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École Doctorale de Sciences Mathématiques de Paris Centre (ED 386)
Institut de Mathématiques de Jussieu - Paris Rive Gauche (UMR 7586)

Thèse de doctorat

Discipline : Mathématiques

Présentée par

Antoine Julia

Théorème de Stokes et intégration sur les courants entiers

Dirigée par **Thierry De Pauw**

Soutenue publiquement le 9 octobre 2018 devant le jury composé de :

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Résumé

Les méthodes d'intégration de jauge, telle que l'intégrale de Pfeffer sur les ensembles bornés de périmètre fini sont particulièrement adaptées à l'étude des grands théorèmes d'intégration que sont le Théorème Fondamental de l'Analyse, le Théorème de la Divergence et le Théorème de Stokes. Dans cette thèse, ces outils sont transposés à l'intégration sur des domaines singuliers, vus comme des courants entiers au sens de Federer et Fleming. On obtient un critère d'effaçabilité pour les singularités des courants considérés : les courants ayant un ensemble singulier de *contenu de Minkowski relatif* fini satisfont un Théorème de Stokes général, c'est le cas notamment des courants définissables dans une structure o-minimale quelconque, c'est aussi le cas de courants minimiseurs de masse sans singularité au bord. A contrario, on construit un courant de dimension 2 dans $\mathbb{R}^3$ ayant un ensemble singulier réduit à un point, qui ne vérifie pas ce Théorème de Stokes général. Cette thèse contient aussi les définitions de méthodes d'intégration non absolument convergentes sur tout courant entier de dimension 1, ainsi que sur les courants entiers de dimension quelconque dans un espace euclidien dont les singularités sont effaçables.

Mots-clés

Intégration de jauge, Théorème de Stokes, courants entiers, singularités effaçables.

Abstract

Methods of gauge integration, like those developped by W. F. Pfeffer on bounded sets of finite perimeter, are well suited to the study of integration theorems, such as the Fundamental Theorem of Calculus, The Divergence Theorem and Stokes' Theorem. In this thesis, Pfeffer Integration is transposed to the context of integral currents in the sense of Federer and Fleming. Not all integral currents are adapted to this type of gauge integration and a criterion on the singular set of the current is obtained. Well behaved currents include all 1-dimensional integral currents, integral currents definable in an o-minimal structure and mass minimizing integral currents whenever the boundary singularities are controlled. All those currents are shown to satisfy a general Stokes' Theorem. On the other hand, an example is given of an integral current of dimension 2 in $\mathbb{R}^3$ with only one singular point, which does not satisfy such a general Stokes-Cartan Theorem. This thesis also contains the definitions of non-absolutely convergent integrations methods on 1-dimensional integral currents as well as on integral currents of any dimension in Euclidean space, whenever their singular set has controlled relative Minkowski content.

Keywords

Gauge integration, Stokes' Theorem, integral currents, removable singularities.

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Chapter 1

Introduction

This thesis is centered on the question of Integration on singular domains. The goal is to define a notion of integral which leads to a general Stokes' Theorem. This is motivated by the study of removable singularities. We build on the works of R. Henstock, J. Kurzweil, W. Pfeffer and many others, and extend some of the key notions to develop an integration method on integral currents in Euclidean space. We also give an example of current on which a Generalized Stokes' Theorem does not hold, which implies that it does not behave well with respect to our notion of integration. In the following section, we give an account of the various preexisting integration theories and results we have used and translated to our setting when it was possible.

1.1 Fundamental Theorems and singularities

1.1.1 Fundamental Theorem of Calculus

At the heart of calculus is the complementarity of the integration and differentiation operations. This complementarity is referred to as the Fundamental Theorem of Calculus, which can be stated in modern terms as
The integral on an interval of the derivative of a gentle *function is equal to the variation of this function on the same interval, i.e. if F is continuous and regular enough on $[a, b]$, then*

$$\int_a^b F' = F(b) - F(a) \tag{1.1}$$

Here, we use the imprecise word *gentle*, however this is not the only imprecision in the statement: the integral sign in (1.1) may refer to several types of integrals. Depending on the method of integration we choose, we should change the meaning of *gentle* so that the statement above is valid.

The modern study of integration usually starts with the Riemann Integral defined in 1854, following the works of Cauchy and others. Riemann defined a rigorous integration method, at the heart of which is the discretization of the integration domain into small intervals leading to what we now call Riemann sums. The Fundamental Theorem of the Riemann Integral holds if the function F is differentiable at each point and its derivative, F', is Riemann integrable. In particular, F' should be bounded and continuous almost everywhere.

