \widehat{S} &= t \partial_t + x^i \partial_i + v^i \partial_{v^i} \end{aligned}$$

et peut vérifier que l'on a bien

$$\mathbf{T}_1(\widehat{\Omega}_{ij} f) = 0 \quad \text{et} \quad \mathbf{T}_1(\widehat{\Omega}_{0k} f) = 0.$$

Le champ de vecteurs S étant seulement conforme Killing, on a $\mathbf{T}_1(\widehat{S} f) \neq 0$. Cependant, dans le cas où les particules sont sans masse, on a $[\mathbf{T}_0, \widehat{S}] = 0$. On choisit donc

1. comme commutateurs les éléments de

$$\widehat{\mathbb{P}}_0 := \{\partial_\mu, 0 \leq \mu \leq 3\} \cup \{\widehat{\Omega}_{ij}, 1 \leq i < j \leq 3\} \cup \{\widehat{\Omega}_{0k}, 1 \leq k \leq 3\} \cup \{S\}.$$

Bien que S ne soit pas un lift complet, on utilisera souvent \widehat{Z} pour désigner un champ de vecteurs de $\widehat{\mathbb{P}}_0$.

2. Comme inégalité d'énergie, on peut utiliser la loi de conservation approchée de la norme L^1 de toute solution suffisamment régulière de $\mathbf{T}_1(g) = G$, i.e.

$$\forall t \in [0, +\infty[, \quad \|g\|_{L^1_{x,v}}(t) = \|g\|_{L^1_{x,v}}(0) + \int_0^t \left\| \int_{v \in \mathbb{R}^3} |\mathbf{T}_1(g)| \frac{dv}{v^0} \right\|_{L^1_x}(s) ds. \quad (1.13)$$

3. Enfin, en tant que solution d'une équation de transport, la norme $L^\infty_{x,v}$ de f est conservée au cours du temps et ne peut donc décroître. Il est par contre bien connu que les moyennes en vitesse de f elles décroissent. Fajman-Joudioux-Smulevici ont donc généralisé l'inégalité de Klainerman-Sobolev L^2_x aux moyennes en vitesse dans [18] et ont prouvé que

$$\forall [0, T[\times \mathbb{R}^3, \quad \int_{v \in \mathbb{R}^3} |g|(t, x, v) dv \lesssim \frac{1}{(1+t+r)^2(1+|t-r|)} \sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|} \\ |\beta| \leq 3}} \|\widehat{Z}^\beta g\|_{L^1_{x,v}}(t), \quad (1.14)$$

pour toute fonction suffisamment régulière g .

Remarque 1.2.1. *L'inégalité (1.14) donne, tout comme (1.6), un taux de décroissance qui dégénère près du cône de lumière $t = r$. Le problème ici, contrairement au cas de l'équation des ondes, est que si $f(0, \cdot, \cdot)$ est suffisamment régulière, alors*

$$\forall [0, T[\times \mathbb{R}^3, \quad \int_{v \in \mathbb{R}^3} |f|(t, x, v) dv \leq \frac{C_f}{(1+t+r)^3},$$

où C_f est une constant dépendant uniquement de $f(0, \cdot, \cdot)$. Par contre, dans le cas où les particules sont sans masse, (1.14) fournit le taux de décroissance attendu.

Enfin, l'utilisation des éléments de

$$\begin{aligned} \mathbf{k}_1 &:= \left\{ \frac{v^\mu}{v^0}, 0 \leq \mu \leq 3 \right\} \cup \{x^\mu v^\nu - x^\nu v^\mu, 0 \leq \mu < \nu \leq 3\}, \\ \mathbf{k}_0 &:= \mathbf{k}_1 \cup \{tv^0 - x^i v_i\} \end{aligned}$$

⁹Prendre le lift complet d'un champ de vecteur est une opération classique en géométrie différentielle. Les informations présentées ici, dans un cas particulier, sont suffisantes pour le contenu de cette thèse.

pourra également s'avérer utile. Cela est lié aux propriétés suivantes, valables pour tout $m \in \{0, 1\}$,

$$\forall z \in \mathbf{k}_m, \quad \mathbf{T}_m(z) = 0 \quad \text{et} \quad \forall (z, \widehat{Z}) \in \mathbf{k}_m \times \widehat{\mathbb{P}}_0, \quad \widehat{Z}(v^0 z) \in v^0 \mathbf{k}_m \cup \{0\}.$$

Par conséquent, si $z \in \mathbf{k}_1$, $\mathbf{T}_1(zf) = 0$ et si f décroît suffisamment initialement, on a

$$\|zf\|_{L_{x,v}^1}(t) = \|zf\|_{L_{x,v}^1}(0),$$

ce qui nous permet de propager des normes à poids de f . On pourra en trouver une application dans le théorème 1.2.2 ci-dessous. Dans le cas où les particules sont sans masse, l'inégalité

$$1 + |t - r| \leq \sum_{z \in \mathbf{k}_0} |z|,$$

analogue à (1.9), illustre comment ces poids peuvent apporter de la décroissance.

1.2.4 Résultat : inégalité de décroissance pour une solution de $\mathbf{T}_1(f) = G$

En raison de l'utilisation d'un feuilletage hyperbolique lors de leurs études des solutions à données petites des systèmes de Vlasov-Nordström et d'Einstein-Vlasov, Fajman-Joudioux-Smulevici n'ont pas eu à faire face au problème exposé dans la Remarque 1.2.1. L'analogue de (1.14) pour un tel feuilletage fournit un meilleur taux de décroissance mais le contrepoint de l'utilisation d'un tel procédé réside dans la nécessité de supposer que les données initiales sont à support compact en espace. Afin d'éviter une telle restriction, la première étape de cette thèse consista à prouver le résultat suivant, issu de [4].

Théorème 1.2.2. *Soit $T > 0$ et $f : [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ une fonction suffisamment régulière. Alors, pour tout $(t, x) \in [0, T] \times \mathbb{R}^3$,*

$$\int_{v \in \mathbb{R}^3} |f|(t, x, v) \frac{dv}{(v^0)^2} \lesssim \frac{1}{(1+t+r)^3} \sum_{\substack{|\beta| \leq 3 \\ z \in \mathbf{k}}} \left(\|z \widehat{Z}^\beta f\|_{L_{x,v}^1}(0) + \int_0^t \left\| \int_{v \in \mathbb{R}^3} |\mathbf{T}_1(z \widehat{Z}^\beta f)| \frac{dv}{v^0} \right\|_{L_x^1}(s) ds \right).$$

On peut alors formuler plusieurs remarques.

- Cette inégalité est spécifique au cas massif. Si on considérait une masse $m \in \mathbb{R}_+^*$ quelconque, la constante cachée dans \lesssim serait proportionnelle à m^{-2} .
- Le taux de décroissance est conforme à celui attendu pour une solution régulière de $\mathbf{T}_1(f) = 0$ et meilleur que celui donné par l'inégalité de type Klainerman-Sobolev (1.14).
- On a une perte de deux puissances de v^0 . De plus, contrairement à (1.14), ce n'est pas une inégalité de Sobolev au sens classique du terme car on utilise l'opérateur \mathbf{T}_1 afin de l'obtenir.

L'idée de la démonstration est de découper l'espace-temps en trois domaines.

1. Si $t + |x| \leq 1$ ou $|x| \leq \frac{t}{2}$, on applique l'inégalité de type Klainerman-Sobolev (1.14) ainsi que l'inégalité d'énergie (1.13). Il suffit ensuite de remarquer que dans cette région, $1 + t + r \leq C(1 + |t - r|)$.
2. Si $t + |x| \geq 1$ et $t \leq |x|$, on applique cette fois l'inégalité de type Klainerman-Sobolev (1.14) (puis l'inégalité d'énergie (1.13)) à zf , avec $z \in \mathbf{k}_1$. Il convient ensuite de remarquer que

$$|x| \geq t \quad \Rightarrow \quad \frac{|x|}{2v^0} \leq \frac{|x|}{v^0(v^0 + |v|)} \leq |x| - t \frac{|v|}{v^0} \leq \left| x - t \frac{v}{v^0} \right| \leq \sum_{z \in \mathbf{k}_1} |z|$$

ainsi que $1 + t + r \leq C|x|$ dans cette région.

3. Enfin, pour la dernière région, on travaille sur des parties de \mathbb{R}^{3+1} composées d'une partie d'un hyperboloïde ainsi que d'une partie d'une hypersurface du type $t = \text{constant}$. On mélange ainsi les techniques usuellement utilisées pour de tels problèmes.

1.3 Étude des solutions à données petites pour des systèmes de Vlasov

Le système de Vlasov-Poisson

Le premier résultat fut obtenu par Bardos et Degond dans [2]. Sous une hypothèse de petitesse, ils ont prouvé que l'unique solution du système de Vlasov-Poisson est globale en temps et que

$$\int_{v \in \mathbb{R}^3} f dv \lesssim \frac{1}{(1+t)^3}.$$

Ils ont aussi pu estimer la vitesse de décroissance du champ électrique ainsi que de ses dérivées d'ordre 1 et 2 mais n'ont par contre pas obtenu d'informations sur les dérivées d'ordres supérieurs. Il a fallu attendre attendre les travaux [27] de Hwang, Rendall et Velázquez pour que les taux de décroissance optimaux

$$\left| \partial_{t,x}^\beta \int_{v \in \mathbb{R}^3} f dv \right| \lesssim \frac{1}{(1+t)^{3+|\beta|}}$$

soient obtenus. Notons que tant pour [2] que pour [27], le cœur de la démonstration consiste à contrôler le plus précisément possible les caractéristiques de l'équation de Vlasov.

Récemment, Smulevici a prouvé dans [46] un résultat similaire en utilisant des méthodes de champs de vecteurs. Comme les équations ne sont pas relativistes, les rotations hyperboliques (ou leur lift complet) ont été remplacées par $t\partial_i$ (ou $t\partial_i + \partial_{v^i}$). La méthode employée a notamment permis d'obtenir la décroissance en espace-temps, i.e.

$$\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad \left| \partial_{t,x}^\beta \int_{v \in \mathbb{R}^3} f(t, x, v) dv \right| \lesssim \frac{1}{(1+t+|x|)^{3+|\beta|}}.$$

Un élément crucial de la démonstration, propre à la dimension 3, fut de modifier légèrement les commutateurs $\widehat{\Gamma}$ de l'opérateur de transport classique $\partial_t + v^i \partial_i$ en raison de termes sources non intégrables. L'idée générale est de considérer des champs de vecteurs de la forme

$$\widehat{\Gamma} + \Phi^\nu \partial_\nu,$$

où les coefficients Φ^ν dépendent de la solution elle-même et sont définis par une équation de transport afin de compenser les pires termes d'erreurs dans les formules de commutations. Nous verrons ci-dessous qu'un procédé similaire sera utilisé pour étudier d'autres systèmes de Vlasov.

1.3.1 Système de Vlasov-Nordström

Dans leur article fondateur [18], David Fajman, Jérémie Joudioux et Jacques Smulevici ont utilisé la méthode des champs de vecteurs qu'ils ont développée afin d'étudier les solutions à données petites du système de Vlasov-Nordström dans le cas massif pour les dimensions $d \geq 4$ et dans le cas sans masse pour les dimensions $d \geq 3$. Ce système est donné par

$$\begin{aligned} v^\mu \partial_\mu f - (v^\mu v^i \partial_\mu \phi + \delta_1^m \partial^i \phi) \partial_{v^i} f &= 4\delta_1^m f v^\mu \partial_\mu \phi \\ \square \phi &= \delta_1^m \int_{v \in \mathbb{R}^3} f \frac{dv}{v^0}, \end{aligned}$$

où $m \in \{0, 1\}$ est la masse des particules et δ le symbole de Kronecker. On peut ainsi voir que le cas sans masse $m = 0$ est relativement simple à traiter étant donné que l'équation d'onde se résume à $\square \phi = 0$. Néanmoins, en dimension 3, exploiter la structure isotrope de l'équation de Vlasov s'est avérée être cruciale afin d'obtenir le taux de décroissance optimale de la densité de particules. Plus précisément, à l'image de $L\phi$ (voir (1.9)), $v^\mu \partial_\mu \phi$ décroît plus rapidement que prévu proche du cône de lumière. Pour le cas massif, l'étude est plus compliquée et est pour cela restreinte aux grandes dimensions $d \geq 4$. Bien que les solutions aient des taux de décroissance plus importants qu'en dimension 3, la méthode utilisée a tout de même nécessité d'exploiter une partie de la structure isotrope du système¹⁰. Remarquons également que Fajman-Joudioux-Smulevici utilisent un feuilletage hyperbolique pour étudier les solutions, ce qui, comme mentionné dans la sous-section 1.2.4, implique de supposer que les données initiales sont à support compact en espace. Les solutions sont donc étudiées sur les hyperboloïdes

$$\mathcal{H}_\rho := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 / \rho^2 = t^2 - |x|^2\}, \quad \rho \geq 1.$$

¹⁰En comparaison, nous n'avons pas besoin d'utiliser la condition isotrope pour prouver la proposition 1.8 en dimension 4.

Pour ce faire, ils supposent qu'une énergie bien choisie de (f, ϕ) est finie initialement, i.e. qu'une certaine norme L^1 à poids de f et une certaine norme L^2 à poids de ϕ sont finies sur \mathcal{H}_1 , et ils prouvent que cette énergie peut être propagée sur chaque hyperboloïde \mathcal{H}_ρ .

Un résultat similaire pour le système massif en tridimensionnel fut obtenu plus tard (voir [16]). Étant donné le faible taux de décroissance des solutions comparé aux grandes dimensions, une meilleure compréhension de la structure isotrope du système fut nécessaire. La différence majeure cependant fut de modifier les commutateurs de l'opérateur de transport relativiste \mathbf{T}_1 afin de compenser les pires termes d'erreurs dans les formules de commutations du champ de Vlasov (comme [46] l'a fait dans le cadre du système de Vlasov-Poisson). Ces champs de vecteurs sont définis à partir des éléments $\widehat{Z} \in \widehat{\mathbb{P}}_0$ et sont de la forme

$$Y = \widehat{Z} + \Phi_{\widehat{Z}}^i X_i, \quad \text{où}^{11} \quad X_i := \partial_i + \frac{v^i}{v^0} \partial_t$$

et où les coefficients $\Phi_{\widehat{Z}}^i$ dépendent de la solution (f, ϕ) du système et sont définis comme solutions d'équations de transport. Une des difficultés liées à l'utilisation des champs de vecteurs Y réside dans la réécriture de la formule de commutation pour l'équation d'onde. Schématiquement, on a

$$\square_Z \phi = \int_v \widehat{Z} f \frac{dv}{v^0} = \int_v (Y f - \Phi^i X_i(f)) \frac{dv}{v^0}.$$

Le problème causé par le terme $\Phi^i X_i(f)$ vient de la croissance des coefficients Φ , en $\sqrt{\epsilon \rho}$, où ρ désigne le temps hyperbolique. On transforme alors le terme source de la façon suivante,

$$\square_Z \phi = \int_v \widehat{Z} f \frac{dv}{v^0} = \int_v Y f \frac{dv}{v^0} - \int_v X_i(\Phi^i f) \frac{dv}{v^0} + \int_v X_i(\Phi^i) f \frac{dv}{v^0},$$

et deux éléments sont alors importants pour pouvoir boucler les estimations d'énergie.

- On a, toujours en désignant par ρ le temps hyperbolique, $|\partial_{t,x} \Phi^i| \sim \sqrt{\epsilon} \log(\rho)$, ce qui permet de bien mieux contrôler

$$\int_v X_i(\Phi^i) f \frac{dv}{v^0}$$

que

$$\int_v \Phi^i X_i(f) \frac{dv}{v^0}.$$

- Le champ de vecteur X_i permet sous certaines conditions de gagner en décroissance. Cela est lié à la relation

$$tX_i = t\partial_i + t\frac{v^i}{v^0}\partial_t = \Omega_{0i} + \left(t\frac{v^i}{v^0} - x^i\right)\partial_t, \quad (1.15)$$

où $t\frac{v^i}{v^0} - x^i \in \mathbf{k}_1$. Cela nous indique donc que l'on pourra mieux contrôler

$$\int_v X_i(\Phi^i f) \frac{dv}{v^0} \quad \text{que} \quad \int_v \Phi^i X_i(f) \frac{dv}{v^0}.$$

La formule (1.15) suggère pourquoi il a fallu considérer des modifications de la forme $\widehat{Z} + \Phi^i X_i$ et non simplement de la forme $\widehat{Z} + \Phi^\nu \partial_\nu$. Il fut également nécessaire d'appliquer (1.15) à ϕ afin de boucler les estimations d'énergie pour le champ de Vlasov f . Du fait de la présence des poids $z \in \mathbf{k}_1$, différentes hiérarchies dans les équations commutées ont ainsi dû être considérées, évoquant notamment [35] et la proposition 1.2.3.

1.3.2 Le système d'Einstein-Vlasov

L'objectif initial de Fajman-Joudioux-Smulevici lorsqu'ils ont commencé à développer une méthode de champ de vecteurs pour les équations de Vlasov était de prouver la stabilité de l'espace-temps de Minkowski pour le système d'Einstein-Vlasov, dont les équations de Vlasov-Nordström en sont un modèle simplifié. Leur résultat établi dans [17] constitue donc en quelque sorte l'aboutissement de leurs travaux et prouve que

¹¹En réalité les translations ∂_μ sont remplacées par $\partial_\mu - \partial_\mu \phi v^i \partial_{v^i}$ et les champs de vecteurs X_i sont modifiés en conséquence.

- la solution triviale du système d'Einstein-Vlasov est asymptotiquement stable. Plus précisément, étant donné des données initiales coïncidant avec les données initiales d'un espace-temps de Schwarzschild en dehors d'un compact et suffisamment proche de celles de l'espace-temps de Minkowski, alors l'unique solution (g, f) du système est globale en temps. De plus,
- le taux de décroissance du champ de Vlasov f et de ses dérivées est presque optimal. Il existe une petite constante $\delta > 0$ tel que pour tout (t, x) dans le futur de l'hyperboloïde \mathcal{H}_1 ,

$$\forall |\beta| \leq N - 3, \quad \int_{v \in \mathbb{R}^3} |\partial_{t,x}^\beta f| v^0 dv \lesssim \epsilon \frac{\rho^\delta}{(1+t)^3}, \quad \rho = \sqrt{t^2 - |x|^2}.$$

- La déviation de la métrique g de la métrique de l'espace-temps de Minkowski η vérifie, pour tout (t, x) dans le futur de l'hyperboloïde \mathcal{H}_1 ,

$$\forall |\beta| \leq N - 3, \quad |g - \eta|(t, x) \lesssim \sqrt{\epsilon} \frac{\rho^{D\epsilon^{\frac{1}{2}}}}{1+t}, \quad D > 0.$$

Notons que

- l'utilisation d'un feuilletage hyperbolique impose de supposer que les données initiales coïncident avec les données initiales d'un espace-temps de Schwarzschild en dehors d'un compact.
- Cela implique en particulier que le champ de Vlasov est initialement à support compact.

De nombreux ingrédients de la démonstration sont analogues à ceux utilisés dans [16] pour le système de Vlasov-Nordström. En particulier,

- les commutateurs de l'opérateur de transport relativiste \mathbf{T}_1 sont modifiés afin de compenser les pires termes sources dans l'équation de Vlasov commutée.
- Plusieurs hiérarchies entre différentes normes de f et g sont exploitées pour boucler les estimations d'énergie.
- Comprendre et exploiter la structure isotrope du système d'Einstein-Vlasov, plus complexe que celle des équations de Vlasov-Nordström, constitue un élément clé de la démonstration.

Un résultat similaire fut indépendamment obtenu par Lindblad et Taylor en utilisant aussi des méthodes de champs de vecteurs (voir [26]). Notons que

1. les commutateurs pour l'équation de Vlasov sont construits à partir des éléments de $\widehat{\mathbb{P}}_0$ d'une tout autre manière que dans [17].
2. Une des étapes de la démonstration consiste à contrôler les dérivées des caractéristiques de l'équation de Vlasov.
3. La densité de particules est initialement à support compact en espace et en vitesse.

Enfin, mentionnons aussi le travail de Taylor [49] qui a prouvé la stabilité de l'espace-temps de Minkowski pour le système d'Einstein-Vlasov sans masse sous une contrainte de support compact en espace et en vitesse pour la densité de particules.

1.4 Derniers prérequis et présentation de la structure isotrope du système de Vlasov-Maxwell

1.4.1 Quelques notations

On ordonne les ensemble \mathbb{K} et $\widehat{\mathbb{P}}_0$ de sorte que

$$\mathbb{K} = \{Z^1, \dots, Z^{|\mathbb{K}|} = S\} \quad \text{et} \quad \widehat{\mathbb{P}}_0 = \{\widehat{Z}^1, \dots, \widehat{Z}^{|\mathbb{K}|-1}, Z^{|\widehat{\mathbb{P}}_0|} = S\}.$$

Étant donné un multi-indice $\beta \in \llbracket 1, |\mathbb{K}| \rrbracket^p$ de longueur $p \in \mathbb{N}$, on définit Z^β et \widehat{Z}^β de la façon suivante :

$$Z^\beta := Z^{\beta_1} \dots Z^{\beta_p} \quad \text{et} \quad \widehat{Z}^\beta := \widehat{Z}^{\beta_1} \dots \widehat{Z}^{\beta_p}.$$

Soit $(v^L, v^{\underline{L}}, v^{e_1}, v^{e_2})$ les composantes isotropes du vecteur vitesse v . Elles vérifient

$$v = v^L L + v^{\underline{L}} \underline{L} + v^A e_A, \quad \text{où } v^A = v^{e_A},$$

et on a en particulier

$$v^L = \frac{1}{2} \left(v^0 + \frac{x_i}{r} v^i \right), \quad v^{\underline{L}} = \frac{1}{2} \left(v^0 - \frac{x_i}{r} v^i \right).$$

Selon le contexte (étude de particules massives ou sans masse), nous utiliserons la notation \mathbb{R}_v^3 pour dénoter \mathbb{R}^3 ou $\mathbb{R}^3 \setminus \{0\}$. Enfin, pour $(t, r) \in \mathbb{R}_+ \times \mathbb{R}_+^*$, nous désignerons l'hypersurface $t \times \mathbb{R}^3$ par Σ_t .

1.4.2 Aspect général des normes utilisées

Dans la suite nous allons propager des normes L^1 à poids sur la densité de particules et ses dérivées ainsi que des normes L^2 à poids sur le champ électromagnétique et ses dérivées. Pour f ,

- les dérivées seront toutes de la forme $\widehat{Z}^\beta f$, où $\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|}$, sauf pour le résultat de la sous-section 1.5.3 où on modifiera les commutateurs de l'opérateur \mathbf{T}_1 .
- Les poids proviendront soit de \widehat{Z}^β ou seront de la forme $|z|^a$, où $a \geq 0$ et $z \in \mathbf{k}_1$ ou $z \in \mathbf{k}_0$.
- Un exemple simple d'une telle norme est

$$\sum_{0 \leq k \leq 10} \sum_{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^k} \sum_{z \in \mathbf{k}_1} \int_{\Sigma_t} \int_{\mathbb{R}_v^3} |z \widehat{Z}^\beta f| dv dx.$$

Pour F ,

- les dérivées seront de la forme $\mathcal{L}_{Z^\gamma}(F)$, avec $Z^\gamma \in \mathbb{K}^{|\gamma|}$,

$$\mathcal{L}_{Z^\gamma} := \mathcal{L}_{Z^{\gamma_1}} \dots \mathcal{L}_{Z^{\gamma_{|\gamma|}}}$$

et où $\mathcal{L}_{Z^{\gamma_i}}$ désigne la dérivée de Lie selon le champ de vecteur Z^{γ_i} .

- Les poids proviendront soit de Z^γ ou seront des puissances de

$$\tau_+ := \sqrt{1 + (t+r)^2} \sim 1 + t + r \quad \text{ou} \quad \tau_- := \sqrt{1 + (t-r)^2} \sim 1 + |t-r|.$$

- Par exemple, nous utiliserons régulièrement une norme similaire à

$$\sum_{0 \leq k \leq 8} \sum_{\widehat{Z}^\gamma \in \widehat{\mathbb{P}}_0^k} \int_{\Sigma_t} \tau_+^2 |\alpha(\mathcal{L}_{Z^\gamma}(F))|^2 + \tau_-^2 |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|^2 + (\tau_+^2 + \tau_-^2) (|\rho(\mathcal{L}_{Z^\gamma}(F))|^2 + |\sigma(\mathcal{L}_{Z^\gamma}(F))|^2) dx.$$

1.4.3 Commutation des équations et conservation de la structure

Afin d'utiliser des méthodes de champs de vecteurs, nous serons amenés à commuter les équations (1.2)-(1.4) par les éléments de \mathbb{K} ou $\widehat{\mathbb{P}}_0$. Fixons pour le reste de cette section un champ de vecteur¹² $Z \in \mathbb{K} \setminus \{S\}$ et désignons par \widehat{Z} son lift complet. Bien que \widehat{Z} commute avec \mathbf{T}_1 ou \mathbf{T}_0 , nous allons voir que la situation est légèrement différente pour l'opérateur T_F . Si (f, F) est une solution du système de Vlasov-Maxwell (1.2)-(1.4), alors

$$T_F(\widehat{Z}f) = -\mathcal{L}_Z(F)(v, \nabla_v f) \tag{1.16}$$

$$\nabla^\mu \mathcal{L}_Z(F)_{\mu\nu} = \int_{\mathbb{R}_v^3} \frac{v_\nu}{v^0} \widehat{Z}f dv \tag{1.17}$$

$$\nabla^{\mu*} \mathcal{L}_Z(F)_{\mu\nu} = 0. \tag{1.18}$$

On voit donc que

- l'équation de transport commutée (1.16) comporte un terme d'erreur. Cependant, ce dernier a la même structure que la non linéarité $F(v, \nabla_v f)$ présente dans $T_F(f) = 0$.

¹²Le cas du champ de vecteur S est légèrement différent mais se traite de manière similaire et sans aucune difficulté supplémentaire.

- Le champ $\mathcal{L}_Z(F)$ est solution des équations de Maxwell, dont le terme source est donné par le courant

$$\int_{\mathbb{R}_v^3} \frac{v_\nu}{v^0} \widehat{Z} f dv.$$

Notons que le terme source

$$\int_{\mathbb{R}_v^3} \frac{v_\nu}{v^0} f dv,$$

associé au champ électromagnétique F a une forme analogue.

Au vu de ces observations, on peut ainsi dire que la structure du système de Vlasov-Maxwell est conservée par commutation. Il semble donc pertinent de s'y intéresser de plus près.

1.4.4 La charge totale du plasma

La densité de charge du plasma étant donnée par $\int_{\mathbb{R}_v^3} f dv$, sa charge totale, qui est une quantité conservée au cours du temps, est donnée par

$$\int_{\mathbb{R}_x^3} \int_{\mathbb{R}_v^3} f dv dx.$$

Le théorème de la divergence nous donne alors, pour $\mathbb{S}_{t,r}^2$ la sphère de centre 0 et de rayon r sur Σ_t ,

$$-\int_{\mathbb{S}_{t,r}^2} \rho(G) d\mathbb{S}_{t,r}^2 = \int_{\mathbb{S}_{t,r}^2} \frac{x^i}{r} F_{0i} d\mathbb{S}_{t,r}^2 = \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_v^3} f(t, x, v) dv dx = \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_v^3} f(0, x, v) dv dx.$$

On définit ainsi plus généralement la charge Q_G d'une 2-forme G par

$$Q_G(t) := -\int_{\mathbb{S}_{t,r}^2} \rho(G) d\mathbb{S}_{t,r}^2.$$

Dans le cas où $Q_G(0) \neq 0$, $G(0, \cdot)$ ne peut décroître plus vite que $(1+r)^{-2}$ et l'énergie

$$\int_{\mathbb{R}^3} (1+r) |G(0, x)|^2 dx$$

est infinie. Dans le contexte du système de Vlasov-Maxwell, une charge totale non nulle nous empêcherait de propager certaines normes à poids du champ électromagnétique et notamment celle permettant d'appliquer le théorème 1.2.5. Ce problème peut être évité en supposant le plasma électriquement neutre ou si la propagation de normes L^2 à faibles poids suffisent à établir l'existence globale des solutions. Toutefois, dans le résultat majeur de cette thèse, de telles conditions ne sont pas réunies. Le principe consiste alors à décomposer le champ électromagnétique en deux parties $\overline{F} + \widetilde{F}$, avec

$$\overline{F}(t, x) := \chi(t-r) \frac{Q_F}{4\pi r^2} \frac{x_i}{r} dx^i \wedge dt, \quad \widetilde{F} := F - \overline{F}$$

et χ une fonction vérifiant

$$\forall s \leq -2, \quad \chi(s) = 1 \quad \text{and} \quad \forall s \geq -1, \quad \chi(s) = 0.$$

On remarque alors que la charge de \widetilde{F} est nulle, i.e.

$$\forall t, \quad Q_{\widetilde{F}}(t) = 0 \quad \text{et} \quad Q_{\overline{F}}(t) = Q_F.$$

Il est par conséquent cohérent de supposer que

$$\int_{\mathbb{R}^3} (1+r) |\widetilde{F}(0, x)|^2 dx < +\infty,$$

ce qui nous permettra de propager des normes L^2 sur \widetilde{F} avec des poids en $t+r$ et en $t-r$, fournissant ainsi de forts taux de décroissance sur les composantes isotropes de \widetilde{F} . Bien que l'on ne puisse pas appliquer ce raisonnement à \overline{F} , on utilise sa forme explicite pour obtenir son comportement asymptotique et ainsi en déduire celui de F .

1.4.5 Contrôler les grandes vitesses en exploitant la structure isotrope du système

Nous considérons ici uniquement des particules massives, i.e. $v^0 = \sqrt{1 + |v|^2}$, bien que le cas sans masse présente bien des similarités. Les démonstrations de nos résultats sont toutes basées sur le principe de continuité, décrit plus en détail dans la sous-section 1.5.1, et consiste à améliorer des inégalités d'énergie. Or, en utilisant (1.16), on a la loi de conservation approchée suivante

$$\left\| \widehat{Z}f \right\|_{L_v^1 L^1(\Sigma_t)} \lesssim \left\| \widehat{Z}f \right\|_{L_v^1 L^1(\Sigma_0)} + \int_0^t \int_{\Sigma_s} \int_{\mathbb{R}_v^3} |\mathcal{L}_Z(F)(v, \nabla_v f)| \frac{dv}{v^0} dx ds. \quad (1.19)$$

Le but sera alors de montrer que l'intégrale dans le terme de droite de (1.19) peut être bornée indépendamment du temps t par une constante suffisamment petite.

Étude d'un problème plus simple

Afin de nous focaliser ici sur les difficultés causées par les particules ayant une grande vitesse, nous allons considérer $g : \mathbb{R}_+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ et G une 2-forme, toutes deux suffisamment régulières, vérifiant

$$\begin{aligned} v^\mu \partial_\mu (g) &= \mathbf{T}_1(g) = 0 \\ \nabla^\mu G_{\mu\nu} &= 0 \\ \nabla^\mu *G_{\mu\nu} &= 0 \end{aligned}$$

et montrons que

$$I := \int_{t=0}^{+\infty} \int_{\Sigma_t} \int_{\mathbb{R}_v^3} |G(v, \nabla_v g)| \frac{dv}{v^0} dx dt < +\infty. \quad (1.20)$$

Autrement dit, on considère une situation considérablement simplifiée puisque g et G sont complètement indépendantes mais elle nous permettra de cerner les difficultés évoquées. Par suffisamment régulière, nous entendons ici que l'on peut appliquer le théorème 1.2.5 à G , i.e.

$$\begin{aligned} |\underline{\alpha}(G)_i|(t, x) &:= |G_{e_i \underline{L}}|(t, x) \lesssim \frac{1}{(1+t+r)(1+|t-r|)^{\frac{3}{2}}} \\ |\alpha(G)_i|(t, x) &:= |G_{e_i L}|(t, x) \lesssim \frac{1}{(1+t+r)^{\frac{5}{2}}} \\ |\rho(G)|(t, x) &:= \left| \frac{1}{2} G_{LL} \right|(t, x) \lesssim \frac{1}{(1+t+r)^2 (1+|t-r|)^{\frac{1}{2}}} \\ |\sigma(G)|(t, x) &:= |G_{e_1 e_2}|(t, x) \lesssim \frac{1}{(1+t+r)^2 (1+|t-r|)^{\frac{1}{2}}} \end{aligned}$$

et que la décroissance initiale de g assure que

$$\|g\|_{L_v^1 L^1(\Sigma_t)} + \sum_{\widehat{\Gamma} \in \widehat{\mathbb{P}}_0} \|\widehat{\Gamma}g\|_{L_v^1 L^1(\Sigma_t)} = \|g\|_{L_v^1 L^1(\Sigma_0)} + \sum_{\widehat{\Gamma} \in \widehat{\mathbb{P}}_0} \|\widehat{\Gamma}g\|_{L_v^1 L^1(\Sigma_0)} < +\infty.$$

Remarque 1.4.1. *En fait, le raisonnement qui suit permet de borner*

$$\int_{t=0}^{+\infty} \int_{\Sigma_t} \int_{\mathbb{R}_v^3} |G(v, \nabla_v g)| dv dx dt.$$

Néanmoins, la présence du facteur $\frac{1}{v^0}$ dans I s'avèrera être d'une importance capitale car les solutions du système de Vlasov-Maxwell que l'on considèrera ne se comporteront pas toujours aussi bien que g et G .

Majoration naïve et étude du cas où les données initiales sont à support compact

Commençons dans un premier temps par majorer $G(v, \nabla_v g)$ sans tenir compte de sa structure. Remarquons que ∂_{v^i} ne fait pas partie des commutateurs de l'opérateur \mathbf{T}_1 et que

$$\partial_{v^i} g = \frac{1}{v^0} \widehat{\Omega}_{0i} g - \frac{t}{v^0} \partial_i - \frac{x^i}{v^0} \partial_t, \quad \text{d'où} \quad \partial_{v^i} g \sim \frac{t+r}{v^0} \partial_{t,x} g.$$

On voit ainsi que les dérivées en vitesse se comportent mal car elles causent une perte de l'ordre de $t + r$. Par conséquent, comme

$$|G|(t, x) \lesssim \frac{1}{(1 + t + r)(1 + |t - r|)^{\frac{3}{2}}},$$

on a

$$|G(v, \nabla_v g)| \sim (t + r)|G||\partial_{t,x}g| \lesssim \frac{1}{(1 + |t - r|)^{\frac{3}{2}}}|\partial_{t,x}g|. \quad (1.21)$$

Le problème ici est que

$$\|(1 + |t - r|)^{-\frac{3}{2}}\|_{L^\infty(\Sigma_t)} = 1,$$

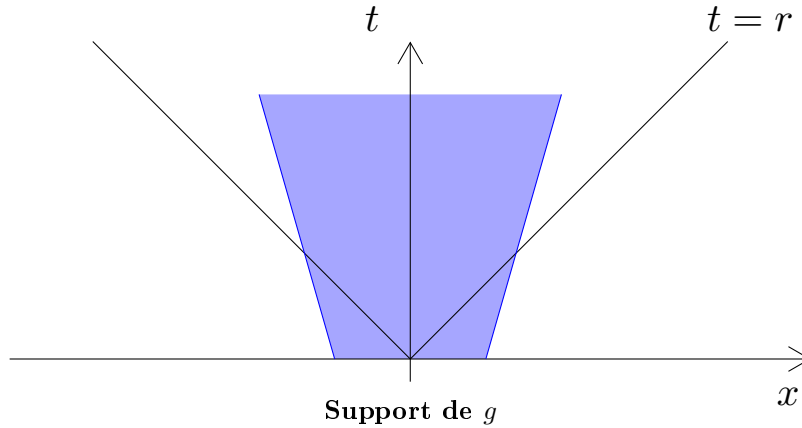
et donc que (1.21) ne nous permet pas de prouver (1.20) :

$$I \lesssim \int_{t=0}^{+\infty} \|(1 + |t - r|)^{-\frac{3}{2}}\|_{L^\infty(\Sigma_t)} \left\| \int_{\mathbb{R}_v^3} |\partial_{t,x}g| dv \right\|_{L^1(\Sigma_t)} ds = \int_{t=0}^{+\infty} \left\| \int_{\mathbb{R}_v^3} |\partial_{t,x}g| dv \right\|_{L^1(\Sigma_0)} ds = +\infty.$$

Idéalement, il faudrait être en mesure de transformer la décroissance en $t - r$ dans (1.21) en une décroissance en t . Essayons maintenant de comprendre comment s'en sont sortis Glassey-Strauss dans [24]. Bien que les méthodes qu'ils ont employées soient différentes des nôtres, notre exemple permet tout de même de voir en quoi avoir des solutions à support compact apporte des simplifications notables. En utilisant la méthode des caractéristiques, on a

$$\forall (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3, \quad g(t, x, v) = g\left(0, x - \frac{v}{v_0}t, v\right)$$

et donc si g est initialement à support compact en (x, v) , son support spatial sera de la forme suivante :



En particulier, la région où g est non nulle est située loin du cône de lumière, ce qui nous donne

$$\frac{1}{(1 + |t - r|)^{\frac{3}{2}}}|\partial_{t,x}g| \lesssim \frac{1}{(1 + t)^{\frac{3}{2}}}|\partial_{t,x}g| \quad (1.22)$$

et nous permet alors de prouver que I est une intégrale convergente.

Nécessité d'exploiter la structure isotrope

Lorsque les données initiales ne sont pas à support compact¹³, les particules peuvent avoir une vitesse arbitrairement proche de la vitesse de la lumière et (1.22) n'est plus vérifiée. Nous allons donc utiliser la structure isotrope de $G(v, \nabla_v g)$ afin d'améliorer la majoration (1.21). Cela consiste à

1. transformer de la décroissance en $t - r$ en de la décroissance en $t + r$.
2. Transformer une perte en $t + r$ en une perte en $t - r$.
3. Tirer profit de la décroissance en $t - r$.

¹³Rappelons qu'un de nos objectifs principaux consiste à éliminer toute restriction liée au support des données initiales.

Au vu des taux de décroissance des bonnes composantes isotropes du champ électromagnétique G , développer $G(v, \nabla_v g)$ dans une base isotrope semble être un bon point de départ. Cela nous amène ainsi à étudier les composantes isotropes de $\nabla_v g$ et du vecteur vitesse v . Remarquons tout d'abord que

$$(\nabla_v g)^L = -(\nabla_v g)^{\underline{L}} = \frac{x^i}{r} \partial_{v^i} g = \frac{x^i}{v^0 r} \widehat{\Omega}_{0i} g - \frac{x^i}{v^0 r} x_i \partial_t g + \frac{x^i}{v^0 r} t \partial_i g = \frac{x^i}{v^0 r} \widehat{\Omega}_{0i} g - \frac{1}{v^0} S g + \frac{t-r}{v^0} \underline{L} g,$$

d'où

$$(\nabla_v g)^L = -(\nabla_v g)^{\underline{L}} \sim \frac{|t-r|}{v^0} |\partial_{t,x} g|. \quad (1.23)$$

En exprimant $G(v, \nabla_v g)$ dans une base isotrope¹⁴, nous sommes amenés à borner les termes suivants,

$$\bullet \quad v^L \rho(G) (\nabla_v g)^{\underline{L}}, \quad v^{\underline{L}} \rho(G) (\nabla_v g)^L \quad (1.24)$$

$$\bullet \quad v^L \alpha_A (\nabla_v g)^A, \quad v^A \alpha_A (\nabla_v g)^L \quad (1.25)$$

$$\bullet \quad v^B \varepsilon_{BA} \sigma (\nabla_v g)^A, \quad v^{\underline{L}} \underline{\alpha}_A (\nabla_v g)^A, \quad v^A \underline{\alpha}_A (\nabla_v g)^{\underline{L}}. \quad (1.26)$$

Nous nous contenterons de majorer un seul terme pour chacun des groupes (1.24), (1.25) et (1.26) (les autres pouvant être traités de manière analogue). En utilisant que

$$|\rho(G)|(t, x) \lesssim (1+t+r)^{-2} (1+|t-r|)^{-\frac{1}{2}}$$

et (1.23), on obtient

$$\left| v^L \rho(G) (\nabla_v g)^{\underline{L}} \right| \lesssim v^L \frac{1}{(1+t+r)^2 (1+|t-r|)^{-\frac{1}{2}}} \frac{|t-r|}{v^0} |\partial_{t,x} g| \lesssim \frac{1}{(1+t+r)^{\frac{3}{2}}} |\partial_{t,x} g|. \quad (1.27)$$

En utilisant cette fois le fort taux de décroissance de $|\alpha(G)|(t, x) \lesssim (1+t+r)^{-\frac{5}{2}}$, il vient

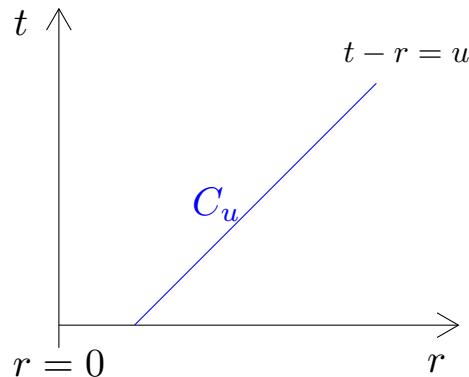
$$\left| v^L \alpha_A (\nabla_v g)^A \right| \lesssim v^L \frac{1}{(1+t+r)^{-\frac{5}{2}}} \frac{t+r}{v^0} |\partial_{t,x} g| \lesssim \frac{1}{(1+t+r)^{\frac{3}{2}}} |\partial_{t,x} g|. \quad (1.28)$$

On peut donc observer que pour les termes des groupes (1.24) et (1.25), le taux de décroissance obtenu est bien meilleur au niveau du cône de lumière que celui obtenu par une majoration naïve, à savoir (1.21). Concentrons-nous dorénavant sur le dernier groupe de termes (1.26). On a¹⁵

$$\left| v^{\underline{L}} \underline{\alpha}_A (\nabla_v g)^A \right| \lesssim v^{\underline{L}} \frac{1}{(1+t+r)(1+|t-r|)^{-\frac{3}{2}}} \frac{t+r}{v^0} |\partial_{t,x} g| \lesssim \frac{1}{(1+|t-r|)^{\frac{3}{2}}} \frac{v^{\underline{L}}}{v^0} |\partial_{t,x} g|,$$

étant donné que les composantes angulaires de $\nabla_v g$ n'ont pas un meilleur comportement que $\partial_{v^i} g$ et que $\underline{\alpha}(G)$ est la pire composante isotrope du champ électromagnétique G . Nous ne sommes cette fois pas en mesure d'obtenir un taux de décroissance en $(1+t)^{-\frac{3}{2}}$, ce qui nous empêche pour le moment de prouver que $I < +\infty$. La dernière option restante est donc de réussir à exploiter la décroissance en $t-r$ et nous allons pour cela effectuer un changement de variables et tirer profit de la composante $v^{\underline{L}}$ du vecteur vitesse. Introduisons les cônes suivants, pour $u \in \mathbb{R}$,

$$C_u := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 / t - |x| = u\}.$$



¹⁴Dans (1.2), la non linéarité $F(v, \nabla_v f) = v^\mu F_\mu^j \partial_{v^j} f$ est développée en coordonnées cartésiennes.

¹⁵La majoration des autres termes s'effectue légèrement différemment. On peut notamment utiliser que $|v^A| \lesssim \sqrt{v^0 v^{\underline{L}}}$.

La propriété qui nous sera utile est que $t - r$ est constant sur chacun de ces cônes. Le changement de variables $(\underline{u}, u) = (t + r, t - r)$ donne, avec $dC_u = r^2 dS^2 du$,

$$\begin{aligned}
\int_{t=0}^{+\infty} \int_{\Sigma_t} \int_{\mathbb{R}^3} \frac{1}{(1 + |t - r|)^{\frac{3}{2}}} \frac{v^L}{v^0} |\partial_{t,x} g| \frac{dv}{v^0} dx ds &= \frac{1}{2} \int_{u=-\infty}^{+\infty} \int_{C_u} \int_{\mathbb{R}^3} \frac{1}{(1 + |u|)^{\frac{3}{2}}} \frac{v^L}{v^0} |\partial_{t,x} g| \frac{dv}{v^0} dC_u du \\
&\leq \int_{u=-\infty}^{+\infty} \frac{1}{(1 + |u|)^{\frac{3}{2}}} \int_{C_u} \int_{\mathbb{R}^3} \frac{v^L}{v^0} |\partial_{t,x} g| \frac{dv}{v^0} dC_u du \\
&\leq \int_{u=-\infty}^{+\infty} \frac{du}{(1 + |u|)^{\frac{3}{2}}} \sup_{u \in \mathbb{R}} \int_{C_u} \int_{\mathbb{R}^3} \frac{v^L}{v^0} |\partial_{t,x} g| \frac{dv}{v^0} dC_u \\
&\leq \sup_{u \in \mathbb{R}} \int_{C_u} \int_{\mathbb{R}^3} \frac{v^L}{v^0} |\partial_{t,x} g| \frac{dv}{v^0} dC_u. \tag{1.29}
\end{aligned}$$

La dernière étape consiste donc à borner (1.29) et c'est ici que la présence de la composante v^L s'avère être cruciale. En effet, le théorème de la divergence, appliqué à $\int_{\mathbb{R}^3} |f| \frac{v^\mu}{v^0} dv$ dans le domaine

$$\{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 / t - |x| \leq u\}, \quad u \in \mathbb{R},$$

nous permet d'obtenir, comme $\mathbf{T}_1(\partial_{t,x} g) = 0$,

$$\begin{aligned}
\int_{C_u} \int_{\mathbb{R}^3} \frac{v^L}{v^0} |\partial_{t,x} g| dv dC_u &\leq \int_{|x| \geq -u} \int_{\mathbb{R}^3} |\partial_{t,x} g|(0, x, v) dv dx - \int_{|x| \geq t-u} \int_{\mathbb{R}^3} |\partial_{t,x} g|(t, x, v) dv dx \\
&\leq \left\| \int_{\mathbb{R}^3} |\partial_{t,x} g| dv \right\|_{L^1(\Sigma_0)}.
\end{aligned}$$

En combinant cela avec (1.27) et (1.28), on peut finalement prouver (1.20), i.e. $I < +\infty$.

Plus généralement (et notamment dans le cadre du système de Vlasov-Maxwell), lorsque les supports des solutions ne sont pas compacts, la structure isotrope est ce qui permet de prouver des bornes uniformes en temps sur certaines normes des solutions.

1.5 Résultats : système de Vlasov-Maxwell massif

Dans toute la suite, nous dirons que (f_0, F_0) constitue des données initiales pour le système de Vlasov-Maxwell si $f_0 : \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbf{T}$ et F_0 sont suffisamment régulières et vérifient les équations de contraintes

$$\nabla^i (F_0)_{i0} = - \int_{\mathbb{R}^3} f_0 dv \quad \text{et} \quad \nabla^i (*F_0)_{i0} = 0.$$

1.5.1 Le cas des grandes dimensions

Dû à des effets dispersifs, le taux de décroissance des solutions de l'équation des ondes et de l'équation de transport relativiste est d'autant plus important que la dimension de l'espace ambiant est grande. C'est pourquoi nous nous sommes d'abord intéressés aux solutions à données petites du système de Vlasov-Maxwell en dimension $n \geq 4$.

Remarque 1.5.1. *Pour des raisons techniques mentionnées ci-dessous, nous considérerons le système de Vlasov-Maxwell dans la jauge de Lorentz. Plus précisément, nous introduirons une 1-forme A bien choisie et vérifiant*

$$dA = F, \quad \text{i.e.} \quad \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}, \quad \text{et} \quad \partial^\mu A_\mu = 0.$$

Notre résultat obtenu dans [4] et présenté dans le chapitre 2 de cette thèse s'énonce ainsi.

Theorem 1.5.1. *Considérons $n \geq 4$ et $N \geq \frac{5}{2}n + 1$. Soit (f_0, F_0) des données initiales pour le système de Vlasov-Maxwell massif et (f, F) l'unique solution classique du système vérifiant $(f, F)(t=0) = (f_0, F_0)$. Soit également A un potentiel satisfaisant la jauge de Lorenz. Il existe $\epsilon > 0$ tel que, si¹⁶*

$$\tilde{\mathcal{E}}_N[A](0) \leq \epsilon, \quad \mathcal{E}_N[F](0) \leq \epsilon, \quad \mathbb{E}_{N+n,1}^2[f](0) \leq \epsilon,$$

alors (f, F) est une solution globale en temps et vérifie les propriétés suivantes.

¹⁶Les normes considérées ici sont analogues à celles présentées dans la sous-section 1.4.2. De plus, une hypothèse de petitesse sur F assure que $\tilde{\mathcal{E}}_N[A](0) \leq \epsilon$ (voir [4] pour plus de détails).

- *Propagation de normes de F et f : $\forall t \in \mathbb{R}_+$,*

$$\mathcal{E}_N[F](t) \lesssim \epsilon \log^3(3+t), \quad \mathbb{E}_N^2[f](t) \lesssim \epsilon \quad \text{et} \quad \mathbb{E}_{N,1}^2[f](t) \lesssim \epsilon \log^{\frac{1}{2}}(3+t).$$

- *Décroissance ponctuelle pour les composantes isotropes de $\mathcal{L}_{Z^\beta}(F)$: $\forall |\beta| \leq N-n, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$,*

$$\begin{aligned} |\alpha(L_{Z^\beta}F)| &\lesssim \sqrt{\epsilon} \log^2(3+t) \tau_+^{-\frac{n+2}{2}}, & |\underline{\alpha}(L_{Z^\beta}F)| &\lesssim \sqrt{\epsilon} \log^2(3+t) \tau_+^{-\frac{n-1}{2}} \tau_-^{-\frac{3}{2}}, \\ |\rho(L_{Z^\beta}F)| &\lesssim \sqrt{\epsilon} \log^2(3+t) \tau_+^{-\frac{n+1}{2}} \tau_-^{-\frac{1}{2}}, & |\sigma(L_{Z^\beta}F)| &\lesssim \sqrt{\epsilon} \log^2(3+t) \tau_+^{-\frac{n+1}{2}} \tau_-^{-\frac{1}{2}}. \end{aligned}$$

- *Décroissance ponctuelle pour $\int_{v \in \mathbb{R}^n} |\widehat{Z}^\beta f| dv$:*

$$\forall |\beta| \leq N - \frac{3n+2}{2}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad \int_{v \in \mathbb{R}^n} |\widehat{Z}^\beta f|(t, x, v) dv \lesssim \frac{\epsilon}{\tau_+}.$$

Décrivons maintenant le principe de la démonstration ainsi que les arguments clés. Comme pour de nombreux résultats de ce type, la démonstration est basée sur le principe de continuité. Par des arguments classiques, on peut montrer que le problème est bien posé et qu'il existe une unique solution maximale définie sur $[0, T^0[$, avec $T^0 > 0$. De plus, il existe une certaine constante $C > 0$ et un temps maximal $T^* > 0$ tel que pour tout $t \in [0, T^*[$,

$$\mathcal{E}_N[F](t) \leq C\epsilon \log^3(3+t)(t), \quad \mathbb{E}_N^2[f](t) \leq C\epsilon \quad \text{et} \quad \mathbb{E}_{N,1}^2[f](t) \leq C\epsilon \log^{\frac{1}{2}}(3+t).$$

Le but est alors de montrer qu'on a en fait, pour ϵ assez petit et C suffisamment grande,

$$\mathcal{E}_N[F](t) \leq \frac{C}{2}\epsilon\chi(t), \quad \mathbb{E}_N^2[f](t) \leq \frac{C}{2}\epsilon \quad \text{et} \quad \mathbb{E}_{N,1}^2[f](t) \leq \frac{C}{2}\epsilon\chi^{\frac{1}{6}}(t),$$

pour tout $t \in [0, T^*[$. Par un argument de connexité, on montre qu'on peut alors prendre $T^* = T^0$ et l'existence globale, i.e. $T^0 = +\infty$, découle alors d'un critère de non explosion. On comprend donc que le cœur de la démonstration consiste à établir des inégalités d'énergies, tant pour la densité de particules que pour le champ électromagnétique. Les trois difficultés majeures rencontrées sont les suivantes.

1. *Traiter les grandes vitesses.* Cet obstacle apparaît essentiellement lorsqu'on étudie l'équation de Vlasov et est traité en exploitant la structure isotrope du système. Le principe a déjà été donné plus haut dans la sous-section 1.4.5. De plus, le fort taux de décroissance des solutions en dimension $n \geq 4$ nous évite d'avoir à exploiter toute la structure du système.
2. *Obtenir le taux de décroissance optimal de $\int_v |f| dv$.* Pour cela, nous appliquons le théorème 1.2.2.
3. *Les équations de Maxwell ne sont pas conformément invariantes en dimension $n \neq 3$.* Dans [11], une norme L^2 à poids est propagée sur le champ électromagnétique en contractant son tenseur d'énergie impulsion

$$T[F]_{\mu\nu} := F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}\eta_{\mu\nu}F_{\lambda\xi}F^{\lambda\xi}$$

par le multiplicateur de Morawetz

$$\overline{K} := \frac{(t+r)^2}{2}L + \frac{(t-r)^2}{2}\underline{L},$$

qui est un champ de vecteur conforme Killing de l'espace-temps de Minkowski. Si F est solution des équations de Maxwell dans le vide, on a

$$\nabla^\mu \left(T[F]_{\mu\nu} \overline{K}^\nu \right) = \frac{3-n}{2} t F_{\lambda\xi} F^{\lambda\xi} \quad (1.30)$$

et on voit que la dimension 3 est un cas particulier permettant de propager aisément la norme

$$\int_{\mathbb{R}^3} T[F]_{0\nu} \overline{K}^\nu + T[F]_{00} dx \geq \int_{\mathbb{R}^3} (1+t+r)^2 (|\alpha(F)|^2 + |\rho(F)|^2 + |\sigma(F)|^2) + (1+|t-r|)^2 |\underline{\alpha}(F)|^2 dx. \quad (1.31)$$

Afin de contourner cette difficulté en dimension $n \geq 4$, nous étudions le système de Vlasov-Maxwell dans la jauge de Lorenz. En considérant

$$P_\mu := T[F]_{\mu\nu} \overline{K}^\nu - (n-3) \left(t A_\beta \partial_\mu A^\beta - \frac{1}{2} \partial_\mu(t) A_\beta A^\beta - t A_\beta \partial^\beta A_\mu - A_0 A_\mu \right),$$

on a (rappelons que pour simplifier, F est ici solution des équations de Maxwell dans le vide),

$$\nabla^\mu P_\mu = (n-3)tA_\lambda \square A^\lambda = 0.$$

En appliquant le théorème de la divergence puis en effectuant quelques manipulations algébriques sur P_0 , nous pouvons alors contrôler la norme (1.31) au cours du temps.

Remarque 1.5.2. *Ce procédé n'est pas sans rappeler celui employé afin d'utiliser le champ de vecteur de Morawetz comme multiplicateur pour l'étude de solutions de l'équation des ondes. Le potentiel A permet ici de compenser le mauvais terme obtenu dans (1.30), au prix d'un terme ayant un bon comportement, $(n-3)tA_\lambda \square A^\lambda$.*

Remarquons que, comme [18] pour le système de Vlasov-Nordström,

- nous avons eu besoin d'utiliser (une partie de) la structure isotrope des équations.
- Le fort taux de décroissance des solutions nous a permis d'éviter de modifier les commutateurs de l'opérateur \mathbf{T}_1 .

1.5.2 Étude des solutions à l'extérieur d'un cône de lumière en dimension 3

Du fait que l'extérieur d'un cône $V_b := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 / r > t - b\}$, pour disons $b \leq -1$, soit globalement hyperbolique nous permet d'étudier les solutions du système de Vlasov-Maxwell dans une telle région de l'espace sans nous soucier de ce qu'il se passe dans la partie intérieure. Les motivations sont les suivantes.

- Comparé à notre résultat ci-dessous concernant l'espace entier, les hypothèses sur la décroissance initiale du champ électromagnétique sont bien plus faibles. Plus précisément, on impose que

$$\int_{\mathbb{R}^3} |F|^2(0, x) dx$$

soit petit alors que dans le théorème 1.5.3 nous aurons besoin de propager des normes nécessitant que

$$\int_{\mathbb{R}^3} (1+r)|F|^2(0, x) dx \leq \epsilon.$$

Notons en particulier que cela nous évite certaines difficultés apparaissant lorsque la charge totale du plasma est non nulle (voir la sous-section 1.4.4).

- La démonstration est bien plus simple que celle du théorème 1.5.3 traitant le même problème dans l'espace-temps entier. Soulignons notamment qu'aucune modification des commutateurs de \mathbf{T}_1 , i.e. des éléments de $\widehat{\mathbb{P}}_0$, n'est nécessaire.
- Cette approche permet de compléter une étude des solutions basée sur un feuilletage hyperbolique. En particulier, adapter ce résultat au système de Vlasov-Nordström et le combiner avec [16] permettrait d'éviter toute hypothèse de support compact sur les données initiales et d'obtenir le comportement asymptotique des solutions dans tout l'espace-temps.

Le chapitre 5 est consacré à la démonstration du résultat suivant, établi dans [7].

Theorem 1.5.2. *Soit $N \geq 8$, $b \leq -1$, $0 < \eta < \frac{1}{16N}$, $\epsilon > 0$, (f_0, F_0) des données initiales pour les équations de Vlasov-Maxwell (1.2)-(1.4) satisfaisant*

$$\sum_{|\beta|+|\kappa| \leq N+3} \int_{|x| \geq b} \int_{v \in \mathbb{R}^3} (1+|x|)^{\frac{N+14+|\beta|}{2}} (1+|v|)^{|\kappa|} |\partial_x^\beta \partial_v^\kappa f_0| dv dx \leq \epsilon$$

$$\sum_{|\gamma| \leq N+2} \int_{|x| \geq b} (1+|x|)^{2|\gamma|} |\nabla_{\partial_x^2} F_0|^2 dx \leq \epsilon$$

et (f, F) l'unique solution classique du système vérifiant $f(t=0) = f_0$ et $F(t=0) = F_0$. Il existe alors $C > 0$ et $\epsilon_0 > 0$, ne dépendant que de N et η , tels que, si $0 \leq \epsilon \leq \epsilon_0$, (f, F) est définie sur $V_b = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 / r > t - b\}$ et vérifie les propriétés suivantes.

- Bornes L^2 pour le champ électromagnétique F :

$$\forall t \in \mathbb{R}_+, \quad \sum_{0 \leq k \leq N} \sum_{Z^\gamma \in \mathbb{K}^k} \int_{|x| \geq t-b} |\mathcal{L}_{Z^\gamma}(F)(t, x)|^2 dx \leq C\epsilon,$$

- Décroissance ponctuelle pour les composantes isotropes de $\mathcal{L}_{Z^\gamma}(F)$: $\forall |\gamma| \leq N-2, (t, x) \in V_b,$

$$\begin{aligned} |\alpha(\mathcal{L}_{Z^\gamma}(F))(t, x)| &\lesssim \frac{\sqrt{\epsilon}}{\tau_+^{\frac{3}{2}}} \\ |\rho(\mathcal{L}_{Z^\gamma}(F))(t, x) + |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))(t, x) + |\sigma(\mathcal{L}_{Z^\gamma}(F))(t, x)| &\lesssim \frac{\sqrt{\epsilon}}{\tau_+ \tau_-^{\frac{1}{2}}}. \end{aligned}$$

- Bornes L^1 sur la densité de particules:

$$\forall t \in \mathbb{R}_+, \quad \sum_{0 \leq k \leq N} \sum_{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^k} \int_{|x| \geq t-b} \int_{v \in \mathbb{R}^3} |\widehat{Z}^\beta f|(t, x, v) dv dx \leq C\epsilon(1+t)^{(N+1)\eta}.$$

- Décroissance ponctuelle pour les moyennes en vitesse de $\widehat{Z}^\beta f$: $\forall |\beta| \leq N-3, (t, x) \in V_b,$

$$\forall a \in \left[0, \frac{9}{2}\right], \quad \int_{v \in \mathbb{R}^3} |\widehat{Z}^\beta f|(t, x, v) \frac{dv}{(v^0)^{2a}} \lesssim \frac{\epsilon}{\tau_+^{2+a-(N+1)\eta} \tau_-}.$$

Remarque 1.5.3. Ce résultat permet d'obtenir des informations sur les solutions classiques à grandes données dans un domaine du type V_b , pour $b \leq -1$ suffisamment petit.

De la même façon que pour le théorème 1.5.1, la démonstration consiste à améliorer des estimations portant sur normes L^1 ou L^2 à poids des solutions. Pour ce faire, trois éléments se sont montrés être déterminants.

1. *Le fort taux de décroissance de $\int_v |f| dv$ à l'extérieur du cône de lumière.* Les solutions de l'équation de Vlasov ont un meilleur comportement dans la région V_0 , i.e. pour $|x| \geq t$, qu'à l'intérieur du cône de lumière. Pour illustrer cela, considérons une solution suffisamment régulière de $\mathbf{T}_1(g) = v^\mu \partial_\mu(g) = 0$. Alors,

$$\forall |x| \geq t, \quad \int_{\mathbb{R}_v^3} |g|(t, x, v) dv \lesssim \sum_{|\beta| \leq 3} \frac{\|(v^0)^{2k} (1+r)^{|\beta|+k+q} \partial_{t,x}^\beta g\|_{L_{x,v}^1}(t=0)}{(1+t+r)^{2+k} (1+|t-r|)^{1+q}} \quad (1.32)$$

$$\forall |x| \leq t, \quad \int_{\mathbb{R}_v^3} |g|(t, x, v) dv \lesssim \sum_{|\beta| \leq 3} \frac{\|(v^0)^{2k} (1+r)^{|\beta|+k} \partial_{t,x}^\beta g\|_{L_{x,v}^1}(t=0)}{(1+t+r)^{2+k} (1+|t-r|)^{1-k}}. \quad (1.33)$$

C'est ce fort taux de décroissance, donné par (1.32), qui nous permet d'affaiblir considérablement les hypothèses sur le champ électromagnétique comparé à notre étude sur l'espace entier. C'est également pour cela que nous n'avons pas besoin d'introduire de champs de vecteurs modifiés. Notons tout de même que nos hypothèses sur la variable vitesse de la densité de particules nous empêchent de pleinement utiliser (1.32). Aucun gain de décroissance en $t+r$ ne pourra être obtenu par ce procédé, ce qui nous oblige à exploiter au maximum la structure isotrope du système.

2. *Contrôler les grandes vitesses en utilisant la structure isotrope du système.* Pour plus de détails à ce propos, voir la sous-section 1.4.5.
3. *Exploiter des hiérarchies dans les équations commutées.* Les faibles hypothèses de décroissance sur le champ électromagnétique nous obligent à identifier et exploiter plusieurs hiérarchies entre les différentes normes du champ de Vlasov. Illustrons par un exemple la manière dont de tels échelonnages apparaissent.

- L'un des pires termes sources de $T_F(\widehat{Z}f)$, où \widehat{Z} est le lift complet du champ de Killing Z , est borné par $(1+t+r) \frac{v^k}{v^0} |\mathcal{L}_Z(F)| |\partial_{t,x} f|$.

- Or $|\mathcal{L}_Z(F)|$ décroît seulement, au vu de nos hypothèses sur F_0 , comme $(1+t+r)^{-1}(1+|t-r|)^{-\frac{1}{2}}$. Afin d'obtenir un taux de décroissance presque intégrable, on utilise alors l'inégalité $1+|t-r| \lesssim \sum_{z \in \mathbf{k}_1} |z|$, valide à l'extérieur du cône de lumière, de sorte que

$$(1+t+r) \frac{v^L}{v^0} |\mathcal{L}_Z(F)| |\partial_{t,x} f| \lesssim \frac{1}{1+|t-r|} \frac{v^L}{v^0} \sqrt{|z|} |\partial_{t,x} f|.$$

- On obtient ainsi $\|\widehat{Z}f\|_{L_{x,v}^1}(t) \lesssim \|\sqrt{|z|} \partial_{t,x} f\|_{L_{x,v}^1}(t) \log(3+t)$, ce qui nous amène à considérer des normes de la forme¹⁷ $\| |z|^{\frac{N-\beta_P}{2}} \widehat{Z}^\beta f \|_{L_{x,v}^1}$ où β_P est le nombre de champs de vecteurs homogènes¹⁸ composant \widehat{Z}^β .

Remarque 1.5.4. *On ne peut pas, avec notre raisonnement, considérer des normes du type*

$$\| |z|^{N-\beta_P} \widehat{Z}^\beta f \|_{L_{x,v}^1}$$

et ainsi gagner plus de décroissance par l'inégalité $1+|t-r| \lesssim \sum_{z \in \mathbf{k}_1} |z|$. En effet, lorsque $\beta_P = 0$ nous ne pouvons pas utiliser l'inégalité précédente¹⁹ et nous sommes même amenés pour certains termes d'erreurs à majorer

$$|z|^{\frac{N}{2}} \quad \text{par} \quad \sqrt{1+t+r} |z|^{\frac{N-1}{2}}.$$

Une perte supplémentaire en $t+r$ nous empêcherait de boucler nos estimations d'énergie.

1.5.3 Étude des solutions à données petites dans l'espace-temps de Minkowski en dimension 3

Comme mentionné précédemment, nous aurons besoin ici d'utiliser d'autres champs de vecteurs pour étudier la densité de particules. Chacun de ces commutateurs $Y \in \mathbb{Y}$ est construit à partir d'un et un seul élément $\widehat{Z} \in \widehat{\mathbb{P}}_0$ et dépend de la solution elle-même (leur construction sera décrite plus en détail ci-dessous). Cela fournit naturellement un ordre sur \mathbb{Y} et nous désignerons ainsi par Y^β la combinaison des $|\beta|$ champ de vecteurs modifiés $Y^{\beta_1} \dots Y^{\beta_{|\beta|}}$. Présentons maintenant le résultat principal de cette thèse [5], que l'on démontrera au cours du chapitre 4.

Theorem 1.5.3. *Soit $N \geq 11$, $\epsilon > 0$, (f_0, F_0) des données initiales pour les équations de Maxwell (1.2)-(1.4) et (f, F) l'unique solution du système vérifiant $(f, F)(t=0) = (f_0, F_0)$. Si*

$$\sum_{|\beta|+|\kappa| \leq N+3} \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_v^3} (1+|x|)^{2N+3} (1+|v|)^{|\kappa|} |\partial_x^\beta \partial_v^\kappa f_0| \, dv dx + \sum_{|\gamma| \leq N+2} \int_{\mathbb{R}_x^3} (1+|x|)^{2|\gamma|+1} \left| \nabla_x^\gamma \widetilde{F}_0 \right|^2 \, dx \leq \epsilon,$$

il existe $C > 0$, $M \in \mathbb{N}$ et $\epsilon_0 > 0$ ne dépendant que de N tels que, si $\epsilon \leq \epsilon_0$, alors (f, F) est une solution globale du système de Vlasov-Maxwell et vérifie les propriétés suivantes.

- *Bornes L^2 pour la partie électriquement neutre de F : $\forall t \in \mathbb{R}_+$,*

$$\sum_{\substack{Z^\gamma \in \mathbb{K}^{|\gamma|} \\ |\gamma| \leq N}} \int_{|x| \geq t} \tau_+ \left(|\alpha(\mathcal{L}_{Z^\gamma}(\widetilde{F}))|^2 + |\rho(\mathcal{L}_{Z^\gamma}(\widetilde{F}))|^2 + |\sigma(\mathcal{L}_{Z^\gamma}(\widetilde{F}))|^2 \right) + \tau_- |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(\widetilde{F}))|^2 \, dx \leq C\epsilon,$$

$$\sum_{\substack{Z^\gamma \in \mathbb{K}^{|\gamma|} \\ |\gamma| \leq N}} \int_{|x| \leq t} \tau_+ \left(|\alpha(\mathcal{L}_{Z^\gamma}(F))|^2 + |\rho(\mathcal{L}_{Z^\gamma}(F))|^2 + |\sigma(\mathcal{L}_{Z^\gamma}(F))|^2 \right) + \tau_- |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|^2 \, dx$$

$$\leq C\epsilon \log^{2M}(3+t).$$

- *Décroissance ponctuelle pour les composantes isotropes de $\mathcal{L}_{Z^\gamma}(F)$: $\forall |\gamma| \leq N-5$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,*

$$|\alpha(\mathcal{L}_{Z^\gamma}(F))|(t, x) \lesssim \sqrt{\epsilon} \frac{\log(3+t)}{\tau_+^2}, \quad |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|(t, x) \lesssim \sqrt{\epsilon} \frac{\log(3+t)}{\tau_+ \tau_-},$$

$$|\rho(\mathcal{L}_{Z^\gamma}(F))|(t, x) \lesssim \sqrt{\epsilon} \frac{\log(3+t)}{\tau_+^{\frac{3}{2}} \tau_-^{\frac{1}{2}}}, \quad |\sigma(\mathcal{L}_{Z^\gamma}(F))|(t, x) \lesssim \sqrt{\epsilon} \frac{\log(3+t)}{\tau_+^{\frac{3}{2}} \tau_-^{\frac{1}{2}}}.$$

¹⁷En réalité nous considérerons des normes légèrement différentes en raison de certains autres termes d'erreurs.

¹⁸Cela revient à compter le nombre de champs de vecteurs qui ne sont pas des translations.

¹⁹Le gain de décroissance en $t-r$ provient alors du champ électromagnétique, qui a été dérivé au moins une fois par une translation, et de la proposition 1.9.

- Bornes L^1 pour le champ de Vlasov: $\forall t \in \mathbb{R}_+$,

$$\sum_{\substack{Y^\beta \in \mathbb{Y}^{|\beta|} \\ |\beta| \leq N}} \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} |Y^\beta f| dv dx \leq C\epsilon.$$

- Décroissance ponctuelle pour les moyennes en vitesse de $Y^\beta f$: $\forall |\beta| \leq N - 3, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$\int_{v \in \mathbb{R}^3} |Y^\beta f| dv \lesssim \frac{\epsilon}{\tau_+^2 \tau_-} \quad \text{et} \quad \int_{v \in \mathbb{R}^3} |Y^\beta f| \frac{dv}{(v^0)^2} \lesssim \epsilon \frac{1}{\tau_+^3} \mathbf{1}_{t \geq |x|} + \epsilon \frac{\log^2(3+t)}{\tau_+^3 \tau_-} \mathbf{1}_{|x| \geq t}$$

Discutons maintenant des principales difficultés de la preuve (qui consiste ici aussi à améliorer des inégalités d'énergie).

1. *La structure isotrope du système.* La discussion menée dans la sous-section 1.4.5 donne les grandes lignes de ce point majeur de la démonstration.
2. *Le champ électromagnétique et la non neutralité du plasma.* On utilise ici la stratégie décrite dans la sous-section 1.4.4.
3. *Les champs de vecteurs modifiés.* En dimension 3, la moyenne en vitesse du champ de Vlasov ne peut décroître plus vite que $(1+t+r)^{-3}$ à l'intérieur du cône de lumière. Cela nous empêche de propager une norme L^2 de \tilde{F} contenant des poids de l'ordre de $(t+r)^2$ et $(t-r)^2$ et d'obtenir un fort taux de décroissance sur F . La meilleure estimation que l'on peut espérer est donc

$$|F|(t, x) \lesssim \frac{\sqrt{\epsilon}}{(1+t+r)(1+|t-r|)}.$$

En reprenant le cheminement de la sous-section 1.4.5 et en supposant que l'on ait une parfaite compréhension de la structure isotrope du système²⁰, on obtiendrait alors

$$\begin{aligned} \|\widehat{Z}f\|_{L_v^1 L^1(\Sigma_t)} - \|\widehat{Z}f\|_{L_v^1 L^1(\Sigma_0)} &\leq \int_0^t \int_{\Sigma_s} \int_{\mathbb{R}_v^3} |\mathcal{L}_Z(F)(v, \nabla_v f)| \frac{dv}{v^0} dx ds \\ &\lesssim \sum_{\widehat{\Gamma} \in \widehat{\mathbb{P}}_0} \int_0^t \|(1+t+r)\mathcal{L}_Z(F)\|_{L^\infty(\Sigma_s)} \left\| \int_{\mathbb{R}_v^3} |\widehat{\Gamma}f| \frac{dv}{v^0} \right\|_{L^1(\Sigma_s)} ds \\ &\lesssim \sum_{\widehat{\Gamma} \in \widehat{\mathbb{P}}_0} \int_0^t \frac{\sqrt{\epsilon}}{1+s} \left\| \int_{\mathbb{R}_v^3} |\widehat{\Gamma}f| \right\|_{L^1(\Sigma_s)} ds. \end{aligned}$$

Le taux de décroissance non intégrable $(1+s)^{-1}$ entraîne une croissance logarithmique de $\|\widehat{Z}f\|_{L_v^1 L^1(\Sigma_t)}$ qui nous empêche d'améliorer les estimations d'énergie²¹. Cela nous amène donc, comme [16] pour le système de Vlasov-Nordström, à commuter l'équation de Vlasov par des champs de vecteurs de la forme

$$Y = \widehat{Z} + \Phi_{\widehat{Z}}^i X_i, \quad \widehat{Z} \in \widehat{\mathbb{P}}_0, \quad X_i = \partial_i + \frac{v^i}{v^0} \partial_t.$$

Les coefficients $\Phi_{\widehat{Z}}^i$ sont définis comme solutions d'équations de transport, dépendent de la solution (f, F) du système et permettent de compenser les pires termes sources de $T_F(\widehat{Z}^\beta f)$. Précisons que l'on ne modifie pas les translations, i.e. $\Phi_{\partial_{t,x}}^i = 0$.

4. *Hierarchies entre les différentes normes de la densité de particules.* Afin de prouver que $\|Y^\beta f\|_{L_{x,v}^1}$ reste bornée au cours du temps, certains termes sources des équations commutées requièrent un contrôle sur des normes L^1 de f qui, quant à elles, croissent. Pour boucler nos estimations d'énergie, nous aurons alors besoin d'identifier plusieurs hiérarchies entre les normes considérées. Montrons comment de tels échelonnages apparaissent naturellement.

²⁰Autrement dit, que l'on sache transformer toute la décroissance en $t-r$ en de la décroissance en $t+r$.

²¹Une perte sur l'énergie de la densité de particules se répercuterait sur celle du champ électromagnétique, ce qui n'était pas le cas dans la démonstration du théorème 1.5.2.

- Les pires termes sources de l'équation de transport vérifiée par Yf sont de la forme $tX_i(F_{\mu\nu})\partial_{t,x}f$.
- En utilisant la bonne structure du champ de vecteurs X_i (voir (1.15)), on a

$$|(t+r)X_i(F_{\mu\nu})\partial_{t,x}f| \lesssim \sum_{Z \in \mathbb{K}} |\nabla_Z F| \sum_{z \in \mathbf{k}} |z\partial_{t,x}f|.$$

- On obtient donc que $\|Yf\|_{L_{x,v}^1}$ est bornée uniformément en temps sous réserve que $\|z\partial_{t,x}f\|_{L_{x,v}^1}$ ne croisse pas trop vite. On considèrera donc des normes de la forme

$$\sum_{|\beta| \leq N} \|z^{N_0 - \beta_P} Y^\beta f\|_{L_v^1 L^1(\Sigma_t)},$$

où β_P dénombre les champs de vecteurs composant Y^β qui ne sont pas des translations.

- Si $\beta_P = 0$, nous devons contrôler des termes d'erreurs de la forme $tz^{N_0}\partial_{t,x}^\gamma(F_{\mu\nu})\partial_{t,x}^\beta f$ et nous utiliserons cette fois la proposition 1.9 pour gagner de la décroissance en $t-r$ grâce à $\partial_{t,x}^\gamma$.

1.6 Résultats : système de Vlasov-Maxwell sans masse

Intéressons-nous maintenant aux plasmas avec des particules sans masse, qui ne correspondent à aucun modèle physique étant donné qu'aucune particule chargée sans masse n'est connue. Néanmoins, d'un point de vue mathématique, l'étude d'un tel système est intéressant car

- pour les grandes vitesses, $\sqrt{1+|v|^2} \sim |v|$. On peut donc s'attendre à ce que les solutions de $\mathbf{T}_1(g) = 0$ et de $\mathbf{T}_0(g) = 0$ aient un comportement comparable pour $|v| \rightarrow +\infty$.
- L'étude du comportement asymptotique des solutions du système de Vlasov-Maxwell sans masse est plus simple et moins technique que le cas massif en dimension 3. Les idées clés de la démonstration ainsi que la structure isotrope des équations apparaissent ainsi plus clairement.
- Le système de Vlasov-Maxwell sans masse peut être vu comme un modèle simplifié du système d'Einstein-Vlasov sans masse, qui est quant à lui utilisé en physique.

Un des avantages des méthodes de champs de vecteurs est que les démonstrations de nos résultats sur le système de Vlasov-Maxwell massif peuvent être adaptées au cas sans masse, ce qui n'est pas le cas des preuves de Glassey-Strauss [24], Schaeffer [44] et Wang [50]. Cependant, nous devons faire face à deux nouvelles difficultés spécifiques aux particules sans masse.

- Soit g et h deux fonctions suffisamment régulières telles que $\mathbf{T}_1(g) = 0$ et $\mathbf{T}_0(h) = 0$. Alors, avec C_g (respectivement C_h) une constante ne dépendant que de $g(0, \cdot, \cdot)$ (respectivement $h(0, \cdot, \cdot)$), on a, pour tout $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$\int_{\mathbb{R}_v^3} |g|(t, x, v) dv \lesssim \frac{C_g}{(1+t+r)^3} \quad \text{et} \quad \int_{\mathbb{R}_v^3} |h|(t, x, v) dv \lesssim \frac{C_h}{(1+t+r)^2}.$$

Les taux de décroissance en $t+r$ donnés ici sont optimaux (on peut par contre améliorer la décroissance en $t-r$ pour $\int_v |h| dv$) et on voit donc qu'à ce niveau, un champ de Vlasov massif a un meilleur comportement.

- Les petites vitesses posent des problèmes de régularité pour les solutions du système de Vlasov-Maxwell sans masse. La sous-section suivante contient plus de détails à ce propos.

Afin de compenser le manque de décroissance en $t+r$, il est crucial de remarquer que le taux de décroissance de solutions d'équations de Vlasov sans masse peut être amélioré dans les directions isotropes. Par exemple, si h vérifie $\mathbf{T}_0(h) = 0$, alors

$$\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad \int_{\mathbb{R}_v^3} \left| \frac{v^L}{|v|} \right|^p \left| \frac{v^A}{|v|} \right|^k \left| \frac{v^{\underline{L}}}{|v|} \right|^q |h|(t, x, v) dv \lesssim \sum_{|\beta| \leq 3} \frac{\|(1+r)^{|\beta|+p+k+q} \partial_x^\beta h\|_{L_{x,v}^1}(t=0)}{(1+t+r)^{2+k+q} (1+|t-r|)^{1+p}}. \quad (1.34)$$

Ces gains de décroissance fournis par les composantes isotropes du vecteur vitesse sont à comparer à ceux donnés par les dérivées isotropes pour une solution d'une équation d'onde (voir proposition 1.9). Cela est lié aux poids préservés par le flot de \mathbf{T}_0 :

$$(1+t+r) \left| \frac{v^A}{|v|} \right| + (1+t+r) \left| \frac{v^{\underline{L}}}{|v|} \right| + (1+|t-r|) \left| \frac{v^L}{|v|} \right| \lesssim \sum_{z \in \mathbf{k}_0} |z|.$$

Pour l'ensemble de cette section, v^0 désignera $|v|$.

1.6.1 Le problème causé par les petites vitesses

La difficulté liée aux petites vitesses provient de la différence entre le degré d'homogénéité dans la variable v entre $\mathbf{T}_0 = v^\mu \partial_\mu$ et $F(v, \nabla_v)$, couplé au fait que $v^0 = |v|$ ne soit pas minoré par une constante strictement positive. La partie vitesse V des caractéristiques de l'équation de Vlasov

$$\partial_t f + \frac{v^i}{|v|} \partial_i f + \frac{v^\mu}{|v|} F_\mu^j \partial_{v^j} f = 0$$

est solution de l'équation différentielle

$$\dot{V}^j = \frac{V^\mu}{|V|} F_\mu^j,$$

et peut donc atteindre la valeur $V = 0$ en temps fini.

Remarque 1.6.1. *Aucune difficulté de cette nature n'est présente dans l'étude du système Einstein-Vlasov. En effet, l'équation de Vlasov peut être écrite, pour une métrique g , sous la forme*

$$v_\nu g^{\nu\mu} \partial_\mu f - \frac{1}{2} v_\mu v_\nu \partial_i g^{\mu\nu} \partial_{v_i} f = 0,$$

de telle sorte que l'homogénéité en v de la non linéarité est la même que celle de \mathbf{T}_0 . Ces deux situations peuvent être comparées aux deux équations différentielles

$$\dot{y} = 1 \quad \text{et} \quad \dot{y} = -y.$$

Dans nos démonstrations basées sur l'amélioration d'inégalités d'énergie, ce phénomène se manifeste lorsque l'on doit borner des quantités du type (voir (1.19))

$$\|\widehat{Z}f\|_{L_v^1 L^1(\Sigma_t)} - \|\widehat{Z}f\|_{L_v^1 L^1(\Sigma_0)} \leq \int_0^t \int_{\Sigma_s} \int_{\mathbb{R}_v^3} |\mathcal{L}_Z(F)(v, \nabla_v f)| \frac{dv}{v^0} dx ds.$$

En utilisant la relation $v^0 \partial_{v^i} f = \widehat{\Omega}_{0i} - t \partial_i - x^i \partial_t$, on obtient

$$\|\widehat{Z}f\|_{L^1(\Sigma_t)} - \|\widehat{Z}f\|_{L^1(\Sigma_0)} \lesssim \sum_{\widehat{\Gamma} \in \widehat{\mathbb{F}}_0} \int_0^t \|(1+t+r)\mathcal{L}_Z(F)\|_{L^\infty(\Sigma_s)} \left\| \int_{\mathbb{R}_v^3} |\widehat{\Gamma}g| \frac{dv}{v^0} \right\|_{L^1(\Sigma_s)} ds$$

et on voit qu'une borne sur $\| |v|^{-1} \widehat{Z}f \|_{L^1}$ semble nécessaire afin d'améliorer celle sur $\|\widehat{Z}f\|_{L^1}$. Afin d'éviter un tel problème, nous allons supposer que f est initialement nulle dans un voisinage de $v = 0$ et une étape importante de la démonstration consistera à montrer qu'une telle propriété reste vraie au cours du temps.

Enfin, mentionnons notre résultat obtenu dans [4] qui implique qu'une telle hypothèse semble être nécessaire.

Proposition 1.6.2. *Il existe des données initiales régulières²² telles que le système de Vlasov-Maxwell sans masse n'admette pas de solution locale de classe C^1 .*

1.6.2 Le cas des grandes dimensions

Les normes considérées ici sont analogues à celles présentées dans la sous-section 1.4.2. Notre résultat, présenté plus en détail au chapitre 2, concernant les dimensions $n \geq 4$ obtenu dans [4] s'énonce ainsi.

Theorem 1.6.1. *Soit $n \geq 4$, $N \geq 6n + 3$, $0 < \eta < \frac{1}{2}$ et (f_0, F_0) des données initiales pour le système de Vlasov-Maxwell sans masse. Soit (f, F) l'unique solution classique du système et A un potentiel du champ électromagnétique satisfaisant la jauge de Lorenz. Il existe $\epsilon > 0$ tel que si*

$$\widetilde{\mathcal{E}}_N[A](0) \leq \epsilon, \quad \mathcal{E}_N[F](0) \leq \epsilon, \quad \mathbb{E}_{N+n,1}[f](0) \leq \epsilon$$

et

$$\text{supp}(f_0) \subset \{(x, v) \in \mathbb{R}_x^n \times \mathbb{R}_v^n \setminus \{0\} / |v| \geq 2\},$$

alors (f, F) est une solution globale et vérifie les propriétés suivantes.

²²Pour l'ensemble des données initiales considérées, le champ de Vlasov est non nul sur un voisinage de $v = 0$.

- La densité de particules f s'annule pour les petites vitesses :

$$\text{supp}(f) \subset \{(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}_x^n \times \mathbb{R}_v^n \setminus \{0\} / |v| \geq 1\}.$$

- Propagation de normes L^2 pour F et L^1 pour f : $\forall t \in \mathbb{R}_+$,

$$\mathcal{E}_N[F](t) \lesssim \epsilon(1+t)^{1+n}, \quad \mathbb{E}_N[f](t) \lesssim \epsilon \log(3+t), \quad \mathbb{E}_{N-n,1}[f](t) \lesssim \epsilon.$$

- Décroissance ponctuelle pour les composantes isotropes de $\mathcal{L}_{Z^\beta}(F)$: $\forall |\beta| \leq N - \frac{5n+4}{2}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\begin{aligned} |\alpha(\mathcal{L}_{Z^\beta} F)| &\lesssim \sqrt{\epsilon} \tau_+^{-\frac{n+1}{2}}, & |\underline{\alpha}(\mathcal{L}_{Z^\beta} F)| &\lesssim \sqrt{\epsilon} \tau_+^{-\frac{n-1}{2}} \tau_-^{-1}, \\ |\rho(\mathcal{L}_{Z^\beta} F)| &\lesssim \sqrt{\epsilon} \tau_+^{-\frac{n}{2}} \tau_-^{-\frac{1}{2}}, & |\sigma(\mathcal{L}_{Z^\beta} F)| &\lesssim \sqrt{\epsilon} \tau_+^{-\frac{n}{2}} \tau_-^{-\frac{1}{2}}. \end{aligned}$$

- Décroissance ponctuelle pour $\int_{v \in \mathbb{R}^n \setminus \{0\}} |z \widehat{Z}^\beta f| dv$:

$$\forall |\beta| \leq N - 2n, z \in \mathbf{k}_0, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad \int_{v \in \mathbb{R}^n \setminus \{0\}} |z \widehat{Z}^\beta f| dv \lesssim \frac{\epsilon}{\tau_+^{n-1} \tau_-}.$$

Les trois principales difficultés rencontrées lors de la démonstration ont déjà été évoquées.

1. *Le manque de décroissance en $t-r$ des solutions.* Tant pour boucler les estimations d'énergie pour le champ de Vlasov que pour le champ électromagnétique, le manque de décroissance des solutions en $t+r$ nous force à exploiter la structure isotrope du système. Pour borner $\mathcal{E}_N[F]$ nous prouvons et utilisons une inégalité analogue à (1.34). Pour la densité de particules, on tire profit du bon comportement de certaines composantes isotropes de F , v et $\nabla_v f$ comme illustré dans la sous-section 1.4.5.
2. *La densité de particules reste nulle dans un voisinage de $v = 0$.* Pour montrer cette propriété nous avons étudié la partie vitesse V des caractéristiques de l'équation de Vlasov. L'hypothèse de petitesse sur F ainsi que son fort taux de décroissance (lié à la dimension $n \geq 4$) nous a permis de montrer que

$$|V(t=0)| \geq 2 \Rightarrow \inf_{t \in \mathbb{R}_+} |V(t)| \geq 1.$$

3. *Les équations de Maxwell ne sont pas conformément invariantes en dimension $n \neq 3$.* On procède alors de la même façon que pour le cas massif.

1.6.3 Le cas 3d

Contrairement au cas massif, l'étude des solutions ne nécessite pas de travailler avec des modifications des champs de vecteurs de $\widehat{\mathbb{P}}_0$. Notre résultat obtenu dans [6] et exposé au chapitre 3 est le suivant.

Theorem 1.6.2. *Soit $N \geq 10$, $\epsilon > 0$ et (f^0, F^0) des données initiales pour le système de Vlasov-Maxwell sans masse (1.2)-(1.4) satisfaisant l'hypothèse de petitesse*

$$\sum_{|\beta|+|\kappa| \leq N+3} \int_{\mathbb{R}_x^3} \int_{\mathbb{R}_v^3} (1+|x|)^{|\beta|+2} (1+|v|)^{|\kappa|} |\partial_x^\beta \partial_v^\kappa f^0| dv dx + \sum_{|\gamma| \leq N+2} \int_{\mathbb{R}_x^3} (1+|x|)^{2|\gamma|+2} |\nabla_{\partial_x} F^0|^2 dx \leq \epsilon,$$

la propriété de neutralité

$$\int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} f^0 dv dx = 0$$

et qui s'annule pour les petites vitesses

$$\forall 0 < |v| \leq 3, \quad f^0(\cdot, v) = 0.$$

Il existe $C > 0$ et $\epsilon_0 > 0$ ne dépendant que de N tels que si $0 \leq \epsilon \leq \epsilon_0$, alors l'unique solution classique (f, F) du système satisfaisant $f(t=0) = f^0$ et $F(t=0) = F^0$ est globale en temps et vérifie les propriétés suivantes.

- Bornes L^2 pour le champ électromagnétique F : $\forall t \in \mathbb{R}_+$,

$$\sum_{\substack{Z^\gamma \in \mathbb{K}^{|\gamma|} \\ |\gamma| \leq N}} \int_{\Sigma_t} \tau_+^2 \left(|\alpha(\mathcal{L}_{Z^\gamma}(F))|^2 + |\rho(\mathcal{L}_{Z^\gamma}(F))|^2 + |\sigma(\mathcal{L}_{Z^\gamma}(F))|^2 \right) + \tau_-^2 |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|^2 dx \leq C \epsilon \log^4(3+t).$$

- *Décroissance ponctuelle pour les composantes isotropes de $\mathcal{L}_{Z^\gamma}(F)$* : $\forall |\gamma| \leq N - 2, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$\begin{aligned} |\rho(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log^2(3+t)}{\tau_+^2 \tau_-^{\frac{1}{2}}}, & |\alpha(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log^2(3+t)}{\tau_+^{\frac{5}{3}}}, \\ |\sigma(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log^2(3+t)}{\tau_+^2 \tau_-^{\frac{1}{2}}}, & |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log^{\frac{5}{3}}(1+\tau_-)}{\tau_+ \tau_-^{\frac{3}{2}}}. \end{aligned}$$

- *Bornes L^1 pour le champ de Vlasov f* : $\forall t \in \mathbb{R}_+$,

$$\sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|} \\ |\beta| \leq N}} \sum_{z \in \mathbf{k}_0} \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} |z \widehat{Z}^\beta f|(t, x, v) dv dx \leq C \epsilon \log(3+t).$$

- *La densité de particules s'annule pour les petites vitesses* :

$$\forall (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}), \quad |v| \leq 1 \Rightarrow f(t, x, v) = 0.$$

- *Décroissance optimale pour les moyennes en vitesse de $\widehat{Z}^\beta f$* : $\forall |\beta| \leq N - 5, z \in \mathbf{k}_0$,

$$\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad \int_{v \in \mathbb{R}^3} |z^2 \widehat{Z}^\beta f| dv \lesssim \frac{\epsilon}{\tau_+^2 \tau_-}.$$

Les deux ingrédients clés de la démonstration sont

1. *l'exploitation de la structure isotrope des équations*. Le principe est ici le même que pour les grandes dimensions. Mentionnons tout de même que le faible taux de décroissance des solutions nous a amenés à effectuer une étude bien plus fine de cette structure.
2. *Prouver que la densité de particules s'annule pour les petites vitesses*. Le point de départ est le même que pour les grandes dimensions et consiste à étudier la partie vitesse des caractéristiques de l'équation de Vlasov. Néanmoins, le taux de décroissance en $t+r$ du champ électromagnétique, qui est non intégrable en temps, rend caduc le reste du raisonnement effectué lors de la preuve du théorème 1.6.1. On force alors l'apparition de la composante isotrope $\rho(F)$ dans l'équation différentielle satisfaite par $|V|^2$,

$$\frac{d}{dt} (|V|^2) = V^i F_{0i} = |V| \frac{X^i}{|X|} F_{0i} + |V| \left(\frac{V^i}{|V|} - \frac{X^i}{|X|} \right) F_{0i} = |V| \rho(F) + |V| \left(\frac{V^i}{|V|} - \frac{X^i}{|X|} \right) F_{0i}.$$

On peut alors utiliser le fort taux de décroissance en $t+r$ de $\rho(F)$ et le reste de la démonstration consiste à majorer convenablement

$$\left(\frac{V^i}{|V|} - \frac{X^i}{|X|} \right) \tag{1.35}$$

lorsque V s'approche de 0. L'équation différentielle vérifiée par X

$$\frac{dX}{dt} = \frac{V}{|V|}$$

ainsi que des considérations géométriques indiquent que dans une telle situation, X et V sont presque parallèles, d'où la petitesse de (1.35).

1.7 Perspectives

Les équations de Maxwell ayant des propriétés relativement proches de celles des équations d'Einstein, un prolongement naturel de nos travaux pourrait être de prouver la stabilité de l'espace-temps de Minkowski pour le système d'Einstein-Vlasov sans aucune hypothèse de support compact, tant en espace qu'en vitesse, sur les données initiales. Une solution du système d'Einstein-Vlasov est un triplet (\mathcal{M}, g, f) , où \mathcal{M} est une variété munie de la métrique lorentzienne g , orientée en temps, et f est la fonction de distribution d'un amas de particules de masse $m \in \{0, 1\}$, définie sur

$$\mathcal{P} := \{(x, v) \in T\mathcal{M}^* / g_x^{-1}(v, v) = -m^2 \text{ et } v \text{ dirigée vers le futur}\},$$

où TM^* est le fibré cotangent de \mathcal{M} . Introduisons la notation

$$\mathcal{P}_x := \{v / (x, v) \in \mathcal{P}\}$$

et désignons par $Ric(g)$ (respectivement $R(g)$) la courbure de Ricci (respectivement la courbure scalaire) de \mathcal{M} . Plaçons nous dans un système de coordonnées (x^0, \dots, x^3) , de sorte que

$$v = v_\mu dx^\mu,$$

et paramétrisons \mathcal{P}_x par²³ (v_1, v_2, v_3) . La métrique g et la fonction de distribution f sont alors solutions de

$$\begin{aligned} Ric(g)_{\mu\nu} - \frac{1}{2}R(g)g_{\mu\nu} &= T_{\mu\nu}[f], & T_{\mu\nu}[f](x) &:= \int_{\mathcal{P}_x} f(x, v) v_\mu v_\nu \frac{\sqrt{|\det(g_x)|}}{v^0} dv_1 dv_2 dv_3, \\ v^\mu \partial_{x^\mu} f - \frac{1}{2} v_\nu v_\lambda \partial_{x^i} g^{\nu\lambda} \cdot \partial_{v_i} f &= 0, & \text{où} & \quad v^\lambda = v_\sigma g^{\sigma\lambda}. \end{aligned}$$

En coordonnées d'onde, i.e. dans un système de coordonnées vérifiant, pour $0 \leq \mu \leq 3$,

$$\square_g x^\mu = 0, \quad \text{avec} \quad \square_g = \frac{1}{\sqrt{|\det(g)|}} \partial_{x^\lambda} g^{\lambda\nu} \sqrt{|\det(g)|} \partial_{x^\nu},$$

et en introduisant $h = g - \eta$, où η est la métrique de l'espace-temps de Minkowski, les équations d'Einstein prennent la forme suivante :

$$\square_g h_{\mu\nu} = P_{\mu\nu}(\partial h, \partial h) + Q_{\mu\nu}(\partial h, \partial h) + G(h)(\partial h, \partial h) - 2T_{\mu\nu}[f] - \int_{\mathcal{P}_x} f g_{\mu\nu} \frac{\sqrt{|\det(g_x)|}}{v^0} dv_1 dv_2 dv_3, \quad (1.36)$$

où

- $Q_{\mu\nu}(\xi, \zeta)$ est une combinaison linéaire de formes quadratiques, telle que $\xi^\mu \zeta_\mu$, satisfaisant la condition isotrope (voir le théorème 1.2.2).
- $G(h)(\partial h, \partial h)$ est une forme quadratique en ∂h dont les coefficients dépendent de h de façon lisse et telle que

$$G(0)(\partial h, \partial h) = 0.$$

- $P_{\mu\nu}$ est une combinaison de formes quadratiques ne satisfaisant pas la condition isotrope mais ayant tout de même une structure particulière, permettant à (1.36) de vérifier la condition isotrope faible (voir la proposition 1.2.3).

On voit donc que l'on a affaire, comme pour les équations de Vlasov-Maxwell, à un système ondes/équation de transport relativiste. Cependant, une nouvelle difficulté propre à ce système réside dans le fait que v_0 et v^μ , pour $0 \leq \mu \leq 3$, dépendent de la métrique g , ce qui engendre de nombreux termes d'erreurs dans les formules de commutations.

On pourrait commencer par étudier le cas où les particules sont sans masse afin de bien comprendre la structure isotrope du système car on s'attend, au vu de nos résultats sur les équation de Vlasov-Maxwell, que l'étude soit plus simple que pour le cas massif. En particulier, un objectif pourrait être de s'astreindre à ne pas modifier les champs de vecteurs de $\widehat{\mathbb{P}}_0$ afin d'éviter de nombreux calculs et d'avoir une démonstration aussi accessible que possible. Dans un second temps, on pourrait considérer des particules massives et ainsi généraliser les travaux de [17] et [26]. Une éventuelle piste pourrait être d'étudier uniquement les solutions en dehors d'un cône de lumière et de combiner les estimations obtenues sur la solution avec celles de [17] afin de couvrir tout l'espace-temps.

Une toute autre direction possible pour de futures recherches pourrait être d'étudier des équations de Vlasov sur des variétés courbes tel que l'espace-temps de Schwarzschild. Par exemple, dans l'esprit de [13] pour l'équation des ondes, il serait intéressant de prouver que les moyennes en vitesse des solutions de l'équation de transport relativiste décroissent pour des particules sans masse sur un trou noir de type Schwarzschild.

Finalement, la structure isotrope du système de Vlasov-Maxwell semblant être nécessaire afin de prouver l'existence globale de ses solutions à données petites, on peut s'attendre à des résultats d'explosion en temps

²³On utilise pour cela que $v_0 = -(g^{00})^{-1}(g^{0j}v_j - \sqrt{(g^{0j}v_j)^2 - g^{00}(1 + g^{ij}v_i v_j)})$.

fini pour des systèmes couplant une équation d'onde à une équation cinétique et où une telle structure est absente. On pourrait par exemple s'intéresser à un système de la forme

$$\begin{aligned}
\mathbf{T}_1(f_\epsilon) + \partial_t u_\epsilon \cdot \partial_{v^1} f_\epsilon &= 0, \\
\Box u_\epsilon &= \int_{\mathbb{R}^3} f_\epsilon dv, \\
f_\epsilon(0, \cdot, \cdot) &= \epsilon g, & g &\in C_c^\infty(\mathbb{R}_x^3 \times \mathbb{R}_v^3, \mathbb{R}), \\
u_\epsilon(0, \cdot) &= \epsilon v, & v &\in C_c^\infty(\mathbb{R}^3, \mathbb{R}), \\
\partial_t u_\epsilon(0, \cdot) &= \epsilon w, & w &\in C_c^\infty(\mathbb{R}^3, \mathbb{R}),
\end{aligned}$$

et essayer de prouver que

$$\forall \epsilon > 0, \exists T_\epsilon > 0, \quad \left\| \int_{\mathbb{R}^3} f_\epsilon dv \right\|_{L^\infty([0, T_\epsilon] \times \mathbb{R}_x^3)} + \|u_\epsilon\|_{L^\infty([0, T_\epsilon] \times \mathbb{R}_x^3)} = +\infty.$$

On aurait alors une situation analogue à celle des équations d'onde semi-linéaires (voir les théorèmes 1.2.1 et 1.2.2 ci-dessus ainsi que le résultat de Fritz John [28]).

Chapter 2

Asymptotic properties of small data solutions of the Vlasov-Maxwell system in high dimensions

Abstract

We prove almost sharp decay estimates for the small data solutions and their derivatives of the Vlasov-Maxwell system in dimension $n \geq 4$. The smallness assumption concerns only certain weighted L^1 or L^2 norms of the initial data. In particular, no compact support assumption is required on the Vlasov or the Maxwell fields. The main ingredients of the proof are vector field methods for both the kinetic and the wave equations, null properties of the Vlasov-Maxwell system to control high velocities and a new decay estimate for the velocity average of the solution of the relativistic massive transport equation.

We also consider the massless Vlasov-Maxwell system under a lower bound on the velocity support of the Vlasov field. As we prove in this paper, the velocity support of the Vlasov field needs to be initially bounded away from 0. We compensate the weaker decay estimate on the velocity average of the massless Vlasov field near the light cone by an extra null decomposition of the velocity vector.

2.1 Introduction

In this paper, we study the asymptotic properties of the small data solutions of the Vlasov-Maxwell system in dimensions $n \geq 4$. For K species, this system is given by¹

$$\sqrt{m_k^2 + |v|^2} \partial_t f_k + v^i \partial_i f_k + e_k v^\mu F_\mu^j \partial_{v^j} f_k = 0, \quad (2.1)$$

$$\nabla^\mu F_{\mu\nu} = e^k J(f_k)_\nu, \quad (2.2)$$

$$\nabla^{\mu*} F_{\mu\alpha_1 \dots \alpha_{n-2}} = 0, \quad (2.3)$$

with initial data,

$$f_k(0, \cdot, \cdot) = f_{0k}, \quad (2.4)$$

$$F(0, \cdot) = F_0. \quad (2.5)$$

This is a classical model in plasma physics and we refer to [21] for an introduction to its analysis. Here,

- $m_k \in \mathbb{R}_+$ and $e_k \in \mathbb{R}^*$ are the mass and the charge of the particles of the species $k \in \llbracket 1, K \rrbracket$. The function $f_k(t, x, v)$ is their velocity distribution, which is a non-negative function.
- The Maxwell field is described in geometric form by the 2-form $F(t, x)$ and its Hodge dual $*F(t, x)$.
- The $(n+1)$ -current $J(f_k)_\nu$ in equation (2) is given by

$$J(f_k)^\nu(t, x) = \int_{v \in \mathbb{R}^n} \frac{v^\nu}{v_k^0} f_k(t, x, v) dv, \quad \text{where} \quad v_k^0 := \sqrt{m_k^2 + |v|^2}.$$

¹During this article, we will use the Einstein summation convention. For instance, $e^k J(f_k)_\nu = \sum_{k=1}^K e^k J(f_k)_\nu$. Roman indices goes from 1 to n and greek indices from 0 to n . Moreover, we raise and lower indices with respect to the Minkowski metric.

- The variable t will be taken in \mathbb{R}_+ , x will be taken in \mathbb{R}^n and for the species k , v will be taken either in \mathbb{R}^n , if $m_k \neq 0$, or in $\mathbb{R}^n \setminus \{0\}$, if $m_k = 0$.

In the 3 dimensional case, we can express the system in terms of the electric and the magnetic vector fields through the relations

$$E^i = F_{0i} \quad \text{and} \quad B^i = -{}^*F_{0i}$$

so that the Vlasov-Maxwell equations take the familiar form

$$\begin{aligned} \sqrt{m_k^2 + |v|^2} \partial_t f_k + v^i \partial_i f_k + e_k (E + v \times B) \cdot \nabla_v f_k &= 0, \\ \nabla \cdot E &= e^k J(f_k)^0, \quad \partial_t E^j = (\nabla \times B)^j - e^k J(f_k)^j, \\ \nabla \cdot B &= 0, \quad \partial_t B = -\nabla \times E. \end{aligned}$$

2.1.1 Global in time solutions for the Vlasov-Maxwell system

The global existence for classical solutions to the Vlasov-Maxwell system is still an open problem. They are known to be global in certain particular cases such as under a translation symmetry hypothesis on the initial data in one of the space variables. The pioneer works on this two and one half dimensional case originated from Glassey-Schaeffer in [20] and required a compact support assumption in v . The result obtained recently by Luk-Strain allows data with non-compact velocity support [37]. The solutions to the Vlasov-Maxwell system also appear to be global when they arise from perturbation of spherically symmetric initial data² (see [41]).

For the general case, several continuation criteria are known. The first one, obtained by Glassey and Strauss in [23] (see also [8] and [32] for alternative proofs), expresses that C^1 solutions to the Vlasov-Maxwell system arising from compactly initial data do not develop singularities as long as the velocity supports of the particle densities f_k remain bounded. An improved continuation criteria requires the finiteness of

$$\left\| \sqrt{1 + |v|^2}^\theta f_k \right\|_{L^\infty([0, T^*[, L_x^2 L_v^1)} \tag{2.6}$$

for a certain q and θ , in order to extend the solution beyond $T^* > 0$. Let us mention [38] for the cases $6 \leq q \leq \infty$ and $\theta > \frac{4}{q}$, [48] for $q = +\infty$ and $\theta = 0$ as well as [39] for $q = 6$ and $\theta = 0$. Earlier results of Glassey-Strauss required the boundedness of (2.6) for $q = \infty$ and $\theta = 1$ and cover data with non-compact support in v (see [25]). Recently, Luk-Strain removed in [37] all compact support assumptions and extend the continuation criteria (2.6) for $2 < q \leq +\infty$ and $\theta > \frac{2}{q}$.

2.1.2 Previous work on small data solutions of the Vlasov-Maxwell system

Global existence for small data in dimension 3 was first established by Glassey-Strauss in [24] under a compact support assumption (in x and in v). In particular they proved $\int_v f dv \lesssim \frac{1}{(1+t)^3}$, coinciding with the linear decay, but they did not control $\partial_{\mu_1} \dots \partial_{\mu_p} \int_v f dv$. They also proved decay estimates for the electromagnetic field and its derivatives of first order, but not for the derivatives of higher order. A similar result was proved in [22] for the nearly neutral case, i.e. $\sum_k e_k m_k^3 f_{0k}(x, m_k v)$ has to be small and not the individual particle densities. The result established by Schaeffer in [44] allowed particles with high velocity but still requires the data to be compactly supported in space³. Finally, let us also mention the earlier result of Bardos-Degond for the more classical Vlasov-Poisson system [2]. Under a smallness assumption, they established that the solution of the system is global in time and proved that $\int_v f dv \lesssim \frac{1}{(1+t)^3}$ but they did not obtain informations on the derivatives of f . They also proved decay estimates for the electric field up to second order.

2.1.3 Optimal gradient estimates for Vlasov systems

Due to the linearity of the Maxwell equations and the elliptic nature of the Poisson equation or a nonresonant phenomenon⁴, the previous results are established without essentially commuting the Vlasov equation and

²Recall that for spherically symmetric solutions, the Vlasov-Maxwell system reduces to the relativistic Vlasov-Poisson system.

³Note also that when the Vlasov field is not compactly supported (in v), the decay estimate obtained in [44] for its velocity average contains a loss.

⁴According to [9], the velocity averages of the solutions of a system coupling a linear wave equation with a transport equation are such that the velocity averages are more regular than expected if the speed of propagation of the wave equation is strictly larger than the speed of the particles governed by the transport equation.

controlling higher derivatives of the solutions. For the Vlasov-Poisson system with small data, the sharp time decay estimates $|\int_v \partial_{\mu_1} \dots \partial_{\mu_p} f dv| \lesssim \frac{1}{(1+t)^{3+p}}$ was proved⁵ in [27]. A similar result was obtained in [46] using a vector field method which led to global bounds for the solution and optimal space and time decay rates for the velocity averages. In the same spirit, optimal decay estimates was proved for the derivatives of the solutions of the Vlasov-Nordström system in [18] and [16]. The stability of the Minkowski space for the Einstein-Vlasov system was recently, and independently, proved in [17] and [26]. For both of them, vector field methods was a crucial point in the proof and led in particular to almost optimal decay rates for the derivatives of the solutions.

The goal of this paper is to prove almost optimal decay for the small data solutions and their derivatives of the Vlasov-Maxwell system in dimension $n \geq 4$ without any compact support assumption on the initial data.

2.1.4 The vector field method for Vlasov fields

In this paper, we will use vector field methods to derive decay estimates for both the electromagnetic field and the Vlasov field. The vector field method was originally developed by Klainerman in [29] to study wave equations and was adapted to cover the Maxwell equations (and the spin 2 equations) in $3d$ in [11]. More recently, the method was extended for the free relativistic transport equation in [18].

As in [29], these methods are based on energy estimates, commutation vector fields and weighted Sobolev inequalities. For the transport operator $v^\mu \partial_\mu$, the set $\widehat{\mathbb{P}}_0$ of commutation vector fields used in [18] is composed of the scaling vector field $S = x^\mu \partial_\mu$ and the complete lifts of the generators of the Poincaré group, that is to say the translations

$$\partial_\mu, \quad 0 \leq \mu \leq n,$$

the complete lifts of the rotations

$$\widehat{\Omega}_{ij} = x^i \partial_j - x^j \partial_i + v^i \partial_{v^j} - v^j \partial_{v^i}, \quad 1 \leq i < j \leq n$$

and the complete lifts of the Lorentz boosts

$$\widehat{\Omega}_{0k} = t \partial_k + x^k \partial_t + v^0 \partial_{v^k}, \quad 1 \leq k \leq n.$$

In [18], vector field methods are applied to derive the behavior of the solutions to the Vlasov-Nordström system in the future the hyperboloid⁶ $t^2 - r^2 = 1$. However, without any compact support assumption, one cannot reduce the study of a solution in the future of a $t = \text{constant}$ slice to its study in the future of a hyperboloidal slice (see for instance [18], Appendix A, for more details). In order to remove all compact support assumption on the data, one of the goal of this paper is to start from a $t = 0$ slice and adapt the vector field method for transport equations to the more common foliation $(\{t\} \times \mathbb{R}^n)_{t \geq 0}$. Note that [16] (respectively [17]) use slight modifications of the commutation vector fields⁷ of the operator $v^\mu \partial_\mu$ in order to study the small data solutions of the Vlasov-Nordström (respectively Einstein-Vlasov) system in $3d$. They also use a hyperboloidal foliation and restrict the study of the solutions to the future of a hyperboloid.

2.1.5 The Lorenz gauge

Recall that a 1-form A is a potential of the electromagnetic field F if $F = dA$ (or $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$). If moreover

$$\partial^\mu A_\mu = 0,$$

we say that A satisfies the Lorenz gauge condition. As the energy momentum tensor of F is not traceless in dimension $n \geq 4$ and the Morawetz vector field $K_0 := (t^2 + r^2)\partial_t + 2tr\partial_r$ is merely a *conformal* Killing vector field, we encounter the same difficulty in using it as for the wave equation in $3d$ (see [47], Chapter II for more details). To circumvent this difficulty, we will consider in this paper the Vlasov-Maxwell system in the Lorenz gauge. A_μ will then satisfy the equation

$$\square A_\mu = e^k \int_v \frac{v_\mu}{v^0} f_k dv. \quad (2.7)$$

We also make fundamental use of the Lorenz gauge to establish the optimal decay rate on the component α of the null decomposition of the electromagnetic field, as the method used in [11] cannot be reproduced in dimension $n \neq 3$.

⁵A similar result is established in [10], using the same techniques, for the Vlasov-Yukawa system in dimension 2.

⁶The use of a hyperboloidal foliation in order to establish decay estimates was introduced in [31] in the context of the Klein-Gordon equation.

⁷The modified vector fields are built in order to compensate the worst source terms in the commuted transport equations.

2.1.6 Results for the massive Vlasov-Maxwell system

We will consider weighted L^2 norms to control A and F such as⁸

$$\tilde{\mathcal{E}}_N[A] = \sum_{\mu=0}^n \sum_{|\beta| \leq 1} \sum_{|\gamma| \leq N} \|Z^\beta \mathcal{L}_{Z^\gamma}(A)_\mu\|_{L^2(\mathbb{R}^n)}^2$$

as well as weighted L^1 norms for the Vlasov field, such as

$$\mathbb{E}_N^q[g](t) = \sum_{|\beta| \leq N} \int_{\mathbb{R}_x^n} \int_{\mathbb{R}_v^n} (v^0)^q |\widehat{Z}^\beta g| dv dx + \int_{C_u(t)} \int_{\mathbb{R}_v^n} (v^0)^{q-1} v^L |\widehat{Z}^\beta g| dv dC_u(t),$$

where $C_u(t) := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^n / s \leq t, s - |y| = u\}$. For the Vlasov field, we also use extra norms with the additional weights v^μ , $x^i v^j - x^j v^i$ or $t v^i - x^i v^0$. See Definitions 2.3.2, 2.3.20 and Section 2.2.4 for an introduction to the other norms and the weights.

We are now ready to present our main result for the massive Vlasov-Maxwell system (for a detailed version, see Theorem 2.6.1).

Theorem 2.1.1. *Let $n \geq 4$, $K \geq 2$, and $N \geq \frac{5}{2}n + 1$. Let (f_0, F_0) be an initial data set for the massive Vlasov-Maxwell system with K species. Let (f, F) be the unique classical solution to the system and let A be a potential in the Lorenz gauge. There exists $\epsilon > 0$ such that⁹, if*

$$\tilde{\mathcal{E}}_N[A](0) + \mathcal{E}_N[F](0) + \sum_{k=1}^K \mathbb{E}_{N+n,1}^2[f_k](0) \leq \epsilon,$$

then (f, F) exists globally in time and verifies the following estimates.

- Energy bounds for A , F and f_k such as $\mathbb{E}_N^2[f_k] \lesssim \epsilon$ on \mathbb{R}_+ .
- Sharp pointwise decay estimates on the null decomposition of $\mathcal{L}_{Z^\beta}(F)$ and for the velocity average of f_k and its derivatives. For instance,

$$\forall |\beta| \leq N - \frac{3n+2}{2}, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad \int_{v \in \mathbb{R}^n} |\widehat{Z}^\beta f_k| dv \lesssim \frac{\epsilon}{(1+t+|x|)^n}.$$

Remark 2.1.2. *Since we suppose that the initial energy on F is finite, we are necessarily in the neutral case¹⁰ when the dimension is $n = 4$. On the other hand, when the total charge is non zero, Gauss's law implies that the energy $\mathcal{E}_N[F]$ is infinite. We refer to [3] and [51] for a study of the Maxwell-Klein-Gordon system with a non-zero total charge.*

Remark 2.1.3. *Thanks to the vector field method and in view of the definition of our norms, we do not need any compact support assumption on the initial data. We also automatically obtain improved decay rates for the derivatives of both the electromagnetic field and the velocity averages of the particle densities. For instance, for all $|\beta| \leq N - \frac{3n+2}{2}$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$,*

$$\left| \partial_{t,x}^\beta \int_{v \in \mathbb{R}^n} f_k dv \right| \lesssim \frac{\epsilon}{(1+t+|x|)^n (1+|t-|x||)^{|\beta|}}$$

and (see Section 2.5.4), assuming more decay on the initial data,

$$\left| \partial_{t,x}^\beta \int_{v \in \mathbb{R}^n} f_k dv \right| \lesssim \frac{\epsilon}{(1+t+|x|)^{n+|\beta|}}.$$

Remark 2.1.4. *Notice that in dimension $n = 4$, it is sufficient for $\widehat{Z}^\beta f_k$ to initially decay faster than $(1+|v|)^{-6-\delta}(1+|x|)^{-5-\delta}$, with $\delta > 0$, for our theorem to apply. In [44], which concerns the 3d case, the main result requires the initial particle densities to be compactly supported in x and to decay faster than $(1+|v|)^{-q}$, with $q > 60 + 12\sqrt{17}$.*

⁸For a tensor G and a multi-index $\beta = (\beta_1, \dots, \beta_p)$,

$$\mathcal{L}_{Z^\beta} G = \mathcal{L}_{Z^{\beta_1}} \dots \mathcal{L}_{Z^{\beta_p}} G,$$

while

$$Z \mathcal{L}_{Z^\beta}(G)_\mu = Z[\mathcal{L}_{Z^\beta}(G)_\mu].$$

⁹A smallness condition on F , which implies $\tilde{\mathcal{E}}_N[A](0) \leq \epsilon$, is given in Proposition 2.2.20.

¹⁰In other words, the total charge verifies

$$e^k \int_{x \in \mathbb{R}^n} \int_{v \in \mathbb{R}^n} f_{0k} dv dx = 0.$$

2.1.7 Results for the massless Vlasov-Maxwell system

We now present an elusive version of our main result for massless particles (we refer to Theorem 2.7.1 for more details).

Theorem 2.1.5. *Let $n \geq 4$, $K \geq 2$, $N \geq 6n + 3$ and $R > 0$. Let (f_0, F_0) be an initial data set for the massless Vlasov-Maxwell system with K species, (f, F) be the unique classical solution to the system and A be a potential in the Lorenz gauge. There exists $\epsilon > 0$ and $R > 0$ such that, if*

$$\tilde{\mathcal{E}}_N[A](0) + \mathcal{E}_N[F](0) + \sum_{k=1}^K \mathbb{E}_{N+n,1}^0[f_k](0) \leq \epsilon,$$

$$\forall 1 \leq k \leq K, \quad \text{supp}(f_{0k}) \subset \{(x, v) \in \mathbb{R}_+^n \times \mathbb{R}_v^n \setminus \{0\} \mid |v| \geq R\},$$

then (f, F) exists globally in time and verifies the following properties.

- $f_k(\cdot, \cdot, v)$ vanishes for all $|v| \leq \frac{R}{2}$.
- Energy bounds are propagated for F and f_k . For instance, if $n = 4$, $\mathcal{E}_{N-8}[F](t) \lesssim \epsilon(1+t)$ for all $t \in [0, T]$.
- Pointwise decay estimates on the null decomposition of $\mathcal{L}_{Z^\beta}(F)$ and for the velocity average of f and its derivatives. For instance,

$$\forall |\beta| \leq N - 2n, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad \int_{v \in \mathbb{R}^n \setminus \{0\}} |\widehat{Z}^\beta f_k| dv \lesssim \frac{\epsilon}{\tau_+^{n-1} \tau_-}.$$

Remark 2.1.6. *The hypotheses on the velocity supports of the particle densities appear to be necessary (see Section 2.8).*

Remark 2.1.7. *We are not able to obtain optimal decay estimates for the electromagnetic field in dimension $n = 4$ with our reasoning since the velocity average of the Vlasov field has a weaker decay rate near the light cone when the mass is zero (this is related to the estimate (2.10) mentioned below, which only applies to massive particles).*

2.1.8 The main difficulties and ingredients of our proof

High velocities and null properties of the system

As we use vector field methods, we are brought to commute the equations and prove global bounds on the solutions through energy estimates. After commuting the Vlasov equation once, we are led to estimate terms that could be written schematically as

$$\int_0^t \int_x \int_v |v \mathcal{L}_Z(F) \partial_v f| dv dx ds.$$

Unfortunately, ∂_{v^i} , for $1 \leq i \leq n$, is not part of the commutation vector fields for the transport equation. We rewrite them in terms of the elements of $\widehat{\mathbb{P}}_0$ as

$$\partial_{v^i} f_k = \frac{1}{v_0} (\widehat{\Omega}_{0i} f_k - t \partial_i f_k - x^i \partial_t f_k), \quad (2.8)$$

so that $\partial_v f$ essentially behaves like $t \partial_\mu f$, which is consistent with the behavior of solutions to the free transport equation. This leads us to estimate

$$\int_0^t \int_x \int_v (s + |x|) |\mathcal{L}_Z(F) \partial f| dv dx ds. \quad (2.9)$$

As a solution to a wave equation, $\mathcal{L}_Z(F)$ only decays near the light cone as $\frac{1}{(1+t+|x|)^{\frac{n-1}{2}}}$ and we cannot prove by a naive estimate that, in dimensions $n \leq 5$, (2.9) is uniformly bounded. However, if f is initially compactly supported, one can expect (for, say, sufficiently small data) the characteristics of the transport equation to have velocities bounded away from 1, and thus the Vlasov field support (in x) to be ultimately remote from the light cone. Now, assuming enough initial decay on the Maxwell field, one can prove that

$$|\mathcal{L}_Z(F)| \lesssim (1 + s + |x|)^{-\frac{n-1}{2}} (1 + |s - |x||)^{-\frac{3}{2}}$$

which, combined with the support properties of f , leads to

$$\int_x \int_v (s + |x|) |\mathcal{L}_Z(F) \partial f| dv dx \lesssim (1 + s)^{-\frac{n}{2}}$$

and (2.9) is then uniformly bounded in dimensions¹¹ $n \geq 4$.

In our work, we do not make any compact support assumption. Instead, we make crucial use of null properties of the Vlasov-Maxwell system¹² to deal with the high velocities. More precisely, certain null components of the velocity vector v , the derivatives of the electromagnetic field (as $\mathcal{L}_Z(F)$) or the vector $(0, \partial_{v^1} f, \dots, \partial_{v^n} f)$ behave better than others and the structure of the system is such that there is no product involving only terms with a bad behavior. Taking advantage of the null structure allows us either

- to transform a $t - r$ decay in a $t + r$ one. For instance,

$$|\rho(\mathcal{L}_Z(F))| = \left| \frac{x^i}{r} \mathcal{L}_Z(F)_{i0} \right| \lesssim (1 + s + |x|)^{-\frac{n+1}{2}} (1 + |s - |x||)^{-\frac{1}{2}}.$$

- To transform a $t + r$ loss in a $t - r$ loss using $\frac{x^i}{r} \partial_{v^i} f \sim (t - r) \partial f$.
- Or to exploit the $t - r$ decay. For instance, we will control

$$\int_{C_u(t)} \int_{v \in \mathbb{R}^n} \frac{v^L}{v^0} |\partial f| dv dC_u(t) \leq \mathbb{E}[f](t),$$

so that, by the change of variables $(\underline{u}, u) = (s + r, s - r)$,

$$\int_0^t \int_x \frac{1}{(1 + |s - |x||)^2} \int_v \frac{v^L}{v^0} |\partial f| dv dx ds \leq \int_{u=-\infty}^t \frac{\mathbb{E}[f](t)}{(1 + |u|)^2} du.$$

Improved decay estimates

- *For massive particles.* In [18], two Klainerman-Sobolev inequalities for velocity averages of Vlasov fields were obtained. They imply in particular that, for g a solution to $v^\mu \partial_\mu(g) = 0$,

$$\begin{aligned} \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \int_{\mathbb{R}_v^n} |g|(t, x, v) dv &\lesssim \sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^k \\ |\beta| \leq 3}} \frac{\int_{\mathbb{R}_y^n} \int_{\mathbb{R}_v^n} |\widehat{Z}^\beta g|(0, y, v) dv dy}{(1 + t + |x|)^{n-1} (1 + |t - |x||)}, \\ \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad t^2 - |x|^2 \geq 1, \\ \int_{\mathbb{R}_v^n} |g|(t, x, v) dv &\lesssim \frac{1}{(1 + t)^n} \sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^k \\ |\beta| \leq 3}} \int_{\mathbb{R}_y^n} \int_{\mathbb{R}_v^n} |\widehat{Z}^\beta g|(\sqrt{1 + |y|^2}, y, v) \frac{dv dy}{\sqrt{1 + |y|^2}}. \end{aligned}$$

The first one has the advantage to be based on the foliation $(\{t\} \times \mathbb{R}^n)_{t \geq 0}$ but provides a weak estimate near the light cone. The second one gives a stronger decay rate near the light cone but is based on a hyperboloidal foliation. In this paper, in order to remove all compact support assumption on the data and start from a $t = 0$ slice, we will prove and use a refined version of the Klainerman-Sobolev inequalities of [18]. Our estimate will imply that, for g a solution to $v^\mu \partial_\mu(g) = 0$ and for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\int_{\mathbb{R}_v^n} |g|(t, x, v) dv \lesssim \sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^k \\ |\beta| \leq 3}} \frac{\int_{\mathbb{R}_y^n} \int_{\mathbb{R}_v^n} |v^0|^2 (1 + |y|) |\widehat{Z}^\beta g|(0, y, v) dv dy}{(1 + t + |x|)^n}. \quad (2.10)$$

Compared to the Klainerman-Sobolev inequalities proved in [18], (2.10) cumulates the advantages of giving a strong decay in the whole spacetime and being adapted to the foliation $(\{t\} \times \mathbb{R}^n)_{t \geq 0}$. On the other hand, our estimate is not a pure Sobolev inequality (we used the transport equation satisfied by g to establish it).

¹¹Note that our proof would lead to a \sqrt{t} -loss in dimension 3 which is not sufficient to prove the uniform boundedness of (2.9).

¹²The null structure of the Vlasov-Nordström system is also a main ingredient of the proof of [18] for the dimension $n = 4$.

Remark 2.1.8. *In the exterior of the light cone (where $t \leq |x|$), one can in fact obtain arbitrary decay provided we consider additional decay on the initial data (see Section 2.5.3).*

• *For massless particles.* Unfortunately, (2.10) does not apply to massless particles. Instead, we use weights $z \in \mathbf{k}_0$ preserved by the relativistic transport operator $|v|\partial_t + v^i\partial_i$ in order to gain additional decay. More precisely, in the same spirit as the derivative $\partial_t + \partial_r$ (respectively $\partial_t - \partial_r$) provides an extra decay in $t+r$ (respectively $t-r$) for, say, a solution to $\square u = 0$, one has

$$v^0 - \frac{x^i}{r}v_i \leq \frac{|v|}{1+t+r} \sum_{z \in \mathbf{k}_0} |z| \quad \text{and} \quad v^0 + \frac{x^i}{r}v_i \leq \frac{|v|}{1+|t-r|} \sum_{z \in \mathbf{k}_0} |z|.$$

The problem of the small velocities

For the massless Vlasov-Maxwell system, another problem arises from the small velocities since $v^0 = |v|$ is not bounded by below. The velocity part V of the characteristics of

$$\partial_t f + \frac{v^i}{|v|}\partial_i f + \frac{v^\mu}{|v|}F_\mu^j \partial_{v^j} f = 0$$

solves the ordinary differential equation

$$\dot{V}^j = \frac{V^\mu}{|V|}F_\mu^j,$$

which can lead to $V = 0$ in finite time¹³. More precisely, we prove in Section 2.8 that there exists smooth initial data such that the particle densities f_k do not vanish for small velocities and for which the massless Vlasov-Maxwell system does not even admit a local classical solution.

An important step of the proof of Theorem 2.1.5 then consists in proving that the velocity supports of f_k remain bounded by below. For this, we make crucial use of the smallness of assumption on the electromagnetic field as well as its strong decay rate¹⁴

The perspective of the three dimensional case

Nevertheless, even in making use of the null properties of the system, our proof does not work in dimension 3. One way to treat the $3d$ massive case would be to use modified vector fields in the spirit of [46] for the Vlasov-Poisson system and [16] for the Vlasov-Nordström system. This method led to the proof of the stability of the Minkowski space for the Einstein-Vlasov system (cf [17], [26]), providing sharp estimates on both the Vlasov field and the metric.

2.1.9 Structure of the paper

In section 2.2 we introduce the notations used in the paper, the commutation vector fields and the Vlasov-Maxwell system. In Section 2.3 we establish various energy estimates for solutions to the relativistic transport equation or the Maxwell equations. Section 2.4 contains an integral estimate, some ways to estimate the v derivatives and the tools to prove pointwise decay estimates for the electromagnetic field. Section 2.5 is devoted to our new decay estimate for the solution of a massive relativistic transport equation. In Section 2.6 (respectively 2.7), we prove the global existence and the asymptotic properties of the small data solutions of the massive (respectively massless) Vlasov-Maxwell system, which is Theorem 2.1.1 (respectively Theorem 2.1.5). In Section 2.8, we prove that there exists smooth initial data which do not vanish for small velocities and for which the massless Vlasov-Maxwell system do not admit a local classical solution.

2.1.10 Acknowledgements

This work forms part of my Ph.D. thesis and I obviously would like to thank my advisor Jacques Smulevici for his guidance.

¹³Note that this difficulty does not appear in the Einstein-Vlasov system. Indeed, as in [17], the Vlasov equation can be written, for a metric g , as

$$v_\mu g^{\mu\nu} \partial_\nu f - \frac{1}{2} v_\mu v_\nu \partial_i g^{\mu\nu} \partial_{v^i} f = 0,$$

so these situations should be compared to the two ordinary differential equations

$$\dot{y} = 1 \quad \text{and} \quad \dot{y} = y.$$

¹⁴In dimension $n \geq 4$, $\|F\|_{L_x^\infty} \lesssim (1+t)^{-\frac{3}{2}}$ is time integrable.

2.2 Notations and preliminaries

2.2.1 Basic notations

Throughout this article we work on the $n + 1$ dimensional Minkowski spacetime $(\mathbb{R}^{n+1}, \tilde{\eta})$ and we consider two types of coordinates on it. The Cartesian coordinates (t, x) , in which $\tilde{\eta} = \text{diag}(-1, 1, \dots, 1)$, and null coordinates which are defined by

$$u = t - r, \quad \underline{u} = t + r,$$

and spherical variables (B, C, D, \dots) (always denoted by capital Latin letters¹⁵) which are spherical coordinates on the $(n - 1)$ -dimensional spheres $t, r = \text{constant}$. These coordinates are defined globally on \mathbb{R}^{n+1} apart from the usual degeneration of spherical coordinates and at $r = 0$. The null derivatives L and \underline{L} are defined as

$$L = \partial_t + \partial_r, \quad \underline{L} = \partial_t - \partial_r,$$

and we designate by (e_B, e_C, e_D, \dots) an orthonormal basis on the spheres $(t, r) = \text{constant}$. We will use the weights

$$\tau_+^2 = 1 + \underline{u}^2 \quad \text{and} \quad \tau_-^2 = 1 + u^2.$$

For a 2-form $F_{\mu\nu}$, its Hodge dual is denoted by $*F$, with

$$*F_{\lambda_1 \dots \lambda_{n-1}} = \frac{1}{2} F^{\mu\nu} \varepsilon_{\mu\nu\lambda_1 \dots \lambda_{n-1}}, \quad (2.11)$$

where $\varepsilon_{\lambda_1 \dots \lambda_{n+1}}$ is the Levi-Civita symbol. As in [11], we consider its null-decomposition given by

$$\alpha_B(F) = F_{BL}, \quad \underline{\alpha}_B(F) = F_{B\underline{L}}, \quad \rho(F) = \frac{1}{2} F_{L\underline{L}}, \quad \sigma_{BD}(F) = F_{BD}.$$

We also associate to a 2-form F its energy-momentum tensor

$$T[F]_{\mu\nu} = F_{\mu\beta} F_{\nu}{}^{\beta} - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}.$$

We use Greek letters to denote spacetime indices and Latin letters for space indices. The velocity vector $(v^\beta)_{0 \leq \beta \leq n}$ is parametrized by $(v^i)_{1 \leq i \leq n}$ and $v^0 = \sqrt{m^2 + |v|^2}$. When we study massive particles, we often take $m = 1$ for simplicity so that $v^0 = \sqrt{1 + |v|^2}$. On the other hand, for massless particles $v^0 = |v|$.

We designate the null components of the velocity vector $(v^\beta)_{0 \leq \beta \leq n}$ by $(v^L, v^{\underline{L}}, v^B, \dots)$, i.e.

$$v = v^L L + v^{\underline{L}} \underline{L} + v^B e_B.$$

In particular,

$$v^L = \frac{v^0 + v^r}{2} \quad \text{and} \quad v^{\underline{L}} = \frac{v^0 - v^r}{2}.$$

We now introduce several subsets of $\mathbb{R}_+ \times \mathbb{R}^n$ depending on $t \in \mathbb{R}_+$ or $u \in \mathbb{R}$. Let $\Sigma_t, C_u(t)$ and $V_u(t)$ be defined as

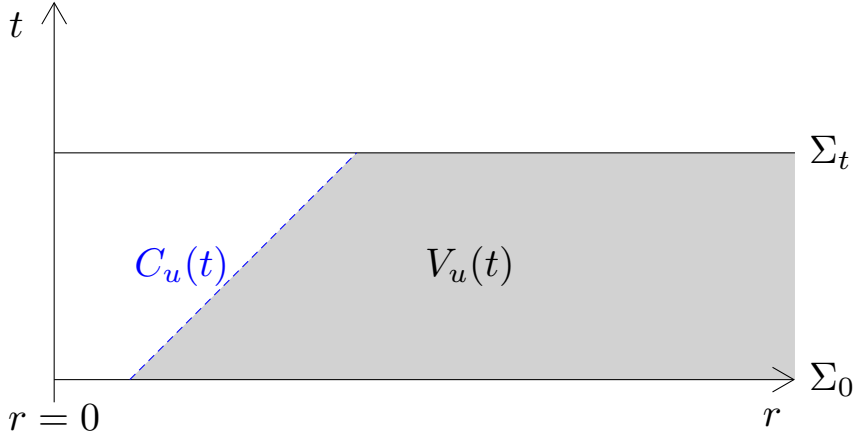
$$\Sigma_t := \{t\} \times \mathbb{R}^n, \quad C_u(t) := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^n / s \leq t, s - |y| = u\}$$

and

$$V_u(t) := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^n / s \leq t, s - |y| \leq u\}.$$

The volum form on $C_u(t)$ is given by $dC_u(t) = \sqrt{2} r^{n-1} du d\underline{u} d\underline{S}^{n-1}$, where $d\underline{S}^{n-1}$ is the standard volume form on the $n - 1$ dimensional unit sphere.

¹⁵The letter A will be reserved for the potential of the electromagnetic.



The sets Σ_t , $C_u(t)$ and $V_u(t)$

We will use the notation $Q_1 \lesssim Q_2$ for an inequality of the form $Q_1 \leq CQ_2$, where $C > 0$ is a positive constant independent of the solutions but which could depend on $N \in \mathbb{N}$, the maximal order of commutation, or fixed parameters. Finally we will raise and lower indices with respect to the Minkowski metric $\tilde{\eta}$. For instance, $F_\mu^j = \tilde{\eta}^{j\nu} F_{\mu\nu}$ so that $F_\mu^j = F_{\mu j}$ for all $1 \leq j \leq n$.

2.2.2 The relativistic transport operator

For $m > 0$, we use the notation T_m to refer to the operator defined, for all $v \in \mathbb{R}^n$, by

$$T_m = v^0 \partial_t + v^i \partial_i,$$

with $v^0 = \sqrt{m^2 + |v|^2}$.

For the massless case ($m = 0$), the relativistic transport operator T_0 is only defined for all $v \in \mathbb{R}^n \setminus \{0\}$ and we have

$$T_m = v^0 \partial_t + v^i \partial_i,$$

with $v^0 = |v|$.

To simplify the notation, we will most of the time take either $m = 1$ or $m = 0$ and we will only use T_1 and T_0 .

2.2.3 Vector fields

The conformal isometries and their complete lifts

Let us consider the set \mathbb{K} composed by the generators of the isometries group of Minkowski spacetime and by the scaling vector field. \mathbb{K} contains

the translations ¹⁶	$\partial_\mu, \quad 0 \leq \mu \leq n,$
the rotations	$\Omega_{ij} = x^i \partial_j - x^j \partial_i, \quad 1 \leq i < j \leq n,$
the hyperbolic rotations	$\Omega_{0k} = t \partial_k + x^k \partial_t, \quad 1 \leq k \leq n,$
the scaling	$S = x^\mu \partial_\mu.$

Sometimes we will only use the Poincaré group $\mathbb{P} := \mathbb{K} \setminus \{S\}$ or the set of the generators of the rotation group, \mathbb{O} (composed of the Ω_{ij}). These vector fields will be used as commutators whereas ∂_t , S and the vector field \bar{K}_0 , defined by

$$\bar{K}_0 = K_0 + \partial_t = \frac{1}{2} \tau_-^2 L + \frac{1}{2} \tau_+^2 L,$$

will be used as multipliers as in [11].

These vector fields are well known to commute with the wave operator, i.e. if a smooth function u satisfies $\square u = 0$, then,

$$\forall Z \in \mathbb{K}, \quad \square Z u = 0.$$

¹⁶In this paper, we will denote ∂_{x^i} , for $1 \leq i \leq n$, by ∂_i and sometimes ∂_t by ∂_0 .

We will use them to commute the Maxwell equations. However, as in [18], we use another set of vector fields to study the Vlasov equation. For this, we use the *complete lift* of a vector field, a classical operation in differential geometry (see [18], Appendix C for more details). For us, the following definition in coordinates will be sufficient.

Definition 2.2.1. *Let Γ be a vector field of the form $\Gamma^\beta \partial_\beta$. Then, the complete lift $\widehat{\Gamma}$ of Γ is defined by*

$$\widehat{\Gamma} = \Gamma^\beta \partial_\beta + v^\gamma \frac{\partial \Gamma^i}{\partial x^\gamma} \partial_{v^i}.$$

We then consider $\widehat{\mathbb{P}}$ the set of the complete lifts of \mathbb{P} given by

$$\widehat{\mathbb{P}} = \{\widehat{Z} / Z \in \mathbb{P}\}.$$

The last set of vector fields required is the following

$$\widehat{\mathbb{P}}_0 = \widehat{\mathbb{P}} \cup \{S\}.$$

We can list the complete lifts that we will manipulate.

Lemma 2.2.2. *For $0 \leq \mu \leq n$,*

$$\widehat{\partial}_\mu = \partial_\mu.$$

For $1 \leq i < j \leq n$,

$$\widehat{\Omega}_{ij} = x^i \partial_j - x^j \partial_i + v^i \partial_{v^j} - v^j \partial_{v^i}.$$

Finally, for $1 \leq k \leq n$,

$$\widehat{\Omega}_{0k} = t \partial_k + x^k \partial_t + v^0 \partial_{v^k}.$$

The following lemma is used in [18] to prove a Klainerman-Sobolev inequality.

Lemma 2.2.3. *Let $f : [0, T] \times \mathbb{R}_x^n \times P \rightarrow \mathbb{R}$, with $P = \mathbb{R}_v^n$ or $P = \mathbb{R}_v^n \setminus \{0\}$, be a sufficiently regular function. Almost everywhere, we have*

$$\begin{aligned} \forall Z \in \mathbb{P}, \quad \left| Z \left(\int_{v \in P} |f| dv \right) \right| &\lesssim \int_{v \in P} |\widehat{Z} f| dv + \int_{v \in P} |f| dv, \\ \left| S \left(\int_{v \in P} |f| dv \right) \right| &\lesssim \int_{v \in P} |S f| dv. \end{aligned}$$

Similar estimates exist for $\int_{v \in \mathbb{R}^n} (v^0)^k |f| dv$. For instance,

$$\left| S \left(\int_{v \in P} v^0 |f| dv \right) \right| \lesssim \int_{v \in P} v^0 |S f| dv.$$

Remark 2.2.4. *When $Z \in \mathbb{P}$ is not a Lorentz boost, we have*

$$\left| Z \left(\int_{v \in P} |f| dv \right) \right| \lesssim \int_{v \in P} |\widehat{Z} f| dv.$$

We consider an ordering on each of the following sets of vector fields : \mathbb{O} , \mathbb{P} , \mathbb{K} , $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{P}}_0$. For simplicity, we introduce \mathbb{L} which represents one of those sets. We can suppose that

$$\mathbb{L} = \{L^i / 1 \leq i \leq |\mathbb{L}|\}.$$

Let $\beta \in \{1, \dots, |\mathbb{L}|\}^r$, with $r \in \mathbb{N}^*$. Then we will denote the differential operator $Z^{\beta_1} \dots Z^{\beta_r}$ by Z^β . For a vector field Y , we will denote by \mathcal{L}_Y the Lie derivative with respect to Y and if $Z^\gamma \in \mathbb{K}^q$, we will write \mathcal{L}_{Z^γ} for $\mathcal{L}_{Z^{\gamma_1}} \dots \mathcal{L}_{Z^{\gamma_q}}$. We can suppose that the orderings are such that if

$$\mathbb{P} = \{Z^i / 1 \leq i \leq |\mathbb{P}|\},$$

then

$$\widehat{\mathbb{P}} = \{\widehat{Z}^i / 1 \leq i \leq |\mathbb{P}|\} \quad \text{and} \quad \widehat{\mathbb{P}}_0 = \{\widehat{Z}^i / 1 \leq i \leq |\mathbb{P}| + 1\}, \quad \text{with} \quad \widehat{Z^{|\mathbb{P}|+1}} = S.$$

Note that even if the scaling is not a complete lift, we will for simplicity denote any vector field of $\widehat{\mathbb{P}}_0$ by \widehat{Z} .

We now introduce some pointwise norms.

Definition 2.2.5. Let U be a smooth p -covariant tensor field defined in \mathbb{R}^n or in \mathbb{R}^{1+n} . For $k \in \mathbb{N}$, the pointwise norm of U with respect to \mathbb{O} , of order k , is defined by

$$|U|_{\mathbb{O},k} = \left(\sum_{|\beta| \leq k} |\mathcal{L}_{Z^\beta} U|^2 \right)^{\frac{1}{2}},$$

with $Z^\beta \in \mathbb{O}^{|\beta|}$ and where

$$|\mathcal{L}_{Z^\beta} U|^2 = \sum_{\lambda_1, \dots, \lambda_p} |\mathcal{L}_{Z^\beta}(U)_{\lambda_1 \dots \lambda_p}|^2,$$

with $\mathcal{L}_{Z^\beta}(U)_{\lambda_1 \dots \lambda_p}$ the Cartesian components of $\mathcal{L}_{Z^\beta}(U)$.

Commutation properties

We have the following commutation relations :

Lemma 2.2.6. Let \mathbb{L} be either \mathbb{O} , \mathbb{P} , \mathbb{K} , $\widehat{\mathbb{P}}$ or $\widehat{\mathbb{P}}_0$. Then

$$\forall L, L' \in \mathbb{L}, \exists C_{LL'\Gamma} \in \mathbb{R}, \text{ such that } [L, L'] = \sum_{\Gamma \in \mathbb{L}} C_{LL'\Gamma} \Gamma.$$

The commutation relations between the vector fields of $\widehat{\mathbb{P}}_0$ and the massive transport operator T_1 (or the massless relativistic transport operator) are similar to those between the vector fields of \mathbb{K} and the wave operator.

Lemma 2.2.7. We have, for $m \in \{0, 1\}$,

$$\forall \widehat{Z} \in \widehat{\mathbb{P}}, \quad [T_m, \widehat{Z}] = 0 \quad \text{and} \quad [T_m, S] = T_m.$$

Proof. This follows easily from Lemma 2.2.2 and the definition of the relativistic transport operator. \square

2.2.4 Weights preserved by the flow

We define, as in [18], the two following sets of weights

$$\mathbf{k}_1 = \left\{ \frac{v^\mu}{v^0} / 0 \leq \mu \leq n \right\} \cup \left\{ x^\mu \frac{v^\nu}{v^0} - x^\nu \frac{v^\mu}{v^0} / \mu \neq \nu \right\},$$

$$\mathbf{k}_0 = \mathbf{k}_1 \cup \left\{ x^\mu \frac{v_\mu}{v^0} \right\}.$$

These weights are solutions to the free transport equation, i.e.

$$\forall z \in \mathbf{k}_0, \quad T_0(z) = 0, \tag{2.12}$$

and

$$\forall z \in \mathbf{k}_1, \quad T_1(z) = 0. \tag{2.13}$$

Thus, if f is a regular function and if $z \in \mathbf{k}_1$, then $T_1(zf) = zT_1(f)$.

Moreover, these weights have also good interactions with the vector fields of $\widehat{\mathbb{P}}_0$.

Lemma 2.2.8. If $\widehat{Z} \in \widehat{\mathbb{P}}_0$, $m \in \{0, 1\}$ and $z \in \mathbf{k}_m$, then either

$$\widehat{Z}(v^0 z) = 0 \quad \text{or} \quad \widehat{Z}(v^0 z) \in v^0 \mathbf{k}_m.$$

This leads to

$$\forall \widehat{Z} \in \widehat{\mathbb{P}}_0, z \in \mathbf{k}_m, \quad |\widehat{Z}(|z|)| \leq \sum_{z \in \mathbf{k}_m} |z|.$$

Proof.

Consider for instance $\widehat{\Omega}_{01}$ and $x^1 v^2 - x^2 v^1$ or $x^2 v^3 - x^3 v^2$. We have

$$\widehat{\Omega}_{01}(x^1 v^2 - x^2 v^1) = t v^2 - x^2 v^0$$

as well as

$$\widehat{\Omega}_{01}(x^2 v^3 - x^3 v^2) = 0.$$

All the other cases are similar. \square

The next proposition shows how these weights can be used to provide us extra decay (at least in the massless case).

Proposition 2.2.9. *Denoting $x^\mu v_\mu$ by s and $x^\nu v^\mu - x^\mu v^\nu$ by $z_{\mu\nu}$, we have*

$$2(t-r)v^L = -\frac{x^i}{r}z_{0i} - s,$$

and

$$2(t+r)v^L = \frac{x^i}{r}z_{0i} - s.$$

We also have

$$\frac{|v^B|}{v^0} \lesssim \frac{1}{\tau_+} \sum_{z \in \mathbf{k}_1} |z|, \quad |v^B| \lesssim \sqrt{v^L v^{\underline{L}}} \quad \text{and} \quad \frac{m^2}{4v^0} \leq v^{\underline{L}}.$$

Remark 2.2.10. *This result should be compared with the identities*

$$(t-r)\underline{L} = S - \frac{x^i}{r}\Omega_{0i},$$

$$(t+r)L = S + \frac{x^i}{r}\Omega_{0i},$$

and

$$re_B = \sum_{1 \leq i < j \leq n} C_B^{i,j} \Omega_{ij},$$

where $C_B^{i,j}$ are bounded functions on the sphere.

Proof.

Let us start by the first two equations. On the one hand,

$$(t^2 - r^2)v^0 = -x^i z_{0i} - ts.$$

On the other hand,

$$(t^2 - r^2)v^r = -t \frac{x^i}{r} z_{0i} - rs.$$

It only remains to take the sum and the difference of these two equations. For the third one, use $|v^B| \leq v^0$ and that $rv^B = C_B^{i,j} z_{ij}$, which implies

$$\begin{aligned} |v^B| &\lesssim \frac{v^0}{r} \sum_{1 \leq i < j \leq n} |z_{ij}| \\ &= \frac{v^0}{tr} \sum_{1 \leq i < j \leq n} \left| x^i \left(\frac{v^j}{v^0} t - x^j + x^j \right) - x^j \left(\frac{v^i}{v^0} t - x^i + x^i \right) \right| \lesssim \frac{v^0}{t} \sum_{q=1}^n |z_{0q}|. \end{aligned}$$

The fourth inequality ensues from $rv^B = C_B^{i,j} z_{ij}$ and

$$\begin{aligned} 4r^2 v^L v^{\underline{L}} &= m^2 r^2 + r^2 |v|^2 - |x^i|^2 |v_i|^2 - 2 \sum_{1 \leq k < l \leq n} x^k x^l v^k v^l \\ &= m^2 r^2 + \sum_{1 \leq k < l \leq n} |z_{kl}|^2, \end{aligned}$$

since $v^0 = \sqrt{m^2 + |v|^2}$. Finally, using the Cauchy-Schwarz inequality,

$$2v^{\underline{L}} = v^0 - \frac{x^i}{r} v_i \geq \frac{m^2}{v^0 + |v|} \geq \frac{m^2}{2v^0}.$$

□

As for the sets of vector fields, we consider an ordering on \mathbf{k}_0 with $x^\mu \frac{v_\mu}{v^0}$ being the last weight. It then gives an ordering on \mathbf{k}_1 too. If $\mathbf{k}_0 = \{z^i / 1 \leq i \leq |\mathbf{k}_0|\}$ and $\beta \in \{1, \dots, |\mathbf{k}_0|\}^r$ with $r \in \mathbb{N}^*$, we denote $z^{\beta_1} \dots z^{\beta_r}$ by z^β .

2.2.5 Decay estimates

Norms

With the vector field method, the pointwise decay estimates are obtained through weighted Sobolev inequalities. In view of the above definitions of the vector fields and weights, we are naturally brought to define the following weighted L^1 and L^2 norms.

Definition 2.2.11. *Let $u : [0, T[\times \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function. For $k \in \mathbb{N}$, we define for all $t \in [0, T[$,*

$$\|u\|_{\mathbb{K},k}(t) := \sum_{\mu=0}^n \sum_{|\beta| \leq k} \|\partial_\mu Z^\beta u(t, \cdot)\|_{L^2(\mathbb{R}^n)},$$

with $Z^\beta \in \mathbb{K}^{|\beta|}$.

Let $f : [0, T[\times \mathbb{R}_x^n \times P \rightarrow \mathbb{R}$ be a smooth function, with $P = \mathbb{R}_v^n$ or $P = \mathbb{R}_v^n \setminus \{0\}$. For $k \in \mathbb{N}$, we define for all $t \in [0, T[$,

$$\|f\|_{\widehat{\mathbb{P}}_0,k}(t) := \sum_{|\beta| \leq k} \|\widehat{Z}^\beta f(t, \cdot, \cdot)\|_{L^1_{x,v}},$$

with $\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|}$.

We also define, for $q \in \mathbb{N}$ and $m \in \{0, 1\}$,

$$\|f\|_{\widehat{\mathbb{P}}_0,k,q,m}(t) := \sum_{|\beta| \leq k} \sum_{|\gamma| \leq q} \|z^\gamma \widehat{Z}^\beta f(t, \cdot, \cdot)\|_{L^1_{x,v}},$$

with $\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|}$ and $z^\gamma \in \mathbf{k}_m^{|\gamma|}$.

Note that $\|u\|_{\mathbb{K},0}$ corresponds to the energy $\sum_{\mu=0}^n \|\partial_\mu u\|_{L^2(\mathbb{R}^n)}$.

Decay estimates for the velocity averages

Recall the Klainerman-Sobolev inequality (see [47], Chapter II). For u a sufficiently regular function such that for all $t \in [0, T[$, $\|u\|_{\mathbb{K}, \frac{n+2}{2}}(t) < +\infty$, we have

$$\forall (t, x) \in [0, T[\times \mathbb{R}^n, \quad |\nabla_{t,x} u(t, x)| \lesssim \frac{\|u\|_{\mathbb{K}, \frac{n+2}{2}}(t)}{(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}}. \quad (2.14)$$

In particular, if $\square u = 0$ then $\|u\|_{\mathbb{K}, \frac{n+2}{2}}$ is constant, as $\square Z^\beta u = 0$ for all $Z^\beta \in \mathbb{K}^{|\beta|}$. It gives us a decay estimate for $\nabla_{t,x} u$.

However if f is a solution to a relativistic transport equation, we cannot expect decay on $\|f\|_{L^\infty_{x,v}}$ (even for the free transport equation $T_1(f) = 0$ or $T_0(f) = 0$). It is only the velocity averages of f , such as $\int_v f dv$, that decay. For instance, we have the following classical estimate.

Lemma 2.2.12. *Let f be the solution of $T_1(f) = 0$ which satisfies $f(0, \cdot, \cdot) = f_0$, with f_0 a smooth function compactly supported in v . Then, if R is such that $f_0(\cdot, v) = 0$ for all $|v| \geq R$,*

$$\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad \int_{v \in \mathbb{R}^n} |f(t, x, v)| dv \leq \frac{\sqrt{1+R^2}^{n+2}}{t^n} \|f_0\|_{L^1_x L^\infty_v}.$$

Proof. We fix $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$. By the method of characteristics, we obtain that

$$\forall v \in \mathbb{R}^n, \quad f(t, x, v) = f_0\left(x - \frac{v}{v_0} t, v\right).$$

We now use the change of variables $y = \frac{v}{v_0}$. Then,

$$\int_{v \in \mathbb{R}^n} |f(t, x, v)| dv = \int_{|y| < 1} |f_0(x - ty, \frac{y}{\sqrt{1-|y|^2}})| \frac{1}{\sqrt{1-|y|^2}^{n+2}} dy.$$

Using the hypothesis on the support of f_0 , we have

$$\int_{v \in \mathbb{R}^n} |f(t, x, v)| dv \leq \sqrt{1+R^2}^{n+2} \int_{|y| < \frac{R}{\sqrt{1+R^2}}} \|f_0(x - ty, \cdot)\|_{L^\infty_v} dy.$$

A last change of variables ($z = x - ty$) gives the result. □

Klainerman-Sobolev inequalities for velocity averages

As we can expect decay on the velocity average of a solution of a relativistic transport equation (and not on the solution itself), we will then use the following Klainerman-Sobolev inequalities.

Theorem 2.2.13. *Let $T > 0$ and f be a smooth function defined on $[0, T[\times \mathbb{R}_x^n \times \mathbb{R}_v^n$ or $[0, T[\times \mathbb{R}_x^n \times (\mathbb{R}_v^n \setminus \{0\})$. Then*

$$\forall (t, x) \in [0, T[\times \mathbb{R}^n, \quad \int_{v \in \mathbb{R}^n} |f(t, x, v)| dv \lesssim \frac{\|f\|_{\widehat{\mathbb{P}}_{0,n}}(t)}{\tau_+^{n-1} \tau_-}.$$

A proof of this inequality can be found in [18] (see Theorem 7). We then deduce the following result.

Corollary 2.2.14. *Let $T > 0$, $q \in \mathbb{N}$, $m \in \{0, 1\}$ and f be a smooth function defined on $[0, T[\times \mathbb{R}_x^n \times \mathbb{R}_v^n$ or $[0, T[\times \mathbb{R}_x^n \times (\mathbb{R}_v^n \setminus \{0\})$. Then*

$$\forall |\gamma| \leq q, (t, x) \in [0, T[\times \mathbb{R}^n, \quad \int_{v \in \mathbb{R}^n} |z^\gamma f(t, x, v)| dv \lesssim \frac{\|f\|_{\widehat{\mathbb{P}}_{0,n,q,m}}(t)}{\tau_+^{n-1} \tau_-},$$

with $z^\gamma \in \mathbf{k}_m^{|\gamma|}$.

Proof.

Let $|\beta| \leq n$, $|\gamma| \leq q$, $\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|}$ and $z^\gamma \in \mathbf{k}_m^{|\gamma|}$. By Lemma 2.2.8, we have

$$|\widehat{Z}^\beta(z^\gamma f)| \lesssim \sum_{|\beta_0| \leq |\beta|} \sum_{|\gamma_0| \leq |\gamma|} |w^{\gamma_0} \widehat{\Gamma}^{\beta_0} f|, \quad (2.15)$$

with $w^{\gamma_0} \in \mathbf{k}_m^{|\gamma_0|}$ and $\widehat{\Gamma}^{\beta_0} \in \widehat{\mathbb{P}}_0^{|\beta_0|}$. It only remains to apply Theorem 2.2.13. \square

Remark 2.2.15. *All the results of this section are true if we add a v^0 -weight (we can for instance study $\int_{v \in \mathbb{R}^n} (v^0)^k |f| dv$, for $k \in \mathbb{Z}$). We just need to modify the norms in the same way. For instance,*

$$\forall (t, x) \in [0, T[\times \mathbb{R}^n, \quad \int_{v \in \mathbb{R}^n} (v^0)^k |f| dv \lesssim \frac{\sum_{|\beta| \leq n} \|(v^0)^k \widehat{Z}^\beta f(t, \cdot, \cdot)\|_{L_{x,v}^1}}{\tau_+^{n-1} \tau_-}.$$

2.2.6 The Vlasov-Maxwell system

Presentation

In order to introduce the Vlasov-Maxwell system, we abusively use the notation

$$\nabla_v f = \begin{pmatrix} 0 \\ \frac{\partial f}{\partial v^1} \\ \vdots \\ \frac{\partial f}{\partial v^n} \end{pmatrix}.$$

For a sufficiently regular function f , we recall that

$$(J(f)^\nu)_{0 \leq \nu \leq n} = \begin{pmatrix} \int_v f dv \\ \int_v f \frac{v^1}{v^0} dv \\ \vdots \\ \int_v f \frac{v^n}{v^0} dv \end{pmatrix},$$

with $v^0 = \sqrt{m^2 + |v|^2}$, where the mass m depends on the species considered.

