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Some Aspects of Growth-Fragmentation

Benjamin Dadoun

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Some Aspects of Growth-Fragmentation

Dissertation

zur

**Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)**

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von

Benjamin Dadoun

aus

Frankreich

Promotionskommission

Prof. Dr. **Jean Bertoin** (Vorsitz)

Prof. Dr. **Bénédicte Haas** (Gutachterin)

Dr. **Bastien Mallein** (Gutachter)

Prof. Dr. **Rémi Abgrall**

Prof. Dr. **Ashkan Nikeghbali**

Zürich, 2019

Some Aspects of Growth-Fragmentation

Benjamin Dadoun

Ph. D. thesis

Abstract

This thesis treats stochastic aspects of fragmentation processes when growth and/or immigration of particles are incorporated as a compensating phenomenon. In a first part, we study the asymptotic behavior of self-similar growth-fragmentation processes, extending the results related to pure fragmentations. In a second part, we prove that self-similar growth-fragmentations arise as scaling limits of truncated Markov branching processes and we provide a rather general criterion. This bolsters the conviction that growth-fragmentations appear in many discrete Markovian structures, as already observed in random planar geometry. Lastly, we study a growth-fragmentation with immigration equation. In particular, we investigate the asymptotic behavior of the solution by relating it to a stochastic particle system in which immigrate copies of a certain growth-fragmentation process.

À mes frères,

Chaque fois que la science avance d'un pas,
c'est qu'un imbécile la pousse,
sans le faire exprès.

Émile Zola, *La Joie de vivre*

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.

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(Zut, ça ne me semble pas aussi indéchiffrable que ça...)

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¹J’ai une preuve vidéo incontestable.

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Si tout est moyen

Si la vie est un film de rien

Ce passage-là était vraiment bien

Ce passage-là était bien.

Introduction

The word *fragmentation* designates a process in which an object is shattered into many smaller pieces called *fragments*. One may think of those resulting from the collision between two celestial objects of comparable sizes, or from the fragmentation of a meteorite entering the atmosphere. In cell biology, one may picture mitosis as a kind of fragmentation. Other instances where fragmentations occur include earthquakes in geophysics, fission in biology or in nuclear physics, sequencing in mass spectrometry, crushing in mineral processing, data transmission in telecommunications, *etc.*

As a simple and first, chronological example, let us take a pile of (fine) sand, split it in a truly random way, and repeat the operation independently on each two subpiles. Obviously, there are 2^n piles after the n -th iteration. But what is the repartition of their respective volumes? Such a question was answered in 1941 by A. N. Kolmogorov, in the paper [74] “Übers logarithmisch normale Verteilungsgesetz von der Dimensionen von der Teiler bei der Zersücklig”. As the title may suggest, the logarithms of the volumes follow a Normal distribution (the well-known bell-shaped curve) when n is sufficiently large; see Figure 0.1a for a simulation. More generally, Kolmogorov’s result applies to the situation where “particles” may be divided into a random number of fragments with random “sizes”, possibly in a non-conservative, but always *homogeneous*, way.

The question of inhomogeneous fragmentation, which he asked at the end of his work, addresses the situation where the rate of grinding may depend on the particle size (referring to our example, the splitting scheme can then be a function of the volume of sand being split). More precisely, one could imagine that particles with size x fragmentate x^α “faster” than particles with size 1 (so the original formulation is enclosed in the *homogeneous case*

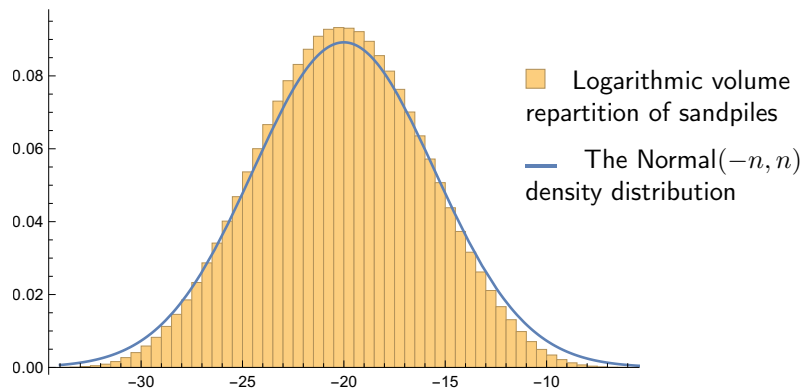


Figure 0.1a. Empirical distribution of sandpiles after $n = 20$ steps.

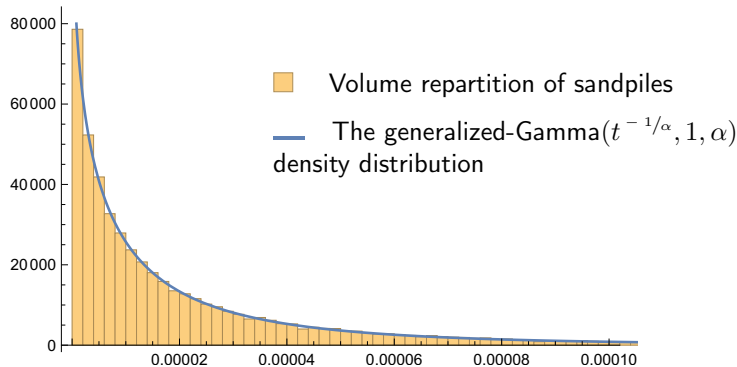


Figure 0.1b. Empirical distribution of sandpiles at large time for $\alpha = 1/2$.

$\alpha = 0$). Kolmogorov predicted that in the nowadays called *self-similar case* $\alpha \neq 0$, “the logarithmic normal law is no longer applicable”. This discrepancy was confirmed twenty years later by one of his former students, A. F. Filippov² [60]. If $u_t(x)$ denotes the average density of sandpiles of volume x at time t , then u_t solves the integro-differential equation

$$\frac{\partial}{\partial t} u_t(x) = -x^\alpha u_t(x) + 2 \int_x^\infty y^{\alpha-1} u_t(y) dy. \quad (0.1)$$

For $\alpha > 0$, the analysis of this equation shows that when t is sufficiently large, u_t is close to a generalized Gamma distribution³; see Figure 0.1b. For $\alpha < 0$, Filippov shone a light on the phenomenon of “formation of dust”: because particles fragmentate at higher rates as they get smaller, the total mass of the system decreases continuously and vanishes within a finite amount of time (although the mass involved in each individual fragmentation is conserved).

There is a natural genealogy induced by a fragmentation process, where “daughter particles” are related to the “mother particle” they originate from. In the self-similar setting, we easily picture a sort of phylogenetic tree where the length of a mother-daughter branch is proportional to the daughter’s lifetime. When $\alpha < 0$, because of the total extinction of mass at large times, this *fragmentation tree* is “compact”. In many cases (typically, when the “macroscopic” fragmentations into large comparable fragments are rare), its global shape does further not depend heavily on the exact splitting scheme, and we often observe so called *continuum random trees*, which bear interesting fractal properties. See Figure 0.2.

Fragmentation equations like (0.1) appear in a large variety of biological or physical models of particle systems and constitute an object of study as a whole, generally (at least, originally) from a non-probabilistic point of view. Besides the customary questions of existence and uniqueness of solutions, one typically wants to describe the stationary regime

²Incidentally, Filippov was twenty years younger than Kolmogorov, whom he acknowledges for his assistance and advice. Both had the same academic grandfather, D. F. Egorov.

³For this simple example, Filippov even provides a semi-explicit expression for u_t at any time t .

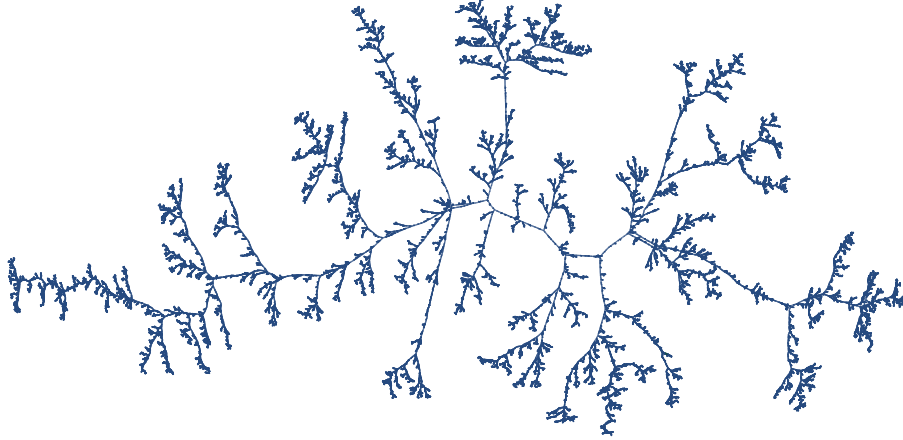


Figure 0.2. Large fragmentation tree with binary splits $n \mapsto \{k, n-k\}$, $1 \leq k < n$, occurring at rate $\propto n^\alpha$ for $\alpha = -1/2$, and with probability $\propto \binom{n}{k} \int_{1/2}^1 x^{k-3/2} (1-x)^{n-k-3/2} dx$.

and estimate the speed of convergence toward the asymptotic profile. Equilibrium may arise when fragmentation is compensated by growth of particles or immigration of new particles.

This thesis treats some of the aforementioned aspects when growth and possibly immigration is added to the picture. In a growth-fragmentation process, particles are not only subject to (random) fragmentation but may as well grow larger or smaller in a continuous (also random) way. In [Chapter 1](#), we look at asymptotics of empirical measures associated with the fragments, extending prior results in this vein (Kolmogorov [\[74\]](#), Filippov [\[60\]](#), Baryshnikov and Gnedin [\[16\]](#), Bertoin and al. [\[23, 26, 30, 32\]](#), Kyprianou et al. [\[77\]](#)). Then, in [Chapter 2](#), motivated by the appearance of self-similar growth-fragmentation processes in random planar geometry [\[25, 24, 81\]](#), we consider a simple Markovian model acting as discrete growth-fragmentation and we discuss its self-similar scaling limit by amending a criterion of Bertoin and Kortchemski [\[27\]](#). This work can be seen as a humble addition to the general and various criteria of Haas, Miermont et al. [\[86, 65, 87, 66, 63\]](#) for self-similar pure-fragmentations. Lastly, in [Chapter 3](#), we focus on a growth-fragmentation equation as studied by Bertoin and Watson [\[34, 20, 36\]](#) but with an additional term accounting for immigration.

In the remaining of this general introduction, we formally set up the growth-fragmentation processes and present more precisely the different questions that we alluded above.

0.1 Fragmentation processes

A first, fundamental property of fragmentation processes is the so called *branching property*: fragments split and evolve independently of one another. As such, we may define them within the framework of branching processes on $(0, \infty)$, which we now introduce.

0.1.1 General framework

For any measurable function $f: (0, \infty) \rightarrow (0, \infty)$, we write $f \in \mathcal{B}_{0<}$ if it is bounded from below away from zero, that is $\inf\{f(x): x \geq a\} > 0$ for every $a > 0$. We set $f(0) := 0$ and let \mathcal{M}_f be the space of non-increasing sequences $\mathbf{x} := (x_1, x_2, \dots)$ on $[0, \infty)$ such that

$$\langle \mathbf{x}, f \rangle := \sum_{j \geq 1} f(x_j) < \infty.$$

(Null values in \mathbf{x} are disregarded; their presence is merely needed to include finite sequences.) We see \mathcal{M}_f as a subspace of \mathcal{M}_0 , the space of non-increasing null sequences endowed with the topology of pointwise convergence, which because of monotonicity is also that of uniform convergence. Equivalently, we may view elements $\mathbf{x} \in \mathcal{M}_0$ as point measures $\sum_{j \geq 1} \delta_{x_j}$ on $(0, \infty)$ under the topology of vague convergence. This makes \mathcal{M}_0 a Polish space which we naturally equip with its Borel σ -field.

For the remaining of this section, we fix $f \in \mathcal{B}_{0<} \cup \{0\}$ and note that for point measures \mathbf{x} and $\mathbf{x}^{(i)}$, $i \geq 1$, we can always define the scalar multiplication $\lambda \mathbf{x} := \sum_{j \geq 1} \delta_{\lambda x_j}$, $\lambda > 0$, and the sum $\biguplus_{i \geq 1} \mathbf{x}^{(i)} := \sum_{i, j \geq 1} \delta_{x_j^{(i)}}$ (not necessarily in \mathcal{M}_0).

Definition 0.1.1. Let $\mathbf{X} := (\mathbf{X}(t): t \geq 0)$ be a \mathcal{M}_f -valued stochastic process, which is continuous in probability and whose conditional law given $\mathbf{X}(0) = \delta_x$, $x > 0$, is denoted \mathbf{P}_x .

- We call \mathbf{X} a *branching process* (in \mathcal{M}_f) if it has the (temporal) *branching property*: for every $s \geq 0$, the family $(\mathbf{X}(t+s): t \geq 0)$ given $\mathbf{X}(s) = (x_1, x_2, \dots)$ is independent of $(\mathbf{X}(r): r \leq s)$ and distributed like $(\biguplus_{i \geq 1} \mathbf{X}^{(i)}(t): t \geq 0)$, where the $\mathbf{X}^{(i)}$ are independent processes with law \mathbf{P}_{x_i} , respectively.
- If further there exists $\alpha \in \mathbb{R}$ such that for every $x > 0$, the law of $(x\mathbf{X}(x^\alpha t): t \geq 0)$ under \mathbf{P}_1 is \mathbf{P}_x , then we say that \mathbf{X} is *self-similar*; α is the *index of self-similarity*. When $\alpha = 0$, the process is said *homogeneous*.

Additive martingales and supermartingales associated with branching processes constitute one essential tool. Let us state a first elementary fact in this direction. Hereafter, \mathbf{E}_x denotes the expectation under \mathbf{P}_x .

Proposition 0.1.2 (Corollary 1 in [19]; see also Biggins and Kyprianou [42]). Let $C > 0$ and suppose that f is C -excessive for \mathbf{X} , in the sense that for all $x > 0, t \geq 0$,

$$\mathbf{E}_x[\langle \mathbf{X}(t), f \rangle] \leq C^t f(x)$$

(when there is equality, we rather say that f is C -invariant). Then the process

$$C^{-t} \langle \mathbf{X}(t), f \rangle, \quad t \geq 0,$$

is a supermartingale under \mathbf{P}_x for every $x > 0$. (It is in fact a martingale if f is C -invariant.)

Often, typically for self-similar fragmentations, we restrict $\mathcal{B}_{0<}$ to power functions $f_q: x \mapsto x^q$ with $q \geq 0$ and identify \mathcal{M}_{f_q} as a closed subspace of the sequence space $\ell^q(\mathbb{N})$.

0.1.2 Pure fragmentations

Definition 0.1.1 embraces processes that may be far from the intuitive notion of fragmentation. In a truly (*i.e.*, pure) fragmentation process, one indeed also expects $\mathbf{X}(t)$ to be “finer” than $\mathbf{X}(s)$ for all $t \geq s$.

Definition 0.1.3. A branching process is called a *pure-fragmentation process* (or simply a *pure fragmentation*), if its sample paths are non-increasing with respect to the lexicographic ordering \leq_{lex} on \mathcal{M}_0 .

It is clear from the branching property that a branching process is a pure fragmentation if and only if under $\mathbf{P}_x, x > 0$, it has values in the subspace

$$\mathcal{S}_x := \left\{ \mathbf{s} \in \mathcal{M}_0 : \mathbf{s} \leq_{\text{lex}} (x, 0, \dots) \right\}.$$

We call $\mathbf{s} \in \mathcal{S} := \mathcal{S}_1$ a *configuration*, giving the different arrangements $x \cdot \mathbf{s} := (xs_1, xs_2, \dots)$ of fragments which may result from the fragmentation of a particle with initial size $x > 0$. This includes the *trivial configuration* $\mathbf{1} := (1, 0, \dots)$ and the *dust* $\mathbf{0} := (0, 0, \dots)$. Often, the size parameter x rather corresponds to a “mass” or any quantity which cannot increase under fragmentation. Then the total mass $\langle \mathbf{X}(t), f_1 \rangle$ is also non-increasing over time, so that under $\mathbf{P}_x, x > 0$, the sample paths have values in $\{x \cdot \mathbf{p} : \mathbf{p} \in \mathcal{P}\}$, where

$$\mathcal{P} := \left\{ \mathbf{p} \in \mathcal{M}_0 : \sum_{i \geq 1} p_i \leq 1 \right\}$$

is the space of *mass-partitions*. In this situation, f_1 is 1-excessive for \mathbf{X} (so $\sum_{i \geq 1} X_i(t), t \geq 0$, is a supermartingale). Of course, there are other situations where this is not true, for instance when the size of a particle is measured by its diameter.

Note that a fragmentation event $x \mapsto x \cdot \mathbf{p}$ specified by $\mathbf{p} \in \mathcal{P}$ may induce a loss of mass, that is a possibly positive fraction $p_0 := 1 - \sum_{i \geq 1} p_i$ of the mass x may be reduced to dust. We say that \mathbf{p} (resp. a measure ν on \mathcal{P}) is *conservative* if $p_0 = 0$ (resp. $\nu(p_0 > 0) = 0$).

In the next sections we briefly sketch two constructions of pure fragmentations, and forward the interested reader to the first three chapters of the monograph [21].

0.1.2.1 Fragmentation chains

Let ν_x , $x > 0$, be a *finite* measure on \mathcal{M}_0 , with support in \mathcal{S}_x . We assume that the family $(\nu_x)_{x>0}$ depends in a measurable way on the variable x . A (*pure-*)*fragmentation chain with kernel* $(\nu_x)_{x>0}$ is a system of non-interacting particles which, at rate $\nu_x(\mathcal{S}_x)$ according to their respective size x , are each replaced by a cloud of particles with law $\nu_x(\cdot)/\nu_x(\mathcal{S}_x)$.

It is formally defined as a continuous-time branching Markov chain $(\mathbf{X}(t): t \geq 0)$ with intensity kernel q on \mathcal{M}_0 given by $q(\mathbf{0}, \cdot) := 0$ and

$$q(\mathbf{x}, d\mathbf{y}) := \sum_{i \geq 1} \nu_{x_i}(\mathbf{y} + \delta_{x_i} - \mathbf{x} \in ds), \quad \mathbf{x} \neq \mathbf{0}.$$

As particles evolve independently, we think of $q_{\mathbf{x}} := q(\mathbf{x}, \mathcal{M}_0) = \sum_{i \geq 1} \nu_{x_i}(\mathcal{S}_{x_i})$ as the rate of first fragmentation from the configuration \mathbf{x} ; the next configuration \mathbf{y} having law $q(\mathbf{x}, d\mathbf{y})/q_{\mathbf{x}}$. Of course it really makes sense only if $q_{\mathbf{x}} < \infty$, but this ‘‘hold-jump’’ description can be made rigorous (using a truncation argument) with a small assumption [21, Chapter 1]: that for every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that

$$\nu_x(\mathcal{S}_x) < c_\varepsilon \quad \text{and} \quad \int \mathbf{s}((\varepsilon, \infty)) \nu_x(ds) < c_\varepsilon \nu_x(\mathcal{S}_x), \quad x > \varepsilon.$$

It is then checked that \mathbf{X} is a pure-fragmentation process. In particular, if ν_x , $x > 0$, is the image measure of $x^\alpha \nu$ by the map $\mathbf{s} \mapsto x \cdot \mathbf{s}$, where $\alpha \in \mathbb{R}$ and ν is any finite measure on \mathcal{S} with $\int \mathbf{s}((\varepsilon, \infty)) \nu(ds) < \infty$ for every $\varepsilon > 0$, then the above condition holds, and so does the self-similarity property of Definition 0.1.1; we say that \mathbf{X} is the (α, ν) -*self-similar fragmentation chain* (or simply the ν -*homogeneous fragmentation chain* when $\alpha = 0$).

0.1.2.2 Self-similar pure fragmentations

Naturally, a *self-similar pure-fragmentation process* is a self-similar branching process that is also a pure fragmentation. One example is the (α, ν) -self-similar fragmentation chain of the previous section, where the measure ν was such that $\nu(\mathcal{S}) < \infty$ and $\int \mathbf{s}((\varepsilon, \infty)) \nu(ds) < \infty$ for every $\varepsilon > 0$. In fact, we can also make sense of a self-similar fragmentation process with

infinite dislocation rates. Suppose instead that ν has support in \mathcal{P} , that $\nu(\{\mathbf{1}\}) = 0$ and

$$\int_{\mathcal{P}} (1 - p_1) \nu(d\mathbf{p}) < \infty. \quad (0.2)$$

While allowing $\nu(\mathcal{P}) = \infty$, condition (0.2) limits the intensity of “macroscopic” dislocations (those given by the mass-partitions which are far from $\mathbf{1}$). More precisely, if a measurable set $A \subset \mathcal{M}_0$ is at distance $d(\mathbf{1}, A) > 0$ from $\mathbf{1}$, then $\nu(A) \leq d(\mathbf{1}, A)^{-1} \int (1 - p_1) \nu(d\mathbf{p}) < \infty$.

A fundamental result [22, 17], is that the law $\mathbf{P} := \mathbf{P}_1$ of a self-similar \mathcal{P} -valued fragmentation process is uniquely determined by its self-similarity index $\alpha \in \mathbb{R}$, its erosion coefficient $\mathfrak{c} \in [0, \infty)$, and its so called *dislocation measure* ν , which is a measure on \mathcal{P} satisfying to $\nu(\{\mathbf{1}\}) = 0$ and (0.2); we shall hence refer to the $(\alpha, \mathfrak{c}, \nu)$ -self-similar fragmentation process (or rather the (\mathfrak{c}, ν) -homogeneous fragmentation process when $\alpha = 0$). Considering a Poisson point process \mathbf{M} on $\mathcal{P} \times \mathbb{N}$ with intensity $\nu \otimes \#$, where $\#$ is the counting measure on \mathbb{N} , the $(0, \nu)$ -homogeneous fragmentation process \mathbf{X} can be constructed from \mathbf{M} in such a way that \mathbf{X} only jumps when some atom (\mathbf{p}, k) of \mathbf{M} occurs, and if it happens at time t , then $\mathbf{X}(t)$ is obtained from $\mathbf{X}(t-)$ by replacing its k^{th} largest particle $X_k(t-)$ by the cloud of particles $X_k(t-) \cdot \mathbf{p}$, leaving the other particles unchanged. Erosion simply corresponds to a continuous decay in the fragment masses, to the extent that $(\exp(-ct)\mathbf{X}(t) : t \geq 0)$ is a version of the (\mathfrak{c}, ν) -homogeneous fragmentation process. There also exists some procedure to change the index of self-similarity, transforming the (\mathfrak{c}, ν) -homogeneous fragmentation process into the $(\alpha, \mathfrak{c}, \nu)$ -self-similar fragmentation. Explaining this transformation as well as the bijection between \mathbf{P} and $(\alpha, \mathfrak{c}, \nu)$ requires that we enrich the fragmentation with a genealogical structure. One way to achieve this is via so called *interval representations*: one can couple any self-similar pure fragmentation \mathbf{X} with some process G onto the usual topology of the open interval $(0, 1)$ such that for all $s \leq t$, $G(t) \subseteq G(s)$ and $\mathbf{X}(t)$ coincides with the non-increasing rearrangements of the lengths of the interval components of $G(t)$. We refer to [22, Section 3.2] for greater detail.

Example 0.1.4. Let $\alpha \in \mathbb{R}$ and ν be the law of $(U, 1 - U, 0, \dots)$, where U is uniformly sampled on $[\frac{1}{2}, 1)$. Then the $(\alpha, 0, \nu)$ -self-similar pure-fragmentation process matches the dividing sandpile process with uniform binary splitting that started this introduction.

0.1.3 Growth-fragmentation processes

Allowing particles to vary continuously between fragmentation events raises intricate questions as well as important applications. The equilibrium between growth and fragmentation has first been studied “deterministically” by analysts. The first stochastic models were introduced recently by Bertoin [18, 19].

0.1.3.1 Cell systems

We start by constructing a rather general system of non-interacting particles, called *cell system* in [19]. Our approach is a bit more general and bears similarities with [29, Section 4.2]. Informally, a cell system consists in a family $\mathcal{X} := \{(\mathcal{X}_u(t - b_u) : b_u \leq t) : u \in \mathcal{U}\}$ of processes on $(0, \infty)$, recorded since their birth times b_u (which we implicitly encode in the notation \mathcal{X}_u) and whose negative jumps produce birth events. These processes are assumed to be càdlàg and either converging to or absorbed at 0, so that their negative jumps can be easily enumerated. In [19], the j -th negative jump (ordered by decreasing absolute size) of \mathcal{X}_u , occurring say at time $t \geq 0$ and with size $-y < 0$, is the cause of a single daughter cell \mathcal{X}_{uj} born at $b_{uj} := t$ with $\mathcal{X}_{uj}(0) := y$. There, processes were indexed by the usual Harris–Ulam tree $\mathcal{U} := \bigcup_{n \geq 0} \mathbb{N}^n$, where $\mathbb{N}^0 := \{\emptyset\}$ is reduced to the root of \mathcal{U} , which labels the Eve cell \mathcal{X}_\emptyset .

We can slightly generalize the preceding idea and imagine that the cell material $y > 0$ lost by \mathcal{X}_u at time t during its j -th negative jump serves to the creation of *several* (zero, one, or more) daughter cells $\mathcal{X}_{u,(j,1)}, \mathcal{X}_{u,(j,2)}, \text{ etc.}$, all with birth times $b_{u,(j,k)} := t$ and respective sizes at birth $y_{s_1}, y_{s_2}, \text{ etc.}$, for some random configuration $\mathbf{s} \in \mathcal{S}$ whose law may depend on both $\mathcal{X}_u(t-)$ and y . Thus, we rather choose the indexing set as

$$\mathcal{U} := \bigcup_{n \geq 0} (\mathbb{N}^2)^n = \bigcup_{n \geq 0} \mathbb{N}^{2n}.$$

In this slightly unconventional Harris–Ulam tree, each node $u := (u_1, \dots, u_{2n}) \in \mathcal{U}$ at a given generation $|u| := n$ has children $ujk := (u_1, \dots, u_{2n}, j, k)$, $j, k \geq 1$, labelling at the next generation all daughter cells $\mathcal{X}_{ujk} := \mathcal{X}_{u,(j,k)}$ originating from the j^{th} negative jump of \mathcal{X}_u , respectively, where these jumps are enumerated in the non-increasing order of their absolute sizes, and chronologically in case of *ex aequo*. More precisely, if $(t_1, -y_1), (t_2, -y_2), \dots$ denotes this enumeration of negative jump times and sizes, then either $y_j > y_{j+1}$ or $y_j = y_{j+1}$ and $t_j < t_{j+1}$, for all j . The children labelled ujk , $k \geq 1$, corresponding to the jump $(t_j, -y_j)$, depend on a certain configuration \mathbf{s}^{uj} prescribing the relative sizes at birth, namely $(\mathcal{X}_{ujk}(0) : k \geq 1) = y_j \cdot \mathbf{s}^{uj}$ (when $\mathbf{s}^{uj} = \mathbf{0}$, no particle is born). We set $b_\emptyset := 0$, $b_{ujk} := b_u + t_j$, and agree with the conventions $b_{ujk} := \infty$ and $\mathcal{X}_{ujk} \equiv 0$ if $p_k^{uj} = 0$ or \mathcal{X}_u has fewer than j negative jumps. [Figure 0.3](#) sets up the notation.

The law \mathcal{P}_x of \mathcal{X} started from a single particle \mathcal{X}_\emptyset with initial size $x \geq 0$ is constructed to ensure the *genealogical branching property*: for all $u \in \mathcal{U}$ and $i \geq 1$, conditionally on $\sigma((\mathcal{X}_u, \mathbf{s}^{uj}) : |u| \leq i, j \geq 1)$, the cell systems $\mathcal{X}^{ujk} := \{\mathcal{X}_{ujkv} : v \in \mathcal{U}\}$, $|u| = i, j, k \geq 1$, are independent; further, each \mathcal{X}^{ujk} has law $\mathcal{P}_{y_{s_k}^{uj}}$, where y is the j^{th} negative jump of \mathcal{X}_u (according to the above enumeration).

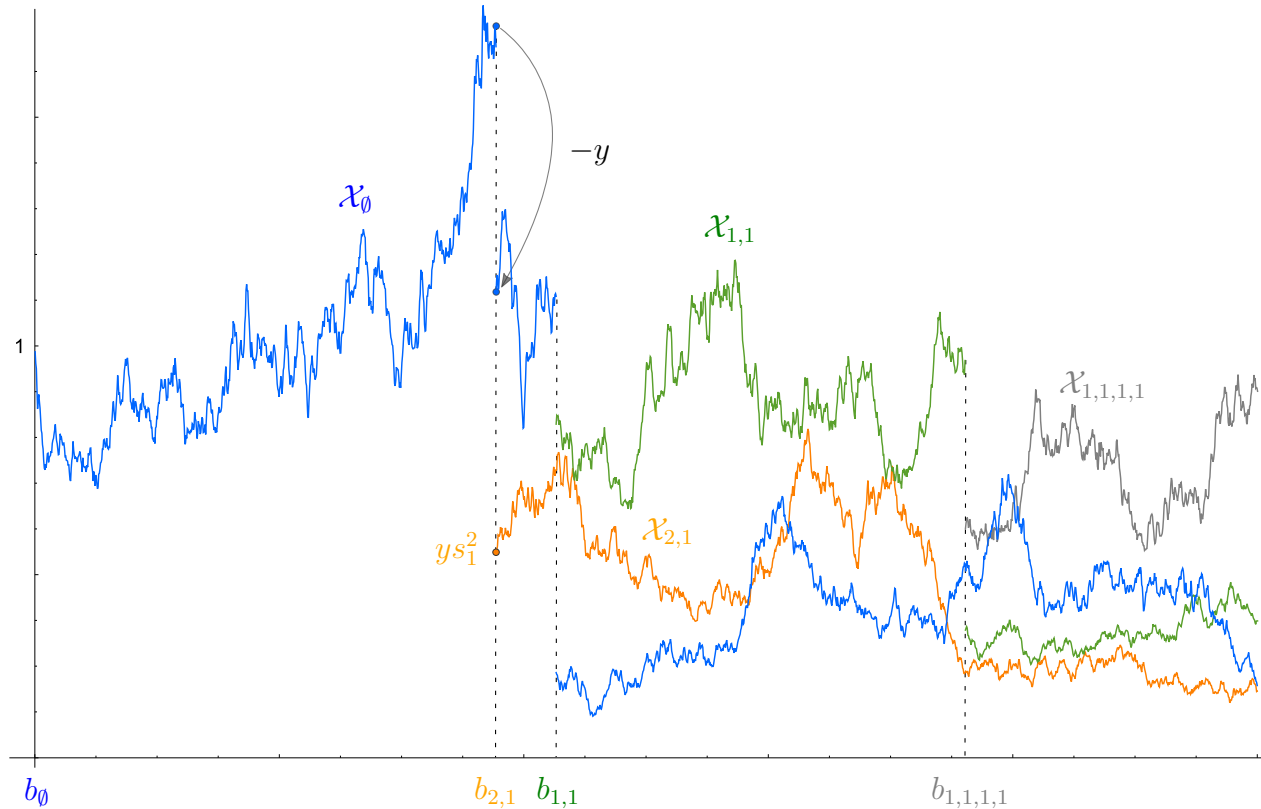


Figure 0.3. Illustrating example of cell system.

If \mathcal{X}_u has its j^{th} negative jump $-y < 0$ at time t , then independent daughter-cells \mathcal{X}_{ujk} , $k \geq 1$, with respective sizes at birth ys_k^{uj} , $k \geq 1$, are born at $b_{ujk} := t$, where \mathbf{s}^{uj} is a $\varrho(\mathcal{X}_u(t-), y/\mathcal{X}_u(t-), \cdot)$ -distributed configuration. (For the picture we chose $\varrho \equiv \delta_1$, so each cell \mathcal{X}_u begets at its j^{th} negative jump a unique child \mathcal{X}_{uj1} , with size at birth equal to the jump size.)

Formally, \mathcal{P}_x depends on a probability kernel $\varrho(x, r, ds)$ from $(0, \infty) \times (0, 1]$ to \mathcal{S} , the *fragmentation kernel*, and on a càdlàg Markov process $(Y, (P_y)_{y \geq 0})$ on $(0, \infty)$, the *cell process*, which, under the initial distribution P_y , $y \geq 0$ ($P_y(Y(0) = y) = 1$ and $P_0(Y \equiv 0) = 1$), is either eventually absorbed at 0 or converging to 0 as $t \rightarrow \infty$. It is defined with the help of Ionescu-Tulcea's theorem as the unique distribution on $[0, \infty)^{\mathcal{U}}$ such that \mathcal{X}_\emptyset has law P_x and, conditionally on \mathcal{X}_\emptyset , the enumeration $(t_1, -y_1), (t_2, -y_2), \dots$ of its negative jumps, and on an independent family $(\mathbf{s}^j : j \geq 1)$ of independent configurations with respective laws $\varrho(\mathcal{X}_\emptyset(t_j-), y_j/\mathcal{X}_\emptyset(t_j-), \cdot)$, the cell systems $\{\mathcal{X}_{jkv} : v \in \mathcal{U}\}$, $j, k \geq 1$, are independent $\mathcal{P}_{y_j s_k^j}$ -distributed variables. We forward the reader to [70] for more rigorous details on the construction of this type of branching processes.

Note that we may as for pure fragmentations give a sense to a dislocation measure ν : if negative jumps $-y < 0$ of Y from a certain size x occur at rate $\theta(x, dy)$, then particles with size x fragmentate at rate $\theta(x, dy)\varrho(x, y/x, ds)$ into a cloud of particles with initial sizes

$x - y, ys_1, ys_2, ys_3 \dots$; see the connection in [Example 0.1.6](#). We also underline the “binary case” $x \mapsto \{x - y, y\}$ (for $\varrho \equiv \delta_1$), which reduces the construction presented here to that treated in [\[19\]](#).

We call \mathcal{X} the *cell system driven by* (Y, ϱ) , but we are in fact more interested in the point process of alive cells:

$$\mathbf{X}(t) := \sum_{u \in \mathcal{U}} \mathbb{1}_{\{b_u \leq t\}} \delta_{\mathcal{X}_u(t-b_u)}, \quad t \geq 0.$$

When $(\mathbf{X}(t) : t \geq 0)$ is a branching process (in the sense of [Definition 0.1.1](#)), it is called the *growth-fragmentation process driven by* (Y, ϱ) . Note that in general, \mathbf{X} might very well not be \mathcal{M}_0 -valued, nor might it enjoy the *temporal* branching property. However, the existence of some excessive function is sufficient to enforce this. In this direction, write $\Delta^- y(s) := \max(y(s-) - y(s), 0)$ for $s > 0$, $y : [0, \infty) \rightarrow \mathbb{R}$ and suppose that there exist $C > 0$ and $f \in \mathcal{B}_{0<}$ such that for all $x > 0$ and $t \geq 0$,

$$E_x \left[C^{-t} f(Y(t)) + \sum_{0 < r \leq t} \int_{\mathcal{S}} \sum_{i \geq 1} C^{-r} f(\Delta^- Y(r) s_i) \varrho \left(Y(r-), \frac{\Delta^- Y(r)}{Y(r-)}, ds \right) \right] \leq f(x). \quad (0.3)$$

Then⁴ \mathbf{X} is a growth-fragmentation process in \mathcal{M}_f . Moreover, f is a C -excessive function for \mathbf{X} , and so $C^{-t} \langle \mathbf{X}(t), f \rangle$, $t \geq 0$, is a supermartingale.

Of course, if \mathbf{X} is a growth-fragmentation driven by (Y, ϱ) where the cell process Y has non-increasing sample paths, then \mathbf{X} corresponds to a pure fragmentation. When further the fragmentation kernel ϱ has support in \mathcal{P} , we can check that [\(0.3\)](#) trivially holds for $C = 1$ and $f = f_1$.

0.1.3.2 Self-similar growth-fragmentations

In this section, we assume that the fragmentation kernel $\varrho(x, r, ds)$ does not depend on x , and by a slight abuse of notation we set $\varrho(x, r, ds) \equiv: \varrho(r, ds)$. A *self-similar growth-fragmentation process* is a growth-fragmentation fulfilling the self-similarity property of [Definition 0.1.1](#). It must be driven by a cell process Y which is itself self-similar (with index α):

$$\text{for all } x > 0, \text{ the law of } (xY(x^\alpha t) : t \geq 0) \text{ under } P_1 \text{ is } P_x.$$

Indeed, by [\[19, Lemma 1\]](#), self-similarity then extends to the cell system \mathcal{X} driven by (Y, ϱ) : for $y > 0$ arbitrary and $b'_u := y^{-\alpha} b_u$, $u \in \mathcal{U}$, $\mathcal{X}'_u(t) := y \mathcal{X}_u(y^\alpha t)$, $u \in \mathcal{U}, t \geq 0$, the law of $((\mathcal{X}'_u, b'_u) : u \in \mathcal{U})$ under \mathcal{P}_1 is the same as the law of $((\mathcal{X}_u, b_u) : u \in \mathcal{U})$ under \mathcal{P}_y .

⁴This criterion is easily adapted from [\[19, Proposition 2 & Theorem 1\]](#) proved for $C = 1$ and $\varrho \equiv \delta_1$. By the Markov property, it is fulfilled if and only if the process $f(\Delta^- Y(r) s_i) \varrho(Y(r-), \frac{\Delta^- Y(r)}{Y(r-)}, ds)$, $t \geq 0$, is a supermartingale under P_x , for every $x > 0$. When Y is a semimartingale, this can be investigated for smooth enough functions f by stochastic calculus.