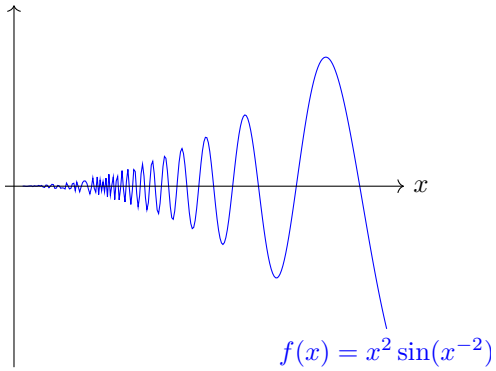

Figure 1.1: A function differentiable everywhere

In the second half of the 19th century, it became clear that Riemann's method was too restrictive, with the invention (or the discovery) of increasingly irregular functions, in particular the Dirichlet Function, which is discontinuous at every point. This led Lebesgue to use a radically different approach in [47]: instead of discretizing the domain of definition of the function, he discretized the image of the function and studied the measure of the level sets. This method, called Lebesgue Integration, is more involved that Riemann's, as it relies on measure theory, but also much more powerful. The Fundamental Theorem for the Lebesgue Integral is then valid whenever F is continuous on $[a, b]$ and differentiable at all points and its derivative is Lebesgue integrable. There are more general statements, but this one is sufficient for now. In the following, $\int f$ will denote the Lebesgue integral of f when no integration method is precised.

The Lebesgue integrability condition is much weaker than Riemann Integrability and the corresponding Fundamental Theorem is therefore much more general. There are however examples of functions which are differentiable at all points, but whose derivative is not integrable, as for instance the function plotted in figure 1.1:

$$f : [0, 1] \to \mathbb{R}; x \mapsto \begin{cases} x^2 \sin(1/x^2) \text{ if } x > 0, \\ 0 \text{ if } x = 0. \end{cases}$$

This function is differentiable everywhere, but its derivative is not Lebesgue integrable (its absolute value is comparable with $x \mapsto 1/|x|$ on large portions of each interval containing 0).

In 1912, Denjoy attempted to solve this problem by defining an extension of the Lebesgue integral, his method involves transfinite induction and is not very practical. Two years later, Perron defined an equivalent integral in a simpler way. The Denjoy-Perron Integral was later found to be equivalent to a simple modification of the Riemann Integral. Independently, Kurzweil in 1957 ([46])and Henstock in 1961 ([37]), defined a process of integration using gauges to refine the discretization process of Riemann.

The first section of chapter 2 in this thesis contains a brief review of the Henstock-Kurzweil (HK) method. For more details on the history of the Integral as well as a complete treatment, we refer the reader to [64, 31, 53]. For now, we will be very brief: the Henstock Kurweil Integral relies on a discretization of the domain into a family of pairs (I, x) where I is an interval containing x with length no more than $\delta(x)$ and δ is a positive function on the domain, called a **gauge**. The

introduction of the gauge is the only difference between this method and the Riemann Integral. However this difference is key, as in this context, the Fundamental Theorem holds without any integrability condition - Henstock-Kurzweil integrability is indeed a consequence of the hypothesis - and furthermore we can also allow the function to lack differentiability at some points.

How many such points can we have? In order to integrate the derivative of a function this derivative should of course exist almost everywhere (with respect to the Lebesgue measure on the interval). However, the example of Cantor's "Devil's Staircase" proves that "almost everywhere" is not sufficient. In fact, constructing a Devil's Staircase from a Cantor set of Hausdorff dimension 0, one sees that even asking that the set of non differentiability points be 0 dimensional is not sufficient. Yet, one can allow for a finite set of non differentiability points for F. How about countable infinite sets? We have the following Fundamental Theorem for the Henstock Kurzweil integral:

Theorem. *Let F be a continuous function on an interval $[a, b]$. If F is differentiable at all points except for a countable set in $[a, b]$, then its derivative F' is HK integrable on $[a, b]$ and*

$$F(b) - F(a) = (HK) \int_a^b F'.$$

It is, in fact, possible to do slightly better using a sort of generalized absolute continuity: the notion of AC_* function, and Stepanoff's theorem, we will discuss this in more details in chapter 2.

Countable turns out to be the optimal hypothesis for the singular set. Indeed, in the paper [76], Z. Zahorski characterizes completely the set of points of non differentiability of a continuous function on the line and proves that it is always a countable union of G_δ sets. Consider an uncountable G_δ, E of Lebesgue measure 0 in the interval $[0, 1]$, it contains a compact nowhere dense set of measure 0 which is homeomorphic to a Cantor set (see for instance Lemma 5.1 in [57]). Using the Haar measure on the abstract Cantor set $\{0, 1\}^{\mathbb{N}}$, it is therefore possible to define a "Devil's staircase" with a singular set contained in E.

Let us finally mention a general question: Is there a minimal integration theory? It should be linear, all Lebesgue integrable functions should be integrable for this integral and the integral should satisfy a change of variables theorem and a general Fundamental Theorem of Integration? It is possible to give an abstract answer to this question (see [18, section 1.2] for a description in French). Bongiorno, Di Piazza and Preiss defined in [8] and [7] an integral which solves this problem and has a constructive definition: the C-integral. It has a formulation in terms of Riemann sums which is a compromise between the Henstock Kurzweil Integral and the McShane Integral, the latter being equivalent to the Lebesgue integral and the former being too general.

For now we turn to the question of integration in higher dimensional domains.