Let $K \in \mathbb{N}^*$. The equation (1) of the Vlasov-Maxwell system, for the species k , can be rewritten as

$$T_{m_k}(f_k) + e_k F(v, \nabla_v f_k) = 0. \quad (2.16)$$

Note that the initial data needs to satisfy

$$\nabla^i(F_0)_{i0} = e^k J(f_{0k})_0 \quad \text{and} \quad \nabla^i(*F_0)_{i\alpha_1\dots\alpha_{n-3}0} = 0.$$

It is well known that in $3d$ the electric field and the magnetic field are solutions to a wave equation. In dimension n and in the context of the Vlasov-Maxwell system (and more precisely, with equations (2.2) and (2.3)), we have

$$\forall 1 \leq i \leq n, \quad \square E^i = \sum_{k=1}^K e_k \int_{v \in \mathbb{R}^n} \partial_i f_k + \frac{v^i}{\sqrt{m_k^2 + |v|^2}} \partial_t f_k dv, \quad (2.17)$$

with $E^i = F_{0i}$, and¹⁷

$$\forall 1 \leq i < j \leq n, \quad \square F_{ij} = \sum_{k=1}^K e_k \int_{v \in \mathbb{R}^n} \frac{v^j}{\sqrt{m_k^2 + |v|^2}} \partial_i f_k - \frac{v^i}{\sqrt{m_k^2 + |v|^2}} \partial_j f_k dv. \quad (2.18)$$

We end this subsection by the following proposition, which gives an alternative form of the Maxwell equation.

Proposition 2.2.16. *The Maxwell equations*

$$\begin{cases} \nabla^\mu G_{\mu\nu} &= M_\nu \\ \nabla^\mu *G_{\mu\alpha_1\dots\alpha_{n-2}} &= 0, \end{cases}$$

for a 2-form G and a 1-form M , are equivalent to

$$\begin{cases} \nabla_{[\lambda} G_{\mu\nu]} &= 0 \\ \nabla_{[\lambda} *G_{\alpha_1\dots\alpha_{n-1}]} &= (-1)^{n+1} \frac{(n-1)!}{2} \varepsilon_{\lambda\alpha_1\dots\alpha_n} M^{\alpha_n}, \end{cases}$$

Proof.

That ensues from straightforward calculations. Let us consider the equation $\nabla^i G_{i0} = M_0$. For $1 \leq i \leq n$, we denote by $(m_j^i)_{1 \leq j \leq n-1}$ the $n-1$ integers of $\llbracket 1, n \rrbracket \setminus \{i\}$ ranked in ascending order. We have, without any summation,

$$*G_{m_1^i\dots m_{n-1}^i} = G^{0i} \varepsilon_{0im_1^i\dots m_{n-1}^i} = G_{i0} \varepsilon_{im_1^i\dots m_{n-1}^i}.$$

Hence,

$$\nabla^i G_{i0} = \sum_{i=1}^n \varepsilon_{im_1^i\dots m_{n-1}^i} \nabla^i *G_{m_1^i\dots m_{n-1}^i} = \frac{2}{(n-1)!} \nabla_{[1} *G_{2\dots n]}.$$

It only remains to remark that

$$M_0 = (-1)^{n+1} \varepsilon_{1\dots n0} M^0.$$

For the equation $\nabla^\mu *G_{\mu 3\dots n} = 0$, we note that

$$*G_{03\dots n} = G_{12}, \quad *G_{13\dots n} = G_{02}, \quad *G_{2\dots n} = G_{10}.$$

So

$$\nabla^\mu *G_{\mu 3\dots n} = \nabla_0 G_{21} + \nabla_1 G_{02} + \nabla_2 G_{10}.$$

It then comes that

$$\nabla_{[0} G_{12]} = 0.$$

The remaining equations can be treated similarly. □

For the remaining of this section, we consider the maximal smooth solution $(f := (f_1, \dots, f_K), F)$ to the Vlasov-Maxwell system, defined on $[0, T[$, arising from initial data (f_0, F_0) , so that f is a vector valued field (f_1, \dots, f_K) . However, to lighten the notations, we will often denote (by a small abuse of notation) by f only one of the f_i and we will suppose, without loss of generality for the results establish below, that the charge of the species associated to f is equal to 1.

¹⁷In dimension $n > 3$, the magnetic field is a 2-form defined by $B_{ij} = -F_{ij}$ but we make the choice to work with F_{ij} .

The electromagnetic potential

In order to establish energy estimates for the electromagnetic field, it is useful to introduce a potential in the Lorenz gauge.

Definition 2.2.17. A 1-form A is said to be a potential of the electromagnetic field F if

$$F = dA \quad \text{or, in coordinates,} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

A satisfies the Lorenz gauge condition if moreover

$$\partial^\mu A_\mu = 0.$$

Every electromagnetic field F defined on \mathbb{R}^{n+1} , which is contractible, has a potential since $dF = 0$. Furthermore, if A is a potential then, for χ a regular function, $A + d\chi$ is also a potential. In particular, if A is a potential and χ solves

$$\square\chi = -\partial^\mu A_\mu$$

then $A + d\chi$ is a new potential satisfying the Lorenz gauge. The following lemma will be useful to study the derivatives of F in the Lorenz gauge.

Lemma 2.2.18. If A is a potential satisfying the Lorenz gauge for an electromagnetic field G , i.e.

$$dA = G \quad \text{and} \quad \partial^\mu A_\mu = 0,$$

then, for all $Z \in \mathbb{K}$,

$$d\mathcal{L}_Z(A) = \mathcal{L}_Z G \quad \text{and} \quad \partial^\mu \mathcal{L}_Z(A)_\mu = 0.$$

Let us mention the wave equation satisfied by the potential in the Lorenz gauge.

Proposition 2.2.19. Let (f, F) be a solution to the Vlasov-Maxwell system and A be a potential of the electromagnetic field F which satisfies the Lorenz gauge. Then, for all $Z^\beta \in \mathbb{K}^{|\beta|}$ and $0 \leq \mu \leq n$, there exists constants C_γ^μ such that

$$\square \mathcal{L}_{Z^\beta} A_\mu = \sum_{|\gamma| \leq |\beta|} C_\gamma^\mu e^k \int_{v \in \mathbb{R}^n} \frac{v^\mu}{v^0} \widehat{Z}^\gamma f_k dv,$$

with $\widehat{Z}^\gamma \in \widehat{\mathbb{P}}_0^{|\gamma|}$.

Proof.

As

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{and} \quad \partial^\mu A_\mu = 0,$$

we have for $0 \leq \nu \leq n$

$$\partial^\mu \partial_\mu A_\nu = \nabla^\mu F_{\mu\nu}.$$

It remains to apply this to $\mathcal{L}_{Z^\beta} A$ (see Lemma 2.2.18) and to use Proposition 2.2.30 below. □

The following proposition shows how we can construct a potential in the Lorenz gauge which is initially controlled by the energy (at the time 0) of the electromagnetic field.

Proposition 2.2.20. We suppose here that $n \geq 4$. Let $N \in \mathbb{N}$ and let F be a closed 2-form such that all the norms considered below are finite and $F(0, \cdot) \in L^2(\mathbb{R}^n)$. Then, there exists a potential A in the Lorenz gauge such that, for all $|\beta| \leq N$,

$$\begin{aligned} \|\mathcal{L}_{Z^\beta} A\|_{L^2(\mathbb{R}^n)}(0) &\lesssim \sum_{\substack{|\gamma| \leq N-1 \\ 1 \leq i \leq n}} \|(1 + |x|)^{|\gamma|+1} \partial^\gamma F_{0i}(0, \cdot)\|_{L^2(\mathbb{R}^n)} \\ &+ \sum_{\substack{|\gamma| \leq N \\ 1 \leq i \leq n}} \left(\|(1 + |x|)^{|\gamma|} \partial^\gamma \partial^j F_{ji}(0, \cdot)\|_{L_x^2} + \|(1 + |x|)^{|\gamma|+1} \partial^\gamma \partial^j F_{ji}(0, \cdot)\|_{L_x^1} \right), \end{aligned}$$

with $Z^\beta \in \mathbb{K}^{|\beta|}$.

We start by a technical lemma.

Lemma 2.2.21. *Let G such that*

$$\|(1 + |x|)G\|_{L^1(\mathbb{R}^n)} + \|G\|_{L^2(\mathbb{R}^n)} < +\infty \quad \text{and} \quad \int_{\mathbb{R}^n} G dx = 0.$$

Then, denoting by \mathcal{F} the Fourier transform (in x),

$$\left\| \mathcal{F}^{-1} \left(\frac{-1}{|\xi|^2} \mathcal{F}(G) \right) \right\|_{L^2(\mathbb{R}^n)} \lesssim \|(1 + |x|)G\|_{L^1(\mathbb{R}^n)} + \|G\|_{L^2(\mathbb{R}^n)}.$$

Proof. We have

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left(\frac{-1}{|\xi|^2} \mathcal{F}(G) \right) \right\|_{L^2(\mathbb{R}^n)} &= \left\| \frac{1}{|\xi|^2} \mathcal{F}(G) \right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|\mathcal{F}(G)\|_{L^2(|\xi| \geq 1)} + \left\| \frac{1}{|\xi|^4} \mathcal{F}(G)^2 \right\|_{L^1(|\xi| \leq 1)}^{\frac{1}{2}}. \end{aligned}$$

Note now that $\|\mathcal{F}(G)\|_{L^2(|\xi| \geq 1)} \leq \|G\|_{L^2(\mathbb{R}^n)}$. Finally, as $\|(1 + |x|)G\|_{L^1(\mathbb{R}^n)}$ is finite, $\mathcal{F}(G)$ is of class C^1 and vanishes at 0, so, using the mean value theorem,

$$\begin{aligned} \left\| \frac{\mathcal{F}(G)}{|\xi|^4} \right\|_{L^1(|\xi| \leq 1)}^{\frac{1}{2}} &\lesssim \|\nabla_{\xi} \mathcal{F}(G)\|_{L_{\xi}^{\infty}} \left\| \frac{1}{|\xi|^3} \right\|_{L^1(|\xi| \leq 1)}^{\frac{1}{2}} \\ &\lesssim \| |x|G \|_{L^1(\mathbb{R}^n)}, \end{aligned}$$

since $\|F(g)\|_{L_{\xi}^{\infty}} \leq \|g\|_{L_x^1}$ for any L^1 function g . □

The first step of the construction of the suitable potential is contained in the following lemma.

Lemma 2.2.22. *There exists a potential A of the electromagnetic field F satisfying the Lorenz gauge and such that*

$$A_0(0, \cdot) = 0, \quad \partial_t A_0(0, \cdot) = 0,$$

and

$$\forall 1 \leq k \leq n, \quad \|A_k\|_{H^2(\mathbb{R}^n)}(0) \leq \|\partial^j F_{jk}(0, \cdot)\|_{L_x^2} + \|(1 + |x|)\partial^j F_{jk}(0, \cdot)\|_{L_x^1}.$$

This implies in particular that

$$\forall 1 \leq k \leq n, \quad \partial_t A_k(0, \cdot) = F_{0k}(0, \cdot) \quad \text{and} \quad \Delta A_k(0, \cdot) = \partial^i F_{ik}(0, \cdot). \quad (2.19)$$

Proof.

Suppose that A exists. As $\partial_t A_0(0, \cdot) = 0$ and $\partial^{\mu} A_{\mu} = 0$, we have $\partial^i A_i(0, \cdot) = 0$. Combined with $\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} = F_{\mu\nu}$ and $A_0(0, \cdot) = 0$, it comes that at $t = 0$,

$$\forall 1 \leq k \leq n, \quad \partial_t A_k = F_{0k} \quad \text{and} \quad \Delta A_k = \partial^i F_{ik}. \quad (2.20)$$

Moreover, recall from the proof of Proposition 2.2.19 that

$$\forall 0 \leq \nu \leq n, \quad \square A_{\nu} = \nabla^{\mu} F_{\mu\nu}. \quad (2.21)$$

We then define A_{ν} as the solution of the wave equation (2.21) such that $A_0(0, \cdot) = 0$, $\partial_t A_0(0, \cdot) = 0$ and, for all $1 \leq k \leq n$,

$$\partial_t A_k(0, \cdot) = F_{0k}(0, \cdot) \quad \text{and} \quad A_k(0, \cdot) = \mathcal{F}^{-1} \left(\frac{-1}{|\xi|^2} \mathcal{F}(\partial^j F_{jk}) \right) (0, \cdot).$$

Consequently, according to Lemma 2.2.21, $\Delta A_k(0, \cdot) = \partial^j F_{jk}$ and

$$\|A_k\|_{L^2(\mathbb{R}^n)}(0) \leq \|\partial^j F_{jk}(0, \cdot)\|_{L_x^2} + \|(1 + |x|)\partial^j F_{jk}(0, \cdot)\|_{L_x^1}.$$

From classical elliptic equations theory, we have

$$\|\nabla^2 A_k\|_{L^2(\mathbb{R}^n)} = \|\partial^j F_{jk}\|_{L^2(\mathbb{R}^n)}$$

and

$$\nabla A_k \in L^2(\mathbb{R}^n), \quad \text{with} \quad \|\nabla A_k\|_{L^2(\mathbb{R}^n)} \lesssim \|A_k\|_{L^2(\mathbb{R}^n)} + \|\nabla^2 A_k\|_{L^2(\mathbb{R}^n)},$$

which concludes the proof. □

Proof. [Proof of Proposition 2.2.20]

We consider the potential A constructed in

Lemma 2.2.22. In what follows, we omit to specify that all the quantities are considered at $t = 0$. Since, for instance,

$$\forall 1 \leq i, j \leq n, \quad \Omega_{0i}\Omega_{0j}A = x^i\partial_j A + x^i x^j \partial_t \partial_t A,$$

we have (and it is sufficient) to estimate $\|x^\beta \partial_{t,x}^\gamma A\|_{L^2(\mathbb{R}^n)}$, with $|\beta| \leq |\gamma| \leq N$, in order to control

$$\sum_{\substack{Z^\xi \in \mathbb{K}^{|\xi|} \\ |\xi| \leq N}} \|\mathcal{L}_{Z^\xi} A\|_{L^2(\mathbb{R}^n)}(0).$$

Note that, as $\partial^\mu A_\mu = 0$,

$$\|x^\beta \partial_{t,x}^\gamma \partial_t A_0\|_{L^2(\mathbb{R}^n)} \leq \sum_{k=1}^n \|x^\beta \partial_{t,x}^\gamma \partial_k A_k\|_{L^2(\mathbb{R}^n)},$$

so that, since $A_0 = 0$, we only have to bound $\|x^\beta \partial_{t,x}^\gamma A_k\|_{L^2(\mathbb{R}^n)}$, for all $1 \leq k \leq n$.

Let $1 \leq k \leq n$, $|\gamma| \leq N - 1$ and $|\beta| \leq |\gamma| + 1$. Then, since $\partial_t A_k = F_{0k}$ (see Lemma 2.2.22),

$$x^\beta \partial_{t,x}^\gamma \partial_t A_k = x^\beta \partial_{t,x}^\gamma F_{0k}, \quad \text{so} \quad \|x^\beta \partial_{t,x}^\gamma \partial_t A_k\|_{L_x^2} \lesssim \|(1 + |x|)^{|\gamma|+1} \partial^\gamma F_{0k}\|_{L_x^2}.$$

The remaining case, where there are only spatial translations, is treated in the following lemma.

Lemma 2.2.23. *For all $1 \leq k \leq n$, $|\gamma| \leq N$ and $|\beta| \leq |\gamma|$,*

$$\|x^\beta \partial^\gamma A_k\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{|\beta_0| \leq |\gamma_0| \leq N} \|x^{\beta_0} \partial^{\gamma_0} \partial^j F_{jk}\|_{L_x^2} + \|(1 + |x|) x^{\beta_0} \partial^{\gamma_0} \partial^j F_{jk}\|_{L_x^1},$$

where $\gamma, \beta \in \mathbb{N}^n$, $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$ and $\partial^\gamma = \partial_1^{\gamma_1} \dots \partial_n^{\gamma_n}$, so there are no time derivatives.

Proof. We fix $1 \leq k \leq n$ and we proceed by induction on $|\beta|$. As $\Delta A_k = \partial^j F_{jk}$, we have, for all $|\gamma| \leq N - 2$,

$$\forall 1 \leq k \leq n, \quad \Delta \partial^\gamma A_k = \partial^\gamma \partial^j F_{jk}$$

So, by classical elliptic equations theory,

$$\forall |\gamma| \leq N - 2, \quad \|\nabla^2 \partial^\gamma A_k\|_{L^2(\mathbb{R}^n)} = \|\partial^\gamma \partial^j F_{jk}\|_{L^2(\mathbb{R}^n)},$$

implying the result for $|\beta| = 0$ (the case of the lower order derivatives is treated in Lemma 2.2.22).

Let $1 \leq |\beta| \leq N$. We suppose that for all $|\delta| \leq |\gamma| \leq N$ and $|\delta| \leq |\beta| - 1$,

$$\|x^\delta \partial^\gamma A_k\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{|\beta_0| \leq |\gamma_0| \leq N} \|x^{\beta_0} \partial^{\gamma_0} \partial^j F_{jk}\|_{L_x^2} + \|(1 + |x|) x^{\beta_0} \partial^{\gamma_0} \partial^j F_{jk}\|_{L_x^1}.$$

Let γ be a multi-index such that $|\beta| \leq |\gamma| \leq N$. We have

$$\Delta x^\beta \partial^\gamma A_k = \Delta(x^\beta) \partial^\gamma A_k + 2\partial_j(x^\beta) \partial^j \partial^\gamma A_k + x^\beta \partial^\gamma \partial^j F_{jk}. \quad (2.22)$$

The first two terms of the right hand side are equal to zero or can be rewritten as a linear combination of terms like

$$\partial^{\gamma_2}(x^\delta \partial^{\gamma_1} A_k), \quad (2.23)$$

with $|\gamma_2| = 2$, $|\gamma_1| \leq |\gamma| - 1$ and $|\delta| \leq |\gamma_1|$. For instance,

$$2\partial_j(x_1^q) \partial^j \partial_2^q A_k = 2q\partial_1 \partial_2(x_1^{q-1} \partial_2^{q-1} A_k) - 2q(q-1) \partial_2^2(x_1^{q-2} \partial_2^{q-2} A_k).$$

Let B be the right hand side of (2.22) and $G = x^\beta \partial^\gamma \partial^j F_{jk}$. G satisfies the hypothesis of Lemma 2.2.21 and $B - G$ is a linear combination of terms such as (2.23), which implies

$$\left\| \mathcal{F}^{-1} \left(\frac{-1}{|\xi|^2} \mathcal{F}(B - G) \right) \right\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{\substack{|\gamma_1| \leq |\gamma| - 1 \\ |\delta| \leq |\gamma_1|}} \|x^\delta \partial^{\gamma_1} A_k\|_{L^2(\mathbb{R}^n)}.$$

So we only have to prove that

$$x^\beta \partial^\gamma A_k = \mathcal{F}^{-1} \left(\frac{-1}{|\xi|^2} \mathcal{F}(B) \right), \quad (2.24)$$

or (it is equivalent) that $x^\beta \partial^\gamma A_k$ is the L^2 solution of $\Delta \varphi = B$. Recall that the difference of two solutions of this equation is an harmonic polynomial, so that there exists exactly one L^2 solution, given by the right hand side of (2.24). Consequently, there exists $Q_{k,\beta,\gamma} \in L^2(\mathbb{R}^n)$ and $P_{k,\beta,\gamma}$ an harmonic polynomial function such that

$$x^\beta \partial^\gamma A_k = Q_{k,\beta,\gamma} + P_{k,\beta,\gamma}.$$

By the induction hypothesis, $x^\delta \partial^\gamma A_k \in L^2(\mathbb{R}^n)$ for all $|\delta| = |\beta| - 1$, so

$$\frac{P_{k,\beta,\gamma}}{1+|x|} = \frac{x^\beta}{1+|x|} \partial^\gamma A_k - \frac{1}{1+|x|} Q_{k,\beta,\gamma} \in L^2(\mathbb{R}^n).$$

As the dimension is $n \geq 4 > 1$, $P_{k,\beta,\gamma}$ is necessarily zero. □

□

If the dimension n is at least 5, we can do better.

Proposition 2.2.24. *We suppose here that $n \geq 5$. Let $N \in \mathbb{N}$ and let F be a 2-form such that all the norms considered below are finite. There exists a potential in the Lorenz gauge such that, for all $|\beta| \leq N$,*

$$\begin{aligned} \|\mathcal{L}_{Z^\beta} A\|_{L^2(\mathbb{R}^n)}(0) &\lesssim \sum_{\substack{|\gamma| \leq N-1 \\ 1 \leq i \leq n}} \|(1+|x|)^{|\gamma|+1} \partial^\gamma F_{0i}(0, \cdot)\|_{L^2(\mathbb{R}^n)} \\ &\quad + \sum_{\substack{|\gamma| \leq N \\ 1 \leq i \leq n}} \left(\|(1+|x|)^{|\gamma|} \partial^\gamma \partial^j F_{ji}(0, \cdot)\|_{L^2(\mathbb{R}^n)} + \|(1+|x|)^{|\gamma|} \partial^\gamma \partial^j F_{ji}(0, \cdot)\|_{L^1(\mathbb{R}^n)} \right), \end{aligned}$$

with $Z^\beta \in \mathbb{K}^{|\beta|}$.

Proof.

The proof is similar to the previous one. The difference comes from the fact $\xi \mapsto \frac{1}{|\xi|^4}$ is integrable around 0 in \mathbb{R}^n , with $n \geq 5$, which allows us to lower the hypothesis of Lemma 2.2.21. □

Commutation properties

Commutation of the transport equation

We fix the mass $m \in \mathbb{R}_+$ and we denote by T_F the operator¹⁸

$$T_F : g \mapsto T_m(g) + F(v, \nabla_v g),$$

so that $T_F(f) = 0$. We are now interested by the nature of the source terms of the equation $T_F(\widehat{Z}f) = G$.

Lemma 2.2.25. *If $\widehat{Z} \in \widehat{\mathbb{P}}$, then*

$$T_F(\widehat{Z}f) = -\mathcal{L}_Z(F)(v, \nabla_v f).$$

For the scaling, we have

$$T_F(Sf) = 2F(v, \nabla_v f) - \mathcal{L}_S(F)(v, \nabla_v f).$$

¹⁸Note that if the charge e of the species considered is not equal to 1, one just has to consider T_{eF} (in other words, one just has to replace F by eF).

Proof.

First of all, let us consider the scaling. According to Lemma 2.2.7,

$$T_m(Sf) = -S(F(v, \nabla_v f)) + T_m(f).$$

But,

$$S(F(v, \nabla_v f)) = \mathcal{L}_S(F)(v, \nabla_v f) + F([S, v], \nabla_v f) + F(v, [S, \nabla_v f]).$$

Since

$$[S, v] = -v \quad \text{and} \quad [S, \nabla_v f] = \nabla_v S(f) - \nabla_v f,$$

we obtain

$$T_F(Sf) = 2F(v, \nabla_v f) - \mathcal{L}_S(F)(v, \nabla_v f).$$

Now, let $\widehat{Z} \in \widehat{\mathbb{P}}$ and consider $Z_v = \widehat{Z} - Z$. According to lemma 2.2.7,

$$T_m(\widehat{Z}f) = -Z(F(v, \nabla_v f)) - Z_v(F(v, \nabla_v f)).$$

On the one hand, we have

$$Z_v(F(v, \nabla_v f)) = F(Z_v(v), \nabla_v f) + F(v, Z_v(\nabla_v f)).$$

On the other hand we have

$$Z(F(v, \nabla_v f)) = \mathcal{L}_Z(F)(v, \nabla_v f) + F([Z, v], \nabla_v f) + F(v, [Z, \nabla_v f]).$$

As $[Z, v] = -Z_v(v)$, $F(Z_v(v), \nabla_v f)$ and $F([Z, v], \nabla_v f)$ cancel.

If \widehat{Z} is a translation (we denote it by ∂), then $Z_v = 0$ and $[Z, \nabla_v f] = \nabla_v \partial(f)$. Thus

$$T_F(\partial f) = -\mathcal{L}_\partial(F)(v, \nabla_v f).$$

If $\widehat{Z} = \widehat{\Omega}_{ij}$, then

$$Z_v(\nabla_v f) = \nabla_v Z_v(f) + \partial_{v^i} f \partial_j - \partial_{v^j} f \partial_i$$

and

$$[Z, \nabla_v f] = \nabla_v Z(f) - \partial_{v^i} f \partial_j + \partial_{v^j} f \partial_i.$$

Therefore

$$T_F(\widehat{\Omega}_{ij}f) = -\mathcal{L}_{\Omega_{ij}}(F)(v, \nabla_v f).$$

Finally, if $\widehat{Z} = \widehat{\Omega}_{0i}$, then

$$Z_v(\nabla_v f) = \nabla_v Z_v(f) - \partial_{v^i} f \frac{v^k}{v^0} \partial_k \quad \text{and} \quad [Z, \nabla_v f] = \nabla_v Z(f) - \partial_{v^i} f \partial_0.$$

It comes that

$$T_F(\widehat{\Omega}_{0i}f) = -\mathcal{L}_{\Omega_{0i}}(F)(v, \nabla_v f) + \frac{\partial_{v^i} f}{v^0} F(v, v).$$

It remains to remark that $F(v, v) = 0$ for all $v \in \mathbb{R}^n$, as F is a 2-form. □

Iterating the above, one obtains

Corollary 2.2.26. *If $\beta \in \{1, \dots, |\widehat{\mathbb{P}}_0|\}^r$, with $r \geq 0$, there exist integers $C_{\gamma, \delta}^\beta$ such that*

$$T_F(\widehat{Z}^\beta f) = \sum_{\substack{|\gamma| + |\delta| \leq r \\ |\delta| \leq r-1}} C_{\gamma, \delta}^\beta \mathcal{L}_{Z^\gamma}(F)(v, \nabla_v \widehat{Z}^\delta(f)),$$

with $\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^r$, $\widehat{Z}^\delta \in \widehat{\mathbb{P}}_0^{|\delta|}$ and $Z^\gamma \in \mathbb{K}^{|\gamma|}$.

Remark 2.2.27. *If there is a source term G (such that $T_F(f) = G$), then we need to add a linear combination of terms such as $\widehat{Z}^\beta G$, with $|\beta| \leq r$, on the right hand side.*

Commutation of the Maxwell equations

Before studying specifically the Vlasov-Maxwell system, we recall the following general result.

Proposition 2.2.28. *Let M_ν be a smooth 1-form and $G_{\mu\nu}$ a 2-form satisfying*

$$\begin{cases} \nabla^\mu G_{\mu\nu} &= M_\nu \\ \nabla^{\mu*} G_{\mu\alpha_1 \dots \alpha_{n-2}} &= 0. \end{cases}$$

Then, for all $Z \in \mathbb{P}$,

$$\begin{cases} \nabla^\mu \mathcal{L}_Z(G)_{\mu\nu} &= \mathcal{L}_Z(M)_\nu \\ \nabla^{\mu*} \mathcal{L}_Z(G)_{\mu\alpha_1 \dots \alpha_{n-2}} &= 0. \end{cases}$$

For the scaling, we have

$$\begin{cases} \nabla^\mu \mathcal{L}_S(G)_{\mu\nu} &= \mathcal{L}_S(M)_\nu + 2M_\nu \\ \nabla^{\mu*} \mathcal{L}_S(G)_{\mu\alpha_1 \dots \alpha_{n-2}} &= 0. \end{cases}$$

In the Vlasov-Maxwell system, the source term is $e^k J(f_k)_\nu$ (see (2.2)), with

$$(J(f_k)^\nu)_{0 \leq \nu \leq 3} = \begin{pmatrix} \int_v f_k dv \\ \int_v f_k \frac{v^1}{v^0} dv \\ \vdots \\ \int_v f_k \frac{v^n}{v^0} dv \end{pmatrix},$$

so we need to compute $\mathcal{L}_Z(J(f))$, with $Z \in \mathbb{K}$ and f a regular function.

Proposition 2.2.29. *For all $Z \in \mathbb{P}$,*

$$\mathcal{L}_Z(J(f)_\nu) = J(\widehat{Z}f)_\nu.$$

For the scaling, we have

$$\mathcal{L}_S(J(f)_\nu) = J(Sf)_\nu + J(f)_\nu.$$

Proof.

Let $Z \in \mathbb{K}$,

$$\mathcal{L}_Z J(f)_\nu = Z J(f)_\nu + J(f)_\mu \frac{\partial Z^\mu}{\partial x^\nu}.$$

So

$$L_\partial J(f) = J(\partial f), \quad L_S J(f) = J(Sf) + J(f).$$

If Z is a Lorentz boost, say $x^1 \partial_t + t \partial_1$, then, as

$$\begin{aligned} \int_v v^0 \partial_{v^1} f dv &= - \int_v f \frac{v^1}{v^0} dv = -J(f)_1, \\ \int_v v^0 \frac{v^i}{v^0} \partial_{v^1} f dv &= -\delta_{1,i} \int_v f dv = \delta_{1,i} J(f)_0 \end{aligned}$$

and

$$J(f)_\mu \frac{\Omega_{01}^\mu}{\partial x^\nu} = J(f)_1 \delta_{\nu,0} + J(f)_0 \delta_{\nu,1},$$

it comes that

$$\mathcal{L}_{\Omega_{01}} J(f) = J(\widehat{\Omega_{01}}f).$$

The case where Z is a rotation is similar. □

Iterating the above, we obtain the following proposition.

Proposition 2.2.30. *Let (f, F) be a smooth solution of the Vlasov-Maxwell system. For all $\beta \in \{1, \dots, |\mathbb{K}|\}^r$, with $r \in \mathbb{N}$, there exist integers C_γ^β such that*

$$\begin{aligned}\nabla^\mu \mathcal{L}_{Z^\beta}(F)_{\mu\nu} &= e^k J(\widehat{Z}^\beta f_k)_\nu + \sum_{|\gamma| \leq |\beta| - 1} C_\gamma^\beta e^k J(\widehat{Z}^\gamma f_k)_\nu, \\ \nabla^{\mu*} \mathcal{L}_{Z^\beta}(F)_{\mu\alpha_1 \dots \alpha_{n-2}} &= 0,\end{aligned}$$

with $Z^\beta \in \mathbb{K}^r$ and $\widehat{Z}^\gamma \in \widehat{\mathbb{P}}_0^{|\gamma|}$.

2.3 Energy estimates for the Vlasov-Maxwell system

For all this section, we consider a sufficiently regular solution (f, F) , on $[0, T[$, to the Vlasov-Maxwell system arising from smooth initial data (f_0, F_0) .

2.3.1 Energy estimates for the transport equation

We treat here the massless and the massive case together. As the set $\{v = 0\}$ is of measure zero, we write $\int_{v \in \mathbb{R}^n} h dv$, or merely $\int_v h dv$, even when the function h is not defined for $v = 0$. We start by introducing the vector field $N^\mu(g)$ defined by, for a function $g : [0, T[\times \mathbb{R}_x^n \times \mathbb{R}_v^n \rightarrow \mathbb{R}$,

$$N^\mu(g) := \int_{v \in \mathbb{R}^n} g v^\mu \frac{dv}{v^0}.$$

We have the following energy estimates.

Proposition 2.3.1. *Let g and H be two smooth functions defined on $[0, T[\times \mathbb{R}_x^n \times \mathbb{R}_v^n$ such that $T_F(g) = H$ and $k \in \mathbb{Z}$. Then, for all $t \in [0, T[$,*

$$\int_{\Sigma_t} \int_{\mathbb{R}_v^n} |g| dv dx + \sqrt{2} \sup_{u \leq t} \int_{C_u(t)} \int_{\mathbb{R}_v^n} |g| \frac{v^L}{v^0} dv dC_u(t) \leq 2 \int_{\Sigma_0} \int_{\mathbb{R}_v^n} |g| dv dx + 2 \int_0^t \int_{\Sigma_s} \int_{\mathbb{R}_v^n} |H| \frac{dv}{v^0} dx ds.$$

Proof. First, let us compute the (euclidian) divergence of $N^\mu(|g|)$. Start by noticing that, in $W^{1,1}$,

$$T_m(|g|) = v^\mu \partial_\mu |g| = \frac{g}{|g|} H - F(v, \nabla_v |g|).$$

By integrations by parts and using $F_{jj} = 0$ as well as $v^i v^j F_{ij} = 0$ (recall that F is a 2-form), we have

$$\int_v F(v, \nabla_v |g|) \frac{dv}{v^0} = \int_v \frac{v^\mu}{v^0} F_\mu^j \partial_{v^j} |g| dv = \int_v \frac{v^i v^j}{(v^0)^3} F_{ij} |g| dv = 0.$$

Consequently,

$$\partial_\mu N^\mu(|g|) = \int_{v \in \mathbb{R}^n} \left(\frac{g}{|g|} H - F(v, \nabla_v |g|) \right) \frac{dv}{v^0} = \int_{v \in \mathbb{R}^n} \frac{g}{|g|} H \frac{dv}{v^0}. \quad (2.25)$$

We now apply the divergence theorem to $N^\mu(|g|)$ in several region. Applied to $[0, t] \times \mathbb{R}^n$, it gives

$$\int_{\Sigma_t} \int_v |g| dv dx \leq \int_{\Sigma_0} \int_v |g| dv dx + \int_0^t \int_{\Sigma_s} \int_v |H| \frac{dv}{v^0} dx ds.$$

Applied to $V_u(t)$ and using that $\frac{1}{\sqrt{2}}(\partial_t - \partial_r)$ is the outward pointing unit normal field to $C_u(t)$, it gives

$$\sqrt{2} \int_{C_u(t)} \int_{v \in \mathbb{R}^n} |g| \frac{v^L}{v^0} dv dC_u(t) \leq \int_{\Sigma_0} \int_v |g| dv dx + \int_0^t \int_{\Sigma_s} \int_v |H| \frac{dv}{v^0} dx ds.$$

The estimate then ensues from the combination of the two inequalities. \square

This estimate invites us to consider the following energies.

Definition 2.3.2. For $N \in \mathbb{N}$ and $k \in \mathbb{Z}$, we define, for g a sufficiently regular function,

$$\mathbb{E}_N^k[g](t) = \sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|} \\ |\beta| \leq N}} \|(v^0)^k \widehat{Z}^\beta g\|_{L_{x,v}^1}(t) + \sup_{u \in \mathbb{R}} \int_{C_u(t)} \int_{\mathbb{R}^n} |\widehat{Z}^\beta g|(v^0)^k \frac{v^L}{v^0} dv dC_u(t).$$

We also need the following norms. For $q \in \mathbb{N}$ and $m \in \{0, 1\}$,

$$\begin{aligned} \mathbb{E}_{N,q,m}^k[g](t) &= \sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|} \\ |\beta| \leq N}} \sum_{\substack{z^\gamma \in \mathbf{k}_m^{|\gamma|} \\ |\gamma| \leq q}} \|(v^0)^k z^\gamma \widehat{Z}^\beta g\|_{L_{x,v}^1}(t) \\ &\quad + \sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|} \\ |\beta| \leq N}} \sum_{\substack{z^\gamma \in \mathbf{k}_m^{|\gamma|} \\ |\gamma| \leq q}} \sup_{u \in \mathbb{R}} \int_{C_u(t)} \int_{v \in \mathbb{R}^n} |z^\gamma \widehat{Z}^\beta g| \frac{v^L}{v^0} (v^0)^k dv dC_u(t). \end{aligned}$$

When $k = 0$, we drop the dependance in k of the energy norm. For instance, $\mathbb{E}_N^0[g]$ is denoted by $\mathbb{E}_N[g]$.

The following energy estimates hold.

Proposition 2.3.3. Let g and H be such that $T_F(g) = H$. Then, assuming that g and H are sufficiently regular, we have for all $N \in \mathbb{N}$ and for all $t \in [0, T[$,

$$\begin{aligned} \mathbb{E}_N[g](t) - 2\mathbb{E}_N[g](0) &\lesssim \sum_{|\beta| \leq N} \int_0^t \left\| \frac{1}{v^0} \widehat{Z}^\beta H \right\|_{L_{x,v}^1}(s) ds \\ &\quad + \sum_{\substack{|\gamma|+|\delta| \leq N \\ |\delta| \leq N-1}} \int_0^t \left\| \mathcal{L}_{Z^\gamma}(F) \left(\frac{v}{v^0}, \nabla_v \widehat{Z}^\delta(g) \right) \right\|_{L_{x,v}^1}(s) ds \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}_N^2[g](t) - 2\mathbb{E}_N^2[g](0) &\lesssim \sum_{|\beta| \leq N} \int_0^t \|v^0 \widehat{Z}^\beta H\|_{L_{x,v}^1}(s) + \|v^i F_{i0} \widehat{Z}^\beta g\|_{L_{x,v}^1}(s) ds \\ &\quad + \sum_{\substack{|\gamma|+|\delta| \leq N \\ |\delta| \leq N-1}} \int_0^t \|\mathcal{L}_{Z^\gamma}(F)(v, \nabla_v \widehat{Z}^\delta(g)) v^0\|_{L_{x,v}^1}(s) ds, \end{aligned}$$

with $\widehat{Z}^\delta \in \widehat{\mathbb{P}}_0^{|\delta|}$, $\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|}$ and $Z^\gamma \in \mathbb{K}^{|\gamma|}$.

Proof.

The first estimate follows from Corollary 2.2.26, Remark 2.2.27 and Proposition 2.3.1, applied to $\widehat{Z}^\beta g$ for $|\beta| \leq N$. For the second one, apply the same results to $(v^0)^2 \widehat{Z}^\beta g$ and note that

$$T_F((v^0)^2) = F(v, \nabla_v (v^0)^2) = 2v^\mu v^i F_{\mu i} = -2v^\mu v^0 F_{\mu 0} = -2v^i v^0 F_{i0}.$$

□

Remark 2.3.4. Assuming enough decay on the data, similar inequalities holds for $\mathbb{E}_N^k[g]$.

We also have an energy estimates which implies the weights transported by the flow.

Proposition 2.3.5. Let g and H be two sufficiently regular functions such that $T_F(g) = H$. For all $N \in \mathbb{N}$, $m \in \{0, 1\}$ and $t \in [0, T[$, we have

$$\mathbb{E}_{N,1,m}[g](t) - 2\mathbb{E}_{N,1,m}[g](0) \lesssim \sum_{z \in \mathbf{k}_m} \sum_{|\beta| \leq N} \int_0^t \left\| \frac{z}{v^0} \widehat{Z}^\beta H \right\|_{L_{x,v}^1}(s) ds$$

$$\begin{aligned}
& + \sum_{z \in \mathbf{k}_m} \sum_{|\beta| \leq N} \int_0^t \left\| F \left(\frac{v}{v^0}, \nabla_v z \right) \widehat{Z}^\beta g \right\|_{L_{x,v}^1} (s) ds \\
& + \sum_{z \in \mathbf{k}_m} \sum_{\substack{|\gamma|+|\delta| \leq N \\ |\delta| \leq N-1}} \int_0^t \left\| z \mathcal{L}_{Z^\gamma} (F) \left(\frac{v}{v^0}, \nabla_v \widehat{Z}^\delta g \right) \right\|_{L_{x,v}^1} (s) ds
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_{N,1,m}^2[g](t) - 2\mathbb{E}_{N,1,m}^2[g](0) & \lesssim \sum_{z \in \mathbf{k}_m} \sum_{|\beta| \leq N} \int_0^t \left\| v^0 z \widehat{Z}^\beta H \right\|_{L_{x,v}^1} (s) ds \\
& + \sum_{z \in \mathbf{k}_m} \sum_{|\beta| \leq N} \int_0^t \left\| z v^i F_{i0} \widehat{Z}^\beta g \right\|_{L_{x,v}^1} (s) + \left\| v^0 F(v, \nabla_v z) \widehat{Z}^\beta g \right\|_{L_{x,v}^1} (s) ds \\
& + \sum_{z \in \mathbf{k}_m} \sum_{\substack{|\gamma|+|\delta| \leq N \\ |\delta| \leq N-1}} \int_0^t \left\| v^0 z \mathcal{L}_{Z^\gamma} (F)(v, \nabla_v \widehat{Z}^\delta g) \right\|_{L_{x,v}^1} (s) ds,
\end{aligned}$$

with $\widehat{Z}^\delta \in \widehat{\mathbb{P}}_0^{|\delta|}$, $\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|}$ and $Z^\gamma \in \mathbb{K}^{|\gamma|}$.

Proof. Note that, for $z \in \mathbf{k}_m$ and according to equations (2.12) and (2.13),

$$T_F \left(z \widehat{Z}^\beta g \right) = z T_F \left(\widehat{Z}^\beta g \right) + T_F(z) \widehat{Z}^\beta g = z T_F \left(\widehat{Z}^\beta g \right) + F(v, \nabla_v z) \widehat{Z}^\beta g.$$

The remaining of the proof is then similar to the one of Proposition 2.3.3. \square

2.3.2 Energy estimates for the wave equation

Recall that a potential A in the Lorenz gauge satisfies the wave equation 2.7. In order to bound its L^2 norm, we recall here a classical energy estimates for the wave equation using the vector field \overline{K}_0 . We mostly follow [47], Chapter II.

During this subsection, we consider $u : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ a smooth function such that

$$\|u\|_{L^2(\mathbb{R}^n)}(0) + \sum_{Z \in \mathbb{K}} \|Zu\|_{L^2(\mathbb{R}^n)}(0) < +\infty.$$

We also introduce its energy momentum tensor

$$T_{\mu\nu}[u] = \partial_\mu u \partial_\nu u - \frac{1}{2} \eta_{\mu\nu} \eta^{\sigma\rho} \partial_\sigma u \partial_\rho u.$$

Since \overline{K}_0 is merely a conformal Killing vector field and as $T_{\mu\nu}[u]$ is not traceless, $T_{\mu\nu}[u] \overline{K}_0^\nu$ is not divergence free when $\square u = 0$. In fact

$$\nabla^\mu (T_{\mu\nu}[u] \overline{K}_0^\nu) = \square u \overline{K}_0 u + \frac{1}{2} T_{\mu\nu}[u] \pi^{\mu\nu},$$

with

$$\pi^{\mu\nu} = \partial^\mu \overline{K}_0^\nu + \partial^\nu \overline{K}_0^\mu.$$

Since \overline{K}_0 is a conformal vector field of conformal factor $4t$, $\pi^{\mu\nu} = 4t \eta^{\mu\nu}$. So

$$\nabla^\mu (T_{\mu\nu}[u] \overline{K}_0^\nu) = \square u \overline{K}_0 u + (1-n)t \partial^\mu u \partial_\mu u.$$

The equality

$$t \partial^\mu u \partial_\mu u = \partial_\mu (tu \partial^\mu u) - \partial_\mu (t) u \partial^\mu u - tu \square u = \partial^\mu \left(tu \partial_\mu u - \frac{1}{2} u^2 \partial_\mu t \right) - tu \square u,$$

suggests us to introduce the 1-form

$$P_\mu = T_{\mu\nu}[u] \overline{K}_0^\nu + (n-1)tu \partial_\mu u - \frac{n-1}{2} u^2 \partial_\mu t.$$

Applying the divergence theorem on $[0, t] \times \mathbb{R}^n$ to $T_{\mu 0}[u]$ and P_μ , we obtain

Proposition 2.3.6. $\forall t \in [0, T[$,

$$\sum_{\mu=0}^n \|\partial_\mu u\|_{L^2(\Sigma_t)} \leq \sum_{\mu=0}^n \|\partial_\mu u\|_{L^2(\Sigma_0)} + \int_0^t \int_{\Sigma_s} |\square u| d\Sigma_s ds$$

and

$$\int_{\Sigma_t} P_0 d\Sigma_t \leq \int_{\Sigma_0} P_0 d\Sigma_0 + \int_0^t \int_{\Sigma_s} |\square u| |\bar{K}_0 u + (n-1)tu| d\Sigma_s ds.$$

The first thing to verify is that $\int_{\Sigma_t} P_0 d\Sigma_t$ can be compared with the L^2 norm of u (and of its derivatives).

Proposition 2.3.7. *We suppose that $n \geq 3$. We have, for all $t \in [0, T[$,*

$$\sum_{|\beta| \leq 1} \|Z^\beta u\|_{L^2(\mathbb{R}^n)}^2(t) \lesssim \int_{\Sigma_t} P_0 d\Sigma_t \lesssim \sum_{|\beta| \leq 1} \|Z^\beta u\|_{L^2(\mathbb{R}^n)}^2(t).$$

Proof.

Let us first remark that

$$P_0 = \frac{1}{2}(1 + |x|^2 + t^2)|\nabla_{t,x} u|^2 + 2tx^i \partial_i u \partial_t u + (n-1)tu \partial_t u - \frac{n-1}{2}u^2.$$

Moreover,

$$(1 + |x|^2 + t^2)|\nabla_{t,x} u|^2 + 4tx^i \partial_i u \partial_t u = |\nabla_{t,x} u|^2 + |Su|^2 + \sum_{0 \leq \mu < \nu \leq n} |\Omega_{\mu\nu} u|^2$$

together with

$$\int_{\mathbb{R}^n} 2tu \partial_t u dx = \int_{\mathbb{R}^n} 2uSu - x^i \partial_i (u^2) dx = \int_{\mathbb{R}^n} 2uSu + nu^2 dx \quad (2.26)$$

gives

$$\int_{\Sigma_t} P_0 d\Sigma_t = \frac{1}{2} \int_{\Sigma_t} |\nabla_{t,x} u|^2 + |Su + (n-1)u|^2 + \sum_{0 \leq \mu < \nu \leq n} |\Omega_{\mu\nu} u|^2 d\Sigma_t. \quad (2.27)$$

This proves the second inequality and reduces the first one to

$$\|u\|_{L^2(\mathbb{R}^n)}^2(t) + \|Su\|_{L^2(\mathbb{R}^n)}^2(t) \lesssim \int_{\Sigma_t} P_0 d\Sigma_t.$$

In order to transform $\int_{\mathbb{R}^n} 2tu \partial_t u dx$ in an alternative expression, we remark that

$$2u \partial_t u = 2u \frac{1}{r} \Omega_{0r} u - \frac{t}{r^2} x^i \partial_i (u^2), \quad \text{with } \Omega_{0r} = \frac{x^i}{r} \Omega_{0i}.$$

So, by integration by parts,

$$\int_{\mathbb{R}^n} 2tu \partial_t u dx = \int_{\mathbb{R}^n} \left(2\frac{t}{r} u \Omega_{0r} u + (n-2) \frac{t^2}{r^2} u^2 \right) dx.$$

Combined with equation (2.26), we get

$$\begin{aligned} \int_{\mathbb{R}^n} (2(n-1)tu \partial_t u - (n-1)u^2) dx &= \frac{2n-3}{2} \int_{\mathbb{R}^n} (2uSu + nu^2) dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} \left(2\frac{t}{r} u \Omega_{0r} u + (n-2) \frac{t^2}{r^2} u^2 \right) dx - (n-1) \int_{\mathbb{R}^n} u^2 dx. \end{aligned}$$

It then comes that

$$\begin{aligned} 2 \int_{\Sigma_t} P_0 dx &= \int_{\Sigma_t} |Su|^2 + 2\frac{2n-3}{2} uSu + \frac{2n^2-5n+2}{2} u^2 dx \\ &\quad + \int_{\Sigma_t} |\nabla_{t,x} u|^2 + \sum_{\mu < \nu} |\Omega_{\mu\nu} u|^2 - |\Omega_{0r} u|^2 + |\Omega_{0r} u|^2 + \frac{t}{r} u \Omega_{0r} u + (n-2) \frac{t^2}{2r^2} u^2 dx. \end{aligned} \quad (2.28)$$

The integral in (2.28) is nonnegative since

$$|\Omega_{0r}u|^2 = \left| \frac{x^i}{r} \Omega_{0i}u \right|^2 \leq \sum_{i=1}^n |\Omega_{0i}u|^2$$

and

$$|\Omega_{0r}u|^2 + \frac{t}{r}u\Omega_{0r}u + (n-2)\frac{t^2}{2r^2}u^2 = \left(\Omega_{0r}u + \frac{t}{2r}u \right)^2 + (2n-5)\frac{t^2}{4r^2}u^2.$$

Consequently,

$$\|u\|_{L^2(\mathbb{R}^n)}^2(t) + \|Su\|_{L^2(\mathbb{R}^n)}^2(t) \lesssim \int_{\Sigma_t} P_0 d\Sigma_t$$

comes from

$$|Su|^2 + 2\frac{2n-3}{2}uSu + \frac{2n^2-5n+2}{2}u^2 = \left(Su + \frac{2n-3}{2}u \right)^2 + \frac{2n-5}{4}u^2$$

and from

$$\begin{aligned} |Su|^2 + 2\frac{2n-3}{2}uSu + \frac{2n^2-5n+2}{2}u^2 = \\ \left(\frac{2n-3}{\sqrt{4n^2-10n+4}}Su + \left(n^2 - \frac{5}{2}n + 1 \right)^{\frac{1}{2}} u \right)^2 + \frac{2n-5}{4n^2-10n+4}|Su|^2. \end{aligned}$$

□

Remark 2.3.8. *We also proved that*

$$\int_{\Sigma_t} \frac{t^2}{r^2}u^2 d\Sigma_t \lesssim \int_{\Sigma_t} P_0 d\Sigma_t.$$

Finally, we obtain the expected estimate.

Proposition 2.3.9. *We have, for all $t \in [0, T[$,*

$$\sum_{|\beta| \leq 1} \|Z^\beta u\|_{L^2(\mathbb{R}^n)}^2(t) \lesssim \sum_{|\beta| \leq 1} \|Z^\beta u\|_{L^2(\mathbb{R}^n)}^2(0) + \sum_{|\beta| \leq 1} \int_0^t \int_{\Sigma_s} |Z^\beta u| |\tau_+ \square u| dx ds,$$

with $Z^\beta \in \mathbb{K}$ if $|\beta| = 1$, leading to, for all $t \in [0, T[$,

$$\sum_{|\beta| \leq 1} \|Z^\beta u\|_{L^2(\mathbb{R}^n)}(t) \lesssim \sum_{|\beta| \leq 1} \|Z^\beta u\|_{L^2(\mathbb{R}^n)}(0) + \int_0^t \|\tau_+ \square u\|_{L^2(\mathbb{R}^n)} ds.$$

Proof.

We have, according to Propositions 2.3.6 and 2.3.7,

$$\sum_{|\beta| \leq 1} \|Z^\beta u\|_{L^2(\mathbb{R}^n)}^2(t) \lesssim \sum_{|\beta| \leq 1} \|Z^\beta u\|_{L^2(\mathbb{R}^n)}^2(0) + \int_0^t \int_{\Sigma_s} |\square u| |\overline{K}_0 u + (n-1)tu| dx ds.$$

The result then follows from Remark 2.2.10, which gives us

$$|\overline{K}_0 u| \lesssim \tau_+^2 |Lu| + \tau_-^2 |\underline{L}u| \lesssim \tau_+ \sum_{Z \in \mathbb{K}} |Zu|.$$

□

We now apply this to the electromagnetic potential in the Lorenz gauge. Since we will need to estimate $\|S(\mathcal{L}_{Z^\beta}(A))\|_{L^2(\mathbb{R}^n)}$ in order to bound the energy of the electromagnetic field F (see Proposition 2.3.21 below), we consider the following norms.

Definition 2.3.10. *Let A be a sufficiently regular 1-form defined on $[0, T] \times \mathbb{R}^n$. We define, for $N \in \mathbb{N}$ and all $t \in [0, T]$,*

$$\tilde{\mathcal{E}}_N[A](t) = \sum_{\mu=0}^n \sum_{|\beta| \leq 1} \sum_{|\gamma| \leq N} \|Z^\beta(\mathcal{L}_{Z^\gamma}(A)_\mu)\|_{L^2(\mathbb{R}^n)}^2(t).$$

Remark 2.3.11. *Note that*

$$\sum_{\mu=0}^n \sum_{|\beta| \leq N+1} \|Z^\beta A_\mu\|_{L^2(\mathbb{R}^n)}^2 \lesssim \tilde{\mathcal{E}}_N[A] \lesssim \sum_{\mu=0}^n \sum_{|\beta| \leq N+1} \|Z^\beta A_\mu\|_{L^2(\mathbb{R}^n)}^2.$$

We work with $\tilde{\mathcal{E}}_N[A]$ as we will apply Proposition 2.3.9 to $\mathcal{L}_{Z^\beta}(A)_\mu$.

Using Proposition 2.3.9, we get the following result.

Proposition 2.3.12. *Let $N \in \mathbb{N}$ and A_μ be a sufficiently regular 1-form, defined on $[0, T] \times \mathbb{R}^n$, such that $\tilde{\mathcal{E}}_N[A](0) < +\infty$. Then, $\forall t \in [0, T]$,*

$$\sqrt{\tilde{\mathcal{E}}_N[A]}(t) \lesssim \sqrt{\tilde{\mathcal{E}}_N[A]}(0) + \sum_{\mu=0}^n \sum_{|\gamma| \leq N} \int_0^t \|\tau_+ \square \mathcal{L}_{Z^\gamma}(A)_\mu\|_{L^2(\Sigma_s)} ds.$$

2.3.3 Energy estimates for the Maxwell equations

We prove three conservation laws for the Maxwell equations, using each time a different multiplier (∂_t , \bar{K}_0 or S). In the study of the massive case, we will mostly use the one associated to the Morawetz vector field.

For the remaining of this section, we consider a 2-form G and a current J , sufficiently regular and defined on $[0, T]$, such that

$$\begin{cases} \nabla^\mu G_{\mu\nu} = J_\nu \\ \nabla^{\mu*} G_{\mu\lambda_1 \dots \lambda_{n-2}} = 0. \end{cases}$$

The following lemmas hold.

Lemma 2.3.13. *We have, for all $0 \leq \nu \leq n$,*

$$\nabla^\mu T[G]_{\mu\nu} = G_{\nu\rho} J^\rho.$$

Proof.

According to Proposition 2.2.16,

$$\begin{aligned} G_{\mu\rho} \nabla^\mu G_{\nu}{}^\rho &= G^{\mu\rho} \nabla_\mu G_{\nu\rho} \\ &= \frac{1}{2} G^{\mu\rho} (\nabla_\mu G_{\nu\rho} - \nabla_\rho G_{\nu\mu}) \\ &= \frac{1}{2} G^{\mu\rho} \nabla_\nu G_{\mu\rho} \\ &= \frac{1}{4} \nabla_\nu (G^{\mu\rho} G_{\mu\rho}). \end{aligned}$$

So,

$$\nabla^\mu T[G]_{\mu\nu} = \nabla^\mu (G_{\mu\rho}) G_{\nu}{}^\rho + \frac{1}{4} \nabla_\nu (G^{\mu\rho} G_{\mu\rho}) - \frac{1}{4} \eta_{\mu\nu} \nabla^\mu (G^{\sigma\rho} G_{\sigma\rho}) = G_{\nu\rho} J^\rho.$$

□

Lemma 2.3.14. *We have, denoting by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of G ,*

$$T[G]_{LL} = |\alpha|^2, \quad T[G]_{\underline{LL}} = |\underline{\alpha}|^2 \quad \text{and} \quad T[G]_{LL} = |\rho|^2 + |\sigma|^2.$$

Using ∂_t as a multiplier

As we use here the multiplier ∂_t , we work with $T[G]_{\mu 0}$. Applying the divergence theorem to $T[G]_{\mu 0}$ on $[0, t] \times \mathbb{R}^n$ and $V_u(t)$, we obtain the following result.

Proposition 2.3.15. *For all $t \in [0, T]$,*

$$\int_{\Sigma_t} |\alpha|^2 + |\underline{\alpha}|^2 + 2|\rho|^2 + 2|\sigma|^2 dx = \int_{\Sigma_0} |\alpha|^2 + |\underline{\alpha}|^2 + 2|\rho|^2 + 2|\sigma|^2 dx + 4 \int_0^t \int_{\Sigma_s} G_{0\mu} J^\mu dx ds$$

and

$$\sqrt{2} \sup_{u \leq t} \int_{C_u(t)} |\alpha|^2 + |\rho|^2 + |\sigma|^2 dC_u(t) \leq \int_{\Sigma_0} |\alpha|^2 + |\underline{\alpha}|^2 + 2|\rho|^2 + 2|\sigma|^2 dx + 4 \int_0^t \int_{\Sigma_s} |G_{0\mu} J^\mu| dx ds.$$

This explains the introduction of the following norms.

Definition 2.3.16. *Let $N \in \mathbb{N}$. We define, for $t \in [0, T]$,*

$$\mathcal{E}^0[G](t) = \int_{\Sigma_t} (|\alpha|^2 + |\underline{\alpha}|^2 + 2|\rho|^2 + 2|\sigma|^2) dx + \sup_{u \leq t} \int_{C_u(t)} (|\alpha|^2 + |\rho|^2 + |\sigma|^2) dC_u(t)$$

and

$$\mathcal{E}_N^0[G](t) = \sum_{|\beta| \leq N} \mathcal{E}_N^0[\mathcal{L}_{Z^\beta}(G)](t),$$

with $Z^\beta \in \mathbb{K}^{|\beta|}$.

Using the previous energy identities and commutation formula of Proposition 2.2.30, we obtain

Proposition 2.3.17. *For all $N \in \mathbb{N}$ and all $t \in [0, T]$, we have*

$$\mathcal{E}_N^0[F](t) - 2\mathcal{E}_N^0[F](0) \lesssim \sum_{|\beta|, |\gamma| \leq N} \int_0^t \int_{\Sigma_s} |e^k \mathcal{L}_{Z^\beta}(F)_{0\mu} J(\widehat{Z}^\gamma f_k)^\mu| dx ds,$$

with $Z^\beta \in \mathbb{K}^{|\beta|}$ and $\widehat{Z}^\gamma \in \widehat{\mathbb{P}}_0^{|\gamma|}$.

Using \overline{K}_0 as a multiplier

As $T[G]$ is not traceless in dimension $n \geq 4$, $\nabla^\mu(T[G]_{\mu\nu} \overline{K}_0^\nu)$ does not necessarily vanishes when G solves the free Maxwell equations. We then consider, in the spirit of what is done for the wave equation, for A a sufficiently regular potential of G in the Lorenz gauge, the current

$$P_\mu = T[G]_{\mu\nu} \overline{K}^\nu + (n-3) \left(t A_\beta \partial_\mu A^\beta - \frac{1}{2} \partial_\mu(t) A_\beta A^\beta - t A_\beta \partial^\beta A_\mu + \partial^\beta(t) A_\beta A_\mu \right).$$

In order to establish an energy estimate for the electromagnetic field, we compute the divergence of P_μ .

Lemma 2.3.18. *We have*

$$\nabla^\mu P_\mu = G_{\mu\nu} \overline{K}_0^\nu J^\mu + (n-3) t A_\beta \square A^\beta.$$

Proof. We have

$$\nabla^\mu(T[G]_{\mu\nu} \overline{K}_0^\nu) = \nabla^\mu(T[G]_{\mu\nu}) \overline{K}_0^\nu + T[G]_{\mu\nu} \nabla^\mu \overline{K}_0^\nu.$$

Since $T[G]$ is symmetric,

$$T[G]_{\mu\nu} \nabla^\mu \overline{K}_0^\nu = \frac{1}{2} T[G]_{\mu\nu} \pi^{\mu\nu},$$

with $\pi^{\mu\nu} = \nabla^\mu \overline{K}_0^\nu + \nabla^\nu \overline{K}_0^\mu$. As \overline{K}_0 is a conformal vector field (of conformal factor $4t$), we have

$$\pi_{\mu\nu} = 4t \eta_{\mu\nu}.$$

Thus,

$$T[G]_{\mu\nu}\nabla^\mu\bar{K}_0^\nu = 2tT(G)_\mu{}^\mu = \frac{3-n}{2}tG_{\sigma\rho}G^{\sigma\rho}.$$

Now, according to Lemma 2.3.13, we obtain that

$$\nabla^\mu(T[G]_{\mu\nu}\bar{K}_0^\nu) = G_{\nu\rho}\bar{K}_0^\nu J^\rho + \frac{3-n}{2}tG_{\sigma\rho}G^{\sigma\rho}.$$

We now compute the divergence of

$$(n-3)\left(tA_\beta\partial_\mu A^\beta - \frac{1}{2}\partial_\mu(t)A_\beta A^\beta - tA_\beta\partial^\beta A_\mu + \partial^\beta(t)A_\beta A_\mu\right).$$

First,

$$\nabla^\mu(tA_\beta\partial_\mu A^\beta) = -A_\beta\partial_0 A^\beta + t\partial^\mu A_\beta\partial_\mu A^\beta + tA_\beta\Box A^\beta.$$

Secondly,

$$\nabla^\mu\left(\frac{1}{2}\partial_\mu(t)A_\beta A^\beta\right) = A_\beta\partial^0 A^\beta.$$

We also have, using in particular that in Lorenz gauge $\partial^\mu A_\mu = 0$,

$$\begin{aligned}\nabla^\mu(tA_\beta\partial^\beta A_\mu) &= -A_\beta\partial^\beta A_0 + t\partial^\mu(A_\beta)\partial^\beta A_\mu + tA_\beta\partial^\beta\partial^\mu A_\mu \\ &= -A_\beta\partial^\beta A_0 + t\partial^\mu(A_\beta)\partial^\beta A_\mu.\end{aligned}$$

Finally

$$\begin{aligned}\nabla^\mu(\partial^\beta(t)A_\beta A_\mu) &= -\partial^\mu(A_0)A_\mu - A_0\partial^\mu A_\mu \\ &= -\partial^\mu(A_0)A_\mu.\end{aligned}$$

Hence,

$$\begin{aligned}(n-3)\nabla^\mu\left(tA_\beta\partial_\mu A^\beta - \frac{1}{2}\partial_\mu(t)A_\beta A^\beta - tA_\beta\partial^\beta A_\mu - \partial^\beta(t)A_\beta A_\mu\right) &= \\ &= (n-3)tA_\beta\Box A^\beta + (n-3)t(\partial_\mu A_\beta\partial^\mu A^\beta - \partial_\mu A_\beta\partial^\beta A^\mu).\end{aligned}$$

And, since $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

$$\frac{1}{2}G_{\mu\beta}G^{\mu\beta} = \partial_\mu A_\beta\partial^\mu A^\beta - \partial_\mu A_\beta\partial^\beta A^\mu,$$

which gives us the result. □

We are now ready to prove the following energy estimate.

Proposition 2.3.19. *For all $t \in [0, T[$,*

$$\begin{aligned}\int_{\Sigma_t} \tau_+^2|\alpha|^2 + \tau_-^2|\underline{\alpha}|^2 + (\tau_+^2 + \tau_-^2)(|\rho|^2 + |\sigma|^2)d\Sigma_t + (n-3)^2\sum_{\mu=0}^n\|A_\mu\|_{L^2(\Sigma_t)}^2 &\leq \\ \int_{\Sigma_0} (1+r^2)(|\alpha|^2 + |\underline{\alpha}|^2 + |\rho|^2 + |\sigma|^2)d\Sigma_0 + 4\int_0^t\int_{\Sigma_s} |\bar{K}_0^\nu G_{\nu\mu}J^\mu|dxds & \\ + (n-3)\sum_{\mu=0}^n\|SA_\mu\|_{L^2(\Sigma_t)}^2 + 4(n-3)\int_0^t\int_{\Sigma_s} s|A_\mu\Box A^\mu|dxds. &\end{aligned}$$

Proof.

In order to apply the divergence theorem to P_μ in $[0, t] \times \mathbb{R}^n$, we transform

$$\int_{\mathbb{R}^n} \left(tA_\beta\partial_t A^\beta - \frac{1}{2}A_\beta A^\beta - tA_\beta\partial^\beta A_0 - A_0^2\right) dx.$$

On the one hand, let us notice that

$$-\frac{1}{2}A_\beta A^\beta - A_0^2 = -\frac{1}{2}\sum_{\beta=0}^n A_\beta^2.$$

On the other hand,

$$-t \int_{\mathbb{R}^n} A_\beta \partial^\beta A_0 dx = -t \int_{\mathbb{R}^n} A_0 \partial^0 A_0 dx + t \int_{\mathbb{R}^n} \partial^j (A_j) A_0 dx = t \partial_t \int_{\mathbb{R}^n} A_0^2 dx, \quad (2.29)$$

since $\partial^\mu A_\mu = 0$ in the Lorenz gauge. As

$$t \int_{\mathbb{R}^n} A_\beta \partial_t A^\beta dx = \frac{t}{2} \partial_t \int_{\mathbb{R}^n} A_\beta A^\beta dx, \quad (2.30)$$

we finally obtain that

$$\int_{\mathbb{R}^n} \left(t A_\beta \partial_t A^\beta - \frac{1}{2} A_\beta A^\beta - t A_\beta \partial^\beta A_0 - A_0^2 \right) dx = \frac{1}{2} \sum_{\beta=0}^n (t \partial_t - 1) \|A_\beta\|_{L^2(\mathbb{R}^n)}^2.$$

The divergence theorem applied to P_μ in $[0, t] \times \mathbb{R}^n$ gives, using Lemma 2.3.14 and 2.3.18,

$$\begin{aligned} \int_{\Sigma_t} \tau_+^2 |\alpha|^2 + \tau_-^2 |\underline{\alpha}|^2 + (\tau_+^2 + \tau_-^2) (|\rho|^2 + |\sigma|^2) dx &\leq 4 \int_0^t \int_{\Sigma_s} |\nabla^\mu P_\mu| dx ds \\ &+ \int_{\Sigma_0} (1 + r^2) (|\alpha|^2 + |\underline{\alpha}|^2 + 2|\rho|^2 + 2|\sigma|^2) dx + 2(n-3) \sum_{\mu=0}^n \left(\int_{\Sigma_t} (1 - t \partial_t) A_\mu^2 dx \right). \end{aligned}$$

It only remains to use the last lemma and the inequality

$$-t \partial_t \int_{\Sigma_t} A_\mu^2 dx \leq \frac{1-n}{2} \|A_\mu\|_{L^2(\Sigma_t)}^2 + \frac{1}{2} \|SA_\mu\|_{L^2(\Sigma_t)}^2$$

which ensues from (2.26). □

This estimate justifies the introduction of the following norms.

Definition 2.3.20. *Let G be a 2-form defined on $[0, T[$ and $N \in \mathbb{N}$. We define, for all $t \in [0, T[$,*

$$\mathcal{E}[G](t) = \int_{\Sigma_t} \tau_+^2 |\alpha(G)|^2 + \tau_-^2 |\underline{\alpha}(G)|^2 + (\tau_+^2 + \tau_-^2) (|\rho(G)|^2 + |\sigma(G)|^2) dx$$

and

$$\mathcal{E}_N[G](t) = \sum_{|\beta| \leq N} \mathcal{E}[\mathcal{L}_{Z^\beta} G](t),$$

with $Z^\beta \in \mathbb{K}^{|\beta|}$.

We then deduce, using Propositions 2.2.30, 2.3.19 and Lemma 2.2.18, an energy estimate for the electromagnetic field F .

Proposition 2.3.21. *Let A be a sufficiently regular potential in the Lorenz gauge of F . We have, for all $N \in \mathbb{N}$ and all $t \in [0, T[$,*

$$\begin{aligned} \mathcal{E}_N[F](t) - \mathcal{E}_N[F](0) - (n-3) \sum_{|\kappa| \leq N} \sum_{\mu=0}^n \|S\mathcal{L}_{Z^\kappa}(A)_\mu\|_{L^2(\Sigma_t)}^2 &\lesssim + \sum_{|\kappa| \leq N} \int_0^t \int_{\Sigma_s} s |\mathcal{L}_{Z^\kappa}(A)_\mu \square \mathcal{L}_{Z^\kappa}(A)^\mu| dx ds \\ &+ \sum_{|\beta|, |\gamma| \leq N} \int_0^t \int_{\Sigma_s} |e^k \bar{K}_0^\nu \mathcal{L}_{Z^\beta}(F)_{\nu\mu} J^\mu(\widehat{Z}^\gamma f_k)| dx ds \end{aligned}$$

with $Z^\beta \in \mathbb{K}^{|\beta|}$, $Z^\kappa \in \mathbb{K}^{|\kappa|}$ and $\widehat{Z}^\gamma \in \widehat{\mathbb{P}}_0^{|\gamma|}$.

Using S as a multiplier

The main difference with the previous case comes from the fact that the scaling is not a timelike vector field. Because of that we are not able to estimate all the null components of the electromagnetic field with this energy estimate. We start by introducing, for A a potential of G satisfying the Lorenz gauge,

$$Q_\mu = T(G)_{\mu\nu} S^\nu + \frac{n-3}{2}(A_\beta \partial_\mu A^\beta - A_\beta \partial^\beta A_\mu).$$

As the potential A satisfies the Lorenz gauge and since the conformal factor of the scaling is 2, we have

$$\nabla^\mu Q_\mu = G_{\mu\nu} S^\nu J^\mu + \frac{n-3}{2} A_\beta \square A^\beta. \quad (2.31)$$

We can now state the energy estimate.

Proposition 2.3.22. *For all $t \in [0, T]$,*

$$\begin{aligned} \int_{\Sigma_t} (t+r)|\alpha|^2 + (t-r)|\underline{\alpha}|^2 + 2t(|\rho|^2 + |\sigma|^2) dx + (n-3) \partial_t \sum_{\beta=0}^n \|A_\beta\|_{L^2(\Sigma_t)}^2 &= 4 \int_0^t \int_{\Sigma_s} \nabla^\mu Q_\mu dx ds \\ &+ \int_{\Sigma_0} r(|\alpha|^2 - |\underline{\alpha}|^2) dx + (n-3) \partial_t \sum_{\beta=0}^n \|A_\beta\|_{L^2(\Sigma_0)}^2. \end{aligned}$$

Proof.

Note first that we proved, during the proof of Proposition 2.3.19 (see Equations (2.29) and (2.30)),

$$\int_{\mathbb{R}^n} A_\beta \partial_0 A^\beta - A_\beta \partial^\beta A_0 dx = \frac{\partial_t}{2} \sum_{\beta=0}^n \|A_\beta\|_{L^2(\mathbb{R}^n)}^2.$$

It then remains to apply the divergence theorem to Q_μ on $[0, T] \times \mathbb{R}^n$ (recall that $2S = (t+r)L + (t-r)\underline{L}$). \square

Note that $(t-r)|\underline{\alpha}|^2$ is not necessarily non negative, which invites us to transform the equality in the following estimate.

Proposition 2.3.23. *For all $t \in [0, T]$,*

$$\begin{aligned} \int_{\Sigma_t} (1+|t-r|)|\underline{\alpha}|^2 dx &\leq \int_{\Sigma_0} (1+r)(|\alpha|^2 + |\underline{\alpha}|^2) + 2|\rho|^2 + 2|\sigma|^2 dx \\ &+ (n-3)(n+2)\tilde{\mathcal{E}}_0[A](0) + \frac{2}{1+t} \left(\mathcal{E}[F](t) + \frac{(n-3)(n+2)}{2} \tilde{\mathcal{E}}_0[A](t) \right) \\ &+ 4 \int_0^t \int_{\Sigma_s} |G_{0\mu} J^\mu| + |S^\nu G_{\mu\nu} J^\mu| dx ds + 2(n-3) \int_0^t \int_{\Sigma_s} |A_\mu \square A^\mu| dx ds. \end{aligned}$$

Proof. Adding the energy identities of Propositions 2.3.22 and 2.3.15, we can obtain,

$$\begin{aligned} \int_{\Sigma_t} (1+|t-r|)|\underline{\alpha}|^2 dx &\leq \int_{\Sigma_0} (1+r)(|\alpha|^2 + |\underline{\alpha}|^2) + 2|\rho|^2 + 2|\sigma|^2 dx \\ &+ \int_{\Sigma_t} (t+r)|\alpha|^2 + 2t(|\rho|^2 + |\sigma|^2) dx + 4 \int_0^t \int_{\Sigma_s} |G_{0\mu} J^\mu| + |\nabla^\mu Q_\mu| dx ds \\ &+ (n-3) \left| \partial_t \sum_{\beta=0}^n \left(\|A_\beta\|_{L^2(\mathbb{R}^n)}^2(0) - \|A_\beta\|_{L^2(\mathbb{R}^n)}^2(t) \right) \right|. \end{aligned}$$

The result then ensues from the three following inequalities. Using Definition 2.3.20, one has

$$(1+t) \int_{\Sigma_t} (t+r)|\alpha|^2 + 2t(|\rho|^2 + |\sigma|^2) dx \leq 2\mathcal{E}[F](t).$$

According to (2.31), we have

$$\int_0^t \int_{\Sigma_s} |\nabla^\mu Q_\mu| dx ds \leq \int_0^t \int_{\Sigma_s} |S^\nu G_{\mu\nu} J^\mu| + \frac{(n-3)}{2} |A_\mu \square A^\mu| dx ds.$$

Finally, Equation (2.26) gives us

$$(1+t) \left| \partial_t \|A_\mu\|_{L^2(\Sigma_s)}^2 \right| \leq \|SA_\mu\|_{L^2(\Sigma_s)}^2 + \|\partial_t A_\mu\|_{L^2(\Sigma_s)}^2 + (n+2) \|A_\mu\|_{L^2(\Sigma_s)}^2.$$

□

Let us introduce the following norms.

Definition 2.3.24. We define, for $N \in \mathbb{N}$ and $t \in [0, T[$,

$$\mathcal{E}_N^S[F](t) = \sum_{\substack{Z^\beta \in \mathbb{K}^{|\beta|} \\ |\beta| \leq N}} \int_{\Sigma_t} \tau_- |\underline{\alpha}(\mathcal{L}_{Z^\beta}(F))|^2 dx.$$

Commuting the equation satisfied by the electromagnetic field F and using the previous energy estimate, we get the following proposition (see the commutation formulas of Proposition 2.2.30 and Lemma 2.2.18).

Proposition 2.3.25. Let A a sufficiently regular potential of the the electromagnetic field F in the Lorenz gauge. Then, for $N \in \mathbb{N}$, we have, for all $t \in [0, T[$,

$$\begin{aligned} \mathcal{E}_N^S[F](t) - \mathcal{E}_N[F](0) &\lesssim \tilde{\mathcal{E}}_N[A](0) + \frac{1}{1+t} \left(\tilde{\mathcal{E}}_N[A](t) + \mathcal{E}_N[F](t) \right) \\ &+ \sum_{|\beta|, |\gamma| \leq N} |e^k| \int_0^t \int_{\Sigma_s} |\mathcal{L}_{Z^\beta}(F)_{0\mu} J^\mu(\widehat{Z}^\gamma f_k)| + |S^\nu \mathcal{L}_{Z^\beta}(F)_{\nu\mu} J^\mu(\widehat{Z}^\gamma f_k)| dx ds \\ &+ \sum_{|\beta| \leq N} \int_0^t \int_{\Sigma_s} |\mathcal{L}_{Z^\beta}(A)_\mu \square \mathcal{L}_{Z^\beta}(A)^\mu| dx ds, \end{aligned}$$

with $Z^\beta \in \mathbb{K}^{|\beta|}$ and $\widehat{Z}^\gamma \in \widehat{\mathbb{P}}_0^{|\gamma|}$.

Later, we will have, in the 4 dimensional massless case, a strong loss on $\mathcal{E}_N[F]$ which will lead to a poor pointwise decay estimate on $|\underline{\alpha}|$. With this inequality, we will avoid the τ_+ -loss and we will have an extra τ_- -decay (which is not given by Proposition 2.3.17).

2.4 Some technical results

2.4.1 An integral estimate

The following lemma is useful so as to estimate a quantity like

$$\int_0^t \int_{\mathbb{R}^n} |u(s, x)| \int_v |f(s, x, v)| dv dx ds,$$

where we already have a bound on $\|u(s, \cdot)\|_{L^2}$ and a pointwise decay estimate on $\int_v |f(s, x, v)| dv$.

Lemma 2.4.1. Let $m \in \mathbb{N}^*$ and let $a, b \in \mathbb{R}$, such that $a + b > m$ and $b \neq 1$. Then

$$\exists C_{a,b,m} > 0, \forall t \in \mathbb{R}_+ \int_0^{+\infty} \frac{r^{m-1}}{\tau_+^a \tau_-^b} dr \leq C_{a,b,m} \frac{1 + t^{b-1}}{1 + t^{a+b-m}}.$$

A proof of this estimate can be found in [18], Appendix B.

2.4.2 The null coordinates of $\nabla_v f$

Let $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ be a smooth function. We designate by $((\nabla_v f)^L, (\nabla_v f)^{\underline{L}}, (\nabla_v f)^B, \dots)$ the null components of $\nabla_v f$. Later, we will have to transform the v -derivatives in combinations of $\widehat{\mathbb{P}}_0$ -derivatives. If we only use the relation

$$v^0 \partial_{v^k} = \widehat{\Omega}_{0k} - t \partial_k - x^k \partial_t, \quad (2.32)$$

we get that

$$\left| (\nabla_v f)^L \right|, \left| (\nabla_v f)^{\underline{L}} \right|, \left| (\nabla_v f)^B \right| \leq \frac{\tau_+}{v^0} \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |\widehat{Z} f|, \quad (2.33)$$

which will not be good enough to close the energy estimates (for the Vlasov-Maxwell system).

We then use the following lemma.

Lemma 2.4.2. *Let f be a smooth function. We have*

$$\left| (\nabla_v f)^L \right|, \left| (\nabla_v f)^{\underline{L}} \right| \leq \frac{\tau_-}{v^0} \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |\widehat{Z} f|.$$

Proof.

Since $(\nabla_v f)^0 = 0$ (by definition),

$$(\nabla_v f)^L = \frac{x^i}{r} \partial_{v^i} f.$$

Now, we use $\partial_{v^i} = \frac{1}{v^0} (\widehat{\Omega}_{0i} - t \partial_i - x_i \partial_t)$. As

$$\frac{x^i}{r v^0} (t \partial_i + x_i \partial_t) = \frac{1}{v^0} (t \partial_r + r \partial_t) = \frac{1}{v^0} (S + (r - t) \underline{L}),$$

we have

$$(\nabla_v f)^L = \frac{x^i}{r v^0} \widehat{\Omega}_{0i} f - \frac{1}{v^0} S f + \frac{t - r}{v^0} \underline{L} f.$$

It only remains to notice that $(\nabla_v f)^{\underline{L}} = -(\nabla_v f)^L$, since $(\nabla_v f)^0 = 0$. □

We are now interested in $(\nabla_v f)^B$. During the study of the Vlasov equation, each time that (2.33) is not sufficient to close the estimates, $(\nabla_v f)^B$ is multiplied by $v^{\underline{L}}$, which reflects the null structure of the system. This leads us to study $v^{\underline{L}} (\nabla_v f)^B$.

Lemma 2.4.3. *For $1 \leq i \leq n$, we have*

$$2v^{\underline{L}} \frac{x^i}{r} = \frac{v^0 x^i}{r} - v^i + \frac{z_{ij} x^j}{r^2},$$

where $z_{\mu\nu} = x^\nu v^\mu - x^\mu v^\nu$.

Remark 2.4.4. *If $\mu \neq \nu$, $\frac{z_{\mu\nu}}{v^0} \in \mathbf{k}_1$ and if $\mu = \nu$, then $z_{\mu\mu} = 0$.*

Proof.

For simplicity, we take $i = 1$. We have

$$\begin{aligned} 2v^{\underline{L}} \frac{x^1}{r} &= \frac{x^1 v^0}{r} - \frac{x^1}{r^2} x_i v^i \\ &= \frac{x^1 v^0}{r} - v^1 + \frac{z_{1j} x^j}{r^2}. \end{aligned}$$

□

And we obtain

Corollary 2.4.5. *Let $i, j \in \llbracket 1, n \rrbracket$ such that $i \neq j$. We have*

$$\begin{aligned} 2v^{\underline{L}} \left(\frac{x^i}{r} \partial_{v^j} - \frac{x^j}{r} \partial_{v^i} \right) &= \left(\frac{x^i}{r} + \frac{z_{ik} x^k}{v^0 r^2} \right) \widehat{\Omega}_{0j} - \left(\frac{x^j}{r} + \frac{z_{jk} x^k}{v^0 r^2} \right) \widehat{\Omega}_{0i} - \widehat{\Omega}_{ij} \\ &\quad - \left(\frac{x^i(t-r)}{r} + \frac{t x^k z_{ik}}{r^2 v^0} \right) \partial_j + \left(\frac{x^j(t-r)}{r} + \frac{t x^k z_{jk}}{r^2 v^0} \right) \partial_i - \frac{z_{ij}}{v^0} \partial_t. \end{aligned}$$

Proof. By the previous lemma,

$$2v^{\underline{L}} \left(\frac{x^i}{r} \partial_{v^j} - \frac{x^j}{r} \partial_{v^i} \right) = \left(\frac{v^0 x^i}{r} + \frac{z_{ik} x^k}{r^2} \right) \partial_{v^j} - \left(\frac{v^0 x^j}{r} + \frac{z_{jk} x^k}{r^2} \right) \partial_{v^i} - \widehat{\Omega}_{ij} + x^i \partial_j - x^j \partial_i.$$

Now, using the relation $v^0 \partial_{v^k} = \widehat{\Omega}_{0k} - t \partial_k - x^k \partial_t$, we have

$$\begin{aligned} 2v^{\underline{L}} \left(\frac{x^i}{r} \partial_{v^j} - \frac{x^j}{r} \partial_{v^i} \right) &= \left(\frac{x^i}{r} + \frac{z_{ik} x^k}{v^0 r^2} \right) \widehat{\Omega}_{0j} - \left(\frac{x^j}{r} + \frac{z_{jk} x^k}{v^0 r^2} \right) \widehat{\Omega}_{0i} - \widehat{\Omega}_{ij} \\ &\quad + x^i \partial_j - x^j \partial_i - \frac{t}{v^0} \left(\frac{v^0 x^i}{r} + \frac{z_{ik} x^k}{r^2} \right) \partial_j + \frac{t}{v^0} \left(\frac{v^0 x^j}{r} + \frac{z_{jk} x^k}{r^2} \right) \partial_i - \frac{z_{ij}}{v^0} \partial_t. \end{aligned}$$

It remains to remark that $t \frac{v^0 x^i}{r} - v^0 x^i = v^0 \frac{x^i}{r} (t-r)$. □

The naive estimation gave us

$$\left| v^{\underline{L}} (\nabla_v f)^B \right| \lesssim |x| |\partial_t f| + \sum_{k=1}^n \left(|\widehat{\Omega}_{0k} f| + t |\partial_k f| \right),$$

whereas, with this lemma and the fact that $(\nabla_v f)^B$ is a combination with bounded coefficients of

$$\left(\frac{x^i}{r} \partial_{v^j} f - \frac{x^j}{r} \partial_{v^i} f \right)_{1 \leq i < j \leq n},$$

we have

$$\left| v^{\underline{L}} (\nabla_v f)^B \right| \lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}} |\widehat{Z} f| + \sum_{1 \leq i < j \leq n} \frac{|z_{ij}|}{v^0} |\partial_t f| + \sum_{k=1}^n \left(\tau_- + \frac{t \sum_{i=1}^n |z_{ki}|}{r v^0} \right) |\partial_k f|. \quad (2.34)$$

Therefore, with the last corollary, we transform a t -loss (and a $|x|$ -loss) in a τ_- -loss and a $\frac{t}{r}$ -loss (thanks, among others, to the weights transported by the flow). It is particularly useful when we look for an estimate of $\| \int_v |v^{\underline{L}} (\nabla_v \widehat{Z}^\beta f)^B | dv \|_{L_x^2}$ and we already have an estimate of $\int_v v^0 |z \widehat{Z}^\delta f| dv$. We can then use Lemma 2.4.1. One can also transform the $\frac{t}{r}$ -loss.

Lemma 2.4.6. *For $1 \leq j \leq n$,*

$$\left| \frac{x^j(t-r)}{r} + \frac{t x^k z_{jk}}{r^2 v^0} \right| \lesssim \tau_- \sum_{z \in \mathbf{k}_1} |z|.$$

Proof. We obviously have $\tau_-^{-1} \left| \frac{x^j(t-r)}{r} \right| \leq \frac{v^0}{v^0}$. For the second term, we need to study different cases.

If $r \leq 1$, then

$$\tau_-^{-1} \left| \frac{t x^k z_{jk}}{r^2 v^0} \right| = \frac{t}{\tau_-} \frac{x_k}{r} \frac{x^k v^j - x^j v^k}{r v^0} \lesssim \frac{1}{v^0} \sum_{i=1}^n |v^i|.$$

Otherwise, $r \geq 1$, and

$$\tau_-^{-1} \left| \frac{t x^k z_{jk}}{r^2 v^0} \right| \leq \frac{t}{\tau_- r} \frac{x^k}{r} \frac{|z_{jk}|}{v^0}.$$

It remains to note that if $r \leq \frac{t}{2}$, $\tau_- \geq \frac{t}{2}$ and if $r \geq \frac{t}{2}$, then $\frac{t}{r} \leq 2$. □

One then obtains the following result.

Proposition 2.4.7. *We have*

$$\left| v^L (\nabla_v f)^B \right| \lesssim \tau_- \sum_{\widehat{Z} \in \widehat{\mathbb{P}}} \sum_{z \in \mathbf{k}_1} |z \widehat{Z} f|.$$

Note that later, in Sections 2.6.6 and 2.7.6, when we will establish an estimate on $\left\| \int_v |\widehat{Z}^\beta f| dv \right\|_{L_x^2}$, we will not be able to apply Propositions 2.4.7 or 2.4.2. A vector X will contain various derivatives of f and we will split it in two vectors $H + G$ such that

$$T_F(H) = 0, \quad \text{with } H(0) = X(0), \quad \text{and } T_F(G) = T_F(X), \quad \text{with } G(0) = 0.$$

Note yet that, for instance, if X_μ is $\partial_\mu f$ and X_S is $S(f)$, we have $x^\mu X_\mu = X_S$ whereas we do not necessarily have $x^\mu G_\mu = G_S$.

2.4.3 Some Sobolev inequalities

The following results come from [11] and in order to be self-sufficient, we also recall their proof. We will use them to prove pointwise decay estimate for the electromagnetic field.

We first recall two classical Sobolev inequalities.

Lemma 2.4.8. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}$ be a sufficiently regular function. We have*

$$\forall x \in \mathbb{R}^n, \quad |u(x)| \lesssim \sum_{|\beta| \leq \frac{n+2}{2}} \|\partial^\beta u\|_{L_y^2(|y-x| \leq 1)}.$$

Let $v : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ a sufficiently regular function (where \mathbb{S}^{n-1} is the unit sphere in \mathbb{R}^n). We have

$$\forall \xi \in \mathbb{S}^{n-1}, \quad |v(\xi)| \lesssim \sum_{|\beta| \leq \frac{n}{2}} \|\nabla_{Z^\beta} v\|_{L^2(\mathbb{S}^{n-1})},$$

with $Z^\beta \in \mathbb{O}^{|\beta|}$.

In order to treat the interior of the light cone (or rather the domain in which $|x| \leq 1 + \frac{1}{2}t$), we will use.

Lemma 2.4.9. *Let U be a smooth tensor field defined in the Euclidian space \mathbb{R}^n . Then,*

$$\forall t \in \mathbb{R}_+, \quad \sup_{|x| \leq 1 + \frac{t}{2}} |U(x)| \lesssim \frac{1}{(1+t)^{\frac{n}{2}}} \sum_{k=0}^{\frac{n+2}{2}} (1+t)^k \|\nabla^k U\|_{L^2(\{|y| \leq 3 + \frac{3}{4}t\})}.$$

Proof.

As it suffices to prove the result for each component of the tensor, we assume that U is a scalar function. Let $t \in \mathbb{R}_+$ and $|x| \leq 1 + \frac{1}{2}t$. If $t \leq 1$, then $|x| \leq 2$, so, according to Lemma 2.4.8,

$$|U(x)| \lesssim \sum_{|\beta| \leq \frac{n+2}{2}} \|\nabla^\beta U\|_{L_y^2(|y| \leq 3)}.$$

Now, if $t \geq 1$, we apply Lemma 2.4.8 to $y \mapsto U(x + \frac{t}{4}y)$. It comes that (after a change of variables)

$$|U(x)| \lesssim \left(\frac{t}{4}\right)^{-\frac{n}{2}} \sum_{|\beta| \leq \frac{n+2}{2}} \left(\frac{t}{4}\right)^{|\beta|} \|\nabla^\beta U\|_{L_y^2(|y-x| \leq \frac{t}{4})}.$$

It remains to observe that $|y-x| \leq \frac{t}{4}$ imply $|y| \leq 1 + \frac{3}{4}t$. □

For the other region ($|x| \geq 1 + \frac{1}{2}t$), we have the following inequality.

Lemma 2.4.10. *Let U be a sufficiently regular tensor field, which in particular vanishes at ∞ , defined in the euclidian space \mathbb{R}^n . Then, for $t \in \mathbb{R}_+$,*

$$\forall x \neq 0, \quad |U(x)| \lesssim \frac{1}{|x|^{\frac{n-1}{2}} \tau_-^{\frac{1}{2}}} \left(\int_{|y| \geq |x|} |U(y)|_{\mathbb{O}, \frac{n}{2}}^2 + \tau_-^2 |\nabla_{\partial_r} U(y)|_{\mathbb{O}, \frac{n}{2}}^2 dy \right)^{\frac{1}{2}}.$$

Proof.

As $\sum_{|\beta| \leq k} |\nabla_{Z^\beta} U|^2 \lesssim |U|_{\mathbb{O}, k}^2$, for $Z^\beta \in \mathbb{O}^{|\beta|}$, we only have to prove the result for each component of U and we can assume that U is a scalar function.

Let $x \neq 0$ such that $x = r\xi$, with $r = |x|$ and $\xi \in \mathbb{S}^{n-1}$. Since $\partial_r((\sqrt{\tau_-}U)^2) = 2\sqrt{\tau_-}U\partial_r(\sqrt{\tau_-}U)$,

$$\tau_- |U(r\xi)|^2 \lesssim r^{-(n-1)} \int_r^{+\infty} |\sqrt{\tau_-}U(\lambda\xi)| \partial_r(\sqrt{\tau_-}U)(\lambda\xi) |\lambda^{n-1} d\lambda|.$$

Therefore, an integration over \mathbb{S}^{n-1} and the inequality $2|ab| \leq a^2 + b^2$ gives us

$$\|U(r\xi)\|_{L_\xi^2(\mathbb{S}^{n-1})} \lesssim r^{-\frac{n-1}{2}} \tau_-^{-\frac{1}{2}} \left(\int_{|y| \geq r} |U(y)|^2 + \tau_-^2 |\partial_r U(y)|^2 dy \right)^{\frac{1}{2}}.$$

As every vector field of \mathbb{O} commute with ∂_r , we obtain, using Lemma 2.4.8,

$$|U(x)| \lesssim r^{-\frac{n-1}{2}} \tau_-^{-\frac{1}{2}} \left(\int_{|y| \geq r} |U(y)|_{\mathbb{O}, \frac{n}{2}}^2 + \tau_-^2 |\partial_r U(y)|_{\mathbb{O}, \frac{n}{2}}^2 dy \right)^{\frac{1}{2}}.$$

□

2.4.4 Pointwise decay estimate for the null decomposition of the electromagnetic field

In this section, we recall some inequalities coming from [11] between quantites linked to the null decomposition of a 2-form (see Section 2.2.1 for its definition) and we then prove pointwise decay estimates on it. However, we cannot adapt the method used in [11] to establish, in dimension 3, the optimal decay estimate on the null component α . To circumvent this difficulty, we make crucial use of an electromagnetic potential satisfying the Lorenz gauge. We first introduce some notations.

Definition 2.4.11. *Let F be a 2-form. We define its pointwise norm $|F|^\#$ by*

$$|F|^\# = \sqrt{\tau_+^2 |\alpha|^2 + \tau_-^2 |\underline{\alpha}|^2 + (\tau_-^2 + \tau_+^2)(|\rho|^2 + |\sigma|^2)},$$

which is also equal to $\sqrt{4T[F](\overline{K}_0, \partial_t)}$.

We also define, for $\mathbb{L} = \mathbb{O}$ or $\mathbb{L} = \mathbb{K}$ and $k \in \mathbb{N}$,

$$|F|_{\mathbb{L}, k}^\# = \sqrt{\sum_{|\beta| \leq k} (|\mathcal{L}_{Z^\beta} F|^\#)^2},$$

with $Z^\beta \in \mathbb{L}^{|\beta|}$.

Similarly, we define

$$|F| = \sqrt{|\alpha|^2 + |\underline{\alpha}|^2 + 2(|\rho|^2 + |\sigma|^2)}$$

and

$$|F|_{\mathbb{L}, k} = \sqrt{\sum_{\substack{Z^\beta \in \mathbb{L}^{|\beta|} \\ |\beta| \leq k}} |\mathcal{L}_{Z^\beta} F|^2}.$$

Remark 2.4.12. *By definition of $|F|^\#$, it comes that $\tau_- |F| \leq |F|^\#$.*

We have the following inequality (cf Remark 2.2.10).

Lemma 2.4.13. *Let F be a 2-form and k a non-negative integer. Then*

$$\forall |\beta| = k, \quad |\nabla^\beta F|^\# \lesssim \tau_-^{-k} |F|_{\mathbb{K},k}^\#.$$

We also have, according to [11].

Lemma 2.4.14. *Let F be a 2-form and $(\alpha, \underline{\alpha}, \rho, \sigma)$ its null decomposition. Then, for all $k \in \mathbb{N}$,*

$$\sum_{l=0}^k \sum_{i+j=l} \tau_-^{2i} r^{2j} \left(|\nabla_{\underline{L}}^i \nabla_L^j \underline{\alpha}|_{\mathbb{O},k-i-j}^2 + |\nabla_{\underline{L}}^i \nabla_L^j \alpha|_{\mathbb{O},k-i-j}^2 + |\nabla_{\underline{L}}^i \nabla_L^j \rho|_{\mathbb{O},k-i-j}^2 + |\nabla_{\underline{L}}^i \nabla_L^j \sigma|_{\mathbb{O},k-i-j}^2 \right) \lesssim |F|_{\mathbb{K},k}^2$$

and

$$\sum_{l=0}^k \sum_{i+j=l} \tau_-^{2i} r^{2j} \left(\tau_-^2 |\nabla_{\underline{L}}^i \nabla_L^j \underline{\alpha}|_{\mathbb{O},k-i-j}^2 + r^2 (|\nabla_{\underline{L}}^i \nabla_L^j \alpha|_{\mathbb{O},k-i-j}^2 + |\nabla_{\underline{L}}^i \nabla_L^j \rho|_{\mathbb{O},k-i-j}^2 + |\nabla_{\underline{L}}^i \nabla_L^j \sigma|_{\mathbb{O},k-i-j}^2) \right) \lesssim \left(|F|_{\mathbb{K},k}^\# \right)^2.$$

The first inequality is not proved in [11] but can be treated similarly as the second one.

The following corollary will be useful, particularly for the massless case in dimension 4, to obtain an extra decay on $\underline{\alpha}$ away from the light cone.

Corollary 2.4.15. *Using the same notations as in the previous lemma, we have, for F a 2-form,*

$$|\sqrt{\tau_-} \underline{\alpha}|_{\mathbb{O},k}^2 + \tau_-^2 |\nabla_r(\sqrt{\tau_-} \underline{\alpha})|_{\mathbb{O},k-1}^2 \lesssim \tau_- |F|_{\mathbb{K},k}^2.$$

Proof. One only has to use that

$$|\nabla_r \sqrt{\tau_-}| \leq \tau_-^{-\frac{1}{2}}, \quad 2\nabla_r = L - \underline{L}$$

and the previous lemma. □

Let us show how to establish pointwise decay estimates on the null decomposition of the electromagnetic field with these inequalities.

Proposition 2.4.16. *Let G be a 2-form and J be a 1-form, both defined on $[0, T[\times \mathbb{R}^n$, such that*

$$\begin{aligned} \nabla^\mu G_{\mu\nu} &= J_\nu, \\ \nabla^{\mu*} G_{\mu\lambda_1 \dots \lambda_{n-2}} &= 0. \end{aligned}$$

If G and J are sufficiently regular, we have, for all $(t, x) \in [0, T[\times \mathbb{R}^n$,

$$\begin{aligned} |\alpha(G)|(t, x), \quad |\rho(G)|(t, x), \quad |\sigma(G)|(t, x) &\lesssim \frac{\sqrt{\mathcal{E}_{\frac{n+2}{2}}[G](t)}}{\tau_+^{\frac{n+1}{2}} \tau_-^{\frac{1}{2}}}, \\ |\underline{\alpha}(G)|(t, x) &\lesssim \frac{\sqrt{\mathcal{E}_{\frac{n+2}{2}}[G](t)}}{\tau_+^{\frac{n-1}{2}} \tau_-^{\frac{3}{2}}} \end{aligned} \tag{2.35}$$

and

$$|\alpha(G)|(t, x) \lesssim \frac{\sqrt{\frac{\mathcal{E}_{\frac{n+2}{2}}[G](t)}{1+t}} + \sqrt{\mathcal{E}_{\frac{n+2}{2}}^S[G](t)}}{\tau_+^{\frac{n-1}{2}} \tau_-}. \tag{2.36}$$

Remark 2.4.17. *When we will study the massless Vlasov-Maxwell system in dimension $n = 4$, a strong t -loss on $\mathcal{E}_{\frac{n+2}{2}}[G]$ will lead to a strong τ_+ -loss on the pointwise estimate (2.35). Since we will not need all the τ_- decay rate of (2.35), we will rather use (2.36).*

Proof.

Let us denote the null decomposition of G by $(\alpha, \underline{\alpha}, \rho, \sigma)$. Let $(t, x) \in [0, T] \times \mathbb{R}^n$.

First, we consider the case $|x| \leq 1 + \frac{1}{2}t$.

As

$$\int_{\Sigma_t} \left(|G|_{\mathbb{K}, \frac{n+2}{2}}^\# \right)^2 dx \lesssim \mathcal{E}_{\frac{n+2}{2}}[G](t),$$

Lemma 2.4.13 and Remark 2.4.12 give us

$$\sum_{|\beta| \leq \frac{n+2}{2}} \int_{\Sigma_t} \tau_-^{2|\beta|+2} |\nabla^\beta G|^2 dx \lesssim \mathcal{E}_{\frac{n+2}{2}}[G](t).$$

Moreover,

$$\forall (t, y) \in [0, T] \times \mathbb{R}^n \text{ such that } |y| \leq 3 + \frac{3}{4}t, \quad \tau_-(t, y) \gtrsim 1 + t.$$

Hence,

$$\sum_{|\beta| \leq \frac{n+2}{2}} \int_{|y| \leq 3 + \frac{3}{4}t} (1+t)^{2|\beta|+2} |\nabla^\beta G|^2 dy \lesssim \mathcal{E}_{\frac{n+2}{2}}[G](t).$$

Using Lemma 2.4.9, we obtain

$$|G(t, x)| \lesssim \frac{\sqrt{\mathcal{E}_{\frac{n+2}{2}}[G](t)}}{(1+t)^{\frac{n+2}{2}}}.$$

We consider now the case $|x| \geq 1 + \frac{1}{2}t$.

According to Lemma 2.4.14,

$$\begin{aligned} \sum_{l=0}^1 \sum_{i+j=l} \int_{|y| \geq 1 + \frac{1}{2}t} \tau_-^{2i} r^{2j} \left(\tau_-^2 |\nabla_{\underline{L}}^i \nabla_L^j \alpha|_{\mathbb{O}, \frac{n+2}{2} - i - j}^2 + r^2 (|\nabla_{\underline{L}}^i \nabla_L^j \alpha|_{\mathbb{O}, \frac{n+2}{2} - i - j}^2 \right. \\ \left. + |\nabla_{\underline{L}}^i \nabla_L^j \rho|_{\mathbb{O}, \frac{n+2}{2} - i - j}^2 + |\nabla_{\underline{L}}^i \nabla_L^j \sigma|_{\mathbb{O}, \frac{n+2}{2} - i - j}^2) \right) dy \lesssim \mathcal{E}_{\frac{n+2}{2}}[G](t). \end{aligned}$$

Let w be either $r\alpha$, $r\rho$, $r\sigma$ or $\tau_- \alpha$. Since $\partial_r = \frac{L-\underline{L}}{2}$ and $|\partial_r(\tau_-)| \leq 1$, we have

$$\int_{|y| \geq 1 + \frac{1}{2}t} |w|_{\mathbb{O}, \frac{n+2}{2}}^2 + \tau_-^2 |\nabla_{\partial_r} w|_{\mathbb{O}, \frac{n}{2}}^2 dy \lesssim \mathcal{E}_{\frac{n+2}{2}}[G](t).$$

Lemma 2.4.10 then gives us

$$|w(t, x)| \lesssim \frac{\sqrt{\mathcal{E}_{\frac{n+2}{2}}[G](t)}}{|x|^{\frac{n-1}{2}} \tau_-^{\frac{1}{2}}}.$$

Thus,

$$|\alpha(t, x)|, |\rho(t, x)|, |\sigma(t, x)| \lesssim \frac{\sqrt{\mathcal{E}_{\frac{n+2}{2}}[G](t)}}{|x|^{\frac{n+1}{2}} \tau_-^{\frac{1}{2}}} \quad \text{and} \quad |\underline{\alpha}(t, x)| \lesssim \frac{\sqrt{\mathcal{E}_{\frac{n+2}{2}}[G](t)}}{|x|^{\frac{n-1}{2}} \tau_-^{\frac{3}{2}}}.$$

We now prove (2.36). Using Corollary 2.4.15, we have

$$\begin{aligned} \int_{|y| \geq 1 + \frac{1}{2}t} |\sqrt{\tau_-} \underline{\alpha}|_{\mathbb{O}, \frac{n+2}{2}}^2 + \tau_-^2 |\nabla_{\partial_r}(\sqrt{\tau_-} \underline{\alpha})|_{\mathbb{O}, \frac{n}{2}}^2 dy \lesssim \\ \sum_{|\beta| \leq \frac{n+2}{2}} \int_{\Sigma_t} \tau_- (|\alpha(\mathcal{L}_{Z^\beta} G)|^2 + |\underline{\alpha}(\mathcal{L}_{Z^\beta} G)|^2 + |\rho(\mathcal{L}_{Z^\beta} G)|^2 + |\sigma(\mathcal{L}_{Z^\beta} G)|^2) dx. \end{aligned}$$

As, by Definition 2.3.20,

$$\sum_{|\beta| \leq \frac{n+2}{2}} \int_{\Sigma_t} \tau_- (|\alpha(\mathcal{L}_{Z^\beta} G)|^2 + |\rho(\mathcal{L}_{Z^\beta} G)|^2 + |\sigma(\mathcal{L}_{Z^\beta} G)|^2) dx \lesssim \frac{\mathcal{E}_{\frac{n+2}{2}}[G](t)}{1+t}$$

and, by Definition 2.3.24

$$\sum_{|\beta| \leq \frac{n+2}{2}} \int_{\Sigma_t} \tau_- |\underline{\alpha}(\mathcal{L}_{Z^\beta} G)|^2 dx \lesssim \mathcal{E}_{\frac{n+2}{2}}^S[G](t),$$

we obtain, again by Lemma 2.4.10

$$|\underline{\alpha}(t, x)| \lesssim \frac{\sqrt{\frac{\mathcal{E}_{\frac{n+2}{2}}[G](t)}{1+t}} + \sqrt{\mathcal{E}_{\frac{n+2}{2}}^S[G](t)}}{|x|^{\frac{n-1}{2}} \tau_-}.$$

□

Our goal now is to show how to improve the decay estimate on α , in the Lorenz gauge, near the light cone (we cannot reproduce the method used by [11] to treat the 3d case). We start by the following lemma.

Lemma 2.4.18. *Let \mathcal{A} be a sufficiently regular current, defined on $[0, T] \times \mathbb{R}^n$, such that*

$$\partial^\mu \mathcal{A}_\mu = 0 \quad \text{and} \quad \forall t \in [0, T], \quad \tilde{\mathcal{E}}_{\frac{n+2}{2}}[\mathcal{A}](t) \leq \mathcal{E}(t),$$

with $\mathcal{E} : [0, T] \rightarrow \mathbb{R}_+$ an increasing function. Then

$$|\mathcal{A}_{\underline{L}}|(t, x), \quad |\mathcal{A}_B|(t, x) \lesssim \frac{\mathcal{E}(t)}{\tau_+^{\frac{n-1}{2}} \tau_-^{\frac{1}{2}}} \quad \text{and} \quad |\mathcal{A}_L|(t, x) \lesssim \frac{\mathcal{E}(t)}{\tau_+^{\frac{n}{2}}}.$$

Proof. Using a classical L^2 -Klainerman-Sobolev inequality, we have, $\forall |\gamma| \leq 1, 1 \leq \mu \leq n, (t, x) \in [0, T] \times \mathbb{R}^n$,

$$|Z^\gamma \mathcal{A}_\mu|(t, x) \lesssim \frac{\sqrt{\tilde{\mathcal{E}}_{\frac{n+2}{2}}[\mathcal{A}](t)}}{\tau_+^{\frac{n-1}{2}} \tau_-^{\frac{1}{2}}}.$$

We then have

$$|(Z^\gamma \mathcal{A})_L|, |(Z^\gamma \mathcal{A})_{\underline{L}}|, |(Z^\gamma \mathcal{A})_B| \lesssim \frac{\sqrt{\mathcal{E}(t)}}{\tau_+^{\frac{n-1}{2}} \tau_-^{\frac{1}{2}}}. \quad (2.37)$$

It then remains to improve the decay estimate on \mathcal{A}_L near the light cone. Since, $\partial^\mu \mathcal{A}_\mu = 0$,

$$(\nabla^L \mathcal{A})_L + (\nabla^{\underline{L}} \mathcal{A})_{\underline{L}} + (\nabla^B \mathcal{A})_B = 0.$$

So, as $\nabla_{\underline{L}} L = 0$,

$$-\nabla_{\underline{L}} \mathcal{A}_L - (\nabla_L \mathcal{A})_{\underline{L}} + (\nabla^B \mathcal{A})_B = 0. \quad (2.38)$$

If $r \leq \frac{t}{2}$ or $r \geq \frac{t}{2}$ and $t \leq 1$, the result comes from (2.37). For the remaining case, $r \geq \frac{t}{2}$ and $t \geq 1$, note first that

$$|\underline{L}(\mathcal{A}_L)| \lesssim \frac{\sqrt{\mathcal{E}(t)}}{\tau_+^{\frac{n-1}{2}} \tau_-^{\frac{1}{2}} r} \lesssim \frac{\sqrt{\mathcal{E}(t)}}{\tau_+^{\frac{n+1}{2}} \tau_-^{\frac{1}{2}}}.$$

Indeed, using Remark 2.2.10 and (2.37), we have

$$|(\nabla_L \mathcal{A})_{\underline{L}}|(t, x), \quad |(\nabla^B \mathcal{A})_B|(t, x) \lesssim \frac{\sqrt{\mathcal{E}(t)}}{\tau_+^{\frac{n-1}{2}} \tau_-^{\frac{1}{2}} r}$$

so that (using (2.38)), $\underline{L}(\mathcal{A}_L)$ satisfies also this decay rate. As for a sufficiently regular function g ,

$$g(t, r) = g(0, t+r) + \int_{u=-t-r}^{t-r} \underline{L}(g) du,$$

and since \mathcal{E} is a increasing function, we have

$$\begin{aligned} |\mathcal{A}_L|(t, r) &\leq |\mathcal{A}_L|(0, t+r) + \int_{u=-t-r}^{t-r} |\underline{L}(\mathcal{A}_L)| du \\ &\lesssim \frac{\sqrt{\mathcal{E}(0)}}{\tau_+^{\frac{n}{2}}} + \frac{\sqrt{\mathcal{E}(t)}}{\tau_+^{\frac{n+1}{2}}} \int_{u=-t-r}^{t-r} \tau_-^{-\frac{1}{2}} du \\ &\lesssim \frac{\sqrt{\mathcal{E}(t)}}{\tau_+^{\frac{n}{2}}}. \end{aligned}$$

□

Finally, we obtain the following pointwise decay on α .

Proposition 2.4.19. *Let G and J be a sufficiently regular 2-form and 1-form (respectively), defined on $[0, T] \times \mathbb{R}^n$, such that*

$$\begin{aligned}\nabla^\mu G_{\mu\nu} &= J_\nu, \\ \nabla^{\mu*} G_{\mu\lambda_1 \dots \lambda_{n-2}} &= 0.\end{aligned}$$

Let A be a potential of G in the Lorenz gauge such that $\tilde{\mathcal{E}}_{\frac{n+4}{2}}[A](t) \leq \mathcal{E}(t)$. We suppose that

$$|J|(t, x) \lesssim \frac{\theta(t)}{\tau_+^{n-1} \tau_-}$$

and that \mathcal{E} and θ are increasing functions. Then,

$$\forall (t, x) \in [0, T] \times \mathbb{R}^n, \quad |\alpha(G)|(t, x) \lesssim \frac{\sqrt{\mathcal{E}(t)}}{\tau_+^{\frac{n+2}{2}}} + \frac{\theta(t) \log(\tau_-)}{\tau_+^{n-1}}.$$

Remark 2.4.20. *the functions \mathcal{E} and θ will later be of the form $t \mapsto (1+t)^a$ or $t \mapsto \log^k(1+t)$.*

Proof. We consider a spherical variable B . We have

$$\begin{aligned}\alpha_B(G) &= (\partial_\mu A_\nu - \partial_\nu A_\mu)_{BL} \\ &= e_B(A)_L - L(A)_B \\ &= \mathcal{L}_{e_B}(A)_L - \frac{1}{r} A_B - L(A)_B,\end{aligned}$$

since $e_B(A)_L = \mathcal{L}_{e_B}(A)_L - \frac{1}{r} A_B$. Indeed, as e_B can be written as a linear combination of rescaled rotations (namely $\frac{\Omega_{ij}}{r}$), we only have to prove

$$\Omega_{ij}(A)_L = \mathcal{L}_{\Omega_{ij}}(A)_L - \frac{1}{r} A_{\Omega_{ij}} \quad \text{for all } 1 \leq i < j \leq n.$$

Consider for instance Ω_{12} . As $\Omega_{12} = x^1 \partial_2 - x^2 \partial_1$, we have

$$\mathcal{L}_{\Omega_{12}}(A)_1 = \Omega_{12} A_1 + A_2, \quad \mathcal{L}_{\Omega_{12}}(A)_2 = \Omega_{12} A_2 - A_1$$

and

$$\mathcal{L}_{\Omega_{12}}(A)_k = \Omega_{12} A_k \quad \text{if } k \geq 3,$$

so that

$$\mathcal{L}_{\Omega_{12}}(A)_L = \Omega_{12}(A)_L + \frac{1}{r} A_{\Omega_{12}}.$$

As $4\tau_- \geq \tau_+$ if $t \geq 2r$ or $t+r \leq 2$, we only have to consider the case¹⁹ where $2r \geq t$ and $t+r \geq 2$, so that $3r \geq \tau_+$. Recall from Lemma 2.2.18 that

$$\forall \Omega \in \mathbb{O}, \quad \partial^\mu \mathcal{L}_\Omega A_\mu = 0.$$

So, using Lemma 2.4.18 and that e_B can be written as a linear combination of rescaled rotations, we have

$$|\mathcal{L}_{e_B}(A)_L|(t, x) \lesssim \frac{\sqrt{\mathcal{E}(t)}}{r\tau_+^{\frac{n}{2}}} \lesssim \frac{\sqrt{\mathcal{E}(t)}}{\tau_+^{\frac{n+2}{2}}}.$$

For the remaining term, rewritting the wave equation (2.7) satisfied by A in null coordinates, we have, for $0 \leq \mu \leq n$,

$$-\underline{L}L A_\mu + \nabla^C \nabla_C A_\mu + \frac{1}{r} L A_\mu - \frac{1}{r} \underline{L} A_\mu = J_\mu.$$

Hence

$$\underline{L} \left(\left(L + \frac{1}{r} \right) A_\mu \right) = \nabla^C \nabla_C A_\mu + \frac{1}{r} L A_\mu + \underline{L} \left(\frac{1}{r} \right) A_\mu - J_\mu.$$

¹⁹When $4\tau_- \geq \tau_+$, the result comes from Proposition 2.4.16 or from $|\partial A| \lesssim \tau_+^{-\frac{n-1}{2}} \tau_-^{-\frac{3}{2}} \sum_{Z \in \mathbb{K}} |ZA|$ (cf Remark 2.2.10).

Now, note that, using a classical L^2 Klainerman-Sobolev inequality and Remark 2.2.10,

$$|\underline{L}\left(\frac{1}{r}\right)A_\mu|, |\nabla^C \nabla_C A_\mu| \lesssim \frac{\sqrt{\mathcal{E}(t)}}{r^2 \tau_+^{\frac{n-1}{2}} \tau_-^{\frac{1}{2}}}, \quad \left|\frac{1}{r}LA_\mu\right| \lesssim \frac{\sqrt{\mathcal{E}(t)}}{r\tau_+^{\frac{n+1}{2}} \tau_-^{\frac{1}{2}}}, \quad |J^\mu| \lesssim \frac{\theta(t)}{\tau_+^{n-1} \tau_-}$$

so that, as $3r \geq \tau_+$,

$$\left|\underline{L}\left(\left(L + \frac{1}{r}\right)A_\mu\right)\right| \lesssim \frac{\sqrt{\mathcal{E}(t)}}{\tau_+^{\frac{n+3}{2}} \tau_-^{\frac{1}{2}}} + \frac{\theta(t)}{\tau_+^{n-1} \tau_-}.$$

Hence, as for a sufficiently regular function g ,

$$g(t, r) = g(0, t+r) + \int_{u=-t-r}^{t-r} \underline{L}(g) du,$$

we have (using that \mathcal{E} and θ are increasing functions)

$$\begin{aligned} \left|\left(L + \frac{1}{r}\right)A_\mu\right|(t, x) &\lesssim \frac{\sqrt{\mathcal{E}(0)}}{\tau_+^{\frac{n+2}{2}}} + \frac{\sqrt{\mathcal{E}(t)}}{\tau_+^{\frac{n+3}{2}}} \int_{-t-r}^{t-r} \frac{1}{\tau_-^{\frac{1}{2}}} du + \frac{\theta(t)}{\tau_+^{n-1}} \int_{-t-r}^{t-r} \frac{1}{\tau_-} du \\ &\lesssim \frac{\sqrt{\mathcal{E}(t)}}{\tau_+^{\frac{n+2}{2}}} + \frac{\theta(t) \log(\tau_-)}{\tau_+^{n-1}}, \end{aligned}$$

implying

$$\left|\frac{1}{r}A_B + L(A)_B\right|(t, x) \lesssim \frac{\sqrt{\mathcal{E}(t)}}{\tau_+^{\frac{n+2}{2}}} + \frac{\theta(t) \log(\tau_-)}{\tau_+^{n-1}}.$$

□

Remark 2.4.21. *In the context of the Vlasov-Maxwell system, using the null component v^B of the velocity vector, we have a better pointwise estimate on the component J_B of the source term, as J_B is a linear combination of the terms $\int_v \frac{v^B}{v^0} \widehat{Z}^\beta f dv$. Since the dimension n is such that $n \geq 4$, we do not need this extra decay (and we then worked with the Cartesian components of the source term in the proof of the previous proposition).*

2.4.5 A Grönwall inequality

Later, when we will study the velocity support of the scalar field in the massless case, we will need the following variant of Grönwall's lemma.

Lemma 2.4.22. *Let $T > 0$, f and g two continuous nonnegatives functions defined on $[0, T]$ and $C \geq 0$. If*

$$\forall t \in [0, T], f(t) \leq C + 2 \int_0^t g(s) \sqrt{f(s)} ds,$$

then

$$\forall t \in [0, T], f(t) \leq \left(\sqrt{C} + \int_0^t g(s) ds\right)^2.$$

Proof.

First, we suppose that $C > 0$. Let $F : t \mapsto C + 2 \int_0^t g(s) \sqrt{f(s)} ds$. We have

$$F'(t) \leq 2g(t) \sqrt{F(t)}.$$

Since $C > 0$, F is nonnegative and we can divide by $2\sqrt{F(t)}$. Integrating the above, we obtain

$$\sqrt{F(t)} \leq \sqrt{C} + \int_0^t g(s) ds,$$

which implies the result. If $C = 0$, then, for all $\epsilon > 0$,

$$\forall t \in [0, T], f(t) \leq \epsilon + 2 \int_0^t g(s) \sqrt{f(s)} ds.$$

It only remains to apply the inequality in the case $C \neq 0$ and let ϵ tends to zero.

□

2.5 Decay estimate for the massive case

Recall that, as we study massive particles in this section, $v^0 = \sqrt{1 + |v|^2}$. We will use the commutation vector fields of $\widehat{\mathbb{P}}_0$ and the weights of \mathbf{k}_1 preserved by the operator T_1 (see Subsections 2.2.3 and 2.2.4 for their definitions). We fix for all this section a sufficiently regular 2-form F defined on $[0, T^*[\times\mathbb{R}^n$ and we recall that we defined T_F as the operator

$$T_F : g \mapsto v^\mu \partial_\mu g + F(v, \nabla_v g)$$

and that $\nabla_v g = (0, \partial_{v^1} g, \dots, \partial_{v^n} g)$. The main result of this section is the following estimate.

Theorem 2.5.1. *Let $T^* > 0$ and $f : [0, T^*[\times\mathbb{R}_x^n \times \mathbb{R}_v^n \rightarrow \mathbb{R}$ be a sufficiently regular function such that*

$$\sum_{z \in \mathbf{k}_1} \sum_{|\beta| \leq n} \int_{\Sigma_0} \int_{\mathbb{R}_v^n} |z \widehat{Z}^\beta f| \, dv dx < +\infty,$$

Then, for all $(t, x) \in [0, T^[\times\mathbb{R}^n$,*

$$\int_{\mathbb{R}_v^n} |f(t, x, v)| \frac{dv}{(v^0)^2} \lesssim \frac{1}{\tau_+^n} \sum_{z \in \mathbf{k}_1} \sum_{|\beta| \leq n} \left(\int_{\Sigma_0} \int_{\mathbb{R}_v^n} |z \widehat{Z}^\beta f| \, dv dx + \int_0^t \int_{\Sigma_s} \int_{\mathbb{R}_v^n} |T_F(z \widehat{Z}^\beta f)| \frac{dv}{v^0} dx ds \right).$$

Remark 2.5.2. *Compared to Theorem 8 in [18], the advantage is that the L^1 norms on the right hand side are taken on $\{t\} \times \mathbb{R}^n$ (or $\{0\} \times \mathbb{R}^n$) and not on a hyperboloid. On the other hand, our estimate is not a pure Sobolev inequality (we applied the operator T_F to $\widehat{Z}^\beta f$ to establish it).*

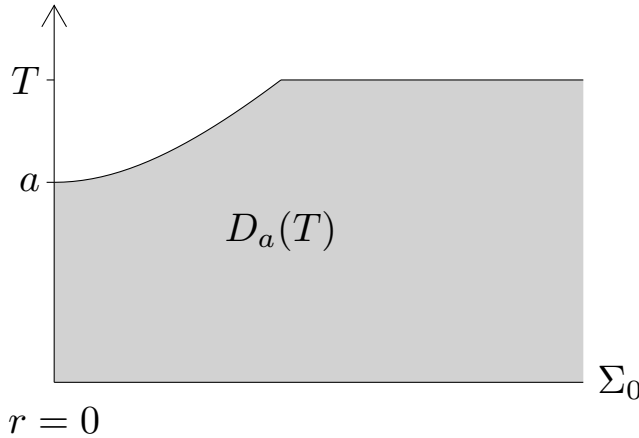
Remark 2.5.3. *To simplify the notation, we took the mass to be 1, but the estimate is true as long as the mass is strictly positive (the constant hidden in \lesssim is however proportional to $\frac{1}{m^2}$).*

Remark 2.5.4. *As we will need an estimate on $\int_v |f| dv$ in this article, we will apply Theorem 2.5.1 to $(v^0)^2 f$. Note that, since $T_1((v^0)^2 z) = 0$, the spacetime integral given by Theorem 2.5.1 can be bounded, in that case, by*

$$\int_0^t \int_{\Sigma_s} \int_{\mathbb{R}_v^n} \left| v^0 z T_F(\widehat{Z}^\beta f) \right| + \left| v^0 F(v, \nabla_v z) \widehat{Z}^\beta f \right| + \left| z v^i F_{i0} \widehat{Z}^\beta f \right| \, dv dx ds.$$

One can then use commutation formula of Proposition 2.2.26 in order to compute $T_F(\widehat{Z}^\beta f)$.

The proof is based on a partition of the spacetime. In the interior ($|x| \leq \frac{t}{2}$) and the exterior ($t \leq |x|$) of the light cone, the proof relies on the Klainerman-Sobolev inequality of Theorem 2.2.13. In the exterior region, the lack of decay is compensated by using the weights $(x^i - t \frac{v^i}{v^0}) \in \mathbf{k}_1$ defined in Section 2.2.4. For the remaining region, we work on subsets of \mathbb{R}^{n+1} composed of a piece of an hyperboloid and a piece of a slice $t = \text{constant}$ as [19] for the Klein-Gordon equation, mixing what is usually done for such problems.



The set $D_a(T)$ and its boundary

2.5.1 Sobolev inequalities

We start by a Sobolev inequality independent of time.

Lemma 2.5.5. *Let $g : \mathbb{R}_x^n \times \mathbb{R}_v^n \rightarrow \mathbb{R}$ be a sufficiently regular function. Then, for all $x \in \mathbb{R}^n$,*

$$|x|^n \int_{v \in \mathbb{R}^n} |g(x, v)| dv \lesssim \sum_{\substack{|\beta| \leq n-1 \\ j \leq 1}} \left\| \int_{v \in \mathbb{R}^n} (r \partial_r)^j (|\widehat{\Omega}^\beta g|)(y, v) dv \right\|_{L^1(|y| \leq |x|)},$$

where $\Omega^\beta \in \mathbb{O}^{|\beta|}$.

During the proof of this lemma, we will use many time the following one dimensional Sobolev inequality. For $w \in W^{1,1}$, we have, for all $a \in \mathbb{R}$ and all $\delta \geq \eta > 0$,

$$|w(a)| \leq C_\eta (\|w(y)\|_{L^1(a-\delta \leq y \leq a)} + \|w'(y)\|_{L^1(a-\delta \leq y \leq a)}),$$

with C_η a positive constant depending only on η .

Proof. As there is nothing to prove when $x = 0$, we suppose $x \neq 0$. We start by introducing spherical coordinates. A point $y \in \mathbb{R}^n$ has for coordinates (r, θ) , with $r = |y|$ and $\theta \in \mathbb{S}^{n-1}$. We denote by $(|x|, \omega)$ the spherical coordinates of x and by $(\theta_1, \dots, \theta_{n-1})$ a local coordinate map in a neighbourhood of $\omega \in \mathbb{S}^{n-1}$ (by the symmetry of the sphere, we can suppose that the θ_i take their values in an interval of a size independent of ω). Let h be the function defined by $h(r, \theta, v) = g(|x|r\theta, v)$. By a one dimensional Sobolev inequality,

$$\begin{aligned} \int_{v \in \mathbb{R}^n} |h(1, \omega, v)| dv &\lesssim \int_{\theta_1} \left| \int_{v \in \mathbb{R}^n} |h|(1, \omega_1 + \theta_1, \omega_2, \dots, \omega_n, v) dv \right| \\ &\quad + \left| \partial_{\theta_1} \int_{v \in \mathbb{R}^n} |h|(1, \omega_1 + \theta_1, \omega_2, \dots, \omega_n, v) dv \right| d\theta_1. \end{aligned}$$

As ∂_{θ_1} is a linear combination of the rotation vector fields, Remark 2.2.4 gives us

$$\left| \partial_{\theta_1} \int_{v \in \mathbb{R}^n} |h|(1, \omega_1 + \theta_1, \omega_2, \dots, \omega_n, v) dv \right| \lesssim \sum_{\Omega \in \mathbb{O}} \int_{v \in \mathbb{R}^n} |\widehat{\Omega} h|(1, \omega_1 + \theta_1, \omega_2, \dots, \omega_n, v) dv.$$

Thus,

$$\int_{v \in \mathbb{R}^n} |h(1, \omega, v)| dv \lesssim \sum_{\substack{\Omega^\beta \in \mathbb{O}^{|\beta|} \\ |\beta| \leq 1}} \int_{\theta_1} \int_{v \in \mathbb{R}^n} |\widehat{\Omega}^\beta h|(1, \omega_1 + \theta_1, \omega_2, \dots, \omega_n, v) dv d\theta_1.$$

Using the same argument for the variables $\theta_2, \dots, \theta_{n-2}$ and θ_{n-1} , it comes

$$\int_{v \in \mathbb{R}^n} |h(1, \omega, v)| dv \lesssim \sum_{\substack{\Omega^\beta \in \mathbb{O}^{|\beta|} \\ |\beta| \leq n-1}} \int_{\theta \in \mathbb{S}^{n-1}} \int_{v \in \mathbb{R}^n} |\widehat{\Omega}^\beta h|(1, \theta, v) dv d\theta.$$

The one dimensional Sobolev inequality, applied this time to the first variable, gives

$$\int_{v \in \mathbb{R}^n} |h(1, \omega, v)| dv \lesssim \sum_{\substack{j \leq 1 \\ |\beta| \leq n-1}} \sum_{\Omega^\beta \in \mathbb{O}^{|\beta|}} \int_{\frac{1}{2}}^1 \left| \partial_r^j \int_{\theta \in \mathbb{S}^{n-1}} \int_{v \in \mathbb{R}^n} |\widehat{\Omega}^\beta h|(r, \theta, v) dv d\theta \right| dr.$$

Hence, as $\frac{1}{2} \leq r$,

$$\int_{v \in \mathbb{R}^n} |h(1, \omega, v)| dv \lesssim \sum_{\substack{j \leq 1 \\ |\beta| \leq n-1}} \sum_{\Omega^\beta \in \mathbb{O}^{|\beta|}} \int_{\frac{1}{2}}^1 \int_{\theta \in \mathbb{S}^{n-1}} \left| (r \partial_r)^j \int_{v \in \mathbb{R}^n} |\widehat{\Omega}^\beta h|(r, \theta, v) dv \right| d\theta r^{n-1} dr,$$

which implies

$$\int_{v \in \mathbb{R}^n} |g(x, v)| dv \lesssim \sum_{j \leq 1, |\beta| \leq n-1} \left\| \int_{v \in \mathbb{R}^n} (r \partial_r)^j (|\widehat{\Omega}^\beta (g(|x|y, v))|) dv \right\|_{L^1(|y| \leq 1)},$$

It only remains to remark that, as $r \partial_r$ and Ω are homogeneous vector fields,

$$(r \partial_r)^j \left(|\widehat{\Omega}^\beta (g(|x|y, v))| \right) = (r \partial_r)^j \left(|\widehat{\Omega}^\beta g| \right) (|x|y, v)$$

and to make the change of variables $y' = |x|y$. □

We are now able to prove the following time dependent Sobolev inequality.

Lemma 2.5.6. *Let $g : [0, T^*] \times \mathbb{R}_x^n \times \mathbb{R}_v^n \rightarrow \mathbb{R}$ a sufficiently regular function. For all $(t, x) \in [0, T^*] \times \mathbb{R}^n$ such that $|x| \leq t$, we have*

$$|x|^n \int_{v \in \mathbb{R}^n} |g(t, x, v)| dv \lesssim \sum_{|\beta| \leq n} \|\widehat{Z}^\beta g(\sqrt{|y|^2 + a^2}, y, v)\|_{L^1(|y| \leq |x|) L_v^1},$$

with $a^2 = t^2 - |x|^2$ and $\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|}$ (more precisely the vector fields involved are either rotations or Lorentz boosts).

Proof. Let $(t, x) \in [0, T^*] \times \mathbb{R}^n$ such that $|x| \leq t$ and $a^2 = t^2 - |x|^2$. We apply the previous lemma to $(y, v) \mapsto g(\sqrt{|y|^2 + a^2}, y, v)$ to get

$$|x|^n \int_{v \in \mathbb{R}^n} |g(t, x, v)| dv \lesssim \sum_{j \leq 1, |\beta| \leq n-1} \left\| \int_{v \in \mathbb{R}^n} (r \partial_r)^j \left(|\widehat{\Omega}^\beta g(\sqrt{|y|^2 + a^2}, y, v) \right) dv \right\|_{L^1(|y| \leq |x|)},$$

where we used that

$$\widehat{\Omega}^\beta \left(g(\sqrt{|y|^2 + a^2}, y, v) \right) = \widehat{\Omega}^\beta (g) (\sqrt{|y|^2 + a^2}, y, v),$$

since $\Omega(\sqrt{|y|^2 + a^2}) = 0$ for all $\Omega \in \mathcal{O}$.

Now, we remark that

$$r \partial_r \left(|\widehat{\Omega}^\beta g(\sqrt{|y|^2 + a^2}, y, v) \right) = \frac{\widehat{\Omega}^\beta g(\sqrt{|y|^2 + a^2}, y, v)}{|\widehat{\Omega}^\beta g(\sqrt{|y|^2 + a^2}, y, v)|} \frac{y^i}{\sqrt{|y|^2 + a^2}} \Omega_{0i} \widehat{\Omega}^\beta g(\sqrt{|y|^2 + a^2}, y, v).$$

Note also that, dropping the dependance in $(\sqrt{|y|^2 + a^2}, y, v)$ of the functions considered,

$$\int_{v \in \mathbb{R}^n} \frac{\widehat{\Omega}^\beta g}{|\widehat{\Omega}^\beta g|} \frac{y^i}{\sqrt{|y|^2 + a^2}} v^0 \partial_{v^i} \widehat{\Omega}^\beta g dv = - \int_{v \in \mathbb{R}^n} \frac{v^i}{v^0} \frac{y_i}{\sqrt{|y|^2 + a^2}} |\widehat{\Omega}^\beta g| dv.$$

It then comes that

$$|x|^n \int_{v \in \mathbb{R}^n} |g(t, x, v)| dv \lesssim \sum_{|\beta| \leq n} \left\| \left(1 + \frac{|y|}{\sqrt{|y|^2 + a^2}} \right) \int_{v \in \mathbb{R}^n} |\widehat{Z}^\beta g(\sqrt{|y|^2 + a^2}, y, v)| dv \right\|_{L^1(|y| \leq |x|)},$$

which allows us to deduce the result. \square

2.5.2 An energy estimate

Before starting the proof of Theorem 2.5.1, we establish the following lemma, which combined with our last Sobolev inequality, will give us the expected decay on the velocity average of the Vlasov field for a spacetime region.

Lemma 2.5.7. *Let $a > 0$, $T \in]a, T^*[$ and $g : [0, T^*] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ be a sufficiently regular function. Then,*

$$\int_{|y| \leq \sqrt{T^2 - a^2}} \int_{\mathbb{R}_v^n} |g(\sqrt{|y|^2 + a^2}, y, v)| \frac{dv}{(v^0)^2} dy \leq 2 \int_{\Sigma_0} \int_{\mathbb{R}_v^n} |g| dx dv + 2 \int_0^T \int_{\Sigma_s} \int_{\mathbb{R}_v^n} |T_F(g)| \frac{dv}{v^0} dx ds.$$

Proof. We use again the vector field $N^\mu(g) := \int_{\mathbb{R}_v^n} g \frac{v^\mu}{v^0} dv$ and recall from (2.25) that

$$\partial_\mu N^\mu(|g|) = \int_{v \in \mathbb{R}^n} \frac{g}{|g|} \frac{T_F(g)}{v^0} - \frac{g}{|g|} F \left(\frac{v}{v^0}, \nabla_v g \right) dv = \int_{v \in \mathbb{R}^n} \frac{g}{|g|} T_F(g) \frac{dv}{v^0}.$$

We now introduce the following subset of $\mathbb{R}_+ \times \mathbb{R}^n$:

$$D_a(T) = \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^n / a^2 \geq s^2 - |y|^2, 0 \leq s \leq T\}.$$

Denoting by ν is the outward pointing unit normal field to $\partial D_a(T)$, the divergence theorem (in $W^{1,1}$, for the euclidian space \mathbb{R}^{n+1}) gives us

$$\int_{\partial D_a(T)} \nu_\mu N^\mu(|g|) d\partial D_a(T) = \int_{D_a(T)} \int_{v \in \mathbb{R}^n} \frac{g}{|g|} T_F(g) \frac{dv}{v^0} dy dD_a(T).$$

The boundary term is equal to

$$\int_{|y| \leq \sqrt{T^2 - a^2}} \int_{v \in \mathbb{R}^n} \nu_\mu \frac{v^\mu}{v^0} |g|(\sqrt{|y|^2 + a^2}, y, v) dv d\lambda(y) + \|g\|_{L_x^1(|y| \geq \sqrt{T^2 - a^2}) L_v^1}(T) - \|g\|_{L_x^1 L_v^1}(0),$$

where $d\lambda(y)$ is the surface measure on the hyperboloid $\{s^2 - |y|^2 = a^2\}$. More precisely, on this hyperboloid²⁰,

$$d\lambda(y) = \sqrt{\det \left(I_n + \frac{1}{|y|^2 + a^2} {}^t y y \right)} dy = \sqrt{\frac{2|y|^2 + a^2}{|y|^2 + a^2}}$$

and

$$\nu(y) = \frac{1}{\sqrt{2|y|^2 + a^2}} (\sqrt{|y|^2 + a^2}, -y).$$

We then deduce, as $D_a(T) \subset [0, T] \times \mathbb{R}^n$,

$$\begin{aligned} \int_{|y| \leq \sqrt{T^2 - a^2}} \int_{\mathbb{R}_v^n} \left(\sqrt{|y|^2 + a^2} v^0 - y_i v^i \right) |g|(\sqrt{|y|^2 + a^2}, y, v) \frac{dv dy}{v^0 \sqrt{|y|^2 + a^2}} \leq \\ \|g\|_{L_x^1 L_v^1}(0) + \int_0^T \int_{\Sigma_s} \int_{\mathbb{R}_v^n} |T_F(g)| \frac{dv}{v^0}. \end{aligned}$$

Finally, note that for $s = \sqrt{|y|^2 + a^2} \geq |y|$,

$$\frac{sv^0 - y_i v^i}{s} \geq \frac{sv^0 - |y||v|}{s} \geq s \frac{(v^0 - |v|)(v^0 + |v|)}{s(v^0 + |v|)} \geq \frac{1}{2v^0}.$$

The result follows from a combination of the last two inequalities. \square

Remark 2.5.8. *The lemma is also valid on the cone $s = |y|$, which means that the result is true for $a = 0$, but we already knew it with Proposition 2.3.1.*

2.5.3 Proof of Theorem 2.5.1

We consider a partition of the spacetime into four regions.

- The bounded region, $t + |x| \leq 2$, where a standard Sobolev inequality gives the result.
- The interior of the light cone, where $|x| \leq \frac{t}{2}$.
- The exterior of the light cone, where $t \leq |x|$ and $|x| \geq 1$.
- The remaining region where $\frac{t}{2} \leq |x| \leq t$ and $t \geq 1$.

The interior of the light cone

Let $(t, x) \in [0, T^*[\times \mathbb{R}^n$ such that $|x| \leq \frac{t}{2}$. Thus, $\tau_- \geq \frac{1}{3}\tau_+$ and the Klainerman-Sobolev inequality of Theorem 2.2.13 gives

$$\int_{v \in \mathbb{R}^n} |f(t, x, v)| dv \lesssim \frac{1}{\tau_+^n} \sum_{|\beta| \leq n} \left\| \widehat{Z}^\beta f \right\|_{L_{x,v}^1}(t).$$

It only remains to apply Proposition 2.3.1, which gives us

$$\left\| \widehat{Z}^\beta f \right\|_{L_{x,v}^1}(t) \lesssim \left\| \widehat{Z}^\beta f \right\|_{L_{x,v}^1}(0) + \int_0^t \int_{\Sigma_s} \int_{\mathbb{R}_v^n} |T_F(\widehat{Z}^\beta f)| \frac{dv}{v^0} dx ds.$$

²⁰Here, ${}^t y$ denotes the transpose of y .

The exterior of the light cone

We use $(x^i - t \frac{v^i}{v^0}) \in \mathbf{k}_1$, for $1 \leq i \leq n$, which are solutions to the homogeneous relativistic transport equation. Let $(t, x) \in [0, T^*[\times \mathbb{R}^n$ such that $t \leq |x|$ and $|x| \geq 1$. By Theorem 2.2.13, we have

$$\int_{\mathbb{R}_v^n} \left| x^i - t \frac{v^i}{v^0} \right| |f|(t, x, v) dv \lesssim \frac{1}{\tau_+^{n-1}} \sum_{z \in \mathbf{k}_1} \sum_{|\beta| \leq n} \|z \widehat{Z}^\beta f\|_{L_{x,v}^1}(t).$$

Since $|xv^0 - tv| \geq v^0|x| - t|v| \geq \frac{|x|}{2v^0}$, it comes

$$\begin{aligned} |x| \int_v |f|(t, x, v) \frac{dv}{(v^0)^2} &\lesssim \int_v \left| x - t \frac{v}{v^0} \right| |f|(t, x, v) dv \\ &\lesssim \sum_{i=1}^n \int_v \left| x^i - t \frac{v^i}{v^0} \right| |f|(t, x, v) dv. \end{aligned}$$

Hence, using that $|x| \gtrsim \tau_+$ (recall that $|x| \geq 1$ and $t \leq |x|$ in the region studied) and applying Proposition 2.3.1, we finally obtain

$$\begin{aligned} \int_v |f|(t, x, v) \frac{dv}{(v^0)^2} &\lesssim \frac{1}{|x| \tau_+^{n-1}} \sum_{z \in \mathbf{k}_1} \sum_{|\beta| \leq n} \|z \widehat{Z}^\beta f\|_{L_{x,v}^1}(t) \\ &\lesssim \frac{1}{\tau_+^n} \sum_{\substack{|\beta| \leq n \\ z \in \mathbf{k}_1}} \|z \widehat{Z}^\beta f\|_{L_{x,v}^1}(0) + \int_0^t \int_{\Sigma_s} \int_v |T_F(z \widehat{Z}^\beta f)| \frac{dv}{v^0} dx ds. \end{aligned}$$

The remaining region

Let $(t, x) \in [0, T^*[\times \mathbb{R}^n$ such that $\frac{t}{2} \leq |x| \leq t$ and $t \geq 1$. We start by applying Lemma 2.5.7 to $\widehat{Z}^\beta f$, for all $|\beta| \leq n$, with $T = t$ and $a^2 = t^2 - |x|^2$. We have

$$\begin{aligned} \sum_{|\beta| \leq n} \int_{|y| \leq |x|} \int_{\mathbb{R}_v^n} |\widehat{Z}^\beta f(\sqrt{|y|^2 + a^2}, y, v)| \frac{dv}{(v^0)^2} dy &\lesssim \\ &\sum_{|\beta| \leq n} \left(\int_{\Sigma_0} \int_{\mathbb{R}_v^n} |\widehat{Z}^\beta f| dv dx + \int_0^t \int_{\Sigma_s} \int_{\mathbb{R}_v^n} T_F(\widehat{Z}^\beta f) \frac{dv}{v^0} dx ds \right). \end{aligned}$$

As $|\widehat{Z}^\gamma((v^0)^{-2})| \lesssim (v^0)^{-2}$, Lemma 2.5.6 applied to $g = (v^0)^{-2}f$ allows us to bound by below the left hand side of the previous inequality by

$$|x|^n \int_v |f|(t, x, v) \frac{dv}{(v^0)^2}.$$

The result follows from $|x|^n \gtrsim \tau_+^n$ (as $|x| \geq \frac{t}{2} \geq \frac{1}{2}$).

2.5.4 Improved decay for the derivatives of the velocity averages

Let us introduce the following vector fields.

Definition 2.5.9. For $1 \leq i \leq n$ and $1 \leq k, l \leq n$, with $k \neq l$, we consider

$$X_i = \frac{v^i}{v^0} \partial_t + \partial_i \quad \text{and} \quad Y_{kl} = \frac{v^k}{v^0} \partial_l - \frac{v^l}{v^0} \partial_k.$$

Proposition 2.5.10. The vector fields $\frac{1}{v^0} T_1$, X_i and Y_{kl} are good derivatives (as the derivatives tangential to the light cone L and e_B , see Remark 2.2.10), which means that if W denotes one of them, we have, for a smooth function f ,

$$\left| \int_v W f dv \right| \lesssim \frac{1}{\tau_+} \left(\sum_{\widehat{Z} \in \mathbb{P}_0} \left| \int_v \widehat{Z} f dv \right| + \sum_{z \in \mathbf{k}_1} \int_v |z| |\nabla_{t,x} f| dv \right)$$

Proof. For T_1 , we remark that

$$tT_1 = v^0 S + (tv^i - x^i v^0) \partial_i, \quad rT_1 = tT_1 + (r-t)T_1$$

and that

$$\left| (r-t) \int_v \partial f dv \right| \lesssim \sum_{\widehat{Z} \in \mathbb{K}} \left| \int_v Z f dv \right| \lesssim \sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|} \\ |\beta| \leq 1}} \left| \int_v \widehat{Z}^\beta f dv \right|.$$

For X_i , that ensues from

$$tv^0 X_i = v^0 \Omega_{0i} + (tv^i - x^i v^0) \partial_t \quad \text{and} \quad rX_i = tX_i + (r-t)X_i.$$

For Y_{kl} , that follows from

$$tv^0 Y_{kl} = v^0 \Omega_{kl} + (tv^k - x^k v^0) \partial_l - (tv^l - x^l v^0) \partial_k \quad \text{and} \quad rY_{kl} = tY_{kl} + (r-t)Y_{kl}.$$

□

Finally, let us show how we can obtain extra decay on $\partial \int_v f dv$ if f solves an equation such as $T_F(f) = 0$.

Proposition 2.5.11. *Let $f : [0, T] \times \mathbb{R}_x^n \times \mathbb{R}_v^n \rightarrow \mathbb{R}$ be a function such that*

$$\forall \widehat{Z} \in \widehat{\mathbb{P}}_0, z \in \mathbf{k}_1, \quad \left| \int_v (v^0)^2 z \widehat{Z} f dv \right| \lesssim \tau_+^{-n}.$$

Then, for all $0 \leq \mu \leq n$,

$$\left| \partial_\mu \int_v f dv \right| \lesssim \tau_+^{-n-1}.$$

Proof. As

$$T_1 = v^\mu \partial_\mu = v^0 \partial_t + v^i X_i - \frac{|v|^2}{v^0} \partial_t = v^i X_i + \frac{1}{v^0} \partial_t,$$

we have

$$\partial_t = v^0 T_1 - v^0 v^i X_i.$$

Similarly

$$\partial_i = (v^0)^2 X_i - v^i T_1 - v^0 v^k Y_{ki}.$$

□

Remark 2.5.12. *We can prove a similar proposition for derivatives of higher orders.*

2.6 The massive Vlasov-Maxwell equations

2.6.1 Global existence for small data

The aim of this section is to prove Theorem 2.1.1. We suppose that the dimension n is at least 4 and we consider the massive Vlasov-Maxwell system (2.1)-(2.3) with at least two species, so that $K \geq 2$. For simplicity, we suppose that $m_k = 1$ for all $1 \leq k \leq K$.

To simplify the notation, we denote during this chapter the energy norm $\mathbb{E}_{M,q,1}^k$, introduced previously in Definition 2.3.2, by $\mathbb{E}_{M,q}^k$. We also introduce the function χ defined on \mathbb{R}_+ by

$$\chi(s) = \log^3(3+s) \quad \text{if } n=4 \quad \text{and} \quad \chi(s) = 1 \quad \text{if } n \geq 5.$$

This is a more precise version of Theorem 2.1.1.

Theorem 2.6.1. *Let $n \geq 4$, $K \geq 2$ and $N \geq \frac{5}{2}n + 1$. Let (f_0, F_0) be an initial data set for the massive Vlasov-Maxwell system. Let (f, F) be the unique classical solution to the system and let A be a potential in the Lorenz gauge. There exists $\epsilon > 0$ such that²¹, if*

$$\tilde{\mathcal{E}}_N[A](0) \leq \epsilon, \quad \mathcal{E}_N[F](0) \leq \epsilon$$

and if, for all $1 \leq k \leq K$,

$$\mathbb{E}_{N+n,1}^2[f_k](0) \leq \epsilon,$$

then (f, F) exists globally in time and verifies the following estimates.

- Energy bounds for A , F and f_k : $\forall 1 \leq k \leq K$ and $\forall t \in \mathbb{R}_+$,

$$\begin{aligned} \tilde{\mathcal{E}}_N[A](t) &\lesssim \epsilon \chi(t), & \mathcal{E}_N[F](t) &\lesssim \epsilon \chi(t), \\ \mathbb{E}_N^2[f_k](t) &\lesssim \epsilon & \text{and} & \mathbb{E}_{N,1}^2[f_k](t) &\lesssim \epsilon \chi^{\frac{1}{6}}(t). \end{aligned}$$

- Pointwise decay for the null decomposition of $\mathcal{L}_{Z^\beta}(F)$: $\forall |\beta| \leq N - n$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\begin{aligned} |\alpha(L_{Z^\beta} F)| &\lesssim \sqrt{\epsilon} \sqrt{\chi(t)} \tau_+^{-\frac{n+2}{2}}, & |\underline{\alpha}(L_{Z^\beta} F)| &\lesssim \sqrt{\epsilon} \sqrt{\chi(t)} \tau_+^{-\frac{n-1}{2}} \tau_-^{-\frac{3}{2}}, \\ |\rho(L_{Z^\beta} F)| &\lesssim \sqrt{\epsilon} \sqrt{\chi(t)} \tau_+^{-\frac{n+1}{2}} \tau_-^{-\frac{1}{2}}, & |\sigma(L_{Z^\beta} F)| &\lesssim \sqrt{\epsilon} \sqrt{\chi(t)} \tau_+^{-\frac{n+1}{2}} \tau_-^{-\frac{1}{2}}. \end{aligned}$$

- Pointwise decay for $\int_{v \in \mathbb{R}^n} |\widehat{Z}^\beta f_k| dv$:

$$\forall |\beta| \leq N - \frac{3n+2}{2}, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad \int_{v \in \mathbb{R}^n} |\widehat{Z}^\beta f_k| dv \lesssim \frac{\epsilon}{\tau_+^n}.$$

- Pointwise decay for $\int_{v \in \mathbb{R}^n} |\widehat{Z}^\beta f_k|(v^0)^2 dv$ and $\int_{v \in \mathbb{R}^n} |z \widehat{Z}^\beta f_k|(v^0)^2 dv$:

$$\begin{aligned} \forall |\beta| \leq N - n, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, & \quad \int_{v \in \mathbb{R}^n} |\widehat{Z}^\beta f_k|(v^0)^2 dv \lesssim \frac{\epsilon}{\tau_+^{n-1} \tau_-}, \\ \forall |\beta| \leq N - \frac{3n+2}{2}, z \in \mathbf{k}_1, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, & \quad \int_{v \in \mathbb{R}^n} |z \widehat{Z}^\beta f_k|(v^0)^2 dv \lesssim \frac{\epsilon}{\tau_+^{n-1} \tau_-}. \end{aligned}$$

- L^2 estimates on $\int_{v \in \mathbb{R}^n} |\widehat{Z}^\beta f_k| dv$:

$$\forall |\beta| \leq N, t \in \mathbb{R}_+, \quad \left\| \int_{v \in \mathbb{R}^n} |\widehat{Z}^\beta f_k| dv \right\|_{L^2(\Sigma_t)} \lesssim \frac{\epsilon \chi^{\frac{1}{6}}(t)}{(1+t)^{\frac{n}{2}}}.$$

Remark 2.6.2. *In dimension 4, if $N \geq 14$, we can take $\chi(t) = \log^2(3+t)$ and avoid the $\log^{\frac{1}{2}}(3+t)$ -loss on the L^2 estimate on $\int_v |\widehat{Z}^\beta f_k| dv$.*

2.6.2 Structure and beginning of the proof

Let (f_0, F_0) be an initial data set satisfying the assumptions of Theorem 2.6.1. By a standard local well-posedness argument, there exists a unique maximal solution (f, F) of the massive Vlasov-Maxwell system defined on $[0, T^*[$, with $T^* \in \mathbb{R}_+^* \cup \{+\infty\}$.

We consider the following bootstrap assumptions. Let T be the largest time such that, $\forall 1 \leq k \leq K$ and $\forall t \in [0, T]$,

$$\mathcal{E}_N[F](t) \leq 2C\epsilon\chi(t), \quad \mathcal{E}_N^S[F](t) \leq 2\bar{C}\epsilon, \quad (2.39)$$

$$\mathbb{E}_N^2[f_k](t) \leq 4\epsilon, \quad \mathbb{E}_{N-\frac{n+2}{2},1}^2[f_k](t) \leq 4\epsilon \quad \text{and} \quad \mathbb{E}_{N,1}^2[f_k](t) \leq 4\epsilon\chi^{\frac{1}{6}}(t), \quad (2.40)$$

where C and \bar{C} are positive constants which will be specified during the proof. Note that by continuity, $T > 0$. We now present our strategy to improve these bootstrap assumptions.

²¹A smallness condition on F , which implies $\tilde{\mathcal{E}}_N[A](0) \leq \epsilon$, is given in Proposition 2.2.20.

1. First, using the bootstrap assumptions, we obtain decay estimates for the null decomposition of F (and its Lie derivatives) and for velocity averages of derivatives of f_k .
2. Next, we improve the bounds on the Vlasov fields energies by means of the energy estimates proved in Propositions 2.3.3 and 2.3.5. To bound the right hand side in these energy estimates, we make fundamental use of the null structure of the system and the pointwise decay estimates on ρ , σ , α , $\underline{\alpha}$ and $\int_{v \in \mathbb{R}^n} |z \widehat{Z}^\beta f_k| dv$.
3. Then, using Theorem 2.5.1, we improve the decay estimate on $\int_v |\widehat{Z}^\beta f_k| dv$ near the light cone.
4. In order to improve the estimates on the electromagnetic field energies, we establish an L_x^2 estimate for the velocity averages of the Vlasov fields (and its derivatives). For this purpose, we follow [18] and we rewrite all the transport equations as an inhomogeneous system of transport equations. The velocity averages of the homogeneous part of the solution verify strong pointwise decay estimates (we use particularly the control that we have at our disposal on the initial data of f , for derivatives of order $N + n$ or less). The inhomogeneous part is decomposed into a product of an integrable function and a pointwise decaying function which gives us the expected estimate.
5. Finally, we bound the energy of the electromagnetic potential (which satisfy the Lorenz gauge) and we improve the estimates on the electromagnetic field energies with the energy estimates for the Maxwell equations (Propositions 2.3.21 and 2.3.25). We use again the null decomposition of F (and its Lie derivatives), which, combined by the estimates on $\left\| \tau_+ \int_{\mathbb{R}^n} |\widehat{Z}^\beta f_k| dv \right\|_{L_x^2}$, gives us the improvement.

2.6.3 Step 1: Decay estimates

Using the bootstrap assumption on $\mathcal{E}_N[F]$ and Proposition 2.4.16, one immediately obtains the following pointwise decay estimates on the electromagnetic field.

Proposition 2.6.3. *For all $t \in [0, T]$, $|\beta| \leq N - \frac{n+2}{2}$, we have*

$$\begin{aligned} |\alpha(\mathcal{L}_{Z^\beta} F)| &\lesssim \sqrt{\epsilon} \sqrt{\chi(t)} \tau_+^{-\frac{n+1}{2}} \tau_-^{-\frac{1}{2}}, & |\underline{\alpha}(\mathcal{L}_{Z^\alpha} F)| &\lesssim \sqrt{\epsilon} \sqrt{\chi(t)} \tau_+^{-\frac{n-1}{2}} \tau_-^{-\frac{3}{2}}, \\ |\rho(\mathcal{L}_{Z^\alpha} F)| &\lesssim \sqrt{\epsilon} \sqrt{\chi(t)} \tau_+^{-\frac{n+1}{2}} \tau_-^{-\frac{1}{2}}, & |\sigma(\mathcal{L}_{Z^\alpha} F)| &\lesssim \sqrt{\epsilon} \sqrt{\chi(t)} \tau_+^{-\frac{n+1}{2}} \tau_-^{-\frac{1}{2}}. \end{aligned}$$

Remark 2.6.4. *We will improve later the decay estimate on $\alpha(\mathcal{L}_{Z^\beta} F)$, for $|\beta| \leq N - n$, near the light cone (see Section 2.6.7).*

The pointwise decay estimates on the velocity averages of the Vlasov fields are given by Klainerman-Sobolev inequalities and the bootstrap assumptions on the f_k energy norms. Using Theorem 2.2.13, we have that $\forall |\beta| \leq N - n$, $(t, x) \in [0, T] \times \mathbb{R}^n$, $1 \leq k \leq K$

$$\int_{\mathbb{R}^n} |\widehat{Z}^\beta f_k|(v^0)^2 dv \lesssim \frac{\mathbb{E}_N^2[f_k](t)}{\tau_+^{n-1} \tau_-} \lesssim \frac{\epsilon}{\tau_+^{n-1} \tau_-}. \quad (2.41)$$

In the same spirit, using Corollary 2.2.14, we have that $\forall |\beta| \leq N - \frac{3n+2}{2}$, $z \in \mathbf{k}_1$, $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |z \widehat{Z}^\beta f_k|(v^0)^2 dv \lesssim \frac{\mathbb{E}_{N - \frac{n+2}{2}, 1}^2[f_k](t)}{\tau_+^{n-1} \tau_-} \lesssim \frac{\epsilon}{\tau_+^{n-1} \tau_-}, \quad (2.42)$$

2.6.4 Step 2: Improving the energy estimates for the transport equation

We fix, for this section, $1 \leq k \leq K$. According to Proposition 2.3.3, $\mathbb{E}_N^2[f_k] \leq 3\epsilon$ on $[0, T]$, for ϵ small enough, would follow if we prove

$$\int_0^t \int_{\Sigma_s} \int_{v \in \mathbb{R}^n} |\mathcal{L}_{Z^{\beta_1}}(F)(v, \nabla_v \widehat{Z}^{\beta_2} f_k)| v^0 dv dx ds \lesssim \epsilon^{\frac{3}{2}}, \quad (2.43)$$

for all $|\beta_1| + |\beta_2| \leq N$, with $|\beta_2| \leq N - 1$, and

$$\int_0^t \int_{\Sigma_s} \int_{v \in \mathbb{R}^n} |v^i F_{i0} \widehat{Z}^\beta f_k| dv dx ds \lesssim \epsilon^{\frac{3}{2}},$$

for all $|\beta| \leq N$. The second integral is easy to bound. Using Proposition 2.6.3 and the bootstrap assumption on $\mathbb{E}_N^2[f_k]$, we have

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \int_{\mathbb{R}^n} |v^i F_{i0} \widehat{Z}^\beta f_k| dv dx ds &\lesssim \int_0^t \|F\|_{L^\infty(\Sigma_s)} \mathbb{E}_N^2[f_k](s) ds \\ &\lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

Similarly, according to Proposition 2.3.5, $\mathbb{E}_{N,1}^2[f_k] \leq 3\epsilon \chi^{\frac{1}{6}}(t)$ on $[0, T]$, for ϵ small enough, would follow if we prove

$$\int_0^t \int_{\Sigma_s} \int_{v \in \mathbb{R}^n} |z v^0 \mathcal{L}_{Z^{\beta_1}}(F)(v, \nabla_v \widehat{Z}^{\beta_2} f_k)| dv dx ds \lesssim \epsilon^{\frac{3}{2}} \chi^{\frac{1}{6}}(t), \quad (2.44)$$

for all $z \in \mathbf{k}_1$ and $|\beta_1| + |\beta_2| \leq N$, with $|\beta_2| \leq N - 1$,

$$\int_0^t \int_{\Sigma_s} \int_{v \in \mathbb{R}^n} |v^0 F(v, \nabla_v z) \widehat{Z}^\beta f_k| dv dx ds \lesssim \epsilon^{\frac{3}{2}} \chi^{\frac{1}{6}}(t), \quad (2.45)$$

for all $z \in \mathbf{k}_1$, $|\beta| \leq N$ and

$$\int_0^t \int_{\Sigma_s} \int_{v \in \mathbb{R}^n} |z v^i F_{i0} \widehat{Z}^\beta f_k| dv dx ds \lesssim \epsilon^{\frac{3}{2}},$$

for all $|\beta| \leq N$. Again, the last integral is easy to bound.