Lamperti [78] characterized all positive self-similar Markov processes (for short, *pssMp*). Suppose Y is a *pssMp* with index $\alpha \in \mathbb{R}$ which is either absorbed at or converging to 0 a.s. Then $\theta(t) := \inf\{s \geq 0: \int_0^s Y(r)^\alpha dr > t\}$, $t \geq 0$, is increasing for $t < \int_0^\infty Y(s)^\alpha ds =: \zeta$, and equals ∞ if $t \geq \zeta$. Lamperti's transformation says that under P_1 , $\xi(t) := \log Y(\theta(t))$, $t \geq 0$ (with the convention $\xi(t) := -\infty$ for $t \geq \zeta$), is a Lévy process (a càdlàg process with independent and stationary increments), which is either absorbed at or diverging to $-\infty$. Conversely, for such a Lévy process ξ , we can define Y as having under P_y the law of $(y \exp(\xi(\tau_{y^\alpha t})) : t \geq 0)$, where $\tau_t := \inf\{s \geq 0: \int_0^s \exp(-\alpha \xi(r)) dr > t\}$. In other words, a self-similar cell process Y is determined by (α, ξ) , where $\alpha \in \mathbb{R}$ and ξ is a Lévy process which is either absorbed at or diverging to $-\infty$. In turn, ξ is characterized by a quadruple $(\mathbf{k}, b, \sigma^2, \Lambda)$, where $\mathbf{k} \geq 0$ is the killing rate, $b \in \mathbb{R}$ the drift coefficient, $\sigma^2 \geq 0$ the Gaussian component, and Λ is the so called *Lévy measure*, that is a measure on $\mathbb{R} \setminus \{0\}$ with $\int (1 \wedge y^2) \Lambda(dy) < \infty$. Since it describes the intensity of jumps in the cell process, Λ will often be supported on the negative half-line. However, allowing cells to encounter sudden growth can be relevant especially in applications to random planar maps (see [24] or the forthcoming Section 0.3), and it is actually only assumed that $\int \mathbf{1}_{\{y>1\}} e^y \Lambda(dy) < \infty$. Then the distribution of ξ is identified by the *Lévy-Khintchine formula* $E[\exp(q\xi(t))] = \exp(t\Psi(q))$, $t \geq 0$, through a Laplace exponent Ψ of the form

$$\Psi(q) := -\mathbf{k} + bq + \frac{1}{2}\sigma^2 q^2 + \int_{\mathbb{R}} (e^{qy} - 1 + q(1 - e^y)) \Lambda(dy), \quad q \in \mathbb{C}, \quad (0.4)$$

which makes sense at least for $q \in [0, \infty)$. In particular, $\Psi: [0, \infty) \rightarrow (-\infty, \infty]$ is convex.

In view of Lamperti's transformation, we shall in this setting refer to Y and \mathcal{X} as, respectively, the (α, Ψ) -*self-similar cell process* and the (α, Ψ, ϱ) -*self-similar cell system*. When further \mathbf{X} is a branching process (typically, when (0.3) applies), \mathbf{X} is called the (α, Ψ, ϱ) -*self-similar growth-fragmentation* (or rather, if $\alpha = 0$, the (Ψ, ϱ) -*homogeneous growth-fragmentation*).

Remark 0.1.5. Different (Ψ, ϱ) may lead to the same self-similar growth-fragmentation process with index α [99].

Example 0.1.6. Let $c \geq 0$ and ϱ have support in \mathcal{P} . Suppose Λ has support in $[-\log 2, 0)$ with $\int (1 \wedge |x|) \Lambda(dx) < \infty$, and that $-\xi$ is a subordinator, *i.e.*, Ψ has the form

$$\Psi(q) := -\mathbf{k} - cq + \int_{\mathbb{R}} (e^{qx} - 1) \Lambda(dx), \quad q \geq 0 \quad (0.5)$$

(which means that Y is non-increasing and never jumps lower than half its current size). Then the (α, Ψ, ϱ) -self-similar growth-fragmentation is a version of the (α, c, ν) -self-similar

pure fragmentation, where the dislocation measure ν is given by

$$\int f(\mathbf{p}) \nu(d\mathbf{p}) = \mathbf{k}f(\mathbf{0}) + \iint f(e^x, p_1(1 - e^x), p_2(1 - e^x), \dots) \Lambda(dx) \varrho(1 - e^x, d\mathbf{p}), \quad (0.6)$$

for every measurable function $f: \mathcal{P} \rightarrow [0, \infty)$. Conversely, suppose ν is a dislocation measure, that is $\nu(\{\mathbf{1}\}) = 0$ and $\int_{\mathcal{P}} (1 - p_1) \nu(d\mathbf{p}) < \infty$. Then (0.6) holds for $\mathbf{k} = \nu(\{\mathbf{0}\})$, Λ the image measure of ν by the map $\mathbf{p} \mapsto \log p_1$, and, by disintegration⁵ [72, Corollary 1.23], some probability kernel $\varrho(r, d\mathbf{p})$, $r \in (0, 1]$, on \mathcal{P} which we can interpret as the image measure of $\nu(\cdot \mid p_1 = 1 - r)$ by the map $\mathbf{p} \mapsto (\frac{p_j}{1-p_1})_{j \geq 2}$. Further, the $(\alpha, \mathbf{c}, \nu)$ -self-similar pure fragmentation coincides with the (α, Ψ, ϱ) -self-similar growth-fragmentation with Ψ as in (0.5).

This example shows that the greater generality of self-similar growth-fragmentations with respect to self-similar pure fragmentations lies essentially in the fact that ξ need no longer be the negative of a subordinator; it may for instance have a Brownian component and jumps of unbounded variation.

When \mathcal{X} is driven by a general (α, Ψ) -self-similar cell process Y with Laplace exponent Ψ as in (0.4), there is the following specialization of (0.3) to prove excessiveness. Suppose first $\alpha = 0$, so that $Y = xe^\xi$ under P_x . By stochastic calculus (adapting the proof of [19, Lemma 2]) we have, for all $x > 0$ and $t, q \geq 0$,

$$E_x \left[Y(t)^q + \sum_{0 < r \leq t} \int_{\mathcal{J}} \sum_{i \geq 1} (\Delta^- Y(r) s_i)^q \varrho \left(\frac{\Delta^- Y(r)}{Y(r-)}, ds \right) \right] = x^q + x^q \kappa(q) \int_0^t e^{r\Psi(q)} dr,$$

where

$$\kappa(q) := \Psi(q) + \int_{(-\infty, 0)} (1 - e^y)^q \Lambda(dy) \int_{\mathcal{J}} \sum_{i \geq 1} s_i^q \varrho(1 - e^y, ds), \quad q \geq 0. \quad (0.7)$$

Hence (by a supermartingale argument) the inequality (0.3) with $\kappa(q) \leq 0$, $f = f_q$ and $C = 1$ will hold for all $x > 0$ and $t \geq 0$, and (by Doob's optional stopping theorem) even if $\alpha \neq 0$. We can here already see that κ , known as the *cumulant function*, plays a crucial rôle.

Lemma 0.1.7. Suppose $q \geq 0$ is such that $\kappa(q) \leq 0$. Then (0.3) holds for $f = f_q$ and $C = 1$, and the process $\langle \mathbf{X}(t), f_q \rangle$, $t \geq 0$, is a supermartingale under \mathbf{P}_x for every $x > 0$.

As a matter of fact, Bertoin and Stephenson [33] have shown that for $\alpha \neq 0$, the condition of Lemma 0.1.7 is necessary for \mathbf{X} to be \mathcal{M}_0 -valued.

⁵We view ν as a measure on $[-\infty, 0) \times \mathcal{P}$ via the bijective transformation $\mathbf{p} \mapsto (\log p_1, \frac{p_2}{1-p_1}, \frac{p_3}{1-p_1}, \dots)$.

Naturally, the presence of power functions in additive (super)martingales is expected because of the multiplicative structure of self-similar fragmentations. Let us derive here $f(t) := f(t; x, q) := \mathbf{E}_x[\langle \mathbf{X}(t), f_q \rangle]$ in the homogeneous case $\alpha = 0$. Then, the cell process Y under P_x is simply represented as xe^ξ , where ξ is a Lévy process with Laplace exponent Ψ . By the Lévy–Khintchine formula, the infinitesimal generator L of Y fulfills $Lf_q(x) = x^q\Psi(q)$. Now, consider the variation of f from $f(0) = x^q$ after an infinitesimal amount of time dt . On the one hand, the variation due to the growth of the mother particle is $Lf_q(x)dt$ (in the first order). On the other hand, the amount of negative jumps with relative size e^y that have occurred on this time interval is roughly $\Lambda_{|(-\infty, 0)}(dy) dt$, and each begets a random cloud $x(1 - e^y) \cdot \mathbf{p}$ of daughter particles where \mathbf{p} has law $\varrho(1 - e^y, \cdot)$, bringing thus a contribution of $\int \sum_{i \geq 1} (x(1 - e^y)p_i)^q \varrho(1 - e^y, d\mathbf{p})$ to $f(dt) - f(0)$. Putting pieces together, it follows that $\partial_t f(0) = x^q \kappa(q)$ (at least when $\kappa(q) < \infty$). Since the branching property easily entails $f(t+s) = f(t)f(s)$ for all $s, t \geq 0$, we conclude that $f(t; x, q) = e^{t\kappa(q)}x^q$. This means that f_q is $e^{\kappa(q)}$ -invariant for \mathbf{X} , and therefore (by [Proposition 0.1.2](#)) $e^{-t\kappa(q)}\langle \mathbf{X}(t), f_q \rangle$, $t \geq 0$, is a martingale under \mathbf{P}_x for every $x > 0$. We refer to [[18](#), Theorem 1] for a rigorous derivation of this fact.

Example 0.1.8. Let ν be a dislocation measure. With the notation of [Example 0.1.6](#), the cumulant function for the $(0, \nu)$ -homogeneous pure-fragmentation process \mathbf{X} is

$$\begin{aligned} \kappa(q) &= \int_{\mathbb{R}} (e^{qy} - 1) \Lambda(dy) + \int_{(-\infty, 0)} (1 - e^y)^q \Lambda(dy) \int_{\mathcal{P}} \sum_{i \geq 1} p_i^q \varrho(1 - e^y, d\mathbf{p}) \\ &= \iint_{[0, \infty) \times \mathcal{P}} \left(x^q - 1 + (1 - x)^q \sum_{i \geq 2} \left(\frac{p_i}{1 - p_1} \right)^q \right) \nu(p_1 \in dx) \nu(d\mathbf{p} \mid p_1 = x) \\ &= \int_{\mathcal{P}} \left(\sum_{i \geq 1} p_i^q - 1 \right) \nu(d\mathbf{p}). \end{aligned}$$

In particular if ν is conservative, then $\kappa(1) = 0$ (and the martingale $\langle \mathbf{X}, f_1 \rangle$ is trivial). For the example starting this introduction, we have $\nu(p_1 \in dx) = 2\mathbf{1}_{[\frac{1}{2}, 1)}(x) dx$ and $\nu(p_1 + p_2 = 1) = 1$, so $\kappa(q) = \frac{1-q}{1+q}$.

The strictly self-similar case $\alpha \neq 0$ is somewhat different. There, it is generally assumed that *Cramér’s hypothesis* is satisfied [[24](#), Section 3]:

$$\text{There exist } 0 < \omega_- < \omega_+ \text{ such that } \kappa(\omega_-) = \kappa(\omega_+) = 0 \text{ and } \kappa'(\omega_-) > -\infty. \quad (0.8)$$

Under this hypothesis, $\langle \mathbf{X}, f_{\omega_+} \rangle$ is a martingale when $\alpha \leq 0$ [[24](#), Corollary 3.5], and $\langle \mathbf{X}, f_{\omega_-} \rangle$ is a uniformly integrable martingale when $\alpha \geq 0$ [[24](#), Theorem 3.7].

Remark 0.1.9. Additive martingales cannot be uniformly integrable when $\alpha < 0$, because the self-similar fragmentation gets eventually extinct almost surely [60, 22, 19]: $\mathbf{X}(t) = \mathbf{0}$ after a \mathbf{P} -almost surely finite time t .

Remark 0.1.10 (Compensated fragmentation processes; general branching Lévy processes).

- The first construction of homogeneous growth-fragmentations (in [18]) was not made from the point of view of cell systems, but rather directly using projective limits of general branching Lévy processes (see the second point below). More conceptually, one may interpret them as limits of dilated homogeneous pure fragmentations. Suppose indeed that \mathbf{X}_n , $n \in \mathbb{N}$, are (c_n, ν_n) -homogeneous pure-fragmentation processes such that there is the weak convergence of finite measures on \mathcal{P}

$$(1 - p_1)^2 \nu_n(d\mathbf{p}) \Longrightarrow \sigma^2 \delta_{\mathbf{1}}(d\mathbf{p}) + (1 - p_1)^2 \nu(d\mathbf{p}),$$

for some $\sigma^2 \geq 0$ and measure ν on \mathcal{P} fulfilling $\nu(\{\mathbf{1}\}) = 0$ and

$$\int_{\mathcal{P}} (1 - p_1)^2 \nu(d\mathbf{p}) < \infty, \tag{0.9}$$

and suppose further that $c_n - \frac{\sigma^2}{2}$ converges as $n \rightarrow \infty$ to some constant $b \in \mathbb{R}$ (if $c_n \equiv 0$, this is automatically verified with $b = -\frac{\sigma^2}{2}$). Then there exist a sequence $(d_n)_{n \geq 0}$ of nonnegative numbers and a non-trivial process \mathbf{X} such that for every $q > 2$, the convergence in distribution

$$(\exp(d_n t) \mathbf{X}_n(t) : t \geq 0) \xrightarrow[n \rightarrow \infty]{(d)} \mathbf{X},$$

holds in the space of \mathcal{M}_{f_q} -valued càdlàg functions endowed with Skorokhod's J_1 -topology. Moreover, the dilation coefficients may be chosen as $d_n := \int_{\mathcal{P}} (1 - p_1) \nu_n(d\mathbf{p})$; if so, then \mathbf{X} is the *compensated fragmentation process with characteristics* (σ^2, b, ν) . If we disintegrate ν as in (0.6) and define Ψ by (0.4) (with $\mathbf{k} := \nu(\{\mathbf{0}\})$), then the compensated fragmentation process with characteristics (σ^2, b, ν) coincides with the (Ψ, ϱ) -homogeneous growth-fragmentation.

- Condition (0.9) is weaker than the integrability requirement (0.2) for the dislocation measure of a self-similar pure fragmentation. When the latter fails, the too strong accumulation of “microscopic” dislocations would instantaneously shatter the mass into dust, so it must be compensated by a suitable dilation of the fragments. This is of course reminiscent of the construction of Lévy processes as compensated Poisson integrals. The analogy is not coincidental [29]: in the same way that Lévy processes characterize infinitely divisible distributions, processes of the form

$$\sum_{i \geq 1} \delta_{\log X_i(t)}, \quad t \geq 0,$$

where \mathbf{X} is a homogeneous growth-fragmentation, are called *branching Lévy processes* because they identify the random point measures which can be written for any $n \geq 1$ as the n -th generation of some branching random walk. These processes are studied in greater detail in [29, 28].

0.2 Asymptotics of self-similar fragmentations

Here, we summarize the large-time asymptotics of several quantities related to homogeneous, self-similar, pure- and growth-fragmentation processes, such as empirical distributions associated with the fragments and the size of the largest fragment.

0.2.1 Homogeneous fragmentations

The bottom line is that for homogeneous fragmentations, the empirical measure of the fragments exhibits a log-Normal distribution.

Theorem 0.2.1 (Kolmogorov [74]). Let \mathbf{X} be the ν -homogeneous fragmentation chain, with ν a probability measure on $\mathcal{S} \cap \mathcal{M}_{f_0}$. Suppose that $Q(1) > 1$ and $\int_0^1 |\log t|^3 dQ(t) < \infty$, where for $t \in (0, 1]$, $Q(t) := \int \#\{i \in \mathbb{N}: 0 < s_i \leq t\} \nu(ds)$. Then as $t \rightarrow \infty$, the quantity

$$\sup_{x \in \mathbb{R}} \left| \frac{1}{\langle \mathbf{X}(t), f_0 \rangle} \sum_{i \geq 1} \mathbf{1}_{\{\log X_i(t) \leq x\}} - F\left(\frac{x - \mu t}{\sigma \sqrt{t}}\right) \right|$$

converges to 0 in probability, where F is the cumulative distribution function of the standard Normal distribution, $\mu := Q(1)^{-1} \int_0^1 \log t dQ(t)$, and $\sigma^2 := Q(1)^{-1} \int_0^1 (\log t - \mu)^2 dQ(t)$.

Nowadays, Kolmogorov's theorem should be seen as a version of a central limit theorem for branching random walks [9, 10, 40]. Applying additive martingale techniques due to Biggins [39, 41], Bertoin [23] and Bertoin and Rouault [32] then established asymptotics for possibly infinite dislocation measures:

Theorem 0.2.2 (Bertoin [23], [21, Theorem 1.2]). Let ν have support in \mathcal{P} with $\nu(\{\mathbf{1}\}) = 0$ and $\int (1 - p_1) \nu(d\mathbf{p}) < \infty$, and let \mathbf{X} be the $(0, \nu)$ -homogeneous pure-fragmentation process. Suppose that the cumulant function

$$\kappa(q) := \int_{\mathcal{S}} \left(\sum_{i \geq 1} s_i^q - 1 \right) \nu(ds), \quad q \geq 0, \quad (0.10)$$

has a (necessarily unique) zero $\omega \geq 0$ and that, for some $r > 1$,

$$\int_{\mathcal{S}} \left(\sum_{i \geq 1} s_i^\omega \right)^r \nu(ds) < \infty. \quad (0.11)$$

Then as $t \rightarrow \infty$, the empirical measures

$$\sum_{i \geq 1} X_i(t)^\omega \delta_{\frac{1}{t} \log X_i(t)} \quad \text{and} \quad \sum_{i \geq 1} X_i(t)^\omega \delta_{\sqrt{t} \left(\frac{1}{t} \log X_i(t) - \mu \right)} \quad (0.12)$$

converge in $L^1(\mathbf{P})$ to $M_\infty \delta_\mu$ and $M_\infty \cdot \sigma N$ respectively, in the sense of weak convergence of measures, where $\mu := \kappa'(\omega)$, $\sigma^2 := \kappa''(\omega)$, N is a standard Gaussian random variable, and M_∞ is the terminal value of the uniformly integrable martingale $\langle \mathbf{X}, f_\omega \rangle$.

The local central limit theorem for branching random walks (Stone [101], Biggins [41]) specializes to homogeneous pure fragmentations as follows.

Theorem 0.2.3 (Bertoin and Rouault [32]). Let the dislocation measure ν , the homogeneous pure-fragmentation \mathbf{X} , and the cumulant function κ as above, and suppose further that ν is conservative and non-geometric. Then for every Riemann integrable function $f: (0, \infty) \rightarrow \mathbb{R}$ with compact support, there is as $t \rightarrow \infty$ the \mathbf{P} -almost sure convergence of

$$q \mapsto \sqrt{t} e^{-(\kappa(q) - q\kappa'(q))t} \sum_{i \geq 1} f\left(X_i(t) e^{-\kappa'(q)t}\right)$$

toward

$$q \mapsto \frac{M_\infty(q)}{\sqrt{2\pi\kappa''(q)}} \int_0^\infty \frac{f(y)}{y^{q+1}} dy,$$

locally uniformly in $U := \{q \geq 0: |\kappa(q)| < \infty \text{ and } q\kappa'(q) - \kappa(q) < 0\}$, where $M_\infty(q)$ is the terminal value of the uniformly integrable martingale $e^{-t\kappa(q)} \langle \mathbf{X}(t), f_q \rangle$, $t \geq 0$.

Asymptotics for the largest fragment $X_1(t)$ are also tractable:

Proposition 0.2.4 (Bertoin [23]). Assuming further that U has no empty interior, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log X_1(t) = \kappa'(\bar{q})$$

\mathbf{P} -almost surely on the non-extinction event $\{\forall t \geq 0: \mathbf{X}(t) \neq \mathbf{0}\}$, where $\bar{q} := \sup U$.

We show in [Chapter 1](#) that [Theorem 0.2.3](#) and [Proposition 0.2.4](#) also hold when \mathbf{X} is a compensated fragmentation process. Actually, we augment [Proposition 0.2.4](#) by deriving further asymptotic orders for the largest fragment X_1 using results that have since appeared in the literature on branching random walks [[3](#), [100](#)].

Theorem 0.2.5

Let \mathbf{X} be the compensated fragmentation with characteristics (σ^2, b, ν) and cumulant function

$$\kappa(q) := \frac{1}{2}\sigma^2 q^2 + bq + \int_{\mathcal{P}} \left(\sum_{i=1}^{\infty} p_i^q - 1 + q(1 - p_1) \right) \nu(d\mathbf{p}), \quad q \geq 0.$$

Suppose $\kappa(0) > 0$, ν non-geometric and $\nu(p_2 > 0, \sum_{i \geq 1} p_i^q > 1) < \infty$ whenever $\kappa(q) < \infty$ and $q < 1$. Then:

- The conclusion of [Theorem 0.2.3](#) holds.
- If $U := \{q \geq 0: \kappa(q) < \infty \text{ and } q\kappa'(q) - \kappa(q) < 0\}$ has no empty interior, then there exist a constant $C^* > 0$ and a nonnegative random variable D_∞ such that, for every $x > 0$,

$$\lim_{t \rightarrow \infty} \mathbf{P} \left(t^{3/2\bar{q}} e^{-\kappa'(\bar{q})t} X_1(t) \leq x \right) = \mathbf{E} [e^{-C^* D_\infty / x}].$$

Moreover, $D_\infty > 0$ \mathbf{P} -almost surely on the non-extinction event.

0.2.2 Self-similar fragmentations

Recall that a self-similar fragmentation process with negative index gets almost surely extinct. In this section, we summarize asymptotics of self-similar fragmentations when $\alpha > 0$. As before, we are interested in empirical measures of the fragments and in the largest fragment.

Theorem 0.2.6 (Bertoin and Gneden [26]). Let \mathbf{X} be the (α, ν) -self-similar fragmentation chain with index $\alpha > 0$ and non-geometric dislocation measure ν on \mathcal{S} with $\nu(\mathcal{S}) = 1$ and $\nu(\{\mathbf{0}\}) = 0$. Suppose that

$$\kappa(q) := \int_{\mathcal{S}} \left(\sum_{i \geq 1} s_i^q - 1 \right) \nu(ds), \quad q \in \mathbb{C},$$

has a positive root $\omega \in (0, \infty)$, and define the random finite measures

$$\rho_t := \sum_{i \geq 1} X_i(t)^\omega \delta_{t^{1/\alpha} X_i(t)}, \quad t \geq 0.$$

(i) If $\mu := \kappa'(\omega) < \infty$, then as $t \rightarrow \infty$, ρ_t converges in \mathbf{P} -mean to a probability measure ρ_∞ , in the sense of weak convergence of measures. Setting $\phi(q) := -\kappa(q)$, the limit ρ_∞ is uniquely determined by the moments

$$\int_0^\infty y^{k\alpha} \rho_\infty(dy) = \frac{(k-1)!}{\alpha \mu \phi(\omega + \alpha) \cdots \phi(\omega + \alpha(k-1))}, \quad k \geq 1. \quad (0.13)$$

(ii) If $q := \inf\{q \in \mathbb{R} : \kappa(q) < \infty\} < \omega$ and (0.11) holds for $r = 2$, then for every $\varepsilon > 0$ and every measurable function $f : (0, 1] \rightarrow \mathbb{R}$ dominated by $f_{q-\omega+\varepsilon}$, there is as $t \rightarrow \infty$ the convergence in $L^2(\mathbf{P})$ of $\langle \rho_t, f \rangle$ toward $M_\infty \langle \rho_\infty, f \rangle$, where M_∞ is the terminal value of the uniformly integrable martingale $\langle \mathbf{X}, f_\omega \rangle$.

Example 0.2.7. Suppose $\nu(p_1 \in dx) = 2\mathbf{1}_{[\frac{1}{2}, 1)}(x) dx$ and $\nu(p_1 + p_2 = 1) = 1$, so $\kappa(q) = \frac{1-q}{1+q}$ with $q = -1 < 1 = \omega$, $\mu = \frac{1}{2}$, and ρ_∞ is the law of $G^{1/\alpha}$ where G is Gamma($2/\alpha$)-distributed. Consequently, the average density $u_t(dx) := \mathbf{E}[\langle \mathbf{X}(t), f_0 \rangle]^{-1} \mathbf{E}[\mathbf{X}(t)(dx)]$ of particles with size x at a large time t approximately follows the generalized Gamma($t^{-1/\alpha}, 1, \alpha$)-distribution $E[G^{-1/\alpha}]^{-1} E[G^{-1/\alpha}, (G/t)^{1/\alpha} \in dx]$. This is consistent with the observations of Filippov [60]; see also Brennan and Durrett [46].

A similar statement is valid for infinite dislocation measures:

Theorem 0.2.8 (Bertoin [23], [21, Theorem 1.3]). Let \mathbf{X} be the $(\alpha, 0, \nu)$ -self-similar pure fragmentation with index $\alpha > 0$ and dislocation measure ν that is not geometric (*i.e.*, $\mathbf{X}(t)$ is not supported on a set of the form $\{se^{-kr} : k \geq 1\}$, for any $r, s > 0$). Suppose that the cumulant function (0.10) has a zero $\omega \geq 0$, that (0.11) holds for some $r > 1$, and that $\mu := \kappa'(\omega) < \infty$. Then as $t \rightarrow \infty$ (with the same notations as above), ρ_t converges in $L^1(\mathbf{P})$ toward $M_\infty \rho$, in the sense of weak convergence of measures.

Unlike in the homogeneous case, the asymptotic velocity of the largest fragment is not of exponential order:

Proposition 0.2.9 (Bertoin [23]). Under the assumptions of Theorem 0.2.8,

$$\lim_{t \rightarrow \infty} \frac{1}{\log t} \log X_1(t) = -\frac{1}{\alpha}$$

in \mathbf{P} -probability, conditionally on non-extinction.

In Chapter 1, we extend these last two results to self-similar growth-fragmentations, under Cramér's hypothesis (0.8). There, the rôle of ω is played by ω_- , and ρ_∞ is not given by (0.13), but rather in terms of the exponential functional of a certain Lévy process.

Theorem 0.2.10

Let \mathbf{X} be the binary (α, Ψ, δ_1) -self-similar growth-fragmentation process with cumulant function

$$\kappa(q) := \Psi(q) + \int_{(-\infty, 0)} (1 - e^y)^q \Lambda(dy), \quad q \geq 0.$$

Suppose that Cramér's hypothesis (0.8) holds. Then for every $0 \leq q < (\omega_+ - \omega_-)/\alpha$, every $1 < p < \omega_+/(\omega_- + q\alpha)$, and every measurable function $f: (0, \infty) \rightarrow \mathbb{R}$ dominated by $f_{q\alpha}$,

$$\lim_{t \rightarrow \infty} \sum_{i \geq 1} X_i(t)^{\omega_-} f(t^{1/\alpha} X_i(t)) = M_\infty \int_0^\infty f(y) \rho_\infty(dy), \quad \text{in } L^p(\mathbf{P}),$$

with M_∞ the terminal value of the uniformly integrable *intrinsic* martingale $\langle \mathbf{X}, f_{\omega_-} \rangle$, and where ρ_∞ is defined in terms of the exponential functional $I := \int_0^\infty \exp(\alpha\eta(t)) dt$ for the Lévy process η with characteristic exponent $\kappa(\cdot + \omega_-)$, by

$$\rho_\infty(dy) := -\frac{1}{\alpha\kappa'(\omega_-)} E[I^{-1}; I^{1/\alpha} \in dy].$$

We might expect that growth have some influence on the speed of decay for the largest fragment X_1 . At least in the first order, this is not the case:

Theorem 0.2.11

Under the assumptions and notations of [Theorem 0.2.10](#), suppose further that $\Lambda((0, \infty)) = 0$. Then

$$\lim_{t \rightarrow \infty} \frac{1}{\log t} \log X_1(t) = -\frac{1}{\alpha}$$

in \mathbf{P} -probability, conditionally on non-extinction.

To conclude, we mention another asymptotic result for an empirical measure of particles which are “frozen” once they fall below some vanishing threshold. Specifically, we can go back over the construction of \mathbf{X} , with the difference that when a particles reaches the interval $(0, \varepsilon]$ (which may happen at birth), it is stopped and thus no longer grows, splits, or produces children. We denote by $\{\chi_{i,\varepsilon}\}_{i=1}^\infty$ the state of the system once all particles have been frozen below ε . Note that it does not depend on the index α , since self-similarity only affects the time when particles get frozen.

Theorem 0.2.12 (Bertoin and Martínez [30], Bertoin [21, Proposition 1.12]). Adopting either the notations and assumptions of [Theorem 0.2.6](#), or those of [Theorem 0.2.8](#), and supposing further that $\omega > \inf\{q \in \mathbb{R}: \kappa(q) < \infty\}$ and (0.11) for some $r > 1$, the random

point measures

$$\sum_{i=1}^{\infty} \chi_{i,\varepsilon}^{\omega} \delta_{\frac{1}{\varepsilon} \chi_{i,\varepsilon}}, \quad \varepsilon > 0,$$

converge in $L^1(\mathbf{P})$ as $\varepsilon \rightarrow 0$ toward $M_{\infty} \varphi$, in the sense of weak convergence of measures, where φ is a deterministic probability measure on $[0, 1]$ given by

$$\varphi(da) := \left(\int_{\mathcal{S}} \sum_{i \geq 1} \mathbf{1}_{\{s_i < a\}} s_i^{\omega} \nu(ds) \right) \frac{da}{a \kappa'(\omega)}.$$

We should also cite Harris, Knobloch and Kyprianou [67] who completed this result by establishing an almost sure convergence. A statement analogous to [Theorem 0.2.12](#) is proved in [Chapter 1](#) for self-similar growth-fragmentations.

Theorem 0.2.13

In the setting of [Theorem 0.2.10](#), suppose further that $\Lambda((0, \infty)) = 0$ and η is not arithmetic, and let $\{\chi_{i,\varepsilon}\}_{i=1}^{\infty}$ be the final state of the growth-fragmentation when particles are frozen below $\varepsilon > 0$. Then as $\varepsilon \rightarrow 0$, the random point measure

$$\sum_{i=1}^{\infty} \chi_{i,\varepsilon}^{\omega} \delta_{\frac{1}{\varepsilon} \chi_{i,\varepsilon}}$$

converges in \mathbf{P} -probability to $M_{\infty} \varphi$, where φ is a deterministic probability measure on $[0, 1]$ given by

$$\langle \varphi, f \rangle := \frac{\omega_+ - \omega_-}{-\kappa'(\omega_-)} \iint_{(-\infty, 0)^2} f(e^x) e^{(\omega_+ - \omega_-)y} \Lambda^-((-\infty, x + y)) dx dy,$$

for $f: [0, 1] \rightarrow [0, \infty)$ measurable and Λ^- the jump measure of the Lévy process η .

0.3 Self-similar fragmentations as scaling limits

Self-similarity stipulates a *scale invariance* property in time and space. Self-similar processes thus naturally “attract” limits of rescaled dynamics. Doubtless the most classical example is that of Brownian motion, which as proved by Donsker [53] is the continuous limit of rescaled random walks with finite variance. A few years later, Lamperti [78] fully characterized all real Markov processes that arise as weak limits of suitably normalized processes (so called *scaling limits*). Among these, and besides Brownian motion, are notably stable processes, stable Lévy processes, Bessel processes, stable Lévy processes conditioned to stay positive, etc. For more recent results on self-similar Markov processes, see the survey [92].

Haas, Miermont, and al. [86, 65, 87, 66, 63] have investigated scaling limits in (pure) fragmentation theory. Mainly, they derive general conditions under which discrete branching mechanisms converge (after suitable renormalization) to self-similar pure fragmentations. This convergence is stated within the realm of *random trees*, in terms of natural genealogies associated with the fragmentation processes, where fragments are respectively linked to the particle they originate from by an edge whose length is proportional to their lifetime.

To fix ideas, imagine a discrete fragmentation as pictured in Figure 0.4a with the following mechanism. Each cell, say with integer size k , may independently of the other cells divide into daughter cells with respective sizes $k_1 \geq \dots \geq k_r \geq 1$ forming a random integer-partition⁶ of k , which we can write $k \cdot \mathbf{p}$ for a certain mass-partition $\mathbf{p} \in \mathcal{P}$ whose law we denote \bar{q}_k . To ensure finiteness of trees, we suppose that $\bar{q}_k(\mathbf{1}) < 1$ for every $k \geq 1$, so cells repeatedly divide (conservatively) into eventually two or more daughter cells until only cells with size 1 remain, which are then stopped at a geometric rate. Let $\mathcal{T}^{(n)}$ stand for the random tree with n leaves obtained when the initial cell has size n . We ask whether $\mathcal{T}^{(n)}$ admits a non-trivial limit of the form $\mathcal{T}^{(n)}/a_n \rightarrow \mathcal{T}$ for some scaling sequence $(a_n)_{n \geq 0}$ of positive numbers, where by $\mathcal{T}^{(n)}/a_n$ we mean that the branch lengths of $\mathcal{T}^{(n)}$ are divided by a_n . Note that self-similarity for the limit \mathcal{T} implies that $(a_n)_{n \geq 0}$ must be *regularly-varying* with some index $-\alpha \in \mathbb{R}$, *i.e.*, we assume that $a_{\lfloor nx \rfloor}/a_n$ tends to $x^{-\alpha}$ as $n \rightarrow \infty$, for every $x > 0$.

Theorem 0.3.1 (Haas and Miermont [63]). Suppose $\alpha < 0$ and “macroscopic” fragmentation events are rare, in the sense that there is as $n \rightarrow \infty$ the weak convergence of finite measures

$$a_n(1 - p_1) \bar{q}_n(d\mathbf{p}) \Longrightarrow (1 - p_1) \nu(d\mathbf{p}),$$

for some dislocation measure ν on \mathcal{P} with $\nu(\{\mathbf{1}\}) = 0$ and $\int_{\mathcal{P}} (1 - p_1) \nu(d\mathbf{p}) < \infty$. Then $\mathcal{T}^{(n)}/a_n$ converges in distribution as $t \rightarrow \infty$ toward the genealogical tree \mathcal{T} associated with the (α, ν) -self-similar pure-fragmentation.

The *self-similar fragmentation tree* \mathcal{T} at the limit belongs to the class of *continuum random trees* as first introduced by Aldous [5, 4, 6]. Technically, $\mathcal{T}^{(n)}$ and \mathcal{T} are viewed as a compact metric space which we can basically embed in $\ell^1(\mathbb{N})$ by gluing together closed segments with variable lengths, and the above convergence holds in the Gromov–Hausdorff topology [59].

In Chapter 2, we consider Markov branching trees which incorporate growth, so that cells may not only split but also grow. For simplicity, we focus on binary fragmentations as illustrated in Figure 0.4b. The Markovian structure allows us to encode such a tree by a probability kernel $(p_{n,m})_{n,m \geq 0}$ on \mathbb{N} with the following interpretation: with probability $p_{n,m}$,

⁶more precisely, $k_1 + \dots + k_r = k$ except possibly for $k = 1$, where $r = 0$ means that the cell is killed.

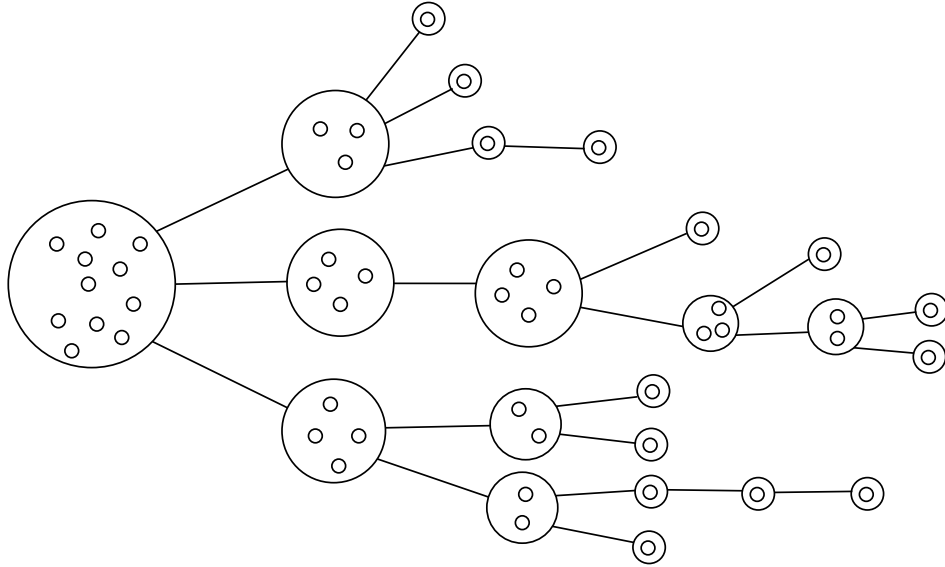


Figure 0.4a. A sample from $\mathcal{T}^{(11)}$ (illustration from [63]).

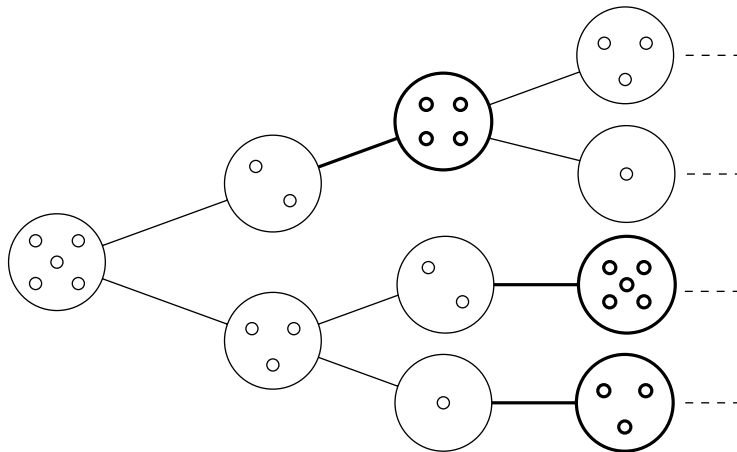


Figure 0.4b. A discrete growth-fragmentation tree (growth in bold).

each cell with size n grows either larger ($m > n$), or smaller ($n/2 \leq m < n$), and in the latter case an independent daughter cell with initial size $n - m$ is born. Our purpose is to find conditions on $p_{n,m}$ for $n \rightarrow \infty$ such that the tree constructed in this way looks (after rescaling) like the genealogical tree corresponding to a self-similar growth-fragmentation process. This will again be a continuum random tree; we refer to Rembart and Winkel [98] for a description using a contraction method.