1.1.2 The Divergence Theorem and Pfeffer integration

Let us consider a higher dimensional space. If we are looking for an equivalent to the Fundamental Theorem of Calculus, the best candidate is probably the Divergence Theorem (also known as the Gauss-Green Theorem):

Suppose $A \subseteq \mathbb{R}^n$ is a friendly *set and $\mathbf{v} : A \to \mathbb{R}^n$ is a* gentle *vector field, then the flux of $\mathbf{v}$ across*

the boundary ∂A of A is equal to the integral of the divergence of $\mathbf{v}$ in A, i.e.

$$\int_{\partial A} \mathbf{v} \cdot \nu_A = \int_A \operatorname{div} \mathbf{v},$$

where ν_A is the outer normal to A on its boundary.
Letting $n = 1$, we notice that the above statement boils down to the Fundamental Theorem, provided A is a nonempty closed interval.

Now, we can ask ourselves the same questions as in the one dimensional case: how *gentle* does $\mathbf{v}$ have to be for the above to hold, and for which integral? However another question arises: what does a *friendly* set mean? It is clear that we should restrict ourselves to domains on which integration has a meaning (in particular they should be Lebesgue measurable domains), but they also need to have a well defined boundary, with an outer normal.

The theory of sets of finite perimeter is particularly well suited to this. Indeed, their modern definition, due in particular to De Giorgi relies on the Divergence Theorem:

Definition 1.1.1. A Lebesgue measurable set $A \subset \mathbb{R}^n$ has **finite perimeter** if

$$Per(A) := \sup \left\{ (L) \int_A \operatorname{div} \mathbf{v} \, \mathrm{d}\mathcal{L}^n, \mathbf{v} \in \mathcal{C}_c^1(\mathbb{R}^n, \mathbb{R}^n), |\mathbf{v}| \leqslant 1 \right\} < +\infty.$$

For such an A, there exists a $\mathcal{H}^{n-1}$ measurable set $\partial_* A$, the **reduced boundary** of A and a normal unit vector field ν_A defined $\mathcal{H}^{n-1} \llcorner \partial_* A$ almost everywhere and oriented "outwards" such that for all $\mathbf{v} \in \mathcal{C}_c^1(\mathbb{R}^n, \mathbb{R}^n)$,

$$(L) \int_A \operatorname{div} \mathbf{v} \, \mathrm{d}\mathcal{L}^n = (L) \int_{\partial_* A} \mathbf{v} \cdot \nu_A. \tag{1.2}$$

This result was generalized by De Giorgi and Federer to get

Theorem 1.1.2. *If A is a bounded set of finite perimeter in $\mathbb{R}^n$ and $\mathbf{v}$ is a Lipschitz vector field, then* (1.2) *holds.*

We note that fine properties of functions of bounded variation and sets of finite perimeter have been established by Vol'pert in [69]. For the Divergence Theorem, we will not try to study wilder classes of sets, however, there are results for sets with fractal boundaries, which require more regularity than C^1 on the vector field (see, e.g. [36, 49]).

Leaving the context of Lebesgue integration, a lot of work has been done to get a general Divergence Theorem using gauge integrals (see [43, 52]). Note that variants in infinite dimensional spaces have been studied by Henstock and Muldowney, see for instance [38, 55]. There are, as in the previous paragraph, two reasons for this: removing the integrability condition and weakening the conditions on the regularity of the vector field. Naturally, the first idea was to generalize the Henstock Kurzweil Integral to higher dimensions, in the same way as we usually generalize the Riemann Integral. There is no obstacle to this generalization as such: to integrate on a box (a multi dimensional interval) we can consider Riemann sums with smaller boxes, controlling the diameters of these boxes with a gauge. This yields a well defined Integral which satisfies Fubini's Theorem, i.e. if f is an HK integrable function on $[0,1]^2$, almost all sections f_x are HK integrable

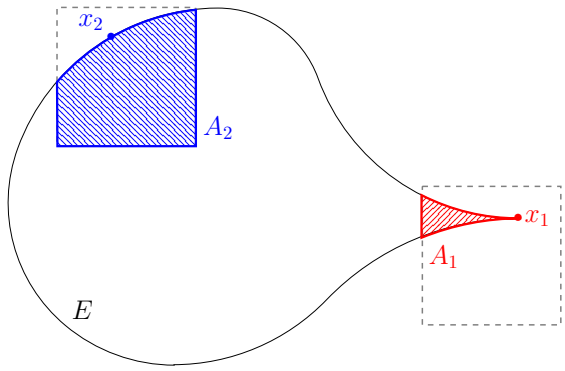

Figure 1.2: (x_1, B_1) is regular, but (x_2, B_2) is not.

on $[0,1]$ with:

$$(HK)\int_{[0,1]^2} f = (HK)\int_0^1 \left((HK)\int_0^1 f_x(y)\,\mathrm{d}y\right)\mathrm{d}x.$$

However, this integral is impractical for two reasons. First, it does not behave well with respect to changes of variables which do not transform all boxes into boxes, that is, changes of variables which are not compositions of homotheties and translations. Second, it is incompatible with a General Divergence Theorem. This happens to be a general phenomenon:

An integration method (I) has a **General Divergence Theorem** if given a continuous vector field $\mathbf{v}$ defined in a cube $[0,1]^d$ with $d \geqslant 1$, such that $\mathbf{v}$ is differentiable at all points of $(0,1)^d$,

$$(I)\int_{[0,1]^d} \operatorname{div}\mathbf{v} = \int_{\partial([0,1]^d)} \mathbf{v}\cdot\nu,$$

where ν denotes the outer unit normal vector to $[0,1]^d$.