We fix $|\beta_1| + |\beta_2| \leq N$ (with $|\beta_2| \leq N - 1$), $|\beta| \leq N$ and $z \in \mathbf{k}_1$. We denote respectively $\rho(\mathcal{L}_{Z^{\beta_1}}(F))$, $\sigma(\mathcal{L}_{Z^{\beta_1}}(F))$, $\alpha(\mathcal{L}_{Z^{\beta_1}}(F))$ and $\underline{\alpha}(\mathcal{L}_{Z^{\beta_1}}(F))$ by ρ , σ , α and $\underline{\alpha}$. We denote also $\widehat{Z}^{\beta_2} f_k$ by g and $\widehat{Z}^\beta f_k$ by h . To unify the study of the remaining integrals, we introduce b , which could be equal to 0 or 1, $z_0 = v^0$ and $z_b = v^0 z$. The null decomposition of $\mathcal{L}_{Z^{\beta_1}}(F)(v, \nabla_v g)$ (for (2.43) and (2.44)) or $F(v, \nabla_v z)$ (for (2.45)) brings us to control the integral of the following terms.

The good terms

$$\left| z_b v^{\underline{L}} \rho(\nabla_v g)^{\underline{L}} \right|, \quad \left| v^0 v^{\underline{L}} h \rho(F)(\nabla_v z)^{\underline{L}} \right|, \quad (2.46)$$

$$\left| z_b v^{\underline{L}} \rho(\nabla_v g)^{\underline{L}} \right|, \quad \left| v^0 v^{\underline{L}} h \rho(F)(\nabla_v z)^{\underline{L}} \right|, \quad (2.47)$$

$$\left| z_b v^B \sigma_{BD}(\nabla_v g)^D \right|, \quad \left| v^0 v^B h \sigma(F)_{BD}(\nabla_v z)^D \right|, \quad (2.48)$$

$$\left| z_b v^{\underline{L}} \alpha_B(\nabla_v g)^B \right|, \quad \left| v^0 v^{\underline{L}} h \alpha(F)_B(\nabla_v z)^B \right|, \quad (2.49)$$

$$\left| z_b v^B \alpha_B(\nabla_v g)^{\underline{L}} \right|, \quad \left| v^0 v^B h \alpha(F)_B(\nabla_v z)^{\underline{L}} \right|, \quad (2.50)$$

and the bad terms

$$\left| z_b v^{\underline{L}} \underline{\alpha}_B(\nabla_v g)^B \right|, \quad \left| v^0 v^{\underline{L}} h \underline{\alpha}(F)_B(\nabla_v z)^B \right|, \quad (2.51)$$

$$\left| z_b v^B \underline{\alpha}_B(\nabla_v g)^{\underline{L}} \right|, \quad \left| v^0 v^B h \underline{\alpha}(F)_B(\nabla_v z)^{\underline{L}} \right|. \quad (2.52)$$

The study of $\mathbb{E}_N^2[f_k]$ corresponds to $b = 0$ and, in this case, we only have to estimate the spacetime integral of each of the first terms of (2.46)-(2.52). The study of $\mathbb{E}_{N,1}^2[f_k]$ corresponds to $b = 1$. For both of them, when $|\beta_1| \leq N - \frac{n+2}{2}$ we can use the pointwise decay estimates on the electromagnetic field given by Proposition 2.6.3. When $|\beta_1| > N - \frac{n+2}{2}$, $|\beta_2| \leq N - \frac{3n+2}{2}$ (since $N \geq \frac{5}{2}n + 1$), and we can then use the pointwise estimates (2.41) and (2.42) on the velocity averages of the Vlasov field.

For the part where $|\beta_1| \leq N - \frac{n+2}{2}$, our proof leads also to $\mathbb{E}_{N - \frac{n+2}{2}, 1}^2[f_k] \leq 3\epsilon$, for ϵ small enough, on $[0, T]$.

Remark 2.6.5. *To simplify the argument we will sometimes denote $\mathbb{E}_N^2[f_k]$ by $\mathbb{E}_{N,0}^2[f_k]$.*

Estimating the v derivatives

To deal with the v derivatives of the Vlasov field, which do not commute with the relativistic transport operator, we recall (2.33)

$$\left|(\nabla_v \psi)^L\right|, \left|(\nabla_v \psi)^{\underline{L}}\right|, \left|(\nabla_v \psi)^B\right| \lesssim \frac{\tau_+}{v^0} \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |\widehat{Z}\psi|. \quad (2.53)$$

We will also use

$$\left|(\nabla_v \psi)^L\right|, \left|(\nabla_v \psi)^{\underline{L}}\right| \lesssim \frac{\tau_-}{v^0} \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |\widehat{Z}\psi| \quad (2.54)$$

and

$$\left|v^{\underline{L}}(\nabla_v \psi)^B\right| \lesssim \tau_- \sum_{\widehat{Z} \in \widehat{\mathbb{P}}} \sum_{z \in \mathbf{k}_1} |z\widehat{Z}\psi|, \quad (2.55)$$

which come from Lemma 2.4.2 and Proposition 2.4.7. In order to reutilize certain estimates of this section, we will not use inequalities (2.54) and (2.55) in the case where we have a pointwise estimate on the electromagnetic field. We make this choice because we do not identify such null structures in the equations studied in Section 2.6.6, where we will make similar computations as in Subsection 2.6.4.

If $|\beta_1| \leq N - \frac{n+2}{2}$

We start by treating the good terms. We use ζ to denote α , ρ or σ . Thus, according to Proposition 2.6.3,

$$|\zeta| \lesssim \frac{\sqrt{\epsilon} \sqrt{\chi(t)}}{\tau_+^{\frac{n+1}{2}} \tau_-^{\frac{1}{2}}}.$$

Using (2.53), we can bound by $\sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \tau_+ |\zeta| |z_b \widehat{Z}g|$ each first term of (2.46)-(2.50) so that their integrals on $[0, t] \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ are bounded by

$$\sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^t \int_{\Sigma_s} \tau_+ |\zeta| \int_v |z_b \widehat{Z}g| dv dx ds. \quad (2.56)$$

It remains to notice that

$$\int_0^t \int_{\Sigma_s} \tau_+ |\zeta| \int_v |z_b \widehat{Z}g| dv dx ds \lesssim \int_0^t \sqrt{\epsilon} \frac{\log^{\frac{3}{2}}(3+s)}{(1+s)^{\frac{3}{2}}} \mathbb{E}_{N,b}^1[f_k](s) ds \lesssim \epsilon^{\frac{3}{2}},$$

since $\mathbb{E}_{N,b}^1[f_k](s) \leq \mathbb{E}_{N,1}^2[f_k](s) \leq 4\epsilon \log^{\frac{1}{2}}(3+s)$ for all $s \in [0, T]$. Similarly, each second term of (2.46)-(2.50) is bounded by $\sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} v^0 \tau_+ |\zeta(F)| |h| |\widehat{Z}(z)|$ and, using Lemma 2.2.8, their integral on $[0, t] \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ are bounded by

$$\sum_{z' \in \mathbf{k}_1} \int_0^t \int_{\Sigma_s} \tau_+ |\zeta(F)| \int_v v^0 |z'h| dv dx ds.$$

Using the pointwise estimate on ζ and the bootstrap assumption (2.40), one has

$$\int_0^t \int_{\Sigma_s} \tau_+ |\zeta(F)| \int_v v^0 |z'h| dv dx ds \lesssim \int_0^t \sqrt{\epsilon} \frac{\log^{\frac{3}{2}}(3+s)}{(1+s)^{\frac{3}{2}}} \mathbb{E}_{N,1}^1[f_k](s) ds \lesssim \epsilon^{\frac{3}{2}}.$$

We now study the bad terms. Recall that, according to Proposition 2.6.3,

$$|\alpha| \lesssim \frac{\sqrt{\epsilon} \sqrt{\chi(t)}}{\tau_+^{\frac{n-1}{2}} \tau_-^{\frac{3}{2}}}.$$

Using (2.53), $\int_{C_u(t)} \int_v v^{\underline{L}} |z_b \widehat{Z} g| dv dC_u(t) \leq \mathbb{E}_{N,b}^2[f_k](t)$ and the bootstrap assumption (2.40), we have,

$$\begin{aligned}
\int_0^t \int_{\Sigma_s} \int_v |z_b v^{\underline{L}} \underline{\alpha}_B (\nabla_v g)^B | dv dx ds &\lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^t \int_{\Sigma_s} \tau_+ |\underline{\alpha}| \int_v \frac{v^{\underline{L}}}{v^0} |z_b \widehat{Z} g| dv dx ds \\
&\lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_{u=-\infty}^t \int_{C_u(t)} \tau_+ |\underline{\alpha}| \int_v v^{\underline{L}} |z_b \widehat{Z} g| dv dC_u(t) du \\
&\lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_{u=-\infty}^t \frac{1}{\tau_-^{\frac{3}{2}}} \int_{C_u(t)} \int_v v^{\underline{L}} |z_b \widehat{Z} g| dv dC_u(t) du \\
&\lesssim \epsilon^{\frac{1}{2}} \mathbb{E}_{N,b}^2[f_k](t) \int_{u=-\infty}^{+\infty} \frac{1}{\tau_-^{\frac{3}{2}}} du \\
&\lesssim \epsilon^{\frac{3}{2}} \text{ if } b=0, \quad \epsilon^{\frac{3}{2}} \chi^{\frac{1}{6}}(t) \text{ if } b=1.
\end{aligned}$$

Finally, for the first term of (2.52), we use successively (2.53), the inequality $|v^B| \lesssim v^0 v^{\underline{L}}$ (which ensues from Proposition 2.2.9) as well as the bootstrap assumptions (2.40) to get

$$\begin{aligned}
\int_0^t \int_{\Sigma_s} \int_v |z_b v^B \underline{\alpha}_B (\nabla_v g)^{\underline{L}} | dv dx ds &\lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^t \int_{\Sigma_s} \int_v |\underline{\alpha}| v^0 v^{\underline{L}} \frac{\tau_+}{v^0} |z_b \widehat{Z} g| dv dx ds \\
&\lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_{u=-\infty}^t \int_{C_u(t)} \tau_+ |\underline{\alpha}| \int_v v^{\underline{L}} |z_b \widehat{Z} g| dv dC_u(t) du \\
&\lesssim \epsilon^{\frac{1}{2}} \int_{u=-\infty}^t \tau_-^{-\frac{3}{2}} \mathbb{E}_{N,b}^2[f_k](t) du \\
&\lesssim \epsilon^{\frac{1}{2}} \mathbb{E}_{N,b}^2[f_k](t).
\end{aligned}$$

The integrals of the second terms of (2.51) and (2.52) are treated similarly. For instance, as $\sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |\widehat{Z}(z)| \lesssim \sum_{z' \in \mathbf{k}_1} |z'|$, we have

$$\begin{aligned}
\int_0^t \int_{\Sigma_s} \int_v |v^0 v^B h \underline{\alpha}_B (\nabla_v z)^{\underline{L}} | dv dx ds &\lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^t \int_{\Sigma_s} \tau_+ |\underline{\alpha}| \int_v \sqrt{v^{\underline{L}} v^{\underline{L}}} |h \widehat{Z}(z)| dv dx ds \\
&\lesssim \sum_{z' \in \mathbf{k}_1} \int_{-\infty}^t \int_{C_u(t)} \tau_+ |\underline{\alpha}| \int_v v^{\underline{L}} v^0 |z' h| dv dC_u(t) du \\
&\lesssim \epsilon^{\frac{1}{2}} \mathbb{E}_{N,1}^2[f_k](t).
\end{aligned}$$

If $|\beta_1| > N - \frac{n+2}{2}$

In this case we cannot use Proposition 2.6.3 anymore. As $|\beta_2| \leq \frac{n}{2}$, we can however use the pointwise estimates on the velocity averages of $v^0 z_b \widehat{Z}^\beta g$ given by (2.42). This time, we only have to bound the first terms of (2.46)-(2.52).

Again, we start by studying the good terms. Let us denote again α , ρ or σ by ζ . Then, according to Definition 2.3.20, for all $s \in [0, T]$,

$$\int_{\Sigma_s} \tau_+^2 |\zeta|^2 dx \leq \mathcal{E}_N[F](s) \lesssim \epsilon \chi(s).$$

Recall that the integral of each first term of (2.46)-(2.50) can be bounded by (2.56). Using the Cauchy-Schwarz inequality and

$$\left\| \int_v |z_b \widehat{Z} g| dv \right\|_{L^2(\Sigma_s)}^2 \lesssim \epsilon^2 \int_0^{+\infty} \frac{r^{n-1}}{\tau_+^{2n-2} \tau_-^2} dr \lesssim \epsilon^2 (1+s)^{-(n-1)},$$

which comes from (2.42) and Lemma 2.4.1, we have

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \tau_+ |\zeta| \int_v |z_b \widehat{Z}g| dv dx ds &\lesssim \int_0^t \|\tau_+ \zeta\|_{L^2(\Sigma_s)} \left\| \int_v |z_b \widehat{Z}g| dv \right\|_{L^2(\Sigma_s)} ds \\ &\lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

In order to close the estimates for the bad terms, we use (2.54) or (2.55). The integral of the first term of (2.52) is then bounded by

$$\sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^{+\infty} \|\tau_{-\underline{\alpha}}\|_{L^2(\Sigma_s)} \left\| \int_v \left| \frac{v^B}{v^0} z_b \widehat{Z}g \right| dv \right\|_{L^2(\Sigma_s)} ds.$$

Now, using (2.42) and Lemma 2.4.1, we have

$$\left\| \int_v \left| \frac{v^B}{v^0} z_b \widehat{Z}g \right| dv \right\|_{L^2(\Sigma_s)} \lesssim \left\| \int_v |z_b \widehat{Z}g| dv \right\|_{L^2(\Sigma_s)} \lesssim \epsilon (1+s)^{-\frac{3}{2}}.$$

Since $\|\tau_{-\underline{\alpha}}\|_{L^2(\Sigma_s)}^2 \leq \mathcal{E}_N[F](s) \lesssim \epsilon \chi(t)$,

$$\int_0^t \int_{\Sigma_s} \int_v |z_b v^B \underline{\alpha}_B (\nabla_v g)^{\underline{L}}| dv dx ds \lesssim \epsilon^{\frac{3}{2}}.$$

For the remaining term, $z_b v^{\underline{L}} \underline{\alpha}_B (\nabla_v g)^B$, we treat the two cases separately. First, if $b = 0$, then

$$\int_0^t \int_{\Sigma_s} \int_v v^0 v^{\underline{L}} |\underline{\alpha}_B (\nabla_v g)^B| dv dx ds \lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \sum_{z' \in \mathbf{k}_1} \int_0^{+\infty} \|\tau_{-\underline{\alpha}}\|_{L^2(\Sigma_s)} \left\| \int_v v^0 |z' \widehat{Z}g| dv \right\|_{L^2(\Sigma_s)} ds.$$

Now, using (2.42) and Lemma 2.4.1, we have

$$\left\| \int_v v^0 |z' \widehat{Z}g| dv \right\|_{L^2(\Sigma_s)}^2 \lesssim \epsilon^2 (1+s)^{-(n-1)}.$$

Hence, as $\|\tau_{-\underline{\alpha}}\|_{L^2(\Sigma_s)}^2 \leq \mathcal{E}_N[F](s) \lesssim \epsilon \chi(t)$, we obtain

$$\int_0^t \int_{\Sigma_s} \int_v v^0 v^{\underline{L}} |\underline{\alpha}_B (\nabla_v g)^B| dv dx ds \lesssim \epsilon^{\frac{3}{2}}.$$

Finally, if $b = 1$, we have, by (2.53),

$$\int_0^t \int_{\Sigma_s} \int_v v^0 v^{\underline{L}} |z \underline{\alpha}_B (\nabla_v g)^B| dv dx ds \lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^t \int_{\Sigma_s} \frac{\tau_-^{\frac{1}{2}}}{(1+s)^{\frac{n-3}{2}}} |\underline{\alpha}|^{\frac{\tau_+(1+s)^{\frac{n-3}{2}}}{\tau_-^{\frac{1}{2}}}} \int_v v^{\underline{L}} |z \widehat{Z}g| dv dx ds.$$

By the Cauchy-Schwarz inequality (in (s, x)), the right-hand side of the previous inequality is bounded by

$$\left(\int_0^t \frac{\|\sqrt{\tau_-} |\underline{\alpha}|\|_{L^2(\Sigma_s)}^2}{(1+s)^{n-3}} ds \right)^{\frac{1}{2}} \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \left(\int_0^t \int_{\Sigma_s} \frac{\tau_+^{n-1}}{\tau_-} \left(\int_v v^{\underline{L}} |z \widehat{Z}g| dv \right)^2 dx ds \right)^{\frac{1}{2}}. \quad (2.57)$$

By the bootstrap assumption²² (2.39), $\|\sqrt{\tau_-} |\underline{\alpha}|\|_{L^2(\Sigma_s)}^2 \lesssim \epsilon$, so

$$\int_0^t \frac{\|\sqrt{\tau_-} |\underline{\alpha}|\|_{L^2(\Sigma_s)}^2}{(1+s)^{n-3}} ds \lesssim \epsilon \chi^{\frac{1}{3}}(t).$$

The second factor of (2.57) is bounded by ϵ^2 . Indeed, as, by (2.42),

$$\frac{\tau_+^{n-1}}{\tau_-} \left(\int_v v^{\underline{L}} |z \widehat{Z}g| dv \right)^2 \lesssim \frac{\epsilon}{\tau_-^2} \int_v v^{\underline{L}} |z \widehat{Z}g| dv,$$

²²Note that if we used the bound on $\|\tau_{-\underline{\alpha}}\|_{L^2(\Sigma_s)}$ we would have in $4d$ an extra loss on $\mathbb{E}_{N,1}^2[f_k]$ which would lead to a $(1+t)^\eta$ -loss for the electromagnetic energy.

we have

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \frac{\tau_+^{n-1}}{\tau_-} \left(\int_v v^L |z \widehat{Z} g| dv \right)^2 dx ds &\lesssim \epsilon \int_{-\infty}^t \tau_-^{-2} \int_{C_u(t)} \int_v v^L |z \widehat{Z} g| dv dC_u(t) du \\ &\lesssim \epsilon \int_{u=-\infty}^t \tau_-^{-2} \mathbb{E}_{N-\frac{n+2}{2},1}^2 [f_k](t) du \\ &\lesssim \epsilon^2, \end{aligned}$$

since $\mathbb{E}_{N-\frac{n+2}{2},1}^2 [f_k](t) \leq 4\epsilon$ by the bootstrap assumption (2.40). Thus

$$\int_0^t \int_{\Sigma_s} \int_v v^0 v^L |z \underline{\alpha}_B (\nabla_v g)^B| dv dx ds \lesssim \epsilon^2 \chi^{\frac{1}{6}}(t).$$

This concludes the improvement of the bootstrap assumption (2.40).

2.6.5 Step 3: Improved decay estimates for velocity averages

In this section, we improve the pointwise decay estimate on $\int_v |\widehat{Z}^\beta f_k| dv$ near the lightcone.

Proposition 2.6.6. *We have, for all $1 \leq k \leq K$,*

$$\forall (t, x) \in [0, T] \times \mathbb{R}^n, |\beta| \leq N - \frac{3n+2}{2}, \quad \int_{v \in \mathbb{R}^n} |\widehat{Z}^\beta f_k| dv \lesssim \frac{\epsilon}{\tau_+^n}$$

and

$$\forall (t, x) \in [0, T] \times \mathbb{R}^n, |\beta| \leq N - n, \quad \int_{v \in \mathbb{R}^n} |\widehat{Z}^\beta f_k| dv \lesssim \epsilon \frac{\chi^{\frac{1}{6}}(t)}{\tau_+^n}.$$

Proof. This ensues from Theorem 2.5.1, Remark 2.5.4 and the estimations made in Section 2.6.4. The loss for the derivatives of higher order is linked to the loss on $\mathbb{E}_{N,1}^2 [f_k]$. \square

2.6.6 Step 4: L^2 estimates for the velocity averages

In view of commutation formula of Propositions 2.2.19 and the energy estimates of Propositions 2.3.12, 2.3.21, we need to prove enough decay on

$\|\tau_+ \int_{\mathbb{R}^n} |\widehat{Z}^\beta f_k| dv\|_{L_x^2}$ for all $|\beta| \leq N$. The goal of this section is to prove the following proposition.

Proposition 2.6.7. *We have, for all $1 \leq k \leq K$, $|\beta| \leq N$ and for all $t \in [0, T]$,*

$$\left\| \tau_+ \int_{\mathbb{R}^n} |\widehat{Z}^\beta f_k| dv \right\|_{L^2(\Sigma_t)} \lesssim \epsilon \frac{\chi^{\frac{1}{6}}(t)}{(1+t)^{\frac{n}{2}-1}}.$$

The $\log^{\frac{1}{2}}(3+t)$ -loss (specific to the dimension 4) can be removed for $|\beta| \leq N - \frac{3n+2}{2}$ or improved in a $\log^{\frac{1}{4}}(3+t)$ -loss for $|\beta| \geq N - n + 1$.

Note that if $|\beta| \leq N - n$, that ensues from Proposition 2.6.6 and Lemma 2.4.1. For the higher order derivatives, we follow the strategy used in [18], in Section 4.5.7, to prove similar L^2 estimates. Let²³ $1 \leq k \leq K$ and $M \in \mathbb{N}$ such that $\frac{3n+4}{2} \leq M \leq N - n + 1$. Let I_1 and I_2 be two sets defined as

$$I_1 = \{\beta \text{ multi-index} / M \leq |\beta| \leq N\} \text{ and } I_2 = \{\beta \text{ multi-index} / |\beta| \leq M - 1\}.$$

We consider an ordering on I_i , for $1 \leq i \leq 2$, so that $I_i = \{\beta_{i,1}, \dots, \beta_{i,|I_i|}\}$ and two vector valued fields X and Y , of respective length $|I_1|$ and $|I_2|$, such that

$$X^j = \widehat{Z}^{\beta_{1,j}} f_k \quad \text{and} \quad Y^j = \widehat{Z}^{\beta_{2,j}} f_k.$$

²³If $n = 4$ and $N \geq 14$, we can take $8 \leq M \leq N - 6$ and avoid the $\log^{\frac{1}{2}}(3+t)$ -loss for all derivatives.

Lemma 2.6.8. *There exists three matrices valued functions $A_1 : [0, T] \times \mathbb{R}^n \rightarrow \mathfrak{M}_{|I_1|}(\mathbb{R})$, $A_2 : [0, T] \times \mathbb{R}^n \rightarrow \mathfrak{M}_{|I_2|}(\mathbb{R})$ and $B : [0, T] \times \mathbb{R}^n \rightarrow \mathfrak{M}_{|I_1|, |I_2|}(\mathbb{R})$ such that*

$$T_F(X) + A_1 X = BY, \quad \text{and} \quad T_F(Y) = A_2 Y.$$

If $1 \leq j \leq I_1$, A_1 and B are such that $T_F(X^j)$ is a linear combination of

$$\begin{aligned} & \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_1}}(F)_{\mu m} X^{\beta_{1,q}}, \quad t \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_1}}(F)_{\mu m} X^{\beta_{1,q}}, \quad \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_1}}(F)_{\mu i} x^i X^{\beta_{1,q}}, \\ & \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_2}}(F)_{\mu m} Y^{\beta_{2,l}}, \quad t \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_2}}(F)_{\mu m} Y^{\beta_{2,l}} \quad \text{and} \quad \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_2}}(F)_{\mu i} x^i Y^{\beta_{2,l}}, \end{aligned}$$

with $|\gamma_1| \leq N - \frac{3n+2}{2}$, $|\gamma_2| \leq N$, $1 \leq m \leq n$, $1 \leq q \leq |I_1|$ and $1 \leq l \leq |I_2|$. Similarly, if $1 \leq j \leq I_2$, A_2 is such that $T_F(Y^j)$ is a linear combination of

$$\frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu m} Y^{\beta_{2,l}}, \quad t \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu m} Y^{\beta_{2,l}} \quad \text{and} \quad \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu i} x^i Y^{\beta_{2,l}},$$

with $|\gamma| \leq N - n$, $1 \leq m \leq n$ and $1 \leq l \leq |I_2|$. Note also, using Proposition 2.6.6, that

$$\int_v |Y|_\infty dv \lesssim \epsilon \frac{\chi^{\frac{1}{6}}(t)}{\tau_+^n}.$$

Proof.

Let $|\beta| \leq N$. According to commutation formula of Lemma 2.2.26, $T_F(\widehat{Z}^\beta f_k)$ is a linear combination of terms such as $\mathcal{L}_{Z^\gamma}(F)(v, \nabla_v \widehat{Z}^\delta(f_k))$, with $|\gamma| + |\delta| \leq |\beta|$ and $|\delta| \leq |\beta| - 1$. Replacing each $\partial_{v^i} \widehat{Z}^\delta f_k$ by $\frac{1}{v^0}(\widehat{\Omega}_{0i} \widehat{Z}^\beta f_k - t \partial_i \widehat{Z}^\beta f_k - x^i \partial_t \widehat{Z}^\beta f_k)$, the matrices naturally appear. \square

Now, we split X in $G + H$ where G is the solution of the homogeneous system and H is the solution to the inhomogeneous system,

$$\begin{cases} T_F(H) + AH = 0 & , \quad H(0, \dots) = X(0, \dots), \\ T_F(G) + AG = BY & , \quad G(0, \dots) = 0. \end{cases}$$

The goal now is to prove L^2 estimates on the velocity averages of H and G .

The homogeneous part

We start by the following commutation formula.

Lemma 2.6.9. *Let $1 \leq i \leq |I_1|$ and consider $\widehat{Z}^\delta \in \widehat{\mathbb{P}}_0^{|\delta|}$, with $|\delta| \leq n$. Then, $T_F(\widehat{Z}^\delta H^i)$ can be written as a linear combination of terms of the form*

$$\mathcal{L}_{Z^\gamma}(F)(v, W),$$

where W is such that

$$\forall 0 \leq \mu \leq n, \quad |W^\mu| \lesssim \frac{\tau_+}{v^0} \sum_{|\theta| \leq n} \sum_{q=1}^{|I_1|} |\widehat{Z}^\theta H^q|,$$

and where $|\gamma| \leq N - \frac{n+2}{2}$, so that the electromagnetic field can be estimated pointwise.

Proof. The proof is similar to the ones of Lemma 2.2.25 and Corollary 2.2.26. \square

We introduce the energy $\widetilde{\mathbb{E}}[H]$ of H .

$$\widetilde{\mathbb{E}}[H] = \sum_{i=1}^{|I_1|} \mathbb{E}_n^2[H^i] + \mathbb{E}_{n,1}^2[H^i]$$

and we have the following lemma.

Lemma 2.6.10. *If ϵ is small enough, we have, for all $t \in [0, T]$,*

$$\tilde{\mathbb{E}}[H](t) \leq 8\epsilon \quad \text{and} \quad \forall 1 \leq i \leq |I_1|, \quad \left\| \int_v \tau_+ |H^i| dv \right\|_{L^2(\Sigma_t)} \lesssim \frac{\epsilon}{(1+t)^{\frac{n-2}{2}}}.$$

Proof. We follow here what we have done in Section 2.6.4. Since $\tilde{\mathbb{E}}[H](0) \leq \frac{3}{2}\mathbb{E}_{N+n}^2[f_k](0) + \frac{3}{2}\mathbb{E}_{N+n,1}^2[f_k](0) \leq 3\epsilon$ for ϵ small enough, there exists $0 < \tilde{T} \leq T$ such that

$$\forall t \in [0, \tilde{T}], \quad \tilde{\mathbb{E}}[H](t) \leq 8\epsilon.$$

To improve this bootstrap assumption, for ϵ small enough, we only have to use the previous lemma and to follow Section 2.6.4 (as we always estimated $|(\nabla_v w)^L|$, $|(\nabla_v w)^{\underline{L}}|$ and $|(\nabla_v w)^B|$ by $\frac{\tau_+}{v^0} \sum_{\hat{Z} \in \hat{\mathbb{P}}_0} |\hat{Z}w|$). We can then take $\tilde{T} = T$ and obtain, as in Section 2.6.5, that

$$\forall 1 \leq i \leq |I_1|, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \quad \int_{v \in \mathbb{R}^n} |H^i(t, x, v)| dv \lesssim \frac{\epsilon}{\tau_+^n}.$$

The L^2 estimate then ensues from Lemma 2.4.1. □

The inhomogeneous part

Let us introduce K , the solution of $T_F(K) + A_1 K + K A_2 = B$ which verifies $K(0, \dots) = 0$, and the function

$$|KKY|_\infty = \sum_{\substack{1 \leq i \leq |I_1| \\ 1 \leq j, p \leq |I_2|}} |K_i^j|^2 |Y_p|.$$

KY and G are solutions of the same system,

$$\begin{aligned} T_F(KY) = T_F(K)Y + K T_F(Y) &= BY - A_1 KY - K A_2 Y + \bar{K} A_2 Y \\ &= BY - A_1 KY. \end{aligned}$$

As $KY(0, \dots) = 0$ and $G(0, \dots) = 0$, it comes that $KY = G$. For $1 \leq i \leq |I_1|$ and $1 \leq j, p \leq |I_2|$, $|K_i^j|^2 Y_q$ satisfies the equation

$$T_F(|K_i^j|^2 Y_p) = |K_i^j|^2 (A_2)_p^q Y_q - 2((A_1)_i^q K_q^j + K_i^q (A_2)_q^j) K_i^j Y_p + 2B_i^j K_i^j Y_p,$$

which will allow us to estimate

$$\mathbb{E}[|KKY|_\infty] := \mathbb{E}_0^0[|KKY|_\infty].$$

We will then be able to bound $\|\tau_+ \int_{v \in \mathbb{R}^n} |G| dv\|_{L^2(\Sigma_t)}$ thanks to the estimates on $\int_{v \in \mathbb{R}^n} |Y| dv$ and $\mathbb{E}[|KKY|_\infty]$.

Lemma 2.6.11. *We have,*

$$\forall t \in [0, T], \quad \mathbb{E}[|KKY|_\infty] \leq \epsilon.$$

Proof. We use again the continuity method. Let T_0 be the largest time such that $\mathbb{E}[|KKY|_\infty] \leq 2\epsilon$ for all $t \in [0, T_0]$ and let us prove, with the energy estimate of Proposition 2.3.1, that for ϵ small enough, $\mathbb{E}[|KKY|_\infty] \leq \epsilon$ on $[0, T_0]$. Let $t \in [0, T_0]$.

As for the estimate of $\tilde{\mathbb{E}}[H]$ in the proof of Lemma 2.6.10, we have

$$\int_0^t \int_{\Sigma_s} \int_v \frac{1}{v^0} \left| |K_i^j|^2 (A_2)_p^q Y_q - 2((A_1)_i^q K_q^j + K_i^q (A_2)_q^j) K_i^j Y_p \right| dv dx ds \lesssim \epsilon^{\frac{3}{2}}.$$

Next, we need to estimate the following integral,

$$\int_0^t \int_{\Sigma_s} \int_v \frac{1}{v^0} |B_i^j K_i^j Y_p| dv dx. \tag{2.58}$$

The components of the matrix B involve terms in which the electromagnetic field has too many derivatives to be estimated pointwise. Indeed, recall from Lemma 2.6.8 that

$$|B_i^j K_i^j Y_p| \lesssim \sum_{m=1}^n \sum_{|\gamma| \leq N} \tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu m} K_i^j Y_p \right|.$$

We fix $|\gamma|$ and we denote the null decomposition of $\mathcal{L}_{Z^\gamma}(F)$ by $(\alpha, \underline{\alpha}, \rho, \sigma)$. In order to bound (2.58), we bound the integral of the five following terms, given by the null decomposition of the velocity vector v and $\mathcal{L}_{Z^\gamma}(F)$.

- The good terms

$$\tau_+ |\alpha| \frac{|KY|}{v^0}, \quad \tau_+ |\rho| \frac{|KY|}{v^0} \quad \text{and} \quad \tau_+ |\sigma| \frac{|KY|}{v^0}.$$

- The bad terms

$$\tau_+ \frac{v^{\underline{L}}}{(v^0)^2} |\underline{\alpha}| |KY| \quad \text{and} \quad \tau_+ \frac{|v^B|}{(v^0)^2} |\underline{\alpha}| |KY|.$$

We start by bounding the integral on $\Sigma_s \times \mathbb{R}_v^n$ of the good terms. We use ζ to denote either α , ρ or σ . Using twice the Cauchy-Schwarz inequality (in x and then in v), we have

$$\begin{aligned} \int_{\Sigma_s} \int_v \tau_+ |\zeta| \frac{|KY|}{v^0} dv dx &\lesssim \|\tau_+ |\zeta|\|_{L^2(\Sigma_s)} \left(\int_{\Sigma_s} \left(\int_v |KY| dv \right)^2 dx \right)^{\frac{1}{2}} \\ &\lesssim \sqrt{\mathcal{E}_N[F](s)} \left(\int_{\Sigma_s} \int_v |Y| dv \int_v |KKY|_\infty dv dx \right)^{\frac{1}{2}} \\ &\lesssim \sqrt{\mathcal{E}_N[F](s)} \left\| \int_v |Y| dv \right\|_{L^\infty(\Sigma_s)}^{\frac{1}{2}} \mathbb{E}[|KKY|_\infty]^{\frac{1}{2}}. \end{aligned}$$

Using the bootstrap assumptions, on $\mathcal{E}_N[F]$ and $\mathbb{E}[|KKY|_\infty]$, and the pointwise estimate $\int_v |Y| dv \lesssim \epsilon \log^{\frac{1}{2}}(3+t) \tau_+^{-n}$ given in Lemma 2.6.8, we obtain

$$\int_0^t \int_{\Sigma_s} \int_v \tau_+ |\zeta| \frac{|KY|}{v^0} dv dx ds \lesssim \int_0^t \frac{\epsilon^{\frac{3}{2}} \log^2(3+t)}{(1+s)^2} ds \lesssim \epsilon^{\frac{3}{2}}.$$

To unify the study of the bad terms, we use \tilde{v} to denote $v^{\underline{L}}$ or v^B . Using the Cauchy-Schwarz inequality (in (s, x)), the integral on $[0, t] \times \Sigma_s \times \mathbb{R}_v^n$ of a bad term is bounded by

$$\left(\int_0^t \int_{\Sigma_s} \frac{\tau_-^2 |\underline{\alpha}|^2}{(1+s)^{\frac{3}{2}}} dx ds \int_0^t \int_{\Sigma_s} \frac{\tau_+^2 (1+s)^{\frac{3}{2}}}{\tau_-^2} \left(\int_v \left| \frac{\tilde{v}}{v^0} \right| |KY| dv \right)^2 dx ds \right)^{\frac{1}{2}}. \quad (2.59)$$

As $\|\tau_- |\underline{\alpha}|\|_{L^2(\Sigma_s)}^2 \lesssim \epsilon \log^3(3+t)$, we have

$$\int_0^t \int_{\Sigma_s} \frac{\tau_-^2 |\underline{\alpha}|^2}{(1+s)^{\frac{3}{2}}} dx ds \lesssim \epsilon.$$

For the second factor of the product in (2.59), we first note that, by the Cauchy-Schwarz inequality,

$$\left(\int_v \left| \frac{\tilde{v}}{v^0} \right| |KY| dv \right)^2 \leq \int_v |Y| dv \int_v \left| \frac{\tilde{v}}{v^0} \right|^2 |KKY|_\infty dv.$$

Now, recall from Proposition 2.2.9 that $|v^B| \lesssim \sqrt{v^L v^{\underline{L}}}$ so that $\left| \frac{\tilde{v}}{v^0} \right|^2 \lesssim \frac{v^{\underline{L}}}{v^0}$. Using the pointwise decay estimate $\int_v |Y| dv \lesssim \epsilon \log^{\frac{1}{2}}(3+t) \tau_+^{-n}$, it comes

$$\left(\int_v \left| \frac{\tilde{v}}{v^0} \right| |KY| dv \right)^2 \leq \epsilon \frac{\log^{\frac{1}{2}}(3+t)}{\tau_+^n} \int_v \frac{v^{\underline{L}}}{v^0} |KKY|_\infty dv.$$

As $\int_{C_u(t)} \int_v \frac{v^{\underline{L}}}{v^0} |KKY|_\infty dC_u(t) dv \leq \mathbb{E}[|KKY|_\infty](t) \leq 2\epsilon$, we obtain

$$\int_0^t \int_{\Sigma_s} \frac{\tau_+^2 (1+s)^{\frac{3}{2}}}{\tau_-^2} \left(\int_v \left| \frac{\tilde{v}}{v^0} \right| |KY| dv \right)^2 dx ds \lesssim \epsilon^2 \int_{u=-\infty}^{t+\infty} \tau_-^{-2} du \lesssim \epsilon^2.$$

Hence,

$$\int_0^t \int_{\Sigma_s} \int_v \tau_+ \frac{|\tilde{v}|}{(v^0)^2} |\underline{\alpha}| |KY| dv dx ds \lesssim \epsilon^{\frac{3}{2}}$$

and the energy estimate of Proposition 2.3.1 gives that, for ϵ small enough, $\mathbb{E}[|KKY|_\infty] \leq \epsilon$ on $[0, T_0]$. \square

Remark 2.6.12. A naive estimation of the bad terms in the previous lemma would lead to a $(1+t)^n$ -loss which would affect the electromagnetic energy.

We are now able to prove the expected L^2 estimate on $\int_v |G| dv$.

Lemma 2.6.13. *If ϵ is small enough, we have,*

$$\forall t \in [0, T], 1 \leq i \leq |I_1|, \left\| \int_v \tau_+ |G^i| dv \right\|_{L^2(\Sigma_t)} \lesssim \frac{\epsilon \chi^{\frac{1}{12}}(t)}{(1+t)^{\frac{n-2}{2}}}.$$

Proof. Let $1 \leq i \leq |I_1|$. The Cauchy-Schwarz inequality (in v) gives us

$$\left\| \tau_+ \int_v |G^i| dv \right\|_{L^2(\Sigma_t)} \lesssim \sum_{j=1}^{|I_2|} \left\| \tau_+^2 \int_v |Y_j| dv \int_v |(K_i^j)^2 Y_j| dv \right\|_{L^1(\Sigma_t)}^{\frac{1}{2}}.$$

Thus, using once again that $\int_v |Y_j| dv \lesssim \epsilon \chi^{\frac{1}{6}}(t) \tau_+^{-n}$, we obtain

$$\left\| \tau_+ \int_v |G^i| dv \right\|_{L^2(\Sigma_t)} \lesssim \frac{\epsilon \chi^{\frac{1}{12}}(t)}{(1+t)^{\frac{n}{2}-1}}.$$

□

We can now conclude this section. *Proof.* [Proof of Proposition 2.6.7]

As mentioned earlier, for $|\beta| \leq M-1$, the estimate ensues from Proposition 2.6.6 and Lemma 2.4.1. If $M \leq |\beta| \leq N$, as there exists $1 \leq i \leq |I_1|$ such that $\widehat{Z}^\beta f_k = H^i + G^i$, we have

$$\left\| \tau_+ \int_v |\widehat{Z}^\beta f_k| dv \right\|_{L^2(\Sigma_t)} \leq \left\| \tau_+ \int_v |H^i| dv \right\|_{L^2(\Sigma_t)} + \left\| \tau_+ \int_v |G^i| dv \right\|_{L^2(\Sigma_t)}.$$

It then remains to use Lemmas 2.6.10 and 2.6.13. □

2.6.7 Step 5: Improvement of the electromagnetic field energy estimates

The bound on the potential energy

According to the energy estimate given by Proposition 2.3.12 and the commutation formula of Proposition 2.2.19, we have, for all $t \in [0, T]$,

$$\sqrt{\widetilde{\mathcal{E}}_N[A]}(t) \lesssim \sqrt{\widetilde{\mathcal{E}}_N[A]}(0) + \sum_{|\gamma| \leq N} \int_0^t |e^k| \left\| \tau_+ \int_{\mathbb{R}^n} |\widehat{Z}^\gamma f_k| dv \right\|_{L^2(\Sigma_s)} ds.$$

Using the L^2 decay estimate of Proposition 2.6.7 and $\widetilde{\mathcal{E}}_N[A](0) \leq \epsilon$, we obtain, for ϵ small enough and if the constant C is large enough, that

$$\forall t \in [0, T], \quad \widetilde{\mathcal{E}}_N[A](t) \leq \frac{C}{2(n-3)} \epsilon \log^3(3+t) \quad \text{if } n = 4$$

and

$$\forall t \in [0, T], \quad \widetilde{\mathcal{E}}_N[A](t) \leq \frac{C}{2(n-3)} \epsilon \quad \text{if } n \geq 5.$$

We are now able, using Proposition 2.4.19, to improve the pointwise decay estimate on α .

$$\forall |\beta| \leq N-n, (t, x) \in [0, T] \times \mathbb{R}^n, \quad |\alpha(\mathcal{L}_{Z^\beta}(F))|(t, x) \lesssim \sqrt{\epsilon} \frac{\sqrt{\chi(t)}}{\tau_+^{\frac{n+2}{2}}}.$$

Improvement of the electromagnetic field energy estimates

Recall from Proposition 2.3.21 that

$$\mathcal{E}_N[F](t) \leq \mathcal{E}_N[F](0) + (n-3)\tilde{\mathcal{E}}_N[A](t) + \varphi(t),$$

where $\varphi(t)$ is a linear combination of terms such that

$$\int_0^t \int_{\Sigma_s} |\overline{K}_0^\nu \mathcal{L}_{Z^\beta}(F)_{\mu\nu} J(\widehat{Z}^\gamma f_k)^\mu| dx ds \quad \text{and} \quad \int_0^t \int_{\Sigma_s} s |\mathcal{L}_{Z^\delta} A_\mu \square \mathcal{L}_{Z^\delta} A^\mu| dx ds, \quad (2.60)$$

with $|\beta|, |\gamma|, |\delta| \leq N$ and $1 \leq k \leq K$. Then, if we could prove that each integrals of (2.60) is bounded by $\epsilon^{\frac{3}{2}}\chi(t)$, we would have, for ϵ small enough and if the constant C is large enough, $\mathcal{E}_N[F] \leq C\epsilon\chi(t)$ on $[0, T]$ since $\mathbb{E}_N[F](0) \leq \epsilon$ and $(n-3)\tilde{\mathcal{E}}_N[A](t) \leq \frac{C}{2}\epsilon\chi(t)$.

We start by bounding the integrals involving the potential. Using Proposition 2.2.19 and the Cauchy-Schwarz inequality, we have, for $|\delta| \leq N$,

$$\int_0^t \int_{\Sigma_s} s |\mathcal{L}_{Z^\delta} A_\mu \square \mathcal{L}_{Z^\delta} A^\mu| dx ds \lesssim \sum_{k=1}^K \sum_{|\gamma| \leq |\delta|} \int_0^t \sqrt{\tilde{\mathcal{E}}_N[A](s)} \left\| \tau_+ \int_v |\widehat{Z}^\gamma f_k| dv \right\|_{L^2(\Sigma_s)} ds.$$

Using the L^2 estimate of Proposition 2.6.7 and that $\tilde{\mathcal{E}}_N[A](s) \lesssim \epsilon\chi(s)$, it comes

$$\begin{aligned} \sum_{|\delta| \leq N} \int_0^t \int_{\Sigma_s} s |\mathcal{L}_{Z^\delta} A_\mu \square \mathcal{L}_{Z^\delta} A^\mu| dx ds &\lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{\log^2(3+s)}{(1+s)^{\frac{n-2}{2}}} ds \\ &\lesssim \epsilon^{\frac{3}{2}} \chi(t). \end{aligned}$$

In order to estimate the remaining integrals of (2.60), we express $\overline{K}_0^\nu \mathcal{L}_{Z^\beta}(F)_{\mu\nu} J(\widehat{Z}^\gamma f_k)^\mu$ in null coordinates. Dropping the dependance in $\mathcal{L}_{Z^\beta}(F)$ or $\widehat{Z}^\gamma f_k$, this gives us the four following terms :

$$\tau_+^2 \rho J^{\underline{L}}, \quad \tau_-^2 \rho J^L, \quad \tau_+^2 \alpha_B J^B, \quad \text{and} \quad \tau_-^2 \alpha_B J^B. \quad (2.61)$$

As

$$J^{\underline{L}} = \int_v \frac{v^{\underline{L}}}{v^0} \widehat{Z}^\gamma f_k dv, \quad J^L = \int_v \frac{v^L}{v^0} \widehat{Z}^\gamma f_k dv \quad \text{and} \quad J^B = \int_v \frac{v^B}{v^0} \widehat{Z}^\gamma f_k dv,$$

we have,

$$|J^{\underline{L}}|, |J^L|, |J^B| \lesssim \int_{v \in \mathbb{R}^n} |\widehat{Z}^\gamma f_k| dv.$$

The integrals (on $[0, T] \times \mathbb{R}_x^n \times \mathbb{R}_v^n$) of each of the four terms of (2.61) are then bounded, using the Cauchy-Schwarz inequality, by

$$\int_0^t \sqrt{\mathcal{E}_N[F](s)} \left\| \tau_+ \int_v |\widehat{Z}^\gamma f_k| dv \right\|_{L^2(\Sigma_s)} ds.$$

By Proposition 2.6.7 and the bootstrap assumption (2.39),

$$\begin{aligned} \int_0^t \sqrt{\mathcal{E}_N[F](s)} \left\| \tau_+ \int_v |\widehat{Z}^\gamma f_k| dv \right\|_{L^2(\Sigma_s)} ds &\lesssim \int_0^t \sqrt{\epsilon\chi(s)} \frac{\epsilon \log^{\frac{1}{2}}(3+s)}{(1+s)^{\frac{n-2}{2}}} ds \\ &\lesssim \epsilon^{\frac{3}{2}} \chi(t). \end{aligned}$$

Hence, $\mathcal{E}_N[F](t) \leq C\epsilon\chi(t)$ for all $t \in [0, T]$ if ϵ is small enough.

We can prove in the same way, using in particular the energy estimate of Proposition 2.3.25 and

$$\left\| \int_v |\widehat{Z}^\beta f_k| dv \right\|_{L^2(\Sigma_t)} \leq \frac{1}{1+t} \left\| \tau_+ \int_v |\widehat{Z}^\beta f_k| dv \right\|_{L^2(\Sigma_t)},$$

that $\mathcal{E}_N^S[F] \leq \overline{C}\epsilon$ on $[0, T]$ if ϵ is small enough and the constant \overline{C} is large enough. We then improve the bootstrap assumption (2.39).

2.7 The massless Vlasov-Maxwell equations

2.7.1 Global existence for small data

The aim of this section is to prove Theorem 2.1.5. We then consider the massless Vlasov-Maxwell system (2.1)-(2.3), with at least two species²⁴, in dimension $n \geq 4$. This means that $K \geq 2$ and $m_k = 0$ for all $1 \leq k \leq K$.

To simplify the notation, we denote, during this chapter, $\mathbb{E}_M^0[f]$ by $\mathbb{E}_M[f]$ and $\mathbb{E}_{M,1,0}^0$ by $\mathbb{E}_{M,1}$. In view of Definition 2.3.2 and $1 \in \mathbf{k}_0$, we have

$$\mathbb{E}_M[f] \leq \mathbb{E}_{M,1}[f].$$

We introduce the functions χ , defined on \mathbb{R}_+ by

$$\chi(s) = 1 + s \text{ if } n = 4, \quad \chi(s) = \log^2(3 + s) \text{ if } n = 5 \text{ and } \chi(s) = 1 \text{ if } n \geq 6,$$

and \log^* , defined on \mathbb{R}_+ by

$$\log^* = \log \text{ if } n = 4 \text{ and } \log^* = 1 \text{ if } n \geq 5.$$

We give a more precise version of Theorem 2.1.5.

Theorem 2.7.1. *Let $n \geq 4$, $K \geq 2$, $N \geq 6n + 2$ if n is even and $N \geq 6n + 3$ if n is odd, $0 < \eta < \frac{1}{2}$ if $n = 4$ and $\eta = 0$ if $n \geq 5$ and $R > 0$. Let (f_0, F_0) be an initial data set for the massless Vlasov-Maxwell system. Let (f, F) be the unique classical solution to the system and let A be a potential in the Lorenz gauge. There exists $\epsilon > 0$ such that²⁵, if*

$$\tilde{\mathcal{E}}_N[A](0) \leq \epsilon, \quad \mathcal{E}_N[F](0) \leq \epsilon$$

and if, for all $1 \leq k \leq K$,

$$\text{supp}(f_{0k}) \subset \{(x, v) \in \mathbb{R}_x^n \times \mathbb{R}_v^n \setminus \{0\} / |v| \geq R\},$$

$$\mathbb{E}_{N+n,1}[f_k](0) \leq \epsilon,$$

then (f, F) exists globally in time and verifies the following estimates.

- *Vanishing property for small velocities : for all $1 \leq k \leq K$,*

$$\text{supp}(f_k) \subset \left\{ (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}_x^n \times \mathbb{R}_v^n \setminus \{0\} / |v| \geq \frac{R}{2} \right\}.$$

- *Energy bounds for F and f_k : $\forall 1 \leq k \leq K$ and $\forall t \in \mathbb{R}_+$,*

$$\begin{aligned} \mathcal{E}_N[F](t) &\lesssim \epsilon \chi(t) (1+t)^\eta, & \mathcal{E}_{N-2n}[F](t) &\lesssim \epsilon \chi(t), \\ \mathbb{E}_N[f_k](t) &\lesssim \epsilon \log^*(3+t), & \mathbb{E}_{N-n,1}[f_k](t) &\lesssim \epsilon. \end{aligned}$$

- *Pointwise decay for the null decomposition of $\mathcal{L}_{Z^\beta}(F)$: $\forall |\beta| \leq N - \frac{5n+4}{2}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$,*

$$\begin{aligned} |\alpha(\mathcal{L}_{Z^\beta} F)| &\lesssim \sqrt{\epsilon \chi(t)} \tau_+^{-\frac{n+2}{2}}, & |\underline{\alpha}(\mathcal{L}_{Z^\beta} F)| &\lesssim \sqrt{\epsilon \chi(t)} \tau_+^{-\frac{n-1}{2}} \tau_-^{-\frac{3}{2}}, \\ |\rho(\mathcal{L}_{Z^\beta} F)| &\lesssim \sqrt{\epsilon \chi(t)} \tau_+^{-\frac{n+1}{2}} \tau_-^{-\frac{1}{2}}, & |\sigma(\mathcal{L}_{Z^\beta} F)| &\lesssim \sqrt{\epsilon \chi(t)} \tau_+^{-\frac{n+1}{2}} \tau_-^{-\frac{1}{2}} \end{aligned}$$

and

$$|\underline{\alpha}(\mathcal{L}_{Z^\beta} F)| \lesssim \sqrt{\epsilon} \tau_+^{-\frac{n-1}{2}} \tau_-^{-1}.$$

²⁴We recall that we take $K \geq 2$ since we suppose that the initial energy $\mathcal{E}[F]$ is finite, which implies that the plasma is electrically neutral (see Remark 2.1.2 for more details).

²⁵We recall that a smallness condition on F , which implies $\tilde{\mathcal{E}}_N[A](0) \leq \epsilon$, is given in Proposition 2.2.20.

- Pointwise decay for $\int_{v \in \mathbb{R}^n \setminus \{0\}} |z \widehat{Z}^\beta f_k| dv$:

$$\forall |\beta| \leq N - 2n, z \in \mathbf{k}_0, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad \int_v |z \widehat{Z}^\beta f_k| dv \lesssim \frac{\epsilon}{\tau_+^{n-1} \tau_-}.$$

- L^2 estimates on $\int_{v \in \mathbb{R}^n \setminus \{0\}} |\widehat{Z}^\beta f_k| dv$:

$$\forall |\beta| \leq N, t \in \mathbb{R}_+, \quad \left\| \int_{v \in \mathbb{R}^n \setminus \{0\}} |\widehat{Z}^\beta f_k| dv \right\|_{L^2(\Sigma_t)} \lesssim \frac{\epsilon}{(1+t)^{\frac{n-1-n}{2}}}.$$

- Energy bound for a potential A satisfying the Lorenz gauge :

$$\forall t \in \mathbb{R}_+, \quad \widetilde{\mathcal{E}}_N[A](t) \lesssim \epsilon \chi(t) (1+t)^\eta \quad \text{and} \quad \widetilde{\mathcal{E}}_{N-2n}[A](t) \lesssim \epsilon \chi(t).$$

2.7.2 Structure and beginning of the proof

Let (f_0, F_0) be an initial data set satisfying the assumptions of Theorem 2.7.1. By a standard local well-posedness argument, there exists a unique maximal solution (f, F) of the massless Vlasov-Maxwell system defined on $[0, T^*[$, with $T^* \in \mathbb{R}_+^* \cup \{+\infty\}$.

We consider the following bootstrap assumptions. Let T be the largest time such that, $\forall 1 \leq k \leq K$ and $\forall t \in [0, T]$,

$$\mathcal{E}_N[F](t) \leq 2C\epsilon\chi(t)(1+t)^\eta, \quad \mathcal{E}_{N-2n}[F](t) \leq 2C\epsilon\chi(t), \quad (2.62)$$

$$\mathcal{E}_N^0[F](t) \leq 4\epsilon, \quad \mathcal{E}_N^S[F](t) \leq 2\overline{C}\epsilon(1+t)^\eta, \quad \mathcal{E}_{N-2n}^S[F](t) \leq 2\overline{C}, \quad (2.63)$$

$$\widetilde{\mathcal{E}}_N[A](t) \leq 2C\epsilon\chi(t)(1+t)^\eta, \quad \widetilde{\mathcal{E}}_{N-2n}[A](t) \leq 2C\epsilon\chi(t), \quad (2.64)$$

$$\mathbb{E}_N[f_k](t) \leq 4\log^*(3+t) \quad \text{and} \quad \mathbb{E}_{N-n,1}[f_k](t) \leq 4\epsilon, \quad (2.65)$$

where C and \overline{C} are positive constants which will be specified during the proof. Note that by continuity, $T > 0$. We now present our strategy to improve these bootstrap assumptions.

1. First, using the bootstrap assumptions, we obtain decay estimates for the null decomposition of F (and its Lie derivatives) and for velocity averages of derivatives of f_k .
2. Then, we prove that 0 is not in the closure of the Vlasov fields v -support. This follows from the study of the characteristics of the transport equation.
3. Next, we improve the bounds on the Vlasov fields energies by means of the energy estimates proved in Propositions 2.3.3 and 2.3.5. To bound the right hand side in these energy estimates, we make fundamental use of the null structure of the system and the pointwise decay estimates on ρ , σ , α , $\underline{\alpha}$ and $\int_{\mathbb{R}^n \setminus \{0\}} |z \widehat{Z}^\beta f_k| dv$.
4. In order to improve the estimates on the electromagnetic field energies, we establish an L_x^2 estimate for the velocity averages of the Vlasov fields (and its derivatives). For this purpose, we follow [18] and we rewrite all the transport equations as an inhomogeneous system of transport equations. The velocity averages of the homogeneous part of the solution verify strong pointwise decay (we use particularly the control that we have at our disposal on the initial data of f_k , for derivatives of order $N+n$ or less). The inhomogeneous part is decomposed into a product of an integrable function and a pointwise decaying function which gives us the expected estimate.
5. Finally, we improve the estimates on the energies of the electromagnetic potential and the electromagnetic field, with the energy estimate for the Maxwell equations (using in particular Propositions 2.3.21 and 2.3.25). We use the null decomposition of $J(\widehat{Z}^\gamma f_k)^\mu \mathcal{L}_{Z^\beta}(F)_{\mu\nu} \overline{K}_0^\nu$, which, combined with L_x^2 estimates on quantities such as $\int_{\mathbb{R}^n} |\widehat{Z}^\gamma f_k| dv$, gives us the improvement.

2.7.3 Step 1: Decay estimates

By the Klainerman-Sobolev inequality of Theorem 2.2.13 and the bootstrap assumption (2.65), we have $\forall |\beta| \leq N - n$, $(t, x) \in [0, T] \times \mathbb{R}^n$, $1 \leq k \leq K$,

$$\int_v |\widehat{Z}^\beta f_k| dv \lesssim \frac{\mathbb{E}_N[f_k](t)}{\tau_+^{n-1} \tau_-} \lesssim \frac{\epsilon \log^*(3+t)}{\tau_+^{n-1} \tau_-}. \quad (2.66)$$

In the same spirit²⁶, using Corollary 2.2.14, we have $\forall |\beta| \leq N - 2n$, $z \in \mathbf{k}_0$, $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$\int_v |z \widehat{Z}^\beta f_k| dv \lesssim \frac{\mathbb{E}_{N-n,1}[f_k](t)}{\tau_+^{n-1} \tau_-} \lesssim \frac{\epsilon}{\tau_+^{n-1} \tau_-}. \quad (2.67)$$

We have improved decay estimates for the null components of the current $M_\mu := \int_v \frac{v_\mu}{v^0} \widehat{Z}^\beta f_k dv$. For all $|\beta| \leq N - 2n$, we have

$$\int_v \frac{v^L}{v^0} |\widehat{Z}^\beta f_k| dv \lesssim \frac{\epsilon}{\tau_+ \tau_-}, \quad (2.68)$$

$$\int_v \frac{v^L}{v^0} |\widehat{Z}^\beta f_k| dv \lesssim \frac{\epsilon}{\tau_+^{n-1} \tau_-^2} \quad (2.69)$$

and

$$\int_v \frac{|v^B|}{v^0} |\widehat{Z}^\beta f_k| dv \lesssim \frac{\epsilon}{\tau_+ \tau_-}. \quad (2.70)$$

This results from (see Proposition 2.2.9)

$$\frac{v^L}{v^0} \lesssim \frac{1}{\tau_+} \sum_{z \in \mathbf{k}_0} |z|, \quad \frac{v^L}{v^0} \lesssim \frac{1}{\tau_-} \sum_{z \in \mathbf{k}_0} |z| \quad \text{and} \quad \left| \frac{v^B}{v^0} \right| \lesssim \frac{1}{\tau_+} \sum_{z \in \mathbf{k}_0} |z|.$$

Using the bootstrap assumptions (2.62), (2.63), (2.64), Propositions 2.4.16, 2.4.19 and the pointwise decay estimate (2.66), we obtain.

Proposition 2.7.2. *For all $t \in [0, T]$, $|\beta| \leq N - n$, we have*

$$\begin{aligned} |\alpha(\mathcal{L}_{Z^\beta} F)| &\lesssim \frac{\sqrt{\epsilon \chi(t)(1+t)^\eta}}{\tau_+^{\frac{n+2}{2}}}, & |\underline{\alpha}(\mathcal{L}_{Z^\beta} F)| &\lesssim \frac{\sqrt{\epsilon \chi(t)(1+t)^\eta}}{\tau_+^{\frac{n-1}{2}} \tau_-^{\frac{3}{2}}}, \\ |\rho(\mathcal{L}_{Z^\beta} F)| &\lesssim \frac{\sqrt{\epsilon \chi(t)(1+t)^\eta}}{\tau_+^{\frac{n+1}{2}} \tau_-^{\frac{1}{2}}}, & |\sigma(\mathcal{L}_{Z^\beta} F)| &\lesssim \frac{\sqrt{\epsilon \chi(t)(1+t)^\eta}}{\tau_+^{\frac{n+1}{2}} \tau_-^{\frac{1}{2}}} \end{aligned}$$

and

$$|\underline{\alpha}(\mathcal{L}_{Z^\gamma} F)| \lesssim \frac{\sqrt{\epsilon(1+t)^\eta}}{\tau_+^{\frac{n-1}{2}} \tau_-}.$$

For all $t \in [0, T]$, $|\beta| \leq N - \frac{5n+2}{2}$, we have

$$\begin{aligned} |\alpha(\mathcal{L}_{Z^\gamma} F)| &\lesssim \sqrt{\epsilon} \tau_+^{-\frac{n-1}{2}} \tau_-^{-1}, & |\underline{\alpha}(\mathcal{L}_{Z^\beta} F)| &\lesssim \sqrt{\epsilon \chi(t)} \tau_+^{-\frac{n-1}{2}} \tau_-^{-\frac{3}{2}}, \\ |\rho(\mathcal{L}_{Z^\beta} F)| &\lesssim \sqrt{\epsilon \chi(t)} \tau_+^{-\frac{n+1}{2}} \tau_-^{-\frac{1}{2}}, & |\sigma(\mathcal{L}_{Z^\beta} F)| &\lesssim \sqrt{\epsilon \chi(t)} \tau_+^{-\frac{n+1}{2}} \tau_-^{-\frac{1}{2}}. \end{aligned}$$

Finally, for all $t \in [0, T]$, $|\beta| \leq N - \frac{5n+4}{2}$,

$$|\alpha(\mathcal{L}_{Z^\gamma} F)| \lesssim \sqrt{\epsilon \chi(t)} \tau_+^{-\frac{n+2}{2}}.$$

Remark 2.7.3. *The alternative estimate on $\underline{\alpha}$ is useful to avoid a τ_+ -loss when $n \leq 5$ and is particularly used in Section 2.7.6.*

Remark 2.7.4. *We also have pointwise decay estimates if $|\beta| \leq N - \frac{n+2}{2}$ but the one on α is worse near the light cone (see Proposition 2.4.16).*

²⁶Note that the pointwise decay estimate (2.67) implies (2.66) for the lower order derivatives, taking $z = 1$.

2.7.4 Step 2: the Vlasov fields vanishes for small velocities

We recall that

$$\forall 1 \leq i \leq n, \quad E^i = F_{0i},$$

and that the transports equation of the Vlasov-Maxwell system can be rewritten

$$v^\mu \partial_\mu f_k + v^0 E^i \partial_{v^i} f_k + v^j F_j^i \partial_{v^i} f_k = 0.$$

We now fix $1 \leq k \leq K$ and we prove, under the bootstrap assumption, that if $f_k(t, x, v) \neq 0$, with $(t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$, then $|v| \geq \frac{R}{2}$. During the argument, we will use various constants and we will all call them C for simplicity. These constants will not depend on ϵ or on T .

Let $x \in \mathbb{R}^n$ and $|v| \geq R$. Let (X, V) be the characteristics of the transport equation such that $(X(0), V(0)) = (x, v)$. In particular

$$\forall 1 \leq i \leq n, \quad \frac{dV^i}{ds} = E^i(s, X) + \frac{V^j}{V^0} F_{ji}(s, X).$$

It follows that

$$\frac{d(|V|^2)}{ds} = 2 \langle E(s, X), V \rangle.$$

So,

$$|V(t)|^2 = |v|^2 + 2 \int_0^t \langle E(s, X(s)), V(s) \rangle ds. \quad (2.71)$$

We denote $|V(s)|^2$ by $g(s)$. By the Cauchy-Schwarz inequality, we have

$$g(t) \leq |v|^2 + 2 \int_0^t |E(s, X(s))| \sqrt{g(s)} ds.$$

We now use a Grönwall inequality (Lemma 2.4.22) and $|E(s, X(s))| \leq \frac{C\sqrt{\epsilon}}{(1+s)^{\frac{n-1}{2}}}$ (which come from Proposition 2.7.2) to obtain

$$g(s) \leq \left(|v| + \int_0^s \frac{C\sqrt{\epsilon} ds}{(1+s)^{\frac{3}{2}}} \right)^2.$$

Thus,

$$|V(s)| \leq |v| + C\sqrt{\epsilon}.$$

Returning to (2.71), we obtain

$$|V(s)|^2 \geq |v|^2 - 2 \int_0^s |E(s, X(s))| |V(s)| ds.$$

Therefore, using again the pointwise estimate on E ,

$$|V(s)|^2 \geq |v|^2 - 2C\sqrt{\epsilon}(|v| + C\sqrt{\epsilon}).$$

Finally,

$$|V(s)|^2 \geq |v|(|v| - C\sqrt{\epsilon}) - C\epsilon \geq \frac{1}{4}|v|^2,$$

if ϵ is sufficiently small so that $C\epsilon \leq \frac{R}{4}$ and $C\sqrt{\epsilon} \leq \frac{R}{2}$.

Then, if (x, v) is such that $|v| \geq R$, (X, V) is well defined on $[0, T]$ (X is also bounded since $|\frac{dX}{ds}| = 1$) and $|V| \geq \frac{R}{2}$. Consequently, we obtain.

Lemma 2.7.5.

$$\text{supp}(f_{k|_{[0, T]}}) \subset \{(t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} / |v| \geq \frac{R}{2}\}.$$

In the remainder, we will then be able to use inequalities like

$$\frac{1}{v^0} |f_k(t, x, v)| \lesssim |f_k(t, x, v)|.$$

Sometimes, we will abusively use inequalities such that

$$\frac{1}{v^0} \sum_{z \in \mathbf{k}_0} |z| \lesssim \sum_{z \in \mathbf{k}_0} |z|$$

because these quantities are always multiplied by $\widehat{Z}^\beta f_k$.

2.7.5 Step 3: Improving the Energy estimates for the transport equations

We fix for all this section $1 \leq k \leq K$. According to Proposition 2.3.3, $\mathbb{E}_N[f_k] \leq 3\epsilon \log^*(3+t)$ on $[0, T]$, for ϵ small enough, follows from

$$\int_0^t \int_{\Sigma_s} \int_v \left| \mathcal{L}_{Z^{\beta_1}}(F) \left(\frac{v}{v^0}, \nabla_v \widehat{Z}^{\beta_2} f_k \right) \right| dv dx ds \lesssim \epsilon^{\frac{3}{2}} \log^*(3+t),$$

for all $|\beta_1| + |\beta_2| \leq N$, with $|\beta_2| \leq N-1$.

Similarly, according to Proposition 2.3.5, $\mathbb{E}_{N-n,1}[f_k] \leq 3\epsilon$ on $[0, T]$, for ϵ small enough, follows from

$$\int_0^t \int_{\Sigma_s} \int_v |z| \left| \mathcal{L}_{Z^{\beta_1}}(F) \left(\frac{v}{v^0}, \nabla_v \widehat{Z}^{\beta_2} f_k \right) \right| dv dx ds \lesssim \epsilon^{\frac{3}{2}},$$

and

$$\int_0^t \int_{\Sigma_s} \int_v \left| F \left(\frac{v}{v^0}, \nabla_v z \right) \widehat{Z}^\beta f_k \right| dv dx ds \lesssim \epsilon^{\frac{3}{2}},$$

for all $z \in \mathbf{k}_0$, $|\beta_1| + |\beta_2| \leq N-n$ (with $|\beta_2| \leq N-n-1$) and $|\beta| \leq N-n$.

To unify the study of $\mathbb{E}_N[f_k]$ and $\mathbb{E}_{N-n,1}[f_k]$, we consider b , which could be equal to 0 or to 1, $N_0 = N$ and $N_1 = N-n$. Now, we fix $z \in \mathbf{k}_0$, $|\beta_1| + |\beta_2| \leq N_b$ (with $|\beta_2| \leq N_b-1$) and $|\beta| \leq N-n$. We denote $\rho(\mathcal{L}_{Z^{\beta_1}}(F))$, $\sigma(\mathcal{L}_{Z^{\beta_1}}(F))$, $\alpha(\mathcal{L}_{Z^{\beta_1}}(F))$ and $\underline{\alpha}(\mathcal{L}_{Z^{\beta_1}}(F))$ by ρ , σ , α and $\underline{\alpha}$ (respectively). We also denote $\widehat{Z}^{\beta_2} f_k$ by g and $\widehat{Z}^\beta f_k$ by h . The null decomposition of $\mathcal{L}_{Z^{\beta_1}}(F)(v, \nabla_v g)$ or $F(v, \nabla_v z)$ brings us to control the integral of the following terms, with $z_0 = 1$ and $z_1 = z$.

The terms involving L or \underline{L} components of $\nabla_v g$ or $\nabla_v z$

$$\left| z_b \frac{v^{\underline{L}}}{v^0} \rho(\nabla_v g)^L \right|, \quad \left| h \frac{v^{\underline{L}}}{v^0} \rho(F)(\nabla_v z)^L \right|, \quad (2.72)$$

$$\left| z_b \frac{v^L}{v^0} \rho(\nabla_v g)^{\underline{L}} \right|, \quad \left| h \frac{v^L}{v^0} \rho(F)(\nabla_v z)^{\underline{L}} \right|, \quad (2.73)$$

$$\left| z_b \frac{v^B}{v^0} \alpha_B(\nabla_v g)^L \right|, \quad \left| h \frac{v^B}{v^0} \alpha_B(F)(\nabla_v z)^L \right|, \quad (2.74)$$

$$\left| z_b \frac{v^B}{v^0} \underline{\alpha}_B(\nabla_v g)^{\underline{L}} \right|, \quad \left| h \frac{v^B}{v^0} \underline{\alpha}_B(F)(\nabla_v z)^{\underline{L}} \right|. \quad (2.75)$$

The terms involving angular components of $\nabla_v g$ or $\nabla_v z$

$$\left| z_b \frac{v^L}{v^0} \alpha_B(\nabla_v g)^B \right|, \quad \left| h \frac{v^L}{v^0} \alpha_B(F)(\nabla_v z)^B \right|, \quad (2.76)$$

$$\left| z_b \frac{v^B}{v^0} \sigma_{BD}(\nabla_v g)^D \right|, \quad \left| h \frac{v^B}{v^0} \sigma(F)_{BD}(\nabla_v z)^D \right|, \quad (2.77)$$

$$\left| z_b \frac{v^{\underline{L}}}{v^0} \underline{\alpha}_B(\nabla_v g)^B \right|, \quad \left| h \frac{v^{\underline{L}}}{v^0} \underline{\alpha}_B(F)(\nabla_v z)^B \right|. \quad (2.78)$$

The study of $\mathbb{E}_N[f_k]$ corresponds to $b = 0$. In this case, we only have to estimate the spacetime integral of each of the first terms of (2.72)-(2.78), but we need to consider two cases. When $|\beta_1| \leq N-n$ we can use the pointwise decay estimates on the electromagnetic field given by Proposition 2.7.2. When $|\beta_1| > N-n$, $|\beta_2| \leq N-2n$ (since $N \geq 6n+2$), and we can then use the pointwise decay estimates on the velocity averages of the Vlasov field given in Section 2.7.3.

In the study of $\mathbb{E}_{N-n,1}[f_k]$ (which corresponds to $b = 1$ and where z can be any weights of \mathbf{k}_0), we can always use a pointwise estimate on the electromagnetic field (as $|\beta_1| \leq N-n$), but we need to estimate the spacetime integral of all the terms of (2.72)-(2.78).

Remark 2.7.6. *To simplify the argument we will sometimes denote $\mathbb{E}_N[f_k]$ by $\mathbb{E}_{N_0,0}[f_k]$ and $\mathbb{E}_{N-n,1}[f_k]$ by $\mathbb{E}_{N_1,1}[f_k]$.*

Estimating the v derivatives

In order to eliminate the v derivatives, we use, as in Section 2.6.4,

$$|\nabla_v w| \lesssim \frac{\tau_+}{v^0} \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |\widehat{Z}w| \quad (2.79)$$

and

$$\left| (\nabla_v w)^L \right|, \left| (\nabla_v w)^{\underline{L}} \right| \lesssim \frac{\tau_-}{v^0} \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |\widehat{Z}w|. \quad (2.80)$$

If $|\beta_1| \leq N - n$

We start by the terms involving L or \underline{L} components of $\nabla_v g$ or $\nabla_v z$. We use ζ to denote α , $\underline{\alpha}$ or ρ . Thus, by Proposition 2.7.2,

$$|\zeta| \lesssim \frac{\sqrt{\epsilon}(1+t)^{\frac{\eta}{2}}}{\tau_+^{\frac{3}{2}} \tau_-}.$$

The integral on $[0, t] \times \mathbb{R}_x^n \times (\mathbb{R}_v^n \setminus \{0\})$ of each of the first terms of (2.72)-(2.75) are bounded by

$$\sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^t \int_{\Sigma_s} \tau_- |\zeta| \int_v |z_b \widehat{Z}g| dv dx ds,$$

where we use in particular (2.80) and the fact that $\frac{1}{v^0} \lesssim 1$ on the support of g . Using the bootstrap assumption (2.65), we obtain

$$\int_0^t \int_{\Sigma_s} \tau_- |\zeta| \int_v |z_b \widehat{Z}g| dv dx ds \lesssim \int_0^t \sqrt{\epsilon}(1+s)^{-\frac{3-\eta}{2}} \mathbb{E}_{N_b, b}[f_k](s) ds \lesssim \epsilon^{\frac{3}{2}}.$$

Similarly, the integrals of each of the second terms of (2.72)-(2.75) are bounded by

$$\sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^t \int_{\Sigma_s} \tau_- |\zeta(F)| \int_v |h \widehat{Z}(|z|)| \frac{dv}{v^0} dx ds.$$

Using again the bootstrap assumption (2.65) and $\frac{1}{v^0} \lesssim 1$ on the support of h , one has

$$\int_0^t \int_{\Sigma_s} \tau_- |\zeta(F)| \int_v |h \widehat{Z}(|z|)| \frac{dv}{v^0} dx ds \lesssim \int_0^t \sqrt{\epsilon}(1+s)^{-\frac{3-\eta}{2}} \mathbb{E}_{N_{1,1}}[f_k](s) ds \lesssim \epsilon^{\frac{3}{2}},$$

since $|\widehat{Z}(|z|)| \lesssim \sum_{w \in \mathbf{k}_0} |w|$ by Lemma 2.2.8.

We now study the remaining terms. Using (2.79), the pointwise decay estimates of Proposition 2.7.2, that $\frac{1}{v^0} \lesssim 1$ on the support of g and the bootstrap assumption (2.65), we have

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \int_v \left| z_b \frac{v^L}{v^0} \alpha_B (\nabla_v g)^B \right| dv dx ds &\lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^t \int_{\Sigma_s} \tau_+ |\alpha| \int_v |z_b \widehat{Z}g| dv dx ds \\ &\lesssim \sqrt{\epsilon} \int_0^t (1+s)^{-\frac{3-\eta}{2}} \mathbb{E}_{N_b, b}[f_k](s) ds \\ &\lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

The second term of (2.76) can be treated similarly. For the second term of (2.77) (as the first one can be treated in a similar way), we have, using (2.79), Lemma 2.2.8 and $|v^B| \lesssim \sqrt{v^L v^{\underline{L}}}$ (which comes from Proposition 2.2.9),

$$\int_0^t \int_{\Sigma_s} \int_v \left| h \frac{v^B}{v^0} \sigma(F)_{BD} (\nabla_v |z|)^D \right| dv dx ds \lesssim \sum_{z' \in \mathbf{k}_0} \int_0^t \int_{\Sigma_s} \tau_+ |\sigma(F)| \int_v \frac{\sqrt{v^L v^{\underline{L}}}}{(v^0)^2} |z'| |h| dv dx ds.$$

Since, by Proposition 2.7.2, $|\sigma(F)| \lesssim \sqrt{\epsilon} \tau_+^{-2} \tau_-^{-\frac{1}{2}} (1+t)^{\frac{n}{2}}$, one has, using the Cauchy-Schwarz inequality (in (s, x, v)), that the right hand side of the previous inequality is bounded by the product of

$$\sum_{z' \in \mathbf{k}_0} \left(\sqrt{\epsilon} \int_0^t (1+s)^{-\frac{3-n}{2}} \int_{\Sigma_s} \int_v \frac{v^L}{(v^0)^3} |z'| |h| dv dx ds \right)^{\frac{1}{2}}$$

with

$$\sum_{z' \in \mathbf{k}_0} \left(\sqrt{\epsilon} \int_{u=-\infty}^t \tau_-^{-\frac{3-n}{2}} \int_{C_u(t)} \int_v \frac{v^L}{v^0} |z'| |h| dv dC_u(t) du \right)^{\frac{1}{2}}.$$

The first factor is bounded by $\epsilon^{\frac{3}{4}}$ since $\frac{v^L}{(v^0)^3} \lesssim 1$ on the support of h and $\int_{\Sigma_s} \int_v |z'| |h| dv dx \leq 4\epsilon$ by the bootstrap assumption 2.65. The same is true for the second factor since $\int_{C_u(t)} \int_v \frac{v^L}{v^0} |z'| |h| dv dC_u(t) \leq \mathbb{E}_{N-n,1}[f_k](t) \leq 4\epsilon$, still by the bootstrap assumption (2.65).

Finally, let us treat, for instance, the first term of (2.78). Using the same ingredients as before, namely (2.79), that $\frac{1}{v^0} \lesssim 1$ on the support of g and the bootstrap assumption (2.65), we have,

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \int_v \left| z_b \frac{v^L}{v^0} \underline{\alpha}_B (\nabla_v g)^B \right| ds dx dv &\lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^t \int_{\Sigma_s} \tau_+ |\underline{\alpha}| \int_v \frac{v^L}{(v^0)^2} |z_b \widehat{Z} g| dv dx ds \\ &\lesssim \int_{u=-\infty}^t \|\tau_+ |\underline{\alpha}|\|_{L^\infty(C_u(t))} \mathbb{E}_{N,b}[f_k](t) du \\ &\lesssim \epsilon^{\frac{3}{2}} \int_{-\infty}^{+\infty} \frac{1}{\tau_-^{\frac{3-n}{2}}} du (\delta_{1,b} + \delta_{0,b} \log^*(3+t)). \end{aligned}$$

Remark 2.7.7. *If we used (2.79) instead of (2.80) to estimate (2.73) and (2.75), it would give us a $(1+t)^n$ -loss on the energies (as in the proof of Lemma 2.7.10 below). The weight v^B could be used to avoid this loss in (2.75).*

If $|\beta_1| > N - n$

We study again the integrals of the first terms of (2.72)-(2.78), but this time when $|\beta_1| > N - n$, so that $|\beta_2| \leq N - 2n$, and $z_b = 1$. We then use the pointwise estimate on the velocity averages of the Vlasov fields. This time, we study the terms involving α , ρ and σ together²⁷ and we finish with the two terms involving $\underline{\alpha}$. Note that as we use the extra decay given by v^L , $v^{\underline{L}}$ and v^B , we cannot close the estimate for $\mathbb{E}_{N,1}[f_k]$ with our method.

Let us denote this time α , ρ or σ by ζ . Then, by the bootstrap assumption (2.62),

$$\forall t \in [0, T], \quad \int_{C_u(t)} |\zeta|^2 dx \lesssim \epsilon.$$

All first terms of (2.72)-(2.78) involving α , ρ or σ have their integral on $[0, t] \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ bounded by

$$M := \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^t \int_{\Sigma_s} |\zeta| \tau_+ \int_v \left| \frac{\tilde{v}}{v^0} \widehat{Z} g \right| dv dx ds,$$

where we used (2.79), that $\frac{1}{v^0} \lesssim 1$ on the support of g and where \tilde{v} denotes either v^L , $v^{\underline{L}}$ or v^B .

Using the pointwise decay estimate on $\int_v \left| \frac{\tilde{v}}{v^0} \widehat{Z} g \right| dv$, given by (2.68), (2.69) or (2.70), and the Cauchy-Schwarz inequality (on the $u = \text{constant}$ integrals), we have

²⁷Note that except for (2.76), we could bound all this terms without the $\log^*(1+t)$ -loss.

$$\begin{aligned}
M &\lesssim \int_{u=-\infty}^t \left(\int_{C_u(t)} |\zeta|^2 dx \right)^{\frac{1}{2}} \left(\int_{C_u(t)} \frac{\epsilon^2}{\tau_+^{2n-4} \tau_-^4} dC_u(t) \right)^{\frac{1}{2}} du \\
&\lesssim \int_{u=-\infty}^t \frac{\epsilon^{\frac{3}{2}}}{\tau_-^2} \left| \int_{\underline{u}=0}^{2t-u} \frac{r^{n-1}}{\tau_+^{2n-4}} du \right|^{\frac{1}{2}} du \\
&\lesssim \epsilon^{\frac{3}{2}} \int_{u=-\infty}^t \frac{\log^*(1+2t-u)}{\tau_-^2} du \\
&\lesssim \epsilon^{\frac{3}{2}} \log^*(1+t) \int_{u=-\infty}^{+\infty} \tau_-^{-\frac{3}{2}} du.
\end{aligned}$$

We now study the two remaining terms, which involve $\underline{\alpha}$. We start by (2.75). Using (2.80), we obtain that $\int_0^t \int_{\Sigma_s} \int_v \left| \frac{v^B}{v^0} \underline{\alpha}_B (\nabla_v g)^L \right| dv dx ds$ is bounded by

$$\sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^{+\infty} \|\tau_- \underline{\alpha}\|_{L^2(\Sigma_s)} \left\| \int_v \left| \frac{v^B}{(v^0)^2} \widehat{Z} g \right| dv \right\|_{L^2(\Sigma_s)} ds.$$

Using (2.70), Lemma 2.4.1 and that $\frac{1}{v^0} \lesssim 1$ on the support of $\widehat{Z}g$, we have

$$\left\| \int_v \left| \frac{v^B}{(v^0)^2} \widehat{Z} g \right| dv \right\|_{L^2(\Sigma_s)} \lesssim \frac{\epsilon}{(1+s)^{\frac{n+1}{2}}}.$$

By the bootstrap assumption 2.62, $\|\tau_- \underline{\alpha}\|_{L^2(\Sigma_s)} \lesssim \sqrt{\epsilon \chi(t)(1+t)^n}$, so

$$\int_0^t \int_{\Sigma_s} \int_v \left| \frac{v^B}{v^0} \underline{\alpha}_B (\nabla_v g)^L \right| dv dx ds \lesssim \epsilon^{\frac{3}{2}}.$$

Finally,

$$\int_0^t \int_{\Sigma_s} \int_v \left| \frac{v^L}{v^0} \underline{\alpha}_B (\nabla_v g)^B \right| dv dx ds \lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_0^t \|\underline{\alpha}\|_{L^2(\Sigma_s)} \left\| \tau_+ \int_v \left| \frac{v^L}{(v^0)^2} \widehat{Z} g \right| dv \right\|_{L^2(\Sigma_s)} ds.$$

Now, using the pointwise estimates (2.68) and Lemma 2.4.1, we have

$$\left\| \tau_+ \int_v \frac{v^L}{(v^0)^2} |\widehat{Z} g| dv \right\|_{L^2(\Sigma_s)}^2 \lesssim \epsilon^2 \int_0^{+\infty} \frac{r^{n-1}}{\tau_+^{2n-2} \tau_-^2} dr \lesssim \epsilon (1+s)^{-(n-1)}.$$

As, by the bootstrap assumption (2.63), $\|\underline{\alpha}\|_{L^2(\Sigma_s)}^2 \lesssim \epsilon$,

$$\int_0^t \int_{\Sigma_s} \int_v \left| \frac{v^L}{v^0} \underline{\alpha}_B (\nabla_v g)^B \right| dv dx ds \lesssim \epsilon^{\frac{3}{2}}.$$

This concludes the improvement of the bootstrap assumption (2.65).

2.7.6 Step 4: L^2 estimates for the velocity averages

As for the massive case, to close the energy estimates on the electromagnetic field, we need enough decay on quantities such as $\|\int_v |\widehat{Z}^\beta f_k| dv\|_{L_x^2}$ for all $|\beta| \leq N$. If $|\beta| \leq N - 2n$, strong L^2 decay estimates can already be obtained on $\int_v \frac{v^L}{v^0} |\widehat{Z}^\beta f_k| dv$, for instance, combining (2.68) and Lemma 2.4.1.

We fix, for the remaining of this section, $1 \leq k \leq K$. Following the strategy of [18] (see Section 4.5.7), for a similar problem, we introduce $M \in \mathbb{N}$ such that $\frac{7n+4}{2} \leq M \leq N - \frac{5}{2}n$. Let I_1 and I_2 be defined as

$$I_1 = \{\beta \text{ multi-index} / M \leq |\beta| \leq N\} \text{ and } I_2 = \{\beta \text{ multi-index} / |\beta| \leq M - 1\}.$$

We consider an ordering on I_i , for $1 \leq i \leq 2$, so that $I_i = \{\beta_{i,1}, \dots, \beta_{i,|I_i|}\}$ and two vector valued fields X and Y , of respective length $|I_1|$ and $|I_2|$, such that

$$X^j = \widehat{Z}^{\beta_{1,j}} f_k \quad \text{and} \quad Y^j = \widehat{Z}^{\beta_{2,j}} f_k.$$

Lemma 2.7.8. *There exists three matrices valued functions $A_1 : [0, T] \times \mathbb{R}^n \rightarrow \mathfrak{M}_{|I_1|}(\mathbb{R})$, $A_2 : [0, T] \times \mathbb{R}^n \rightarrow \mathfrak{M}_{|I_2|}(\mathbb{R})$ and $B : [0, T] \times \mathbb{R}^n \rightarrow \mathfrak{M}_{|I_1|, |I_2|}(\mathbb{R})$ such that*

$$T_F(X) + A_1 X = BY, \quad \text{and} \quad T_F(Y) = A_2 Y.$$

If $1 \leq j \leq I_1$, A_1 and B are such that $T_F(X^j)$ is a linear combination of

$$\begin{aligned} & \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_1}}(F)_{\mu m} X^{\beta_{1,q}}, \quad t \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_1}}(F)_{\mu m} X^{\beta_{1,q}}, \quad \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_1}}(F)_{\mu i} x^i X^{\beta_{1,q}}, \\ & \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_2}}(F)_{\mu m} Y^{\beta_{2,l}}, \quad t \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_2}}(F)_{\mu m} Y^{\beta_{2,l}} \quad \text{and} \quad \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_2}}(F)_{\mu i} x^i Y^{\beta_{2,l}}, \end{aligned}$$

with $|\gamma_1| \leq N - \frac{7n+2}{2}$, $|\gamma_2| \leq N$, $1 \leq m \leq n$, $1 \leq q \leq |I_1|$ and $1 \leq l \leq |I_2|$. Similarly, if $1 \leq j \leq I_2$, A_2 is such that $T_F(Y^j)$ is a linear combination of

$$\frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu m} Y^{\beta_{2,l}}, \quad t \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu m} Y^{\beta_{2,l}} \quad \text{and} \quad \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu i} x^i Y^{\beta_{2,l}},$$

with $|\gamma| \leq N - \frac{5n+2}{2}$, $1 \leq m \leq n$ and $1 \leq l \leq |I_2|$. Moreover,

$$\forall z \in \mathbf{k}_0, \quad \int_v |z| |Y|_\infty dv \lesssim \frac{\epsilon}{\tau_+^{n-1} \tau_-}.$$

Proof.

Let $|\beta| \leq N$. According to commutation formula of Lemma 2.2.26, $T_F(\widehat{Z}^\beta f_k)$ is a linear combination of terms such as $\mathcal{L}_{Z^\gamma}(F)(v, \nabla_v \widehat{Z}^\delta(f_k))$, with $|\gamma| + |\delta| \leq |\beta|$ and $|\delta| \leq |\beta| - 1$. Replacing each $\partial_{v^i} \widehat{Z}^\delta f_k$ by $\frac{1}{v^0} (\widehat{\Omega}_{0i} \widehat{Z}^\beta f_k - t \partial_i \widehat{Z}^\beta f_k - x^i \partial_t \widehat{Z}^\beta f_k)$, the matrices naturally appear. The decay estimates ensue from the definition of Y and (2.67). \square

Now, we split X in $G + H$ where G is the solution of the homogeneous system and H is the solution to the inhomogeneous system,

$$\begin{cases} T_F(H) + AH = 0, & H(0, \cdot, \cdot) = X(0, \cdot, \cdot), \\ T_F(G) + AG = BY, & G(0, \cdot, \cdot) = 0. \end{cases}$$

We will prove below that $G = KY$ (with K a well chosen matrix), which implies, in view of the velocity support of X and Y , that H and G vanish if $|v| \leq \frac{R}{2}$.

The goal now is to prove L^2 estimates on the velocity averages of H and G .

The homogeneous part

As for the massive case, we have the following commutation formula.

Lemma 2.7.9. *Let $1 \leq i \leq |I_1|$ and consider $\widehat{Z}^\delta \in \widehat{\mathbb{P}}_0^{|\delta|}$, with $|\delta| \leq n$. Then, $T_F(\widehat{Z}^\delta H^i)$ can be written as a linear combination of terms of the form*

$$\mathcal{L}_{Z^\gamma}(F)(v, W),$$

where W is such that

$$\forall 0 \leq \mu \leq n, \quad |W^\mu| \lesssim \frac{\tau_+}{v^0} \sum_{|\theta| \leq n} \sum_{q=1}^{|I_1|} |\widehat{Z}^\theta H^q|,$$

and where $|\gamma| \leq N - \frac{5n+2}{2}$, so that we can use the sharpest estimates of Proposition 2.7.2, except for α .

We introduce the energy $\widetilde{\mathbb{E}}_1[H]$ defined by

$$\widetilde{\mathbb{E}}_1[H] = \sum_{q=1}^{|I_1|} \mathbb{E}_{n,1}[H^q].$$

Note that for ϵ small enough,

$$\widetilde{\mathbb{E}}_1[H](0) \leq 2\mathbb{E}_{N+n,1}[f](0) \leq 2\epsilon.$$

Lemma 2.7.10. *If ϵ is small enough, we have*

$$\forall t \in [0, T], \quad \widetilde{E}_1[H](t) \leq 6\epsilon(1+t)^{\frac{3}{2}}.$$

Moreover,

$$\forall 1 \leq i \leq |I_1|, z \in \mathbf{k}_0, (t, x) \in [0, T] \times \mathbb{R}^n, \quad \int_v |zH^i| dv \lesssim \epsilon \frac{(1+t)^{\frac{3}{2}}}{\tau_+^{n-1}\tau_-}.$$

Proof. We use again the continuity method. Since, for ϵ small enough, $\widetilde{\mathbb{E}}_1[H](0) \leq 2\epsilon$, there exists a larger time $0 < \widetilde{T} \leq T$ such that

$$\forall t \in [0, \widetilde{T}], \quad \widetilde{\mathbb{E}}_1[H](t) \leq 6\epsilon(1+t)^{\frac{3}{2}}.$$

Following the argument of Section 2.7.5, we almost get that for ϵ small enough, $\widetilde{\mathbb{E}}_1[H] \leq 5\epsilon(1+t)^\eta$ on $[0, \widetilde{T}]$. In fact, using Lemma 2.7.9, we have that $T_F(H^\beta)$ is a linear combination of terms like $\mathcal{L}_{Z^\gamma}(F)(v, W)$, with $|\gamma| \leq N - \frac{5n+2}{2}$. Thus we can use the null decomposition of the velocity vector and the electromagnetic field (and use its pointwise estimates) and then make similar computations as in Section 2.7.5. As we cannot use (2.80) (the algebraic relations between $S\widehat{Z}^\beta f$ and $\partial_\mu \widehat{Z}^\beta f$ ($\mu \in \llbracket 0, n \rrbracket$), for instance, are not necessarily conserved by the decomposition $X = H + G$), we need to reexamine the terms corresponding to (2.72)-(2.75). For instance, for the terms analogous to one of (2.75), we have to prove, for $z \in \mathbf{k}_0$,

$$\int_0^t \int_{\Sigma_s} \int_v \tau_+ |\underline{\alpha}| \left| z \frac{v^B}{(v^0)^2} \widehat{Z}^\theta H^q \right| dv dx ds \lesssim \epsilon^{\frac{3}{2}} (1+t)^{\frac{3}{2}}. \quad (2.81)$$

As $|v^B| \lesssim \sqrt{|v^L v^{\underline{L}}|}$ by Proposition 2.2.9 and as $\tau_+ |\underline{\alpha}| \lesssim \frac{\sqrt{\epsilon}}{\tau_+^{\frac{n-3}{2}} \tau_-}$, we have, by the Cauchy-Schwarz inequality (in (s, x, v)), that (2.81) is bounded by the product of

$$\left(\int_0^t \int_{\Sigma_s} \frac{\epsilon}{\tau_+^{n-3}} \int_v \left| z \widehat{Z}^\theta H^q \right| dv dx ds \right)^{\frac{1}{2}}$$

with

$$\left(\int_{u=-\infty}^t \frac{1}{\tau_-^2} \int_{C_u(t)} \int_v \frac{v^L v^{\underline{L}}}{(v^0)^4} \left| z \widehat{Z}^\theta H^q \right| dv dC_u(t) du \right)^{\frac{1}{2}}.$$

The first factor is bounded by

$$\left(\int_0^t \frac{\epsilon}{1+s} \widetilde{\mathbb{E}}_1[H](s) ds \right)^{\frac{1}{2}} \lesssim \epsilon(1+t)^{\frac{3}{4}},$$

and the other one, since $\frac{v^L}{(v^0)^3} \lesssim 1$ on the support of H , by

$$\sqrt{\widetilde{\mathbb{E}}_1[H](t)} \left(\int_{u=-\infty}^{+\infty} \frac{1}{\tau_-^2} du \right)^{\frac{1}{2}} \lesssim \sqrt{\epsilon}(1+t)^{\frac{3}{4}}.$$

The other terms are easier to bound. Let us study also the terms analogous to one of (2.73), as there are also the cause of the $(1+t)^{\frac{\eta}{2}}$ -loss²⁸.

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \int_v \tau_+ |\rho| \left| z \frac{v^L}{(v^0)^2} \widehat{Z}^\theta H^\xi \right| dv dx ds &\lesssim \sqrt{\epsilon} \int_0^t (1+s)^{-\frac{n-2}{2}} \widetilde{\mathbb{E}}_1[H](s) ds \\ &\lesssim \epsilon^{\frac{3}{2}} (1+t)^{\frac{3}{2}}. \end{aligned}$$

The pointwise estimate on $\int_v |z||H^i| dv$ then ensues from the Klainerman-Sobolev inequality of Corollary 2.2.14. □

²⁸Note that we could use that $\sqrt{\tau_+ \tau_-} |v^B| \lesssim v^0 \sum_{z \in \mathbf{k}_0} |z|$ in (2.75) to obtain a better bound in (2.81) for an other energy of H . On the other hand, the loss coming from (2.73) could not be avoided with such techniques.

The inhomogeneous part

As in the massive case, let us introduce K , the solution of $T_F(K) + A_1K + KA_2 = B$ which verifies $K(0, \cdot, \cdot) = 0$, and the function

$$|KKY|_\infty = \sum_{\substack{1 \leq i \leq |I_1| \\ 1 \leq j, p \leq |I_2|}} |K_i^j|^2 |Y_p|.$$

KY and G are solutions of the same system,

$$\begin{aligned} T_F(KY) = T_F(K)Y + KT_F(Y) &= BY - A_1KY - KA_2Y + KA_2Y \\ &= BY - A_1KY. \end{aligned}$$

As $KY(0, \cdot, \cdot) = 0$ and $G(0, \cdot, \cdot) = 0$, it comes that $KY = G$. For $1 \leq i \leq |I_1|$ and $1 \leq j, p \leq |I_2|$, $|K_i^j|^2 Y_p$ satisfies the equation

$$T_F\left(|K_i^j|^2 Y_p\right) = |K_i^j|^2 (A_2)_p^q Y_q - 2\left((A_1)_i^q K_q^j + K_i^q (A_2)_q^j\right) K_i^j Y_p + 2B_i^j K_i^j Y_p,$$

which will allow us to estimate

$$\mathbb{E}[|KKY|_\infty] := \mathbb{E}_{0,1}[|KKY|_\infty].$$

We will then be able to prove L^2 estimates for $\int_{v \in \mathbb{R}^n} |G| dv$ thanks to the estimates on $\int_{v \in \mathbb{R}^n} |Y| dv$ and on $\mathbb{E}[|KKY|_\infty]$.

Lemma 2.7.11. *We have, if ϵ is small enough,*

$$\forall t \in [0, T], \quad \mathbb{E}[|KKY|_\infty] \leq \epsilon(1+t)^\eta.$$

Proof. Let $\tilde{T} > 0$ be the largest time such that $\mathbb{E}[|KKY|_\infty](t) \leq 2\epsilon(1+t)^\eta$ for all $t \in [0, \tilde{T}]$ and let us prove, with Proposition 2.3.1, that for ϵ small enough, $\mathbb{E}[|KKY|_\infty](t) \leq \epsilon(1+t)^\eta$ for all $t \in [0, \tilde{T}]$. It will follow that $\tilde{T} = T$. As for the estimate of $\tilde{\mathbb{E}}_1[H]$ in the proof of Lemma 2.7.10,

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \int_v \left| |K_i^j|^2 (A_2)_p^q Y_q - 2\left((A_1)_i^q K_q^j + K_i^q (A_2)_q^j\right) K_i^j Y_p \right| \frac{|z|}{v^0} dv dx ds &\lesssim \\ &\epsilon^{\frac{3}{2}} (1+t)^\eta \end{aligned}$$

and

$$\int_0^t \int_{\Sigma_s} \int_v \left| F\left(\frac{v}{v^0}, \nabla_v(|z|)\right) \right| |KKY|_\infty dv dx ds \lesssim \epsilon^{\frac{3}{2}}.$$

Next, we need to estimate the following integral,

$$\int_0^t \int_{\Sigma_s} \int_v \frac{|z|}{v^0} |B_i^j K_i^j Y_p| dv dx. \quad (2.82)$$

Recall from Lemma 2.7.8 that

$$|B_i^j K_i^j Y_p| \lesssim \sum_{m=1}^n \sum_{|\gamma| \leq N} \tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu m} K_i^j Y_p \right|.$$

The components of the matrix B involve terms in which the electromagnetic field has too many derivatives to be estimated pointwise. We fix $|\gamma|$ and we denote the null decomposition of $\mathcal{L}_{Z^\gamma}(F)$ by $(\alpha, \underline{\alpha}, \rho, \sigma)$. In order to bound (2.82), we bound the integral of the five following terms, given by the null decomposition of the velocity vector v and $\mathcal{L}_{Z^\gamma}(F)$.

- The terms which do not involve $\underline{\alpha}$

$$\tau_+ |\alpha| |z| \frac{|KY|}{v^0}, \quad \tau_+ |\rho| |z| \frac{|KY|}{v^0} \quad \text{and} \quad \tau_+ |\sigma| |z| \frac{|KY|}{v^0}.$$

- The terms involving $\underline{\alpha}$

$$\tau_+ |\underline{\alpha}| \frac{v^{\underline{L}}}{(v^0)^2} |z| |KY| \quad \text{and} \quad \tau_+ |\underline{\alpha}| \frac{|v^B|}{(v^0)^2} |z| |KY|.$$

We start by bounding the integral on $\Sigma_s \times (\mathbb{R}_v^n \setminus \{0\})$ of the good terms. We use ζ to denote either α , ρ or σ . Using twice the Cauchy-Schwarz inequality (in x and then in v) and that $\frac{1}{v^0} \lesssim 1$ on the support of Y , we have

$$\begin{aligned} \int_{\Sigma_s} \int_v \tau_+ |\zeta| \frac{|zKY|}{v^0} dv dx &\lesssim \|\tau_+ |\zeta|\|_{L^2(\Sigma_s)} \left(\int_{\Sigma_s} \left(\int_v |zKY| dv \right)^2 dx \right)^{\frac{1}{2}} \\ &\lesssim \left| \mathcal{E}_N[F](s) \int_{\Sigma_s} \int_v |zY| dv \int_v |zKKY| dv dx \right|^{\frac{1}{2}} \\ &\lesssim \left| \mathcal{E}_N[F](s) \left\| \int_v |zY| dv \right\|_{L^\infty(\Sigma_s)} \mathbb{E}[|KKY|_\infty] \right|^{\frac{1}{2}}. \end{aligned}$$

Using the bootstrap assumptions, on $\mathcal{E}_N[F]$ and $\mathbb{E}[|KKY|_\infty]$, and the pointwise decay estimate $\int_v |zY| dv \lesssim \epsilon \tau_+^{-n+1} \tau_-^{-1}$ given in Lemma 2.7.8, we obtain

$$\int_0^t \int_{\Sigma_s} \int_v \tau_+ |\zeta| \frac{|zKY|}{v^0} dv dx ds \lesssim \int_0^t \frac{\epsilon^{\frac{3}{2}} \sqrt{\chi(t)}}{(1+s)^{\frac{3}{2}-\eta}} ds \lesssim \epsilon^{\frac{3}{2}} (1+t)^\eta.$$

As in the massive case, to unify the study of the terms involving $\underline{\alpha}$, we use \tilde{v} to denote $v^{\underline{L}}$ or v^B . Using the Cauchy-Schwarz inequality (in (s, x)), we have

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \tau_+ |\underline{\alpha}| \int_v \frac{|\tilde{v}|}{(v^0)^2} |z| |KY| dv dx ds &\lesssim \\ \left| \int_0^t \int_{\Sigma_s} \frac{\tau_- |\underline{\alpha}|^2}{(1+s)^{n-3}} dx ds \int_0^t \int_{\Sigma_s} \frac{\tau_+^2 (1+s)^{n-3}}{\tau_-} \left| \int_v \frac{\tilde{v}z}{(v^0)^2} KY \right| dv \right|^2 dx ds &\Big|^{\frac{1}{2}}. \end{aligned} \quad (2.83)$$

As, by the bootstrap assumption 2.63, $\|\sqrt{\tau_-} |\underline{\alpha}|\|_{L^2(\Sigma_s)}^2 \lesssim \epsilon(1+s)^\eta$, we have

$$\int_0^t \int_{\Sigma_s} \frac{\tau_- |\underline{\alpha}|^2}{(1+s)^{n-3}} dx ds \lesssim \epsilon(1+t)^\eta.$$

For the second factor of the product in (2.83), we first note that, by the Cauchy-Schwarz inequality and that $\frac{1}{(v^0)^2} \lesssim 1$ on the support of Y ,

$$\left(\int_v \left| \frac{\tilde{v}z}{(v^0)^2} KY \right| dv \right)^2 \leq \int_v |zY| dv \int_v \left| \frac{\tilde{v}}{v^0} \right|^2 |z| |KKY|_\infty dv.$$

Now, recall from Proposition 2.2.9 that $|v^B| \lesssim \sqrt{v^{\underline{L}} v^{\underline{L}}}$ so that $\left| \frac{\tilde{v}}{v^0} \right|^2 \lesssim \frac{v^{\underline{L}}}{v^0}$. Using the pointwise estimate $\int_v |zY| dv \lesssim \epsilon \tau_+^{-n+1} \tau_-^{-1}$, it comes

$$\left(\int_v \left| \frac{\tilde{v}z}{(v^0)^2} KY \right| dv \right)^2 \lesssim \frac{\epsilon}{\tau_+^{n-1} \tau_-} \int_v \frac{v^{\underline{L}}}{v^0} |z| |KKY|_\infty dv.$$

As $\int_{C_u(t)} \int_v \frac{v^{\underline{L}}}{v^0} |z| |KKY|_\infty dC_u(t) dv \leq \mathbb{E}[|KKY|_\infty](t) \leq 2\epsilon(1+t)^\eta$, we obtain

$$\int_0^t \int_{\Sigma_s} \frac{\tau_+^2 (1+s)^{n-3}}{\tau_-} \left| \int_v \left| \frac{\tilde{v}z}{(v^0)^2} KY \right| dv \right|^2 dx ds \lesssim \epsilon^2 (1+t)^\eta \int_{u=-\infty}^t \tau_-^{-2} du.$$

Hence,

$$\int_0^t \int_{\Sigma_s} \int_v \tau_+ \frac{|\tilde{v}|}{(v^0)^2} |\underline{\alpha}| |zKY| dv dx ds \lesssim \epsilon^2 (1+t)^\eta$$

and the energy estimate of Proposition 2.3.1 gives that, for ϵ small enough, $\mathbb{E}[|KKY|_\infty] \leq \epsilon(1+t)^\eta$ on $[0, \tilde{T}]$.

□

The L^2 estimates

We start with the following proposition.

Proposition 2.7.12. *We have,*

$$\forall |\beta| \leq N, t \in [0, T], \left\| \int_{v \in \mathbb{R}^n \setminus \{0\}} |\widehat{Z}^\beta f_k| dv \right\|_{L^2(\Sigma_t)} \lesssim \frac{\epsilon}{(1+t)^{\frac{n-1-\eta}{2}}}$$

and

$$\forall |\beta| \leq N, t \in [0, T], \left\| \tau_+ \int_{v \in \mathbb{R}^n \setminus \{0\}} |\widehat{Z}^\beta f_k| dv \right\|_{L^2(\Sigma_t)} \lesssim \frac{\epsilon}{(1+t)^{\frac{n-3-\eta}{2}}}$$

We can remove the $(1+t)^{\frac{3}{2}}$ -loss if $|\beta| \leq N - 2n$.

Proof. Let $1 \leq k \leq K$. The first inequality ensues from the second one since $1+t \leq \tau_+$. If $|\beta| \leq N - 2n$, we only have to use the pointwise estimate (2.66) and Lemma 2.4.1. If $|\beta| > N - 2n$, recall that there exists $1 \leq i \leq |I_1|$ such that $\widehat{Z}^\beta f_k = H^i + G^i$. For $1 \leq i \leq |I_1|$, Lemmas 2.7.10 and 2.4.1 imply

$$\left\| \tau_+ \int_v |H^i| dv \right\|_{L^2(\Sigma_t)} \lesssim \frac{\epsilon}{(1+t)^{\frac{n-3-\eta}{2}}}.$$

Moreover, as $G = KY$, we have, by the Cauchy-Schwarz inequality (in v),

$$\left\| \tau_+ \int_v |G^i| dv \right\|_{L^2(\Sigma_t)} \leq \left\| \tau_+^2 \int_v |Y|_\infty dv \int_v |K_i^j|^2 |Y_j| dv \right\|_{L^1(\Sigma_t)}^{\frac{1}{2}}.$$

As, by Lemmas 2.7.8 and 2.7.11,

$$\left\| \tau_+^2 \int_v |Y|_\infty dv \right\|_{L^\infty(\Sigma_t)} \lesssim \frac{\epsilon}{(1+t)^{n-3}} \text{ and } \left\| \int_v |K_i^j|^2 |Y_j| dv \right\|_{L^1(\Sigma_t)}^{\frac{1}{2}} \leq \epsilon^{\frac{1}{2}} (1+t)^{\frac{\eta}{2}},$$

we have

$$\left\| \tau_+ \int_v |G^i| dv \right\|_{L^2(\Sigma_t)} \lesssim \frac{\epsilon}{(1+t)^{\frac{n-3-\eta}{2}}}.$$

□

These inequalities will not be sufficient to close the estimate on the energy $\mathcal{E}_{N-\frac{n+2}{2}}^S[F]$ in the next section. This is why we prove the following proposition.

Proposition 2.7.13. *For all $|\beta| \leq N$ and all $t \in [0, T]$, we have :*

$$\begin{aligned} \left\| \int_v \frac{v^L}{v^0} |\widehat{Z}^\beta f_k| dv \right\|_{L^2(\Sigma_t)} &\lesssim \frac{\epsilon}{(1+t)^{\frac{n+1-\eta}{2}}}, \\ \left\| \frac{\tau_-}{\tau_+} \int_v \frac{v^L}{v^0} |\widehat{Z}^\beta f_k| dv \right\|_{L^2(\Sigma_t)} &\lesssim \frac{\epsilon}{(1+t)^{\frac{n+1-\eta}{2}}}, \\ \left\| \int_v \left| \frac{v^B}{v^0} \widehat{Z}^\beta f_k \right| dv \right\|_{L^2(\Sigma_t)} &\lesssim \frac{\epsilon}{(1+t)^{\frac{n+1-\eta}{2}}}. \end{aligned}$$

We can remove the $(1+t)^{\frac{3}{2}}$ -loss if $|\beta| \leq N - 2n$.

Proof. If $|\beta| \leq N - 2n$, these inequalities are implied by the pointwise estimates (2.68), (2.69) and (2.70) and Lemma 2.4.1.

If $|\beta| > N - 2n$, we prove in the same way that these inequalities are true if we replace $\widehat{Z}^\beta f_k$ by H^i , with $1 \leq i \leq |I_1|$ such that $\widehat{Z}^\beta f_k = H^i + G^i$. It then only remains to consider G^i . Recall that by Proposition 2.2.9 and Lemma 2.7.8,

$$\frac{v^L}{v^0} \leq \tau_+^{-1} \sum_{z \in \mathbf{k}_0} |z|, \quad \frac{v^L}{v^0} \leq \tau_-^{-1} \sum_{z \in \mathbf{k}_0} |z| \quad \text{and} \quad \int_v |z| |Y|_\infty dv \lesssim \frac{\epsilon}{\tau_+^{n-1} \tau_-}.$$

Hence, using also $G = KY$, the Cauchy-Schwarz inequality (in v) and $\mathbb{E}[|KKY|_\infty](t) \leq 2\epsilon(1+t)^\eta$, we have

$$\begin{aligned} \left\| \int_v \frac{v^L}{v^0} |G^i| dv \right\|_{L^2(\Sigma_t)} &\lesssim \left\| \int_v \frac{v^L}{v^0} |Y|_\infty dv \int_v \frac{v^L}{v^0} |(K_i^j)^2 Y_j| dv \right\|_{L^1(\Sigma_t)}^{\frac{1}{2}} \\ &\lesssim \sum_{z \in \mathbf{k}_0} \left\| \tau_+^{-2} \int_v |z| |Y|_\infty dv \int_v |z| |KKY|_\infty dv \right\|_{L^1(\Sigma_t)}^{\frac{1}{2}} \\ &\lesssim \epsilon(1+t)^{-\frac{n+1-\eta}{2}}, \end{aligned}$$

$$\begin{aligned} \left\| \frac{\tau_-}{\tau_+} \int_v \frac{v^L}{v^0} |G^i| dv \right\|_{L^2(\Sigma_t)} &\lesssim \left\| \frac{\tau_-^2}{\tau_+^2} \int_v \frac{v^L}{v^0} |Y|_\infty dv \int_v \frac{v^L}{v^0} |(K_i^j)^2 Y_j| dv \right\|_{L^1(\Sigma_t)}^{\frac{1}{2}} \\ &\lesssim \sum_{z \in \mathbf{k}_0} \left\| \tau_+^{-2} \int_v |z| |Y|_\infty dv \int_v |z| |KKY|_\infty dv \right\|_{L^1}^{\frac{1}{2}} \\ &\lesssim \epsilon(1+t)^{-\frac{n+1-\eta}{2}}. \end{aligned}$$

The remaining estimate can be proved in a similar way, using $|v^B| \lesssim \frac{v^0}{\tau_+} \sum_{z \in \mathbf{k}_0} |z|$ (see Proposition 2.2.9). \square

2.7.7 Step 5: Improvement of the electromagnetic field estimates

Improvement of the energies estimates for the potential

According to the energy estimate of Proposition 2.3.12 and the commutation formula of Proposition 2.2.19, one has, for all $t \in [0, T]$,

$$\sqrt{\tilde{\mathcal{E}}_N[A](t)} \lesssim \sqrt{\tilde{\mathcal{E}}_N[A](0)} + \sum_{|\gamma| \leq N} \int_0^t \left\| \tau_+ e^k \int_{\mathbb{R}^n} |\widehat{Z}^\gamma f_k| dv \right\|_{L^2(\mathbb{R}^n)} ds.$$

As $\tilde{\mathcal{E}}_N[A](0) \leq \epsilon$ and $\left\| \tau_+ e^k \int_{\mathbb{R}^n} |\widehat{Z}^\gamma f_k| dv \right\|_{L^2(\mathbb{R}^n)} \lesssim \epsilon(1+t)^{-\frac{n-3-\eta}{2}}$ (see Proposition 2.7.12), we have, for ϵ small enough and if the constant C is large enough,

$$\forall t \in [0, T], \quad \tilde{\mathcal{E}}_N[A](t) \leq \frac{C}{2(n-3)} \epsilon \chi(t) (1+t)^\eta,$$

with χ such that

$$\chi(s) = 1 + s \text{ if } n = 4, \quad \chi(s) = \log^2(3+s) \text{ if } n = 5 \text{ and } \chi(s) = 1 \text{ if } n \geq 6.$$

Similarly, using (2.66) and Lemma 2.4.1, we obtain

$$\forall t \in [0, T], \quad \tilde{\mathcal{E}}_{N-2n}[A](t) \leq \frac{C}{2(n-3)} \epsilon \chi(t).$$

This concludes the improvement of the bootstrap assumption (2.64).

Improvement of the estimate on $\mathcal{E}_N^0[F]$

Recall from Proposition 2.3.17 that, for all $t \in [0, T]$,

$$\mathcal{E}_N^0[F](t) - 2\mathcal{E}_N^0[F](0) \lesssim \sum_{|\beta|, |\gamma| \leq N} |e^k| \int_0^t \int_{\Sigma_s} |\mathcal{L}_{Z^\beta}(F)_{0\nu} J(\widehat{Z}^\gamma f_k)^\nu| dx ds.$$

As, by the Cauchy-Schwarz inequality, the bootstrap assumption (2.63) and the L^2 estimates of Proposition 2.7.12,

$$\begin{aligned}
\int_0^t \int_{\Sigma_s} |\mathcal{L}_{Z^\beta}(F)_{0\nu} J(\widehat{Z}^\gamma f_k)^\nu| dx ds &\lesssim \int_0^t \|\mathcal{L}_{Z^\beta}(F)\|_{L^2(\Sigma_s)} \|J(\widehat{Z}^\gamma f_k)\|_{L^2(\Sigma_s)} ds \\
&\lesssim \int_0^t \sqrt{\mathcal{E}_N^0[F](s)} \left\| \int_v \widehat{Z}^\gamma f_k dv \right\|_{L^2(\Sigma_s)} ds \\
&\lesssim \int_0^t \epsilon^{\frac{1}{2}} \epsilon (1+s)^{\frac{n-1-\eta}{2}} ds \\
&\lesssim \epsilon^{\frac{3}{2}},
\end{aligned}$$

we have, for ϵ small enough, $\mathcal{E}_N^0[F] \leq 3\epsilon$ on $[0, T]$.

Improvement of the estimates on $\mathcal{E}_N[F]$ and $\mathcal{E}_{N-2n}[F]$

Recall from Proposition 2.3.21 that

$$\mathcal{E}_N[F](t) \leq \mathcal{E}_N[F](0) + (n-3)\widetilde{\mathcal{E}}_N[A](t) + \varphi(t),$$

where $\varphi(t)$ is a linear combination of terms such that

$$\int_0^t \int_{\Sigma_s} |\overline{K}_0^\nu \mathcal{L}_{Z^\beta} F_{\mu\nu} J(\widehat{Z}^\gamma f_k)^\mu| dx ds \quad \text{and} \quad \int_0^t \int_{\Sigma_s} s |\mathcal{L}_{Z^\delta} A_\mu \square \mathcal{L}_{Z^\delta} A^\mu| dx ds, \quad (2.84)$$

with $|\beta|, |\gamma|, |\delta| \leq N$ and $1 \leq k \leq K$. Then, if we could prove that each integrals of (2.84) is bounded by $\epsilon^{\frac{3}{2}} \chi(t)(1+t)^\eta$, we would have, for ϵ small enough and C large enough, $\mathcal{E}_N[F] \leq C\epsilon \chi(t)(1+t)^\eta$ on $[0, T]$ since $\mathbb{E}_N[F](0) \leq \epsilon$ and $(n-3)\widetilde{\mathcal{E}}_N[A](t) \leq \frac{C}{2}\epsilon \chi(t)(1+t)^\eta$.

Remark 2.7.14. *We could estimate the integrals of (2.84) with a better bound (the computations are similar to those done below in Section 2.7.7, but this would not give us a better estimate on $\mathcal{E}_N[F]$ because of the $\chi(t)(1+t)^\eta$ -loss on $\widetilde{\mathcal{E}}_N[A]$.*

We start by bounding the integrals involving the potential. Using Proposition 2.2.19 and the Cauchy-Schwarz inequality, we have, for $|\delta| \leq N$,

$$\int_0^t \int_{\Sigma_s} s |\mathcal{L}_{Z^\delta} A_\mu \square \mathcal{L}_{Z^\delta} A^\mu| dx ds \lesssim \sum_{k=1}^K \sum_{|\gamma| \leq |\delta|} \int_0^t \sqrt{\widetilde{\mathcal{E}}_N[A](s)} \left\| \tau_+ \int_v \widehat{Z}^\gamma f_k dv \right\|_{L^2(\Sigma_s)} ds.$$

Using the L^2 estimates of Proposition 2.7.12 and that $\widetilde{\mathcal{E}}_N[A](s) \lesssim \epsilon \chi(s)(1+t)^\eta$, it comes

$$\begin{aligned}
\sum_{|\delta| \leq N} \int_0^t \int_{\Sigma_s} s |\mathcal{L}_{Z^\delta} A_\mu \square \mathcal{L}_{Z^\delta} A^\mu| dx ds &\lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{\sqrt{\chi(s)}}{(1+s)^{\frac{n-3}{2}-\eta}} ds \\
&\lesssim \epsilon^{\frac{3}{2}} \chi(t)(1+t)^\eta.
\end{aligned}$$

In order to estimate the remaining integrals of (2.84), we express $\overline{K}_0^\nu \mathcal{L}_{Z^\beta}(F)_{\mu\nu} J(\widehat{Z}^\gamma f_k)^\mu$ in null coordinates. Dropping the dependance in $\mathcal{L}_{Z^\beta}(F)$ or $\widehat{Z}^\gamma f_k$, this gives us the four following terms :

$$\tau_+^2 \rho J^{\underline{L}}, \quad \tau_-^2 \rho J^L, \quad \tau_+^2 \alpha_B J^B, \quad \text{and} \quad \tau_-^2 \alpha_B J^B. \quad (2.85)$$

As

$$J^{\underline{L}} = \int_v \frac{v^{\underline{L}}}{v^0} \widehat{Z}^\gamma f_k dv, \quad J^L = \int_v \frac{v^L}{v^0} \widehat{Z}^\gamma f_k dv \quad \text{and} \quad J^B = \int_v \frac{v^B}{v^0} \widehat{Z}^\gamma f_k dv,$$

we have,

$$|J^{\underline{L}}|, |J^L|, |J^B| \lesssim \int_v |\widehat{Z}^\gamma f_k| dv.$$

The integrals (on $[0, T] \times \mathbb{R}_x^n \times (\mathbb{R}_v^n \setminus \{0\})$) of each of the four terms of (2.85) are then bounded, using the Cauchy-Schwarz inequality (in x), by

$$\int_0^t \sqrt{\mathcal{E}_N[F](s)} \left\| \tau_+ \int_v |\widehat{Z}^\gamma f_k| dv \right\|_{L^2(\Sigma_s)} ds.$$

By Proposition 2.7.12 and the bootstrap assumption (2.62),

$$\begin{aligned} \int_0^t \sqrt{\mathcal{E}_N[F](s)} \left\| \tau_+ \int_v |\widehat{Z}^\gamma f_k| dv \right\|_{L^2(\Sigma_s)} ds &\lesssim \int_0^t \sqrt{\epsilon \chi(s)} \frac{\epsilon}{(1+s)^{\frac{n-3}{2}-\eta}} ds \\ &\lesssim \epsilon^{\frac{3}{2}} \chi(t) (1+t)^\eta. \end{aligned}$$

Hence, $\mathcal{E}_N[F](t) \leq C\epsilon\chi(t)(1+t)^\eta$ for all $t \in [0, T]$ if ϵ is small enough. We can prove exactly in the same way that $\mathcal{E}_{N-2n}[F](t) \leq C\epsilon\chi(t)$ for all $t \in [0, T]$ if ϵ is small enough.

We then improve the bootstrap assumption (2.62).

Improvement of the estimates on $\mathcal{E}_N^S[F]$ and $\mathcal{E}_{N-2n}^S[F]$

Recall from Propositions 2.3.25 and 2.2.19 that, for all $t \in [0, T]$,

$$\mathcal{E}_N^S[F](t) \leq \mathcal{E}_N[F](0) + C_n \left(\widetilde{\mathcal{E}}_N[A](0) + \frac{\widetilde{\mathcal{E}}_N[A](t)}{1+t} + \frac{\mathcal{E}_N[F](t)}{1+t} \right) + \psi(t),$$

where C_n is a positive constant and where $\psi(t)$ is a linear combination of terms such as

$$\int_0^t \int_{\Sigma_s} |\mathcal{L}_{Z^\beta}(F)_{0\mu} J^\mu(\widehat{Z}^\gamma f_k)| + |S^\nu \mathcal{L}_{Z^\beta}(F)_{\nu\mu} J^\mu(\widehat{Z}^\gamma f_k)| dx ds, \quad (2.86)$$

with $|\beta|, |\gamma| \leq N$ and $1 \leq k \leq K$, and

$$\int_0^t \int_{\Sigma_s} \left| \mathcal{L}_{Z^\beta}(A)_\mu \int_v \frac{v^\mu}{v^0} \widehat{Z}^\gamma f_k \right| dx ds, \quad (2.87)$$

with $|\beta|, |\gamma| \leq N$ and $1 \leq k \leq K$.

Let $|\beta| + |\gamma| \leq N$ and $1 \leq k \leq K$. We denote the null decomposition of $\mathcal{L}_{Z^\beta}(F)$ by $(\alpha, \underline{\alpha}, \rho, \sigma)$, $\widehat{Z}^\gamma f_k$ by g and $J(\widehat{Z}^\gamma f_k)$ by J . Expressing $\mathcal{L}_{Z^\beta}(F)_{0\mu} J^\mu(g)$ and $S^\nu \mathcal{L}_{Z^\beta}(F)_{\nu\mu} J^\mu(g)$ in null components, (2.86) would be bounded by $\epsilon^{\frac{3}{2}}$ if

$$\begin{aligned} \int_0^t \int_{\Sigma_s} |\tau_+ \rho J^L| dx ds &\lesssim \epsilon^{\frac{3}{2}}, & \int_0^t \int_{\Sigma_s} |\tau_- \rho J^L| dx ds &\lesssim \epsilon^{\frac{3}{2}}, \\ \int_0^t \int_{\Sigma_s} |\tau_+ \alpha J^B| dx ds &\lesssim \epsilon^{\frac{3}{2}} & \text{and} & \int_0^t \int_{\Sigma_s} |\tau_+ \underline{\alpha} J^B| dx ds &\lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\int_0^t \int_{\Sigma_s} \tau_+ |\rho J^L| dx ds \lesssim \int_0^t \|\tau_+ \rho\|_{L^2(\Sigma_s)} \left\| \int_v \frac{v^L}{v^0} |g| dv \right\|_{L^2(\Sigma_s)} ds.$$

Since, by the bootstrap assumption (2.62), $\|\tau_+ \rho\|_{L^2(\Sigma_s)}^2 \lesssim \epsilon \chi(s) (1+s)^\eta$ and, according to Proposition 2.7.13, $\left\| \int_v \frac{v^L}{v^0} |g| dv \right\|_{L^2(\Sigma_s)} \lesssim \epsilon (1+s)^{-\frac{n+1}{2}-\eta}$, it comes that

$$\int_0^t \int_{\Sigma_s} \tau_+^2 |\rho J^L| dx ds \lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{\sqrt{\chi(t)}}{(1+s)^{\frac{n+1}{2}-\eta}} ds \lesssim \epsilon^{\frac{3}{2}}.$$

The other terms are treated similarly.

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \tau_- |\rho J^L| dx ds &\lesssim \int_0^t \|\tau_- \rho\|_{L^2(\Sigma_s)} \left\| \frac{\tau_-}{\tau_+} \int_v \frac{v^L}{v^0} |g| dv \right\|_{L^2(\Sigma_s)} ds \lesssim \epsilon^{\frac{3}{2}}, \\ \int_0^t \int_{\Sigma_s} \tau_+ |\alpha_B J^B| dx ds &\lesssim \int_0^t \|\tau_+ \alpha\|_{L^2(\Sigma_s)} \left\| \int_v \frac{v^B}{v^0} |g| dv \right\|_{L^2(\Sigma_s)} ds \lesssim \epsilon^{\frac{3}{2}}, \end{aligned}$$

$$\int_0^t \int_{\Sigma_s} \tau_- |\alpha_B J^B| dx ds \lesssim \int_0^t \|\tau_- \alpha\|_{L^2(\Sigma_s)} \left\| \int_v \frac{v^B}{v^0} |g| dv \right\|_{L^2(\Sigma_s)} ds \lesssim \epsilon^{\frac{3}{2}}.$$

Denoting $\mathcal{L}_{Z^\beta}(A)$ by B , (2.87) would be bounded by $\epsilon^{\frac{3}{2}}(1+t)^\eta$ if we prove that

$$\int_0^t \int_{\Sigma_s} |B_L J^L| dx ds \lesssim \epsilon^{\frac{3}{2}}(1+t)^\eta, \quad (2.88)$$

$$\int_0^t \int_{\Sigma_s} |B_{\underline{L}} J^{\underline{L}}| dx ds \lesssim \epsilon^{\frac{3}{2}} \quad \text{and} \quad \int_0^t \int_{\Sigma_s} |B_D J^D| dx ds \lesssim \epsilon^{\frac{3}{2}}. \quad (2.89)$$

Let us show (2.89) first. Using Proposition 2.7.13 and the bound on $\tilde{\mathcal{E}}_N[A]$, we have

$$\|B\|_{L^2(\Sigma_s)} \lesssim \sqrt{\epsilon \chi(s)} (1+t)^\eta \quad \text{and} \quad \|J^D\|_{L^2(\Sigma_s)} + \|J^{\underline{L}}\|_{L^2(\Sigma_s)} \lesssim \frac{\epsilon}{(1+s)^{\frac{n-\eta}{2}}}.$$

Hence, by the Cauchy-Schwarz inequality,

$$\int_0^t \int_{\Sigma_s} |B_{\underline{L}} J^{\underline{L}}| + |B_D J^D| dx ds \lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{\sqrt{\chi(s)}}{(1+s)^{2-\eta}} ds \lesssim \epsilon^{\frac{3}{2}}.$$

For (2.88), we have

$$\begin{aligned} \int_0^t \int_{\Sigma_s} |B_L J^L| dx ds &\lesssim \int_0^t \|B_L\|_{L^2(\Sigma_s)} \left\| \int_v |g| dv \right\|_{L^2(\Sigma_s)} ds \\ &\lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{\sqrt{\chi(s)}}{(1+s)^{\frac{n-1}{2}-\eta}} ds. \\ &\lesssim \epsilon^{\frac{3}{2}}(1+t)^\eta. \end{aligned}$$

Hence, if ϵ is small enough and \bar{C} large enough, we have $\mathcal{E}_N^S[F] \leq \bar{C}\epsilon(1+t)^\eta$ for all $t \in [0, T]$.

In view of the above, $\mathcal{E}_{N-2n}^S[F] \leq \bar{C}\epsilon$ on $[0, T]$, for ϵ small enough, would follow if we improve the bound in (2.88) from $\epsilon^{\frac{3}{2}}(1+t)^\eta$ to $\epsilon^{\frac{3}{2}}$, when $|\beta| \leq N - \frac{n+2}{2}$. To do this, we use a pointwise estimate on B_L and we keep J^L in L^1 -norm. By Lemma 2.4.18, we have

$$|B_L(t, x)| \lesssim \frac{\sqrt{\epsilon \chi(t)} (1+t)^\eta}{\tau_+^{\frac{n}{2}}},$$

which implies

$$\begin{aligned} \int_0^t \int_{\Sigma_s} |B_L J^L| dx ds &\lesssim \int_0^t \|B_L\|_{L^\infty(\Sigma_s)} \|g\|_{L^1(\Sigma_s)} ds \\ &\lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{\sqrt{\chi(s)} \log^*(3+s)}{(1+s)^{\frac{n-n}{2}}} ds \lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

This concludes the improvement of the bootstrap assumption (2.63).

2.8 Non existence

We show in this chapter the following proposition. Let us denote $(1, \dots, 1)$ by \bar{u} and we recall that $E^i = F_{0i}$.

Proposition 2.8.1. *Let the dimension n be such that $n \geq 2$ and let $\chi : \mathbb{R} \rightarrow \mathbb{R}_+$ be a function of class C^∞ such that $\chi = 1$ on $]-\infty, 1]$ and $\chi = 0$ on $[3, +\infty[$. We suppose also that χ is decreasing on $[1, 3]$. Let also $M \in \mathbb{R}_+$ such that $M^{-1} = \int_{v \in \mathbb{R}^n} \chi(|v|^2) dv$.*

The Vlasov-Maxwell system (2.1)-(2.3), with two species ($K = 2$), $e_1 = 1$, $e_2 = -1$, $m_1 = 0$, $m_2 \in \mathbb{R}_+$ and the initial data

$$E_0 : x \mapsto 10\chi(2)^{-1}\chi\left(2\frac{r^2}{n}\right)\bar{u}, \quad F_{0ij} = 0 \quad \text{for all } 1 \leq i, j \leq n,$$

$$f_{01} = M \left(\operatorname{div}(E_0) + \|\operatorname{div}(E_0)\|_{L^\infty(\mathbb{R}^n)} \right) \chi \left(\frac{2r^2}{3n} \right) \chi(|v|^2),$$

and

$$f_{02} = M \|\operatorname{div}(E_0)\|_{L^\infty(\mathbb{R}^n)} \chi \left(\frac{2r^2}{3n} \right) \chi(|v|^2),$$

do not admit a C^1 local solution, provided²⁹ $w \mapsto w\chi'(2w^2)$ is not constant on a neighborhood of 1.

Remark 2.8.2. Note that the initial data satisfy the constraint equations. Indeed,

$$\int_v f_{01} - f_{02} dv = \operatorname{div}(E_0) \chi \left(\frac{2r^2}{3n} \right)$$

and $x \mapsto \chi \left(\frac{2r^2}{3n} \right)$ is equal to 1 on the support of E_0 . The other ones, $\nabla_{[i} F_{0jk]} = 0$, are obvious to check.

Remark 2.8.3. There is uniqueness for a such Cauchy problem in the class of the local C^1 functions. Indeed, let (f_1, f_2, F) and (g_1, g_2, G) be two such solutions on $[0, T]$. As f_i and g_i are the unique C^1 solution of $T_{(-1)^{i+1}F}(h) = 0$ and $T_{(-1)^{i+1}G}(h) = 0$ on $[0, T]$, respectively, we obtain with the method of characteristics that they both vanish for $|x| \geq \frac{3}{\sqrt{2}}\sqrt{n} + T$. In view of the wave equations (2.17) and (2.18), the same is true for F and G . All the integrals considered below will then be finite. We have

$$\begin{aligned} T_F(f_q - g_q) &= (G - F)(v, \nabla_v g_q), \\ \nabla^\mu (F - G)_{\mu\nu} &= e^q J(f_q - g_q), \\ \nabla^{\mu*} (F - G)_{\mu\alpha_1 \dots \alpha_{n-2}} &= 0. \end{aligned}$$

Using Propositions 2.3.17 and 2.3.1, we obtain

$$\begin{aligned} h(t) &:= \sum_{q=1}^2 \mathbb{E}_0[f_q - g_q](t) + \sqrt{\mathcal{E}_0^0[F - G](t)} \\ &\lesssim \int_0^t h(s) \left(1 + \left\| \int_v |e^k \nabla_v g_k| dv \right\|_{L^2(\Sigma_s)} \right) ds. \end{aligned}$$

The Grönwall lemma gives us that $h = 0$ on $[0, T]$, implying $(f, F) = (g, G)$.

The strategy of the proof of Proposition 2.8.1 is to construct, for all $T_0 > 0$, a characteristic of the system such that its velocity part vanish in a time less than T_0 . For this, we make crucial use of the colinearity of $y \mapsto E(t, y\vec{u})$ and \vec{u} which is a corollary of the following subsection.

2.8.1 A symmetry property for the Vlasov-Maxwell system

To lighten the notations, we use $x_{(ij)}$, if $i \neq j$, to denote $(x^1, \dots, x^{i-1}, x^j, x^{i+1}, \dots, x^{j-1}, x^i, x^{j+1}, \dots, x^n)$.

Proposition 2.8.4. We consider the n dimensional Vlasov-Maxwell system, with K species,

$$\begin{aligned} T_{m_q}(f_q) + e_q v^0 E^i \partial_{v^i} f_q + e_q v^i F_i^j \partial_{v^j} f_q &= 0, \\ \nabla^\mu F_{\mu\nu} &= e^q J(f_q)_\nu, \\ \nabla^{\mu*} F_{\mu\lambda_1 \dots \lambda_{n-2}} &= 0, \end{aligned}$$

with the initial smooth data $f_q(0, \cdot, \cdot) = f_{q0}$, $F(0, \cdot) = F_0$. We suppose that the initial data satisfy the following symmetry relations

$$\begin{aligned} f_{q0}(x_{(ik)}, v_{(ik)}) &= f_{q0}(x, v), \quad i \neq j, \\ E_0^i(x_{(ik)}) &= E_0^k(x), \quad i \neq k, \\ E_0^i(x_{(kl)}) &= E_0^i(x), \quad l \neq i, \quad k \neq i. \\ (F_{kl})_0(x_{(kl)}) &= -(F_{kl})_0(x), \\ (F_{kl})_0(x_{(ik)}) &= (F_{il})_0(x), \quad l \neq k, i, \\ (F_{kl})_0(x_{(ij)}) &= (F_{kl})_0(x), \quad i \neq k, l, \quad j \neq k, l. \end{aligned}$$

²⁹Note that such a function χ exists. Recall for instance the classical construction of cut-off functions

If there is a unique classical solution (f_1, \dots, f_K, F) on $[0, T[$, then $(f_1(t, \cdot, \cdot), \dots, f_K(t, \cdot, \cdot), F(t, \cdot))$ satisfies also these symmetries.

Proof.

To simplify the notation, we suppose that $K = 1$, $e_q = 1$ and we consider the transposition $\tau = (12)$. We denote $(y^2, y^1, y^3, \dots, y^n)$ by y_τ , m_1 by m and f_1 by f . Let g and G be defined by

$$\begin{aligned} g(t, x, v) &:= f(t, x_\tau, v_\tau), \\ G_{02}(t, x) &:= E^1(t, x_\tau), \\ G_{01}(t, x) &:= E^2(t, x_\tau), \\ G_{0k}(t, x) &:= E^k(t, x_\tau), \quad k \geq 3, \\ G_{12}(t, x) &:= -F_{12}(t, x_\tau), \\ G_{1k}(t, x) &:= F_{2k}(t, x_\tau), \quad k \neq 1, \quad k \neq 2, \\ G_{2k}(t, x) &:= F_{1k}(t, x_\tau), \quad k \neq 1, \quad k \neq 2, \\ G_{kl}(t, x) &:= F_{kl}(t, x_\tau), \quad k, l \geq 3 \end{aligned}$$

and let $D^k = G_{0k}$. We want to prove that $(g, G) = (f, F)$. By assumption, this is true for $t = 0$ and, by uniqueness, it will be true for $t < T$ if we can prove that (g, G) is solution to the same system as (f, F) .

Propagation of symmetry for the Maxwell equations

Let us prove first that $\nabla^\mu G_{\mu\nu} = J(g)_\nu$. As $J(h)^\nu = \int_v \frac{v^\nu}{v^0} h dv$, we have, by the change of variables $v' = v_\tau$,

$$J(g)^1(t, x) = J(f)^2(t, x_\tau), \quad J(g)^2(t, x) = J(f)^1(t, x_\tau)$$

and

$$J(g)^\nu(t, x) = J(f)^\nu(t, x_\tau) \quad \text{if } \nu \neq 1, 2.$$

The equation $\nabla^i G_{i0} = J(g)_0$ then comes from

$$\partial_1 D^1(t, x) = \partial_2 E^2(t, x_\tau), \quad \partial_2 D^2(t, x) = \partial_1 E^1(t, x_\tau) \quad \text{and} \quad \nabla_i E^i = J(f)_0.$$

As

$$\begin{aligned} \nabla^\mu G_{\mu 1}(t, x) &= -\partial_t E^2(t, x_\tau) - \partial_2 (F_{21}(t, x_\tau)) + \sum_{i=3}^n \nabla^i (F_{i2}(t, x_\tau)) \\ &= -\partial_t E^2(t, x_\tau) - \partial_1 F_{21}(t, x_\tau) + \sum_{i=3}^n \nabla^i F_{i2}(t, x_\tau) \\ &= \nabla^\mu F_{\mu 2}(t, x_\tau), \end{aligned}$$

we have $\nabla^\mu G_{\mu 1} = J(g)_1$. The equation $\nabla^\mu G_{\mu 2}(t, x) = J(g)_2$ can be obtained similarly. The remaining equations are obtained from

$$\begin{aligned} \nabla^j G_{jk}(t, x) &= \partial_1 (F_{2k}(t, x_\tau)) + \partial_2 (F_{1k}(t, x_\tau)) + \sum_{i=3}^n \nabla^i (F_{ik}(t, x_\tau)) \\ &= \partial_2 F_{2k}(t, x_\tau) + \partial_1 F_{1k}(t, x_\tau) + \sum_{i=3}^n \nabla^i F_{i2}(t, x_\tau) \\ &= \nabla^j F_{jk}(t, x_\tau) \end{aligned}$$

and $\partial_t D^k(t, x) = \partial_t E^k(t, x_\tau)$, for $k \geq 3$. For the other part of the Maxwell equations, recall from Proposition 2.2.16 that it is equivalent to prove

$$\nabla_{[\lambda} G_{\mu\nu]} = 0.$$

We have

$$\begin{aligned} \nabla_{[1} G_{23]}(t, x) &= \partial_1 (F_{13}(t, x_\tau)) + \partial_2 (F_{32}(t, x_\tau)) - \partial_3 (F_{12}(t, x_\tau)) \\ &= \partial_2 F_{13}(t, x_\tau) + \partial_1 F_{32}(t, x_\tau) + \partial_3 F_{21}(t, x_\tau) \\ &= \nabla_{[2} F_{13]}(t, x_\tau) \\ &= 0. \end{aligned}$$

The other equations can be obtained in the same way.

Propagation of symmetry for the Vlasov equation

We have

$$\begin{aligned} T_m(g)(t, x, v) &= v^1 \partial_2 f(t, x_\tau, v_\tau) + v^2 \partial_1 f(t, x_\tau, v_\tau) + \sum_{\substack{\mu=0 \\ \mu \neq 1,2}}^n v^\mu \partial_\mu f(t, x_\tau, v_\tau) \\ &= T_m(f)(t, x_\tau, v_\tau). \end{aligned}$$

Moreover, as

$$\begin{aligned} \partial_{v^1} g(t, x, v) &= \partial_{v^2} f(t, x_\tau, v_\tau) \quad \text{and} \quad \partial_{v^2} g(t, x, v) = \partial_{v^1} f(t, x_\tau, v_\tau), \\ D^i(t, x) \partial_{v^i} g(t, x, v) &= E^i(t, x_\tau) \partial_{v^i} f(t, x_\tau, v_\tau). \end{aligned}$$

Finally,

$$\begin{aligned} (v^k G_{k1} \partial_{v^1} g)(t, x, v) &= \left(-v^2 F_{21}(t, x_\tau) + \sum_{k=3}^n v^k F_{k2}(t, x_\tau) \right) \partial_{v^2} f(t, x_\tau, v_\tau) \\ &= (v^k F_{k2} \partial_{v^2} f)(t, x_\tau, v_\tau), \end{aligned}$$

and more generally

$$(v^k G_{kj} \partial_{v^j} g)(t, x, v) = (v^k F_{k\tau(j)} \partial_{v^{\tau(j)}} f)(t, x_\tau, v_\tau).$$

We then deduce,

$$T_G(g) = 0, \quad \text{as} \quad T_F(f)(t, x_\tau, v_\tau) = 0.$$

The symmetries are propagated over time

We then proved that (g, G) satisfies the same system as (f, F) . As $(f, F) = (g, G)$ at $t = 0$, we have, by the uniqueness of the solution, that $(f, F) = (g, G)$ for all $t \in [0, T]$. \square

Remark 2.8.5. *More generally, from the above proof, $(f, F) \mapsto (g, G)$ maps C^1 solutions of the Vlasov-Maxwell system to C^1 solutions of the Vlasov-Maxwell system.*

2.8.2 Proof of Proposition 2.8.1

Let us suppose that the system admits a local C^1 solution on $[0, T]$, with $T > 0$, which is then necessarily unique. We will reduce T later if necessary, but we already assume that $T \leq 1$.

Some informations on the electromagnetic fields around \vec{u}

We start by the study of the solution around \vec{u} . Let us introduce $M_0 := 20\chi(2)^{-1}$ and $(B_{ij})_{1 \leq i, j \leq n}$ the 2-form defined by $B_{ij} = F_{ij}$.

Proposition 2.8.6. *Reducing T if necessary, we have the following properties.*

1. *Local bounds on the field: $\forall t \in [0, T]$,*

$$\forall |x| \leq \sqrt{n} + 2T, \quad 1 \leq i \leq 3, \quad 5 \leq E^i(t, x) \leq M_0, \quad |\partial_t E(t, x)| \leq 1$$

and

$$\forall t \in [0, T], \quad |x - \vec{u}| \leq 2T, \quad |B(t, x)| \leq \frac{1}{4}. \quad (2.90)$$

2. *The field is locally-Lipschitz: $\exists L > 0, \forall t \in [0, T], |x|, |y| \leq \sqrt{n} + 2T$,*

$$|E(t, x) - E(t, y)| + |B(t, x) - B(t, y)| \leq L|x - y|. \quad (2.91)$$

3. *Specific behaviour along the \vec{u} -direction:*

$$\forall y \in \mathbb{R}, \quad t \in [0, T], \quad E(t, y\vec{u}) = E^1(t, y\vec{u})\vec{u} \quad \text{and} \quad B(t, y\vec{u}) = 0.$$

Proof. In view of the initial data, we have $B(0, \vec{u}) = 0$ and

$$\forall |y| \leq \sqrt{n}, 1 \leq i \leq n, \quad 10 \leq E^i(0, y) \leq \frac{M_0}{2}, \quad \partial_t E(0, y) = 0.$$

The point 1 then ensues, taking T smaller if necessary, from the uniform continuity of the electromagnetic field on every compact subset of $[0, T] \times \mathbb{R}^n$. The point 2 comes from the mean value theorem, as E and B are C^1 and the point 3 follows from Proposition 2.8.4. \square

The method of characteristics fails

Let us denote by $(X(s, t, x, v), V(s, t, x, v))$ the value at s of the characteristic, for the transport equation (2.1) satisfied by f_1 , which was equal to (x, v) at t . Let $\eta \in]0, T[$ and

$$X_\eta : (s, t) \mapsto X(s, t, \vec{u}, \eta \vec{u}), \quad V_\eta : (s, t) \mapsto V(s, t, \vec{u}, \eta \vec{u}).$$

We now fix $t \in [0, T[$. $(X_\eta(\cdot, t), V_\eta(\cdot, t))$ is well defined on a neighborhood of t and we have, denoting $\frac{v}{|v|}$ by \widehat{v} ,

$$\frac{dX_\eta(\cdot, t)}{ds}(s) = \widehat{V}_\eta(s), \tag{2.92}$$

$$\frac{dV_\eta^j(\cdot, t)}{ds}(s) = E^j(s, X_\eta(s, t)) + \widehat{V}_\eta^i(s) F_i^j(s, X_\eta(s, t)). \tag{2.93}$$

Lemma 2.8.7. $X_\eta(\cdot, t)$, $V_\eta(\cdot, t)$ and E (along $X_\eta(\cdot, t)$) stay collinear to \vec{u} . We have, as long as V_η stay positive, $X_\eta(s, t) = \left(1 + \frac{1}{\sqrt{n}}(s - t)\right) \vec{u}$ and

$$V_\eta(s, t) = \eta \vec{u} + \int_t^s E^1\left(s', \left(1 + \frac{1}{\sqrt{n}}(s - t)\right) \vec{u}\right) ds' \vec{u}.$$

Proof. We start by a change of coordinates. We consider an orthonormal system $(u_i)_{1 \leq i \leq n}$ such that $u_1 = \frac{1}{\sqrt{n}} \vec{u}$ and we denote by \widetilde{X}^i and \widetilde{V}^i the coordinates of $X_\eta(\cdot, t)$ and $V_\eta(\cdot, t)$ in this basis. Then, for all $1 \leq i \leq n$,

$$\frac{d\widetilde{X}^i}{ds}(s) = \frac{\widetilde{V}^i(s)}{|V|(s)}$$

and, for $i \geq 2$, $\widetilde{X}^i(0) = 0$ and $\widetilde{V}^i(0) = 0$. We remark, using Proposition 2.8.6, that if $\widetilde{X}^i = 0$ for $i \geq 2$, then $E(s, X_\eta(s, t)) = E^1(s, X_\eta(s, t)) \vec{u}$ and $B(s, X_\eta(s, t)) = 0$. Consider now the solution of the following system

$$\begin{aligned} \frac{dr}{ds} &= \frac{w}{|w|}, \\ \frac{dw}{ds} &= \sqrt{n} E^1\left(s, \frac{r}{\sqrt{n}}, \dots, \frac{r}{\sqrt{n}}\right), \end{aligned}$$

with the initial data $r(t) = \sqrt{n}$ and $w(t) = \eta \sqrt{n}$. The solution exists as long long as $w \neq 0$ and we have

$$\frac{r(s)}{\sqrt{n}} = 1 - \frac{t - s}{\sqrt{n}} \quad \text{and} \quad \frac{w(s)}{\sqrt{n}} = \eta + \int_t^s E^1\left(s', 1 - \frac{t - s'}{\sqrt{n}}, \dots, 1 - \frac{t - s'}{\sqrt{n}}\right) ds'.$$

By uniqueness of the solution of the system (2.92)-(2.93), we have

$$(\widetilde{X}^1, \dots, \widetilde{X}^n, \widetilde{V}^1, \dots, \widetilde{V}^n) = (r, 0, \dots, 0, w, 0, \dots, 0),$$

which implies the result. \square

We now try to estimate the time when V_η vanishes.

Proposition 2.8.8. *The exists $0 < \eta_0 < T$ such that for all $\eta \in]0, \eta_0[$, there exists T_η such that if $t < T_\eta$, $(X_\eta(\cdot, t), V_\eta(\cdot, t))$ is well defined on $[0, t]$ and if $T_\eta \leq t < T$ there exists $\tau_\eta(t) \leq t$ such that $(X_\eta(\cdot, t), V_\eta(\cdot, t))$ is well defined on $]t - \tau_\eta(t), t]$ and*

$$\lim_{s \rightarrow (t - \tau_\eta(t))^+} V_\eta(s, t) = 0, \quad \lim_{s \rightarrow (t - \tau_\eta(t))^+} X_\eta(s, t) = \left(1 - \frac{\tau_\eta(t)}{\sqrt{n}}\right) \vec{u}.$$

Moreover, $t \mapsto t - \tau_\eta(t)$ is in $C^0([T_\eta, T]) \cap C^1(]T_\eta, T[)$, vanishes at T_η , and such that

$$\forall t \in]T_\eta, T[, \quad \frac{4}{M_0} \leq \frac{\partial(t - \tau_\eta)}{\partial t}(t) \leq \frac{M_0 + 1}{5}.$$

Proof.

We fix $\eta \in]0, T[$. Noting, by (2.92), that

$$|X_\eta(s, t) - \vec{u}| \leq |t - s| \leq T,$$

we obtain by Proposition 2.8.6, as X_η and \vec{u} are collinear, that $E(s, X_\eta(s, t)) = E^1(s, X_\eta(s, t))\vec{u}$. Hence, if $t \in [0, T[$, only two situations can occur. Either $(X_\eta(\cdot, t), V_\eta(\cdot, t))$ is well defined on $[0, t]$, or there exists $\tau_\eta(t) < t$ such that

$$\lim_{s \rightarrow (t - \tau_\eta(t))^+} V_\eta(s, t) = 0,$$

and the characteristic is well defined on $]t - \tau_\eta(t), t]$. Now, consider

$$g_\eta : (s, t) \mapsto \eta + \int_t^s E^1\left(s', \left(1 - \frac{t - s'}{\sqrt{n}}\right) \vec{u}\right) ds'$$

so that, by Lemma 2.8.7, if $t \in]0, T[$ and s is near to t , $g_\eta(s, t)$ is equal to $V_\eta^i(s, t)$ (for all $1 \leq i \leq n$). For all $t \in]0, T[$, $g_\eta(\cdot, t)$ strictly increases on $[0, t]$, as $E^1 > 0$ by Proposition 2.8.6. As

$$\int_t^s E^1\left(s', \left(1 - \frac{t - s'}{\sqrt{n}}\right) \vec{u}\right) ds' = - \int_0^{t-s} E^1\left(t - s', \left(1 - \frac{s'}{\sqrt{n}}\right) \vec{u}\right) ds'$$

and

$$\frac{\partial g_\eta}{\partial t}(s, t) = -E^1\left(s, \left(1 - \frac{t - s}{\sqrt{n}}\right) \vec{u}\right) - \int_0^{t-s} \partial_t E^1\left(t - s', \left(1 - \frac{s'}{\sqrt{n}}\right) \vec{u}\right) ds',$$

we have

$$\frac{\partial g_\eta}{\partial t}(s, t) \leq -4,$$

so that $g_\eta(s, \cdot)$ is strictly decreasing on $[s, T[$. Moreover, by the bounds given on E^1 in Proposition 2.8.6, if $t < \frac{\eta}{M_0}$, $g_\eta(\cdot, t)$ does not vanish on $[0, t]$ and vanishes exactly one time, in $t - \tau_\eta(t)$, if $t \geq \frac{\eta}{5}$. Then, if η is small enough, there exists $t \in]0, T[$ such that $g_\eta(\cdot, t)$ vanishes in $t - \tau_\eta(t)$. Let t_1 be a such time and let $t_2 > t_1$. We have

$$0 = g_\eta(t_1 - \tau_\eta(t_1), t_1) > g_\eta(t_1 - \tau_\eta(t_1), t_2),$$

implying the existence of $t_2 - \tau_\eta(t_2)$ and $t_1 - \tau_\eta(t_1) < t_2 - \tau_\eta(t_2)$, since

$$g_\eta(t_1 - \tau_\eta(t_1), t_2) < 0 = g_\eta(t_2 - \tau_\eta(t_2), t_2).$$

Hence, T_η exists³⁰ and $t \mapsto t - \tau_\eta(t)$ strictly increases on $[T_\eta, T[$, vanishes in T_η and tends to zero as $t \rightarrow T_\eta$. The fact that $t \mapsto t - \tau_\eta(t)$ is in $C^1(]T_\eta, T[)$ follows from the implicit function theorem, as $g_\eta(t - \tau_\eta(t), t) = 0$ and $\frac{\partial g_\eta}{\partial s}(s, t) \geq 5$. Furthermore, dropping the dependance in t of τ_η ,

$$\frac{\partial(t - \tau_\eta)}{\partial t}(t) = \frac{E^1\left(t - \tau_\eta, \left(1 - \frac{\tau_\eta}{\sqrt{n}}\right) \vec{u}\right) + \int_0^{\tau_\eta} \partial_t E^1\left(t - s', \left(1 - \frac{s'}{\sqrt{n}}\right) \vec{u}\right) ds'}{E^1\left(t - \tau_\eta, \left(1 - \frac{\tau_\eta}{\sqrt{n}}\right) \vec{u}\right)},$$

which, by Proposition 2.8.6, implies the last statement. □

Remark 2.8.9. *Note that, if $0 < \eta < \eta_0$, $\tau_\eta(T_\eta) = T_\eta$ and then $g_\eta(0, T_\eta) = 0$.*

Later, we will use again that $g_\eta(0, \cdot)$ is strictly decreasing on $[0, T[$.

³⁰More precisely, $T_\eta = \sup\{t \in]0, T[\mid g_\eta(\cdot, t) > 0 \text{ on } [0, t]\}$.

The contradiction

We fix again $\eta \in]0, \eta_0[$. As $V_\eta(\cdot, t)$ is not defined on $[0, t - \tau_\eta(t)]$ if $t > T_\eta$, we cannot directly express $f_1(t, \vec{u}, \eta \vec{u})$ in terms of f_{01} by the method of the characteristics.

If $t \geq T_\eta$, we extend $X_\eta(\cdot, t)$ and $V_\eta(\cdot, t)$ on $[0, t - \tau_\eta(t)]$ by

$$X_\eta(s, t) = \left(1 + \frac{t - s - 2\tau_\eta(t)}{\sqrt{n}}\right) \vec{u} \quad \text{and} \quad V_\eta(s, t) = \eta \vec{u} + \int_t^s E(s', X_\eta(s', t)) ds'.$$

Remark 2.8.10. If $t > t - \tau_\eta(t)$, $\frac{dX_\eta}{ds}(s, t) = \frac{\vec{u}}{\sqrt{n}}$. We extend $X_\eta(\cdot, t)$ on $[0, t - \tau_\eta(t)]$ in order that

$$\frac{dX_\eta}{ds}(s, t) = -\frac{\vec{u}}{\sqrt{n}}.$$

We then extend $V_\eta(\cdot, t)$ such that (2.93) remains true on $[0, t - \tau_\eta(t)]$.

We have the following result.