One major difference induced by growth is that the system may be subject to explosion. To prevent this, we require that cells cease to evolve when they become too small, which we enforce mathematically by setting $p_{n,n} := 1$ for $n \leq M$ and some constant threshold M that

will be fixed accordingly⁷. We let $\mathcal{X}^{(n)}$ denote the resulting tree rooted at a cell with size n and $\mathbf{X}^{(n)}(k)$ the non-increasing rearrangement of the cells' sizes at height $k \geq 0$ in $\mathcal{X}^{(n)}$, and we consider as before a regularly-varying sequence $(a_n)_{n \geq 0}$ with index $-\alpha$.

Theorem 0.3.2

Let $\alpha < 0$, and suppose there exist $q > -\alpha$ and a Lévy process diverging to $-\infty$, whose characteristic exponent and Lévy measure we respectively denote Ψ and Λ , such that

$$\forall t \in \mathbb{R}, \quad \lim_{n \rightarrow \infty} a_n \sum_{m=1}^{\infty} p_{n,m} \left[\left(\frac{m}{n} \right)^{it} - 1 \right] = \Psi(it),$$

$$\limsup_{n \rightarrow \infty} a_n \sum_{m=2n}^{\infty} p_{n,m} \left(\frac{m}{n} \right)^q < \infty,$$

$$\Psi(q) + \int_{(-\infty, 0)} (1 - e^y)^q \Lambda(dy) < 0,$$

and for some $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} a_n \sum_{m=1}^{n-1} p_{n,m} \left(1 - \frac{m}{n} \right)^{q-\varepsilon} = \int_{(-\infty, 0)} (1 - e^y)^{q-\varepsilon} \Lambda(dy).$$

Then we can fix M sufficiently large so that as $n \rightarrow \infty$, the process

$$\left(\frac{1}{n} \mathbf{X}^{(n)}(\lfloor a_n t \rfloor) : t \geq 0 \right)$$

converges in distribution to the binary (α, Ψ, δ_1) -self-similar growth-fragmentation \mathbf{Y} , in the sense of \mathcal{M}_0 -valued càdlàg processes. Furthermore, there is the convergence in distribution $\mathcal{X}^{(n)}/a_n \rightarrow \mathcal{Y}$ for the corresponding random trees seen as compact metric spaces.

One motivation for [Theorem 0.3.2](#) stems from the evidence that growth-fragmentations arise quite naturally in statistics of random planar objects, for instance Boltzmann triangulations [[25](#), [24](#)]. Consider a random triangulation of the n -gon which, given the number of internal vertices, is picked uniformly at random. Revealing (*i.e.*, conditioning on) the faces at distance at most $r \geq 1$ from the boundary, the unexplored areas are independent Boltzmann triangulations; we let $\mathbf{X}^{(n)}(r)$ denote the family of the lengths of their boundaries ranked in the non-increasing order — see [Figure 0.5](#). It is proved in [[25](#)] that $(\mathbf{X}^{(n)}(r))_{r \geq 1}$ admits as $n \rightarrow \infty$ a scaling limit toward a self-similar growth-fragmentation process.

⁷We stress that the process depends on the threshold M , even though this dependency does not appear in the subsequent notation.

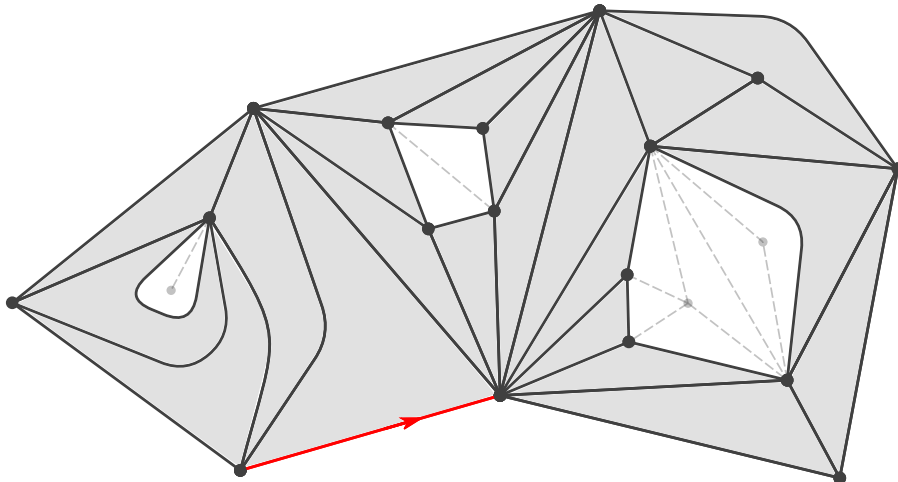


Figure 0.5. The ball of radius 2 in this random Boltzmann triangulation of the 7-gon displays three holes (in white) with perimeters 5, 4, and 1: $\mathbf{X}^{(7)}(2) = (5, 4, 1, 0, \dots)$.

0.4 Fragmentation equations

In this section, and in [Chapter 3](#), we consider a simpler model of growth-fragmentation where, essentially, particles grow deterministically and thus randomness only occurs in the fragmentation process. We further drop the assumption of self-similarity, so the rate at which a particle grows or splits may depend on its size in a fairly general manner.

Specifically, suppose there is an average density of $u(t, x)$ particles around size x and time t . If particles with size x grow at rate $\tau(x)$ and fragmentate at rate $B(x)$ in a conservative way, which means that fragments with size $y < x$ occur at rate $b(x, y)$ such that $\int y b(x, y) dy = xB(x)$, then u solves the (linear) *growth-fragmentation equation*

$$\frac{\partial}{\partial t} u(t, x) + \frac{\partial}{\partial x} (\tau(x)u(t, x)) + B(x)u(t, x) = \int_x^\infty b(y, x)u(t, y) dy. \quad (0.14)$$

This equation is often subject to initial and boundary conditions, such as $u(0, x) = u_0(x)$ and $u(t, 0) \equiv 0$. When the measure of particles is not necessarily given by a density (through the expression $\mu_t(dx) = u(t, x)dx$), one may instead of (0.14) analyze the weak formulation

$$\partial_t \langle \mu_t, f \rangle = \langle \mu_t, \mathcal{A}f \rangle, \quad (0.15)$$

for appropriate test-functions $f: (0, \infty) \rightarrow \mathbb{R}$, where the operator \mathcal{A} is defined by

$$\mathcal{A}f(x) := \tau(x)f'(x) + \int_0^x f(y) b(x, y) dy - B(x)f(x).$$

We shall call $(\mu_t)_{t \geq 0}$ a solution if (0.15) holds for all $t \geq 0$ and f in the domain $D(\mathcal{A})$ of \mathcal{A} .

Self-similarity, mirroring the random self-similar growth-fragmentations of the preceding sections, is achieved here by setting $\tau(x) := x^\alpha c(x)$, $B(x) := x^\alpha$, and $b(x, y) := x^{\alpha-1} \theta(\frac{y}{x})$. In particular, the self-similar fragmentation model with uniform binary splitting (Eq. (0.1)) corresponds to $c \equiv 0$ and $\theta \equiv 2$. More generally, the self-similar pure-fragmentation equation with possibly erosion ($c(x) := cx$ with $c \leq 0$) has been studied by Haas [62]. Bertoin and Watson [35], and Doumic and Escobedo [55] considered the self-similar growth-fragmentation equation with linear growth ($c(x) := cx$ with $c > 0$).

Under fairly general regularity assumptions on τ , b , and B (see for instance [95] or [96, Theorem 4.3]), the growth-fragmentation equation admits a unique solution (in a sense to be specified), and one sometimes may ascertain the time asymptotic behavior. Namely, the global exponential rate and the asymptotic profile of the solution appertain to an eigenvalue problem for the operator \mathcal{A} [54], and under the existence of a spectral gap the rescaled solution converges exponentially fast toward the asymptotic profile [94, 79, 47, 13]. The literature we just cited covers these questions via analytic techniques, *e.g.* entropy methods [85] or splitting of operators [88].

A deeper probabilistic treatment has however been developed recently by Bertoin and Watson [34, 36, 20]. In few words, they connect the eigenproblem to the ergodicity of a certain Markov process $(X, (\mathbb{P}_x)_{x>0})$ whose infinitesimal generator \mathcal{G} is similar to \mathcal{A} :

$$\mathcal{G}f(x) := \frac{1}{x} \mathcal{A}\bar{f}(x) - \underline{c}(x) = \tau(x)f'(x) + \int (f(y) - f(x)) \frac{y}{x} b(x, y) dy,$$

with the notation $\bar{g}(x) := xg(x)$ and $\underline{g}(x) := x^{-1}g(x)$ for any function $g: (0, \infty) \rightarrow \mathbb{R}$. Their results are summarized below.

Theorem 0.4.1 (from Bertoin and Watson [34]). Suppose that X is irreducible, *i.e.*, the hitting time $H_y := \inf\{t \geq 0: X_t = y\}$ is not \mathbb{P}_x -a.s. infinite for all $x, y > 0$, and that the maps $x \mapsto \frac{\cdot}{x} b(x, \cdot)$ from $(0, \infty)$ to $L^1(dy)$ and $\underline{\tau}: (0, \infty) \rightarrow (0, \infty)$ are continuous and bounded. Then \mathcal{A} is the infinitesimal generator of a unique positive, strongly continuous semigroup $(T_t)_{t \geq 0}$ on $\bar{\mathcal{C}}_b := \{f: (0, \infty) \rightarrow \mathbb{R} \text{ continuous with } \underline{f} \text{ bounded}\}$, and the solution to (0.15) subject to $\mu_0 = \delta_x$ is given for every $t \geq 0$ and $f \in \bar{\mathcal{C}}_b$ by the *Feynman–Kac formula*

$$\langle \mu_t, f \rangle = T_t f(x) = x \mathbb{E}_x [\underline{f}(X_t) \mathcal{E}_t], \quad \text{where } \mathcal{E}_t := \exp\left(\int_0^t \underline{\tau}(X_s) ds\right).$$

Furthermore, defining the Laplace transform $L_{x,y}(q) := \mathbb{E}_x[e^{-qH(y)} \mathcal{E}_{H(y)}; H(y) < \infty]$, $q \in \mathbb{R}$, the parameter

$$\lambda := \inf\{q \in \mathbb{R}: L_{x_0, x_0}(q) < 1\} \tag{0.16}$$

does not depend on $x_0 > 0$, and:

(a) If

$$L_{x_0, x_0}(\lambda) = 1 \quad \text{and} \quad L'_{x_0, x_0}(\lambda) > -\infty, \quad (0.17)$$

is fulfilled for some (then all) $x_0 > 0$, then for every f with compact support

$$\lim_{t \rightarrow \infty} e^{-\lambda t} \langle \mu_t, f \rangle = \langle \nu, f \rangle h(x), \quad (0.18)$$

where

$$h(x) := x L_{x, x_0}(\lambda) \quad \text{and} \quad \nu(dx) := \frac{dx}{h(x)\tau(x)|L'_{x, x}(\lambda)|}, \quad x > 0. \quad (0.19)$$

(b) If instead of (0.17) holds the stronger condition

$$L_{x_0, x_0}(q) < \infty \quad \text{for some } q < \lambda \text{ and } x_0 > 0,$$

then the convergence (0.18) occurs exponentially fast.

Eq. (0.18) is referred to as a *Malthusian behavior*. In [36], Bertoin and Watson linked the solution $(\mu_t)_{t \geq 0}$ to the growth-fragmentation equation (0.15) to the intensity measure of a growth-fragmentation process $(\mathbf{Z}(t) : t \geq 0)$, namely $\langle \mu_t, f \rangle = \int \mu_0(dx) \mathbf{E}_x[\langle \mathbf{Z}(t), f \rangle]$ for all $t \geq 0$ and $f : \mathbb{R} \rightarrow [0, \infty)$ measurable.

Theorem 0.4.2 (Bertoin and Watson [36]). Under the main assumptions of Theorem 0.4.1, and

$$\limsup_{x \rightarrow 0+} \tau(x) < \lambda \quad \text{and} \quad \limsup_{x \rightarrow \infty} \tau(x) < \lambda, \quad (0.20)$$

with λ defined by (0.16), the process \mathbf{Z} exhibits strong Malthusian behavior: for all $x > 0$ and continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ with $f = O(h)$, one has

$$\lim_{t \rightarrow \infty} e^{-\lambda t} \langle \mathbf{Z}(t), f \rangle = \langle \nu, f \rangle W_\infty, \quad \text{in } L^1(\mathbf{P}_x),$$

where W_∞ is the limit of the uniformly integrable martingale $W_t := e^{-\lambda t} \langle \mathbf{Z}_t, h \rangle$, $t \geq 0$, and the pair (h, ν) is given by (0.19).

Condition (0.20) implies $\lambda > 0$ and [20, Theorem 2] is a stronger requirement than (0.17).

Many models in natural science are concerned with *open systems*. As such, one may for instance want to take into account the introduction of new particles coming from the environment. Perhaps the simplest kind of immigration is when individuals arrive at a constant rate independently of the current population. This situation is described by the equation (compare with (0.15))

$$\partial_t \langle \mu_t, f \rangle = \langle \mu_t, \mathcal{A}f \rangle + \int I(dy) f(y), \quad (0.21)$$

where I is a measure on $(0, \infty)$ (called the *immigration measure*). Haas [61] investigated a self-similar pure-fragmentation with immigration equation like (0.21) where $\tau(x) = -cx^{\alpha+1}$ for $\alpha \in \mathbb{R}$ and $c > 0$. She discussed the equilibrium between fragmentation and immigration and gave the expression of the unique stationary solution when equilibrium happens. In Chapter 3, we look at the same problem but for positive τ (growth), in the setting of [34].

Theorem 0.4.3

We take the same assumptions and notations of Theorem 0.4.1.(a), and we suppose further that $m := \langle I, f_1 \rangle < \infty$ and $\langle I, h \rangle < \infty$. Then the growth-fragmentation with immigration equation (0.21) admits a unique solution $(\mu_t^\ddagger)_{t \geq 0}$ started from $\mu_0 = \delta_x$, and for every continuous function $f: (0, \infty) \rightarrow \mathbb{R}$ with compact support,

$$\langle \mu_t^\ddagger, f \rangle = \begin{cases} e^{\lambda t} \langle \nu, f \rangle (h(x) + \frac{1}{\lambda} \langle I, h \rangle) + o(e^{\lambda t}), & \text{if } \lambda > 0, \\ \frac{1}{-\lambda} \langle \nu, f \rangle \langle I, h \rangle + o(1), & \text{if } \lambda < 0, \\ \langle \nu, f \rangle (h(x) + t \langle I, h \rangle) + o(t), & \text{if } \lambda = 0. \end{cases}$$

Further, we can relate μ_t^\ddagger to the intensity measure of a so called *growth-fragmentation with immigration process* $(\mathbf{Z}_t^\ddagger: t \geq 0)$, which when (0.20) holds also exhibits a strong Malthusian behavior.

Informally, \mathbf{Z}^\ddagger is the superposition of \mathbf{Z} together with independent copies of \mathbf{Z} immigrating at rate I .

One can imagine a second kind of immigration where each individual in the current population independently attracts new immigrants. Then the system is instead ruled by the equation

$$\partial_t \langle \mu_t, f \rangle = \langle \mu_t, \mathcal{A}^\ddagger f \rangle, \quad (0.22)$$

where $\mathcal{A}^\ddagger := \mathcal{A} + \langle I, \cdot \rangle$. This situation is briefly discussed in Chapter 3.

Asymptotics of self-similar growth-fragmentation processes

This chapter is a reproduction of the article [49].

Markovian growth-fragmentation processes introduced in [18, 19] extend the pure-fragmentation model by allowing the fragments to grow larger or smaller between dislocation events. What becomes of the known asymptotic behaviors of self-similar pure fragmentations [23, 26, 30, 32] when growth is added to the fragments is a natural question that we investigate in this paper. Our results involve the terminal value of some additive martingales whose uniform integrability is an essential requirement. Dwelling first on the homogeneous case [18], we exploit the connection with branching random walks and in particular the martingale convergence of Biggins [39, 41] to derive precise asymptotic estimates. The self-similar case [19] is treated in a second part; under the so called Malthusian hypotheses and with the help of several martingale-flavored features recently developed in [24], we obtain limit theorems for empirical measures of the fragments.

1.1 Introduction

Fragmentation processes are meant to describe the evolution of an object which is subject to random and repeated dislocations over time. The way the mass is spread into smaller fragments during a dislocation event is usually given by a (random) *mass-partition*, that is an element of the space

$$\mathcal{P} := \left\{ \mathbf{p} := (p_i, i \in \mathbb{N}) : p_1 \geq p_2 \geq \dots \geq 0 \text{ and } \sum_{i=1}^{\infty} p_i \leq 1 \right\}, \quad (1.1)$$

where the total mass need not be conserved, i.e. a positive proportion $1 - \sum_{i \geq 1} p_i$ may disintegrate into dust. The first probabilistic models of fragmentations go back at least to Kolmogorov [74]. Roughly, Kolmogorov imagined a discrete branching system in which particles get fragmented according to a conservative distribution ν on \mathcal{P} and in a *homogeneous* manner, that is to say the rate at which a particle splits does not depend on its mass. Under this essential assumption of homogeneity, Kolmogorov showed that a simple rescaling of the empirical measure of the logarithms of the fragments converges with probability one toward the Gaussian distribution. Later, a student of his, Filippov [60] investigated mass-dependent dislocation rates and more precisely the *self-similar* case, in

the sense that a particle with size m splits at speed m^α for some fixed constant $\alpha \in \mathbb{R}$ (the homogeneous case then corresponds to $\alpha = 0$). Most notably he discovered a limit theorem for a weighted version of the empirical measure of the fragments when $\alpha > 0$. The special but common *binary* situation, where particles always split into two smaller fragments, has been emphasized by Brennan and Durrett [45, 46], and later reconsidered by Baryshnikov and Gnedin [16] in some variant of the car packing problem. Further extensions and other asymptotic properties in the non-conservative case have also been derived by Bertoin and Gnedin [26] by means of complex analysis and contour integrals.

In the 2000s (see [21, Chapters 1-3] for a comprehensive summary), Bertoin extended and theorized the construction of general fragmentation processes in continuous time. In particular the *dislocation measure* ν need no longer be a probability distribution, as there is only the integrability requirement

$$\int_{\mathcal{P}} (1 - p_1) \nu(d\mathbf{p}) < \infty. \quad (1.2)$$

While permitting infinite dislocation rates (so infinitely many dislocation events may occur in a bounded time interval), this condition prevents the total mass from being immediately shattered into dust and leads to a nondegenerate fragmentation process $\mathbf{X}(t) := (X_1(t), X_2(t), \dots)$, $t \geq 0$, with values in \mathcal{P} . When $\alpha = 0$, fragmentation processes can be related (via a simple logarithmic transformation) to *branching random walks*, for which fruitful literature is available, see e.g. the works of Biggins and Uchiyama [39, 102, 41], and [100]. Especially, *additive martingales*, which are processes of the form

$$\mathbb{E} \left[\sum_{i=1}^{\infty} X_i^q(t) \right]^{-1} \sum_{i=1}^{\infty} X_i^q(t), \quad t \geq 0, \quad (1.3)$$

for some parameter $q > 0$, play a key role and the question of their uniform integrability inquired by Biggins has successfully led to the asymptotic behavior of homogeneous conservative fragmentations [31, 32]. More generally, in the self-similar case, some specific so called *Malthusian hypotheses* guarantee the existence of an intrinsic martingale associated with the fragmentation and whose convergence again yields many interesting asymptotic results. Among others the results of Kolmogorov and Filippov have been revisited [23], applying known statistics of self-similar Markov processes to the process of a randomly tagged fragment.

More recently, Bertoin [18, 19] introduced a new type of fragmentation processes in which the fragments are allowed to grow during their lifetimes. We expect that most of the aforementioned asymptotic properties extend to these *growth-fragmentation* processes, and it is the main purpose of the present work to derive some of them. We shall first give a bit more

description and explain why our task is not completely straightforward. Like in the pure (i.e. without growth) setting, we are interested in the process which describes the (sizes of the) fragments as time passes. For homogeneous growth-fragmentations, namely the *compensated fragmentations* of [18], the basic prototype is simply a dilated homogeneous fragmentation, that is a pure homogeneous fragmentation affected by a deterministic exponential drift. However, there exist much more general compensated fragmentations, where the dislocation measure ν has only to fulfill

$$\int_{\mathcal{P}} (1 - p_1)^2 \nu(d\mathbf{p}) < \infty, \quad (1.4)$$

so that the process is nondegenerate and can still be encoded at any time by a non-increasing null sequence. Condition (1.4) is weaker than the necessary and sufficient condition (1.2) for ν to be the dislocation measure of a homogeneous fragmentation, and both are reminiscent of those concerning the jump intensities of Lévy processes, respectively subordinators. Incidentally, it was the main motivation of [18] to establish that, just like the Lévy–Itô construction of Lévy processes in terms of compensated Poisson integrals, compensated fragmentations naturally arise as limits of suitably dilated homogeneous fragmentations [18, Theorem 2]. Though asymptotic properties of pure homogeneous fragmentations immediately transfer to the dilated ones, extending them to general compensated fragmentations would correspond to interchanging two limits, which does not seem obvious at first sight. This is without to mention the self-similar case, that is for the growth-fragmentations in [19], where things look even more complicated.

There, and unlike the compensated fragmentations which are constructed directly as processes in time, the self-similar *cell systems* are rather built from a genealogical point of view: roughly, the (size of the) mother cell evolves like a Markov process on the positive half-line where each negative jump $-y$ is interpreted as a splitting event, giving birth to a daughter cell with initial size y and which then grows independently of the mother particle and according to the same dynamics, i.e. producing in turn granddaughters, and so on. Bertoin focused in particular on the situation where the associated growth-fragmentation process $\mathbf{X} := (\mathbf{X}(t), t \geq 0)$, that is the process recording the sizes of all alive cells in the system, fulfills a self-similarity property, namely when there exists $\alpha \in \mathbb{R}$ such that for each $x > 0$, the process $(x\mathbf{X}(x^\alpha t), t \geq 0)$ has the same law as \mathbf{X} started from a cell whose initial size is x . In the homogeneous case $\alpha = 0$, these growth-fragmentations correspond to the compensated fragmentations of [18] for which the dislocation measure is binary, see [19, Proposition 3]. In the self-similar case $\alpha < 0$, they have been proved to be eventually extinct [19, Corollary 3], an observation which was already made by Filippov [60] in the context of pure fragmentations.

Both for homogeneous and for self-similar fragmentations, the additive martingales (1.3) and more precisely their uniform integrability have turned out to be of greatest importance in the study of asymptotic behaviors. We stress that sufficient conditions to this uniform integrability appear less easily for growth-fragmentations, as they non longer take values in the space of mass-partitions \mathcal{P} .

Our work is organized in two independent parts. In Section 1.2, we deal with the homogeneous case $\alpha = 0$ in the slightly more general setting of compensated fragmentations [18]. With the help of a well-known theorem due to Biggins [41] and by adapting arguments of Bertoin and Rouault [32], we prove the uniform convergence of additive martingales from which, in the realm of branching random walks, we infer precise estimates for the empirical measure of the fragments and the asymptotic behavior of the largest one. This part can be viewed as an application to the study of extremal statistics in certain branching random walks, see e.g. the recent developments by Aïdékon [2], Aïdékon et al. [1], Arguin et al. [7] and Hu et al. [68]. The self-similar case is considered in Section 1.3 within the framework of [19]. Relying on recent results in [24] and in particular on the uniform integrability of the Malthusian martingale, we establish for $\alpha > 0$ the convergence in probability of the empirical measure of the fragments and that of the largest fragment. In a concluding section we also address the convergence of another empirical measure where fragments are stopped as soon as they become smaller than a vanishing threshold.

1.2 Compensated fragmentations

1.2.1 Prerequisites

Recall the space of mass-partitions \mathcal{P} defined in (1.1) and denote by \mathcal{P}_1 the subspace of mass-partitions $\mathbf{p} := (p_1, 0, \dots) \in \mathcal{P}$ having only one single fragment $p_1 \in (0, 1]$. A compensated fragmentation process $\mathbf{Z}(t) := (Z_1(t), Z_2(t), \dots)$, $t \geq 0$, is a stochastic process whose distribution is characterized by a triple (σ^2, c, ν) where $\sigma^2 \geq 0$ is a diffusion coefficient, $c \in \mathbb{R}$ is a growth rate, and ν is a nontrivial measure on $\mathcal{P} \setminus \{(1, 0, \dots)\}$ such that (1.4) holds. It can be seen as the decreasing rearrangement of the exponential of the atoms of a branching process in continuous time. Namely, the process giving the empirical measure of the logarithms of the fragments at time t ,

$$\mathcal{Z}^t := \sum_{i=1}^{\infty} \delta_{\log Z_i(t)},$$

is called a *branching Lévy process* in [18], to which we refer for background. In the basic case where $\nu(\mathcal{P} \setminus \mathcal{P}_1) < \infty$, i.e. the fragmentation rates are finite, \mathcal{Z} is a generalization of the

branching random walk in continuous time introduced by Uchiyama [102]: more precisely, \mathcal{Z} is a branching particle system in which each atom, during its lifetime, is allowed to move in \mathbb{R} independently of the other atoms and according to the dynamics of a spectrally negative Lévy process¹ η with Laplace transform

$$E[\exp(q\eta(t))] = \exp(t\psi(q)), \quad t \geq 0, q \geq 0,$$

where under (1.4) the Laplace exponent²

$$\psi(q) := \frac{1}{2}\sigma^2 q^2 + \left(c + \int_{\mathcal{P} \setminus \mathcal{P}_1} (1 - p_1) \nu(d\mathbf{p}) \right) q + \int_{\mathcal{P}_1} (p_1^q - 1 + q(1 - p_1)) \nu(d\mathbf{p}) \quad (1.5)$$

is finite for all $q \geq 0$. In words, when $\nu(\mathcal{P} \setminus \mathcal{P}_1) < \infty$, the system can be described as follows. It starts at the origin of space and time with a single particle which evolves like η . Each particle dies after a random exponential time with intensity $\nu(\mathcal{P} \setminus \mathcal{P}_1)$, giving birth to a random family of children (η_1, η_2, \dots) whose initial position $(\Delta a_1, \Delta a_2, \dots)$ relative to the mother particle at its death is such that $(e^{\Delta a_1}, e^{\Delta a_2}, \dots)$ has the conditional distribution $\nu(\cdot \mid \mathcal{P} \setminus \mathcal{P}_1)$.

In the general situation where the dislocation rate $\nu(\mathcal{P} \setminus \mathcal{P}_1)$ may be infinite, the construction is achieved by approximation from compensated fragmentations with finite dislocation rates, using a monotonicity argument (see [18, Lemma 3] recalled in the proof of Proposition 1.2.8 below).

Let us denote by

$$\mu(dx) := \mathbb{E}[\mathcal{Z}^1(dx)] \quad (1.6)$$

the mean intensity of the point process \mathcal{Z}^1 , so that

$$m(q) := \int e^{qx} \mu(dx), \quad q \geq 0,$$

is the Laplace transform of μ . An important fact (cf. [18, Theorem 1]) is that, for every $t \geq 0$ and every $q \geq 0$,

$$m(q)^t = \mathbb{E} \left[\sum_{i=1}^{\infty} Z_i^q(t) \right] = \exp(t\kappa(q)), \quad (1.7)$$

¹That is a càdlàg stochastic process with stationary and independent increments which has only negative jumps. The results of this section could be quite straightforwardly adapted to also handle positive jumps in the particle motions; we shall however not do so as this would burden the expository and was anyway not considered in [18].

²Formula (1.5) is designed in such a way that if $\sigma^2 = 0$, $c = 0$, and $D := \int_{\mathcal{P}} (1 - p_1) \nu(d\mathbf{p}) < \infty$, then \mathbf{Z} simply is a pure homogeneous fragmentation \mathbf{X} with dislocation measure ν affected by a dilatation with coefficient D , i.e. $\mathbf{Z}(t) = e^{Dt} \mathbf{X}(t)$, $t \geq 0$. In this case η is a compound Poisson process with jump measure $(\log p_1) \nu_{|\mathcal{P}_1}(d\mathbf{p})$ and drift D , but we stress that $\psi(q) < \infty$ holds in greater generality, namely under (1.4) and $\nu(\mathcal{P} \setminus \mathcal{P}_1) < \infty$. See [18] for details.

where

$$\kappa(q) := \frac{1}{2}\sigma^2 q^2 + cq + \int_{\mathcal{P}} \left(\sum_{i=1}^{\infty} p_i^q - 1 + q(1 - p_1) \right) \nu(d\mathbf{p})$$

defines a convex function $\kappa: [0, \infty) \rightarrow (-\infty, \infty]$. We mention that under $\nu(\mathcal{P} \setminus \mathcal{P}_1) < \infty$, there is the identity

$$\kappa(q) = \psi(q) + \int_{\mathcal{P} \setminus \mathcal{P}_1} \left(\sum_{i=1}^{\infty} p_i^q - 1 \right) \nu(d\mathbf{p}), \quad q \geq 0. \quad (1.8)$$

As we shall explain in the forthcoming [Lemma 1.2.7](#), the first summand describes the motion of a particle, while the second outlines the branching mechanism. In better words, κ is merely the log-Laplace transform of the cloud of particles at first generation (i.e. after the first branching event), which is a key feature of branching random walks.

Since under [\(1.4\)](#),

$$p_1^q - 1 + q(1 - p_1) = O((1 - p_1)^2)$$

is integrable with respect to ν , we easily observe that, if we set

$$\underline{q} := \inf \{ q \geq 0 : \kappa(q) < \infty \} = \inf \left\{ q \geq 0 : \int_{\mathcal{P} \setminus \mathcal{P}_1} \sum_{i=2}^{\infty} p_i^q \nu(d\mathbf{p}) < \infty \right\},$$

then κ takes finite values and is analytic on the open interval (\underline{q}, ∞) . Note that [\(1.4\)](#) also implies $\kappa(2) < \infty$, so $\underline{q} \leq 2$. Let us introduce the subspace

$$\ell^{q\downarrow} := \left\{ \mathbf{z} := (z_1, z_2, \dots) : z_1 \geq z_2 \geq \dots \geq 0 \text{ and } \sum_{i=1}^{\infty} z_i^q < \infty \right\}$$

of the space ℓ^q of q -summable sequences endowed with the distance $\|\mathbf{z} - \mathbf{z}'\|_{\ell^q}^q := \sum_{i=1}^{\infty} |z_i - z'_i|^q$. We also denote $\ell^{\infty\downarrow}$ the space of bounded, non-increasing sequences of nonnegative real numbers endowed with the uniform norm $\|\cdot\|_{\ell^\infty}$. We see by [\(1.7\)](#) that the compensated fragmentation $\mathbf{Z} := (\mathbf{Z}(t), t \geq 0)$ is a $\ell^{q\downarrow}$ -valued process for every $q \in (\underline{q}, \infty]$, and in particular for $q = 2$. Further if $\mathbf{z} := (z_1, z_2, \dots)$ is in $\ell^{2\downarrow}$ and $\mathbf{Z}^{[1]}, \mathbf{Z}^{[2]}, \dots$ are independent copies of \mathbf{Z} , then the process of the family $(z_j Z_i^{[j]}(t), i, j \in \mathbb{N}), t \geq 0$, rearranged in the non-increasing order is again in $\ell^{2\downarrow}$, and we denote its distribution by $\mathbb{P}_{\mathbf{z}}$. It has been proved in [\[18\]](#) that $(\mathbf{Z}, (\mathbb{P}_{\mathbf{z}})_{\mathbf{z} \in \ell^{2\downarrow}})$ is a Markov process which fulfills the so called *branching property*: for all $s \geq 0$, the conditional law of $(\mathbf{Z}(t + s))_{t \geq 0}$ given $(\mathbf{Z}(r))_{0 \leq r \leq s}$ is $\mathbb{P}_{\mathbf{z}}$, where $\mathbf{z} = \mathbf{Z}(s)$. Without loss of generality we shall assume in the sequel that the fragmentation starts with a single mass with unit size, i.e. $\mathbb{P} := \mathbb{P}_{(1,0,\dots)}$.

Equation (1.7) and the branching property yield an important family of additive martingales. Namely, the \mathbb{R} -valued process

$$M(t; q) := \exp(-t\kappa(q)) \sum_{i=1}^{\infty} Z_i^q(t), \quad t \geq 0, \quad (1.9)$$

is a martingale for every $q \in (\underline{q}, \infty)$. As a first consequence [18, Proposition 2], the compensated fragmentation \mathbf{Z} possesses a càdlàg version in $\ell^{2\downarrow}$, that is a version in the Skorokhod space $D([0, \infty), \ell^{2\downarrow})$ of right continuous with left limits, $\ell^{2\downarrow}$ -valued functions. Working with such a version from now on, \mathbf{Z} has actually càdlàg paths in $\ell^{q\downarrow}$ for every $q \in (\underline{q}, \infty]$.

Proposition 1.2.1. Almost surely, for every $q \in (\underline{q}, \infty]$, \mathbf{Z} has càdlàg paths in $\ell^{q\downarrow}$.

Proof. Recall that $\|\cdot\|_{\ell^{q'}} \leq \|\cdot\|_{\ell^q}$ whenever $q \leq q' \leq \infty$. Since \mathbf{Z} has càdlàg paths in $\ell^{2\downarrow}$, it has in particular càdlàg paths in $\ell^{\infty\downarrow}$. Let $(q_k, k \in \mathbb{N})$ be a sequence decreasing to \underline{q} , and define

$$T_m^{(k)} := \inf \{t \geq 0: M(t; q_k) > m\} = \inf \left\{ t \geq 0: \|\mathbf{Z}(t)\|_{\ell^{q_k}}^{q_k} > m e^{t\kappa(q_k)} \right\}$$

for $k, m \in \mathbb{N}$. Applying Doob's maximal inequality to the martingale (1.9) we have that almost surely, for all $k \in \mathbb{N}$, $T_m^{(k)} \uparrow \infty$ as $m \rightarrow \infty$. Thus, almost surely, for every $q \in (\underline{q}, \infty]$ and $T \geq 0$ we can find a $k \in \mathbb{N}$ such that $\underline{q} < q_k < q$ and then a $m \in \mathbb{N}$ such that $T < T_m^{(k)}$, whence

$$\begin{aligned} \|\mathbf{Z}(t) - \mathbf{Z}(s)\|_{\ell^q}^q &\leq \|\mathbf{Z}(t) - \mathbf{Z}(s)\|_{\ell^\infty}^{q-q_k} \|\mathbf{Z}(t) - \mathbf{Z}(s)\|_{\ell^{q_k}}^{q_k} \\ &\leq m 2^{1+q_k} \left(1 + e^{T_m^{(k)} \kappa(q_k)}\right) \|\mathbf{Z}(t) - \mathbf{Z}(s)\|_{\ell^\infty}^{q-q_k} \end{aligned}$$

for all $0 \leq s, t < T$. The fact that \mathbf{Z} has càdlàg paths in $\ell^{\infty\downarrow}$ completes the proof. \blacksquare

We first would like to extend to the compensated fragmentation \mathbf{Z} the asymptotic results obtained by Bertoin and Rouault [31, 32] for pure homogeneous fragmentations. They strongly rely on the work of [39, 41] about the uniform integrability of additive martingales. Essentially, the martingales $(M(t; q))_{t \geq 0}$ will be uniformly integrable if $q\kappa'(q) - \kappa(q) < 0$ and $M(1; q) \in L^\gamma(\mathbb{P})$ for some $\gamma > 1$. With this in mind, let us introduce

$$\bar{q} := \sup \left\{ q > \underline{q}: q\kappa'(q) - \kappa(q) < 0 \right\}.$$

First note that $\bar{q} < \infty$, because

$$q\kappa'(q) - \kappa(q) = \frac{1}{2}\sigma^2 q^2 + \int_{\mathcal{P}} (1 - p_1^q(1 - \log p_1^q)) \nu(d\mathbf{p}) - \int_{\mathcal{P} \setminus \mathcal{P}_1} \sum_{i=2}^{\infty} p_i^q (1 - \log p_i^q) \nu(d\mathbf{p}),$$

which, by Fatou's lemma, is at least $\nu(\mathcal{P})$ as $q \rightarrow \infty$. Second, we have $\bar{q} > \underline{q}$ as soon as $q\kappa'(q) - \kappa(q) < 0$ for some $q \geq 0$ such that $\kappa(q) < \infty$ (e.g. for $q = 2$), which is realized when

$$\int_{\mathcal{P} \setminus \mathcal{P}_1} \sum_{i=2}^{\infty} p_i^q (1 - \log p_i^q) \nu(d\mathbf{p}) \geq \left(\frac{1}{2} \sigma^2 + \int_{\mathcal{P}} (1 - p_1)^2 \nu(d\mathbf{p}) \right) q^2.$$

We distinguish two different regimes for the function $q \mapsto \kappa(q)/q$:

Lemma 1.2.2. The function $q \mapsto \kappa(q)/q$ is decreasing on (\underline{q}, \bar{q}) and increasing on (\bar{q}, ∞) . Further, $\bar{q}\kappa'(\bar{q}) = \kappa(\bar{q})$ when $\bar{q} > \underline{q}$.

In the context of branching random walks, the value $\kappa'(\bar{q})$ is the asymptotic velocity of the maximal displacement $\log Z_1(t)$; see [Figure 1.1](#) and [Section 1.2.3](#).

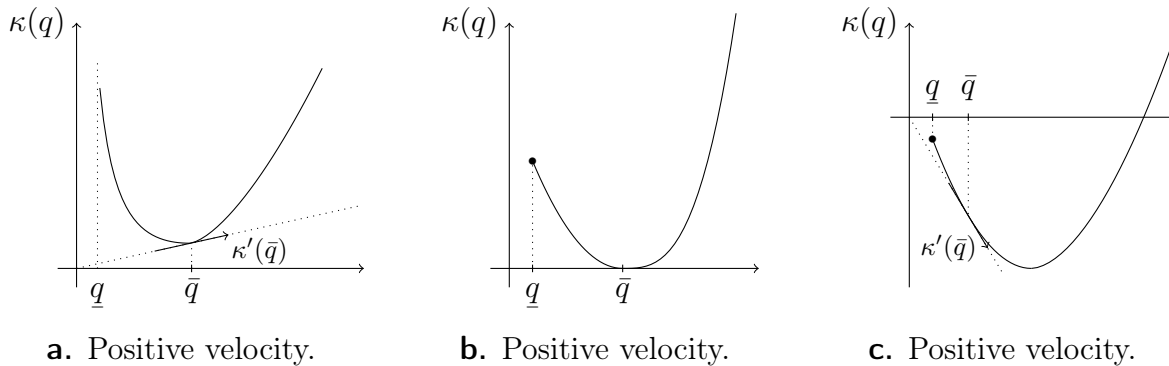


Figure 1.1. The cumulant function κ , the points \underline{q}, \bar{q} , and the velocity $\kappa'(\bar{q}) = \kappa(\bar{q})/\bar{q}$.