W. Pfeffer proved that Fubini's Theorem for a multidimensional version of the Henstock Kurzweil Integral and a General Divergence Theorem are incompatible: If Fubini's Theorem holds for such an integral, there exists a vector field which is continuous and everywhere differentiable but whose divergence is not integrable (see [60, section 11.1] for the construction of the example).

To obtain a General Divergence Theorem, one needs a minima to impose a regularity condition on the elements of the partition involved in the Riemann sum: the sides of the box should have comparable length as this ensures that the divergence at the point of evaluation is a correct approximation for the flux of the vector field across the boundary of this element.

The main issue is then decomposing a domain into regular subsets. In one dimension, the Henstock Kurzweil Integral relies on Cousin's Lemma, which states that given any positive function on an interval, a partition of this interval subordinate to this gauge exists. A similar result is needed for decomposition of sets of finite perimeter. The example of a set of finite perimeter with an outward cusp (see figure 1.2), proves that it is not in general possible to partition the whole set, but that one should accept to leave small pieces out of the decomposition. This leads to the question of estimating the error made when doing so and, in turn to the study of charges, inspired by Mařik ([51]). Given a bounded set of finite perimeter A, a **charge** on A is a function on the space of subsets of finite perimeter of A, which is both additive under finite disjoint unions and continuous with respect to the convergence for the topology of functions of bounded variation: a

sequence of sets converges in this sense if it converges in L^1, has uniformly bounded perimeter and if all sets are contained in a compact subset. For instance, if $\mathbf{v}$ is a continuous vector field defined in $\mathbb{R}^n$, the function

$$F_v : B \subseteq A \mapsto (L) \int_{\partial_* B} \mathbf{v} \cdot \nu_B,$$

is a charge on A, called the **flux** of $\mathbf{v}$.

This theory enabled Howard and Pfeffer to prove a decomposition result now called the Howard-Cousin Lemma (of which lemma 4.1.8 is a variant). Using this key result, Pfeffer then defined a nonabsolutely convergent integral (which we will call the Pfeffer Integral, or $\mathfrak{P}$ Integral) on any bounded set of finite perimeter in [59]. He was able to state a general Gauss Green Theorem:

Theorem (Gauss-Green Theorem for the Pfeffer Integral)**.** *If A is a bounded set of finite perimeter in $\mathbb{R}^m$ and $\mathbf{v}$ is a continuous vector field in $\mathrm{cl}(\mathrm{cl}_e A)$ (the closure of the essential closure of A), which is differentiable at each point of $\mathrm{cl}_e A \backslash E$, where E has σ-finite $\mathcal{H}^{m-1}$ measure, then $\operatorname{div} \mathbf{v}$ is Pfeffer integrable on A and*

$$(\mathfrak{P}) \int_A \operatorname{div} \mathbf{v} = \int_{\partial_* A} \mathbf{v} \cdot \nu_A.$$

Here $\mathrm{cl}_e A$ stands for the set of density points of A. The condition of having σ-finite codimension 1 Hausdorff measure will come up very often in the following and we will now say that such a set is **thin**.

Note that there are variants of this theorem with different conditions, for instance in [20], the authors replace the condition that E be thin, by a similar integral-geometric condition and the additional assumption that $\mathrm{D}\mathbf{v} \in L^1(A)$). It is also possible to relax the continuity condition on the vector field $\mathbf{v}$ to continuity $\mathcal{H}^{m-1}$ almost everywhere, under the assumption that $\mathbf{v}$ is bounded. Finally, one can relax the differentiability condition, asking instead that the field be pointwise Lipschitz except on a thin set. The Stepanoff Theorem then ensures that the $\mathbf{v}$ is differentiable almost everywhere and a Divergence Theorem holds.

This last result is invariant by diffeomorphisms, or bilipschitz homeomorphisms (Lipeomorphisms), thus it seems reasonable that such a theorem is valid on C^1 or Lipschitz submanifolds with boundary. It is in fact more natural to consider the "dual" result to the Divergence Theorem: Stokes' Theorem ([58]). The aim of this thesis is to study the validity of Stokes' Theorem on more singular objects, which we represent by integral currents. The next paragraph is a review of both Stokes' Theorem and the theory of integral currents.