Lemma 2.8.11.

$$\forall t \in [0, T[, \quad f_1(t, \vec{u}, \eta \vec{u}) = f_{01}(X_\eta(0, t), V_\eta(0, t)). \quad (2.94)$$

Proof. If $t < T_\eta$, this follows from the method of characteristics. In order to prove the result for $t \geq T_\eta$, we consider $\epsilon > 0$, $v_\epsilon = (0, \dots, 0, \epsilon)$,

$$X_{\eta,t}^\epsilon : s \mapsto X(s, t - \tau_\eta(t), X_\eta(t - \tau_\eta(t), t), v_\epsilon)$$

and

$$V_{\eta,t}^\epsilon : t \mapsto V(s, t - \tau_\eta(t), X_\eta(t - \tau_\eta(t), t), v_\epsilon).$$

Proposition 2.8.6 gives us that $X_{\eta,t}^\epsilon$ and $V_{\eta,t}^\epsilon$ are well defined on $[0, T[$. Indeed, as, by (2.92),

$$|X_{\eta,t}^\epsilon(s) - \vec{u}| \leq |X_{\eta,t}^\epsilon(s) - X_\eta(t - \tau_\eta(t), t)| + |X_\eta(t - \tau_\eta(t), t) - \vec{u}| \leq 2T,$$

it comes

$$\forall 1 \leq i \leq n, \quad 5 \leq E^i(s, X_{\eta,t}^\epsilon(s)) \leq M_0 \quad \text{and} \quad |B(s, X_{\eta,t}^\epsilon(s))| \leq \frac{1}{4},$$

so that $V_{\eta,t}^\epsilon$ cannot vanish. Now, the method of characteristics gives us, for all $t \in [0, T[$,

$$f_1(t, X_{\eta,t}^\epsilon(t), V_{\eta,t}^\epsilon(t)) = f_{01}(X_{\eta,t}^\epsilon(0), V_{\eta,t}^\epsilon(0)).$$

Then, the result, for $t \geq T_\eta$, follows from the continuity of f_1 and the following proposition. □

Proposition 2.8.12. We have

$$\lim_{\epsilon \rightarrow 0} \|X_\eta(\cdot, t) - X_{\eta,t}^\epsilon\|_{L^\infty([0,t])} + \|V_\eta(\cdot, t) - V_{\eta,t}^\epsilon\|_{L^\infty([0,t])} = 0.$$

Proof.

On the one hand, as

$$\forall v, w \in \mathbb{R}^n \setminus \{0\}, \quad \left| \frac{v}{|v|} - \frac{w}{|w|} \right| \leq \frac{2}{|w|} |v - w|, \quad (2.95)$$

we have

$$|X_\eta(s, t) - X_{\eta,t}^\epsilon(s)| \leq \left| \int_{t-\tau_\eta(t)}^s \frac{2}{|V_{\eta,t}^\epsilon(w)|} |V_\eta(w, t) - V_{\eta,t}^\epsilon(w)| dw \right|.$$

On the other hand, note first that for $s < T$ and $|x|, |y| \leq \sqrt{n} + 2T$, we have, by the local Lipschitz property of the electromagnetic field (2.91),

$$|E(s, x) - E(s, y) + \widehat{v}^i B_i(s, x) - \widehat{w}^i B_i(s, y)| \leq L|x - y| + |\widehat{v} - \widehat{w}| |B(s, x)|.$$

Then, using (2.93), (2.95) and the bound (2.90) on the magnetic field,

$$|V_\eta(s, t) - V_{\eta,t}^\epsilon(s)| \leq \epsilon + \left| \int_{t-\tau_\eta(t)}^s L |X_\eta(w, t) - X_{\eta,t}^\epsilon(w)| + \frac{1}{2|V_{\eta,t}^\epsilon(w)|} |V_\eta(w, t) - V_{\eta,t}^\epsilon(w)| dw \right|.$$

Hence, by the Grönwall lemma, for all $s \in [0, T[$,

$$|X_\eta(s, t) - X_{\eta, t}^\epsilon(s)| + |V_\eta(s, t) - V_{\eta, t}^\epsilon(s)| \leq \epsilon \exp \left(\left| \int_{t-\tau_\eta(t)}^s L + \frac{5}{2|V_{\eta, t}^\epsilon(w)|} dw \right| \right). \quad (2.96)$$

We now prove that, $\exists a > 0, b > \frac{5}{2}$ such that $\forall w \in [0, T[$,

$$|V_{\eta, t}^\epsilon(w)| \geq a\epsilon + b|t - \tau_\eta(t) - w|. \quad (2.97)$$

Recall that $5 \leq E^j \leq M_0$ and $|B| \leq 1$ around \vec{u} (see Proposition 2.8.6) and

$$V_{\eta, t}^{\epsilon, j}(w) = v_\epsilon^j + \int_{t-\tau_\eta(t)}^w E^j(s, X_{\eta, t}^\epsilon(s)) + \widehat{V}_{\eta, t}^{\epsilon, i}(s) B_i^j(s, X_{\eta, t}^\epsilon(s)) ds.$$

Hence, we have.

- If $w \geq t - \tau_\eta(t)$,

$$V_{\eta, t}^{\epsilon, j}(w) \geq \delta_{j, n} \epsilon + (5 - 1)(w - t + \tau_\eta(t))$$

so that

$$|V_{\eta, t}^\epsilon(w)|^2 \geq \epsilon^2 + n(5 - 1)^2(w - t + \tau_\eta(t))^2 \geq \frac{1}{2}(\epsilon + 4\sqrt{n}|w - t + \tau_\eta(t)|)^2.$$

- If $t - \tau_\eta(t) - \frac{\epsilon}{2(M_0+1)} \leq w \leq t - \tau_\eta(t)$,

$$V_{\eta, t}^{\epsilon, j}(w) \leq -(5 - 1)(t - \tau_\eta(t) - w) \text{ for } 1 \leq j \leq n - 1 \text{ and } V_{\eta, t}^{\epsilon, n}(w) \geq \frac{\epsilon}{2},$$

so

$$\begin{aligned} |V_{\eta, t}^\epsilon(w)|^2 &\geq \frac{\epsilon^2}{4} + (n - 1)(5 - 1)^2(t - \tau_\eta(t) - w)^2 \\ &\geq \frac{1}{2} \left(\frac{\epsilon}{2} + 4\sqrt{n-1}|t - \tau_\eta(t) - w| \right)^2 \end{aligned}$$

- If $w \leq t - \tau_\eta(t) - \frac{\epsilon}{2(M_0+1)}$, then, for $1 \leq j \leq n - 1$,

$$V_{\eta, t}^{\epsilon, j}(w) \leq -(5 - 1)(t - \tau_\eta(t) - w) \leq -\frac{4}{3} \left| \frac{\epsilon}{2(M_0 + 1)} + 2(t - \tau_\eta(t) - w) \right|.$$

It comes,

$$|V_{\eta, t}^\epsilon(w)|^2 \geq \frac{16}{9}(n - 1) \left(\frac{\epsilon}{2(M_0 + 1)} + 2|t - \tau_\eta(t) - w| \right)^2.$$

Inequality (2.97) then holds with

$$a = \min \left(\frac{1}{2\sqrt{2}}, \frac{2\sqrt{n-1}}{3(M_0 + 1)} \right) \text{ and } b = \frac{8}{3}\sqrt{n-1}.$$

We now prove that the right hand side of (2.96) tends uniformly to zero in s , on $[0, t]$. As, by (2.97),

$$\left| \int_{t-\tau_\eta(t)}^s \frac{5}{2|V_{\eta, t}^\epsilon(w)|} dw \right| \leq \frac{5}{2b} \log \left(1 + \frac{b \max(t - \tau_\eta(t), \tau_\eta(t))}{a\epsilon} \right),$$

we have, since $\max(t - \tau_\eta(t), \tau_\eta(t)) \leq t$,

$$\epsilon \exp \left(\left| \int_{t-\tau_\eta(t)}^s \frac{5}{2|V_{\eta, t}^\epsilon(w)|} dw \right| \right) \leq \exp \left(\frac{2b - 5}{2b} \log(\epsilon) + \frac{5}{2b} \log \left(\epsilon + \frac{bt}{a} \right) \right).$$

We then deduce, as $2b > 5$, that

$$\lim_{\epsilon \rightarrow 0} \max_{s \in [0, t]} \epsilon \exp \left(\left| \int_{t-\tau_\eta(t)}^s L + \frac{5}{2|V_{\eta, t}^\epsilon(w)|} dw \right| \right) = 0,$$

which implies the result. \square

Differentiating (2.94) in t for $t < T_\eta$ gives us

$$\partial_t f_1(t, \bar{u}, \eta \bar{u}) = \sum_{i=1}^n -\frac{1}{\sqrt{n}} \partial_i f_{01} \left(\left(1 - \frac{t}{\sqrt{n}}\right) \bar{u}, V_\eta(0, t) \right) + \frac{dV_\eta^i}{dt}(0, t) \partial_{v^i} f_{01} \left(\left(1 - \frac{t}{\sqrt{n}}\right) \bar{u}, V_\eta(0, t) \right).$$

Doing the same for $t > T_\eta$ gives

$$\begin{aligned} \partial_t f_1(t, \bar{u}, \eta \bar{u}) &= \sum_{i=1}^n \frac{dV_\eta^i}{dt}(0, t) \partial_{v^i} f_{01} \left(\left(1 + \frac{t - 2\tau_\eta(t)}{\sqrt{n}}\right) \bar{u}, V_\eta(0, t) \right) \\ &\quad + \frac{1}{\sqrt{n}} \left(2 \frac{\partial(t - \tau_\eta)}{\partial t}(t) - 1 \right) \partial_i f_{01} \left(\left(1 + \frac{t - 2\tau_\eta(t)}{\sqrt{n}}\right) \bar{u}, V_\eta(0, t) \right). \end{aligned}$$

Recall from Proposition 2.8.8 that $t \mapsto \frac{\partial(t - \tau_\eta)}{\partial t}(t)$ is defined on $]T_\eta, T[$ and takes its values in $[\frac{4}{M_0}, \frac{M_0+1}{5}]$. Hence, there exists a sequence (t_n) , with $t_n \rightarrow T_\eta$, such that,

$$\exists C > 0, \quad \lim_{t_n \rightarrow T_\eta} \frac{\partial(t - \tau_\eta)}{\partial t}(t_n) = C.$$

Using that f_1 and f_{01} are C^1 and taking the limit $t_n \rightarrow T_\eta$ in the two last equations, we obtain

$$2C \sum_{i=1}^n \partial_i f_{01} \left(\left(1 - \frac{T_\eta}{\sqrt{n}}\right) \bar{u}, 0 \right) = 0$$

and thus

$$\sum_{i=1}^n \partial_i f_{01} \left(\left(1 - \frac{T_\eta}{\sqrt{n}}\right) \bar{u}, 0 \right) = 0. \quad (2.98)$$

Finally, we need the following proposition.

Proposition 2.8.13. *The function $\eta \mapsto T_\eta$ is defined on $]0, \eta_0[$, strictly increasing, continuous and such that*

$$\lim_{\eta \rightarrow 0} T_\eta = 0.$$

Proof.

We recall (see Remark (2.8.9)) that T_η is defined by the implicit equation

$$g_\eta(0, T_\eta) = \eta - \int_0^{T_\eta} E^1 \left(w, \left(1 - \frac{T_\eta - w}{\sqrt{n}}\right) \bar{u} \right) dw = 0.$$

Let $0 < \eta_1 < \eta_2 < T$. We have

$$g_{\eta_1}(0, T_{\eta_2}) < g_{\eta_2}(0, T_{\eta_2}) = 0,$$

so

$$g_{\eta_1}(0, T_{\eta_2}) < g_{\eta_1}(0, T_{\eta_1}) = 0.$$

Since $g_{\eta_1}(0, \cdot)$ strictly decreases (see again Remark (2.8.9)), $T_{\eta_2} > T_{\eta_1}$, which means that $\eta \mapsto T_\eta$ is strictly increasing. As E^1 is bounded away from 0 on the domain of integration, T_η tends to 0 as $\eta \rightarrow 0$. The continuity ensues from the implicit function theorem. \square

Using Equation (2.98) and the last proposition, we can find $T^* > 0$ such that $w \mapsto f_{01}((1-w)\bar{u}, 0)$ is constant on $]0, T^*[$. However, there exists $C_0 > 0$ and $C_1 > 0$ such that

$$f_{01}((1-w)\bar{u}, 0) = C_0 + C_1(1-w)\chi'(2(1-w)^2)$$

for all $0 < w < T^*$, and $w \mapsto (1-w)\chi'(2(1-w)^2)$ is not constant around 0.

Chapter 3

Sharp asymptotics for the solutions of the three-dimensional massless Vlasov-Maxwell system with small data

Abstract

This paper is concerned with the asymptotic properties of the small data solutions to the massless Vlasov-Maxwell system in $3d$. We use vector field methods to derive almost optimal decay estimates in null directions for the electromagnetic field, the particle density and their derivatives. No compact support assumption in x or v is required on the initial data and the decay in v is in particular initially optimal. Consistently with Proposition 8.1 of [4], the Vlasov field is supposed to vanish initially for small velocities. In order to deal with the slow decay rate of the solutions near the light cone and to prove that the velocity support of the particle density remains bounded away from 0, we make crucial use of the null properties of the system.

3.1 Introduction

This article is part of a series of works concerning the asymptotic behavior of small data solutions to the Vlasov-Maxwell equations. The system is a classical model for collisionless plasma and is given, for K species of particles, by¹

$$\begin{aligned} \sqrt{m_k^2 + |v|^2} \partial_t f_k + v^i \partial_i f_k + e_k \left(\sqrt{m_k^2 + |v|^2} F_0^j + v^q F_q^j \right) \partial_{v^j} f_k &= 0, \\ \nabla^\mu F_{\mu\nu} &= \sum_{k=1}^K e_k \int_{v \in \mathbb{R}^3} \frac{v_\nu}{\sqrt{m_k^2 + |v|^2}} f_k dv, \\ \nabla^{\mu*} F_{\mu\nu} &= 0, \end{aligned}$$

where

- $m_k \geq 0$ is the mass of the particles of the species k and $e_k \neq 0$ is their charge.
- The function $f_k(t, x, v)$ is the particle density of the species k , where $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$ if $m_k = 0$ and $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ otherwise.
- The 2-form $F(t, x)$, with $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, is the electromagnetic field and $*F(t, x)$ is its Hodge dual.

In [4], we studied the massless Vlasov-Maxwell system in high dimensions ($n \geq 4$) and we proved that if the particle densities initially vanish for small velocities and if certain weighted L^1 and L^2 norms of the initial data are small enough, then the unique classical solution to the system exists globally in time. Moreover, as the smallness assumption only concerns L^1 and L^2 norms, no compact support assumption in x or v was required. We also obtained optimal pointwise decay estimates on the velocity averages of f_k and their derivatives as well as improved decay estimates on the null components of the electromagnetic field and

¹We will, throughout this article, use the Einstein summation convention so that $v^i \partial_i f = \sum_{i=1}^3 v^i \partial_i f$. A sum on latin letters starts from 1 whereas a sum on greek letters starts from 0.

its derivatives. In the same article, we also proved that there exists smooth initial data such that the particle densities do not vanish for small velocities and for which (3.1)-(3.3) does not admit a local classical solution².

Similar results for the massive Vlasov-Maxwell system in high dimensions are also obtained in [4]. A main difference however is that f_k does not have to be supported away from $v = 0$. The $3d$ massive case requires a refinement of our method and will be treated in [5]. We will also study the solutions of (3.1)-(3.3) in the exterior of a light cone. The strong decay satisfied by f_k in such a region will allow us to lower the initial decay hypothesis on the electromagnetic field and to obtain asymptotics on the solutions in a simpler way than for the whole spacetime. This will be done in [7].

In this paper, we study the asymptotic properties of the small data solutions to the three-dimensional massless Vlasov-Maxwell, so that $m_k = 0$. We start with optimal decay in v on the particle densities in the sense that we merely suppose $f_k(0, x, \cdot)$ to be integrable in v , which is a necessary condition for the source term of the Maxwell equations to be well defined. In massive Vlasov systems, powers of $|v|$ are often lost in order to gain time decay or to exploit null properties³. Our assumptions will force us to better understand the null structure of the equations. In fact, one of the goal of this article is to describe in full details the null structure of the system, which appears to be fundamental for proving integrability and controlling the velocity support of the particle density.

In view of their physical meaning, the functions f_k are usually supposed non negative. However, as their signs play no role in this paper and since we will consider neutral plasmas, we suppose for simplicity that $K = 1$ and we do not restrict the values of f_1 to \mathbb{R}_+ . We also normalize the charge e_1 to 1 and we denote f_1 by f . The system can then be rewritten as

$$|v|\partial_t f + v^i \partial_i f + (|v|F_0^j + v^q F_q^j) \partial_{v^j} f = 0, \quad (3.1)$$

$$\nabla^\mu F_{\mu\nu} = J(f)_\nu := \int_{v \in \mathbb{R}^3} \frac{v_\nu}{|v|} f dv, \quad (3.2)$$

$$\nabla^\mu {}^*F_{\mu\nu} = 0. \quad (3.3)$$

Note that we can recover the more common form of the Vlasov-Maxwell system using the relations

$$E^i = F_{0i} \quad \text{and} \quad B^i = -{}^*F_{0i},$$

so that the equations (3.1)-(3.3) can be rewritten as

$$\begin{aligned} |v|\partial_t f + v^i \partial_i f + (|v|E + v \times B) \cdot \nabla_v f &= 0, \\ \nabla \cdot E &= \int_{v \in \mathbb{R}^3} f dv, & \partial_t E^j &= (\nabla \times B)^j - \int_{v \in \mathbb{R}^3} \frac{v^j}{|v|} f dv, \\ \nabla \cdot B &= 0, & \partial_t B &= -\nabla \times E. \end{aligned}$$

We choose to work with a neutral plasma to simplify the proof but the case of a non zero total charge will be covered in [5] and [7].

3.1.1 Previous results on small data solutions for the massive Vlasov-Maxwell system

Global existence for small data in dimension 3 was first established by Glassey-Strauss in [24] under a compact support assumption (in space and in velocity). In [22], a similar result is obtained for the nearly neutral case. The compact support assumption in v is removed in [44] but the data still have to be compactly supported in space. Note that none of these results contain estimates on $\partial_{\mu_1} \dots \partial_{\mu_k} \int_v f dv$ and the optimal decay rate on $\int_v f dv$ is not obtained by the method of [44]. They all proved decay estimates on the electromagnetic field up to first order derivatives.

In [4], we used vector field methods, developed in [11] for the electromagnetic field and [18] for the Vlasov field, in order to remove all compact support assumptions for the dimensions $n \geq 4$. We then derived (almost) optimal decay on the solutions of the system and their derivatives and we described precisely the behavior of the null components of F .

Recently, Wang proved in [50] a similar result for the $3d$ case. Using both vector field method and Fourier analysis, he replaced the compact support assumption by strong polynomial decay hypotheses in (x, v) on f and obtained optimal pointwise decay estimates on $\int_v f dv$ and its derivatives.

²Note that this result holds for dimensions $n \geq 2$.

³See for instance [4] for the Vlasov-Maxwell system, where we used the inequality $1 \leq 4v^0 v^L$ in order to take advantage of decay in $t - r$.

3.1.2 Previous works on Vlasov systems using vector field methods

Properties of small data solutions of other Vlasov systems were obtained recently using vector field methods. First on the Vlasov-Nordström system, in [18] and [17], and the Vlasov-Poisson system (see [46]). Vector field methods led to a proof of the stability of the Minkowski spacetime for the Einstein-Vlasov system, obtained independently by [16] and [26].

Note that vector field methods can also be used to derive integrated decay for solutions to the massless Vlasov equation on curved background such as slowly rotating Kerr spacetime (see [1]).

3.1.3 Statement of the main result

The following theorem is the main result of this paper. For the notations not yet defined, see Section 3.2.

Theorem 3.1.1. *Let $N \geq 10$, $\epsilon > 0$ and (f^0, F^0) an initial data set for the Vlasov-Maxwell equations (3.1)-(3.3) satisfying the smallness assumption⁴*

$$\sum_{|\beta|+|\kappa|\leq N+3} \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} (1+|x|)^{|\beta|+2} (1+|v|)^{|\kappa|} |\partial_x^\beta \partial_v^\kappa f^0| dv dx + \sum_{|\gamma|\leq N+2} \int_{x \in \mathbb{R}^3} (1+|x|)^{2|\gamma|+2} |\nabla_{\partial_x^2} F^0|^2 dx \leq \epsilon,$$

the neutral hypothesis

$$\int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} f^0 dv dx = 0 \quad (3.4)$$

and the support assumption

$$\forall 0 < |v| \leq 3, \quad f^0(\cdot, v) = 0.$$

There exists $C > 0$ and $\epsilon_0 > 0$ such that if $0 \leq \epsilon \leq \epsilon_0$, then the unique classical solution (f, F) of the system which satisfies $f(t=0) = f^0$ and $F(t=0) = F^0$ is a global solution and verifies the following estimates.

- Energy bound for the electromagnetic field F : $\forall t \in \mathbb{R}_+$,

$$\sum_{\substack{Z^\gamma \in \mathbb{K}^{|\gamma|} \\ |\gamma| \leq N}} \int_{\mathbb{R}^3} \tau_+^2 \left(|\alpha(\mathcal{L}_{Z^\gamma}(F))|^2 + |\rho(\mathcal{L}_{Z^\gamma}(F))|^2 + |\sigma(\mathcal{L}_{Z^\gamma}(F))|^2 \right) + \tau_-^2 |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|^2 dx \leq C\epsilon \log^4(3+t).$$

- Sharp pointwise decay estimates for the null components of $\mathcal{L}_{Z^\gamma}(F)$: $\forall |\gamma| \leq N-2$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$\begin{aligned} |\rho(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log^2(3+t)}{\tau_+^2 \tau_-^{\frac{1}{2}}}, & |\alpha(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log^2(3+t)}{\tau_+^{\frac{5}{2}}}, \\ |\sigma(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log^2(3+t)}{\tau_+^2 \tau_-^{\frac{1}{2}}}, & |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log^{\frac{5}{2}}(1+\tau_-)}{\tau_+ \tau_-^{\frac{3}{2}}}. \end{aligned}$$

- Energy bound for the particle density: $\forall t \in \mathbb{R}_+$,

$$\sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|} \\ |\beta| \leq N}} \sum_{z \in \mathbf{k}_0} \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} |z \widehat{Z}^\beta f|(t, x, v) dv dx \leq C\epsilon \log(3+t).$$

- Vanishing property for small velocities:

$$\forall (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}), \quad |v| \leq 1 \Rightarrow f(t, x, v) = 0.$$

- Sharp pointwise decay estimates for the velocity averages of $\widehat{Z}^\beta f$: $\forall |\beta| \leq N-5$, $z \in \mathbf{k}_0$,

$$\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad \int_{v \in \mathbb{R}^3} |z^2 \widehat{Z}^\beta f| dv \lesssim \frac{\epsilon}{\tau_+^2 \tau_-}.$$

Remark 3.1.2. *One can prove a similar result if f^0 vanishes for the velocities v such that $|v| \leq R$, with $R > 0$ (ϵ_0 would then also depend on R).*

⁴We could avoid any hypotheses on the derivatives of order $N+1$ and $N+2$ of F^0 (see Remark 3.7.6 for more details).

Remark 3.1.3. We say that (f^0, F^0) is an initial data set for the Vlasov-Maxwell system if the function $f^0 : \mathbb{R}_x^3 \times (\mathbb{R}_v^3 \setminus \{0\}) \rightarrow \mathbb{R}$ and F^0 are both sufficiently regular and satisfy the constraint equations

$$\nabla^i (F^0)_{i0} = - \int_{v \in \mathbb{R}^3} f^0 dv \quad \text{and} \quad \nabla^i (*F^0)_{i0} = 0.$$

Remark 3.1.4. The neutral hypothesis (3.4) is a necessary condition for $\int_{\mathbb{R}^3} (1+r)^2 |F|^2 dx$ to be finite. This means that, for a sufficiently regular solution to the Vlasov-Maxwell system (f, F) , the total electromagnetic charge

$$Q(t) := \lim_{r \rightarrow +\infty} \int_{\mathbb{S}_{t,r}} \frac{x^i}{r} F_{0i} d\mathbb{S}_{t,r} = \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} f dv dx,$$

which is a conserved quantity in t , vanishes. More precisely, if $Q(0) \neq 0$, then

$$\int_{\mathbb{R}^3} r |\rho(F^0)|^2 dx = +\infty, \quad \text{where} \quad \rho(F^0) := \frac{x^i}{r} (F^0)_{i0}.$$

We prove in Appendix 3.C that the derivatives of F are automatically chargeless, whether or not Q vanishes.

3.1.4 Strategy of the proof and main difficulties

The proof of Theorem 3.1.1 is based on energy and vector field methods and essentially relies on bounding sufficiently well the spacetime integrals of the commuted equations. The solutions of the massless Vlasov equation enjoy improved decay estimates in the null directions. More precisely, one can already see that with the following estimate (see Lemma 3.2.11 and Proposition 3.3.6), for g a solution to the free transport equation $|v|\partial_t g + v^i \partial_i g = 0$,

$$\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad \int_{v \in \mathbb{R}^3} \left| \frac{v^L}{|v|} \right|^p \left| \frac{v^A}{|v|} \right|^k \left| \frac{v^{\underline{L}}}{|v|} \right|^q |g|(t, x, v) dv \lesssim \sum_{|\beta| \leq 3} \frac{\|(1+r)^{|\beta|+p+k+q} \partial_x^\beta g\|_{L_{x,v}^1}(t=0)}{(1+t+r)^{2+k+q}(1+|t-r|)^{1+p}}. \quad (3.5)$$

This strong decay is a key element of our proof. Without it, we would have to consider modifications of the commutation vector fields of the free transport operator as in [17], [46], [16] and [26] for, respectively, the Vlasov-Nordström, the Vlasov-Poisson and the Einstein-Vlasov systems. As the particles are massless, the characteristics of the transport equation and those of the Maxwell equations have the same velocity⁵. The consequence is that, in a product such as $\mathcal{L}_{Z^\gamma}(F) \cdot \widehat{Z}^\beta f$, we cannot transform a $|t-r|$ decay in a $t+r$ one as it is done, in view of support consideration, for the massive case with compactly supported initial data. We are then led to carefully study the null structure of the equations, and in particular of the non linearities such as

$$v^\mu \mathcal{L}_{Z^\gamma}(F)_{\mu}{}^i \partial_{v^i} \widehat{Z}^\beta f, \quad (3.6)$$

with Z a Killing vector field and \widehat{Z} its complete lift⁶. The problem is that, for g solution to $|v|\partial_t + v^i \partial_i g = 0$, $\partial_{v^i} g$ essentially behaves as $(1+t+r)\partial_{t,x} g$ and the electromagnetic field, as a solution of a wave equation, only decay with a rate of $(1+t+r)^{-1}$ in the $t+r$ direction. However, from [11], we know that certain null components of the Maxwell field are expected to behave better than others. The same is true for the null components of the velocity vector v as it is suggested by (3.5). Moreover, we also know from [4] that $v^{\underline{L}}$ allows us to take advantage of the $t-r$ decay as it permits to estimate spacetime integrals by using a null foliation. Finally, the radial component of $(0, \partial_{v^1} \widehat{Z}^\beta f, \partial_{v^2} \widehat{Z}^\beta f, \partial_{v^3} \widehat{Z}^\beta f)$ costs a power of $t-r$ instead of $t+r$. The null structure of (3.6) is then studied in Lemma 3.4.1 and we can observe that each term contains at least one good component.

Another problem, specific to massless particles, arises from small velocities. We already observed in Section 8 of [4] that the velocity part V of the characteristics of

$$\partial_t + \frac{v^i}{|v|} \partial_i f + \left(F_{0i} + \frac{v^j}{|v|} F_{ji} \right) \partial_{v^i} f = 0 \quad (3.7)$$

can reach 0 in finite time. The consequence is that if f does not initially vanish for small velocities, the Vlasov-Maxwell system could not admit a local classical solution. This issue is reflected in the energy estimates through, schematically,

$$\left\| \widehat{Z} f \right\|_{L_{x,v}^1}(t) \leq 2 \left\| \widehat{Z} f \right\|_{L_{x,v}^1}(0) + \int_0^t \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} |\psi(t, x, v)| \frac{|\widehat{Z} f|}{|v|} dv dx ds,$$

⁵Note that this is not the case for particles of mass $m > 0$ since the free transport operator is then $\sqrt{m^2 + |v|^2} \partial_t + v^i \partial_i$.

⁶The expression of the complete lift of a vector field of the Minkowski space is presented in Definition 3.2.4.

where ψ is a homogeneous function of degree 0 in v . One cannot hope to close such an estimate using say Grönwall inequality due to the factor of $\frac{1}{|v|}$ appearing in the error term on the right hand side. In [4], we take advantage of the strong decay rate of the electromagnetic field, given by the high dimensions, to prove that the velocity support of f remains bounded away from 0 if initially true. The slow decay of F in dimension 3 forces us to exploit the null structure of the equations satisfied by the characteristics of (3.7) in order to recover this result. The strong decay rate satisfied by the radial component of the electric field $\rho(F)$ plays a fundamental role here. We point out that this difficulty is not present in the Einstein-Vlasov system as the Vlasov equation can be written, for a metric g and defined in terms of the cotangent variables, as

$$v_\mu g^{\mu\nu} \partial_{x^\nu} f - \frac{1}{2} v_\mu v_\nu \partial_i g^{\mu\nu} \partial_{v_i} f.$$

One can observe that the homogeneity in v of the non linearity of the Vlasov equation is the same than the one of $|v|\partial_t + v^i\partial_i$, so that the velocity part of the characteristics cannot reach 0 in finite time.

3.1.5 Structure of the paper

Section 3.2 presents the notations used in this article, basic results on the electromagnetic field and its null decomposition. The commutation vector fields are introduced in Subsection 3.2.4 and the source terms of the commuted equations are described in Subsection 3.2.5. Subsection 3.2.6 contains fundamental properties on the null components of the velocity vector. In Section 3.3, we introduce the norms used to study the Vlasov-Maxwell system and we present energy estimates in order to control them. We then exploit these energy norms to obtain pointwise decay estimates on both fields through Klainerman-Sobolev type inequalities. Lemma 3.4.1, proved in Section 3.4, is of fundamental importance in this work since it depicts the null structure of the non linearities of the transport equations. In section 3.5, we set up the bootstrap assumptions, discuss their immediate consequences and describe the main steps of the proof of Theorem 3.1.1. Sections 3.6 to 3.8 concern respectively the improvement of the bounds on the distribution function, the proof of L^2 estimates for the velocity averages of its higher order derivatives and the improvement of the estimates on the electromagnetic field energies. In Appendix 3.A, we prove that the Vlasov field vanishes for small velocities. In Appendix 3.B we expose how to bound the energy norms of f and F in terms of weighted L^1 and L^2 norms of the initial data. We prove in Appendix 3.C that the derivatives of F , for (f, F) a sufficiently regular solution to the Vlasov-Maxwell system, are automatically chargeless. Finally, Appendix 3.D contains the proof of certain results concerning the null decomposition of the electromagnetic field.

3.1.6 Acknowledgements

This article forms part of my Ph.D. thesis and I am grateful to my advisor Jacques Smulevici for his support and fruitful discussions. Part of this work was funded by the European Research Council under the European Union's Horizon 2020 research and innovation program (project GEOWAKI, grant agreement 714408).

3.2 Notations and preliminaries

3.2.1 Basic notations

In this paper we work on the $3 + 1$ dimensional Minkowski spacetime (\mathbb{R}^{3+1}, η) . We will use two sets of coordinates, the Cartesian (t, x^1, x^2, x^3) , in which $\eta = \text{diag}(-1, 1, 1, 1)$, and null coordinates $(\underline{u}, u, \omega_1, \omega_2)$, where

$$\underline{u} = t + r, \quad u = t - r$$

and (ω_1, ω_2) are spherical variables, which are spherical coordinates on the spheres $(t, r) = \text{constant}$. These coordinates are defined globally on \mathbb{R}^{3+1} apart from the usual degeneration of spherical coordinates and at $r = 0$. We will also use the following classical weights,

$$\tau_+ := \sqrt{1 + \underline{u}^2} \quad \text{and} \quad \tau_- := \sqrt{1 + u^2}.$$

We denote by (e_1, e_2) an orthonormal basis on the spheres and by ∇ (respectively d/v) the intrinsic covariant differentiation (respectively divergence operator) on the spheres $(t, r) = \text{constant}$. Capital Latin indices (such as A or B) will always correspond to spherical variables. The null derivatives are defined by

$$L = \partial_t + \partial_r \quad \text{and} \quad \underline{L} = \partial_t - \partial_r, \quad \text{so that} \quad L(\underline{u}) = 2, \quad L(u) = 0, \quad \underline{L}(\underline{u}) = 0 \quad \text{and} \quad \underline{L}(u) = 2.$$

The velocity vector $(v^\mu)_{0 \leq \mu \leq 3}$ is parametrized by $(v^i)_{1 \leq i \leq 3}$ and $v^0 = |v|$ since we study massless particles. We introduce T , the operator defined, for all sufficiently regular function $f : [0, T] \times \mathbb{R}_x^3 \times (\mathbb{R}_v^3 \setminus \{0\})$, by

$$T : f \mapsto v^\mu \partial_\mu f.$$

We will use the notation $\nabla_v g := (0, \partial_{v^1} g, \partial_{v^2} g, \partial_{v^3} g)$ so that (3.1) can be rewritten

$$T_F(f) := v^\mu \partial_\mu f + F(v, \nabla_v f) = 0.$$

Remark 3.2.1. *As we study massless particles, the functions considered in this paper will not be defined for $v = 0$. However, for simplicity and since $\{v = 0\}$ has Lebesgue measure 0, we will consider integrals over \mathbb{R}_v^3 . Moreover, the distribution function f will be supported away from $v = 0$ during the proof of Theorem 3.1.1.*

We will use the notation $D_1 \lesssim D_2$ for an inequality such as $D_1 \leq CD_2$, where $C > 0$ is a positive constant independent of the solutions but which could depend on $N \in \mathbb{N}$, the maximal order of commutation. Finally we will raise and lower indices using the Minkowski metric η . For instance, $v_\mu = v^\nu \eta_{\nu\mu}$ so that $v_0 = -v^0$ and $v_i = v^i$ for all $1 \leq i \leq 3$.

3.2.2 The problem of the small velocities

For technical reasons, we will use all along this paper a fixed cutoff function χ such that $\chi = 1$ on $[1, +\infty[$ and $\chi = 0$ on $] -\infty, \frac{1}{2}]$. We introduce the operator

$$T_F^\chi : g \mapsto v^\mu \partial_\mu g + \chi(|v|)F(v, \nabla g). \quad (3.8)$$

As mentioned earlier, we proved in Section 8 of [4] that because of the small velocities, there exists initial data sets for which the Vlasov-Maxwell system does not admit a local classical solution. The main idea of the proof consists in studying characteristics such that their velocity part reaches 0 in finite time. This is why we suppose in Theorem 3.1.1 that the Vlasov field vanishes initially for small velocities and one step of the proof will be to verify that this property remains true for all $t \in \mathbb{R}_+$. To circumvent difficulties related to characteristics reaching $v = 0$, we will rather first define (f, F) as the solution to (3.2) – (3.3) and $T_F^\chi(f) = 0$. Notice that none of the characteristics of the operator T_F^χ reaches $v = 0$. Indeed, if (X, V) is one of them, we have

$$\frac{dV^j}{dt}(s) = \chi(|V|(s)) \frac{V^\mu(s)}{|V|(s)} F_{\mu}^j(s, X(s)).$$

Consequently, if $|V(s)| < \frac{1}{2}$, then $V(t) = V(s)$ for all $t \geq s$. The goal will then to prove that if $f(0, \dots)$ vanishes for all $|v| \leq 3$, so does $f(t, \dots)$ for all $|v| \leq 1$, implying that $T_F(f) = 0$ and that (f, F) is a solution to the Vlasov-Maxwell system (3.1)-(3.3).

3.2.3 Basic tools for the study of the electromagnetic field

As we describe the electromagnetic field in geometric form, it will be represented throughout this article by a 2-form. Let F be a 2-form defined on $[0, T] \times \mathbb{R}_x^3$. Its null decomposition $(\alpha(F), \underline{\alpha}(F), \rho(F), \sigma(F))$, introduced by [11], is defined by

$$\alpha_A(F) = F_{AL}, \quad \underline{\alpha}_A(F) = F_{A\underline{L}}, \quad \rho(F) = \frac{1}{2} F_{LL} \quad \text{and} \quad \sigma(F) = F_{12}.$$

The Hodge dual $*F$ of F is the 2-form given by

$$*F_{\mu\nu} = \frac{1}{2} F^{\lambda\sigma} \varepsilon_{\lambda\sigma\mu\nu},$$

where $\varepsilon_{\lambda\sigma\mu\nu}$ are the components of the Levi-Civita symbol, and its energy-momentum tensor is

$$T[F]_{\mu\nu} := F_{\mu\beta} F_\nu^\beta - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}.$$

Note that $T[F]_{\mu\nu}$ is symmetric and traceless, i.e. $T[F]_{\mu\nu} = T[F]_{\nu\mu}$ and $T[F]_\mu{}^\mu = 0$. This last point is specific to the dimension 3 and engenders additional difficulties in the analysis of the Maxwell equations in high dimensions (see Section 3.3.2 of [4] for more details). We have an alternative form of the Maxwell equations.

Lemma 3.2.2. *Let G be a 2-form and J be a 1-form both sufficiently regular. Then,*

$$\begin{cases} \nabla^\mu G_{\mu\nu} = J_\nu \\ \nabla^{\mu*} G_{\mu\nu} = 0 \end{cases} \Leftrightarrow \begin{cases} \nabla_{[\lambda} G_{\mu\nu]} = 0 \\ \nabla_{[\lambda} {}^*G_{\mu\nu]} = \varepsilon_{\lambda\mu\nu\kappa} J^\kappa, \end{cases}$$

where $\nabla_{[\lambda} H_{\mu\nu]} := \nabla_\lambda H_{\mu\nu} + \nabla_\mu H_{\nu\lambda} + \nabla_\nu H_{\lambda\mu}$.

Proof. Consider for instance $\nabla^\mu G_{\mu 1} = J_1$ and $\nabla^{i*} G_{i0} = 0$. As

$$\begin{aligned} G^{01} &= {}^*G_{23}\varepsilon_{0123} = {}^*G_{23}, & G^{21} &= {}^*G_{03}\varepsilon_{2103} = {}^*G_{30}, & G^{31} &= {}^*G_{02}\varepsilon_{3102} = G_{02}, \\ {}^*G_{10} &= G_{23}\varepsilon_{2310} = -G_{23}, & {}^*G_{20} &= G_{31}\varepsilon_{3120} = -G_{31} & \text{and} & {}^*G_{30} = G_{12}\varepsilon_{1230} = -G_{12}, \end{aligned}$$

we have

$$\nabla^\mu G_{\mu 1} = J_1 \Leftrightarrow \nabla_0 {}^*G_{23} + \nabla_2 {}^*G_{30} + \nabla_3 {}^*G_{02} = J_1 \quad \text{and} \quad \nabla^{i*} G_{i0} = 0 \Leftrightarrow \nabla_1 G_{23} + \nabla_2 G_{31} + \nabla_3 G_{12} = 0.$$

The equivalence of the two systems can be obtained by similar computations. \square

We can then compute the divergence of the energy momentum tensor of an electromagnetic field.

Corollary 3.2.3. *Let G and J be as in the previous lemma. Then, $\nabla^\mu T[G]_{\mu\nu} = G_{\nu\lambda} J^\lambda$.*

Proof. Using the previous lemma, we have

$$G_{\mu\rho} \nabla^\mu G_{\nu}{}^\rho = G^{\mu\rho} \nabla_\mu G_{\nu\rho} = \frac{1}{2} G^{\mu\rho} (\nabla_\mu G_{\nu\rho} - \nabla_\rho G_{\nu\mu}) = \frac{1}{2} G^{\mu\rho} \nabla_\nu G_{\mu\rho} = \frac{1}{4} \nabla_\nu (G^{\mu\rho} G_{\mu\rho}).$$

Hence,

$$\nabla^\mu T[G]_{\mu\nu} = \nabla^\mu (G_{\mu\rho}) G_{\nu}{}^\rho + \frac{1}{4} \nabla_\nu (G^{\mu\rho} G_{\mu\rho}) - \frac{1}{4} \eta_{\mu\nu} \nabla^\mu (G^{\sigma\rho} G_{\sigma\rho}) = G_{\nu\rho} J^\rho.$$

\square

Finally, the null components of the energy-momentum tensor of a 2-form G are given by

$$T[G]_{LL} = |\alpha(G)|^2, \quad T[G]_{\underline{L}\underline{L}} = |\underline{\alpha}(G)|^2 \quad \text{and} \quad T[G]_{L\underline{L}} = |\rho(G)|^2 + |\sigma(G)|^2. \quad (3.9)$$

3.2.4 The vector fields of the Poincaré group and their complete lifts

We present in this section the commutation vector fields for the Maxwell equations and those for the relativistic transport operator. Let \mathbb{P} be the generators of the Poincaré algebra, i.e. the set containing

$$\begin{aligned} \text{the translations}^7 & \quad \partial_\mu, \quad 0 \leq \mu \leq 3, \\ \text{the rotations} & \quad \Omega_{ij} = x^i \partial_j - x^j \partial_i, \quad 1 \leq i < j \leq 3, \\ \text{the hyperbolic rotations} & \quad \Omega_{0k} = t \partial_k + x^k \partial_t, \quad 1 \leq k \leq 3. \end{aligned}$$

We also consider $\mathbb{K} := \mathbb{P} \cup \{S\}$, where $S = x^\mu \partial_\mu$ is the scaling vector field and $\mathbb{O} := \{\Omega_{12}, \Omega_{13}, \Omega_{23}\}$, the set of the rotational vector fields. The vector fields of \mathbb{K} are well known for commuting with the wave and the Maxwell equations (see Proposition 3.2.8 below). However, to commute the operator $T = v^\mu \partial_\mu$, one should consider, as in [18], the complete lifts of the vector fields of \mathbb{P} .

Definition 3.2.4. *Let Γ be a vector field of the form $\Gamma^\beta \partial_\beta$. Then, the complete lift $\widehat{\Gamma}$ of Γ is defined by*

$$\widehat{\Gamma} = \Gamma^\beta \partial_\beta + v^\gamma \frac{\partial \Gamma^i}{\partial x^\gamma} \partial_{v^i}.$$

We then have $\widehat{\partial}_\mu = \partial_\mu$ for all $0 \leq \mu \leq 3$,

$$\widehat{\Omega}_{ij} = x^i \partial_j - x^j \partial_i + v^i \partial_{v^j} - v^j \partial_{v^i}, \quad \text{for } 1 \leq i < j \leq 3, \quad \text{and} \quad \widehat{\Omega}_{0k} = t \partial_k + x^k \partial_t + v^0 \partial_{v^k}, \quad \text{for } 1 \leq k \leq 3.$$

⁷In this article, we will denote ∂_{x^i} , for $1 \leq i \leq 3$, by ∂_i and sometimes ∂_t by ∂_0 .

One can check that $[T, \widehat{Z}] = 0$ for all $Z \in \mathbb{P}$. As we also have $[T, S] = T$, we consider

$$\widehat{\mathbb{P}}_0 := \{\widehat{Z} / Z \in \mathbb{P}\} \cup \{S\}$$

and we will, for simplicity, denote by \widehat{Z} an arbitrary vector field of $\widehat{\mathbb{P}}_0$, even if S is not a complete lift. These vector fields and the averaging in v almost commute in the following sense.

Lemma 3.2.5. *Let $f : [0, T] \times \mathbb{R}_x^3 \times (\mathbb{R}_v^3 \setminus \{0\}) \rightarrow \mathbb{R}$ be a sufficiently regular function. We have, almost everywhere,*

$$\forall Z \in \mathbb{K}, \quad \left| Z \left(\int_{v \in \mathbb{R}^3} |f| dv \right) \right| \leq \int_{v \in \mathbb{R}^3} |f| dv + \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_{v \in \mathbb{R}^3} |\widehat{Z} f| dv.$$

Proof. Let us consider, for instance, the case where $Z = \Omega_{12} = x^1 \partial_2 - x^2 \partial_1$. Then, integrating by parts in v , we have almost everywhere

$$\begin{aligned} \left| \Omega_{12} \left(\int_{v \in \mathbb{R}^3} |f| dv \right) \right| &= \left| \int_{v \in \mathbb{R}^3} \widehat{\Omega}_{12}(|f|) dv - \int_{v \in \mathbb{R}^3} \left(v^1 \partial_{v^2}(|f|) - v^2 \partial_{v^1}(|f|) \right) dv \right| \\ &= \left| \int_{v \in \mathbb{R}^3} \frac{f}{|f|} \widehat{\Omega}_{12}(f) dv + 0 \right| \leq \int_{v \in \mathbb{R}^3} \left| \widehat{\Omega}_{12}(f) \right| dv. \end{aligned}$$

□

The vector space engendered by each of the sets defined in this section is an algebra.

Lemma 3.2.6. *Let \mathbb{L} be either $\widehat{\mathbb{P}}_0$, \mathbb{K} , \mathbb{P} or \mathbb{O} . Then for all $(Z_1, Z_2) \in \mathbb{L}^2$, $[Z_1, Z_2]$ is a linear combination of vector fields of \mathbb{L} .*

We consider an ordering on each of the sets \mathbb{O} , \mathbb{P} , \mathbb{K} and $\widehat{\mathbb{P}}_0$. We take orderings such that, if $\mathbb{P} = \{Z^i / 1 \leq i \leq |\mathbb{P}|\}$, then $\mathbb{K} = \{Z^i / 1 \leq i \leq |\mathbb{K}|\}$, with $Z^{|\mathbb{K}|} = S$, and

$$\widehat{\mathbb{P}}_0 = \left\{ \widehat{Z}^i / 1 \leq i \leq |\widehat{\mathbb{P}}_0| \right\}, \quad \text{with } \left(\widehat{Z}^i \right)_{1 \leq i \leq |\mathbb{P}|} = \left(\widehat{Z}^i \right)_{1 \leq i \leq |\mathbb{P}|} \quad \text{and } \widehat{Z}^{|\widehat{\mathbb{P}}_0|} = S.$$

If \mathbb{L} denotes \mathbb{O} , \mathbb{P} , \mathbb{K} or $\widehat{\mathbb{P}}_0$, and $\beta \in \{1, \dots, |\mathbb{L}|\}^q$, with $q \in \mathbb{N}^*$, we will denote the differential operator $\Gamma^{\beta_1} \dots \Gamma^{\beta_r} \in \mathbb{L}^{|\beta|}$ by Γ^β . For a vector field Y , we will denote by \mathcal{L}_Y the Lie derivative with respect to Y and if $Z^\gamma \in \mathbb{K}^q$, we will write \mathcal{L}_{Z^γ} for $\mathcal{L}_{Z^{\gamma_1}} \dots \mathcal{L}_{Z^{\gamma_q}}$.

Let us recall, by the following classical result, that the derivatives tangential to the cone behave better than others.

Lemma 3.2.7. *The following relations hold,*

$$(t-r)\underline{L} = S - \frac{x^i}{r} \Omega_{0i}, \quad (t+r)L = S + \frac{x^i}{r} \Omega_{0i} \quad \text{and} \quad re_A = \sum_{1 \leq i < j \leq 3} C_A^{i,j} \Omega_{ij},$$

where the $C_A^{i,j}$ are uniformly bounded and depend only on spherical variables. We also have

$$(t-r)\partial_t = \frac{t}{t+r} S - \frac{x^i}{t+r} \Omega_{0i} \quad \text{and} \quad (t-r)\partial_i = \frac{t}{t+r} \Omega_{0i} - \frac{x^i}{t+r} S - \frac{x^j}{t+r} \Omega_{ij}.$$

Finally, we introduce the vector field

$$\overline{K}_0 := \frac{1}{2} \tau_+^2 L + \frac{1}{2} \tau_-^2 \underline{L},$$

which will be used as a multiplier.

3.2.5 Commutation of the Vlasov-Maxwell system

Let us start by proving the following result. For convenience, we extend the Kronecker symbol to vector fields, i.e. $\delta_{X,Y} = 1$ if $X = Y$ and $\delta_{X,Y} = 0$ otherwise.

Lemma 3.2.8. *Let G be a 2-form and g a function, both sufficiently regular. For all $\widehat{Z} \in \widehat{\mathbb{P}}_0$,*

$$\widehat{Z}(G(v, \nabla_v g)) = \mathcal{L}_Z(G)(v, \nabla_v g) + G(v, \nabla_v \widehat{Z}g) - 2\delta_{\widehat{Z}, S}G(v, \nabla_v g).$$

If $\nabla^\mu G_{\mu\nu} = J(g)_\nu$ and $\nabla^{\mu*}G_{\mu\nu} = 0$, then

$$\forall Z \in \mathbb{K}, \quad \nabla^\mu \mathcal{L}_Z(G)_{\mu\nu} = J(\widehat{Z}g)_\nu + 3\delta_{Z, S}J(g)_\nu \quad \text{and} \quad \nabla^{\mu*} \mathcal{L}_Z(G)_{\mu\nu} = 0.$$

Proof. Let $\widehat{Z} \in \widehat{\mathbb{P}}_0$ and define $Z_v := \widehat{Z} - Z$. Then,

$$\widehat{Z}(G(v, \nabla_v g)) = \mathcal{L}_Z(G)(v, \nabla_v g) + G([Z, v], \nabla_v g) + G(v, [Z, \nabla_v g]) + G(Z_v(v), \nabla_v g) + G(v, Z_v(\nabla_v g)).$$

Note now that

- $S_v = 0$ and $[S, v] = -v$,
- $[Z, v] = -Z_v(v)$ if $Z \in \mathbb{P}$.

The first identity is then implied by

- $[\partial, \nabla_v g] = \nabla_v \partial(g)$ and $[S, \nabla_v g] = \nabla_v S(g) - \nabla_v g$.
- $[Z, \nabla_v g] + Z_v(\nabla_v g) = \nabla_v \widehat{Z}(g)$, if $Z \in \mathbb{O}$.
- $[Z, \nabla_v g] + Z_v(\nabla_v g) = \nabla_v \widehat{Z}(g) - \frac{v}{v^0} \partial_{v^i} g$ and $G(v, v) = 0$, if $Z = \Omega_{0i}$.

Recall now that if⁸ $Z \in \mathbb{K}$,

$$\nabla^\mu \mathcal{L}_Z(G)_{\mu\nu} = \mathcal{L}_Z(J(g))_\nu + 2\delta_{Z, S}J(g)_\nu \quad \text{and} \quad \nabla^{\mu*} \mathcal{L}_Z(G)_{\mu\nu} = 0.$$

One then only have to notice that

$$\mathcal{L}_S(J(g)) = J(Sg) + J(g) \quad \text{and} \quad \forall Z \in \mathbb{P}, \quad \mathcal{L}_Z(J(g)) = J(\widehat{Z}g).$$

This follows from $\mathcal{L}_Z(J(g))_\nu = Z(J(g)_\nu) + \partial_\nu(Z^\lambda)J(g)_\lambda$ and integration by parts in v . For instance,

$$\begin{aligned} \mathcal{L}_{\Omega_{12}}(J(g))_\nu &= \int_{v \in \mathbb{R}^3} \frac{v_\nu}{v^0} \left(\widehat{\Omega}_{12} - v^1 \partial_{v^2} + v^2 \partial_{v^1} \right) g dv + \delta_{1, \nu} \int_{v \in \mathbb{R}^3} \frac{v_2}{v^0} g dv - \delta_{2, \nu} \int_{v \in \mathbb{R}^3} \frac{v_1}{v^0} g dv \\ &= J\left(\widehat{\Omega}_{12}g\right) + \delta_{2, \nu} \int_{v \in \mathbb{R}^3} \frac{v^1}{v^0} g dv - \delta_{1, \nu} \int_{v \in \mathbb{R}^3} \frac{v^1}{v^0} g dv + \delta_{1, \nu} \int_{v \in \mathbb{R}^3} \frac{v_2}{v^0} g dv - \delta_{2, \nu} \int_{v \in \mathbb{R}^3} \frac{v_1}{v^0} g dv. \end{aligned}$$

□

Iterating Lemma 3.2.8, we can describe the form of the source terms of the commuted Vlasov-Maxwell equations.

Proposition 3.2.9. *Let (f, F) be a sufficiently regular solution to the Vlasov-Maxwell system (3.1)-(3.3) and $Z^\beta \in \mathbb{K}^{|\beta|}$. There exists integers $n_{\gamma, \kappa}^\beta$ and m_ξ^β such that*

$$\begin{aligned} T_F\left(\widehat{Z}^\beta f\right) &= \sum_{\substack{|\gamma|+|\kappa| \leq |\beta| \\ |\kappa| \leq |\beta|-1}} n_{\gamma, \kappa}^\beta \mathcal{L}_{Z^\gamma}(F)\left(v, \nabla_v \widehat{Z}^\kappa(f)\right), \\ \nabla^\mu \mathcal{L}_{Z^\beta}(F)_{\mu\nu} &= \sum_{|\xi| \leq |\beta|} m_\xi^\beta J\left(\widehat{Z}^\xi f\right)_\nu, \\ \nabla^{\mu*} \mathcal{L}_{Z^\beta}(F)_{\mu\nu} &= 0. \end{aligned}$$

The main observation is that the structure of the non-linearity $F(v, \nabla_v f)$ is conserved after commutation, which is important since if the source terms of the Vlasov equation behaved as $v^0|F||\partial_v f|$, we would not be able to close the energy estimates for the Vlasov field. The other conserved structure is $J(f)$, which is also crucial since a source term behaving as $\int_v |f| dv$ would prevent us to close the energy estimates for the electromagnetic field.

⁸This can be obtained by straightforward computations in cartesian coordinates. We also refer to Proposition 3.3 of [11].

3.2.6 Weights preserved by the flow and null components of the velocity vector

We designate the null components of the velocity vector by $(v^L, v^{\underline{L}}, v^{e_1}, v^{e_2})$, so that

$$v = v^L L + v^{\underline{L}} \underline{L} + v^{e_A} e_A, \quad v^L = \frac{v^0 + v^r}{2} \quad \text{and} \quad v^{\underline{L}} = \frac{v^0 - v^r}{2}.$$

For simplicity we will write v^A instead of v^{e_A} . We introduce, as in [18], the following set of weights

$$\mathbf{k}_0 := \left\{ \frac{v^\mu}{v^0} \mid 0 \leq \mu \leq 3 \right\} \cup \left\{ x^\mu \frac{v^\nu}{v^0} - x^\nu \frac{v^\mu}{v^0} \mid \mu \neq \nu \right\} \cup \left\{ x^\mu \frac{v_\mu}{v^0} \right\}$$

and we will denote $x^\mu \frac{v^\nu}{v^0} - x^\nu \frac{v^\mu}{v^0}$ by $z_{\mu\nu}$ and $x^\mu \frac{v_\mu}{v^0}$ by s_0 . They are preserved by the flow of T and by the action of $\widehat{\mathbb{P}}_0$. More precisely, we have the following result.

Lemma 3.2.10. *Let $z \in \mathbf{k}_0$ and $\widehat{Z} \in \widehat{\mathbb{P}}_0$. Then,*

$$T(z) = 0, \quad \widehat{Z}(v^0 z) \in v^0 \mathbf{k}_0 \cup \{0\} \quad \text{and} \quad \left| \widehat{Z}(z) \right| \leq \sum_{w \in \mathbf{k}_0} |w|.$$

Proof. The first property ensues from straightforward computations. For the second one, let us consider for instance $tv^1 - x^1 v^0$, $x^1 v^2 - x^2 v^1$, $\widehat{\Omega}_{12}$ and $\widehat{\Omega}_{02}$. We have

$$\begin{aligned} \widehat{\Omega}_{12}(tv^1 - x^1 v^0) &= -tv^2 - x^2 v^0, & \widehat{\Omega}_{12}(x^1 v^2 - x^2 v^1) &= 0, \\ \widehat{\Omega}_{02}(tv^1 - x^1 v^0) &= x^2 v^1 - x^1 v^2 \quad \text{and} & \widehat{\Omega}_{02}(x^1 v^2 - x^2 v^1) &= x^1 v^0 - tv^1. \end{aligned}$$

The other cases are similar and the third property follows directly from the second one. \square

The following inequalities, which should be compared to those of Lemma 3.2.7, suggest how we will use these weights.

Lemma 3.2.11. *We have,*

$$\frac{v^L}{v^0} \lesssim \frac{1}{\tau_-} \sum_{z \in \mathbf{k}_0} |z|, \quad \frac{v^{\underline{L}}}{v^0} + \frac{|v^A|}{v^0} \lesssim \frac{1}{\tau_+} \sum_{z \in \mathbf{k}_0} |z| \quad \text{and} \quad |v^A| \lesssim \sqrt{v^0 v^{\underline{L}}}.$$

Proof. Note first that

$$2(t-r) \frac{v^L}{v^0} = -s_0 - \frac{x^i}{r} z_{0i}, \quad 2(t+r) \frac{v^{\underline{L}}}{v^0} = -s_0 + \frac{x^i}{r} z_{0i} \quad \text{and} \quad rv_A = v^0 C_A^{i,j} z_{ij},$$

where $C_A^{i,j}$ are bounded functions depending only on the spherical variables such as $re_A = C_A^{i,j} \Omega_{i,j}$. This gives the first two estimates. For the last one, use also that $4r^2 v^L v^{\underline{L}} = \sum_{k < l} |v^0 z_{kl}|^2$, which comes from

$$\begin{aligned} 4r^2 v^L v^{\underline{L}} &= (rv^0)^2 - (x^i v_i)^2 = \sum_{i=1}^3 (r^2 - |x^i|^2) |v_i|^2 - 2 \sum_{1 \leq k < l \leq 3} x^k x^l v_k v_l, \\ \sum_{1 \leq k < l \leq 3} |v^0 z_{kl}|^2 &= \sum_{1 \leq k < l \leq 3} |x^k|^2 |v_l|^2 + |x^l|^2 |v_k|^2 - 2x^k x^l v_k v_l = \sum_{i=1}^3 \sum_{j \neq i} |x^j|^2 |v_i|^2 - 2 \sum_{1 \leq k < l \leq n} x^k x^l v_k v_l. \end{aligned}$$

\square

Remark 3.2.12. *There are certain differences with the massive case, where $v^0 = \sqrt{m^2 + |v|^2}$ and $m > 0$.*

- *The inequality $1 \lesssim v^0 v^{\underline{L}}$ does not hold.*
- *As $x^i v_i - tv^0$ does not commute with the massive relativistic transport operator, we rather consider the set of weights $\mathbf{k}_1 := \mathbf{k}_0 \setminus \{s_0\}$ in this context. Then, the estimate $\tau_- v^L + \tau_+ v^{\underline{L}} \lesssim v^0 \sum_{z \in \mathbf{k}_1} |z|$ is merely satisfied in the exterior of the light cone.*

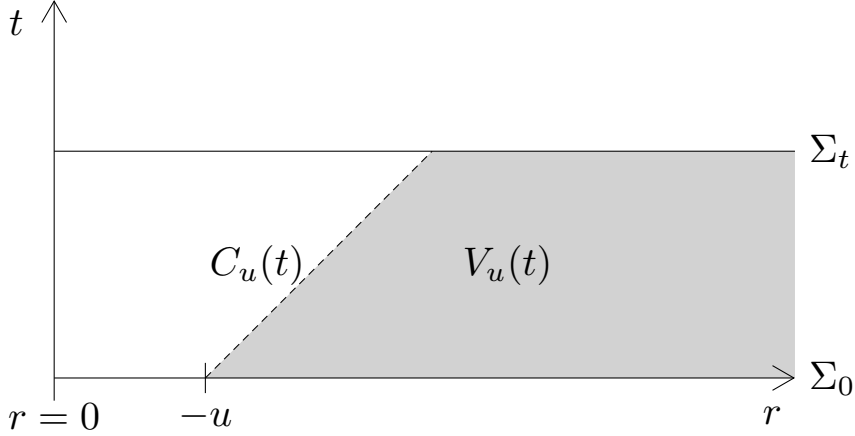
3.2.7 Various subsets of the Minkowski spacetime

We introduce here several subsets of the Minkowski space depending on $t \in \mathbb{R}_+$, $r \in \mathbb{R}_+$, $u \in \mathbb{R}$. Let $\mathbb{S}_{t,r}$, Σ_t , $C_u(t)$ and $V_u(t)$, be the sets defined as

$$\begin{aligned} \mathbb{S}_{t,r} &:= \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / (s, |y|) = (t, r)\}, & C_u(t) &:= \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s \leq t, s - |y| = u\}, \\ \Sigma_t &:= \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s = t\}, & V_u(t) &:= \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s \leq t, s - |y| \leq u\}. \end{aligned}$$

The volum form on $C_u(t)$ is given by $dC_u(t) = \sqrt{2}^{-1} r^2 du d\mathbb{S}^2$, where $d\mathbb{S}^2$ is the standard metric on the 2 dimensional unit sphere. In view of applying the divergence theorem, we also introduce

$$\Sigma_t^u := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s = t, |y| \geq s - u\}.$$



The sets Σ_t , $C_u(t)$ and $V_u(t)$

We also introduce a dyadic partition of \mathbb{R}_+ by considering the sequence $(t_i)_{i \in \mathbb{N}}$ and the functions $(T_i(t))_{i \in \mathbb{N}}$ defined by

$$t_0 = 0, \quad t_i = 2^i \quad \text{if } i \geq 1, \quad \text{and} \quad T_i(t) = t \mathbb{1}_{t \leq t_i}(t) + t_i \mathbb{1}_{t > t_i}(t).$$

We then define the troncated cones $C_u^i(t)$ adapted to this partition by

$$C_u^i(t) := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / t_i \leq s \leq T_{i+1}(t), s - |y| = u\} = \{(s, y) \in C_u(t) / t_i \leq s \leq T_{i+1}(t)\}.$$

The following lemma will be used several times during this paper. It depicts that we can foliate $[0, t] \times \mathbb{R}^3$ by $(\Sigma_s)_{0 \leq s \leq t}$, $(C_u(T))_{u \leq t}$ or $(C_u^i(T))_{u \leq t, i \in \mathbb{N}}$.

Lemma 3.2.13. *Let $t > 0$ and $g \in L^1([0, t] \times \mathbb{R}^3)$. Then*

$$\int_0^t \int_{\Sigma_s} g dx ds = \int_{u=-\infty}^t \int_{C_u(t)} g dC_u(t) \frac{du}{\sqrt{2}} = \sum_{i=0}^{+\infty} \int_{u=-\infty}^t \int_{C_u^i(t)} g dC_u^i(t) \frac{du}{\sqrt{2}}.$$

Note that the sum over i is in fact finite. The second foliation is useful to take advantage of decay in the $t - r$ direction since $\|\tau_-^{-1}\|_{L^\infty(C_u(t))} = \tau_-^{-1}$, whereas $\|\tau_-^{-1}\|_{L^\infty(\Sigma_s)} = 1$. The last foliation will be used to take advantage of time decay on $C_u(t)$ as we merely have $\|\tau_+^{-1}\|_{L^\infty(C_u(t))} = \tau_-^{-1}$, whereas $\|\tau_+^{-1}\|_{L^\infty(C_u^i(t))} \leq (1 + t_i)^{-1} \leq 3(1 + t_{i+1})^{-1}$.

3.3 Energy and pointwise decay estimates

In this section, we recall classical energy estimates for both the electromagnetic field and the Vlasov field and how obtain pointwise decay estimates from them.

3.3.1 Energy estimates

For the Vlasov field, we will use the following energy estimate.

Proposition 3.3.1. *Let $H : [0, T[\times \mathbb{R}_x^3 \times (\mathbb{R}_v^3 \setminus \{0\}) \rightarrow \mathbb{R}$ and $g_0 : \mathbb{R}_x^3 \times (\mathbb{R}_v^3 \setminus \{0\}) \rightarrow \mathbb{R}$ be two sufficiently regular functions and F a sufficiently regular 2-form. Then, g , the unique classical solution of*

$$\begin{aligned} T_F(g) &= H \\ g(0, \cdot, \cdot) &= g_0, \end{aligned}$$

satisfies, for all $t \in [0, T[$, the following estimates,

$$\left\| \int_{v \in \mathbb{R}^3} |g| dv \right\|_{L^1(\Sigma_t)} + \sup_{u \leq t} \left\| \int_{v \in \mathbb{R}^3} \frac{v^L}{v^0} |g| dv \right\|_{L^1(C_u(t))} \leq 2 \left\| \int_{v \in \mathbb{R}^3} |g_0| dv \right\|_{L^1(\Sigma_0)} + 2 \int_0^t \int_{\Sigma_s} \int_{v \in \mathbb{R}^3} |H| \frac{dv}{v^0} dx ds.$$

Proof. Note first that as $T(|g|) = \frac{g}{|g|} H - \frac{g}{|g|} F(v, \nabla_v g)$ and since F is a 2-form, integration by parts in v gives us

$$\partial_\mu \int_v |g| \frac{v^\mu}{v^0} dv = \int_v \left(\frac{g}{|g|} \frac{H}{v^0} - \frac{g}{|g|} F \left(\frac{v}{v^0}, \nabla_v g \right) \right) dv = \int_v \left(\frac{g}{|g|} \frac{H}{v^0} - \frac{v^j v^i}{(v^0)^3} F_{ji} |g| \right) dv = \int_v \frac{g}{|g|} H \frac{dv}{v^0}.$$

Hence, the divergence theorem applied to $\int_v |g| \frac{v^\mu}{v^0} dv$ in the regions $[0, t] \times \mathbb{R}_x^3$ and $V_u(t)$, for all $u \leq t$, gives

$$\begin{aligned} \int_{\Sigma_t} \int_v |g| dv dx &\leq \int_{\Sigma_0} \int_v |g| dv dx + \int_0^t \int_{\Sigma_s} \left| \int_v \frac{g}{|g|} H \frac{dv}{v^0} \right| dx ds, \\ \int_{\Sigma_t^u} \int_v |g| dv dx + \sqrt{2} \int_{C_u(t)} \int_v \frac{v^L}{v^0} |g| dv dC_u(t) &\leq \int_{\Sigma_0^u} \int_v |g| dv dx + \int_0^t \int_{\Sigma_s^u} \left| \int_v \frac{g}{|g|} H \frac{dv}{v^0} \right| dx ds, \end{aligned}$$

which implies the result. \square

We then define, for $(Q, q) \in \mathbb{N}^2$,

$$\mathbb{E}[g](t) := \left\| \int_{v \in \mathbb{R}^3} |g| dv \right\|_{L^1(\Sigma_t)} + \sup_{u \leq t} \left\| \int_{v \in \mathbb{R}^3} \frac{v^L}{v^0} |g| dv \right\|_{L^1(C_u(t))}, \quad (3.10)$$

$$\mathbb{E}_Q^q[g](t) := \sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|} \\ |\beta| \leq Q}} \sum_{z \in \mathbf{k}_0} \mathbb{E} \left[z^q \widehat{Z}^\beta g \right] (t). \quad (3.11)$$

We now introduce the energy norms, related to the electromagnetic field, used in this paper. We consider, for the remaining of this section, G a sufficiently regular 2-form defined on $[0, T[\times \mathbb{R}^3$ and we denote by $(\alpha, \underline{\alpha}, \rho, \sigma)$ its null decomposition. We moreover suppose that G satisfies

$$\begin{aligned} \nabla^\mu G_{\mu\nu} &= J_\nu \\ \nabla^{\mu*} G_{\mu\nu} &= 0, \end{aligned}$$

with J be a sufficiently regular 1-form defined on $[0, T[\times \mathbb{R}^3$.

Definition 3.3.2. *Let $k \in \mathbb{N}$. We define, for $t \in [0, T[$,*

$$\begin{aligned} \mathcal{E}^{\overline{K}_0}[G](t) &:= 4 \int_{\Sigma_t} T[G]_{0\nu} \overline{K}_0^\nu dx + 2 \sup_{u \leq t} \int_{C_u(t)} T[G]_{L\nu} \overline{K}_0^\nu dC_u(t) \\ &= \int_{\Sigma_t} \tau_+^2 |\alpha|^2 + \tau_-^2 |\underline{\alpha}|^2 + (\tau_+^2 + \tau_-^2) (|\rho|^2 + |\sigma|^2) dx + \sup_{u \leq t} \int_{C_u(t)} \tau_+^2 |\alpha|^2 + \tau_-^2 (|\rho|^2 + |\sigma|^2) dC_u(t), \\ \mathcal{E}^{\partial_t, k}[G](t) &= \int_{\Sigma_t} \tau_-^2 \log^{-k} (1 + \tau_-) (|\alpha|^2 + |\underline{\alpha}|^2 + 2|\rho|^2 + 2|\sigma|^2) dx. \end{aligned}$$

For $N \in \mathbb{N}^*$, we also introduce

$$\mathcal{E}_N[G] := \sum_{\substack{Z^\gamma \in \mathbb{K}^{|\gamma|} \\ |\gamma| \leq N}} \mathcal{E}^{\overline{K}_0}[\mathcal{L}_{Z^\gamma}(G)] \quad \text{and} \quad \mathcal{E}_N^k[G] := \sum_{\substack{Z^\gamma \in \mathbb{K}^{|\gamma|} \\ |\gamma| \leq N}} \mathcal{E}^{\partial_t, k}[\mathcal{L}_{Z^\gamma}(G)].$$

Remark 3.3.3. *During the proof of Theorem 3.1.1, we will have a small growth on $\mathcal{E}_N[G]$ and not on $\mathcal{E}_N^k[G]$. The second energy norm will then permit us to obtain the optimal decay rate in the $t + r$ direction on $\underline{\alpha}$, which will be crucial for closing the energy estimates for the Vlasov field.*

The following energy estimates hold.

Proposition 3.3.4. *We have, with $\bar{C} > 0$ a constant depending on k , for all $t \in [0, T]$,*

$$\begin{aligned}\mathcal{E}^{\bar{K}_0}[G](t) &\leq 2\mathcal{E}^{\bar{K}_0}[G](0) + 8 \int_0^t \int_{\Sigma_s} |G_{\mu\nu} J^\mu \bar{K}_0^\nu| dx ds, \\ \mathcal{E}^{\partial_t, k}[G](t) &\leq \bar{C} \mathcal{E}^{\bar{K}_0}[G](0) + \bar{C} \int_0^t \int_{\Sigma_s} \frac{\tau_-^2}{\log^k(1 + \tau_-)} |G_{\mu 0} J^\mu| dx ds + \bar{C} \sup_{0 \leq s \leq t} \left(\log^{1-k}(2 + s) \mathcal{E}^{\bar{K}_0}[G](s) \right).\end{aligned}$$

Proof. Denoting $T[G]$ by T and using Corollary 3.2.3, we have, as $\nabla^\mu \bar{K}_0^\nu + \nabla^\nu \bar{K}_0^\mu = 4t\eta^{\mu\nu}$ and $T_\mu^\mu = 0$,

$$\nabla^\mu \left(T_{\mu\nu} \bar{K}_0^\nu \right) = \nabla^\mu T_{\mu\nu} \bar{K}_0^\nu + T_{\mu\nu} \nabla^\mu \bar{K}_0^\nu = G_{\nu\lambda} J^\lambda \bar{K}_0^\nu + \frac{1}{2} T_{\mu\nu} \left(\nabla^\mu \bar{K}_0^\nu + \nabla^\nu \bar{K}_0^\mu \right) = G_{\nu\lambda} J^\lambda \bar{K}_0^\nu.$$

Applying the divergence theorem in $[0, t] \times \mathbb{R}^3$ and in $V_u(t)$, for all $u \leq t$, it comes

$$\int_{\Sigma_t} T_{0\nu} \bar{K}_0^\nu dx = \int_{\Sigma_0} T_{0\nu} \bar{K}_0^\nu dx - \int_0^t \int_{\Sigma_s} G_{\mu\nu} J^\mu \bar{K}_0^\nu dx ds, \quad (3.12)$$

$$\int_{\Sigma_u^u} T_{0\nu} \bar{K}_0^\nu dx + \frac{1}{\sqrt{2}} \int_{C_u(t)} T_{L\nu} \bar{K}_0^\nu dC_u(t) = \int_{\Sigma_0^u} T_{0\nu} \bar{K}_0^\nu dx - \int_0^t \int_{\Sigma_s^u} G_{\mu\nu} J^\mu \bar{K}_0^\nu dx ds. \quad (3.13)$$

Notice, using (3.9) and $2\bar{K}_0 = \tau_+^2 L + \tau_-^2 \underline{L}$, that

$$4T_{0\nu} \bar{K}_0^\nu = \tau_+^2 |\alpha| + \tau_-^2 |\underline{\alpha}| + (\tau_+^2 + \tau_-^2)(|\rho| + |\sigma|) \quad \text{and} \quad 2T_{L\nu} \bar{K}_0^\nu = \tau_+^2 |\alpha|^2 + \tau_-^2 |\rho|^2 + \tau_-^2 |\sigma|^2.$$

It then only remains, to obtain the first estimate, to take the supremum over all $u \leq t$ in (3.13) and to combine it with (3.12). For the other one, note first using Corollary 3.2.3 and (3.9) that

$$\begin{aligned}\left| \nabla^\mu \left(\tau_-^2 \log^{-k}(1 + \tau_-) T_{\mu 0} \right) \right| &= \left| \tau_-^2 \log^{-k}(1 + \tau_-) \nabla^\mu T_{\mu 0} - \frac{1}{2} \underline{L} \left(\tau_-^2 \log^{-k}(1 + \tau_-) \right) T_{L0} \right| \\ &= \left| \tau_-^2 \log^{-k}(1 + \tau_-) \nabla^\mu T_{\mu 0} - u \log^{-k}(1 + \tau_-) \left(2 - \frac{k\tau_-}{1 + \tau_-} \log^{-1}(1 + \tau_-) \right) T_{L0} \right| \\ &\lesssim \tau_-^2 \log^{-k}(1 + \tau_-) |G_{0\lambda} J^\lambda| + \frac{\tau_+^2}{\tau_+ \log^{k+1}(1 + \tau_+)} \left(|\alpha|^2 + |\rho|^2 + |\sigma|^2 \right).\end{aligned}$$

Consequently, applying the divergence theorem in $[0, t] \times \mathbb{R}^3$, we obtain

$$\int_{\Sigma_t} \frac{\tau_-^2}{\log^k(1 + \tau_-)} T_{00} dx \lesssim \int_{\Sigma_0} (1+r)^2 T_{00} dx + \int_0^t \int_{\Sigma_s} \frac{\tau_-^2}{\log^k(1 + \tau_-)} |G_{0\nu} J^\nu| dx ds + \int_0^t \frac{\mathcal{E}^{\bar{K}_0}[G](s)}{(1+s) \log^{k+1}(2+s)} ds.$$

The result then follows from $4T_{00} = |\alpha|^2 + |\underline{\alpha}|^2 + 2|\rho|^2 + 2|\sigma|^2$ and $\int_0^{+\infty} (1+s)^{-1} \log^{-2}(2+s) ds < +\infty$. \square

3.3.2 Decay estimates

Decay estimates for velocity averages

We prove in this subsection an $L^\infty - L^1$ and an $L^2 - L^1$ Klainerman-Sobolev inequality for velocity averages. The $L^\infty - L^1$ one was originally proved in [18] (see Theorem 6) and we propose here a shorter proof. Let us start with the following lemma.

Lemma 3.3.5. *Let $g : \mathbb{S}^2 \times (\mathbb{R}_v^3 \setminus \{0\}) \rightarrow \mathbb{R}$ a sufficiently regular function. Then, with $\Omega^\beta \in \mathbb{O}^{|\beta|}$,*

$$\left\| \int_{v \in \mathbb{R}^3} |g| dv \right\|_{L^\infty(\mathbb{S}^2)} \lesssim \sum_{|\beta| \leq 2} \left\| \int_{v \in \mathbb{R}^3} |\widehat{\Omega}^\beta g| dv \right\|_{L^1(\mathbb{S}^2)}, \quad \left\| \int_{v \in \mathbb{R}^3} |g| dv \right\|_{L^2(\mathbb{S}^2)} \lesssim \sum_{|\beta| \leq 1} \left\| \int_{v \in \mathbb{R}^3} |\widehat{\Omega}^\beta g| dv \right\|_{L^1(\mathbb{S}^2)}.$$

Proof. Let $\omega \in \mathbb{S}^2$ and (θ, φ) a local coordinate map in a neighborhood of ω . By the symmetry of the sphere we can suppose that θ and φ take their values in an interval of a size independent of ω . Using a one dimensional Sobolev inequality, that $|\partial_\theta u| \lesssim \sum_{\Omega \in \mathbb{O}} |\Omega u|$ and Lemma 3.2.5, we have,

$$\int_v |g|(\omega_1, \omega_2, v) dv \lesssim \int_\theta \left(\left| \int_v |g|(\theta, \omega_2, v) dv \right| + \left| \partial_\theta \int_v |g|(\theta, \omega_2, v) dv \right| \right) d\theta \lesssim \sum_{\substack{\Omega^\kappa \in \mathbb{O}^{|\kappa|} \\ |\kappa| \leq 1}} \int_\theta \int_v |\widehat{\Omega}^\kappa g|(\theta, \omega_2, v) dv d\theta.$$

We obtain similarly that

$$\int_{\theta} \int_v |\widehat{\Omega}^{\kappa} g|(\theta, \omega_2, v) dv d\theta \lesssim \sum_{\substack{\Omega^{\gamma} \in \mathbb{Q}^{|\gamma|} \\ |\gamma| \leq 1}} \int_{\theta} \int_{\varphi} \int_v |\widehat{\Omega}^{\gamma} \widehat{\Omega}^{\kappa} g|(\theta, \varphi, v) dv d\varphi d\theta \lesssim \sum_{\substack{\Omega^{\beta} \in \mathbb{Q}^{|\beta|} \\ |\beta| \leq 2}} \left\| \int_v |\widehat{\Omega}^{\beta} g| dv \right\|_{L^1(\mathbb{S}^2)},$$

which implies the first inequality. For the other one, by a standard $L^2 - L^1$ Sobolev inequality, one have

$$\left\| \int_{v \in \mathbb{R}^3} |g| dv \right\|_{L^2(\mathbb{S}^2)} \lesssim \left\| \int_{v \in \mathbb{R}^3} |g| dv \right\|_{L^1(\mathbb{S}^2)} + \left\| \partial_{\theta} \int_{v \in \mathbb{R}^3} |g| dv \right\|_{L^1(\mathbb{S}^2)} + \left\| \partial_{\varphi} \int_{v \in \mathbb{R}^3} |g| dv \right\|_{L^1(\mathbb{S}^2)}.$$

It then remains to apply Lemma 3.2.5 again. \square

Proposition 3.3.6. *Let $f : [0, T] \times \mathbb{R}^3 \times (\mathbb{R}_v^3 \setminus \{0\})$ be a sufficiently regular function, $z \in \mathbf{k}_0$ and $j \in \mathbb{N}$. Then,*

$$\forall (t, x) \in [0, T] \times \mathbb{R}^3, \quad \int_{v \in \mathbb{R}^3} |z^j f|(t, x, v) dv \lesssim \frac{(j+1)^3}{\tau_+^2 \tau_-} \sum_{\substack{\widehat{Z}^{\beta} \in \widehat{\mathbb{P}}_0^{|\beta|} \\ |\beta| \leq 3}} \sum_{w \in \mathbf{k}_0} \left\| \int_{v \in \mathbb{R}^3} |w^j \widehat{Z}^{\beta} f| dv \right\|_{L^1(\Sigma_t)}.$$

Proof. Note first that if $j \geq 1$, the inequality follows from the case $j = 0$ as, using Lemma 3.2.10,

$$\left| \widehat{Z}^{\beta} (z^j f) \right| \lesssim j^{|\beta|} \sum_{w \in \mathbf{k}_0} \sum_{|\kappa| \leq |\beta|} |w^j \widehat{Z}^{\kappa} f|.$$

Suppose now that $j = 0$ and consider $(t, x) = (t, |x|\omega) \in [0, T] \times \mathbb{R}^3$.

- If $1 + t \leq 2|x|$, one have, using Lemmas 3.2.7 and 3.2.5,

$$\begin{aligned} |x|^2 \tau_- \int_v |f|(t, |x|\omega, v) dv &= -|x|^2 \int_{r=|x|}^{+\infty} \partial_r \left(\tau_- \int_v |f|(t, r\omega, v) dv \right) dr \\ &\lesssim |x|^2 \sum_{Z \in \mathbb{K}} \int_{r=|x|}^{+\infty} \left(\int_v |f|(t, r\omega, v) dv + Z \left(\int_v |f|(t, r\omega, v) dv \right) \right) dr \\ &\lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_{r=|x|}^{+\infty} \left(\int_v |f|(t, r\omega, v) dv + \int_v |\widehat{Z} f|(t, r\omega, v) dv \right) r^2 dr. \end{aligned}$$

It then remains to apply Lemma 3.3.5 and to remark that $\tau_+ \lesssim r$ in the region considered.

- Otherwise $1 + t \geq 2|x|$, so that, with $\tau := 1 + t$,

$$\forall |y| \leq \frac{1}{4}, \quad \tau \leq 10(1 + |t - |x + \tau y||).$$

Thus, for all sufficiently regular function h , $1 \leq i \leq 3$ and almost all $|y| \leq \frac{1}{4}$, we have, using Lemmas 3.2.7 and then 3.2.5,

$$\begin{aligned} \left| \partial_{y^i} \left(\int_v |h|(t, x + \tau y, v) dv \right) \right| &= \left| \tau \int_v (\partial_i |h|)(t, x + \tau y, v) dv \right| \\ &\lesssim \left| (1 + |t - |x + \tau y||) \int_v (\partial_i |h|)(t, x + \tau y, v) dv \right| \\ &\lesssim \sum_{Z \in \mathbb{K}} \left| \int_v (Z |h|)(t, x + \tau y, v) dv \right| \\ &\lesssim \sum_{\substack{\widehat{Z}^{\kappa} \in \widehat{\mathbb{P}}_0^{|\kappa|} \\ |\kappa| \leq 1}} \int_v \left| \widehat{Z}^{\kappa} h \right|(t, x + \tau y, v) dv. \end{aligned} \tag{3.14}$$

Hence, using alternatively three times a one dimensional Sobolev inequality and then (3.14), it comes,

$$\begin{aligned}
\int_v |f|(t, x, v) dv &\lesssim \sum_{n=0}^1 \int_{|y^1| \leq \frac{1}{4\sqrt{3}}} \left| (\partial_{y^1})^n \left(\int_v |f|(t, x^1 + \tau y^1, x^2, x^3, v) dv \right) \right| dy^1 \\
&\lesssim \sum_{|\kappa| \leq 1} \int_{|y^1| \leq \frac{1}{4\sqrt{3}}} \int_v |\widehat{Z}^\kappa f|(t, x^1 + \tau y^1, x^2, x^3, v) dv dy^1 \\
&\lesssim \sum_{n=0}^1 \sum_{|\kappa| \leq 1} \int_{|y^1| \leq \frac{1}{4\sqrt{3}}} \int_{|y^2| \leq \frac{1}{4\sqrt{3}}} \left| (\partial_{y^2})^n \left(\int_v |\widehat{Z}^\kappa f|(t, x + \tau(y^1, y^2, 0), v) dv \right) \right| dy^2 dy^1 \\
&\lesssim \sum_{|\kappa| \leq 2} \int_{|y^1| \leq \frac{1}{4\sqrt{3}}} \int_{|y^2| \leq \frac{1}{4\sqrt{3}}} \int_v |\widehat{Z}^\kappa f|(t, x + \tau(y^1, y^2, 0), v) dv dy^2 dy^1 \\
&\lesssim \sum_{n=0}^1 \sum_{|\kappa| \leq 2} \int_{|y| \leq \frac{1}{4}} \left| (\partial_{y^3})^n \left(\int_v |\widehat{Z}^\kappa f|(t, x + \tau y, v) dv \right) \right| dy \\
&\lesssim \sum_{|\kappa| \leq 3} \int_{|y| \leq \frac{1}{4}} \int_v |\widehat{Z}^\kappa f|(t, x + \tau y, v) dv dy.
\end{aligned}$$

The result then follows from the change of variables $z = \tau y$ and that $\tau_- \leq \tau_+ \lesssim \tau$ in the region studied. \square

We now turn on the $L^2 - L^1$ Klainerman-Sobolev inequality.

Proposition 3.3.7. *Let $f : [0, T[\times \mathbb{R}^3 \times (\mathbb{R}_v^3 \setminus \{0\})$ be a sufficiently regular function, $z \in \mathbf{k}_0$ and $j \in \mathbb{N}$. Then,*

$$\forall t \in [0, T[, \quad \left\| \tau_+ \tau_-^{\frac{1}{2}} \int_{v \in \mathbb{R}^3} |z^j f| dv \right\|_{L^2(\Sigma_t)} \lesssim (j+1)^2 \sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{\beta 1} \\ |\beta| \leq 2}} \sum_{w \in \mathbf{k}_0} \left\| \int_{v \in \mathbb{R}^3} |w^j \widehat{Z}^\beta f| dv \right\|_{L^1(\Sigma_t)}.$$

Proof. As previously, we can restrict the proof to the case $j = 0$. We introduce $\delta = \frac{1}{4}$ for convenience and we suppose first that $t \geq 1$. The idea is classical and consists in splitting Σ_t into the three domains, $|x| \leq \frac{t}{2}$, $|x| \geq \frac{3}{2}t$ and $\frac{1}{2}t \leq |x| \leq \frac{3}{2}t$.

• *Step 1, the interior region.* Applying a local two-dimensional $L^2 - L^1$ Sobolev inequality to the function $x \mapsto \int_v |f|(t, tx, v) dv$, we get

$$\int_{|x| \leq \frac{1}{2}} \left| \int_v |f|(t, tx, v) dv \right|^2 dx_1 dx_2 dx_3 \lesssim \sum_{q=0}^1 \int_{|x_3| \leq \frac{1}{2}} \left| \int_{x_1^2 + x_2^2 \leq \frac{1}{4} - x_3^2 + \delta^2} \int_v ((t \partial_{x_1, x_2})^q |f|)(t, tx, v) dv dx_1 dx_2 \right|^2 dx_3.$$

As $t - |tx| \geq \frac{1}{4}t$ on the domain of integration since $|x| \leq \frac{1}{2} + \delta \leq \frac{3}{4}$, Lemmas 3.2.7 and 3.2.5 gives us

$$\int_{|x| \leq \frac{1}{2}} \left| \int_v |f|(t, tx, v) dv \right|^2 dx_1 dx_2 dx_3 \lesssim \sum_{|\beta| \leq 1} \int_{|x_3| \leq \frac{1}{2}} \left| \int_{x_1^2 + x_2^2 \leq \frac{1}{4} - x_3^2 + \delta^2} \int_v |\widehat{Z}^\beta f|(t, tx, v) dv dx_1 dx_2 \right|^2 dx_3.$$

Now, one can obtain similarly, using a one-dimensional $L^2 - L^1$ Sobolev inequality in the variable x_3 , that

$$\int_{|x| \leq \frac{1}{2}} \left| \int_v |f|(t, tx, v) dv \right|^2 dx_1 dx_2 dx_3 \lesssim \sum_{|\beta| \leq 2} \left| \int_{|x_3| \leq \frac{1}{2} + \delta} \int_{x_1^2 + x_2^2 \leq \frac{1}{4} - x_3^2 + \delta^2} \int_v |\widehat{Z}^\beta f|(t, tx, v) dv dx_1 dx_2 dx_3 \right|^2.$$

Since $\tau_+^2 \tau_- \lesssim t^3$ if $|x| \leq \frac{1}{2}t$, we finally obtain, by the change of variables $y = tx$,

$$\left\| \tau_+ \tau_-^{\frac{1}{2}} \int_{v \in \mathbb{R}^3} |f|(t, y, v) dv \right\|_{L^2(|y| \leq \frac{1}{2}t)} \lesssim t^3 \left\| \int_{v \in \mathbb{R}^3} |f|(t, tx, v) dv \right\|_{L^2(|x| \leq \frac{1}{2})} \lesssim \sum_{|\beta| \leq 2} \left\| \int_v |\widehat{Z}^\beta f| dv \right\|_{L^1(\Sigma_t)}. \quad (3.15)$$

• *Step 2, the exterior region.* Let us introduce, for $i \in \mathbb{N}$, the following sets⁹

$$X_i := \{y \times \Sigma_t / 3t \times 2^{i-1} \leq |y| < 3t \times 2^i\}, \quad \text{and} \quad Y_i := \{y \times \Sigma_t / 5t \times 2^i \leq 4|y| < 13t \times 2^i\}.$$

In the domain considered here, where $|x| \geq \frac{3}{2}t$, we have $\tau_+ \lesssim |x|$ but we cannot follow exactly what we have done for the interior region as we cannot view $|x|$ as a parameter. However, as for $i \in \mathbb{N}$, $2^i t \sim \tau_+$ on X_i and

$$\forall \frac{3}{2} - \delta \leq |x| \leq 3 + \delta, \quad |2^i t x| - t \geq \left(2^i \frac{5}{4} - 1\right) t \geq \frac{1}{4} \times 2^i t,$$

we can apply similar operations to $x \mapsto \int_v |f|(t, 2^i t x, v) dv$ as to $x \mapsto \int_v |f|(t, t x, v) dv$ previously and obtain

$$\begin{aligned} \int_{\frac{3}{2} \leq |x| \leq 3} \left| \int_v |f|(t, 2^i t x, v) dv \right|^2 dx &\lesssim \sum_{|\beta| \leq 2} \left| \int_{|x_3| \leq 3 + \delta} \int_{\frac{9}{4} - x_3^2 - \delta^2 \leq x_1^2 + x_2^2 \leq 9 - x_3^2 + \delta^2} \int_v \left| \widehat{Z}^\beta g \right|(t, 2^i t x, v) dv dx \right|^2 \\ &\lesssim \sum_{|\beta| \leq 2} \left| \int_{\frac{3}{4} \leq |x| \leq \frac{13}{4}} \int_v \left| \widehat{Z}^\beta g \right|(t, 2^i t x, v) dv dx \right|^2. \end{aligned}$$

As $\tau_+^2 \tau_- \lesssim 2^{3i} t^3$ on X_i , we finally obtain by the change of variables $y = 2^i t x$,

$$\left\| \tau_+ \tau_-^{\frac{1}{2}} \int_{v \in \mathbb{R}^3} |f| dv \right\|_{L^2(X_i)} \lesssim 2^{3i} t^3 \left\| \int_{v \in \mathbb{R}^3} |f|(t, 2^i t x, v) dv \right\|_{L^2(\frac{3}{2} \leq |x| \leq 3)} \lesssim \sum_{|\beta| \leq 2} \left\| \int_v \left| \widehat{Z}^\beta f \right| dv \right\|_{L^1(Y_i)}.$$

As $Y_i \cap Y_j = \emptyset$ if $|i - j| \geq 2$, $X_i \cap X_j = \emptyset$ if $i \neq j$ and since $\{y \in \Sigma_t / |y| \geq \frac{3}{2}t\} = \cup_{i=0}^{+\infty} X_i$, we get

$$\left\| \tau_+ \tau_-^{\frac{1}{2}} \int_{v \in \mathbb{R}^3} |f|(t, y, v) dv \right\|_{L^2(|y| \geq \frac{3}{2}t)} \lesssim \sum_{|\beta| \leq 2} \left\| \int_v \left| \widehat{Z}^\beta f \right| dv \right\|_{L^1(\Sigma_t)}. \quad (3.16)$$

• *Step 3, the remaining domain.* We now focus on the region $\frac{1}{2}t \leq |x| \leq \frac{3}{2}t$. We will obtain the τ_+ integrated decay with the rotational vector fields through Sobolev inequalities on the spheres. To obtain the $\sqrt{\tau_-}$ decay, note first that $|u| \leq \frac{1}{2}t$ in this region (recall that $u = t - |x|$). The idea to capture the decay in u will then be to divide the domain in the disjoint union of the sets

$$V_i := \{y \in \Sigma_t / 2^{-i-1}t < |t - |y|| \leq 2^{-i}t\}, \quad i \in \mathbb{N}^*.$$

Let $\omega \in \mathbb{S}^2$. Applying a $L^2 - L^1$ Sobolev inequality to $g : s \mapsto \int_v |f|(t, t(1 - 2^{-i}s)\omega, v) dv$, we obtain

$$\int_{\frac{1}{2} \leq |s| \leq 1} \left| \int_v |f|(t, t(1 - 2^{-i}s)\omega, v) dv \right|^2 ds \lesssim \sum_{j=0}^1 \left| \int_{\frac{1}{4} \leq |s| \leq \frac{5}{4}} \left| \int_v (t 2^{-i} \partial_r |f|)^j(t, t(1 - 2^{-i}s)\omega, v) dv \right| ds \right|^2.$$

Since $\frac{1}{4} 2^{-i}t \leq |t - |t - 2^{-i}ts||$ for all $i \in \mathbb{N}^*$ and $\frac{1}{4} \leq |s| \leq \frac{5}{4}$, it comes, using Lemmas 3.2.7 and 3.2.5 that

$$\int_{\frac{1}{2} \leq |s| \leq 1} \left| \int_v |f|(t, t(1 - 2^{-i}s)\omega, v) dv \right|^2 ds \lesssim \sum_{|\kappa| \leq 1} \left| \int_{\frac{1}{4} \leq |s| \leq \frac{5}{4}} \left| \int_v \left| \widehat{Z}^\kappa f \right|(t, t(1 - 2^{-i}s)\omega, v) dv \right| ds \right|^2.$$

The change of variables $r = t(1 - 2^{-i}s)$ gives

$$t 2^{-i} \int_{2^{-i-1}t \leq |t-r| \leq 2^{-i}t} \left| \int_v |f|(t, r\omega, v) dv \right|^2 dr \lesssim \sum_{|\kappa| \leq 1} \left| \int_{2^{-i-2}t \leq |t-r| \leq 5 \times 2^{-i-2}t} \left| \int_v \left| \widehat{Z}^\kappa f \right|(t, r\omega, v) dv \right| dr \right|^2.$$

As previously with the domains X_i and Y_i , we take the sum over $i \in \mathbb{N}^*$ and we get

$$\int_{\frac{1}{2}t \leq r \leq \frac{3}{2}t} |t - r| \left| \int_v |f|(t, r\omega, v) dv \right|^2 dr \lesssim \sum_{|\kappa| \leq 1} \left| \int_{\frac{3}{8}t \leq r \leq \frac{13}{8}t} \left| \int_v \left| \widehat{Z}^\kappa f \right|(t, r\omega, v) dv \right| dr \right|^2.$$

⁹The constants hidden in \lesssim in the upcoming computations will not depend on i .

By simpler operations, one can also obtain that

$$\int_{t-\frac{1}{2} \leq r \leq t+\frac{1}{2}} \left| \int_v |f|(t, r\omega, v) dv \right|^2 dr \lesssim \sum_{|\kappa| \leq 1} \left| \int_{t-\frac{5}{8} \leq r \leq t+\frac{5}{8}} \left| \int_v |\widehat{Z}^\kappa f|(t, r\omega, v) dv \right| dr \right|^2.$$

Integrating each side of these inequalities over \mathbb{S}^2 and applying Proposition 3.3.5 to the right hand sides, we get

$$\int_{\frac{1}{2}t \leq r \leq \frac{3}{2}t} \int_{\omega \in \mathbb{S}^2} \tau_- \left| \int_v |f|(t, r\omega, v) dv \right|^2 d\mathbb{S}^2 dr \lesssim \sum_{|\kappa| \leq 2} \left| \int_{\frac{3}{8}t \leq r \leq \frac{13}{8}t} \int_{\omega \in \mathbb{S}^2} \left| \int_v |\widehat{Z}^\kappa f|(t, r\omega, v) dv \right| d\mathbb{S}^2 dr \right|^2.$$

Finally, multiply both side of the inequality by t^2 and use $\tau_+ \lesssim t \leq 2r$ on the domain of integration in order to obtain

$$\int_{\frac{1}{2}t \leq r \leq \frac{3}{2}t} \int_{\omega \in \mathbb{S}^2} \tau_+^2 \tau_- \left| \int_v |f|(t, r\omega, v) dv \right|^2 d\mathbb{S}^2 r^2 dr \lesssim \sum_{|\kappa| \leq 2} \left| \int_{\frac{3}{8}t \leq r \leq \frac{13}{8}t} \int_{\omega \in \mathbb{S}^2} \left| \int_v |\widehat{Z}^\kappa f|(t, r\omega, v) dv \right| d\mathbb{S}^2 r^2 dr \right|^2 \quad (3.17)$$

The result then follows from (3.15), (3.16) and (3.17). The case $t \leq 1$ can be treated similarly, repeating the arguments of Steps 1 and 2 since in that case $\tau_+^2 \tau_- \lesssim (1+r)^3$ and

$$\Sigma_t = \{y \in \Sigma_t / |y| \leq 2^{-1}\} \cup (\cup_{i=0}^{+\infty} \{y \in \Sigma_t / 2^{i-1} \leq |y| < 2^i\}).$$

□

Pointwise Decay estimates for the electromagnetic field

In this section, we follow mostly [11]. We first present certain identities and inequalities between quantities linked to the null decomposition of a 2-form (see Section 3.2.3 for its definition), then we recall Sobolev inequalities and, finally, we prove the desired pointwise decay estimates for the electromagnetic field.

For the remaining of this section, we consider G a 2-form and J a 1-form, both sufficiently regular and defined on $[0, T] \times \mathbb{R}^3$, such that

$$\begin{aligned} \nabla^\mu G_{\mu\nu} &= J_\nu, \\ \nabla^\mu * G_{\mu\nu} &= 0. \end{aligned}$$

Aside from Lemma 3.3.10 and the estimate on $\alpha(G)$ in Proposition 3.3.13, all the result of this subsection apply to a general 2-form.

• Preparatory results.

To lighten the presentation, we prove the three upcoming lemmas in Appendix 3.D.

Lemma 3.3.8. *Let $\Omega \in \mathbb{O}$. Then, the operators \mathcal{L}_Ω and ∇_{∂_r} commute with the null decomposition of G as well as with each other, i.e., denoting by ζ any of the null component α , $\underline{\alpha}$, ρ or σ ,*

$$[\mathcal{L}_\Omega, \nabla_{\partial_r}]G = 0, \quad \mathcal{L}_\Omega(\zeta(G)) = \zeta(\mathcal{L}_\Omega(G)) \quad \text{and} \quad \nabla_{\partial_r}(\zeta(G)) = \zeta(\nabla_{\partial_r}(G)).$$

Similar results hold for \mathcal{L}_Ω and ∇_{∂_t} , ∇_L or $\nabla_{\underline{L}}$. For instance, $\nabla_L(\zeta(G)) = \zeta(\nabla_L(G))$.

We now give a more precise version of Lemma 3.3 of [11].

Lemma 3.3.9. *Denoting by ζ any of the null component α , $\underline{\alpha}$, ρ or σ , we have*

$$\tau_- |\nabla_{\underline{L}} \zeta(G)| + \tau_+ |\nabla_L \zeta(G)| \lesssim \sum_{|\gamma| \leq 1} |\zeta(\mathcal{L}_{Z^\gamma}(G))| \quad \text{and} \quad (1+r) |\nabla \zeta(G)| \lesssim |\zeta(G)| + \sum_{\Omega \in \mathbb{O}} |\zeta(\mathcal{L}_\Omega(G))|.$$

The following equation will be useful in order to obtain a strong decay estimate on $\alpha(G)$.

Lemma 3.3.10. *Denoting by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of G , we have*

$$\nabla_{\underline{L}} \alpha_A - \frac{\alpha_A}{r} + \nabla_{e_A} \rho + \varepsilon_{BA} \nabla_{e_B} \sigma = J_A. \quad (3.18)$$

The following result will allow us to treat part of the interior of the light cone.

Lemma 3.3.11. *Let U be a smooth tensor field defined on $[0, T[\times \mathbb{R}^3$. Then,*

$$\forall t \in [0, T[, \quad \sup_{|x| \leq 1 + \frac{t}{2}} |U(t, x)| \lesssim \frac{1}{(1+t)^{\frac{5}{2}}} \sum_{|\gamma| \leq 2} \|\tau_- \mathcal{L}_{Z^\gamma}(U)(t, y)\|_{L^2(|y| \leq 2 + \frac{3}{4}t)}.$$

Proof. As $|\mathcal{L}_{Z^\gamma}(U)| \lesssim \sum_{|\beta| \leq |\gamma|} \sum_{\mu, \nu} |Z^\beta(U_{\mu\nu})|$, it suffices to prove the result for each component of the tensor and we can assume that U is a scalar function. Let $(t, x) \in [0, T[\times \mathbb{R}^3$ such that $|x| \leq 1 + \frac{t}{2}$. Apply a standard L^2 Sobolev inequality to $V : y \mapsto U(t, x + \frac{1+t}{4}y)$ and then make a change of variables to get

$$|U(t, x)| = |V(0)| \lesssim \sum_{|\beta| \leq 2} \|\partial_x^\beta V\|_{L_y^2(|y| \leq 1)} \lesssim \left(\frac{1+t}{4}\right)^{-\frac{3}{2}} \sum_{|\beta| \leq 2} \left(\frac{1+t}{4}\right)^{|\beta|} \|\partial_x^\beta U(t, \cdot)\|_{L_y^2(|y-x| \leq \frac{1+t}{4})}.$$

Observe now that $|y-x| \leq \frac{1+t}{4}$ implies $|y| \leq 2 + \frac{3}{4}t$ and that $1+t \lesssim \tau_-$ on that domain. Consequently, using Lemma 3.2.7 and that $[Z, \partial]$, for $Z \in \mathbb{K}$, is either 0 or a translation, we have

$$(1+t)^{|\beta|+1} \|\partial_x^\beta U(t, \cdot)\|_{L_y^2(|y-x| \leq \frac{1+t}{4})} \lesssim \|\tau_-^{|\beta|+1} \partial_x^\beta U(t, \cdot)\|_{L_y^2(|y| \leq 2 + \frac{3}{4}t)} \lesssim \sum_{|\gamma| \leq |\beta|} \|\tau_- Z^\gamma U(t, \cdot)\|_{L_y^2(|y| \leq 2 + \frac{3}{4}t)}.$$

□

We refer to Lemma 2.3 of [11] for a proof of the following two Sobolev inequalities, which will permit us to deal with the remaining region.

Lemma 3.3.12. *Let U be a sufficiently regular tensor field defined on \mathbb{R}^3 and denote $\sum_{|\beta| \leq k} |\mathcal{L}_{\Omega^\beta}(U)|^2$, where $\Omega^\beta \in \mathbb{O}^{|\beta|}$, by $|U|_{\mathbb{O}, k}^2$. There exists an absolute constant $C > 0$, independent of U , such that*

$$\begin{aligned} \forall t \in \mathbb{R}_+, \quad \forall |x| \geq \frac{1}{2}t + 1, \quad |U(x)| &\leq \frac{C}{|x|^{\frac{3}{2}}} \left(\int_{|y| \geq \frac{1}{2}t + 1} |U(y)|_{\mathbb{O}, 2}^2 + \tau_-^2 |\nabla_{\partial_r} U(y)|_{\mathbb{O}, 1}^2 dy \right)^{\frac{1}{2}}, \\ \forall x \neq 0, \quad |U(x)| &\leq \frac{C}{|x|^{\frac{3}{2}}} \left(\int_{|y| \geq |x|} |U(y)|_{\mathbb{O}, 2}^2 + |y|^2 |\nabla_{\partial_r} U(y)|_{\mathbb{O}, 1}^2 dy \right)^{\frac{1}{2}}. \end{aligned}$$

• **Decay estimates for G .**

We are now ready to prove the pointwise decay estimates on the electromagnetic field.

Proposition 3.3.13. *Let $k \in \mathbb{N}^*$. Then, we have for all $(t, x) \in [0, T[\times \mathbb{R}^3$,*

$$\begin{aligned} |\rho|(t, x) + |\sigma|(t, x) &\lesssim \frac{\sqrt{\mathcal{E}_2[G](t)}}{\tau_+^2 \tau_-^{\frac{1}{2}}}, & |\alpha|(t, x) &\lesssim \frac{\sqrt{\mathcal{E}_2^k[G](t)} \log^{\frac{k}{2}}(1 + \tau_-)}{\tau_+ \tau_-^{\frac{3}{2}}}, \\ |\alpha|(t, x) &\lesssim \frac{\sqrt{\mathcal{E}_2[G](t)} + \sum_{|\kappa| \leq 1} \|r^2 \mathcal{L}_{Z^\kappa}(J)_A\|_{L^2(\Sigma_t)}}{\tau_+^{\frac{5}{2}}}. \end{aligned}$$

Proof. We fix for all this proof $(t, x) \in [0, T[\times \mathbb{R}^3$. If $|x| \leq 1 + \frac{1}{2}t$, the result follows from Proposition 3.3.11. We then suppose $|x| \geq 1 + \frac{t}{2}$. During this proof, Ω^β will always denote a combination of rotational vector fields, i.e. $\Omega^\beta \in \mathbb{O}^{|\beta|}$. Let ζ be either α , ρ or σ . As ∇_{∂_r} and \mathcal{L}_Ω commute with the null decomposition (see Lemma 3.3.8), Lemma 3.3.12 gives us

$$r^4 \tau_- |\zeta|^2 \lesssim \int_{|y| \geq \frac{t}{2} + 1} |r\zeta|_{\mathbb{O}, 2}^2 + \tau_-^2 |\nabla_{\partial_r}(r\zeta)|_{\mathbb{O}, 1}^2 dy \lesssim \sum_{|\beta| \leq 1} \sum_{|\gamma| \leq 2} \int_{|y| \geq \frac{t}{2} + 1} r^2 |\zeta(\mathcal{L}_{Z^\gamma}(G))|^2 + \tau_-^2 r^2 |\zeta(\mathcal{L}_{\Omega^\beta}(\nabla_{\partial_r} G))|^2 dy.$$

As ∇_{∂_r} commute with \mathcal{L}_Ω as well as with the null decomposition (see Lemma 3.3.8), we have, using $2\partial_r = L - \underline{L}$ and Lemma 3.3.9,

$$|\zeta(\mathcal{L}_\Omega(\nabla_{\partial_r} G))| + |\zeta(\nabla_{\partial_r} G)| \lesssim |\nabla_L \zeta(\mathcal{L}_\Omega(G))| + |\nabla_{\underline{L}} \zeta(\mathcal{L}_\Omega(G))| + |\nabla_L \zeta(G)| + |\nabla_{\underline{L}} \zeta(G)| \lesssim \frac{1}{\tau_-} \sum_{|\gamma| \leq 2} |\zeta(\mathcal{L}_{Z^\gamma}(G))|. \quad (3.19)$$

Since $\tau_+ \lesssim r \leq \tau_+$ in the region considered, we finally obtain

$$\tau_+^4 \tau_- |\zeta|^2 \lesssim \sum_{|\gamma| \leq 2} \int_{|y| \geq \frac{t}{2} + 1} \tau_+^2 |\zeta(\mathcal{L}_{Z^\gamma}(G))|^2 dx \lesssim \mathcal{E}_2[G](t).$$

We improve now the estimate on α . As $\nabla^\mu \mathcal{L}_\Omega(G)_{\mu\nu} = \mathcal{L}_\Omega(J)_\nu$ and $\nabla^{\mu*} \mathcal{L}_\Omega(G)_{\mu\nu} = 0$ for all $\Omega \in \mathbb{O}$, we have according to (3.18) that

$$\forall |\beta| \leq 1, \quad \nabla_{\underline{L}} \alpha(\mathcal{L}_{\Omega^\beta}(G))_A = \frac{1}{r} \alpha(\mathcal{L}_{\Omega^\beta}(G))_A - \nabla_{e_A} \rho(\mathcal{L}_{\Omega^\beta}(G)) + \varepsilon_{AB} \nabla_{e_B} \sigma(\mathcal{L}_{\Omega^\beta}(G)) + \mathcal{L}_{\Omega^\beta}(J)_A.$$

Consequently, we get using Lemma 3.3.9 that for all $\Omega \in \mathbb{O}$,

$$|\alpha(\nabla_{\partial_r} G)| + |\alpha(\mathcal{L}_\Omega(\nabla_{\partial_r} G))| \lesssim |J_A| + |\mathcal{L}_\Omega(J)_A| + \frac{1}{r} \sum_{|\gamma| \leq 2} |\alpha(\mathcal{L}_{Z^\gamma}(G))| + |\rho(\mathcal{L}_{Z^\gamma}(G))| + |\sigma(\mathcal{L}_{Z^\gamma}(G))|. \quad (3.20)$$

Applying the second inequality of Lemma 3.3.12 and using this time (3.20) instead of (3.19), it comes

$$\tau_+^5 |\alpha|^2 \lesssim |x|^5 |\alpha|^2 \lesssim \int_{|y| \geq |x|} |r\alpha|_{\mathbb{O},2}^2 + r^2 |\nabla_{\partial_r}(r\alpha)|_{\mathbb{O},1}^2 dy \lesssim \mathcal{E}_2[G](t) + \sum_{|\kappa| \leq 1} \|r^2 \mathcal{L}_{Z^\kappa}(J)_A\|_{L^2(\Sigma_t)}^2.$$

Applying the first inequality of Lemma 3.3.12 to $\tau_- \log^{-\frac{k}{2}}(1 + \tau_-) \underline{\alpha}$ and using the same arguments as previously, one have

$$\frac{r^2 \tau_-^3}{\log^k(1 + \tau_-)} |\underline{\alpha}|^2 \lesssim \int_{|y| \geq \frac{t}{2} + 1} \left| \frac{\tau_-}{\log^{\frac{k}{2}}(1 + \tau_-)} \underline{\alpha} \right|_{\mathbb{O},2}^2 + \tau_-^2 \left| \nabla_{\partial_r} \left(\frac{\tau_-}{\log^{\frac{k}{2}}(1 + \tau_-)} \underline{\alpha} \right) \right|_{\mathbb{O},1}^2 dy \lesssim \mathcal{E}_2^k[G](t).$$

□

3.4 The null structure of the non linearity $\mathcal{L}_{Z^\gamma}(F) \left(v, \nabla_v \widehat{Z}^\beta f \right)$

In order to take advantage of the null structure of the Vlasov equation, we will expand quantities such as $\mathcal{L}_{Z^\gamma}(F)(v, \nabla_v g)$, with g a regular function, in null coordinates. We then use the following lemma.

Lemma 3.4.1. *Let G be a sufficiently regular 2-form, $(\alpha, \underline{\alpha}, \rho, \sigma)$ its null components and g a sufficiently regular function. Then,*

$$|G(v, \nabla_v g)| \lesssim \left(\tau_- |\rho| + \tau_+ |\alpha| + \tau_+ \frac{|v^A|}{v^0} |\sigma| + \tau_- \frac{|v^A|}{v^0} |\underline{\alpha}| + \tau_+ \frac{v^{\underline{L}}}{v^0} |\alpha| \right) \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |\widehat{Z}g|.$$

Proof. Expanding $G(v, \nabla_v g)$ with null components, we obtain

$$\begin{aligned} G(v, \nabla_v g) &= 2\rho \left(v^{\underline{L}} (\nabla_v g)^{\underline{L}} - v^{\underline{L}} (\nabla_v g)^{\underline{L}} \right) + v^B \varepsilon_{BA} \sigma (\nabla_v g)^A - v^{\underline{L}} \alpha_A (\nabla_v g)^A + v^A \alpha_A (\nabla_v g)^{\underline{L}} \\ &\quad - v^{\underline{L}} \underline{\alpha}_A (\nabla_v g)^A + v^A \underline{\alpha}_A (\nabla_v g)^{\underline{L}}. \end{aligned} \quad (3.21)$$

We bound the angular components of $\nabla_v g$ by merely using $v^0 \partial_{v^i} = \widehat{\Omega}_{0i} - t \partial_i - x^i \partial_t$. The radial component¹⁰ have a better behavior since

$$v^0 (\nabla_v g)^r = \frac{x^i}{r} v^0 \partial_{v^i} g = \frac{x^i}{r} \widehat{\Omega}_{0i} g - Sg + (t - r) \underline{L}g. \quad (3.22)$$

□

Remark 3.4.2. *Let us explain how this lemma reflects the null structure of the system. For this, we write $D_1 \prec D_2$ if D_2 is expected to behave better than D_1 . Recall that we have the following hierarchies between the null components of G , v and $\nabla_v g$.*

- $\underline{\alpha} \prec \rho \sim \sigma \prec \alpha$,
- $v^{\underline{L}} \prec v^A \prec v^{\underline{L}}$,
- $(\nabla_v g)^A \prec (\nabla_v g)^r$.

We can then notice that $\underline{\alpha}$ is hit by $v^{\underline{L}}$ or $v^A (\nabla_v g)^r$, ρ by $(\nabla_v g)^r$ and σ by v^A .

¹⁰Note that $(\nabla_v g)^{\underline{L}} = -(\nabla_v g)^{\underline{L}} = (\nabla_v g)^r$.

3.5 Bootstrap assumptions and strategy of the proof

Let $N \geq 10$ and (f^0, F^0) be an initial data set satisfying the assumptions of Theorem 3.1.1. Then, by a local well-posedness argument, there exists a unique maximal solution (f, F) arising from this data to the system¹¹

$$\begin{aligned} T_F^\chi(f) &= 0, \\ \nabla^\mu F_{\mu\nu} &= J(f)_\nu, \\ \nabla^{\mu*} F_{\mu\nu} &= 0. \end{aligned}$$

Applying Proposition 3.B.1 and considering possibly $\epsilon_1 = C_1\epsilon$, with C_1 a constant depending only on N , we can suppose without loss of generality that $\mathbb{E}_N^2[f](0) \leq \epsilon$ and $\mathcal{E}_N[F](0) \leq \epsilon$. Let $T^* > 0$ such that $[0, T^*[$ is the maximum domain of (f, F) and $T \in]0, T^*[$ be the largest time such that¹², for all $t \in [0, T]$,

$$\mathbb{E}_{N-2}^2[f](t) \leq 4\epsilon, \quad (3.23)$$

$$\mathbb{E}_N^1[f](t) \leq 4\epsilon \log(3+t), \quad (3.24)$$

$$\sum_{|\beta|=N-1} \left\| r^2 \int_v \frac{v^A}{v^0} \widehat{Z}^\beta f dv \right\|_{L^2(\Sigma_t)} \leq \sqrt{\epsilon} \log(3+t), \quad (3.25)$$

$$\mathcal{E}_N[F](t) \leq 4\epsilon \log^4(3+t), \quad (3.26)$$

$$\mathcal{E}_N^5[F](t) \leq 2\underline{C}\epsilon, \quad (3.27)$$

where $\underline{C} > 0$ is a positive constant which will be specified later. The third bootstrap assumption is here for convenience, we could avoid it but it would complicate the proof. Before presenting our strategy, let us write the immediate consequences of these bootstrap assumptions. Using the Klainerman-Sobolev inequality of Proposition 3.3.6 and the bootstrap assumption (3.23), one have

$$\forall (t, x) \in [0, T[\times \mathbb{R}^3, \quad z \in \mathbf{k}_0, \quad |\beta| \leq N-5, \quad \int_v \left| z^2 \widehat{Z}^\beta f \right| dv \lesssim \frac{\epsilon}{\tau_+^2 \tau_-}. \quad (3.28)$$

Applying the Klainerman-Sobolev inequality of Proposition 3.3.7, Lemma 3.2.11 and using (3.23) and (3.24), we get

$$\forall t \in [0, T[, \quad \sum_{\substack{|\beta| \leq N-4 \\ z \in \mathbf{k}_0}} \left\| \tau_+ \sqrt{\tau_-} \int_v \left| z^2 \widehat{Z}^\beta f \right| dv \right\|_{L^2(\Sigma_t)} \lesssim \epsilon, \quad \sum_{|\beta| \leq N-2} \left\| r^2 \int_v \frac{v^A}{v^0} \widehat{Z}^\beta f dv \right\|_{L^2(\Sigma_t)} \lesssim \epsilon \log(3+t). \quad (3.29)$$

By Proposition 3.3.13, commutation formula of Proposition 3.2.9, the bootstrap assumptions (3.26), (3.27), (3.25) and the estimate (3.29), we obtain that, for all $(t, x) \in [0, T[\times \mathbb{R}^3$ and $|\gamma| \leq N-2$,

$$|\rho(\mathcal{L}_{Z^\gamma}(F))|(t, x) \lesssim \sqrt{\epsilon} \frac{\log^2(3+t)}{\tau_+^2 \tau_-^{\frac{1}{2}}}, \quad |\alpha(\mathcal{L}_{Z^\gamma}(F))|(t, x) \lesssim \sqrt{\epsilon} \frac{\log^2(3+t)}{\tau_+^{\frac{5}{2}}}, \quad (3.30)$$

$$|\sigma(\mathcal{L}_{Z^\gamma}(F))|(t, x) \lesssim \sqrt{\epsilon} \frac{\log^2(3+t)}{\tau_+^2 \tau_-^{\frac{1}{2}}}, \quad |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|(t, x) \lesssim \sqrt{\epsilon} \frac{\log^{\frac{5}{2}}(1+\tau_-)}{\tau_+ \tau_-^{\frac{3}{2}}}. \quad (3.31)$$

Applying Proposition 3.A.1, one obtain that f vanishes for small velocities, i.e.

$$\forall t \in [0, T[, \quad x \in \mathbb{R}^3, \quad 0 < |v| \leq 1, \quad f(t, x, v) = 0. \quad (3.32)$$

In view of the support of χ , we then obtain that $T_F(f) = 0$ on $[0, T[$, so that (f, F) is the unique classical solution to the Vlasov-Maxwell system (3.1)-(3.3) on $[0, T[$. The remaining of the article is devoted to the improvement of the bootstrap assumptions (3.23)-(3.27), which will imply Theorem 3.1.1 as it will prove that $T = T^*$ and then $T^* = +\infty$. The proof is divided in three parts.

1. First, we improve the bootstrap assumptions (3.23) and (3.24) by using Proposition 3.3.1. To bound the spacetime integrals arising from this energy estimate, we make crucial use of the null structure of the non linearity $\mathcal{L}_{Z^\gamma}(F)(v, \nabla_v \widehat{Z}^\beta f)$ as well as (3.30), (3.31) and (3.28).

¹¹We refer to Subsection 3.2.2 for the reasons which bring us to define (f, F) as a solution to these equations rather than the Vlasov-Maxwell system.

¹²Notice that such a $T > 0$ exists by a standard continuity argument.

2. Then, in view of improving (3.25), (3.26) and (3.27), the next step consists in proving L^2 estimates on quantities such as $\int_v |z \widehat{Z}^\beta f| dv$. To treat the higher order derivatives, we rewrite all transport equations as an inhomogeneous system of Vlasov equations. To handle the homogenous part, we take advantage of the smallness assumption on the $N + 3$ derivatives of f at $t = 0$, (3.30) and (3.31). The inhomogeneous part G will be schematically decomposed as a product KY , with $\int_v |Y| dv$ a decaying function and $|K|^2 Y$ an integrable function in (x, v) .
3. Finally, we improve the bounds on the energy norms of the electromagnetic field through Proposition 3.3.4. The null structure of the source terms of the Maxwell equations is fundamental for us here.

3.6 Improvement of the energy bound on the particle density

The purpose of this section is to improve the bootstrap assumptions (3.23) and (3.24). Note first that

$$\forall z \in \mathbf{k}_0, \quad q \in \{1, 2\}, \quad |\beta| \leq N, \quad T_F(z^q \widehat{Z}^\beta f) = T_F(z^q) \widehat{Z}^\beta f + z^q T_F(\widehat{Z}^\beta f) = qz^{q-1} F(v, \nabla_v z) \widehat{Z}^\beta f + z^q T_F(\widehat{Z}^\beta f)$$

and recall from (3.32) that $T_F(f) = 0$ on $[0, T[$. Thus, in view of the energy estimate of Proposition 3.3.1, $\mathbb{E}_{N-2}^2[f](0) \leq \epsilon$ and commutation formula of Proposition 3.2.9, $\mathbb{E}_{N-2}^2[f] \leq 3\epsilon$ on $[0, T[$ ensues, if ϵ is small enough, from the following proposition.

Proposition 3.6.1. *Let $z \in \mathbf{k}_0$, $|\zeta| \leq N - 2$, $|\gamma| \leq N - 2$ and $|\xi| \leq N - 3$. Then,*

$$\begin{aligned} I_\zeta^{z,2} &:= \int_0^t \int_{\Sigma_s} \int_v \left| z F(v, \nabla_v z) \widehat{Z}^\zeta f \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}, \\ K_{\gamma,\xi}^{z,2} &:= \int_0^t \int_{\Sigma_s} \int_v \left| z^2 \mathcal{L}_{Z^\gamma}(F) \left(v, \nabla_v \widehat{Z}^\xi f \right) \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

Similarly, the following result implies, if ϵ is small enough, that $\mathbb{E}_N^1[f](t) \leq 3\epsilon \log(3+t)$ for all $t \in [0, T[$.

Proposition 3.6.2. *Let $z \in \mathbf{k}_0$, $|\zeta| \leq N$, γ and ξ such that $|\gamma| + |\xi| \leq N$ and $|\xi| \leq N - 1$. We have,*

$$\begin{aligned} I_\zeta^{z,1} &:= \int_0^t \int_{\Sigma_s} \int_v \left| F(v, \nabla_v z) \widehat{Z}^\zeta f \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} \log(3+t), \\ K_{\gamma,\xi}^{z,1} &:= \int_0^t \int_{\Sigma_s} \int_v \left| z \mathcal{L}_{Z^\gamma}(F) \left(v, \nabla_v \widehat{Z}^\xi f \right) \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} \log(3+t). \end{aligned}$$

The proofs are based on the analysis, through Lemma 3.4.1, of quantities such as $\mathcal{L}_{Z^\gamma}(F) \left(v, \nabla_v \widehat{Z}^\beta f \right)$. We then prove the following preparatory lemma.

Lemma 3.6.3. *Let $|\gamma| \leq N - 2$ and $h : [0, T[\times \mathbb{R}_x^3 \times (\mathbb{R}^3 \setminus \{0\})$ be a sufficiently regular function. Then,*

$$|\mathcal{L}_{Z^\gamma}(F) \left(v, \nabla_v h \right)| \lesssim \left(\frac{\sqrt{\epsilon}}{\tau_+^{\frac{5}{4}}} + \frac{\sqrt{\epsilon v^L}}{\tau_-^{\frac{5}{4}} v^0} \right) \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |\widehat{Z}h| \quad \text{and} \quad |F(v, \nabla_v z)| \lesssim \left(\frac{\sqrt{\epsilon}}{\tau_+^{\frac{5}{4}}} + \frac{\sqrt{\epsilon v^L}}{\tau_-^{\frac{5}{4}} v^0} \right) \sum_{w \in \mathbf{k}_0} |w|.$$

Proof. Let $(\alpha, \underline{\alpha}, \rho, \sigma)$ be the null decomposition of $\mathcal{L}_{Z^\gamma}(F)$. Using Lemma 3.4.1, we have

$$|\mathcal{L}_{Z^\gamma}(F) \left(v, \nabla_v h \right)| \lesssim \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \left(\tau_- |\rho| + \tau_+ |\alpha| + \tau_+ \frac{|v^A|}{v^0} |\sigma| + \tau_- \frac{|v^A|}{v^0} |\underline{\alpha}| + \tau_+ \frac{v^L}{v^0} |\underline{\alpha}| \right) |\widehat{Z}h|.$$

According to the pointwise estimates (3.30), (3.31) and the inequality $|v^A| \lesssim \sqrt{v^0 v^L}$ (see Lemma 3.2.11), one have

$$\tau_- |\rho| + \tau_+ |\alpha| \lesssim \frac{\sqrt{\epsilon}}{\tau_+^{\frac{5}{4}}}, \quad \tau_+ \frac{v^L}{v^0} |\underline{\alpha}| \lesssim \frac{\sqrt{\epsilon v^L}}{\tau_-^{\frac{5}{4}} v^0}, \quad \frac{|v^A|}{v^0} (\tau_+ |\sigma| + \tau_- |\underline{\alpha}|) \lesssim \frac{\sqrt{\epsilon} \sqrt{v^L}}{\tau_+^{\frac{3}{4}} \tau_-^{\frac{1}{2}} \sqrt{v^0}} \lesssim \frac{\sqrt{\epsilon}}{\tau_+^{\frac{5}{4}}} + \frac{\sqrt{\epsilon v^L}}{\tau_-^{\frac{5}{4}} v^0}, \quad (3.33)$$

which implies the first inequality. The second one follows directly since, by Lemma 3.2.10, $\sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |\widehat{Z}(z)| \lesssim \sum_{w \in \mathbf{k}_0} |w|$. \square

The remaining of the section is devoted to the proof of Propositions 3.6.1 and 3.6.2.

3.6.1 Proof of Proposition 3.6.1

Let $z \in \mathbf{k}_0$, $|\zeta| \leq N - 2$, $|\gamma| \leq N - 2$ and $|\xi| \leq N - 3$. Using successively Lemma 3.6.3, that $1 \leq v^0$ on the support of f (see (3.32)), that $[0, t] \times \mathbb{R}^3$ can be foliated by $(C_u(t))_{u \leq t}$ (see Lemma 3.2.13) and $\mathbb{E}_{N-2}^2[f] \leq 4\epsilon$, which comes from the bootstrap assumption (3.23), we have,

$$\begin{aligned}
I_\zeta^{z,2} + K_{\gamma,\xi}^{z,2} &\lesssim \sum_{|\beta| \leq N-2} \sum_{w \in \mathbf{k}_0} \int_0^t \int_{\Sigma_s} \int_v \frac{\sqrt{\epsilon}}{\tau_+^{\frac{5}{4}}} \left| w^2 \widehat{Z}^\beta f \right| \frac{dv}{v^0} dx ds + \int_0^t \int_{\Sigma_s} \int_v \frac{\sqrt{\epsilon} v^L}{\tau_-^{\frac{5}{4}} v^0} \left| w^2 \widehat{Z}^\beta f \right| \frac{dv}{v^0} dx ds \\
&\lesssim \sum_{\substack{|\beta| \leq N-2 \\ w \in \mathbf{k}_0}} \int_0^t \frac{\sqrt{\epsilon}}{(1+s)^{\frac{5}{4}}} \int_{\Sigma_s} \int_v \left| w^2 \widehat{Z}^\beta f \right| dv dx ds + \int_{u=-\infty}^t \int_{C_u(t)} \int_v \frac{\sqrt{\epsilon} v^L}{\tau_-^{\frac{5}{4}} v^0} \left| w^2 \widehat{Z}^\beta f \right| dv dC_u(t) du \\
&\lesssim \sqrt{\epsilon} \sum_{|\beta| \leq N-2} \sum_{w \in \mathbf{k}_0} \int_0^t \frac{\mathbb{E}[w^2 \widehat{Z}^\beta f](s)}{(1+s)^{\frac{5}{4}}} ds + \int_{u=-\infty}^t \frac{1}{\tau_-^{\frac{5}{4}}} \int_{C_u(t)} \int_v \frac{v^L}{v^0} \left| w^2 \widehat{Z}^\beta f \right| dv dC_u(t) du \\
&\lesssim \sqrt{\epsilon} \int_0^t \frac{\mathbb{E}_{N-2}^2[f](s)}{(1+s)^{\frac{5}{4}}} ds + \sqrt{\epsilon} \int_{u=-\infty}^t \frac{1}{\tau_-^{\frac{5}{4}}} \mathbb{E}_{N-2}^2[f](t) du \lesssim \epsilon^{\frac{3}{2}} \int_0^{+\infty} \frac{ds}{(1+s)^{\frac{5}{4}}} + \epsilon^{\frac{3}{2}} \int_{u=-\infty}^{+\infty} \frac{du}{\tau_-^{\frac{5}{4}}}.
\end{aligned}$$

We then deduce that $I_\zeta^{z,2} + K_{\gamma,\xi}^{z,2} \lesssim \epsilon^{\frac{3}{2}}$, which concludes the proof of Proposition 3.6.1.

3.6.2 Proof of Proposition 3.6.2

We fix $z \in \mathbf{k}_0$, ζ , γ and ξ satisfying $|\zeta| \leq N$, $|\gamma| + |\xi| \leq N$ and $|\xi| \leq N - 1$. Suppose first that $|\gamma| \leq N - 2$. Hence, following the computations of Subsection 3.6.1 and using that $\mathbb{E}_N^1[f](t) \leq 4 \log(3+t)$ by the bootstrap assumption (3.24), we get

$$\begin{aligned}
I_\zeta^{z,1} + K_{\gamma,\xi}^{z,1} &\lesssim \sqrt{\epsilon} \sum_{|\beta| \leq N} \sum_{w \in \mathbf{k}_0} \int_0^t \frac{\mathbb{E}[w \widehat{Z}^\beta f](s)}{(1+s)^{\frac{5}{4}}} ds + \int_{u=-\infty}^t \frac{1}{\tau_-^{\frac{5}{4}}} \int_{C_u(t)} \int_v \frac{v^L}{v^0} \left| w \widehat{Z}^\beta f \right| dv dC_u(t) du \\
&\lesssim \sqrt{\epsilon} \int_0^t \frac{\mathbb{E}_N^1[f](s)}{(1+s)^{\frac{5}{4}}} ds + \sqrt{\epsilon} \int_{u=-\infty}^t \frac{1}{\tau_-^{\frac{5}{4}}} \mathbb{E}_N^1[f](t) du \\
&\lesssim \epsilon^{\frac{3}{2}} \int_0^{+\infty} \frac{\log(3+s)}{(1+s)^{\frac{5}{4}}} ds + \epsilon^{\frac{3}{2}} \log(3+t) \int_{u=-\infty}^{+\infty} \frac{du}{\tau_-^{\frac{5}{4}}} \lesssim \epsilon^{\frac{3}{2}} \log(3+t).
\end{aligned}$$

We now consider the cases where $|\gamma| \geq N - 1$, so that $|\xi| \leq 1$. Let us denote the null decomposition of $\mathcal{L}_{Z^\gamma}(F)$ by $(\alpha, \underline{\alpha}, \rho, \sigma)$. Using Lemma 3.4.1 and that $1 \leq v^0$ on the support of f , we are led to bound, for all $|\beta| = |\kappa| + 1 \leq 2$, the following integrals,

$$\begin{aligned}
I_F &:= \int_0^t \int_{\Sigma_s} \int_v \left(\tau_- |\rho| + \tau_+ |\alpha| + \tau_+ |\sigma| \frac{|v^A|}{v^0} \right) \left| z \widehat{Z}^\beta f \right| dv dx ds, \\
I_\alpha &:= \int_0^t \int_{\Sigma_s} \int_v |\underline{\alpha}| \frac{\tau_- |v^A| + \tau_+ v^L}{v^0} \left| z \widehat{Z}^\beta f \right| dv dx ds.
\end{aligned}$$

Using $\tau_+ v^L + \tau_+ |v^A| \lesssim v^0 \sum_{w \in \mathbf{k}_0} |w|$ (which comes from Lemmas 3.2.11), the Cauchy-Schwarz inequality, the bootstrap assumption (3.27) and the estimate (3.29), it comes

$$I_\alpha \lesssim \sum_{w \in \mathbf{k}_0} \int_0^t \|\underline{\alpha}\|_{L^2(\Sigma_s)} \left\| \int_v \left| (w^2 + z^2) \widehat{Z}^\beta f \right| dv \right\|_{L^2(\Sigma_s)} ds \lesssim \int_0^t \sqrt{\mathcal{E}_N^5[F](s)} \frac{ds}{1+s} \lesssim \epsilon^{\frac{3}{2}} \log(3+t).$$

For I_F , in order to apply Lemma 3.2.13, notice first that we have by the estimate (3.28), for $u \leq t$ and $i \in \mathbb{N}$,

$$\left\| \int_v \left| w^2 \widehat{Z}^\beta f \right| dv \right\|_{L^2(C_u^i(t))}^2 \lesssim \int_{C_u^i(t)} \frac{\epsilon^2}{\tau_+^4 \tau_-^2} dC_u^i(t) \lesssim \frac{\epsilon^2}{\tau_-^{\frac{9}{4}} (1+t_i)^{\frac{1}{4}}} \int_{u=2t_i-u}^{2t_{i+1}-u} \frac{r^2}{\tau_+^{\frac{7}{2}}} du \lesssim \frac{\epsilon^2}{\tau_-^{\frac{9}{4}} (1+2^i)^{\frac{1}{4}}}.$$

Hence, using $\tau_+ |v^A| \lesssim v^0 \sum_{w \in \mathbf{k}_0} |w|$, the Cauchy-Schwarz inequality and the bootstrap assumption (3.26), it comes

$$\begin{aligned}
I_F &= \sum_{w \in \mathbf{k}_0} \sum_{i=0}^{+\infty} \int_{u=-\infty}^t \int_{C_u^i(t)} (\tau_- |\rho| + \tau_+ |\alpha| + |\sigma|) \int_v \left| w^2 \widehat{Z}^\beta f \right| dv dC_u(t) du \\
&\lesssim \sum_{w \in \mathbf{k}_0} \sum_{i=0}^{+\infty} \int_{u=-\infty}^t \|\tau_- |\rho| + \tau_+ |\alpha| + |\sigma|\|_{L^2(C_u^i(t))} \left\| \int_v \left| w^2 \widehat{Z}^\beta f \right| dv \right\|_{L^2(C_u^i(t))} du \\
&\lesssim \sum_{i=0}^{+\infty} \int_{u=-\infty}^t \sqrt{\mathcal{E}_N[F](T_{i+1}(t))} \frac{\epsilon}{\tau_-^{\frac{9}{8}} (1+2^i)^{\frac{1}{8}}} du \lesssim \epsilon^{\frac{3}{2}} \sum_{i=0}^{+\infty} \frac{\log^2(3+2^{i+1})}{(1+2^i)^{\frac{1}{8}}} \int_{u=-\infty}^{+\infty} \frac{du}{\tau_-^{\frac{9}{8}}} \lesssim \epsilon^{\frac{3}{2}}.
\end{aligned}$$

This concludes the proof and the improvement of the bootstrap assumptions (3.23) and (3.24).

3.7 L^2 estimates on the velocity averages of the Vlasov field

In view of the energy estimate of Proposition 3.3.4, we have to prove L_x^2 estimates on quantities such as $\int_v |z \widehat{Z}^\beta f| dv$, for $|\beta| \leq N$. If $|\beta| \leq N - 2$, we can use a Klainerman-Sobolev inequality to obtain a sufficient decay rate (see Proposition 3.7.9 below). The main part of this section then consists in deriving such estimates for $|\beta| \geq N - 1$. For this purpose, we follow the strategy used in [18] (Section 4.5.7) and adapted in [4] for the Vlasov-Maxwell system. Contrary to [4], we will have to keep more of the null structure of the system. This will force us to add a new hierarchy on the functions studied here. Let us first rewrite the system and then we will explain how we will proceed. Let I_1 and I_2 be the following ordered sets,

$$\begin{aligned}
I_1 &:= \{\beta \text{ multi-index} / N - 5 \leq |\beta| \leq N\} = \{\beta_{1,1}, \dots, \beta_{1,|I_1|}\}, \\
I_2 &:= \{\beta \text{ multi-index} / |\beta| \leq N - 5\} = \{\beta_{2,1}, \dots, \beta_{2,|I_2|}\}.
\end{aligned}$$

Remark 3.7.1. *Contrary to [4], we have $I_1 \cap I_2 \neq \emptyset$.*

We also consider, for $N - 5 \leq k \leq N$, $I_1^k := \{\beta \in I_1 / |\beta| = k\}$, and two vector valued fields R and W of respective length $|I_1|$ and $|I_2|$ such that

$$R_i = \widehat{Z}^{\beta_{1,i}} f \quad \text{and} \quad W_i = \widehat{Z}^{\beta_{2,i}} f.$$

We will sometimes abusively write $i \in I_1^k$ instead of $\beta_{1,i} \in I_1^k$. Let us denote by \mathbb{V} the module over the ring $C^0([0, T[\times \mathbb{R}_x^3 \times (\mathbb{R}_v^3 \setminus \{0\})])$ engendered by $(\partial_{v^l})_{1 \leq l \leq 3}$. We now rewrite the Vlasov equations satisfied by R and W .

Lemma 3.7.2. *There exists three matrices valued functions $A : [0, T[\times \mathbb{R}^3 \times (\mathbb{R}_v^3 \setminus \{0\}) \rightarrow \mathfrak{M}_{|I_1|}(\mathbb{V})$, $D : [0, T[\times \mathbb{R}^3 \times (\mathbb{R}_v^3 \setminus \{0\}) \rightarrow \mathfrak{M}_{|I_2|}(\mathbb{V})$ and $B : [0, T[\times \mathbb{R}^3 \times (\mathbb{R}_v^3 \setminus \{0\}) \rightarrow \mathfrak{M}_{|I_1|, |I_2|}(\mathbb{V})$ such that*

$$T_F(R) + AR = BW \quad \text{and} \quad T_F(W) = DW.$$

Moreover, if $1 \leq i \leq |I_1|$, A and B are such that $T_F(R_i)$ is a linear combination of

$$\begin{aligned}
\mathcal{L}_{Z^\gamma}(F)(v, \nabla_v R_j), & \quad \text{with} \quad |\beta_{1,j}| < |\beta_{1,i}| \quad \text{and} \quad |\gamma| + |\beta_{1,j}| \leq |\beta_{1,i}|, \\
\mathcal{L}_{Z^\xi}(F)(v, \nabla_v W_j), & \quad \text{with} \quad |\beta_{2,j}| \leq N - 6 \quad \text{and} \quad |\xi| \leq N.
\end{aligned}$$

Similarly, if $1 \leq i \leq |I_2|$, D is such that $T_F(W_i)$ is a linear combination of

$$\mathcal{L}_{Z^\gamma}(F)(v, \nabla_v W_j), \quad \text{with} \quad |\beta_{2,j}| \leq N - 6 \quad \text{and} \quad |\gamma| \leq N - 5.$$

Note also, using (3.28), that

$$\forall (t, x) \in [0, T[\times \mathbb{R}^3, \quad z \in \mathbf{k}_0, \quad 1 \leq q \leq |I_2|, \quad \int_v |z^2 W_q| dv \lesssim \frac{\epsilon}{\tau_+^2 \tau_-}.$$

Remark 3.7.3. *Notice that if $\beta_{1,i} \in I_1^{N-5}$, then $A_i^q = 0$ for all $1 \leq q \leq |I_1|$. Note also that if $p \geq 1$ and $\beta_{1,i} \in I_1^{N-5+p}$, we have $|\gamma| \leq p$.*

Proof. One only has to apply the commutation formula of Proposition 3.2.9 to $\widehat{Z}^{\beta_{1,i}} f$ or $\widehat{Z}^{\beta_{2,i}} f$ and to replace each quantity such as $\widehat{Z}^\kappa f$, for $|\kappa| \neq N - 5$, by the corresponding component of R or W . If $|\kappa| = N - 5$, we replace it by the corresponding component of R . \square

The goal is to obtain an L^2 estimate on R . For this, let us split it in $R := H + G$, where

$$\begin{cases} T_F^\chi(H) + AH = 0 & , \quad H(0, \cdot, \cdot) = R(0, \cdot, \cdot), \\ T_F^\chi(G) + AG = BW & , \quad G(0, \cdot, \cdot) = 0 \end{cases}$$

and then prove L^2 estimates on the velocity averages of H and G . To do it, we will schematically establish that $G = KW$, with K a matrix such that $\mathbb{E}[KKW]$ do not growth too fast, and then use the pointwise decay estimates on $\int_v |z^2 W| dv$ to obtain the expected decay rate on $\|\int_v |G| dv\|_{L_x^2}$. For $\|\int_v |H| dv\|_{L_x^2}$, we will make crucial use of Klainerman-Sobolev inequalities so that we will need to commute the transport equation satisfied by H and prove L^1 bounds such as we made in the proof of Proposition 3.6.1. Contrary to what we did in [4], we keep the v derivatives in order to take advantage of the good behavior of radial component of $\nabla_v g$. This is why we put the derivatives of order $N - 5$ in both R and W .

Remark 3.7.4. *If we proceed as in [4], we would not be able to use the estimate $(\nabla_v g)^r \sim \tau_- \widehat{Z}g$ and an analogous result to Lemma 3.4.1 would give the term $\tau_+ |\alpha| \frac{|v^A|}{|v^0|} |\widehat{Z}g|$. In our case (the three dimensional one), a lack of decay in the $t + r$ direction prevents us to deal with it.*

3.7.1 The homogeneous system

In order to obtain L^∞ , and then L^2 , estimates on $\int_v |H| dv$, we will have to commute at least three times the transport equation satisfied by each component of H . However, if $\beta_{1,i} \in I_1^k$, with $k \geq N - 4$, we need to control the L^1 norm of $\widehat{Z}^\kappa H_j$, with $|\kappa| = 4$ and $j \in I_1^{k-1}$, to bound $\|\widehat{Z}^\xi H_i\|_{L_{x,v}^1}$, with $|\xi| = 3$. We then consider the following energy norm

$$\mathbb{E}_H := \sum_{z \in \mathbf{k}_0} \sum_{k=0}^5 \sum_{|\beta| \leq 3+k} \sum_{i \in I_1^{N-k}} \mathbb{E}[z^2 \widehat{Z}^\beta H_i].$$

We have the following commutation formula.

Lemma 3.7.5. *Let $0 \leq k \leq 5$, $|\beta| \leq 3 + k$ and $i \in I_1^{N-k}$. Then, if H vanishes for all $|v| \leq 1$, $T_F(\widehat{Z}^\beta H_i)$ can be written as a linear combination of terms of the form*

$$\mathcal{L}_{Z^\gamma}(F) \left(v, \nabla_v \widehat{Z}^\xi H_j \right), \quad \text{with} \quad |\gamma| \leq 8 \leq N - 2, \quad |\xi| \leq |\beta|, \quad |\beta_{1,j}| \leq |\beta_{1,i}|, \quad |\xi| + |\beta_{1,j}| < |\beta| + |\beta_{1,i}|.$$

Proof. If H vanishes for all $|v| \leq 1$, we have $T_F(H) + AH = 0$. Hence, according to Proposition 3.2.9, the source terms which arise from the commutator $[T_F, \widehat{Z}^\beta]$ are such as those described in this lemma, with $j = i$. The other ones come from $\widehat{Z}^\beta(T_F(H_i))$ (use Lemma 3.7.2, Remark 3.7.3 and Lemma 3.2.8 to check that they are of the researched form). \square

As $H(0, \cdot, \cdot) = R(0, \cdot, \cdot)$, it then comes that $H(0, \cdot, v) = 0$ for all $|v| \leq 3$ and, applying Proposition¹³ 3.B.2, that there exists $C_H > 0$ such that $\mathbb{E}_H(0) \leq C_H \epsilon$. Consequently, using Corollary 3.A.5 and following the proof of Proposition 3.6.1, one can prove that, for ϵ small enough,

$$\forall t \in [0, T[, \quad \mathbb{E}_H(t) \leq 3C_H \epsilon \quad \text{and} \quad \forall (t, x) \in [0, T[\times \mathbb{R}^3, \quad 0 < |v| \leq 1, \quad H(t, x, v) = 0.$$

By Proposition 3.3.6, we then obtain, for $0 \leq k \leq 5$,

$$\forall (t, x) \in [0, T[\times \mathbb{R}^3, \quad z \in \mathbf{k}_0, \quad 1 \leq j \leq |I_1^{N-k}|, \quad |\beta| \leq k, \quad \int_v |z^2 \widehat{Z}^\beta H_j| dv \lesssim \frac{\epsilon}{\tau_+^2 \tau_-}. \quad (3.34)$$

Remark 3.7.6. *Proceeding as in Subsection 17.2 of [16], we could avoid any hypothesis on the higher order derivatives of F^0 .*

¹³Strictly speaking, one cannot simply apply Proposition 3.B.2 since $T_F(H_i) \neq 0$ if $i \in I_1^k$ and $k \geq N - 4$. However, in view of Proposition 3.7.5, one can easily adapt it to our context.

3.7.2 The inhomogenous system

Start by noticing that G vanishes for all $|v| \leq 1$ since $G = R - H$. We then deduce from $\chi(|v|) = 1$ for all $|v| \geq 1$ that G satisfies $T_F(G) + AG = BW$. To derive an L^2 estimate on G , we cannot commute the transport equation because B contains top order derivatives of the electromagnetic field. Instead, we follow the methodology of [18] (see Subsection 4.5.7). We kept the v derivatives of G in the matrix A so that we could better use the null structure. In order to obtain L^1 bounds on quantities introduced below, we now need to rewrite these v derivatives. This is the purpose of the following lemma.

Lemma 3.7.7. *There exists $p \geq 1$, a vector valued field Y of length p , which vanishes for $|v| \leq 1$, and three matrices valued functions $A : [0, T[\times \mathbb{R}^3 \times (\mathbb{R}_v^3 \setminus \{0\}) \rightarrow \mathfrak{M}_{|I_1|}(\mathbb{R})$, $\bar{B} : [0, T[\times \mathbb{R}^3 \times (\mathbb{R}_v^3 \setminus \{0\}) \rightarrow \mathfrak{M}_{|I_1|, p}(\mathbb{R})$, $\bar{D} : [0, T[\times \mathbb{R}^3 \times (\mathbb{R}_v^3 \setminus \{0\}) \rightarrow \mathfrak{M}_p(\mathbb{R})$ such that*

$$T_F(G) + \bar{A}G = \bar{B}Y, \quad T_F(Y) = \bar{D}Y \quad \text{and} \quad \sum_{z \in \mathbf{k}_0} \int_v |z^2 Y| dv \lesssim \frac{\epsilon}{\tau_+^2 \tau_-}.$$

Moreover, \bar{A} and \bar{B} are such that, if $i \in \llbracket 1, |I_1| \rrbracket$, $T_F(G_i)$ can be bounded by a linear combination of terms of the form,

$$\left(\tau_- |\rho(\mathcal{L}_{Z^\gamma}(F))| + \tau_+ |\alpha(\mathcal{L}_{Z^\gamma}(F))| + \tau_+ \frac{|v^A|}{v^0} |\sigma(\mathcal{L}_{Z^\gamma}(F))| + \frac{\tau_- |v^A| + \tau_+ v^L}{v^0} |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))| \right) |G_j| \quad \text{and} \\ (\tau_- |\rho(\mathcal{L}_{Z^\xi}(F))| + \tau_+ |\alpha(\mathcal{L}_{Z^\xi}(F))| + |\sigma(\mathcal{L}_{Z^\xi}(F))| + |\underline{\alpha}(\mathcal{L}_{Z^\xi}(F))|) |z Y_q|,$$

where $j \in \llbracket 1, |I_1| \rrbracket$, $|\gamma| \leq 5$, $q \in \llbracket 1, p \rrbracket$, $|\xi| \leq N$ and $z \in V$. Similarly, \bar{D} is such that, if $i \in \llbracket 1, p \rrbracket$, $T_F(Y_i)$ can be bounded by a linear combination of terms of the form,

$$\left(\tau_- |\rho(\mathcal{L}_{Z^\gamma}(F))| + \tau_+ |\alpha(\mathcal{L}_{Z^\gamma}(F))| + \tau_+ \frac{|v^A|}{v^0} |\sigma(\mathcal{L}_{Z^\gamma}(F))| + \frac{\tau_- |v^A| + \tau_+ v^L}{v^0} |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))| \right) |Y_j|,$$

where $j \in \llbracket 1, p \rrbracket$ and $|\gamma| \leq N - 5$.

Proof. The strategy of the proof is the following. If $\partial_{v^k} G_j$ appears in $T_F(G) + AG = BW$, then, by Lemma 3.7.2, $j \in I_1^k$, with $N - 5 \leq k \leq N - 1$. We then transform it with $v^0 \partial_{v^k} = \widehat{\Omega}_{0k} - x^k \partial_t - t \partial_k$ and express it, with controllable error terms, as a combination of $(G_l)_{l \in I_1^{k+1}}$. The other manipulations are similar to those made in Section 3.6 when we applied Lemma 3.4.1. Let us denote, for $j \in I_1 \setminus I_1^N$ and $\widehat{Z} \in \widehat{\mathbb{P}}_0$, by $j_{\widehat{Z}}$ the index such that $R_{j_{\widehat{Z}}} = \widehat{Z} \widehat{Z}^{\beta_{1,j}} f = \widehat{Z} R_j$. Hence, by (3.34) and since $R = H + G$, we have, for all $j \in I_1 \setminus I_1^N$,

$$\forall (t, x) \in [0, T[\times \mathbb{R}^3, \quad (z, \widehat{Z}) \in \mathbf{k}_0 \times \widehat{\mathbb{P}}_0, \quad \int_v |z|^2 |G_{j_{\widehat{Z}}} - \widehat{Z} G_j| dv = \int_v |z|^2 |H_{j_{\widehat{Z}}} - \widehat{Z} H_j| dv \lesssim \frac{\epsilon}{\tau_+^2 \tau_-}. \quad (3.35)$$

Let $p^0 := |I_2| + |I_1 \setminus I_1^N|$ and Y^0 a vector valued field¹⁴ of length p^0 containing each component of W and each $G_{j_{\widehat{Z}}} - \widehat{Z} G_j$, for $j \in I_1 \setminus I_1^N$. We order the components of Y^0 such as $Y_{j_{\widehat{Z}}}^0 = G_{j_{\widehat{Z}}} - \widehat{Z} G_j$. In view of (3.35) and Lemma 3.7.2, $\int_v |z^2 Y^0| dv$ satisfies the desired pointwise decay estimate on $\int_v |z^2 Y| dv$. We now fix $i \in I_1$. Applying Lemma 3.7.2, one can see that $T_F(G_i)$ can be written as a linear combination of the following terms.

- Those coming from BW ,

$$\mathcal{L}_{Z^\xi}(F)(v, \nabla_v W_j), \quad \text{with} \quad |\beta_{2,j}| \leq N - 6 \quad \text{and} \quad |\xi| \leq N,$$

leading, by Lemma 3.4.1 and $\tau_+ v^L + \tau_+ |v^A| \lesssim v^0 \sum_{w \in \mathbf{k}_0}$ (see Lemma 3.2.11), to the announced terms involving Y .

- Those coming from AW ,

$$\mathcal{L}_{Z^\gamma}(F)(v, \nabla_v G_j), \quad \text{with} \quad |\beta_{1,j}| < |\beta_{1,i}| \quad \text{and} \quad |\gamma| + |\beta_{1,j}| \leq |\beta_{1,i}|.$$

Then, expand $\mathcal{L}_{Z^\gamma}(F)(v, \nabla_v G_j)$ in null components using formula (3.21). We now rewrite the angular components of $\nabla_v G_j$ using $v^0 \partial_{v^k} = \widehat{\Omega}_{0k} - x^k \partial_t - t \partial_k$, so that

$$v^0 \partial_{v^k} G_j = G_{j_{\widehat{\Omega}_{0k}}} - x^k G_{j_{\partial_t}} - t G_{j_{\partial_k}} - Y_{j_{\widehat{\Omega}_{0k}}}^0 + x^k Y_{j_{\partial_t}}^0 + t Y_{j_{\partial_k}}^0.$$

¹⁴ Y^0 will be a subvector of the vector Y of the lemma.