Proof. On the one hand,

$$\frac{d}{dq} \left[\frac{\kappa(q)}{q} \right] = \frac{q\kappa'(q) - \kappa(q)}{q^2}.$$

On the other hand, the map $q \mapsto q\kappa'(q) - \kappa(q)$ is increasing on (\underline{q}, ∞) since κ is convex, so it has at most one sign change, occurring at \bar{q} if $\bar{q} > \underline{q}$. ■

Our main result provides sufficient conditions for the convergence of the martingales $(M(t; q))_{t \geq 0}$ uniformly in $q \in (\underline{q}, \bar{q})$, both almost surely and in $L^1(\mathbb{P})$. Most of the coming section is devoted to a precise statement and a proof. As consequences, we ascertain the convergence of a rescaled version of the empirical measure \mathcal{Z}^t and in [Section 1.2.3](#) we expand on the asymptotic behavior of the largest fragment. One last application is exposed in [Section 1.2.4](#).

1.2.2 Uniform convergence of the additive martingales

In the remaining of [Section 1.2](#) we will make, in addition to [\(1.4\)](#), the assumption that the dislocation measure ν fulfills

$$\kappa(0) \in (0, \infty], \quad (1.10)$$

and, for all $\underline{q} < \bar{q} < 1$,

$$\nu_{|\mathcal{P} \setminus \mathcal{P}_1} \left(\sum_{i=1}^{\infty} p_i^q < 1 \right) < \infty. \quad (1.11)$$

Condition [\(1.10\)](#) holds e.g. when $\nu(p_2 > 0) > \nu(p_1 = 0)$ and merely rephrases that the mean number $\mu(\mathbb{R}) = m(0)$ of offspring of particles is greater than 1, i.e. the branching process \mathcal{Z} is supercritical. This implies that the *non-extinction* event $\{\forall t \geq 0, Z_1(t) > 0\}$ occurs with positive probability. Condition [\(1.11\)](#) is just a minor technical requirement for the possible values $q < 1$ and is fulfilled in many situations: when $\underline{q} \geq 1$, when $\nu(\mathcal{P} \setminus \mathcal{P}_1) < \infty$, or more importantly when the measure ν is *conservative*, i.e. $\sum_{i \geq 1} p_i = 1$ for ν -almost every $\mathbf{p} \in \mathcal{P}$. Observe also that in the conservative case, $\underline{q} < 1$ is possible only if [\(1.2\)](#) holds, i.e. \mathbf{Z} is essentially a dilated pure fragmentation.

We may now state:

Theorem 1.2.3. Suppose [\(1.10\)](#) and [\(1.11\)](#). Then the following assertions hold almost surely:

- (i) On (\underline{q}, \bar{q}) , $M(t; \cdot)$ converges locally uniformly as $t \rightarrow \infty$. More precisely, there exists a random continuous function $M(\infty; \cdot): (\underline{q}, \bar{q}) \rightarrow [0, \infty)$ such that, for any compact subset K of (\underline{q}, \bar{q}) ,

$$\lim_{t \rightarrow \infty} \sup_{q \in K} |M(t; q) - M(\infty; q)| = 0,$$

and this convergence also holds in mean. Furthermore for every $q \in (\underline{q}, \bar{q})$, $M(\infty; q) > 0$ conditionally on non-extinction.

- (ii) For every $q \in [\bar{q}, \infty)$,

$$\lim_{t \rightarrow \infty} M(t; q) = 0.$$

As a first important consequence, we derive uniform estimates for the empirical measure of the fragments, which echo those determined by Bertoin and Rouault [[32](#), Corollary 3]. We shall assume here that the mean intensity measure μ in [\(1.6\)](#) is *non-lattice*, in that it is not supported on $r\mathbb{Z} + s$ for any $r > 0$, $s \in \mathbb{R}$.

Corollary 1.2.4. Suppose (1.10), (1.11), and μ non-lattice. Then for any Riemann integrable function $f: (0, \infty) \rightarrow \mathbb{R}$ with compact support and for all compact subset K of (q, \bar{q}) ,

$$\lim_{t \rightarrow \infty} \sqrt{t} e^{-(\kappa(q) - q\kappa'(q))t} \sum_{i=1}^{\infty} f\left(Z_i(t) e^{-\kappa'(q)t}\right) = \frac{M(\infty; q)}{\sqrt{2\pi\kappa''(q)}} \int_0^{\infty} \frac{f(y)}{y^{q+1}} dy$$

uniformly in $q \in K$, almost surely.

Remark 1.2.5. We stress that condition (1.11) is unnecessary if we only deal with $q \geq 1$. In particular it may be removed from the above statements provided that we replace \underline{q} by $\underline{q} \vee 1$.

Before proving these two results, let us give a quick summary on the sizes of particles in a compensated fragmentation. On the one hand, it is easy (see e.g. [21, Corollary 1.4]) to derive from Theorem 1.2.3.(i) that in the first order, the largest particle $Z_1(t)$ evolves like $e^{\kappa'(\bar{q})t}$ as $t \rightarrow \infty$, and we will have a look at the second and third asymptotic orders in Section 1.2.3. On the other hand, Corollary 1.2.4 provides the local density of particles at intermediate scales: if $\kappa'(\underline{q}) < a < \kappa'(\bar{q})$ and $\kappa^*(a) := \kappa(q) - q\kappa'(q)$ for $\kappa'(q) := a$, then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \#\{i \in \mathbb{N} : e^{at-\varepsilon} \leq Z_i(t) \leq e^{at+\varepsilon}\} = \kappa^*(a)$$

for every $\varepsilon > 0$, almost surely (just take $f(x) := \mathbb{1}_{[-\varepsilon, \varepsilon]}(\log x)$ above). Lastly, we shall observe in Section 1.2.4 that fragments at untypical levels $a > \kappa'(\bar{q})$ appear with a probability that is roughly of the same order as their expected number (Corollary 1.2.13.(ii)).

Theorem 1.2.3 is essentially a version of a theorem of Biggins [41] in the context of compensated fragmentations. In this respect, one important requirement to derive part (i) is that $\mathbb{E}[M(1; q)^\gamma] < \infty$ for some $\gamma > 1$. We start with a lemma controlling the finiteness of

$$W_{\nu, q}^\gamma := \int_{\mathcal{P} \setminus \mathcal{P}_1} \left| 1 - \sum_{i=1}^{\infty} p_i^q \right|^\gamma \nu(d\mathbf{p}).$$

Lemma 1.2.6. Let $q > \underline{q}$ and suppose either (1.11) or $q \geq 1$. Then $W_{\nu, q}^\gamma < \infty$ for some $\gamma \in (1, 2]$.

Proof. Suppose first $q \geq 1$. Then for $\gamma := 2$ and for all $\mathbf{p} \in \mathcal{P}$,

$$0 \leq \left(1 - \sum_{i=1}^{\infty} p_i^q \right)^2 \leq (1 - p_1^q)^2 \leq q^2 (1 - p_1)^2$$

(the last inequality resulting from the convexity of $x \mapsto x^q$), so $W_{\nu,q}^2 < \infty$ by (1.4). Suppose now $q < 1$. Then

$$W_{\nu,q}^\gamma \leq \nu_{|\mathcal{P} \setminus \mathcal{P}_1} \left(\sum_{i=1}^{\infty} p_i^q < 1 \right) + \int_{\mathcal{P} \setminus \mathcal{P}_1} \mathbb{1}_{\left\{ \sum_{i=1}^{\infty} p_i^q \geq 1 \right\}} \left(\sum_{i=1}^{\infty} p_i^q - 1 \right)^\gamma \nu(d\mathbf{p}).$$

Under (1.11), $W_{\nu,q}^\gamma$ is finite as soon as the latter integral is finite. But by Jensen's inequality, the integrand is bounded from above by

$$\left(\sum_{i=2}^{\infty} p_i p_i^{q-1} \right)^\gamma \leq \sum_{i=2}^{\infty} p_i^{1+\gamma(q-1)},$$

which is ν -integrable if $1 + \gamma(q-1) = q - (\gamma-1)(1-q) > q$, i.e. provided $\gamma \in (1, 2]$ is close enough to 1. \blacksquare

We then derive an upper bound for $\mathbb{E}[M(t; q)^\gamma]$ in terms of $W_{\nu,q}^\gamma$:

Lemma 1.2.7. Suppose that $\nu(\mathcal{P} \setminus \mathcal{P}_1) < \infty$. Then for every $q \in (q, \infty)$, $\gamma \in (1, 2]$ and $t \geq 0$,

$$\mathbb{E}[M(t; q)^\gamma] \leq \mathbf{c}_\gamma W_{\nu,q}^\gamma f(t, \psi(\gamma q) - \gamma\psi(q), \kappa(\gamma q) - \gamma\kappa(q)), \quad (1.12)$$

where ψ is given by (1.5), $f(t, x, y) := (e^{tx} - e^{ty})/(x - y)$, and \mathbf{c}_γ is a finite constant depending only on γ .

Proof. Lemma 2 in [18] states that the branching Lévy process \mathcal{Z} can be obtained by superposing independent spatial Lévy motions to a “steady” branching random walk. Specifically, for each $t \geq 0$,

$$\mathbf{Z}(t) \stackrel{d}{=} (e^{\beta_1} X_1(t), e^{\beta_2} X_2(t), \dots),$$

where $\mathbf{X}(t) := (X_1(t), X_2(t), \dots)$ are the atoms at time t of a homogeneous fragmentation \mathbf{X} with dislocation measure $\nu_{|\mathcal{P} \setminus \mathcal{P}_1}$ and $(\beta_i)_{i \in \mathbb{N}}$ is an independent sequence of random variables with Laplace transform $\mathbb{E}[\exp(q\beta_i)] = \exp(t\psi(q))$, $q \geq 0$. Applying Jensen's inequality and conditioning on $\mathbf{X}(t)$ produce

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{i=1}^{\infty} Z_i^q(t) \right)^\gamma \right] &= \mathbb{E} \left[\left(\sum_{j=1}^{\infty} X_j^q(t) \right)^\gamma \left(\sum_{i=1}^{\infty} e^{q\beta_i} \frac{X_i^q(t)}{\sum_j X_j^q(t)} \right)^\gamma \right] \\ &\leq \mathbb{E} \left[\left(\sum_{j=1}^{\infty} X_j^q(t) \right)^{\gamma-1} \sum_{i=1}^{\infty} e^{\gamma q \beta_i} X_i^q(t) \right] \\ &= \exp(t\psi(\gamma q)) \mathbb{E} \left[\left(\sum_{i=1}^{\infty} X_i^q(t) \right)^\gamma \right]. \end{aligned} \quad (1.13)$$

We now recall from [23] (see the proof of its Theorem 2) how to estimate the latter expectation. Denoting

$$\phi(q) := \int_{\mathcal{P} \setminus \mathcal{P}_1} \left(\sum_{i=1}^{\infty} p_i^q - 1 \right) \nu(d\mathbf{p}) < \infty,$$

the process

$$N(t; q) := \exp(-t\phi(q)) \sum_{i=1}^{\infty} X_i^q(t), \quad t \geq 0, \quad (1.14)$$

is a purely discontinuous martingale. It is then deduced from an inequality of Burkholder–Davis–Gundy that

$$\mathbb{E}[N(t; q)^\gamma] \leq \mathbf{c}_\gamma \mathbb{E}[V^\gamma(t; q)],$$

where $\mathbf{c}_\gamma < \infty$ is some constant, and V^γ is the γ -variation process of N :

$$V^\gamma(t; q) := \sum_{0 < s \leq t} |N(s; q) - N(s-; q)|^\gamma.$$

Since in this setting

$$|N(s; q) - N(s-; q)|^\gamma = \exp(-s\gamma\phi(q)) X_k^{\gamma q}(s-) \left| 1 - \sum_{i=1}^{\infty} p_i^q \right|^\gamma,$$

$(s, \mathbf{p}, k) \in (0, t] \times \mathcal{P} \times \mathbb{N}$ being the atoms of a Poisson random measure with intensity $dt \otimes \nu|_{\mathcal{P} \setminus \mathcal{P}_1} \otimes \sharp$, it follows that V^γ has predictable compensator

$$\left(\int_{\mathcal{P} \setminus \mathcal{P}_1} \left| \sum_{i=1}^{\infty} 1 - p_i^q \right|^\gamma \nu(d\mathbf{p}) \right) \int_0^t \exp(-s\gamma\phi(q)) \sum_{i=1}^{\infty} X_i^{\gamma q}(s) ds,$$

and therefore

$$\mathbb{E}[N(t; q)^\gamma] \leq \mathbf{c}_\gamma W_{\nu, q}^\gamma f(t, 0, \phi(\gamma q) - \gamma\phi(q)). \quad (1.15)$$

Now recall (1.9), (1.14) and the identity $\phi(q) + \psi(q) = \kappa(q)$ already observed in (1.8). Multiplying (1.13) by $e^{-t\gamma\kappa(q)}$ and then reporting the bound (1.15), we end up with

$$\mathbb{E}[M(t; q)^\gamma] \leq \mathbf{c}_\gamma W_{\nu, q}^\gamma f(t, \psi(\gamma q) - \gamma\psi(q), \kappa(\gamma q) - \gamma\kappa(q)),$$

as desired. ■

Putting the previous results together now yields:

Proposition 1.2.8. Let $q > \underline{q}$ and suppose either (1.11) or $q \geq 1$. Then there exists $\gamma \in (1, 2]$ such that $M(t; q) \in L^\gamma(\mathbb{P})$ for all $t \geq 0$.

Proof. By [Lemma 1.2.6](#) we can choose $\gamma \in (1, 2]$ such that $W_{\nu, q}^\gamma < \infty$. Let us first assume $\nu(\mathcal{P} \setminus \mathcal{P}_1) < \infty$, so that we may apply [Lemma 1.2.7](#). Note that

$$\psi(\gamma q) - \gamma\psi(q) = \frac{1}{2}\sigma^2(\gamma - 1)\gamma q^2 + \int_{\mathcal{P}_1} (p_1^{\gamma q} - \gamma p_1^q + \gamma - 1) \nu(d\mathbf{p}),$$

with

$$0 \leq p_1^{\gamma q} - \gamma p_1^q + \gamma - 1 = O((1 - p_1)^2).$$

Then the inequality [\(1.12\)](#) is

$$\mathbb{E}[M(t; q)^\gamma] \leq c_\gamma W_{\nu, q}^\gamma f\left(t, \frac{1}{2}\sigma^2(\gamma - 1)\gamma q^2 + \int_{\mathcal{P}_1} (p_1^{\gamma q} - \gamma p_1^q + \gamma - 1) \nu(d\mathbf{p}), \kappa(\gamma q) - \gamma\kappa(q)\right),$$

where f is a continuous function. This bound is finite for each $t \geq 0$, and we shall show by approximation that this also holds when $\nu(\mathcal{P} \setminus \mathcal{P}_1) = \infty$. The measures $\nu^{(b)}$, images of ν by the maps

$$\mathbf{p} \longmapsto (p_1, p_2 \mathbb{1}_{\{p_2 > e^{-b}\}}, p_3 \mathbb{1}_{\{p_3 > e^{-b}\}}, \dots), \quad b > 0,$$

define a consistent family of dislocation measures such that $\nu^{(b)}(\mathcal{P} \setminus \mathcal{P}_1) < \infty$. Thanks to [\[18, Lemma 3\]](#) we can consider that \mathbf{Z} arises from the inductive limit $\mathbf{Z} := \limup \mathbf{Z}^{(b)}$ as $b \uparrow \infty$, where the $\mathbf{Z}^{(b)}$, $b > 0$, are suitably embedded compensated fragmentations with characteristics $(\sigma^2, c, \nu^{(b)})$. With obvious notations, we deduce from the monotone convergence theorem that $\mathbb{E}[\|\mathbf{Z}^{(b)}(t)\|_{\ell^q}^{\gamma q}] \rightarrow \mathbb{E}[\|\mathbf{Z}(t)\|_{\ell^q}^{\gamma q}]$ as $b \rightarrow \infty$, and from the dominated convergence theorem that $\kappa^{(b)}(q) \rightarrow \kappa(q)$ and $W_{\nu^{(b)}, q}^\gamma \rightarrow W_{\nu, q}^\gamma$ (working like in the proof of [Lemma 1.2.6](#)). The proof is then completed by Fatou's lemma. \blacksquare

We are finally ready to tackle the proof of the convergence of the martingale $(M(t; q))_{t \geq 0}$.

Proof of [Theorem 1.2.3](#). Since $(M(t; q))_{t \geq 0}$ is a nonnegative càdlàg martingale, its limit $M(\infty; q)$ as $t \rightarrow \infty$ exists almost surely. If $q \in [\bar{q}, \infty)$, then $q\kappa'(q) - \kappa(q) \geq 0$ so that condition [\(3.3\)](#) in [\[39\]](#) fails, and therefore $M(\infty; q) = 0$. This proves [\(ii\)](#). For [\(i\)](#), we follow the lines of [\[32\]](#). From [Proposition 1.2.1](#) we know that almost surely, for every $t \geq 0$ and $(t_n, n \in \mathbb{N})$ such that $t_n \downarrow t$ as $n \rightarrow \infty$, the sequence of random functions on K

$$q \mapsto (1 + Z_1(t_n))^{-q} \sum_{i=1}^{\infty} Z_i^q(t_n), \quad n \in \mathbb{N},$$

which all are non-increasing because of the leading factors $(1 + Z_1(t_n))^{-q}$, converges pointwise to the random continuous function

$$q \mapsto (1 + Z_1(t))^{-q} \sum_{i=1}^{\infty} Z_i^q(t).$$

By a classical counterpart of Dini's theorem (see e.g. [97, Problem II.3.127]), the convergence is actually uniform in $q \in K$. Multiplying by the continuous function $q \mapsto (1 + Z_1(t))^q \exp(-t\kappa(q))$ and dealing similarly with the left limits of \mathbf{Z} , we can therefore view $(M(t; \cdot))_{t \geq 0}$ as a martingale with càdlàg paths in the Banach space $\mathcal{C}(K, \mathbb{R})$ of continuous functions on K .

We now observe that the process

$$\mathcal{Z}^n = \sum_{i=1}^{\infty} \delta_{\log Z_i(n)}, \quad n \in \mathbb{N},$$

is a branching random walk (in discrete time), and check the two conditions to apply the results of Biggins [41, Theorems 1 & 2]: first, if $q \in (\underline{q}, \bar{q})$ then by Proposition 1.2.8 we have $\mathbb{E}[M(1; q)^\gamma] < \infty$ for some $\gamma \in (1, 2]$; second, using Lemma 1.2.2 we can find $\alpha \in (1, \gamma]$ such that $\alpha q \in (q, \bar{q})$, hence

$$\frac{m(\alpha q)}{m(q)^\alpha} = \exp \left\{ \alpha q \left(\frac{\kappa(\alpha q)}{\alpha q} - \frac{\kappa(q)}{q} \right) \right\} < 1. \quad (1.16)$$

It thus follows that the $\mathcal{C}(K, \mathbb{R})$ -valued discrete-time martingale

$$M(n; \cdot): q \mapsto \exp(-n\kappa(q)) \int e^{qx} \mathcal{Z}^n(dx), \quad n \in \mathbb{N},$$

converges as $n \rightarrow \infty$ to a random function $M(\infty; \cdot) \in \mathcal{C}(K, \mathbb{R})$, almost surely and in mean. Now, the uniform norm $\|\cdot\|$ of $\mathcal{C}(K, \mathbb{R})$ is a convex map and thus for any integer $n \geq 0$ the process $(\|M(t; \cdot) - M(n; \cdot)\|)_{t \geq n}$ is a nonnegative submartingale with càdlàg paths. If $t \geq 0$ and n is chosen so that $n \leq t < n + 1$, we have in particular

$$\mathbb{E}[\|M(t; \cdot) - M(n; \cdot)\|] \leq \mathbb{E}[\|M(n + 1; \cdot) - M(n; \cdot)\|],$$

and consequently

$$\mathbb{E}[\|M(t; \cdot) - M(\infty; \cdot)\|] \leq \mathbb{E}[\|M(n + 1; \cdot) - M(n; \cdot)\|] + \mathbb{E}[\|M(n; \cdot) - M(\infty; \cdot)\|].$$

The convergence in $L^1(\mathbb{P})$ of the continuous-time martingale $(M(t; \cdot))_{t \geq 0}$ then follows from the one in discrete time. The almost sure convergence is established by applying Doob's maximal inequality and the Borel–Cantelli lemma, like in the proof of [32]: indeed for every $\varepsilon > 0$,

$$\mathbb{P}(\exists t \geq n: \|M(t; \cdot) - M(n; \cdot)\| > \varepsilon) \leq \varepsilon^{-1} \mathbb{E}[\|M(\infty; \cdot) - M(n; \cdot)\|] \xrightarrow[n \rightarrow \infty]{} 0.$$

We finally deal with the almost sure positivity of the terminal value $M(\infty; q)$ conditionally on non-extinction. We derive from the branching property at time n that, for every $q \in (\underline{q}, \bar{q})$,

$$\mathbb{P}(M(\infty; q) = 0 \mid \mathcal{Z}^n) = \prod_{z \in \mathcal{Z}^n} \mathbb{P}_z(M(\infty; q) = 0),$$

where by scaling, the probability $\mathbb{P}_z(M(\infty; q) = 0)$ does actually not depend on the initial size z . Hence $\rho := \mathbb{P}(M(\infty; q) = 0) = \mathbb{E}[\rho^{\#(n)}]$, where $\#(n) := \mathcal{Z}^n(\mathbb{R})$, the number of particles at time $n \in \mathbb{N}$, defines a supercritical Galton–Watson process. Since $\rho < 1$ (because $\mathbb{E}[M(\infty; q)] = \mathbb{E}[M(0; q)] = 1$), ρ is its probability of extinction. The two events $\{\text{extinction}\} \subseteq \{M(\infty; q) = 0\}$ having thus the same probability we conclude that they coincide up to a negligible event. \blacksquare

Remark 1.2.9. Under the conditions of [Theorem 1.2.3](#) we have also from [[41](#), Theorem 5] that for each $q \in (q, \bar{q})$ and $\alpha \in (1, \gamma]$ as in [\(1.16\)](#), the martingale $(M(t; q))_{t \geq 0}$ converges in $L^\alpha(\mathbb{P})$.

We close this section with the proof of [Corollary 1.2.4](#).

Proof of [Corollary 1.2.4](#). Let us define the tilted measures

$$\mathcal{Z}_q^t(dx) := \frac{e^{qx}}{m(q)^t} \mathcal{Z}^t(dx), \quad t \geq 0, \quad \text{and} \quad \mu_q(dx) := \frac{e^{qx}}{m(q)} \mu(dx).$$

Using [\(1.7\)](#), μ_q is a probability measure with mean

$$c_q := m(q)^{-1} \mathbb{E} \left[\sum_{i=1}^{\infty} Z_i^q(1) \log Z_i(1) \right] = \kappa'(q),$$

and variance

$$\sigma_q^2 := m(q)^{-1} \mathbb{E} \left[\sum_{i=1}^{\infty} Z_i^q(1) \log^2 Z_i(1) \right] - c_q^2 = \kappa''(q).$$

On the one hand, we observe that for every $n \in \mathbb{N}$,

$$e^{-(\kappa(q) - q\kappa'(q))n} \sum_{i=1}^{\infty} f\left(Z_i(n) e^{-\kappa'(q)n}\right) = \int_{\mathbb{R}} f(e^x) e^{-qx} \mathcal{Z}_q^n(nc_q + dx).$$

On the other hand, by a local limit theorem due to Stone [[101](#), Theorem 2],

$$\sqrt{n} \mu_q^{(*n)}(nc_q + dx) \approx p_q\left(\frac{x}{\sqrt{n}}\right) dx, \quad n \rightarrow \infty,$$

uniformly for $x \in \mathbb{R}$ and q in compact subsets of (q, ∞) , where $\mu_q^{(*n)}$, $n \in \mathbb{N}$, is the n^{th} convolution of μ_q with itself and $p_q(x)$ denotes the Gaussian density with mean 0 and variance σ_q^2 . Thanks to the uniform convergence in [Theorem 1.2.3](#), this in terms of the branching random walk translates into

$$\sqrt{n} \mathcal{Z}_q^n(nc_q + dx) \approx M(\infty; q) p_q\left(\frac{x}{\sqrt{n}}\right) dx, \quad n \rightarrow \infty,$$

uniformly for $x \in \mathbb{R}$ and q in compact subsets of (q, \bar{q}) , almost surely. The corollary then results from a Riemann sum argument. We leave details and refer the interested reader to [Corollary 4](#) in [[41](#)] and its “continuous-time” extension discussed on page 150 there. \blacksquare

1.2.3 On the largest fragment

Alike the observation made by Bertoin [23, Equation (9)] for pure homogeneous fragmentations, [Theorem 1.2.3](#) readily reveals the asymptotic velocity of the largest fragment Z_1 : if \mathbb{P}^* denotes the probability \mathbb{P} conditionally on non-extinction, then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log Z_1(t) = \kappa'(\bar{q}), \quad \mathbb{P}^*\text{-almost surely,}$$

where $\kappa'(\bar{q}) = \kappa(\bar{q})/\bar{q}$ and provided that $\bar{q} > \underline{q}$. We shall now delve deeper into the analogy with branching random walks and tell a bit more about the asymptotic expansion of $Z_1(t)$. To this end, we proceed to a renormalization of the branching process \mathcal{Z}^t : specifically, for

$$\tilde{\mathcal{Z}}^t := \sum_{i=1}^{\infty} \delta_{\kappa(\bar{q})t - \bar{q} \log Z_i(t)},$$

which has the log-Laplace transform

$$\tilde{\kappa}(q) := \frac{1}{t} \log \mathbb{E} \left[\int_{\mathbb{R}} e^{-qy} \tilde{\mathcal{Z}}^t(dy) \right] = \kappa(q\bar{q}) - q\kappa(\bar{q}), \quad q \geq 0,$$

we are now in the so called *boundary case*, namely $\tilde{\kappa}(0) > 0$ and $\tilde{\kappa}(1) = \tilde{\kappa}'(1) = 0$. Let us also introduce the process

$$D(t) := \int_{\mathbb{R}} ye^{-y} \tilde{\mathcal{Z}}^t(dy) = -\bar{q} \left. \frac{d}{dq} M(t; q) \right|_{q=\bar{q}}, \quad t \geq 0,$$

which is easily seen from the branching property to be a martingale (rightly called the *derivative martingale*) and will serve our purpose.

Corollary 1.2.10. Suppose (1.10), (1.11), and $\bar{q} > \underline{q}$.

(a) Then

$$\lim_{t \rightarrow \infty} \frac{\log Z_1(t) - \kappa'(\bar{q})t}{\log t} = -\frac{3}{2\bar{q}}, \quad \text{in } \mathbb{P}^*\text{-probability.} \quad (1.17)$$

(b) If further μ is non-lattice, then there exist a constant $C^* > 0$ and a nonnegative random variable D_∞ such that, for every $x > 0$,

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(t^{3/2\bar{q}} e^{-\kappa'(\bar{q})t} Z_1(t) \leq x \right) = \mathbb{E} [e^{-C^* D_\infty/x}]. \quad (1.18)$$

Moreover $D_\infty > 0$, \mathbb{P}^* -almost surely.

Remark 1.2.11. (i) Kyprianou et al. [77] recently derived an analogue of (a) for pure homogeneous fragmentations. However the method we employ here (for both statements)

is different: basically, we directly transfer the known results on branching random walks to discrete skeletons of the growth-fragmentation, and then infer the behavior of the whole process with the help of [Lemma 1.2.12](#) below.

(ii) The logarithmic fluctuations [[100](#), Theorem 5.23] also show that

$$\limsup_{t \rightarrow \infty} \frac{\log Z_1(t) - \kappa'(\bar{q})t}{\log t} \geq -\frac{1}{2\bar{q}}, \quad \mathbb{P}^*\text{-almost surely}$$

(we conjecture that there is in fact equality), so the convergence [\(1.17\)](#) cannot be strengthened.

(iii) Other interesting facts from the literature of branching random walks could be inherited. For instance, by specializing a recent result due to Aidékon and Shi [[100](#), Theorem 5.29] one infers a so called Seneta–Heyde renormalization for the convergence of $M(t; \bar{q})$ in [Theorem 1.2.3\(ii\)](#): namely

$$\lim_{t \rightarrow \infty} \sqrt{t} M(t; \bar{q}) = \sqrt{\frac{2}{\pi \bar{q}^2 \kappa''(\bar{q})}} D_\infty, \quad \text{in } \mathbb{P}^*\text{-probability}$$

(with D_∞ as above), and again this convergence cannot be strengthened (the lim sup is infinite \mathbb{P}^* -almost surely).

Lemma 1.2.12 (Croft–Kingman, [[73](#), Theorem 2]). Let $f: (0, \infty) \rightarrow \mathbb{R}$ be a continuous function such that for every $h > 0$, the sequence $f(nh)$, $n \in \mathbb{N}$, converges. Then $f(x)$ has a limit as $x \rightarrow \infty$.

Proof of [Corollary 1.2.10](#). Let $h > 0$ be any fixed time mesh. It is plain from the branching property that $\tilde{\mathcal{Z}}^{nh}$ corresponds to the individuals at generation $n \in \mathbb{N}$ of a branching random walk on \mathbb{R} whose offspring point process is distributed like $\tilde{\mathcal{Z}}^h$. On the one hand, there is

$$\mathbb{E} \left[\int_{\mathbb{R}} y^2 e^{-y} \tilde{\mathcal{Z}}^h(dy) \right] = h \tilde{\kappa}''(1) = h \bar{q}^2 \kappa''(\bar{q}) < \infty.$$

On the other hand, with the notation $u_+ := \max(u, 0)$ for any $u \in \mathbb{R}$ and

$$X := \int_{\mathbb{R}} e^{-y} \tilde{\mathcal{Z}}^h(dy) = M(h; \bar{q}), \quad \tilde{X} := \int_{\mathbb{R}} y_+ e^{-y} \tilde{\mathcal{Z}}^h(dy),$$

[Proposition 1.2.8](#) readily entails that

$$\mathbb{E} [X (\log X)_+^2] < \infty \quad \text{and} \quad \mathbb{E} [\tilde{X} (\log \tilde{X})_+] < \infty.$$

(For the second, we use that $|\log(x)| \leq (x^\varepsilon + x^{-\varepsilon})/\varepsilon$ for every $x > 0$ and any $0 < \varepsilon < \bar{q} - \underline{q}$.) As a result, Assumption (H) of [100, § 5.1] is fulfilled³. From Theorem 5.12 there, we obtain that for every $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\frac{\log Z_1(nh) - \kappa'(\bar{q})nh}{\log nh} + \frac{3}{2\bar{q}}\right| > \varepsilon\right) \xrightarrow{n \rightarrow \infty} 0, \quad \text{for each } h > 0.$$

As the left-hand side is a continuous function of $t := nh$, the proof of (a) follows from Lemma 1.2.12. Similarly, when μ is non-lattice, then $\tilde{\mathcal{Z}}^h$ is non-lattice as well and Theorem 5.15 of [100] (likewise, Theorem 1.1 of [2]) applies: for every $y \in \mathbb{R}$ and every $h > 0$, the quantity

$$\mathbb{P}\left(\log Z_1(nh) - \kappa'(\bar{q})nh + \frac{3}{2\bar{q}} \log nh \leq y\right)$$

has a limit as $n \rightarrow \infty$. Applying Croft–Kingman’s lemma once more gives the convergence (1.18), where the limit is e.g. that for $h = 1$. In (b), the random variable D_∞ can be taken as the terminal value of the derivative martingale $(D(n), n \in \mathbb{N})$: Theorem 5.2 of [100] shows that it exists almost surely and is positive on non-extinction. That D_∞ is at least nonnegative holds simply because the smallest atom of $\tilde{\mathcal{Z}}^n$,

$$\kappa(\bar{q})n - \bar{q} \log Z_1(n),$$

is bounded from below by $-\log M(n; \bar{q})$, which tends to ∞ a.s. due to Theorem 1.2.3.(ii). ■

1.2.4 On abnormally large fragments

In this last section we give an estimation for the probability of presence of fragments at scale greater than $\kappa'(\bar{q})$ in a compensated fragmentation. We simply perform the very same analysis as done in [32] for homogeneous pure fragmentations. Let us fix two real numbers $\alpha < \beta$ and introduce

$$U(t, x) := \mathbb{P}\left(\mathcal{Z}^t([x + \alpha, x + \beta]) > 0\right),$$

$$V(t, x) := \mathbb{E}\left[\mathcal{Z}^t([x + \alpha, x + \beta])\right],$$

for every $t \geq 0$ and $x \in \mathbb{R}$.

Corollary 1.2.13. Let $q > \underline{q}$. Suppose (1.10), μ non-lattice, and either (1.11) or $q \geq 1$.

³Strictly speaking, [100] also requires a *finite* branching ($\tilde{\mathcal{Z}}^t(\mathbb{R}) < \infty$ a.s.), but this condition turns out to be unnecessary (see e.g. [84]; besides, it is not needed in the latest version of [2] that we invoke to prove (b), and the conclusion of (b) obviously implies (a)).

(i) Then

$$\lim_{t \rightarrow \infty} \sqrt{t} e^{-(\kappa(q) - q\kappa'(q))t} V(t, t\kappa'(q)) = \frac{e^{-q\alpha} - e^{-q\beta}}{q\sqrt{2\pi\kappa''(q)}}.$$

(ii) If further $q > \bar{q}$ (so that $\kappa(q) - q\kappa'(q) < 0$), then

$$\lim_{t \rightarrow \infty} \frac{U(t, t\kappa'(q))}{V(t, t\kappa'(q))} = K_q,$$

where K_q is some positive finite constant.

Remark 1.2.14. In the range $q \in (\underline{q}, \bar{q})$, (i) is the counterpart in mean of the convergence stated in [Corollary 1.2.4](#) for $f := \mathbb{1}_{[\alpha, \beta]}$. This convergence thus holds in $L^1(\mathbb{P})$ thanks to the Riesz–Scheffé lemma.

Proof. The proof is a straightforward adaptation of that of Theorem 5 in [\[32\]](#). In our setting, we have

$$a := \kappa'(q), \quad \Lambda^*(a) := q\kappa'(q) - \kappa(q),$$

and for any time mesh $h > 0$,

$$\Lambda_h(q) := \log \mathbb{E} \left[\sum_{i=1}^{\infty} Z_i^q(h) \right] = h\kappa(q),$$

by [\(1.7\)](#). From Equation (12) in [\[31\]](#) we readily get

$$\sqrt{nh} e^{nh\Lambda^*(a)} V(nh, anh) \xrightarrow{n \rightarrow \infty} \frac{e^{-q\alpha} - e^{-q\beta}}{q\sqrt{2\pi\kappa''(q)}}. \quad (1.19)$$

If furthermore $\Lambda^*(a) > 0$ (i.e. $q > \bar{q}$), then [Proposition 1.2.8](#) ensures that the conditions of Theorem 2 in [\[31\]](#) are fulfilled and therefore

$$\frac{U(nh, anh)}{V(nh, anh)} \xrightarrow{n \rightarrow \infty} K_q^{(h)}, \quad (1.20)$$

where $K_q^{(h)}$ is a positive constant. Besides, the time mesh $h > 0$ in [\(1.19\)](#) and [\(1.20\)](#) is arbitrary and the left-hand sides are both continuous functions of the variable $t := nh$. The existence of limits as $t \rightarrow \infty$ then comes again from [Lemma 1.2.12](#). (In particular, the constant $K_q^{(h)}$ in [\(1.20\)](#) does actually not depend on h .) ■

1.3 Self-similar growth-fragmentations

As opposed to the previous part, a *self-similar* growth-fragmentation will now allow inhomogeneous fragmentation rates. Loosely speaking, one can picture it as a homogeneous fragmentation where each fragment is “sped up” all along its history by a fixed power $\alpha \in \mathbb{R}$ of its current size. If as before the Laplace transform of the fragment sizes at genealogical births may be related through a cumulant function κ , self-similarity induces significant changes when we look at processes over time. Mainly, in the case $\alpha > 0$ we shall mostly focus on, where positive growth in the fragments is thus compensated by higher dislocation rates, the typical sizes will no longer be of exponential order (given through the derivative κ'), but will instead encounter a polynomial decay of the type $t^{-1/\alpha}$. Another side effect is that additive martingales appear less nicely, so specific assumptions will be needed.

1.3.1 Prerequisites

We begin with a quick summary of the construction and important properties of self-similar growth-fragmentations processes. These were introduced in [19]; greater details as well as some applications to random planar maps can be found in [24].