1.1.3 Stokes' Theorem and integral currents

Right after learning of the Divergence Theorem in an early mathematics or physics class, one usually hears Stokes' Theorem stated as follows:

Theorem 1.1.3. *Let $\mathbf{v}$ be a smooth vector field in $\mathbb{R}^3$ and S be a smooth surface (of dimension 2) with smooth boundary ∂S. Then the circulation of $\mathbf{v}$ along ∂S is related to its curl, $\mathbf{curl}\ \mathbf{v}$, by the formula*

$$\int_{\partial S} \mathbf{v} \cdot \mathrm{d}\overrightarrow{l} = \int_S \mathbf{curl}\ \mathbf{v} \cdot \mathrm{d}\overrightarrow{S}.$$

This result is used in particular in the study of classical electromagnetism and fluid dynamics. However, the fact that $\mathbf{v}$, $\mathbf{curl}\ \mathbf{v}$, $\mathrm{d}l$ and $\mathrm{d}S$ are all vectors is misleading and may hide a fact that

becomes clear after a first class on differential geometry: the Fundamental Theorem of Calculus, the Gauss Green Theorem and Stokes' Theorem can be stated as a more general principle, known as the Generalized Stokes' Theorem, or as the Stokes-Cartan Theorem:

Theorem 1.1.4. *Let M be a smooth bounded submanifold of dimension m in $\mathbb{R}^n$ with boundary, and let ω be a smooth differential form of degree $m-1$ in a neighbourhood of M, then*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

(See for instance Theorem 4.9 in [70]) To see that the previous formulations are consequences of this theorem, it suffices to notice that there is a correspondence between forms of degree $n-1$ in $\mathbb{R}^n$ and vector fields and between forms of degree n and functions. By this correspondance and the change of variables formula proved by Pfeffer in [58, 61], a general Stokes' Theorem for Pfeffer integration applies to smooth closed oriented submanifolds. This relies on the triangulation theorems for C^1 manifolds. See for instance Theorem 12A [1] in Whitney's book [73].

Stokes' theorem for smooth differential forms holds on much more singular sets than smooth manifolds, for instance on singular chains (see Theorem 4.7 in [70]). For more general, non parametric domains, like currents, Stokes' Theorem holds for C^∞ forms with compact support by definition of the boundary of a current (see [28, 2.3] or [26, 4.1.7]). If the currents are normal or integral, this extends to C^1 forms. For integral currents, this extends also to Lipschitz forms by approximation in C^1, using the theory of flat cochains.

Before the study of currents by Federer and Fleming, H. Whitney defined flat chains and cochains in [73]. Flat cochains are identified with flat forms by a theorem of Wolfe. Flat forms are essentially bounded forms whose weak differential is also essentially bounded. In particular a bounded Lipschitz function is a flat form of degree 0. This allowed Federer to state a general Stokes' Theorem for flat forms on flat chains (see 4.6 in [27]). For even more singular objects studied from an integration point of view, we refer to the work of Züst [77, 78].

Finally we mention that the notion of charge, initially defined for sets of finite perimeter, has a generalization to higher codimensional objects that makes the space of m-charges in $\mathbb{R}^n$ the dual of the space of normal currents of dimension m for a topology which is suited to our integration purpose, see [19]. This has a generalization to normal chains and cochains in metric spaces in [17].

1.1.4 The "nice singularities": O-minimal geometry

Currents form a very general family of objects, which is well suited to the Calculus of Variations. However, there are families of interesting singular domains which are much more civilized. In particular the family of algebraic varieties which is of interest for other domains of mathematics. More generally, a general Stokes' Theorem on semialgebraic and subanalytic sets might be of interest. These categories have been studied in the common framework of *o-minimal geometry*. This field, which originates in Model Theory can be approached from more geometric or analytic viewpoints, see [21] for a detailed and accessible exposition of the theory..

[1] This theorem is generally stated for C^1 manifolds without boundary with a remark that it can be easily extended to the case of manifolds with boundary. This is part of folklore in differential geometry and implies that a Pfeffer-like Theorem holds on C^1 manifolds with boundary in the generality of Pfeffer's Divergence Theorem. We will not use triangulations, but prove that this theorem holds in section 4.3

O-minimal geometry is also called *tame geometry* after the *topologie modérée* sketched by Grothendieck in [32]. This makes it into a topological and geometric paradise, but not easy to study with analytic tools. In particular, integrating and taking limits of sequences are not well-behaved operations with respect to o-minimality. However, most of the pathologies we encounter in Geometric Measure Theory are absent from this "paradise", while one can still work with large range of singularities. This is a motivation to study o-minimal objects from a GMT point of view. Without sequences, one has to use definable families, which makes it possible to define limits in Hausdorff distance ([44]) and get upper bounds in Hausdorff measure ([48]).

Federer and Hardt have studied analytic varieties as currents in [26] and [35] respectively. Analytic and semi-algebraic chains and sets have been used in homotopy and homology theories in [33] and [30]. Objects definable in o-minimal structures have good topological and geometric properties and this makes the theory of definable chains in an o-minimal structure interesting.