For the radial component, use (3.22) to obtain

$$v^0 (\nabla_v G_j)^r = \frac{x^q}{r} \left(G_{j\hat{\alpha}_{0q}} - Y_{j\hat{\alpha}_{0q}}^0 \right) - G_{j_s} + Y_{j_s}^0 + (t-r) \left(G_{j\partial_t} - Y_{j\partial_t}^0 - \frac{x^q}{r} G_{j\partial_q} + \frac{x^q}{r} Y_{j\partial_q}^0 \right).$$

This concludes the construction of \bar{A} , \bar{B} . To obtain an equation on Y^0 , we will see that we need to consider a bigger vector than Y^0 . Let $i \in \llbracket 1, p^0 \rrbracket$. If $Y_i^0 = W_q$, with $q \in I_2$, we can build the line i of \bar{D} using Lemmas 3.7.2 and 3.4.1. Otherwise, $Y_i^0 = \widehat{Z}H_j - H_{j\widehat{z}}$ and by Lemma 3.7.5 we see that functions such as $\partial_v \widehat{Z}H_r$, with $|\beta_{1,r}| < |\beta_{1,j}|$, appear in certain source terms of $T_F(Y_i^0)$. We then consider the vector valued field Y containing Y^0 and all the quantities $\widehat{Z}^\kappa H_j$ such as $\beta_{1,j} \in I_1^{N-5+k}$ and $|\kappa| + k \leq 5$. It remains to use (3.34) and Lemmas 3.7.5, 3.4.1. \square

Consider now K satisfying $T_F^\chi(K) + \chi \bar{A}K + \chi K \bar{D} = \chi \bar{B}$ and $K(0, \dots) = 0$. Hence, $KY = G$ since they both initially vanish and $T_F(KY) + \bar{A}KY = \bar{B}Y$ in view of the velocity support of Y . The goal now is to control the energy

$$\mathbb{E}_G := \sum_{i=0}^{|I_1|} \sum_{j=0}^p \sum_{q=0}^p \mathbb{E} \left[\left| K_i^j \right|^2 Y_q \right].$$

We will then be naturally led to use that

$$T_F \left(\left| K_i^j \right|^2 Y_q \right) = \left| K_i^j \right|^2 \bar{D}_q^r Y_r - 2 \left(\bar{A}_i^r K_r^j + K_i^r \bar{D}_r^j \right) K_i^j Y_q + 2 \bar{B}_i^j K_i^j Y_q. \quad (3.36)$$

Proposition 3.7.8. *If ϵ is small enough, we have $\mathbb{E}_G(t) \lesssim \epsilon \log^2(3+t)$ for all $t \in [0, T[$.*

Proof. Let $T_0 \in [0, T[$ the largest time such that $\mathbb{E}_G(t) \lesssim \epsilon \log^2(3+t)$ for all $t \in [0, T_0[$. By continuity, $T_0 > 0$. The remaining of the proof consists in improving this bootstrap assumption, which would imply the result. The computations will be similar as those of the proof of Proposition 3.6.2. Let $i \in \llbracket 1, |I_1| \rrbracket$ and $(j, q) \in \llbracket 1, p \rrbracket^2$. According to the energy estimate of Proposition 3.3.1 and (3.36), it suffices to prove that

$$I_{\bar{A}, \bar{D}} := \int_0^t \int_{\Sigma_s} \int_v \left| \left| K_i^j \right|^2 \bar{D}_q^r Y_r - 2 \left(\bar{A}_i^r K_r^j + K_i^r \bar{D}_r^j \right) K_i^j Y_q \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} \log^2(3+t), \quad (3.37)$$

$$I_{\bar{B}} := \int_0^t \int_{\Sigma_s} \int_v \left| \bar{B}_i^j K_i^j Y_q \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} \log^2(3+t). \quad (3.38)$$

According to Proposition 3.7.7 and (3.33), one have, using $\mathbb{E}_G(t) \lesssim \epsilon \log^2(3+t)$ and $1 \leq v^0$ on the support of Y ,

$$I_{\bar{A}, \bar{D}} \lesssim \int_0^t \int_{\Sigma_s} \int_v \left(\frac{\sqrt{\epsilon}}{\tau_+^{\frac{5}{4}}} + \frac{\sqrt{\epsilon v^L}}{\tau_-^{\frac{5}{4}} v^0} \right) |K|^2 |Y| dv dx ds \lesssim \sqrt{\epsilon} \int_0^t \frac{\mathbb{E}_G(s)}{(1+s)^{\frac{5}{4}}} ds + \sqrt{\epsilon} \int_{u=-\infty}^t \frac{\mathbb{E}_G(t)}{\tau_-^{\frac{5}{4}}} du \lesssim \epsilon^{\frac{3}{2}} \log^2(3+t).$$

We now turn on (3.38), where the electromagnetic field is differentiated too many times to be estimated pointwise. According to Proposition 3.7.7, $1 \leq v^0$ on the support of Y and using the Cauchy-Schwarz inequality in v , we can bound $\int_v \bar{B}_i^j K_i^j Y_q \frac{dv}{v^0}$ by a linear combination of terms of the form

$$\bullet \Phi_F^\xi := \left| \int_v |z^2 Y| dv \int_v |K|^2 |Y| dv \right|^{\frac{1}{2}} (\tau_- |\rho_\xi| + \tau_+ |\alpha_\xi| + |\sigma_\xi|), \quad \bullet \Phi_{\underline{\alpha}}^\xi := \left| \int_v |z^2 Y| dv \int_v |K|^2 |Y| dv \right|^{\frac{1}{2}} |\underline{\alpha}_\xi|,$$

where $|\xi| \leq N$ and $(\alpha_\xi, \underline{\alpha}_\xi, \rho_\xi, \sigma_\xi)$ is the null decomposition of $\mathcal{L}_{Z^\xi}(F)$. Now, fix $|\xi| \leq N$. Using the Cauchy-Schwarz inequality in x , the bootstrap assumption (3.27) and $\mathbb{E}_G(t) \leq \epsilon \log^2(3+t)$, we have

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \Phi_{\underline{\alpha}}^\xi dx ds &\lesssim \int_0^t \|\underline{\alpha}_\xi\|_{L^2(\Sigma_s)} \left\| \left\| \int_v |z^2 Y| dv \int_v |K|^2 |Y| dv \right\|^{\frac{1}{2}} \right\|_{L^2(\Sigma_s)} ds \\ &\lesssim \int_0^t \sqrt{\mathcal{E}_N^5[F](s)} \left\| \int_v |z^2 Y| dv \right\|_{L^\infty(\Sigma_s)}^{\frac{1}{2}} \left\| \int_v |K|^2 |Y| dv \right\|_{L^1(\Sigma_s)}^{\frac{1}{2}} ds \\ &\lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{\sqrt{\mathbb{E}_G(s)}}{1+s} ds \lesssim \epsilon^{\frac{3}{2}} \log^2(3+t). \end{aligned}$$

By the inequality $2ab \leq a^2 + b^2$ and $\tau_+^2 \tau_- \int_v |z^2 Y| dv \lesssim \epsilon$, one have

$$\Phi_F^\xi \leq \frac{\epsilon}{\tau_+^{\frac{5}{4}}} \int_v |K|^2 |Y| dv + \frac{\epsilon}{\tau_+^{\frac{3}{4}} \tau_-} (\tau_- |\rho_\xi| + \tau_+ |\alpha_\xi| + |\sigma_\xi|)^2,$$

so that, by Lemma 3.2.13 and the bootstrap assumption (3.26)

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \Phi_F^\xi dx ds &\lesssim \epsilon \int_0^t \frac{\mathbb{E}_G(s)}{(1+s)^{\frac{5}{4}}} ds + \sum_{i=0}^{+\infty} \int_{u=-\infty}^t \frac{\epsilon}{\tau_-^{\frac{5}{4}}} \int_{C_u^i(t)} \frac{1}{\sqrt{\tau_+}} (\tau_- |\rho_\xi| + \tau_+ |\alpha_\xi| + |\sigma_\xi|)^2 dC_u^i(t) du \\ &\lesssim \epsilon^{\frac{3}{2}} + \epsilon \sum_{i=0}^{+\infty} \int_{-\infty}^t \frac{\mathcal{E}_N[F](T_{i+1}(t))}{\tau_-^{\frac{5}{4}} \sqrt{1+t_i}} du \lesssim \epsilon^{\frac{3}{2}} + \epsilon^2 \sum_{i=0}^{+\infty} \frac{\log^4(1+2^{i+1})}{\sqrt{1+2^i}} \int_{-\infty}^{+\infty} \frac{du}{\tau_-^{\frac{5}{4}}} \lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

This concludes the improvement of the bootstrap assumption on \mathbb{E}_G and then the proof. \square

3.7.3 The L^2 estimates

In order to improve the bound on the electromagnetic field energy, we will use the following estimates.

Proposition 3.7.9. *Let $z \in \mathbf{k}_0$ and $|\beta| \leq N$. Then,*

$$\forall t \in [0, T[, \quad \left\| \tau_+ \sqrt{\tau_-} \int_v |z \widehat{Z}^\beta f| dv \right\|_{L^2(\Sigma_t)} \lesssim \epsilon \log(3+t).$$

The logarithmical growth can be removed for $|\beta| \leq N-4$.

Proof. The cases $|\beta| \leq N-4$ ensue from (3.29). Suppose now that $|\beta| \geq N-3$, so that there exists $j \in \llbracket 1, |I_1| \rrbracket$ such that $\widehat{Z}^\beta f = H_j + G_j$. It then suffices to prove that both H_j and G_j satisfy such L^2 estimates. For H_j , one only has to use $\mathbb{E}_H \leq 3\epsilon$ on $[0, T[$ and the Klainerman-Sobolev inequality of Proposition 3.3.7. For G_j , recall that $G_j = K_j^q Y_q$ and use $\int_v |z^2 Y| dv \lesssim \epsilon \tau_+^{-2}$, which comes from Proposition 3.7.7, and the Cauchy-Schwarz inequality in v in order to obtain

$$\left\| \int_v |z G_j| dv \right\|_{L^2(\Sigma_t)} = \left\| \int_v |z| |K_j^q Y_q| dv \right\|_{L^2(\Sigma_t)} \lesssim \left\| \int_v |z Y| dv \right\|_{L^\infty(\Sigma_t)}^{\frac{1}{2}} \left\| \int_v |K_j^q|^2 |Y_q| dv \right\|_{L^1(\Sigma_t)}^{\frac{1}{2}} \lesssim \frac{\sqrt{\epsilon}}{1+t} \sqrt{\mathbb{E}_G(t)}.$$

It then remains to use Proposition 3.7.8, which gives $\mathbb{E}_G(t) \lesssim \epsilon \log^2(3+t)$. \square

Combining this Proposition with the inequality $r|v^A| \lesssim v^0 \sum_{w \in \mathbf{k}_0} |w|$ (see Lemma 3.2.11), one can then improve the bootstrap assumption (3.25) if ϵ is small enough.

3.8 The energy bounds of the electromagnetic field

The last part of the proof consists in improving the bootstrap assumptions (3.26) and (3.27). According to the energy estimate of Proposition 3.3.4, commutation formula of Proposition 3.2.9 and $\mathcal{E}_N[F](0) \leq \epsilon$, $\mathcal{E}_N[F](t) \leq 3\epsilon \log^4(3+t)$ and $\mathcal{E}^5[F](t) \leq \underline{C}\epsilon$ for all $t \in [0, T[$ follow, if ϵ is small enough and \underline{C} chosen large enough, from

$$\begin{aligned} \sum_{|\gamma| \leq N} \sum_{|\beta| \leq N} \int_0^t \int_{\Sigma_s} \left| \overline{K}_0^\mu \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \int_v \frac{v^\nu}{v^0} \widehat{Z}^\beta f dv \right| dx ds &\lesssim \epsilon^{\frac{3}{2}} \log^4(3+t), \\ \sum_{|\gamma| \leq N} \sum_{|\beta| \leq N} \int_0^t \int_{\Sigma_s} \frac{\tau_-^2}{\log^5(1+\tau_-)} \left| \mathcal{L}_{Z^\gamma}(F)_{0\nu} \int_v \frac{v^\nu}{v^0} \widehat{Z}^\beta f dv \right| dx ds &\lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

Fix $|\beta| \leq N$, $|\gamma| \leq N$, denote by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of $\mathcal{L}_{Z^\gamma}(F)$ and recall that $\overline{K}_0^L = \frac{1}{2}\tau_+^2$ and $\overline{K}_0^{\underline{L}} = \frac{1}{2}\tau_-^2$. Expanding $\overline{K}_0^\mu \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} J(\widehat{Z}^\beta f)^\nu$ and $\mathcal{L}_{Z^\gamma}(F)_{0\nu} J(\widehat{Z}^\beta f)^\nu$ in null coordinates, we can observe that it suffices to prove that,

$$\begin{aligned} I &:= \int_0^t \int_{\Sigma_s} \int_v \left(\tau_+^2 |\rho| \frac{v^{\underline{L}}}{v^0} + \tau_+^2 |\alpha| \frac{|v^A|}{v^0} + \tau_-^2 |\rho| \frac{v^L}{v^0} + \tau_-^2 |\underline{\alpha}| \frac{|v^A|}{v^0} \right) |\widehat{Z}^\beta f| dv dx ds \lesssim \epsilon^{\frac{3}{2}} \log^4(3+t), \\ I_0 &:= \int_0^t \int_{\Sigma_s} \int_v \frac{\tau_-^2}{\log^5(1+\tau_-)} \left(|\rho| + |\alpha| + |\underline{\alpha}| \frac{|v^A|}{v^0} \right) |\widehat{Z}^\beta f| dv dx ds \lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

Using the Cauchy-Schwarz inequality in x , $\tau_+ v^L + \tau_+ |v^A| + \tau_- v^L \lesssim v^0 \sum_{w \in \mathbf{k}_0} |w|$ (see Lemmas 3.2.11), the bootstrap assumption (3.26) and Proposition 3.7.9, we have

$$\begin{aligned} I &\lesssim \sum_{w \in \mathbf{k}_0} \int_0^t \|\tau_+ |\rho| + \tau_+ |\alpha| + \tau_- \underline{\alpha}\|_{L^2(\Sigma_s)} \left\| \int_v |w \widehat{Z}^\beta f| dv \right\|_{L^2(\Sigma_s)} ds \\ &\lesssim \epsilon \int_0^t \sqrt{\mathcal{E}_N[F](s)} \frac{\log(3+s)}{1+s} ds \lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{\log^3(3+s)}{1+s} ds \lesssim \epsilon^{\frac{3}{2}} \log^4(3+t). \end{aligned}$$

Similarly, using $\tau_+ |v^A| + \tau_- v^0 \lesssim v^0 \sum_{w \in \mathbf{k}_0} |w|$, we obtain

$$\begin{aligned} I_0 &\lesssim \sum_{w \in \mathbf{k}_0} \int_0^t \|\tau_+ |\rho| + \tau_+ |\alpha| + \tau_- \underline{\alpha}\|_{L^2(\Sigma_s)} \left\| \frac{\tau_-}{\tau_+ \log^5(1+\tau_-)} \int_v |w \widehat{Z}^\beta f| dv \right\|_{L^2(\Sigma_s)} ds \\ &\lesssim \int_0^t \sqrt{\mathcal{E}_N[F](s)} \left\| \frac{\tau_+ \sqrt{\tau_-}}{\tau_+^{\frac{3}{2}}} \int_v |w \widehat{Z}^\beta f| dv \right\|_{L^2(\Sigma_s)} ds \lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{\log(3+s)}{(1+s)^{\frac{3}{2}}} ds \lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

These two estimates allow us to improve the bootstrap assumptions (3.26) and (3.27) if ϵ is small enough, which concludes the proof.

3.A The Vlasov field vanishes for small velocities

Let F be a smooth 2-form defined on $[0, T[\times \mathbb{R}^3$ which satisfies

$$\forall (t, x) \in [0, T[\times \mathbb{R}^3, \quad |F|(t, x) \lesssim \frac{\sqrt{\epsilon}}{\tau_+ \tau_-} \quad \text{and} \quad |\rho(F)|(t, x) \lesssim \frac{\sqrt{\epsilon}}{\tau_+^{\frac{3}{2}}}. \quad (3.39)$$

The aim of this section is to prove the following result.

Proposition 3.A.1. *Let f be a classical solution to $T_F^\chi(f) = 0$ such that $f(0, \cdot, v) = 0$ for all $|v| \leq 3$. Then if ϵ is small enough, we have*

$$\forall (t, x, v) \in [0, T[\times \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}), \quad |v| \leq 1 \Rightarrow f(t, x, v) = 0.$$

The proof is based on the study of the characteristics of the system. As $f_0(\cdot, v) = 0$ for all $|v| \leq 3$, we consider $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $|v| \geq 3$ and (X, V) the characteristic of the operator T_F^χ such that $(X(0), V(0)) = (x, v)$. Our goal is to prove $\inf_{[0, T[} |V| \geq 1$, which would imply Proposition 3.A.1. Then, suppose that $|V|$ reaches the value 1 and define

$$t_1 := \inf\{s \in [0, T[\mid |V(s)| = 1\}, \quad t_0 := \sup\{s \in [0, t_1] \mid |V(s)| = 2\}.$$

As V is continuous, t_0 and t_1 are well defined. In view of the support of χ , (X, V) satisfies the following system of ODE on $[t_0, t_1]$,

$$\forall 1 \leq i \leq 3, \quad \frac{dX^i}{ds}(s) = \frac{V^i(s)}{|V(s)|} \quad \text{and} \quad \frac{dV^i}{ds}(s) = F_{0i}(s, X(s)) + \frac{V^j(s)}{|V(s)|} F_{ji}(s, X(s)). \quad (3.40)$$

We then deduce, since F is a 2-form, that $\frac{d(|V|^2)}{ds} = 2V^i F_{0i}$, which implies

$$\forall t_0 \leq t \leq t_1, \quad |V(t)|^2 \geq |V(t_0)|^2 - 2 \int_{t_0}^t |V^i(s) F_{0i}(s, X(s))| ds. \quad (3.41)$$

Before presenting the strategy of the proof, let us introduce certain subsets of $[t_0, t_1]$ and two constants. Note that if ϵ is small enough, we can suppose that $22 \leq t_0 < t_1$. We can then introduce two constants $\delta > 0$ and $K > 0$ independent of ϵ and satisfying

$$4\delta \leq 1 + \delta \leq K < \frac{\pi}{4\sqrt{2}} t_0^{\frac{1}{4}} \quad \text{and} \quad 2^{-\frac{5}{2}} K^2 - \delta > 2\delta. \quad (3.42)$$

We also consider, for $Q > 0$, the following subsets of $[t_0, t_1]$,

$$\mathcal{A}_Q := \{s \in [t_0, t_1] \mid |s - |X(s)|| \geq Qs^{\frac{1}{4}}\} \quad \text{and} \quad \overline{\mathcal{A}}_Q := [t_0, t_1] \setminus \mathcal{A}_Q.$$

Then, using (3.41) and $\sup_{[t_0, t_1]} |V| \leq 2$, we have for all $t \in [t_0, t_1]$,

$$\begin{aligned} |V(t)|^2 &\geq 4 - 4 \int_{\mathcal{A}_\delta} \left| \frac{V^i(s)}{|V(s)} F_{0i}(s, X(s)) \right| ds - 4 \int_{\overline{\mathcal{A}}_\delta} \left| \frac{V^i(s)}{|V(s)} - \frac{X^i(s)}{s} \right| |F_{0i}(s, X(s))| ds \\ &\quad - 4 \int_{\overline{\mathcal{A}}_\delta} \frac{|X(s)|}{s} \left| \frac{X^i(s)}{|X(s)} F_{0i}(s, X(s)) \right| ds. \end{aligned}$$

The result would ensue if we could bound the three integrals on the right hand side of the last inequality by $C\sqrt{\epsilon}$, with $C > 0$ a constant independent of T and (x, v) . Indeed, we would then obtain, for $\epsilon < (2C)^{-2}$,

$$\forall t \in [t_0, t_1], \quad |V(t)| \geq \sqrt{2},$$

which would contradict $|V(t_1)| = 1$. We can easily bound two of these integrals, using either the strong decay rate of F away from the light cone or the strong decay rate satisfied by the null component ρ and that $\frac{|X(s)|}{s}$ is bounded near the light cone. More precisely, using (3.39), the definition of \mathcal{A}_δ and $t_0 \geq 1$, one have

$$\begin{aligned} \int_{\mathcal{A}_\delta} \left| \frac{V^i(s)}{|V(s)} F_{0i}(s, X(s)) \right| ds &\lesssim \int_0^{+\infty} \frac{\sqrt{\epsilon} ds}{(1+s)(1+\delta s^{\frac{1}{4}})} \lesssim \sqrt{\epsilon}, \\ \int_{\overline{\mathcal{A}}_\delta} \frac{|X(s)|}{s} \left| \frac{X^i(s)}{|X(s)} F_{0i}(s, X(s)) \right| ds &\lesssim \int_{\overline{\mathcal{A}}_\delta} (1+\delta) |\rho(F)(s, X(s))| ds \lesssim \int_0^{+\infty} \frac{\sqrt{\epsilon}}{(1+s)^{\frac{3}{2}}} ds \lesssim \sqrt{\epsilon}. \end{aligned}$$

For the last integral, observe, in view of (3.39), that

$$I_1 := \int_{\overline{\mathcal{A}}_\delta} \left| \frac{V^i(s)}{|V(s)} - \frac{X^i(s)}{s} \right| |F_{0i}(s, X(s))| ds \lesssim \sqrt{\epsilon} \int_{\overline{\mathcal{A}}_\delta} \left| \frac{V(s)}{|V(s)} - \frac{X(s)}{s} \right| \frac{1}{1+|s-|X(s)||} \frac{ds}{1+s}. \quad (3.43)$$

As $|s - |X(s)||$ is small on $\overline{\mathcal{A}}_\delta$, the goal is to obtain enough decay from $\frac{V(s)}{|V(s)} - \frac{X(s)}{s}$. The rough idea behind the following computations is the following. As $|s - X(s)|$ is small, then, by (3.40), $s \sim |X(s)| \sim \left| \int_0^s \frac{V(\tau)}{|V(\tau)} d\tau \right|$ and we almost have equality in the triangular inequality

$$\left| \int_0^s \frac{V(\tau)}{|V(\tau)} d\tau \right| \leq \int_0^s \left| \frac{V(\tau)}{|V(\tau)} \right| d\tau = s.$$

$\frac{V}{|V|}$ then almost keep a constant direction \vec{u} , so that

$$\frac{X(s)}{s} \sim \frac{1}{s} \int_0^s \frac{V(\tau)}{|V(\tau)} d\tau \sim \vec{u} \sim \frac{V}{|V|}.$$

In order to bound (3.43), let us introduce, for $Q > 0$ and $\delta_0 > 0$, the following subsets of $[t_0, t_1]$,

$$\mathcal{B}_Q := \left\{ s \in [t_0, t_1] / \left| \frac{V(s)}{|V(s)} - \frac{X(s)}{s} \right| > \frac{Q}{s^{\frac{1}{4}}} \right\}, \quad \overline{\mathcal{B}}_Q := [t_0, t_1] \setminus \mathcal{B}_Q \quad \text{and} \quad \mathcal{C}_Q^{\delta_0} := \overline{\mathcal{A}}_{\delta_0} \cap \mathcal{B}_Q.$$

In view of the definition of $\mathcal{C}_{4K}^{\delta_0}$ and (3.43), $I_1 \lesssim \sqrt{\epsilon}$ would ensue if we prove

$$I := \int_{\mathcal{C}_{4K}^{\delta_0}} \left| \frac{V^i(s)}{|V(s)} - \frac{X^i(s)}{s} \right| |F_{0i}(s, X(s))| ds \lesssim \sqrt{\epsilon}. \quad (3.44)$$

From now, we suppose that $\mathcal{C}_{4K}^{\delta_0} \neq \emptyset$ as otherwise, $I = 0$. We start by the following two results.

Lemma 3.A.2. *Let $s \in \mathcal{C}_{4K}^{\delta_0}$. Then, $[s, \min(t_1, s + s^{\frac{3}{4}})] \subset \mathcal{B}_{2K}$.*

Proof. Let $t \in [s, \min(t_1, s + s^{\frac{3}{4}})]$. As $s \in \mathcal{B}_{4K}$ and by the triangle inequality, one has

$$\frac{4K}{s^{\frac{1}{4}}} \leq \left| \frac{V^i(s)}{|V(s)} - \frac{X^i(s)}{s} \right| \leq \left| \frac{V^i(t)}{|V(t)} - \frac{X^i(t)}{t} \right| + \left| \frac{V^i(s)}{|V(s)} - \frac{V^i(t)}{|V(t)} \right| + \left| \frac{X^i(s)}{s} - \frac{X^i(t)}{t} \right|.$$

By the mean value theorem applied to the function $\frac{V}{|V|}$ and using the estimate (3.39), we have

$$\left| \frac{V^i(s)}{|V(s)} - \frac{V^i(t)}{|V(t)} \right| \leq C_1 \sup_{\tau \in [s, t]} |F(\tau, X(\tau))| |t - s| \leq \frac{C_0 \sqrt{\epsilon}}{1+s} |t - s| \leq \frac{C_0 \sqrt{\epsilon}}{s} |t - s|.$$

Using $|X(s)| \leq s + \delta s^{\frac{1}{4}}$ and $|X(t) - X(s)| \leq |t - s|$, which comes from (3.40), we have

$$\left| \frac{X^i(s)}{s} - \frac{X^i(t)}{t} \right| \leq \frac{(t-s)|X^i(s)| + s|X^i(t) - X^i(s)|}{ts} \leq \frac{2+\delta}{t}|t-s| \leq \frac{2+\delta}{s}|t-s|.$$

Thus, as $s \leq t \leq s + s^{\frac{3}{4}}$ and $2K \geq 2 + 2\delta$ (see (3.42)), it comes, for ϵ small enough,

$$\frac{4K}{s^{\frac{1}{4}}} - \frac{C_0\sqrt{\epsilon} + 2 + \delta}{s}|t-s| \geq \frac{4K - C_0\sqrt{\epsilon} - 2 - \delta}{s^{\frac{1}{4}}} > \frac{2K}{t^{\frac{1}{4}}}, \quad \text{so that} \quad t \in \mathcal{B}_{2K}.$$

□

Lemma 3.A.3. *Suppose that $s \in \mathcal{C}_{4K}^\delta$ and let $t_*(s)$ be equal to $\inf\{t \in [s, t_1] / t \notin \mathcal{C}_{2K}^{2\delta}\}$ if it is well defined and $t_*(s) = t_1$ otherwise. Then,*

$$\forall t \in [s, t_*(s)], \quad t - |X(t)| \geq s - |X(s)| + K^2\sqrt{t} - K^2\sqrt{s}.$$

Proof. Let $g : t \mapsto t - |X(t)|$, so that $g'(t) = 1 - \langle \frac{V(t)}{|V(t)|}, \frac{X(t)}{|X(t)|} \rangle$. Let us estimate $\theta \in [0, \pi[$, the angle between V and X . For $t \in [s, t_*(s)]$, we have

- $\left| \frac{V(t)}{|V(t)|} - \frac{X(t)}{|X(t)|} \right| \leq |\theta(t)|$ since $\frac{V}{|V|}$ and $\frac{X}{|X|}$ are unit vectors.
- $\frac{2K}{t^{\frac{1}{4}}} \leq \left| \frac{V(t)}{|V(t)|} - \frac{X(t)}{|X(t)|} \right|$ as $t \in \mathcal{B}_{2K}$ and $\left| \frac{|X(t)|}{t} - 1 \right| \leq \frac{2\delta}{t^{\frac{3}{4}}}$ since $t \in \overline{\mathcal{A}_{2\delta}}$.

We then obtain, using $4\delta \leq K$, the trivial fact $\left| \frac{X(t)}{t} - \frac{X(t)}{|X(t)|} \right| = \left| \frac{|X(t)|}{t} - 1 \right|$ and the triangle inequality, that

$$\frac{\sqrt{2}K}{t^{\frac{1}{4}}} \leq \frac{2K}{t^{\frac{1}{4}}} - \frac{2\delta}{t^{\frac{3}{4}}} \leq \left| \frac{V(t)}{|V(t)|} - \frac{X(t)}{t} \right| - \left| \frac{X(t)}{t} - \frac{X(t)}{|X(t)|} \right| \leq \left| \frac{V(t)}{|V(t)|} - \frac{X(t)}{|X(t)|} \right| \leq |\theta(t)|.$$

Consequently, as $1 - \frac{1}{4}\phi^2 \geq \cos \phi$ for $\phi \in [0, \frac{\pi}{4}]$ and since $\sqrt{2}K \leq \frac{\pi}{4}t_0^{\frac{1}{4}}$, we obtain

$$g'(t) = 1 - \cos \theta(t) \geq 1 - \cos \left(\frac{\sqrt{2}K}{t^{\frac{1}{4}}} \right) \geq \frac{K^2}{2\sqrt{t}} \quad \text{and then} \quad g(t) \geq g(0) + K^2(\sqrt{t} - \sqrt{s}).$$

□

The strategy now is to prove that \mathcal{C}_{4K}^δ is composed of pieces sufficiently well separated for (3.44) to holds. It is then convenient to consider a dyadic partition of $[t_0, t_1]$, which leads us to introduce, for all $i \in \mathbb{N}$,

$$\mathcal{C}_{4K}^i := \mathcal{C}_{4K}^\delta \cap [2^i, 2^{i+1}[\quad \text{and, if } \mathcal{C}_{4K}^i \neq \emptyset, \quad s^i = \inf \mathcal{C}_{4K}^i \quad \text{and} \quad t_*^i = t_*(s^i).$$

Corollary 3.A.4. *Let $i \in \mathbb{N}$ such that $\mathcal{C}_{4K}^i \neq \emptyset$. Then, $\mathcal{C}_{4K}^i \subset [s^i, \min(t_*^i, 2^{i+1})]$. Moreover, if $|X(s^i)| - s^i > 0$, $t \mapsto t - |X(t)|$ is positive on $[2^{i+2}, t_1]$.*

Proof. Let $i \in \mathbb{N}$ and suppose that $\mathcal{C}_{4K}^i \neq \emptyset$. We assume moreover that $t_*^i < t_1$ since there is nothing to prove otherwise and we introduce $T^i = \min(t_1, 2^{i+2})$. We will use several times that $s - |X(s)| > 0$, for $s \in [t_0, t_1]$, implies that $t \mapsto t - |X(t)|$ is positive on $[s, t_1]$ since it is an increasing function. As $t_*^i \in \mathcal{A}_{2\delta} \cup \overline{\mathcal{B}_{2K}}$ by definition, we have two cases to study.

- If $t_*^i \in \overline{\mathcal{B}_{2K}}$, then, by Lemma 3.A.2, $t_*^i \geq \tau^i := s^i + |s^i|^{\frac{3}{4}}$. Hence, using Lemma 3.A.3, it comes

$$\tau^i - |X(\tau^i)| \geq s^i - |X(s^i)| + K^2\sqrt{\tau^i} - K^2\sqrt{s^i} \geq -\delta|s^i|^{\frac{1}{4}} + K^2\sqrt{s^i} \left(\sqrt{1 + |s^i|^{-\frac{1}{4}}} - 1 \right).$$

Since $4\sqrt{1+h} - 4 \geq h$ for all $h \in [0, 1]$ and $t \mapsto t - |X(t)|$ increases, we obtain, for all $t \in [\tau^i, T^i]$,

$$t - |X(t)| \geq \tau^i - |X(\tau^i)| \geq -\delta 2^{\frac{i+2}{4}} + K^2\sqrt{s^i} \frac{1}{4} |s^i|^{-\frac{1}{4}} \geq (2^{-\frac{10}{4}}K^2 - \delta) 2^{\frac{i+2}{4}} \geq 2\delta |T^i|^{\frac{1}{4}} \geq 2\delta t^{\frac{1}{4}}. \quad (3.45)$$

so that¹⁵ $[\tau^i, T^i] \subset \mathcal{A}_{2\delta}$. As $\mathcal{C}_{4K}^i \subset \overline{\mathcal{A}_\delta}$ and $\mathcal{A}_{2\delta} \cap \overline{\mathcal{A}_\delta} = \emptyset$, we obtain $\mathcal{C}_{4K}^i \subset [s^i, \tau^i] \subset [s^i, t_*^i]$. Observe also that $\tau^i - |X(\tau^i)| > 0$, which implies that $t \mapsto t - |X(t)|$ is positive on $[\tau^i, t_1] \subset [2^{i+2}, t_1]$.

¹⁵We then necessarily have $\tau^i = t_*^i$.

- Otherwise $t_*^i \in \mathcal{A}_{2\delta}$. If $t_*^i \geq T^i$, we have $\tau^i \leq 2^{i+2} \leq t_*^i$ so that $C_{4K}^i \subset [s^i, 2^{i+1}] \subset [s^i, t_*^i]$. The positivity of $t \mapsto t - |X(t)|$ on $[2^{i+2}, t_1]$ then follows from (3.45). Otherwise, as $|t_*^i - |X(t_*^i)|| \geq 2\delta |t_*^i|^{\frac{1}{4}}$ and $t \mapsto t - |X(t)|$ increases, we necessarily have $t_*^i - |X(t_*^i)| \geq 0$ since $s^i - |X(s^i)| \geq -\delta |s^i|^{\frac{1}{4}}$. Hence, using again that $t \mapsto t - |X(t)|$ increases, it comes

$$\forall t \in [t_*^i, T^i], \quad t - |X(t)| \geq 2\delta |t_*^i|^{\frac{1}{4}} > \delta t^{\frac{1}{4}}, \quad \text{as} \quad 2\delta |t_*^i|^{\frac{1}{4}} \geq 2\delta 2^{\frac{i}{4}} > \delta 2^{\frac{i+2}{4}}.$$

We can then conclude that $[t_*^i, T^i] \subset \mathcal{A}_\delta$, implying that $C_{4K}^i \subset [s^i, t_*^i]$ since $A_\delta \cap C_{4K}^i = \emptyset$. We also proved that $t \mapsto t - |X(t)|$ on $[2^{i+2}, t_1]$ as $t_*^i - |X(t_*^i)| \geq 0$ and $t_*^i \leq 2^{i+2}$. □

We are now able to bound I . Let $\mathcal{D} := \{i \in \mathbb{N} / C_{4K}^i \neq \emptyset\}$. Suppose first that $t \geq |X(t)|$ for all $t \in C_{4K}^\delta$. Then, using (3.39), Lemma 3.A.3 and Corollary 3.A.4,

$$\begin{aligned} I &\lesssim \sum_{i \in \mathcal{D}} \int_{C_{4K}^i} \frac{\sqrt{\epsilon}}{(1+s)(1+|s-|X(s)||)} ds \lesssim \sum_{i \in \mathcal{D}} \int_{s^i}^{2^{i+1}} \frac{\sqrt{\epsilon}}{(1+s)(1+K^2\sqrt{s}-K^2\sqrt{s^i})} ds \\ &\lesssim \sum_{i \in \mathcal{D}} \sqrt{\epsilon} \int_{s^i}^{2^{i+1}} \frac{1+\sqrt{s}+\sqrt{s^i}}{(1+s)(1+s-s^i)} ds \lesssim \sum_{i=0}^{+\infty} \frac{\sqrt{\epsilon}}{2^{\frac{i}{2}}} \int_{2^i}^{2^{i+1}} \frac{1}{(1+s-2^i)} ds \\ &\lesssim \sqrt{\epsilon} \sum_{i=0}^{+\infty} \frac{\log(1+2^i)}{2^{\frac{i}{2}}} \lesssim \sqrt{\epsilon}. \end{aligned}$$

Otherwise, with $p = \min \mathcal{D}$ and according to Corollary 3.A.4, we have $t \geq |X(t)|$ for all $t \geq 2^{p+2}$ and the result then follows from

$$\sqrt{\epsilon} \int_{C_{4K}^p \cup C_{4K}^{p+1}} \frac{ds}{1+s} \leq \sqrt{\epsilon} \int_{2^p}^{2^{p+2}} \frac{ds}{1+s} \leq \sqrt{\epsilon} \log(4).$$

In order to apply this result in Subsection 3.7.1, we need to adapt it to an echeloned system of transport equations.

Corollary 3.A.5. *Let $k \in \mathbb{N}^*$ and, for all $1 \leq j < i \leq k$ and $1 \leq q \leq 3$, let $A_i^{q,j}$ be a sufficiently regular matrix valued function defined on $[0, T[\times \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$. Consider $g = (g_1, \dots, g_k)$, where each g_i is a vector valued field, a classical solution on $[0, T[$ to the system*

$$T_F^\chi(g_1) = 0, \quad T_F^\chi(g_i) = A_i^{q,j} \partial_{v^q} g_j \quad 2 \leq i \leq k.$$

If $g(0, \cdot, v) = 0$ for all $|v| \leq 3$, then $g(t, \cdot, v) = 0$ for all $|v| \leq 1$.

Proof. Let $(t, x, v) \in [0, T[\times \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$ such that $|v| < 1$. We denote by (X_s, V_s) the value in s of the characteristic of the operator T_F^χ which was equal to (x, v) in $s = t$. By Duhamel's formula, we have

$$\begin{aligned} g_1(t, x, v) &= g_1(s, X_s, V_s) = g_1(0, X_0, V_0), \\ g_i(t, x, v) &= g_i(0, X_0, V_0) + \int_0^t A_i^{q,j}(s, X_s, V_s) \partial_{v^q} g_j(s, X_s, V_s) ds \quad \text{for all } 2 \leq i \leq k. \end{aligned} \quad (3.46)$$

According to the proof of Proposition 3.A.1,

$$|V_0| < 3, \quad \text{so that} \quad \forall 1 \leq i \leq k, \quad g_i(0, X_0, V_0) = 0 \quad (3.47)$$

since otherwise we would have $|V_t| = |v| \geq 1$. Fix now $s \in [0, t]$ and consider $w \in \mathbb{R}^3$ such that $|w| < \frac{1}{2}|v|$. We denote by $(X_\tau^{w,s}, V_\tau^{w,s})$ the value in τ of the characteristic of T_F^χ which was equal to $(X_s, V_s + w)$ in $\tau = s$. Then,

$$g_1(s, X_s, V_s + w) = g_1(0, X_0^{w,s}, V_0^{w,s}).$$

By continuous dependence on the initial condition of the solutions to

$$\frac{dX^i}{ds}(s) = \frac{V^i(s)}{|V(s)|}, \quad \frac{dV^i}{ds}(s) = \chi(|V(s)|) F_{0i}(s, X(s)) + \frac{V^j(s)}{|V(s)|} \chi(|V(s)|) F_{ji}(s, X(s)), \quad 1 \leq i \leq 3,$$

and since $|V_0| < 3$, there exists $\bar{\delta} > 0$ (depending on (t, s, x, v)) such that $|V_0^{w,s}| < 3$ for all $|w| < \bar{\delta}$. Hence,

$$\forall |w| < \bar{\delta}, \quad g_1(s, X_s, V_s + w) = 0, \quad \text{so that} \quad \forall 1 \leq q \leq 3, \quad \partial_{v^q} g_1(s, X_s, V_s) = 0. \quad (3.48)$$

Repeating the argument, one can obtain

$$\forall |w| \leq \bar{\delta}, \quad \forall \tau \in [0, s], \quad \exists \underline{\delta} > 0, \quad \forall |w| \leq \underline{\delta}, \quad g_1(\tau, X_\tau^{w,s}, V_\tau^{w,s} + \underline{w}) = 0,$$

which implies

$$\forall |w| \leq \bar{\delta}, \quad \forall \tau \in [0, s], \quad \forall 1 \leq q \leq 3, \quad \partial_{v^q} g_1(\tau, X_\tau^{w,s}, V_\tau^{w,s}) = 0. \quad (3.49)$$

Combining (3.46), (3.47) and (3.48), we get $g_2(t, x, v) = 0$. Using (3.49) and $|V_0^{w,s}| < 3$ for all $|w| < \bar{\delta}$, it comes, in view of the support of $g_2(0, \cdot, \cdot)$,

$$\forall |w| < \bar{\delta}, \quad g_2(s, X_s, V_s + w) = g_2(0, X_0^{w,s}, V_0^{w,s}) + \int_0^s A_i^{q,j}(\tau, X_\tau^{w,s}, V_\tau^{w,s}) \partial_{v^q} g_1(\tau, X_\tau^{w,s}, V_\tau^{w,s}) d\tau = 0.$$

We then deduce that $\partial_{v^q} g_2(s, X_s, V_s) = 0$ for all $q \in \llbracket 1, 3 \rrbracket$ and $s \in [0, t]$, so that, by (3.46), $g_3(t, x, v) = 0$. We then proved the desired result if $k = 3$. The general case can be treated similarly, by a tedious induction. \square

3.B Bounding the initial norms

We consider in this section (f^0, F^0) satisfying the hypotheses of Theorem 3.1.1 and (f, F) the unique classical solution of (3.1)-(3.3) arising from these data.

Proposition 3.B.1. *There exists a constant C_1 , depending only on N , such that*

$$\mathcal{E}_N[F](0) \leq C_1 \epsilon \quad \text{and} \quad \mathbb{E}_{N+3}^2[f](0) \leq C_1 \epsilon.$$

The proof is a corollary of the following proposition and that, for all $\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|}$ and $Z \in \mathbb{K}^{|\gamma|}$,

$$|\widehat{Z}^\beta f| \lesssim \sum_{|\alpha_2| + |\alpha_1| + p \leq |\beta|} \tau_+^{|\alpha_1| + p} |v|^{|\alpha_2|} |\partial_t^p \partial_x^{\alpha_1} \partial_v^{\alpha_2} f|, \quad |\mathcal{L}_{Z^\gamma}(F)| \lesssim \sum_{|\kappa| + q \leq |\gamma|} \tau_+^{|\kappa| + q} |\nabla_{\partial_t}^q \nabla_x^\kappa F|.$$

Proposition 3.B.2. *We have, for all $|\alpha_2| + |\alpha_1| + p \leq N + 3$ and $|\kappa| + q \leq N + 2$,*

$$\left\| (1 + |x|)^{|\alpha_1| + p + 2} |v|^{|\alpha_2|} \partial_t^p \partial_x^{\alpha_1} \partial_v^{\alpha_2} f \right\|_{L_v^1 L^1(\Sigma_0)} \lesssim \epsilon, \quad \text{and} \quad \left\| (1 + |x|)^{|\kappa| + q + 1} \nabla_{\partial_t}^q \nabla_x^\kappa F \right\|_{L^2(\Sigma_0)}^2 \lesssim \epsilon. \quad (3.50)$$

Proof. Note that $\tau_+^2 = 1 + |x|^2$ on Σ_0 . The proof consists in an induction on $\max(p, q)$. The result holds for $\max(p, q) = 0$ in view of the hypotheses on (f^0, F^0) . Let $r \in \llbracket 1, N + 2 \rrbracket$ and suppose that (3.50) is satisfied for all $p \leq r$ and all $q \leq r$. If $r < N + 2$, fix $|\kappa| \leq N + 2 - (r + 1)$ and notice that, using (1.3) and Lemma 3.2.2,

$$\partial_t F_{0i} = \partial^j F_{ji} + \int_{v \in \mathbb{R}^3} \frac{v^i}{v^0} f dv \quad \text{and} \quad \partial_t F_{ij} = \partial_i F_{0j} + \partial_j F_{i0}.$$

Then, by a standard $L_x^2 - L_x^1$ Sobolev inequality and using the induction hypothesis twice, we get

$$\begin{aligned} \left\| \tau_+^{|\kappa| + r + 2} \nabla_{\partial_t}^{r+1} \nabla_x^\kappa F \right\|_{L^2(\Sigma_0)}^2 &\leq 2 \left\| \tau_+^{|\kappa| + r + 2} \nabla_x \nabla_{\partial_t}^r \nabla_x^\kappa F \right\|_{L^2(\Sigma_0)}^2 + \left\| \tau_+^{|\kappa| + r + 2} \int_v \partial_t^r \partial_x^\kappa f dv \right\|_{L^2(\Sigma_0)}^2 \\ &\lesssim \epsilon + \sum_{|\beta| \leq 2} \left\| \tau_+^{|\kappa| + r + 2} \int_v \partial_x^\beta \partial_t^r \partial_x^\kappa f dv \right\|_{L^1(\Sigma_0)}^2 \lesssim \epsilon + \epsilon^2, \end{aligned}$$

since $r + |\kappa| + |\beta| \leq N + 3$. We now turn on the Vlasov field and we do not suppose $r < N + 2$ anymore. Start by fixing α_1 and α_2 such that $|\alpha_1| + |\alpha_2| + r + 1 \leq N + 3$. Iterating the commutation formula $T_F(\partial_\mu f) = -\mathcal{L}_{\partial_\mu}(F)(v, \nabla_v f)$ and using $\mathcal{L}_{\partial_\mu} = \nabla_{\partial_\mu}$, it comes

$$\partial_t \partial_t^r \partial_x^{\alpha_1} f = -\frac{v^i}{|v|} \partial_i \partial_t^r \partial_x^{\alpha_1} f + \sum_{q+p=r} \sum_{\substack{|\kappa| + |\beta_1| = |\alpha_1| \\ p + |\beta_1| \leq r + |\alpha_1| - 1}} \frac{v^\mu}{|v|} \nabla_{\partial_t}^q \nabla_x^\kappa F_\mu^j \partial_{v^j} \partial_t^p \partial_x^{\beta_1} f.$$

Taking the $\partial_v^{\alpha_2}$ derivative of each side and multiplying them by $\tau_+^{|\alpha_1|+r+3}|v|^{|\alpha_2|}$, one obtain

$$\begin{aligned} \tau_+^{|\alpha_1|+r+1+2}|v|^{|\alpha_2|}|\partial_t^{r+1}\partial_x^{\alpha_1}\partial_v^{\alpha_2}f| &\leq \sum_{|\alpha_3|+n=|\alpha_2|} \tau_+^{|\alpha_1|+1+r+2}|v|^{|\alpha_2|-n}|\partial_x\partial_t^r\partial_x^{\alpha_1}\partial_v^{\alpha_3}f| + \\ &\sum_{\substack{|\kappa|+|\beta_1|=|\alpha_1| \\ p+|\beta_1|\leq r+|\alpha_1|-1}} \sum_{\substack{q+p=r \\ |\alpha_3|+n=|\alpha_2|}} \tau_+^{|\alpha_1|+r+3}|v|^{|\alpha_2|-n}|\nabla_{\partial_t}^q\nabla_x^\kappa F\partial_v\partial_t^p\partial_x^{\beta_1}\partial_v^{\alpha_3}f| \end{aligned} \quad (3.51)$$

By the induction hypothesis, the $L_v^1L^1(\Sigma_0)$ norm of the terms of (3.51) are bounded by ϵ . Consider parameters such as in the sum in (3.52).

• If $q+|\kappa|\leq N$, then, using a standard $L^\infty-L^2$ Sobolev inequality on $\tau_+^{q+|\kappa|+1}\nabla_{\partial_t}^q\nabla_x^\kappa F$ and that $|v|\geq 3$ on the support of f^0 , we get

$$\tau_+^{|\alpha_1|+r+3}|v|^{|\alpha_2|-n}|\nabla_{\partial_t}^q\nabla_x^\kappa F\partial_v\partial_t^p\partial_x^{\beta_1}\partial_v^{\alpha_3}f| \lesssim \tau_+^{|\beta_1|+p+2}|v|^{|\alpha_3|+1}|\partial_t^p\partial_x^{\beta_1}\partial_v\partial_v^{\alpha_3}f| \sum_{|\gamma|\leq|\kappa|+2} \left\| \tau_+^{|\kappa|+q+1}\nabla_{\partial_t}^q\nabla_x^\gamma F \right\|_{L_x^2}.$$

The $L_v^1L^1(\Sigma_0)$ norm of the left hand side of the previous inequality is then bounded by $\epsilon^{\frac{3}{2}}$ according to the induction hypothesis and $p+q=r$.

• Otherwise $|\beta_1|+p\leq 2$ and, by the Cauchy-Schwarz inequality in x ,

$$\left\| \tau_+^{|\alpha_1|+r+3}|v|^{|\alpha_2|-n}\nabla_{\partial_t}^q\nabla_x^\kappa F\partial_v\partial_t^p\partial_x^{\beta_1}\partial_v^{\alpha_3}f \right\|_{L_{v,x}^1} \leq \left\| \tau_+^{|\kappa|+q+1}\nabla_{\partial_t}^q\nabla_x^\kappa F \right\|_{L_x^2} \left\| \tau_+^{|\beta_1|+p+2}|v|^{|\alpha_3|}\partial_t^p\partial_x^{\beta_1}\partial_v\partial_v^{\alpha_3}f \right\|_{L_v^1L_x^2}.$$

The left hand side of the previous inequality can be bounded by $\epsilon^{\frac{3}{2}}$. Indeed, as $|v|\geq 3$ on the support of f^0 and using a $L_x^2-L_x^1$ Sobolev inequality¹⁶, it comes

$$\left\| \tau_+^{|\beta_1|+p+2}|v|^{|\alpha_3|}\partial_t^p\partial_x^{\beta_1}\partial_v\partial_v^{\alpha_3}f \right\|_{L_v^1L^2(\Sigma_0)} \lesssim \sum_{|\beta|\leq|\beta_1|+2} \left\| \tau_+^{|\beta_1|+p+2}|v|^{|\alpha_3|+1}\partial_t^p\partial_x^\beta\partial_v\partial_v^{\alpha_3}f \right\|_{L_v^1L^1(\Sigma_0)}.$$

It remains to use the induction hypothesis twice. This concludes the induction and then the proof. \square

3.C All derivatives of F are chargeless

The aim of this section is to prove the following result, which also applies to massive particles.

Proposition 3.C.1. *Let $N_0\geq 2$ and (f, F) be a sufficiently regular solution to the Vlasov-Maxwell system on $[0, T]$ such that*

$$\forall t \in [0, T], \quad \sum_{|\beta|\leq N_0} \int_{\Sigma_t} \int_{v \in \mathbb{R}^3} |\widehat{Z}^\beta f| dv dx + \sum_{|\gamma|\leq N_0} \int_{\Sigma_t} |\mathcal{L}_{Z^\gamma}(F)|^2 dx < +\infty.$$

Then, for all $1\leq|\gamma|\leq N_0$, $\mathcal{L}_{Z^\gamma}(F)$ is chargeless, i.e.

$$-\lim_{r\rightarrow+\infty} \int_{\mathbb{S}_{0,r}} \rho(\mathcal{L}_{Z^\gamma}(F)) d\mathbb{S}_{0,r} = 0.$$

This ensues from commutation formula of Proposition 3.2.9 and the following lemma.

Lemma 3.C.2. *Fix $p\geq 1$ and let H_0, H_1, \dots, H_p be sufficiently regular 2-forms defined on $[0, T] \times \mathbb{R}^3$ and h_0, h_1, \dots, h_p be sufficiently regular functions defined on $[0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$ such that*

$$\nabla^\mu(H_0)_{\mu\nu} = J(h_0)_\nu \quad \text{and} \quad T(h_0) = \sum_{1\leq i\leq p} H_i(v, \nabla_v h_i).$$

Suppose moreover that

$$\forall t \in [0, T], \quad \sum_{\lambda=0}^p \|H_\lambda\|_{L_x^2(t)} + \|h_\lambda\|_{L_{x,v}^1(t)} + \|(1+r)\nabla_{t,x}h_\lambda\|_{L_{x,v}^1(t)} + \|(1+|v|)\nabla_v h_\lambda\|_{L_{x,v}^1(t)} < +\infty.$$

Then, $\mathcal{L}_Z(H_0)$ is chargeless for all $Z \in \mathbb{K}$.

¹⁶To deal with the lack of regularity of the absolute value of, say, $\int_v |g|dv$ one may apply first a L^2-L^1 Sobolev inequality in the variables (x^1, x^2) and then in the variable x^3 , as we made in the proof of Proposition 3.3.7.

Remark 3.C.3. Note that in dimension $n \neq 3$, we merely have that $\mathcal{L}_S(H_0)$ is chargeless if and only if H_0 is.

Proof. To lighten the proof, we suppose that $p = 1$ and we denote (H_0, h_0) by (G, g) and (h_1, H_1) by (H, h) . Consider first $Z \in \mathbb{P}$. Then, by Lemma 3.2.8, $\nabla^\mu \mathcal{L}_Z(G)_{\mu 0} = J(\widehat{Z}g)_0$ so that, using the divergence theorem,

$$Q(t) := \lim_{r \rightarrow +\infty} \int_{\mathbb{S}_{t,r}} \rho(\mathcal{L}_Z(G)) d\mathbb{S}_{t,r} = - \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} \widehat{Z}g dv dx. \quad (3.53)$$

- If $Z = \Omega \in \mathbb{O}$ is a rotational vector field, the result does not depend of the source term of the Maxwell equations. Indeed, using Lemma 3.3.8 and the divergence theorem on $\mathbb{S}_{t,r}$, one have,

$$\int_{\mathbb{S}_{t,r}} \rho(\mathcal{L}_\Omega(G)) d\mathbb{S}_{t,r} = \int_{\mathbb{S}_{t,r}} \Omega(\rho(G)) d\mathbb{S}_{t,r} = \int_{\mathbb{S}_{t,r}} d\sharp v(\Omega)\rho(G) d\mathbb{S}_{t,r}.$$

As Ω is tangential to $\mathbb{S}_{t,r}$, we have $d\sharp v(\Omega)(t, r, \omega_1, \omega_2) = \text{div}_{\mathbb{R}^3}(\Omega)(t, r, \omega_1, \omega_2) = 0$, so that $Q(t) = 0$.

- If $Z = \partial_i$ is a spatial translation, an integration by parts on the right hand side of (3.53) gives the result.
- If $Z = \partial_t$, then, as $\partial_t g = -\frac{v^j}{v^0} \partial_j g + H\left(\frac{v}{v^0}, \nabla_v h\right)$, integrations by parts (in x and in v) gives us, as H is a 2-form,

$$Q(t) = \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} \frac{v^i}{v^0} \partial_i g - H\left(\frac{v}{v^0}, \nabla_v h\right) dv dx = \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} \frac{v^i v^j}{(v^0)^3} H_{ij} h dv dx = 0. \quad (3.54)$$

- If $Z = \Omega_{0i}$ is a Lorentz boost, then an integration by parts in x on $t\partial_i g$ and in v on $v^0 \partial_{v^i} g$ gives

$$Q(t) = - \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} (t\partial_i g + x^i \partial_t g + v^0 \partial_{v^i} g) dv dx = \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} \frac{v^i}{v^0} g dv dx - \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} x^i \partial_t g dv dx.$$

Using again $\partial_t g = -\frac{v^j}{v^0} \partial_j g + H\left(\frac{v}{v^0}, \nabla_v h\right)$ and integrating by parts, we have

$$\begin{aligned} \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} x^i \partial_t g dv dx &= - \int_{v \in \mathbb{R}^3} \frac{v^j}{v^0} \int_{x \in \mathbb{R}^3} x^i \partial_j g dx dv + \int_{x \in \mathbb{R}^3} x^i \int_{v \in \mathbb{R}^3} H\left(\frac{v}{v^0}, \nabla_v h\right) dv dx \\ &= \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} \frac{v^i}{v^0} g dv dx - \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} x^i \frac{v^k v^j}{(v^0)^3} H_{kj} h dv dx. \end{aligned}$$

As H is a 2-form, we finally obtain that $Q(t) = 0$.

For the case of the scaling vector field, note first by Lemma 3.2.8 that $\nabla^\mu \mathcal{L}_S(G)_{\mu 0} = J(Sg)_0 + 3J(g)_0$. Hence,

$$\lim_{r \rightarrow +\infty} \int_{\mathbb{S}_{t,r}} \rho(\mathcal{L}_S(G)) d\mathbb{S}_{t,r} = - \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} (x^\mu \partial_\mu g + 3g) dv dx = -t \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} \partial_t g dv dx.$$

Recall from (3.54) that the integral on the right hand side of the last equation is equal to 0. This concludes the proof. \square

3.D Proof of Lemmas 3.3.8, 3.3.9 and 3.3.10

Let G be a 2-form and J be a 1-form, both sufficiently regular and defined on $[0, T[\times \mathbb{R}^3$, such that

$$\begin{aligned} \nabla^\mu G_{\mu\nu} &= J_\nu, \\ \nabla^\mu *G_{\mu\nu} &= 0. \end{aligned}$$

Let us successively prove Lemmas 3.3.8, 3.3.9 and 3.3.10.

Lemma 3.D.1. Let $\Omega \in \mathbb{O}$. Then, denoting by ζ any of the null component α , $\underline{\alpha}$, ρ or σ ,

$$[\mathcal{L}_\Omega, \nabla_{\partial_r}]G = 0, \quad \mathcal{L}_\Omega(\zeta(G)) = \zeta(\mathcal{L}_\Omega(G)) \quad \text{and} \quad \nabla_{\partial_r}(\zeta(G)) = \zeta(\nabla_{\partial_r}(G)).$$

Similar results hold for \mathcal{L}_Ω and ∇_{∂_t} , ∇_L or $\nabla_{\underline{L}}$. For instance, $\nabla_L(\zeta(G)) = \zeta(\nabla_L(G))$.

Proof. Let $\Omega \in \mathbb{O}$. The property $[\mathcal{L}_\Omega, \nabla_{\partial_r}]G = 0$ follows from $[\Omega, \partial_r] = 0$, straightforward computations and that, in cartesian coordinates and for a vector field X ,

$$\mathcal{L}_X(G)_{\mu\nu} = X(G_{\mu\nu}) + \partial_\mu(X^\lambda)G_{\lambda\nu} + \partial_\nu(X^\lambda)G_{\mu\lambda} \quad \text{and} \quad \nabla_X(G)_{\mu\nu} = X(G_{\mu\nu}). \quad (3.55)$$

Aside from $\mathcal{L}_\Omega(\sigma(G)) = \sigma(\mathcal{L}_\Omega(G))$, the remaining identities ensue from $[\Omega, L] = [\Omega, \underline{L}] = \nabla_{\partial_r}L = \nabla_{\partial_r}\underline{L} = \nabla_{\partial_r}e_A = 0$ since, for instance,

$$2\nabla_{\partial_r}\rho(G) = (\nabla_{\partial_r}G)(L, \underline{L}) + G(\nabla_{\partial_r}L, \underline{L}) + G(L, \nabla_{\partial_r}\underline{L}) \quad \text{and} \quad \mathcal{L}_\Omega(\alpha) = \mathcal{L}_\Omega(G)(L, \cdot) + G([\Omega, L], \cdot).$$

Using that $[\Omega, e_A] = C_\Omega^B(\omega_1, \omega_2)e_B$, where C_Ω^B are bounded functions on the sphere, we directly obtain $|\Omega\sigma(G)| \lesssim |\sigma(\mathcal{L}_\Omega(G))| + |\sigma(G)|$, which is good enough for proving the following results of this appendix. To obtain $\Omega(\sigma(G)) = \sigma(\mathcal{L}_\Omega(G))$, one can check, with straightforward computations that, for a 2-form H ,

$$\rho(*H) = -\sigma(H) \quad \text{and} \quad *\mathcal{L}_\Omega(H) = \mathcal{L}_\Omega(*H).$$

It then comes

$$\Omega\sigma(G) = -\Omega\rho(*G) = -\rho(\mathcal{L}_\Omega(*H)) = -\rho(*\mathcal{L}_\Omega(G)) = \sigma(\mathcal{L}_\Omega(G)).$$

For the results concerning the operator ∇_{∂_t} , use $\nabla_{\partial_t}L = \nabla_{\partial_t}\underline{L} = \nabla_{\partial_t}e_A = 0$. Finally, for ∇_L and $\nabla_{\underline{L}}$, recall that $L = \partial_t + \partial_r$ and $\underline{L} = \partial_t - \partial_r$. \square

Lemma 3.D.2. *Denoting by ζ any of the null component α , $\underline{\alpha}$, ρ or σ , we have*

$$\tau_- |\nabla_{\underline{L}}\zeta(G)| + \tau_+ |\nabla_L\zeta(G)| \lesssim \sum_{|\gamma| \leq 1} |\zeta(\mathcal{L}_{Z^\gamma}(G))| \quad \text{and} \quad (1+r) |\nabla\zeta(G)| \lesssim |\zeta(G)| + \sum_{\Omega \in \mathbb{O}} |\zeta(\mathcal{L}_\Omega(G))|.$$

Proof. Let $\zeta \in \{\alpha, \underline{\alpha}, \rho, \sigma\}$. As $\nabla_{\underline{L}}$ commute with the null decomposition (see the previous Lemma) and $(t-r)\underline{L} = S - \frac{x^i}{r}\Omega_{0i}$ (see Lemma 3.2.7), we have,

$$(t-r)\nabla_{\underline{L}}\zeta(G) = \zeta(\nabla_{(t-r)\underline{L}}G) = \zeta\left(\nabla_S G - \frac{x^i}{r}\nabla_{\Omega_{0i}}G\right) = \zeta(\mathcal{L}_S(G)) - 2\zeta(G) - \frac{x^i}{r}\zeta(\mathcal{L}_{\Omega_{0i}}(G)) + \frac{x^i}{r}\zeta(H^i),$$

since $\mathcal{L}_S(G) = \nabla_S(G) + 2G$ and where $H^i = \mathcal{L}_{\Omega_{0i}}(G) - \nabla_{\Omega_{0i}}(G)$. We have, using (3.55),

$$H_{\mu\nu}^i = 0 \quad \text{if} \quad \mu, \nu \notin \{0, i\} \quad \text{or} \quad \mu, \nu \in \{0, i\}. \quad \text{If} \quad \nu \notin \{0, i\}, \quad H_{0\nu}^i = G_{i\nu} \quad \text{and} \quad H_{i\nu}^i = G_{0\nu}. \quad (3.56)$$

Consequently,

$$\frac{x_i}{r}\rho(H^i) = \frac{x_i}{r}H_{r0}^i = \frac{x^i}{r}\frac{x^j}{r}G_{ji} = 0, \quad \text{so that} \quad |\underline{L}\rho(G)| \lesssim \frac{1}{\tau_-} \sum_{|\gamma| \leq 1} |\rho(\mathcal{L}_{Z^\gamma}(G))|.$$

It remains to study $x_i\zeta(H^i)$ for $\zeta \in \{\alpha, \underline{\alpha}, \sigma\}$. Let us treat together the case of α and $\underline{\alpha}$ by computing $x_i H_{A0}^i$ and $x_i H_{Ar}^i$. Recall that $e_A = \sum_{1 \leq k < l \leq 3} C^{k,l}(\omega_1, \omega_2) \frac{\Omega_{kl}}{r}$ where $C^{k,l}$ are bounded functions on the sphere. As

$$\begin{aligned} x_i H^i\left(\frac{\Omega_{kl}}{r}, \partial_t\right) &= \frac{x_i x^k}{r} H_{l0}^i - \frac{x_i x^l}{r} H_{k0}^i = \frac{x^i x^k}{r} G_{li} - \frac{x^i x^l}{r} G_{ki} = rG\left(\frac{\Omega_{kl}}{r}, \partial_r\right), \\ x_i H^i\left(\frac{\Omega_{kl}}{r}, \partial_r\right) &= \frac{x_i x^k x^j}{r^2} H_{lj}^i - \frac{x_i x^l x^j}{r^2} H_{kj}^i \\ &= x^k \left(\frac{x_i x^i - (x^l)^2}{r^2} G_{l0} + \frac{x_l x^j}{r^2} G_{0j} - \frac{(x^l)^2}{r^2} G_{0l} \right) - x^l \left(\frac{x_i x^i - (x^k)^2}{r^2} G_{k0} + \frac{x_k x^j}{r^2} G_{0j} - \frac{(x^k)^2}{r^2} G_{0k} \right) \\ &= rG\left(\frac{\Omega_{kl}}{r}, \partial_t\right), \end{aligned}$$

it comes

$$|\nabla_{\underline{L}}(\alpha(G))_A| \lesssim \frac{1}{\tau_-} \sum_{|\gamma| \leq 1} |\alpha(\mathcal{L}_{Z^\gamma}(G))_A| \quad \text{and} \quad |\nabla_{\underline{L}}(\underline{\alpha}(G))_A| \lesssim \frac{1}{\tau_-} \sum_{|\gamma| \leq 1} |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(G))_A|.$$

For the remaining case, $\zeta = \sigma$, straightforward computations give

$$x_i H^i \left(\frac{\Omega_{kl}}{r}, \frac{\Omega_{pq}}{r} \right) = \frac{x^k x^p x_i}{r^2} H_{lq}^i + \frac{x^l x^q x_i}{r^2} H_{kp}^i - \frac{x^k x^q x_i}{r^2} H_{lp}^i - \frac{x^l x^p x_i}{r^2} H_{kq}^i = 0, \quad \text{so that} \quad x_i H_{AB}^i = 0.$$

The proof for ∇_L is similar as it also commutes with the null decomposition and since $(t+r)L = S + \frac{x^i}{r}\Omega_{0i}$. Finally, for the angular derivatives, use that \mathcal{L}_Ω commute with the null decomposition and that for a function u and a tensor U ,

$$r |\nabla u| \leq \sum_{\Omega \in \mathbb{O}} |\Omega u| \quad \text{and} \quad r |\nabla U| \leq \sum_{\Omega \in \mathbb{O}} |\nabla_\Omega U| \lesssim |U| + \sum_{\Omega \in \mathbb{O}} |\mathcal{L}_\Omega U|.$$

□

Lemma 3.D.3. *Denoting by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of G , we have*

$$\nabla_L \alpha_A - \frac{\alpha_A}{r} + \nabla_{e_A} \rho + \varepsilon_{BA} \nabla_{e_B} \sigma = J_A.$$

Proof. Let us start by proving, for $B \neq A$,

$$(\nabla_{e_B} G)(e_B, e_A) = \varepsilon_{BA} \nabla_{e_B} \sigma - \frac{1}{2r} \alpha(e_A) + \frac{1}{2r} \underline{\alpha}(e_A). \quad (3.57)$$

Suppose for instance that $e_B = e_1$ and $e_A = e_2$. Then, as $\nabla_{e_C} e_D = \nabla_{e_C} e_D - \frac{\delta_{C,D}}{r} \nabla_{\partial_r}$,

$$\begin{aligned} \varepsilon_{BA} \nabla_{e_B} \sigma &= e_1(G(e_1, e_2)) = (\nabla_{e_1} G)(e_1, e_2) + G(\nabla_{e_1} e_1, e_2) + G(e_1, \nabla_{e_1} e_2) \\ &= (\nabla_{e_1} G)(e_1, e_2) + \frac{1}{2r} \alpha(e_2) - \frac{1}{2r} \underline{\alpha}(e_2) + G(\nabla_{e_1} e_1, e_2) + G(e_1, \nabla_{e_1} e_2). \end{aligned}$$

It remains to notice that, as G is a 2-form and $2 \langle \nabla_{e_1} e_A, e_A \rangle = \nabla_{e_1} (\langle e_A, e_A \rangle) = 0$,

$$G(\nabla_{e_1} e_1, e_2) + G(e_1, \nabla_{e_1} e_2) = G(\langle \nabla_{e_1} e_1, e_1 \rangle e_1, e_2) + G(e_1, \langle \nabla_{e_1} e_2, e_2 \rangle e_2) = 0.$$

Recall now from Lemma 3.2.2 that $\nabla_{[\lambda} G_{\mu\nu]} = 0$. Taking the (L, \underline{L}, A) component of this tensorial equation, we get, as $\nabla_L \underline{L} = \nabla_L L = 0$ and $\nabla_{e_A} L = -\nabla_{e_A} \underline{L} = \frac{1}{r} e_A$,

$$\begin{aligned} \nabla_{[L} G_{\underline{L}A]} = 0 &\Leftrightarrow (\nabla_L G)(\underline{L}, e_A) + (\nabla_{\underline{L}} G)(e_A, L) + (\nabla_{e_A} G)(L, \underline{L}) = 0 \\ &\Leftrightarrow -(\nabla_L \underline{\alpha})(e_A) + (\nabla_{\underline{L}} \alpha)(e_A) + 2\nabla_{e_A} \rho - G(\nabla_{e_A} L, \underline{L}) - G(L, \nabla_{e_A} \underline{L}) = 0 \\ &\Leftrightarrow -(\nabla_L \underline{\alpha})(e_A) + (\nabla_{\underline{L}} \alpha)(e_A) + 2\nabla_{e_A} \rho - \frac{1}{r} \underline{\alpha}(e_A) - \frac{1}{r} \alpha(e_A) = 0. \end{aligned} \quad (3.58)$$

Similarly, taking $\nu = A$ in $\nabla^\mu G_{\mu\nu} = J_\nu$, we obtain, using (3.57) and since $\nabla^L = -\frac{1}{2} \nabla_{\underline{L}}$ and $\nabla^{\underline{L}} = -\frac{1}{2} \nabla_L$,

$$\begin{aligned} \nabla^\mu G_{\mu A} = J_A &\Leftrightarrow -\frac{1}{2} (\nabla_L G)(\underline{L}, e_A) - \frac{1}{2} (\nabla_{\underline{L}} G)(L, e_A) + (\nabla_{e_B} G)(e_B, e_A) = J_A \\ &\Leftrightarrow \frac{1}{2} (\nabla_L \underline{\alpha})(e_A) + \frac{1}{2} (\nabla_{\underline{L}} \alpha)(e_A) + \varepsilon_{BA} \nabla_{e_B} \sigma - \frac{1}{2r} \alpha(e_A) + \frac{1}{2r} \underline{\alpha}(e_A) = J_A. \end{aligned} \quad (3.59)$$

It remains to add half of (3.58) to (3.59). □

Chapter 4

Sharp asymptotic behavior of solutions of the 3d Vlasov-Maxwell system with small data

Abstract

We study the asymptotic properties of the small data solutions of the Vlasov-Maxwell system in dimension three. No neutral hypothesis nor compact support assumptions are made on the data. In particular, the initial decay in the velocity variable is optimal. We use vector field methods to obtain sharp pointwise decay estimates in null directions on the electromagnetic field and its derivatives. For the Vlasov field and its derivatives, we obtain, as in [17], optimal pointwise decay estimates by a vector field method where the commutators are modification of those of the free relativistic transport equation. In order to control high velocities and to deal with non integrable source terms, we make fundamental use of the null structure of the system and of several hierarchies in the commuted equations.

4.1 Introduction

This article is concerned with the asymptotic behavior of small data solutions to the three-dimensional Vlasov-Maxwell system. These equations, used to model collisionless plasma, describe, for one species of particles¹, a distribution function f and an electromagnetic field which will be represented by a two form $F_{\mu\nu}$. The equations are given by²

$$v^0 \partial_t f + v^i \partial_i f + ev^\mu F_{\mu}{}^j \partial_{v^j} f = 0, \quad (4.1)$$

$$\nabla^\mu F_{\mu\nu} = eJ(f)_\nu := e \int_{v \in \mathbb{R}^3} \frac{v_\nu}{v^0} f dv, \quad (4.2)$$

$$\nabla^\mu {}^*F_{\mu\nu} = 0, \quad (4.3)$$

where $v^0 = \sqrt{m^2 + |v|^2}$, $m > 0$ is the mass of the particles and $e \in \mathbb{R}^*$ their charge. For convenience, we will take $m = 1$ and $e = 1$ for the remaining of this paper. The particle density f is a non-negative³ function of $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$, while the electromagnetic field F and its Hodge dual *F are 2-forms depending on $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$. We can recover the more common form of the Vlasov-Maxwell system using the relations

$$E^i = F_{0i} \quad \text{and} \quad B^i = -{}^*F_{0i},$$

¹Our results can be extended without any additional difficulty to several species of particles.

²We will use all along this paper the Einstein summation convention so that, for instance, $v^i \partial_i f = \sum_{i=1}^3 v^i \partial_i f$ and $\nabla^\mu F_{\mu\nu} = \sum_{\mu=0}^3 \nabla^\mu F_{\mu\nu}$. The latin indices goes from 1 to 3 and the greek indices from 0 to 3.

³In this article, the sign of f does not play any role.

so that the equations can be rewritten as

$$\begin{aligned} \sqrt{1+|v|^2}\partial_t f + v^i\partial_i f + (\sqrt{1+|v|^2}E + v \times B) \cdot \nabla_v f &= 0, \\ \nabla \cdot E &= \int_{v \in \mathbb{R}^3} f dv, & \partial_t E^j &= (\nabla \times B)^j - \int_{v \in \mathbb{R}^3} \frac{v^j}{\sqrt{1+|v|^2}} f dv, \\ \nabla \cdot B &= 0, & \partial_t B &= -\nabla \times E. \end{aligned}$$

We refer to [21] for a detailed introduction to this system.

4.1.1 Small data results for the Vlasov-Maxwell system

The first result on global existence with small data for the Vlasov-Maxwell system in $3d$ was obtained by Glassey-Strauss in [24] and then extended to the nearly neutral case in [44]. This result required compactly supported data (in x and in v) and shows that $\int_v f dv \lesssim \frac{\epsilon}{(1+t)^3}$, which coincides with the linear decay. They also obtain estimates for the electromagnetic field and its derivatives of first order, but they do not control higher order derivatives of the solutions. The result established by Schaeffer in [44] allows particles with high velocity but still requires the data to be compactly supported in space⁴.

In [4], using vector field methods, we proved optimal decay estimates on small data solutions and their derivatives of the Vlasov-Maxwell system in high dimensions $d \geq 4$ without any compact support assumption on the initial data. We also obtained that similar results hold when the particles are massless ($m = 0$) under the additional assumption that f vanishes for small velocities⁵.

A better understanding of the null condition of the system led us in our recent work [6] to an extension of these results to the massless $3d$ case. In our forthcoming paper [7] we will study the asymptotic properties of solutions to the massive Vlasov-Maxwell in the exterior of a light cone for mildly decaying initial data. Due to the strong decay satisfied by the particle density in such a region we will be able to lower the initial decay hypothesis on the electromagnetic field and then avoid any difficulty related to the presence of a non-zero total charge.

The results of this paper establish sharp decay estimates on the small data solutions to the three-dimensional Vlasov-Maxwell system. The hypotheses on the particle density in the variable v are optimal in the sense that we merely suppose f (as well as its derivatives) to be initially integrable in v , which is a necessary condition for the source term of the Maxwell equations to be well defined.

Recently, Wang proved independently in [50] a similar result for the $3d$ massive Vlasov-Maxwell system. Using both vector field methods and Fourier analysis, he does not require compact support assumptions on the initial data but strong polynomial decay hypotheses in (x, v) on f and obtained optimal pointwise decay estimates on $\int_v f dv$ and its derivatives.

4.1.2 Vector fields and modified vector fields for the Vlasov equations

The vector field method of Klainerman was first introduced in [29] for the study of nonlinear wave equations. It relies on energy estimates, the algebra \mathbb{P} of the Killing vector fields of the Minkowski space and conformal Killing vector fields, which are used as commutators and multipliers, and weighted functional inequalities now known as Klainerman-Sobolev inequalities.

In [18], the vector field method was adapted to relativistic transport equations and applied to the small data solutions of the Vlasov-Nordström system in dimensions $d \geq 4$. It provided sharp asymptotics on the solutions and their derivatives. Key to the extension of the method is the fact that even if $Z \in \mathbb{P}$ does not commute with the free transport operator $T := v^\mu \partial_\mu$, its complete lift⁶ \widehat{Z} does. The case of the dimension 3, studied in [16], required to consider modifications of the commutation vector fields of the form $Y = \widehat{Z} + \Phi^\nu \partial_\nu$, where \widehat{Z} is a complete lift of a Killing field (and thus commute with the free transport operator) while the coefficients Φ are constructed by solving a transport equation depending on the solution itself. In [46], similar results was proved for the Vlasov-Poisson equations and, again, the three-dimensional case required to modify the set of commutation vector fields in order to compensate the worst source terms in the commuted transport equations. Vector field methods led to a proof of the stability of the Minkowski spacetime for the Einstein-Vlasov system, obtained independently by [17] and [26].

Note that vector field methods can also be used to derive integrated decay for solutions to the the massless Vlasov equation on curved background such as slowly rotating Kerr spacetime (see [1]).

⁴Note also that when the Vlasov field is not compactly supported (in v), the decay estimate obtained in [44] on its velocity average contains a loss.

⁵Note that there exists initial data violating this condition and such that the system does not admit a local classical solution (see Section 8 of [4]).

⁶The expression of the complete lift of a vector field of the Minkowski space is presented in Definition 4.3.1.

4.1.3 Charged electromagnetic field

In order to present our main result, we introduce in this subsection the pure charge part and the chargeless part of a 2-form.

Definition 4.1.1. *Let G be a sufficiently regular 2-form defined on $[0, T] \times \mathbb{R}^3$. The total charge $Q_G(t)$ of G is defined as*

$$Q_G(t) = \lim_{r \rightarrow +\infty} \int_{\mathbb{S}_{t,r}} \frac{x^i}{r} G_{0i} d\mathbb{S}_{t,r},$$

where $\mathbb{S}_{t,r}$ is the sphere of radius r of the hypersurface $\{t\} \times \mathbb{R}^3$ which is centered at the origin $x = 0$.

If (f, F) is a sufficiently regular solution to the Vlasov-Maxwell system, Q_F is a conserved quantity. More precisely,

$$\forall t \in [0, T], \quad Q_F(t) = Q_F(0) = \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} f(0, x, v) dv dx.$$

Note that the derivatives of F are automatically chargeless (see Appendix C of [6]). The presence of a non-zero charge implies $\int_{\mathbb{R}^3} r|F|^2 dx = +\infty$ and prevents us from propagating strong weighted L^2 norms on the electromagnetic field. This leads us to decompose 2-forms into two parts. For this, let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a cut-off function such that

$$\forall s \leq -2, \quad \chi(s) = 1 \quad \text{and} \quad \forall s \geq -1, \quad \chi(s) = 0.$$

Definition 4.1.2. *Let G be a sufficiently regular 2-form with total charge Q_G . We define the pure charge part \overline{G} and the chargeless part \tilde{G} of G as*

$$\overline{G}(t, x) := \chi(t - r) \frac{Q_G(t)}{4\pi r^2} \frac{x_i}{r} dt \wedge dx^i \quad \text{and} \quad \tilde{G} := G - \overline{G}.$$

One can then verify that $Q_{\overline{G}} = Q_G$ and $Q_{\tilde{G}} = 0$, so that the hypothesis $\int_{\mathbb{R}^3} r|\tilde{G}|^2 dx = +\infty$ is consistent. Notice moreover that $G = \tilde{G}$ in the interior of the light cone.

The study of non linear systems with a presence of charge was initiated by [45] in the context of the Maxwell-Klein Gordon equations. The first complete proof of such a result was given by Lindblad and Sterbenz in [36] and improved later by Yang (see [51]). Let us also mention the work of [3].

4.1.4 Statement of the main result

Definition 4.1.3. *We say that (f_0, F_0) is an initial data set for the Vlasov-Maxwell system if $f_0 : \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ and the 2-form F_0 are both sufficiently regular and satisfy the constraint equations*

$$\nabla^i (F_0)_{i0} = - \int_{v \in \mathbb{R}^3} f_0 dv \quad \text{and} \quad \nabla^{i*} (F_0)_{i0} = 0.$$

The main result of this article is the following theorem.

Theorem 4.1.4. *Let $N \geq 11$, $\epsilon > 0$, (f_0, F_0) an initial data set for the Vlasov-Maxwell equations (4.1)-(4.3) and (f, F) be the unique classical solution to the system arising from (f_0, F_0) . If*

$$\sum_{|\beta|+|\kappa| \leq N+3} \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} (1+|x|)^{2N+3} (1+|v|)^{|\kappa|} |\partial_x^\beta \partial_v^\kappa f_0| dv dx + \sum_{|\gamma| \leq N+2} \int_{x \in \mathbb{R}^3} (1+|x|)^{2|\gamma|+1} \left| \nabla_x^\gamma \tilde{F}_0 \right|^2 dx \leq \epsilon,$$

then there exists $C > 0$, $M \in \mathbb{N}$ and $\epsilon_0 > 0$ such that, if $\epsilon \leq \epsilon_0$, (f, F) is a global solution to the Vlasov-Maxwell system and verifies the following estimates.

- Energy bounds for the chargeless part of F : $\forall t \in \mathbb{R}_+$,

$$\sum_{\substack{Z^\gamma \in \mathbb{K}^{|\gamma|} \\ |\gamma| \leq N}} \int_{|x| \geq t} \tau_+ \left(|\alpha(\mathcal{L}_{Z^\gamma}(\tilde{F}))|^2 + |\rho(\mathcal{L}_{Z^\gamma}(\tilde{F}))|^2 + |\sigma(\mathcal{L}_{Z^\gamma}(\tilde{F}))|^2 \right) + \tau_- |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(\tilde{F}))|^2 dx \leq C\epsilon,$$

$$\sum_{\substack{Z^\gamma \in \mathbb{K}^{|\gamma|} \\ |\gamma| \leq N}} \int_{|x| \leq t} \tau_+ \left(|\alpha(\mathcal{L}_{Z^\gamma}(F))|^2 + |\rho(\mathcal{L}_{Z^\gamma}(F))|^2 + |\sigma(\mathcal{L}_{Z^\gamma}(F))|^2 \right) + \tau_- |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|^2 dx \leq C\epsilon \log^{2M}(3+t).$$

- *Pointwise decay estimates for the null components of $\mathcal{L}_{Z^\gamma}(F)$: $\forall |\gamma| \leq N - 5, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,*

$$\begin{aligned} |\alpha(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log(3+t)}{\tau_+^2}, & |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log(3+t)}{\tau_+ \tau_-}, \\ |\rho(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log(3+t)}{\tau_+^{\frac{3}{2}} \tau_-^{\frac{1}{2}}}, & |\sigma(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log(3+t)}{\tau_+^{\frac{3}{2}} \tau_-^{\frac{1}{2}}}. \end{aligned}$$

- *Energy bounds for the Vlasov field: $\forall t \in \mathbb{R}_+$,*

$$\sum_{\substack{Y^\beta \in \mathbb{Y}^{|\beta|} \\ |\beta| \leq N}} \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} |Y^\beta f| \, dv dx \leq C\epsilon.$$

- *Pointwise decay estimates for the velocity averages of $Y^\beta f$: $\forall |\beta| \leq N - 3, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,*

$$\int_{v \in \mathbb{R}^3} |Y^\beta f| \, dv \lesssim \frac{\epsilon}{\tau_+^2 \tau_-} \quad \text{and} \quad \int_{v \in \mathbb{R}^3} |Y^\beta f| \frac{dv}{(v^0)^2} \lesssim \epsilon \frac{1}{\tau_+^3} \mathbb{1}_{t \geq |x|} + \epsilon \frac{\log^2(3+t)}{\tau_+^3 \tau_-} \mathbb{1}_{|x| \geq t}$$

Remark 4.1.5. For the highest derivatives of f_0 , those of order at least $N - 2$, we could save four powers of $|x|$ in the condition on the initial norm and even more for those of order at least $N + 1$. We could also avoid any hypothesis on the derivatives of order $N + 1$ and $N + 2$ of F_0 (see Remark 4.9.9).

Remark 4.1.6. Assuming more decay on \tilde{F} and its derivatives at $t = 0$, we could use the Morawetz vector field as a multiplier, propagate a stronger energy norm and obtain better decay estimates on its null components. In the exterior of the lightcone, we could recover the decay rates of the free Maxwell equations (see [11]) on $\alpha(F)$, $\underline{\alpha}(F)$ and $\sigma(F)$ and obtain that $|\rho(F)| \lesssim \sqrt{\epsilon} \tau_+^{-2}$. We cannot obtain a better decay rate on $\rho(F)$ because of the presence of the charge. In the interior⁸, we could improve the estimates on ρ and σ up to a rate of $\sqrt{\epsilon} \log(3+t) \tau_+^{-2}$. With our approach, we cannot recover the sourceless behavior in this region because of the slow decay of $\int_v f \, dv$. Under these hypotheses, one can check that the number of derivatives can be reduced to $N = 9$.

4.1.5 Key elements of the proof

Modified vector fields

In [4], we observed that commuting (4.1) with the complete lift of a Killing vector field gives problematic source terms. More precisely, if $Z \in \mathbb{P}$,

$$[T_F, \widehat{Z}]f = -v^\mu \mathcal{L}_Z(F)_{\mu^j} \partial_{v^j} f, \quad \text{with } T_F = v^\mu \partial_\mu + v^\mu F_{\mu^j} \partial_{v^j}. \quad (4.4)$$

The difficulty comes from the presence of ∂_v , which is not part of the commutation vector fields, since in the linear case ($F = 0$) $\partial_v f$ essentially behaves as $t \partial_{t,x} f$. However, one can see that the source term has the same form as the non-linearity $v^\mu F_{\mu^j} \partial_{v^j} f$. In [4], we controlled the error terms by taking advantage of their null structure and the strong decay rates given by high dimensions. Unfortunately, our method does not apply in dimension 3 since even assuming a full understanding of the null structure of the system, we would face logarithmic divergences. The same problem arises for others Vlasov systems and were solved using modified vector fields in order to cancel the worst source terms in the commutation formula. Let us mention again the works of [16] for the Vlasov-Nordström system, [46] for the Vlasov-Poisson equations, [17] and [26] for the Einstein-Vlasov system. We will thus consider vector fields of the form $Y = \widehat{Z} + \Phi^\nu \partial_\nu$, where the coefficients Φ^ν are themselves solutions to transport equations, growing logarithmically. As a consequence, we will need to adapt the Klainerman-Sobolev inequalities for velocity averages and the result of Theorem 1.1 of [4] in order to replace the original vector fields by the modified ones.

⁷If $|x| \geq t + 1$, the $\log(3+t)$ -loss can be removed.

⁸The multiplier for this region would rather be $(1 + |t - r|^2)^{-1} \overline{K}_0$, where \overline{K}_0 is the Morawetz vector field.

The electromagnetic field and the non-zero total charge

Because of the presence of a non-zero total charge, i.e. $\lim_{r \rightarrow +\infty} \int_{\mathbb{S}_{0,r}} \frac{x^i}{r} (F_0)_{0i} d\mathbb{S}_{0,r} \neq 0$, we have, at $t = 0$,

$$\int_{\mathbb{R}^3} (1+r) \left| \frac{x^i}{r} F_{0i} \right|^2 dx = \int_{\mathbb{R}^3} (1+r) |\rho(F)|^2 dx = +\infty$$

and we cannot propagate L^2 bounds on $\int_{\mathbb{R}^3} (1+t+r) |\rho(F)(t,x)|^2 dx$. However, provided that we can control the flux of the electromagnetic field on the light cone $t = r$, we can propagate weighted L^2 norms of F in the interior region. To deal with the exterior of the light cone, recall from Definition 4.1.2 the decomposition

$$F = \widetilde{F} + \overline{F}, \quad \text{with} \quad \overline{F}(t,x) := \chi(t-r) \frac{Q_F}{4\pi r^2} dr \wedge dt. \quad (4.5)$$

The hypothesis $\int_{\mathbb{R}^3} (1+|x|) |\widetilde{F}(0,\cdot)| dx < +\infty$ is consistent with the chargelessness of \widetilde{F} and we can then propagate weighted energy norms of \widetilde{F} and bound the flux of F on the light cone. On the other hand, we have at our disposal pointwise estimates on \overline{F} and its derivatives through the explicit formula (4.5). These informations will allow us to deduce pointwise decay estimates on the null components of F in both the exterior and the interior regions.

An other problem arises from the source terms of the commuted Maxwell equations, which need to be written with our modified vector fields. This leads us, as [16] and [17], to rather consider them of the form $Y = \widehat{Z} + \Phi^i X_i$, where $X_i = \partial_i + \frac{v^i}{v^0} \partial_t$. The X_i vector fields enjoy a kind of null condition⁹ and allow us to avoid a small growth on the electromagnetic field norms which would prevent us to close our energy estimates¹⁰. However, at the top order, a loss of derivative do not allow us to take advantage of them and creates a t^η -loss, with $\eta > 0$ a small constant. A key step is to make sure that $\| |Y^\kappa \Phi|^2 Y f \|_{L^1_{x,v}}$, for $|\kappa| = N-1$, does not grow faster than t^η .

High velocities and null structure of the system

After commuting the transport equation satisfied by the coefficients Φ^i and in order to prove energy estimates, we are led to control integrals such as

$$\int_0^t \int_{\mathbb{R}^3} \int_{v \in \mathbb{R}^3} (s+|x|) |\mathcal{L}_Z(F) f| dv dx ds.$$

If f vanishes for high velocities, the characteristics of the transport equations have velocities bounded away from 1. If f is moreover initially compactly supported in space, its spatial support is ultimately disjoint from the light cone and, assuming enough decay on the Maxwell field, one can prove

$$|\mathcal{L}_Z(F) f| \lesssim (1+t+r)^{-1} (1+|t-r|)^{-1} |f| \lesssim (1+t+r)^{-2} |f|,$$

so that

$$\int_0^t \int_{\mathbb{R}^3} \int_{v \in \mathbb{R}^3} (s+|x|) |\mathcal{L}_Z(F) f| dv dx ds \lesssim \int_0^t (1+s)^{-1} ds, \quad (4.6)$$

which is almost uniformly bounded in time¹¹. As we do not make any compact support assumption on the initial data, we cannot expect f to vanish for high velocities and certain characteristics of the transport operator ultimately approach those of the Maxwell equations. We circumvent this difficulty by taking advantage of the null structure of the error term given in (4.4), which, in some sense, allows us to transform decay in $|t-r|$ into decay in $t+r$. The key is that certain null components of v , $\mathcal{L}_Z(F)$ and $\nabla_v f := (0, \partial_{v^1} f, \partial_{v^2} f, \partial_{v^3} f)$ behave better than others and we will see in Lemma 4.3.28 that no product of three bad components appear. More precisely, noting $c \prec d$ if d is expected to behave better than c , we have,

$$v^L \prec v^A, \quad v^{\underline{L}}, \quad \underline{\alpha}(\mathcal{L}_Z(F)) \prec \rho(\mathcal{L}_Z(F)) \sim \sigma(\mathcal{L}_Z(F)) \prec \alpha(\mathcal{L}_Z(F)) \quad \text{and} \quad (\nabla_v f)^A \prec (\nabla_v f)^{\underline{L}}.$$

In the exterior of the light cone (and for the massless relativistic transport operator), we have $v^A \prec v^{\underline{L}}$ since $v^{\underline{L}}$ permits to integrate along outgoing null cones¹² and they are both bounded by $(1+t+r)^{-1} v^0 \sum_{z \in \mathbf{k}_1} |z|$,

⁹Note that they were also used in [4] to improve the decay estimate on $\partial \int_v f ds$.

¹⁰We make similar manipulations to recover the standard decay rate on the modified Klainerman-Sobolev inequalities.

¹¹Dealing with these small growth is the next problem addressed.

¹²The angular component v^A can, in some sense, merely do half of it since $|v^A| \lesssim \sqrt{v^0 v^{\underline{L}}}$.

where \mathbf{k}_1 is a set of weights preserved by the free transport operator. In the interior region, the angular components still satisfies the same properties whereas $v^{\underline{L}}$ merely satisfies the inequality

$$v^{\underline{L}} \lesssim \frac{|t-r|}{1+t+r} v^0 + \frac{v^0}{1+t+r} \sum_{z \in \mathbf{k}} |z| \quad (\text{see Lemma 4.2.4}). \quad (4.7)$$

This inequality is crucial for us to close the energy estimates on the electromagnetic field without assuming more initial decay in v on f . It gives a decay rate of $(1+t+r)^{-3}$ on $\int_v \frac{v^{\underline{L}}}{v^0} |f| dv$ by only using a Klainerman-Sobolev inequality (Theorem 4.4.9 and Proposition 4.4.10 would cost us two powers of v^0). As $1 \lesssim v^0 v^{\underline{L}}$ for massive particles, we obtain, combining (4.7) and Theorem 4.4.9, for g a solution to $v^\mu \partial_\mu g = 0$,

$$\forall t \geq |x|, \quad \int_{v \in \mathbb{R}^3} |g|(t, x, v) dv \lesssim \frac{(1+|t-r|)^k}{(1+t+r)^{3+k}} \sum_{|\beta| \leq 3} \left\| (v^0)^{2k+2} (1+r)^k \widehat{Z}^\beta g \right\|_{L_{x,v}^1} (t=0).$$

In the exterior region, the estimate can be improved by removing the factor $(1+|t-r|)^k$ (however one loses one power of r in the initial norm). This remarkable behavior reflects that the particles do not reach the speed of light so that $\int_{v \in \mathbb{R}^3} |g| dv$ enjoys much better decay properties along null rays than along time-like directions and should be compared with solutions to the Klein-Gordon equation (see [31]).

Hierarchy in the equations

Because of certain source terms of the commuted transport equation, we cannot avoid a small growth on certain L^1 norms as it is suggested by (4.6). In order to close the energy estimates, we then consider several hierarchies in the energy norms of the particle density, in the spirit of [35] for the Einstein equations or [17] for the Einstein-Vlasov system. Let us show how a hierarchy related to the weights $z \in \mathbf{k}_1$ preserved by the free massive transport operator (which are defined in Subsection 4.2.3) naturally appears.

- The worst source terms of the transport equation satisfied by Yf are of the form $(t+r)X_i(F_{\mu\nu})\partial_{t,x}f$.
- Using the improved decay properties given by X_i (see (4.11)), we have

$$|(t+r)X_i(F_{\mu\nu})\partial_{t,x}f| \lesssim \sum_{Z \in \mathbb{K}} |\nabla_Z F| \sum_{z \in \mathbf{k}_1} |z \partial_{t,x} f|.$$

- Then, we can obtain a good bound on $\|Yf\|_{L_{x,v}^1}$ provided we have a satisfying one on $\|z \partial_{t,x} f\|_{L_{x,v}^1}$. We will then work with energy norms controlling $\|z^{N_0 - \beta_P} Y^\beta f\|_{L_{x,v}^1}$, where β_P is the number of non-translations composing Y^β .
- At the top order, we will have to deal with terms such as $(t+r)z^{N_0} \partial_{t,x}^\gamma (F_{\mu\nu}) \partial_{t,x}^\beta f$ and we will this time use the extra decay $(1+|t-r|)^{-1}$ given by the translations $\partial_{t,x}^\gamma$.

4.1.6 Structure of the paper

In Section 4.2 we introduce the notations used in this article. Basic results on the electromagnetic field as well as fundamental relations between the null components of the velocity vector v and the weights preserved by the free transport operator are also presented. Section 4.3 is devoted to the commutation vector fields. The construction and basic properties of the modified vector fields are in particular presented. Section 4.4 contains the energy estimates and the pointwise decay estimates used to control both fields. Section 4.5 is devoted to properties satisfied by the pure charge part of the electromagnetic field. In Section 4.6 we describe the main steps of the proof of Theorem 4.1.4 and present the bootstrap assumptions. In Section 4.7, we derive pointwise decay estimates on the solutions and the Φ coefficients of the modified vector fields using only the bootstrap assumptions. Section 4.8 (respectively Section 4.10) concerns the improvement of the bootstrap assumptions on the norms of the particle density (respectively the electromagnetic field). A key step consists in improving the estimates on the velocity averages near the light cone (cf. Proposition 4.8.11). In Section 4.9, we prove L^2 estimates for $\int_v |Y^\beta f| dv$ in order to improve the energy estimates on the Maxwell field.

4.2 Notations and preliminaries

4.2.1 Basic notations

In this paper we work on the 3 + 1 dimensional Minkowski spacetime (\mathbb{R}^{3+1}, η) . We will use two sets of coordinates, the Cartesian (t, x^1, x^2, x^3) , in which $\eta = \text{diag}(-1, 1, 1, 1)$, and null coordinates $(\underline{u}, u, \omega_1, \omega_2)$, where

$$\underline{u} = t + r, \quad u = t - r$$

and (ω_1, ω_2) are spherical variables, which are spherical coordinates on the spheres $(t, r) = \text{constant}$. These coordinates are defined globally on \mathbb{R}^{3+1} apart from the usual degeneration of spherical coordinates and at $r = 0$. We will also use the following classical weights,

$$\tau_+ := \sqrt{1 + \underline{u}^2} \quad \text{and} \quad \tau_- := \sqrt{1 + u^2}.$$

We denote by (e_1, e_2) an orthonormal basis on the spheres and by ∇ the intrinsic covariant differentiation on the spheres $(t, r) = \text{constant}$. Capital Latin indices (such as A or B) will always correspond to spherical variables. The null derivatives are defined by

$$L = \partial_t + \partial_r \quad \text{and} \quad \underline{L} = \partial_t - \partial_r, \quad \text{so that} \quad L(\underline{u}) = 2, \quad L(u) = 0, \quad \underline{L}(\underline{u}) = 0 \quad \text{and} \quad \underline{L}(u) = 2.$$

The velocity vector $(v^\mu)_{0 \leq \mu \leq 3}$ is parametrized by $(v^i)_{1 \leq i \leq 3}$ and $v^0 = \sqrt{1 + |v|^2}$ since we take the mass to be 1. We introduce the operator

$$T : f \mapsto v^\mu \partial_\mu f,$$

defined for all sufficiently regular function $f : [0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$, and we denote $(0, \partial_{v^1} g, \partial_{v^2} g, \partial_{v^3} g)$ by $\nabla_v g$ so that (4.1) can be rewritten

$$T_F(f) := v^\mu \partial_\mu f + F(v, \nabla_v f) = 0.$$

We will use the notation $D_1 \lesssim D_2$ for an inequality such as $D_1 \leq CD_2$, where $C > 0$ is a positive constant independent of the solutions but which could depend on $N \in \mathbb{N}$, the maximal order of commutation. Finally we will raise and lower indices using the Minkowski metric η . For instance, $\nabla^\mu = \eta^{\nu\mu} \nabla_\nu$ so that $\nabla^{\partial_t} = -\nabla_{\partial_t}$ and $\nabla^{\partial_i} = \nabla_{\partial_i}$ for all $1 \leq i \leq 3$.

4.2.2 Basic tools for the study of the electromagnetic field

As we describe the electromagnetic field in geometric form, it will be represented, throughout this article, by a 2-form. Let F be a 2-form defined on $[0, T] \times \mathbb{R}_x^3$. Its Hodge dual $*F$ is the 2-form given by

$$*F_{\mu\nu} = \frac{1}{2} F^{\lambda\sigma} \varepsilon_{\lambda\sigma\mu\nu},$$

where $\varepsilon_{\lambda\sigma\mu\nu}$ are the components of the Levi-Civita symbol. The null decomposition of F , introduced by [11], is denoted by $(\alpha(F), \underline{\alpha}(F), \rho(F), \sigma(F))$, where

$$\alpha_A(F) = F_{AL}, \quad \underline{\alpha}_A(F) = F_{A\underline{L}}, \quad \rho(F) = \frac{1}{2} F_{L\underline{L}} \quad \text{and} \quad \sigma(F) = F_{12}.$$

Finally, the energy-momentum tensor of F is

$$T[F]_{\mu\nu} := F_{\mu\beta} F_\nu^\beta - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}.$$

Note that $T[F]_{\mu\nu}$ is symmetric and traceless, i.e. $T[F]_{\mu\nu} = T[F]_{\nu\mu}$ and $T[F]_\mu^\mu = 0$. This last point is specific to the dimension 3 and engenders additional difficulties in the analysis of the Maxwell equations in high dimension (see Section 3.3.2 in [4] for more details).

We have the following alternative form of the Maxwell equations (for a proof, see [11] or Lemmas 2.2 and D.3 of [6]).

Lemma 4.2.1. *Let G be a 2-form and J be a 1-form both sufficiently regular and such that*

$$\begin{aligned} \nabla^\mu G_{\mu\nu} &= J_\nu \\ \nabla^\mu *G_{\mu\nu} &= 0. \end{aligned}$$

Then,

$$\nabla_{[\lambda} G_{\mu\nu]} = 0 \quad \text{and} \quad \nabla_{[\lambda} *G_{\mu\nu]} = \varepsilon_{\lambda\mu\nu\kappa} J^\kappa.$$

We also have, if $(\alpha, \underline{\alpha}, \rho, \sigma)$ is the null decomposition of G ,

$$\nabla_{\underline{L}}\alpha_A - \frac{\alpha_A}{r} + \nabla_{e_A}\rho + \varepsilon_{BA}\nabla_{e_B}\sigma = J_A.$$

We can then compute the divergence of the energy momentum tensor of a 2-form.

Corollary 4.2.2. *Let G and J be as in the previous lemma. Then, $\nabla^\mu T[G]_{\mu\nu} = G_{\nu\lambda}J^\lambda$.*

Proof. Using the previous lemma, we have

$$\begin{aligned} G_{\mu\rho}\nabla^\mu G_{\nu}{}^\rho &= G^{\mu\rho}\nabla_\mu G_{\nu\rho} \\ &= \frac{1}{2}G^{\mu\rho}(\nabla_\mu G_{\nu\rho} - \nabla_\rho G_{\nu\mu}) \\ &= \frac{1}{2}G^{\mu\rho}\nabla_\nu G_{\mu\rho} \\ &= \frac{1}{4}\nabla_\nu(G^{\mu\rho}G_{\mu\rho}). \end{aligned}$$

Hence,

$$\nabla^\mu T[G]_{\mu\nu} = \nabla^\mu(G_{\mu\rho})G_{\nu}{}^\rho + \frac{1}{4}\nabla_\nu(G^{\mu\rho}G_{\mu\rho}) - \frac{1}{4}\eta_{\mu\nu}\nabla^\mu(G^{\sigma\rho}G_{\sigma\rho}) = G_{\nu\rho}J^\rho.$$

□

Finally, we recall the values of the null components of the energy-momentum tensor of a 2-form.

Lemma 4.2.3. *Let G be 2-form. We have*

$$T[G]_{LL} = |\alpha(G)|^2, \quad T[G]_{\underline{L}\underline{L}} = |\underline{\alpha}(G)|^2 \quad \text{and} \quad T[G]_{L\underline{L}} = |\rho(G)|^2 + |\sigma(G)|^2.$$

4.2.3 Weights preserved by the flow and null components of the velocity vector

Let $(v^L, v^{\underline{L}}, v^A, v^B)$ be the null components of the velocity vector, so that

$$v = v^L L + v^{\underline{L}} \underline{L} + v^A e_A, \quad v^L = \frac{v^0 + \frac{x_i}{r}v^i}{2} \quad \text{and} \quad v^{\underline{L}} = \frac{v^0 - \frac{x_i}{r}v^i}{2}.$$

As in [18], we introduce the following set of weights,

$$\mathbf{k}_1 := \left\{ \frac{v^\mu}{v^0} / 0 \leq \mu \leq 3 \right\} \cup \{z_{\mu\nu} / \mu \neq \nu\}, \quad \text{with} \quad z_{\mu\nu} := x^\mu \frac{v^\nu}{v^0} - x^\nu \frac{v^\mu}{v^0}.$$

Note that

$$\forall z \in \mathbf{k}_1, \quad T(z) = 0. \quad (4.8)$$

Recall that if $\mathbf{k}_0 := \mathbf{k}_1 \cup \{x^\mu v_\mu\}$, then $v^{\underline{L}} \lesssim \tau_+^{-1} \sum_{w \in \mathbf{k}_0} |w|$. Unfortunately, $x^\mu v_\mu$ is not preserved by¹³ T so we will not be able to take advantage of this inequality in this paper. In the following lemma, we try to recover (part of) this extra decay. We also recall inequalities involving other null components of v , which will be used all along this paper.

Lemma 4.2.4. *The following estimates holds,*

$$1 \leq 4v^0 v^{\underline{L}}, \quad |v^A| \lesssim \sqrt{v^L v^{\underline{L}}}, \quad |v^A| \lesssim \frac{v^0}{\tau_+} \sum_{z \in \mathbf{k}_1} |z|, \quad \text{and} \quad v^{\underline{L}} \lesssim \frac{\tau_-}{\tau_+} v^0 + \frac{v^0}{\tau_+} \sum_{z \in \mathbf{k}_1} |z|.$$

Proof. Note first that, as $v^0 = \sqrt{1 + |v|^2}$,

$$4r^2 v^L v^{\underline{L}} = r^2 + r^2 |v|^2 - |x^i|^2 |v_i|^2 - 2 \sum_{1 \leq k < l \leq n} x^k x^l v^k v^l = r^2 + \sum_{1 \leq k < l \leq n} |z_{kl}|^2.$$

¹³Note however that $x^\mu v_\mu$ is preserved by $|v|\partial_t + x^i \partial_i$, the massless relativistic transport operator.

It gives us the first inequality since $v^L \leq v^0$. For the second one, use also that $rv^A = v^0 C_A^{i,j} z_{ij}$, where $C_A^{i,j}$ are bounded functions on the sphere such that $re_A = C_A^{i,j} (x^i \partial_j - x^j \partial_i)$. The third one follows from $|v^A| \leq v^0$ and

$$|v^A| \lesssim \frac{v^0}{r} \sum_{1 \leq i < j \leq 3} |z_{ij}| = \frac{v^0}{tr} \sum_{1 \leq i < j \leq 3} \left| x^i \left(\frac{v^j}{v^0} t - x^j + x^j \right) - x^j \left(\frac{v^i}{v^0} t - x^i + x^i \right) \right| \lesssim \frac{v^0}{t} \sum_{q=1}^3 |z_{0q}|.$$

For the last inequality, note first that $v^L \leq v^0$, which treats the case $t + |x| \leq 2$. Otherwise, use

$$2tv^L = tv^0 - \frac{x^i}{r} tv_i = tv^0 - v^0 \frac{x^i z_{0i}}{r} - v^0 r = (t-r)v^0 - \frac{x^i}{r} z_{0i} v^0 \quad \text{and} \quad rv^L = (r-t)v^L + tv^L.$$

□

Remark 4.2.5. Note that $v^L \lesssim \frac{v^0}{\tau_+} \sum_{z \in \mathbf{k}_1} |z|$ holds in the exterior region. Indeed, if $r \geq t$,

$$v^0(r-t) \leq v^0|x| - |v|t \leq |v^0x - tv| \leq \sum_{i=1}^3 |v^0x^i - tv^i| = v^0 \sum_{i=1}^3 |z_{0i}|.$$

We also point out that $1 \lesssim v^0 v^L$ is specific to massive particles.

Finally, we consider an ordering on \mathbf{k}_1 such that $\mathbf{k}_1 = \{z_i / 1 \leq i \leq |\mathbf{k}_1|\}$.

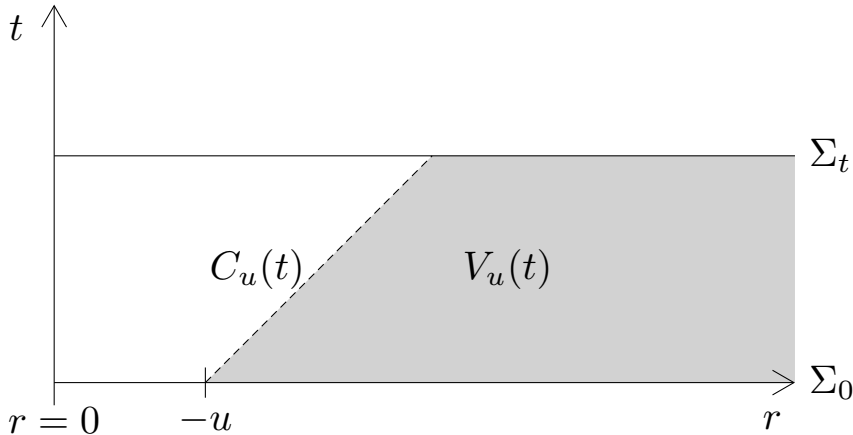
Definition 4.2.6. If $\kappa \in \llbracket 1, |\mathbf{k}_1| \rrbracket^r$, we define $z^\kappa := z_{\kappa_1} \dots z_{\kappa_r}$.

4.2.4 Various subsets of the Minkowski spacetime

We now introduce several subsets of $\mathbb{R}_+ \times \mathbb{R}^3$ depending on $t \in \mathbb{R}_+$, $r \in \mathbb{R}_+$ or $u \in \mathbb{R}$. Let Σ_t , $\mathbb{S}_{t,r}$, $C_u(t)$ and $V_u(t)$ be defined as

$$\begin{aligned} \Sigma_t &:= \{t\} \times \mathbb{R}^n, & C_u(t) &:= \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s \leq t, s - |y| = u\}, \\ \mathbb{S}_{t,r} &:= \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / (s, |y|) = (t, r)\} & \text{and} & \quad V_u(t) := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s \leq t, s - |y| \leq u\}. \end{aligned}$$

The volum form on $C_u(t)$ is given by $dC_u(t) = \sqrt{2}^{-1} r^2 du d\mathbb{S}^2$, where $d\mathbb{S}^2$ is the standard metric on the 2 dimensional unit sphere.



The sets Σ_t , $C_u(t)$ and $V_u(t)$

We will use the following subsets, given for $\underline{u} \in \mathbb{R}_+$, specifically in the proof of Proposition 4.7.4,

$$\underline{V}_{\underline{u}}(t) := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s \leq t, s + |y| \leq \underline{u}\}.$$

For $b \geq 0$ and $t \in \mathbb{R}_+$, define Σ_t^b and $\bar{\Sigma}_t^b$ as

$$\Sigma_t^b := \{t\} \times \{x \in \mathbb{R}^3 / |x| \leq t - b\} \quad \text{and} \quad \bar{\Sigma}_t^b := \{t\} \times \{x \in \mathbb{R}^3 / |x| \geq t - b\}.$$

We also introduce a dyadic partition of \mathbb{R}_+ by considering the sequence $(t_i)_{i \in \mathbb{N}}$ and the functions $(T_i(t))_{i \in \mathbb{N}}$ defined by

$$t_0 = 0, \quad t_i = 2^i \quad \text{if } i \geq 1, \quad \text{and} \quad T_i(t) = t \mathbf{1}_{t \leq t_i}(t) + t_i \mathbf{1}_{t > t_i}(t).$$

We then define the troncated cones $C_u^i(t)$ adapted to this partition by

$$C_u^i(t) := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 \mid t_i \leq s \leq T_{i+1}(t), \quad s - |y| = u\} = \{(s, y) \in C_u(t) \mid t_i \leq s \leq T_{i+1}(t)\}.$$

The following lemma will be used several times during this paper. It depicts that we can foliate $[0, t] \times \mathbb{R}^3$ by $(\Sigma_s)_{0 \leq s \leq t}$, $(C_u(t))_{u \leq t}$ or $(C_u^i(t))_{u \leq t, i \in \mathbb{N}}$.

Lemma 4.2.7. *Let $t > 0$ and $g \in L^1([0, t] \times \mathbb{R}^3)$. Then*

$$\int_0^t \int_{\Sigma_s} g dx ds = \int_{u=-\infty}^t \int_{C_u(t)} g dC_u(t) \frac{du}{\sqrt{2}} = \sum_{i=0}^{+\infty} \int_{u=-\infty}^t \int_{C_u^i(t)} g dC_u^i(t) \frac{du}{\sqrt{2}}.$$

Note that the sum over i is in fact finite. The second foliation will allow us to exploit $t - r$ decay since $\|\tau_-^{-1}\|_{L^\infty(C_u(t))} = \tau_-^{-1}$ whereas $\|\tau_-^{-1}\|_{L^\infty(\Sigma_s)} = 1$. The last foliation will be used to take advantage of time decay on $C_u(t)$ (the problem comes from $\|\tau_+^{-1}\|_{L^\infty(C_u(t))} = \tau_+^{-1}$). More precisely, let $0 < \delta < a$ and suppose for instance that,

$$\forall t \in [0, T], \quad \int_{C_u(t)} g dC_u(t) \leq C(1+t)^\delta, \quad \text{so that} \quad \int_{C_u^i(t)} g dC_u^i(t) \leq C(1+T_{i+1}(t))^\delta \leq C(1+t_{i+1})^\delta.$$

Then,

$$\int_{C_u(t)} \tau_+^{-a} g dC_u(t) \leq \sum_{i=0}^{+\infty} \int_{C_u^i(t)} (1+s)^{-a} g dC_u^i(t) \leq \sum_{i=0}^{+\infty} (1+t_i)^{-a} \int_{C_u^i(t)} g dC_u^i(t) \leq 3^a C \sum_{i=0}^{+\infty} (1+2^{i+1})^{\delta-a}.$$

As $\delta - a < 0$, we obtain a bound independent of T .

4.2.5 An integral estimate

A proof of the following inequality can be found in the appendix B of [18].

Lemma 4.2.8. *Let $m \in \mathbb{N}^*$ and let $a, b \in \mathbb{R}$, such that $a + b > m$ and $b \neq 1$. Then*

$$\exists C_{a,b,m} > 0, \forall t \in \mathbb{R}_+, \quad \int_0^{+\infty} \frac{r^{m-1}}{\tau_+^a \tau_-^b} dr \leq C_{a,b,m} \frac{1+t^{b-1}}{1+t^{a+b-m}}.$$

4.3 Vector fields and modified vector fields

For all this section, we consider F a sufficiently regular 2-form.

4.3.1 The vector fields of the Poincaré group and their complete lift

We present in this section the commutation vector fields of the Maxwell equations and those of the relativistic transport operator (we will modified them to study the Vlasov equation). Let \mathbb{P} be the generators of Poincaré group of the Minkowski spacetime, i.e. the set containing

- the translations¹⁴ $\partial_\mu, \quad 0 \leq \mu \leq 3,$
- the rotations $\Omega_{ij} = x^i \partial_j - x^j \partial_i, \quad 1 \leq i < j \leq 3,$
- the hyperbolic rotations $\Omega_{0k} = t \partial_k + x^k \partial_t, \quad 1 \leq k \leq 3.$

We also consider $\mathbf{T} := \{\partial_t, \partial_1, \partial_2, \partial_3\}$ and $\mathbb{O} := \{\Omega_{12}, \Omega_{13}, \Omega_{23}\}$, the subsets of \mathbb{P} containing respectively the translations and the rotational vector fields as well as $\mathbb{K} := \mathbb{P} \cup \{S\}$, where $S = x^\mu \partial_\mu$ is the scaling vector field. The set \mathbb{K} is well known for commuting with the wave and the Maxwell equations (see Subsection 4.3.6). However, to commute the operator $T = v^\mu \partial_\mu$, one should consider the complete lifts of the elements of \mathbb{P} .

¹⁴In this article, we will denote $\partial_{x,i}$, for $1 \leq i \leq 3$, by ∂_i and sometimes ∂_t by ∂_0 .

Definition 4.3.1. Let $\Gamma = \Gamma^\beta \partial_\beta$ be a vector field. Then, the complete lift $\widehat{\Gamma}$ of Γ is defined by

$$\widehat{\Gamma} = \Gamma^\beta \partial_\beta + v^\gamma \frac{\partial \Gamma^\gamma}{\partial x^\gamma} \partial_{v^i}.$$

We then have $\widehat{\partial}_\mu = \partial_\mu$ for all $0 \leq \mu \leq 3$ and

$$\widehat{\Omega}_{ij} = x^i \partial_j - x^j \partial_i + v^i \partial_{v^j} - v^j \partial_{v^i}, \quad \text{for } 1 \leq i < j \leq 3, \quad \text{and} \quad \widehat{\Omega}_{0k} = t \partial_k + x^k \partial_t + v^0 \partial_{v^k}, \quad \text{for } 1 \leq k \leq 3.$$

One can check that $[T, \widehat{Z}] = 0$ for all $Z \in \mathbb{P}$. Since $[T, S] = T$, we consider

$$\widehat{\mathbb{P}}_0 := \{\widehat{Z} / Z \in \mathbb{P}\} \cup \{S\}$$

and we will, for simplicity, denote by \widehat{Z} an arbitrary vector field of $\widehat{\mathbb{P}}_0$, even if S is not a complete lift. The weights introduced in Subsection 4.2.3 are, in a certain sense, preserved by the action of $\widehat{\mathbb{P}}_0$.

Lemma 4.3.2. Let $z \in \mathbf{k}_1$, $\widehat{Z} \in \widehat{\mathbb{P}}_0$ and $j \in \mathbb{N}$. Then

$$\widehat{Z}(v^0 z) \in v^0 \mathbf{k}_1 \cup \{0\} \quad \text{and} \quad \left| \widehat{Z}(z^j) \right| \leq 3j \sum_{w \in \mathbf{k}_1} |w|^j.$$

Proof. Let us consider for instance $tv^1 - x^1 v^0$, $x^1 v^2 - x^2 v^1$, $\widehat{\Omega}_{01}$ and $\widehat{\Omega}_{02}$. We have

$$\begin{aligned} \widehat{\Omega}_{01}(x^1 v^2 - x^2 v^1) &= tv^2 - x^2 v^0, & \widehat{\Omega}_{01}(tv^1 - x^1 v^0) &= 0, \\ \widehat{\Omega}_{02}(x^1 v^2 - x^2 v^1) &= x^1 v^0 - tv^1 & \text{and} & \quad \widehat{\Omega}_{02}(tv^1 - x^1 v^0) = x^2 v^1 - x^1 v^2. \end{aligned}$$

The other cases are similar. Consequently,

$$\left| \widehat{Z}(z^j) \right| = \left| \widehat{Z} \left(\frac{1}{(v^0)^j} (v^0 z)^j \right) \right| \leq j |z|^j + \frac{j}{(v^0)^j} \left| \widehat{Z}(v^0 z) \right| |v^0 z|^{j-1} \leq j |z|^j + j \frac{|\widehat{Z}(v^0 z)|^j}{(v^0)^j} + j |z|^j,$$

since $|w||z|^{a-1} \leq |w|^a + |z|^a$ when $a \geq 1$. □

The vector fields introduced in this section and the averaging in v almost commute in the following sense (we refer to [18] or to Lemma 4.3.20 below for a proof).

Lemma 4.3.3. Let $f : [0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ be a sufficiently regular function. We have, almost everywhere,

$$\forall Z \in \mathbb{K}, \quad \left| Z \left(\int_{v \in \mathbb{R}^3} |f| dv \right) \right| \lesssim \sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|} \\ |\beta| \leq 1}} \int_{v \in \mathbb{R}^3} |\widehat{Z}^\beta f| dv.$$

Similar estimates hold for $\int_{v \in \mathbb{R}^3} (v^0)^k |f| dv$. For instance,

$$\left| S \left(\int_{v \in \mathbb{R}^3} (v^0)^{-2} |f| dv \right) \right| \lesssim \int_{v \in \mathbb{R}^3} (v^0)^{-2} |Sf| dv.$$

The vector spaces engendered by each of the sets defined in this section are actually algebras.

Lemma 4.3.4. Let \mathbb{L} be either \mathbb{K} , \mathbb{P} , \mathbb{O} , \mathbf{T} or $\widehat{\mathbb{P}}_0$. Then for all $(Z_1, Z_2) \in \mathbb{L}^2$, $[Z_1, Z_2]$ is a linear combinations of vector fields of \mathbb{L} . Note also that if $Z_2 = \partial \in \mathbf{T}$, then $[Z_1, \partial]$ can be written as a linear combination of translations.

We consider an ordering on each of the sets \mathbb{O} , \mathbb{P} , \mathbb{K} and $\widehat{\mathbb{P}}_0$. We take orderings such that, if $\mathbb{P} = \{Z^i / 1 \leq i \leq |\mathbb{P}|\}$, then $\mathbb{K} = \{Z^i / 1 \leq i \leq |\mathbb{K}|\}$, with $Z^{|\mathbb{K}|} = S$, and

$$\widehat{\mathbb{P}}_0 = \left\{ \widehat{Z}^i / 1 \leq i \leq |\widehat{\mathbb{P}}_0| \right\}, \quad \text{with} \quad \left(\widehat{Z}^i \right)_{1 \leq i \leq |\mathbb{P}|} = \left(\widehat{Z}^i \right)_{1 \leq i \leq |\mathbb{P}|} \quad \text{and} \quad \widehat{Z}^{|\widehat{\mathbb{P}}_0|} = S.$$

If \mathbb{L} denotes \mathbb{O} , \mathbb{P} , \mathbb{K} or $\widehat{\mathbb{P}}_0$, and $\beta \in \{1, \dots, |\mathbb{L}|\}^r$, with $r \in \mathbb{N}^*$, we will denote the differential operator $\Gamma^{\beta_1} \dots \Gamma^{\beta_r} \in \mathbb{L}^{|\beta|}$ by Γ^β . For a vector field W , we denote the Lie derivative with respect to W by \mathcal{L}_W and if $Z^\gamma \in \mathbb{K}^r$, we will write \mathcal{L}_{Z^γ} for $\mathcal{L}_{Z^{\gamma_1}} \dots \mathcal{L}_{Z^{\gamma_r}}$. The following definition will be useful to lighten the notations in the presentation of commutation formulas.

Definition 4.3.5. We call good coefficient $c(t, x, v)$ any function c of (t, x, v) such that

$$\forall Q \in \mathbb{N}, \exists C_Q > 0, \forall |\beta| \leq Q, (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \setminus \{0\} \times \{0\} \times \mathbb{R}_v^3, \quad \left| \widehat{Z}^\beta (c(t, x, v)) \right| \leq C_Q.$$

Similarly, we call good coefficient $c(v)$ any function c such that

$$\forall Q \in \mathbb{N}, \exists C_Q > 0, \forall |\beta| \leq Q, v \in \mathbb{R}^3, \quad \left| \widehat{Z}^\beta (c(v)) \right| \leq C_Q.$$

Finally, we will say that B is a linear combination, with good coefficients $c(v)$, of $(B^i)_{1 \leq i \leq M}$ if there exists good coefficients $(c_i(v))_{1 \leq i \leq M}$ such that $B = c_i B^i$. We define similarly a linear combination with good coefficients $c(t, x, v)$.

The sets of functions introduced here are to be thought as bounded functions which remain bounded when they are differentiated (by $\widehat{\mathbb{P}}_0$ derivatives) or multiplied between them. In the remaining of this paper, we will denote by $c(t, x, v)$ (or $c_Z(t, x, v)$, $c_i(t, x, v)$) any such functions. Note that $\widehat{Z}^\beta (c(t, x, v))$ is not necessarily defined on $\{0\} \times \{0\} \times \mathbb{R}_v^3$ as, for instance, $c(t, x, v) = \frac{x^1 v^2}{t+r v^0}$ satisfies these conditions. Typically, the good coefficients $c(v)$ will be of the form $\widehat{Z}^\gamma \left(\frac{v^i}{v^0} \right)$.

Let us recall, by the following classical result, that the derivatives tangential to the cone behave better than others.

Lemma 4.3.6. The following relations hold,

$$(t-r)\underline{L} = S - \frac{x^i}{r} \Omega_{0i}, \quad (t+r)L = S + \frac{x^i}{r} \Omega_{0i} \quad \text{and} \quad re_A = \sum_{1 \leq i < j \leq 3} C_A^{i,j} \Omega_{ij},$$

where the $C_A^{i,j}$ are uniformly bounded and depends only on spherical variables. In the same spirit, we have

$$(t-r)\partial_t = \frac{t}{t+r} S - \frac{x^i}{t+r} \Omega_{0i} \quad \text{and} \quad (t-r)\partial_i = \frac{t}{t+r} \Omega_{0i} - \frac{x^i}{t+r} S - \frac{x^j}{t+r} \Omega_{ij}.$$

As mentionned in the introduction, we will crucially use the vector fields $(X_i)_{1 \leq i \leq 3}$, defined by

$$X_i := \partial_i + \frac{v^i}{v^0} \partial_t. \tag{4.9}$$

They provide extra decay in particular cases since

$$X_i = \frac{1}{t} (\Omega_{0i} + z_{0i} \partial_t). \tag{4.10}$$

We also have, using Lemma 4.3.6 and $(1+t+r)X_i = X_i + 2tX_i + (r-t)X_i$, that there exists good coefficients $c_Z(t, x, v)$ such that

$$(1+t+r)X_i = 2z_{0i} \partial_t + \sum_{Z \in \mathbb{K}} c_Z(t, x, v) Z. \tag{4.11}$$

By a slight abuse of notation, we will write $\mathcal{L}_{X_i}(F)$ for $\mathcal{L}_{\partial_i}(F) + \frac{v^i}{v^0} \mathcal{L}_{\partial_t}(F)$. We are now interested in the compatibility of these extra decay with the Lie derivative of a 2-form and its null decomposition.

Proposition 4.3.7. Let G be a sufficiently regular 2-form. Then, with $z = t \frac{v^i}{v^0} - x^i$ if $X = X_i$ and $\zeta \in \{\alpha, \underline{\alpha}, \rho, \sigma\}$, we have

$$|\mathcal{L}_\partial(G)| \lesssim \frac{1}{\tau_-} \sum_{Z \in \mathbb{K}} |\nabla_Z G| \lesssim \frac{1}{\tau_-} \sum_{|\gamma| \leq 1} |\mathcal{L}_{Z^\gamma}(G)|, \tag{4.12}$$

$$|\mathcal{L}_X(G)| \lesssim \frac{1}{\tau_+} \left(|z| |\nabla_{\partial_t} G| + \sum_{Z \in \mathbb{K}} |\nabla_Z G| \right), \tag{4.13}$$

$$\tau_- |\nabla_{\underline{L}} \zeta| + \tau_+ |\nabla_L \zeta| + (1+r) |\nabla \zeta| \lesssim \sum_{|\gamma| \leq 1} |\zeta(\mathcal{L}_{Z^\gamma}(G))|, \tag{4.14}$$

$$|\zeta(\mathcal{L}_\partial(G))| \lesssim \sum_{|\gamma| \leq 1} \frac{1}{\tau_-} |\zeta(\mathcal{L}_{Z^\gamma}(G))| + \frac{1}{\tau_+} |\mathcal{L}_{Z^\gamma}(G)|. \tag{4.15}$$

Proof. To obtain the first two identities, use Lemma 4.3.6 as well as (4.11) and then remark that if Γ is a translation or an homogeneous vector field,

$$|\nabla_{\Gamma}(G)| \lesssim |\mathcal{L}_{\Gamma}(G)| + |G|.$$

For (4.14), we refer to Lemma D.2 of [6]. Finally, the last inequality comes from (4.12) if $2t \leq \max(r, 1)$ and from

$$\partial_i = \frac{\Omega_{0i}}{t} - \frac{x^i}{2t}L - \frac{x^i}{2t}L \quad \text{and} \quad (4.14) \quad \text{if } 2t \geq \max(r, 1).$$

□

Remark 4.3.8. *We do not have, for instance, $|\rho(\mathcal{L}_{\partial_k}(G))| \lesssim \sum_{|\gamma| \leq 1} \tau^{-1} |\rho(\mathcal{L}_{Z^\gamma}(G))|$, for $1 \leq k \leq 3$.*

Remark 4.3.9. *If G solves the Maxwell equations $\nabla^\mu G_{\mu\nu} = J_\nu$ and $\nabla^\mu *G_{\mu\nu} = 0$, a better estimate can be obtained on $\alpha(\mathcal{L}_{\partial}(G))$. Indeed, as $|\nabla_{\partial}\alpha| \leq |\nabla_L\alpha| + |\underline{L}\alpha| + |\nabla\alpha|$, (4.15) and Lemma 4.2.1 gives us,*

$$\forall |x| \geq 1 + \frac{t}{2}, \quad |\alpha(\mathcal{L}_{\partial}(G))|(t, x) \lesssim |J_A| + \frac{1}{\tau_+} \sum_{|\gamma| \leq 1} \left(|\alpha(\mathcal{L}_{Z^\gamma}(G))|(t, x) + |\sigma(\mathcal{L}_{Z^\gamma}(G))|(t, x) + |\rho(\mathcal{L}_{Z^\gamma}(G))|(t, x) \right).$$

We make the choice to work with (4.15) since it does not directly require a bound on the source term of the Maxwell equation, which lighten the proof of Theorem 4.1.4 (otherwise we would have, among others, to consider more bootstrap assumptions).

4.3.2 Modified vector field and the first order commutation formula

We start this section with the following commutation formula and we refer to Lemma 2.8 of [6] for a proof¹⁵.

Lemma 4.3.10. *If $\widehat{Z} \in \widehat{\mathbb{P}}_0 \setminus \{S\}$, then*

$$[T_F, \widehat{Z}](f) = -\mathcal{L}_Z(F)(v, \nabla_v f) \quad \text{and} \quad [T_F, S](f) = F(v, \nabla_v f) - \mathcal{L}_S(F)(v, \nabla_v f).$$

In order to estimate quantities such as $\mathcal{L}_Z(F)(v, \nabla_v f)$, we rewrite $\nabla_v f$ in terms of the commutation vector fields (i.e. the elements of $\widehat{\mathbb{P}}_0$). Schematically, if we neglect the null structure of the system, we have, since $v^0 \partial_{v^i} = \widehat{\Omega}_{0i} - t\partial_i - x^i \partial_t$,

$$\begin{aligned} |\mathcal{L}_Z(F)(v, \nabla_v f)| &\lesssim v^0 |\mathcal{L}_Z(F)| |\partial_v f| \\ &\sim \tau_+ |\mathcal{L}_Z(F)| |\partial_{t,x} f| + \text{l.o.t.}, \end{aligned}$$

so that the v derivatives engender a τ_+ -loss. The modified vector fields, constructed below, will allow us to absorb the worst terms in the commuted equations.

Definition 4.3.11. *Let \mathbb{Y}_0 be the set of vector fields defined by*

$$\mathbb{Y}_0 := \{ \widehat{Z} + \Phi_{\widehat{Z}}^j X_j \mid \widehat{Z} \in \widehat{\mathbb{P}}_0 \setminus \mathbf{T} \},$$

where $\Phi_{\widehat{Z}}^j : [0, T] \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ are smooth functions which will be specified below and the X_j are defined in (4.9).

We will denote $\widehat{\Omega}_{0k} + \Phi_{\widehat{\Omega}_{0k}}^j X_j$ by Y_{0k} and, more generally, $\widehat{Z} + \Phi_{\widehat{Z}}^j X_j$ by $Y_{\widehat{Z}}$. We also introduce the sets

$$\mathbb{Y} := \mathbb{Y}_0 \cup \mathbf{T} \quad \text{and} \quad \mathbb{Y}_X := \mathbb{Y} \cup \{X_1, X_2, X_3\}.$$

We consider an ordering on \mathbb{Y} and \mathbb{Y}_X compatible with $\widehat{\mathbb{P}}_0$ in the sense that if $\mathbb{Y} = \{Y^i \mid 1 \leq i \leq |\mathbb{Y}|\}$, then $Y^i = \widehat{Z}^i + \Phi_{\widehat{Z}^i}^k X_k$ or $Y^i = \partial_\mu = \widehat{Z}^i$. We suppose moreover that X_j is the $(|\mathbb{Y}| + j)^{\text{th}}$ element of \mathbb{Y}_X . Most of the time, for a vector field $Y \in \mathbb{Y}_0$, we will simply write $Y = \widehat{Z} + \Phi X$.

Let $\widehat{Z} \in \widehat{\mathbb{P}}_0 \setminus \{S\}$ and $1 \leq k \leq 3$. $\Phi_{\widehat{Z}}^k$ and Φ_S^k are defined such as

$$T_F(\Phi_{\widehat{Z}}^k) = -t \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k}, \quad T_F(\Phi_S^k) = t \frac{v^\mu}{v^0} (F_{\mu k} - \mathcal{L}_S(F)_{\mu k}) \quad \text{and} \quad \Phi_{\widehat{Z}}^k(0, \cdot, \cdot) = \Phi_S^k(0, \cdot, \cdot) = 0. \quad (4.16)$$

¹⁵Note that a similar result is proved in Lemma 4.3.22 below.

As explained during the introduction, we consider the X_i vector fields rather than translations in view of (4.11). We are then led to compute $[T_F, X_i]$.

Lemma 4.3.12. *Let $1 \leq i \leq 3$. We have*

$$[T_F, X_i] = -\mathcal{L}_{X_i}(F)(v, \nabla_v) + \frac{v^\mu}{v^0} F_{\mu X_i} \partial_t.$$

Proof. One juste has to notice that

$$[T_F, X_i] = \frac{v^i}{v^0} [T_F, \partial_t] + [T_F, \partial_i] + F \left(v, \nabla_v \left(\frac{v^i}{v^0} \right) \right) \partial_t$$

and $v^\mu v^j F_{\mu j} = -v^\mu v^0 F_{\mu 0}$, as F is a 2-form. \square

Finally, we study the commutator between the transport operator and these modified vector fields. The following relation,

$$\partial_{v^i} = \frac{1}{v^0} \left(Y_{0i} - \Phi_{\Omega_{0i}}^j X_j - t X_i + z_{0i} \partial_t \right), \quad (4.17)$$

will be useful to express the v derivatives in terms of the commutation vector fields

Proposition 4.3.13. *Let $Y \in \mathbb{Y}_0 \setminus \{Y_S\}$. we have, using (4.16)*

$$\begin{aligned} [T_F, Y] &= -\frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu}{}^j \left(Y_{0j} - \Phi_{\Omega_{0j}}^k X_k + z_{0j} \partial_t \right) - \Phi_{\widehat{Z}}^j \mathcal{L}_{X_j}(F)(v, \nabla_v) + \Phi_{\widehat{Z}}^j \frac{v^\mu}{v^0} F_{\mu X_j} \partial_t, \\ [T_F, Y_S] &= \frac{v^\mu}{v^0} \left(F_{\mu}{}^j - \mathcal{L}_S(F)_{\mu}{}^j \right) \left(Y_{0j} - \Phi_{\Omega_{0j}}^k X_k + z_{0j} \partial_t \right) - \Phi_S^j \mathcal{L}_{X_j}(F)(v, \nabla_v) + \Phi_S^j \frac{v^\mu}{v^0} F_{\mu X_j} \partial_t. \end{aligned}$$

Proof. We only treat the case $Y \in \mathbb{Y}_0 \setminus \{Y_S\}$ (the computations are similar for Y_S). Using Lemmas 4.3.10 and 4.3.12 as well as (4.17), we have

$$\begin{aligned} [T_F, Y] &= [T_F, \widehat{Z}] + [T_F, \Phi_{\widehat{Z}}^j X_j] \\ &= -\mathcal{L}_Z(F)(v, \nabla_v) + T_F(\Phi_{\widehat{Z}}^j) X_j + \Phi_{\widehat{Z}}^j [T_F, X_j]. \\ &= -\mathcal{L}_Z(F)(v, \nabla_v) + T_F(\Phi_{\widehat{Z}}^j) X_j - \Phi_{\widehat{Z}}^j \mathcal{L}_{X_j}(F)(v, \nabla_v) + \Phi_{\widehat{Z}}^j \frac{v^\mu}{v^0} F_{\mu X_j} \partial_t \\ &= -\frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu}{}^j \left(Y_{0j} - \Phi_{\Omega_{0j}}^k X_k + z_{0j} \partial_t \right) + \left(t \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu}{}^j + T_F(\Phi_{\widehat{Z}}^j) \right) X_j \\ &\quad - \Phi_{\widehat{Z}}^j \mathcal{L}_{X_j}(F)(v, \nabla_v) + \Phi_{\widehat{Z}}^j \frac{v^\mu}{v^0} F_{\mu X_j} \partial_t. \end{aligned}$$

To conclude, recall from (4.16) that $t \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu}{}^j + T_F(\Phi_{\widehat{Z}}^j) = 0$. \square

Remark 4.3.14. *As we will have $|\Phi| \lesssim \log^2(1 + \tau_+)$, a good control on $z_{0j} \partial_t f$ and in view of the improved decay given by X_j (see Proposition 4.3.7), it holds schematically*

$$|[T_F, Y](f)| \lesssim \log^2(1 + \tau_+) |\mathcal{L}_Z(F)| |Yf|,$$

which is much better than $\left| [T_F, \widehat{Z}](f) \right| \lesssim \tau_+ |\mathcal{L}_Z(F)| |\partial_{t,x} f|$.

Let us introduce some notations for the presentation of the higher order commutation formula.

Definition 4.3.15. *Let $Y^\beta \in \mathbb{Y}^{|\beta|}$. We denote by β_T the number of translations composing Y^β and by β_P the number of modified vector fields (the elements of \mathbb{Y}_0). Note that β_T denote also the number of translations composing \widehat{Z}^β and Z^β and β_P the number of elements of $\widehat{\mathbb{P}}_0 \setminus \mathbf{T}$ or $\mathbb{K} \setminus \mathbf{T}$. We have*

$$|\beta| = \beta_T + \beta_P$$

and, for instance, if $Y^\beta = \partial_t Y_1 \partial_3$, $|\beta| = 3$, $\beta_T = 2$ and $\beta_P = 1$. We define similarly β_X if $Y^\beta \in \mathbb{Y}_X^{|\beta|}$.

Definition 4.3.16. Let $k = (k_T, k_P) \in \mathbb{N}^2$ and $p \in \mathbb{N}$. We will denote by $P_{k,p}(\Phi)$ any linear combination of terms such as

$$\prod_{j=1}^p Y^{\beta_j}(\Phi), \quad \text{with } Y^{\beta_j} \in \mathbb{Y}^{|\beta_j|}, \quad \sum_{j=1}^p |\beta_j| = |k|, \quad \sum_{j=1}^p (\beta_j)_P = k_P$$

and where Φ denotes any of the Φ coefficients. Note that $\sum_{j=1}^p (\beta_j)_T = k_T$. Finally, if $\min_j |\beta_j| \geq 1$, we will denote $\prod_{j=1}^p Y^{\beta_j}(\Phi)$ by $P_\beta(\Phi)$, where $\beta = (\beta_1, \dots, \beta_p)$.

Definition 4.3.17. Let $k = (k_T, k_P, k_X) \in \mathbb{N}^3$ and $p \in \mathbb{N}$. We will denote by $P_{k,p}^X(\Phi)$ any linear combination of terms such as

$$\prod_{j=1}^p Y^{\beta_j}(\Phi), \quad \text{with } Y^{\beta_j} \in \mathbb{Y}^{|\beta_j|}, \quad \sum_{j=1}^p |\beta_j| = |k|, \quad \sum_{j=1}^p (\beta_j)_P = k_P, \quad \sum_{j=1}^p (\beta_j)_X = k_X \quad \text{and} \quad \min_{1 \leq j \leq p} (\beta_j)_X \geq 1.$$

We will also denote $\prod_{j=1}^p Y^{\beta_j}(\Phi)$ by $P_\beta^X(\Phi)$.

Remark 4.3.18. For convenience, if $p = 0$, we will take $P_{k,p}(\Phi) = 1$. Similarly, if $|\beta| = 0$, we will take $P_\beta(\Phi) = P_\beta^X(\Phi) = 1$.

In view of presenting the higher order commutation formulas, let us gather the source terms in different categories.

Proposition 4.3.19. Let $Y \in \mathbb{Y} \setminus \mathbf{T}$. In what follows, $0 \leq \nu \leq 3$. The commutator $[T_F, Y]$ can be written as a linear combination, with $c(\nu)$ coefficients, of terms such as

- $\frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \Gamma$, where $|\gamma| \leq 1$ and $\Gamma \in \mathbb{Y}_0$.
- $\Phi \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \partial_{t,x}$, where $|\gamma| \leq 1$.
- $z \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \partial_{t,x}$, where $|\gamma| \leq 1$ and $z \in \mathbf{k}_1$.
- $\Phi \mathcal{L}_X(F)(v, \nabla_v)$.

Finally, let us adapt Lemma 4.3.3 to our modified vector fields.

Lemma 4.3.20. Let $f : [0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ be a sufficiently regular function and suppose that for all $|\beta| \leq 1$, $|Y^\beta \Phi| \lesssim \log^{\frac{7}{2}}(1 + \tau_+)$. Then, we have, almost everywhere,

$$\forall Z \in \mathbb{K}, \quad \left| Z \left(\int_{v \in \mathbb{R}^3} |f| dv \right) \right| \lesssim \sum_{\substack{Y \in \mathbb{Y} \\ z \in \mathbf{k}_1}} \int_{v \in \mathbb{R}^3} \left(|Yf| + |f| + |X(\Phi)f| + \frac{\log^7(1 + \tau_+)}{\tau_+} (|z \partial_t f| + |zf|) \right) dv.$$

Proof. Consider, for instance, the rotation Ω_{12} . We have by integration by parts, as $\Omega_{12} = \widehat{\Omega}_{12} - v^1 \partial_{v^2} + v^2 \partial_{v^1}$,

$$\Omega_{12} \left(\int_{v \in \mathbb{R}^3} |f| dv \right) = \int_{v \in \mathbb{R}^3} \widehat{\Omega}_{12}(|f|) dv - \int_{v \in \mathbb{R}^3} (v^1 \partial_{v^2} - v^2 \partial_{v^1})(|f|) dv = \int_{v \in \mathbb{R}^3} \widehat{\Omega}_{12}(|f|) dv.$$

This proves Lemma 4.3.3 for Ω_{12} since $|\widehat{\Omega}_{12}(|f|)| = \left| \frac{f}{|f|} \widehat{\Omega}_{12}(f) \right| \leq |\widehat{\Omega}_{12}(f)|$. On the other hand,

$$\int_{v \in \mathbb{R}^3} \widehat{\Omega}_{12}(|f|) dv = \int_{v \in \mathbb{R}^3} \left(\widehat{\Omega}_{12} + \Phi_{\widehat{\Omega}_{12}}^k X_k - \Phi_{\widehat{\Omega}_{12}}^k X_k \right) (|f|) dv \quad (4.18)$$

$$= \int_{v \in \mathbb{R}^3} \frac{f}{|f|} Y_{\widehat{\Omega}_{12}} f dv + \int_{v \in \mathbb{R}^3} X_k \left(\Phi_{\widehat{\Omega}_{12}}^k \right) |f| dv - \int_{v \in \mathbb{R}^3} X_k \left(\Phi_{\widehat{\Omega}_{12}}^k |f| \right) dv. \quad (4.19)$$

(4.18) implies the result if $t + r \leq 1$. Otherwise, if $t \geq r$, note that by (4.10),

$$\begin{aligned} \int_{v \in \mathbb{R}^3} X_k \left(\Phi_{\widehat{\Omega}_{12}}^k |f| \right) dv &= \frac{1}{t} \int_{v \in \mathbb{R}^3} (\Omega_{0k} + z_{0k} \partial_t) \left(\Phi_{\widehat{\Omega}_{12}}^k |f| \right) dv \\ &= \frac{1}{t} \int_{v \in \mathbb{R}^3} \left(Y_{0k} - v^0 \partial_{v^k} - \Phi_{\Omega_{0k}}^q X_q + z_{0k} \partial_t \right) \left(\Phi_{\widehat{\Omega}_{12}}^k |f| \right) dv \\ &= \frac{1}{t} \int_{v \in \mathbb{R}^3} \left(Y_{0k} + \frac{v_k}{v^0} - \Phi_{\Omega_{0k}}^q X_q + z_{0k} \partial_t \right) \left(\Phi_{\widehat{\Omega}_{12}}^k |f| \right) dv. \end{aligned}$$

Consequently, in view of the bounds on $Y^\beta \Phi$ for $|\beta| \leq 1$,

$$\left| \int_{v \in \mathbb{R}^3} X_k \left(\Phi_{\Omega_{12}}^k |f| \right) dv \right| \lesssim \sum_{Y \in \mathbb{Y}} \sum_{z \in \mathbf{k}_1} \int_{v \in \mathbb{R}^3} |Yf| + |f| + \frac{|z| \log^7(1+t)}{t} (|\partial_t f| + |f|) dv,$$

and it remains to combine it with (4.19). When $t \leq r$, one can use $rX_k = tX_k + (r-t)X_k$ and Lemma 4.3.6. \square

Remark 4.3.21. *If moreover $|\Phi| \lesssim \log^2(1+\tau_+)$, one can prove similarly that, for $Z \in \mathbb{K}$, $z \in \mathbf{k}_1$ and $j \in \mathbb{N}^*$,*

$$\left| Z \left(\int_v |z^j f| dv \right) \right| \lesssim j \sum_{\substack{|\xi|+|\beta| \leq 1 \\ w \in \mathbf{k}_1}} \int_v |w^j P_\xi^X(\Phi) Y^\beta f| + \log^2(3+t) |w^{j-1} f| + \frac{\log^7(1+\tau_+) |w|^{j+1}}{\tau_+} (|\partial_t f| + |f|) dv.$$

To prove this inequality, apply Lemma 4.3.20 to $z^j f$ and use the two following properties,

$$|Y(z^j)| \leq |\widehat{Z}(z^j)| + |\Phi X(z^j)| \lesssim j \left(\sum_{w \in \mathbf{k}_1} |w|^j + \log^2(1+\tau_+) |z|^{j-1} \right) \quad \text{and} \quad \sum_{w \in \mathbf{k}_1} |w| |z|^j \lesssim \sum_{w \in \mathbf{k}_1} |w|^{j+1}.$$

It remains to apply Remark 4.2.5 in order to get

$$\forall |x| \geq 1 + 2t, \quad \log^2(1+\tau_+) |z|^{j-1} \lesssim \frac{\log^2(3+r)}{r} \sum_{w \in \mathbf{k}_1} |w z^{j-1}| \lesssim \sum_{w \in \mathbf{k}_1} |w|^j$$

and to note that $\log(1+\tau_+) \lesssim \log(3+t)$ if $|x| \leq 1 + 2t$.

4.3.3 Higher order commutation formula

The following lemma will be useful for upcoming computations.

Lemma 4.3.22. *Let G be a sufficiently regular 2-form and g a sufficiently regular function defined respectively on $[0, T] \times \mathbb{R}^3$ and $[0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$. Let also $Y = \widehat{Z} + \Phi X \in \mathbb{Y}_0$ and $\nu \in \llbracket 0, 3 \rrbracket$. We have, with $n_Z = 0$ is $Z \in \mathbb{P}$ and $n_S = -1$,*

$$\begin{aligned} Y(v^\mu G_{\mu\nu}) &= v^\mu \mathcal{L}_Z(G)_{\mu\nu} + n_Z v^\mu G_{\mu\nu} + \Phi v^\mu \mathcal{L}_X(G)_{\mu\nu} + v^\mu G_{\mu[Z, \partial_\nu]}, \\ Y(G(v, \nabla_v g)) &= \mathcal{L}_Z(G)(v, \nabla_v g) + 2n_Z G(v, \nabla_v g) + \Phi \mathcal{L}_X(G)(v, \nabla_v g) + G(v, \nabla_v \widehat{Z}g) + c(v) \Phi G(v, \nabla_v \partial g). \end{aligned}$$

For $i \in \llbracket 1, 3 \rrbracket$, $Y(v^\mu \mathcal{L}_{X_i}(G)_{\mu\nu})$ can be written as a linear combination, with $c(v)$ coefficients, of terms of the form

$$\Phi^p v^\mu \mathcal{L}_{XZ^\gamma}(G)_{\mu\theta}, \quad \text{with } 0 \leq \theta \leq 3 \quad \text{and} \quad \max(p, |\gamma|) \leq 1.$$

Finally, $Y(\mathcal{L}_{X_i}(G)(v, \nabla_v g))$ can be written as a linear combination, with $c(v)$ coefficients, of terms of the form

$$\Phi^p \mathcal{L}_{XZ^\gamma}(G) \left(v, \nabla \left(\widehat{Z}^\kappa g \right) \right), \quad \text{with } \max(|\gamma| + |\kappa|, p + \kappa_P) \leq 1.$$

Proof. Let $Z_v = \widehat{Z} - Z$ so that $Y = Z + Z_v + \Phi X$. We prove the second and the fourth properties (the first and the third ones are easier). We have

$$\begin{aligned} Y(G(v, \nabla_v g)) &= \mathcal{L}_Z(G)(v, \nabla_v g) + G([Z, v], \nabla_v g) + G(v, [Z, \nabla_v g]) + G(Z_v(v), \nabla_v g) + G(v, Z_v(\nabla_v g)) \\ &\quad + \Phi \mathcal{L}_X(G)(v, \nabla_v g) + c(v) \Phi G(v, \nabla_v \partial g). \end{aligned}$$

Note now that

- $S_v = 0$ and $[S, v] = -v$,
- $[Z, v] = -Z_v(v)$ if $Z \in \mathbb{P}$.

The second identity is then implied by

- $[\partial, \nabla_v g] = \nabla_v \partial(g)$ and $[S, \nabla_v g] = \nabla_v S(g) - \nabla_v g$.

- $[Z, \nabla_v g] + Z_v (\nabla_v g) = \nabla_v \widehat{Z}(g)$ if $Z \in \mathbb{O}$.
- $[\Omega_{0i}, \nabla_v g] + (\Omega_{0i})_v (\nabla_v g) = \nabla_v \widehat{Z}(g) - \frac{v^i}{v^0} \partial_{v^i}$ and $G(v, v) = 0$ as G is a 2-form.

We now prove the fourth identity. We treat the case $Y = \widehat{Z} + \Phi X \in \mathbb{Y}_0 \setminus \{Y_S\}$ as the computations are similar for Y_S . On the one hand, since $[\partial, X_i] = 0$ and $X_k = \partial_k + \frac{v^k}{v^0} \partial_t$, one can easily check that $\Phi X_k (\mathcal{L}_{X_i}(G)(v, \nabla_v g))$ gives four terms of the expected form. On the other hand,

$$\widehat{Z} (\mathcal{L}_{X_i}(G)(v, \nabla_v g)) = \widehat{Z} (\mathcal{L}_{\partial_i}(G)(v, \nabla_v g)) + \widehat{Z} \left(\frac{v^i}{v^0} \mathcal{L}_{\partial_t}(G)(v, \nabla_v g) \right).$$

Applying the second equality of this Lemma to $\mathcal{L}_{\partial}(G)$, g and \widehat{Z} (which is equal to Y when $\Phi = 0$), we have

$$\begin{aligned} \widehat{Z} (\mathcal{L}_{\partial_i}(G)(v, \nabla_v g)) &= \mathcal{L}_{Z\partial_i}(G)(v, \nabla_v g) + \mathcal{L}_{\partial_i}(G)(v, \nabla_v \widehat{Z}g) \\ \widehat{Z} \left(\frac{v^i}{v^0} \mathcal{L}_{\partial_t}(G)(v, \nabla_v g) \right) &= \widehat{Z} \left(\frac{v^i}{v^0} \right) \mathcal{L}_{\partial_t}(G)(v, \nabla_v g) + \frac{v^i}{v^0} \mathcal{L}_{Z\partial_t}(G)(v, \nabla_v g) + \frac{v^i}{v^0} \mathcal{L}_{\partial_t}(G)(v, \nabla_v \widehat{Z}g) \end{aligned}$$

The sum of the last terms of these two identities is of the expected form. The same holds for the sum of the three other terms since

$$\begin{aligned} [\Omega_{0j}, \partial_i] + \frac{v^i}{v^0} [\Omega_{0j}, \partial_t] + v^0 \partial_{v^j} \left(\frac{v^i}{v^0} \right) \partial_t &= -\delta_j^i \partial_t - \frac{v^i}{v^0} \partial_j - \frac{v^i v^j}{(v^0)^2} \partial_t + \delta_j^i \partial_t = -\frac{v^i}{v^0} X_j = c(v) X_j, \\ [\Omega_{kj}, \partial_i] + \frac{v^i}{v^0} [\Omega_{kj}, \partial_t] + (v^k \partial_{v^j} - v^j \partial_{v^k}) \left(\frac{v^i}{v^0} \right) \partial_t &= \delta_j^i \partial_k - \delta_k^i \partial_j + \left(\frac{v^k \delta_j^i - v^j \delta_k^i}{v^0} \right) \partial_t = \delta_j^i X_k - \delta_k^i X_j, \\ [S, \partial_i] + \frac{v^i}{v^0} [S, \partial_t] &= -\partial_i - \frac{v^i}{v^0} \partial_t = -X_i. \end{aligned}$$

□

We are now ready to present the higher order commutation formula. To lighten its presentation and facilitate its future usage, we introduce $\mathbb{G} := \widehat{\mathbb{P}}_0 \cup \mathbb{Y}_0$, on which we consider an ordering. A combination of vector fields of \mathbb{G} will always be denoted by Γ^σ and we will also denote by σ_T its number of translations and by $\sigma_P = |\sigma| - \sigma_T$ its number of homogeneous vector fields. In Lemma 4.3.30 below, we will express Γ^σ in terms of Φ coefficients and \mathbb{Y} vector fields.

Proposition 4.3.23. *Let β be a multi-index. In what follows, $\nu \in \llbracket 0, 3 \rrbracket$. The commutator $[T_F, Y^\beta]$ can be written as a linear combination, with $c(v)$ coefficients, of the following terms.*

- $$z^d P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} Y^\sigma, \quad (\text{type 1-}\beta)$$

where $z \in \mathbf{k}_1$, $d \in \{0, 1\}$, $|\sigma| \geq 1$ $\max(|\gamma|, |k| + |\gamma|, |k| + |\sigma|) \leq |\beta|$, $|k| + |\gamma| + |\sigma| \leq |\beta| + 1$ and $p + k_P + \sigma_P + d \leq \beta_P$. Note also that, as $|\sigma| \geq 1$, $|k| \leq |\beta| - 1$.
- $$P_{k,p}(\Phi) \mathcal{L}_{XZ^\gamma_0}(F)(v, \nabla_v \Gamma^\sigma), \quad (\text{type 2-}\beta)$$

where $|k| + |\gamma_0| + |\sigma| \leq |\beta| - 1$, $p + k_P + \sigma_P \leq \beta_P$ and $p \geq 1$.
- $$P_{k,p}(\Phi) \mathcal{L}_{\partial Z^\gamma_0}(F)(v, \nabla_v \Gamma^\sigma), \quad (\text{type 3-}\beta)$$

where $|k| + |\gamma_0| + |\sigma| \leq |\beta| - 1$, $p + |\gamma_0| \leq |\beta| - 1$ and $p + k_P + \sigma_P \leq \beta_P$.

Proof. The result follows from an induction on $|\beta|$, Proposition 4.3.19 (which treat the case $|\beta| = 1$) and

$$[T_F, YY^{\beta_0}] = Y[T_F, Y^{\beta_0}] + [T_F, Y]Y^{\beta_0}.$$

Let $Q \in \mathbb{N}$ and suppose that the commutation formula holds for all $|\beta_0| \leq Q$. We then fix a multi-index $|\beta_0| = Q$, consider $Y \in \mathbb{Y}$ and denote the multi-index corresponding to YY^{β_0} by β . Then, $|\beta| = |\beta_0| + 1$.

Suppose first that $Y = \partial$ is a translation so that $\beta_P = (\beta_0)_P$. Then, using Lemma 4.3.10, we have

$$[T_F, \partial]Y^{\beta_0} = -\mathcal{L}_{\partial}(F)(v, \nabla_v Y^{\beta_0}),$$

which is a term of (type 3- β) as $|\beta_0| = |\beta| - 1$ and $(\beta_0)_P = \beta_P$. Using the induction hypothesis, $\partial[T_F, Y^{\beta_0}]$ can be written as a linear combination with good coefficients $c(v)$ of terms of the form¹⁶

¹⁶We do not mention the $c(v)$ coefficients here since $\partial(c(v)) = 0$.

- $\partial(z^d P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} Y^\sigma)$, with $z \in \mathbf{k}_1$, $d \in \{0, 1\}$, $|\sigma| \geq 1$, $\max(|\gamma|, |k| + |\gamma|, |k| + |\sigma|) \leq |\beta_0|$, $|k| + |\gamma| + |\sigma| \leq |\beta_0| + 1$ and $p + k_P + \sigma_P + d \leq (\beta_0)_P$. This leads to the sum of the following terms.
 - $\partial(z^d) P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} Y^\sigma$, which is of (type 1- β) since $\partial(z) = 0$ or $\frac{v^\lambda}{v^0}$.
 - $z^d P_{(k_T+1, k_P), p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} Y^\sigma + z^d P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{\partial Z^\gamma}(F)_{\mu\nu} Y^\sigma + z^d P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \partial Y^\sigma$, which is the sum of terms of (type 1- β) (as, namely, k_P does not increase and $(\sigma_0)_P = \sigma_P$ if $Y^{\sigma_0} = \partial Y^\sigma$).
- $\partial(P_{k,p}(\Phi) \mathcal{L}_{\partial Z^{\gamma_0}}(F)(v, \nabla_v \Gamma^\sigma))$, with $|k| + |\gamma_0| + |\sigma| \leq |\beta_0| - 1$, $p + |\gamma_0| \leq |\beta_0| - 1$ and $p + k_P + \sigma_P \leq (\beta_0)_P$. We then obtain

$$P_{(k_T+1, k_P), p}(\Phi) \mathcal{L}_{\partial Z^{\gamma_0}}(F)(v, \nabla_v \Gamma^\sigma), \quad P_{k,p}(\Phi) \mathcal{L}_{\partial \partial Z^{\gamma_0}}(F)(v, \nabla_v \Gamma^\sigma) \quad \text{and} \quad P_{k,p}(\Phi) \mathcal{L}_{\partial Z^{\gamma_0}}(F)(v, \nabla_v \partial \Gamma^\sigma),$$

which are all of (type 3- β) since $|k| + |\gamma_0| + |\sigma| + 1 \leq |\beta_0| = |\beta| - 1$, $p + |\gamma_0| + 1 \leq |\beta| - 1$ and, if $\Gamma^\sigma = \partial \Gamma^\sigma$, $p + k_P + \sigma_P = p + k_P + \sigma_P \leq (\beta_0)_P = \beta_P$.

- $\partial(P_{k,p}(\Phi) \mathcal{L}_{X Z^{\gamma_0}}(F)(v, \nabla_v \Gamma^\sigma))$, with $|k| + |\gamma_0| + |\sigma| \leq |\beta_0| - 1$, $p + k_P + \sigma_P \leq (\beta_0)_P$ and $p \geq 1$. We then obtain, as $[\partial, X] = 0$,
- $$P_{(k_T+1, k_P), p}(\Phi) \mathcal{L}_{X Z^{\gamma_0}}(F)(v, \nabla_v \Gamma^\sigma), \quad P_{k,p}(\Phi) \mathcal{L}_{X \partial Z^{\gamma_0}}(F)(v, \nabla_v \Gamma^\sigma) \quad \text{and} \quad P_{k,p}(\Phi) \mathcal{L}_{X Z^{\gamma_0}}(F)(v, \nabla_v \partial \Gamma^\sigma),$$
- which are all of (type 2- β) since, for instance, $|k| + |\gamma_0| + |\sigma| + 1 \leq |\beta_0| = |\beta| - 1$.

We now suppose that $Y \in \mathbb{Y} \setminus \mathbf{T}$, so that $\beta_P = (\beta_0)_P + 1$. We will write schematically that $Y = \widehat{Z} + \Phi X$. Using Proposition 4.3.19, we have that $[T_F, Y] Y^{\beta_0}$ can be written as a linear combination, with $c(v)$ coefficients, of the following terms.

- $\frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \Gamma Y^{\beta_0}$, where $|\gamma| \leq 1$ and $\Gamma \in \mathbb{Y}$, which is of (type 1- β).
- $\Phi^{1-d} z^d \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \partial Y^{\beta_0}$, where $|\gamma| \leq 1$, $d \in \{0, 1\}$ and $z \in \mathbf{k}_1$, which is of (type 1- β) since, if ξ is the multi-index corresponding to ∂Y^{β_0} , $\xi_P = (\beta_0)_P < \beta_P$.
- $\Phi \mathcal{L}_X(F)(v, \nabla_v Y^{\beta_0})$, which is of (type 2- β) since $|\beta_0| \leq |\beta| - 1$ and $1 + (\beta_0)_P \leq \beta_P$.