Let $\xi := (\xi(t), t \geq 0)$ be a possibly killed Lévy process and $(\sigma^2, b, \Lambda, \mathbf{k})$ denote its characteristic quadruple in the following sense. The Gaussian coefficient $\sigma^2 \geq 0$, the drift coefficient $b \in \mathbb{R}$, the Lévy measure Λ (that is, a measure on \mathbb{R} with $\int (1 \wedge y^2) \Lambda(dy) < \infty$), and the killing rate $\mathbf{k} \in [0, \infty)$ may be recovered from this slight variation of the Lévy–Khinchin formula:

$$E[\exp(q\xi(t))] = \exp(t\Psi(q)), \quad t, q \geq 0,$$

the Laplace exponent Ψ being written in the form

$$\Psi(q) := -\mathbf{k} + \frac{1}{2}\sigma^2 q^2 + bq + \int_{\mathbb{R}} (e^{qy} - 1 + q(1 - e^y)) \Lambda(dy), \quad q \geq 0.$$

The case $\Lambda((-\infty, 0)) = 0$ will be uninteresting and is therefore excluded. We shall also assume that $\int_{(1, \infty)} e^y \Lambda(dy) < \infty$ (which always holds when the support of Λ is bounded from above), and that

$$\mathbf{k} > 0 \quad \text{or} \quad (\mathbf{k} = 0 \quad \text{and} \quad \Psi'(0+) \in [-\infty, 0)) \quad (1.21)$$

(in other words, that $\Psi(q) < 0$ for some $q > 0$). This latter condition means that $\xi(t)$ either has a finite lifetime or tends to $-\infty$ as $t \rightarrow \infty$, almost surely. Let now $\alpha \in \mathbb{R}$ and, for each $x > 0$, P_x be the law of the process

$$X(t) := x \exp\{\xi(\tau_{x^\alpha t})\}, \quad t \geq 0,$$

where

$$\tau_t := \inf \left\{ u \geq 0 : \int_0^u \exp(-\alpha\xi(s)) ds \geq t \right\},$$

and with the convention that $X(t) := \partial$ for $t \geq \zeta := x^{-\alpha} \int_0^\infty \exp(-\alpha\xi(s)) ds$. This *Lamperti transform* ([78]; see also [76, Theorem 13.1]) makes $(X, (P_x)_{x>0})$ be a positive self-similar Markov process (for short, *pssMp*), in the sense that:

$$\text{For all } x > 0, \quad \text{the law of } (xX(x^\alpha t), t \geq 0) \text{ under } P_1 \text{ is } P_x. \quad (1.22)$$

(Following the terminology in [38], we say that X is a pssMp with index $1/(-\alpha)$.) Moreover, this transformation is reversible, and since the law of the Lévy process ξ is uniquely determined by its Laplace exponent Ψ , the pair (Ψ, α) characterizes the law of X ; we call it the characteristics of the pssMp X . Note that under (1.21), X either is eventually absorbed to the cemetery point ∂ added to the positive half-line $(0, \infty)$, or it converges to 0 as $t \rightarrow \infty$.

The process X above will portray the typical size of a cell in the system and is thus referred as the *cell process*. Specifically, a cell system is a process $((\mathcal{X}_u, b_u), u \in \mathbb{U})$ indexed on the Ulam–Harris tree

$$\mathbb{U} := \bigcup_{i=0}^{\infty} \mathbb{N}^i,$$

with the following classical notations: \mathbb{N}^0 is reduced to the *root* of \mathbb{U} , labeled \emptyset , and for any *node* $u := u_1 u_2 \cdots u_i \in \mathbb{U}$ in this tree, $|u| := i \in \{0, 1, 2, \dots\}$ refers to its *generation* (or *height*), and $u1, u2, \dots$ to its children. For each $u \in \mathbb{U}$, $(\mathcal{X}_u(t), t \geq 0)$ is a càdlàg process on $(0, \infty) \cup \{\partial\}$ driven by X and recording the size of the cell labeled by u since its birth time b_u , which shall be implicitly encoded in the notation \mathcal{X}_u . In this system \mathcal{X}_\emptyset refers to Eve cell, born at time $b_\emptyset := 0$, and each negative⁴ jump of a cell is interpreted as the birth of a daughter cell. More precisely for every $u \in \mathbb{U}$ and $j \in \mathbb{U}$, \mathcal{X}_{uj} is the process of the j^{th} daughter cell of u , born at the absolute time $b_{uj} := b_u + \beta_{uj}$, where β_{uj} is the instant of the j^{th} biggest positive jump⁵ of $-\mathcal{X}_u$. The law \mathcal{P}_x of \mathcal{X} is then defined recursively as the unique probability distribution such that \mathcal{X}_\emptyset has the law P_x and, conditionally on \mathcal{X}_\emptyset , the processes $(\mathcal{X}_{iu}, u \in \mathbb{U}), i \in \mathbb{N}$, are independent with respective laws $\mathcal{P}_{x_i}, i \in \mathbb{N}$, where $(x_1, \beta_1), (x_2, \beta_2), \dots$ is the sequence of positive jump sizes and times of $-\mathcal{X}_\emptyset$, sorted by decreasing sizes (with $\beta_i < \beta_{i+1}$ if $x_i = x_{i+1}$).

⁴In [19], the author only considered spectrally negative Lévy processes so jumps were always of negative sign. However, allowing the cells to have sudden positive growths during their lifetimes is relevant in some applications such as those exposed in [24]. Their slightly more general setting, which we have also chosen to adopt, does not invalidate the results of [19] — the significant point being that only the negative jumps correspond to division events while the possible positive jumps just remain part of the trajectories of the cells.

⁵Recall that the cell process is either absorbed in finite time or converges to 0, so the positive jumps of $-\mathcal{X}_u$ may indeed be ranked in the decreasing order.

Here, we agree that $x_i := \partial$ and $\beta_i := \infty$ if \mathcal{X}_\emptyset has less than i negative jumps, and we let \mathcal{P}_∂ denote the law of the degenerate cell system where $\mathcal{X}_u := \partial$ for every $u \in \mathbb{U}$, $b_\emptyset := 0$ and $b_u := \infty$ for $u \neq \emptyset$.

The associated growth-fragmentation process is the process of the family of (the sizes of) all alive cells in the system:

$$\mathbf{X}(t) := \left\{ \left\{ \mathcal{X}_u(t - b_u) : u \in \mathbb{U}, b_u \leq t < d_u \right\} \right\}, \quad t \geq 0$$

(with b_u and d_u denoting respectively the birth time and the death time of the cell labeled by u). Additionally to the scaling parameter α , one other specific quantity is

$$\kappa(q) := \Psi(q) + \int_{(-\infty, 0)} (1 - e^y)^q \Lambda(dy), \quad q \geq 0.$$

If $\alpha = 0$ and Λ has support in $[-\log 2, 0]$, then [19, Proposition 3] \mathbf{X} is merely a compensated fragmentation of the type considered in Section 1.2, and the notation κ there is compatible with the one we use here: more precisely, \mathbf{X} has diffusion coefficient σ^2 , growth rate b and dislocation measure $\nu := \mathbf{k} \delta_{\mathbf{0}} + \nu_2$, where $\mathbf{0} := (0, 0, \dots) \in \mathcal{P}$ is the null mass-partition and ν_2 is the image of Λ by the map $x \in [-\log 2, 0] \mapsto (e^x, 1 - e^x, 0, \dots) \in \mathcal{P}$ (the fragmentation is binary).

We shall work under the assumption

$$\exists q \geq 0, \kappa(q) \leq 0, \tag{1.23}$$

see [19, Theorem 2]. Then for each time t , the family $\mathbf{X}(t)$ may be ranked in the non-increasing order, i.e. $\mathbf{X}(t) := (X_1(t), X_2(t), \dots)$ with $X_1(t) \geq X_2(t) \geq \dots \geq 0$. Further, the self-similarity property (1.22) extends to the process $\mathbf{X} := (\mathbf{X}(t), t \geq 0)$, and there is the branching property. Formally, if \mathbf{P}_x denotes the law of \mathbf{X} under \mathcal{P}_x , then firstly, for every $x > 0$, the distribution of $(x\mathbf{X}(x^\alpha t), t \geq 0)$ under \mathbf{P}_1 is \mathbf{P}_x , and secondly, for each $s \geq 0$, conditionally on $\mathbf{X}(s) = (x_1, x_2, \dots)$, the process $(\mathbf{X}(t + s), t \geq 0)$ is independent of $(\mathbf{X}(r), 0 \leq r \leq s)$ and has the same law as the non-increasing rearrangement of the family $(X_j^{(i)}, i, j \in \mathbb{N})$, where the $\mathbf{X}^{(i)}$ are independent self-similar growth-fragmentations with respective laws \mathbf{P}_{x_i} .

In the sequel we mainly focus on large time asymptotics for the growth-fragmentation process \mathbf{X} . Since we can refer to Section 1.2 when $\alpha = 0$, and because the growth-fragmentation is eventually extinct when the scaling parameter α is negative [19, Corollary 3], we will mostly suppose $\alpha > 0$. Note in this case that (1.23) is a necessary and sufficient condition preventing local explosion of the fragmentation [33], that is a phenomenon causing infinitely many particles of arbitrary large sizes to be produced in almost surely finite time

(which in particular would impede us to list the elements of $\mathbf{X}(t)$ in the non-increasing order). Like in [Section 1.2](#), the function $\kappa: [0, \infty) \rightarrow (-\infty, \infty]$ will be of greatest importance in the study. It is clearly convex; therefore the equation $\kappa(q) = 0$ has at most two solutions. We assume from here on that these two solutions exist — more precisely that the *Malthusian hypotheses* hold:

$$\text{there exist } 0 < \omega_- < \omega_+ \text{ such that } \kappa(\omega_-) = \kappa(\omega_+) = 0 \text{ and } \kappa'(\omega_-) > -\infty \quad (1.24)$$

(note then that $\kappa'(\omega_-) < 0$, by convexity). Condition [\(1.24\)](#) implies that $\kappa(q) < 0$ for some $q > 0$, which in turn implies [\(1.23\)](#), and [\(1.21\)](#) (because $\Psi \leq \kappa$).

As before, limit theorems for the growth-fragmentation process \mathbf{X} will involve the terminal value of some additive martingale, namely the *Malthusian martingale*

$$M^-(t) := \sum_{i=1}^{\infty} X_i^{\omega_-}(t), \quad t \geq 0.$$

In this direction, results of [\[24\]](#) will be of fundamental use; we restate some of them here for sake of reference.

Proposition 1.3.1 (from [\[24\]](#), Theorem 3.10.(ii), Corollaries 3.7.(ii) and 3.9).

Suppose $\alpha > 0$.

- (i) The process $(M^-(t), t \geq 0)$ under \mathbf{P}_x is a uniformly integrable martingale; more precisely it is bounded in $L^p(\mathbf{P}_x)$ for every $1 < p < \omega_+/\omega_-$.
- (ii) For every $0 < q < (\omega_+ - \omega_-)/\alpha$, the process

$$\sum_{i=1}^{\infty} X_i^{q\alpha + \omega_-}(t), \quad t \geq 0,$$

is a supermartingale converging to 0 in $L^1(\mathbf{P}_x)$: more precisely,

$$\mathbf{E}_x \left[\sum_{i=1}^{\infty} X_i^{q\alpha + \omega_-}(t) \right] \sim c(q) x^{\omega_-} t^{-q}$$

as $t \rightarrow \infty$, for some constant $c(q) > 0$.

Remark 1.3.2. We find relevant to mention that [\[24\]](#) also introduced the genealogical martingale

$$\mathcal{M}^-(n) := \sum_{|u|=n+1} \mathcal{X}_u^{\omega_-}(0), \quad n \geq 0,$$

called the *intrinsic martingale*, which under \mathcal{P}_x is always uniformly integrable. When $\alpha \geq 0$, there is the remarkable fact

$$M^-(t) = \mathcal{E}_x[\mathcal{M}^-(\infty) \mid \mathcal{F}_t], \quad t \geq 0,$$

with $(\mathcal{F}_t)_{t \geq 0}$ the canonical filtration of \mathbf{X} . In particular, $M^-(\infty) = \mathcal{M}^-(\infty)$ almost surely.

Additive martingales — in the present context, the Malthusian martingale $(M^-(t))_{t \geq 0}$, are of important interest since the celebrated work of Lyons et al. [83]. Roughly speaking, one can perform a change of probability measure in terms of the terminal value $M^-(\infty)$ so that the genealogical system may be observed from the point of view of a randomly tagged branch. Specifically, write $\partial\mathbb{U}$ for the set of *leaves* of \mathbb{U} , each of which determines a unique branch from the root. For every leaf $\ell \in \partial\mathbb{U}$, let $\ell(n)$ denote its unique ancestor at generation $n \geq 0$, and $\mathcal{X}_\ell := (\mathcal{X}_\ell(t), t \geq 0)$ be the process of the cell on the branch from \emptyset to ℓ :

$$\mathcal{X}_\ell(t) := \mathcal{X}_{\ell[t]}(t - b_{\ell[t]}), \quad t \geq 0,$$

where $\ell[t]$ labels the cell in this branch which is alive at time t (i.e. $\ell[t]$ is the unique ancestor u of ℓ such that $b_u \leq t < b_{\ell(|u|+1)}$), with the convention that $\mathcal{X}_\ell(t) := \partial$ for $t > \limsup_{n \rightarrow \infty} b_{\ell(n)} =: b_\ell$. Next we consider a random leaf $\mathcal{L} \in \partial\mathbb{U}$ and we define for every $x > 0$ the joint distribution $\widehat{\mathcal{P}}_x^-$ of $(\mathcal{X}, \mathcal{L})$ as follows. Under $\widehat{\mathcal{P}}_x^-$, the law of $\mathcal{X} := (\mathcal{X}_u, u \in \mathbb{U})$ is absolutely continuous with respect to \mathcal{P}_x with density $x^{-\omega^-} M^-(\infty)$, and the law of \mathcal{L} conditionally on \mathcal{X} is

$$\widehat{\mathcal{P}}_x^-(u \text{ ancestor of } \mathcal{L} \mid \mathcal{X}) := \lim_{n \rightarrow \infty} \frac{1}{\mathcal{M}^-(\infty)} \sum_{|v|=n} \mathcal{X}_{uv}^{\omega^-}(0). \quad (1.25)$$

Denoting $\widehat{\mathcal{X}} := \mathcal{X}_{\mathcal{L}}$ the *randomly tagged cell*, Bertoin et al. [24] derived:

Proposition 1.3.3 (from [24, Theorem 4.7 and Proposition 4.6]).

- (i) The process $(\widehat{\mathcal{X}}, (\widehat{\mathcal{P}}_x^-)_{x>0})$ is a pssMp with characteristics (Φ^-, α) , where

$$\Phi^-(q) := \kappa(q + \omega_-), \quad q \geq 0. \quad (1.26)$$

- (ii) *Many-to-one formula.* For every $x > 0$, every $t \geq 0$, and every measurable function $f: (0, \infty) \rightarrow (0, \infty)$, we have

$$\mathbf{E}_x \left[\sum_{i=1}^{\infty} X_i^{\omega^-}(t) f(X_i(t)) \right] = x^{\omega^-} \widehat{\mathcal{E}}_x^- \left[f(\widehat{\mathcal{X}}(t)) \right], \quad (1.27)$$

with the convention $f(\partial) := 0$.

Formula (1.27) will be a key ingredient for our purpose. Roughly speaking, it says that the intensity of the weighted point measure $\sum_{z \in \mathbf{X}} z^{\omega_-} \delta_z$ is captured by the law of the randomly tagged cell $\widehat{\mathcal{X}}$ (hence the denomination “many-to-one”).

When $\alpha > 0$, unlike in the homogeneous case, a *polynomial* decrease in the size of the fragments is expected. Large-time asymptotics for their empirical measure will be retrieved in the next section. In Section 1.3.3 we center our attention on the largest fragment. Lastly, in Section 1.3.4, we discuss the convergence of the empirical measure of the fragments taken at the instant when they become smaller than a vanishing threshold.

1.3.2 Convergence of the empirical measure

We are here especially interested in the convergence of the empirical measure $\rho_t^{(\alpha)}$ given by

$$\langle \rho_t^{(\alpha)}, f \rangle := \sum_{i=1}^{\infty} X_i^{\omega_-}(t) f(t^{1/\alpha} X_i(t)),$$

for $\alpha > 0$. From now on, we shall suppose that the Lévy process ξ^- associated with the tagged cell $\widehat{\mathcal{X}}$ via Lamperti’s transformation is not *arithmetic*, in the sense that there is no $r > 0$ such that $\mathbb{P}(\xi^-(t) \in r\mathbb{Z}) = 1$ for all $t \geq 0$. To state our result, let us define the probability distribution ρ on $(0, \infty)$ by

$$\int_0^{\infty} f(y) \rho(dy) := \frac{-1}{\alpha \kappa'(\omega_-)} E[I^{-1} f(I^{1/\alpha})],$$

where

$$I := \int_0^{\infty} \exp(\alpha \xi^-(s)) ds \tag{1.28}$$

is the so called *exponential functional* of $\alpha \xi^-$. The following completes the results of Bertoin [23] and Bertoin and Gneden [26] relative to self-similar pure fragmentations, and differs substantially from the homogeneous case (Corollary 1.2.4).

Theorem 1.3.4. For every $1 < p < \omega_+/\omega_-$ and for every bounded continuous function $f: (0, \infty) \rightarrow \mathbb{R}$,

$$\lim_{t \rightarrow \infty} \langle \rho_t^{(\alpha)}, f \rangle = M^-(\infty) \int_0^{\infty} f(y) \rho(dy), \quad \text{in } L^p(\mathbf{P}_1).$$

Consequently, the random measure $\rho_t^{(\alpha)}$ converges in \mathbf{P}_1 -probability to $M^-(\infty) \rho$ as $t \rightarrow \infty$, in the space of finite measures on $(0, \infty)$ endowed with the topology of weak convergence.

Remark 1.3.5. Note the presence of the random factor $M^-(\infty)$, which does not appear in [23] because the Malthusian martingale is trivial for *conservative* pure fragmentations⁶.

⁶Since then the cumulant function vanishes at 1 and the total mass of the fragments at any generation is constant.

It does nonetheless appear in the non-conservative case [26]; however the method used there leads to a L^2 -convergence that we cannot hope for growth-fragmentations when $\omega_+/\omega_- < 2$, and it seems anyway difficult to generalize.

Exponential functionals of Lévy processes such as (1.28) arise in a variety of contexts and their laws have been widely studied, see the survey [37] and the recent works [90, 93, 91, 8]. In particular, Pardo et al. [90] showed that under mild assumptions, they can be factorized into the product of two independent exponential functionals associated with companion Lévy processes, and the distributions of both these functionals are uniquely determined by either their positive or their negative moments. To name just one example, in the common situation where ξ^- is spectrally negative ($\Lambda((0, \infty)) = 0$), we have [90, Corollary 2.1] that $I \stackrel{d}{=} J/\Gamma$, with J the exponential functional of the descending ladder height process of $\alpha\xi^-$ and Γ an independent Gamma random variable with parameter $(\omega_+ - \omega_-)/\alpha$. Further, the density of I has a polynomial tail of order $1 + (\omega_+ - \omega_-)/\alpha$ and admits a semi-explicit series expansion.

The distribution of I (likewise, ρ) naturally takes part in asymptotics of the tagged cell $\widehat{\mathcal{X}}$:

Lemma 1.3.6. As $t \rightarrow \infty$, the random variable $t^{1/\alpha}\widehat{\mathcal{X}}(t)$ under $\widehat{\mathcal{P}}_1^-$ converges in distribution to ρ . Moreover, $\int_0^\infty y^{q\alpha} \rho(dy) < \infty$ for every $0 \leq q < 1 + (\omega_+ - \omega_-)/\alpha$.

Proof. Clearly, $(1/\widehat{\mathcal{X}}(t), t \geq 0)$ is a pssMp with self-similarity index $1/\alpha$ associated with $-\xi^-$, where ξ^- is a Lévy process with the Laplace exponent Φ^- in (1.26). According to [38, Theorem 1], all we need to check to prove the first part of the statement is that $-\xi^-(1)$ admits a finite and positive first moment, which is implied by the Malthusian hypotheses (1.24): indeed,

$$E[-\xi^-(1)] = -(\Phi^-)'(0+) = -\kappa'(\omega_-) \in (0, \infty).$$

The existence of moments is quite straightforwardly adapted from the proof of [37, Theorem 3]. ■

We now turn to the proof of Theorem 1.3.4, arguing along the lines of [21, Theorem 1.3]. The main idea is that, by the branching and scaling properties, the empirical measure of the fragments can be rewritten as the sum of identically distributed pieces arising from an intermediate (arbitrary large) time, which are all independent conditionally on the past. With the help of a (conditional) law of large numbers, we are then reduced to a first moment estimate for some additive functional of the growth-fragmentation, which we can work out thanks to the many-to-one formula and the asymptotic behavior of the tagged fragment above.

Proof of Theorem 1.3.4. Using the branching property at time t and the self-similarity of \mathbf{X} we can write, on the event $\{\mathbf{X}(t) = (x_1, x_2, \dots)\}$,

$$\langle \rho_{t+t^2}^{(\alpha)}, f \rangle := \sum_{i=1}^{\infty} X_i^{\omega^-}(t+t^2) f\left((t+t^2)^{1/\alpha} X_i(t+t^2)\right) = \sum_{i=1}^{\infty} \lambda_i(t) Y_i(t),$$

where $\lambda_i(t) := X_i^{\omega^-}(t) = x_i^{\omega^-}$ and

$$Y_i(t) := \sum_{j=1}^{\infty} X_{i,j}^{\omega^-}(x_i^\alpha t^2) f\left((t+t^2)^{1/\alpha} x_i X_{i,j}(x_i^\alpha t^2)\right), \quad (1.29)$$

the families $(X_{i,1}, X_{i,2}, \dots)$, $i \geq 1$, being i.i.d. copies independent of \mathbf{X} , having all the same law \mathbf{P}_1 . Clearly, the Y_i are independent conditionally on $\boldsymbol{\lambda}(t) := (\lambda_1(t), \lambda_2(t), \dots)$ and, if we introduce

$$\bar{Y}_i := \|f\|_\infty \sup_{t \geq 0} \sum_{j=1}^{\infty} X_{i,j}^{\omega^-}(t),$$

then thanks to Proposition 1.3.1.(i) and Doob's maximal inequality, the \bar{Y}_i are i.i.d. random variables in $L^p(\mathbf{P}_1)$ such that $|Y_i(t)| \leq \bar{Y}_i$ for all $t \geq 0$. For the same reason,

$$\sup_{t \geq 0} \mathbf{E}_1 \left[\left(\sum_{i=1}^{\infty} \lambda_i(t) \right)^p \right] < \infty$$

and further, using Proposition 1.3.1.(ii),

$$\lim_{t \rightarrow \infty} \mathbf{E}_1 \left[\sum_{i=1}^{\infty} \lambda_i^p(t) \right] = 0.$$

By a variation of the law of large numbers ([89]; see also [21, Lemma 1.5]) we then have

$$\lim_{t \rightarrow \infty} \sum_{i=1}^{\infty} \lambda_i(t) \left(Y_i(t) - \mathbf{E}_1[Y_i(t) \mid \boldsymbol{\lambda}(t)] \right) = 0, \quad \text{in } L^p(\mathbf{P}_1).$$

Consequently, the proof boils down to showing that

$$\lim_{t \rightarrow \infty} \sum_{i=1}^{\infty} \lambda_i(t) \mathbf{E}_1[Y_i(t) \mid \boldsymbol{\lambda}(t)] = M^-(\infty) \int_0^\infty f(y) \rho(dy), \quad \text{in } L^p(\mathbf{P}_1), \quad (1.30)$$

where, applying the many-to-one formula (1.27),

$$\mathbf{E}_1[Y_i(t) \mid \boldsymbol{\lambda}(t)] = \widehat{\mathcal{E}}_1^- \left[f\left((1+t^{-1})^{1/\alpha} x_i t^{2/\alpha} \widehat{\mathcal{X}}(x_i^\alpha t^2)\right) \right].$$

But we know from Lemma 1.3.6 that as $s \rightarrow \infty$, the law of $s^{1/\alpha} \widehat{\mathcal{X}}(s)$ under $\widehat{\mathcal{P}}_1^-$ converges weakly to ρ . On the one hand, it thus follows that

$$\widehat{\mathcal{E}}_1^- \left[f\left((1+t^{-1})^{1/\alpha} x_i t^{2/\alpha} \widehat{\mathcal{X}}(x_i^\alpha t^2)\right) \right] \xrightarrow{t \rightarrow \infty} \int_0^\infty f(y) \rho(dy)$$

uniformly in i such that, say, $x_i^\alpha t^2 > \sqrt{t}$, i.e. $x_i > t^{-3/2\alpha}$. On the other hand, applying again (1.27), the quantity

$$\sum_{i=1}^{\infty} X_i^{\omega_-}(t) \mathbf{1}_{\{X_i(t) \leq t^{-3/2\alpha}\}} \quad (1.31)$$

has, under \mathbf{P}_1 , mean

$$\widehat{\mathcal{P}}_1^-(t^{1/\alpha} \widehat{\mathcal{X}}(t) < t^{-1/2\alpha}),$$

which tends to 0 as $t \rightarrow \infty$. Since (1.31) is bounded in $L^q(\mathbf{P}_1)$ for every $p < q < \omega_+/\omega_-$, it also converges to 0 in $L^p(\mathbf{P}_1)$ (by Hölder's inequality). Putting everything together yields (1.30), and thus the first part of the statement.

The second part is derived from standard arguments: the space $\mathcal{C}_c((0, \infty))$ of continuous functions on $(0, \infty)$ with compact support being separable, a diagonal extraction procedure easily entails, for every sequence $t_n \rightarrow \infty$, that there exists an extraction $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ such that, almost surely,

$$\forall f \in \mathcal{C}_c((0, \infty)), \quad \langle \rho_{t_{\sigma(n)}}^{(\alpha)}, f \rangle \xrightarrow{n \rightarrow \infty} M^-(\infty) \int_0^\infty f(y) \rho(dy),$$

i.e. $\rho_{t_{\sigma(n)}}^{(\alpha)}$ converges vaguely to $M^-(\infty) \rho$, a.s. Since the total mass is conserved, that is

$$\langle \rho_t^{(\alpha)}, 1 \rangle = \sum_{i=1}^{\infty} X_i^{\omega_-}(t) \xrightarrow{t \rightarrow \infty} M^-(\infty) = \langle M^-(\infty) \rho, 1 \rangle \quad \text{a.s.},$$

the convergence of $\rho_{t_{\sigma(n)}}^{(\alpha)}$ toward $M^-(\infty) \rho$ is actually weak. The conclusion follows easily. ■

The existence of moments for ρ (Lemma 1.3.6) allows us to strengthen Theorem 1.3.4:

Corollary 1.3.7. For every $0 < q < (\omega_+ - \omega_-)/\alpha$, every measurable function $f: (0, \infty) \rightarrow \mathbb{R}$ such that $f(y) = O(y^{q\alpha})$, and every $1 < p < \omega_+/(q\alpha + \omega_-)$,

$$\lim_{t \rightarrow \infty} \langle \rho_t^{(\alpha)}, f \rangle = M^-(\infty) \int_0^\infty f(y) \rho(dy), \quad \text{in } L^p(\mathbf{P}_1).$$

Proof. Approximating $y \mapsto y^{-q\alpha} f(y)$ by simple functions, it is enough to do the proof for $f = f_q: y \mapsto y^{q\alpha}$, that is to prove:

$$\lim_{t \rightarrow \infty} t^q \sum_{i=1}^{\infty} X_i^{q\alpha + \omega_-}(t) = M^-(\infty) \int_0^\infty y^{q\alpha} \rho(dy), \quad \text{in } L^p(\mathbf{P}_1).$$

This is of course not a direct consequence to Theorem 1.3.4 because f_q is not a bounded continuous function; nevertheless we can repeat the argument used in the previous proof. Observing that $q\alpha + \omega_- \in (\omega_-, \omega_+)$ and defining $Y_i(t)$ as in (1.29) but with f_q in place of f , we

easily check with the help of [Proposition 1.3.1](#) and Hölder's inequality that conditionally on \mathbf{X} , the Y_i are independent supermartingales bounded in $L^p(\mathbf{P}_1)$ for every $1 < p < \omega_+/(q\alpha + \omega_-)$. Therefore, all that remains to show is the convergence

$$t^q \mathbf{E}_1 \left[\sum_{i=1}^{\infty} X_i^{q\alpha + \omega_-}(t) \right] = \widehat{\mathcal{E}}_1^- \left[\left(t^{1/\alpha} \widehat{\mathcal{X}}(t) \right)^{q\alpha} \right] \xrightarrow[t \rightarrow \infty]{} \int_0^\infty y^{q\alpha} \rho(dy),$$

where the equality is just an application of the many-to-one formula [\(1.27\)](#). Since we already know that $t^{1/\alpha} \widehat{\mathcal{X}}(t)$ converges in distribution toward ρ ([Lemma 1.3.6](#)), it suffices to show that $((t^{1/\alpha} \widehat{\mathcal{X}}(t))^{q\alpha})_{t \geq 0}$ is bounded in $L^r(\widehat{\mathcal{P}}_1^-)$ for some $r > 1$, which is immediate using again the many-to-one formula and the convergence rate in [Proposition 1.3.1\(ii\)](#) (we can take $1 < r < (\omega_+ - \omega_-)/q\alpha$). \blacksquare

1.3.3 Asymptotic behavior of the largest fragment

For pure self-similar fragmentations with scaling parameter $\alpha > 0$, it is known [\[23\]](#) that the size of the largest fragment decreases like $t^{-1/\alpha}$ as $t \rightarrow \infty$. The same holds for growth-fragmentations⁷:

Theorem 1.3.8. Assume again [\(1.24\)](#), $\alpha > 0$, and that ξ^- is not arithmetic, and suppose further that $\Lambda((0, \infty)) = 0$. Let $S := \{\forall t \geq 0, \mathbf{X}(t) \neq \emptyset\}$ be the non-extinction event, and $\mathcal{P}^* := \mathcal{P}_1(\cdot | S)$. Then

$$\lim_{t \rightarrow \infty} \frac{\log X_1(t)}{\log t} = -\frac{1}{\alpha}, \quad \text{in } \mathcal{P}^*\text{-probability.}$$

Proof of the lower bound. The fact that the \mathcal{P}^* -liminf of $\log X_1(t)/\log t$ as $t \rightarrow \infty$ is at least $-1/\alpha$ follows by comparison with the randomly tagged cell $\widehat{\mathcal{X}}$. Indeed, we know by [Lemma 1.3.6](#) that $\log \widehat{\mathcal{X}}(t)/\log t$ converges to $-1/\alpha$ in $\widehat{\mathcal{P}}_1^-$ -probability. Because $X_1(t)$ is the size of the largest fragment and $\widehat{\mathcal{X}}(t)$ is that of some other fragment in the system, we deduce that for every $\eta > 0$,

$$\widehat{\mathcal{P}}_1^- \left(\frac{\log X_1(t)}{\log t} + \frac{1}{\alpha} < -\eta \right) \leq \widehat{\mathcal{P}}_1^- \left(\frac{\log \widehat{\mathcal{X}}(t)}{\log t} + \frac{1}{\alpha} < -\eta \right) \xrightarrow[t \rightarrow \infty]{} 0.$$

Since $d\widehat{\mathcal{P}}_1^-/d\mathcal{P}_1 = M^-(\infty)$, which by the branching property is positive \mathcal{P}_1 -a.s. on S , the latter convergence also holds with \mathcal{P}^* in place of $\widehat{\mathcal{P}}_1^-$. \blacksquare

⁷For simplicity, we suppose that the cell process has no positive jumps, though this restriction is probably superfluous.

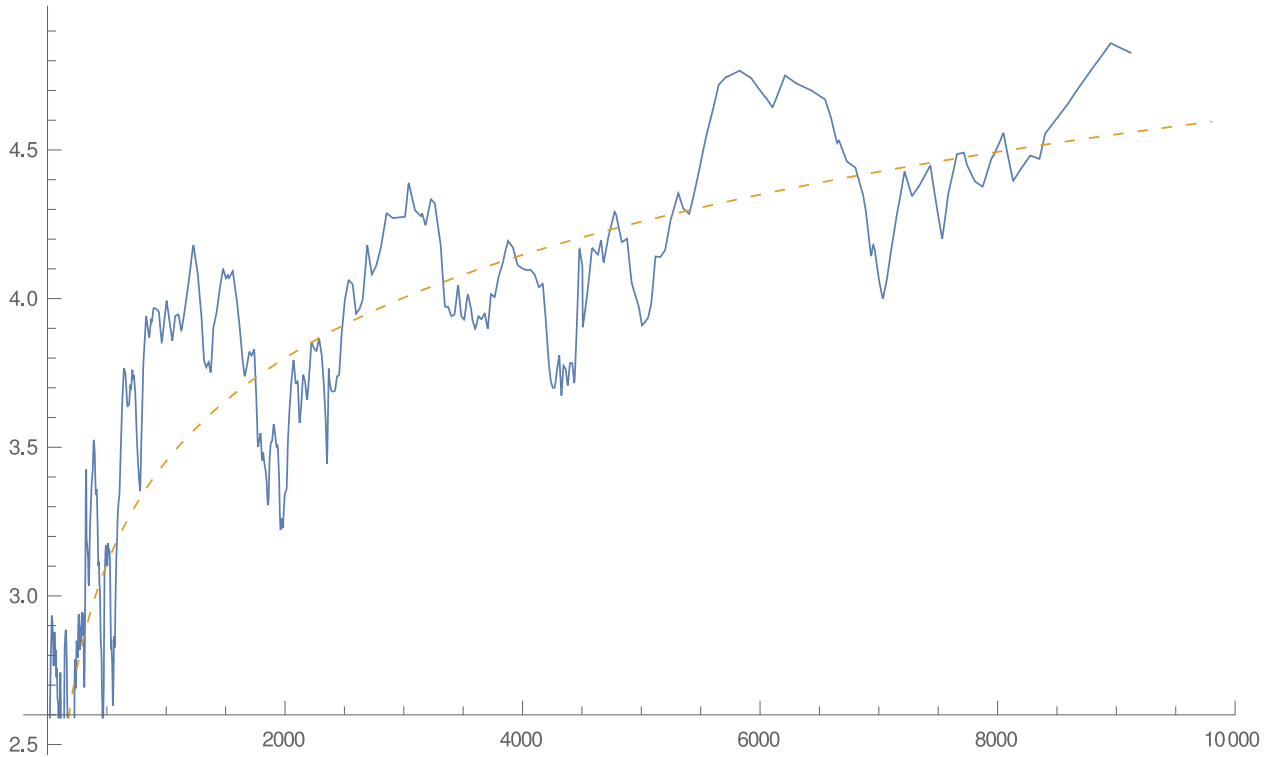


Figure 1.2. Simulation of $-\log X_1$ in a self-similar growth-fragmentation process with scaling $\alpha = 2$. (The dashed line represents the map $t \mapsto \frac{1}{\alpha} \log t$.)

For the other direction, we need to make sure that asymptotically, if the largest fragment ever exceeds the level $t^{-1/\alpha}$, it is unlikely that one of its parents has gone far below this level before t . To this end, we write $X_1(t) := \mathcal{X}_{u^*(t)}(t - b_{u^*(t)})$ with $u^*(t) := \arg \max_{u \in \mathbb{U}, b_u \leq t < d_u} \mathcal{X}_u(t - b_u)$ (in case of *ex aequo*, we choose $u^*(t)$ to be minimal in lexicographic order), and introduce the event

$$\mathcal{H}_t(\varepsilon): \quad \mathcal{X}_v(s - b_v) < \varepsilon, \quad \text{for some time } s \text{ and ancestor } v \preceq u^*(t) \text{ with } b_v \leq s < d_v \wedge t.$$

The following statement is tailored for our purpose.

Proposition 1.3.9. There exists $\theta \geq \omega_+$ such that

$$\left\{ \sup_{x>1}, \lim_{x \rightarrow \infty} \right\} \frac{1}{\log x} \log \mathcal{P}_1 \left(\sup_{s \geq 0} X_1(s) > x \right) = -\theta. \quad (1.32)$$

Furthermore, for every $\gamma, \varepsilon \in (0, 1)$ and every $t > 0$,

$$\mathcal{P}_1 \left(\mathcal{H}_t(\varepsilon), X_1(t) > \varepsilon^\gamma \right) \leq \varepsilon^{(1-\gamma)\theta}.$$

Proof. We may assume $\alpha = 0$ as the statement does not depend on α . The first assertion is a large deviation estimate for the probability $F(x)$ that $T^+(x) < \infty$, where $T^+(x) := \inf\{s \geq$

0: $X_1(s) > x$ }. To eventually obtain a fragment larger than xy in the growth-fragmentation, for $x, y > 1$, it is enough that the largest particle X_1 first reaches some level $z > x$, and that the subsequent fragmentation of this particle produces a fragment with size larger than xy . But by scaling we have, for any $z > x$,

$$\mathcal{P}_z(T^+(xy) < \infty) = \mathcal{P}_1(T^+(xy/z) < \infty) = F(xy/z) \geq F(y) = \mathcal{P}_1(T^+(y) < \infty),$$

so that the branching property at $T^+(x)$ yields, since $z := X_1(T^+(x)) > x$ on the event $\{T^+(x) < \infty\}$,

$$F(xy) = \mathcal{P}_1(T^+(xy) < \infty) \geq \mathcal{E}_1[\mathbf{1}_{\{T^+(x) < \infty\}} F(xy/X_1(T^+(x)))] \geq F(x)F(y).$$

Eq. (1.32) then arises from the subadditive lemma (see e.g. [75, Theorem 16.2.9]). The lower bound $\theta \geq \omega_+$ is just a consequence of Doob's maximal inequality applied to the process

$$M^+(s) := \sum_{i=1}^{\infty} X_i^{\omega_+}(s), \quad s \geq 0,$$

which [24, Corollary 3.7.(i)] is a martingale (for $\alpha = 0$): namely

$$F(x) = \mathcal{P}_1\left(\sup_{s \geq 0} X_1(s) > x\right) \leq \mathcal{P}_1\left(\sup_{s \geq 0} M^+(s) > x^{\omega_+}\right) \leq x^{-\omega_+}, \quad x > 1.$$

Next, we take $x := \varepsilon^{\gamma-1}$ and apply again the scaling property: we deduce that, for every $0 < y < \varepsilon$,

$$F(\varepsilon^\gamma/y) = \mathcal{P}_y\left(\sup_{s \geq 0} X_1(s) > \varepsilon^\gamma\right) \leq \varepsilon^{(1-\gamma)\theta}.$$

But the event $\mathcal{H}_t(\varepsilon)$ holds precisely when the cell process following the ancestral lineage of $u^*(t)$ has reached a value $0 < y < \varepsilon$ before t . Using the branching property at the first time this happens, the second assertion is then easily proved. \blacksquare

We can now derive the upper bound and complete the proof of [Theorem 1.3.8](#).