1.2 Summary of the thesis

This work consists in three main parts. Chapter 2 is devoted to the definition of a non-absolutely converging integral on integral currents of dimension 1 in $\mathbb{R}^n$ for $n \geqslant 1$. The case of the dimension 1 is specific, as every current in $\mathbb{I}_1(\mathbb{R}^n)$ can be decomposed in pieces whose size is controlled by a gauge, a feature we call the Howard-Cousin Property. This is due to decomposability of integral currents into countable sums of curves. This implies that we can reduce most of the problem to integration along a curve of finite length.

This chapter starts by a review of Henstock-Kurzweil integration on an interval (section 2.1). We then give a more robust, equivalent definition of this integral which inspired from both Henstock-Kurzweil Integration and Pfeffer Integration (proposition 2.1.14). And then turn to integration on an integral current T of dimension 1. If T corresponds to a simple Lipschitz curve, then there is a natural Henstock-Kurzweil Integral on T by a simple change of variable.

After defining the space of **pieces** of an integral current $T \in \mathbb{I}_1(\mathbb{R}^n)$ in section 2.2 and studying continuous additive functions on this space, we define a Pfeffer type integration method, which we call the HKP integral along T (definition 2.3.4). We use the decomposability property of 1 dimensional normal current to view T as a countable sum of simple Lipschitz curves. Since all but a finite number of those curves are cycles, we can remove uncountably many cycles whose contribution we controlled as in Pfeffer Integration, and study only a finite sum of simple Lipschitz curves on which we simply use the Henstock Kurzweil integral as above.

In proposition 2.3.12, we prove that any Lebesgue integrable function is HKP integrable along T, and that the integrals take the same value. We thus prove a general **Fundamental Theorem of Integration** for currents of dimension 1:

Theorem (2.3.19)**.** *Given an integral current $T \in \mathbb{I}_1(\mathbb{R}^n)$, suppose that the function $u : \mathbb{R}^n \to \mathbb{R}$ is*

1. *continuous on* $\operatorname{spt} T$*,*
2. *pointwise Lipschitz on* $\operatorname{set}_1 \|T\|$ *at all but countably many points of* $\operatorname{set}_1 \|T\|$*,*
3. *differentiable along T at $\|T\|$ almost all points of* $\operatorname{set}_1 \|T\|$*.*

Then the function $x \mapsto \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle$ *is* HKP *integrable along* T *and there holds*

$$(HKP)\int_T \langle \mathrm{D}u, \overrightarrow{T}\rangle = \partial T(u).$$

This chapter also contains a Monotone Convergence Theorem for the HKP Integral: theorem 2.3.16, which we prove using standard techniques of gauge integration. We conclude by a few questions concerning differentiation and derivation along an integral current of dimension one.

The second part consists of chapters 3, 4 and 5. We define tools for a gauge integration theory and investigate the validity of a general Stokes' Theorem on an integral current T of dimension m in $\mathbb{R}^n$. In chapter 3, we define the space $\mathcal{S}(T)$ of **subcurrents** of an integral current T, these subcurrents will be the elements of the Riemann sums. We compare the general case to the codimension 0 case of bounded sets of finite perimeter and their subsets. This allows us to point out the important properties of sets of finite perimeter which play a role in the Pfeffer integral and are not shared by general integral currents. In particular we prove that subcurrents do not always have good intersection or approximation properties. However we prove a compactness theorem for subcurrents in the topology of flat convergence with controled boundary mass (theorem 3.1.6).

We then define the continuous additive functions on this space, which are the equivalent to charges in the sense of [61] and contain the restrictions of the m-charges of Moonens [19, 17] to the space of subcurrent. Then we define and study the derivation of these functions and give a criterion for derivability of functions on $\mathcal{S}(T)$ associated to continuous differential forms. Denoting by $\operatorname{set}_k \mu$ the set of points of positive k-dimensional upper density of a measure μ, we have:

Lemma (3.2.15)**.** *Let* T *be an integral current of dimension* m *in* $\mathbb{R}^n$ *and* ω *a continuous differential form of degree* $m-1$ *on* $\operatorname{spt} T$*. Suppose that the tangent vector to* T *has a continuous representative at the point* $x \in \operatorname{set}_m \|T\| \setminus \operatorname{set}_{m-1} \|\partial T\|$*, and* ω *is differentiable at* x*, then the function* Θ_ω *on* $\mathcal{S}(T)$ *is derivable at* x *along* T *and we have*

$$\mathfrak{D}_T\, \Theta_\omega(x) = \langle \mathrm{d}\omega(x), \overrightarrow{T}(x)\rangle.$$

A remark on the terminology: We will use the terms **derivate** and **derivable** following Federer [26, section 2.9] and Pfeffer [61] instead of the more general *differential* and *differentiable*. This is perhaps not very common, but we believe it is important to distinguish the differential of a function, or form in Euclidean space, from the derivate of a function on a space of sets or on currents. In this sense, the **density** of a measure at a point is a derivate.