It then remains to compute $Y[T_F, Y^{\beta_0}]$. Using the induction hypothesis, it can be written as a linear combination of terms of the form

- $Y(c(v) z^d P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} Y^\sigma)$, with $z \in \mathbf{k}_1$, $d \in \{0, 1\}$, $|\sigma| \geq 1$, $\max(|\gamma|, |k| + |\gamma|, |k| + |\sigma|) \leq |\beta_0|$, $|k| + |\gamma| + |\sigma| \leq |\beta_0| + 1$ and $p + k_P + \sigma_P + d \leq (\beta_0)_P$. It leads to the following error terms.
 - $Y\left(\frac{c(v)}{v^0}\right) z^d P_{k,p}(\Phi) v^\mu \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} Y^\sigma$, which is of (type 1- β) since $Y\left(\frac{c(v)}{v^0}\right) = \widehat{Z}\left(\frac{c(v)}{v^0}\right) = \frac{c_0(v)}{v^0}$.
 - $c(v) Y(z^d) P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} Y^\sigma$, which is a linear combination of terms of (type 1- β) since, by Lemma 4.3.2,
$$Y(z) = \widehat{Z}(z) + \Phi_{\widehat{Z}}^i X_i(z) = c_0(v) z + z' + \Phi_{\widehat{Z}}^i c_i(v), \quad \text{where } z' \in \mathbf{k}_1, \text{ and } p + 1 + k_P + \sigma_P + 1 \leq \beta_P.$$
 - $c(v) z^d P_{(k_T, k_P+1), p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} Y^\sigma + c(v) z^d P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} Y Y^\sigma$, which is the sum of terms of (type 1- β), since $p + k_P + \sigma_P + d + 1 \leq (\beta_0)_P + 1 = \beta_P$.
 - $c(v) z^d P_{k, p+p_0}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\xi Z^\gamma}(F)_{\mu\theta} Y^\sigma$, with $\max(p_0, |\xi|) \leq 1$, which is given by the first identity of Lemma 4.3.22. These terms are of (type 1- β) since $|k| + |\gamma| + |\xi| + |\sigma| \leq |\beta_0| + 2 = |\beta| + 1$ and $|\gamma| + |\xi| \leq |\beta|$.

For the remaining terms, we suppose for simplicity that $c(v) = 1$, as we have just seen that $Y(c(v))$ is a good coefficient.

- $Y\left(P_{k,p}(\Phi) \mathcal{L}_{X Z^{\gamma_0}}(F)(v, \nabla_v \Gamma^\sigma)\right)$, with $|k| + |\gamma_0| + |\sigma| \leq |\beta_0| - 1$, $p + k_P + \sigma_P \leq (\beta_0)_P$ and $p \geq 1$. It gives us

$$P_{(k_T, k_P+1), p}(\Phi) \mathcal{L}_{X Z^{\gamma_0}}(F)(v, \nabla_v \Gamma^\sigma),$$

which is of (type 2- β) since, $p + k_P + 1 + \sigma_P \leq (\beta_0)_P + 1 = \beta_P$. We also obtain, using the fourth identity of Lemma 4.3.22,

$$c(v) P_{k, p+p_0}(\Phi) \mathcal{L}_{X Z^\delta Z^{\gamma_0}}(F)\left(v, \nabla_v \widehat{Z}^\xi \Gamma^\sigma\right), \quad \text{with } \max(|\delta| + |\xi|, p_0 + \xi_P) \leq 1.$$

They are all of (type 2- β) since $|k| + |\gamma_0| + |\delta| + |\sigma| + |\xi| \leq |\beta_0| = |\beta| - 1$, $p + p_0 + k_P + \sigma_P + \xi_P \leq (\beta_0)_P + 1 = \beta_P$ and $p + p_0 \geq p \geq 1$.

- $Y \left(P_{k,p}(\Phi) \mathcal{L}_{\partial Z^{\gamma_0}}(F)(v, \nabla_v \Gamma^\sigma) \right)$, with $|k| + |\gamma_0| + |\sigma| \leq |\beta_0| - 1$, $p + |\gamma_0| \leq |\beta_0| - 1$ and $p + k_P + \sigma_P \leq (\beta_0)_P$.
We obtain

- $P_{(k_T, k_P+1), p}(\Phi) \mathcal{L}_{\partial Z^{\gamma_0}}(F)(v, \nabla_v \Gamma^\sigma)$, clearly of (type 3- β),

and, using the second identity of Lemma 4.3.22,

- $P_{k, p+1}(\Phi) \mathcal{L}_{X \partial Z^{\gamma_0}}(F)(v, \nabla_v \Gamma^\sigma)$, which is of (type 2- β), and

$$c(v) P_{k, p+p_0}(\Phi) \mathcal{L}_{Z^\delta \partial Z^{\gamma_0}}(F) \left(v, \nabla_v \widehat{Z}^\xi \Gamma^\sigma \right), \quad \text{with } |\delta| + |\xi| \leq 1, \quad p_0 + |\delta| \leq 1 \quad \text{and} \quad p_0 + \xi_P \leq 1.$$

As $p + p_0 + |\gamma_0| + |\delta| \leq p + |\gamma_0| + 1 \leq |\beta| - 1$, $p + p_0 + k_P + \sigma_P + \xi_P \leq (\beta_0)_P + 1 = \beta_P$ and, if $|\delta| = 1$, $[Z^\delta, \partial] \in \mathbf{T} \cup \{0\}$, we can conclude that these terms are of (type 3- β).

□

Remark 4.3.24. To deal with the weight τ_+ in the terms of (type 2- β) and (type 3- β) (hidden by the v derivatives), we will take advantage of the extra decay given by the X vector fields or the translations ∂_μ through Proposition 4.3.7. To deal with the terms of (type 1- β), when $d = 1$, we will need to control the L^1 norm of $\sum_{w \in \mathbf{k}_1} |w|^{q+1} P_{k,p}(\Phi) Y^\sigma f$, with $k_P + \sigma_P < \beta_P$, in order to control $\| |z|^q Y^\beta f \|_{L^1_{x,v}}$.

As we will need to bound norms such as $\| P_\xi(\Phi) Y^\beta f \|_{L^1_{x,v}}$, we will apply Proposition 4.3.23 to Φ and we then need to compute the derivatives of $T_F(\Phi)$. This is the purpose of the next proposition.

Proposition 4.3.25. Let $Y^\beta \in \mathbb{Y}^{|\beta|}$ and $Z^{\gamma_1} \in \mathbb{K}^{|\gamma_1|}$ (we will apply the result for $|\gamma_1| \leq 1$). Then,

$$Y^\beta \left(t \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_1}}(F)_{\mu\zeta} \right)$$

can be written as a linear combination, with $c(v)$ coefficients, of the following terms, with $0 \leq \theta, \nu \leq 3$ and $p \leq |\beta|$.

$$x^\theta \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}, \quad \text{where } |\gamma| \leq |\beta| \quad \text{and} \quad \gamma_T = \beta_T. \quad (\text{family } \beta - 1)$$

$$P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}, \quad \text{where } |k| + |\gamma| \leq |\beta| - 1 \quad \text{and} \quad k_P \leq \beta_P. \quad (\text{family } \beta - 2)$$

$$x^\theta P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{X Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}, \quad \text{where } |k| + |\gamma| \leq |\beta| - 1 \quad \text{and} \quad k_P < \beta_P. \quad (\text{family } \beta - 3)$$

Proof. Let us prove this by induction on $|\beta|$. The result holds for $|\beta| = 0$. We then consider $Y^\beta = Y Y^{\beta_0} \in \mathbb{Y}^{|\beta|}$ and we suppose that the Proposition holds for β_0 . Suppose first that $Y = \partial$, so that $\beta_P = (\beta_0)_P$. Using the induction hypothesis, $\partial Y^{\beta_0} \left(t \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_1}}(F)_{\mu\nu} \right)$ can be written as a linear combination, with good coefficients $c(v)$, of the following terms.

- $\partial(x^\theta) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}$, with $|\gamma| \leq |\beta_0| < |\beta|$, which is part of (family $\beta - 2$).
- $x^\theta \frac{v^\mu}{v^0} \mathcal{L}_{\partial Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}$, with $1 + |\gamma| \leq 1 + |\beta_0| = |\beta|$. Denoting ∂Z^γ by Z^ξ , we have $\xi_T = 1 + \gamma_T = 1 + (\beta_0)_T = \beta_T$ and this term is part of (family $\beta - 1$).
- $P_{(k_T+1, k_P), p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}$, with $|k| + 1 + |\gamma| \leq |\beta| - 1 + 1 = |\beta| - 1$ and $k_P \leq (\beta_0)_P = \beta_P$, which is part of (family $\beta - 2$).
- $P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{\partial Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}$, with $|k| + |\gamma| + 1 \leq |\beta_0| - 1 + 1 = |\beta| - 1$ and $k_P \leq (\beta_0)_P = \beta_P$, which is part of (family $\beta - 2$).
- $\partial(x^\theta) P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{X Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}$, with $|k| + |\gamma| \leq |\beta_0| - 1 \leq |\beta| - 2$ and $k_P < (\beta_0)_P = \beta_P$, which is then equal to 0 or part of (family $\beta - 2$).
- $x^\theta P_{(k_T+1, k_P), p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{X Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}$, with $|k| + 1 + |\gamma| \leq |\beta_0| - 1 + 1 = |\beta| - 1$ and $k_P < (\beta_0)_P = \beta_P$, which is then part of (family $\beta - 3$).
- $x^\theta P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{\partial X Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}$, with $|k| + |\gamma| + 1 \leq |\beta| - 1$ and $k_P < \beta_P$, which is part of (family $\beta - 3$), as $[\partial, X] = 0$.

Suppose now that $Y = \widehat{Z} + \Phi X \in \mathbb{Y}_0$. We then have $\beta_P = (\beta_0)_P + 1$ and $(\beta_0)_T = \beta_T$. In the following, we will skip the case where Y hits $c(v)(v^0)^{-1}$ and we suppose for simplicity that $c(v) = 1$. Note however that this case is straightforward since

$$Y \left(\frac{c(v)}{v^0} \right) = \widehat{Z} \left(\frac{c(v)}{v^0} \right) = \frac{\widehat{Z}(c(v))}{v^0} + c(v) \widehat{Z} \left(\frac{1}{v^0} \right) = \frac{c_1(v)}{v^0}.$$

Using again the induction hypothesis, $Y Y^{\beta_0} (t \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_1}}(F)_{\mu\zeta})$ can be written as a linear combination of the following terms.

- $Y(x^\theta) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}$, with $|\gamma| \leq |\beta_0| < |\beta|$ and $\gamma_T = (\beta_0)_T = \beta_T$. As, schematically (with $\delta = 0$ or $\delta = 1$),

$$Y(x^\theta) = \widehat{Z}(x^\theta) + \Phi X(x^\theta) = \delta x^\kappa + c(v)\Phi, \quad (4.20)$$

This leads to terms of (family $\beta - 1$) and (family $\beta - 2$).

- $x^\theta \frac{1}{v^0} Y (v^\mu \mathcal{L}_{Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu})$, with $|\gamma| \leq |\beta_0|$ and $\gamma_T = (\beta_0)_T = \beta_T$. Using the first identity of Lemma 4.3.22, we have that $Y (v^\mu \mathcal{L}_{Z^\gamma Z^{\gamma_1}}(F)_{\mu\theta})$ is a linear combination of terms such as

$$v^\mu \mathcal{L}_{Z^{\gamma_0} Z^\gamma Z^{\gamma_1}}(F)_{\mu\lambda}, \quad \text{with } |\gamma_0| \leq 1, \quad (\gamma_0)_T = 0, \quad \text{and } 0 \leq \lambda \leq 3,$$

leading to terms of (family $\beta - 1$), and

$$\Phi v^\mu \mathcal{L}_{X Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu},$$

giving terms of (family $\beta - 3$), as $|\gamma| \leq |\beta_0| = |\beta| - 1$.

- $\frac{1}{v^0} Y (P_{k,p}(\Phi)) v^\mu \mathcal{L}_{Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}$, with $|k| + |\gamma| \leq |\beta_0| - 1$ and $k_P \leq \beta_P$. We obtain terms of (family $\beta - 2$), since

$$Y (P_{k,p}(\Phi)) = P_{(k_T, k_P+1), p}(\Phi), \quad |k| + 1 + |\gamma| \leq |\beta| - 1 \quad \text{and} \quad k_P + 1 \leq (\beta_0)_P + 1 = \beta_P.$$

- $\frac{1}{v^0} P_{k,p}(\Phi) Y (v^\mu \mathcal{L}_{Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu})$, with $|k| + |\gamma| \leq |\beta_0| - 1$ and $k_P \leq (\beta_0)_P$. Using the first identity of Lemma 4.3.22, we have that $Y (v^\mu \mathcal{L}_{Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu})$ is a linear combination of terms of the form

$$c(v) \Phi^r v^\mu \mathcal{L}_{Z^{\gamma_0} Z^\gamma Z^{\gamma_1}}(F)_{\mu\lambda}, \quad \text{with } \max(r, |\gamma_0|) \leq 1 \quad \text{and} \quad 0 \leq \lambda \leq 3.$$

We then obtain terms of (family $\beta - 2$), as $|k| + |\gamma| + |\gamma_0| \leq |\beta_0| = |\beta| - 1$ and $k_P \leq \beta_P$.

- $Y(x^\theta) P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{X Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}$, with $|k| + |\gamma| \leq |\beta_0| - 1$ and $k_P < (\beta_0)_P$, which, using (4.20), gives terms of (family $\beta - 2$) and (family $\beta - 3$).
- $x^\theta P_{(k_T, k_P+1), p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{X Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}$, with $|k| + 1 + |\gamma| \leq |\beta_0| - 1 + 1 = |\beta| - 1$ and $k_P + 1 < (\beta_0)_P + 1 = \beta_P$, which is part of (family $\beta - 3$).
- $x^\theta P_{k,p}(\Phi) \frac{1}{v^0} Y (v^\mu \mathcal{L}_{X Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu})$, with $|k| + |\gamma| \leq |\beta_0| - 1$ and $k_P < (\beta_0)_P$. By the third point of Lemma 4.3.22, we can write $Y (v^\mu \mathcal{L}_{X Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu})$ as a linear combination of terms such as

$$c(v) \Phi^r v^\mu \mathcal{L}_{X Z^{\gamma_0} Z^\gamma Z^{\gamma_1}}(F)_{\mu\lambda}, \quad \text{with } \max(r, |\gamma_0|) \leq 1 \quad \text{and} \quad 0 \leq \lambda \leq 3.$$

It gives us terms of (family $\beta - 3$), as $|k| + |\gamma_0| + |\gamma| \leq |\beta_0| = |\beta| - 1$ and $k_P < \beta_P$.

□

The worst terms are those of (family $\beta - 1$). They do not appear in the source term of $T_F (P_\zeta^X(\Phi))$, which explains why our estimate on $\|P_\zeta^X(\Phi) Y^\beta f\|_{L_{x,v}^1}$ will be better than the one on $\|P_\zeta(\Phi) Y^\beta f\|_{L_{x,v}^1}$.

Proposition 4.3.26. *Let $Y^{\bar{\beta}} \in \mathbb{Y}_X^{|\bar{\beta}|}$, with $\bar{\beta}_X \geq 1$, $Z^{\gamma_1} \in \mathbb{K}^{|\gamma_1|}$ and β be a multi-index associated to \mathbb{Y} such that $\beta_P = \bar{\beta}_P$ and $\beta_T = \bar{\beta}_T + \bar{\beta}_X$. Then, $Y^{\bar{\beta}} (t \frac{v^\mu}{v^0} \mathcal{L}_{Z^{\gamma_1}}(F)_{\mu\zeta})$ can be written as a linear combination of terms of (family $\beta - 2$), (family $\beta - 3$) and,*

$$\text{if } \beta_P = 0, \quad x^\theta \frac{v^\mu}{v^0} \mathcal{L}_{X Z^\gamma Z^{\gamma_1}}(F)_{\mu\nu}, \quad \text{where } |\gamma| \leq |\beta| - 1. \quad (\text{family } \beta - 3 - \text{bis})$$

Proof. The proof is similar to the previous one. The difference comes from the fact a X vector field necessarily have to hit a term of the first family, giving either a term of the second family or of the third-bis family, where we do not have the condition $k_P < \beta_P$ since k_P and β_P could be both equal to 0. □

4.3.4 The null structure of $G(v, \nabla_v g)$

In this subsection, we consider G , a 2-form defined on $[0, T] \times \mathbb{R}^3$, and g , a function defined on $[0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$, both sufficiently regular. We investigate in this subsection the null structure of $G(v, \nabla_v g)$ in view of studying the error terms obtained in Proposition 4.3.23. Let us denote by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of G . Then, expressing $G(v, \nabla_v g)$ in null coordinates, we obtain a linear combination of the following terms.

- The terms with the radial component of $\nabla_v g$ (remark that $(\nabla_v g)^L = -(\nabla_v g)^{\underline{L}} = (\nabla_v g)^r$),

$$v^L \rho (\nabla_v g)^{\underline{L}}, \quad v^{\underline{L}} \rho (\nabla_v g)^L, \quad v^A \alpha_A (\nabla_v g)^L \quad \text{and} \quad v^A \underline{\alpha}_A (\nabla_v g)^{\underline{L}}. \quad (4.21)$$

- The terms with an angular component of ∇g ,

$$\varepsilon_{BA} v^B \sigma (\nabla_v g)^A, \quad v^L \alpha_A (\nabla_v g)^A \quad \text{and} \quad v^{\underline{L}} \underline{\alpha}_A (\nabla_v g)^A. \quad (4.22)$$

We are then led to bound the null components of $\nabla_v g$. A naive estimate, using $v^0 \partial_{v^k} = Y_k - \Phi X - t \partial_k - x^k \partial_t$, gives

$$\left| (\nabla_v g)^L \right|, \left| (\nabla_v g)^{\underline{L}} \right|, \left| (\nabla_v g)^A \right| \leq |\nabla_v g| \lesssim \frac{\tau_+ + |\Phi|}{v^0} |\nabla_{t,x} g| + \frac{1}{v^0} \sum_{Y \in \mathbb{Y}} |Yg|. \quad (4.23)$$

With these inequalities, using our schematic notations $c \prec d$ if d is expected to behave better than c , we have $v^L \rho (\nabla_v g)^{\underline{L}} \prec \varepsilon_{BA} v^B \sigma (\nabla_v g)^A$, since $v^L \prec v^B$ and $\rho \sim \sigma$. The purpose of the following result is to improve (4.23) for the radial component in order to have a better control on $v^L \rho (\nabla_v g)^{\underline{L}}$.

Lemma 4.3.27. *Let g be a sufficiently regular function, $z \in \mathbf{k}_1$ and $j \in \mathbb{N}^*$. We have*

$$|(\nabla_v g)^r| \lesssim \frac{\tau_- + |\Phi|}{v^0} |\nabla_{t,x} g| + \frac{1}{v^0} \sum_{Y \in \mathbb{Y}} |Yg| \quad \text{and} \quad \left| (\nabla_v z^j)^r \right| \lesssim \frac{\tau_-}{v^0} |z|^{j-1} + \frac{1}{v^0} \sum_{w \in \mathbf{k}_1} |w|^j.$$

Proof. We have

$$(\nabla_v g)^r = \frac{x^i}{r} \partial_{v^i} g \quad \text{and} \quad \frac{x^i}{r v^0} (t \partial_i + x_i \partial_t) = \frac{1}{v^0} (t \partial_r + r \partial_t) = \frac{1}{v^0} (S + (r-t) \underline{L}),$$

so that, using $\partial_{v^i} = \frac{1}{v^0} (\widehat{\Omega}_{0i} - t \partial_i - x_i \partial_t)$,

$$(\nabla_v g)^r = \frac{x^i}{r v^0} \widehat{\Omega}_{0i} (g) - \frac{1}{v^0} S (g) + \frac{t-r}{v^0} \underline{L} (g). \quad (4.24)$$

To prove the first inequality, it only remains to write schematically that $\widehat{\Omega}_{0i} = Y_{0i} - \Phi X$, $S = Y_S - \Phi X$ and to use the triangle inequality. To complete the proof of the second inequality, apply (4.24) to $g = z^j$, recall from Lemma 4.3.2 that $|\widehat{Z}(z^j)| \lesssim \sum_{z \in \mathbf{k}_1} |w|^j$ and use that $|\underline{L}(z^j)| \lesssim |z|^{j-1}$. \square

For the terms containing an angular component, note that they are also composed by either α , the better null component of the electromagnetic field, v^A or $v^{\underline{L}}$. The following lemma is fundamental for us to estimate the energy norms of the Vlasov field.

Lemma 4.3.28. *We can bound $|G(v, \nabla_v g)|$ either by*

$$(|\rho| + |\underline{\alpha}|) \left(\sum_{Y \in \mathbb{Y}} |Y(g)| + \left(\tau_- + |\Phi| + \sum_{w \in \mathbf{k}_1} |w| \right) |\nabla_{t,x} g| \right) + \left(|\alpha| + \sqrt{\frac{v^{\underline{L}}}{v^0}} |\sigma| \right) \left(\sum_{Y \in \mathbb{Y}} |Y(g)| + (\tau_+ + |\Phi|) |\nabla_{t,x} g| \right)$$

or by

$$\left(|\alpha| + |\rho| + \sqrt{\frac{v^{\underline{L}}}{v^0}} |\sigma| + \sqrt{\frac{v^{\underline{L}}}{v^0}} |\underline{\alpha}| \right) \left(\sum_{Y \in \mathbb{Y}} |Y(g)| + (\tau_+ + |\Phi|) |\nabla_{t,x} g| \right)$$

Proof. The proof consists in bounding the terms given in (4.21) and (4.22). By Lemma 4.3.27 and $|v^A| \lesssim \sqrt{v^0 v^{\underline{L}}}$, one has

$$\left| v^L \rho (\nabla_v g)^{\underline{L}} - v^{\underline{L}} \rho (\nabla_v g)^L + v^A \underline{\alpha}_A (\nabla_v g)^{\underline{L}} \right| \lesssim \left(|\rho| + \sqrt{\frac{v^{\underline{L}}}{v^0}} |\underline{\alpha}| \right) \left(\sum_{Y \in \mathbb{Y}} |Y(g)| + (\tau_- + |\Phi|) |\nabla_{t,x} g| \right).$$

As $v^0 \partial_{v^i} = Y_i - \Phi X - x^i \partial_t - t \partial_i$ and $|v^B| \lesssim \sqrt{v^0 v^L}$, we obtain

$$\left| v^L \alpha_A (\nabla_v g)^A + v^A \alpha_A (\nabla_v g)^L + v^B \sigma_{BA} (\nabla_v g)^A \right| \lesssim \left(|\alpha| + \sqrt{\frac{v^L}{v^0}} |\sigma| \right) \left(\sum_{Y \in \mathbb{Y}} |Y(g)| + (\tau_+ + |\Phi|) |\nabla_{t,x} g| \right).$$

Finally, using $v^0 \partial_{v^i} = Y_i - \Phi X - x^i \partial_t - t \partial_i$ and Lemma 4.2.4 (for the first inequality), it comes

$$\begin{aligned} \left| v^L \underline{\alpha}_A (\nabla_v g)^A \right| &\lesssim |\alpha| \left(\sum_{Y \in \mathbb{Y}} |Y(g)| + \left(\tau_- + |\Phi| + \sum_{w \in \mathbf{k}_1} |w| \right) |\nabla_{t,x} g| \right) \\ \left| v^L \underline{\alpha}_A (\nabla_v g)^A \right| &\lesssim \sqrt{\frac{v^L}{v^0}} |\alpha| \left(\sum_{Y \in \mathbb{Y}} |Y(g)| + (\tau_+ + |\Phi|) |\nabla_{t,x} g| \right). \end{aligned}$$

□

Remark 4.3.29. *The second inequality will be used in extremal cases of the hierarchies considered, where we will not be able to take advantage of the weights $w \in \mathbf{k}_1$ in front of $|\nabla_{t,x} g|$ and where the terms $\sum_{Y \in \mathbb{Y}_0} |Yg|$ will force us to estimate a weight $z \in \mathbf{k}_1$ by τ_+ (see Proposition 4.3.31 below).*

4.3.5 Source term of $T_F(z^j P_\xi(\Phi) Y^\beta f)$

In view of Remark 4.3.24, we will consider hierarchised energy norms controlling, for Q a fixed integer, $\|z^{Q-\xi_P-\beta_P} P_\xi(\Phi) Y^\beta f\|_{L^1_{x,v}}$. In order to estimate them, we compute in this subsection the source term of $T_F(z^j P_\xi(\Phi) Y^\beta f)$. We start by the following technical result.

Lemma 4.3.30. *Let $h : [0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ be a sufficiently regular function and $\Gamma^\sigma \in \mathbb{G}^{|\sigma|}$. Then,*

$$\begin{aligned} \Gamma^\sigma h &= \sum_{\substack{|g|+|\bar{\sigma}| \leq |\sigma| \\ |g| \leq |\sigma|-1 \\ r+g_P+\bar{\sigma}_P \leq \sigma_P}} c_{\bar{\sigma}}^{g,r}(v) P_{g,r}(\Phi) Y^{\bar{\sigma}} h, \\ |\partial_{v^i}(\Gamma^\sigma h)| &\lesssim \sum_{\delta=0}^1 \sum_{\substack{|g|+|\bar{\sigma}| \leq |\sigma|+1 \\ |g| \leq |\sigma| \\ r+g_P+\bar{\sigma}_P+\delta \leq \sigma_P+1}} \tau_+^\delta |P_{g,r}(\Phi) Y^{\bar{\sigma}} h|. \end{aligned}$$

Proof. The first formula can be proved by induction on $|\sigma|$, using that $\widehat{Z} = Y - \Phi X$ for each \widehat{Z} composing Γ^σ . The inequality then follows using $v^0 \partial_{v^i} = Y_i - \Phi X - t \partial_i - x^i \partial_t$. □

Proposition 4.3.31. *Let $N \in \mathbb{N}$ and $N_0 \geq N$. Consider ζ^0 and β multi-indices such that $|\zeta^0| + |\beta| \leq N$ and $|\zeta^0| \leq N - 1$. Let also $z \in \mathbf{k}_1$ and $j \leq N_0 - \zeta_P^0 - \beta_P$. Then, $T_F(z^j P_{\zeta^0}(\Phi) Y^\beta f)$ can be bounded by a linear combination of the following terms, where $|\gamma| + |\zeta| \leq |\zeta^0| + |\beta|$.*

- $|F(v, \nabla_v(z^j)) P_{\zeta^0}(\Phi) Y^\beta f|$. (category 0)

- $\left(|\nabla_{Z^\gamma} F| + \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(F))| + \frac{\tau_+}{\tau_-} \sqrt{\frac{v^L}{v^0}} |\sigma(\mathcal{L}_{Z^\gamma}(F))| \right) |\Phi|^n |w^i P_\zeta(\Phi) Y^\kappa f|$, (category 1)

where $n \leq 2N$, $w \in \mathbf{k}_1$, $|\zeta| + |\gamma| + |\kappa| \leq |\zeta^0| + |\beta| + 1$, $i \leq N_0 - \zeta_P - \kappa_P$, $\max(|\gamma|, |\zeta| + |\kappa|) \leq |\zeta^0| + |\beta|$ and $|\zeta| \leq N - 1$.

- $\frac{\tau_+}{\tau_-} |\rho(\mathcal{L}_{Z^\gamma}(F))| |z^{j-1} P_\zeta(\Phi) Y^\sigma f|$ and $\frac{\tau_+}{\tau_-} \sqrt{\frac{v^L}{v^0}} |\alpha(\mathcal{L}_{Z^\gamma}(F))| |z^i P_\zeta(\Phi) Y^\kappa f|$, (category 2)

where $|\zeta| + |\gamma| + |\kappa| \leq |\zeta^0| + |\beta| + 1$, $j - 1, i \leq N_0 - \zeta_P - \kappa_P$, $\max(|\gamma|, |\zeta| + |\kappa|) \leq |\zeta^0| + |\beta|$ and $|\zeta| \leq N - 1$. Moreover, we have $i \leq j$.

$$\tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\theta} z^j P_\zeta(\Phi) Y^\beta f \right|, \quad (\text{category 3})$$

with $|\zeta| < |\zeta^0|$, $\zeta_T + \gamma_T = \zeta_T^0$, $\zeta_P \leq \zeta_P^0$, and $|\zeta| + |\gamma| \leq |\zeta^0| + 1$. This implies $j \leq N_0 - \zeta_P - \beta_P$.

Note that the terms of (category 2) only appears when $j = N_0 - k_P - \beta_P$ and the ones of (category 3) when $|\zeta^0| \geq 1$.

Proof. The first thing to remark is that

$$T_F(z^j P_{\zeta^0}(\Phi) Y^\beta f) = F(v, \nabla_v(z^j)) P_{\zeta^0}(\Phi) Y^\beta f + z^j T_F(P_{\zeta^0}(\Phi)) Y^\beta f + z^j P_{\zeta^0}(\Phi) T_F(Y^\beta f).$$

We immediately obtain the terms of (category 0). Let us then consider $z^j P_{\zeta^0}(\Phi) T_F(Y^\beta f)$. Using Proposition 4.3.23, it can be written as a linear combination of terms of (type 1- β), (type 2- β) or (type 3- β) (applied to f), multiplied by $z^j P_{\zeta^0}(\Phi)$. Consequently, $|z^j P_{\zeta^0}(\Phi) T_F(Y^\beta f)|$ can be bounded by a linear combination of

- $|z|^j |w^d Z^\gamma(F_{\mu\nu})| |P_{k,p}(\Phi) P_{\zeta^0}(\Phi) Y^\kappa f|$, with $d \in \{0, 1\}$, $w \in \mathbf{k}_1$, $|\sigma| \geq 1$, $\max(|\gamma|, |k| + |\gamma|, |k| + |\kappa|, |k| + 1) \leq |\beta|$, $|k| + |\gamma| + |\kappa| \leq |\beta| + 1$ and $p + k_P + \kappa_P + d \leq \beta_P$. Now, note that

$$\exists n, \zeta \text{ such that } P_{k,p}(\Phi) P_{\zeta^0}(\Phi) = \Phi^n P_\zeta(\Phi), \quad n \leq |\beta|, \quad \zeta_T = k_T + \zeta_T^0 \quad \text{and} \quad \zeta_P = k_P + \zeta_P^0.$$

Consequently, $|\zeta| = |k| + |\zeta^0| \leq |\zeta^0| + |\beta| - 1 \leq N - 1$, $|\zeta| + |\gamma| = |k| + |\zeta^0| + |\gamma| \leq |\zeta^0| + |\beta|$,

$$|\zeta| + |\kappa| = |k| + |\zeta^0| + |\kappa| \leq |\zeta^0| + |\beta| \quad \text{and} \quad |\zeta| + |\gamma| + |\kappa| \leq |k| + |\zeta^0| + |\gamma| + |\kappa| \leq |\zeta^0| + |\beta| + 1.$$

Since

$$k_P + \kappa_P + d \leq \beta_P \quad \text{and} \quad \zeta_P = k_P + \zeta_P^0, \quad \text{we have} \quad j + d \leq N_0 - \zeta_P - \kappa_P.$$

Finally, as $|z^j w^d| \leq |z|^{j+d} + |w|^{j+d}$, we obtain terms of (category 1).

- $|z|^j |P_{k,p}(\Phi) \mathcal{L}_{XZ^\gamma}(F)(v, \nabla_v(\Gamma^\sigma f)) P_{\zeta^0}(\Phi)|$, with $|k| + |\gamma_0| + |\sigma| \leq |\beta| - 1$, $p + k_P + \sigma_P \leq \beta_P$ and $p \geq 1$. Then, apply Lemma 4.3.30 in order to get

$$|\nabla_v(\Gamma^\sigma f)| \lesssim \sum_{\delta=0}^1 \sum_{\substack{|g|+|\bar{\sigma}| \leq |\sigma|+1 \\ |g| \leq |\sigma| \\ r+g_P+\bar{\sigma}_P+\delta \leq \sigma_P+1}} \tau_+^\delta |P_{g,r}(\Phi) Y^{\bar{\sigma}} f|.$$

Fix parameters $(\delta, g, r, \bar{\sigma})$ as in the right hand side of the previous inequality and consider first the case $\delta = 0$. Then, $|z|^j |\mathcal{L}_{XZ^\gamma}(F)| |P_{k,p}(\Phi) P_{g,r}(\Phi) P_{\zeta^0}(\Phi) Y^{\bar{\sigma}} f|$ can be bounded by terms such as

$$|z|^j |Z^\gamma(F_{\mu\nu})| |\Phi^n P_\zeta(\Phi) Y^{\bar{\sigma}} f|, \quad \text{with } |\gamma| \leq |\gamma_0| + 1, \quad n \leq p + r, \quad \zeta_T = k_T + g_T + \zeta_T^0, \quad \zeta_P = k_P + g_P + \zeta_P^0.$$

We then have $n \leq 2|\beta|$, $|\zeta| + |\gamma| + |\bar{\sigma}| \leq |k| + |g| + |\zeta^0| + |\gamma_0| + 1 + |\bar{\sigma}| \leq |\zeta^0| + |\beta| + 1$, $|\zeta| + |\bar{\sigma}| \leq |\zeta^0| + |\beta|$ and $|\zeta| \leq |\zeta^0| + |\beta| - 1$. As

$$\zeta_P + \bar{\sigma}_P = k_P + g_P + \zeta_P^0 + \bar{\sigma}_P \leq k_P + \sigma_P + 1 + \zeta_P^0 \leq \zeta_P^0 + \beta_P,$$

we have $j \leq N_0 - \zeta_P - \bar{\sigma}_P$. If $\delta = 1$, use the inequality (4.13) of Proposition 4.3.7 to compensate the weight τ_+ . The only difference is that it brings a weight $w \in \mathbf{k}_1$. To handle it, use $|z^j w| \leq |z|^{j+1} + |w|^{j+1}$ and

$$\zeta_P + \bar{\sigma}_P = k_P + g_P + \zeta_P^0 + \bar{\sigma}_P \leq k_P + \sigma_P + 1 - \delta + \zeta_P^0 \leq \zeta_P^0 + \beta_P - 1,$$

so that $j + 1 \leq N_0 - \zeta_P - \beta_P$. In both cases, we then have terms of (category 1).

- $|z|^j |P_{k,p}(\Phi) \mathcal{L}_{\partial Z^\gamma}(F)(v, \nabla_v(\Gamma^{\sigma^0} f)) P_{\zeta^0}(\Phi)|$, with $|k| + |\gamma_0| + |\sigma^0| \leq |\beta| - 1$, $p + |\gamma_0| \leq |\beta| - 1$ and $p + k_P + \sigma_P^0 \leq \beta_P$, which arises from a term of (type 3- β). Applying Lemma 4.3.30, we can schematically suppose that

$$\Gamma^{\sigma^0} = c(v) \Phi^r P_\chi(\Phi) Y^\kappa \quad \text{with} \quad |\chi| + |\kappa| \leq |\sigma^0|, \quad |\chi| \leq |\sigma^0| - 1 \quad \text{and} \quad r + r_\chi + \chi_P + \kappa_P \leq \sigma_P^0,$$

where r_χ is the number of Φ coefficients in $P_\chi(\Phi)$. As $Y(c(v))$ is a good coefficient, $c(v)$ does not play any role in what follows and we then suppose for simplicity that $c(v) = 1$. We suppose moreover, in order to not have a weight in excess, that

$$j + k_P + \chi_P + \kappa_P < N_0 - \zeta_P^0 \quad (4.25)$$

and we will treat the remaining cases below. Using the first inequality of Lemma 4.3.28 and denoting by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of $\mathcal{L}_{\partial Z^{\gamma_0}}(F)$, we can bound the quantity considered here by the sum of the three following terms

$$|z|^j |P_{k,p}(\Phi)P_{\zeta_0}(\Phi)| \left(|\alpha| + |\rho| + \sqrt{\frac{v\mathcal{L}}{v^0}} |\sigma| + |\underline{\alpha}| \right) \sum_{Y \in \mathbb{Y}_0} |Y(\Phi^r P_\chi(\Phi) Y^\kappa f)|, \quad (4.26)$$

$$|z|^j |P_{k,p}(\Phi)P_{\zeta_0}(\Phi)| (|\rho| + |\underline{\alpha}|) \left(\tau_- + |\Phi| + \sum_{w \in \mathbf{k}_1} |w| \right) |\nabla_{t,x}(\Phi^r P_\chi(\Phi) Y^\kappa f)|, \quad (4.27)$$

$$|z|^j |P_{k,p}(\Phi)P_{\zeta_0}(\Phi)| (\tau_+ + |\Phi|) \left(|\alpha| + \sqrt{\frac{v\mathcal{L}}{v^0}} |\sigma| \right) |\nabla_{t,x}(\Phi^r P_\chi(\Phi) Y^\kappa f)|. \quad (4.28)$$

Let us start by (4.26). We have schematically, for $Y \in \mathbb{Y}_0$, $Y^{\kappa^1} = Y^\kappa$ and $Y^{\kappa^2} = YY^\kappa$,

$$P_{k,p}(\Phi)P_{\zeta_0}(\Phi)Y(\Phi^r P_\chi(\Phi)Y^\kappa f) = \Phi^{n_1} P_{\zeta^1}(\Phi)Y^{\kappa^1} f + \Phi^{n_2} P_{\zeta^2}(\Phi)Y^{\kappa^2} f,$$

$$\text{with } |n_i| \leq p+r, \quad |\zeta^i| = |k| + |\zeta^0| + |\chi| + \delta_1^i \quad \text{and} \quad \zeta_P^i = k_P + \zeta_P^0 + \chi_P + \delta_1^i.$$

We have, according to (4.25),

$$j + \zeta_P^i + \kappa_P^i = \zeta_P^0 + j + k_P + \chi_P + \kappa_P + 1 \leq N_0.$$

Consequently, as

$$|\alpha| + |\rho| + \sqrt{\frac{v\mathcal{L}}{v^0}} |\sigma| + |\underline{\alpha}| \lesssim |\mathcal{L}_{\partial Z^\gamma}(F)| \lesssim \sum_{|\gamma| \leq |\gamma_0|+1} |\nabla_{Z^\gamma} F| \quad \text{and} \quad |\zeta^i| + |\gamma| + |\kappa^i| \leq |\beta| + |\zeta^0| + 1, \quad (4.29)$$

we obtain terms of (category 1) (the other conditions are easy to check).

Let us focus now on (4.27) and (4.28). Defining $Y^{\kappa^3} = Y^\kappa$ and $Y^{\kappa^4} = \partial Y^\kappa$, we have schematically

$$P_{k,p}(\Phi)P_{\zeta_0}(\Phi)\partial(\Phi^r P_\chi(\Phi)Y^\kappa f) = \Phi^{n_3} P_{\zeta^3}(\Phi)Y^{\kappa^3} f + \Phi^{n_4} P_{\zeta^4}(\Phi)Y^{\kappa^4} f,$$

$$\text{with } |n_i| \leq p+r \leq 2|\beta| - 2, \quad |\zeta^i| = |k| + |\zeta^0| + |\chi| + \delta_i^3 \quad \text{and} \quad \zeta_P^i = k_P + \zeta_P^0 + \chi_P.$$

This time, one obtains $j+1 \leq N_0 - \zeta_P^i - \kappa_P^i$. As, by inequality (4.15) of Proposition 4.3.7,

$$(|\rho| + |\underline{\alpha}|) \lesssim \frac{1}{\tau_-} \sum_{|\gamma| \leq |\gamma_0|+1} |\nabla_{Z^\gamma} F|, \quad |\alpha| \lesssim \sum_{|\gamma| \leq |\gamma_0|+1} \frac{1}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(F))| + \frac{1}{\tau_+} |\nabla_{Z^\gamma} F|,$$

$$|\sigma| \lesssim \sum_{|\gamma| \leq |\gamma_0|+1} \frac{1}{\tau_-} |\sigma(\mathcal{L}_{Z^\gamma}(F))| + \frac{1}{\tau_+} |\nabla_{Z^\gamma} F| \quad \text{and} \quad |z^j w| \leq |z|^{j+1} + |w|^{j+1},$$

(4.27) and (4.28) also give us terms of (category 1).

- We now treat the remaining terms arising from those of (type 3- β), for which

$$j + k_P + \chi_P + \kappa_P = N_0 - \zeta_P^0.$$

This equality can only occur if $j = N_0 - \zeta_P^0 - \beta_P$ and $k_P + \chi_P + \kappa_P = \beta_P$. It implies $p+r+r_\chi = 0$ and we then have to study terms of the form

$$|z|^j |\mathcal{L}_{\partial Z^{\gamma_0}}(F)(v, \nabla_v(Y^\kappa f)) P_{\zeta^0}(\Phi)|, \quad \text{with } |\gamma_0| + |\kappa| \leq |\beta| - 1.$$

Using the second inequality of Lemma 4.3.28, and denoting again the null decomposition of $\mathcal{L}_{\partial Z^{\gamma_0}}(F)$ by $(\alpha, \underline{\alpha}, \rho, \sigma)$, we can bound it by quantities such as

$$\begin{aligned} & |\Phi| |\mathcal{L}_{\partial Z^{\gamma_0}}(F)| |z^j P_{\zeta^0}(\Phi) \partial Y^\kappa f|, \quad \text{leading to terms of (category 1),} \\ & |\rho| |P_{\zeta^0}(\Phi)| (\tau_+ |z|^{j-1} |YY^\sigma f| + \tau_- |z|^j |\partial Y^\kappa f|), \quad \text{with } Y \in \mathbb{Y}_0, \quad \text{and} \end{aligned} \quad (4.30)$$

$$\left(|\alpha| + \sqrt{\frac{v\mathcal{L}}{v^0}} |\sigma| + \sqrt{\frac{v\mathcal{L}}{v^0}} |\underline{\alpha}| \right) |P_{\zeta^0}(\Phi)| (\tau_+ |z|^{j-1} |YY^\kappa f| + \tau_+ |z|^j |\partial Y^\kappa f|), \quad \text{with } Y \in \mathbb{Y}_0. \quad (4.31)$$

If $YY^\kappa = Y\chi^1$ and $\partial Y^\kappa = Y\chi^2$, we have

$$|\zeta^0| + |\chi^i| \leq |k| + |\beta|, \quad j-1 = N_0 - \zeta_P^0 - \chi_P^1 \quad \text{and} \quad j = N_0 - \zeta_P^0 - \chi_P^2.$$

Thus, (4.30) and (4.31) give terms of (category 1) and (category 2) since we have, according to inequality (4.15) of Proposition 4.3.7 and for $\varphi \in \{\alpha, \underline{\alpha}, \rho, \sigma\}$,

$$|\varphi| \lesssim \sum_{|\gamma| \leq |\gamma_0| + 1} \tau_-^{-1} |\varphi(\mathcal{L}_{Z^\gamma}(F))| + \tau_+^{-1} |\nabla_{Z^\gamma} F|.$$

It then remains to bound $T_F(P_{\zeta^0}(\Phi))z^j Y^\beta f$. If $|\zeta^0| \geq 1$, there exists $1 \leq p \leq |\zeta^0|$ and $(\xi^i)_{1 \leq i \leq p}$ such that

$$P_{\zeta^0}(\Phi) = \prod_{i=1}^p Y^{\xi^i} \Phi, \quad \min_{1 \leq i \leq p} |\xi^i| \geq 1, \quad \sum_{i=1}^p |\xi^i| = |k| \quad \text{and} \quad \sum_{i=1}^p (\xi^i)_T = k_T.$$

Then, $T_F(P_{\zeta^0}(\Phi)) = \sum_{i=1}^p T_F(Y^{\xi^i} \Phi) \prod_{j \neq i} Y^{\xi^j} \Phi$ and let us, for instance, bound $T_F(Y^{\xi^1} \Phi) Y^\beta f \prod_{j=2}^p Y^{\xi^j} \Phi$. To lighten the notation, we define χ such that

$$P_\chi(\Phi) = \prod_{j=2}^p Y^{\xi^j} \Phi, \quad \text{so that} \quad (\chi_T, \chi_P) = (\zeta_T^0 - \xi_T^1, \zeta_P^0 - \xi_P^1).$$

Using Propositions 4.3.23 and 4.3.25 (with $|\gamma_1| \leq 1$), $T_F(Y^{\xi^1} \Phi) P_\chi(\Phi) Y^\beta f$ can be written as a linear combination of terms of (*type 1* - ξ_1), (*type 2* - ξ_1), (*type 3* - ξ_1) (applied to Φ), (*family 1* - ξ_1), (*family 2* - ξ_1) and (*family 3* - ξ_1), multiplied by $P_\chi(\Phi) Y^\beta f$. The treatment of the first three type of terms is similar to those which arise from $z^j P_{\zeta^0}(\Phi) T_F(Y^\beta f)$, so we only give details for the first one. We then have to bound

- $|z|^j |Z^\gamma(F_{\mu\nu})| |w^d P_{k,p}(\Phi) Y^\kappa \Phi P_\chi(\Phi) Y^\beta f|$, with $d \in \{0, 1\}$, $w \in \mathbf{k}_1$, $|\kappa| \geq 1 \max(|\gamma|, |k| + |\gamma|, |k| + |\kappa|) \leq |\xi^1|$, $|k| + |\gamma| + |\kappa| \leq |\xi^1| + 1$ and $p + k_P + \kappa_P + d \leq \xi_P^1$. Note now that

$$P_{k,p}(\Phi) Y^\kappa \Phi P_\chi(\Phi) = \Phi^n P_\zeta(\Phi), \quad \text{with } n \leq p \leq |\xi^1|, \quad \zeta_T = k_T + \kappa_T + \chi_T \quad \text{and} \quad \zeta_P = k_P + \kappa_P + \chi_P.$$

Note moreover that

$$|\zeta| + |\gamma| + |\beta| = |k| + |\gamma| + |\kappa| + |\chi| + |\beta| \leq |\xi^1| + |\chi| + |\beta| + 1 = |\zeta^0| + |\beta| + 1, \quad |\zeta| + |\beta| \leq |\zeta^0| + |\beta|$$

and $\zeta_P + \beta_P + d = k_P + \kappa_P + d + \chi_P + \beta_P \leq \xi_P^1 + \chi_P + \beta_P = \zeta_P^0 + \beta_P$, which proves that this is a term of (category 1).

- $\tau_+ |z|^j \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\theta} P_\chi(\Phi) Y^\beta f \right|$, with $|\gamma| \leq |\xi^1| + 1$ and $\gamma_T = \xi_T^1$. It is part of (category 3) as

$$|\chi| < |k|, \quad \chi_T + \gamma_T = \chi_T + \xi_T^1 = \zeta_T^0, \quad \chi_P \leq \zeta_P^0 \quad \text{and} \quad |\chi| + |\gamma| \leq |\chi| + |\xi^1| + 1 = |\zeta^0| + 1.$$
- $|Z^\gamma(F_{\mu\nu})| |z^j P_{k,p}(\Phi) P_\chi(\Phi) Y^\beta f|$, with $|k| + |\gamma| \leq |\xi^1| - 1$, $k_P \leq \xi_P^1$ and $p \leq |\xi^1|$, which is part of (category 1). Indeed, we can write

$$P_{k,p}(\Phi) P_\chi(\Phi) = \Phi^r P_\zeta(\Phi), \quad \text{with } r \leq p \leq |\xi^1|, \quad (\zeta_T, \zeta_P) = (k_T + \chi_T, k_P + \chi_P)$$

and we then have $|\zeta| + |\gamma| = |k| + |\gamma| + |\chi| \leq |\xi^1| + |\chi| \leq |\zeta^0|$,

$$|\zeta| + |\gamma| + |\beta| \leq |\xi^1| + |\chi| + |\beta| \leq |\zeta^0| + |\beta| \quad \text{and} \quad \zeta_P + \beta_P \leq \xi_P^1 + \chi_P + \beta_P = \zeta_P^0 + \beta_P$$

- $\tau_+ |\mathcal{L}_{XZ^{\gamma_0}}(F)| |z^j P_{k,p}(\Phi) P_\chi(\Phi) Y^\beta f|$, with $|k| + |\gamma_0| \leq |\xi_1| - 1$, $k_P < \xi_P^1$ and $p \leq |\xi^1|$. By inequality (4.13) of Proposition 4.3.7

$$\exists w \in \mathbf{k}_1, \quad \tau_+ |\mathcal{L}_{XZ^{\gamma_0}}(F)| \lesssim (1 + |w|) \sum_{|\gamma| \leq |\gamma_0| + 1} |\nabla_{Z^\gamma} F|.$$

Note moreover that $k_P + \chi_P + \beta_P \leq \xi_P^1 - 1 + \chi_P + \beta_P < \zeta_P^0 + \beta_P$, as¹⁷ $k_P < \xi_P^1$. We then have $j+1 \leq N_0 - k_P - \chi_P - \beta_P$ and we obtain, using $|z^j w| \leq |z|^{j+1} + |w|^{j+1}$ and writing again $P_{k,p}(\Phi) P_\chi(\Phi) = \Phi^r P_\zeta(\Phi)$, terms which are in (category 1) (the other conditions can be checked as previously). \square

¹⁷Note that this term could appear only if $\xi_P^1 \geq 1$.

Remark 4.3.32. *There is three types of terms which bring us to consider a hierarchy on the quantities of the form $z^j P_\xi(\Phi) Y^\beta f$.*

- *Those of (category 0), as $\nabla_v(z^j)$ creates (at least) a τ_- -loss and since $\tau_- F \sim \tau_+^{-1}$.*
- *The first ones of (category 2). Indeed, we will have $|\rho| \lesssim \tau_+^{-\frac{3}{2}} \tau_-^{-\frac{1}{2}}$, so, using¹⁸ $1 \lesssim \sqrt{v^0 v^\underline{L}}$,*

$$\frac{\tau_+}{\tau_-} |\rho| \lesssim \frac{v^0}{\tau_+} + \frac{v^\underline{L}}{\tau_-^3}.$$

$v^\underline{L} \tau_-^{-3}$ will give an integrable term, as the component $v^\underline{L}$ will allow us to use the foliation $(u, C_u(t))$ of $[0, t] \times \mathbb{R}_x^3$. However, $v^0 \tau_+^{-1}$ will create a logarithmical growth.

- *The ones of (category 3), because of the τ_+ weight and the fact that even the better component of $\mathcal{L}_{Z^\gamma}(F)$ will not have a better decay rate than τ_+^{-2} .*

We will then classify them by $|\xi| + |\beta|$ and j , as one of these quantities is lowered in each of these terms.

Remark 4.3.33. *Let β and, for $i \in \{1, 2\}$, ζ^i be multi-indices such that $|\zeta^i| + |\beta| \leq N$, $|\zeta^1| \leq N - 1$ and $N_0 \geq 2N - 1$. We can adapt the previous proposition to $T_F(z^j P_{\zeta_1}(\Phi) P_{\zeta_2}(\Phi) Y^\beta f)$. One just has*

- *to add the factor $P_{\zeta_2}(\Phi)$ (or $P_{\zeta_1}(\Phi)$) in the terms of each categories and*
- *to replace conditions such as $j \leq N_0 - \zeta_P - \sigma_P$ by $j \leq N_0 - \zeta_P - \zeta_P^2 - \sigma_P$ (or $j \leq N_0 - \zeta_P - \zeta_P^1 - \sigma_P$).*

The worst terms are those of (category 3) as they are responsible for the stronger growth of the top order energy norms. However, as suggested by the following proposition, we will have better estimates on $\|z^j P_\xi^X(\Phi) Y^\beta\|_{L_{x,v}^1}$.

Proposition 4.3.34. *Let $N \in \mathbb{N}$, $z \in \mathbf{k}_1$, $N_0 \geq N$, ξ^0 , β and $j \in \mathbb{N}$ be such that $|\xi^0| \leq N - 1$, $|\xi^0| + |\beta| \leq N$ and $j \leq N_0 - \xi_P^0 - \beta_P$. Then, $T_F(z^j P_{\xi^0}^X(\Phi) Y^\beta f)$ can be bounded by a linear combination of terms of (category 0), (category 1), (category 2) and*

$$\frac{\tau_+}{\tau_-} \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} w^j P_\xi^X(\Phi) Y^\beta f \right|, \quad (\text{category } 3 - X)$$

with $\xi_X < \xi_X^0$, $\xi_T \leq \xi_T^0$, $\xi_P \leq \xi_P^0$, $|\xi| + |\gamma| + |\beta| \leq |\xi| + |\beta| + 1$, $|\gamma| \leq |\xi| + 1$, $w \in \mathbf{k}_1$ and $j = N_0 - \zeta_P - \beta_P$.

Note that the terms of (category 2) only appear when $j = N_0 - \xi_P^0 - \beta_P$ and those of (category 3 - X) if $j = N_0 - \xi_P^0 - \beta_P$ and $|\xi^0| \geq 1$.

Proof. Proposition 4.3.23 also holds for $Y^\beta \in \mathbb{Y}_X$ in view of Lemma 4.3.12 and the fact that X can be considered as $c(v)\partial$. Then, one only has to follow the proof of the previous proposition and to apply Proposition 4.3.26 where we used Proposition 4.3.25. Hence, instead of terms of (category 3), we obtain

$$\tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{XZ^\gamma}(F)_{\mu\nu} z^j P_X^X(\Phi) Y^\beta f \right|, \quad \text{with } |\gamma| \leq |\xi^1|, \quad \chi_X < \xi_X^0, \quad \chi_T \leq \xi_T^0 \quad \text{and} \quad \chi_P \leq \xi_P^0.$$

Apply now the second and then the first inequality of Proposition 4.3.7 to obtain that

$$\tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{XZ^\gamma}(F)_{\mu\theta} z^j P_X^X(\Phi) Y^\beta f \right| \lesssim |P_X^X(\Phi) Y^\beta f| \sum_{|\delta| \leq |\xi^1| + 1} \left(\sum_{w \in \mathbf{k}_1} \frac{|w|^{j+1}}{\tau_-} \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\delta}(F)_{\mu\theta} \right| + |z|^j |\mathcal{L}_{Z^\delta}(F)| \right)$$

which leads to terms of (category 3 - X) (if $j = N_0 - \chi_P - \beta_P$) and (category 1) (as $P_X^X(\Phi)$ can be bounded by a linear combination of $P_{\chi^0}(\Phi)$ with $\chi_T^0 = \chi_T + \chi_X$ and $\chi_P^0 \leq \chi_P$). \square

Remark 4.3.35. *As we will mostly apply this commutation formula with a lower N_0 than for our utilizations of Proposition 4.3.31 or for $|\xi^0| = 0$, we will have to deal with terms of (category 3 - X) only once (for (4.76)).*

¹⁸We will be able to lose one power of v^0 as it is suggested by the energy estimate of Proposition 4.4.1.

4.3.6 Commutation of the Maxwell equations

We recall the following property (see Lemma 2.8 of [6] for a proof).

Lemma 4.3.36. *Let G and M be respectively a 2-form and a 1-form such that $\nabla^\mu G_{\mu\nu} = M_\nu$. Then,*

$$\forall Z \in \mathbb{P}, \quad \nabla^\mu \mathcal{L}_Z(G)_{\mu\nu} = \mathcal{L}_Z(M)_\nu \quad \text{and} \quad \nabla^\mu \mathcal{L}_S(G)_{\mu\nu} = \mathcal{L}_S(M)_\nu + 2M_\nu.$$

If g is a sufficiently regular function such that $\nabla^\mu G_{\mu\nu} = J(g)_\nu$, then

$$\forall Z \in \mathbb{P}, \quad \nabla^\mu \mathcal{L}_Z(G)_{\mu\nu} = J(\widehat{Z}g)_\nu \quad \text{and} \quad \nabla^\mu \mathcal{L}_S(G)_{\mu\nu} = J(Sg)_\nu + 3J(g)_\nu.$$

We need to adapt this formula since we will control Yf and not $\widehat{Z}f$. We cannot close the estimates using only the formula

$$J(\widehat{Z}f) = J(Yf) - J(\Phi^k X_k f)$$

as we will have $\|\Phi\|_{L^\infty} \lesssim \log^2(\tau_+)$ and since this small loss would prevent us to close the energy estimates.

Proposition 4.3.37. *Let $Z \in \mathbb{K}$. Then, for $0 \leq \nu \leq 3$, $\nabla^\mu \mathcal{L}_Z(F)_{\mu\nu}$ can be written as a linear combination of the following terms.*

- $\int_v \frac{v_\nu}{v^0} (X\Phi)^j Y^\kappa f dv$, with $j + |\kappa| \leq 1$.
- $\frac{1}{\tau_+} \int_v c(t, x, v) z P_{k,p}(\Phi) Y^\kappa f dv$, with $z \in \mathbf{k}_1$, $p + |k| + |\kappa| \leq 3$ and $|k| + |\kappa| \leq 1$.

Remark 4.3.38. *We would obtain a similar proposition if $J(f)_\nu$ was equal to $\int_v c_\nu(v) f dv$, excepted that we would have to replace $\frac{v_\nu}{v^0}$, in the first terms, by certain good coefficients $c(v)$.*

Proof. If $Z \in \mathbf{T}$, the result ensues from Lemma 4.3.36. Otherwise, we have, using (4.11)

$$\begin{aligned} J(\widehat{Z}f) &= J(Yf) - J(\Phi^k X_k f) \\ &= J(Yf)_\nu + J(X_k(\Phi^k)f)_\nu - J(X_k(\Phi^k f)) \\ &= J(Yf) + J(X_k(\Phi^k)f) - \frac{1}{1+t+r} \sum_{k=1}^3 J \left(\left(2z_{0k} \partial_t + \sum_{Z \in \mathbb{K}} c_Z(t, x, v) Z \right) (\Phi^k f) \right). \end{aligned}$$

Now, note that $J(z_{0k} \partial_t(\Phi^k f)) = J(z_{0k} \Phi \partial_t f + z_{0k} \partial_t(\Phi) f)$ and, for $Z \in \mathbb{K} \setminus \mathbf{T}$ (in the computations below, we consider $Z = \Omega_{0i}$, but the other cases are similar), by integration by parts in v ,

$$\begin{aligned} J(Z(\Phi^k f)) &= J((Y - v^0 \partial_{v^i} - \Phi^q X_q)(\Phi^k f)) \\ &= J(Y(\Phi^k)f + \Phi^k Y(f) - \Phi^q X_q(\Phi^k)f + \Phi^q \Phi^k X_q(f)) + \left(\int_v \Phi^k f dv \right) dx^i - \left(\int_v \Phi^k f \frac{v_i}{v^0} dv \right) dx^0, \end{aligned}$$

where dx^μ is the differential of x^μ . □

We are now ready to establish the higher order commutation formula.

Proposition 4.3.39. *Let $R \in \mathbb{N}$ and $Z^\beta \in \mathbb{K}^R$. Then, for all $0 \leq \nu \leq 3$, $\nabla^\mu \mathcal{L}_{Z^\beta}(F)_{\mu\nu} = \mathcal{L}_{Z^\beta}(F)_\nu$ can be written as a linear combination of terms such as*

$$\int_v \frac{v_\nu}{v^0} P_\xi^X(\Phi) Y^\kappa f dv, \quad \text{with } |\xi| + |\kappa| \leq R, \quad (\text{type 1} - R)$$

$$\frac{1}{\tau_+} \int_v c(t, x, v) z P_{k,p}(\Phi) Y^\kappa f dv, \quad \text{with } p + |k| + |\kappa| \leq 3R \quad \text{and} \quad k + |\kappa| \leq R. \quad (\text{type 2} - R)$$

Proof. We will use during the proof the following properties, arising from Lemma 4.3.2 and the definition of the X_i vector field,

$$\forall (Y, z) \in \mathbb{Y} \times \mathbf{k}_1, \quad \exists z' \in \mathbf{k}_1, \quad Y(z) = c_1(v)z + z' + c_2(v)\Phi, \quad (4.32)$$

$$P_\xi^X(\Phi) = \sum_{\substack{\zeta_T = \xi_T + \xi_X \\ \zeta_P \leq \xi_P}} c^\zeta(v) P_\zeta(\Phi). \quad (4.33)$$

Let us suppose that the formula holds for all $|\beta_0| \leq R - 1$, with $R \geq 2$ (for $R - 1 = 1$, see Proposition 4.3.37). Let $(Z, Z^{\beta_0}) \in \mathbb{K} \times \mathbb{K}^{|\beta_0|}$ with $|\beta_0| = R - 1$ and consider the multi-index β such that $Z^\beta = ZZ^{\beta_0}$. We fix $\nu \in \llbracket 0, 3 \rrbracket$. By the first order commutation formula, Remark 4.3.38 and the induction hypothesis, $\nabla^\mu \mathcal{L}_{Z^\beta}(F)_{\mu\nu}$ can be written as a linear combination of the following terms (to lighten the notations, we drop the good coefficients $c(t, x, v)$ in the integrands of the terms given by Proposition 4.3.37).

- $\int_v \frac{v_\nu}{v^0} (X\Phi)^j Y^{\kappa^0} \left(P_\xi^X(\Phi) Y^\kappa f \right) dv$, with $j + |\kappa^0| \leq 1$ and $|\xi| + |\kappa| \leq R - 1$. It leads to $\int_v \frac{v_\nu}{v^0} P_\xi^X(\Phi) Y^\kappa f dv$,

$$\int_v \frac{v_\nu}{v^0} X(\Phi) P_\xi^X(\Phi) Y^\kappa f dv, \quad \int_v \frac{v_\nu}{v^0} Y(P_\xi^X(\Phi)) Y^\kappa f dv \quad \text{and} \quad \int_v \frac{v_\nu}{v^0} P_\xi^X(\Phi) Y^{\kappa^0} Y^\kappa f dv,$$

which are all of (type 1 – R) since $Y(P_\xi^X(\Phi)) = P_\zeta^X(\Phi)$, with $|\zeta| = |\xi| + 1$, and $|\xi| + 1 + |\kappa| \leq R$.

- $\int_v c(v) (X\Phi)^j Y^{\kappa^0} \left(\frac{z}{\tau_+} c(t, x, v) P_{k,p}(\Phi) Y^\kappa f \right) dv$, with $j + |\kappa^0| \leq 1$, $z \in \mathbf{k}_1$, $p + |k| + |\kappa| \leq 3R - 3$ and $|k| + |\kappa| \leq R - 1$. For simplicity, we suppose $c(v) = 1$. As

$$Y \left(\frac{1}{\tau_+} c(t, x, v) \right) = \frac{1}{\tau_+} c_1(t, x, v) + \frac{1}{\tau_+} c_2(t, x, v) \Phi,$$

we obtain, dropping the dependance in (t, x, v) of the good coefficients, the following terms (with the first one corresponding to $j = 1$ and the other ones to $j = 0$).

$$\begin{aligned} & \frac{1}{\tau_+} \int_v cz P_{(k_T+1, k_P), p+1}(\Phi) Y^\kappa f dv, & \frac{1}{\tau_+} \int_v (c + c_1) z P_{k,p}(\Phi) Y^\kappa f dv, & \frac{1}{\tau_+} \int_v c_2 z P_{k,p+1}(\Phi) Y^\kappa f dv, \\ & \frac{1}{\tau_+} \int_v cz P_{(k_T+\kappa_T^0, k_P+\kappa_P^0), p}(\Phi) Y^\kappa f dv, & \frac{1}{\tau_+} \int_v c Y(z) P_{k,p}(\Phi) Y^\kappa f dv, & \frac{1}{\tau_+} \int_v cz P_{k,p}(\Phi) Y^{\kappa^0} Y^\kappa f dv. \end{aligned}$$

It is now easy to check that all these terms are of (type 2 – R) (for the before last term, recall in particular (4.32)). For instance, for the first one, we have

$$(p+1) + (|k|+1) + |\kappa| = (p+|k|+|\kappa|) + 2 \leq 3R - 1 \leq 3R \quad \text{and} \quad (|k|+1) + |\kappa| \leq (|k|+|\kappa|) + 1 \leq R.$$

- $\frac{1}{\tau_+} \int_v z P_{k^0, p^0}(\Phi) Y^{\kappa^0} \left(P_\xi^X(\Phi) Y^\kappa f \right) dv$, with $p^0 + |k^0| + |\kappa^0| \leq 3$, $|k^0| + |\kappa^0| \leq 1$ and $|\xi| + |\kappa| \leq R - 1$. According to (4.33), we can suppose without loss of generality that $P_\xi^X(\Phi) = c(v) P_\zeta(\Phi)$, with $|\zeta| \leq |\xi|$. If $|k^0| = 1$, we obtain

$$\frac{1}{\tau_+} \int_v c(v) z P_{(\zeta_T+k_T^0, \zeta_P+k_P^0), r}(\Phi) Y^\kappa f dv, \quad \text{with} \quad r \leq |\zeta| + p^0,$$

which is of (type 2 – R) since

$$(|\zeta| + p^0) + (|\zeta| + |k^0|) + |\kappa| \leq (p^0 + |k^0|) + 2(|\xi| + |\kappa|) \leq 2R + 1 \leq 3R \quad \text{and} \quad (|\zeta| + |k^0|) + |\kappa| \leq R.$$

If $|k^0| = 0$, we obtain, with $r \leq |\zeta| + p^0$ and since $Y^{\kappa^0}(c(v)) = c_1(v)$,

$$\frac{1}{\tau_+} \int_v (c + c_1)(v) z P_{(\zeta_T, \zeta_P), r}(\Phi) Y^\kappa f dv, \quad \frac{1}{\tau_+} \int_v c(v) z P_{(\zeta_T, \zeta_P), r}(\Phi) Y^{\kappa^0} Y^\kappa f dv \quad \text{and}$$

$$\frac{1}{\tau_+} \int_v c(v) z P_{(\zeta_T+\kappa_T^0, \zeta_P+\kappa_P^0), r}(\Phi) Y^\kappa f dv,$$

which are of (type 2 – R) since

$$|\zeta| + 1 + |\kappa| \leq R \quad \text{and} \quad |\zeta| + p^0 + |\zeta| + |\kappa^0| + |\kappa| \leq 3 + 2R - 2 \leq 3R.$$

- $\frac{1}{\tau_+} \int_v w P_{k^0, p^0}(\Phi) Y^{\kappa^0} \left(\frac{z}{\tau_+} c(t, x, v) P_{k,p}(\Phi) Y^\kappa f \right) dv$, with $(w, z) \in \mathbf{k}_1^2$, $p^0 + |k^0| + |\kappa^0| \leq 3$, $|k^0| + |\kappa^0| \leq 1$, $p + |k| + |\kappa| \leq 3R - 3$ and $|k| + |\kappa| \leq R - 1$.

If $|k^0| = 1$, we obtain the term

$$\frac{1}{\tau_+} \int_v c_0(t, x, v) w P_{k+k^0, p+p^0}(\Phi) Y^\sigma f dv, \quad \text{where} \quad c_0(t, x, v) := c(t, x, v) \frac{z}{\tau_+},$$

which is of (type 2 – R) since

$$|k+k^0| + (p+p^0) + |\kappa| \leq (p+|k|+|\kappa|) + (p^0+|k^0|) \leq 3R \quad \text{and} \quad |k+k^0| + |\kappa| = (|k|+|\kappa|) + 1 \leq R.$$

If $|k_0| = 0$, using that

$$\frac{z}{\tau_+} c(t, x, v) + Y^{\kappa^0} \left(\frac{z}{\tau_+} c(t, x, v) \right) = c_3(t, x, v) + c_4(t, x, v) \Phi,$$

we obtain the following terms of (type 2 - R),

$$\begin{aligned} & \frac{1}{\tau_+} \int_v (c_3(t, x, v) P_{k, p+p^0}(\Phi) + c_4(t, x, v) P_{k, p+p^0+1}(\Phi)) w Y^\kappa f dv, \\ & \frac{1}{\tau_+} \int_v c_0(t, x, v) w P_{k, p+p^0}(\Phi) Y^{\kappa^0} Y^\kappa f dv \quad \text{and} \quad \frac{1}{\tau_+} \int_v c_0(t, x, v) w P_{(k_T, \kappa_T^0, k_P + \kappa_P^0), p+p^0}(\Phi) Y^\kappa f. \end{aligned}$$

□

Recall from the transport equation satisfied by the Φ coefficients that, in order to estimate $Y^\gamma \Phi$, we need to control $\mathcal{L}_{Z^\beta}(F)$ with $|\beta| = |\gamma| + 1$. Consequently, at the top order, we will rather use the following commutation formula.

Proposition 4.3.40. *Let $Z^\beta \in \mathbb{K}^{|\beta|}$. Then,*

$$\nabla^\mu \mathcal{L}_{Z^\beta}(F)_{\mu\nu} = \sum_{\substack{|q|+|\kappa| \leq |\beta| \\ |q| \leq |\beta|-1 \\ p \leq q_X + \kappa_T}} J(c_\kappa^{k,q}(v) P_{q,p}(\Phi) Y^\kappa f),$$

where $P_{q,p}(\Phi)$ can contain \mathbb{Y}_X , and not merely \mathbb{Y} , derivatives of Φ . We then denote by q_X its number of X derivatives.

Proof. Iterating Lemma 4.3.36, we have

$$\nabla^\mu \mathcal{L}_{Z^\beta}(F)_{\mu\nu} = \sum_{|\gamma| \leq |\beta|} C_\gamma^\beta J(\widehat{Z}^\gamma f). \quad (4.34)$$

The result then follows from an induction on $|\gamma|$. Indeed, write $\widehat{Z}^\gamma = \widehat{Z} \widehat{Z}^{\gamma_0}$ and suppose that

$$\widehat{Z}^{\gamma_0} f = \sum_{\substack{|q|+|\kappa| \leq |\gamma_0| \\ |q| \leq |\gamma_0|-1 \\ p \leq q_X + \kappa_T}} c_\kappa^{k,q}(v) P_{q,p}(\Phi) Y^\kappa f. \quad (4.35)$$

If $\widehat{Z} = \partial \in \mathbf{T}$, then

$$\widehat{Z}^\gamma f = \sum_{\substack{|q|+|\kappa| \leq |\gamma_0| \\ |q| \leq |\gamma_0|-1 \\ p \leq q_X + \kappa_T}} c_\kappa^{k,q}(v) P_{(q_T+1, q_P, q_X), p}(\Phi) Y^\kappa f + c_\kappa^{k,q}(v) P_{q,p}(\Phi) \partial Y^\kappa f = \sum_{\substack{|q|+|\kappa| \leq |\gamma| \\ |q| \leq |\gamma|-1 \\ p \leq q_X + \kappa_T}} c_\kappa^{k,q}(v) P_{q,p}(\Phi) Y^\kappa f.$$

Otherwise $\gamma_P = (\gamma_0)_P + 1$ and write $\widehat{Z} = Y - \Phi X$ with $Y \in \mathbb{Y}_0$. It comes, using $XY^\kappa f = c(v) \partial Y^\kappa f$,

$$\begin{aligned} \widehat{Z}^\gamma f &= \sum_{\substack{|q|+|\kappa| \leq |\gamma_0| \\ |q| \leq |\gamma_0|-1 \\ p \leq q_X + \kappa_T}} \left(Y(c_\kappa^{k,q}(v)) P_{q,p}(\Phi) Y^\kappa f + c_\kappa^{k,q}(v) P_{(q_T, q_P+1, q_X), p}(\Phi) Y^\kappa f + c_\kappa^{k,q}(v) P_{q,p}(\Phi) Y Y^\kappa f \right. \\ &\quad \left. + c_\kappa^{k,q}(v) P_{(q_T, q_P, q_X+1), p+1}(\Phi) Y^\kappa f + c_\kappa^{k,q}(v) P_{(q_T, q_P, q_X), p+1}(\Phi) c(v) \partial Y^\kappa \right) \lesssim \sum_{\substack{|q|+|\kappa| \leq |\gamma| \\ |q| \leq |\gamma|-1 \\ p \leq q_X + \kappa_T}} c_\kappa^{k,q}(v) P_{q,p}(\Phi) Y^\kappa f. \end{aligned}$$

□

4.4 Energy and pointwise decay estimates

In this section, we recall classical energy estimates for both the electromagnetic field and the Vlasov field and how to obtain pointwise decay estimates from them. For that purpose, we need to prove Klainerman-Sobolev inequalities for velocity averages, similar to Theorem 8 of [18] or Theorem 1.1 of [4], adapted to modified vector fields.

4.4.1 Energy estimates

For the particle density, we will use the following approximate conservation law.

Proposition 4.4.1. *Let $H : [0, T[\times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ and $g_0 : \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ be two sufficiently regular functions and F a sufficiently regular 2-form defined on $[0, T[\times \mathbb{R}^3$. Then, g , the unique classical solution of*

$$\begin{aligned} T_F(g) &= H \\ g(0, \cdot, \cdot) &= g_0, \end{aligned}$$

satisfies the following estimate,

$$\forall t \in [0, T[, \quad \|g\|_{L^1_{x,v}}(t) + \sup_{u \in \mathbb{R}} \left\| \frac{v^L}{v^0} g \right\|_{L^1(C_u(t))L^1_v} \leq 2\|g_0\|_{L^1_{x,v}} + 2 \int_0^t \int_{\Sigma_s} \int_v |H| \frac{dv}{v^0} dx ds.$$

Proof. The estimate follows from the divergence theorem, applied to $\int_v \frac{v^\mu}{v^0} |f| dv$ in $[0, t] \times \mathbb{R}^3$ and $V_u(t)$, for all $u \leq t$. We refer to Proposition 3.1 of [6] for more details. \square

We consider, for the remaining of this section, a 2-form G and a 1-form J , both defined on $[0, T[\times \mathbb{R}^3$ and sufficiently regular, such that

$$\begin{aligned} \nabla^\mu G_{\mu\nu} &= J_\nu \\ \nabla^\mu *G_{\mu\nu} &= 0. \end{aligned}$$

We denote by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of G . As $\int_{\Sigma_0} r\rho(G)|(0, x)dx = +\infty$ when the total charge is non-zero, we cannot control norms such as $\|\sqrt{\tau_+}\rho\|_{L^2(\Sigma_t)}$ and we then separate the study of the electromagnetic in two parts.

- The exterior of the light cone, where we propagate L^2 norms on the chargeless part \tilde{F} of F (introduced, as \bar{F} , in Definition 4.1.2), which has a finite initial weighted energy norm. The pure charge part \bar{F} is given by an explicit formula, which describes directly its asymptotic behavior. As $F = \tilde{F} + \bar{F}$, we are then able to obtain pointwise decay estimates on the null components of F .
- The interior of the light cone, where we can propagate L^2 weighted norms of F since we control its flux on $C_0(t)$ with the bounds obtained on \tilde{F} in the exterior region.

We then introduce the following energy norms.

Definition 4.4.2. *Let $N \in \mathbb{N}$. We define, for $t \in [0, T[$,*

$$\begin{aligned} \mathcal{E}^0[G](t) &:= \int_{\Sigma_t} (|\alpha|^2 + |\underline{\alpha}|^2 + 2|\rho|^2 + 2|\sigma|^2) dx + \sup_{u \leq t} \int_{C_u(t)} (|\alpha|^2 + |\rho|^2 + |\sigma|^2) dC_u(t), \\ \mathcal{E}_N^0[G](t) &:= \sum_{\substack{Z^\gamma \in \mathbb{K}^{|\gamma|} \\ |\gamma| \leq N}} \mathcal{E}_N^0[\mathcal{L}_{Z^\gamma}(G)](t), \\ \mathcal{E}^{S, u \geq 0}[G](t) &:= \int_{\Sigma_t^0} \tau_+ (|\alpha|^2 + |\rho|^2 + |\sigma|^2) + \tau_- |\underline{\alpha}| dx + \sup_{0 \leq u \leq t} \int_{C_u(t)} \tau_+ |\alpha|^2 + \tau_- (|\rho|^2 + |\sigma|^2) dC_u(t). \\ \mathcal{E}_N^{S, u \geq 0}[G](t) &:= \sum_{\substack{Z^\gamma \in \mathbb{K}^{|\gamma|} \\ |\gamma| \leq N}} \mathcal{E}_N^{S, u \geq 0}[\mathcal{L}_{Z^\gamma}(G)](t) \\ \mathcal{E}^{S, u \leq 0}[G](t) &:= \int_{\Sigma_t^0} \tau_+ (|\alpha|^2 + |\rho|^2 + |\sigma|^2) + \tau_- |\underline{\alpha}| dx + \sup_{u \leq 0} \int_{C_u(t)} \tau_+ |\alpha|^2 + \tau_- (|\rho|^2 + |\sigma|^2) dC_u(t) \\ \mathcal{E}_N^{E, xt}[G](t) &:= \sum_{\substack{Z^\gamma \in \mathbb{K}^{|\gamma|} \\ |\gamma| \leq N}} \mathcal{E}_N^{S, u \leq 0}[\mathcal{L}_{Z^\gamma}(G)](t). \end{aligned}$$

The following estimates hold.

Proposition 4.4.3. *Let $\bar{S} := S + \partial_t \mathbf{1}_{u>0} + 2\tau_- \partial_t \mathbf{1}_{u\leq 0}$. For all $t \in [0, T]$,*

$$\begin{aligned}\mathcal{E}^0[G](t) &\leq 2\mathcal{E}^0[G](0) + 8 \int_0^t \int_{\Sigma_s} |G_{\mu 0} J^\mu| dx ds \\ \mathcal{E}^{S, u \leq 0}[G](t) &\leq 6\mathcal{E}^{S, u \leq 0}[G](0) + 8 \int_0^t \int_{\Sigma_s^0} \left| \bar{S}^\nu G_{\nu \mu} J^\mu \right| dx ds \\ \mathcal{E}^{S, u \geq 0}[G](t) &\leq 3\mathcal{E}^{S, u \leq 0}[\tilde{G}](t) + 8 \int_0^t \int_{\Sigma_s^0} \left| \bar{S}^\nu G_{\nu \mu} J^\mu \right| dx ds.\end{aligned}$$

Proof. For the first inequality, apply the divergence theorem to $T_{\mu 0}[G]$ in $[0, t] \times \mathbb{R}^3$ and $V_u(t)$, for all $u \leq t$. Let us give more details for the other ones. Denoting $T[G]$ by T and using Lemma 4.2.3, we have, if $u \leq 0$,

$$\begin{aligned}\nabla^\mu (\tau_- T_{\mu 0}) &= \tau_- \nabla^\mu T_{\mu 0} - \frac{1}{2} \underline{L}(\tau_-) T_{L 0} \\ &= \tau_- \nabla^\mu T_{\mu 0} - \frac{u}{2\tau_-} \left(|\alpha|^2 + |\rho|^2 + |\sigma|^2 \right) \geq \tau_- \nabla^\mu T_{\mu 0}.\end{aligned}$$

Consequently, applying Corollary 4.2.2 and the divergence theorem in $V_{u_0}(t)$, for $u_0 \leq 0$, we obtain

$$\int_{\Sigma_t^{u_0}} \tau_- T_{00} dx + \frac{1}{\sqrt{2}} \int_{C_{u_0}(t)} \tau_- T_{L 0} dC_{u_0}(t) \leq \int_{\Sigma_{0^+}^{u_0}} \sqrt{1+r^2} T_{00} dx - \int_0^t \int_{\Sigma_s^{u_0}} \tau_- G_{0\nu} J^\nu dx ds. \quad (4.36)$$

On the other hand, as $\nabla^\mu S^\nu + \nabla^\nu S^\mu = 2\eta^{\mu\nu}$ and $T_\mu{}^\mu = 0$, we have

$$\begin{aligned}\nabla^\mu (T_{\mu\nu} S^\nu) &= \nabla^\mu T_{\mu\nu} S^\nu + T_{\mu\nu} \nabla^\mu S^\nu \\ &= G_{\nu\lambda} J^\lambda S^\nu + \frac{1}{2} T_{\mu\nu} (\nabla^\mu S^\nu + \nabla^\nu S^\mu) \\ &= G_{\nu\lambda} J^\lambda S^\nu.\end{aligned}$$

Applying again the divergence theorem in $V_{u_0}(t)$, for all $u_0 \leq 0$, it comes

$$\int_{\Sigma_t^{u_0}} T_{0\nu} S^\nu dx + \frac{1}{\sqrt{2}} \int_{C_{u_0}(t)} T_{L\nu} S^\nu dC_{u_0}(t) = \int_{\Sigma_{0^+}^{u_0}} T_{0\nu} S^\nu dx - \int_0^t \int_{\Sigma_s^{u_0}} G_{\mu\nu} J^\mu S^\nu dx ds. \quad (4.37)$$

Using Lemma 4.2.3 and $2S = (t+r)L + (t-r)\underline{L}$, notice that

$$\begin{aligned}4\tau_- T_{00} &= \tau_- (|\alpha|^2 + |\underline{\alpha}|^2 + 2|\rho|^2 + 2|\sigma|^2), & 4T_{0\nu} S^\nu &= (t+r)|\alpha| + (t-r)|\underline{\alpha}| + 2t(|\rho| + |\sigma|), \\ 2\tau_- T_{L 0} &= \tau_- (|\alpha|^2 + |\rho|^2 + |\sigma|^2), & 2T_{L\nu} S^\nu &= (t+r)|\alpha|^2 + (t-r)|\rho|^2 + (t-r)|\sigma|^2,\end{aligned}$$

and then add twice (4.36) to (4.37). The second estimate then follows and we now turn on the last one. Recall that $\nabla^\mu T_{\mu\nu} G = G_{\nu\lambda} J^\lambda$ and $\nabla^\mu (T_{\mu\nu} S^\nu) = G_{\nu\lambda} J^\lambda S^\nu$. Hence, by the divergence theorem applied in $[0, t] \times \mathbb{R}^3 \setminus V_0(t)$, we obtain

$$\int_{\Sigma_t^0} (T_{00} + T_{0\nu} S^\nu) dx = \frac{1}{\sqrt{2}} \int_{C_0(t)} (T_{L 0} + T_{L\nu} S^\nu) dC_0(t) - \int_0^t \int_{\Sigma_s^0} G_{0\nu} J^\nu + S^\nu G_{\nu\mu} J^\mu dx ds. \quad (4.38)$$

By Lemma 4.2.3, we have $4T_{00} = (|\alpha|^2 + |\underline{\alpha}|^2 + 2|\rho|^2 + 2|\sigma|^2)$, so that

$$4T_{00} + 4T_{0\nu} S^\nu \geq \tau_+ |\alpha|^2 + \tau_- |\underline{\alpha}|^2 + \tau_+ |\rho|^2 + \tau_+ |\sigma|^2 \geq 0 \quad \text{on } \Sigma_t^0. \quad (4.39)$$

Consequently, the divergence theorem applied in $V_u(t) \setminus V_0(t)$, for $0 \leq u \leq t$, gives

$$\frac{1}{\sqrt{2}} \int_{C_u(t)} (T_{L 0} + T_{L\nu} S^\nu) dC_u(t) \leq \frac{1}{\sqrt{2}} \int_{C_0(t)} (T_{L 0} + T_{L\nu} S^\nu) dC_0(t) - \int_{V_u(t) \setminus V_0(t)} (G_{0\nu} J^\nu + S^\nu G_{\nu\mu} J^\mu). \quad (4.40)$$

Not now that $T_{L 0} + T_{L\nu} S^\nu \geq \tau_+ |\alpha|^2 + \tau_- |\rho|^2 + \tau_- |\sigma|^2$ if $u \geq 0$ since

$$2T_{L 0} = |\alpha|^2 + |\rho|^2 + |\sigma|^2 \quad \text{and} \quad 2T_{L\nu} S^\nu = (t+r)|\alpha|^2 + (t-r)|\rho|^2 + (t-r)|\sigma|^2.$$

It then remains to take the sup over all $0 \leq u \leq t$ in (4.40), to combine it with (4.38), (4.39) and to remark that

$$\begin{aligned}2 \int_{C_0(t)} T_{L 0} + T_{L\nu} S^\nu dC_0(t) &\leq \int_{C_0(t)} |\rho|^2 + |\sigma|^2 dC_0(t) + \int_{C_0(t)} \tau_+ |\alpha|^2 dC_0(t) \\ &\leq \mathcal{E}^{S, u \leq 0}[\tilde{G}](t),\end{aligned}$$

since $G = \tilde{G}$ on $C_0(t)$. □

4.4.2 Pointwise decay estimates

Decay estimates for velocity averages

As the set of our commutation vector fields is not $\widehat{\mathbb{P}}_0$, we need to modify the following standard Klainerman-Sobolev inequality, which was proved in [18] (see Theorem 8).

Proposition 4.4.4. *Let g be a sufficiently regular function defined on $[0, T[\times\mathbb{R}_x^3 \times \mathbb{R}_v^3$. Then, for all $(t, x) \in [0, T[\times\mathbb{R}^3$,*

$$\forall (t, x) \in [0, T[\times\mathbb{R}^3, \quad \int_{v \in \mathbb{R}^3} |g(t, x, v)| dv \lesssim \frac{1}{\tau_+^2 \tau_-} \sum_{\substack{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^{|\beta|} \\ |\beta| \leq 3}} \|\widehat{Z}^\beta g\|_{L_{x,v}^1}(t).$$

We need to rewrite it using the modified vector fields. For the remaining of this section, g will be a sufficiently regular function defined on $[0, T[\times\mathbb{R}_x^3 \times \mathbb{R}_v^3$. We also consider F , a regular 2-form, so that we can consider the Φ coefficients introduced in Definition 4.3.11 and we suppose that they satisfy the following pointwise estimates, with $M_1 \geq 7$ a fixed integer. For all $(t, x, v) \in [0, T[\times\mathbb{R}^3 \times \mathbb{R}^3$,

$$|Y\Phi|(t, x, v) \lesssim \log^{\frac{7}{2}}(1 + \tau_+), \quad |\Phi|(t, x, v) \lesssim \log^2(1 + \tau_+) \quad \text{and} \quad \sum_{|\kappa| \leq 3} |Y^\kappa \Phi|(t, x, v) \lesssim \log^{M_1}(1 + \tau_+).$$

Proposition 4.4.5. *For all $(t, x) \in [0, T[\times\mathbb{R}^3$,*

$$\tau_+^2 \tau_- \int_{v \in \mathbb{R}^3} |g(t, x, v)| dv \lesssim \sum_{|\xi| + |\beta| \leq 3} \|P_\xi^X(\Phi)Y^\beta g\|_{L_{x,v}^1}(t) + \sum_{|\kappa| \leq \min(2 + \kappa_T, 3)} \sum_{z \in \mathbf{k}_1} \frac{\log^{6M_1}(3 + t)}{1 + t} \|zY^\kappa g\|_{L_{x,v}^1}(t).$$

Remark 4.4.6. *This inequality is suitable for us since we will bound $\|P_\xi^X(\Phi)Y^\beta g\|_{L_{x,v}^1}$ without any growth in t . Moreover, observe that Y^κ contains at least a translation if $|\kappa| = 3$, which is compatible with our hierarchy on the weights $z \in \mathbf{k}_1$ (see Remark 4.3.24).*

Proof. Let $(t, x) \in [0, T[\times\mathbb{R}^n$. Consider first the case $|x| \leq \frac{1+t}{2}$, so that, with $\tau := 1 + t$,

$$\forall |y| \leq \frac{1}{4}, \quad \tau \leq 10(1 + |t - |x + \tau y||).$$

For a sufficiently regular function h , we then have, using Lemmas 4.3.6 and then 4.3.20,

$$\begin{aligned} \left| \partial_{y^i} \left(\int_v |h|(t, x + \tau y, v) dv \right) \right| &= \left| \tau \partial_i \int_v |h|(t, x + \tau y, v) dv \right| \\ &\lesssim \left| (1 + |t - |x + \tau y||) \partial_i \int_v |h|(t, x + \tau y, v) dv \right| \\ &\lesssim \sum_{Z \in \mathbb{K}} \left| Z \int_v |h|(t, x + \tau y, v) dv \right| \\ &\lesssim \sum_{\substack{|\xi| + |\beta| \leq 1 \\ p \leq 1}} \sum_{z \in \mathbf{k}_1} \int_v \left(|P_\xi^X(\Phi)Y^\beta h| + \frac{\log^7(1 + \tau_+)}{\tau_+} |z \partial_t^p h| \right) (t, x + \tau y, v) dv. \end{aligned}$$

Using a one dimensional Sobolev inequality, it comes, for $\delta = \frac{1}{4\sqrt{3}}$ (so that $|y| \leq \frac{1}{4}$ if $|y^i| \leq \delta$ for all $1 \leq i \leq 3$),

$$\begin{aligned} \int_v |g|(t, x, v) dv &\lesssim \sum_{n=0}^1 \int_{|y^1| \leq \delta} \left| (\partial_{y^1})^n \int_v |g|(t, x + \tau(y^1, 0, 0), v) dv \right| dy^1 \\ &\lesssim \sum_{\substack{|\xi| + |\beta| \leq 1 \\ p \leq 1 \\ z \in \mathbf{k}_1}} \int_{|y^1| \leq \delta} \int_v \left(|P_\xi^X(\Phi)Y^\beta g| + \frac{\log^7(3 + t)}{1 + t} |z \partial_t^p g| \right) (t, x + \tau(y^1, 0, 0), v) dv dy^1. \end{aligned}$$

Repeating the argument for y^2 and the functions $\int_v P_\xi^X(\Phi)Y^\beta g dv$ and $\int_v z \partial_t^p g dv$, it comes, as $|z| \leq 2t$ in the region considered and dropping the dependence in $(t, x + \tau(y^1, y^2, 0), v)$ of the functions in the integral,

$$\int_v |g|(t, x, v) dv \lesssim \sum_{\substack{|\xi| + |\beta| \leq 2 \\ z \in \mathbf{k}_1}} \sum_{\substack{|\zeta| + |\kappa| \leq 2 \\ |\kappa| \leq 1 + \kappa_T}} \int_{|y^1| \leq \delta} \int_{|y^2| \leq \delta} \int_v \left(|P_\xi^X(\Phi)Y^\beta g| + \frac{\log^{14}(3 + t)}{1 + t} |z P_\zeta^X(\Phi)Y^\kappa g| \right) dv dy^1 dy^2.$$

Repeating again the argument for the variable y^3 , we finally obtain

$$\int_v |g|(t, x, v) dv \lesssim \sum_{\substack{|\xi|+|\beta|\leq 3 \\ z \in \mathbf{k}_1}} \sum_{\substack{|\zeta|+|\kappa|\leq 3 \\ |\kappa|\leq 2+\kappa_T}} \int_{|y|\leq \frac{1}{4}} \int_v |P_\xi^X(\Phi)Y^\beta g| + \frac{\log^{21}(3+t)}{1+t} |zP_\zeta^X(\Phi)Y^\kappa g| dv(t, x + \tau y) dy.$$

It then remains to remark that $|P_\zeta^X(\Phi)| \lesssim \log^{3M_1}(3+t)$ on the domain of integration and to make the change of variables $z = \tau y$. Note now that one can prove similarly that, for a sufficiently regular function h ,

$$\int_v |h|(t, r, \theta, \phi) dv \lesssim \sum_{\substack{|\xi|+|\beta|\leq 2 \\ z \in \mathbf{k}_1}} \sum_{|\kappa|\leq \min(1+\kappa_T, 2)} \int_{\mathbb{S}^2} \int_v |P_\xi^X(\Phi)Y^\beta h| + \frac{\log^{14+2M_1}(1+\tau_+)}{\tau_+} |zY^\kappa h| dv d\mathbb{S}^2(t, r). \quad (4.41)$$

Indeed, by a one dimensional Sobolev inequality, we have

$$\int_v |f|(t, r, \theta, \phi, v) dv \lesssim \sum_{r=0}^1 \int_{\omega_1} \left| (\partial_{\omega_1})^r \int_v |f|(t, r, \theta + \omega_1, \phi, v) dv \right| d\omega_1.$$

Then, since ∂_{ω_1} (and ∂_{ω_2}) can be written as a combination with bounded coefficients of the rotational vector fields Ω_{ij} , we can repeat the previous argument. Finally, let us suppose that $\frac{1+t}{2} \leq |x|$. We have, using again Lemmas 4.3.6 and 4.3.20,

$$\begin{aligned} |x|^2 \tau_- \int_v |g|(t, x, v) dv &= -|x|^2 \int_{|x|}^{+\infty} \partial_r \left(\tau_- \int_v |g|(t, r, \theta, \phi, v) dv \right) dr \\ &\lesssim \int_{|x|}^{+\infty} \int_v |g|(t, r, \theta, \phi, v) dv r^2 dr + \int_{|x|}^{+\infty} \left| \tau_- \partial_r \int_v |g|(t, r, \theta, \phi, v) dv \right| r^2 dr \\ &\leq \sum_{\substack{|\xi|+|\beta|\leq 1 \\ p\leq 1}} \sum_{w \in \mathbf{k}_1} \int_0^{+\infty} \int_v \left(|P_\xi^X(\Phi)Y^\beta g| + \frac{\log^7(3+t)}{1+t} |w\partial_t^p g| \right) (t, r, \theta, \phi, v) dv r^2 dr. \end{aligned}$$

It then remains to apply (4.41) to the functions $P_\xi^X(\Phi)Y^\beta g$ and $z\partial_t^p g$ and to remark that $|z| \leq 2\tau_+$. \square

A similar, but more general, result holds.

Corollary 4.4.7. *Let $z \in \mathbf{k}_1$ and $j \in \mathbb{N}$. Then, for all $(t, x) \in [0, T] \times \mathbb{R}^3$,*

$$\int_{v \in \mathbb{R}^n} |z|^j |g|(t, x, v) dv \lesssim \frac{1}{\tau_+^2 \tau_-} \sum_{w \in \mathbf{k}_1} \left(\sum_{d=0}^{\min(3, j)} \sum_{|\xi|+|\beta|\leq 3-d} \log^{2d}(3+t) \|w^{j-d} P_\xi^X(\Phi)Y^\beta g\|_{L_{x,v}^1(t)} \right. \\ \left. + \frac{\log^{6M_1}(3+t)}{1+t} \sum_{|\kappa|\leq \min(2+\kappa_T, 3)} \|w^{j+1} Y^\kappa f\|_{L_{x,v}^1(t)} \right).$$

Proof. One only has to follow the proof of Proposition 4.4.5 and to use Remark (4.3.21) instead of Lemma 4.3.20). \square

A weaker version of this inequality will be used in Subsection 4.9.1.

Corollary 4.4.8. *Let $z \in \mathbf{k}_1$ and $j \in \mathbb{N}$. Then, for all $(t, x) \in [0, T] \times \mathbb{R}^3$,*

$$\int_{v \in \mathbb{R}^n} |z|^j |g|(t, x, v) dv \lesssim \frac{1}{\tau_+^2 \tau_-} \sum_{w \in \mathbf{k}_1} \left(\sum_{d=0}^{\min(3, j)} \sum_{|\beta|\leq 3-d} \log^{2d+M_1}(3+t) \|w^{j-d} Y^\beta g\|_{L_{x,v}^1(t)} \right. \\ \left. + \frac{\log^{6M_1}(3+t)}{1+t} \sum_{|\kappa|\leq \min(2+\kappa_T, 3)} \|w^{j+1} Y^\kappa f\|_{L_{x,v}^1(t)} \right).$$

Proof. Start by applying Corollary 4.4.7. It remains to bound the terms of the form

$$\|w^{j-d} P_\xi^X(\Phi)Y^\beta g\|_{L_{x,v}^1(\Sigma_t)}, \quad \text{with} \quad d \leq \min(3, j), \quad |\xi| + |\beta| \leq 3 - d \quad \text{and} \quad |\xi| \geq 1.$$

For this, we divide Σ_t in two regions, the one where $r \leq 1+2t$ and its complement. As $|P_\xi^X(\Phi)| \lesssim \log^{M_1}(1+\tau_+)$ and $\tau_+ \lesssim 1+t$ if $r \leq 1+2t$, we have

$$\|w^{j-d}P_\xi^X(\Phi)Y^\beta g\|_{L_v^1 L^1(|y| \leq 2t)} \lesssim \log^{M_1}(3+t) \|w^{j-d}Y^\beta g\|_{L_v^1 L^1(\Sigma_t)}.$$

Now recall from Remark 4.2.5 that $1+r \lesssim \sum_{z_0 \in \mathbf{k}_1} |z_0|$ and $|P_\xi^X(\Phi)|(1+r)^{-1} \lesssim \frac{\log^{M_1}(3+t)}{1+t}$ if $r \geq 1+2t$, so that

$$\|w^{j-d}P_\xi^X(\Phi)Y^\beta g\|_{L_v^1 L^1(|y| \geq 2t)} \lesssim \frac{\log^{M_1}(3+t)}{1+t} \sum_{z_0 \in \mathbf{k}_1} \|z_0^{j+1}Y^\beta g\|_{L_v^1 L^1(\Sigma_t)}.$$

The result follows from $|\beta| \leq 2-d \leq 2+\beta_T$. \square

We are now interested in adapting Theorem 1.1 of [4] to the modified vector fields.

Theorem 4.4.9. *Suppose that $\sum_{|\kappa| \leq 3} \|Y^\kappa \Phi\|_{L_{x,v}^\infty(0)} \lesssim 1$. Let $H : [0, T[\times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ and $h_0 : \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ be two sufficiently regular functions and h the unique classical solution of*

$$\begin{aligned} T_F(h) &= H \\ h(0, \cdot, \cdot) &= h_0. \end{aligned}$$

Consider also $z \in \mathbf{k}_1$ and $j \in \mathbb{N}$. Then, for all $(t, x) \in [0, T[\times \mathbb{R}^3$ such that $t \geq |x|$,

$$\begin{aligned} \tau_+^3 \int_v |z^j h|(t, x, v) \frac{dv}{(v^0)^2} &\lesssim \sum_{|\beta| \leq 3} \|(1+r)^{|\beta|+j} \partial_{t,x}^\beta h\|_{L_x^1 L_v^1(0)} \\ &+ \sum_{\substack{|\xi|+|\beta| \leq 3 \\ w \in \mathbf{k}_1}} \sum_{\substack{0 \leq d \leq 3 \\ \delta \in \{0,1\}}} \frac{\log^{2d}(3+t)}{\sqrt{1+t}^\delta} \int_0^t \int_{\Sigma_s} \int_v |T_F(w^{j-d+\delta} P_\xi^X(\Phi) Y^\beta h)| \frac{dv}{v^0} dx ds, \end{aligned}$$

where $|\xi| = 0$ and $|\beta| \leq \min(2+\beta_T, 3)$ if $\delta = 1$.

Proof. If $|x| \leq \frac{t}{2}$, the result follows from Corollary 4.4.7 and the energy estimate of Proposition 4.4.1. If $\frac{t}{2} \leq |x| \leq t$, we refer to Section 5 of [4], where Lemma 5.2 can be rewritten in the same spirit as we rewrite Proposition 4.4.4 with modified vector field. \square

To deal with the exterior, we use the following result.

Proposition 4.4.10. *For all $(t, x) \in [0, T[\times \mathbb{R}^3$ such that $|x| \geq t$, we have*

$$\int_v |g|(t, x, v) \frac{dv}{(v^0)^2} \lesssim \frac{1}{\tau_+} \sum_{w \in \mathbf{k}_1} \int_v |w| |g|(t, x, v) dv.$$

Proof. Let $|x| \geq t$. If $|x| \leq 1$, $\tau_+ \leq 3$ and the estimate holds. Otherwise, $\tau_+ \leq 3|x|$ so, as $(x^i - t \frac{v^i}{v^0}) \in \mathbf{k}_1$ and

$$\left| x - t \frac{v}{v^0} \right| \geq |x| - t \frac{|v|}{v^0} \geq |x| \frac{(v^0)^2 - |v|^2}{v^0(v^0 + |v|)} \geq \frac{|x|}{2(v^0)^2}, \quad \text{we have} \quad \int_v |g|(t, x, v) \frac{dv}{(v^0)^2} \lesssim \frac{1}{|x|} \sum_{w \in \mathbf{k}} \int_v |w| |g|(t, x, v) dv.$$

\square

Remark 4.4.11. *Using $1 \lesssim v^0 v^\perp$ and Lemma 4.2.4, we can obtain a similar inequality for the interior of the light cone, at the cost of a τ_- -loss. Note however that because of the presence of the weights $w \in \mathbf{k}_1$, this estimate, combined with Corollary 4.4.7, is slightly weaker than Theorem 4.4.9. During the proof, this difference will lead to a slower decay rate insufficient to close the energy estimates.*

Decay estimates for the electromagnetic field

We start by presenting weighted Sobolev inequalities for general tensor field. Then we will use them in order to obtain improved decay estimates for the null components of a 2-form¹⁹. In order to treat the interior of the light cone (or rather the domain in which $|x| \leq 1 + \frac{1}{2}t$), we will use the following result.

Lemma 4.4.12. *Let U be a smooth tensor field defined on $[0, T[\times \mathbb{R}^3$. Then,*

$$\forall t \in [0, T[, \quad \sup_{|x| \leq 1 + \frac{t}{2}} |U(t, x)| \lesssim \frac{1}{(1+t)^2} \sum_{|\gamma| \leq 2} \|\sqrt{\tau_-} \mathcal{L}_{Z^\gamma}(U)(t, y)\|_{L^2(|y| \leq 2 + \frac{3}{4}t)}.$$

Proof. As $|\mathcal{L}_{Z^\gamma}(U)| \lesssim \sum_{|\beta| \leq |\gamma|} \sum_{\mu, \nu} |Z^\beta(U_{\mu\nu})|$, we can restrict ourselves to the case of a scalar function. Let $t \in \mathbb{R}_+$ and $|x| \leq 1 + \frac{1}{2}t$. Apply a standard L^2 Sobolev inequality to $V : y \mapsto U(t, x + \frac{1+t}{4}y)$ and then make a change of variables to get

$$|U(t, x)| = |V(0)| \lesssim \sum_{|\beta| \leq 2} \|\partial_x^\beta V\|_{L_y^2(|y| \leq 1)} \lesssim \left(\frac{1+t}{4}\right)^{-\frac{3}{2}} \sum_{|\beta| \leq 2} \left(\frac{1+t}{4}\right)^{|\beta|} \|\partial_x^\beta U(t, \cdot)\|_{L_y^2(|y-x| \leq \frac{1+t}{4})}.$$

Observe now that $|y-x| \leq \frac{1+t}{4}$ implies $|y| \leq 2 + \frac{3}{4}t$ and that $1+t \lesssim \tau_-$ on that domain. By Lemma 4.3.6 and since $[Z, \partial] \in \mathbf{T} \cup \{0\}$, it comes

$$(1+t)^{|\beta| + \frac{1}{2}} \|\partial_x^\beta U(t, \cdot)\|_{L_y^2(|y-x| \leq \frac{1+t}{4})} \lesssim \|\tau_-^{|\beta| + \frac{1}{2}} \partial_x^\beta U(t, \cdot)\|_{L_y^2(|y| \leq 2 + \frac{3}{4}t)} \lesssim \sum_{|\gamma| \leq |\beta|} \|\sqrt{\tau_-} Z^\gamma U(t, \cdot)\|_{L_y^2(|y| \leq 2 + \frac{3}{4}t)}.$$

□

For the remaining region, we have the two following inequalities, coming from Lemma 2.3 (or rather from its proof for the second estimate) of [11].

Lemma 4.4.13. *Let U be a sufficiently regular tensor field defined on \mathbb{R}^3 . Then, for $t \in \mathbb{R}_+$,*

$$\begin{aligned} \forall |x| \geq \frac{t}{2} + 1, \quad |U(x)| &\lesssim \frac{1}{|x| \tau_-^{\frac{1}{2}}} \left(\int_{|y| \geq \frac{t}{2} + 1} |U(y)|_{\mathbb{O},2}^2 + \tau_-^2 |\nabla_{\partial_r} U(y)|_{\mathbb{O},1}^2 dy \right)^{\frac{1}{2}}, \\ \forall |x| > t, \quad |U(x)| &\lesssim \frac{1}{|x| \tau_-^{\frac{1}{2}}} \left(\int_{|y| \geq t} |U(y)|_{\mathbb{O},2}^2 + \tau_-^2 |\nabla_{\partial_r} U(y)|_{\mathbb{O},1}^2 dy \right)^{\frac{1}{2}}, \\ \forall x \neq 0, \quad |U(x)| &\lesssim \frac{1}{|x|^{\frac{3}{2}}} \left(\int_{|y| \geq |x|} |U(y)|_{\mathbb{O},2}^2 + |y|^2 |\nabla_{\partial_r} U(y)|_{\mathbb{O},1}^2 dy \right)^{\frac{1}{2}}. \end{aligned}$$

Recall that G and J satisfy

$$\begin{aligned} \nabla^\mu G_{\mu\nu} &= J_\nu \\ \nabla^\mu * G_{\mu\nu} &= 0 \end{aligned}$$

and that $(\alpha, \underline{\alpha}, \rho, \sigma)$ denotes the null decomposition of G . Before proving pointwise decay estimates on the components of G , we recall the following classical result and we refer, for instance, to Lemma D.1 of [6] for a proof. Concretely, it means that \mathcal{L}_Ω , for $\Omega \in \mathbb{O}$, ∇_{∂_r} , $\nabla_{\underline{L}}$ and ∇_L commute with the null decomposition.

Lemma 4.4.14. *Let $\Omega \in \mathbb{O}$. Then, denoting by ζ any of the null component α , $\underline{\alpha}$, ρ or σ ,*

$$[\mathcal{L}_\Omega, \nabla_{\partial_r}]G = 0, \quad \mathcal{L}_\Omega(\zeta(G)) = \zeta(\mathcal{L}_\Omega(G)) \quad \text{and} \quad \nabla_{\partial_r}(\zeta(G)) = \zeta(\nabla_{\partial_r}(G)).$$

Similar results hold for \mathcal{L}_Ω and ∇_{∂_t} , ∇_L or $\nabla_{\underline{L}}$. For instance, $\nabla_L(\zeta(G)) = \zeta(\nabla_L(G))$.

¹⁹Note hower that our estimate on the component α require the 2-form G to satisfy $\nabla^\mu * G_{\mu\nu} = 0$.

Proposition 4.4.15. *We have, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,*

$$\begin{aligned} |\rho|(t, x), \quad |\sigma|(t, x) &\lesssim \frac{\sqrt{\mathcal{E}_2[G](t) + \mathcal{E}_2^{Ext}[G](t)}}{\tau_+^{\frac{3}{2}}\tau_-^{\frac{1}{2}}}, \\ |\alpha|(t, x) &\lesssim \frac{\sqrt{\mathcal{E}_2[G](t) + \mathcal{E}_2^{Ext}[G](t)} + \sum_{|\kappa| \leq 1} \|r^{\frac{3}{2}} \mathcal{L}_{Z^\kappa}(J)_A\|_{L^2(\Sigma_t)}}{\tau_+^2}, \\ |\underline{\alpha}|(t, x) &\lesssim \min \left(\frac{\sqrt{\mathcal{E}_2[G](t) + \mathcal{E}_2^{Ext}[G](t)}}{\tau_+ \tau_-}, \frac{\sqrt{\mathcal{E}_2^0[G](t)}}{\tau_+ \tau_-^{\frac{1}{2}}} \right). \end{aligned}$$

Moreover, if $|x| \geq \max(t, 1)$, the term involving $\mathcal{E}_2[G](t)$ on the right hand side of each of these three estimates can be removed.

Remark 4.4.16. *As we will have a small loss on $\mathcal{E}_2[F]$ and not on $\mathcal{E}_2^0[F]$, the second estimate on $\underline{\alpha}$ is here for certain situations, where we will need a decay rate of degree at least 1 in the $t + r$ direction.*

Proof. Let $(t, x) \in [0, T] \times \mathbb{R}^3$. If $|x| \leq 1 + \frac{1}{2}t$, $\tau_- \leq \tau_+ \leq 2 + 2t$ so the result immediately follows from Lemma 4.4.12. We then focus on the case $|x| \geq 1 + \frac{t}{2}$. During this proof, Ω^β will always denote a combination of rotational vector fields, i.e. $\Omega^\beta \in \mathbb{O}^{|\beta|}$. Let ζ be either α , ρ or σ . As, by Lemma 4.4.14, ∇_{∂_r} and \mathcal{L}_Ω commute with the null decomposition, we have, applying Lemma 4.4.13,

$$r^3 \tau_- |\zeta|^2 \lesssim \int_{|y| \geq \frac{t}{2} + 1} |\sqrt{r} \zeta|_{\mathbb{O}, 2}^2 + \tau_-^2 |\nabla_{\partial_r}(\sqrt{r} \zeta)|_{\mathbb{O}, 1}^2 dy \lesssim \sum_{\substack{|\gamma| \leq 2 \\ |\beta| \leq 1}} \int_{|y| \geq \frac{t}{2} + 1} r |\zeta(\mathcal{L}_{Z^\gamma}(G))|^2 + r \tau_-^2 |\zeta(\mathcal{L}_{\Omega^\beta}(\nabla_{\partial_r} G))|^2 dy.$$

As ∇_{∂_r} commute with \mathcal{L}_Ω and since ∇_{∂_r} commute with the null decomposition (see Lemma 4.4.14), we have, using $2\partial_r = L - \underline{L}$ and (4.14),

$$|\zeta(\mathcal{L}_\Omega(\nabla_{\partial_r} G))| + |\zeta(\nabla_{\partial_r} G)| \lesssim |\nabla_{\partial_r} \zeta(\mathcal{L}_\Omega(G))| + |\nabla_{\partial_r} \zeta(G)| \lesssim \frac{1}{\tau_-} \sum_{|\gamma| \leq 2} |\zeta(\mathcal{L}_{Z^\gamma}(G))|. \quad (4.42)$$

As $\tau_+ \lesssim r \leq \tau_+$ in the region considered, it finally comes

$$\tau_+^3 \tau_- |\zeta|^2 \lesssim \sum_{|\gamma| \leq 2} \int_{|y| \geq \frac{t}{2} + 1} \tau_+ |\zeta(\mathcal{L}_{Z^\gamma}(G))|^2 dx \lesssim \mathcal{E}_2[G](t) + \mathcal{E}_2^{Ext}[G](t).$$

Let us improve now the estimate on α . As, by Lemma 4.3.36, $\nabla^\mu \mathcal{L}_\Omega(G)_{\mu\nu} = \mathcal{L}_\Omega(J)_\nu$ and $\nabla^{\mu*} \mathcal{L}_\Omega(G)_{\mu\nu} = 0$ for all $\Omega \in \mathbb{O}$, we have according to Lemma 4.2.1 that

$$\forall |\beta| \leq 1, \quad \nabla_{\underline{L}} \alpha(\mathcal{L}_{\Omega^\beta}(G))_A = \frac{1}{r} \alpha(\mathcal{L}_{\Omega^\beta}(G))_A - \nabla_{e_A} \rho(\mathcal{L}_{\Omega^\beta}(G)) + \varepsilon_{AB} \nabla_{e_B} \sigma(\mathcal{L}_{\Omega^\beta}(G)) + \mathcal{L}_{\Omega^\beta}(J)_A.$$

Thus, using (4.14), we obtain, for all $\Omega \in \mathbb{O}$,

$$|\alpha(\nabla_{\partial_r} G)| + |\alpha(\mathcal{L}_\Omega(\nabla_{\partial_r} G))| \lesssim |J_A| + |\mathcal{L}_\Omega(J)_A| + \frac{1}{r} \sum_{|\gamma| \leq 2} |\alpha(\mathcal{L}_{Z^\gamma}(G))| + |\rho(\mathcal{L}_{Z^\gamma}(G))| + |\sigma(\mathcal{L}_{Z^\gamma}(G))|. \quad (4.43)$$

Hence, utilizing this time the third inequality of Lemma 4.4.13 and (4.43) instead of (4.42), it comes

$$\tau_+^4 |\alpha|^2 \lesssim r^4 |\alpha|^2 \lesssim \int_{|y| \geq |x|} |\sqrt{r} \alpha|_{\mathbb{O}, 2}^2 + r^2 |\nabla_{\partial_r}(\sqrt{r} \alpha)|_{\mathbb{O}, 1}^2 dy \lesssim \mathcal{E}_2[G](t) + \mathcal{E}_2^{Ext}[G](t) + \sum_{|\kappa| \leq 1} \|r^{\frac{3}{2}} \mathcal{L}_{Z^\kappa}(J)_A\|_{L^2(\Sigma_t)}^2.$$

Using the same arguments as previously, one has

$$\begin{aligned} \int_{|y| \geq \frac{t}{2} + 1} |\underline{\alpha}|_{\mathbb{O}, 2}^2 + \tau_-^2 |\nabla_{\partial_r} \underline{\alpha}|_{\mathbb{O}, 1}^2 dy &\lesssim \mathcal{E}_2^0[G](t), \\ \int_{|y| \geq \frac{t}{2} + 1} |\sqrt{\tau_-} \underline{\alpha}|_{\mathbb{O}, 2}^2 + \tau_-^2 |\nabla_{\partial_r}(\sqrt{\tau_-} \underline{\alpha})|_{\mathbb{O}, 1}^2 dy &\lesssim \mathcal{E}_2[G](t) + \mathcal{E}_2^{Ext}[G](t) \end{aligned}$$

and a last application of Lemma 4.4.13 gives us the result. The estimates for the region $|x| \geq \max(t, 1)$ can be obtained similarly, using the second inequality of Lemma 4.4.13 instead of the first one. \square

4.5 The pure charge part of the electromagnetic field

As we will consider an electromagnetic field with a non-zero total charge, $\int_{\mathbb{R}^3} r|\rho(F)|dx$ will be infinite and we will not be able to apply the results of the previous section to F and its derivatives. As mentioned earlier, we will split F in $\tilde{F} + \bar{F}$, where \tilde{F} and \bar{F} are introduced in Definition 4.1.2. We will then apply the results of the previous section to the chargeless field \tilde{F} , which will allow us to derive pointwise estimates on F since the field \bar{F} is completely determined. More precisely, we will use the following properties of the pure charge part \bar{F} of F .

Proposition 4.5.1. *Let F be a 2-form with a constant total charge Q_F and \bar{F} its pure charge part*

$$\bar{F}(t, x) := \chi(t-r) \frac{Q_F}{4\pi r^2} \frac{x_i}{r} dt \wedge dx^i.$$

Then,

1. \bar{F} is supported in $\cup_{t \geq 0} V_{-1}(t)$ and \tilde{F} is chargeless.
2. $\rho(\bar{F})(t, x) = -\frac{Q_F}{4\pi r^2} \chi(t-r)$, $\alpha(\bar{F}) = 0$, $\underline{\alpha}(\bar{F}) = 0$ and $\sigma(\bar{F}) = 0$.
3. $\forall Z^\gamma \in \mathbb{K}^{|\gamma|}$, $\exists C_\gamma > 0$, $|\mathcal{L}_{Z^\gamma}(\bar{F})| \leq C_\gamma |Q_F| \tau_+^{-2}$.
4. \bar{F} satisfies the Maxwell equations $\nabla^\mu \bar{F}_{\mu\nu} = \bar{J}_\nu$ and $\nabla^\mu \star \bar{F}_{\mu\nu} = 0$, with \bar{J} such that

$$\bar{J}_0(t, x) = \frac{Q_F}{4\pi r^2} \chi'(t-r) \quad \text{and} \quad \bar{J}_i(t, x) = -\frac{Q_F}{4\pi r^2} \frac{x_i}{r} \chi'(t-r).$$

\bar{J} is then supported in $\{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / -2 \leq t - |y| \leq -1\}$ and its derivatives satisfy

$$\forall Z^\gamma \in \mathbb{K}^{|\gamma|}, \exists \tilde{C}_\gamma > 0, \quad |\mathcal{L}_{Z^\gamma}(\bar{J})^L| + \tau_+ |\mathcal{L}_{Z^\gamma}(\bar{J})^A| + \tau_+^2 |\mathcal{L}_{Z^\gamma}(\bar{J})^L| \leq \frac{\tilde{C}_\gamma |Q_F|}{\tau_+^2}.$$

Proof. The first point follows from the definitions of \bar{F} , χ and

$$Q_{\tilde{F}}(t) = Q_F - Q_{\bar{F}}(t) = Q_F - \lim_{r \rightarrow +\infty} \left(\int_{\mathbb{S}_{t,r}} \frac{x^i}{r} \bar{F}_{0i} d\mathbb{S}_{t,r} \right) = Q_F - \frac{Q_F}{4\pi r^2} \int_{\mathbb{S}_{t,r}} d\mathbb{S}_{t,r} = 0.$$

The second point is straightforward and depicts that \bar{F} has a vanishing magnetic part and a radial electric part. The third point can be obtained using that,

- for a 2-form G and a vector field Γ , $\mathcal{L}_\Gamma(G)_{\mu\nu} = \Gamma(G_{\mu\nu}) + \partial_\mu(\Gamma^\lambda)G_{\lambda\nu} + \partial_\nu(\Gamma^\lambda)G_{\mu\lambda}$.
- For all $Z \in \mathbb{K}$, Z is either a translation or a homogeneous vector field.
- For a function $\chi_0 : u \mapsto \chi_0(u)$, we have $\Omega_{ij}(\chi_0(u)) = 0$,

$$\partial_t(\chi_0(u)) = \chi_0'(u), \quad \partial_i(\chi_0(u)) = -\frac{x^i}{r} \chi_0'(u), \quad S(\chi_0(u)) = u \chi_0'(u), \quad \Omega_{0i}(\chi_0(u)) = -\frac{x^i}{r} u \chi_0'(u).$$

- $1+t \leq \tau_+ \lesssim r$ on the support of \bar{F} and $|u| \leq \tau_- \leq \sqrt{5}$ on the support of χ' .

Consequently, one has

$$\forall Z^\xi \in \mathbb{K}^{|\xi|}, \quad Z^\xi \left(\frac{x^i}{r^3} \chi(t-r) \right) \leq C_{\xi, \chi} \tau_+^{-2} \quad \text{and} \quad |\mathcal{L}_{Z^\gamma}(\bar{F})| \lesssim \sum_{|\kappa| \leq |\gamma|} \sum_{\mu=0}^3 \sum_{\nu=0}^3 |Z^\kappa(\bar{F}_{\mu\nu})| \lesssim \frac{C_\gamma}{\tau_+^2}.$$

The equations $\nabla^\mu \star \bar{F}_{\mu\nu}$, equivalent to $\nabla_{[\lambda} \bar{F}_{\mu\nu]} = 0$ by Lemma 4.2.1, follow from $\bar{F}_{ij} = 0$ and that the electric part of \bar{F} is radial, so that $\nabla_i \bar{F}_{0j} - \nabla_j \bar{F}_{0i} = 0$. The other ones ensue from straightforward computations,

$$\begin{aligned} \nabla^i \bar{F}_{i0} &= -\frac{Q_F}{4\pi} \partial_i \left(\frac{x^i}{r^3} \chi(t-r) \right) = -\frac{Q_F}{4\pi} \left(\left(\frac{3}{r^3} - 3 \frac{x_i x^i}{r^5} \right) \chi(t-r) - \frac{x^i}{r^3} \times \frac{x_i}{r} \chi'(t-r) \right) = \frac{Q_F}{4\pi r^2} \chi'(t-r), \\ \nabla^\mu \bar{F}_{\mu i} &= -\partial_t \bar{F}_{0i} = -\frac{Q_F}{4\pi} \frac{x^i}{r^3} \chi'(t-r). \end{aligned}$$

For the estimates on the derivatives of \bar{J} , we refer to [36] (equations (3.52a) – (3.52c)). \square

4.6 Bootstrap assumptions and strategy of the proof

Let, for the remaining of this article, $N \in \mathbb{N}$ such that $N \geq 11$ and $M \in \mathbb{N}$ which will be fixed during the proof. Let also $0 < \eta < \frac{1}{16}$ and (f_0, F_0) be an initial data set satisfying the assumptions of Theorem 4.1.4. By a standard local well-posedness argument, there exists a unique maximal solution (f, F) of the Vlasov-Maxwell system defined on $[0, T^*]$, with $T^* \in \mathbb{R}_+^* \cup \{+\infty\}$. Let us now introduce the energy norms used for the analysis of the particle density.

Definition 4.6.1. *Let $Q \leq N$, $q \in \mathbb{N}$ and $a = M + 1$. For g a sufficiently regular function, we define the following energy norms,*

$$\begin{aligned} \mathbb{E}[g](t) &:= \|g\|_{L_{x,v}^1}(t) + \int_{C_u(t)} \int_v \frac{v^L}{v^0} |g| dv dC_u(t), \\ \mathbb{E}_Q^q[g](t) &:= \sum_{\substack{1 \leq i \leq 2 \\ z \in \mathbf{k}_1}} \sum_{\substack{|\xi^i| + |\beta| \leq Q \\ |\xi^i| \leq Q-1}} \sum_{j=0}^{2N-1+q-\xi_P^1-\xi_P^2-\beta_P} \log^{-(j+|\xi^1|+|\xi^2|+|\beta|)^a} (3+t) \mathbb{E} [z^j P_{\xi^1}(\Phi) P_{\xi^2}(\Phi) Y^\beta f](t), \\ \bar{\mathbb{E}}_N[g](t) &:= \sum_{\substack{1 \leq i \leq 2 \\ z \in \mathbf{k}_1}} \sum_{\substack{|\xi^i| + |\beta| \leq Q \\ |\xi^i| \leq Q-1}} \sum_{j=0}^{2N-1-\xi_P^1-\xi_P^2-\beta_P} \log^{-aj} (3+t) \mathbb{E} [z^j P_{\xi^1}(\Phi) P_{\xi^2}(\Phi) Y^\beta f](t), \\ \mathbb{E}_{N-1}^X[f](t) &:= \sum_{\substack{1 \leq i \leq 2 \\ z \in \mathbf{k}_1}} \sum_{|\zeta^i| + |\beta| \leq N-1} \sum_{j=0}^{2N-2-\zeta_P^1-\zeta_P^2-\beta_P} \log^{-2j} (3+t) \mathbb{E} [z^j P_{\zeta^1}^X(\Phi) P_{\zeta^2}^X(\Phi) Y^\beta f](t), \\ \mathbb{E}_N^X[f](t) &:= \sum_{z \in \mathbf{k}_1} \sum_{\substack{|\zeta| + |\beta| \leq N \\ |\zeta| \leq N-1}} \sum_{j=0}^{2N-2-\zeta_P-\beta_P} \log^{-2j} (3+t) \mathbb{E} [z^j P_\zeta^X(\Phi) Y^\beta f](t). \end{aligned}$$

To understand the presence of the logarithmical weights, see Remark 4.3.32.

In order to control the derivatives of the Φ coefficients and $\bar{\mathbb{E}}_N[f]$ at $t = 0$, we prove the following result.

Proposition 4.6.2. *Let $|\beta| \leq N - 1$ a multi index and $Y^\beta \in \mathbb{Y}^{|\beta|}$. Then, at $t = 0$,*

$$\begin{aligned} \max \left(|Y^\beta \Phi|, |\widehat{Z}^\beta \Phi| \right) &\lesssim \frac{1+r^2}{v^0} \sum_{|\gamma| \leq |\beta|-1} |\mathcal{L}_{Z^\gamma}(F)| \\ &\lesssim \frac{\sqrt{\epsilon}}{v^0}. \end{aligned}$$

Proof. Note that the second inequality ensues from

$$\sum_{|\gamma| \leq N-2} \|\mathcal{L}_{Z^\gamma}(F)\|_{L^\infty(\Sigma_0)} \lesssim \frac{\sqrt{\epsilon}}{1+r^2}, \quad (4.44)$$

which comes from Proposition 4.4.15. Let us now prove the first inequality. Without mention of the opposite (as in (4.48)), all functions considered here will be evaluated at $t = 0$. As $\Phi(0, \dots) = 0$, the result holds for $|\beta| = 0$. Let $1 \leq |\beta| \leq N - 1$ and suppose that the result holds for all $|\sigma| < |\beta|$. Note that, for instance,

$$Y_2 Y_1 \Phi = \widehat{Z}_2 \widehat{Z}_1 \Phi + \Phi X \widehat{Z}_1 \Phi + Y_2(\Phi) X \Phi + \Phi \widehat{Z}_2 X \Phi + \Phi \Phi X X \Phi.$$

More generally, we have,

$$|Y^\beta \Phi| \lesssim \sum_{\substack{p \leq |k| + |\sigma| \leq |\beta| \\ k < |\beta|}} P_{k,p}(\Phi) \widehat{Z}^\sigma \Phi. \quad (4.45)$$

Consequently, using the induction hypothesis, we only have to prove the result for $\widehat{Z}^\beta \Phi$. Indeed, as $|k| < |\beta|$, by (4.44),

$$|P_{k,p}(\Phi) \widehat{Z}^\sigma(\Phi)| \lesssim |\widehat{Z}^\sigma(\Phi)| \left| \frac{1+r^2}{v^0} \right|^p \sum_{|\gamma| \leq N-2} |\mathcal{L}_{Z^\gamma}(F)|^p \lesssim |\widehat{Z}^\sigma(\Phi)|. \quad (4.46)$$

Combining (4.45) and (4.46), we would then obtain the inequality on $|Y^\beta \Phi|$, if we would have it on $\widehat{Z}^\sigma \Phi$ for all $|\sigma| \leq |\beta|$. Let us then prove that the result holds for $\widehat{Z}^\beta \Phi$ and suppose, for simplicity, that $\Phi = \Phi_{\widehat{Z}}^k$, with $\widehat{Z} \neq S$. Remark that

$$|\widehat{Z}^\beta \Phi| \lesssim \sum_{|\alpha_2|+|\alpha_1|+q \leq |\beta|} (1+|x|)^{|\alpha_1|+q} (v^0)^{|\alpha_2|} |\partial_v^{\alpha_2} \partial_x^{\alpha_1} \partial_t^q \Phi|$$

and let us prove by induction on q that

$$\forall |\alpha_2| + |\alpha_1| + q \leq |\beta|, \quad (1+|x|)^{|\alpha_1|+q} (v^0)^{|\alpha_2|} |\partial_v^{\alpha_2} \partial_x^{\alpha_1} \partial_t^q \Phi| \lesssim \frac{1+r^2}{v^0} \sum_{|\gamma| \leq |\beta|-1} |\mathcal{L}_{Z^\gamma}(F)|. \quad (4.47)$$

Recall that for $t \in [0, T^*]$,

$$T_F(\Phi) = v^\mu \partial_\mu \Phi + F(v, \nabla_v \Phi) = -t \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k}. \quad (4.48)$$

As $\Phi(0, \dots) = 0$ and $v^0 \partial_t \Phi = -v^i \partial_i \Phi - F(v, \nabla_v \Phi)$, implying $\partial_t \Phi(0, \dots) = 0$, (4.47) holds for $q \leq 1$. Let $2 \leq q \leq |\beta|$ and suppose that (4.47) is satisfied for all $q_0 < q$. Let $|\alpha_2| + |\alpha_1| \leq |\beta| - q$. Using the commutation formula given by Lemma 4.3.10, we have (at $t = 0$),

$$v^0 \partial_x^{\alpha_1} \partial_t^q \Phi = -v^i \partial_i \partial_x^{\alpha_1} \partial_t^{q-1} \Phi - \frac{v^\mu}{v^0} \mathcal{L}_{\partial_x^{\alpha_1} \partial_t^{q-2} Z}(F)_{\mu k} + \sum_{|\gamma_1|+q_1+|\gamma_2|=|\alpha_1|+q-1} C_{\gamma_1, \gamma_2}^1 \mathcal{L}_{\partial^{\gamma_2}}(F)(v, \nabla_v \partial_x^{\gamma_1} \partial_t^{q_1} \Phi),$$

Dividing the previous equality by v^0 , taking the $\partial_v^{\alpha_2}$ derivatives of each side and using Lemma 4.3.6, we obtain

$$\begin{aligned} |\partial_v^{\alpha_2} \partial_x^{\alpha_1} \partial_t^q \Phi| &\lesssim \sum_{|\alpha_3| \leq |\alpha_2|} (v^0)^{-|\alpha_2|+|\alpha_3|} |\partial_v^{\alpha_3} \partial_x \partial_x^{\alpha_1} \partial_t^{q-1} \Phi| + \sum_{|\gamma| \leq |\alpha_1|+q-2} \frac{1}{(v^0)^{1+|\alpha_2|} (1+r)^{|\alpha_1|+q-2}} |\mathcal{L}_{Z^\gamma Z}(F)| \\ &+ \sum_{\substack{|\gamma_1|+q_1+n=|\alpha_1|+q-1 \\ 1 \leq |\alpha_4| \leq |\alpha_2|+1}} \sum_{|\gamma_2| \leq n} \frac{1}{(v^0)^{|\alpha_2|-|\alpha_4|+1} (1+r)^n} |\mathcal{L}_{Z^{\gamma_2}}(F)| |\partial_v^{\alpha_4} \partial_x^{\gamma_1} \partial_t^{q_1} \Phi|. \end{aligned}$$

It then remains to multiply both side of the inequality by $(v^0)^{|\alpha_2|} (1+r)^{|\alpha_1|+q}$ and

- To bound $(v^0)^{|\alpha_2|} (1+r)^{|\alpha_1|+q} (v^0)^{-|\alpha_2|+|\alpha_3|} |\partial_v^{\alpha_3} \partial_x \partial_x^{\alpha_1} \partial_t^{q-1} \Phi|$ with the induction hypothesis.
- To remark that $(v^0)^{|\alpha_2|} (1+r)^{|\alpha_1|+q} \frac{1}{(v^0)^{1+|\alpha_2|} (1+r)^{|\alpha_1|+q-2}} |\mathcal{L}_{Z^\gamma Z}(F)|$ has the desired form.
- To note that, using $|\gamma_1| + q_1 + 1 = |\alpha_1| + q - n$ and the induction hypothesis,

$$\begin{aligned} \frac{(v^0)^{|\alpha_2|} (1+r)^{|\alpha_1|+q}}{(v^0)^{|\alpha_2|-|\alpha_4|+1} (1+r)^n} |\partial_v^{\alpha_4} \partial_x^{\gamma_1} \partial_t^{q_1} \Phi| |\mathcal{L}_{Z^{\gamma_2}}(F)| &= \frac{1+r}{v^0} (v^0)^{|\alpha_4|} (1+r)^{|\gamma_1|+q_1} |\partial_v^{\alpha_4} \partial_x^{\gamma_1} \partial_t^{q_1} \Phi| |\mathcal{L}_{Z^{\gamma_2}}(F)| \\ &\lesssim \frac{1+r}{v^0} |\mathcal{L}_{Z^{\gamma_2}}(F)| \sum_{|\zeta| \leq |\alpha_4|+|\gamma_1|+q_1-1} \frac{(1+r)^2}{v^0} |\mathcal{L}_{Z^\zeta}(F)| \\ &\lesssim \sum_{|\zeta| \leq |\alpha_2|+|\alpha_1|+q-1} \frac{(1+r)^2}{v^0} |\mathcal{L}_{Z^\zeta}(F)|, \end{aligned}$$

since $|\mathcal{L}_{Z^{\gamma_2}}(F)| \lesssim (1+r)^{-2}$, as $|\gamma_2| \leq |\alpha_1| + q - 1 \leq |\beta| - 1 \leq N - 2$. This concludes the proof of the Proposition. \square

Corollary 4.6.3. *There exists $\tilde{C} > 0$ a constant depending only on N such that $\mathbb{E}_N^4[f](0) \leq \tilde{C}\epsilon = \tilde{\epsilon}$. Without loss of generality and in order to lighten the notations, we suppose that $\mathbb{E}_N^4[f](0) \leq \epsilon$.*

Proof. All the functions considered here are evaluated at $t = 0$. Consider multi-indices ξ_1, ξ_2 and β such that, for $i \in \{1, 2\}$, $\max(|\xi^i| + 1, |\xi^i| + |\beta|) \leq N$ and $j \leq 2N + 3 - \xi_P^1 - \xi_P^2 - \beta_P$. Then,

$$|z^j P_{\xi_1}(\Phi) P_{\xi_2}(\Phi) Y^\beta f| \leq \left| z^j P_{\xi_1}(\Phi) P_{\xi_2}(\Phi) \widehat{Z}^\beta f \right| + \sum_{\substack{|k|+|\kappa| \leq |\beta| \\ |k| \leq |\beta|-1 \\ p+k_P+\kappa_P < \beta_P}} \left| z^j P_{\xi_1}(\Phi) P_{\xi_2}(\Phi) P_{k, p}(\Phi) \widehat{Z}^\kappa f \right|.$$

Using the previous proposition and the assumptions on f_0 , it comes that, with $C_1 > 0$ a constant,

$$\mathbb{E}_N^4[f](0) \leq (1 + C_1\sqrt{\epsilon}) \sum_{\substack{\widehat{Z}^\beta \in \mathbb{P}_0^{|\beta|} \\ |\beta| \leq N}} \|z^{2N+3-\beta_P} \widehat{Z}^\beta f\|_{L_{x,v}^1}(0).$$

By similar computations than in Appendix B of [6], we can bound the right hand side of the last inequality by $\widetilde{C}\epsilon$ using the smallness hypothesis on (f_0, F_0) . \square

By a continuity argument and the previous corollary, there exists a largest time $T \in]0, T^*[$ such that, for all $t \in [0, T[$,

$$\mathbb{E}_{N-3}^4[f](t) \leq 4\epsilon, \quad (4.49)$$

$$\mathbb{E}_{N-1}^0[f](t) \leq 4\epsilon, \quad (4.50)$$

$$\overline{\mathbb{E}}_N[f](t) \leq 4\epsilon(1+t)^\eta, \quad (4.51)$$

$$\sum_{|\beta| \leq N-2} \left\| r^{\frac{3}{2}} \int_v \frac{v^A}{v^0} \widehat{Z}^\beta f dv \right\|_{L^2(\Sigma_t)} \leq \sqrt{\epsilon}, \quad (4.52)$$

$$\mathcal{E}_N^0[F](t) \leq 4\epsilon, \quad (4.53)$$

$$\mathcal{E}_N^{Ext}[\widehat{F}](t) \leq 8\epsilon, \quad (4.54)$$

$$\mathcal{E}_{N-3}[F](t) \leq 30\epsilon \log^2(3+t), \quad (4.55)$$

$$\mathcal{E}_{N-1}[F](t) \leq 30\epsilon \log^{2M}(3+t), \quad (4.56)$$

$$\mathcal{E}_N[F](t) \leq 30\epsilon(1+t)^\eta. \quad (4.57)$$

The remaining of the proof will then consist in improving our bootstrap assumptions, which will prove that (f, F) is a global solution to the 3d massive Vlasov-Maxwell system. The other points of the theorem will be obtained during the proof, which is divided in four main parts.

1. First, we will obtain pointwise decay estimates on the particle density, the electromagnetic field and then on the derivatives of the Φ coefficients, using the bootstrap assumptions.
2. Then, we will improve the bootstrap assumptions (4.49), (4.50) and (4.51) by several applications of the energy estimate of Proposition 4.4.1 and the commutation formula of Proposition 4.3.31. The computations will also lead to optimal pointwise decay estimates on $\int_v |Y^\beta f| \frac{dv}{(v^0)^2}$.
3. The next step consists in proving enough decay on the L^2 norms of $\int_v |zY^\beta f| dv$, which will permit us to improve the bootstrap assumption (4.52).
4. Finally, we will improve the bootstrap assumptions (4.53)-(4.57) by using the energy estimates of Proposition 4.4.3.

4.7 Immediate consequences of the bootstrap assumptions

In this section, we prove pointwise estimates on the Maxwell field, the Φ coefficients and the Vlasov field. We start with the electromagnetic field.

Proposition 4.7.1. *We have, for all $|\gamma| \leq N-3$ and $(t, x) \in [0, T[\times \mathbb{R}^3$,*

$$\begin{aligned} |\alpha(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log^M(3+t)}{\tau_+^2}, & |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \min\left(\frac{1}{\tau_+ \tau_-^{\frac{1}{2}}}, \frac{\log^M(3+t)}{\tau_+ \tau_-}\right), \\ |\sigma(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log^M(3+t)}{\tau_+^{\frac{3}{2}} \tau_-^{\frac{1}{2}}}, & |\rho(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \sqrt{\epsilon} \frac{\log^M(3+t)}{\tau_+^{\frac{3}{2}} \tau_-^{\frac{1}{2}}}. \end{aligned}$$

Moreover, if $|x| \geq t$,

$$\begin{aligned} |\alpha(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \frac{\sqrt{\epsilon}}{\tau_+^2}, & |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \frac{\sqrt{\epsilon}}{\tau_+ \tau_-}, \\ |\sigma(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \frac{\sqrt{\epsilon}}{\tau_+^{\frac{3}{2}} \tau_-^{\frac{1}{2}}}, & |\rho(\mathcal{L}_{Z^\gamma}(F))|(t, x) &\lesssim \frac{\sqrt{\epsilon}}{\tau_+^{\frac{3}{2}} \tau_-^{\frac{1}{2}}}. \end{aligned}$$

We also have

$$\forall (t, x) \in [0, T] \times \mathbb{R}^3, \quad \sum_{|\kappa| \leq N} |\mathcal{L}_{Z^\kappa}(\bar{F})|(t, x) \lesssim \frac{\epsilon}{\tau_+^2}.$$

Remark 4.7.2. If $|\gamma| \leq N - 5$, we can replace the $\log^M(3+t)$ -loss in the interior of the lightcone by a $\log(3+t)$ -loss.

Proof. The last estimate, concerning \bar{F} , ensues from Proposition 4.5.1 and $|Q_F| \leq \|f_0\|_{L_{x,v}^1} \leq \epsilon$. The estimate $\tau_+ \sqrt{\tau_-} |\underline{\alpha}| \lesssim \sqrt{\epsilon}$ follows from Proposition 4.4.15 and the bootstrap assumption (4.53). Note that the other estimates hold with F replaced by \tilde{F} since $\mathcal{E}_{N-1}[F] = \mathcal{E}_{N-1}[\tilde{F}]$ and according to Proposition 4.4.15 and the bootstrap assumptions (4.54), (4.56) and (4.52). It then remains to use $F = \tilde{F} + \bar{F}$ and the estimates obtained on \bar{F} and \tilde{F} . \square

We now turn on the Φ coefficients and start by the following lemma.

Lemma 4.7.3. Let $G, G_1, G_2 : [0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ and $\varphi_0 : \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ be four sufficiently regular functions such that $|G| \leq G_1 + G_2$. Let $\varphi, \tilde{\varphi}, \varphi_1$ and φ_2 be such that

$$T_F(\varphi) = G, \quad \varphi(0, \cdot, \cdot) = \varphi_0, \quad T_F(\tilde{\varphi}) = 0, \quad \tilde{\varphi}(0, \cdot, \cdot) = \varphi_0$$

and, for $i \in \{1, 2\}$,

$$T_F(\varphi_i) = G_i, \quad \varphi_i(0, \cdot, \cdot) = 0.$$

Then, on $[0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$,

$$|\varphi| \leq |\tilde{\varphi}| + |\varphi_1| + |\varphi_2|.$$

Proof. Denoting by $X(s, t, x, v)$ and $V(s, t, x, v)$ the characteristics of the transport operator, we have by Duhamel's formula,

$$\begin{aligned} |\varphi|(t, x, v) &= \left| \tilde{\varphi}(t, x, v) + \int_0^t \frac{G}{v^0}(s, X(s, t, x, v), V(s, t, x, v)) ds \right| \\ &\leq |\tilde{\varphi}|(t, x, v) + \int_0^t \frac{G_1 + G_2}{v^0}(s, X(s, t, x, v), V(s, t, x, v)) ds \\ &= |\tilde{\varphi}|(t, x, v) + |\varphi_1|(t, x, v) + |\varphi_2|(t, x, v). \end{aligned}$$

\square

Proposition 4.7.4. We have, $\forall (t, x, v) \in [0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$

$$|\Phi|(t, x, v) \lesssim \sqrt{\epsilon} \log^2(1 + \tau_+), \quad |\partial_{t,x}\Phi|(t, x, v) \lesssim \sqrt{\epsilon} \log^{\frac{3}{2}}(1 + \tau_+) \quad \text{and} \quad |Y\Phi|(t, x, v) \lesssim \sqrt{\epsilon} \log^{\frac{7}{2}}(1 + \tau_+).$$

Proof. We will obtain this result through the previous Lemma and by parameterizing the characteristics of the operator T_F by t or by u . Let us start by Φ and recall that, schematically, $T_F(\Phi) = -t \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k}$. Denoting by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of $\mathcal{L}_Z(F)$ and using $|v^A| \lesssim \sqrt{v^0 v^{\underline{L}}}$ (see Lemma 4.2.4), we have

$$\begin{aligned} \left| \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k} \right| &\lesssim \frac{v^{\underline{L}} + |v^A|}{v^0} |\alpha| + \frac{v^{\underline{L}} + v^{\underline{L}}}{v^0} |\rho| + \frac{|v^A|}{v^0} |\sigma| + \frac{v^{\underline{L}} + |v^A|}{v^0} |\underline{\alpha}| \\ &\lesssim |\alpha| + |\rho| + |\sigma| + \sqrt{\frac{v^{\underline{L}}}{v^0}} |\underline{\alpha}|. \end{aligned}$$

Using the pointwise estimates given by Remark 4.7.2 as well as the inequalities $1 \lesssim \sqrt{v^0 v^{\underline{L}}}$, which comes from Lemma 4.2.4, and $2ab \leq a^2 + b^2$, we get

$$\tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k} \right| \lesssim \sqrt{\frac{\epsilon v^0 v^{\underline{L}}}{\tau_+ \tau_-}} \log(3+t) + \frac{v^{\underline{L}} \sqrt{\epsilon}}{\tau_-} \log(3+t) \lesssim \frac{v^0 \sqrt{\epsilon}}{\tau_+} \log(3+t) + \frac{v^{\underline{L}} \sqrt{\epsilon}}{\tau_-} \log(3+t). \quad (4.58)$$

Consider now the functions φ_1 and φ_2 such that

$$T_F(\varphi_1) = \frac{v^0 \sqrt{\epsilon}}{\tau_+} \log(3+t), \quad T_F(\varphi_2) = \frac{v^{\underline{L}} \sqrt{\epsilon}}{\tau_-} \log(3+t) \quad \text{and} \quad \varphi_1(0, \cdot, \cdot) = \varphi_2(0, \cdot, \cdot) = 0.$$

According to Lemma 4.7.3, we have $|\Phi| \lesssim |\varphi_1| + |\varphi_2|$. In order to estimate φ_1 , we will parametrize the characteristics of the operator T_F by t . More precisely, let $(X_{s,y,v}(t), V_{s,y,v}(t))$ be the value in t of the characteristic which is equal to (y, v) in $t = s$, with $s < T$. Dropping the indices s, y and v , we have

$$\frac{dX^i}{dt}(t) = \frac{V^i(t)}{V^0(t)} \quad \text{and} \quad \frac{dV^i}{dt}(t) = \frac{V^\mu(t)}{V^0(t)} F_\mu^i(t, X(t)).$$

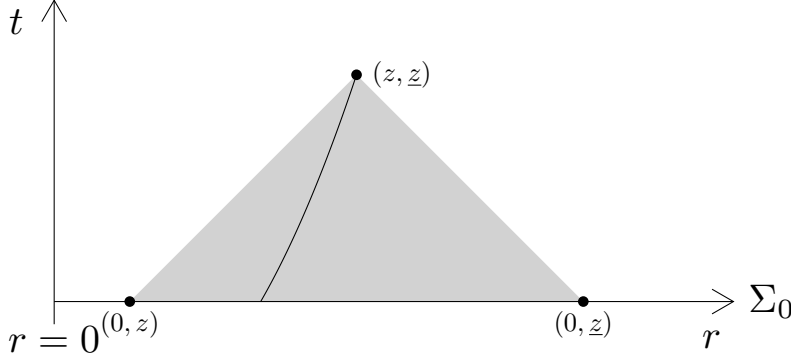
Duhamel's formula gives

$$|\varphi_1|(s, y, v) \lesssim \sqrt{\epsilon} \int_0^s \frac{\log(3+t)}{\tau_+(t, X_{t,y,v}(t))} ds \leq \sqrt{\epsilon} \int_0^s \frac{\log(3+t)}{1+t} ds \leq \sqrt{\epsilon} \log^2(3+s).$$

For φ_2 , we parameterize the characteristics of T_F by²⁰ u . For a point $(s, y) \in [0, T] \times \mathbb{R}^3$, we will write its coordinates in the null fram as $(z, \underline{z}, \omega_1, \omega_2)$. Let $(\underline{U}_{z,\underline{z},\omega_1,\omega_2,v}(u), \Omega_{z,\underline{z},\omega_1,\omega_2,v}^1(u), \Omega_{z,\underline{z},\omega_1,\omega_2,v}^2(u), V_{z,\underline{z},\omega_1,\omega_2,v}(u))$ be the value in u of the characteristic which is equal to $(s, y, v) = (z, \underline{z}, \omega_1, \omega_2, v)$ in $u = z$. Dropping the indices $z, \underline{z}, \omega_1, \omega_2$ and v , we have

$$\frac{d\underline{U}}{du}(u) = \frac{V^L(u)}{V^{\underline{L}}(u)}, \quad \frac{d\Omega^A}{du}(u) = \frac{V^A(u)}{2V^{\underline{L}}(u)} \quad \text{and} \quad \frac{dV^i}{du}(u) = \frac{V^\mu(u)}{2V^{\underline{L}}(u)} F_\mu^i(u, \underline{U}(u), \Omega(u)).$$

Note that $u \mapsto \frac{1}{2}(u + \underline{U}(u))$ vanishes in a unique z_0 such that $-\underline{z} \leq z_0 \leq z$, i.e. the characteristic reaches the hypersurface Σ_0 once and only once, at $u = z_0$. This can be noticed on the following picture, representing a possible trajectory of $(u, \underline{U}(u))$, which has to be in the backward light cone of (z, \underline{z}) by finite time of propagation,



The trajectory of $(u, \underline{U}(u))$ for $u \leq z$.

or by noticing that

$$g(u) := u + \underline{U}(u) \quad \text{satisfies} \quad g'(u) \geq 1 + \frac{V^L(u)}{V^{\underline{L}}(u)} \geq 1$$

so that g vanishes in z_0 such that $-\underline{z} = z - (z + \underline{z}) \leq z_0 \leq z$. Similarly, one can prove (or observe) that $\sup_{z_0 \leq u \leq z} \underline{U}(u) \leq \underline{z}$. It then comes that

$$|\varphi_2|(s, y, v) \lesssim \sqrt{\epsilon} \int_{z_0}^z \frac{\log(3 + \underline{U}(u))}{\tau_-(u, \underline{U}(u))} du \lesssim \sqrt{\epsilon} \log(3 + \underline{z}) \int_{-\underline{z}}^z \frac{1}{1 + |u|} du \lesssim \sqrt{\epsilon} \log^2(1 + \underline{z}), \quad (4.59)$$

which allows us to deduce that $|\Phi|(s, y, v) \lesssim \sqrt{\epsilon} \log^2(3 + s + |y|)$. We prove the other estimates by the continuity method. Let $0 < T_0 < T$ and $\underline{u} > 0$ be the largest time and null ingoing coordinate such that

$$|\nabla_{t,x} \Phi|(t, x, v) \leq C \sqrt{\epsilon} \log^{\frac{3}{2}}(1 + \tau_+) \quad \text{and} \quad \sum_{Y \in \mathbb{Y}_0} |Y \Phi|(t, x, v) \leq C \sqrt{\epsilon} \log^{\frac{7}{2}}(1 + \tau_+) \quad (4.60)$$

hold for all $(t, x, v) \in \underline{V}_{\underline{u}}(T_0) \times \mathbb{R}_v^3$ and where the constant $C > 0$ will be specified below. The goal now is to improve the estimates of (4.60). Using the commutation formula of Lemma 4.3.10 and the definition of Φ , we have (in the case where Φ is not associated to the scaling vector field), for $\partial \in \mathbf{T}$,

$$T_F(\partial \Phi) = -\mathcal{L}_\partial(F)(v, \nabla_v \Phi) - \partial \left(t \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k} \right).$$

²⁰Note that $T_F = 2v^{\underline{L}}\partial_u + 2v^L\partial_{\underline{u}} + v^A e_A + F(v, \nabla_v)$

With $\delta = \partial(t) \in \{0, 1\}$, one has

$$\partial \left(t \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k} \right) = \delta \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k} + t \frac{v^\mu}{v^0} \mathcal{L}_{\partial Z}(F)_{\mu k}.$$

Using successively the inequality (4.15), the pointwise decay estimates²¹ given by Remark 4.7.2 and the inequalities $1 \lesssim \sqrt{v^0 v^L}$, $2ab \leq a^2 + b^2$, it comes

$$\begin{aligned} t \frac{v^\mu}{v^0} \mathcal{L}_{\partial Z}(F)_{\mu k} &\lesssim \tau_+ \left(|\alpha(\mathcal{L}_{\partial Z}(F))| + |\rho(\mathcal{L}_{\partial Z}(F))| + |\sigma(\mathcal{L}_{\partial Z}(F))| + \sqrt{\frac{v^L}{v^0}} |\underline{\alpha}(\mathcal{L}_{\partial Z}(F))| \right) \\ &\lesssim \frac{\tau_+}{\tau_-} \sqrt{v^0 v^L} \sum_{|\beta| \leq 2} \left(\frac{\tau_-}{\tau_+} |\mathcal{L}_{Z^\beta}(F)| + |\alpha(\mathcal{L}_{Z^\beta}(F))| + |\rho(\mathcal{L}_{Z^\beta}(F))| + |\sigma(\mathcal{L}_{Z^\beta}(F))| \right) \\ &\quad + \sqrt{\frac{v^L}{v^0}} |\underline{\alpha}(\mathcal{L}_{Z^\beta}(F))| \\ &\lesssim \frac{\sqrt{v^0 v^L} \sqrt{\epsilon} \log(3+t)}{\tau_+^{\frac{1}{2}} \tau_-^{\frac{3}{2}}} + v^L \frac{\tau_+}{\tau_-} \frac{\sqrt{\epsilon}}{\tau_+ \tau_-^{\frac{1}{2}}} \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+} \log^{\frac{1}{2}}(3+t) + \sqrt{\epsilon} \frac{v^L}{\tau_-^{\frac{3}{2}}} \log^{\frac{3}{2}}(3+t) \end{aligned} \quad (4.61)$$

Similarly,

$$\begin{aligned} \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k} &\lesssim \left(|\alpha(\mathcal{L}_Z(F))| + |\rho(\mathcal{L}_Z(F))| + |\sigma(\mathcal{L}_Z(F))| + \sqrt{\frac{v^L}{v^0}} |\underline{\alpha}(\mathcal{L}_Z(F))| \right) \\ &\lesssim \frac{\sqrt{\epsilon} \log(3+t)}{\tau_+^{\frac{3}{2}} \tau_-^{\frac{1}{2}}} + v^L \frac{\sqrt{\epsilon}}{\tau_+ \tau_-^{\frac{1}{2}}} \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+^{\frac{5}{4}}} + \sqrt{\epsilon} \frac{v^L}{\tau_-^{\frac{5}{4}}}. \end{aligned} \quad (4.62)$$

Expressing $\mathcal{L}_\partial(F)(v, \nabla_v \Phi)$ in null components, denoting by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of $\mathcal{L}_\partial(F)$ and using the inequalities $|v^A| \lesssim \sqrt{v^0 v^L}$, $1 \lesssim \sqrt{v^0 v^L}$ (see Lemma 4.2.4), one has

$$|\mathcal{L}_\partial(F)(v, \nabla_v \Phi)| \lesssim \sqrt{v^0 v^L} |\rho| |(\nabla_v \Phi)^r| + \left(\sqrt{v^0 v^L} |\alpha| + v^L |\underline{\alpha}| + v^L |\sigma| \right) |\nabla_v \Phi|. \quad (4.63)$$

Using Lemma 4.3.27, $v^0 \partial_{v^i} = Y_i - \Phi X - t \partial_i - x^i \partial_t$ and the bootstrap assumption on the Φ coefficients (4.60), it comes

$$\begin{aligned} |(\nabla_v \Phi)^r| &\lesssim \sum_{Y \in \mathbb{Y}_0} |Y\Phi| + |\Phi| |X(\Phi)| + \tau_- |\nabla_{t,x} \Phi| \lesssim C \sqrt{\epsilon} \log^{\frac{7}{2}}(1 + \tau_+) + C \sqrt{\epsilon} \tau_- \log^{\frac{3}{2}}(1 + \tau_+), \\ |\nabla_v \Phi| &\lesssim \sum_{Y \in \mathbb{Y}_0} |Y\Phi| + |\Phi| |X(\Phi)| + \tau_+ |\nabla_{t,x} \Phi| \lesssim C \sqrt{\epsilon} \log^{\frac{7}{2}}(1 + \tau_+) + C \sqrt{\epsilon} \tau_+ \log^{\frac{3}{2}}(1 + \tau_+). \end{aligned}$$

We then deduce, by (4.15) and the pointwise estimates given by Remark 4.7.2,

$$\begin{aligned} \sqrt{v^0 v^L} |\rho| |(\nabla_v \Phi)^r| + \sqrt{v^0 v^L} |\alpha| |\nabla_v \Phi| &\lesssim C \epsilon \frac{\sqrt{v^0 v^L}}{\tau_+ \tau_-} \log^{\frac{5}{2}}(1 + \tau_+) \lesssim C \epsilon \frac{v^0}{\tau_+^{\frac{3}{2}}} + C \epsilon \frac{v^L}{\tau_-^2}, \\ (v^L |\underline{\alpha}| + v^L |\sigma|) |\nabla_v \Phi| &\lesssim C \epsilon \frac{v^L}{\tau_-^{\frac{3}{2}}} \log^{\frac{3}{2}}(1 + \tau_+). \end{aligned}$$

Combining these two last estimates with (4.61) and (4.62), we get

$$|T_F(\partial\Phi)| \lesssim (\sqrt{\epsilon} + C\epsilon) \frac{v^0}{\tau_+} \log^{\frac{1}{2}}(1 + \tau_+) + (\sqrt{\epsilon} + C\epsilon) \frac{v^L}{\tau_-^{\frac{4}{3}}} \log^{\frac{3}{2}}(1 + \tau_+).$$

We then split $\partial\Phi$ in three functions $\tilde{\psi} + \psi_1 + \psi_2$ such that $\psi_1(0, \dots) = \psi_2(0, \dots) = 0$, $\tilde{\psi}(0, \dots) = \partial\Phi(0, \dots)$,

$$T_F(\psi_1) = (\sqrt{\epsilon} + C\epsilon) \frac{v^0}{\tau_+} \log^{\frac{1}{2}}(1 + \tau_+), \quad T_F(\psi_2) = (\sqrt{\epsilon} + C\epsilon) \frac{v^L}{\tau_-^{\frac{4}{3}}} \log^{\frac{3}{2}}(1 + \tau_+) \quad \text{and} \quad T_F(\tilde{\psi}) = 0.$$

²¹Note that we use the estimate $|\underline{\alpha}| \lesssim \sqrt{\epsilon} \tau_+^{-1} \tau_-^{-\frac{1}{2}}$ here in order to obtain a decay rate of τ_+^{-1} in the $t+r$ direction.