Proof of the upper bound. Let $0 < \eta < 1$ and observe that $\delta := \eta - (1 - \eta)(1 - \gamma)/\gamma$ lies in $(0, \eta)$ for any $\gamma \in (1 - \eta, 1)$ arbitrarily fixed. Define $\varepsilon := t^{-(1-\delta)/\alpha}$ for $t > 1$, so that $\varepsilon^\gamma = t^{-(1-\eta)/\alpha}$, and

$$\begin{aligned} \mathcal{P}_1(X_1(t) > t^{-(1-\eta)/\alpha}) &= \mathcal{P}_1(X_1(t) > \varepsilon^\gamma) \\ &= \mathcal{P}_1\left(\mathcal{H}_t(\varepsilon), X_1(t) > \varepsilon^\gamma\right) + \mathcal{P}_1\left(\mathcal{H}_t(\varepsilon)^c, X_1(t) > \varepsilon^\gamma\right). \end{aligned}$$

By [Proposition 1.3.9](#),

$$\mathcal{P}_1\left(\mathcal{H}_t(\varepsilon), X_1(t) > \varepsilon^\gamma\right) \leq t^{-(1-\delta)(1-\gamma)\theta/\alpha} \xrightarrow[t \rightarrow \infty]{} 0.$$

To estimate the second term, we shall exploit the fact that a self-similar growth-fragmentation can be constructed from a homogeneous one by performing an appropriate Lamperti time-substitution on each cell in the system (see [19, Corollary 2] or [33, Section 2.1]). Specifically, there exists a cell system $\mathcal{Z} := ((\mathcal{Z}_u, \beta_u) : u \in \mathbb{U})$, with same cumulant function κ , such that every element in $\mathbf{X}(t)$ with label $v \in \mathbb{U}$ equals $\mathcal{Z}_u(\tau - \beta_u)$ for some $u \in \mathbb{U}$ and $\tau \geq 0$ fulfilling

$$\tau = \int_0^t \left(\mathcal{X}_{\bar{v}(s)}(s - b_{\bar{v}(s)}) \right)^\alpha ds, \quad (1.33)$$

where $\bar{v}(s)$ labels the cell in \mathcal{X} corresponding to the unique ancestor of v that is alive at time s . Further, the connection with compensated fragmentations [19, Proposition 3] entails that for every $q \geq 0$ with $\kappa(q) < \infty$,

$$\mathcal{E}_1 \left[\sum_{u \in \mathbb{U}} \left(\mathcal{Z}_u(\tau - \beta_u) \right)^q \right] = \exp(\tau \kappa(q)).$$

On the one hand, if $Z_1(\tau)$ denotes the size of the largest cell at time τ in \mathcal{Z} , then Markov's inequality yields

$$\mathcal{P}_1(Z_1(\tau) > \varepsilon^\gamma) \leq \varepsilon^{-\gamma q} \exp(\tau \kappa(q)).$$

On the other hand, if we purposely take $v := u^*(t)$ then, on the complementary event of $\mathcal{H}_t(\varepsilon)$, we have $\mathcal{X}_{\bar{v}(s)}(s - b_{\bar{v}(s)}) \geq \varepsilon$ for all $s \in [0, t)$ and thus, by (1.33), $X_1(t) = \mathcal{Z}_u(\tau - \beta_u)$ with $\tau \geq t \varepsilon^\alpha = t^\delta$. Hence, fixing $q \in (\omega_-, \omega_+)$ (so that $\kappa(q) < 0$),

$$\mathcal{P}_1 \left(\mathcal{H}_t(\varepsilon)^c, X_1(t) > \varepsilon^\gamma \right) \leq \mathcal{P}_1(Z_1(\tau) > \varepsilon^\gamma) \leq t^{q(1-\eta)/\alpha} \exp(t^\delta \kappa(q)) \xrightarrow[t \rightarrow \infty]{} 0.$$

Putting the two pieces together we have just showed that, for every $0 < \eta < 1$,

$$\mathcal{P}_1 \left(\frac{\log X_1(t)}{\log t} + \frac{1}{\alpha} > \frac{\eta}{\alpha} \right) \xrightarrow[t \rightarrow \infty]{} 0,$$

which is the upper bound we wanted. ■

1.3.4 Freezing the fragmentation

Suppose now that we “freeze” every cell as soon as its size falls under a fixed diameter $\varepsilon > 0$ (which may occur at birth), in the sense that frozen cells no longer grow or split. To put things more formally we need a more chronological point of view in the cells genealogy. For this reason we suppose that the growth-fragmentation has been constructed as in [33, Section 2.1], where cells are now labeled on the infinite binary tree

$$\mathbb{B} := \bigcup_{n=0}^{\infty} \{1, 2\}^n \subset \mathbb{U}.$$

Roughly speaking, any jump from a size $x > 0$ to some smaller size $x - y \in (0, x)$ of a cell with label, say, $u \in \mathbb{B}$, causes the death of that cell while at the same time two independent cells labeled by $u1$ and $u2$ are born with initial sizes $x - y$ and y respectively. We implicitly reuse the notations of Section 1.3.1 within this new description, e.g. \mathcal{P}_x is the distribution of the cell system $\mathcal{X} := (\mathcal{X}_u : u \in \mathbb{B})$ when the mother cell starts at size $x > 0$ (i.e. has the law P_x). Analogously, $\ell \in \partial\mathbb{B}$ refers to a leaf of \mathbb{B} , and $\ell[t]$ and $\mathcal{X}_\ell(t)$ respectively denote the label and the process of the unique cell in the branch from \emptyset to ℓ that is alive at time t . Let us then introduce the first passage times

$$\mathcal{T}_v(\varepsilon) := \inf \{t \geq 0 : \mathcal{X}_v(t) < \varepsilon\}, \quad v \in \mathbb{B} \cup \partial\mathbb{B},$$

so that the family of frozen cells can be defined as

$$\{\chi_{i,\varepsilon}\}_{i=1}^\infty := \left(\mathcal{X}_u(\mathcal{T}_u(\varepsilon)) : u \in \mathbb{B}(\varepsilon) \right),$$

with $\mathbb{B}(\varepsilon) := \{u \in \mathbb{B} : u = \ell[\mathcal{T}_\ell(\varepsilon)] \text{ for some } \ell \in \partial\mathbb{B}\}$. Note that this procedure of freezing cells does not depend on the scaling parameter α of the growth-fragmentation (changing α just affects the speed at which particles get frozen). It is proved [24, Proposition 2.5] that for each $x > 0$, the process of the sum of the sizes of frozen cells raised to the power ω_- ,

$$\mathcal{M}^-(\varepsilon) := \sum_{i=1}^\infty \chi_{i,\varepsilon}^{\omega_-}, \quad 0 < \varepsilon \leq x,$$

is a backward martingale converging to $M^-(\infty)$ as $\varepsilon \rightarrow 0^+$, almost surely and in $L^1(\mathcal{P}_x)$. In the same vein as in [30], we investigate the empirical measure $\varphi(\varepsilon)$ defined by

$$\langle \varphi(\varepsilon), f \rangle := \sum_{i=1}^\infty \chi_{i,\varepsilon}^{\omega_-} f\left(\frac{\chi_{i,\varepsilon}}{\varepsilon}\right).$$

Again, we let ξ^- denote the Lévy process with Laplace exponent Φ^- associated with the pssMp $\widehat{\mathcal{X}}$ via Lamperti's transformation; see (1.26). We can check that its Lévy measure Λ^- is given by

$$\int_{\mathbb{R}} g(y) \Lambda^-(dy) = \int_{\mathbb{R}} [e^{y\omega_-} g(y) + \mathbf{1}_{\{y < 0\}} (1 - e^y)^{\omega_-} g(\log(1 - e^y))] \Lambda(dy),$$

see [76, Theorem 3.9].

Theorem 1.3.10. Suppose (1.24), $\Lambda((0, \infty)) = 0$, and that ξ^- is not arithmetic. Then as $\varepsilon \rightarrow 0^+$, the random measure $\varphi(\varepsilon)$ converges in \mathcal{P}_1 -probability to $M^-(\infty)\varphi$, where φ is a deterministic probability measure on $(0, 1)$ specified by

$$\langle \varphi, f \rangle := \frac{\omega_+ - \omega_-}{-\kappa'(\omega_-)} \iint_{(-\infty, 0)^2} f(e^x) e^{(\omega_+ - \omega_-)y} \Lambda^-((-\infty, x + y)) dx dy. \quad (1.34)$$

Proof. As said previously we may suppose $\alpha = 0$, so that $\widehat{\mathcal{X}}$ is just the exponential of ξ^- . After Jagers [70], we can see that the random set $\mathbb{B}(\varepsilon) \subset \mathbb{B}$ is a so called *optional line* for which the strong branching property holds — intuitively, freezing the cells below ε is equivalent to freezing those which would descend from a family of cells that have first been frozen below $\varepsilon + \delta$, with $\delta > 0$ fixed. Specifically, by choosing $\delta := \sqrt{\varepsilon} - \varepsilon$ for $0 < \varepsilon < 1$ and scaling, we can write

$$\langle \varphi(\varepsilon), f \rangle = \sum_{i=1}^{\infty} \underbrace{\chi_{i,\sqrt{\varepsilon}}^{\omega_-}}_{\lambda_i(\varepsilon)} \underbrace{\sum_{j=1}^{\infty} \chi_{i,j,\varepsilon_i}^{\omega_-} f\left(\frac{\chi_{i,\sqrt{\varepsilon}} \chi_{i,j,\varepsilon_i}}{\varepsilon}\right)}_{Y_i(\varepsilon)},$$

where conditionally on $\boldsymbol{\lambda}(\varepsilon) := (\lambda_i(\varepsilon))_{i \geq 1}$, the $\{\chi_{i,j,\varepsilon_i}\}_{j=1}^{\infty}$, $i = 1, 2, \dots$, are independent cell families respectively frozen below $\varepsilon_i := \varepsilon/\chi_{i,\sqrt{\varepsilon}}$. For every $1 < p < \omega_+/\omega_-$, (conditional) Jensen's inequality easily shows that the closed martingale $\mathcal{M}^-(\varepsilon)$ is bounded (by $\mathcal{E}_1[M^-(\infty)^p]$) in $L^p(\mathcal{P}_1)$. Hence

$$\sup_{0 < \varepsilon < 1} \mathcal{E}_1 \left[\left(\sum_{i=1}^{\infty} \lambda_i(\varepsilon) \right)^p \right] < \infty,$$

and, because $\mathcal{E}_1[\mathcal{M}^-(\varepsilon)] = \mathcal{E}_1[M^-(\infty)] = 1$,

$$\mathcal{E}_1 \left[\sum_{i=1}^{\infty} \lambda_i^p(\varepsilon) \right] \leq \varepsilon^{(p-1)\omega_-} \xrightarrow{\varepsilon \rightarrow 0^+} 0.$$

The proof then continues like that of [Theorem 1.3.4](#). Similarly to the many-to-one formula, [Lemma 1.3.11](#) below gives

$$\mathcal{E}_1[\langle \varphi(\varepsilon), f \rangle] = \widehat{\mathcal{E}}_1^- \left[f\left(\widehat{\mathcal{X}}(\mathcal{T}_{\mathcal{L}}(\varepsilon))/\varepsilon\right) \right],$$

where $\mathcal{T}_{\mathcal{L}}(\varepsilon) = \inf\{t \geq 0: \widehat{\mathcal{X}}(t) < \varepsilon\}$. It thus remains to find the distributional limit of $\widehat{\mathcal{X}}(\mathcal{T}_{\mathcal{L}}(\varepsilon))/\varepsilon$ as $\varepsilon \rightarrow 0^+$. Observe that up to taking the inverse exponential, this random variable corresponds to the overshoot above $-\log \varepsilon$ of the spectrally positive Lévy process $-\xi^-$, which drifts to ∞ a.s. (since $E[-\xi^-(1)] = -(\Phi^-)'(0+) = -\kappa'(\omega_-) \in (0, \infty)$). By a classical result of renewal theory (see e.g. [43] or [76, Theorem 5.7]) we have, for every continuous function $g: (0, \infty) \rightarrow \mathbb{R}$ with compact support,

$$E[g(-\xi^- - (-\log \varepsilon))] \xrightarrow{\varepsilon \rightarrow 0^+} \frac{1}{\mu} \int_{(0,\infty)^2} g(x) \Pi(y + dx) dy, \quad (1.35)$$

with Π and μ respectively the jump measure and the expectation at time 1 of the ascending ladder height process associated with $-\xi^-$. On the one hand, from [52, Corollary 4.4.4.(iv)] we get

$$\mu = \frac{E[-\xi^-(1)]}{\mathbf{k}^*},$$

where k^* is the killing rate of the ascending ladder height process associated with ξ^- , and equals the right inverse at 0 of the Laplace exponent Φ^- (see for instance [76, Example 6.11]):

$$k^* = \sup \{t \geq 0: \Phi^-(t) = 0\} = \omega_+ - \omega_-.$$

On the other hand, we know since the work of Vigon [103] (see also [76, Corollary 7.9]) that Π fulfills

$$\Pi((y, \infty)) = \int_0^\infty e^{-k^*x} \Lambda^-((-\infty, -x - y)) dx, \quad y > 0.$$

An easy computation then enables us to identify the right-hand sides of (1.34) and (1.35) (with $g(x) := f(e^{-x})$). \blacksquare

Lemma 1.3.11. For every $x > 0$ and every bounded measurable function $f: (0, \infty) \rightarrow \mathbb{R}$,

$$\mathcal{E}_x \left[\sum_{u \in \mathbb{B}(\varepsilon)} \mathcal{X}_u^{\omega^-}(\mathcal{T}_u(\varepsilon)) f\left(\mathcal{X}_u(\mathcal{T}_u(\varepsilon))\right) \right] = x^{\omega^-} \widehat{\mathcal{E}}_x^- \left[f\left(\widehat{\mathcal{X}}(\mathcal{T}_{\mathcal{L}}(\varepsilon))\right) \right],$$

with the usual convention $f(\partial) := 0$.

Proof. We slightly adapt the proof of [24, Proposition 4.1]. To this end, recall the intrinsic martingale \mathcal{M}^- evoked in Remark 1.3.2 and, in the paragraph following that remark, the definition of the randomly tagged branch \mathcal{L} . It is here convenient to write $u \succ \mathbb{B}(\varepsilon)$ if $u \in \mathbb{B}$ stems from a (unique) node in $\mathbb{B}(\varepsilon)$ that we then call \bar{u} (i.e. $\bar{u} \in \mathbb{B}(\varepsilon)$ is a prefix of u). The (conditional) distribution of \mathcal{L} in (1.25) gives

$$\begin{aligned} \widehat{\mathcal{E}}_x^- \left[f\left(\widehat{\mathcal{X}}(\mathcal{T}_{\mathcal{L}}(\varepsilon))\right) \mathbf{1}_{\{\mathcal{L}(k+1) \succ \mathbb{B}(\varepsilon)\}} \right] &= \widehat{\mathcal{E}}_x^- \left[\sum_{|u|=k+1} \mathbf{1}_{\{u \succ \mathbb{B}(\varepsilon)\}} \mathbf{1}_{\{u \text{ ancestor of } \mathcal{L}\}} f\left(\mathcal{X}_{\bar{u}}(\mathcal{T}_{\bar{u}}(\varepsilon))\right) \right] \\ &= \widehat{\mathcal{E}}_x^- \left[\frac{1}{\mathcal{M}^-(\infty)} \lim_{n \rightarrow \infty} \sum_{\substack{|u|=k+1 \\ |v|=n}} \mathbf{1}_{\{u \succ \mathbb{B}(\varepsilon)\}} \mathcal{X}_{uv}^{\omega^-}(0) f\left(\mathcal{X}_{\bar{u}}(\mathcal{T}_{\bar{u}}(\varepsilon))\right) \right]. \end{aligned}$$

Rewriting the latter in terms of \mathcal{P}_x simplifies out $\mathcal{M}^-(\infty)$. The branching property at u and the martingale property of \mathcal{M}^- then entail

$$\widehat{\mathcal{E}}_x^- \left[f\left(\widehat{\mathcal{X}}(\mathcal{T}_{\mathcal{L}}(\varepsilon))\right) \mathbf{1}_{\{\mathcal{L}(k+1) \succ \mathbb{B}(\varepsilon)\}} \right] = x^{-\omega^-} \mathcal{E}_x \left[\sum_{|u|=k+1} \mathbf{1}_{\{u \succ \mathbb{B}(\varepsilon)\}} \mathcal{X}_u^{\omega^-}(0) f\left(\mathcal{X}_{\bar{u}}(\mathcal{T}_{\bar{u}}(\varepsilon))\right) \right].$$

If we now gather the nodes u which have the same ancestor $v := \bar{u} \in \mathbb{B}(\varepsilon)$ and repeat the previous argument, we obtain

$$\widehat{\mathcal{E}}_x^- \left[f\left(\widehat{\mathcal{X}}(\mathcal{T}_{\mathcal{L}}(\varepsilon))\right) \mathbf{1}_{\{\mathcal{L}(k+1) \succ \mathbb{B}(\varepsilon)\}} \right] = x^{-\omega^-} \mathcal{E}_x \left[\sum_{|v| \leq k} \mathbf{1}_{\{v \in \mathbb{B}(\varepsilon)\}} \mathcal{X}_v^{\omega^-}(\mathcal{T}_v(\varepsilon)) f\left(\mathcal{X}_v(\mathcal{T}_v(\varepsilon))\right) \right].$$

Since the event $\{\mathcal{L}(k+1) \succ \mathbb{B}(\varepsilon)\}$ must occur for k large enough when $\lim_{t \rightarrow \infty} \widehat{\mathcal{X}}(t) = 0$ and $f(\partial) = 0$ anyway, letting $k \rightarrow \infty$ yields the result by dominated convergence. ■

Self-similar growth-fragmentations as scaling limits of Markov branching processes

This chapter is a reproduction of the article [50].

We provide explicit conditions, in terms of the transition kernel of its driving particle, for a Markov branching process to admit a scaling limit toward a self-similar growth-fragmentation with negative index. We also derive a scaling limit for the genealogical embedding considered as a compact real tree.

2.1 Introduction

Imagine a bin containing n balls which is repeatedly subject to random (binary) divisions at discrete times, until every ball has been isolated. There is a natural random (binary) tree with n leaves associated with this partitioning process, where the subtrees above a given height $k \geq 0$ represent the different subcollections of all n balls at time k , and the number of leaves of each subtree matches the number of balls in the corresponding subcollection. The habitual *Markov branching property* stipulates that these subtrees must be independent conditionally on their respective size. In the literature on random trees, a central question is the approximation of so called continuum random trees (CRT) as the size of the discrete trees tends to infinity. We mention especially the works of Aldous [5, 4, 6] and Haas, Miermont, et al. [65, 87, 66, 64, 63]. Concerning the above example, Haas and Miermont [63] obtained, under some natural assumption on the splitting laws, distributional scaling limits regarded in the Gromov–Hausdorff–Prokhorov topology. In the Gromov–Hausdorff sense where trees are considered as compact metric spaces, they especially identified the so called *self-similar fragmentation trees* as the scaling limits. The latter describe the genealogy of *self-similar fragmentation processes*, which, reciprocally, are known to record the size of the components of a (continuous) fragmentation tree above a given height [65], and thus correspond to scaling limits for partition sequences of balls as their number n tends to infinity. One key tool in the work of Haas and Miermont [63] is provided by some non-increasing integer-valued Markov chain which, roughly speaking, depicts the size of the subcollection containing a randomly tagged ball. This Markov chain essentially captures the dynamics of the whole fragmentation and, by their previous work [64], itself possesses a scaling limit.

The purpose of the present work is to study more general dynamics which incorporate

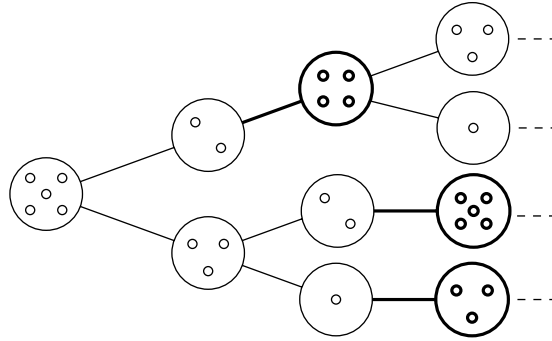


Figure 2.1. An example of dynamics with growth transitions (in bold).

growth, that is the addition of new balls in the system (see Figure 2.1). One example of recent interest lies in the exploration of random planar maps [25, 24], which exhibits “holes” (the yet unexplored areas) that split or grow depending on whether the new edges being discovered belong to an already known face or not. We thus consider a Markov branching system in discrete time and space where at each step, every particle is replaced by either one particle with a bigger size (growth) or by two smaller particles in a conservative way (fragmentation). We condition the system to start from a single particle with size n (we use the superscript $\cdot^{(n)}$ in this respect) and we are interested in its behavior as $n \rightarrow \infty$. Namely, we are looking for:

1. A functional scaling limit for the process in time $(\mathbb{X}(k): k \geq 0)$ of all particle sizes:

$$\left(\frac{\mathbb{X}^{(n)}(\lfloor a_n t \rfloor)}{n} : t \geq 0 \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{Y}(t) : t \geq 0),$$

in some sequence space, where the a_n are positive (deterministic) numbers;

2. A scaling limit for the system’s genealogical tree, seen as a random metric space $(\chi^{(n)}, d_n)$:

$$\left(\chi^{(n)}, \frac{d_n}{a_n} \right) \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{Y},$$

in the Gromov–Hausdorff topology.

Like in the pure-fragmentation setting we may single out some specific integer-valued Markov chain, but which of course is no longer non-increasing. To derive a scaling limit for this chain, a first idea is to apply, as a substitution to [64], the more general criterion of Bertoin and Kortchemski [27] in terms of the asymptotic behavior of its transition kernel at large states. However, this criterion is clearly insufficient for the convergences stated above as it provides no control on the “microscopic” particles. To circumvent this issue, we choose to “prune” the

system by freezing the particles below a (large but fixed) threshold. That is to say, we let the system evolve from a large size n but stop every individual as soon as it is no longer bigger than some threshold $M > 0$ which will be independent of n , and we rather study the modifications $\mathbf{X}^{(n)}$ and $\mathcal{X}^{(n)}$ of the process and the genealogical tree that are induced by this procedure.

The limits \mathbf{Y} and \mathcal{Y} that we aim at are, respectively, a *self-similar growth-fragmentation process* and its associated genealogical structure. Indeed, the scaling limits of integer-valued Markov chains investigated in [27], which we build our work upon, belong to the class of so called *positive self-similar Markov processes* (pssMp), and these processes constitute the cornerstone of Bertoin’s self-similar growth-fragmentations [19, 24]. Besides, in the context of random planar maps [25, 24], they have already been identified as scaling limits for the sequences of perimeters of the separating cycles that arise in the exploration of large Boltzmann triangulations. Informally, a self-similar growth-fragmentation \mathbf{Y} depicts a system of particles which all evolve according to a given pssMp and whose each negative jump $-y < 0$ begets a new independent particle with initial size y . In our setting, the self-similarity property has a negative index and makes the small particles split at higher rates, in such a way that the system becomes eventually extinct [19, Corollary 3]. The genealogical embedding \mathcal{Y} is thus a compact real tree; its formal construction is presented in [98].

Because of growth, one main difference with the conservative case is, of course, that the mass of a particle at a given time no longer equals the size (number of leaves) of the corresponding genealogical subtree. In a similar vein, choosing the uniform distribution to mark a ball at random will appear less relevant than a size-biased pick from an appropriate (nondegenerate) supermartingale. This will highlight a Markov chain admitting a self-similar scaling limit (thanks to the criterion [27]), and which we can plug into a many-to-one formula. Under an assumption preventing an explosive production of relatively small particles, we will then be able to establish our first desired convergence. Concerning the convergence of the (rescaled) trees $\mathcal{X}^{(n)}$, we shall employ a Foster–Lyapunov argument to obtain a uniform control on their heights, which are nothing else than the extinction times of the processes $\mathbf{X}^{(n)}$. Contrary to what one would first expect, it turns out that a good enough Lyapunov function is not simply a power of the size, but merely depends on the scaling sequence (a_n) . This brings a tightness property that, together with the convergence of “finite-dimensional marginals”, will allow us to conclude.

In the next section we set up the notation and the assumptions more precisely, and state our main two results.

2.2 Assumptions and results

Our basic data are probability transitions $p_{n,m}$, $m \geq n/2$ and $n \in \mathbb{N}$ “sufficiently large”, with which we associate a Markov chain, generically denoted X , that governs the law of the particle system \mathbb{X} : at each time $k \in \mathbb{N}$ and with probability $p_{n,m}$, every particle with size n either grows up to a size $m > n$, or fragmentates into two independent particles with sizes $m \in \{\lceil n/2 \rceil, \dots, n-1\}$ and $n-m$. That is to say, $X^{(n)}(0) = n$ is the size of the initial particle in $\mathbb{X}^{(n)}$, and given $X(k)$ for some $k \geq 0$, $X(k+1)$ is the largest among the (one or two) particles replacing $X(k)$. We must emphasize that the transitions $p_{n,m}$ from n “small” are irrelevant since our assumptions shall only rest upon the asymptotic behavior of $p_{n,m}$ as n tends to infinity. Indeed, for the reason alluded in the [Introduction](#) that we explain further below, we rather study the pruned version \mathbf{X} where particles are frozen (possibly at birth) when they become not bigger than a threshold parameter $M > 0$, which we will fix later on. Keeping the same notation, this means that X is a Markov chain stopped upon hitting $\{1, 2, \dots, M\}$. For convenience, *we omit to write the dependency in M* , and set $p_{n,n} := 1$ for $n \leq M$.

In turn, the law of the genealogical tree \mathcal{X} can be defined inductively as follows (we give a more rigorous construction in [Section 2.3](#)). Let $1 \leq k_1 \leq \dots \leq k_p$ enumerate the instants during the lifetime $\zeta^{(n)}$ of $X^{(n)}$ when $n_i := X^{(n)}(k_i - 1) - X^{(n)}(k_i) > 0$. Then $\mathcal{X}^{(n)}$ consists in a branch with length $\zeta^{(n)}$ to which are respectively attached, at positions k_i from the root, independent trees distributed like $\mathcal{X}^{(n_i)}$ (agreeing that $\mathcal{X}^{(n)}$ degenerates into a single vertex for $n \leq M$). We view $\mathcal{X}^{(n)}$ as a metric space with metric denoted by d_n .

Suppose $(a_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers which is regularly varying with index $\gamma > 0$, in the sense that for every $x > 0$,

$$\lim_{n \rightarrow \infty} \frac{a_{\lfloor nx \rfloor}}{a_n} = x^\gamma. \quad (2.1)$$

Our starting requirement will be the convergence in distribution

$$\left(\frac{X^{(n)}(\lfloor a_n t \rfloor)}{n} : t \geq 0 \right) \xrightarrow[n \rightarrow \infty]{(d)} (Y(t) : t \geq 0), \quad (2.2)$$

in the space $\mathbb{D}([0, \infty), \mathbb{R})$ of càdlàg functions on $[0, \infty)$ (endowed with Skorokhod’s J_1 topology), towards a positive strong Markov process $(Y(t) : t \geq 0)$, continuously absorbed at 0 in an almost surely finite time ζ , and with the following self-similarity property:

The law of Y started from $x > 0$ is that of $(xY(x^{-\gamma} t) : t \geq 0)$ when Y starts from 1.

$$(2.3)$$

Since the seminal work of Lamperti [78], this simply means that

$$\log Y(t) = \xi \left(\int_0^t Y(s)^{-\gamma} ds \right), \quad t \geq 0, \quad (2.4)$$

with ξ a Lévy process which drifts to $-\infty$ as $t \rightarrow \infty$. We denote by Ψ the characteristic exponent of ξ (so there is the Lévy–Khinchine formula $E[\exp(q\xi(t))] = \exp(t\Psi(q))$ for every $t \geq 0$ and every $q \in \mathbb{C}$, wherever this makes sense), and by Λ the Lévy measure of its jumps (that is a measure on $\mathbb{R} \setminus \{0\}$ with $\int (1 \wedge y^2) \Lambda(dy) < \infty$).

In order to state precisely our assumptions, we need to introduce some more notation. First, we define the exponent

$$\kappa(q) := \Psi(q) + \int_{(-\infty, 0)} (1 - e^y)^q \Lambda(dy),$$

whose meaning will be discussed shortly (in the paragraph “[Discussion on the assumptions](#)”). Next, we also define, for every $n \in \mathbb{N}$, the discrete versions

$$\Psi_n(q) := a_n \sum_{m=1}^{\infty} p_{n,m} \left[\left(\frac{m}{n} \right)^q - 1 \right], \quad \text{and} \quad \kappa_n(q) := \Psi_n(q) + a_n \sum_{m=1}^{n-1} p_{n,m} \left(1 - \frac{m}{n} \right)^q.$$

Finally, we fix some parameter $q^* > 0$. After [27, Theorem 2], convergence (2.2) holds under the following two assumptions:

(H1) For every $t \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \Psi_n(it) = \Psi(it).$$

(H2) We have

$$\limsup_{n \rightarrow \infty} a_n \sum_{m=2n}^{\infty} p_{n,m} \left(\frac{m}{n} \right)^{q^*} < \infty.$$

Indeed, by [71, Theorem 15.14 & 15.17], [Assumption \(H1\)](#) is essentially equivalent to (A1)&(A2) of [27], while [\(H2\)](#) rephrases Assumption (A3) there. We now introduce the new assumption:

(H3) We have either $\kappa(q^*) < 0$, or $\kappa(q^*) = 0$ and $\kappa'(q^*) > 0$. Moreover, for some $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} a_n \sum_{m=1}^{n-1} p_{n,m} \left(1 - \frac{m}{n} \right)^{q^* - \varepsilon} = \int_{(-\infty, 0)} (1 - e^y)^{q^* - \varepsilon} \Lambda(dy). \quad (2.5)$$

Postponing the description of the limits, we can already state our two convergence results formally:

Theorem 2.2.1. Suppose (H1) to (H3). Then we can fix a freezing threshold M sufficiently large so that, for every $q \geq 1 \vee q^*$, the convergence in distribution

$$\left(\frac{\mathbf{X}^{(n)}(\lfloor a_n t \rfloor)}{n} : t \geq 0 \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{Y}(t) : t \geq 0),$$

holds in the space $\mathbb{D}([0, \infty), \ell^{q\downarrow})$, where \mathbf{Y} is the self-similar growth-fragmentation driven by Y , and

$$\ell^{q\downarrow} := \left\{ \mathbf{x} := (x_1 \geq x_2 \geq \cdots \geq 0) : \sum_{i=1}^{\infty} (x_i)^q < \infty \right\}$$

(that is, the family of particles at a given time is always ranked in the non-increasing order).

Theorem 2.2.2. Suppose (H1) to (H3), and $q^* > \gamma$. Then we can fix a freezing threshold M sufficiently large so that there is the convergence in distribution

$$\left(\mathcal{X}^{(n)}, \frac{d_n}{a_n} \right) \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{Y},$$

in the Gromov–Hausdorff topology, where \mathcal{Y} is the random compact real tree that represents the genealogy of \mathbf{Y} .

Description of the limits. As explained in the [Introduction](#), the process Y portrays the size of particles in the self-similar growth-fragmentation process $\mathbf{Y} := (\mathbf{Y}(t) : t \geq 0)$, whose construction we briefly recall (referring to [\[19, 24\]](#) for more details): The Eve particle Y_\emptyset is distributed like Y . We rank the negative jumps of a particle Y_u in the decreasing order of their absolute sizes (and chronologically in case of *ex aequo*). When this particle makes its j^{th} negative jump, say with size $-y_j < 0$, then a daughter particle Y_{uj} is born at this time and evolves, independently of its siblings, according to the law of Y started from y_j . (Recall that Y is eventually absorbed at 0, so we can indeed rank the negative jumps in this way; for definiteness, we set $b_{uj} := \infty$ and $Y_{uj} \equiv 0$ if Y_u makes less than j negative jumps during its lifetime.) Particles are here labeled on the Ulam–Harris tree $\mathbb{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$, the set of finite words over \mathbb{N} , where $\mathbb{N}^0 = \{\emptyset\}$ is reduced to the root of the tree, and a vertex $u := (u_1, u_2, \dots, u_k) \in \mathbb{U}$, at generation $|u| := k$, has $uj := (u_1, u_2, \dots, u_k, j)$ as j -th descendent. Write b_u for the birth time of Y_u . Then

$$\mathbf{Y}(t) = \left(Y_u(t - b_u) : u \in \mathbb{U}, b_u \leq t \right), \quad t \geq 0.$$

After [\[19, 24\]](#), this process is self-similar with index $-\gamma$. Roughly speaking, this means that a particle with size $x > 0$ evolves $x^{-\gamma}$ times “faster” than a particle with size 1. Since here $-\gamma < 0$, there is the snowball effect that particles get rapidly absorbed toward 0 as time

passes, and it has been shown [19, Corollary 3] that such a growth-fragmentation becomes eventually extinct, namely that $\epsilon := \inf\{t \geq 0: \mathbf{Y}(t) = \emptyset\}$ is almost surely finite.

The extinction time ϵ is also the height of the genealogical structure \mathcal{Y} seen as a compact real tree. Referring to [98] for details, we shall just sketch the construction. Let $\mathcal{Y}_{u,0}$ consists in a segment with length $\zeta_u := \inf\{t \geq 0: Y_u(t) = 0\}$ rooted at a vertex u . Recursively, define $\mathcal{Y}_{u,h+1}$ by attaching to the segment $\mathcal{Y}_{u,0}$ the trees $\mathcal{Y}_{u_j,h}$ at respective distances $b_{u_j} - b_u$ from u , for each born particle u_j , $j \leq h + 1$. The limiting tree $\mathcal{Y} := \lim_{h \rightarrow \infty} \mathcal{Y}_h$, where $\mathcal{Y}_h := \mathcal{Y}_{\emptyset,h}$ fulfills a so called *recursive distributional equation*. Namely, by [98, Corollary 4.2], given the negative jump times and sizes (b_j, y_j) of Y and an independent sequence $\mathcal{Y}^1, \mathcal{Y}^2, \dots$ of copies of \mathcal{Y} , the action of grafting, on a branch with length $\zeta := \inf\{t \geq 0: Y(t) = 0\}$ and at distances b_j from the root, the trees \mathcal{Y}^j rescaled by the multiplicative factor y_j^γ , yields a tree with the same law as \mathcal{Y} . With this connection, Rembart and Winkel [98, Corollary 4.4] proved that ϵ admits moments up to the order $\sup\{q > 0: \kappa(q) < 0\}/\gamma$. When particles do not undergo sudden positive growth (i.e., $\Lambda((0, \infty)) = 0$), Bertoin et al. [24, Corollary 4.5] more precisely exhibited a polynomial tail behavior of this order for the law of ϵ .

Discussion on the assumptions. Observe that (H1) entails (H2) when the Lévy measure Λ of ξ is bounded from above (in particular, when ξ has no positive jumps). By analyticity, Assumptions (H1) and (H2) imply that $\Psi_n(z) \rightarrow \Psi(z)$ as $n \rightarrow \infty$, for $0 \leq \Re z \leq q^*$. Adding the condition (2.5) in (H3) then yields the convergence $\kappa_n(z) \rightarrow \kappa(z)$ for $\Re z$ in a left-neighbourhood of q^* . Lastly, the first condition in (H3) itself implies $\Psi(q^*) < 0$ (since $\Psi < \kappa$) and, together with the other assumptions, that there must exist $q_* \in (0, q^*)$ and some threshold M such that

$$\kappa_n(q) < 0 \text{ and } \kappa(q) < 0, \quad \text{for all } q \in [q_*, q^*) \text{ and } n > M,$$

which is all but a superfluous requirement. Indeed, the condition $\kappa(q) \leq 0$ for some $q > 0$ is necessary (and sufficient) [33] to prevent local explosion of the growth-fragmentation \mathbf{Y} (a phenomenon which would not allow us to view it in some ℓ^q -space). Informally, the *cumulant function* $\kappa(q)$ captures the expected value of the sum of the particle sizes raised to the power q immediately after the first birth event. This function constitutes a key feature of branching processes and, in particular, of self-similar growth-fragmentations [99]. Of course, the meanings of the quantity $\kappa_n(q)$ and of the condition $\kappa_n(q) < 0$ should be regarded the same but at the discrete level (that is, w.r.t. $\mathbf{X}^{(n)}$).

We stress that our assumptions do not provide any control on the “small particles” ($n \leq M$). This explains why we need to “freeze” them (meaning that they no longer grow or beget children); otherwise their number could become quickly very high and make the system

explode, as we illustrate in the example below. We will basically choose M as above, so that $\kappa_n(q) \leq 0$ for some q and all n , once we take the freezing into account (which is tantamount to resetting¹ $\kappa_n := 0$ for $n \leq M$).

Example 2.2.3. Suppose that a particle with size n increases to size $n + 1$ with probability $p < 1/2$ and, at least when n is small, splits into two particles with sizes 1 and $n - 1$ with probability $1 - p$. Thus, at small sizes, the unstopped Markov chain essentially behaves like a simple random walk. On the one hand, we know from Cramér’s theorem (see e.g. [51, Theorem 2.2.3]) that for every $\varepsilon > 0$ sufficiently small,

$$\mathbb{P}\left(X^{(1)}(k) > (1 - 2p + \varepsilon)k\right), \quad k \geq 0,$$

decreases exponentially at a rate $c_p(\varepsilon) > 0$. On the other hand, keeping only track of particles with size 1 or 2, the number of particles with size 1 is bounded from below by $Z^{[1]}$, where $\mathbf{Z} := (Z^{[1]}, Z^{[2]})$ is a 2-type Galton–Watson process whose mean-matrix

$$\begin{pmatrix} 0 & 1 \\ 2(1 - p) & 0 \end{pmatrix}$$

has spectral radius $r_p := \sqrt{2(1 - p)} > 1$, so that by the Kesten–Stigum theorem [12, Theorem V.6.1] the number of particles with size 1 at time $k \rightarrow \infty$ is of order at least r_p^k , almost surely. Consequently the expected number of particles which are above $(1 - 2p + \varepsilon)k$ at time $2k$ is of exponential order at least $m_p(\varepsilon) := \log r_p - c_p(\varepsilon)$. It is easily checked that this quantity may be positive (e.g., $m_{1/4}(1/4) \geq 0.16$). Thus, without any “local” assumption on the reproduction law at small sizes, the number of small particles may grow exponentially and we cannot in general expect $\mathbb{X}^{(n)}(\lfloor a_n \cdot \rfloor) / n$ to be tight in $\ell^{q\downarrow}$, for some $q > 0$. However, this happens to be the case for the perimeters of the cycles in the branching peeling process of random Boltzmann triangulations [25], where versions of Theorems 2.2.1 and 2.2.2 hold for $\gamma = 1/2$, $q^* = 3$, and $M = 0$, although $\kappa_n(3) \leq 0$ seems fulfilled only for $M \geq 3$ (which should mean that the holes with perimeter 1 or 2 do not contribute to a substantial part of the triangulation).