In chapter 4, we give a proof of the Howard-Cousin Lemma for sets of finite perimeter (lemma 4.1.8), in a slightly modified version including a **regularity gauge**, η, instead of a regularity constant. We then define the Howard-Cousin Property (see definition 4.2.1) for integral currents. After giving a notion of **Lipschitz-BV charts** of a current $T \in \mathbb{I}_m(\mathbb{R}^n)$ over a bounded set of finite perimeter in $\mathbb{R}^m$, we obtain a simple criterion for a current to have the Howard-Cousin Property (theorem 4.2.10). This criterion is based on the set of points of $\operatorname{spt} T$ where T does not have a Lipschitz-BV chart: the **nonflat** set of T. This set is called **disposable** if it can be removed without adding too much boundary, which turns out to be a sufficient condition for T to have the Howard-Cousin Property. This implies directly that C^1 submanifolds with boundary have the Howard-Cousin Property and allows us to prove that integral currents definable in an o-

minimal structure - which are the object of chapter 6 - have the hereditary Howard-Cousin Property (proposition 4.2.12). We also define the **hereditary Howard-Cousin Property**, which is strictly stronger (as we will show in section 4.4.3). The word **hereditary** is used to describe a property which is shared between a current and its subcurrents.

In section 4.2.4, we study a stronger condition on the nonflat set of a current in terms of its $m-1$ **dimensional Minkowski content relative to the measure** $\|T\|$ this condition implies hereditary disposability. Then in section 4.2.5, after recalling a few regularity results, we prove that integral currents associated to stationnary varifolds without singular boundary points or double points have the hereditary Howard-Cousin Property. In particular, this is true of mass minimizing currents of codimension 1 with $C^{1,\alpha}$ boundary for $\alpha > 0$.

In the remaining of this chapter, we study the validity of a generalized Stokes' Theorem on integral currents. We prove the following

Theorem (4.3.1)**.** *Given a current $T \in \mathbb{I}_m(\mathbb{R}^n)$ and a differential form ω of degree $m-1$, if*

(i) T has C^1-BV charts at all points of $\operatorname{spt} T$ *except on a disposable set E_T,*

(ii) ω is continuous on $\operatorname{spt} T$

(iii) ω is differentiable at each point of $\operatorname{set}_m \|T\| \backslash E_\omega$*, where E_ω has σ-finite $\mathcal{H}^{m-1}$ measure.*

(iv) $x \mapsto \langle \mathrm{d}\omega(x), \overrightarrow{T}(x)\rangle$ is Lebesgue integrable with respect to $\|T\|$,

then there holds

$$\int \langle \mathrm{d}\omega(x), \overrightarrow{T}(x)\rangle \,\mathrm{d}\|T\|(x) = \int \langle \omega(y), \overrightarrow{\partial T}(y)\rangle \,\mathrm{d}\|\partial T\|(y).$$

The condition that the singular set of T be disposable is necessary, indeed in section 4.4, we give an example of an integral current of dimension 2 in $\mathbb{R}^3$ that is associated to a submanifold (though not a submanifold with boundary) and whose singular set is a single point: $\{0\} \in \mathbb{R}^3$ and a continuous differential form ω of degree 1 which is C^2 at all points $x \in \operatorname{spt} T \backslash \{0\}$ with $\langle \mathrm{D}\omega(x), \overrightarrow{T}(x)\rangle = 0$ but

$$\int \langle \omega(x), \overrightarrow{\partial T}(x)\rangle \,\mathrm{d}\|\partial T\|(x) = 1.$$

(We give a first counter example with a singular set which is a segment and then modify it to get a current with only one singular point.)

In chapter 5 we define $\mathfrak{P}$ integration on a current $T \in \mathbb{I}_m(\mathbb{R}^n)$ which has the Howard-Cousin Property. We prove that any function which is Lebesgue integrable with respect to $\|T\|$ is $\mathfrak{P}$ integrable on T. In the more interesting case where the current has the hereditary Howard-Cousin Property, we prove that the primitive for the Pfeffer integral on T is well defined, additive and continuous and that a Saks-Henstock Lemma holds (proposition 5.1.8). We can then state and prove a general Stokes' Theorem for the $\mathfrak{P}$ integral:

Theorem (5.2.1)**.** *Given a current $T \in \mathbb{I}_m(\mathbb{R}^n)$ and a differential form ω of degree $m-1$, if*

(i) T has C^1-BV charts at all points of $\operatorname{spt} T$ *except on a* **hereditarily disposable** *set E_T (i.e. E_T is disposable for all subcurrents of T).*

(ii) ω is continuous on $\operatorname{spt} T$

(iii) ω is differentiable at each point of $\mathrm{set}_m \|T\| \backslash E_\omega$*, where E_ω has σ-finite $\mathcal{H}^{m-1}$ measure.*

then $x \mapsto \langle \mathrm{d}\omega(x), \overrightarrow{T}(x)\rangle$ is Pfeffer integrable on T and there holds

$$(\mathfrak{P}) \int \langle \mathrm{d}\omega(x), \overrightarrow{T}(x)\rangle \, \mathrm{d}\|T\|(x) = \int \langle \omega(y), \overrightarrow{\partial T}(y)\rangle \, \mathrm{d}\|\partial T\|(y).$$

This theorem differs from the one in the Lebesgue case by the absence of the Lebesgue integrability condition and the requirement that T has an hereditarily disposable singular set.