According to Proposition 4.6.2, we have $\|\tilde{\psi}\|_{L_{t,x,v}^\infty} = \|\partial\Phi(0, \cdot, \cdot)\|_{L_{x,v}^\infty} \lesssim \sqrt{\epsilon}$. Fix now $(s, y, v) \in \underline{V}_u(T_0) \times \mathbb{R}_v^3$ and let $(z, \underline{z}, \omega_1, \omega_2)$ be the coordinates of (s, y) in the null frame. Keeping the notations used previously in this proof, we have

$$\begin{aligned} |\psi_1|(s, y, v) &\lesssim (\sqrt{\epsilon} + C\epsilon) \int_0^s \frac{\log^{\frac{1}{2}}(1 + \tau_+(t, X(t)))}{\tau_+(t, X(t))} dt \\ &\lesssim (\sqrt{\epsilon} + C\epsilon) \int_0^s \frac{\log^{\frac{1}{2}}(3+t)}{1+t} dt \lesssim (\sqrt{\epsilon} + C\epsilon) \log^{\frac{3}{2}}(3+t), \end{aligned} \quad (4.64)$$

$$\begin{aligned} |\psi_2|(s, y, v) &\lesssim (\sqrt{\epsilon} + C\epsilon) \int_{z_0}^z \frac{\log^{\frac{3}{2}}(1 + \tau_+(u, \underline{U}(u)))}{\tau_-^{\frac{5}{4}}(u, \underline{U}(u))} du \\ &\lesssim (\sqrt{\epsilon} + C\epsilon) \log^{\frac{3}{2}}(3+z) \int_{-z}^z \frac{1}{(1+|u|)^{\frac{5}{4}}} du \lesssim (\sqrt{\epsilon} + C\epsilon) \log^{\frac{3}{2}}(3+z). \end{aligned} \quad (4.65)$$

Thus, there exists $C_1 > 0$ such that

$$\forall (s, y, v) \in \underline{V}_u(T_0) \times \mathbb{R}_v^3, \quad |\nabla_{t,x}\Phi|(s, y, v) \leq C_1(\sqrt{\epsilon} + C\epsilon) \log^{\frac{3}{2}}(1 + \tau_+(s, y))$$

and we can then improve the bootstrap assumption on $\nabla_{t,x}\Phi$ if C is chosen large enough and ϵ small enough. It remains to study $Y\Phi$ with $Y \in \mathbb{Y}_0$. Using Lemma 4.3.19, $T_F(Y\Phi)$ can be bounded by a linear combination of terms of the form

$$\left| \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k} Y\Phi \right|, \quad \tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k} \partial_{t,x}\Phi \right|, \quad |\Phi \mathcal{L}_\partial(F)(v, \nabla_v \Phi)| \quad \text{and} \quad \left| Y \left(t \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k} \right) \right|.$$

Using the bootstrap assumption (4.60) in order to estimate $|Y\Phi|$ and reasoning as for (4.62), one obtains

$$\left| \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k} Y\Phi \right| \lesssim C\epsilon \frac{v^0}{\tau_+^{\frac{5}{4}}} + C\epsilon \frac{v^L}{\tau_-^{\frac{5}{4}}}.$$

Bounding $|\partial_{t,x}\Phi|$ with the bootstrap assumption (4.60) and using the inequality (4.58), it comes

$$\tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k} \partial\Phi \right| \lesssim C\epsilon \frac{v^0}{\tau_+} \log^{\frac{5}{2}}(1 + \tau_+) + C\epsilon \frac{v^L}{\tau_-} \log^{\frac{5}{2}}(1 + \tau_+).$$

As $|\Phi| \lesssim \sqrt{\epsilon} \log^2(1 + \tau_+)$, we get, using the bound obtained on the left hand side of (4.63),

$$\Phi \mathcal{L}_\partial(F)(v, \nabla_v \Phi) \lesssim C\epsilon \frac{v^0}{\tau_+^{\frac{3}{2}}} \log^2(1 + \tau_+) + C\epsilon \frac{v^L}{\tau_-^{\frac{3}{2}}} \log^{\frac{7}{2}}(1 + \tau_+).$$

For the remaining term, one has schematically, by the first equality of Lemma 4.3.22,

$$\left| Y \left(t \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu k} \right) \right| \lesssim (\tau_+ + |\Phi|) \left| \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu\theta} \right| + \tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{ZZ}(F)_{\mu k} \right| + \tau_+ |\Phi| \left| \frac{v^\mu}{v^0} \mathcal{L}_{\partial Z}(F)_{\mu k} \right|.$$

Using $|\Phi| \lesssim \log^2(1 + \tau_+) \leq \tau_+$ and following (4.58), we get

$$(\tau_+ + |\Phi|) \left| \frac{v^\mu}{v^0} \mathcal{L}_Z(F)_{\mu\theta} \right| + \tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{ZZ}(F)_{\mu k} \right| \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+} \log(1 + \tau_+) + \sqrt{\epsilon} \frac{v^L}{\tau_-} \log(1 + \tau_+).$$

Combining (4.61) with $|\Phi| \lesssim \log^2(1 + \tau_+)$, we obtain

$$\tau_+ |\Phi| \left| \frac{v^\mu}{v^0} \mathcal{L}_{\partial Z}(F)_{\mu k} \right| \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+} \log^{\frac{5}{2}}(1 + \tau_+) + \sqrt{\epsilon} \frac{v^L}{\tau_-^{\frac{3}{2}}} \log^{\frac{7}{2}}(1 + \tau_+).$$

Consequently, it comes

$$|T_F(Y\Phi)| \lesssim (\sqrt{\epsilon} + C\epsilon) \frac{v^0}{\tau_+} \log^{\frac{5}{2}}(1 + \tau_+) + (\sqrt{\epsilon} + C\epsilon) \frac{v^L}{\tau_-^{\frac{5}{4}}} \log^{\frac{7}{2}}(1 + \tau_+) + (\sqrt{\epsilon} + C\epsilon) \frac{v^L}{\tau_-} \log^{\frac{5}{2}}(1 + \tau_+).$$

One can then split $Y\Phi$ in three functions $\tilde{\zeta}$, ζ_1 and ζ_2 defined as $\tilde{\psi}$, ψ_1 and ψ_2 previously. We have $\|\tilde{\zeta}\|_{L_{t,x,v}^\infty} \lesssim \sqrt{\epsilon}$ since $\|Y\Phi\|_{L_{x,v}^\infty}(0) \lesssim \sqrt{\epsilon}$ (see Proposition 4.6.2) and we can obtain $|\zeta_1| + |\zeta_2| \lesssim (\sqrt{\epsilon} + C\epsilon) \log^{\frac{7}{2}}(1 + \tau_+)$ by similar computations as those of (4.64), (4.65) and (4.59). So, taking C large enough and ϵ small enough, we can improve the bootstrap assumption on $Y\Phi$ and conclude the proof. \square

For the higher order derivatives, we have the following result.

Proposition 4.7.5. *For all $(Q_1, Q_2) \in \llbracket 0, N-4 \rrbracket^2$ satisfying $Q_2 \leq Q_1$, there exists $R(Q_1, Q_2) \in \mathbb{N}$ such that*

$$\forall |\beta| \leq N-4, \quad (t, x) \in [0, T[\times \mathbb{R}^3, \quad |Y^\beta \Phi|(t, x) \lesssim \sqrt{\epsilon} \log^{R(|\beta|, \beta_P)}(1 + \tau_+).$$

Note that $R(Q_1, Q_2)$ is independent of M if $Q_1 \leq N-6$.

Proof. The proof is similar to the previous one and we only sketch it. We proceed by induction on Q_1 and, at Q_1 fixed, we make an induction on Q_2 . Let $|\beta| \leq N-4$ and suppose that the result holds for all $Q_1 \leq |\beta|$ and $Q_2 \leq \beta_P$ satisfying $Q_1 < |\beta|$ or $Q_2 < \beta_P$. Let $0 < T_0 < T$ and $\underline{v} > 0$ be such that

$$\forall (t, x, v) \in \underline{V}_{\underline{v}}(T_0) \times \mathbb{R}_v^3, \quad |Y^\beta \Phi|(t, x, v) \leq C \sqrt{\epsilon} \log^{R(|\beta|, \beta_P)}(1 + \tau_+),$$

with $C > 0$ a constant sufficiently large. We now sketch the improvement of this bootstrap assumption, which will imply the desired result. The source terms of $T_F(Y^\beta \Phi)$, given by Propositions 4.3.23 and 4.3.25, can be gathered in two categories.

- The ones where there is no Φ coefficient derived more than $|\beta| - 1$ times, which can then be bounded by the induction hypothesis and give logarithmical growths, as in the proof of the previous Proposition. We then choose $R(|\beta|, \beta_P)$ sufficiently large to fit with these growths.
- The ones where a Φ coefficient is derived $|\beta|$ times. Note then that they all come from Proposition 4.3.23, when $|\sigma| = |\beta|$ for the quantities of (type 1- β) and when $|\sigma| = |\beta| - 1$ for the other ones. We then focus on the most problematic ones (with a τ_+ or τ_- weight, which can come from a weight $z \in \mathbf{k}_1$ for the terms of (type 1- β)), leading us to integrate along the characteristics of T_F the following expressions.

$$\tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} Y^\kappa \Phi \right|, \quad \text{with} \quad |\gamma| \leq N-3, \quad |\kappa| = |\beta| \quad \text{and} \quad \kappa_P < \beta_P, \quad (4.66)$$

$$|\Phi^p \mathcal{L}_{\partial Z^\gamma}(F)(v, \Gamma^\kappa \Phi)|, \quad \text{with} \quad |\gamma_0| \leq N-4, \quad |\kappa| = |\beta| - 1 \quad \text{and} \quad p + \kappa_P \leq \beta_P. \quad (4.67)$$

To deal with (4.66), use the induction hypothesis, as $\kappa_P < \beta_P$. For the other terms, recall from Lemma 4.3.30 that we can schematically suppose that

$$\Gamma^\kappa \Phi = P_{q,n}(\Phi) Y^\zeta \Phi, \quad \text{with} \quad |q| + |\zeta| \leq |\beta| - 1, \quad |q| \leq |\beta| - 2 \quad \text{and} \quad n + q_P + \zeta_P = \kappa_P.$$

Expressing (4.67) in null coordinates and transforming the v derivatives with Lemma 4.3.27 or $v^0 \partial_{v^i} = Y_i - \Phi X - x^i \partial_t - t \partial_i$, we obtain the following bad terms,

$$\left(\tau_- |\rho| + \tau_+ |\alpha| + \tau_+ \sqrt{\frac{v \underline{L}}{v^0}} (|\sigma| + |\underline{\alpha}|) \right) \Phi^p \partial_{t,x} (P_{q,n}(\Phi) Y^\zeta \Phi).$$

Then, note that there is no derivatives of order $|\beta|$ in $\Phi^p \partial_{t,x} (P_{q,n}(\Phi)) Y^\zeta \Phi$ so that these terms can be handled using the induction hypothesis. It then remains to study the terms related to $P_{q,n+p}(\Phi) \partial_{t,x} Y^\zeta \Phi$. If $\zeta_P < \beta_P$, we can treat them using again the induction hypothesis. Otherwise $p+n=0$ and we can follow the treatment of (4.63). Finally, the fact that $R(|\beta|, \beta_P)$ is independent of M if $|\beta| \leq N-6$ follows from Remark 4.7.2 and that we merely need pointwise estimates on the derivatives of F up to order $N-5$ in order to bound $Y^\zeta \Phi$, with $|\zeta| \leq N-6$. \square

Remark 4.7.6. *There exist $(M_1, M_2) \in \mathbb{N}^2$, with M_1 independent of M , such that, for all $p \leq 3N$ and $(t, x, v) \in [0, T[\times \mathbb{R}^3 \times \mathbb{R}^3$,*

$$\sum_{|k| \leq N-6} |P_{k,p}(\Phi)|(t, x, v) \lesssim \log^{M_1}(1 + \tau_+) \quad \text{and} \quad \sum_{|k| \leq N-4} |P_{k,p}(\Phi)|(t, x, v) \lesssim \log^{M_2}(1 + \tau_+).$$

We are now able to apply the Klainerman-Sobolev inequalities of Proposition 4.4.5 and Corollary 4.4.7. Combined with the bootstrap assumptions (4.49), (4.51) and the estimates on the Φ coefficients, one immediately obtains that, for all $(t, x) \in [0, T[\times \mathbb{R}^3$ and $z \in \mathbf{k}_1$,

$$\forall \max(|\xi| + |\beta|, |\xi| + 1) \leq N-6, \quad j \leq 2N - \xi_P - \beta_P, \quad \int_v |z^j P_\xi(\Phi) Y^\beta f|(t, x, v) dv \lesssim \epsilon \frac{\log^{(j+|\xi|+|\beta|+3)a}(3+t)}{\tau_+^2 \tau_-}, \quad (4.68)$$

4.8 Improvement of the bootstrap assumptions (4.49), (4.50) and (4.51)

As the improvement of all the energy bounds concerning f are similar, we unify them as much as possible. Hence, let us consider

- $Q \in \{N-3, N-1, N\}$, $n_{N-3} = 4$, $n_{N-1} = 0$ and $n_N = 0$.
- Multi-indices β^0 , ξ^0 and ξ^2 such that $\max(|\xi^0| + |\beta^0|, 1 + |\xi^0|) \leq Q$ and $\max(|\xi^2| + |\beta^0|, 1 + |\xi^2|) \leq Q$.
- A weight $z_0 \in \mathbf{k}_1$ and $q \leq 2N - 1 + n_Q - \xi_P^0 - \xi_P^2 - \beta_P^0$.

According to the energy estimate of Propostion 4.4.1, Corollaray 4.6.3 and since ξ^0 and ξ^2 play a symmetric role, we could improve (4.49)-(4.51), for ϵ small enough, if we prove that

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \int_v \left| T_F \left(z_0^q P_{\xi^0}(\Phi) Y^{\beta^0} f \right) P_{\xi^2}(\Phi) \right| \frac{dv}{v^0} dx ds &\lesssim \epsilon^{\frac{3}{2}} (1+t)^\eta \log^{aq} (3+t) \quad \text{if } Q = N, \\ &\lesssim \epsilon^{\frac{3}{2}} \log^{(q+|\xi^0|+|\xi^2|+|\beta^0|)a} (3+t) \quad \text{otherwise.} \end{aligned} \quad (4.69)$$

For that purpose, we will bound the spacetime integral of the terms given by Proposition 4.3.31, applied to $z_0^q P_{\xi^0}(\Phi) Y^{\beta^0} f$. We start, in Subsection 4.8.1, by covering the term of (category 0). Subsection 4.8.2 (respectively 4.8.3) is devoted to the study of the expressions of the other categories for which the electromagnetic field is derived less than $N-3$ times (respectively more than $N-2$ times). Finally, we treat the more critical terms in Subsection 4.8.5. In Subsection 4.8.4, we bound $\mathbb{E}_N^X[f]$, $\mathbb{E}_{N-1}^X[f]$ and we improve the decay estimate of $\int_v (v^0)^{-2} |Y^\beta f| dv$ near the light cone.

4.8.1 The terms of (category 0)

The purpose of this Subsection is to prove the following proposition.

Proposition 4.8.1. *Let ξ^1 , ξ^2 and β such that $\max(1 + |\xi^i|, |\xi^i| + |\beta|) \leq N$ for $i \in \{1, 2\}$. Consider also $z \in \mathbf{k}_1$, $r \in \mathbb{N}^*$, $0 \leq \kappa \leq \eta$, $0 < j \leq 2N + 3 - \xi^1 - \xi_P^2 - \beta_P$ and suppose that,*

$$\forall t \in [0, T], \quad \mathbb{E} [z^j P_{\xi^1}(\Phi) P_{\xi^2}(\Phi) Y^\beta f] (t) + \log^2(3+t) \mathbb{E} [z^{j-1} P_{\xi^1}(\Phi) P_{\xi^2}(\Phi) Y^\beta f] (t) \lesssim \epsilon (1+t)^\kappa \log^r(3+t).$$

Then,

$$\int_0^t \int_{\Sigma_s} \int_v |F(v, \nabla_v z^j) P_{\xi^1}(\Phi) P_{\xi^2}(\Phi) Y^\beta f| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} (1+t)^\kappa \log^r(3+t).$$

Proof. To lighten the notations, we denote $P_{\xi^1}(\Phi) P_{\xi^2}(\Phi) Y^\beta f$ by h and, for $d \in \{0, 1\}$, $\mathbb{E} [z^{j-d} h]$ by H_{j-d} , so that

$$H_{j-d}(t) = \|z^{j-d} h\|_{L_{x,v}^1}(t) + \sup_{u \in \mathbb{R}} \int_{C_u(t)} \int_v \frac{v^L}{v^0} |z^{j-d} h| dv dC_u(t) \lesssim \epsilon (1+t)^\kappa \log^{r-2d}(3+t).$$

Using Lemmas 4.2.4 and 4.3.27, we have

$$\left| (\nabla_v z^j)^L \right|, \left| (\nabla_v z^j)^{\underline{L}} \right|, \frac{|v^A + v^{\underline{L}}|}{v^0} \left| (\nabla_v z^j)^A \right| \lesssim \frac{\tau_-}{v^0} |z|^{j-1} + \frac{1}{v^0} \sum_{w \in \mathbf{k}_1} |w|^j.$$

Hence, the decomposition of $F(v, \nabla_v |z|^j)$ in our null frame brings us to control the integral, over $[0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$, of²²

$$(\tau_- |w|^{j-1} + |w|^j) (|\rho(F)| + |\alpha(F)| + |\sigma(F)| + |\underline{\alpha}(F)|) \frac{|h|}{v^0} \quad \text{and} \quad (\tau_+ |w|^{j-1} + |w|^j) |\alpha(F)| \frac{|h|}{v^0}.$$

According to Remark 4.7.2 and using $1 \lesssim \sqrt{v^0 v^{\underline{L}}}$ (see Lemma 4.2.4), we have

$$\tau_- (|\rho(F)| + |\sigma(F)| + |\underline{\alpha}(F)|) + \tau_+ |\alpha(F)| \lesssim \sqrt{\epsilon} \frac{\log(3+t)}{\tau_+}, \quad |\rho(F)| + |\sigma(F)| + |\underline{\alpha}(F)| + |\alpha(F)| \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+^{\frac{3}{2}}} + \sqrt{\epsilon} \frac{v^{\underline{L}}}{\tau_-^{\frac{3}{2}}}.$$

²²The second term comes from $\alpha(F)_A v^{\underline{L}} (\nabla_v |z|^j)^A$.

The result is then implied by the following two estimates,

$$\begin{aligned}
\int_0^t \int_{\Sigma_s} \int_v \sqrt{\epsilon} |h| \left(\frac{|w|^{j-1}}{1+s} \log(3+s) + \frac{|w|^j}{(1+s)^{\frac{3}{2}}} \right) dv dx ds &\lesssim \sqrt{\epsilon} \int_0^t \frac{\log(3+s)}{1+s} H_{j-1}(s) ds + \int_0^t \frac{H_j(s)}{(1+s)^{\frac{3}{2}}} ds \\
&\lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{\log^{r-1}(3+t)}{(1+s)^{1-\kappa}} + \frac{\log^r(3+t)}{(1+s)^{\frac{5}{4}-\kappa}} ds \\
&\lesssim \epsilon^{\frac{3}{2}} (1+t)^\kappa \log^r(3+t), \\
\int_0^t \int_{\Sigma_s} \frac{\sqrt{\epsilon}}{\tau_-^{\frac{3}{2}}} \int_v \frac{v^{\underline{L}}}{v^0} |w^j h| dv dx ds &= \int_{u=-\infty}^t \frac{\sqrt{\epsilon}}{\tau_-^{\frac{3}{2}}} \int_{C_u(t)} \int_v \frac{v^{\underline{L}}}{v^0} |w^j h| dv dC_u(t) du \\
&\lesssim \sqrt{\epsilon} H_j(t) \int_{u=-\infty}^{+\infty} \frac{du}{\tau_-^{\frac{3}{2}}} \lesssim \epsilon^{\frac{3}{2}} (1+t)^\kappa \log^r(3+t).
\end{aligned}$$

□

4.8.2 Bounds on several spacetime integrals

We estimate in this subsection the spacetime integral of the source terms of (category 1)-(category 3) of $T_F(z_0^q P_{\xi^0}(\Phi) Y^{\beta^0} f)$, multiplied by $(v^0)^{-1} P_{\xi^2}(\Phi)$, where the electromagnetic field is derived less than $N-3$ time. We then fix, for the remaining of the subsection,

- multi-indices γ, β and ξ^1 such that

$$|\gamma| \leq N-3, \quad |\xi^1| + |\gamma| + |\beta| \leq Q+1, \quad |\beta| \leq |\beta^0|, \quad |\xi^1| + |\beta| \leq |\xi^0| + |\beta^0| \leq Q \quad \text{and} \quad |\xi^1| \leq Q-1.$$

- $n \leq 2N$, $z \in \mathbf{k}_1$ and $j \in \mathbb{N}$ such that $j \leq 2N-1+n_Q - \xi_P^1 - \xi_P^2 - \beta_P$.
- We will make more restrictive hypotheses for the study of the terms of (category 2) and (category 3). For instance, for the last ones, we will take $|\xi^1| < |\xi^0|$ and $j = q$. This has to do with their properties described in Proposition 4.3.31.

Note that $|\xi^2| + |\beta| \leq Q$. To lighten the notations, we introduce

$$h := z^j P_{\xi^1}(\Phi) P_{\xi^2}(\Phi) Y^\beta f.$$

We start by treating the terms of (category 1).

Proposition 4.8.2. *Under the bootstrap assumptions (4.49)-(4.51), we have,*

$$I_1 := \int_0^t \int_{\Sigma_s} \int_v |\Phi|^n \left(|\nabla_{Z^\gamma} F| + \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(F))| + \frac{\tau_+}{\tau_-} \sqrt{\frac{v^{\underline{L}}}{v^0}} |\sigma(\mathcal{L}_{Z^\gamma}(F))| \right) |h| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}.$$

Proof. According to Propositions 4.7.4, 4.7.1 and $1 \lesssim \sqrt{v^0 v^{\underline{L}}}$, we have

$$\begin{aligned}
|\Phi|^n |\nabla_{Z^\gamma} F| + |\Phi|^n \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(F))| + |\Phi|^n \frac{\tau_+}{\tau_-} \sqrt{\frac{v^{\underline{L}}}{v^0}} |\sigma(\mathcal{L}_{Z^\gamma}(F))| &\lesssim \sqrt{\epsilon} \log^{4N+M}(3+t) \left(\frac{\sqrt{v^0 v^{\underline{L}}}}{\tau_+ \tau_-} + \frac{v^{\underline{L}}}{\tau_+^{\frac{1}{2}} \tau_-^{\frac{3}{2}}} \right) \\
&\lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+^{\frac{5}{4}}} + \sqrt{\epsilon} \frac{v^{\underline{L}}}{\tau_+^{\frac{1}{4}} \tau_-^{\frac{3}{2}}}.
\end{aligned}$$

Then,

$$\begin{aligned}
I_1 &\lesssim \int_0^t \int_{\Sigma_s} \frac{\sqrt{\epsilon}}{\tau_+^{\frac{5}{4}}} \int_v |h| dv dx ds + \int_0^t \int_{\Sigma_s} \frac{\sqrt{\epsilon}}{\tau_+^{\frac{1}{4}} \tau_-^{\frac{3}{2}}} \int_v \frac{v^{\underline{L}}}{v^0} |h| dv dx ds \\
&\lesssim \sqrt{\epsilon} \int_0^t \frac{\mathbb{E}[h](s)}{(1+s)^{\frac{5}{4}}} ds + \sqrt{\epsilon} \int_{u=-\infty}^t \int_{C_u(t)} \frac{1}{\tau_+^{\frac{1}{4}} \tau_-^{\frac{3}{2}}} \int_v \frac{v^{\underline{L}}}{v^0} |h| dv dC_u(t) du.
\end{aligned}$$

Recall now the definition of $(t_i)_{i \in \mathbb{N}}$, $(T_i(t))_{i \in \mathbb{N}}$ and $C_u^i(t)$ from Subsection 4.2.4. By the bootstrap assumption (4.51) and $2\eta < \frac{1}{8}$, we have

$$\mathbb{E}[h](s) \lesssim \epsilon(1+s)^{\frac{1}{8}} \quad \text{and} \quad \sup_{u \in \mathbb{R}} \int_{C_u^i(t)} \int_v v^0 v^{\underline{L}} |h| dv dC_u^i(t) \lesssim \epsilon(1+T_{i+1}(t))^{2\eta} \lesssim \epsilon(1+t_{i+1})^{\frac{1}{8}},$$

so that, using also²³ $1+t_{i+1} \leq 2(1+t_i)$ and Lemma 4.2.7,

$$\begin{aligned} \sqrt{\epsilon} \int_0^t \frac{\mathbb{E}[h](s)}{(1+s)^{\frac{5}{4}}} &\lesssim \epsilon^{\frac{3}{2}} \int_0^{+\infty} \frac{ds}{(1+s)^{\frac{9}{8}}} \lesssim \epsilon^{\frac{3}{2}}, \\ \sqrt{\epsilon} \int_{u=-\infty}^t \int_{C_u(t)} \frac{1}{\tau_+^{\frac{1}{4}} \tau_-^{\frac{3}{2}}} \int_v \frac{v^{\underline{L}}}{v^0} |h| dv dC_u(t) du &= \sqrt{\epsilon} \int_{u=-\infty}^t \sum_{i=0}^{+\infty} \int_{C_u^i(t)} \frac{1}{\tau_+^{\frac{1}{4}} \tau_-^{\frac{3}{2}}} \int_v \frac{v^{\underline{L}}}{v^0} |h| dv dC_u^i(t) du \\ &\lesssim \sqrt{\epsilon} \int_{u=-\infty}^t \frac{1}{\tau_-^{\frac{3}{2}}} \sum_{i=0}^{+\infty} \frac{1}{(1+t_i)^{\frac{1}{4}}} \int_{C_u^i(t)} \int_v \frac{v^{\underline{L}}}{v^0} |h| dv dC_u^i(t) du \\ &\lesssim \epsilon^{\frac{3}{2}} \int_{u=-\infty}^t \frac{du}{\tau_-^{\frac{3}{2}}} \sum_{i=0}^{+\infty} \frac{(1+t_{i+1})^{\frac{1}{8}}}{(1+t_{i+1})^{\frac{1}{4}}} \\ &\lesssim \epsilon^{\frac{3}{2}} \int_{u=-\infty}^{+\infty} \frac{du}{\tau_-^{\frac{3}{2}}} \sum_{i=0}^{+\infty} 2^{-\frac{i}{8}} \lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

□

We now start to bound the problematic terms.

Proposition 4.8.3. *We study here the terms of (category 2). If, for $\kappa \geq 0$ and $r \in \mathbb{N}$,*

$$\mathbb{E}[h](t) = \|h\|_{L_{x,v}^1} (t) + \sup_{u \in \mathbb{R}} \int_{C_u(t)} \int_v \frac{v^{\underline{L}}}{v^0} |h| dv dC_u(t) \lesssim \epsilon(1+s)^\kappa \log^r(3+t), \quad \text{then} \quad (4.71)$$

$$I_3^1 := \int_0^t \int_{\Sigma_s} \frac{\tau_+}{\tau_-} |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))| \int_v \sqrt{\frac{v^{\underline{L}}}{v^0}} |h| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}(1+s)^\kappa \log^r(3+t) \quad \text{and}$$

$$I_3^2 := \int_0^t \int_{\Sigma_s} \frac{\tau_+}{\tau_-} |\rho(\mathcal{L}_{Z^\gamma}(F))| \int_v |h| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}(1+s)^\kappa \log^{r+a}(3+t).$$

Remark 4.8.4. *The extra $\log^a(3+t)$ -growth on I_3^2 , compared to I_3^1 , will not avoid us to close the energy estimates in view of the hierarchies in the energy norms. Indeed, we have $j = q - 1$ (in I_3^2) according to the properties of the terms of (category 2) (in I_3^1 , we merely have $j \leq q$).*

Proof. Recall first from Lemma 4.2.4 that $1+|v^A| \lesssim \sqrt{v^0 v^{\underline{L}}}$. Then, using Proposition 4.7.1 and the inequality $2CD \leq C^2 + D^2$, one obtains

$$\sqrt{\frac{v^{\underline{L}}}{v^0}} \frac{\tau_+}{\tau_-} |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))| \lesssim \sqrt{\epsilon} \frac{v^{\underline{L}}}{\tau_-^{\frac{3}{2}}} \quad \text{and} \quad \frac{\tau_+}{\tau_-} |\rho(\mathcal{L}_{Z^\gamma}(F))| \lesssim \sqrt{\epsilon} \log^M(3+t) \frac{v^0}{\tau_+} + \sqrt{\epsilon} \log^M(3+t) \frac{v^{\underline{L}}}{\tau_-^{\frac{3}{2}}}.$$

We then have, as $a = M + 1$,

$$\begin{aligned} I_3^1 &\lesssim \int_{u=-\infty}^t \frac{\sqrt{\epsilon}}{\tau_-^{\frac{3}{2}}} \int_{C_u(t)} \int_v \frac{v^{\underline{L}}}{v^0} |h| dv dC_u(t) du \lesssim \epsilon^{\frac{3}{2}} \mathbb{E}[h](t) \int_{u=-\infty}^{+\infty} \frac{du}{\tau_-^{\frac{3}{2}}} \lesssim \epsilon^{\frac{3}{2}}(1+s)^\kappa \log^r(3+t), \\ I_3^2 &\lesssim \sqrt{\epsilon} \int_0^t \int_{\Sigma_s} \frac{\log^M(3+s)}{\tau_+} \int_v |h| dv dx ds + \sqrt{\epsilon} \log^M(3+t) \int_{u=-\infty}^t \frac{\sqrt{\epsilon}}{\tau_-^{\frac{3}{2}}} \int_{C_u(t)} \int_v \frac{v^{\underline{L}}}{v^0} |h| dv dC_u(t) du \\ &\lesssim \sqrt{\epsilon} \int_0^t \frac{\log^{r+M}(3+s)}{(1+s)^{1-\kappa}} ds + \epsilon^{\frac{3}{2}}(1+t)^\kappa \log^{r+M}(3+t) \\ &\lesssim \epsilon^{\frac{3}{2}}(1+t)^\kappa \log^{r+M+1}(3+t). \end{aligned}$$

□

²³Note that the sum over i is actually finite as $C_u^i(t) = \emptyset$ for $i \geq \log_2(1+t)$.

We finally end this subsection by the following estimate.

Proposition 4.8.5. *We suppose here that $\max(|\xi^1| + |\beta|, |\xi^1| + 1) \leq N - 1$. Then,*

$$\begin{aligned} I_4 &:= \int_0^t \int_{\Sigma_s} \tau_+ \int_v \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \right| |h| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} \log^{(1+j+|\xi^1|+|\xi^2|+|\beta|)a}(3+t) & \text{if } |\xi^2| \leq N-2, \\ &\lesssim \epsilon^{\frac{3}{2}} (1+t)^{\frac{3}{4}\eta} & \text{otherwise.} \end{aligned}$$

Remark 4.8.6. *To understand the extra hypothesis made in this proposition, recall from the properties of the terms of (category 3) that we can assume $|\xi^1| < |\xi^0|$, $\beta = \beta^0$ and $j = q$. We then have*

$$1 + j + |\xi^1| + |\xi^2| + |\beta| \leq q + |\xi^0| + |\xi^2| + |\beta^0|.$$

Proof. Let us denote by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of $\mathcal{L}_{Z^\gamma}(F)$. Using $1 + |v^A| \leq \sqrt{v^0 v^L}$ and Proposition 4.7.1, we have

$$\begin{aligned} \tau_+ \left| \frac{v^\mu}{(v^0)^2} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \right| &\lesssim \tau_+ \sqrt{\frac{v^L}{v^0}} (|\alpha| + |\rho| + |\sigma|) + \tau_+ \frac{v^L}{v^0} |\underline{\alpha}| \\ &\lesssim \sqrt{\epsilon} \sqrt{\frac{v^L}{v^0}} \frac{\log^M(3+t)}{\sqrt{\tau_+ \tau_-}} + \sqrt{\epsilon} \frac{v^L}{v^0} \frac{\log^M(3+t)}{\tau_-} \lesssim \sqrt{\epsilon} \frac{\log^M(3+t)}{\tau_+} + \sqrt{\epsilon} \frac{v^L}{v^0} \frac{\log^M(3+t)}{\tau_-}. \end{aligned}$$

As $\tau_- \sim \tau_+$ away from the light cone (for, say²⁴, $u \leq -t$ and $u \geq \frac{t}{2}$), we finally obtain that

$$\begin{aligned} I_4 &= \sqrt{\epsilon} \int_0^t \frac{\log^M(3+s)}{1+s} \int_{\Sigma_s} \int_v |h| dv dx ds + \sqrt{\epsilon} \log^M(3+t) \int_{u=-t}^{\frac{t}{2}} \frac{1}{\tau_-} \int_{C_u(t)} \int_v \frac{v^L}{v^0} |h| dv dC_u(t) du \\ &\lesssim \sqrt{\epsilon} \log^M(3+t) \sup_{[0,t]} \mathbb{E}[h] \int_0^t \frac{ds}{1+s} + \sqrt{\epsilon} \log^M(3+t) \mathbb{E}[h](t) \int_{u=-t}^t \frac{du}{\tau_-} \lesssim \sqrt{\epsilon} \log^a(3+t) \sup_{[0,t]} \mathbb{E}[h]. \end{aligned}$$

If $|\xi^2| \leq N - 2$, the bootstrap assumption (4.49) or (4.50) gives

$$\sup_{[0,t]} \mathbb{E}[h] \leq \epsilon \log^{(j+|\xi^1|+|\xi^2|+|\beta|)a}(3+t)$$

and we can conclude the proof in that case. If $|\xi^2| = N - 1$, we have $j \leq 2N - 1 - \xi_P^1 - \xi_P^2 - \beta_P$ since this case appears only if $Q = N$. Let $(i_1, i_2) \in \mathbb{N}^2$ be such that

$$i_1 + i_2 = 2j, \quad i_1 \leq 2N - 1 - 2\xi_P^1 - \beta_P \quad \text{and} \quad i_2 \leq 2N - 1 - 2\xi_P^2 - \beta_P.$$

Using the bootstrap assumptions (4.50) and (4.51), we have

$$\begin{aligned} \mathbb{E}[h](t) &= \int_{\Sigma_t} \int_v |z^j P_{\xi^1}(\Phi) P_{\xi^2}(\Phi) Y^\beta f| dv dx \\ &\lesssim \left| \int_{\Sigma_t} \int_v |z^{i_1} P_{\xi^1}(\Phi)^2 Y^\beta f| dv dx \int_{\Sigma_t} \int_v |z^{i_2} P_{\xi^2}(\Phi)^2 Y^\beta f| dv dx \right|^{\frac{1}{2}} \\ &\lesssim \left| \log^{(i_1+2|\xi^1|+|\beta|)a}(3+t) \mathbb{E}_{N-1}^0[f](t) \log^{a i_2}(3+t) \bar{\mathbb{E}}_N[f](t) \right|^{\frac{1}{2}} \lesssim \epsilon (1+t)^{\frac{3}{4}\eta}, \end{aligned}$$

which ends the proof. \square

Note now that Propositions 4.3.31, 4.8.1, 4.8.2, 4.8.3 and 4.8.5 imply (4.70) for $Q = N - 3$, so that $\mathbb{E}_{N-3}^4[f] \leq 3\epsilon$ on $[0, T[$.

4.8.3 Completion of the bounds on the spacetime integrals

In this subsection, we bound the spacetime integrals considered previously when the electromagnetic field is differentiated too much time to be estimated pointwise. For this, we make crucial use of the pointwise decay estimates on the velocity averages of $|z^j P_\zeta(\Phi) Y^\beta f|$ which are given by (4.68). The terms studied here appear only if $|\xi^0| + |\beta^0| \geq N - 2$ since otherwise the electromagnetic field would be differentiated at most $N - 3$ times. We then fix, for the remaining of the subsection, $Q \in \{N - 1, N\}$,

²⁴If (s, y) is in one of these regions of $[0, t] \times \mathbb{R}^3$, we have $|y| \geq 2s$ or $|y| \leq \frac{s}{2}$.

- multi-indices γ, β and ξ^1 such that $N - 2 \leq |\gamma| \leq N$,

$$|\gamma| + |\xi^1| \leq Q, \quad |\xi^1| + |\gamma| + |\beta| \leq Q + 1, \quad |\beta| \leq |\beta^0|, \quad |\xi^1| + |\beta| \leq |\xi^0| + |\beta^0| \leq Q \quad \text{and} \quad |\xi^1| \leq Q - 1.$$

- $n \leq 2N$, $z \in \mathbf{k}_1$ and $j \in \mathbb{N}$ such that $j \leq 2N - 1 - \xi_P^1 - \xi_P^2 - \beta_P$.
- Consistently with Proposition 4.3.31, we will, in certain cases, make more assumptions on ξ^1 or j , such as $j \leq q$ for the terms of (category 2).

Note that $|\xi^2| + |\beta| \leq Q$ and that there exists i_1 and i_2 such as

$$i_1 + i_2 = 2j, \quad i_1 \leq 2N - 1 - 2\xi_P^1 - \beta_P \quad \text{and} \quad i_2 \leq 2N - 1 - 2\xi_P^2 - \beta_P.$$

To lighten the notations, we introduce

$$h := z^j P_{\xi^1}(\Phi) P_{\xi^2}(\Phi) Y^\beta f, \quad h_1 := z^{i_1} P_{\xi^1}(\Phi)^2 Y^\beta f \quad \text{and} \quad h_2 := z^{i_2} P_{\xi^2}(\Phi)^2 Y^\beta f,$$

so that $|h| = \sqrt{|h_1 h_2|}$. As $|\gamma| \geq N - 2$, we have $|\xi^1| \leq 2 \leq N - 7$ and $2|\xi^1| + |\beta| \leq 5 \leq N - 6$. Thus, by Lemma 4.2.4 and (4.68), we have, for all $(t, x) \in [0, T[\times \mathbb{R}^3$,

$$\tau_+^3 \int_v |h_1| \frac{dv}{(v^0)^2} + \tau_+^2 \tau_- \int_v |h_1| dv \lesssim \int_v \left(\tau_+^3 \frac{v^L}{v^0} + \tau_+^2 \tau_- \right) |h_1| dv \lesssim \epsilon \log^{(4+i_1+2|\xi^1+|\beta|)^a} (3+t). \quad (4.72)$$

Using Remark 4.2.5, we have,

$$\forall |x| \geq t, \quad \tau_+^3 \tau_- \int_v |h_1| \frac{dv}{(v^0)^2} \lesssim \tau_+^3 \tau_- \int_v \frac{v^L}{v^0} |h_1| dv \lesssim \epsilon \log^{(4+i_1+2|\xi^1+|\beta|)^a} (3+t). \quad (4.73)$$

Proposition 4.8.7. *The following estimates holds,*

$$I_1^1 := \int_0^t \int_{\Sigma_s} \int_v |\Phi|^n |\nabla_{Z^\gamma} F| |h| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}, \quad I_1^2 := \int_0^t \int_{\Sigma_s} \int_v |\Phi|^n \frac{\tau_+}{\tau_-} \sqrt{\frac{v^L}{v^0}} |\sigma(\mathcal{L}_{Z^\gamma}(F))| |h| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}$$

and

$$I_1^3 := \int_0^t \int_{\Sigma_s} \int_v |\Phi|^n \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(F))| |h| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}.$$

Proof. Using the Cauchy-Schwarz inequality twice (in x and then in v), $\|\nabla_{Z^\gamma} F\|_{L^2(\Sigma_t)}^2 \lesssim \mathcal{E}_N^0[F](t) \leq 4\epsilon$, $|\Phi| \lesssim \sqrt{\epsilon} \log^2(1 + \tau_+)$, $\mathbb{E}_N[f](t) \lesssim \epsilon(1+t)^\eta$ and (4.72), we have

$$\begin{aligned} I_1^1 &\lesssim \int_0^t \|\nabla_{Z^\gamma} F\|_{L^2(\Sigma_s)} \left\| \int_v |\Phi|^n |h| \frac{dv}{v^0} \right\|_{L^2(\Sigma_s)} ds \\ &\lesssim \sqrt{\epsilon} \int_0^t \left\| \log^{8N}(1 + \tau_+) \int_v |h_1| \frac{dv}{(v^0)^2} \int_v |h_2| dv \right\|_{L^1(\Sigma_s)}^{\frac{1}{2}} ds \\ &\lesssim \sqrt{\epsilon} \int_0^t \left\| \log^{8N}(1 + \tau_+) \int_v |h_1| \frac{dv}{(v^0)^2} \right\|_{L^\infty(\Sigma_s)}^{\frac{1}{2}} \sqrt{\mathbb{E}[h_2](s)} ds \\ &\lesssim \epsilon \int_0^t \frac{\log^{4N+3Na}(3+s)}{(1+s)^{\frac{3}{2}}} \log^{ai_2}(3+s) \mathbb{E}_N[f](s) ds \lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

For the second one, recall from the bootstrap assumptions (4.53) and (4.51) that for all $t \in [0, T[$ and $i \in \mathbb{N}$,

$$\int_{C_u^i(t)} |\sigma|^2 dC_u^i(t) \leq \mathcal{E}_N^0[F](t_{i+1}(t)) \lesssim \epsilon \quad \text{and} \quad \sup_{u \in \mathbb{R}} \int_{C_u^i(t)} \int_v \frac{v^L}{v^0} |h_2| dv dC_u^i(t) \lesssim \mathbb{E}[h_2](T_{i+1}(t)) \lesssim \epsilon(1+t_{i+1})^{2\eta}.$$

Hence, using this time a null foliation, one has

$$\begin{aligned}
I_1^2 &\lesssim \sum_{i=0}^{+\infty} \int_{u=-\infty}^t \frac{1}{\tau_-} \left| \int_{C_u^i(t)} |\sigma(\mathcal{L}_{Z^\gamma}(F))|^2 dC_u^i(t) \int_{C_u^i(t)} \tau_+^2 \left| \int_v |\Phi|^n \sqrt{\frac{v^{\underline{L}}}{v^0}} |h| \frac{dv}{v^0} \right|^2 dC_u^i(t) \right|^{\frac{1}{2}} du \\
&\lesssim \sqrt{\epsilon} \sum_{i=0}^{+\infty} \int_{u=-\infty}^t \frac{1}{\tau_-} \left| \int_{C_u^i(t)} \tau_+^2 \log^{8N}(1+\tau_+) \int_v |h_1| \frac{dv}{(v^0)^2} \int_v \frac{v^{\underline{L}}}{v^0} |h_2| dv dC_u^i(t) \right|^{\frac{1}{2}} du \\
&\lesssim \sqrt{\epsilon} \sum_{i=0}^{+\infty} \int_{u=-\infty}^t \frac{1}{\tau_-} \left| \int_{C_u^i(t)} \frac{1}{\tau_+^{\frac{3}{4}}} \int_v \frac{v^{\underline{L}}}{v^0} |h_2| dv dC_u^i(t) \right|^{\frac{1}{2}} du \\
&\lesssim \epsilon^{\frac{3}{2}} \int_{u=-\infty}^{+\infty} \frac{du}{\tau_-^{\frac{9}{8}}} \sum_{i=0}^{+\infty} \frac{(1+t_{i+1})^\eta}{(1+t_i)^{\frac{1}{4}}} \lesssim \epsilon^{\frac{3}{2}}.
\end{aligned}$$

For the last one, use first that $F = \tilde{F} + \bar{F}$ to get

$$I_1^3 = I_1^{\tilde{F}} + I_1^{\bar{F}} := \int_0^t \int_{\Sigma_s} \int_v |\Phi|^n \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(\tilde{F}))| |h| \frac{dv}{v^0} dx ds + \int_0^t \int_{\Sigma_s} \int_v |\Phi|^n \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(\bar{F}))| |h| \frac{dv}{v^0} dx ds.$$

By Proposition 4.7.1, we have $|\mathcal{L}_{Z^\gamma}(\bar{F})| \lesssim \epsilon \tau_+^{-2}$. Hence, using $|\Phi| \lesssim \log^2(1+\tau_+)$ and $1 \lesssim \sqrt{v^0 v^{\underline{L}}}$, we have

$$|\Phi|^n \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(\bar{F}))| \lesssim \frac{\epsilon \sqrt{v^0 v^{\underline{L}}}}{\sqrt{v^0} \tau_+^{\frac{3}{4}} \tau_-} \leq \epsilon \frac{v^0}{\tau_+^{\frac{5}{4}}} + \epsilon \frac{v^{\underline{L}}}{\tau_+^{\frac{1}{4}} \tau_-^2}$$

and we can bound $I_1^{\bar{F}}$ by $\epsilon^{\frac{3}{2}}$ as I_1 in Proposition 4.8.2. For $I_1^{\tilde{F}}$, remark first that, by the bootstrap assumptions (4.54), (4.57) and since $F = \tilde{F}$ in the interior of the light cone,

$$\int_{C_u^i(t)} \tau_+ |\alpha(\mathcal{L}_{Z^\gamma}(\tilde{F}))|^2 dC_u^i(t) \lesssim \mathcal{E}_N[F](T_{i+1}(t)) + \mathcal{E}_N^{Ext}[\tilde{F}](T_{i+1}(t)) \lesssim \epsilon(1+t_{i+1})^\eta.$$

It then comes, using $1 \lesssim \sqrt{v^0 v^{\underline{L}}}$, $16\eta < 1$ and $\int_v |\Phi|^n |h_1| dv \lesssim \epsilon \tau_+^{-\frac{3}{2}} \tau_-^{-1}$, that

$$\begin{aligned}
I_1^{\tilde{F}} &\lesssim \sum_{i=0}^{+\infty} \int_{u=-\infty}^t \frac{1}{\tau_-} \left| \int_{C_u^i(t)} \tau_+ |\alpha(\mathcal{L}_{Z^\gamma}(F))|^2 dC_u^i(t) \int_{C_u^i(t)} \tau_+ \left| \int_v |\Phi|^n \sqrt{\frac{v^{\underline{L}}}{v^0}} |h| dv \right|^2 dC_u^i(t) \right|^{\frac{1}{2}} du \\
&\lesssim \sqrt{\epsilon} \sum_{i=0}^{+\infty} (1+t_{i+1})^\eta \int_{u=-\infty}^t \frac{1}{\tau_-} \left| \int_{C_u^i(t)} \tau_+ \int_v |\Phi|^n |h_1| dv \int_v \frac{v^{\underline{L}}}{v^0} |h_2| dv dC_u^i(t) \right|^{\frac{1}{2}} du \\
&\lesssim \sqrt{\epsilon} \sum_{i=0}^{+\infty} \frac{(1+t_{i+1})^{2\eta}}{(1+t_i)^{\frac{1}{4}}} \int_{u=-\infty}^{+\infty} \frac{1}{\tau_-^{\frac{9}{8}}} \lesssim \epsilon^{\frac{3}{2}} \sum_{i=0}^{+\infty} 2^{-\frac{i}{4}(1-8\eta)} \lesssim \epsilon^{\frac{3}{2}}.
\end{aligned}$$

□

We now turn on the problematic terms.

Proposition 4.8.8. *If $|\xi_2| \leq N-2$, we have*

$$\begin{aligned}
I_3^1 &= \int_0^t \int_{\Sigma_s} \frac{\tau_+}{\tau_-} |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))| \int_v \sqrt{\frac{v^{\underline{L}}}{v^0}} |h| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} \log^{(3+j+|\xi_1|+|\xi_2|+|\beta|)a} (3+t) \quad \text{and} \\
I_3^2 &= \int_0^t \int_{\Sigma_s} \frac{\tau_+}{\tau_-} |\rho(\mathcal{L}_{Z^\gamma}(F))| \int_v |h| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} \log^{(2+j+|\xi_1|+|\xi_2|+|\beta|)a} (3+t).
\end{aligned}$$

Otherwise, $|\xi^2| = N-1$ and $I_3^1 + I_3^2 \lesssim \epsilon^{\frac{3}{2}} (1+t)^{\frac{3}{4}\eta}$.

Remark 4.8.9. *Note that these estimates are sufficient to improve the bootstrap assumptions (4.50) and (4.51). Indeed,*

- the case $|\xi^2| = N - 1$ concerns only the study of $\bar{\mathbb{E}}_N[f]$.
- Even if the bound on $I_3^2 + I_3^1$, when $|\xi^2| \leq N - 2$ could seem to possess a factor $\log^{3a}(3+t)$ in excess, one has to keep in mind that $|\gamma| \geq N - 2$, so $|\xi^1| + |\beta| \leq 3$ and $|\xi^0| + |\beta^0| \geq N - 2$. Moreover, by the properties of the terms of (category 2), $j \leq q$. We then have, as $N \geq 8$,

$$j + 3 + |\xi^1| + |\xi^2| + |\beta| \leq q + |\xi^0| + |\xi^2| + |\beta^0|.$$

Proof. Throughout this proof, we will use (4.72) and the bootstrap assumption (4.53), which implies

$$\|\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))\|_{L^2(\Sigma_t)} + \sup_{u \in \mathbb{R}} \|\rho(\mathcal{L}_{Z^\gamma}(F))\|_{L^2(C_u(t))} \lesssim \sqrt{\mathcal{E}_N^0[F](t)} \lesssim \epsilon^{\frac{1}{2}}.$$

Applying the Cauchy-Schwarz inequality twice (in (t, x) and then in v), we get

$$\begin{aligned} I_3^1 &\lesssim \left| \int_0^t \frac{\|\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))\|_{L^2(\Sigma_s)}}{1+s} ds \int_{u=-\infty}^t \int_{C_u(t)} \frac{\tau_+^3}{\tau_-^2} \left| \int_v \sqrt{\frac{v^L}{v^0}} |h| \frac{dv}{v^0} \right|^2 dC_u(t) du \right|^{\frac{1}{2}} \\ &\lesssim \epsilon^{\frac{1}{2}} \log^{\frac{1}{2}}(1+t) \left| \int_{u=-\infty}^t \frac{1}{\tau_-^2} \int_{C_u(t)} \int_v \frac{v^L}{v^0} |h_2| dv dC_u(t) du \right|^{\frac{1}{2}} \sup_{u \in \mathbb{R}} \left\| \tau_+^3 \int_v |h_1| \frac{dv}{(v^0)^2} \right\|_{L^\infty(C_u(t))}^{\frac{1}{2}} \\ &\lesssim \epsilon \log^{\frac{1}{2} + \frac{a}{2}(4+i_1+2|\xi^1|+|\beta|)}(3+t) \sqrt{\mathbb{E}[h_2](t)}. \end{aligned}$$

Using $1 \lesssim \sqrt{v^0 v^L}$ and the Cauchy-Schwarz inequality (this time in $(\underline{u}, \omega_1, \omega_2)$ and then in v), we obtain

$$\begin{aligned} I_3^2 &\lesssim \int_{u=-\infty}^t \|\rho(\mathcal{L}_{Z^\gamma}(F))\|_{L^2(C_u(t))} \left| \int_{C_u(t)} \frac{\tau_+^2}{\tau_-^2} \left| \int_v \sqrt{\frac{v^L}{v^0}} |h| dv \right|^2 dC_u(t) \right|^{\frac{1}{2}} du \\ &\lesssim \epsilon^{\frac{1}{2}} \int_{u=-\infty}^t \frac{1}{\tau_-^{\frac{3}{2}}} \left\| \tau_+^2 \tau_- \int_v |h_1| dv \right\|_{L^\infty(C_u(t))}^{\frac{1}{2}} \left| \int_{C_u(t)} \int_v \frac{v^L}{v^0} |h_2| dv dC_u(t) \right|^{\frac{1}{2}} du \\ &\lesssim \epsilon \log^{\frac{a}{2}(4+i_1+2|\xi^1|+|\beta|)} \sqrt{\mathbb{E}[h_2](t)}. \end{aligned}$$

It then remains to remark that, by the bootstrap assumptions (4.50) and (4.51),

- $\mathbb{E}[h_2](t) \leq \log^{(i_2+2|\xi_2|+|\beta|)a}(3+t) \mathbb{E}_{N-1}^0[f](t) \lesssim \epsilon \log^{(i_2+2|\xi_2|+|\beta|)a}(3+t)$, if $|\xi_2| \leq N - 2$, or
- $\mathbb{E}[h_2](t) \leq \log^{ai_2}(3+t) \bar{\mathbb{E}}_N[f](t) \lesssim \epsilon(1+t)^\eta \log^{ai_2}(3+t)$, if $|\xi_2| = N - 1$.

□

Let us move now on the expressions of (category 3). The ones where $|\gamma| = N$ are the more critical terms and will be treated later.

Proposition 4.8.10. *Suppose that $N - 2 \leq |\gamma| \leq N - 1$. Then, if $|\xi_2| \leq N - 2$,*

$$I_4 = \int_0^t \int_{\Sigma_s} \int_v \tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \right| |h| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} \log^{(3+j+|\xi_1|+|\xi_2|+|\beta|)a}(3+t)$$

and $I_4 \lesssim \epsilon^{\frac{3}{2}}(1+t)^{\frac{3}{4}\eta}$ otherwise.

For similar reasons as those given in Remark 4.8.9, these bounds are sufficient to close the energy estimates on $\bar{\mathbb{E}}_N[f]$ and $\mathbb{E}_{N-1}^0[f]$.

Proof. Denoting by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of $\mathcal{L}_{Z^\gamma}(\tilde{F})$ and using $|v^A| \lesssim \sqrt{v^0 v^L}$, we have

$$\begin{aligned} \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \right| &\lesssim |\alpha(\mathcal{L}_{Z^\gamma}(F))| + |\sigma(\mathcal{L}_{Z^\gamma}(F))| + |\rho(\mathcal{L}_{Z^\gamma}(F))| + \sqrt{\frac{v^L}{v^0}} |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))| \\ &\lesssim |\alpha| + |\rho| + |\sigma| + \sqrt{\frac{v^L}{v^0}} (|\underline{\alpha}| + |\mathcal{L}_{Z^\gamma}(\bar{F})|). \end{aligned}$$

and we can then bound I_4 by $I_{\alpha,\sigma,\rho} + I_{\underline{\alpha}} + I_{\overline{F}}$ (these quantities will be clearly defined below). Note now that

$$\|\sqrt{\tau_+}|\alpha| + \sqrt{\tau_+}|\rho| + \sqrt{\tau_+}|\sigma|\|_{L^2(\Sigma_s)}^2 + \|\sqrt{\tau_-}|\underline{\alpha}|\|_{L^2(\Sigma_s)}^2 \lesssim \mathcal{E}_N^{Ext}[\tilde{F}](s) + \mathcal{E}_{N-1}[F](s) \lesssim \epsilon \log^{2M}(3+s). \quad (4.74)$$

Then, using the Cauchy-Schwarz inequality twice (in (t, x) and then in v), the estimates (4.72) and (4.73) as well as $a = M + 1$, we get

$$\begin{aligned} I_{\underline{\alpha}} &:= \int_0^t \int_{\Sigma_s} \tau_+ |\underline{\alpha}| \int_v \sqrt{\frac{vL}{v^0}} |h| \frac{dv}{v^0} dx ds \\ &\lesssim \left| \int_0^t \frac{\|\sqrt{\tau_-}|\underline{\alpha}|\|_{L^2(\Sigma_s)}^2}{1+s} ds \int_{u=-\infty}^t \int_{C_u(t)} \frac{\tau_+^2(1+s)}{\tau_-} \left| \int_v \sqrt{\frac{vL}{v^0}} |h| \frac{dv}{v^0} \right|^2 dC_u(t) du \right|^{\frac{1}{2}} \\ &\lesssim \sqrt{\epsilon} \log^{M+\frac{1}{2}}(3+t) \left| \int_{u=-\infty}^t \frac{1}{\tau_-} \left\| \tau_+^2(1+s) \int_v |h_1| \frac{dv}{(v^0)^2} \right\|_{L^\infty(C_u(t))} \int_{C_u(t)} \int_v \frac{vL}{v^0} |h_2| dv dC_u(t) du \right|^{\frac{1}{2}} \\ &\lesssim \epsilon^{\frac{3}{2}} \log^{-\frac{1}{2}+\frac{5}{2}(6+i_1+2|\xi^1+|\beta|)}(3+t) \sqrt{\mathbb{E}[h_2](t)} \left| \int_{u=-\infty}^0 \frac{du}{\tau_-^{\frac{3}{2}}} + \int_{u=0}^t \frac{du}{\tau_-} \right|^{\frac{1}{2}} \\ &\lesssim \epsilon^{\frac{3}{2}} \log^{\frac{5}{2}(6+i_1+2|\xi^1+|\beta|)}(3+t) \sqrt{\mathbb{E}[h_2](t)}. \end{aligned}$$

Similarly, one has

$$\begin{aligned} I_{\alpha,\rho,\sigma} &:= \int_0^t \int_{\Sigma_s} \tau_+ (|\alpha| + |\rho| + |\sigma|) \int_v |h| \frac{dv}{v^0} dx ds \\ &\lesssim \int_0^t \|\sqrt{\tau_+}|\alpha| + \sqrt{\tau_+}|\rho| + \sqrt{\tau_+}|\sigma|\|_{L^2(\Sigma_s)} \left\| \sqrt{\tau_+} \int_v |h| \frac{dv}{v^0} \right\|_{L^2(\Sigma_s)} ds \\ &\lesssim \int_0^t \sqrt{\epsilon} \log^M(3+s) \left\| \tau_+ \int_v |h_1| \frac{dv}{(v^0)^2} \right\|_{L^\infty(\Sigma_s)}^{\frac{1}{2}} \left\| \int_v |h_2| dv \right\|_{L^1(\Sigma_s)}^{\frac{1}{2}} ds \\ &\lesssim \epsilon \log^{\frac{5}{2}(6+i_1+2|\xi^1+|\beta|)}(3+t) \|\mathbb{E}[h_2]\|_{L^\infty([0,t])}^{\frac{1}{2}}. \end{aligned}$$

For the last integral, recall from Propositions 4.5.1 and 4.7.1 that $\overline{F}(t, x)$ vanishes for all $t - |x| \geq -1$ and that $|\mathcal{L}_{Z^\gamma}(\overline{F})| \lesssim \epsilon \tau_+^{-2}$. We are then led to bound

$$\begin{aligned} I_{\overline{F}} &:= \int_0^t \int_{|x| \geq s+1} \tau_+ |\mathcal{L}_{Z^\gamma}(\overline{F})| \int_v |h| \frac{dv}{v^0} dx ds \\ &\lesssim \int_0^t \frac{\sqrt{\epsilon}}{1+s} \int_{\Sigma_s} \int_v \sqrt{|h_1 h_2|} dv dx ds \lesssim \int_0^t \frac{\sqrt{\epsilon}}{1+s} \left| \int_{\Sigma_s} \int_v |h_1| dv dx \int_{\Sigma_s} \int_v |h_2| dv dx \right|^{\frac{1}{2}} ds \\ &\lesssim \sqrt{\epsilon} \log(3+t) \|\mathbb{E}[h_1]\|_{L^\infty([0,t])}^{\frac{1}{2}} \|\mathbb{E}[h_2]\|_{L^\infty([0,t])}^{\frac{1}{2}}. \end{aligned}$$

Thus, as $\|\mathbb{E}[h_1]\|_{L^\infty([0,t])} \lesssim \epsilon \log^{(i_1+2|\xi_1+|\beta|)\alpha}(3+t)$ and $i_1 + i_2 = 2j$, it comes

- $I_4 \lesssim \epsilon^{\frac{3}{2}}(1+t)^{\frac{3}{4}\eta}$ if $|\xi_2| = N - 1$, since $\mathbb{E}[h_2](t) \leq \log^{a i_2}(3+t) \overline{\mathbb{E}}_N[f](t) \leq \epsilon(1+t)^\eta \log^{a i_2}(3+t)$, and
- $I_4 \lesssim \epsilon^{\frac{3}{2}} \log^{(3+j+|\xi_1+|\xi_2+|\beta|)\alpha}(3+t)$ otherwise, as $\mathbb{E}[h_2] \leq \log^{(i_2+2|\xi^2+|\beta|)\alpha}(3+t) \mathbb{E}_{N-1}^0[f](t)$.

□

A better pointwise decay estimate on $\int_v |h_1|(v^0)^{-2} dv$ is required to bound sufficiently well I_4 when $|\gamma| = N$. We will then treat this case below, in the last part of this section. However, note that all the Propositions already proved in this section imply (4.70), for $Q = N - 1$, and then $\mathbb{E}_{N-1}^0[f] \leq 3\epsilon$ on $[0, T[$.

4.8.4 Estimates for $\mathbb{E}_{N-1}^X[f]$, $\mathbb{E}_N^X[f]$ and obtention of optimal decay near the light-cone for velocity averages

The purpose of this subsection is to establish that²⁵ $\mathbb{E}_{N-1}^X[f]$, $\mathbb{E}_N^X[f] \leq 3\epsilon$ on $[0, T[$ and then to deduce optimal pointwise decay estimates on the velocity averages of the particle density. Remark that, according

²⁵Note that we cannot unify these norms because of a lack of weights $z \in \mathbf{k}_1$. As we will apply Proposition 4.3.31 with $N_0 = 2N - 1$, we cannot propagate more than $2N - 2$ weights and avoid in the same time the problematics terms.

to the energy estimate of Proposition 4.4.1, ${}^X\mathbb{E}_N[f] \leq 3\epsilon$ follows, if ϵ is small enough, from

$$\int_0^t \int_{\Sigma_s} \int_v |T_F(z^q P_\xi^X(\Phi) Y^\beta f)| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} \log^{2q}(3+t), \quad (4.75)$$

- for all multi-indices β and ξ such that $\max(|\beta| + |\xi|, |\xi| + 1) \leq N$ and
- for all $z \in \mathbf{k}_1$ and $q \in \mathbb{N}$ such that $q \leq 2N - 2 - \xi_P - \beta_P$.

Most of the work has already been done. Indeed, the commutation formula of Proposition 4.3.34 (applied with $N_0 = 2N - 1$) leads us to bound only terms of (category 0) and (category 1) since $q \leq 2N - 2 - \xi_P - \beta_P$. Note that we control quantities of the form

$$z^j P_{\xi^1}(\Phi) Y^{\beta^1} f, \quad \text{with} \quad |\xi^1| + |\beta^1| \leq N, \quad |\xi^1| \leq N - 1 \quad \text{and} \quad j \leq 2N - 1 - \xi_P^1 - \beta_P^1.$$

Consequently, (4.75) ensues from Propositions 4.8.1, 4.8.2 and 4.8.7. $\mathbb{E}_{N-1}^X[f]$ can be estimated similarly since we also control quantities such as

$$z^j P_{\xi^1}(\Phi) P_{\xi^2}(\Phi) Y^\kappa f, \quad \text{with} \quad \max(|\xi^1| + |\kappa|, |\xi^2| + |\kappa|) \leq N - 1 \quad \text{and} \quad j \leq 2N - 1 - \xi_P^1 - \xi_P^2 - \kappa_P.$$

Note that (4.75) also provides us, through Theorem 4.4.9, that, for all $\max(|\xi| + |\beta|, 1 + |\xi|) \leq N - 3$,

$$\forall |x| \leq t < T, \quad z \in \mathbf{k}_1, \quad j \leq 2N - 5 - \xi_P - \beta_P, \quad \int_v |z^j P_\xi^X(\Phi) Y^\beta f| \frac{dv}{(v^0)^2} \lesssim \epsilon \frac{\log^{2j}(3+t)}{\tau_+^3}.$$

For the exterior region, use Proposition 4.4.10 and $\mathbb{E}_N^X[f] \leq 3\epsilon$ to derive, for all $\max(|\xi| + |\beta|, |\xi| + 1) \leq N - 3$,

$$\forall (t, x) \in V_0(T), \quad z \in \mathbf{k}_1, \quad j \leq 2N - 6 - \xi_P - \beta_P, \quad \int_v |z^j P_\xi^X(\Phi) Y^\beta f| \frac{dv}{(v^0)^2} \lesssim \epsilon \frac{\log^{2(j+1)}(3+t)}{\tau_+^3 \tau_-}.$$

We summarize all these results in the following proposition (the last estimate comes from Corollary 4.4.7).

Proposition 4.8.11. *If ϵ is small enough, then $\mathbb{E}_{N-1}^X[f] \leq 3\epsilon$ and $\mathbb{E}_N^X[f] \leq 3\epsilon$ hold on $[0, T]$. Moreover, we have, for all $\max(|\xi| + |\beta|, |\xi| + 1) \leq N - 3$, $z \in \mathbf{k}_1$ and $j \leq 2N - 6 - \xi_P - \beta_P$,*

$$\begin{aligned} \forall (t, x) \in [0, T] \times \mathbb{R}^3, \quad \int_v |z^j P_\xi^X(\Phi) Y^\beta f| \frac{dv}{(v^0)^2} &\lesssim \epsilon \frac{\log^{2j}(3+t)}{\tau_+^3} \mathbb{1}_{t \geq |x|} + \epsilon \frac{\log^{2(j+1)}(3+t)}{\tau_+^3 \tau_-} \mathbb{1}_{|x| \geq t}, \\ \forall (t, x) \in [0, T] \times \mathbb{R}^3, \quad \int_v |z^j P_\xi^X(\Phi) Y^\beta f| dv &\lesssim \epsilon \frac{\log^{2j}(3+t)}{\tau_+^2 \tau_-}. \end{aligned}$$

4.8.5 The critical terms

We finally bound I_4 , defined in Proposition 4.8.10, when $|\gamma| = N$, which concerns only the improvement of the bound of the higher order energy norm $\overline{\mathbb{E}}_N[f]$. We keep the notations introduced in Subsection 4.8.3 and we start by precising them. Using the properties of the terms of (category 3), we remark that we necessary have

$$P_{\xi^0}(\Phi) = Y^{\xi^0} \Phi, \quad |\xi^0| = N - 1, \quad |\beta^0| \leq 1, \quad |\xi^1| = 0, \quad \beta = \beta^0, \quad \gamma_T = \xi_T^0 \quad \text{and} \quad j = q.$$

We are then led to prove

$$I_4 = \int_0^t \int_{\Sigma_s} \int_v \tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \right| \left| z^q P_{\xi^2}(\Phi) Y^{\beta^0} f \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} (1+t)^\eta \log^{aq}(3+t).$$

If $\gamma_T = \xi_T^0 \geq 1$, one can use inequality (4.15) of Proposition 4.3.7 and $|v^A| \lesssim \sqrt{v^0 v^L}$ in order to obtain

$$\tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \right| \lesssim \left(1 + \frac{\sqrt{v^L} \tau_+}{\sqrt{v^0} \tau_-} \right) \sum_{|\gamma_0| \leq N} |\nabla_{Z^{\gamma_0}} F| + \frac{\tau_+}{\tau_-} \sum_{|\gamma_0| \leq N} |\alpha(\mathcal{L}_{Z^{\gamma_0}}(F))| + |\rho(\mathcal{L}_{Z^{\gamma_0}}(F))|$$

and then split I_4 in four parts and bound them by $\epsilon^{\frac{3}{2}}$ or $\epsilon^{\frac{3}{2}}(1+t)^{\frac{3}{4}\eta}$, as I_1^1 , I_1^3 , I_3^1 and I_3^2 in Propositions 4.8.7 and 4.8.8. Otherwise, $\xi_P^0 = N - 1$ and $q \leq N - \xi_P^0 - \beta_P^0$ so that we take $i_2 \leq 2N - 1 - 2\xi_P^0 - \beta_P^0$

and $i_1 \leq 1 - \beta_P^0$. Then, we divide $[0, t] \times \mathbb{R}^3$ in two parts, $V_0(t)$ and its complement. Following the proof of Proposition 4.8.10, one can prove, as $\mathcal{E}_N^{Ext}[\tilde{F}] \lesssim \epsilon$ and $|\mathcal{L}_{Z^\gamma}(\tilde{F})| \lesssim \epsilon \tau_+^{-2}$ on $[0, T[$, that

$$\int_0^t \int_{\Sigma_s^0} \int_v \tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \right| \left| z^q P_{\xi^2}(\Phi) Y^{\beta^0} f \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} (1+t)^{\frac{3}{4}\eta}.$$

To lighten the notations, let us denote the null decomposition of $\mathcal{L}_{Z^\gamma}(F)$ by $(\alpha, \underline{\alpha}, \rho, \sigma)$. Recall from Lemma 4.2.4 that $\tau_+ |v^A| \lesssim v^0 \sum_{w \in \mathbf{k}_1} |w|$ and $\tau_+ v^L \lesssim \tau_- v^0 + v^0 \sum_{w \in \mathbf{k}_1} |w|$, so that

$$\begin{aligned} \tau_+ \left| \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \right| &\lesssim \tau_+ (|\alpha| + |\rho|) + \tau_+ \frac{v^L}{v^0} |\underline{\alpha}| + \tau_+ \frac{|v^A|}{v^0} (|\sigma| + |\underline{\alpha}|) \\ &\lesssim (\tau_+ |\alpha| + \tau_+ |\rho| + \tau_- |\underline{\alpha}|) + \sum_{w \in \mathbf{k}_1} |w| (|\sigma| + |\underline{\alpha}|). \end{aligned}$$

We can then split the remaining part of I_4 in two integrals. The one associated to $\sum_{w \in \mathbf{k}_1} |w| (|\sigma| + |\underline{\alpha}|)$ can be bounded by $\epsilon^{\frac{3}{2}}$ as I_1^1 in Proposition 4.8.7 since $i_1 + 1 \leq 2N - 1 - \beta_P^0$. For the one associated to $(\tau_+ |\alpha| + \tau_+ |\rho| + \tau_- |\underline{\alpha}|)$, \bar{I}_4 , we have

$$\begin{aligned} \bar{I}_4 &:= \int_0^t \int_{\Sigma_s^0} (\tau_+ |\alpha| + \tau_+ |\rho| + \tau_- |\underline{\alpha}|) \int_v \left| z^q P_{\xi^2}(\Phi) Y^{\beta^0} f \right| \frac{dv}{v^0} dx ds \\ &\lesssim \int_0^t \sqrt{\mathcal{E}_N[F](s)} \left\| \sqrt{\tau_+} \int_v \left| z^q P_{\xi^2}(\Phi) Y^{\beta^0} f \right| \frac{dv}{v^0} \right\|_{L^2(\Sigma_s^0)} ds \\ &\lesssim \sqrt{\epsilon} \int_0^t \sqrt{\mathcal{E}_N[F](s)} \left\| \tau_+ \int_v \left| z^{i_1} Y^{\beta^0} f \right| \frac{dv}{(v^0)^2} \right\|_{L^\infty(\Sigma_s^0)}^{\frac{1}{2}} \left\| \int_v \left| z^{i_2} P_{\xi^2}(\Phi)^2 Y^{\beta^0} f \right| dv \right\|_{L^1(\Sigma_s^0)}^{\frac{1}{2}} ds. \end{aligned}$$

Using the bootstrap assumptions (4.51), (4.57) and the pointwise decay estimate on $\int_v \left| z^{i_1} Y^{\beta^0} f \right| \frac{dv}{(v^0)^2}$ given in Proposition 4.8.11, we finally obtain

$$\bar{I}_4 \lesssim \sqrt{\epsilon} \int_0^t (1+s)^{\frac{\eta}{2}} \frac{\sqrt{\epsilon} \log^{i_1}(3+s)}{1+s} \sqrt{\epsilon} (1+s)^{\frac{\eta}{2}} \log^{\frac{\eta}{2} i_2}(3+s) ds \lesssim \epsilon^{\frac{3}{2}} (1+t)^\eta \log^{a q}(3+t),$$

which concludes the improvement of the bootstrap assumption (4.51).

Remark 4.8.12. *In view of the computations made to estimate \bar{I}_4 , note that.*

- *The use of Theorem 4.4.9, instead of (4.68) combined with $1 \lesssim v^0 v^L$ and Lemma 4.2.4, was necessary. Indeed, for the case $q = 0$, a decay rate of $\log^2(3+t) \tau_+^{-3}$ on $\int_v \left| Y^{\beta^0} f \right| \frac{dv}{(v^0)^2}$ would avoid us to close energy estimates on $\mathcal{E}_N[F]$ and $\overline{E}_N[f]$.*
- *Similarly, it was crucial to have a better bound on $\mathcal{E}_N^{Ext}[G](t)$ than $\epsilon(1+t)^\eta$ as the decay rate given by Proposition 4.8.11 on $\int_v \left| Y^{\beta^0} f \right| \frac{dv}{(v^0)^2}$ is weaker, in the $t+r$ direction, outside the light cone.*

Note that Propositions 4.8.2, 4.8.3, 4.8.5, 4.8.7, 4.8.8 and 4.8.10 also prove that

$$\mathbb{A}[f](t) := \sum_{i=1}^2 \sum_{\substack{|\xi^i|+|\beta| \leq N \\ |\xi^i| \leq N-2}} \sum_{\substack{|\zeta^i|+|\beta| \leq N \\ |\zeta^i| \leq N-1}} \mathbb{E} [P_{\xi^1}(\Phi) P_{\xi^2}(\Phi) Y^\beta f](t) + \mathbb{E} [P_{\zeta^1}^X(\Phi) P_{\zeta^2}^X(\Phi) Y^\beta f](t) \lesssim \epsilon (1+t)^{\frac{3}{4}\eta}. \quad (4.76)$$

Indeed, to estimate this energy norm, we do not have to deal with the critical terms of this subsection (as $|\xi^i| \leq N-2$ and according to Proposition 4.3.34).

4.9 L^2 decay estimates for the velocity averages of the Vlasov field

In view of commutation formula of Propositions 4.3.39 and 4.3.40, we need to prove enough decay on quantities such as $\left\| \sqrt{\tau_-} \int_v |Y^\beta f| dv \right\|_{L_x^2}$, for all $|\beta| \leq N$. Applying Proposition 4.8.11, we are already able to obtain such estimates if $|\beta| \leq N-3$ (see Proposition 4.9.14 below). The aim of this section is then to treat the case

of the higher order derivatives. For this, we follow the strategy used in [18] (Section 4.5.7). Before exposing the proceeding, let us rewrite the system. Let I_1, I_2 and I_1^q , for $N - 5 \leq q \leq N$, be the sets defined as

$$\begin{aligned} I_1 &:= \{\beta \text{ multi-index} / N - 5 \leq |\beta| \leq N\} = \{\beta_1^1, \beta_2^1, \dots, \beta_{|I_1|}^1\}, & I_1^q &:= \{\beta \in I_1 / |\beta| = q\}, \\ I_2 &:= \{\beta \text{ multi-index} / |\beta| \leq N - 5\} = \{\beta_1^2, \beta_2^2, \dots, \beta_{|I_2|}^2\}, \end{aligned}$$

and R^1 and R^2 be two vector valued fields, of respective length $|I_1|$ and $|I_2|$, such that

$$R_j^1 = Y^{\beta_j^1} f \quad \text{and} \quad R_j^2 = Y^{\beta_j^2} f.$$

We will sometimes abusevely write $j \in I_i$ instead of $\beta_j^i \in I_i$ (and similarly for $j \in I_1^k$). The goal now is to prove L^2 estimates on $\int_v |R^1| dv$. Finally, we denote by \mathbb{V} the module over the ring $C^0([0, T[\times \mathbb{R}_x^3 \times \mathbb{R}_v^3])$ engendred by $(\partial_{v^i})_{1 \leq i \leq 3}$. In the following lemma, we apply the commutation formula of Proposition 4.3.23 in order to express $T_F(R^1)$ in terms of R^1 and R^2 and we use Lemma 4.3.30 for transforming the vector fields $\Gamma^\sigma \in \mathbb{G}^{|\sigma|}$.

Lemma 4.9.1. *There exists two matrices functions $A : [0, T[\times \mathbb{R}^3 \times \mathbb{R}_v^3 \rightarrow \mathfrak{M}_{|I_1|}(\mathbb{V})$ and $B : [0, T[\times \mathbb{R}^3 \times \mathbb{R}_v^3 \rightarrow \mathfrak{M}_{|I_1|, |I_2|}(\mathbb{V})$ such that $T_F(R^1) + AR^1 = BR^2$. Furthermore, if $1 \leq i \leq |I_1|$, A and B are such that $T_F(R_i^1)$ is a linear combination, with good coefficients $c(v)$, of the following terms, where $r \in \{1, 2\}$ and $\beta_j^r \in I_r$.*

- $$z^d P_{k,p}(\Phi) \frac{v^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} R_j^r, \quad (\text{type 1})$$

where $z \in \mathbf{k}_1$, $d \in \{0, 1\}$, $\max(|\gamma|, |k| + |\beta_j^r|) \leq |\beta_i^1|$, $|k| \leq |\beta_i^1| - 1$, $|k| + |\gamma| + |\beta_j^r| \leq |\beta_i^1| + 1$ and $p + k_P + (\beta_j^r)_P + d \leq (\beta_i^1)_P$.

- $$P_{k,p}(\Phi) \mathcal{L}_{XZ^\gamma_0}(F) \left(v, \nabla_v (c(v) P_{q,s}(\Phi) R_j^r) \right), \quad (\text{type 2})$$

where $|k| + |q| + |\gamma_0| + |\beta_j^r| \leq |\beta_i^1| - 1$, $|q| \leq |\beta_i^1| - 2$, $p + s + k_P + q_P + (\beta_j^r)_P \leq (\beta_i^1)_P$ and $p \geq 1$.

- $$P_{k,p}(\Phi) \mathcal{L}_{\partial Z^\gamma_0}(F) \left(v, \nabla_v (c(v) P_{q,s}(\Phi) R_j^r) \right), \quad (\text{type 3})$$

where $|k| + |q| + |\gamma_0| + |\beta_j^r| \leq |\beta_i^1| - 1$, $|q| \leq |\beta_i^1| - 2$, $p + s + |\gamma_0| \leq |\beta_i^1| - 1$ and $p + s + k_P + q_P + (\beta_j^r)_P \leq (\beta_i^1)_P$.

We also impose that $|\beta_j^2| \leq N - 6$ on the terms of (type 2), (type 3) and that $|\beta_j^1| \geq N - 4$ on the terms of (type 1), which is possible since $\beta \in I_1 \cap I_2$ if $|\beta| = N - 5$.

Remark 4.9.2. *Note that if $\beta_i^1 \in I_1^{N-5}$, then $A_i^q = 0$ for all $q \in \llbracket 1, |I_1| \rrbracket$. If $1 \leq n \leq 5$ and $\beta_i^1 \in I_1^{N-5+n}$, then the terms composing A_i^q are such that $\max(|k| + 1, |\gamma|) \leq n$ or $|k| + |q| + |\gamma_0| \leq n - 1$.*

Let us now write $R = H + G$, where H and G are the solutions to

$$\begin{cases} T_F(H) + AH = 0 & , \quad H(0, \cdot, \cdot) = R(0, \cdot, \cdot), \\ T_F(G) + AG = BR^2 & , \quad G(0, \cdot, \cdot) = 0. \end{cases}$$

The goal now is to prove L^2 estimates on the velocity averages of H and G . As the derivatives of F and Φ composing the matrix A are of low order, we will be able to commute the transport equation satisfied by H and to bound the L^1 norm of its derivatives of order 3 by estimating pointwise the electromagnetic field and the Φ coefficients, as we proceeded in Subsection 4.8.2. The required L^2 estimates will then follow from Klainerman-Sobolev inequalities. Even if we will be lead to modify the form of the equation defining G , the idea is to find a matrix K satisfying $G = KR^2$, such that $\mathbb{E}[KKR^2]$ do not grow too fast, and then to take advantage of the pointwise decay estimates on $\int_v |R^2| dv$ in order to obtain the expected decay rate on $\|\int_v |G| dv\|_{L_x^2}$.

Remark 4.9.3. *As in [6], we keep the v derivatives in the construction of H and G . It has the advantage of allowing us to use Lemma 4.3.27. If we had already transformed the v derivatives, as in [4], we would have obtained terms such as $x^\theta \partial g$ from $(\nabla_v g)^r$. Indeed, Lemma 4.3.27 would have led us to derive coefficients such as $\frac{x^k}{|x|}$ and then to deal, for instance, with factor such as $\frac{t^3}{|x|^3}$ (apply three boost to $\frac{x^k}{|x|}$). We would then*

have to work with an another commutation formula leading to terms such as $x^\theta \frac{v^\mu}{v^0} \partial(F)_{\mu\nu} H_j$ and would then need at least a decay rate of $\tau_+^{-\frac{3}{2}}$ on ρ , in the $t+r$ direction, in order to close the energy estimates on H . This could be obtained by assuming more decay on F initially in order to use the Morawetz vector field \overline{K}_0 or $\tau_-^b \overline{K}_0$ as a multiplier.

However, this creates two technical difficulties compared to what we did in [4]. The first one concerns H and will lead us to consider a new hierarchy (see Subsection 4.9.1). The other one concerns G and we will circumvent it by modifying the source term of the transport equation defining it (see Subsection 4.9.2).

Remark 4.9.4. In Subsection 4.9.2, we will consider a matrix D such that $T_F(R^2) = DR^2$ and we will need to estimate pointwise and independently of M , in order to improve the bootstrap assumption on $\mathcal{E}_{N-1}[F]$, the derivatives of the electromagnetic field of its components. It explains, in view of Remark 4.7.2, why we take I_2 such as $|\beta_j^2| \leq N-5$.

4.9.1 The homogeneous part

The purpose of this subsection is to bound L^1 norms of components of H and their derivatives. We will then be able to obtain the desired L^2 estimates through Klainerman-Sobolev inequalities. For that, we will make use of the hierarchy between the components of H given by $(\beta_i^1)_P$. However, as, for $N-4 \leq q \leq N$ and $\beta_i^1 \in I_1^q$, we need informations on $\|\widehat{Z}^\kappa H_j\|_{L_{x,v}^1}$, with $\beta_j^1 \in I_1^{q-1}$ and $|\kappa| = 4$, in order to close the energy estimate on $\widehat{Z}^\xi H_i$, with $|\xi| = 3$, we will add a new hierarchy in our energy norms. This leads us to define, for $\delta \in \{0, 1\}$,

$$\mathbb{E}_H^\delta(t) := \sum_{z \in \mathbf{k}_1} \sum_{q=0}^5 \sum_{|\beta| \leq 3+q} \sum_{i \in I_1^{N-q}} \sum_{j=0}^{2N+2+\delta-\beta_P-\beta_P^1} \log^{-j(\delta a+2)}(3+t) \mathbb{E} \left[z^j \widehat{Z}^\beta H_i \right] (t).$$

Lemma 4.9.5. Let $\widetilde{N} \geq N+3$, $0 \leq q \leq 5$, $i \in I_1^{N-q}$, $|\beta| \leq 3+q$, $z \in \mathbf{k}_1$ and $j \leq \widetilde{N} - \beta_P - (\beta_i^1)_P$. Then, $T_F(z^j \widehat{Z}^\beta H_i)$ can be bounded by a linear combination of the following terms, where

$$p \leq 3N, \quad \max(|k|+1, |\gamma|) \leq 8, \quad |\kappa| \leq |\beta|+1, \quad |\beta_l^1| \leq |\beta_i^1| \quad \text{and} \quad |\kappa| + |\beta_l^1| \leq |\beta_i^1|.$$

•

$$|F(v, \nabla_v(z^j)) Y^\beta H_i|. \quad (\text{category } 0 - H)$$

•

$$|P_{k,p}(\Phi)| |w^r Y^\kappa H_i| \left(|\nabla_{Z^\gamma} F| + \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(F))| + \frac{\tau_+}{\tau_-} \sqrt{\frac{v^L}{v^0}} |\sigma(\mathcal{L}_{Z^\gamma}(F))| \right), \quad (\text{category } 1 - H)$$

where $w \in \mathbf{k}_1$ and $r \leq \widetilde{N} - k_P - \kappa_P - (\beta_l^1)_P$.

•

$$\frac{\tau_+}{\tau_-} |\rho(\mathcal{L}_{Z^\gamma}(F))| |z^{j-1} Y^\kappa H_i| \quad \text{and} \quad \frac{\tau_+}{\tau_-} \sqrt{\frac{v^L}{v^0}} |\alpha(\mathcal{L}_{Z^\gamma}(F))| |z^r Y^\kappa H_i|, \quad (\text{category } 2 - H)$$

where $j-1, r = \widetilde{N} - \kappa_P - (\beta_l^1)_P$ and $r \leq j$.

The terms of (category 2 - H) can only appear if $j = \widetilde{N} - \beta_P - (\beta_l^1)_P$.

Proof. We merely sketch the proof as it is very similar to previous computations. One can express $T_F(\widehat{Z}^\beta H_i)$ using Lemma 4.9.1 and following what we did in the proof of Proposition 4.3.23. It then remains to copy the proof of Proposition 4.3.31 with $|\zeta_0| = 0$, which explains that we do not have terms of (category 3). Note that $\max(|k|+1, |\gamma|) \leq 8$ comes from Remark 4.9.2 and the fact that $|\kappa|$ can be equal to $|\beta|+1$ ensues from the transformation of the v derivative in the terms obtained from those of (type 2) and (type 3). \square

Remark 4.9.6. As $|\gamma| \leq 8 \leq N-3$, we have at our disposal pointwise decay estimates on the electromagnetic field (see Proposition (4.7.1)). Similarly, as $|k| \leq 7 \leq N-4$, Remark 4.7.6 gives us $|P_{k,p}(\Phi)| \lesssim \log^{M_2}(1+\tau_+)$.

We are now ready to bound \mathbb{E}_H^δ and then to obtain estimates on $\int_v |z^j H_i| dv$.

Proposition 4.9.7. *We have $\mathbb{E}_H^1 + \mathbb{E}_H^0 \lesssim \epsilon$ on $[0, T[$. Moreover, for $0 \leq q \leq 5$ and $|\beta| \leq q$,*

$$\forall (t, x) \in [0, T[\times \mathbb{R}^3, \quad z \in \mathbf{k}_1, \quad i \in I_1^{N-q}, \quad j \leq 2N - 1 - \beta_P - (\beta_i^1)_P, \quad \int_v |z^j Y^\beta H_i| dv \lesssim \epsilon \frac{\log^{2j+M_1}(3+t)}{\tau_+^2 \tau_-}.$$

Proof. In the same spirit as Corollary 4.6.3 and in view of commutation formula of Lemma 4.9.5 (applied with $\tilde{N} = 2N + 3$) as well as the assumptions on f_0 , there exists $C_H > 0$ such that $\mathbb{E}_H^0(0) \leq \mathbb{E}_H^1(0) \leq C_H \epsilon$. We can prove that they both stay bounded by $3C_H \epsilon$ by the continuity method. As it is very similar to what we did previously, we only sketch the proof. Consider $\delta \in \{0, 1\}$, $0 \leq r \leq 5$, $i \in I_1^{N-r}$, $|\beta| \leq 3 + r$, $z \in \mathbf{k}_1$ and $j \leq 2N + 2 + \delta - \beta_P - (\beta_i^1)_P$. The goal is to prove that

$$\int_0^t \int_{\Sigma_s} \int_v |T_F(z^j H_i)| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} \log^{j(\delta a + 2)}(3+t).$$

According to Lemma 4.9.5 (still applied with $\tilde{N} = 2N + 3$), it is sufficient to obtain, if $\delta = 1$, that the integral over $[0, t] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$ of all terms of (category 0 – H)-(category 2 – H) are bounded by $\epsilon^{\frac{3}{2}} \log^{j(a+2)}(3+t)$. If $\delta = 0$, we only have to deal with terms of (category 0 – H) and (category 1 – H) and to estimate their integrals by $\epsilon^{\frac{3}{2}} \log^{2j}(3+t)$. In view of Remark 4.9.6, we only have to apply (or rather follow the computations of) Propositions 4.8.1, 4.8.2 and 4.8.3. The pointwise decay estimates then ensue from the Klainerman-Sobolev inequality of Corollary 4.4.8. \square

Remark 4.9.8. *A better decay rate, $\log^{2j}(3+t)\tau_+^{-2}\tau_-^{-1}$, could be proved in the previous proposition by controlling a norm analogous to $\mathbb{E}_N^X[f]$ but we do not need it to close the energy estimates on F .*

Remark 4.9.9. *We could avoid any hypothesis on the derivatives of order $N + 1$ and $N + 2$ of F^0 (see Subsection 17.2 of [17]).*

4.9.2 The inhomogeneous part

As the matrix B in $T_F(G) + AG = BR^2$ contains top order derivatives of the electromagnetic field, we cannot commute the equation and prove L^1 estimates on $\widehat{Z}G$. Let us explain schematically how we will obtain an L^2 estimate on $\int_v |G| dv$ by recalling how we proceeded in [4]. We did not work with modified vector field and the matrices A and B did not hide v derivatives of G . Then we introduced K the solution of $T_F(K) + AK + KD = B$ which initially vanishes and where $T_F(R^2) = DR^2$. Thus $G = KR^2$ and we proved $\mathbb{E}[|K|^2 |R^2|] \leq \epsilon$ so that the expected L^2 decay estimate followed from

$$\left\| \int_v |G| dv \right\|_{L_x^2} \lesssim \left\| \int_v |R^2| dv \right\|_{L_x^\infty}^{\frac{1}{2}} \mathbb{E}[|K|^2 |R^2|]^{\frac{1}{2}}.$$

The goal now is to adapt this process to our situation. There are two obstacles.

- The v derivatives hidden in the matrix A will then be problematic and we need first to transform them.
- The components of the (transformed) matrix A have to decay sufficiently fast. We then need to consider a larger vector valued field than G by including components such as $z^j G_i$ in order to take advantage of the hierarchies in the source terms already used before.

Recall from Definition 4.2.6 that we considered an ordering on \mathbf{k}_1 and that, if κ is a multi-index, we have

$$z^\kappa = \prod_{i=1}^{|\kappa|} z_{\kappa_i} \quad \text{and} \quad |z^\kappa| \leq \sum_{w \in \mathbf{k}_1} |w|^{|\kappa|}.$$

In this section, we will sometimes have to work with quantities such as z^κ rather than with z^j , where $j \in \mathbb{N}$.

Definition 4.9.10. *Let I and I^q , for $N - 5 \leq q \leq N$, be the sets*

$$I := \{(\kappa, \beta) / N - 5 \leq |\beta| \leq N \text{ and } |\kappa| \leq N - \beta_P\} = \{(\kappa_1, \beta_1), \dots, (\kappa_{|I|}, \beta_{|I|})\}, \quad I^q := \{(\kappa, \beta) \in I / |\beta| = q\}.$$

Define now L , the vector valued fields of length $|I|$, such that

$$L_i = z^{\kappa_i} G_j, \quad \text{with} \quad \beta_j^1 = \beta_i, \quad \text{and} \quad [i]_I := |\kappa_i|.$$

Moreover, for $Y \in \mathbb{Y}$, $1 \leq j \leq |I_1|$ and $1 \leq i \leq |I|$, we define j_Y and i_Y the indices such that

$$R_{j_Y}^1 = Y Y^{\beta_j^1} f \quad \text{and} \quad L_{i_Y} = z^{\kappa_{i_Y}} G_{j_Y}.$$

The following result will be useful for transforming the v derivatives.

Lemma 4.9.11. *Let $Y \in \mathbb{Y}$ and $\beta_i^1 \in I_1 \setminus I_1^N$. Then*

$$YG_i = G_{i_Y} + H_{i_Y} - YH_i.$$

Proof. Recall that $R = H + G$ and remark that $YR_i^1 = YY^{\beta_i^1}f = R_{i_Y}^1$. \square

We now describe the source terms of the equations satisfied by the components of L .

Proposition 4.9.12. *There exists $N_1 \in \mathbb{N}^*$, a vector valued field W and three matrices valued functions $\bar{A} : [0, T[\times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathfrak{M}_{|I|}(\mathbb{R})$, $\bar{B} : [0, T[\times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathfrak{M}_{|I|, N_1}(\mathbb{R})$, $\bar{D} : [0, T[\times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathfrak{M}_{N_1}(\mathbb{R})$ such that*

$$T_F(L) + \bar{A}L = \bar{B}W, \quad T_F(W) = \bar{D}W \quad \text{and} \quad \sum_{z \in \mathbf{k}_1} \int_v |z^2 W| dv \lesssim \epsilon \frac{\log^{3N+M_1}(3+t)}{\tau_+^2 \tau_-}.$$

In order to depict these matrices, we use the quantity $[q]_W$, for $1 \leq q \leq N_1$, which will be defined during the construction of W in the proof. \bar{A} and \bar{B} are such that $T_F(L_i)$ can be bounded, for $1 \leq i \leq |I|$, by a linear combination of the following terms, where $|\gamma| \leq 5$, $1 \leq j, q \leq |I|$ and $1 \leq r \leq N_1$.

$$(\tau_- (|\rho(F)| + |\sigma(F)| + |\underline{\alpha}(F)|) + \tau_+ |\alpha(F)|) |L_j|, \quad \text{with} \quad [j]_I = [i]_I - 1. \quad (\text{category } 0 - \bar{A})$$

$$\log^{M_1}(3+t) |L_j| \left(|\nabla_{Z^\gamma} F| + \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(F))| + \frac{\tau_+}{\tau_-} \sqrt{\frac{v\bar{L}}{v^0}} |\sigma(\mathcal{L}_{Z^\gamma}(F))| \right). \quad (\text{category } 1 - \bar{A})$$

$$\frac{\tau_+}{\tau_-} |\rho(\mathcal{L}_{Z^\gamma}(F))| |L_j| + \frac{\tau_+}{\tau_-} \sqrt{\frac{v\bar{L}}{v^0}} |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))| |L_q|, \quad \text{with} \quad [j]_I + 1, [q]_I \leq [i]_I. \quad (\text{category } 2 - \bar{A})$$

$$|P_{k,p}(\Phi)| |W_r| \left(|\nabla_{Z^\zeta} F| + \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\zeta}(F))| + \frac{\tau_+}{\tau_-} \sqrt{\frac{v\bar{L}}{v^0}} |\sigma(\mathcal{L}_{Z^\zeta}(F))| \right), \quad (\text{category } 1 - \bar{B})$$

where $p \leq 2N$, $|k| \leq N-1$ and $|k| + |\zeta| \leq N$. Moreover, if $|k| \geq 6$, there exists κ and β such that $W_r = z^\kappa Y^\beta f$, $|k| + |\beta| \leq N$ and $|\kappa| \leq N+1 - k_P - \beta_P$.

The matrix \bar{D} is such that, for $1 \leq i \leq N_1$, $T_F(W_i)$ is bounded by a linear combination of the following expressions, where $|\gamma| \leq N-5$ and $1 \leq j, q \leq N_1$.

$$(\tau_- (|\rho(F)| + |\sigma(F)| + |\underline{\alpha}(F)|) + \tau_+ |\alpha(F)|) |W_j|, \quad \text{with} \quad [j]_W = [i]_W - 1. \quad (\text{category } 0 - \bar{D})$$

$$\log^{M_1}(3+t) |W_j| \left(|\nabla_{Z^\gamma} F| + \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(F))| + \frac{\tau_+}{\tau_-} \sqrt{\frac{v\bar{L}}{v^0}} |\sigma(\mathcal{L}_{Z^\gamma}(F))| \right). \quad (\text{category } 1 - \bar{D})$$

$$\frac{\tau_+}{\tau_-} |\rho(\mathcal{L}_{Z^\gamma}(F))| |W_j| + \frac{\tau_+}{\tau_-} \sqrt{\frac{v\bar{L}}{v^0}} |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))| |W_q|, \quad \text{with} \quad [j]_W + 1, [q]_W \leq [i]_W. \quad (\text{category } 2 - \bar{D})$$

Proof. The main idea is to transform the v derivatives in AG , following the proof of Lemma 4.3.28, and then to apply Lemma 4.9.11 in order to eliminate all derivatives of G in the source term of the equations. We then define W as the vector valued field, and N_1 as its length, containing all the following quantities

- $z^j Y^\beta f$, with $z \in \mathbf{k}_1$, $|\beta| \leq N-5$ and $j \leq N+1 - \beta_P$,
- $z^j (H_{i_Y} - YH_i)$, with $z \in \mathbf{k}_1$, $Y \in \mathbb{Y}$, $\beta_i^1 \in I_1 \setminus I_1^N$ and $j \leq N+3 - (\beta_{i_Y}^1)_P$.
- $z^j Y^\beta H_i$, with $z \in \mathbf{k}_1$, $|\beta| + |\beta_i^1| \leq N$ and $j \leq N+3 - \beta_P - (\beta_i^1)_P$.

Let us make three remarks.

- If $1 \leq i \leq N_0$, we can define, in each of the three cases, $[i]_W := j$.
- Including the terms $z^{N+1-\beta_P} Y^\beta f$ and $z^{N+1-(\beta_{i_Y}^1)_P} (H_{i_Y} - YH_i)$ in W allows us to avoid any term of category 2 related to \bar{B} .
- The components such as $z^j Y^\beta H_i$ are here in order to obtain an equation of the form $T_F(W) = \bar{D}W$.

The form of the matrix \overline{D} then follows from Proposition 4.3.31 if $Y_i = z^j Y^\beta f$ and from Lemma 4.9.5, applied with $\tilde{N} = N + 3$, otherwise (we made an additional operation on the terms of category 0 which will be more detailed for the matrix \overline{A}). Note that we use Remark 4.7.6 to estimate all quantities such as $P_{k,p}(\Phi)$. The decay rate on $\int_v |z^2 W| dv$ follows from Proposition 4.8.11 and 4.9.7.

We now turn on the construction of the matrices \overline{A} and \overline{B} . Consider then $1 \leq i \leq |I|$ and $1 \leq q \leq |I_1|$ so that $L_i = z^{\kappa_i} G_q$ and $|\kappa_i| \leq N - (\beta_q^1)_P$. Observe that

$$T_F(L_i) = T_F(z^{\kappa_i} G_q) + z^{\kappa_i} T_F(G_q) = F(v, \nabla_v(z^{\kappa_i})) G_q + z^{\kappa_i} T_F(G_q).$$

The first term on the right hand side gives terms of (category 0 – \overline{A}) and (category 1 – \overline{A}) as, following the computations of Proposition 4.8.1, we have

$$\nabla_v \left(\prod_{r=1}^{|\kappa_i|} z_r \right) = \sum_{p=1}^{|\kappa_i|} \nabla_v(z_p) \prod_{r \neq p} z_r, \quad |F(v, \nabla_v z_p)| \lesssim \tau_- (|\rho(F)| + |\sigma(F)| + |\alpha(F)|) + \tau_+ |\alpha(F)| + \sum_{w \in \mathbf{k}_1} |wF|.$$

The remaining quantity, $z^{\kappa_i} T_F(G_q) = -z^{\kappa_i} A_q^r G_r + z^{\kappa_i} B_q^r R_r^2$, is described in Lemma 4.9.1. Express the terms given by $z^{\kappa_i} A_q^r G_r$ in null components and transform the v derivatives²⁶ of G_r using Lemma 4.9.11, so that, schematically (see (4.24)),

$$\begin{aligned} v^0 (\nabla_v G_r)^r &= Y G_r + (t-r) \partial G_r = G_{r_Y} + H_{r_Y} - Y H_r + (t-r)(G_{r_\partial} + H_{r_\partial} - \partial H_r) \quad \text{and} \\ v^0 \partial_{v^b} G_r &= Y_{0b} G_r + x \partial G_r = G_{r_{Y_{0b}}} + H_{r_{Y_{0b}}} - Y_{0b} H_r + x(G_{r_\partial} + H_{r_\partial} - \partial H_r). \end{aligned}$$

By Remark 4.9.2, the Φ coefficients and the electromagnetic field are both derived less than 5 times. We then obtain, with similar operations as those made in proof of Proposition 4.3.31, the matrix \overline{A} and the columns of the matrix \overline{B} hitting the component of W of the form $z^j (H_{l_Y} - Y H_l)$. For $z^{\kappa_i} B_q^r R_r^2$, we refer to the proof of Proposition 4.3.31, where we already treated such terms. \square

To lighten the notations and since there will be no ambiguity, we drop the index I (respectively W) of $[i]_I$ for $1 \leq i \leq |I|$ (respectively $[j]_W$ for $1 \leq j \leq N_1$). Let us introduce K the solution of $T_F(K) + \overline{A}K + K\overline{D} = \overline{B}$, such as $K(0, \dots) = 0$. Then, $KY = L$ since they are solution of the same system and they both initially vanish. The goal now is to control $\mathbb{E}[|K|^2 |Y|]$. As, for $1 \leq i \leq |I|$ and $1 \leq j, p \leq N_1$,

$$T_F \left(|K_i^j|^2 W_p \right) = |K_i^j|^2 \overline{D}_p^q W_q - 2 \left(\overline{A}_i^k K_k^j + K_i^r \overline{D}_r^j \right) K_i^j W_p + 2 \overline{B}_i^j K_i^j W_p, \quad (4.77)$$

we consider \mathbb{E}_L , the following hierarchied energy norm,

$$\mathbb{E}_L(t) := \sum_{\substack{1 \leq j, p \leq N_1 \\ 1 \leq i \leq |I|}} \log^{-4[i]-2[p]+4[j]}(3+t) \mathbb{E} \left[\left| K_i^j \right|^2 W_p \right] (t).$$

The sign in front of $[j]$ is related to the fact that the hierarchy is inverted on the terms coming from $K\overline{D}$. It prevents us to expect a better estimate than $\mathbb{E}_L(t) \lesssim \log^{4N+12}(3+t)$.

Lemma 4.9.13. *We have, for $M_0 = 4N + 12$ and if ϵ small enough, $\mathbb{E}_L(t) \lesssim \epsilon \log^{M_0}(3+t)$ for all $t \in [0, T[$.*

Proof. We use again the continuity method. Let $T_0 \in [0, T[$ be the largest time such that $\mathbb{E}_L(t) \leq 2\epsilon \log^{M_0}(3+t)$ for all $t \in [0, T_0[$ and let us prove that, if ϵ is small enough,

$$\forall t \in [0, T_0[, \quad \mathbb{E}_L(t) \lesssim \epsilon^{\frac{3}{2}} \log^{M_0}(3+t). \quad (4.78)$$

As $T_0 > 0$ by continuity (K vanishes initially), we would deduce that $T_0 = T$. We fix for the remaining of the proof $1 \leq i \leq |I|$ and $1 \leq j, p \leq N_1$. According to the energy estimate of Proposition 4.4.1, (4.78) would follow if we prove that

$$\begin{aligned} I_{\overline{A}, \overline{D}} &:= \int_0^t \int_{\Sigma_s} \int_v \left| |K_i^j|^2 \overline{D}_p^q W_q - 2 \left(\overline{A}_i^k K_k^j + K_i^r \overline{D}_r^j \right) K_i^j W_p \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} \log^{M_0+4[i]+2[p]-4[j]}(3+t), \\ I_{\overline{B}} &:= \int_0^t \int_{\Sigma_s} \int_v \left| B_i^j \right| \left| K_i^j W_p \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

²⁶Note that this is possible since $\partial_v G_r$ can only appear if $\beta_r^1 \in I_1 \setminus I_1^N$.

Let us start by $I_{\overline{A}, \overline{D}}$ and note that in all the terms given by Proposition 4.9.12, the electromagnetic field is derived less than $N - 5$ times so that we can use the pointwise decay estimates given by Remark 4.7.2. The terms of (category 1 - \overline{A}) and (category 1 - \overline{D}) can be easily handled (as in Proposition 4.8.2). We then only treat the following cases, where $|\gamma| \leq N - 5$ (the other terms are similar).

$$\left| \overline{D}_r^j \right| = \tau_- (|\rho(F)| + |\sigma(F)| + |\underline{\alpha}(F)|) + \tau_+ |\alpha(F)|, \quad \text{with } [j] = [r] - 1,$$

$$\left| \overline{A}_i^k \right| \lesssim \frac{\tau_+ \sqrt{v^L}}{\tau_- \sqrt{v^0}} |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|, \quad \text{with } [k] \leq [i], \quad \text{and} \quad \left| \overline{D}_p^q \right| \lesssim \frac{\tau_+}{\tau_-} |\rho(\mathcal{L}_{Z^\gamma}(F))|, \quad \text{with } [q] < [p].$$

Without any summation on the indices r, k and q , we have, using Remark 4.7.2, $1 \lesssim \sqrt{v^0 v^L}$ and the Cauchy-Schwarz inequality several times,

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \int_v \left| K_i^r \overline{D}_r^j K_i^j W_p \right| \frac{dv}{v^0} dx ds &\lesssim \sqrt{\epsilon} \int_0^t \frac{\log(3+s)}{1+s} \left| \mathbb{E} \left[|K_i^r|^2 W_p \right] (s) \mathbb{E} \left[|K_i^j|^2 W_p \right] (s) \right|^{\frac{1}{2}} ds \\ &\lesssim \epsilon^{\frac{3}{2}} \log^{2+M_0+4[i]+2[p]-2[r]-2[j]}(3+t) \lesssim \epsilon^{\frac{3}{2}} \log^{M_0+4[i]+2[p]-4[j]}(3+t), \\ \int_0^t \int_{\Sigma_s} \int_v \left| \overline{A}_i^k K_k^j K_i^j W_p \right| \frac{dv}{v^0} dx ds &\lesssim \sqrt{\epsilon} \int_{u=-\infty}^t \frac{\tau_+}{\tau_+ \tau_-^{\frac{3}{2}}} \int_{C_u(t)} \int_v \frac{v^L}{v^0} |K_k^j| |W_p|^{\frac{1}{2}} |K_i^j| |W_p|^{\frac{1}{2}} dv dC_u(t) du \\ &\lesssim \sqrt{\epsilon} \left| \mathbb{E} \left[|K_k^j|^2 W_p \right] (t) \mathbb{E} \left[|K_i^j|^2 W_p \right] (t) \right|^{\frac{1}{2}} \int_{u=-\infty}^{+\infty} \frac{du}{\tau_-^{\frac{3}{2}}} \\ &\lesssim \epsilon^{\frac{3}{2}} \log^{M_0+2[k]+2[i]+2[p]-4[j]}(3+t) \lesssim \epsilon^{\frac{3}{2}} \log^{M_0+4[i]+2[p]-4[j]}(3+t), \\ \int_0^t \int_{\Sigma_s} \int_v \left| K_i^j \right|^2 \left| \overline{D}_p^q W_q \right| \frac{dv}{v^0} dx ds &\lesssim \sqrt{\epsilon} \int_0^t \int_{\Sigma_s} \int_v \log(3+s) \frac{\sqrt{v^L v^0}}{\tau_+^{\frac{1}{2}} \tau_-^{\frac{3}{2}}} |K_i^j|^2 |W_q| \frac{dv}{v^0} dx ds \\ &\lesssim \sqrt{\epsilon} \left(\int_0^t \frac{\log(3+s)}{1+s} ds + \log(3+t) \int_{-\infty}^{+\infty} \frac{du}{\tau_-^3} \right) \sup_{[0,t]} \mathbb{E} \left[|K_i^j|^2 W_q \right] \\ &\lesssim \epsilon^{\frac{3}{2}} \log^{2+M_0+4[i]+2[q]-4[j]}(3+t) \lesssim \epsilon^{\frac{3}{2}} \log^{M_0+4[i]+2[p]-4[j]}(3+t). \end{aligned}$$

It remains to study $I_{\overline{B}}$. The form of \overline{B}_i^j is given by Propoposition 4.9.12 and the computations are close to the ones of Proposition 4.8.7. We then only consider the following two cases,

$$\left| \overline{B}_i^j K_i^j W_p \right| \lesssim \log^{M_1} (1 + \tau_+) \frac{\tau_+ \sqrt{v^L}}{\tau_- \sqrt{v^0}} |\sigma(\mathcal{L}_{Z^\zeta}(F))| |K_i^j| |W_p|, \quad \text{with } |\zeta| \leq N \quad \text{and}$$

$$\left| \overline{B}_i^j K_i^j W_p \right| \lesssim |\Phi^r P_\xi(\Phi)| |\nabla_{Z^\gamma} F| |K_i^j W_p|, \quad \text{with } r \leq 2N, \quad |\xi| + |\gamma| \leq N \quad \text{and} \quad 6 \leq |\xi| \leq N - 1.$$

In the first case, using the Cauchy-Schwarz inequality twice (in (t, x) and then in v), we get

$$\begin{aligned} I_{\overline{B}} &\lesssim \int_{u=-\infty}^t \|\sigma(\mathcal{L}_{Z^\zeta}(F))\|_{L^2(C_u(t))} \left| \int_{C_u(t)} \log^{2M_1} (1 + \tau_+) \frac{\tau_+^2}{\tau_-^2} \left| \int_v \sqrt{\frac{v^L}{v^0}} |K_i^j W_p| \frac{dv}{v^0} \right|^2 dC_u(t) \right|^{\frac{1}{2}} du \\ &\lesssim \sqrt{\epsilon} \sum_{q=0}^{+\infty} \int_{u=-\infty}^t \frac{1}{\tau_-^{\frac{3}{2}}} \left\| \tau_+^{\frac{1}{4}} \int_v |W_p| \frac{dv}{(v^0)^2} \right\|_{L^\infty(C_u^q(t))}^{\frac{1}{2}} \left\| \int_v \frac{v^L}{v^0} |K_i^j|^2 |W_p| dv \right\|_{L^1(C_u^q(t))}^{\frac{1}{2}} du \\ &\lesssim \epsilon^{\frac{3}{2}} \int_{-\infty}^{+\infty} \frac{du}{\tau_-^{\frac{3}{2}}} \sum_{q=0}^{+\infty} \frac{\log^{\frac{M_0+4[i]+2[p]+3N+M_1}{2}}(3+t_{q+1})}{(1+t_q)^{\frac{1}{8}}} \lesssim \epsilon^{\frac{3}{2}}, \end{aligned}$$

using the bootstrap assumption on \mathbb{E}_L and $\int_v |W_p| \frac{dv}{(v^0)^2} \lesssim \int_v |W_p| \frac{v^L}{v^0} dv \lesssim \epsilon \log^{3N+M_1}(3+t) \tau_+^{-3}$, which comes from Proposition 4.9.12 and Lemma 4.3.2. For the remaining case, we have $|\gamma| \leq N - 6$ and we can then use the pointwise decay estimates on the electromagnetic field given by Proposition 4.7.1. Moreover, by Proposition 4.9.12, we have that

$$W_p = z^\kappa Y^\beta f, \quad \text{with } |\xi| + |\beta| \leq N \quad \text{and} \quad |\kappa| \leq N + 1 - \beta_P - \xi_P.$$

Suppose first that $|\kappa| \leq 2N - 1 - \beta_P - 2\xi_P$. Then, since $|\Phi|^r |\nabla_{Z^\gamma} F| \lesssim \sqrt{\epsilon} \tau_+^{-\frac{3}{4}} \tau_-^{-1}$ and $1 \lesssim \sqrt{v^0 v^L}$, it comes

$$\left| \overline{B}_i^j K_i^j W_p \right| \lesssim \sqrt{\epsilon} \left(\frac{v^0}{\tau_+^{\frac{5}{4}}} + \frac{v^L}{\tau_+^{\frac{1}{4}} \tau_-^2} \right) \left(|z^\kappa P_\xi(\Phi)^2 Y^\beta f| + |K_i^j|^2 |W_p| \right).$$

Hence, we can obtain $I_{\overline{B}} \lesssim \epsilon^{\frac{3}{2}}$ by following the computations of Proposition 4.8.2, as, by the bootstrap assumptions on $\overline{\mathbb{E}}_N[f]$ and \mathbb{E}_L ,

$$\mathbb{E}[z^\kappa P_\xi(\Phi)^2 Y^\beta f](t) + \mathbb{E} \left[|K_i^j|^2 |W_p| \right](t) \lesssim \epsilon(1+t)^{\frac{1}{8}}.$$

Otherwise, $|\kappa| = 2N - \beta_P - 2\xi_P$ so that $\xi_P = N - 1$, $|\beta| \leq 1$ and $|\kappa| = 2 - \beta_P$. We can then write $z^\kappa = z z^{\kappa_0}$ and find $q \in \llbracket 1, N_1 \rrbracket$ such that $W_q = z^2 z^{\kappa_0} Y^\beta f$. It remains to follow the previous case after noticing that

$$\left| \overline{B}_i^j K_i^j W_p \right| \lesssim \sqrt{\epsilon} \left(\frac{v^0}{\tau_+^{\frac{5}{4}}} + \frac{v^L}{\tau_+^{\frac{1}{4}} \tau_-^2} \right) \left(|z^{\kappa_0} P_\xi(\Phi)^2 Y^\beta f| + |K_i^j|^2 |W_q| \right) \quad \text{and} \quad |\kappa_0| \leq 2N - 1 - 2\xi_P - \beta_P.$$

□

4.9.3 L^2 estimates on the velocity averages of f

We finally end this section by proving several L^2 estimates. The first one is clearly not sharp but is sufficient for us to close the energy estimates for the electromagnetic field.

Proposition 4.9.14. *Let $z \in \mathbf{k}_1$, $p \leq 3N$, $|k| \leq N - 1$ and β such that $|k| + |\beta| \leq N$. Then, for all $t \in [0, T[$,*

$$\left\| \frac{1}{\sqrt{\tau_+}} \int_v |z P_{k,p}(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_t)} \lesssim \frac{1}{1+t} \left\| \sqrt{\tau_+} \int_v |z P_{k,p}(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_t)} \lesssim \frac{\epsilon}{(1+t)^{\frac{5}{4}}}$$

Proof. The first inequality ensues from $1+t \leq \tau_+$ on Σ_t . For the other one, we start by the case $|\beta| \leq N - 3$. Write $P_{k,p}(\Phi) = \Phi^n P_\xi(\Phi)$ and notice that $|\Phi|^n \lesssim \log^{2p}(1+\tau_+)$. Then, using the bootstrap assumption (4.51) and Proposition 4.8.11,

$$\begin{aligned} \left\| \sqrt{\tau_+} \int_v |z P_{k,p}(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_t)}^2 &\lesssim \left\| \tau_+ \log^{4p}(1+\tau_+) \int_v |P_\xi(\Phi)^2 Y^\beta f| dv \int_v |z^2 Y^\beta f| dv \right\|_{L^1(\Sigma_t)} \\ &\lesssim \left\| \tau_+ \log^{4p}(1+\tau_+) \int_v |z^2 Y^\beta f| dv \right\|_{L^\infty(\Sigma_t)} \overline{\mathbb{E}}_N[f](t) \\ &\lesssim \epsilon \frac{\log^{4p+6}(3+t)}{1+t} (1+t)^\eta \lesssim \frac{\epsilon^2}{(1+t)^{\frac{3}{4}}}. \end{aligned}$$

Otherwise, $|\beta| \geq N - 2$ so that $|k| \leq 2$ and, according to Remark 4.7.6, $P_{k,p}(\Phi) \lesssim \tau_+^{\frac{1}{8}}$. Moreover, as there exists $i \in \llbracket 1, |I_1| \rrbracket$ such that $\beta = \beta_i^1$, it comes

$$\left\| \tau_+^{\frac{1}{8}} \int_v |z P_{k,p}(\Phi) Y^\beta| dv \right\|_{L^2(\Sigma_t)} \lesssim \left\| \tau_+^{\frac{5}{8}} \int_v |z H_i| dv \right\|_{L^2(\Sigma_t)} + \left\| \tau_+^{\frac{5}{8}} \int_v |z G_i| dv \right\|_{L^2(\Sigma_t)}.$$

Applying Proposition 4.9.7, one has

$$\left\| \tau_+^{\frac{5}{8}} \int_v |z H_i| dv \right\|_{L^2(\Sigma_t)}^2 \lesssim \left\| \tau_+^{\frac{5}{4}} \int_v |z^2 H_i| dv \right\|_{L^\infty(\Sigma_t)} \left\| \int_v |H_i| dv \right\|_{L^1(\Sigma_t)} \lesssim \frac{\epsilon^2}{(1+t)^{\frac{1}{2}}}.$$

As there exists $q \in \llbracket 1, |I| \rrbracket$ such that $G_i = L_q = K_q^j W_j$, we have, using this time Proposition 4.9.13 and the decay estimate on $\int_v |z^2 W| dv$ given in Proposition 4.9.12,

$$\begin{aligned} \left\| \tau_+^{\frac{5}{8}} \int_v |z G_i| dv \right\|_{L^2(\Sigma_t)}^2 &= \left\| \tau_+^{\frac{5}{8}} \int_v |z K_q^j W_j| dv \right\|_{L^2(\Sigma_t)}^2 \\ &\lesssim \sum_{j=0}^{N_1} \left\| \tau_+^{\frac{5}{4}} \int_v |z^2 W_j| dv \right\|_{L^\infty(\Sigma_t)} \left\| \int_v |K_q^j|^2 |W_j| dv \right\|_{L^1(\Sigma_t)} \\ &\lesssim \epsilon \frac{\log^{3N+M_1}(3+t)}{(1+t)^{\frac{3}{4}}} \log^{4[q]}(3+t) \mathbb{E}_L(t) \lesssim \frac{\epsilon^2}{(1+t)^{\frac{1}{2}}}. \end{aligned}$$

□

This proposition allows us to improve the bootstrap assumption (4.52) if ϵ is small enough. More precisely, the following result holds.

Corollary 4.9.15. *For all $t \in [0, T[$, we have $\sum_{|\beta| \leq N-2} \left\| r^{\frac{3}{2}} \int_v \frac{v^A}{v^0} \widehat{Z}^\beta f dv \right\|_{L^2(\Sigma_t)} \lesssim \epsilon$.*

Proof. Let $t \in [0, T[$. Using $\tau_+ |v^A| \lesssim v^0 \sum_{z \in \mathbf{k}_1} |z|$ and rewritting \widehat{Z}^β in terms of modified vector fields through the identity (4.35), one has

$$\sum_{|\beta| \leq N-2} \left\| r^{\frac{3}{2}} \int_v \frac{v^A}{v^0} \widehat{Z}^\beta f dv \right\|_{L^2(\Sigma_t)} \lesssim \sum_{z \in \mathbf{k}_1} \sum_{p \leq N-2} \sum_{\substack{|q|+|\kappa| \leq N-2 \\ |q| \leq N-3}} \left\| \sqrt{r} \int_v |P_{q,p}(\Phi) Y^\kappa f| dv \right\|_{L^2(\Sigma_t)}.$$

It then only remains to apply the previous proposition. \square

The two following estimates are crucial as a weaker decay rate would prevent us to improve the bootstrap assumptions.

Proposition 4.9.16. *Let β and ξ such that $|\xi| + |\beta| \leq N - 1$. Then, for all $t \in [0, T[$,*

$$\begin{aligned} \left\| \sqrt{\tau_-} \int_v |P_\xi^X(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_t)} &\lesssim \epsilon \frac{1}{1+t} \quad \text{if } |\beta| \leq N-3 \\ &\lesssim \epsilon \frac{\log^M(3+t)}{1+t} \quad \text{otherwise.} \end{aligned}$$

Proof. Suppose first that $|\beta| \leq N - 3$. Then, by Proposition 4.8.11,

$$\begin{aligned} \left\| \sqrt{\tau_-} \int_v |P_\xi^X(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_t)}^2 &\lesssim \left\| \tau_- \int_v |Y^\beta f| dv \right\|_{L^\infty(\Sigma_t)} \left\| \int_v |P_\xi^X(\Phi)^2 Y^\beta f| dv \right\|_{L^1(\Sigma_t)} \\ &\lesssim \frac{\epsilon}{(1+t)^2} \mathbb{E}_{N-1}^X[f](t) \lesssim \left| \frac{\epsilon}{1+t} \right|^2. \end{aligned}$$

Otherwise,

- $|\beta| \geq N - 2$, so $|\xi| \leq 1$ and then $|P_\xi^X(\Phi)| \lesssim \log^{\frac{3}{2}}(1 + \tau_+)$ by Proposition 4.7.4.
- There exists $i \in \llbracket 1, |I_1| \rrbracket$ and $q \in \llbracket 1, |I| \rrbracket$ such that $Y^\beta f = H_i + G_i = H_i + L_q$.

Using Proposition 4.9.7 (for the first estimate) and Propositions 4.9.12, 4.9.13 (for the second one), we obtain

$$\begin{aligned} \left\| \sqrt{\tau_-} \int_v |P_\xi^X(\Phi) H_i| dv \right\|_{L^2(\Sigma_t)}^2 &\lesssim \left\| \tau_- \log^3(1 + \tau_+) \int_v |H_i| dv \right\|_{L^\infty(\Sigma_t)} \left\| \int_v |H_i| dv \right\|_{L^1(\Sigma_t)} \\ &\lesssim \left\| \epsilon \frac{\tau_- \log^{3+M_1}(1 + \tau_+)}{\tau_+^2 \tau_-} \right\|_{L^\infty(\Sigma_t)} \mathbb{E}[H_i](t) \lesssim \epsilon^2 \frac{\log^{3+M_1}(3+t)}{(1+t)^2}, \\ \left\| \sqrt{\tau_-} \int_v |P_\xi^X(\Phi) L_q| dv \right\|_{L^2(\Sigma_t)}^2 &= \left\| \sqrt{\tau_-} \int_v |P_\xi^X(\Phi) K_q^j W_j| dv \right\|_{L^2(\Sigma_t)}^2 \\ &\lesssim \sum_{j=0}^{N_1} \left\| \tau_- \log^3(1 + \tau_+) \int_v |W_j| dv \right\|_{L^\infty(\Sigma_t)} \left\| \int_v |K_q^j|^2 |W_j| dv \right\|_{L^1(\Sigma_t)} \\ &\lesssim \epsilon \frac{\log^{3+3N+M_1}(3+t)}{(1+t)^2} \epsilon \log^{M_0+4[q]}(3+t) \lesssim \epsilon^2 \frac{\log^{M_0+M_1+3N+3}(3+t)}{(1+t)^2}, \end{aligned}$$

since $[q] = 0$. This concludes the proof if M is chosen such that²⁷ $2M \geq M_0 + M_1 + 3N + 3$. \square

²⁷Recall from Remark 4.7.6 that M_1 is independent of M .

The following estimates will be needed for the top order energy norm. As it will be used combined with Proposition 4.3.40, the quantity $P_{q,p}(\Phi)$ will contain \mathbb{Y}_X derivatives of Φ .

Proposition 4.9.17. *Let β , q and p be such as $|q| + |\beta| \leq N$, $|q| \leq N - 1$ and $p \leq q_X + \beta_T$. Then, for all $t \in [0, T[$,*

$$\left\| \sqrt{\tau_-} \int_v |P_{q,p}(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_t^0)} \lesssim \frac{\epsilon}{(1+t)^{1-\frac{\eta}{2}}}.$$

Proof. We consider various cases and, excepted for the last one, the estimates are clearly not sharp. Let us suppose first that $|\beta| \geq N - 2$. Then $|q| \leq 2$ and $|P_{q,p}(\Phi)| \lesssim \log^{M_1}(3+t)$ on Σ_t^0 by Remark 4.7.6, so that, using Proposition 4.9.16,

$$\left\| \sqrt{\tau_-} \int_v |P_{k,p}(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_t^0)} \lesssim \log^{M_1}(3+t) \left\| \sqrt{\tau_-} \int_v |Y^\beta f| dv \right\|_{L^2(\Sigma_t^0)} \lesssim \epsilon \frac{\log^{M+M_1}(3+t)}{1+t}.$$

Let us write $P_{q,p}(\Phi) = \Phi^r P_\xi(\Phi)$ with $r \leq p$ and $(\xi_T, \xi_P, \xi_X) = (q_T, q_P, q_X)$. If $|\beta| \leq N - 3$ and $|q| \leq N - 2$, then by the Cauchy-Schwarz inequality (in v), (4.76) as well as Propositions 4.7.4 and 4.8.11,

$$\begin{aligned} \left\| \sqrt{\tau_-} \int_v |P_{k,p}(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_t)}^2 &\lesssim \left\| \tau_- \int_v |\Phi^{2r} Y^\beta f| dv \right\|_{L^\infty(\Sigma_t)} \left\| \int_v |P_\xi(\Phi)^2 Y^\beta f| dv \right\|_{L^1(\Sigma_t)} \\ &\lesssim \left\| \tau_- \frac{\epsilon \log^{4r}(1+\tau_+)}{\tau_+^2 \tau_-} \right\|_{L^\infty(\Sigma_t)} \mathbb{A}[f](t) \lesssim \epsilon^2 \frac{\log^{8N}(3+t)}{(1+t)^{2-\frac{3}{4}\eta}}. \end{aligned}$$

The remaining case is the one where $|q| = N - 1$ and $|\beta| \leq 1$. Hence, $p \leq k_X + 1$.

- If $p \geq 2$, we have $k_X \geq 1$ and then, schematically, $P_\xi(\Phi) = P_{\xi_1^X}(\Phi) P_{\xi_2}(\Phi)$, with $|\xi^1| \geq 1$ and $|\xi^1| + |\xi^2| = N - 1$. If $|\xi^2| \geq 1$, we have $\min(|\xi^1|, |\xi^2|) \leq \frac{N-1}{2} \leq N - 6$ and one of the two factor can be estimated pointwise, which put us in the context of the case $|k| \leq N - 2$ and $|\beta| \leq N - 3$. Otherwise, $P_{k,p}(\Phi) = \Phi^r P_{\xi_1^X}(\Phi)$ and, using again (4.76),

$$\begin{aligned} \left\| \sqrt{\tau_-} \int_v |P_{k,p}(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_t)}^2 &\lesssim \left\| \tau_- \int_v |\Phi^{2r} Y^\beta f| dv \right\|_{L^\infty(\Sigma_t)} \left\| \int_v |P_{\xi_1^X}(\Phi)^2 Y^\beta f| dv \right\|_{L^1(\Sigma_t)} \\ &\lesssim \left\| \tau_- \frac{\epsilon \log^{4r}(1+\tau_+)}{\tau_+^2 \tau_-} \right\|_{L^\infty(\Sigma_t)} \mathbb{A}[f](t) \lesssim \epsilon^2 \frac{\log^{8N}(3+t)}{(1+t)^{2-\frac{3}{4}\eta}}. \end{aligned}$$

- If $p = 1$, we have $P_{k,p}(\Phi) = Y^\kappa \Phi$ and, using $\bar{\mathbb{E}}_N[f](t) \leq 4\epsilon(1+s)^\eta$,

$$\begin{aligned} \left\| \sqrt{\tau_-} \int_v |P_{k,p}(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_t)}^2 &\lesssim \left\| \tau_- \int_v |Y^\beta f| dv \right\|_{L^\infty(\Sigma_t)} \left\| \int_v |Y^\kappa \Phi|^2 |Y^\beta f| dv \right\|_{L^1(\Sigma_t)} \\ &\lesssim \left\| \tau_- \frac{\epsilon}{\tau_+^2 \tau_-} \right\|_{L^\infty(\Sigma_t)} \mathbb{E} \left[|Y^\kappa \Phi|^2 Y^\beta f \right] (t) \lesssim \epsilon^2 \frac{(1+t)^\eta}{(1+t)^2}. \end{aligned}$$

□

4.10 Improvement of the energy estimates of the electromagnetic field

In order to take advantage of the null structure of the system, we start this section by a preparatory lemma.

Lemma 4.10.1. *Let G be a 2-form and g a function, both sufficiently regular and recall that $J(g)^\nu = \int_v \frac{v^\nu}{v^0} g dv$, $|\bar{S}^L| \lesssim \tau_+$ and $|\bar{S}^L| \lesssim \tau_-$. Then, using several times Lemma 4.2.4 and Remark 4.2.5,*

$$\begin{aligned} |G_{0\nu} J(g)^\nu| &\lesssim |\rho| \int_v |g| dv + (|\alpha_A| + |\underline{\alpha}_A|) \int_v \frac{|v^A|}{v^0} |g| dv \lesssim |\rho| \int_v |g| dv + \frac{1}{\tau_+} \sum_{w \in \mathbf{k}_1} (|\alpha| + |\underline{\alpha}|) \int_v |wg| dv, \\ |\bar{S}^\mu G_{\mu\nu} J(g)^\nu| &\lesssim \tau_+ |\rho| \int_v \frac{v^L}{v^0} |g| dv + \tau_- |\rho| \int_v \frac{v^L}{v^0} |g| dv + \tau_+ |\alpha| \int_v \frac{|v^A|}{v^0} |g| dv + \tau_- |\underline{\alpha}| \int_v \frac{|v^A|}{v^0} |g| dv \\ &\lesssim \left(|\alpha| + |\rho| + \frac{\tau_-}{\tau_+} |\underline{\alpha}| \right) \sum_{z \in \mathbf{k}_1} \int_v |zg| dv \quad \text{if } |x| \geq t, \\ &\lesssim |\rho| \int_v \left(\tau_- + \sum_{z \in \mathbf{k}_1} |z| \right) |g| dv + \left(|\alpha| + \frac{\tau_-}{\tau_+} |\underline{\alpha}| \right) \int_v \sum_{z \in \mathbf{k}_1} |zg| dv \quad \text{otherwise.} \end{aligned}$$

We are now ready to improve the bootstrap assumptions concerning the electromagnetic field.

4.10.1 For $\mathcal{E}_N^0[F]$

Using Proposition 4.4.3 and commutation formula of Proposition 4.3.40, we have, for all $t \in [0, T]$,

$$\mathcal{E}_N^0[F](t) - 2\mathcal{E}_N^0[F](0) \lesssim \sum_{|\gamma| \leq N} \sum_{\substack{p \leq |k| + |\beta| \leq N \\ |k| \leq N-1}} \int_0^t \int_{\Sigma_s} |\mathcal{L}_{Z^\gamma}(F)_{\mu 0} J(P_{k,p}(\Phi) Y^\beta f)^\mu| dx ds. \quad (4.79)$$

We fix $|k| + |\beta| \leq N$, $p \leq N$ and $|\gamma| \leq N$. Denoting the null decomposition of $\mathcal{L}_{Z^\gamma}(F)$ by $(\alpha, \underline{\alpha}, \rho, \sigma)$, $P_{k,p}(\Phi) Y^\beta f$ by g and applying Lemma 4.10.1, it comes

$$\int_0^t \int_{\Sigma_s} |\mathcal{L}_{Z^\gamma}(F)_{\mu 0} J(P_{k,p}(\Phi) Y^\beta f)^\mu| dx ds \lesssim \int_0^t \int_{\Sigma_s} |\rho| \int_v |g| dv + (|\alpha| + |\underline{\alpha}|) \sum_{w \in \mathbf{k}_1} \frac{1}{\tau_+} \int_v |wg| dv dx ds.$$

On the one hand, using Proposition 4.9.14,

$$\begin{aligned} \sum_{w \in \mathbf{k}_1} \int_0^t \int_{\Sigma_s} (|\alpha| + |\underline{\alpha}|) \int_v \frac{1}{\tau_+} |wg| dv dx ds &\lesssim \sum_{w \in \mathbf{k}_1} \int_0^t \sqrt{\mathcal{E}_N^0[F](s)} \left\| \frac{1}{\tau_+} \int_v |wg| dv \right\|_{L^2(\Sigma_s)} ds \\ &\lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

On the other hand, as $\rho = \rho(\mathcal{L}_{Z^\gamma}(\tilde{F})) + \rho(\mathcal{L}_{Z^\gamma}(\bar{F}))$ and $\rho(\mathcal{L}_{Z^\gamma}(\bar{F})) \lesssim \epsilon \tau_+^{-2}$, we have, using Proposition 4.9.14 and the bootstrap assumptions (4.51), (4.54) and (4.57),

$$\begin{aligned} \int_0^t \int_{\Sigma_s} |\rho| \int_v |g| dv dx ds &\lesssim \int_0^t \left\| \sqrt{\tau_+} \rho(\mathcal{L}_{Z^\gamma}(\tilde{F})) \right\|_{L^2(\Sigma_s)} \left\| \frac{1}{\sqrt{\tau_+}} \int_v |g| dv \right\|_{L^2(\Sigma_s)} + \int_{\Sigma_s} \rho(\mathcal{L}_{Z^\gamma}(\bar{F})) \int_v |g| dv dx ds \\ &\lesssim \int_0^t \sqrt{\mathcal{E}_N^{Ext}[\tilde{F}](s) + \mathcal{E}_N[F](s)} \frac{\epsilon}{(1+s)^{\frac{5}{4}}} + \frac{\epsilon}{(1+s)^2} \mathbb{E}[g](s) ds \lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

The right-hand side of (4.79) is then bounded by $\epsilon^{\frac{3}{2}}$, implying that $\mathcal{E}_N^0[f] \leq 3\epsilon$ on $[0, T]$ if ϵ is small enough.

4.10.2 The weighted norm for the exterior region

Applying Proposition 4.4.3 and using $\mathcal{E}_N^{Ext}[\tilde{F}](0) \leq \epsilon$ as well as $\tilde{F} = F - \bar{F}$, we have, for all $t \in [0, T]$,

$$\mathcal{E}_N^{Ext}[\tilde{F}](t) \leq 6\epsilon + \sum_{|\gamma| \leq N} \int_0^t \int_{\Sigma_s^0} \left| \bar{S}^\mu \mathcal{L}_{Z^\gamma}(\tilde{F})_{\mu\nu} \nabla^\lambda \mathcal{L}_{Z^\gamma}(F)_\lambda{}^\nu \right| dx ds + \int_0^t \int_{\Sigma_s^0} \left| \bar{S}^\mu \mathcal{L}_{Z^\gamma}(\tilde{F})_{\mu\nu} \nabla^\lambda \mathcal{L}_{Z^\gamma}(\bar{F})_\lambda{}^\nu \right| dx ds.$$

Let us fix $|\gamma| \leq N$ and denote the null decomposition of $\mathcal{L}_{Z^\gamma}(\tilde{F})$ by $(\alpha, \underline{\alpha}, \rho, \sigma)$. As previously, using Proposition 4.3.40,

$$\int_0^t \int_{\Sigma_s^0} \left| \bar{S}^\mu \mathcal{L}_{Z^\gamma}(\tilde{F})_{\mu\nu} \nabla^\lambda \mathcal{L}_{Z^\gamma}(F)_\lambda{}^\nu \right| dx ds \lesssim \sum_{\substack{p \leq |k| + |\beta| \leq |\gamma| \\ |k| \leq |\gamma| - 1}} \int_0^t \int_{\Sigma_s^0} \left| \bar{S}^\mu \mathcal{L}_{Z^\gamma}(\tilde{F})_{\mu\nu} J(P_{k,p}(\Phi) Y^\beta f)^\nu \right| dx ds.$$

We fix $|k| + |\beta| \leq N$, $p \leq N$ and $|\gamma| \leq N$ and we denote again $P_{k,p}(\Phi)Y^\beta f$ by g . Using successively Lemma 4.10.1, the Cauchy-Schwarz inequality, the bootstrap assumption (4.54) and Proposition 4.9.14, it comes

$$\begin{aligned} \int_0^t \int_{\Sigma_s^0} |\bar{S}^\mu \mathcal{L}_{Z^\gamma}(\tilde{F})_{\mu\nu} J(P_{k,p}(\Phi)Y^\beta f)^\nu| dx ds &\lesssim \int_0^t \int_{\Sigma_s} \left(|\rho| + |\alpha| + \frac{\sqrt{\tau_-}}{\sqrt{\tau_+}} |\underline{\alpha}| \right) \sum_{w \in \mathbf{k}_1} \int_v |wg| dv dx ds. \\ &\lesssim \sum_{w \in \mathbf{V}} \int_0^t \sqrt{\mathcal{E}_N^{Ext}[F](s)} \left\| \frac{1}{\sqrt{\tau_+}} \int_v |wg| dv \right\|_{L^2(\Sigma_s)} ds \\ &\lesssim \epsilon^{\frac{3}{2}} \int_0^{+\infty} \frac{ds}{(1+s)^{\frac{5}{4}}} \lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

Using Proposition 4.5.1 and iterating commutation formula of Proposition 4.3.36, we have,

$$\tau_+^2 \left| \nabla^\mu \mathcal{L}_{Z^\gamma}(\bar{F})_\mu^L \right| (t, x) + \tau_+^4 \left| \nabla^\mu \mathcal{L}_{Z^\gamma}(\bar{F})_\mu^L \right| (t, x) + \tau_+^3 \left| \nabla^\mu \mathcal{L}_{Z^\gamma}(\bar{F})_\mu^A \right| (t, x) \lesssim |Q(F)| \mathbf{1}_{-2 \leq t - |x| \leq -1}(t, x).$$

Consequently, as $|Q(F)| \leq \|f_0\|_{L_{x,v}^1} \leq \epsilon$, $|\bar{S}^L| \lesssim \tau_+$ and $|\bar{S}^L| \lesssim \tau_-$,

$$|\bar{S}^\mu \mathcal{L}_{Z^\gamma}(\tilde{F})_{\mu\nu} \nabla^\lambda \mathcal{L}_{Z^\gamma}(\bar{F})_\lambda^\nu| \lesssim \left(\tau_+ |\rho| \frac{\epsilon}{\tau_+^4} + \tau_- |\rho| \frac{\epsilon}{\tau_+^2} + \tau_+ |\alpha| \frac{\epsilon}{\tau_+^3} + \tau_- |\alpha| \frac{\epsilon}{\tau_+^3} \right) \mathbf{1}_{-2 \leq t - |x| \leq -1}(t, x).$$

Note now that $\tau_- \mathbf{1}_{-2 \leq t - |x| \leq -1} \leq \sqrt{5}$, so that, using the bootstrap assumption (4.54) and the Cauchy-Schwarz inequality,

$$\begin{aligned} \int_0^t \int_{\Sigma_s^0} \left| \bar{S}^\mu \mathcal{L}_{Z^\gamma}(\tilde{F})_{\mu\nu} \nabla^\lambda \mathcal{L}_{Z^\gamma}(\bar{F})_\lambda^\nu \right| dx ds &\lesssim \int_0^t \frac{\epsilon}{(1+s)^{\frac{5}{2}}} \int_{s+1 \leq |x| \leq s+2} \sqrt{\tau_+} |\rho| + \sqrt{\tau_+} |\alpha| + |\underline{\alpha}| dx ds \\ &\lesssim \int_0^t \frac{\epsilon}{(1+s)^{\frac{5}{2}}} \sqrt{\mathcal{E}_N^{Ext}[\tilde{F}](s)} \sqrt{s^2 + 1} ds \lesssim \epsilon^{\frac{3}{2}}. \end{aligned}$$

Thus, if ϵ is small enough, we obtain $\mathcal{E}_N^{Ext}[\tilde{F}] \leq 7\epsilon$ on $[0, T[$ which improves the bootstrap assumption (4.54).

4.10.3 The weighted norms for the interior region

Recall from Proposition 4.4.3 that we have, for $Q \in \{N-3, N-1, N\}$ and $t \in [0, T[$,

$$\mathcal{E}_Q[F](t) \leq 24\epsilon + \sum_{|\gamma| \leq Q} \int_0^t \int_{\Sigma_s^0} \left| \bar{S}^\mu \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \nabla^\lambda \mathcal{L}_{Z^\gamma}(F)_\lambda^\nu \right| dx ds, \quad (4.80)$$

since $\mathcal{E}_N^{Ext}[\tilde{F}] \leq 8\epsilon$ on $[0, T[$ by the bootstrap assumption (4.54)). The remaining of this subsection is divided in two parts. We consider first $Q \in \{N-3, N-1\}$ and we end with $Q = N$ as we need to use in that case a worst commutation formula in order to avoid derivatives of Φ of order N , which is the reason of the stronger loss on the top order energy norm.

The lower order energy norms

Let $Q \in \{N-3, N-1\}$. According to commutation formula of Proposition 4.3.39, we can bound the last term of (4.80) by a linear combination of the following ones.

$$\mathcal{I}_1 := \int_0^t \int_{|x| \leq s} \left| \bar{S}^\mu \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \int_v \frac{v^\nu}{v^0} P_\xi^X(\Phi) Y^\beta f dv \right| dx ds, \quad \text{with } |\gamma|, |\xi| + |\beta| \leq Q, \quad (4.81)$$

$$\mathcal{I}_2 := \int_0^t \int_{|x| \leq s} \left| \bar{S}^\mu \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \int_v \frac{z}{\tau_+} P_{k,p}(\Phi) Y^\beta f dv \right| dx ds, \quad \text{with } |\gamma|, |k| + |\beta| \leq Q, \quad z \in \mathbf{k}_1, \quad (4.82)$$

$0 \leq \nu \leq 3$ and $p \leq 3N$. Fix $|\gamma| \leq Q$ and denote the null decomposition of $\mathcal{L}_{Z^\gamma}(F)$ by $(\alpha, \underline{\alpha}, \rho, \sigma)$. We start by (4.82), which can be estimated independently of Q . Recall that $|\bar{S}^L| \lesssim \tau_+$ and $|\bar{S}^L| \lesssim \tau_-$, so that, using

Proposition 4.9.14 and the bootstrap assumption (4.56),

$$\begin{aligned}
\mathcal{I}_2 &\lesssim \int_0^t \int_{\Sigma_s^0} (\tau_+ |\rho| + \tau_+ |\alpha| + \tau_- |\underline{\alpha}|) \int_v \left| \frac{z}{\tau_+} P_{k,p}(\Phi) Y^\beta f \right| dv dx ds \\
&\lesssim \int_0^t \sqrt{\mathcal{E}_{N-1}[F](s)} \left\| \frac{1}{\sqrt{\tau_+}} \int_v |z P_{k,p}(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_s)} ds \\
&\lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{\log^M(3+s)}{(1+s)^{\frac{5}{4}}} ds \lesssim \epsilon^{\frac{3}{2}}.
\end{aligned}$$

We now turn on (4.81) and we then consider $|\xi| + |\beta| \leq Q$. Start by noticing that, by Lemma 4.10.1,

$$\left| \bar{S}^\mu \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \int_v \frac{v^\nu}{v^0} P_\xi^X(\Phi) Y^\beta f \right| dv \lesssim \tau_- |\rho| \int_v |P_\xi^X(\Phi) Y^\beta f| dv + \left(|\rho| + |\alpha| + \frac{\tau_-}{\tau_+} |\underline{\alpha}| \right) \sum_{w \in \mathbf{k}_1} \int_v |w P_\xi^X(\Phi) Y^\beta f| dv.$$

Consequently, by the bootstrap assumption (4.56) and Proposition 4.9.14,

$$\begin{aligned}
\mathcal{I}_1 &\lesssim \int_0^t \sqrt{\mathcal{E}_Q[F](s)} \left(\left\| \sqrt{\tau_-} \int_v |P_\xi^X(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_s)} + \sum_{w \in \mathbf{k}_1} \left\| \frac{1}{\sqrt{\tau_+}} \int_v |w P_\xi^X(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_s)} \right) ds \\
&\lesssim \epsilon^{\frac{3}{2}} + \int_0^t \sqrt{\mathcal{E}_Q[F](s)} \left\| \sqrt{\tau_-} \int_v |P_\xi^X(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_s)} ds.
\end{aligned}$$

The last integral to estimate is the source of the small growth of $\mathcal{E}_Q[F]$. We can bound it, using again the bootstrap assumptions (4.55), (4.56) and Proposition 4.9.16, by

- $\epsilon^{\frac{3}{2}} \log^2(3+t)$ if $Q = N - 3$ and
- $\epsilon^{\frac{3}{2}} \log^{2M}(3+t)$ otherwise.

Hence, combining this with (4.80) we obtain, for ϵ small enough, that

- $\mathcal{E}_{N-3}[F](t) \leq 25\epsilon \log^2(3+t)$ for all $t \in [0, T[$ and
- $\mathcal{E}_{N-1}[F](t) \leq 25\epsilon \log^{2M}(3+t)$ for all $t \in [0, T[$.

The top order energy norm

We consider here the case $Q = N$ and we then apply this time the commutation formula of Proposition 4.3.40, so that the last term of (4.80) can be bounded by a linear combination of terms of the form

$$\mathcal{I} := \int_0^t \int_{\Sigma_s^0} \left| \bar{S}^\mu \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} \int_v \frac{v^\mu}{v^0} P_{q,p}(\Phi) Y^\beta f dv \right| dx ds,$$

with $|\gamma| \leq N$, $|q| + |\beta| \leq N$, $|q| \leq N - 1$ and $p \leq q_X + \beta_T$. Let us fix such parameters. Following the computations made previously to estimate \mathcal{I}_1 and using $\mathcal{E}_N[F](s) \lesssim \sqrt{\epsilon}(1+s)^\eta \lesssim \sqrt{\epsilon}(1+s)^{\frac{1}{3}}$, we get

$$\begin{aligned}
\mathcal{I}_1 &\lesssim \int_0^t \sqrt{\mathcal{E}_N[F](s)} \left(\left\| \sqrt{\tau_-} \int_v |P_{q,p}(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_s^0)} + \sum_{w \in \mathbf{k}_1} \left\| \frac{1}{\sqrt{\tau_+}} \int_v |w P_{q,p}(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_s)} \right) ds \\
&\lesssim \epsilon^{\frac{3}{2}} + \sqrt{\epsilon} \int_0^t (1+s)^{\frac{\eta}{2}} \left\| \sqrt{\tau_-} \int_v |P_{q,p}(\Phi) Y^\beta f| dv \right\|_{L^2(\Sigma_s^0)} ds. \tag{4.83}
\end{aligned}$$

Applying now Proposition 4.9.17, we can bound (4.83) by $\epsilon^{\frac{3}{2}}(1+t)^\eta$. Thus, if ϵ small enough, we obtain $\mathcal{E}_N[F](t) \leq 25\epsilon(1+t)^\eta$ for all $t \in [0, T[$, which concludes the improvement of the bootstrap assumption (4.57) and then the proof.

Acknowledgements

I am very grateful towards Jacques Smulevici, my Ph.D. advisor, for his support and for giving me precious advice. Part of this work was funded by the European Research Council under the European Union's Horizon 2020 research and innovation program (project GEOWAKI, grant agreement 714408).

Chapter 5

Asymptotic properties of the solutions to the Vlasov-Maxwell system in the exterior of a light cone

Abstract

This paper is concerned with the asymptotic behavior of small data solutions to the three-dimensional Vlasov-Maxwell system in the exterior of a light cone. The plasma does not have to be neutral and no compact support assumptions are required on the data. In particular, the initial decay in the velocity variable of the particle density is optimal and we only require an L^2 bound on the electromagnetic field with no additional weight. We use vector field methods to derive improved decay estimates in null directions for the electromagnetic field, the particle density and their derivatives. In contrast with [5], where we studied the behavior of the solutions in the whole spacetime, the initial data have less decay and we do not need to modify the commutation vector fields of the relativistic transport operator. To control the solutions under these assumptions, we crucially use the strong decay satisfied by the particle density in the exterior of the light cone, null properties of the Vlasov equation and certain hierarchies in the energy norms.

5.1 Introduction

In this article, we study the asymptotic properties of small data solutions of the Vlasov-Maxwell (VM) system in the exterior of a light cone $V_b := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 / |x| > t - b\}$, where, say, $b \leq -1$. More precisely, our main goal is to derive sharp decay estimates. The system, which is of particular importance in plasma physics, is given for one species of particles by^{1,2}

$$v^0 \partial_t f + v^i \partial_i f + e v^\mu F_\mu^j \partial_{v^j} f = 0, \quad (5.1)$$

$$\nabla^\mu F_{\mu\nu} = J(f)_\nu := \int_{v \in \mathbb{R}^3} \frac{v_\nu}{v^0} f dv, \quad (5.2)$$

$$\nabla^\mu {}^* F_{\mu\nu} = 0, \quad (5.3)$$

where

- $v^0 = \sqrt{m^2 + |v|^2}$, $m > 0$ is the mass of the particles and $e \neq 0$ their charge. For the remaining of this paper, we take $m = e = 1$ and we denote $\sqrt{1 + |v|^2}$ by v^0 .
- The function $f(t, x, v)$ is the particle density, the 2-form $F(t, x)$ is the electromagnetic field and ${}^*F(t, x)$ is its Hodge dual.

¹We choose to lighten the notations by considering only one species since the presence of other ones does not complicate the analysis.

²We will, throughout this article, use the Einstein summation convention so that $v^i \partial_i f = \sum_{i=1}^3 v^i \partial_i f$. A sum on latin letters starts from 1 whereas a sum on greek letters starts from 0.

5.1.1 Small data results for the VM system

The study of the small data solutions of the VM system has been initiated in [24] by Glassey-Strauss. Under a compact support assumption in space and in velocity on the initial data, they proved global existence and obtained the optimal decay rate on $\int_v f dv$. The compact support assumption in v is replaced by Schaeffer in [44] by a polynomial decay but the data still have to be compactly supported in space. Moreover, the optimal decay rate on $\int_v f dv$ is not obtained by this method. None of these results contain estimates on the derivatives of $\int_v f dv$ nor on the higher order derivatives of the electromagnetic field.

In [4], we removed all compact support assumptions for the dimensions $d \geq 4$. For this, we used vector field methods, developed in [11] for the electromagnetic field and [18] for relativistic transport equations. We then obtained almost optimal decay on the solutions of the system and their derivatives and we described precisely the behavior each null component of the electromagnetic field. We recently extended these results to the $3d$ case and we also relaxed the assumptions on the initial data, allowing in particular the presence of a non zero total charge. A better understanding of the null structure of the VM system as well as the use of modified vector fields³ were the key for dealing with the slower decay rates of the solutions. We splitted the electromagnetic field into two parts. The chargeless one on which we could then propagate a weighted L^2 norm and the pure charge part, given by an explicit formula, which decays as ϵr^{-2} despite of its infinite energy.

We also investigate the case where the particles are massless (i.e. $m = 0$). First in [4] for the high dimensions, where we proved that similar results to the massive case hold provided that the velocity support of the the particle density is bounded away from 0. These extra hypothesis appears to be necessary since we also proved in [4] that the VM system do not admit a local classical solution for certain smooth initial data which do not vanish for small velocities. Secondly, in our recent work [6], we proved sharp asymptotics on the small data solutions and their derivatives to the massless VM system in $3d$. Contrary to the massive case, the proof does not require the use of modified vector fields but still necessitates a strong understanding of the null properties of the system.

In this article, we study the asymptotic properties of the solutions to the VM system in the exterior of a light cone under a smallness assumption but weaker decay near infinity. We obtain in particular almost optimal pointwise decay estimates on the velocity average of the Vlasov field as well as its derivatives. The hypotheses on the particle density in the variable v are optimal in the sense that we merely suppose f and its derivatives to be initially integrable in v , which is a necessary condition for the source term of the Maxwell equations (1.3) to be well defined. As f strongly decay in the domain studied, our proof merely requires the boundedness of the L^2 norm of the electromagnetic field. This has to be compared with our proof in [5], where we study the same problem in the whole spacetime, which crucially relies on the propagation of a weighted energy norm of F . Another remarkable point, still related to the good behavior of f in the region V_b , concerns the commutation vector fields used to study the Vlasov equation. Contrary to [5], we do not need to modify the commutation vector fields of the relativistic transport operator $v^\mu \partial_\mu$ in order to compensate the worst source terms of the commuted Vlasov equations and then close the energy estimates. This leads in particular to a much simpler proof.

Finally, let us mention the recent result [50] of Wang concerning the small data solutions of the massive $3d$ VM system. Using both vector field methods and Fourier analysis, he proved optimal pointwise decay estimates on $\int_v f dv$ and its derivatives under strong polynomial decay hypotheses in (x, v) on $f(t = 0)$. In particular, the initial data does not have to be compactly supported.

5.1.2 Previous works on Vlasov systems using vector field methods

Results on the asymptotic behavior of solutions of several Vlasov systems were recently derived using vector field methods. Let us mention the pioneer work [18] of Fajman-Joudioux-Smulevici on the Vlasov-Norström system (see also [17]) as well as the results of [46] on the Vlasov-Poisson system. The two different proofs, obtained independently by [16] and [26], of the stability of the Minkowski spacetime as a solution to the Einstein-Vlasov system constitute a culmination of these vector field methods.

³Modified Vector fields, which depend on the solution itself, were already used by [16] (respectively [17]) in the context of the Vlasov-Nordström (respectively the Einstein-Vlasov) system. They are built over the commutation vector fields of the relativistic transport operator $v^\mu \partial_\mu$ in order to compensate the worst source terms of the commuted Vlasov equation.

5.1.3 Statement of the main result

In order to present our main theorem, we call initial data set for the VM system any ordered pair (f_0, F_0) where $f_0 : \mathbb{R}_x^3 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ and F_0 are both sufficiently regular and satisfy the constraint equations

$$\nabla^i (F_0)_{i0} = - \int_{v \in \mathbb{R}^3} f_0 dv \quad \text{and} \quad \nabla^{i*} (F_0)_{i0} = 0.$$

We refer to Section 5.2 for the notations not yet defined.

Theorem 5.1.1. *Let $N \geq 8$, $b \leq -1$, $0 < \eta < \frac{1}{16N}$, $\epsilon > 0$, (f_0, F_0) an initial data set for the Vlasov-Maxwell equations (5.1)-(5.3) satisfying⁴*

$$\sum_{|\beta|+|\kappa| \leq N+3} \int_{|x| \geq b} \int_{v \in \mathbb{R}^3} (1+|x|)^{\frac{N+14+|\beta|}{2}} (1+|v|)^{|\kappa|} |\partial_x^\beta \partial_v^\kappa f_0| dv dx + \sum_{|\gamma| \leq N+2} \int_{|x| \geq b} (1+|x|)^{2|\gamma|} |\nabla_{\partial_x^\gamma} F_0|^2 dx \leq \epsilon$$

and (f, F) be the unique classical solution of the system which satisfies $f(t=0) = f_0$ and $F(t=0) = F_0$. Then, there exists $C > 0$ and $\epsilon_0 > 0$, depending only on N and η , such that, if $0 \leq \epsilon \leq \epsilon_0$, (f, F) is well defined in $V_b = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 / r > t - b\}$ and verifies the following estimates.