We start with the relatively easy convergence of finite-dimensional marginals (Section 2.3). Then, we develop a few key results (Section 2.4) that will be helpful to complete the proofs of Theorem 2.2.1 (Section 2.5) and Theorem 2.2.2 (Section 2.6).

¹We make here a slight abuse on the notation. Again, even though the dependency is not explicitly written, the discrete objects such as $\kappa_n, \mathbf{X}^{(n)}, \dots$ all ultimately depend on the freezing threshold M .

2.3 Convergence of finite-dimensional marginals

In this section, we prove finite-dimensional convergences for both the particle process \mathbf{X} and its genealogical structure \mathcal{X} . (We mention that the freezing procedure is of no relevance here as it will be only useful in the next section to establish tightness results; in particular the freezing threshold M will be fixed later.)

We start by adopting a representation of the particle system \mathbf{X} that better matches that of \mathbf{Y} given above. We define, for every word $u := u_1 \cdots u_i \in \mathbb{N}^i$, the *u-locally largest particle*² $(X_u(k) : k \geq 0)$ by induction on $i = 0, 1, \dots$. Initially, for $i = 0$, there is a single particle X_\emptyset labeled by $u = \emptyset$, born at time $\beta_\emptyset := 0$ and distributed like X . Then, we enumerate the sequence $(\beta_{u_1}, n_1), (\beta_{u_2}, n_2), \dots$ of the negative jump times and sizes of X_u so that $n_1 \geq n_2 \geq \dots$ and $\beta_{u_j} < \beta_{u_{j+1}}$ whenever $n_j = n_{j+1}$. Conditionally on $(n_j)_{j \geq 1}$, the processes X_{u_j} , $j = 1, 2, \dots$ are independent and distributed like $X^{(n_j)}$ respectively (for definiteness, we set $\beta_{u_j} := \infty$ and $X_{u_j} := 0$ if X_u makes less than j negative jumps during its lifetime), and we have

$$(\mathbf{X}(k))_{k \geq 0} \stackrel{d}{=} \left(X_u(k - \beta_u) : u \in \mathbb{U}, \beta_u \leq k \right)_{k \geq 0}.$$

Recall the notation $\cdot^{(n)}$ to stress that the system is started from a particle with size $X_\emptyset(0) = n$.

Lemma 2.3.1. Suppose (H1) and (H2). Then for every finite subset $U \subset \mathbb{U}$, there is the convergence in $\mathbb{D}([0, \infty), \mathbb{R}^U)$:

$$\left(\frac{X_u^{(n)}(\lfloor a_n t \rfloor - \beta_u^{(n)})}{n} : u \in U \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(d)} \left(Y_u(t - b_u) : u \in U \right)_{t \geq 0}. \quad (2.6)$$

Proof. We follow the argument used to prove the second part of [25, Lemma 17]. For $h \geq 0$, let $\mathbb{U}_h := \{u \in \mathbb{U} : |u| \leq h\}$ be the set of vertices with height at most h in the tree \mathbb{U} . It suffices to show

(\mathcal{S}_h) : Convergence (2.6) holds in $\mathbb{D}([0, \infty), \mathbb{R}^U)$ for every finite subset $U \subset \mathbb{U}_h$,

by induction on h . The statement (\mathcal{S}_0) is given by (2.2). Now, if U is a finite subset of \mathbb{U}_{h+1} and F_u , $u \in U$, are continuous bounded functions from $\mathbb{D}([0, \infty), \mathbb{R})$ to \mathbb{R} , then the branching property entails that, for $\hat{X}_u^{(n)} := X_u^{(n)}(\lfloor a_n \cdot \rfloor - \beta_u)/n$,

$$\mathbb{E} \left[\prod_{u \in U} F_u(\hat{X}_u^{(n)}) \mid (X_u^{(n)} : u \in \mathbb{U}_h) \right] = \prod_{u \in U \cap \mathbb{U}_h} F_u(\hat{X}_u^{(n)}) \cdot \prod_{\substack{u \in U \\ |u|=h+1}} E_{\hat{X}_u^{(n)}(0)}^{(n)} [F_u],$$

²In the peeling of random Boltzmann maps [25], the locally-largest cycles are called *left-twigs*.

where $E_x^{(n)}$ stands for expectation under the law $P_x^{(n)}$ of \hat{X}_\emptyset started from x , which by (2.1), (2.2) and (2.3), converges weakly as $n \rightarrow \infty$ to the law P_x of Y started from x . The values $\hat{X}_u^{(n)}(0)$ for $|u| = h + 1$ correspond to (rescaled) negative jump sizes of particles at height h . With [69, Corollary VI.2.8] and our convention of ranking the jump sizes in the non-increasing order, the convergence in distribution $(\hat{X}_u^{(n)}(0): u \in U, |u| = h + 1) \rightarrow (Y_u(0): u \in U, |u| = h + 1)$ as $n \rightarrow \infty$ thus holds jointly with (\mathcal{S}_h) . Further, thanks to the Feller property [78, Lemma 2.1] of Y , its distribution is weakly continuous in its starting point. By the continuous mapping theorem we therefore obtain, applying back the branching property, that

$$\mathbb{E} \left[\prod_{u \in U} F_u(\hat{X}_u^{(n)}) \right] \xrightarrow{n \rightarrow \infty} \mathbb{E} \left[\prod_{u \in U} F_u(Y_u(\cdot - b_u)) \right].$$

A priori, this establishes the convergence in distribution $(\hat{X}_u^{(n)}: u \in U) \rightarrow (Y_u(\cdot - b_u): u \in U)$ only in the product space $\mathbb{D}([0, \infty), \mathbb{R})^U$. By [69, Proposition 2.2] it will also hold in $\mathbb{D}([0, \infty), \mathbb{R}^U)$ provided that the processes Y_u , $u \in U$, almost surely never jump simultaneously. But this is plain since particles evolve independently and the jumps of Y are totally inaccessible. Thus $(\mathcal{S}_h) \implies (\mathcal{S}_{h+1})$. \blacksquare

Next, we proceed to the convergence of the finite-dimensional marginals of \mathcal{X} , which we shall first formally construct. For each $u \in \mathbb{U}$ with $\beta_u < \infty$, let ζ_u denote the lifetime of the stopped Markov chain X_u . Recall the definition in Section 2.2 of the trees $\mathcal{Y}, \mathcal{Y}_h$, $h \geq 0$, related to \mathbf{Y} 's genealogy, that echoes Rembart and Winkel's construction [98]. Similarly, let $\mathcal{X}_{u,0}$ simply consist of an edge with length ζ_u , rooted at a vertex u . Recursively, define $\mathcal{X}_{u,h+1}$ by attaching to the edge $\mathcal{X}_{u,0}$ the trees $\mathcal{X}_{uj,h}$ at a distance $\beta_{uj} - \beta_u$ from the root u , respectively, for each born particle uj , $j \leq h + 1$, descending from u . The tree $\mathcal{X}_h := \mathcal{X}_{\emptyset,h}$ is a finite tree whose vertices are labeled by the set $\mathbb{U}^{(h)}$ of words over $\{1, \dots, h\}$ with length at most h . Plainly, the sequence \mathcal{X}_h , $h \geq 0$, is consistent, in that \mathcal{X}_h is the subtree of \mathcal{X}_{h+1} with vertex set $\mathbb{U}^{(h)}$, and we may consider the inductive limit $\mathcal{X} := \lim_{h \rightarrow \infty} \mathcal{X}_h$. We write $d_n(v, v')$ for the length of the unique path between v and v' in $\mathcal{X}^{(n)}$. All these trees belong to the space \mathcal{T} of (equivalence classes of) compact, rooted, real trees and can be embedded as subspaces of a large metric space (such as, for instance, the space $\ell^1(\mathbb{N})$ of summable sequences [6, Section 2.2]). Irrespectively of the embedding, they can be compared one with each other through the so called Gromov–Hausdorff metric d_{GH} on \mathcal{T} . We forward the reader to [80, 59] and references therein.

Lemma 2.3.2. Suppose (H1) to (H3). Then for all $h \in \mathbb{N}$, there is the convergence in $(\mathcal{T}, d_{\text{GH}})$:

$$\left(\mathcal{X}_h^{(n)}, \frac{d_n}{a_n} \right) \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{Y}_h.$$

Proof. It suffices to show the joint convergence of all branches. The branch going from the root \emptyset through the vertex $u \in \mathbb{U}^{(h)}$ has total length $\mathcal{E}_u^{(n)} := \beta_u^{(n)} + \zeta_u^{(n)}$ in $(\mathcal{X}_h^{(n)}, d_n)$, and length $\epsilon_u := b_u + \zeta_u$ in \mathcal{Y}_h . Recall that conditionally on $\{X_u(0) = n\}$, the random variable ζ_u has the same distribution as $\zeta^{(n)} := \zeta_{\emptyset}^{(n)}$. By [27, Theorem 3.(i)], the convergence

$$\frac{\zeta^{(n)}}{a_n} \xrightarrow[n \rightarrow \infty]{(d)} \zeta := \inf\{t \geq 0: Y(t) = 0\}$$

holds jointly with (2.2). Adapting the proof of Lemma 2.3.1, we can more generally check that for every finite subset $U \subset \mathbb{U}$, we have, jointly with (2.6),

$$\left(\frac{\mathcal{E}_u^{(n)}}{a_n} : u \in U \right) \xrightarrow[n \rightarrow \infty]{(d)} (\epsilon_u : u \in U).$$

In particular, this is true for $U := \mathbb{U}^{(h)}$. ■

To conclude this section, we restate an observation of Bertoin, Curien, and Kortchemski [25, Lemma 21] which results from the convergence of finite-dimensional marginals (Lemma 2.3.1): with high probability as $h \rightarrow \infty$, “non-negligible” particles have their labels in $\mathbb{U}^{(h)}$. Specifically, say that an individual $u \in \mathbb{U}$ is (n, ε) -good, and write $u \in \mathcal{G}(n, \varepsilon)$, if the particles X_v labeled by each ancestor v of u (including u itself) have size at birth at least $n\varepsilon$. Then:

Lemma 2.3.3. We have

$$\lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}^{(n)}(\mathcal{G}(n, \varepsilon) \not\subseteq \mathbb{U}^{(h)}) = 0.$$

2.4 A size-biased particle and a many-to-one formula

We now introduce a “size-biased particle” and relate it to a many-to-one formula. This will help us derive tightness estimates in Sections 2.5 and 2.6, and thus complement the finite-dimensional convergence results of the preceding section. Recall from Assumptions (H1) to (H3) that we can find $q_* \in (0, q^*)$ such that, as $n \rightarrow \infty$, $\kappa_n(q) \rightarrow \kappa(q) < 0$ for every $q \in [q_*, q^*)$. Consequently, *we may and will suppose for the remainder of this section that the freezing threshold M is taken sufficiently large so that $\kappa_n(q_*) \leq 0$ for every $n > M$.* (Note that $\kappa_n(q_*) = 0$ for $n \leq M$, by our convention $p_{n,n} := 1$.)

Lemma 2.4.1. For every $n \in \mathbb{N}$,

$$\mathbb{E}^{(n)} \left[(X(1))^{q_*} + (n - X(1))_+^{q_*} \right] \leq n^{q_*}. \tag{2.7}$$

Therefore, the process³

$$\sum_{u \in \mathbb{U}} (X_u(k - \beta_u))^{q_*}, \quad k \geq 0,$$

is a supermartingale under $\mathbb{P}^{(n)}$.

Proof. The left-hand side of (2.7) is

$$n^{q_*} \sum_{m=0}^{\infty} p_{n,m} \left[\left(\frac{m}{n} \right)^{q_*} + \left(1 - \frac{m}{n} \right)_+^{q_*} \right] = n^{q_*} \left(1 + \frac{\kappa_n(q_*)}{a_n} \right),$$

where $\kappa_n(q_*) \leq 0$. Hence the first part of the statement. The second part follows by applying the branching property at any given time $k \geq 0$:

$$\begin{aligned} \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}} (X_u(k+1 - \beta_u))^{q_*} \mid \mathbf{X}(k) = (x_i : i \in I) \right] &= \sum_{i \in I} \mathbb{E}^{(x_i)} \left[(X(1))^{q_*} + (x_i - X(1))_+^{q_*} \right] \\ &\leq \sum_{i \in I} (x_i)^{q_*} \\ &= \sum_{u \in \mathbb{U}} (X_u(k - \beta_u))^{q_*}. \quad \blacksquare \end{aligned}$$

Remark 2.4.2. Put differently, the condition “ $\kappa_n(q_*) \leq 0$ ” entails that $n \mapsto n^{q_*}$ is superharmonic with respect to the “fragmentation operator”. This map plays the same role as the function f in [25], where it takes the form of a cubic polynomial ($q_* = 3$) and

$$\sum_{u \in \mathbb{U}} f(X_u(k - \beta_u)), \quad k \geq 0,$$

is actually a martingale. More generally, the map $n \mapsto n^{q_*}$ could be replaced by any regularly-varying sequence with index q_* , but probably at the cost of heavier notation.

As we see in the proof of Lemma 2.4.1, the fact that $\kappa_n(q_*) \leq 0$ allows us to introduce a (defective) Markov chain $(\bar{X}(k) : k \geq 0)$ on \mathbb{N} , to which we add 0 as cemetery state, with transition

$$\mathbb{E}^{(n)} [f(\bar{X}(1)); \bar{X}(1) \neq 0] = \sum_{m=1}^{\infty} p_{n,m} \left[\left(\frac{m}{n} \right)^{q_*} f(m) + \left(1 - \frac{m}{n} \right)_+^{q_*} f(n-m) \right]. \quad (2.8)$$

We let $\bar{\zeta} := \inf\{k \geq 0 : \bar{X}(k) = 0\}$ denote its lifetime. Up to a change of probability measure, \bar{X} follows the trajectory of a randomly selected particle in \mathbf{X} , until it is eventually absorbed to the cemetery state 0. It admits the following scaling limit (which could also be seen as a randomly selected particle in \mathbf{Y} ; see [24, Section 4]):

³We set here $X_u(i) := 0$ for $i < 0$ in order to not burden the notation with the indicator $\mathbf{1}_{\{\beta_u \leq k\}}$.

Proposition 2.4.3. There is the convergence in distribution

$$\left(\frac{\bar{X}^{(n)}(\lfloor a_n t \rfloor)}{n} : t \geq 0 \right) \xrightarrow[n \rightarrow \infty]{(d)} (\bar{Y}(t) : t \geq 0) \quad (2.9)$$

in $\mathbb{D}([0, \infty), \mathbb{R})$, where the limit \bar{Y} fulfills the same identity (2.4) as Y , but for a (killed) Lévy process $\bar{\xi}$ with characteristic exponent $\bar{\kappa}(q) := \kappa(q_* + q)$. Further, if $\bar{\zeta}$ denotes the lifetime of \bar{Y} , then the convergence

$$\frac{\bar{\zeta}^{(n)}}{a_n} \xrightarrow[n \rightarrow \infty]{(d)} \bar{\zeta}$$

holds jointly with (2.9).

Proof. Write $\bar{\Lambda}_n$ for the law of $\log(\bar{X}^{(n)}(1)/n)$, with the convention $\log 0 := -\infty$. We see from (2.8) that $a_n \mathbb{P}(\bar{X}(1) = 0) = -\kappa_n(q_*)$, and, for every $0 \leq q \leq q^* - q_*$,

$$\begin{aligned} \int_{\mathbb{R}} (e^{qy} - 1) a_n \bar{\Lambda}_n(dy) &= a_n \sum_{m=0}^{\infty} p_{n,m} \left[\left(\frac{m}{n}\right)^{q_*} \left(\left(\frac{m}{n}\right)^q - 1 \right) + \left(1 - \frac{m}{n}\right)_+^{q_*} \left(\left(1 - \frac{m}{n}\right)^q - 1 \right) \right] \\ &= \kappa_n(q_* + q) - \kappa_n(q_*). \end{aligned}$$

Hence

$$-a_n \bar{\Lambda}_n(\{-\infty\}) + \int_{\mathbb{R}} (e^{qy} - 1) a_n \bar{\Lambda}_n(dy) = \kappa_n(q_* + q) \xrightarrow[n \rightarrow \infty]{} \bar{\kappa}(q).$$

Furthermore, by (H2),

$$\limsup_{n \rightarrow \infty} a_n \int_1^{\infty} e^{(q^* - q_*)y} \bar{\Lambda}_n(dy) \leq \limsup_{n \rightarrow \infty} a_n \sum_{m=2n}^{\infty} p_{n,m} \left(\frac{m}{n}\right)^{q_*} < \infty.$$

In other words, assumptions (A1), (A2) and (A3) of [27] are satisfied (w.r.t the Markov chain \bar{X} and the limiting process \bar{Y}). Our statement thus follows from Theorems 1 and 2 there⁴. ■

Heading now toward pathwise and optional many-to-one formulae, we first set up some notation. Let $\mathbf{A} \subseteq \mathbb{N}$ be a fixed subset of states, and let $\ell \in \partial\mathbb{U}$ refer to an infinite word over \mathbb{N} , which we see as a branch of \mathbb{U} . For every $u \in \mathbb{U} \cup \partial\mathbb{U}$ and every $k \geq 0$, set

$$\tilde{X}_u(k) := X_{u[k]}(k - \beta_{u[k]}),$$

where $u[k]$ is the youngest ancestor v of u with $\beta_v \leq k$, and write $\tau_u^{\mathbf{A}} := \inf\{k \geq 0 : \tilde{X}_u(k) \in \mathbf{A}\}$ for the first hitting time of \mathbf{A} by \tilde{X}_u . Let also $\bar{\tau}^{\mathbf{A}} := \inf\{k \geq 0 : \bar{X}(k) \in \mathbf{A}\}$. Now, imagine

⁴Strictly speaking, the results are only stated when there is no killing, that is $\kappa(q_*) = 0$, but as mentioned by the authors [27, p. 2562, §2], they can be extended using the same techniques to the case where some killing is involved.

that once a particle hits \mathbf{A} , it is stopped and thus has no further progeny. The state when all particles have hit \mathbf{A} in finite time is $x_u^{\mathbf{A}} := \tilde{X}_u(\tau_u^{\mathbf{A}})$, $u \in \mathbb{U}_{\mathbf{A}}$, where $\mathbb{U}_{\mathbf{A}} := \{u \in \mathbb{U}: \ell[\tau_\ell^{\mathbf{A}}] = u \text{ for some } \ell \in \partial\mathbb{U} \text{ with } \tau_\ell^{\mathbf{A}} < \infty\}$.

Lemma 2.4.4 (Many-to-one formula).

(i) For every $n \in \mathbb{N}$, every $k \geq 0$, and every $f: \mathbb{N}^{k+1} \rightarrow \mathbb{R}_+$,

$$\mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}} (X_u(k - \beta_u))^{q^*} f(\tilde{X}_u(i): i \leq k) \right] = n^{q^*} \mathbb{E}^{(n)} [f(\bar{X}(i): i \leq k); \bar{\zeta} > k].$$

(ii) For every $n \in \mathbb{N}$, every $\mathbf{A} \subseteq \mathbb{N}$, and every $f: \mathbb{Z}_+ \times \mathbb{N} \rightarrow \mathbb{R}_+$,

$$\mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}_{\mathbf{A}}} (x_u^{\mathbf{A}})^{q^*} f(\tau_u^{\mathbf{A}}, x_u^{\mathbf{A}}) \right] = n^{q^*} \mathbb{E}^{(n)} [f(\bar{\tau}^{\mathbf{A}}, \bar{X}(\bar{\tau}^{\mathbf{A}})); \bar{\zeta} > \bar{\tau}^{\mathbf{A}}].$$

Proof. (i) The proof is classical (see e.g. [100, Theorem 1.1]) and proceeds by induction on k . The identity clearly holds for $k = 0$. Using (2.8) together with the branching property at time k ,

$$\begin{aligned} & \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}} (X_u(k+1 - \beta_u))^{q^*} f(\tilde{X}_u(i): i \leq k+1) \mid \tilde{X}_u(i) = x_{u,i}, i \leq k \right] \\ &= \sum_{u \in \mathbb{U}} \sum_{m=0}^{\infty} p_{x_{u,k}, m} \left(m^{q^*} f(x_{u,0}, \dots, x_{u,k}, m) + (x_{u,k} - m)_+^{q^*} f(x_{u,0}, \dots, x_{u,k}, x_{u,k} - m) \right) \\ &= \sum_{u \in \mathbb{U}} (x_{u,k})^{q^*} \mathbb{E}^{(x_{u,k})} [f(x_{u,0}, \dots, x_{u,k}, \bar{X}(1)); \bar{X}(1) \neq 0]. \end{aligned}$$

By taking expectations on both sides and applying the induction hypothesis with the function $\tilde{f}(x_0, \dots, x_k) := \mathbb{E}^{(x_k)} [f(x_0, \dots, x_k, \bar{X}(1)); \bar{X}(1) \neq 0]$ on the one hand, and by applying the Markov property of \bar{X} at time k on the other hand, we derive the identity at time $k+1$:

$$\begin{aligned} \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}} (X_u(k+1 - \beta_u))^{q^*} f(\tilde{X}_u(i): i \leq k+1) \right] &= n^{q^*} \mathbb{E}^{(n)} \left[\tilde{f}(\bar{X}(i): i \leq k); \bar{\zeta} > k \right] \\ &= n^{q^*} \mathbb{E}^{(n)} [f(\bar{X}(i): i \leq k+1); \bar{\zeta} > k+1]. \end{aligned}$$

(ii) For every $k \geq 0$ and every $x_0, \dots, x_k \in \mathbb{N}$, we set $f_k^{\mathbf{A}}(x_0, \dots, x_k) := \mathbb{1}_{\{x_0 \notin \mathbf{A}, \dots, x_{k-1} \notin \mathbf{A}, x_k \in \mathbf{A}\}} f(k, x_k)$. Then

$$\begin{aligned} \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}_{\mathbf{A}}} (x_u^{\mathbf{A}})^{q^*} f(\tau_u^{\mathbf{A}}, x_u^{\mathbf{A}}) \right] &= \sum_{k=0}^{\infty} \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}} (X_u(k - b_u))^{q^*} f_k^{\mathbf{A}}(\tilde{X}_u(i) : i \leq k) \right] \\ &= n^{q^*} \sum_{k=0}^{\infty} \mathbb{E}^{(n)} [f(k, \bar{X}(k)); \bar{\tau}^{\mathbf{A}} = k; \bar{\zeta} > k] \\ &= n^{q^*} \mathbb{E}^{(n)} [f(\bar{\tau}^{\mathbf{A}}, \bar{X}(\bar{\tau}^{\mathbf{A}})); \bar{\zeta} > \bar{\tau}^{\mathbf{A}}], \end{aligned}$$

by (i) and the monotone convergence theorem. \blacksquare

We now combine [Proposition 2.4.3](#) and [Lemma 2.4.4](#) to derive the following counterpart of [[25](#), Lemma 14] that we will apply in the next two sections. Consider the hitting set $\mathbf{A} := \{1, \dots, \lfloor n\varepsilon \rfloor\}$ and denote by $x_u^{\leq n\varepsilon} := x_u^{\mathbf{A}}$, $u \in \mathbb{U}^{\leq n\varepsilon} := \mathbb{U}_{\mathbf{A}}$, the population of particles stopped below $n\varepsilon$.

Corollary 2.4.5. We have

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} n^{-q^*} \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}^{\leq n\varepsilon}} (x_u^{\leq n\varepsilon})^{q^*} \right] = 0.$$

Proof. By [Lemma 2.4.4](#),

$$n^{-q^*} \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}^{\leq n\varepsilon}} (x_u^{\leq n\varepsilon})^{q^*} \right] = \mathbb{P}^{(n)}(\bar{\zeta} > \bar{\tau}^{\leq n\varepsilon}),$$

where $\bar{\tau}^{\leq n\varepsilon} := \inf\{k \geq 0 : \bar{X}(k) \leq n\varepsilon\}$. Thus, if $\bar{\zeta}$ is the lifetime of \bar{Y} and $\bar{\tau}^{\leq \varepsilon} := \inf\{t \geq 0 : \bar{Y}(t) \leq \varepsilon\}$, then by [Proposition 2.4.3](#) and the continuous mapping theorem,

$$\limsup_{n \rightarrow \infty} n^{-q^*} \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}^{\leq n\varepsilon}} (x_u^{\leq n\varepsilon})^{q^*} \right] \leq \mathbb{P}(\bar{\zeta} > \bar{\tau}^{\leq \varepsilon}),$$

which tends to 0 as $\varepsilon \rightarrow 0$ (because $\bar{\zeta} < \infty$ and $\bar{Y}(\bar{\zeta}-) > 0$, \mathbb{P} -almost surely). \blacksquare

2.5 Proof of Theorem 2.2.1

We prove [Theorem 2.2.1](#) by combining [Lemma 2.3.1](#) with the next two ‘‘tightness’’ properties. We suppose that [Assumptions \(H1\) to \(H3\)](#) hold and recall that $\mathbb{U}^{(h)} \subset \mathbb{U}$ refers to the set of words over $\{1, \dots, h\}$ with length at most h .

Lemma 2.5.1. For every $\delta > 0$,

$$\lim_{h \rightarrow \infty} \mathbb{P} \left(\sup_{t \geq 0} \sum_{u \in \mathbb{U} \setminus \mathbb{U}^{(h)}} (Y_u(t - b_u))^{q^*} > \delta \right) = 0.$$

Proof. This was already derived in [25, Lemma 20], and results from the following fact [19, Corollary 4]:

$$\mathbb{E} \left[\sum_{u \in \mathbb{U}} \sup_{t \geq 0} (Y_u(t - b_u))^q \right] < \infty \quad \text{for } \kappa(q) < 0. \quad \blacksquare$$

Lemma 2.5.2. If M is sufficiently large, then for every $\delta > 0$,

$$\lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}^{(n)} \left(\sup_{k \geq 0} \sum_{u \in \mathbb{U} \setminus \mathbb{U}^{(h)}} (X_u(k - \beta_u))^{q^*} > \delta n^{q^*} \right) = 0.$$

Proof. Let us first take $q_* < q^*$ and M as in Section 2.4. As in the proof of [25, Lemma 22] and by definition of $\mathcal{G}(n, \varepsilon)$ in Section 2.3, we claim that each particle in $\{X_u(k - \beta_u) : u \in \mathbb{U} \setminus \mathcal{G}(n, \varepsilon)\}$ has an ancestor with size at birth smaller than $n\varepsilon$. Thanks to the branching property, we may therefore consider that these particles derive from a system that has first been “frozen” below the level $n\varepsilon$, that is, with the notations of Section 2.4, from a particle system having $x_u^{\leq n\varepsilon}$, $u \in \mathbb{U}^{\leq n\varepsilon}$, as initial population. Hence, by Lemma 2.4.1 and Doob’s maximal inequality,

$$\mathbb{P}^{(n)} \left(\sup_{k \geq 0} \sum_{u \in \mathbb{U} \setminus \mathcal{G}(n, \varepsilon)} (X_u(k - \beta_u))^{q^*} > \delta n^{q^*} \right) \leq \frac{1}{\delta^{q_*/q^*} n^{q_*}} \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}^{\leq n\varepsilon}} (x_u^{\leq n\varepsilon})^{q_*} \right]$$

(bounding from above the ℓ^{q^*} -norm by the ℓ^{q_*} -norm). We conclude by Corollary 2.4.5 and Lemma 2.3.3. \blacksquare

Proof of Theorem 2.2.1. From Lemmas 2.3.1, 2.5.1 and 2.5.2, we deduce the convergence in distribution

$$\left(\frac{X_u^{(n)}(\lfloor a_n t \rfloor - \beta_u^{(n)})}{n} : u \in \mathbb{U} \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(d)} (Y_u(t - b_u) : u \in \mathbb{U})_{t \geq 0},$$

in the space $\mathbb{D}([0, \infty), \ell^{q^*}(\mathbb{U}))$ of $\ell^{q^*}(\mathbb{U})$ -valued càdlàg functions on $[0, \infty)$, where

$$\ell^{q^*}(\mathbb{U}) := \left\{ \mathbf{x} := (x_u : u \in \mathbb{U}) : \sum_{u \in \mathbb{U}} (x_u)^{q^*} < \infty \right\}.$$

Since for $q \geq 1$, rearranging sequences in the non-increasing order does not increase their q -distance [82, Theorem 3.5], the convergence in $\ell^{q^*}(\mathbb{U})$ implies that in ℓ^{q^\downarrow} , $q \geq 1 \vee q^*$. \blacksquare

2.6 Proof of Theorem 2.2.2

Similarly to the previous section, by [Lemma 2.3.2](#) the proof of [Theorem 2.2.2](#) is complete once we have established that

$$\lim_{h \rightarrow \infty} \mathbb{P}\left(d_{\text{GH}}(\mathcal{Y}, \mathcal{Y}_h) > \delta\right) = 0,$$

and

$$\lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(d_{\text{GH}}(\mathcal{X}^{(n)}, \mathcal{X}_h^{(n)}) > \delta a_n\right) = 0, \quad (2.10)$$

for all $\delta > 0$. The first display is clear since the tree \mathcal{Y} is compact. The second will be a consequence of the following counterpart of [[25](#), Conjecture 1]:

Lemma 2.6.1. Suppose [\(H1\)](#) to [\(H3\)](#), and $q^* > \gamma$. Then for every $q < q^*$, and for M sufficiently large,

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[\left(\frac{\text{ht}(\mathcal{X}^{(n)})}{a_n} \right)^{q/\gamma} \right] < \infty,$$

where $\text{ht}(\mathcal{X}^{(n)}) := \sup_{x \in \mathcal{X}^{(n)}} d_n(\emptyset, x)$ is the height of the tree $\mathcal{X}^{(n)}$.

The proof of [Lemma 2.6.1](#) involves martingale arguments. Prior to writing it, we need a preparatory lemma. Let us define

$$\tilde{\kappa}_n(q) := a_n \sum_{m=1}^{\infty} p_{n,m} \left[\left(\frac{a_m}{a_n} \right)^{q/\gamma} - 1 + \left(\frac{a_{n-m}}{a_n} \right)^{q/\gamma} \right],$$

which slightly differs from $\kappa_n(q)$ to the extent that we have replaced the map $m \mapsto m^q$ by the q -regularly-varying sequence $A_q(m) := a_m^{q/\gamma}$, $m \in \mathbb{N}$ (for convenience, we have set $a_m := 0$, $m \leq 0$). Of course, $\tilde{\kappa}_n = \kappa_n$ if $a_m = m^\gamma$ for every $m \in \mathbb{N}$.

Lemma 2.6.2. Suppose $q^* > \gamma$. Then we can find $q_* \in (0, q^*)$ such that, for every $q \in [q_*, q^*)$,

$$\lim_{n \rightarrow \infty} \tilde{\kappa}_n(q) = \kappa(q) < 0.$$

Proof. We will more generally show that for every q -regularly-varying sequence (r_n) ,

$$\left| a_n \sum_{m=1}^{\infty} p_{n,m} \left[\frac{r_m}{r_n} - \left(\frac{m}{n} \right)^q \right] \right| + \left| a_n \sum_{m=1}^{n-1} p_{n,m} \left[\frac{r_{n-m}}{r_n} - \left(1 - \frac{m}{n} \right)^q \right] \right| \xrightarrow{n \rightarrow \infty} 0,$$

provided $q < q^*$ is close enough to q^* . Denoting by Λ_n the law of $\log(X^{(n)}(1)/n)$, we observe that

$$a_n \sum_{m=1}^{\infty} p_{n,m} \left[\frac{r_m}{r_n} - \left(\frac{m}{n} \right)^q \right] = a_n \int_{-\infty}^{\infty} \left[\left(\frac{r_n e^x}{r_n} \right) - e^{qx} \right] \Lambda_n(dx),$$

which, by repeating the arguments in [27, Proof of Lemma 4.9], tends to 0 as $n \rightarrow \infty$. Next, an appeal to Potter's bounds [44, Theorem 1.5.6] shows that for every $c > 1$ and $\delta > 0$ arbitrary small,

$$\frac{1}{c} \left(\frac{m}{n}\right)^{q+\delta} \leq \frac{r_m}{r_n} \leq c \left(\frac{m}{n}\right)^{q-\delta}$$

whenever $m < n$ are sufficiently large. Thus, recalling that $\Psi_n(q) \rightarrow \Psi(q)$ and $\kappa_n(q) \rightarrow \kappa(q)$ for every q in some left-neighbourhood of q^* , we have

$$\liminf_{n \rightarrow \infty} a_n \sum_{m=1}^{n-1} p_{n,m} \left[\frac{r_{n-m}}{r_n} - \left(1 - \frac{m}{n}\right)^q \right] \geq \frac{1}{c} \left(\kappa(q+\delta) - \Psi(q+\delta) \right) - \left(\kappa(q) - \Psi(q) \right),$$

and

$$\limsup_{n \rightarrow \infty} a_n \sum_{m=1}^{n-1} p_{n,m} \left[\frac{r_{n-m}}{r_n} - \left(1 - \frac{m}{n}\right)^q \right] \leq \frac{1}{c} \left(\kappa(q-\delta) - \Psi(q-\delta) \right) - \left(\kappa(q) - \Psi(q) \right).$$

We conclude by letting $c \rightarrow 1$ and $\delta \rightarrow 0$. ■

We can now prove Lemma 2.6.1.

Proof of Lemma 2.6.1. We shall rely on a Foster-type technique close to the machinery developed in [11]; see in particular the proof of Theorem 2' there. First, observe that $\text{ht}(\mathcal{X})$ is distributed like the extinction time \mathcal{E} of \mathbf{X} :

$$\text{ht}(\mathcal{X}^{(n)}) \stackrel{d}{=} \sup_{u \in \mathbb{U}} \mathcal{E}_u^{(n)} =: \mathcal{E}^{(n)}.$$

Fix $q \in (\gamma, q^*)$ arbitrary close to q^* and set $r := q/\gamma$. By Lemma 2.6.2, suppose M large enough so that $\tilde{\kappa}_m(q) < 0$ for every $m > M$. It is easy to see as in the proof of Lemma 2.4.1 that the process

$$\Gamma(k) := \sum_{u \in \mathbb{U}} A_q(X_u(k - \beta_u)), \quad k \geq 0,$$

is a supermartingale under $\mathbb{P}^{(n)}$ (with respect to the natural filtration $(\mathcal{F}_k)_{k \geq 0}$ of \mathbf{X}): indeed, for $\mathbf{X}(k) = (x_i : i \in I)$,

$$\mathbb{E}^{(n)} \left[\Gamma(k+1) - \Gamma(k) \mid \mathcal{F}_k \right] = \sum_{i \in I} \tilde{\kappa}_{x_i}(q) A_{q-\gamma}(x_i),$$

where the right-hand side is (strictly) negative on the event $\{\mathcal{E} > k\} = \{\exists i \in I : x_i > M\}$.

We will more precisely show the existence of $\eta > 0$ sufficiently small such that the process

$$G(k) := \left(\Gamma(k)^{1/r} + \eta (\mathcal{E} \wedge k) \right)^r, \quad k \geq 0,$$

is a $(\mathcal{F}_k)_{k \geq 0}$ -supermartingale under $\mathbb{P}^{(n)}$, for any $n \in \mathbb{N}$. Then, the result will be readily obtained from $\eta^r \mathbb{E}^{(n)}[(\mathcal{E} \wedge k)^r] \leq \mathbb{E}^{(n)}[G(k)] \leq \mathbb{E}^{(n)}[G(0)] = A_q(n) = a_n^r$ and an appeal to Fatou's lemma.

On the one hand, we have

$$\sigma := \sum_{i \in I} A_{q-\gamma}(x_i) \geq \left(\sum_{i \in I} A_q(x_i) \right)^{1-\gamma/q}$$

because

$$\frac{A_q(x_i)}{\sigma^{q/(q-\gamma)}} = \left(\frac{A_{q-\gamma}(x_i)}{\sigma} \right)^{q/(q-\gamma)} \leq \frac{A_{q-\gamma}(x_i)}{\sigma},$$

where $q/(q-\gamma) > 1$ and the right-hand side sums to 1 as i ranges over I . Then, if we let $\eta > 0$ sufficiently small such that $\tilde{\kappa}_m(q) \leq -r\eta$ for every $m > M$, we deduce that

$$\mathbb{E}^{(n)} \left[\Gamma(k+1) \mid \mathcal{F}_k \right] \leq \Gamma(k) (1 - r\eta \Gamma(k)^{-\gamma/q} \mathbb{1}_{\{\mathcal{E} > k\}}).$$

Raising this to the power $1/r = \gamma/q$ yields

$$\mathbb{E}^{(n)} \left[\Gamma(k+1) \mid \mathcal{F}_k \right]^{1/r} \leq \Gamma(k)^{1/r} (1 - \eta \Gamma(k)^{-\gamma/q} \mathbb{1}_{\{\mathcal{E} > k\}}) = \Gamma(k)^{1/r} - \eta \mathbb{1}_{\{\mathcal{E} > k\}}, \quad (2.11)$$

by concavity of $x \mapsto x^{1/r}$. On the other hand, the supermartingale property also implies that $(\Gamma(k+1)^{1/r} + a)^r$ is integrable for every constant $a > 0$; we may thus apply the generalized triangle inequality [11, Lemma 1] with the positive, convex increasing function $x \mapsto x^r$, the positive random variable $\Gamma(k+1)^{1/r}$, and the probability $\mathbb{P}^{(n)}(\cdot \mid \mathcal{F}_k)$ (under which $\mathcal{E} \wedge (k+1)$ can be seen as a positive constant):

$$\mathbb{E}^{(n)} \left[(\Gamma(k+1)^{1/r} + \eta (\mathcal{E} \wedge (k+1)))^r \mid \mathcal{F}_k \right]^{1/r} \leq \mathbb{E}^{(n)} [\Gamma(k+1) \mid \mathcal{F}_k]^{1/r} + \eta (\mathcal{E} \wedge (k+1)).$$

Reporting (2.11) shows as desired that $(G(k) : k \geq 0)$ is a supermartingale. \blacksquare

We are finally ready to derive (2.10) and complete the proof of Theorem 2.2.2.