- Energy bound for the electromagnetic field F : $\forall t \in \mathbb{R}_+$,

$$\sum_{0 \leq k \leq N} \sum_{Z^\gamma \in \mathbb{K}^k} \int_{|x| \geq t-b} |\mathcal{L}_{Z^\gamma}(F)|^2 dx \leq C\epsilon,$$

- Pointwise decay estimates for the null components of $\mathcal{L}_{Z^\gamma}(F)$: $\forall |\gamma| \leq N-2$, $(t, x) \in V_b$,

$$|\alpha(\mathcal{L}_{Z^\gamma}(F))|(t, x) \lesssim \frac{\sqrt{\epsilon}}{\tau_+^{\frac{3}{2}}}, \quad |\rho(\mathcal{L}_{Z^\gamma}(F))|(t, x) + |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|(t, x) + |\sigma(\mathcal{L}_{Z^\gamma}(F))|(t, x) \lesssim \frac{\sqrt{\epsilon}}{\tau_+ \tau_-^{\frac{1}{2}}}.$$

- Energy bound for the particle density: $\forall t \in \mathbb{R}_+$,

$$\sum_{0 \leq k \leq N} \sum_{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^k} \int_{|x| \geq t-b} \int_{v \in \mathbb{R}^3} |\widehat{Z}^\beta f| dv dx \leq C\epsilon(1+t)^{(N+1)\eta}.$$

- Pointwise decay estimates for the velocity averages of $\widehat{Z}^\beta f$: $\forall |\beta| \leq N-3$, $(t, x) \in V_b$,

$$\int_{v \in \mathbb{R}^3} |\widehat{Z}^\beta f| dv \lesssim \frac{\epsilon}{\tau_+^{2-(N+1)\eta} \tau_-} \quad \text{and} \quad \forall a \in \left[0, \frac{9}{2}\right], \quad \int_{v \in \mathbb{R}^3} |\widehat{Z}^\beta f| \frac{dv}{(v^0)^{2a}} \lesssim \frac{\epsilon}{\tau_+^{2+a-(N+1)\eta} \tau_-}.$$

Remark 5.1.2. *Note that we can study the solutions to the Vlasov-Maxwell equations in the exterior of a light cone, without any information on their behavior in the remaining part of the Minkowski space, by finite speed of propagation. Every inextendible past causal curves of such a region intersect the hypersurface $t=0$ once and only once, i.e. the region is globally hyperbolic.*

Remark 5.1.3. *By a time translation, one can prove a similar result for $b \in \mathbb{R}$ (ϵ_0 would then also depends on b).*

Remark 5.1.4. *Assuming more decay on the electromagnetic field at $t=0$, one could propagate a stronger energy norm as in [5] or [6]. We then could assume less decay decay in x on f_0 and improve the decay rate of the null components of the electromagnetic field. Note however that if the total electromagnetic charge*

$$Q(F)(t) := \lim_{r \rightarrow +\infty} \int_{\mathbb{S}_{t,r}} \frac{x^i}{r} F_{0i} d\mathbb{S}_{t,r} = - \lim_{r \rightarrow +\infty} \int_{\mathbb{S}_{t,r}} \rho(F) d\mathbb{S}_{t,r} = \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} f dx dv,$$

which is a conserved quantity in t , is non zero, we cannot obtain a better decay rate than r^{-2} on $\rho(F)$ and assume that $\int_{\mathbb{R}^3} r|\rho(F)| dx$ is initially finite. We point out that our hypotheses on the electromagnetic field are compatible with the presence of a non zero total charge.

⁴We could save three powers of x in the condition on the initial norm of f_0 with easy but cumbersome modifications of our proof (mostly in Section 5.6.2 and Proposition 5.6.3). Note also that following the strategy used in Subsection 17.2 of [17] to derive L^2 estimates on the Vlasov field, we could avoid any hypotheses on the derivatives of order $N+1$ and $N+2$ of F_0 .

Remark 5.1.5. *The results of [18], [16] and [17] are obtained using a hyperboloidal foliation and then require compactly supported initial data in space. These compact restrictions on the data could be removed by adapting the method used in this article to the Vlasov-Nordström and the Einstein-Vlasov systems.*

Theorem 5.1.1 immediately implies the following result, concerning solutions arising from large data.

Corollary 5.1.6. *Let $N \geq 8$ and (f_0, F_0) an initial data set for the Vlasov-Maxwell equations (5.1)-(5.3) satisfying*

$$\sum_{|\beta|+|\kappa| \leq N+3} \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} (1+|x|)^{\frac{N+14+|\beta|}{2}} (1+|v|)^{|\kappa|} |\partial_x^\beta \partial_v^\kappa f_0| dv dx + \sum_{|\gamma| \leq N} \int_{x \in \mathbb{R}^3} (1+|x|)^{2|\gamma|} |\nabla_{\partial_x^2} F_0|^2 dx < +\infty$$

and (f, F) be the unique local classical solution to the system which satisfies $f(t=0) = f_0$ and $F(t=0) = F_0$. Then, there exists $b \leq -1$ such that (f, F) is well defined in V_b and verifies similar estimates as those presented in Theorem 5.1.1.

Proof. One only has to notice that there exists $b \leq -1$ such that (f_0, F_0) satisfies the hypotheses of Theorem 5.1.1. \square

Global existence in the whole Minkowski spacetime for classical solutions to the VM system with large data still remains an open problem. For the weak solutions, the problem was solved in [15] and revisited in [42].

5.1.4 Main ingredients of the proof

The proof of the main result of this paper is based on vector field methods and then essentially relies on bounding sufficiently well the spacetime integrals of the source terms of the commuted equations. In the exterior of the light cone V_0 , the solutions to the Vlasov equation behave better than in the interior region. One can already see that with the following estimate (see Lemma 5.2.9), for g a solution to the free transport equation $v^\mu \partial_\mu g = 0$,

$$\forall |x| \geq t, \quad \int_v |g|(t, x, v) dv \lesssim \sum_{|\beta| \leq 3} \frac{\|(v^0)^{2k} (1+r)^{|\beta|+k+q} \partial^\beta g\|_{L^1_{x,v}}(t=0)}{(1+t+r)^{2+k} (1+|t-r|)^{1+q}}. \quad (5.4)$$

Contrary to [5], where we study solutions to the VM system in the whole Minkowski spacetime, this strong decay should allow us in principle to avoid the use of modified vector fields. This also allows us to assume less decay on the electromagnetic field and to avoid any difficulty due to the presence of a non zero total charge. However, as we start with optimal decay in v , we cannot fully use (5.4). In particular, no extra decay in the $t+r$ direction can be obtained in that way. Moreover, since the initial data are not compactly supported in v , a problem arises from large velocities, for which $v^0 \sim |v|$, so that the characteristics of the transport equation ultimately approach those of the Maxwell equations. The consequence is that, in a product such as $\mathcal{L}_{Z^\gamma}(F) \cdot \widehat{Z}^\beta f$, one cannot, in view of support considerations, transform a $|t-r|$ decay in a $t+r$ one anymore. To circumvent this difficulty, we take advantage of the null structure of the non linearities such as

$$v^\mu \mathcal{L}_{Z^\gamma}(F)_\mu{}^i \partial_{v^i} \widehat{Z}^\beta f, \quad (5.5)$$

where Z is a Killing vector field and \widehat{Z} its complete lift. The problem is that, for g solution to $v^\mu \partial_\mu g = 0$, $\partial_v g$ essentially behaves as $(1+t+r) \partial_{t,x} g$ and the electromagnetic field, as a solution of a wave equation, only decay with a rate of $(1+t+r)^{-1}$ in the $t+r$ direction. However, from [11] (respectively [5]), we know that certain null components of the Maxwell field (respectively the velocity vector v) are expected to behave better than others. As we propagate a weaker energy norm on F than [11], the null components ρ and σ do not decay faster than $\underline{\alpha}$ but still have a better behavior. Indeed, they allow us to take advantage of the $t-r$ decay as they permit us to estimate spacetime integrals by using a null foliation. For the velocity vector, the component v^L allows us to integrate according to a null foliation and provides, as the angular components, an extra decay in⁵ $t+r$ at the cost of weights preserved by the flow of $v^\mu \partial_\mu$ (see Lemma 5.2.9). Finally, the radial component of $(0, \partial_{v^1} \widehat{Z}^\beta f, \partial_{v^2} \widehat{Z}^\beta f, \partial_{v^3} \widehat{Z}^\beta f)$ costs a power of $t-r$ instead of $t+r$. The null structure of (5.5) is fully depicted in Lemma 5.4.1 and we can observe that each term contains either the better null component α of the electromagnetic field, the better null component v^L of the velocity vector or at least two good components.

Finally, the weak decay assumptions on the electromagnetic field force us to consider several hierarchies in the energy norms of the Vlasov field in order to close the energy estimates. Let us illustrate how appears such a hierarchy by an example.

⁵Note that this property of v^L is specific to the exterior of the light cone. In the whole spacetime, the extra decay is merely $\frac{1+t+r}{1+|t-r|}$.

- One of the worst source term of the transport equation satisfied by $\widehat{Z}f$, where \widehat{Z} is the complete lift of the Killing vector field Z , is bounded by $(1+t+r)\frac{v^L}{v^0}|\mathcal{L}_Z(F)||\partial_{t,x}f|$.
- As $|\mathcal{L}_Z(F)|$ merely decay as $(1+t+r)^{-1}(1+|t-r|)^{-\frac{1}{2}}$, we obtain an (almost) integrable decay rate through the utilization of the inequality $1+|t-r| \lesssim z$, where z is a combination of weights preserved by $v^\mu\partial_\mu$, so that

$$(1+t+r)\frac{v^L}{v^0}|\mathcal{L}_Z(F)||\partial_{t,x}f| \lesssim \frac{1}{1+|t-r|}\frac{v^L}{v^0}\sqrt{z}|\partial_{t,x}f|.$$

- Thus, we schematically have $\|\widehat{Z}f\|_{L^1_{x,v}}(t) \lesssim \|\sqrt{z}\partial_{t,x}f\|_{L^1_{x,v}}(t)\log(3+t)$. This leads us to consider energy norms controlling quantities such as⁶ $\|z^{\frac{N-\beta_P}{2}}\widehat{Z}^\beta f\|_{L^1_{x,v}}$ where β_P is the number of homogeneous vector fields composing \widehat{Z}^β .

5.1.5 Structure of the paper

Section 5.2 contains most of the notations used in this article. The vector fields used in this paper and the commuted equations are presented in Subsection 5.2.3. In Subsection 5.2.4, fundamental properties of the null components of the velocity vector are proved. The energy norms used to study the Vlasov-Maxwell system are introduced in Section 5.3. During this section, we also prove approximate conservation laws as well as Klainerman-Sobolev type inequalities in order to control these norms and derive pointwise decay estimates from them. Section 5.4 is devoted to the study of the null structure of the commuted Vlasov equations. In section 5.5, we set up the bootstrap assumptions, present their immediate consequences and describe the strategy of the proof of our main result. Sections 5.6 (respectively 5.7) concerns the improvement of the energy bounds on the particle density (respectively the electromagnetic field). Finally, we prove in Section 5.8 L^2 estimates for the velocity averages of the higher order derivatives of the Vlasov field.

5.1.6 Acknowledgements

I would like to express my gratitude to my advisor Jacques Smulevici for suggesting me to study this problem and for his valuable comments. Part of this work was funded by the European Research Council under the European Union's Horizon 2020 research and innovation program (project GEOWAKI, grant agreement 714408).

5.2 Preliminaries

5.2.1 Basic notations

In this article we work on the 3 + 1 dimensional Minkowski spacetime $(\mathbb{R}^{3+1}, \widehat{\eta})$ and we will use two sets of coordinates. The Cartesian coordinates $(x^0 = t, x^1, x^2, x^3)$ and null coordinates $(\underline{u}, u, \omega_1, \omega_2)$, where

$$\underline{u} = t + r, \quad u = t - r$$

and (ω_1, ω_2) are spherical variables, which are spherical coordinates on the spheres $(t, r) = \text{constant}$. Apart from $r = 0$ and the usual degeneration of spherical coordinates, these coordinates are defined globally on the Minkowski space. We will also use the following classical weights,

$$\tau_+ := \sqrt{1 + \underline{u}^2} \quad \text{and} \quad \tau_- := \sqrt{1 + u^2}.$$

Remark 5.2.1. *In this paper, we exclusively work in regions where $1 + t \leq \tau_+(t, x) \lesssim |x|$.*

We denote by ∇ the intrinsic covariant differentiation on the spheres $(t, r) = \text{constant}$ and by (e_1, e_2) an orthonormal basis on them. Capital Roman indices such as A or B will always correspond to spherical variables. The null derivatives are defined by

$$L = \partial_t + \partial_r \quad \text{and} \quad \underline{L} = \partial_t - \partial_r, \quad \text{so that} \quad L(\underline{u}) = 2, \quad L(u) = 0, \quad \underline{L}(\underline{u}) = 0 \quad \text{and} \quad \underline{L}(u) = 2.$$

The velocity vector $(v^\mu)_{0 \leq \mu \leq 3}$ is parametrized by $(v^i)_{1 \leq i \leq 3}$ and $v^0 = \sqrt{1 + |v|^2}$ since we normalize the mass of the particles to $m = 1$. Let \mathbf{T} be the operator defined by

$$\mathbf{T} : f \mapsto v^\mu \partial_\mu f,$$

⁶Actually, because of other source terms, we will consider slightly different energy norms.

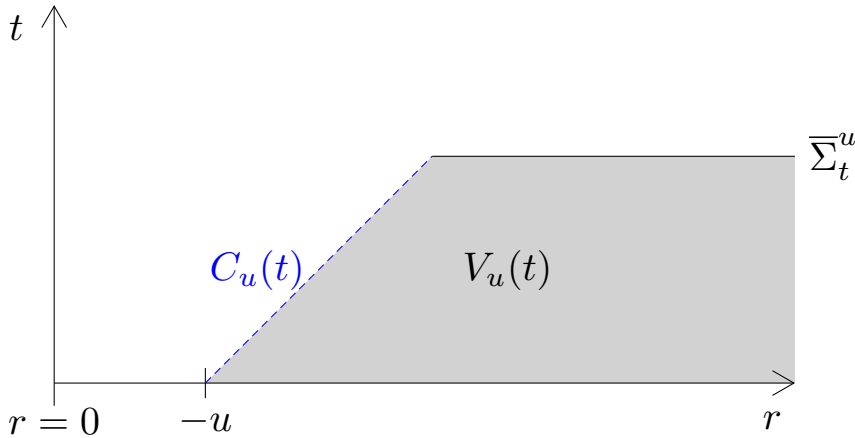
for all sufficiently regular function $f : [0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$. We will raise and lower indices using the metric $\tilde{\eta}$. For instance, $v_0 = v^\mu \tilde{\eta}_{\mu 0} = -v^0$ and $x^1 = x_\mu \tilde{\eta}^{\mu 1} = x_1$. Finally, we will use the notation $Q \lesssim R$ for an inequality of the form $Q \leq CR$, where $C > 0$ is a constant independent of the solutions but which could depend on N , the maximal number of derivatives, or on fixed parameters (δ and η).

5.2.2 A null foliation

We start by presenting various subsets of the Minkowski space which depends on $t \in \mathbb{R}_+$, $r \in \mathbb{R}_+$, $u \in \mathbb{R}$ or $b \in \mathbb{R}$. Let $\mathbb{S}_{t,r}$, $\bar{\Sigma}_t^b$, $C_u(t)$ and $V_u(t)$, be the sets defined as

$$\begin{aligned} \mathbb{S}_{t,r} &:= \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / (s, |y|) = (t, r)\}, & C_u(t) &:= \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s < t, s - |y| = u\}, \\ \bar{\Sigma}_t^b &:= \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s = t, |y| > s - b\}, & V_u(t) &:= \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / s < t, s - |y| < u\}. \end{aligned}$$

The volum form on $C_u(t)$ is given by $dC_u(t) = \sqrt{2}^{-1} r^2 d\underline{u} d\underline{S}^2$, where $d\underline{S}^2$ is the standard metric on the 2 dimensional unit sphere.



The sets $\bar{\Sigma}_t^u$, $C_u(t)$ and $V_u(t)$

The following lemma illustrates that we can foliate $V_b(T)$ by $(\bar{\Sigma}_s^b)_{0 \leq s < T}$ or $(C_u(T))_{u < b}$ and will be used several times during this article.

Lemma 5.2.2. *Let $T > 0$, $b \in \mathbb{R}$ and $g \in L^1(V_b(T))$. Then*

$$\int_{V_b(T)} g dV_b(T) = \int_0^T \int_{\bar{\Sigma}_s^b} g dx ds = \int_{u=-\infty}^b \int_{C_u(T)} g dC_u(T) \frac{du}{\sqrt{2}}.$$

We will use the second foliation in order to take advantage of decay in the $t - r$ direction as $\|\tau_-^{-1}\|_{L^1(C_u(t))} = \tau_-^{-1}$ whereas $\|\tau_-^{-1}\|_{L^\infty(\bar{\Sigma}_s^b)} \geq (1 + b^2)^{-\frac{1}{2}}$.

5.2.3 The commutation vector fields

The aim of this subsection is to introduce the commutation vector fields for the Maxwell equations, those for the relativistic transport operator and certain of their basic properties. Let \mathbb{P} be the generators of the Poincaré algebra, i.e. the set containing

- the translations $\partial_\mu := \partial_{x^\mu}, \quad 0 \leq \mu \leq 3.$
- the rotations $\Omega_{ij} = x^i \partial_j - x^j \partial_i, \quad 1 \leq i < j \leq 3.$
- the Lorentz boosts $\Omega_{0k} = t \partial_k + x^k \partial_t, \quad 1 \leq k \leq 3.$

Let also $\mathbb{O} := \{\Omega_{12}, \Omega_{13}, \Omega_{23}\}$ be the set of the rotational vector fields and $\mathbb{K} := \mathbb{P} \cup \{S\}$, where $S = x^\mu \partial_\mu$ is the scaling vector field. We will use the vector fields of \mathbb{K} for commuting the Maxwell equations. To commute the operator $\mathbf{T} = v^\mu \partial_\mu$, we will rather use the complete lifts of the vector fields of \mathbb{P} .

Definition 5.2.3. *Let Γ be a vector field of the form $\Gamma^\beta \partial_\beta$. Then, the complete lift $\hat{\Gamma}$ of Γ is defined by*

$$\hat{\Gamma} = \Gamma^\beta \partial_\beta + v^\gamma \frac{\partial \Gamma^i}{\partial x^\gamma} \partial_{v^i}.$$

Consequently, for all $\mu \in \llbracket 0, 3 \rrbracket$, $1 \leq i < j \leq 3$ and $k \in \llbracket 1, 3 \rrbracket$,

$$\widehat{\partial}_\mu = \partial_\mu, \quad \widehat{\Omega}_{ij} = x^i \partial_j - x^j \partial_i + v^i \partial_{v^j} - v^j \partial_{v^i} \quad \text{and} \quad \widehat{\Omega}_{0k} = t \partial_k + x^k \partial_t + v^0 \partial_{v^k}.$$

Since $[\mathbf{T}, \widehat{Z}] = 0$ for all $Z \in \mathbb{P}$ and $[\mathbf{T}, S] = \mathbf{T}$, we consider, as [18], the following set

$$\widehat{\mathbb{P}}_0 := \{\widehat{Z} / Z \in \mathbb{P}\} \cup \{S\}.$$

For simplicity, we denote by \widehat{Z} an arbitrary vector field of $\widehat{\mathbb{P}}_0$, even if S is not a complete lift. Note that the vectorial space engendered by each of these sets is an algebra. More precisely, if \mathbb{L} is either \mathbb{K} , \mathbb{P} or \mathbb{O} , then for all $(Z_1, Z_2) \in \mathbb{L}^2$, $[Z_1, Z_2]$ is a linear combinations of vector fields of \mathbb{L} . We also consider an ordering on each of the sets \mathbb{O} , \mathbb{P} , \mathbb{K} and $\widehat{\mathbb{P}}_0$, such that, if $\mathbb{P} = \{Z^i / 1 \leq i \leq |\mathbb{P}|\}$, then $\mathbb{K} = \{Z^i / 1 \leq i \leq |\mathbb{K}|\}$, with $Z^{|\mathbb{K}|} = S$, and

$$\widehat{\mathbb{P}}_0 = \left\{ \widehat{Z}^i / 1 \leq i \leq |\widehat{\mathbb{P}}_0| \right\}, \quad \text{with} \quad \left(\widehat{Z}^i \right)_{1 \leq i \leq |\mathbb{P}|} = \left(Z^i \right)_{1 \leq i \leq |\mathbb{P}|} \quad \text{and} \quad \widehat{Z}^{|\widehat{\mathbb{P}}_0|} = S.$$

If \mathbb{L} denotes \mathbb{O} , \mathbb{P} , \mathbb{K} or $\widehat{\mathbb{P}}_0$, and $\beta \in \{1, \dots, |\mathbb{L}|\}^q$, with $q \in \mathbb{N}^*$, we will denote the differential operator $\Gamma^{\beta_1} \dots \Gamma^{\beta_r} \in \mathbb{L}^{|\beta|}$ by Γ^β . For a vector field X , we denote by \mathcal{L}_X the Lie derivative with respect to X and if $Z^\gamma \in \mathbb{K}^q$, we will write \mathcal{L}_{Z^γ} for $\mathcal{L}_{Z^{\gamma_1}} \dots \mathcal{L}_{Z^{\gamma_q}}$. We denote moreover the number of translations composing Γ^β by β_T and the number of homogeneous vector fields by β_P , so that $|\beta| = \beta_T + \beta_P$.

Let us recall, by the following classical result, that the derivatives tangential to the cone behave better than others.

Lemma 5.2.4. *The following relations hold,*

$$(t-r)\underline{L} = S - \frac{x^i}{r} \Omega_{0i}, \quad (t+r)L = S + \frac{x^i}{r} \Omega_{0i} \quad \text{and} \quad re_A = \sum_{1 \leq i < j \leq 3} C_A^{i,j} \Omega_{ij},$$

where the $C_A^{i,j}$ are uniformly bounded and depend only on spherical variables. Similarly, we have

$$(t-r)\partial_t = \frac{t}{t+r} S - \frac{x^i}{t+r} \Omega_{0i} \quad \text{and} \quad (t-r)\partial_i = \frac{t}{t+r} \Omega_{0i} - \frac{x^i}{t+r} S - \frac{x^j}{t+r} \Omega_{ij}.$$

We introduce now the notation $\nabla_v g := (0, \partial_{v^1} g, \partial_{v^2} g, \partial_{v^3} g)$, so that (5.1) can be rewritten

$$\mathbf{T}_F(f) := v^\mu \partial_\mu f + F(v, \nabla_v f) = 0.$$

In order to commute the Vlasov-Maxwell system, we recall the following result (see Lemma 2.8 of [6] for a proof) where the Kronecker symbol is extended to vector fields, i.e. $\delta_{X,Y} = 1$ if $X = Y$ and $\delta_{X,Y} = 0$ otherwise.

Lemma 5.2.5. *Let G be a 2-form and g a function both sufficiently regular. Then, for all $\widehat{Z} \in \widehat{\mathbb{P}}_0$,*

$$\widehat{Z}(G(v, \nabla_v g)) = \mathcal{L}_{\widehat{Z}}(G)(v, \nabla_v g) + G(v, \nabla_v \widehat{Z}g) - 2\delta_{\widehat{Z}, S} G(v, \nabla_v g).$$

If G and g satisfy $\nabla^\mu G_{\mu\nu} = J(g)_\nu$ and $\nabla^{\mu*} G_{\mu\nu} = 0$, then

$$\forall Z \in \mathbb{K}, \quad \nabla^\mu \mathcal{L}_Z(G)_{\mu\nu} = J(\widehat{Z}g)_\nu + 3\delta_{Z,S} J(g)_\nu \quad \text{and} \quad \nabla^{\mu*} \mathcal{L}_Z(G)_{\mu\nu} = 0.$$

We then deduce the form of the source terms of the commuted Vlasov-Maxwell equations.

Proposition 5.2.6. *Let (f, F) be a sufficiently regular solution to the VM system (5.1)-(5.3) and $Z^\kappa \in \mathbb{K}^{|\kappa|}$. There exists integers $n_{\gamma,\beta}^\kappa$ and m_ξ^κ such that*

$$\begin{aligned} [\mathbf{T}_F, \widehat{Z}^\kappa](f) &= \mathbf{T}_F(\widehat{Z}^\kappa f) = \sum_{\substack{|\gamma|+|\beta| \leq |\kappa| \\ |\beta| \leq |\kappa|-1}} n_{\gamma,\beta}^\kappa \mathcal{L}_{Z^\gamma}(F)(v, \nabla_v \widehat{Z}^\beta(f)), \\ \nabla^\mu \mathcal{L}_{Z^\kappa}(F)_{\mu\nu} &= \sum_{|\xi| \leq |\kappa|} m_\xi^\kappa J(\widehat{Z}^\xi f)_\nu, \\ \nabla^{\mu*} \mathcal{L}_{Z^\kappa}(F)_{\mu\nu} &= 0. \end{aligned}$$

Moreover, the number of homogeneous vector fields β_P of \widehat{Z}^β satisfies the following condition.

- Either $\beta_P < \kappa_P$
- or $\beta_P = \kappa_P$ and $\gamma_T \geq 1$.

Note that the structure of the non-linearity $F(v, \nabla_v f)$ as well as the one of $J(f)$ is preserved by commutation, which reflects the null properties of the system. This is crucial for us since, as mentioned earlier, if the source terms of the Vlasov equation (respectively the Maxwell equations) behaved as $v^0 |F| |\partial_v f|$ (respectively $\int_v |f| dv$), we would not be able to close the energy estimates for the Vlasov field (respectively the electromagnetic field).

Remark 5.2.7. *Let us explain why we count the number of the homogeneous vector fields in the source terms of the Vlasov equation. As $\partial_v f \sim \tau_+ \partial_{t,x} f + \widehat{Z} f$, the decay rate of the solutions will not be strong enough for us to close the energy estimates without using a hierarchy on the derivatives of f . If $\gamma_T \geq 1$, Lemma 5.2.4 will give us an extra decay in the u direction. Otherwise, the worst source terms to control in order to bound $\|\widehat{Z}^\kappa f\|_{L^1_{x,v}}$ will only involve $\widehat{Z}^\beta f$, with $\beta_P < \kappa_P$.*

5.2.4 The null components of the velocity vector and the weights preserved by \mathbf{T}

We denote by $(v^L, v^{\underline{L}}, v^{e_1}, v^{e_2})$ the null components of the velocity vector v , so that

$$v = v^L L + v^{\underline{L}} \underline{L} + v^{e_A} e_A, \quad v^L = \frac{v^0 + v^r}{2} \quad \text{and} \quad v^{\underline{L}} = \frac{v^0 - v^r}{2}.$$

If there is no ambiguity, we will write v^A for v^{e_A} . Let \mathbf{k}_1 and z be defined as

$$\mathbf{k}_1 := \left\{ \frac{v^\mu}{v^0} \mid 0 \leq \mu \leq 3 \right\} \cup \left\{ z_{\mu\nu} \mid \mu \neq \nu \right\}, \quad \text{where} \quad z_{\mu\nu} := x^\mu \frac{v^\nu}{v^0} - x^\nu \frac{v^\mu}{v^0}, \quad \text{and} \quad z^2 := \sum_{w \in \mathbf{k}_1} w^2.$$

Because of regularity issues, we will rather work with z than with the elements of \mathbf{k}_1 . Two fundamental properties of these weights is that they are preserved by the flow of \mathbf{T} and by the action of $\widehat{\mathbb{P}}_0$.

Lemma 5.2.8. *For all $\widehat{Z} \in \widehat{\mathbb{P}}_0$ and $a \in \mathbb{R}_+$, we have*

$$\mathbf{T}(z) = 0 \quad \text{and} \quad \left| \widehat{Z}(z^a) \right| \lesssim a z^a.$$

Proof. Let $w \in \mathbf{k}_1$. By straightforward computations, one can prove that

$$\mathbf{T}(w) = 0 \quad \text{and} \quad \widehat{Z}(v^0 w) \in v^0 \mathbf{k}_1 \cup \{0\}, \quad \text{so that} \quad \left| \widehat{Z}(w) \right| \lesssim \sum_{w_0 \in \mathbf{k}_1} |w_0|.$$

Indeed, considering for instance $tv^1 - x^1 v^0$, $x^1 v^2 - x^2 v^1$, $\widehat{\Omega}_{12}$ and S , we have

$$\begin{aligned} \widehat{\Omega}_{12}(tv^1 - x^1 v^0) &= -tv^2 - x^2 v^0, & \widehat{\Omega}_{12}(x^1 v^2 - x^2 v^1) &= 0, \\ S(tv^1 - x^1 v^0) &= tv^1 - x^1 v^0 & \text{and} & \quad S(x^1 v^2 - x^2 v^1) = x^1 v^2 - x^2 v^1. \end{aligned}$$

Then,

$$\mathbf{T}(z) = \sum_{w \in \mathbf{k}_1} \frac{w}{z} \mathbf{T}(w) = 0 \quad \text{and} \quad \left| \widehat{Z}(z^a) \right| = \left| a z^{a-1} \sum_{w \in \mathbf{k}_1} \frac{w}{z} \widehat{Z}(w) \right| \lesssim a z^{a-1} \sum_{w_0 \in \mathbf{k}_1} |w_0| \lesssim a z^a.$$

□

Recall that if $\mathbf{k}_0 := \mathbf{k}_1 \cup \{x^\mu v_\mu\}$, then $\tau_- v^L + \tau_+ v^{\underline{L}} \lesssim \sum_{w \in \mathbf{k}_0} |w|$. Unfortunately, the weight $x^\mu v_\mu$ is not preserved by \mathbf{T} so we will not be able to take advantage of this inequality during this paper. In the following lemma, which reflects the good behavior of the components $v^{\underline{L}}$ and v^A of the velocity vector, we prove a similar inequality specific to the exterior of the lightcone and adapted to the study of massive particles.

Lemma 5.2.9. *We have, for all $|x| \geq t$,*

$$1 \leq 4v^0 v^{\underline{L}}, \quad |v^A| \lesssim \sqrt{v^0 v^{\underline{L}}} \quad \text{and} \quad \tau_- + (1+r) \frac{v^{\underline{L}}}{v^0} + (1+r) \frac{|v^A|}{v^0} \lesssim z.$$

⁷Note however that $x^\mu v_\mu$ is preserved by the massless relativistic transport operator $|v| \partial_t + v^i \partial_i$.

Proof. Note first that $4r^2v^Lv^{\underline{L}} \geq r^2 + \sum_{k<l} |v^0z_{kl}|^2$. Indeed, as we study massive particles, we have $v^0 = \sqrt{1+|v|^2}$, so that

$$4r^2v^Lv^{\underline{L}} = (rv^0)^2 - (x^iv_i)^2 = r^2 + \sum_{i=1}^3 (r^2 - |x^i|^2) |v_i|^2 - 2 \sum_{1 \leq k < l \leq 3} x^k x^l v_k v_l,$$

$$\sum_{1 \leq k < l \leq 3} |v^0 z_{kl}|^2 = \sum_{1 \leq k < l \leq 3} |x^k|^2 |v_l|^2 + |x^l|^2 |v_k|^2 - 2x^k x^l v_k v_l = \sum_{i=1}^3 \sum_{j \neq i} |x^j|^2 |v_i|^2 - 2 \sum_{1 \leq k < l \leq n} x^k x^l v_k v_l.$$

The first inequality then comes from $v^{\underline{L}} \leq v^0$. The second one and $(1+r)\frac{|v^A|}{v^0} \lesssim z$ then ensue from $rv_A = v^0 C_A^{i,j} z_{ij}$, where $C_A^{i,j}$ are bounded functions depending only on the spherical variables such as $re_A = C_A^{i,j} \Omega_{ij}$. The last part of the third inequality is specific to the exterior of the light cone. Recall that $x^i - t\frac{v^i}{v^0} \in \mathbf{k}_1$. Then, $\tau_- \lesssim z$ follows from $1 \leq z$ and

$$(r-t) \leq r - t\frac{|v|}{v^0} \leq \left| x - t\frac{v}{v^0} \right| \leq \sum_{i=1}^3 \left| x^i - t\frac{v^i}{v^0} \right| = \sum_{i=1}^3 |z_{0i}| \leq z. \quad (5.6)$$

Finally, remark first that $v^{\underline{L}} \leq v^0$, which treats the case $|x| \leq 1$. If $|x| \geq \max(t, 1)$, note that

$$2rv^{\underline{L}} = rv^0 - r\frac{x^i}{r}v_i = rv^0 + (t-r)\frac{x^i}{r}v_i - \frac{x^i}{r}v^0 \left(t\frac{v_i}{v^0} - x_i \right) - rv^0 = (t-r)v_i\frac{x^i}{r} - v^0\frac{x^i}{r}z_{0i} \quad \text{and use (5.6).}$$

□

5.2.5 The null decomposition of the electromagnetic field

In order to capture its geometric properties, the electromagnetic field will be represented all along this paper by a 2-form. Let G be a 2-form defined on $[0, T] \times \mathbb{R}_x^3$. Its Hodge dual $*G$ is the 2-form given by

$$*G_{\mu\nu} = \frac{1}{2} G^{\lambda\sigma} \varepsilon_{\lambda\sigma\mu\nu},$$

where ε is the Levi-Civita symbol, and its energy-momentum tensor is

$$T[G]_{\mu\nu} := G_{\mu\beta} G_{\nu}^{\beta} - \frac{1}{4} \eta_{\mu\nu} G_{\rho\sigma} G^{\rho\sigma}.$$

Note that $T[G]_{\mu\nu}$ is symmetric, i.e. $T[G]_{\mu\nu} = T[G]_{\nu\mu}$. The null decomposition of G , $(\alpha(G), \underline{\alpha}(G), \rho(G), \sigma(G))$, introduced by [11], is defined by

$$\alpha_A(G) = G_{AL}, \quad \underline{\alpha}_A(G) = G_{A\underline{L}}, \quad \rho(G) = \frac{1}{2} G_{LL} \quad \text{and} \quad \sigma(G) = G_{e_1 e_2},$$

so that the null components of $T[G]$ are then given by

$$T[G]_{LL} = |\alpha(G)|^2, \quad T[G]_{\underline{L}\underline{L}} = |\underline{\alpha}(G)|^2 \quad \text{and} \quad T[G]_{LL} = |\rho(G)|^2 + |\sigma(G)|^2. \quad (5.7)$$

For a proof of the following classical results, we refer to [11] or to [6] (Subsection 2.3 and Lemma D.3).

Lemma 5.2.10. *Let G be a 2-form and J be a 1-form both sufficiently regular and such that*

$$\begin{aligned} \nabla^\mu G_{\mu\nu} &= J_\nu \\ \nabla^\mu *G_{\mu\nu} &= 0. \end{aligned}$$

Then, $\nabla^\mu T[G]_{\mu\nu} = G_{\nu\lambda} J^\lambda$ and, denoting by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of G ,

$$\nabla_{\underline{L}} \alpha_A - \frac{\alpha_A}{r} + \nabla_{e_A} \rho + \varepsilon_{BA} \nabla_{e_B} \sigma = J_A.$$

5.3 Energy and pointwise decay estimates

We recall here classical energy estimates for both the Vlasov field and the electromagnetic field and how obtain pointwise decay estimates from them. For all this section, we define $T > 0$ and $b \leq -1$. The energies defined below are adapted to the study of the Vlasov-Maxwell system in the exterior of the light cone $u \geq b$.

5.3.1 Estimates for velocity averages

For the Vlasov field, we will use the following approximate conservation law.

Proposition 5.3.1. *Let $H : V_b(T) \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ and $g_0 : \overline{\Sigma}_0^b \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ be two sufficiently regular functions and F a sufficiently regular 2-form. Then, g , the unique classical solution of*

$$\begin{aligned} \mathbf{T}_F(g) &= H \\ g(0, \cdot, \cdot) &= g_0, \end{aligned}$$

satisfies, for all $t \in [0, T[$, the following estimates,

$$\left\| \int_{v \in \mathbb{R}^3} |g| dv \right\|_{L^1(\overline{\Sigma}_t^b)} + \sup_{u < b} \left\| \int_{v \in \mathbb{R}^3} \frac{v^{\underline{L}}}{v^0} |g| dv \right\|_{L^1(C_u(t))} \leq 2 \left\| \int_{v \in \mathbb{R}^3} |g_0| dv \right\|_{L^1(\overline{\Sigma}_0^b)} + 2 \int_0^t \int_{\overline{\Sigma}_s^b} \int_{v \in \mathbb{R}^3} |H| \frac{dv}{v^0} dx ds.$$

Proof. As $\mathbf{T}(|g|) = \frac{g}{|g|} H - \frac{g}{|g|} F(v, \nabla_v g)$ and since F is a 2-form, integration by parts in v gives us

$$\partial_\mu \int_v |g| \frac{v^\mu}{v^0} dv = \int_v \left(\frac{g}{|g|} \frac{H}{v^0} - \frac{g}{|g|} F \left(\frac{v}{v^0}, \nabla_v g \right) \right) dv = \int_v \left(\frac{g}{|g|} \frac{H}{v^0} - \frac{v^j v^i}{(v^0)^3} F_{ji} |g| \right) dv = \int_v \frac{g}{|g|} H \frac{dv}{v^0}.$$

Apply now the divergence theorem to $\int_v |g| \frac{v^\mu}{v^0} dv$ in the region $V_u(t)$, for $u < b$, in order to get

$$\int_{\overline{\Sigma}_t^u} \int_v |g| dv dx + \sqrt{2} \int_{C_u(t)} \int_v \frac{v^{\underline{L}}}{v^0} |g| dv dC_u(t) = \int_{\overline{\Sigma}_0^u} \int_v |g| dv dx + \int_0^t \int_{\overline{\Sigma}_s^u} \int_v \frac{g}{|g|} H \frac{dv}{v^0} dx ds.$$

We then deduce that

$$\begin{aligned} \int_{\overline{\Sigma}_t^u} \int_v |g| dv dx &= \sup_{u < b} \int_{\overline{\Sigma}_t^u} \int_v |g| dv dx \leq \int_{\overline{\Sigma}_0^b} \int_v |g| dv dx + \int_0^t \int_{\overline{\Sigma}_s^b} \int_v |H| \frac{dv}{v^0} dx ds, \\ \sup_{u < b} \int_{C_u(t)} \int_v \frac{v^{\underline{L}}}{v^0} |g| dv dC_u(t) &\leq \int_{\overline{\Sigma}_0^b} \int_v |g| dv dx + \int_0^t \int_{\overline{\Sigma}_s^b} \int_v |H| \frac{dv}{v^0} dx ds, \end{aligned}$$

which allows us to conclude the proof. \square

In view of Remark 5.2.7 and the previous proposition, we then define hierarchised energy norms. For $(Q, \lambda) \in \mathbb{N} \times [0, \frac{1}{2}]$ and $q \in [Q, +\infty[$, let

$$\mathbb{E}_b[g](t) := \left\| \int_{v \in \mathbb{R}^3} |g| dv \right\|_{L^1(\overline{\Sigma}_t^b)} + \sup_{u < b} \left\| \int_{v \in \mathbb{R}^3} \frac{v^{\underline{L}}}{v^0} |g| dv \right\|_{L^1(C_u(t))}, \quad (5.8)$$

$$\mathbb{E}_{Q,b}^{q,\lambda}[f](t) := \sum_{0 \leq k \leq Q} \sum_{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^k} (1+t)^{-\beta_P \lambda} \mathbb{E}_b[\sqrt{z}^{q-(1-2\lambda)\beta_P} \widehat{Z}^\beta f](t). \quad (5.9)$$

Remark 5.3.2. *As $z \geq 1$, we have $\mathbb{E}[\sqrt{z}^a \widehat{Z}^\beta f](t) \leq (1+t)^{\beta_P \lambda} \mathbb{E}_{Q,b}^{q,\lambda}[f](t)$ for all $0 \leq a \leq q - (1-2\lambda)\beta_P$.*

The remaining of this subsection is devoted to the proof of a Klainerman-Sobolev type inequality. The constants hidden by \lesssim will here depend on a . We start with a commutation property between the vector fields of \mathbb{K} and the averaging in v .

Lemma 5.3.3. *Let $f : V_b(T) \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ be a sufficiently regular function and $a \in \mathbb{R}_+$. We have, almost everywhere,*

$$\forall Z \in \mathbb{K}, \quad \left| Z \left(\int_{v \in \mathbb{R}^3} z^a |f| dv \right) \right| \lesssim \int_{v \in \mathbb{R}^3} z^a |f| dv + \int_{v \in \mathbb{R}^3} z^a |\widehat{Z} f| dv.$$

Proof. Consider for instance the case where $Z = \Omega_{01} = t\partial_1 + x^1\partial_t$. We have, almost everywhere,

$$\begin{aligned} \left| Z \left(\int_{v \in \mathbb{R}^3} |z^a f| dv \right) \right| &= \left| \int_{v \in \mathbb{R}^3} \widehat{\Omega}_{01}(|z^a f|) dv - \int_{v \in \mathbb{R}^3} v^0 \partial_{v^1}(|z^a f|) dv \right| \\ &= \left| \int_{v \in \mathbb{R}^3} \frac{z^a f}{|z^a f|} \widehat{\Omega}_{01}(z^a f) dv + \int_{v \in \mathbb{R}^3} \frac{v^1}{v^0} |z^a f| dv \right| \\ &\leq \int_{v \in \mathbb{R}^3} \left| \widehat{\Omega}_{01}(z^a f) \right| dv + \int_v |z^a f| dv. \end{aligned}$$

It then remains to use $|\widehat{\Omega}_{01}(z^a)| \lesssim az^a$. \square

Before presenting the Klainerman-Sobolev inequality used in this article, we prove the following estimate.

Lemma 5.3.4. *Let $g : \mathbb{S}^2 \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ be a sufficiently regular function and $a \in \mathbb{R}_+$. Then,*

$$\forall \omega \in \mathbb{S}^2, \quad \int_{v \in \mathbb{R}^3} z^a |g|(\omega, v) dv \lesssim \sum_{0 \leq k \leq 2} \sum_{\Omega^\beta \in \mathbb{O}^k} \left\| \int_{v \in \mathbb{R}^3} z^a |\widehat{\Omega}^\beta g| dv \right\|_{L^1(\mathbb{S}^2)}.$$

Proof. Let $\omega \in \mathbb{S}^2$ and (θ, φ) a local coordinate map in a neighborhood of ω . By the symmetry of the sphere we can suppose that θ and φ take their values in an interval of a size independent of ω . Using a one dimensional Sobolev inequality, that $|\partial_\theta u| \lesssim \sum_{\Omega \in \mathbb{O}} |\Omega u|$ and Lemma 5.3.3, we have,

$$\begin{aligned} \int_v z^a |g|(\omega_1, \omega_2, v) dv &\lesssim \int_\theta \left| \int_v z^a |g|(\theta, \omega_2, v) dv \right| + \left| \partial_\theta \int_v z^a |g|(\theta, \omega_2, v) dv \right| d\theta \\ &\lesssim \int_\theta \int_v z^a |g|(\theta, \omega_2, v) dv d\theta + \sum_{\Omega \in \mathbb{O}} \int_\theta \int_v z^a |\widehat{\Omega} g|(\theta, \omega_2, v) dv d\theta. \end{aligned}$$

Doing the same for the second spherical variable of $\int_v z^a |\widehat{\Omega}^\beta g|(\theta, \omega_2, v) dv$, we obtain the result. \square

Proposition 5.3.5. *Let $f : V_b(T) \times \mathbb{R}_v^3 \rightarrow \mathbb{R}$ be a sufficiently regular function and $a \in \mathbb{R}_+$. Then,*

$$\forall (t, x) \in V_b(T), \quad \int_{v \in \mathbb{R}^3} z^a |f|(t, x, v) dv \lesssim \frac{1}{\tau_+^2 \tau_-} \sum_{0 \leq k \leq 3} \sum_{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^k} \int_{|y| \geq |x|} \int_{v \in \mathbb{R}^3} z^a |\widehat{Z}^\beta f| dv dx.$$

Proof. Let $(t, x) = (t, |x|\omega) \in V_b(T)$. One has, using successively Lemmas 5.2.4 and 5.3.3,

$$\begin{aligned} |x|^2 \tau_- \int_v z^a |f|(t, |x|\omega, v) dv &= -|x|^2 \int_{r=|x|}^{+\infty} \partial_r \left(\tau_- \int_v z^a |f|(t, r\omega, v) dv \right) dr \\ &\lesssim |x|^2 \sum_{Z \in \mathbb{K}} \int_{r=|x|}^{+\infty} \left(\int_v z^a |f|(t, r\omega, v) dv + Z \left(\int_v z^a |f|(t, r\omega, v) dv \right) \right) dr \\ &\lesssim \int_{r=|x|}^{+\infty} \int_v z^a |f|(t, r\omega, v) dv r^2 dr + \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \int_{r=|x|}^{+\infty} \int_v z^a |\widehat{Z}^\kappa f|(t, r\omega, v) dv r^2 dr. \end{aligned}$$

It then remains to apply the previous lemma and to recall that $\tau_+ \lesssim r$ in $V_b(T)$. \square

We can improve the decay rate in the u direction if we pay the price in terms of weights in v^0 and z . More precisely, by Lemma 5.2.9, we have $|v^0|^{-a} \lesssim |v^L|^{-a} \lesssim |v^0 z|^a$, so that

$$\forall (t, x) \in V_b(T), \quad \int_{v \in \mathbb{R}^3} |f(t, x, v)| \frac{dv}{(v^0)^{2a}} \lesssim \frac{1}{\tau_+^{2+a} \tau_-} \sum_{0 \leq k \leq 3} \sum_{\widehat{Z}^\beta \in \widehat{\mathbb{P}}_0^k} \left\| \int_{v \in \mathbb{R}^3} z^a |\widehat{Z}^\beta f| dv \right\|_{L^1(\widehat{\Sigma}_t^b)}.$$

5.3.2 Estimates for the electromagnetic field

In this subsection, we introduce first the energy norm used in this paper to study the electromagnetic field and, secondly, we derive pointwise decay estimates from it through Klainerman-Sobolev inequalities. We consider, for the remaining of this section, G a sufficiently regular 2-form defined on $V_b(T)$ and we denote by $(\alpha, \underline{\alpha}, \rho, \sigma)$ its null decomposition. We suppose that G satisfies

$$\begin{aligned} \nabla^\mu G_{\mu\nu} &= J_\nu \\ \nabla^{\mu*} G_{\mu\nu} &= 0, \end{aligned}$$

with J a sufficiently regular 1-form defined on $V_b(T)$.

Definition 5.3.6. *Let $N \in \mathbb{N}$. We define, for $t \in [0, T]$,*

$$\begin{aligned} \mathcal{E}^b[G](t) &:= \int_{\widehat{\Sigma}_t^b} (|\alpha|^2 + |\underline{\alpha}|^2 + 2|\rho|^2 + 2|\sigma|^2) dx + \sup_{u < b} \int_{C_u(t)} (|\alpha|^2 + |\rho|^2 + |\sigma|^2) dC_u(t), \\ \mathcal{E}_N^b[G](t) &:= \sum_{0 \leq k \leq N} \sum_{Z^\gamma \in \mathbb{K}^k} \mathcal{E}^b[\mathcal{L}_{Z^\gamma}(G)](t). \end{aligned}$$

Proposition 5.3.7. *We have, for all $t \in [0, T[$,*

$$\mathcal{E}^b[G](t) \leq 2\mathcal{E}^b[G](0) + 8 \int_0^t \int_{\overline{\Sigma}_s^b} |G_{\mu 0} J^\mu| dx ds.$$

Proof. Recall from Lemma 5.2.10 that $\nabla^\mu T[G]_{\mu 0} = G_{0\nu} J^\nu$. Hence, applying the divergence theorem in $V_u(t)$, for $u < b$, we get

$$\int_{\overline{\Sigma}_t^u} T[G]_{00} dx + \frac{1}{\sqrt{2}} \int_{C_u(t)} T[G]_{L0} dC_u(t) = \int_{\overline{\Sigma}_0^u} T[G]_{00} dx - \int_0^t \int_{\overline{\Sigma}_s^u} G_{0\nu} J^\nu dx ds.$$

We then obtain

$$\begin{aligned} \sup_{u < b} \int_{C_u(t)} T[G]_{L0} dC_u(t) &\leq \int_{\overline{\Sigma}_0^b} |T[G]_{00}| dx + \int_0^t \int_{\overline{\Sigma}_s^b} |G_{0\nu} J^\nu| dx ds, \\ \int_{\overline{\Sigma}_t^b} T[G]_{00} dx &= \sup_{u < b} \int_{\overline{\Sigma}_t^u} T[G]_{00} dx \leq \int_{\overline{\Sigma}_0^b} |T[G]_{00}| dx + \int_0^t \int_{\overline{\Sigma}_s^b} |G_{0\nu} J^\nu| dx ds. \end{aligned}$$

It then remains to add the previous two inequalities and to notice, using (5.7), that

$$4T[G]_{00} = |\alpha|^2 + |\underline{\alpha}|^2 + 2|\rho|^2 + 2|\sigma|^2 \quad \text{and} \quad 2T[G]_{L0} = |\alpha|^2 + |\rho|^2 + |\sigma|^2.$$

□

In order to prove pointwise decay estimates on G , we will use the following three Lemmas. The first one, which is proved in Appendix *D* of [6], extends the results of Lemma 5.2.4 for the null components of a 2-form.

Lemma 5.3.8. *We have, denoting by ζ any of the null component α , $\underline{\alpha}$, ρ or σ ,*

$$\tau_- |\nabla_{\underline{L}} \zeta(G)| + \tau_+ |\nabla_L \zeta(G)| \lesssim \sum_{|\gamma| \leq 1} |\zeta(\mathcal{L}_{Z^\gamma}(G))|, \quad (1+r) |\nabla \zeta(G)| \lesssim |\zeta(G)| + \sum_{\Omega \in \mathbb{O}} |\zeta(\mathcal{L}_\Omega(G))|$$

and, on $V_b(T)$, $\forall \mu \in \llbracket 0, 3 \rrbracket$, $\tau_- |\nabla_{\partial_t} \zeta(G)| \leq \tau_- |\nabla_L \zeta(G)| + \tau_- |\nabla_{\underline{L}} \zeta(G)| + \tau_- |\nabla \zeta(G)| \lesssim \sum_{|\gamma| \leq 1} |\zeta(\mathcal{L}_{Z^\gamma}(G))|$.

The following result, also proved in Appendix *D* of [6], presents commutation properties between \mathcal{L}_Ω , ∇_{∂_r} , ∇_L or $\nabla_{\underline{L}}$ and the null decomposition of G .

Lemma 5.3.9. *Let $\Omega \in \mathbb{O}$. Then, denoting by ζ any of the null component α , $\underline{\alpha}$, ρ or σ ,*

$$[\mathcal{L}_\Omega, \nabla_{\partial_r}]G = 0, \quad \mathcal{L}_\Omega(\zeta(G)) = \zeta(\mathcal{L}_\Omega(G)) \quad \text{and} \quad \nabla_{\partial_r}(\zeta(G)) = \zeta(\nabla_{\partial_r}(G)).$$

Similar results hold for \mathcal{L}_Ω and ∇_{∂_t} , ∇_L or $\nabla_{\underline{L}}$. For instance, $\nabla_L(\zeta(G)) = \zeta(\nabla_L(G))$.

We now recall the Sobolev inequalities which will be used to prove the pointwise decay estimates on the null components of the electromagnetic field. For this, we introduce $|U(y)|_{\mathbb{O}, k}^2 := \sum_{|\beta| \leq k} |\mathcal{L}_{\Omega^\beta}(U)|^2$, where $\Omega^\beta \in \mathbb{O}^{|\beta|}$.

Lemma 5.3.10. *Let U be a sufficiently regular tensor field defined on \mathbb{R}^3 . Then,*

$$\forall x \neq 0, \quad |U(x)| \lesssim \frac{1}{|x|^{\frac{3}{2}}} \left(\int_{|y| \geq |x|} |U(y)|_{\mathbb{O}, 2}^2 + |y|^2 |\nabla_{\partial_r} U(y)|_{\mathbb{O}, 1}^2 dy \right)^{\frac{1}{2}}.$$

If $t \in \mathbb{R}_+$ and $|x| \geq t - b$, we have

$$\forall x \neq 0, \quad |U(x)| \lesssim \frac{1}{|x| \tau_-^{\frac{1}{2}}} \left(\int_{|y| \geq t-b} |U(y)|_{\mathbb{O}, 2}^2 + \tau_-^2 |\nabla_{\partial_r} U(y)|_{\mathbb{O}, 1}^2 dy \right)^{\frac{1}{2}}.$$

Proof. The first inequality is proved in Lemma 2.3 of [11] and the second one can be proved similarly as inequality (ii) of Lemma 2.3 of [11]. □

We now prove the pointwise decay estimates used in this article.

Proposition 5.3.11. *For all $(t, x) \in V_b(T)$, we have*

$$|\rho|(t, x) + |\sigma|(t, x) + |\underline{\alpha}|(t, x) \lesssim \frac{\sqrt{\mathcal{E}_2^b[G](t)}}{\tau_+ \tau_-^{\frac{1}{2}}}, \quad \text{and} \quad |\alpha|(t, x) \lesssim \frac{\sqrt{\mathcal{E}_2^b[G](t)} + \sum_{|\beta| \leq 1} \|r \mathcal{L}_{Z^\beta}(J)_A\|_{L^2(\bar{\Sigma}_t^b)}}{\tau_+^{\frac{3}{2}}}.$$

Proof. Let $(t, x) \in V_b(T)$. In this proof, Ω^β will always be in $\mathbb{O}^{|\beta|}$ and Z^γ in $\mathbb{K}^{|\gamma|}$. Let ζ be either ρ , σ or $\underline{\alpha}$. As ∇_{∂_r} and \mathcal{L}_Ω commute with the null decomposition (see Lemma 5.3.9), Lemma 5.3.10 gives us

$$r^2 \tau_- |\zeta|^2 \lesssim \int_{|y| \geq t-b} |\zeta|_{\mathbb{O}, 2}^2 + \tau_-^2 |\nabla_{\partial_r}(\zeta)|_{\mathbb{O}, 1}^2 dy \lesssim \sum_{|\beta| \leq 1} \sum_{|\gamma| \leq 2} \int_{|y| \geq t-b} |\zeta(\mathcal{L}_{Z^\gamma}(G))|^2 + \tau_-^2 |\zeta(\mathcal{L}_{\Omega^\beta}(\nabla_{\partial_r} G))|^2 dy.$$

Since ∇_{∂_r} commute with \mathcal{L}_Ω and the null decomposition (see Lemma 5.3.9), it comes, using $2\partial_r = L - \underline{L}$ and Lemma 5.3.8,

$$|\zeta(\mathcal{L}_{\Omega^\beta}(\nabla_{\partial_r} G))| \lesssim |\nabla_{\partial_r} \zeta(\mathcal{L}_{\Omega^\beta}(G))| \lesssim |\nabla_L \zeta(\mathcal{L}_{\Omega^\beta}(G))| + |\nabla_{\underline{L}} \zeta(\mathcal{L}_{\Omega^\beta}(G))| \lesssim \frac{1}{\tau_-} \sum_{|\gamma| \leq |\beta|+1} |\zeta(\mathcal{L}_{Z^\gamma}(G))|. \quad (5.10)$$

As $\tau_+ \lesssim r$ in $V_b(T)$, we finally obtain

$$\tau_+^2 \tau_- |\zeta|^2 \lesssim \sum_{|\gamma| \leq 2} \int_{|y| \geq t-b} |\zeta(\mathcal{L}_{Z^\gamma}(G))|^2 dx \lesssim \mathcal{E}_2^b[G](t).$$

We now turn on α . As $\nabla^\mu \mathcal{L}_\Omega(G)_{\mu\nu} = \mathcal{L}_\Omega(J)_\nu$ and $\nabla^{\mu*} \mathcal{L}_\Omega(G)_{\mu\nu} = 0$ for all $\Omega \in \mathbb{O}$, Lemma 5.2.10 gives

$$\forall |\beta| \leq 1, \quad \nabla_{\underline{L}} \alpha(\mathcal{L}_{\Omega^\beta}(G))_A = \frac{1}{r} \alpha(\mathcal{L}_{\Omega^\beta}(G))_A - \nabla_{e_A} \rho(\mathcal{L}_{\Omega^\beta}(G)) + \varepsilon_{AB} \nabla_{e_B} \sigma(\mathcal{L}_{\Omega^\beta}(G)) + \mathcal{L}_{\Omega^\beta}(J)_A.$$

Consequently, we get using Lemma 5.3.8 that for all $\Omega \in \mathbb{O}$,

$$|\alpha(\nabla_{\partial_r} G)| + |\alpha(\mathcal{L}_\Omega(\nabla_{\partial_r} G))| \lesssim |J_A| + |\mathcal{L}_\Omega(J)_A| + \frac{1}{r} \sum_{|\gamma| \leq 2} |\alpha(\mathcal{L}_{Z^\gamma}(G))| + |\rho(\mathcal{L}_{Z^\gamma}(G))| + |\sigma(\mathcal{L}_{Z^\gamma}(G))|. \quad (5.11)$$

Applying the first inequality of Lemma 5.3.10 and using this time (5.11) instead of (5.10), we get

$$\tau_+^3 |\alpha|^2 \lesssim |x|^3 |\alpha|^2 \lesssim \int_{|y| \geq |x|} |\alpha|_{\mathbb{O}, 2}^2 + r^2 |\nabla_{\partial_r}(\alpha)|_{\mathbb{O}, 1}^2 dy \lesssim \mathcal{E}_2^b[G](t) + \sum_{|\kappa| \leq 1} \|r \mathcal{L}_{Z^\kappa}(J)_A\|_{L^2(\bar{\Sigma}_t^b)}^2.$$

□

5.4 Null properties of the Vlasov equation

In order to take advantage of the null structure of the commuted transport equation, we will expand quantities such as $\mathcal{L}_{Z^\gamma}(F)(v, \nabla_v g)$, with g a regular function, in null coordinates. We will then use the following lemma.

Lemma 5.4.1. *Let G be a sufficiently regular 2-form, $(\alpha, \underline{\alpha}, \rho, \sigma)$ its null components and g a sufficiently regular function. Then,*

$$|G(v, \nabla_v g)| \lesssim \left(|\rho| + \frac{|v^A|}{v^0} |\underline{\alpha}| \right) \left(\tau_- |\nabla_{t,x} g| + \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |\widehat{Z} g| \right) + \left(|\alpha| + \frac{|v^A|}{v^0} |\sigma| + \frac{v^{\underline{L}}}{v^0} |\underline{\alpha}| \right) \left(\tau_+ |\nabla_{t,x} g| + \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |\widehat{Z} g| \right).$$

Proof. Expanding $G(v, \nabla_v g)$ with null components, we get

$$\begin{aligned} G(v, \nabla_v g) &= 2\rho \left(v^{\underline{L}} (\nabla_v g)^{\underline{L}} - v^{\underline{L}} (\nabla_v g)^{\underline{L}} \right) + v^B \varepsilon_{BA} \sigma (\nabla_v g)^A - v^{\underline{L}} \alpha_A (\nabla_v g)^A + v^A \alpha_A (\nabla_v g)^{\underline{L}} \\ &\quad - v^{\underline{L}} \underline{\alpha}_A (\nabla_v g)^A + v^A \underline{\alpha}_A (\nabla_v g)^{\underline{L}}. \end{aligned} \quad (5.12)$$

We bound the angular components of $\nabla_v g$ using $v^0 \partial_{v^i} = \widehat{\Omega}_{0i} - t \partial_i - x^i \partial_t$. The radial component $(\nabla_v g)^r = 2(\nabla_v g)^{\underline{L}} = -2(\nabla_v g)^{\underline{L}}$ has a better behavior since

$$v^0 (\nabla_v g)^r = \frac{x^i}{r} v^0 \partial_{v^i} g = \frac{x^i}{r} \widehat{\Omega}_{0i} g - Sg + (t-r) \underline{L}g. \quad (5.13)$$

□

Let us explain how this result reflects the null structure of the Vlasov equation. For this, we use the notation $Q \prec R$ if R is expected to behave better than Q , so that

- $\underline{\alpha} \prec \rho \sim \sigma \prec \alpha$,
- $v^L \prec v^A \prec v^{\underline{L}}$,
- $(\nabla_v g)^A \prec (\nabla_v g)^r$.

Note now that each term given by the previous lemma contains either two good factors, α or $v^{\underline{L}}$.

5.5 Bootstrap assumptions and strategy of the proof

Let $N \geq 8$, $b \leq -1$ and $(\delta, \eta) \in \mathbb{R}_+^2$ be two constants such that $0 < 5\delta < \eta < \frac{1}{16N}$. From now, we drop the dependance in b of all the quantities defined previously (such as the energy norms \mathbb{E}_b and $\mathbb{E}_{Q,b}^{q,\lambda}$ defined in (5.8) and (5.9) or \mathcal{E}_N^b). Let (f_0, F_0) be an initial data set satisfying the assumptions of Theorem 5.1.1. Then, by a local well-posedness argument, there exists a unique maximal solution to the Vlasov-Maxwell system defined in $V_b(T^*)$, with $T^* \in \mathbb{R}_+^* \cup \{+\infty\}$. Let $T \in]0, T^*]$ be the largest time such that⁸, for all $t \in [0, T[$,

$$\mathbb{E}_{N-2}^{N+13,\delta}[f](t) \leq 4\epsilon(1+t)^\delta, \quad (5.14)$$

$$\mathbb{E}_N^{N+9,\eta}[f](t) \leq 4\epsilon(1+t)^\eta, \quad (5.15)$$

$$\mathcal{E}_N[F](t) \leq 4\epsilon, \quad (5.16)$$

$$\sum_{|\beta| \leq N-1} \left\| r \int_v \frac{v^A}{v^0} \widehat{Z}^\beta f dv \right\|_{L^2(\overline{\Sigma}_t^b)} \leq \sqrt{\epsilon}. \quad (5.17)$$

We consider the last bootstrap assumption in order to simplify the proof. The remainder of this paper is devoted to the improvement of these inequalities which will prove that $T = T^*$ and then $T^* = +\infty$, implying Theorem 5.1.1. Let us expose the immediate consequences of the bootstrap assumptions. Using the Klainerman-Sobolev inequality of Proposition 5.3.5 and (5.14) (respectively (5.15)), one has

$$\forall (t, x) \in V_b(T), \quad |\beta| \leq N-5, \quad \int_v \left| \sqrt{z}^{N+10-(1-2\delta)\beta_P} \widehat{Z}^\beta f \right| dv \lesssim \epsilon \frac{(1+t)^{(\beta_P+4)\delta}}{\tau_+^2 \tau_-}, \quad (5.18)$$

$$\forall (t, x) \in V_b(T), \quad |\beta| \leq N-3, \quad \int_v \left| \sqrt{z}^{N+6-(1-2\eta)\beta_P} \widehat{Z}^\beta f \right| dv \lesssim \epsilon \frac{(1+t)^{(\beta_P+4)\eta}}{\tau_+^2 \tau_-}. \quad (5.19)$$

By Proposition 5.3.11, (5.16) and (5.17), we obtain that, for all $(t, x) \in V_b(T)$ and $|\gamma| \leq N-2$,

$$|\alpha(\mathcal{L}_{Z^\gamma}(F))|(t, x) \lesssim \frac{\sqrt{\epsilon}}{\tau_+^{\frac{3}{2}}}, \quad |\rho(\mathcal{L}_{Z^\gamma}(F))|(t, x) + |\sigma(\mathcal{L}_{Z^\gamma}(F))|(t, x) + |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|(t, x) \lesssim \frac{\sqrt{\epsilon}}{\tau_+ \tau_-^{\frac{1}{2}}}. \quad (5.20)$$

The proof is organized as follows:

- We start by improving the bootstrap assumptions (5.14) and (5.15) by several applications of the approximate conservation law of Proposition 5.3.1. Exploiting the null structure of the non linearity $\mathcal{L}_{Z^\gamma}(F)(v, \nabla_v \widehat{Z}^\beta f)$ is then fundamental in order to bound the spacetime integrals arising from the energy estimates.
- Then, we improve the bound on the energy norm of the electromagnetic field (5.16). For this, we use the energy estimate of Proposition 5.3.7 and we make crucial use of the null structure of the source terms of the Maxwell equations.
- The last step consists in proving an estimate on $\| \int_v |z \widehat{Z}^\beta f| dv \|_{L^2(\overline{\Sigma}_t^b)}$ for $|\beta| \geq N-2$. We then rewrite all Vlasov equations as an inhomogeneous system of transport equations. We deal with the homogenous part by taking advantage of the smallness assumption on the $N+3$ derivatives of f at $t=0$ as well as the pointwise decay estimates (5.20). We will decompose the inhomogeneous part as a product KY where $|K|^2 Y \in L_v^1 L^1(\overline{\Sigma}_t^b)$ and $\int_v |Y| dv$ is a decaying function.

⁸Note that $T > 0$ by continuity. Remark also that, considering if necessary $\epsilon_1 = C_1 \epsilon$, with C_1 a constant depending only on N , we can suppose without loss of generality that the energy norms are initially smaller than ϵ . We refer to Appendix B of [6] for the details of the computations for similar energy norms.

5.6 Improvement of the energy bound on the particle density

The aim of this section is to prove that, for ϵ small enough, $\mathbb{E}_N^{N+9,\eta}[f] \leq 3\epsilon(1+t)^\eta$ for all $t \in [0, T[$ (we will sketch the improvement of the estimate on $\mathbb{E}_{N-2}^{N+13,\delta}[f]$ as it is very similar and simpler). For this, recall that $\mathbb{E}_N^{N+9,\eta}[f](0) \leq \epsilon$ and let us prove that

$$\forall |\kappa| \leq N, \quad \forall t \in [0, T[, \quad \mathbb{E}[\sqrt{z}^{N+9-(1-2\eta)\kappa_P} \widehat{Z}^\kappa f](t) - 2\mathbb{E}[\sqrt{z}^{N+9-(1-2\eta)\kappa_P} \widehat{Z}^\kappa f](0) \lesssim \epsilon^{\frac{3}{2}}(1+t)^{(\kappa_P+1)\eta}.$$

We then fix $|\kappa| \leq N$ and we denote $\frac{1}{2}(N+9-(1-2\eta)\kappa_P)$ by a . Note, by Lemma 5.2.8, that

$$\mathbf{T}_F(z^a \widehat{Z}^\kappa f) = F(v, \nabla_v z^a) \widehat{Z}^\kappa f + z^a \mathbf{T}_F(\widehat{Z}^\kappa f). \quad (5.21)$$

Thus, in view of the energy estimate of Proposition 5.3.1 and commutation formula of Proposition 5.2.6, it suffices to prove that

$$\int_0^t \int_{\overline{\Sigma}_s^b} \int_v \left| z^{a-1} F(v, \nabla_v z) \widehat{Z}^\kappa f \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}(1+t)^{(\kappa_P+1)\eta} \quad (5.22)$$

and that the following proposition holds, where $[\gamma] := \max(0, 1 - \gamma_T)$.

Proposition 5.6.1. *Let γ and β be such that $|\gamma| + |\beta| \leq |\kappa|$, $|\beta| \leq |\kappa| - 1$ and $\beta_P + [\gamma] \leq \kappa_P$. Then,*

$$\int_0^t \int_{\overline{\Sigma}_s^b} \int_v \left| z^a \mathcal{L}_{Z^\gamma}(F)(v, \nabla_v \widehat{Z}^\beta f) \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}(1+t)^{(\kappa_P+1)\eta}.$$

The remaining of the section is divided in four parts. The first two ones are devoted to the proof of (5.22) and Proposition 5.6.1. Then, we explain briefly how to improve the bound on $\mathbb{E}_{N-2}^{N+13,\delta}[f]$. Finally, we prove an L^2 estimate on $\int_v z |\widehat{Z}^\beta f| dv$ which will be useful for Section 5.7.

5.6.1 Proof of inequality (5.22)

Note first that we have $|\nabla_{t,x} z| \leq 1$ and, using Lemma 5.2.8, $|\widehat{Z}z| \lesssim z$. Applying Lemma 5.4.1 with $(G, g) = (F, z)$, we can then observe that it suffices to prove that

$$\begin{aligned} I_1 &:= \int_0^t \int_{\overline{\Sigma}_s^b} \int_v \left((\tau_- + z)|\rho(F)| + (\tau_+ + z)|\alpha(F)| \right) \left| z^{a-1} \widehat{Z}^\kappa f \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}(1+t)^{(\kappa_P+1)\eta} \quad \text{and} \\ I_2 &:= \int_0^t \int_{\overline{\Sigma}_s^b} \int_v (\tau_+ + z) \left(\frac{|v^A|}{v^0} |\sigma(F)| + \frac{v^L + |v^A|}{v^0} |\underline{\alpha}(F)| \right) \left| z^{a-1} \widehat{Z}^\kappa f \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}}(1+t)^{(\kappa_P+1)\eta}. \end{aligned}$$

Recall, from Lemma 5.2.9 the inequalities $1 \lesssim \sqrt{v^0 v^L}$, $1 \lesssim \tau_-^{-1} z$ and $v^L + |v^A| \lesssim \tau_+^{-1} v^0 z$, so that

$$1 \lesssim \sqrt{v^0 v^L} \lesssim \frac{v^0 z}{\sqrt{\tau_+ \tau_-}} \quad \text{and} \quad \frac{|v^A| + v^L}{v^0} \lesssim \frac{z}{\tau_+}.$$

Hence, according to (5.20), it comes

$$(\tau_- + z)|\rho(F)|(s, x) + (\tau_+ + z)|\alpha(F)|(s, x) + (\tau_+ + z) \left(\frac{|v^A|}{v^0} |\sigma(F)|(s, x) + \frac{v^L + |v^A|}{v^0} |\underline{\alpha}(F)|(s, x) \right) \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+} z.$$

Consequently, using the bootstrap assumption (5.15),

$$\begin{aligned} I_1 + I_2 &\lesssim \int_0^t \int_{\overline{\Sigma}_s^b} \frac{\sqrt{\epsilon}}{\tau_+} \int_v \left| z^a \widehat{Z}^\kappa f \right| dv dx ds \lesssim \sqrt{\epsilon} \int_0^t \frac{\mathbb{E}[z^a \widehat{Z}^\kappa f](s)}{1+s} ds \\ &\lesssim \sqrt{\epsilon} \int_0^t \frac{(1+s)^{\kappa_P \eta}}{1+s} \mathbb{E}_N^{N+9,\eta}[f](s) ds \lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{(1+s)^{(\kappa_P+1)\eta}}{1+s} ds \lesssim \epsilon^{\frac{3}{2}}(1+t)^{(\kappa_P+1)\eta}. \end{aligned}$$

5.6.2 Proof of Proposition 5.6.1

Let γ and β satisfying $|\beta| + |\gamma| \leq |\kappa|$, $|\beta| \leq |\kappa| - 1$ and $\beta_P + [\gamma] \leq \kappa_P$. Using Lemma 5.4.1, we need to bound by $\epsilon^{\frac{3}{2}}(1+t)^{(\kappa_P+1)\eta}$, for all $\widehat{\Gamma} \in \widehat{\mathbb{P}}_0$, the following integrals,

$$\begin{aligned} I_{\widehat{\Gamma}} &:= \int_0^t \int_{\overline{\Sigma}_s^b} |\mathcal{L}_{Z^\gamma}(F)| \int_v |z^a \widehat{\Gamma} \widehat{Z}^\beta f| \frac{dv}{v^0} dx ds, \\ I_{\rho, \alpha} &:= \int_0^t \int_{\overline{\Sigma}_s^b} (\tau_- |\rho(\mathcal{L}_{Z^\gamma}(F))| + \tau_+ |\alpha(\mathcal{L}_{Z^\gamma}(F))|) \int_v |z^a \nabla_{t,x} \widehat{Z}^\beta f| \frac{dv}{v^0} dx ds, \\ I_{\sigma, \underline{\alpha}} &:= \int_0^t \int_{\overline{\Sigma}_s^b} \tau_+ (|\sigma(\mathcal{L}_{Z^\gamma}(F))| + |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|) \int_v \frac{|v^A| + v^{\underline{L}}}{v^0} |z^a \nabla_{t,x} \widehat{Z}^\beta f| \frac{dv}{v^0} dx ds. \end{aligned}$$

In order to close the energy estimates, we will have to pay attention to the hierarchy discussed in Remark 5.2.7. For the remaining of this subsection, we fix $\widehat{\Gamma} \in \widehat{\mathbb{P}}_0$ and we denote by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of $\mathcal{L}_{Z^\gamma}(F)$. The proof is divided in two parts. First, we treat the case where the electromagnetic field can be estimated pointwise ($|\gamma| \leq N-2$). Otherwise we necessarily have $|\beta| \leq 1$ and we can use the estimate (5.18) on the Vlasov field.

If $|\gamma| \leq N-2$

Suppose first that $\beta_P < \kappa_P$, which implies $a + \frac{1}{2} - \eta \leq \frac{1}{2}(N - (1 - 2\eta)\beta_P)$. The bootstrap assumption (5.15) then gives

$$\mathbb{E}[z^a \widehat{\Gamma} \widehat{Z}^\beta f](t) \lesssim \epsilon(1+t)^{(\beta_P+2)\eta} \quad \text{and} \quad \mathbb{E}[z^{a+\frac{1}{2}-\eta} \nabla_{t,x} \widehat{Z}^\beta f](t) \lesssim \epsilon(1+t)^{(\beta_P+1)\eta}. \quad (5.23)$$

According to the pointwise decay estimates (5.20) we have $|\mathcal{L}_{Z^\gamma}(F)| \lesssim \sqrt{\epsilon} \tau_+^{-1}$, so

$$I_{\widehat{\Gamma}} \lesssim \int_0^t \frac{\sqrt{\epsilon}}{1+s} \int_{\overline{\Sigma}_s^b} \int_v |z^a \widehat{\Gamma} \widehat{Z}^\beta f| dv dx ds \lesssim \int_0^t \frac{\mathbb{E}[z^a \widehat{\Gamma} \widehat{Z}^\beta f](s)}{1+s} ds \lesssim \epsilon^{\frac{3}{2}}(1+t)^{(\beta_P+2)\eta} \lesssim \epsilon^{\frac{3}{2}}(1+t)^{(\kappa_P+1)\eta}.$$

Using again (5.20) as well as $1 \lesssim \sqrt{v^0 v^{\underline{L}}}$, $1 \lesssim \tau_-^{-\frac{1}{2}+\eta} z^{\frac{1}{2}-\eta}$ and $|v^A| \lesssim v^0 v^{\underline{L}}$ (see Lemma 5.2.9), we get

$$\begin{aligned} (\tau_- |\rho(\mathcal{L}_{Z^\gamma}(F))| + \tau_+ |\alpha(\mathcal{L}_{Z^\gamma}(F))|) \int_v |z^a \nabla_{t,x} \widehat{Z}^\beta f| \frac{dv}{v^0} &\lesssim \int_v \sqrt{\frac{\epsilon v^{\underline{L}}}{\tau_+ \tau_-^{1-2\eta} v^0}} |z^{a+\frac{1}{2}-\eta} \nabla_{t,x} \widehat{Z}^\beta f| dv \\ &\lesssim \int_v \left(\frac{\sqrt{\epsilon}}{\tau_+^{1-\eta}} + \frac{\sqrt{\epsilon v^{\underline{L}}}}{\tau_-^{1-\eta} v^0} \right) |z^{a+\frac{1}{2}-\eta} \nabla_{t,x} \widehat{Z}^\beta f| dv, \end{aligned}$$

$$\tau_+ (|\sigma(\mathcal{L}_{Z^\gamma}(F))| + |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))|) \int_v \frac{|v^A| + v^{\underline{L}}}{v^0} |z^a \nabla_{t,x} \widehat{Z}^\beta f| \frac{dv}{v^0} \lesssim \int_v \frac{\sqrt{\epsilon v^{\underline{L}}}}{\tau_-^{1-\eta} v^0} |z^{a+\frac{1}{2}-\eta} \nabla_{t,x} \widehat{Z}^\beta f| dv.$$

In order to lighten the notations, we denote $|z^{a+\frac{1}{2}-\eta} \nabla_{t,x} \widehat{Z}^\beta f|$ by g . We then have

$$I_{\rho, \alpha} + I_{\sigma, \underline{\alpha}} \lesssim \int_0^t \frac{\sqrt{\epsilon}}{(1+s)^{1-\eta}} \int_{\overline{\Sigma}_s^b} \int_v |g| dv dx ds + \int_0^t \int_{\overline{\Sigma}_s^b} \frac{\sqrt{\epsilon}}{\tau_-^{1-\eta}} \int_v \frac{v^{\underline{L}}}{v^0} |g| dv dx ds.$$

To deal with the second integral, we split $V_b(t)$ as follows,

$$V_b(t) = \{(s, x) \in V_b(t) / s - |x| \leq -t\} \cup \{(s, x) \in V_b(t) / -t \leq s - |x| \leq b\} := \mathcal{V}_1 \cup \mathcal{V}_2.$$

Note that if $s \leq t$, then $s - |x| \leq -t$ implies $|x| \geq 2s$ so that $\tau_+ \lesssim \tau_-$ on \mathcal{V}_1 . Consequently, using $\mathcal{V}_1 \subset V_b(t)$, $\mathbb{E}[g](t) \lesssim \epsilon(1+t)^{(\beta_P+1)\eta}$ (see (5.23)) and Lemma 5.2.2, it comes

$$\begin{aligned} I_{\rho, \alpha} + I_{\sigma, \underline{\alpha}} &\lesssim \int_0^t \frac{\sqrt{\epsilon}}{(1+s)^{1-\eta}} \int_{\overline{\Sigma}_s^b} \int_v |g| dv dx ds + \int_{\mathcal{V}_1} \frac{\sqrt{\epsilon}}{\tau_+^{1-\eta}} \int_v \frac{v^{\underline{L}}}{v^0} |g| dv d\mathcal{V}_1 + \int_{\mathcal{V}_2} \frac{\sqrt{\epsilon}}{\tau_-^{1-\eta}} \int_v \frac{v^{\underline{L}}}{v^0} |g| dv d\mathcal{V}_2 \\ &\lesssim \sqrt{\epsilon} \int_0^t \frac{\mathbb{E}[g](s)}{(1+s)^{1-\eta}} ds + \int_{u=-t}^b \frac{\sqrt{\epsilon}}{\tau_-^{1-\eta}} \int_{C_u(t)} \int_v \frac{v^{\underline{L}}}{v^0} |g| dv dC_u(t) du \\ &\lesssim \epsilon^{\frac{3}{2}}(1+t)^{(\beta_P+2)\eta} + \sqrt{\epsilon} \int_{u=-t}^b \mathbb{E}[g](t) \frac{du}{(1+|u|)^{1-\eta}} \lesssim \epsilon^{\frac{3}{2}}(1+t)^{(\kappa_P+1)\eta}. \end{aligned}$$

We suppose now that $\beta_P = \kappa_P$, so that $\gamma_T \geq 1$. Since, for $Z \in \mathbb{K}$ and $0 \leq \mu \leq 3$, $[Z, \partial_\mu]$ is either equal to 0 or $\pm \partial_\nu$ for $\nu \in \llbracket 0, 3 \rrbracket$, we can assume that $Z^\gamma = \partial Z^{\gamma_0}$ with $|\gamma_0| = |\gamma| - 1$. Note also that (5.23) does not hold in that case. The bootstrap assumption (5.15) merely gives us

$$\mathbb{E}[z^{a-\frac{1}{2}+\eta} \widehat{\Gamma} \widehat{Z}^\beta f](t) \lesssim \epsilon(1+t)^{(\beta_P+2)\eta} \quad \text{and} \quad \mathbb{E}[z^a \nabla_{t,x} \widehat{Z}^\beta f](t) \lesssim \epsilon(1+t)^{(\beta_P+1)\eta}. \quad (5.24)$$

Applying Lemma 5.3.8 and using again $1 \lesssim \sqrt{v^0 v^L}$, $|v^A| \lesssim \sqrt{v^0 v^L}$ as well as (5.20), we have

$$\begin{aligned} \tau_- |\rho(\mathcal{L}_{Z^\gamma}(F))| + \tau_+ |\alpha(\mathcal{L}_{Z^\gamma}(F))| &\lesssim \sum_{|\xi| \leq |\gamma|} |\rho(\mathcal{L}_{Z^\xi}(F))| + \sqrt{v^0 v^L} \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\xi}(F))| \\ &\lesssim \frac{\sqrt{\epsilon}}{\tau_+} + \sqrt{\frac{\epsilon v^0 v^L}{\tau_+ \tau_-^2}} \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+} + \sqrt{\epsilon} \frac{v^L}{\tau_-^2}, \\ \tau_+ (|\sigma(\mathcal{L}_{Z^\gamma}(F))| + |\alpha(\mathcal{L}_{Z^\gamma}(F))|) \frac{|v^A| + v^L}{v^0} &\lesssim \sum_{|\xi| \leq |\gamma|} v^L \frac{\tau_+}{\tau_-} (|\sigma(\mathcal{L}_{Z^\xi}(F))| + |\alpha(\mathcal{L}_{Z^\xi}(F))|) \lesssim \sqrt{\epsilon} \frac{v^L}{\tau_-^{\frac{3}{2}}}, \\ \tau_+^{\frac{1}{2}-\eta} |\mathcal{L}_{Z^\gamma}(F)| &\lesssim \sqrt{\epsilon} \frac{\sqrt{v^0 v^L}}{\tau_+^{\frac{1}{2}+\eta} \tau_-^{\frac{3}{2}}} \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+^{1+\eta}} + \sqrt{\epsilon} \frac{v^L}{\tau_+ \tau_-^3}. \end{aligned} \quad (5.25)$$

We then have, using Lemma 5.2.2, (5.24) and $\beta_P = \kappa_P$,

$$\begin{aligned} I_{\rho,\alpha} + I_{\sigma,\alpha} &\lesssim \int_0^t \frac{\sqrt{\epsilon}}{1+s} \int_{\Sigma_s^b} \int_v |z^a \nabla_{t,x} \widehat{Z}^\beta f| \, dv dx ds + \int_0^t \int_{\Sigma_s^b} \frac{\sqrt{\epsilon}}{\tau_-^{\frac{3}{2}}} \int_v \frac{v^L}{v^0} |z^a \nabla_{t,x} \widehat{Z}^\beta f| \, dv dx ds \\ &\lesssim \sqrt{\epsilon} \int_0^t \frac{\mathbb{E}[z^a \nabla_{t,x} \widehat{Z}^\beta f](s)}{1+s} ds + \int_{u=-\infty}^b \frac{\sqrt{\epsilon}}{\tau_-^{\frac{3}{2}}} \int_{C_u(t)} \int_v \frac{v^L}{v^0} |z^a \nabla_{t,x} \widehat{Z}^\beta f| \, dv dC_u(t) du \\ &\lesssim \epsilon^{\frac{3}{2}} \int_0^t \frac{(1+s)^{(\beta_P+1)\eta}}{1+s} ds + \sqrt{\epsilon} \int_{u=-\infty}^b \mathbb{E}[z^a \nabla_{t,x} \widehat{Z}^\beta f](t) \frac{du}{\tau_-^{\frac{3}{2}}} \\ &\lesssim \epsilon^{\frac{3}{2}} (1+t)^{(\beta_P+1)\eta} + \sqrt{\epsilon} \mathbb{E}[z^a \nabla_{t,x} \widehat{Z}^\beta f](t) \int_{u=-\infty}^0 \frac{du}{(1+|u|)^{\frac{3}{2}}} \lesssim \epsilon^{\frac{3}{2}} (1+t)^{(\kappa_P+1)\eta}. \end{aligned}$$

Finally, as $z^a \leq 2\tau_+^{\frac{1}{2}-\eta} z^{a-\frac{1}{2}+\eta}$, we have by (5.25) and (5.24),

$$\begin{aligned} I_{\widehat{\Gamma}} &\lesssim \int_0^t \frac{\sqrt{\epsilon}}{(1+s)^{1+\eta}} \int_{\Sigma_s^b} \int_v |z^{a-\frac{1}{2}+\eta} \widehat{\Gamma} \widehat{Z}^\beta f| \, dv dx ds + \int_0^t \int_{\Sigma_s^b} \frac{\sqrt{\epsilon}}{\tau_+ \tau_-^3} \int_v \frac{v^L}{v^0} |z^{a-\frac{1}{2}+\eta} \widehat{\Gamma} \widehat{Z}^\beta f| \, dv dx ds \\ &\lesssim \epsilon^{\frac{3}{2}} (1+t)^{(\beta_P+2-1)\eta} + \int_0^t \int_{\Sigma_s^b} \frac{\sqrt{\epsilon}}{\tau_+ \tau_-^3} \int_v \frac{v^L}{v^0} |z^{a-\frac{1}{2}+\eta} \widehat{\Gamma} \widehat{Z}^\beta f| \, dv dx ds. \end{aligned}$$

To deal with the remaining integral, let us introduce, for all $u < b$ and $i \in \mathbb{N}$, the following truncated cone

$$C_u^i(t) := \{(s, x) \in C_u(t) \mid 2^i - 1 \leq s \leq T_{i+1}(t)\}, \quad \text{where} \quad T_{i+1}(t) = \min(t, 2^{i+1} - 1). \quad (5.26)$$

Notice that $\|\tau_+^{-\eta}\|_{L^\infty(C_u^i(t))} \leq C 2^{-i\eta}$, with $C > 0$ a constant independant of $i \in \mathbb{N}$, and

$$\int_{C_u^i(t)} \int_v \frac{v^L}{v^0} |z^{a-\frac{1}{2}+\eta} \widehat{\Gamma} \widehat{Z}^\beta f| \, dv dC_u^i(t) \lesssim \mathbb{E}[z^{a-\frac{1}{2}+\eta} \widehat{\Gamma} \widehat{Z}^\beta f](T_{i+1}(t)) \leq 4\epsilon(1+T_{i+1}(t))^{(\beta_P+2)\eta} \leq 8\epsilon 2^{i(\beta_P+2)\eta}.$$

Consequently, as $V_b(t)$ can be foliated by $(C_u^i(t))_{u < b, i \leq \log_2(1+t)}$,

$$\begin{aligned} \int_0^t \int_{\Sigma_s^b} \frac{\sqrt{\epsilon}}{\tau_+ \tau_-^3} \int_v \frac{v^L}{v^0} |z^{a-\frac{1}{2}+\eta} \widehat{\Gamma} \widehat{Z}^\beta f| \, dv dx ds &\lesssim \sum_{i=0}^{\log_2(1+t)} \frac{1}{2^{i\eta}} \int_{-\infty}^b \frac{\sqrt{\epsilon}}{\tau_-^3} \int_{C_u^i(t)} \int_v \frac{v^L}{v^0} |z^{a-\frac{1}{2}+\eta} \widehat{\Gamma} \widehat{Z}^\beta f| \, dv dC_u^i(t) du \\ &\lesssim \sqrt{\epsilon} \sum_{i=0}^{\log_2(1+t)} \frac{\mathbb{E}[z^{a-\frac{1}{2}+\eta} \widehat{\Gamma} \widehat{Z}^\beta f](T_{i+1}(t))}{2^{i\eta}} \int_{u=-\infty}^b \frac{du}{\tau_-^3} \\ &\lesssim \epsilon^{\frac{3}{2}} \sum_{i=0}^{\log_2(1+t)} 2^{i(\beta_P+1)\eta} \lesssim \epsilon^{\frac{3}{2}} (1+t)^{(\beta_P+1)\eta} \end{aligned}$$

and we then deduce that $I_{\widehat{\Gamma}} \lesssim \epsilon^{\frac{3}{2}} (1+t)^{(\kappa_P+1)\eta}$.

When $|\gamma| \geq N - 1$

In that case, we have $|\beta| \leq 1$. Using first the inequality $|v^A|^{\frac{3}{4}} + |v^L|^{\frac{3}{4}} \lesssim \tau_+^{-\frac{3}{4}} z^{\frac{3}{4}}$, coming from Lemma 5.2.9, as well as the Cauchy-Schwarz inequality in (t, x) and secondly the bootstrap assumption (5.16), we get

$$\begin{aligned} I_{\sigma, \underline{\alpha}} &\lesssim \left| \int_0^t \frac{\|\mathcal{L}_{Z^\gamma}(F)\|_{L^2(\Sigma_s^b)}^2}{(1+s)^{\frac{5}{4}}} ds \int_0^t (1+s)^{\frac{5}{4}} \tau_+^{\frac{1}{2}} \int_{\Sigma_s^b} \left| \int_v |z^{a+\frac{3}{4}} \nabla_{t,x} \widehat{Z}^\beta f| \frac{dv}{v^0} \right|^2 dx ds \right|^{\frac{1}{2}} \\ &\lesssim \left| \int_0^{+\infty} \frac{\epsilon}{(1+s)^{\frac{5}{4}}} ds \int_{u=-\infty}^b \frac{1}{\tau_-^{\frac{5}{4}}} \left\| \tau_+^{\frac{7}{8}} \tau_-^{\frac{5}{8}} \int_v |z^{a+\frac{3}{4}} \nabla_{t,x} \widehat{Z}^\beta f| \frac{dv}{v^0} \right\|_{L^2(C_u(t))}^2 du \right|^{\frac{1}{2}}. \end{aligned} \quad (5.27)$$

Using this time Lemma 5.2.2, the inequality $1 \lesssim \tau_-^{-\frac{3}{4}} z^{\frac{3}{4}}$ and the Cauchy-Schwarz inequality in (\underline{u}, ω) , it comes

$$\begin{aligned} I_{\rho, \alpha} &\lesssim \int_{u=-\infty}^b \|\rho\| + |\alpha| \left\| \frac{\tau_+}{\tau_-^{\frac{3}{4}}} \int_v |z^{a+\frac{3}{4}} \nabla_{t,x} \widehat{Z}^\beta f| \frac{dv}{v^0} \right\|_{L^2(C_u(t))} du \\ &\lesssim \sqrt{\epsilon} \int_{u=-\infty}^b \frac{1}{\tau_-^{\frac{5}{4}}} \left\| \tau_+ \tau_-^{\frac{1}{2}} \int_v |z^{a+\frac{3}{4}} \nabla_{t,x} \widehat{Z}^\beta f| \frac{dv}{v^0} \right\|_{L^2(C_u(t))} du. \end{aligned} \quad (5.28)$$

Now, by the Cauchy-Schwarz inequality in v and $1 \lesssim v^0 v^L$, we have

$$\left\| \tau_+ \tau_-^{\frac{1}{2}} \int_v |z^{a+\frac{3}{4}} \nabla_{t,x} \widehat{Z}^\beta f| \frac{dv}{v^0} \right\|_{L^2(C_u(t))}^2 \lesssim \left\| \tau_+^2 \tau_- \int_v |z^a \nabla \widehat{Z}^\beta f| dv \right\|_{L^\infty(C_u(t))} \left\| \int_v \frac{v^L}{v^0} |z^{a+\frac{3}{2}} \nabla \widehat{Z}^\beta f| dv \right\|_{L^1(C_u(t))}.$$

As $\beta_P \leq \kappa_P$, we have $a \leq \frac{1}{2}(N+9 - (1-2\eta)\beta_P) \leq \frac{1}{2}(N+10 - (1-2\delta)\beta_P)$. The pointwise decay estimate (5.18) and the bootstrap assumption (5.14) then gives us

$$\left\| \tau_+ \tau_-^{\frac{1}{2}} \int_v |z^{a+\frac{3}{4}} \nabla_{t,x} \widehat{Z}^\beta f| \frac{dv}{v^0} \right\|_{L^2(C_u(t))}^2 \lesssim \epsilon(1+t)^{(\beta_P+4)\delta} \cdot (1+t)^{\beta_P\delta} \mathbb{E}_{N-2}^{N+13, \delta}[f](t) \lesssim \epsilon^2(1+t)^{7\delta}.$$

Combining the last inequality with (5.27) and (5.28), we finally deduce that $I_{\sigma, \underline{\alpha}} + I_{\rho, \alpha} \lesssim \epsilon^{\frac{3}{2}}(1+t)^\eta$ since $7\delta \leq 2\eta$. Finally, one can obtain that $I_{\widehat{\Gamma}} \lesssim \epsilon^{\frac{3}{2}}(1+t)^{\frac{9}{2}\delta} \lesssim \epsilon^{\frac{3}{2}}(1+t)^\eta$ by simpler considerations, which concludes the proof of Proposition 5.6.1.

5.6.3 The remaining energy norm

For the improvement of $\mathbb{E}_{N-2}^{N+13, \delta}[f] \leq 4\epsilon(1+t)^\delta$ we have, in view of (5.21) as well as Propositions 5.3.1 and 5.2.6, to prove similar estimates than (5.22) and those of Proposition 5.6.1. More precisely, $\mathbb{E}_{N-2}^{N+13, \delta}[f] \leq 3\epsilon(1+t)^\delta$ on $[0, T]$ ensues, for ϵ small enough, from the following proposition.

Proposition 5.6.2. *Let $|\kappa| \leq N - 2$, γ and β be such that $|\gamma| + |\beta| \leq |\kappa|$, $|\beta| \leq |\kappa| - 1$ and $\beta_P + [\gamma] \leq \kappa_P$. Then,*

$$\begin{aligned} \int_0^t \int_{\Sigma_s^b} \int_v z^{\frac{N+13}{2} - (\frac{1}{2}-\delta)\kappa_P - 1} \left| F(v, \nabla_v z) \widehat{Z}^\kappa f \right| \frac{dv}{v^0} dx ds &\lesssim \epsilon^{\frac{3}{2}}(1+t)^{(\kappa_P+1)\delta}. \\ \int_0^t \int_{\Sigma_s^b} \int_v \left| z^{\frac{N+13}{2} - (\frac{1}{2}-\delta)\kappa_P} \mathcal{L}_{Z^\gamma}(F) \left(v, \nabla_v \widehat{Z}^\beta f \right) \right| \frac{dv}{v^0} dx ds &\lesssim \epsilon^{\frac{3}{2}}(1+t)^{(\kappa_P+1)\delta}. \end{aligned}$$

Proof. One only has to follow Subsections 5.6.1 and, as $|\gamma| \leq N-2$, 5.6.2 and to use the bootstrap assumption (5.14) instead of (5.15). \square

5.6.4 L^2 estimates on velocity averages

The following result improves the bootstrap assumption (5.17) if ϵ is small enough and will allow us to improve our estimate on $\mathcal{E}_N[F]$.

Proposition 5.6.3. *We have, for all $t \in [0, T[$,*

$$\sum_{|\beta| \leq N} \left\| r \int_v \frac{v^A}{v^0} \widehat{Z}^\beta f dv \right\|_{L^2(\overline{\Sigma}_t^b)} \lesssim \sum_{|\beta| \leq N} \left\| \int_v z |\widehat{Z}^\beta f| dv \right\|_{L^2(\overline{\Sigma}_t^b)} \lesssim \frac{\epsilon}{(1+t)^{\frac{3}{4}}}.$$

Proof. The first inequality ensues from $r|v^A| \lesssim v^0 z$ (see Lemma 5.2.9). For the second one, we start by considering $|\beta| \leq N-3$. Using successively the Cauchy-Schwarz inequality in v , the pointwise decay estimate (5.19) and the bootstrap assumption (5.15), we get

$$\begin{aligned} \left\| \int_v z |\widehat{Z}^\beta f| dv \right\|_{L^2(\overline{\Sigma}_t^b)}^2 &\lesssim \left\| \int_v |\widehat{Z}^\beta f| dv \int_v z^2 |\widehat{Z}^\beta f| dv \right\|_{L^1(\overline{\Sigma}_t^b)} \lesssim \left\| \int_v |\widehat{Z}^\beta f| dv \right\|_{L^\infty(\overline{\Sigma}_t^b)} \left\| \int_v z^2 |\widehat{Z}^\beta f| dv \right\|_{L^1(\overline{\Sigma}_t^b)} \\ &\lesssim \left\| \frac{\epsilon}{\tau_+^{2-(\beta_P+4)\eta} \tau_-} \right\|_{L^\infty(\overline{\Sigma}_t^b)} (1+t)^{\beta_P \eta} \mathbb{E}_N^{N+9, \eta}[f](t) \lesssim \frac{\epsilon^2}{(1+t)^{2-(2\beta_P+5)\eta}} \lesssim \frac{\epsilon^2}{(1+t)^{\frac{3}{2}}}. \end{aligned}$$

The cases $N-2 \leq |\beta| \leq N$ are the purpose of Section 5.8. \square

5.7 The energy bound on the electromagnetic field

According to the energy estimate of Proposition 5.3.7, commutation formula of Proposition 5.2.6 and since $\mathcal{E}_N[F](0) \leq \epsilon$, we would obtain $\mathcal{E}_N[F] \leq 3\epsilon$ on $[0, T[$ for ϵ small enough if we could prove

$$\sum_{|\gamma| \leq N} \sum_{|\beta| \leq N} \int_0^t \int_{\overline{\Sigma}_s^b} \left| \mathcal{L}_{Z^\gamma}(F)_{0\nu} \int_v \frac{v^\nu}{v^0} \widehat{Z}^\beta f dv \right| dx ds \lesssim \epsilon^{\frac{3}{2}}.$$

We then fix $|\beta| \leq N$, $|\gamma| \leq N$ and we denote by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of $\mathcal{L}_{Z^\gamma}(F)$. Expanding $\mathcal{L}_{Z^\gamma}(F)_{0\nu} \int_v \frac{v^\nu}{v^0} \widehat{Z}^\beta f$ in null coordinates, we can observe that it suffices to prove that

$$I_\rho := \int_0^t \int_{\overline{\Sigma}_s^b} |\rho| \int_v \left| \widehat{Z}^\beta f \right| dv dx ds \lesssim \epsilon^{\frac{3}{2}} \quad \text{and} \quad I_{\alpha, \underline{\alpha}} := \int_0^t \int_{\overline{\Sigma}_s^b} (|\alpha| + |\underline{\alpha}|) \int_v \frac{|v^A|}{v^0} \left| \widehat{Z}^\beta f \right| dv dx ds \lesssim \epsilon^{\frac{3}{2}}.$$

Using successively the Cauchy-Schwarz inequality in (s, x) , the bootstrap assumption (5.16), the inequality $\tau_+ |v^A| \lesssim v^0 z$ which comes from Lemma 5.2.9 and Proposition 5.6.3, we have

$$\begin{aligned} |I_{\alpha, \underline{\alpha}}|^2 &\lesssim \int_0^t \frac{\| |\alpha| + |\underline{\alpha}| \|_{L^2(\overline{\Sigma}_s^b)}^2}{(1+s)^2} ds \int_0^t (1+s)^2 \left\| \int_v \frac{|v^A|}{v^0} \left| \widehat{Z}^\beta f \right| dv \right\|_{L^2(\overline{\Sigma}_s^b)}^2 ds \\ &\lesssim \int_0^t \frac{\mathcal{E}_N[F](s)}{(1+s)^2} ds \int_0^t \left\| \int_v z \left| \widehat{Z}^\beta f \right| dv \right\|_{L^2(\overline{\Sigma}_s^b)}^2 ds \lesssim \int_0^{+\infty} \frac{\epsilon}{(1+s)^2} ds \int_0^{+\infty} \frac{\epsilon^2}{(1+s)^{\frac{3}{2}}} ds \lesssim \epsilon^3. \end{aligned}$$

Similarly, using $\tau_- \lesssim z$ instead of $\tau_+ |v^A| \lesssim v^0 z$ (see also Lemma 5.2.9) and Lemma 5.2.2, it comes

$$\begin{aligned} |I_\rho|^2 &\lesssim \int_0^t \int_{\overline{\Sigma}_s^b} \frac{|\rho|^2}{\tau_-^2} dx ds \int_0^t \int_{\overline{\Sigma}_s^b} \tau_-^2 \left| \int_v \widehat{Z}^\beta f \right| dv dx ds \\ &\lesssim \int_{u=-\infty}^b \int_{C_u(t)} \frac{|\rho|^2}{\tau_-^2} dC_u(t) du \int_0^t \int_{\overline{\Sigma}_s^b} \left| \int_v z \left| \widehat{Z}^\beta f \right| dv \right|^2 dx ds \\ &\lesssim \int_{u=-\infty}^b \frac{\mathcal{E}_N[F](t)}{\tau_-^2} du \int_0^t \left\| \int_v z \left| \widehat{Z}^\beta f \right| dv \right\|_{L^2(\overline{\Sigma}_s^b)}^2 ds \lesssim \int_{u=-\infty}^0 \frac{\epsilon}{\tau_-^2} du \int_0^{+\infty} \frac{\epsilon^2}{(1+s)^{\frac{3}{2}}} ds \lesssim \epsilon^3. \end{aligned}$$

This concludes the improvement of the bootstrap assumption (5.16).

5.8 L^2 estimates for the higher order derivatives of the Vlasov field

In this last section, we complete the proof of Proposition 5.6.3. For this purpose, we slightly modify the strategy used in Section 4.5.7 of [18] in order to keep more of the null structure of the system. The first

step of the proof consists in rewriting all transport equations as a hierarchised system. Let I and \mathcal{J} be the following two ordered sets,

$$\begin{aligned} I &:= \{\beta \text{ multi-index} / N-3 \leq |\beta| \leq N\} = \{\beta^1, \dots, \beta^{|I|}\}, \\ \mathcal{J} &:= \{\xi \text{ multi-index} / |\xi| \leq N-3\} = \{\xi^1, \dots, \xi^{|\mathcal{J}|}\}. \end{aligned}$$

Remark 5.8.1. *Even if it only remains us to estimate $\|\int_v z |\widehat{Z}^\beta f| dv\|_{L^2(\overline{\Sigma}_t^b)}$ for all $|\beta| \geq N-2$, we included the multi-indices of length $N-3$ in I . It will allow us to conserve the null structure of the Vlasov equations.*

We also consider $I^k := \{\beta \in I / |\beta| = k\}$, for $N-3 \leq k \leq N$, and two vector valued fields R and W of respective length $|I|$ and $|\mathcal{J}|$ such that

$$R_i = \widehat{Z}^{\beta^i} f \quad \text{and} \quad W_i = \widehat{Z}^{\xi^i} f.$$

For simplicity, we will sometimes abusively write $i \in I^k$ instead of $\beta^i \in I^k$. We denote by \mathbb{V} the module over the ring $C^0(V_b(T) \times \mathbb{R}_v^3)$ engendered by $(\partial_{v^i})_{1 \leq i \leq 3}$ and we recall that $[\gamma] := \max(0, 1 - \gamma_T)$. Let us now rewrite the Vlasov equations satisfied by the components of R .

Lemma 5.8.2. *There exists two matrices valued functions $A : V_b(T) \times \mathbb{R}_v^3 \rightarrow \mathfrak{M}_{|I|}(\mathbb{V})$ and $B : V_b(T) \times \mathbb{R}_v^3 \rightarrow \mathfrak{M}_{|I|, |\mathcal{J}|}(\mathbb{V})$ such that*

$$\mathbf{T}_F(R) + AR = BW.$$

These two matrices are such that $\mathbf{T}_F(R_i)$, for $1 \leq i \leq |I|$, is a linear combination of

$$\begin{aligned} \mathcal{L}_{Z^\gamma}(F)(v, \nabla_v R_j), \quad & \text{with} \quad |\beta^j| < |\beta^i|, \quad |\gamma| + |\beta^j| \leq |\beta^i| \quad \text{and} \quad \beta_P^j + [\gamma] \leq \beta_P^i, \\ \mathcal{L}_{Z^\zeta}(F)(v, \nabla_v W_j), \quad & \text{with} \quad |\xi^j| \leq N-4, \quad |\zeta| \leq N \quad \text{and} \quad \xi_P^j + [\gamma] \leq \beta_P^i. \end{aligned}$$

Remark 5.8.3. *Note that if $\beta^i \in I^{N-3}$, then $A_i^q = 0$ for all $1 \leq q \leq |I|$. If $p \geq 1$ and $\beta^i \in I^{N-3+p}$, we have $|\gamma| \leq p$. The condition $\beta_P^j + [\gamma] \leq \beta_P^i$ expresses that either Z^γ is composed by a translation, and will then give an extra decay in the u direction, or that the number of homogeneous vector fields composing R_j , β_P^j , is strictly lower than β_P^i .*

Proof. One only has to apply the commutation formula of Proposition 5.2.6 to $\widehat{Z}^{\beta^i} f$ and to replace each quantity such as $\widehat{Z}^\kappa f$, for $|\kappa| \neq N-3$, by the corresponding component of R or W . If $|\kappa| = N-3$, we replace it by the corresponding component of R . \square

In order to establish an L^2 estimate on the velocity average of R , we split it in $R := H + G$, where

$$\begin{cases} \mathbf{T}_F(H) + AH = 0, & H(0, \dots) = R(0, \dots), \\ \mathbf{T}_F(G) + AG = BW, & G(0, \dots) = 0 \end{cases}$$

and then prove L^2 estimates on $\int_v |H| dv$ and $\int_v |G| dv$. For the homogeneous part H , we will commute the transport equation and take advantage of the decaying properties of the matrix A in order to obtain boundedness on a certain L^1 norm as for f in Section 5.6. The L^2 estimate will then follow from a Klainerman-Sobolev inequality and the bound obtained on $\mathbb{E}[H]$. The inhomogeneous part will be schematically decomposed as $G = KW$, with K a matrix such that $\mathbb{E}[|K|^2 |W|](t) \leq \epsilon(1+t)^{\frac{1}{4}}$. The expected decay rate on $\|\int_v |G| dv\|_{L_x^2}$ will then be obtained using the pointwise decay estimates satisfied by the components of W . Note that contrary to [18], we keep the v derivatives in the matrices A and B . This allows us to crucially exploit the good behavior of $(\nabla_v g)^r$ (see (5.13)) but it forces us to put the derivatives of order $N-3$ in both R and W .

5.8.1 The homogeneous system

With the aim of obtaining an L^∞ estimate on $\int_v |H| dv$, we will have to commute at least three times the transport equation satisfied by each component of H . However, in order to control $\|\widehat{Z}^\beta H_i\|_{L_{x,v}^1}$, where $|\beta| = 3$, $\beta^i \in I^k$ and $k \geq N-2$, a bound on the L^1 norm of $\widehat{Z}^\kappa H_j$, with $|\kappa| = 4$ and $j \in I^{k-1}$, is required. This leads us to introduce the following energy norm

$$\mathbb{E}_H(t) := \sum_{k=0}^3 \sum_{|\beta| \leq 3+k} \sum_{i \in I^{N-k}} (1+t)^{-(\beta_P + (\beta_i^1)_P)\delta} \mathbb{E} \left[\sqrt{z}^{N+9-(1-2\eta)(\beta_P + \beta_P^i)} \widehat{Z}^\beta H_i \right] (t).$$

We have the following commutation formula.

Lemma 5.8.4. *Let $0 \leq k \leq 3$, $|\beta| \leq 3 + k$ and $i \in I^{N-k}$. Then, $\mathbf{T}_F(\widehat{Z}^\beta H_i)$ can be written as a linear combination of terms such as*

$$\mathcal{L}_{Z^\gamma}(F) \left(v, \nabla_v \widehat{Z}^\kappa H_j \right), \quad \text{with} \quad |\gamma| \leq 6 \leq N-2, \quad |\kappa| \leq |\beta|, \quad |\beta^j| \leq |\beta^i|, \quad |\kappa| + |\beta^j| < |\beta| + |\beta^i|,$$

- $\beta_P^j + \kappa_P < \beta_P^i + \beta_P$ or
- $\beta_P^j + \kappa_P = \beta_P^i + \beta_P$ and $\gamma_T \geq 1$.

Proof. According to Proposition 5.2.6, the source terms coming from $[\mathbf{T}_F, \widehat{Z}^\beta](H_i)$ are such as those described in this lemma, with $j = i$. The other ones arise from $\widehat{Z}^\beta(\mathbf{T}_F(H_i))$ and the result follows from Lemma 5.8.2 and $|\beta|$ applications of Lemma 5.2.5. \square

Hence, as $R(0, \dots) = H(0, \dots)$, there exists $C_0 > 0$ such that $\mathbb{E}_H(0) \leq C_0 \epsilon$. Following the proof of (5.22) and Proposition 5.6.1 (for the cases where $|\gamma| \leq N-2$), one can prove, if ϵ small enough, that $\mathbb{E}_H(t) \leq 3C_0 \epsilon (1+t)^\eta$ for all $t \in [0, T[$. By Proposition 5.3.5, we then obtain, for $0 \leq k \leq 3$,

$$\forall (t, x) \in V_b(T), \quad 1 \leq j \leq |I^{N-k}|, \quad |\beta| \leq k, \quad \int_v \sqrt{z}^{N+6-(1-2\eta)(\beta_P - \beta_P^j)} |\widehat{Z}^\beta H_j| dv \lesssim \epsilon \frac{(1+t)^{(N+4)\eta}}{\tau_+^2 \tau_-}. \quad (5.29)$$

5.8.2 The inhomogenous system

The purpose of this subsection is to prove an L^2 estimate on $\int_v |G| dv$. We cannot proceed by commuting $\mathbf{T}_F(G) + AG = BW$ since B contains top order derivatives of F and we then follow the strategy exposed earlier in this section. For this, in order to prove L^1 estimates on quantities related to G , we need

- to rewrite the v derivatives hidden in the matrix A .
- to ensure that the (transformed) matrix A decay sufficiently fast. We will then modify each component G_i of G by $z^{a_i} G_i$, with a well chosen exponent a_i , in order to take advantage of similar hierarchies than those used in Section 5.6.

We start by introducing some notations and proving certain preparatory results.

Definition 5.8.5. *Let L be the vector valued field of length $|I|$ such that, for $i \in \llbracket 1, |I| \rrbracket$,*

$$L_i = \sqrt{z}^{N-(1-2\eta)\beta_P^i} G_i \quad .$$

For $\widehat{Z} \in \widehat{\mathbb{P}}_0$ and $i \in I \setminus I^N$, we define $i_{\widehat{Z}}$ such that

$$R_{i_{\widehat{Z}}} = \widehat{Z} \widehat{Z}^{\beta^i} f.$$

We will transform the v derivatives by several applications of the following result.

Lemma 5.8.6. *Let $\widehat{Z} \in \widehat{\mathbb{P}}_0$ and $i \in I \setminus I^N$. Then, as $R = H + G$ and $\widehat{Z} R_i = \widehat{Z} \widehat{Z}^{\beta^i} f = R_{i_{\widehat{Z}}}$,*

$$\widehat{Z} G_i = G_{i_{\widehat{Z}}} + H_{i_{\widehat{Z}}} - \widehat{Z} H_i.$$

The aim of the next lemma is to describe in details the transport equation satisfied by L .

Lemma 5.8.7. *There exists $p \geq 1$, a vector valued field Y of length p and three matrices valued functions $\overline{A} : V_b(T) \times \mathbb{R}_v^3 \rightarrow \mathfrak{M}_{|I|}(\mathbb{R})$, $\overline{B} : V_b(T) \times \mathbb{R}_v^3 \rightarrow \mathfrak{M}_{|I|, p}(\mathbb{R})$ and $\overline{D} : V_b(T) \times \mathbb{R}_v^3 \rightarrow \mathfrak{M}_p(\mathbb{R})$ such that*

$$\mathbf{T}_F(L) + \overline{A}L = \overline{B}Y, \quad \mathbf{T}_F(Y) = \overline{D}Y \quad \text{and} \quad \forall (t, x) \in V_b(T), \quad \int_v z^2 |Y|(t, x, v) dv \lesssim \epsilon \frac{(1+t)^{(N+4)\eta}}{\tau_+^2 \tau_-}.$$

The matrices \overline{A} and \overline{B} are such that $\mathbf{T}_F(L_i)$ can be bounded, for $1 \leq i \leq |I|$, by a linear combination of the following terms,

- $\left(|\rho(\mathcal{L}_{Z^\gamma}(F))| + |\sigma(\mathcal{L}_{Z^\gamma}(F))| + |\alpha(\mathcal{L}_{Z^\gamma}(F))| + \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(F))| \right) |Y|, \quad \text{with} \quad |\gamma| \leq N.$
 - $\sqrt{\epsilon} \frac{v^0}{\tau_+^{1-\eta}} |L_k|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+} |L_q|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+^{1+\eta}} |L_j|, \quad \sqrt{\epsilon} \frac{v^L}{\tau_-^{1-\eta}} |L_k|, \quad \sqrt{\epsilon} \frac{v^L}{\tau_-^{\frac{3}{2}}} |L_q|, \quad \sqrt{\epsilon} \frac{v^L}{\tau_+ \tau_-^2} |L_j|,$
- where $k, q, j \in \llbracket 1, |I| \rrbracket$, $\beta_P^k < \beta_P^i$, $\beta_P^q = \beta_P^i$ and $\beta_P^j = \beta_P^i + 1$.

In order to describe the components of the matrix \bar{D} , we use the quantity $[j]$ which will be defined during the proof for all $j \in \llbracket 1, p \rrbracket$. \bar{D} is such that $\mathbf{T}_F(Y_i)$ can be bounded, for $1 \leq i \leq p$, by a linear combination of the following terms,

$$\bullet \quad \sqrt{\epsilon} \frac{v^0}{\tau_+} |Y_k|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+} |Y_q|, \quad \sqrt{\epsilon} \frac{v^0}{\tau_+^{1+\eta}} |Y_j|, \quad \sqrt{\epsilon} \frac{v^L}{\tau_-^{1-\eta}} |Y_k|, \quad \sqrt{\epsilon} \frac{v^L}{\tau_-^{\frac{3}{2}}} |Y_q|, \quad \sqrt{\epsilon} \frac{v^L}{\tau_+^\eta \tau_-^2} |Y_j|,$$

where $k, q, j \in \llbracket 1, p \rrbracket$, $[k] < [i]$, $[q] = [i]$ and $[j] = [i] + 1$.

Proof. The key element of the proof will be to rewrite all terms of the form $\partial_{v^k} G_j$ appearing in the equation $\mathbf{T}_F(G) + AG = BW$ with the formula $v^0 \partial_{v^k} = \widehat{\Omega}_{0k} - x^k \partial_t - t \partial_k$. As $j \in I \setminus I^N$ by Lemma 5.8.2, we will express $\widehat{Z} G_j$, by Lemma 5.8.6, as a combination of $(G_q)_{q \in I}$, $(H_q)_{q \in I}$ and $\widehat{Z} H_j$. This suggests us to take for Y the vector valued field of length p composed by the following components Y_i , where $i \in \llbracket 1, p \rrbracket$.

- $Y_i = \sqrt{z}^{N+2-(1-2\eta)\beta_P} \widehat{Z}^\beta f$, with $|\beta| \leq N-3$. We then define $[i] := \beta_P$.
- $Y_i = \sqrt{z}^{N+2-(1-2\eta)(\beta_P+\beta_P^j)} \widehat{Z}^\beta H_j$, with $|\beta| + |\beta_P^j| \leq N$. In that case, we define $[i] := \beta_P + \beta_P^j$.

In view of (5.19) and (5.29), $\int_v z^2 |Y| dv$ satisfies the expected pointwise decay estimate. The construction of the matrix \bar{D} is similar to the one of \bar{A} detailed below and then sketched. To obtain it, apply Lemmas 5.2.6 and 5.8.4 and then make similar operations as those made in the proof of (5.22) and Proposition 5.6.1 (for the cases where $|\gamma| \leq N-2$). We now turn on the construction of \bar{A} and \bar{B} . Fix $i \in \llbracket 1, |I| \rrbracket$ and note that

$$\mathbf{T}_F(L_i) = \mathbf{T}_F(\sqrt{z}^{N-(1-2\eta)\beta_P^i} G_i) = \sqrt{z}^{N-(1-2\eta)\beta_P^i-2} F(v, \nabla_v z) G_i + \sqrt{z}^{N-(1-2\eta)\beta_P^i} \mathbf{T}_F(G_i).$$

Following the computations of Subsection 5.6.1, we have

$$\left| F(v, \nabla_v z) \sqrt{z}^{N-(1-2\eta)\beta_P^i-2} G_i \right| \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+} z \sqrt{z}^{N-(1-2\eta)\beta_P^i-2} |G_i| \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+} |L_i|.$$

By Lemma 5.8.2, $\sqrt{z}^{N-(1-2\eta)\beta_P^i} \mathbf{T}_F(G_i)$ can be written as a linear combination of the following terms.

- Those coming from BW ,

$$\sqrt{z}^{N-(1-2\eta)\beta_P^i} \mathcal{L}_{Z^\zeta}(F)(v, \nabla_v W_j), \quad \text{with } |\xi^j| \leq N-4, \quad |\zeta| \leq N \quad \text{and} \quad \xi_P^j \leq \beta_P^i. \quad (5.30)$$

Let $|\zeta| \leq N$ and $(\alpha, \underline{\alpha}, \rho, \sigma)$ be the null decomposition of $\mathcal{L}_{Z^\zeta}(F)$. Using Lemma 5.4.1, we can bound (5.30) by terms of the form

$$|\mathcal{L}_{Z^\zeta}(F)| \sqrt{z}^{N-(1-2\eta)\beta_P^i} \left| \widehat{\Gamma} \widehat{Z}^{\xi^j} f \right|, \quad \left(\tau_- |\rho| + \tau_+ |\alpha| + \tau_+ \frac{|v^A| + v^L}{v^0} (|\sigma| + |\underline{\alpha}|) \right) \sqrt{z}^{N-(1-2\eta)\beta_P^i} \left| \nabla_{t,x} \widehat{Z}^{\xi^j} f \right|, \quad (5.31)$$

where $\widehat{\Gamma} \in \widehat{\mathbb{P}}_0$. As $\xi_P^j \leq \beta_P^i$, there exists κ^1 and κ^2 satisfying

$$\widehat{\Gamma} \widehat{Z}^{\xi^j} f = \widehat{Z}^{\kappa^1} f, \quad |\kappa^1| \leq N-3, \quad \kappa_P^1 \leq \beta_P^i + 1 \quad \text{and} \quad \nabla_{t,x} \widehat{Z}^{\xi^j} f = \widehat{Z}^{\kappa^2} f, \quad |\kappa^2| \leq N-3, \quad \kappa_P^2 \leq \beta_P^i.$$

Consequently, there exists $(j, q) \in \llbracket 1, p \rrbracket$ such that

$$\sqrt{z}^{N-(1-2\eta)\beta_P^i} \left| \widehat{Z}^{\kappa^1} f \right| \leq \sqrt{z}^{N+2-(1-2\eta)\kappa_P^1} \left| \widehat{Z}^{\kappa^1} f \right| = |Y_j| \quad \text{and} \quad \sqrt{z}^{N+2-(1-2\eta)\beta_P^i} \left| \widehat{Z}^{\kappa^2} f \right| \leq |Y_q|. \quad (5.32)$$

Then, combine (5.31) with (5.32) and use the inequality $\tau_- v^0 + \tau_+ |v^A| + \tau_+ v^L \lesssim z$ of Lemma 5.2.9 in order to obtain terms involving Y of the expected form.

- Those coming from AW ,

$$\sqrt{z}^{N-(1-2\eta)\beta_P^i} \mathcal{L}_{Z^\gamma}(F)(v, \nabla_v G_j), \quad \text{with } |\beta^j| < |\beta^i|, \quad |\gamma| \leq 3 \quad \text{and} \quad \beta_P^j + |\gamma| \leq \beta_P^i.$$

In order to rewrite the v derivatives of G , note that Lemma 5.8.6 gives, using $v^0 \partial_{v^k} = \widehat{\Omega}_{0k} - x^k \partial_t - t \partial_k$ and (5.13),

$$\begin{aligned} v^0 \partial_{v^k} G_j &= G_{j_{\widehat{\Omega}_{0k}}} - x^k G_{j_{\partial_t}} - t G_{j_{\partial_k}} + H_{j_{\widehat{\Omega}_{0k}}} - x^k H_{j_{\partial_t}} - t H_{j_{\partial_k}} - \widehat{\Omega}_{0k} H_j + x^k \partial_t H_j + t \partial_k H_j, \\ v^0 (\nabla_v G_j)^r &= \frac{x^q}{r} (G_{j_{\widehat{\Omega}_{0q}}} + H_{j_{\widehat{\Omega}_{0q}}} - \widehat{\Omega}_{0q} H_j) - G_{j_s} - H_{j_s} + S H_j \\ &\quad + (t-r) \left(G_{j_{\partial_t}} + H_{j_{\partial_t}} - \partial_t H_j - \frac{x^q}{r} G_{j_{\partial_q}} - \frac{x^q}{r} H_{j_{\partial_q}} + \frac{x^q}{r} \partial_q H_j \right). \end{aligned}$$

We then have schematically

$$v^0 |(\nabla_v G_j)^r| \lesssim |G_{j\bar{z}}| + \tau_- |G_{j\partial}| + |H_{j\bar{z}}| + |\widehat{Z}H_j| + \tau_- (|H_{j\partial}| + |\partial H_j|), \quad (5.33)$$

$$v^0 |(\nabla_v G_j)^A| \lesssim |G_{j\bar{z}}| + \tau_+ |G_{j\partial}| + |H_{j\bar{z}}| + |\widehat{Z}H_j| + \tau_+ (|H_{j\partial}| + |\partial H_j|). \quad (5.34)$$

Denote by $(\alpha, \underline{\alpha}, \rho, \sigma)$ the null decomposition of $\mathcal{L}_{Z^\gamma}(F)$ and expand $\mathcal{L}_{Z^\gamma}(F)(v, \nabla_v G_j)$ in null components using formula (5.12). As the computations are similar to those made in Subsection 5.6.2, we only bound certain terms given by (5.12). Consider for instance $\sqrt{z}^{N-(1-2\eta)\beta_P^i} \varepsilon_{BA} v^B \sigma (\nabla_v G_j)^A$ and use (5.34) in order to bound it. As $\beta_P^j \leq \beta_P^i$, we have $\beta_P^{j\partial} \leq \beta_P^i$ as well as $\beta_P^{j\bar{z}} \leq \beta_P^i + 1$ and the terms related to H can be estimated, using $\tau_+ |v^B| \leq v^0 z$, as follows

$$\tau_+ |\sigma| \frac{|v^B|}{v^0} \sqrt{z}^{N-(1-2\eta)\beta_P^i} (|H_{j\partial}| + |\partial H_j|) + |\sigma| \sqrt{z}^{N-(1-2\eta)\beta_P^i} (|H_{j\bar{z}}| + |\widehat{Z}H_j|) \lesssim |\sigma| |Y|.$$

For the ones associated to G , suppose first that $\beta_P^j < \beta_P^i$. By Lemma 5.2.9, we have $|v^B| \lesssim v^0 v^{\underline{L}}$ as well as $\tau_-^{\frac{1}{2}-\eta} \lesssim z^{\frac{1}{2}-\eta}$. Hence, using also $|\sigma| \lesssim \sqrt{\varepsilon} \tau_+^{-1} \tau_-^{-\frac{1}{2}}$ which comes from (5.20), we get

$$\begin{aligned} \tau_+ |\sigma| \frac{|v^B|}{v^0} \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\partial}| &\lesssim \frac{\sqrt{\varepsilon}}{\sqrt{\tau_-}} v^{\underline{L}} \tau_-^{-\frac{1}{2}+\eta} \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\partial}| \lesssim \sqrt{\varepsilon} \frac{v^{\underline{L}}}{\tau_-^{1-\eta}} |L_{j\partial}|, & \beta_P^{j\partial} < \beta_P^i, \\ |\sigma| \frac{|v^B|}{v^0} \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\bar{z}}| &\lesssim \frac{\sqrt{\varepsilon}}{\tau_+} \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\bar{z}}| \lesssim \sqrt{\varepsilon} \frac{v^0}{\tau_+} |L_{j\bar{z}}|, & \beta_P^{j\bar{z}} \leq \beta_P^i. \end{aligned}$$

If $\beta_P^j = \beta_P^i$, then $[\gamma] = 0$ and $\gamma_T \geq 1$ so that $|\sigma| \lesssim \sqrt{\varepsilon} \tau_+^{-1} \tau_-^{-\frac{3}{2}}$ by Lemma 5.3.8. We then have

$$\begin{aligned} \tau_+ |\sigma| \frac{|v^B|}{v^0} \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\partial}| &\lesssim \frac{\sqrt{\varepsilon}}{\tau_-^{\frac{3}{2}}} v^{\underline{L}} \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\partial}| \lesssim \sqrt{\varepsilon} \frac{v^{\underline{L}}}{\tau_-^{\frac{3}{2}}} |L_{j\partial}|, & \beta_P^{j\partial} = \beta_P^i, \\ |\sigma| \frac{|v^B|}{v^0} \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\bar{z}}| &\lesssim \frac{\sqrt{\varepsilon}}{\tau_+ \tau_-^{\frac{3}{2}}} v^{\underline{L}} \tau_+^{\frac{1}{2}-\eta} \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\bar{z}}| \lesssim \frac{\sqrt{\varepsilon} v^{\underline{L}}}{\tau_+^{\frac{1}{2}+\eta} \tau_-^{\frac{3}{2}}} |L_{j\bar{z}}|, & \beta_P^{j\bar{z}} = \beta_P^i + 1. \end{aligned}$$

We now treat the terms involving ρ , which can be estimated by $|v^0 \rho (\nabla_v G_j)^r|$, and we use (5.33) to bound them. As $\beta_P^j \leq \beta_P^i$ and $\tau_- \lesssim z$, the terms related to H can be estimated as follows

$$\tau_- |\rho| \sqrt{z}^{N-(1-2\eta)\beta_P^i} (|H_{j\partial}| + |\partial H_j|) + |\rho| \sqrt{z}^{N-(1-2\eta)\beta_P^i} (|H_{j\bar{z}}| + |\widehat{Z}H_j|) \lesssim |\rho| |Y|.$$

For the ones associated to G , start again by assuming $\beta_P^j < \beta_P^i$. As $\tau_-^{\frac{1}{2}-\eta} \lesssim z^{\frac{1}{2}-\eta}$ and $|\rho| \lesssim \sqrt{\varepsilon} \tau_+^{-1} \tau_-^{-\frac{1}{2}}$ by (5.20), we get

$$\begin{aligned} \tau_- |\rho| \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\partial}| &\lesssim \frac{\sqrt{\varepsilon} \sqrt{\tau_-}}{\tau_+} \tau_-^{-\frac{1}{2}+\eta} \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\partial}| \lesssim \frac{\sqrt{\varepsilon}}{\tau_+^{1-\eta}} |L_{j\partial}|, & \beta_P^{j\partial} < \beta_P^i, \\ |\rho| \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\bar{z}}| &\lesssim \frac{\sqrt{\varepsilon}}{\tau_+} \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\bar{z}}| \lesssim \frac{\sqrt{\varepsilon}}{\tau_+} |L_{j\bar{z}}|, & \beta_P^{j\bar{z}} \leq \beta_P^i. \end{aligned}$$

Otherwise, $\beta_P^j = \beta_P^i$ and $\gamma_T \geq 1$, so that $|\rho| \lesssim \sqrt{\varepsilon} \tau_+^{-1} \tau_-^{-\frac{3}{2}}$ by Lemma 5.3.8. We then have, using $1 \lesssim \sqrt{v^0 v^{\underline{L}}}$,

$$\begin{aligned} \tau_- |\rho| \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\partial}| &\lesssim \frac{\sqrt{\varepsilon}}{\tau_+ \tau_-^{\frac{1}{2}}} \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\partial}| \lesssim \frac{\sqrt{\varepsilon}}{\tau_+} |L_{j\partial}|, & \beta_P^{j\partial} = \beta_P^i, \\ |\rho| \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\bar{z}}| &\lesssim \frac{\sqrt{\varepsilon}}{\tau_+ \tau_-^{\frac{3}{2}}} \sqrt{v^0 v^{\underline{L}}} \tau_+^{\frac{1}{2}-\eta} \sqrt{z}^{N-(1-2\eta)\beta_P^i} |G_{j\bar{z}}| \lesssim \sqrt{\varepsilon} \left(\frac{v^0}{\tau_+^{1+\eta}} + \frac{v^{\underline{L}}}{\tau_+ \tau_-^3} \right) |L_{j\bar{z}}|, \end{aligned}$$

$\beta_P^{j\bar{z}} = \beta_P^i + 1$. As the other terms can be handled similarly, this concludes the construction of \bar{A} and \bar{B} . \square

Let K be the solution of $\mathbf{T}_F(K) + \bar{A}K + K\bar{D} = \bar{B}$ satisfying $K(0, \dots) = 0$. Note that KY satisfies $\mathbf{T}_F(KY) + \bar{A}KY = \bar{B}Y$ and initially vanishes, so that $KY = L$. The goal now is to prove a sufficiently good estimate on the energy norm

$$\mathbb{E}_G(t) := \sum_{i=0}^{|I|} \sum_{j=0}^p \sum_{q=0}^p (1+t)^{-\eta(2\beta_P^i + [q] - 2[j])} \mathbb{E} \left[\left| K_i^j \right|^2 Y_q \right] (t).$$

In order to apply Proposition 5.3.1, remark that

$$\mathbf{T}_F \left(|K_i^j|^2 Y_q \right) = |K_i^j|^2 \overline{D}_q^r Y_r - 2 \left(\overline{A}_i^r K_r^j + K_i^r \overline{D}_r^j \right) K_i^j Y_q + 2 \overline{B}_i^j K_i^j Y_q. \quad (5.35)$$

Proposition 5.8.8. *If ϵ is small enough, we have $\mathbb{E}_G(t) \lesssim \epsilon(1+t)^{4N\eta}$ for all $t \in [0, T[$.*

Proof. We use again the continuity method. Let $T_0 \in]0, T[$ be the largest time such that $\mathbb{E}_G(t) \lesssim \epsilon(1+t)^{4N\eta}$ for all $t \in [0, T_0[$. Fix $i \in \llbracket 1, |I| \rrbracket$ and $(j, q) \in \llbracket 1, p \rrbracket^2$. According to the energy estimate of Proposition 5.3.1 and (5.35), we would improve the bootstrap assumption, for ϵ small enough, if we could prove that

$$\int_0^t \int_{\overline{\Sigma}_s^b} \int_v \left| |K_i^j|^2 \overline{D}_q^r Y_r - 2 \left(\overline{A}_i^r K_r^j + K_i^r \overline{D}_r^j \right) K_i^j Y_q \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} (1+t)^{\eta(4N-2[j]+2\beta_P^i+[q])}, \quad (5.36)$$

$$I_{\overline{B}} := \int_0^t \int_{\overline{\Sigma}_s^b} \int_v \left| \overline{B}_i^j K_i^j Y_q \right| \frac{dv}{v^0} dx ds \lesssim \epsilon^{\frac{3}{2}} (1+t)^{\eta(4N-2[j]+2\beta_P^i+[q])}. \quad (5.37)$$

Let us start with (5.36). Since the computations are similar to those of Subsection 5.6.2, we only study certain terms of the integral (they are all descibed in Lemma 5.8.7). Fix $1 \leq r \leq p$ as well as $1 \leq k \leq |I|$ and suppose for instance that

$$|\overline{A}_i^k| \lesssim \sqrt{\epsilon} \frac{v^L}{\tau_-^{\frac{3}{2}}}, \quad \beta_P^k = \beta_P^i, \quad |\overline{D}_r^j| \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+^{1-\eta}}, \quad [j] < [r] \quad \text{and} \quad |\overline{D}_q^r| \lesssim \sqrt{\epsilon} \frac{v^0}{\tau_+^\eta \tau_-^3}, \quad [r] = [q] + 1.$$

Without any sommation in k and by the Cauchy-Schwarz inequality in $(\underline{u}, \omega, v)$,

$$\begin{aligned} \int_0^t \int_{\overline{\Sigma}_s^b} \int_v \left| \overline{A}_i^k K_k^j K_i^j Y_q \right| \frac{dv}{v^0} dx ds &\lesssim \int_{-\infty}^b \frac{\sqrt{\epsilon}}{\tau_-^{\frac{3}{2}}} \left| \int_{C_u(t)} \int_v \frac{v^L}{v^0} |K_k^j|^2 |Y_q| dv dC_u(t) \int_{C_u(t)} \int_v \frac{v^L}{v^0} |K_i^j|^2 |Y_q| dv dC_u(t) \right|^{\frac{1}{2}} du \\ &\lesssim \sqrt{\epsilon \mathbb{E}[|K_k^j|^2 |Y_q|](s) \mathbb{E}[|K_i^j|^2 |Y_q|](s)} \int_{u=-\infty}^0 \frac{du}{\tau_-^{\frac{3}{2}}} \\ &\lesssim \epsilon^{\frac{3}{2}} (1+t)^{\eta(4N-2[j]+[q]+\beta_P^i+\beta_P^k)} = \epsilon^{\frac{3}{2}} (1+t)^{\eta(4N-2[j]+[q]+2\beta_P^i)}. \end{aligned}$$

Without any sommation in r and by the Cauchy-Schwarz inequality in (x, v) as well as $-[r] + 1 \leq -[j]$,

$$\begin{aligned} \int_0^t \int_{\overline{\Sigma}_s^b} \int_v \left| K_i^r \overline{D}_r^j K_i^j Y_q \right| \frac{dv}{v^0} dx ds &\lesssim \int_0^t \frac{\sqrt{\epsilon}}{(1+s)^{1-\eta}} \left| \int_{\overline{\Sigma}_s^b} \int_v |K_i^r|^2 |Y_q| dv dx \int_{\overline{\Sigma}_s^b} \int_v |K_i^j|^2 |Y_q| dv dx \right|^{\frac{1}{2}} ds \\ &\lesssim \epsilon^{\frac{3}{2}} \int_0^t (1+t)^{\eta(4N-[r]-[j]+2\beta_P^i+[q])+\eta} \frac{ds}{1+s} \\ &\lesssim \epsilon^{\frac{3}{2}} (1+t)^{\eta(4N-2[j]+2\beta_P^i+[q])}. \end{aligned}$$

Recall now from (5.26) the definition of $C_u^i(t)$ and $T_{i+1}(t)$. Without any sommation in r and since $[r] = [q] + 1$,

$$\begin{aligned} \int_0^t \int_{\overline{\Sigma}_s^b} \int_v \left| |K_i^j|^2 \overline{D}_q^r Y_r \right| \frac{dv}{v^0} dx ds &\lesssim \sum_{i=0}^{\log_2(1+t)} \int_{u=-\infty}^b \int_{C_u^i(t)} \frac{\sqrt{\epsilon}}{\tau_+^\eta \tau_-^2} \int_v \frac{v^L}{v^0} |K_i^j|^2 |Y_r| dv dC_u^i(t) du \\ &\lesssim \sum_{i=0}^{\log_2(1+t)} \int_{u=-\infty}^b \frac{\sqrt{\epsilon}}{2^{i\eta} \tau_-^2} \int_{C_u^i(t)} \int_v \frac{v^L}{v^0} |K_i^j|^2 |Y_r| dv dC_u^i(t) du \\ &\lesssim \sqrt{\epsilon} \sum_{i=0}^{\log_2(1+t)} \frac{\mathbb{E}[|K_i^j|^2 |Y_r|](T_{i+1}(t))}{2^{i\eta}} \int_{u=-\infty}^0 \frac{du}{\tau_-^2} \\ &\lesssim \epsilon^{\frac{3}{2}} \sum_{i=0}^{\log_2(1+t)} \frac{2^{\eta(4N-2[j]+2\beta_P^i+[r])(i+1)}}{2^{i\eta}} \lesssim (1+t)^{\eta(4N-2[j]+2\beta_P^i+[q])}. \end{aligned}$$

Let us focus now on (5.37). According to Lemma 5.8.7, we have

$$\left| \overline{B}_i^j \right| \lesssim |\rho(\mathcal{L}_{Z^\gamma}(F))| + |\sigma(\mathcal{L}_{Z^\gamma}(F))| + |\underline{\alpha}(\mathcal{L}_{Z^\gamma}(F))| + \frac{\tau_+}{\tau_-} |\alpha(\mathcal{L}_{Z^\gamma}(F))|, \quad \text{where} \quad |\gamma| \leq N.$$

By the bootstrap assumption (5.16) and dropping the dependence of $\alpha, \underline{\alpha}, \rho$ and σ in $\mathcal{L}_{Z^\gamma}(F)$, we have

$$\forall s \in [0, T], \quad \forall u < b, \quad \|\rho\| + \|\sigma\| + \|\underline{\alpha}\|_{L^2(\overline{\Sigma}_s^b)} + \int_{C_u(s)} |\alpha|^2 dC_u(s) \leq \mathcal{E}_N[F](s) \leq 4\epsilon.$$

Consequently, using the Cauchy-Schwarz inequality in v , $\int_v |Y| dv \lesssim \epsilon(1+s)^{2N\eta} \tau_+^{-2} \tau_-^{-1}$ and $1 \lesssim v^0 v^{\frac{L}{2}}$,

$$\begin{aligned} I_{\overline{B}} &\lesssim \int_0^t \int_{\overline{\Sigma}_s^b} \left(|\rho| + |\sigma| + \|\underline{\alpha}\| + \frac{\tau_+}{\tau_-} |\alpha| \right) \left| \int_v |Y_q| dv \int_v |K_i^j|^2 |Y_q| \frac{dv}{(v^0)^2} \right|^{\frac{1}{2}} dx ds \\ &\lesssim (1+t)^{N\eta} \int_0^t \int_{\overline{\Sigma}_s^b} \left(|\rho| + |\sigma| + \|\underline{\alpha}\| + \frac{\tau_+}{\tau_-} |\alpha| \right) \frac{\sqrt{\epsilon}}{\tau_+ \tau_-^{\frac{1}{2}}} \left| \int_v \frac{v^L}{v^0} |K_i^j|^2 |Y_q| dv \right|^{\frac{1}{2}} dx ds. \end{aligned} \quad (5.38)$$

By the Cauchy-Schwarz inequality in x and $\mathbb{E}_G(s) \lesssim \epsilon(1+t)^{4N\eta}$, we get

$$\begin{aligned} \int_0^t \int_{\overline{\Sigma}_s^b} (|\rho| + |\sigma| + \|\underline{\alpha}\|) \frac{\sqrt{\epsilon}}{\tau_+ \tau_-^{\frac{1}{2}}} \left| \int_v \frac{v^L}{v^0} |K_i^j|^2 |Y_q| dv \right|^{\frac{1}{2}} dx ds &\lesssim \int_0^t \frac{\sqrt{\epsilon}}{1+s} \left| \mathcal{E}_N[F](s) \mathbb{E} \left[|K_i^j|^2 |Y_q| \right] (s) \right|^{\frac{1}{2}} ds \\ &\lesssim \epsilon^{\frac{3}{2}} \int_0^t (1+s)^{\eta(2N-[j]+\beta_P^i+\frac{1}{2}[q])} \frac{ds}{1+s} \\ &\lesssim \epsilon^{\frac{3}{2}} (1+t)^{\eta(2N-[j]+\beta_P^i+\frac{1}{2}[q])}. \end{aligned} \quad (5.39)$$

Using the null foliation $(C_u(t))_{u < b}$ of $V_b(t)$ and the Cauchy-Schwarz inequality in (\underline{u}, ω) , it comes

$$\begin{aligned} \int_0^t \int_{\overline{\Sigma}_s^b} |\alpha| \frac{\sqrt{\epsilon}}{\tau_-^{\frac{3}{2}}} \left| \int_v \frac{v^L}{v^0} |K_i^j|^2 |Y_q| dv \right|^{\frac{1}{2}} dx ds &\lesssim \int_{-\infty}^b \frac{\sqrt{\epsilon}}{\tau_-^{\frac{3}{2}}} \left| \int_{C_u(t)} |\alpha|^2 dC_u(t) \int_{C_u(t)} \int_v \frac{v^L}{v^0} |K_i^j|^2 |Y_q| dv dC_u(t) \right|^{\frac{1}{2}} du \\ &\lesssim \sqrt{\epsilon} \left| \mathcal{E}_N[F](t) \mathbb{E} \left[|K_i^j|^2 |Y_q| \right] (t) \right|^{\frac{1}{2}} \int_{u=-\infty}^0 \frac{du}{\tau_-^{\frac{3}{2}}} \\ &\lesssim \epsilon^{\frac{3}{2}} (1+t)^{\eta(2N-[j]+\beta_P^i+\frac{1}{2}[q])}. \end{aligned} \quad (5.40)$$

Combining (5.38), (5.39) and (5.40), we finally obtain, since $[j] \leq N$,

$$I_{\overline{B}} \lesssim \epsilon^{\frac{3}{2}} (1+t)^{\eta(3N-[j]+\beta_P^i+\frac{1}{2}[q])} \lesssim \epsilon^{\frac{3}{2}} (1+t)^{\eta(4N-2[j]+2\beta_P^i+[q])}.$$

This concludes the improvement of the bootstrap assumption and then the proof. \square

5.8.3 End of the proof of Proposition 5.6.3

Let $i \in I$. Using the Cauchy-Schwarz inequality in v , $\mathbb{E}_H(t) \lesssim \epsilon(1+t)^\eta$ and the pointwise decay estimates (5.29), we have

$$\begin{aligned} \left\| \int_v z |H_i| dv \right\|_{L^2(\overline{\Sigma}_t^b)}^2 &\lesssim \left\| \int_v |H_i| dv \right\|_{L^\infty(\overline{\Sigma}_t^b)} \left\| \int_v z^2 |H_i| dv \right\|_{L^1(\overline{\Sigma}_t^b)} \\ &\lesssim \left\| \frac{\epsilon}{\tau_+^{2-(N+4)\eta} \tau_-} \right\|_{L^\infty(\overline{\Sigma}_t^b)} (1+t)^{\beta_P^i \eta} \mathbb{E}_H(t) \lesssim \frac{\epsilon^2}{(1+t)^{2-(2N+5)\eta}} \lesssim \frac{\epsilon^2}{(1+t)^{\frac{3}{2}}}. \end{aligned}$$

As $L_i = K_i^j Y_j$, the Cauchy-Schwarz inequality in v , $\mathbb{E}_G(t) \lesssim \epsilon(1+t)^{4N\eta}$ and $\int_v z^2 |Y| dv \lesssim \epsilon \tau_+^{-2+(N+4)\eta}$ gives

$$\begin{aligned} \left\| \int_v z |L_i| dv \right\|_{L^2(\overline{\Sigma}_t^b)}^2 &\lesssim \left\| \int_v z^2 |Y| dv \int_v |K_i^j|^2 |Y_j| dv \right\|_{L^1(\overline{\Sigma}_t^b)} \lesssim \left\| \int_v z^2 |Y| dv \right\|_{L^\infty(\overline{\Sigma}_t^b)} \left\| \int_v |K_i^j|^2 |Y_j| dv \right\|_{L^1(\overline{\Sigma}_t^b)} \\ &\lesssim \left\| \frac{\epsilon}{\tau_+^{2-(N+4)\eta}} \right\|_{L^\infty(\overline{\Sigma}_t^b)} (1+t)^{\eta(-2[j]+2\beta_P^i+[j])} \mathbb{E}_L(t) \lesssim \frac{\epsilon^2}{(1+t)^{2-8N\eta}} \lesssim \frac{\epsilon^2}{(1+t)^{\frac{3}{2}}}. \end{aligned}$$

To conclude the proof of Proposition 5.6.3, notice that for all $N-2 \leq |\beta| \leq N$, there exists $i \in I$ verifying $\widehat{Z}^\beta f = H_i + G_i$ and that $|G_i| \leq |L_i|$.

Bibliography

- [1] L. Andersson, P. Blue, and J. Joudioux. Hidden symmetries and decay for the Vlasov equation on the Kerr spacetime. *Comm. Partial Differential Equations*, 43(1):47–65, 2018.
- [2] C. Bardos and P. Degond. Global existence for the Vlasov-Poisson equation in 3 space variables with small initial data. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2(2):101–118, 1985.
- [3] L. Bieri, S. Miao, and S. Shahshahani. Asymptotic properties of solutions of the Maxwell Klein Gordon equation with small data. *Comm. Anal. Geom.*, 25(1):25–96, 2017.
- [4] L. Bigorgne. Asymptotic properties of small data solutions of the Vlasov-Maxwell system in high dimensions. arXiv:1712.09698, 2017.
- [5] L. Bigorgne. Sharp asymptotic behavior of solutions of the 3d vlasov-maxwell system with small data. arXiv:1812.11897, 2018.
- [6] L. Bigorgne. Sharp asymptotics for the solutions of the three-dimensional massless Vlasov-Maxwell system with small data. arXiv:1812.09716, 2018.
- [7] L. Bigorgne. Asymptotic properties of the solutions to the Vlasov-Maxwell system in the exterior of a light cone. In preparation, 2019.
- [8] F. Bouchut, F. Golse, and C. Pallard. Classical solutions and the Glassey-Strauss theorem for the 3D Vlasov-Maxwell system. *Arch. Ration. Mech. Anal.*, 170(1):1–15, 2003.
- [9] F. Bouchut, F. Golse, and C. Pallard. Nonresonant smoothing for coupled wave + transport equations and the Vlasov-Maxwell system. *Rev. Mat. Iberoamericana*, 20(3):865–892, 2004.
- [10] S.-H. Choi, S.-Y. Ha, and H. Lee. Dispersion estimates for the two-dimensional Vlasov-Yukawa system with small data. *J. Differential Equations*, 250(1):515–550, 2011.
- [11] D. Christodoulou and S. Klainerman. Asymptotic properties of linear field equations in Minkowski space. *Comm. Pure Appl. Math.*, 43(2):137–199, 1990.
- [12] D. Christodoulou and S. Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.
- [13] M. Dafermos and I. Rodnianski. The red-shift effect and radiation decay on black hole spacetimes. *Comm. Pure Appl. Math.*, 62(7):859–919, 2009.
- [14] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman. Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|a| < M$. *Ann. of Math. (2)*, 183(3):787–913, 2016.
- [15] R. J. DiPerna and P.-L. Lions. Global weak solutions of Vlasov-Maxwell systems. *Comm. Pure Appl. Math.*, 42(6):729–757, 1989.
- [16] D. Fajman, J. Joudioux, and J. Smulevici. Sharp asymptotics for small data solutions of the Vlasov-Nordström system in three dimensions. arXiv:1704.05353, 2017.
- [17] D. Fajman, J. Joudioux, and J. Smulevici. The Stability of the Minkowski space for the Einstein-Vlasov system. arXiv:1707.06141, 2017.
- [18] D. Fajman, J. Joudioux, and J. Smulevici. A vector field method for relativistic transport equations with applications. *Anal. PDE*, 10(7):1539–1612, 2017.

- [19] V. Georgiev. Decay estimates for the Klein-Gordon equation. *Comm. Partial Differential Equations*, 17(7-8):1111–1139, 1992.
- [20] R. Glassey and J. Schaeffer. The “two and one-half-dimensional” relativistic Vlasov Maxwell system. *Comm. Math. Phys.*, 185(2):257–284, 1997.
- [21] R. T. Glassey. *The Cauchy problem in kinetic theory*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.
- [22] R. T. Glassey and J. W. Schaeffer. Global existence for the relativistic Vlasov-Maxwell system with nearly neutral initial data. *Comm. Math. Phys.*, 119(3):353–384, 1988.
- [23] R. T. Glassey and W. A. Strauss. Singularity formation in a collisionless plasma could occur only at high velocities. *Arch. Rational Mech. Anal.*, 92(1):59–90, 1986.
- [24] R. T. Glassey and W. A. Strauss. Absence of shocks in an initially dilute collisionless plasma. *Comm. Math. Phys.*, 113(2):191–208, 1987.
- [25] R. T. Glassey and W. A. Strauss. Large velocities in the relativistic Vlasov-Maxwell equations. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 36(3):615–627, 1989.
- [26] Hans Lindblad and Martin Taylor. Global stability of Minkowski space for the Einstein–Vlasov system in the harmonic gauge. arXiv:1707.06079, 2017.
- [27] H. J. Hwang, A. Rendall, and J. J. L. Velázquez. Optimal gradient estimates and asymptotic behaviour for the Vlasov-Poisson system with small initial data. *Arch. Ration. Mech. Anal.*, 200(1):313–360, 2011.
- [28] F. John. Blow-up for quasilinear wave equations in three space dimensions. *Comm. Pure Appl. Math.*, 34(1):29–51, 1981.
- [29] S. Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.*, 38(3):321–332, 1985.
- [30] S. Klainerman. The null condition and global existence to nonlinear wave equations. In *Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984)*, volume 23 of *Lectures in Appl. Math.*, pages 293–326. Amer. Math. Soc., Providence, RI, 1986.
- [31] S. Klainerman. Remark on the asymptotic behavior of the Klein-Gordon equation in \mathbf{R}^{n+1} . *Comm. Pure Appl. Math.*, 46(2):137–144, 1993.
- [32] S. Klainerman and G. Staffilani. A new approach to study the Vlasov-Maxwell system. *Commun. Pure Appl. Anal.*, 1(1):103–125, 2002.
- [33] M. Kunze. Yet another criterion for global existence in the 3d relativistic vlasov-maxwell system. arXiv:1406.1517, 2014.
- [34] P. G. LeFloch and Y. Ma. *The global nonlinear stability of Minkowski space for self-gravitating massive fields*, volume 3 of *Series in Applied and Computational Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.
- [35] H. Lindblad and I. Rodnianski. The global stability of Minkowski space-time in harmonic gauge. *Ann. of Math. (2)*, 171(3):1401–1477, 2010.
- [36] H. Lindblad and J. Sterbenz. Global stability for charged-scalar fields on Minkowski space. *IMRP Int. Math. Res. Pap.*, pages Art. ID 52976, 109, 2006.
- [37] J. Luk and R. M. Strain. Strichartz estimates and moment bounds for the relativistic Vlasov-Maxwell system. *Arch. Ration. Mech. Anal.*, 219(1):445–552, 2016.
- [38] C. Pallard. On the boundedness of the momentum support of solutions to the relativistic Vlasov-Maxwell system. *Indiana Univ. Math. J.*, 54(5):1395–1409, 2005.
- [39] C. Pallard. A refined existence criterion for the relativistic Vlasov-Maxwell system. *Commun. Math. Sci.*, 13(2):347–354, 2015.
- [40] N. Patel. Three new results on continuation criteria for the 3d relativistic vlasov-maxwell system. arXiv:1607.07416, 2016.

- [41] G. Rein. Generic global solutions of the relativistic Vlasov-Maxwell system of plasma physics. *Comm. Math. Phys.*, 135(1):41–78, 1990.
- [42] G. Rein. Global weak solutions to the relativistic Vlasov-Maxwell system revisited. *Commun. Math. Sci.*, 2(2):145–158, 2004.
- [43] O. Sarbach and T. Zannias. The geometry of the tangent bundle and the relativistic kinetic theory of gases. *Class. Quant. Grav.*, 31:085013, 2014.
- [44] J. Schaeffer. A small data theorem for collisionless plasma that includes high velocity particles. *Indiana Univ. Math. J.*, 53(1):1–34, 2004.
- [45] W.-T. Shu. Global existence of Maxwell-Higgs fields. In *Nonlinear hyperbolic equations and field theory (Lake Como, 1990)*, volume 253 of *Pitman Res. Notes Math. Ser.*, pages 214–227. Longman Sci. Tech., Harlow, 1992.
- [46] J. Smulevici. Small data solutions of the Vlasov-Poisson system and the vector field method. *Ann. PDE*, 2(2):Art. 11, 55, 2016.
- [47] C. D. Sogge. *Lectures on nonlinear wave equations*. Monographs in Analysis, II. International Press, Boston, MA, 1995.
- [48] R. Sospedra-Alfonso and R. Illner. Classical solvability of the relativistic Vlasov-Maxwell system with bounded spatial density. *Math. Methods Appl. Sci.*, 33(6):751–757, 2010.
- [49] M. Taylor. The global nonlinear stability of Minkowski space for the massless Einstein-Vlasov system. *Ann. PDE*, 3(1):Art. 9, 177, 2017.
- [50] X. Wang. Propagation of regularity and long time behavior of the 3D massive relativistic transport equation II: Vlasov-Maxwell system. arXiv:1804.06566, 2018.
- [51] S. Yang. Decay of solutions of Maxwell-Klein-Gordon equations with arbitrary Maxwell field. *Anal. PDE*, 9(8):1829–1902, 2016.

Titre : Propriétés asymptotiques des solutions à données petites du système de Vlasov-Maxwell

Mots Clefs : EDP hyperboliques, système de Vlasov-Maxwell, équations non linéaires, équations d'ondes et de transport, méthode des champs de vecteurs, structure isotrope.

Résumé : L'objectif de cette thèse est de décrire le comportement asymptotique des solutions à données petites du système de Vlasov-Maxwell. En particulier, on s'attachera à étudier tant le champ électromagnétique que le champ de Vlasov par des méthodes de champs de vecteurs, nous permettant ainsi d'éviter toute contrainte de support sur les données initiales. La structure isotrope du système de Vlasov-Maxwell est d'une importance capitale pour compenser le phénomène de résonance causé par les particules approchant la vitesse de propagation du champ électromagnétique. De ce fait, plusieurs parties de ce manuscrit sont dédiées à sa description. Ajoutons également que les méthodes de champs de vecteurs sont connues pour être robustes et s'adapter relativement bien à d'autres situations telles que l'étude des solutions de l'équation des ondes sur un espace-temps courbé. Cette souplesse nous a notamment permis, contrairement aux travaux précédents sur ce sujet, de considérer des plasmas avec des particules sans masse.

Notre étude débute par le cas des grandes dimensions $d \geq 4$ où les effets dispersifs sont plus importants et permettent ainsi d'obtenir de meilleurs taux de décroissance sur les solutions du système et leurs dérivées. Une nouvelle inégalité de décroissance pour les solutions d'une équation de transport relativiste constitue d'ailleurs un élément central de la démonstration. Afin d'établir un résultat analogue dans le cas où les particules sont sans masse, nous avons dû imposer que le champ de Vlasov s'annule initialement pour les petites vitesses puis nous avons ensuite montré que cette hypothèse était nécessaire. Dans un second temps, nous nous intéressons au cas tridimensionnel avec des particules sans masse, où une étude plus poussée de la structure des équations sera nécessaire afin d'obtenir les taux de décroissance optimaux pour les composantes isotropes du champ électromagnétique, les moyennes en vitesse de la fonction de distribution et leurs dérivées. Nous nous concentrons ensuite sur l'étude du comportement asymptotique des solutions à données petites du système de Vlasov-Maxwell massif en dimension 3. Des difficultés spécifiques nous forcent à modifier les champs de vecteurs utilisés précédemment pour l'équation de transport dans le but de compenser les pires termes d'erreurs des équations commutées. Enfin, on considère le même problème en se restreignant à l'étude des solutions à l'extérieur d'un cône de lumière. Les fortes propriétés de décroissance vérifiées par la moyenne en vitesse de la densité de particules dans cette région nous permettent d'affaiblir les hypothèses sur les données initiales et d'avoir une démonstration considérablement plus simple.

Title : Asymptotic properties of the small data solutions of the Vlasov-Maxwell system

Keys words : Hyperbolic PDE, Vlasov-Maxwell system, non linear equations, wave and transport equations, vector field methods, null structure.

Abstract : The purpose of this thesis is to study the asymptotic properties of the small data solutions of the Vlasov-Maxwell system using vector field methods for both the electromagnetic field and the particle density. No compact support assumption is required on the initial data. Instead, we make crucial use of the null structure of the equations in order to deal with a resonant phenomenon caused by the particles approaching the speed of propagation of the Maxwell equations. Due to the robustness of vector field methods and contrary to previous works on this topic, we also study plasmas with massless particles.

We start by investigating the high dimensional cases ($d \geq 4$) where dispersive effects allow us to derive strong decay rate on the solutions of the system and their derivatives. For that purpose, we proved a new decay estimate for solutions to massive relativistic transport equations. In order to obtain an analogous result for massless particles, we required the velocity support of the distribution function to be initially bounded away from 0 and we then proved that this assumption is actually necessary. The second part of this thesis is devoted to the three dimensional massless case, where a stronger understanding of the null structure of the Vlasov-Maxwell system is essential in order to derive the optimal decay rate of the null components of the electromagnetic field, the velocity average of the particle density and their derivatives. We then focus on the asymptotic behavior of the small data solutions of the massive Vlasov-Maxwell system in $3d$. Specific problems force us to modify the vector fields used previously to study the Vlasov field in order to compensate the worst error terms in the commuted transport equations. Finally, still for the massive system in $3d$, we restrict our study of the solutions to the exterior of a light cone. The strong decay properties satisfied by the velocity average of the particle density in such a region permit us to relax the hypothesis on the initial data and lead to a much simpler proof.