Proof of (2.10). We start as in the proof of Theorem 2.2.1: thanks to Lemma 2.3.3 and the branching property, with high probability as $h \rightarrow \infty$, the connected components of $\mathcal{X}^{(n)} \setminus \mathcal{X}_h^{(n)}$ are included in independent copies of \mathcal{X} stemming from the population $x_u^{\leq n\varepsilon}$, $u \in \mathbb{U}^{\leq n\varepsilon}$, of particles frozen below $n\varepsilon$. Specifically,

$$\mathbb{P}^{(n)} \left(d_{\text{GH}}(\mathcal{X}^{(n)}, \mathcal{X}_h^{(n)}) > \delta a_n \right) \leq \mathbb{P}^{(n)}(\mathcal{G}(n, \varepsilon) \not\subseteq \mathbb{U}^{(h)}) + \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}^{\leq n\varepsilon}} \mathbb{P}^{(x_u^{\leq n\varepsilon})}(\text{ht}(\mathcal{X}) > \delta a_n) \right].$$

Now, take $q_* < q < q^*$ and M large enough so that both [Lemma 2.6.1](#) and the results of [Section 2.4](#) hold. So, there exists a constant $C > 0$ such that

$$\begin{aligned} \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}^{\leq n\varepsilon}} \mathbb{P}^{(x_u^{\leq n\varepsilon})} \left(\text{ht}(\mathcal{X}) > \delta a_n \right) \right] &= \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}^{\leq n\varepsilon}} \mathbb{P}^{(x_u^{\leq n\varepsilon})} \left(\text{ht}(\mathcal{X}) > \delta a_{x_u^{\leq n\varepsilon}} \frac{a_n}{a_{x_u^{\leq n\varepsilon}}} \right) \right] \\ &\leq C \mathbb{E}^{(n)} \left[\sum_{u \in \mathbb{U}^{\leq n\varepsilon}} \left(\frac{a_{x_u^{\leq n\varepsilon}}}{a_n} \right)^{q/\gamma} \right]. \end{aligned}$$

But we know, thanks to another application of Potter's bounds, that we may find $c > 0$ such that

$$\left(\frac{a_m}{a_n} \right)^{q/\gamma} \leq c \left(\frac{m}{n} \right)^{q_*},$$

whenever n is sufficiently large and $m \leq n$. Since $x_u^{\leq n\varepsilon} \leq n$ (for $0 < \varepsilon < 1$), we can again conclude by [Corollary 2.4.5](#) and [Lemma 2.3.3](#). ■

A growth-fragmentation with immigration equation

We prove existence and uniqueness of a solution to a growth-fragmentation with immigration equation. The solution corresponds to the intensity measure of a stochastic particle system, in which immigrate copies of a growth-fragmentation process whose intensity measure solves the equation without immigration considered in [34, 20, 36]. This allows us to compare their respective asymptotic behaviors.

3.1 Introduction

Imagine a system of particles, parametrized by their “size” $x > 0$, with the following dynamics:

- particles grow at rate $\tau(x)$,
- fragmentation occurs at rate $B(x)$,
- at rate $I(dy)$, a new particle with size y immigrates into the system.

This means that the measure $\mu_t(dx)$ at time $t \geq 0$ of the average number of particles per unit volume with size $x \pm dx$ fulfills an integro-differential equation of the form¹

$$\partial_t \langle \mu_t, f \rangle = \langle \mu_t, \mathcal{A}f \rangle + \langle I, f \rangle \quad (3.1)$$

for test-functions f such that the right-hand side makes sense. Here, the so called *growth-fragmentation operator* \mathcal{A} may be taken as

$$\mathcal{A}f(x) := \tau(x)f'(x) + \int_0^x f(y)b(x,y)dy - B(x)f(x),$$

where $b(x, y)$ is the rate at which fragments with size $y > 0$ arise from the fragmentation of a particle with size $x > y$.

Let us first recall the *growth-fragmentation equation*, that is (3.1) without the immigration term ($I \equiv 0$), which has been studied extensively by many authors, see the monographs [58, 96] and references therein. Typically, the main purpose is to determine the asymptotic behavior of solutions, such as

$$\langle \mu_t, f \rangle \underset{t \rightarrow \infty}{\sim} e^{\lambda t} \langle \mu_0, h \rangle \langle \nu, f \rangle,$$

¹We employ the usual notation $\langle \nu, g \rangle := \int g(y) \nu(dy)$ for every measure ν and every function g .

also known as the *Malthusian behavior*, for some real $\lambda \in \mathbb{R}$, positive function h and measure ν . In this respect, a key component is the spectral analysis of the operator \mathcal{A} , especially the existence of positive eigenelements (λ, h, ν) . Recently, Bertoin and Watson [34, 20, 36] developed probabilistic tools toward this aim. Roughly, they related the semigroup associated with \mathcal{A} to a certain Markov process through a so called *Feynman–Kac formula*, and then solved the eigenproblem by applying classical results of ergodic theory. Their technique improves on some works in the literature of PDEs which were developed using more analytic tools, see e.g. [94, 54, 13, 55, 88] — although similar approaches, and more generally the use of probability for the study of growth-fragmentation equations and other semigroup-formulated PDEs, appear in earlier works, see e.g. Bansaye et al. [15, 14], Cloez [48], Doumic et al. [56].

In this note, we broaden the problem a bit by taking the immigration term into account ($I \neq 0$). We mention that a similar fragmentation with immigration equation, but without growth (more precisely, with $\tau \leq 0$) was studied by Haas [62, 61]. We shall first prove existence (Section 3.2) and uniqueness (Section 3.3) of the solution to (3.1), and then compare this solution with the one without immigration (Section 3.4), in particular regarding their respective Malthusian behaviors. Then, in Section 3.5, we connect the solution to the intensity measure of a stochastic particle system in which immigrate copies of the growth-fragmentation process of [36], and we also specify the Malthusian behavior for the process. In a concluding section, we discuss a variation of equation (3.1) where immigration no longer appears at constant rate but depends on the population.

Assumptions. We adopt the setting of Bertoin and Watson [34] and thus make the same assumptions. The following is taken for granted throughout this paper.

- Growth is sublinear: $\tau: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ is continuous and

$$\|\tau\| := \sup_{x>0} \frac{\tau(x)}{x} < \infty. \quad (3.2)$$

- Fragmentation is *conservative*: setting $\bar{b}(x, y) := \frac{y}{x}b(x, y)$, we have

$$\int_0^x \bar{b}(x, y) dy = B(x).$$

We also assume that

the map $x \mapsto \bar{b}(x, \cdot)$ from $\mathbb{R}_{>0}$ to $L^1(dy)$ is measurable and bounded.

- Immigration is described by a positive measure I on $\mathbb{R}_{>0}$ with finite mean:

$$m := \int y I(dy) < \infty. \quad (3.3)$$

Then $\langle I, \cdot \rangle$ is defined and finite on $\bar{\mathcal{C}}_b := \{f: \mathbb{R}_{>0} \rightarrow \mathbb{R} \text{ continuous with } \underline{f} \text{ bounded}\}$, where $\underline{f}(x) := f(x)/x$, and the domain $D(\mathcal{A})$ of the operator \mathcal{A} (which we can also define on $\bar{\mathcal{C}}_b$) contains every Lipschitz, continuously differentiable function $f \in \bar{\mathcal{C}}_b$. (In particular, $D(\mathcal{A})$ includes the space \mathcal{C}_c^∞ of smooth, compactly supported functions on $\mathbb{R}_{>0}$ and the identity function $x \mapsto x$.) A solution to the *growth-fragmentation with immigration equation* should then be understood as a family $(\mu_t)_{t \geq 0}$ of measures on $\mathbb{R}_{>0}$ such that (3.1) holds for all $t \geq 0$ and $f \in D(\mathcal{A})$.

3.2 Existence

Let us first provide an explicit solution to the growth-fragmentation with immigration equation (3.1) by integrating results from the no-immigration case. Bertoin and Watson [34] indeed solved the equation in absence of immigration ($I \equiv 0$): there exists a unique positive, strongly continuous semigroup $(T_t)_{t \geq 0}$ on $\bar{\mathcal{C}}_b$ such that, for every $t \geq 0$ and every $f \in D(\mathcal{A})$,

$$\partial_t T_t f = T_t \mathcal{A} f. \quad (3.4)$$

In other words, the family $(\mu_t)_{t \geq 0}$ of measures given by $\langle \mu_t, f \rangle := \langle \mu_0, T_t f \rangle$, $t \geq 0$, fulfills (3.1) when $I \equiv 0$. More precisely, they derived the following Feynman-Kac representation for $(T_t)_{t \geq 0}$: let $(X, (\mathbb{P}_x)_{x > 0})$ denote the Markov process with the infinitesimal generator $(\mathcal{G}, D(\mathcal{G}))$ on the space \mathcal{C}_b of continuous bounded functions on $\mathbb{R}_{>0}$ given by

$$\mathcal{G}f(x) := \tau(x)f'(x) + \int_0^x (f(y) - f(x)) \bar{b}(x, y) dy, \quad f \in D(\mathcal{G}).$$

Then, for $I \equiv 0$, the unique solution to (3.1) subject to $\mu_0 = \delta_x$ is

$$\langle \mu_{t,x}, f \rangle := T_t f(x) = x \mathbb{E}_x [f(X_t) \mathcal{E}_t], \quad \text{where } \mathcal{E}_t := \exp\left(\int_0^t \tau(X_s) ds\right). \quad (3.5)$$

Let us now replace the assumption $I \equiv 0$ by (3.3). Following Haas [61, § 5.1], this integrability condition allows us to define, for every $x > 0$, the measures

$$\langle \mu_{t,x}^{\dot{I}}, f \rangle := T_t f(x) + \int_0^t dr \int T_r f(y) I(dy), \quad t \geq 0. \quad (3.6)$$

(Note that $T_r f(y) = O(y)$ for $f \in \bar{\mathcal{C}}_b$, because $T_r: \bar{\mathcal{C}}_b \rightarrow \bar{\mathcal{C}}_b$, as confirmed by (3.5) and (3.2).)

Proposition 3.2.1. For every $x > 0$, the family $(\mu_{t,x}^{\dot{I}})_{t \geq 0}$ solves (3.1).

Proof. Let $f \in D(\mathcal{A})$. Then

$$\partial_t \langle \mu_{t,x}^{\dot{I}}, f \rangle = T_t \mathcal{A} f(x) + \int T_t f(y) I(dy),$$

where, by integrating (3.4),

$$T_t f(y) = f(y) + \int_0^t T_r \mathcal{A}f(y) \, dr, \quad j \geq 1.$$

Thus, by Fubini's theorem,

$$\begin{aligned} \partial_t \langle \mu_{t,x}^i, f \rangle &= T_t \mathcal{A}f(x) + \int \left(f(y) + \int_0^t T_r \mathcal{A}f(y) \, dr \right) I(dy) \\ &= T_t \mathcal{A}f(x) + \int_0^t dr \int T_r \mathcal{A}f(y) I(dy) + \int f(y) I(dy) \\ &= \langle \mu_{t,x}^i, \mathcal{A}f \rangle + \langle I, f \rangle, \end{aligned}$$

which is (3.1). ■

3.3 Uniqueness

Proposition 3.3.1. For every $x > 0$, the solution $(\mu_{t,x}^i)_{t \geq 0}$ given in (3.6) is the unique solution to (3.1) subject to $\mu_0 = \delta_x$.

Proof. Let $(\nu_t)_{t \geq 0}$ be a solution to (3.1) with $\nu_0 = \delta_x$. This means that, for $f \in D(\mathcal{A})$,

$$\partial_t \langle \nu_t, f \rangle = \langle \nu_t, \mathcal{A}f \rangle + \langle I, f \rangle.$$

We may rewrite this in terms of $\bar{\nu}_t(dy) := y \nu_t(dy)$, $\bar{I}(dy) := y I(dy)$, $\bar{f}(y) := y f(y)$, and $\underline{f} \in D(\mathcal{A})$ as

$$\partial_t \langle \bar{\nu}_t, \bar{f} \rangle = \langle \bar{\nu}_t, \bar{\mathcal{A}}\bar{f} \rangle + \langle \bar{I}, \bar{f} \rangle, \quad (3.7)$$

for the operator $\bar{\mathcal{A}}$ on \mathcal{C}_b defined by

$$\bar{\mathcal{A}}\bar{f}(x) := \frac{1}{x} \mathcal{A}\bar{f}(x) = \tau(x) f'(x) + \int (f(y) - f(x)) \bar{b}(x, y) \, dy + \underline{\tau}(x) f(x).$$

On the one hand, plugging in $f \equiv 1$ in (3.7) yields

$$\partial_t \langle \bar{\nu}_t, 1 \rangle \leq m + \|\underline{\tau}\| \langle \bar{\nu}_t, 1 \rangle,$$

so $\langle \bar{\nu}_t, 1 \rangle = O(\exp(\|\underline{\tau}\|t))$ by Grönwall's lemma. On the other hand, $\bar{\mathcal{A}}$ is known [34, Lemma 2.1] to generate a strongly continuous semigroup, which is also bounded by $\exp(\|\underline{\tau}\|t)$ (for the operator norm on \mathcal{C}_b). In particular, by [57, Theorem II.3.8], $\bar{\mathcal{A}}$ is closed and $p - \bar{\mathcal{A}}$ is surjective for every $p > \|\underline{\tau}\|$. Now, fix $g \in \mathcal{C}_b$ and $p > \|\underline{\tau}\|$, and let $\underline{f} \in D(\mathcal{A})$ such that

$g = (p - \bar{\mathcal{A}})f$. Multiplying both sides of (3.7) by e^{-pt} and integrating by parts, we arrive at the identity

$$\int_0^\infty e^{-pt} \langle \bar{\nu}_t, g \rangle dt = \bar{f}(x) + \frac{1}{p} \langle \bar{I}, f \rangle.$$

Since the right-hand side does not depend on the solution $(\nu_t)_{t \geq 0}$ to (3.6) with $\nu_0 = \delta_x$, we thus see by setting $q := p - \|\mathcal{I}\|$ and $\bar{\mu}_{t,x}^i(dy) := y \mu_{t,x}^i(dy)$ that for every $q > 0$,

$$\int_0^\infty e^{-qt} (e^{-\|\mathcal{I}\|t} \langle \bar{\nu}_t, g \rangle) dt = \int_0^\infty e^{-qt} (e^{-\|\mathcal{I}\|t} \langle \bar{\mu}_{t,x}^i, g \rangle) dt.$$

In other words, the Laplace transforms of the continuous maps $t \mapsto e^{-\|\mathcal{I}\|t} \langle \bar{\nu}_t, g \rangle$ and $t \mapsto e^{-\|\mathcal{I}\|t} \langle \bar{\mu}_{t,x}^i, g \rangle$ coincide. Hence these two maps are equal, and because this equality is valid for every $g \in \mathcal{C}_b$, we conclude that $(\nu_t)_{t \geq 0} = (\mu_{t,x}^i)_{t \geq 0}$. ■

3.4 Comparison of solutions

We now come to the asymptotic behavior as $t \rightarrow \infty$ of the solution to the growth-fragmentation with immigration equation. We shall relate this behavior to (and actually derive it from) the one without immigration. Indeed, Bertoin and Watson [34] provided conditions for which the solution without immigration $(\mu_t)_{t \geq 0}$ has a Malthusian behavior, namely

$$\lim_{t \rightarrow \infty} e^{-\lambda t} \langle \mu_{t,x}, f \rangle = h(x) \langle \nu, f \rangle, \quad x > 0, \quad (3.8)$$

at least for every continuous, compactly supported function f , where $\lambda \in \mathbb{R}$ is the so called *Malthus exponent*, h is a positive function, and ν is a Radon measure on $(0, \infty)$ (called the *asymptotic profile*). In the language of spectral theory, when appropriate definiteness conditions are met (e.g., $h \in D(\mathcal{A})$), λ corresponds to the leading eigenvalue of the operator \mathcal{A} , h is an associated eigenfunction, and ν is an associated eigenmeasure w.r.t. the dual operator.

It is of course expected that, because of immigration, if μ_t and μ_t^i are of order $e^{\lambda t}$ and $e^{\lambda^i t}$ respectively, then $\lambda^i \geq \lambda$. This is indeed true and easy to check because, by (3.6), $\langle \mu_{t,x}^i, f \rangle \geq \langle \mu_{t,x}, f \rangle$ for every nonnegative function f .

Let us recall the necessary [20] and sufficient [34] condition for the solution $(\mu_t)_{t \geq 0}$ to admit a Malthusian behavior. It is expressed using the Markov process $(X, (\mathbb{P}_x)_{x > 0})$ that represents $(\mu_t)_{t \geq 0}$ through the Feynman–Kac formula (3.5). Suppose that X is irreducible, that is, $\mathbb{P}_x(H(y) < \infty) > 0$ for all $x, y > 0$, where $H(y) := \inf\{t \geq 0: X_t = y\}$. Define also the convex function $L_{x,y}: \mathbb{R} \rightarrow (-\infty, \infty]$ by

$$L_{x,y}(q) := \mathbb{E}_x[e^{-qH(y)} \mathcal{E}_{H(y)}; H(y) < \infty], \quad q \in \mathbb{R},$$

with \mathcal{E} the functional defined in (3.5). Then $(\mu_t)_{t \geq 0}$ has a Malthusian behavior if and only if

$$\text{for any (or all) } x > 0, \quad L_{x,x}(\lambda) = 1 \quad \text{and} \quad L'_{x,x}(\lambda) > -\infty, \quad (3.9)$$

where λ is *defined* as

$$\lambda := \inf\{q \in \mathbb{R} : L_{x,x}(q) < 1\}$$

and is proved to not depend on x . More precisely, if (3.9) is in force, then (3.8) holds with the eigenpair

$$h(y) := yL_{y,x}(\lambda) \quad \text{and} \quad \nu(dy) := \frac{dy}{h(y)\tau(y)|L'_{y,y}(\lambda)|}, \quad y > 0,$$

and further

$$\langle \mu_{t,x}, h \rangle = e^{\lambda t} h(x) \quad \text{for all } x > 0, t \geq 0, \quad \text{and} \quad \langle \nu, h \rangle = 1.$$

Proposition 3.4.1. Suppose $(\mu_t)_{t \geq 0}$ admits the Malthusian behavior (3.8) and $\langle I, h \rangle < \infty$. Then as $t \rightarrow \infty$, for every $x > 0$ and every continuous function $f: \mathbb{R}_{>0} \rightarrow \mathbb{R}$ with compact support,

$$\langle \mu_{t,x}^i, f \rangle = \begin{cases} e^{\lambda t} \langle \nu, f \rangle (h(x) + \frac{1}{\lambda} \langle I, h \rangle) + o(e^{\lambda t}), & \text{if } \lambda > 0, \\ \frac{1}{-\lambda} \langle \nu, f \rangle \langle I, h \rangle + o(1), & \text{if } \lambda < 0, \\ \langle \nu, f \rangle (h(x) + t \langle I, h \rangle) + o(t), & \text{if } \lambda = 0. \end{cases}$$

Remark 3.4.2. Unsurprisingly the exponential order is $\lambda^i := \lambda \vee 0$ (immigration prevents the population from extinction).

Proof. Let $C > 0$ such that $|f| \leq Ch$ and recall that $e^{-\lambda t} \langle \mu_{t,x}, h \rangle = h(x)$ for all $x > 0, t \geq 0$. By (3.6),

$$\langle \mu_{t,x}^i, f \rangle = \langle \mu_{t,x}, f \rangle + \int_0^t e^{\lambda r} dr \left(\int e^{-\lambda r} \langle \mu_{r,y}, f \rangle I(dy) \right),$$

where the inner integrand is dominated for all $r \geq 0$ by $Ce^{-\lambda r} \langle \mu_{r,y}, h \rangle = Ch(y)$, which by assumption is integrable with respect to I . Further, (3.8) says that

$$\lim_{r \rightarrow \infty} e^{-\lambda r} \langle \mu_{r,y}, f \rangle = \langle \nu, f \rangle h(y).$$

We easily conclude by the dominated convergence theorem. ■

We mention the sufficient condition [20, Theorem 2] to guarantee that (3.8) holds *exponentially fast*:

$$\limsup_{x \rightarrow 0^+} \frac{\tau(x)}{x} < \lambda \quad \text{and} \quad \limsup_{x \rightarrow \infty} \frac{\tau(x)}{x} < \lambda. \quad (3.10)$$

This condition further implies [34, Lemma 4.6] that $\underline{h} \in \mathcal{C}_b$, so $\langle I, h \rangle = \langle \bar{I}, \underline{h} \rangle < \infty$ and Proposition 3.4.1 applies.

3.5 The growth-fragmentation-immigration process

Recently, Bertoin and Watson [36] showed that the solution without immigration $(\mu_t)_{t \geq 0}$ corresponds to the intensity measure of a so called *growth-fragmentation process* [19], that is a Markov branching process $(\mathbf{Z}(t))_{t \geq 0}$ whose construction in terms of a particle system $\mathcal{Z} := (Z_u(t) : b_u \leq t < d_u, u \in \mathbb{U})$ can be summarized as follows. (We employ the usual Harris–Ulam tree $\mathbb{U} := \bigcup_{n \geq 0} \mathbb{N}^n$ to label the individuals.) Under \mathbb{P}_{x_0} , $x_0 > 0$, there is initially one particle Z_\emptyset , called Eve, which evolves until the first fragmentation event according to the flow velocity

$$\begin{cases} \dot{x}(t) = \tau(x(t)), \\ x(0) = x_0. \end{cases}$$

Thus $Z_\emptyset(t) = x(t)$, $t \in [b_\emptyset, d_\emptyset) := [0, \zeta_\emptyset)$, up to the fission time ζ_\emptyset of Eve. Recalling that $B(y)$ is the rate at which a particle with size y fragmentates, ζ_\emptyset has law

$$\mathbb{P}_{x_0}(\zeta_\emptyset > t) = \exp\left(-\int_0^t B(x(s)) ds\right) = \exp\left(-\int_{x_0}^{x(t)} \frac{B(y)}{\tau(y)} dy\right), \quad t \geq 0.$$

We view $d_\emptyset = \zeta_\emptyset$ as both the death time of Eve and, when $d_\emptyset < \infty$, the birth time b_i of all its daughter particles Z_i , $i \in \mathbb{N}$. Specifically, conditionally on the death $d_\emptyset = s < \infty$ of Eve, a fraction $p_i \in [0, 1)$ of the mass $x = Z_\emptyset(\zeta_\emptyset -) = x(s)$ is given to its i -th child at birth, namely $Z_i(s) = xp_i$, where the (conservative) *mass-partition* $\mathbf{p} = (p_1, p_2, \dots)$ is a random element of the space

$$\mathcal{P} := \left\{ \mathbf{p} \in \ell^1 : p_1 \geq p_2 \geq \dots \geq 0 \text{ and } \sum_{i \geq 1} p_i = 1 \right\},$$

and its law $\varrho(x, \mathbf{dp})$, for some probability kernel ϱ on \mathcal{P} , satisfies to the relation

$$\int_0^x f(y) b(x, y) dy = B(x) \int_{\mathcal{P}} \sum_{i \geq 1} f(xp_i) \varrho(x, \mathbf{dp}). \quad (3.11)$$

Naturally, children evolve independently and according to the same dynamics, which means that conditionally on $b_i = s$ and $Z_i(s) = y$, the system $\mathcal{Z}_i := (Z_{iu}(\cdot - s) : u \in \mathbb{U})$ has the law of \mathcal{Z} under \mathbb{P}_y , and further the family $(\mathcal{Z}_i)_{i \in \mathbb{N}}$ is independent. By definition, the growth-fragmentation process

$$\mathbf{Z}(t) := \sum_{u \in \mathbb{U}} \mathbf{1}_{\{b_u \leq t < d_u\}} \delta_{Z_u(t)}, \quad t \geq 0,$$

is the point process of particles alive at t . We let \mathbf{P}_{x_0} denote the law of \mathbf{Z} under \mathbb{P}_{x_0} , and note that the branching property is fulfilled: for every $s \geq 0$, conditionally on $\mathbf{Z}(s) = \mathbf{z}$,

the process $(\mathbf{Z}(t+s): t \geq 0)$ is independent of $(\mathbf{Z}(r): r \leq s)$ and has the distribution \mathbf{P}_z of the sum $\sum_{i \geq 1} \mathbf{Z}_i$ of independent growth-fragmentation processes \mathbf{Z}_i , $i \geq 1$, with respective laws \mathbf{P}_{z_i} .

It is then fairly easy to see that $f \mapsto \mathbf{E}[\langle \mathbf{Z}, f \rangle]$ enjoys the semigroup property and that for every $f \in D(\mathcal{A})$, we can decompose $\mathbf{E}[\langle \mathbf{Z}(t), f \rangle]$ as

$$\mathbf{P}(\zeta_\emptyset > t) f(x(t)) + \int_0^t \exp\left(-\int_0^s B(x(r)) dr\right) ds \int_0^{x(s)} \mathbf{E}_y[\langle \mathbf{Z}(t-s), f \rangle] b(x(s), y) dy,$$

from which it follows that

$$\partial_t \mathbf{E}[\langle \mathbf{Z}(t), f \rangle] \Big|_{t=0} = \mathcal{A}f.$$

Therefore the intensity measures $\mathbf{E}[\mathbf{Z}(t)] = \mu_t$, $t \geq 0$, solve the growth-fragmentation equation (3.1) without immigration ($I \equiv 0$). Besides, taking $f = f_1: x \mapsto x$ above and applying the branching property show that $e^{-\|x\|t} \langle \mathbf{Z}(t), f_1 \rangle$, $t \geq 0$, is a supermartingale, so we can view \mathbf{Z} in the space \mathbb{D} of càdlàg ℓ^1 -valued functions.

Bertoin and Watson further proved that under the condition (3.10), the Malthusian behavior (3.8) holds strongly: for every $x > 0$ and every continuous function $f: \mathbb{R}_{>0} \rightarrow \mathbb{R}$ such that $f = O(h)$:

$$\lim_{t \rightarrow \infty} e^{-\lambda t} \langle \mathbf{Z}(t), f \rangle = \langle \nu, f \rangle W_\infty, \quad \text{in } L^1(\mathbf{P}_x), \quad (3.12)$$

where W_∞ is the limit of the uniformly integrable martingale $W_t := e^{-\lambda t} \langle \mathbf{Z}(t), h \rangle$, $t \geq 0$, and $\mathbf{E}_x[W_\infty] = h(x)$.

Here, we consider an independent Poisson point process \mathcal{N} on $\mathbb{R}_{\geq 0} \times \mathbb{D}$ with intensity $dt \otimes \mathbf{PI}$, where

$$\mathbf{PI} := \int I(dy) \mathbf{P}_y.$$

We then define the *growth-fragmentation with immigration process* as

$$\mathbf{Z}^i(t) := \mathbf{Z}(t) + \int \mathcal{N}(ds, d\mathbf{z}) \mathbf{1}_{\{s \leq t\}} \mathbf{z}(t-s), \quad t \geq 0.$$

and let $\mathbf{P}_{x_0}^i$ denote its law under \mathbf{P}_{x_0} . We immediately observe that for every $f \in D(\mathcal{A})$,

$$\mathbf{E}_{x_0}^i[\langle \mathbf{Z}^i(t), f \rangle] = \langle \mu_{t, x_0}, f \rangle + \int_0^t ds \int \langle \mu_{t-s, y}, f \rangle I(dy) = \langle \mu_{t, x_0}^i, f \rangle$$

(cf. (3.6)), and so the solution to the growth-fragmentation with immigration equation (3.1) indeed corresponds to the intensity measure of \mathbf{Z}^i . We stress however that, unlike the no-immigration case, the expression “ $T_t^i f(x) := \langle \mu_{t, x}^i, f \rangle$ ” does not define a semigroup, because

the additional immigration term invalidates the branching property. Still, whether the strong Malthusian behavior (3.12) extends to \mathbf{Z}^i is a natural question. In this direction, we map each atom (s, \mathbf{z}) of \mathcal{N} to (s, w) , where $w := \lim_{t \rightarrow \infty} e^{-\lambda t} \langle \mathbf{z}(t), h \rangle$ is a copy of W_∞ , and we let $\tilde{\mathcal{N}}(ds, dw)$ denote the image of $\mathcal{N}(ds, d\mathbf{z})$ by this map.

Proposition 3.5.1. Suppose (3.10). Then for every $x > 0$ and every continuous function $f: \mathbb{R}_{>0} \rightarrow \mathbb{R}$ such that $f = O(h)$,

$$\lim_{t \rightarrow \infty} e^{-\lambda t} \langle \mathbf{Z}^i(t), f \rangle = \langle \nu, f \rangle \left(W_\infty + \int e^{-\lambda s} w \tilde{\mathcal{N}}(ds, dw) \right), \quad \text{in } L^1(\mathbf{P}_x^i).$$

Remark 3.5.2. This strengthens Proposition 3.4.1 in the case $\lambda > 0$, at the cost of the stronger premise (3.10).

Proof. The proof is similar to that of Proposition 3.4.1 (when $\lambda > 0$). Let $C > 0$ such that $|f| \leq Ch$. From the very definitions of \mathbf{Z}^i , \mathcal{N} , $\tilde{\mathcal{N}}$, and \mathbf{PI} , we have that

$$\mathbf{E}_x^i \left[\left| e^{-\lambda t} \langle \mathbf{Z}^i(t), f \rangle - \langle \nu, f \rangle \left(W_\infty + \int e^{-\lambda s} w \tilde{\mathcal{N}}(ds, dw) \right) \right| \right]$$

is bounded from above by

$$\begin{aligned} \mathbf{E}_x \left[\left| e^{-\lambda t} \langle \mathbf{Z}(t), f \rangle - \langle \nu, f \rangle W_\infty \right| \right] \\ + \int_0^\infty e^{-\lambda s} ds \int I(dy) \mathbf{E}_y \left[\left| \mathbf{1}_{\{s \leq t\}} e^{-\lambda(t-s)} \langle \mathbf{Z}(t-s), f \rangle - \langle \nu, f \rangle W_\infty \right| \right]. \end{aligned}$$

Thanks to (3.12), the first term tends to 0 as $t \rightarrow \infty$, as does the integrand in the second term. Moreover, this integrand is dominated by $2Ce^{-\lambda s}h(y)$, which is integrable with respect to $ds \otimes I$. We conclude by another application of the dominated convergence theorem. ■

3.6 Population-dependent immigration

So far immigration has been occurring at a constant rate (independently of the number of particles). This is a natural assumption, however it has caused the branching property to break. Alternatively, one may also consider the situation where each individual independently attracts new immigrants. Assuming

$$\Upsilon := I(\mathbb{R}_{>0}) < \infty,$$

this amounts to repeating the construction of the growth-fragmentation process \mathbf{Z} (as explained in the first paragraph of Section 3.5), except that we replace $B(x)$ by $\tilde{B}(x) := B(x) + \Upsilon$ and the probability kernel $\varrho(x, d\mathbf{p})$ by

$$\tilde{\varrho}(x, ds) = \tilde{B}(x)^{-1} \left(B(x) \varrho(x, d\mathbf{p}) + \mathbf{1}_{\{xs=(x,y,0,\dots)\}} I(dy) \right),$$

for $\mathbf{s} \in \mathcal{S} := \mathcal{P} \cup \{1\} \times \mathbb{R}_{>0} \times \{0\}^{\mathbb{N}}$, with which we associate the fission-immigration rate \tilde{b} defined by

$$\int_0^\infty f(y) \tilde{b}(x, dy) := \tilde{B}(x) \int_{\mathcal{S}} \sum_{i \geq 1} f(xs_i) \tilde{\varrho}(x, ds) = \int_0^x f(y) b(x, y) dy + \int f(y) I(dy) + \Upsilon f(x);$$

see (3.11). We end up with a *growth-fragmentation with population-dependent immigration process* $\tilde{\mathbf{Z}}^i$ with initial laws $(\tilde{\mathbf{P}}_x^i)_{x>0}$, such that

$$\partial_t \tilde{\mathbf{E}}_x^i[\langle \tilde{\mathbf{Z}}^i(t), f \rangle] \Big|_{t=0} = \mathcal{A}^i f(x), \quad t \geq 0, x > 0, f \in D(\mathcal{A}),$$

where

$$\mathcal{A}^i f(x) := \mathcal{A}f(x) + \int f(y) I(dy).$$

In this case the intensity measures $\tilde{\mu}_{t,x}^i := \tilde{\mathbf{E}}_x^i[\tilde{\mathbf{Z}}^i(t)]$, $t \geq 0$, solve the alternative equation

$$\partial_t \langle \mu_t, f \rangle = \langle \mu_t, \mathcal{A}^i f \rangle, \quad (3.13)$$

and, because the branching property holds, the expression $T_t^i f(x) := \langle \tilde{\mu}_{t,x}^i, f \rangle$ defines a semigroup with generator \mathcal{A}^i .

For simplicity, we assume that the fragmentation kernel b is *homogeneous*, meaning that

$$b(x, y) = \frac{1}{x} \rho\left(\frac{y}{x}\right), \quad 0 < y < x,$$

where $\rho: (0, 1) \rightarrow \mathbb{R}_{\geq 0}$ has first positive and negative moments. In particular the fragmentation rate B is constant:

$$B \equiv \int_0^1 u \rho(u) du < \infty.$$

We further assume that

$$\liminf_{x \rightarrow 0^+} \frac{\tau(x)}{x} > \int_0^1 (u^{-1} - 1) \rho(u) du \quad \text{and} \quad \limsup_{x \rightarrow \infty} \frac{\tau(x)}{x} < \int_0^1 (1 - u) \rho(u) du =: \lambda'.$$

Proposition 3.6.1. Under the above assumptions, the solution $(\tilde{\mu}_t^i)_{t \geq 0}$ to (3.13) admits the Malthusian behavior

$$\lim_{t \rightarrow \infty} e^{-(\lambda + \Upsilon)t} \langle \tilde{\mu}_t^i, f \rangle = \Upsilon^{-1} \langle \nu \star I, f \rangle, \quad (3.14)$$

for every $f: \mathbb{R}_{>0} \rightarrow \mathbb{R}$ continuous with compact support, where ν is the asymptotic profile of the solution without immigration.

Proof. By [34, Proposition 7.1], we have $\lambda = \lambda'$ and $h = \mathbf{1}$ (the constant function equal to 1). Then, we can write

$$\mathcal{A}f(x) = \tilde{\mathcal{G}}f(x) + \lambda f(x), \quad \text{and} \quad \mathcal{A}^i f(x) = \tilde{\mathcal{G}}^i f(x) + (\lambda + \Upsilon)f(x),$$

where

$$\tilde{\mathcal{G}}f(x) := \tau(x)f'(x) + \int_0^1 (f(ux) - f(x)) \rho(u)du,$$

and

$$\tilde{\mathcal{G}}^i f(x) := \tilde{\mathcal{G}}f(x) + \int (f(y) - f(x)) I(dy),$$

define infinitesimal generators of some Markov processes. More precisely, if \tilde{X} has generator $\tilde{\mathcal{G}}$ and $(N_t: t \geq 0)$ is an independent Markov jump process on $\mathbb{R}_{>0}$ with infinitesimal generator $\tilde{\mathcal{G}}^i - \tilde{\mathcal{G}}$, then $\tilde{X}^i = \tilde{X} + N$ has generator $\tilde{\mathcal{G}}^i$, and there are the simple Feynman–Kac formulas

$$\langle \mu_{t,x}, f \rangle = e^{\lambda t} \mathbb{E}_x[f(X_t)],$$

and

$$\langle \tilde{\mu}_{t,x}^i, f \rangle = e^{(\lambda+\Upsilon)t} \mathbb{E}_x[f(X_t + N_t)].$$

Clearly, N admits $\Upsilon^{-1}I$ as stationary distribution. Since further X and N are independent, there is the convergence in distribution $(X_t, N_t) \rightarrow (\nu, \Upsilon^{-1}I)$ as $t \rightarrow \infty$. The conclusion follows. \blacksquare

Remark 3.6.2. The Malthusian behavior (3.14) further holds exponentially fast under the additional assumption $r := \int y^{-1} I(dy) < \infty$. Indeed, with $V(x) := x + x^{-1}$, we see that

$$\tilde{\mathcal{G}}^i V(x) = x(\tau(x) - (\lambda + \Upsilon)) + x^{-1} \left(\int_0^1 (u^{-1} - 1) \rho(u)du - (\Upsilon + \tau(x)) \right) + m + r, \quad x > 0,$$

and so there exist $b > 0$ and $\beta > 0$ such that $\tilde{\mathcal{G}}^i V(x) \leq -\beta V(x)$ whenever $x > b$ or $x < 1/b$. This allows us to apply the geometric ergodic theorem, just like in [34, Proof of Proposition 7.2.(ii)].

Remark 3.6.3. We can derive a strong Malthusian behavior for $\tilde{\mathbf{Z}}^i$ by adapting the results of [36]. If

$$\limsup_{x \rightarrow 0^+} \frac{\tau(x)}{x} < \lambda + \Upsilon$$

also holds, then for every $x > 0$ and every continuous bounded function $f: \mathbb{R}_{>0} \rightarrow \mathbb{R}$,

$$\lim_{t \rightarrow \infty} e^{-(\lambda+\Upsilon)t} \langle \tilde{\mathbf{Z}}^i(t), f \rangle = \Upsilon^{-1} \langle \nu \star I, f \rangle \tilde{W}_\infty, \quad \text{in } L^1(\tilde{\mathbf{P}}_x^i),$$

where \tilde{W}_∞ is the limit of the uniformly integrable martingale $\tilde{W}_t := e^{-(\lambda+\Upsilon)t} \langle \tilde{\mathbf{Z}}_t^i, \mathbf{1} \rangle$, $t \geq 0$.

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