Finally, chapter 6 deals with the o-minimal part of the thesis. We define o-minimal structures and give a few of their elementary properties. In particular we give a simpler proof of the uniform control of the Hausdorff measure of members of a bounded definable family (theorem 6.2.9). We go on to define **definable chains**, which are a classes of tame integral currents compatible with operations in the definable category. The properties of these currents imply that their singular sets are hereditarily disposable, which is the property we used in proposition 4.2.12.

1.3 Applications and perspectives

In chapter 2, we would like to obtain a Rademacher/Stepanoff Theorem for "intrinsic differentiation" along an integral current. Similarly it would be useful to have a derivation theorem using indecomposable pieces of currents instead of sets in the Vitali cover. The differentiation theory obtained by G. Alberti and A. Marchese in [1] may be useful.

Concerning chapter 6, we would like to prove that the homology of ominimal chains coincides with the homology of integral currents, on a definable set. A dual question would be to connect this homology with the cohomology of definable forms as defined for instance by L. Shartser and G. Valette in [65] (in the polynomially bounded case).

The main use of gauge integration on currents is the generalization of Stokes' Theorem, even in the case of Lebesgue integration. The Theorem we obtain is well suited to the study of removable singularities, as it has a pointwise formulation. Following [20], one could consider PDE's in *Stokes' form*. Stokes' Theorem is also used in the study of hydrodynamics and electromagnetism, it is possible that in some cases one needs to study the circulation of a field around non-smooth surfaces.

Another important application of Stokes' Theorem is the notion of **calibrations** to characterize minimal surfaces. The results in this thesis, in particular the part on area minimizing currents, could be useful to find non-smooth calibrations. With that in mind, we would like to know how one can work with bounded discontinuous forms for Stokes' Theorem. It is possible that having 0-Minkowski content relative to the supporting current yields a good criterion for the removability of the discontinuity set. In the flat setting, the criterion of is that the discontinuity set has zero $m-1$ dimensional Hausdorff measure (W. Pfeffer calls such sets **slight**).

The nature of singularities which are not removable is also a question. Minkowski content and dimension of the singular set seem to have a strong influence on the analytic properties of the space, for instance in *almost-Riemannian Geometry* we mention a recent unpublished work by Chitour, Prandi and Rizzi.

This thesis will provide the material for two papers : one concerning integration on one dimensional currents [41] and one for general integral currents [42], the latter focusing on Stokes'

Theorem. I have also been working with Katarina Bellova and Felix Otto on an unrelated problem concerning the distribution of electric charges on a hyperplane with both nonlocal and local interactions. [5]

List of Notations

This list describes the main notations used in the text, along with references to the pages where they are defined.

Vectors, covectors and forms

$\lvert x\rvert$	euclidian norm of x,	p. 141
$x \cdot y$	euclidian scalar product,	p. 141
$\Lambda^m(\mathbb{R}^n)$	Space of m-covectors in $\mathbb{R}^n$,	p. 143
$\Lambda_m(\mathbb{R}^n)$	Space of m-vectors in $\mathbb{R}^n$,	p. 143
$\lVert \cdot \rVert$	(co)mass of an m-(co)vector,	p. 144

Sets, topology, functions

$\mathrm{U}(E, r)$	open r-neighbourhood of the set E,	p. 95
$\mathrm{U}(x, r)$, $\mathrm{B}(x, r)$	Euclidean open/closed ball,	p. 141
$\operatorname{int} A, \operatorname{cl} A, \operatorname{bdry} A$	Interior, closure, boundary of $A \subseteq \mathbb{R}^k$,	p. 141
$\subseteq$	inclusion of sets,	p. 141
$A \Delta B$	Symmetric difference of A and B,	p. 141
$f\lvert_E$	Restriction of f to E,	p. 141
$\operatorname{Lip}_x f$	Pointwise Lipschitz constant of F at x,	p. 31

Measures

$\phi_* \mu$	Image measure of μ by ϕ,	p. 59
$\mathcal{M}^m$, $\mathcal{M}^{*m}$, $\mathcal{M}^m_*$	m-dimensional Minkowski content,	p. 96
$\mathcal{H}^s$	s-dimensional Hausdorff measure,	p. 141
$\mathcal{L}$	Lebesgue measure in Euclidian space,	p. 141
$\lvert A\rvert$	Lebesgue measure of $A \subseteq \mathbb{R}^n$,	p. 141
$\mathbf{\Theta}^m(\mu, x)$, $\mathbf{\Theta}^{m*}$, $\mathbf{\Theta}^m_*$	m density of μ at x (upper-, lower-),	p. 143

$\mathrm{set}_m\,\mu$	density set of μ,	p. 143
$\mu \llcorner A$	Restriction of μ to the set A,	p. 144

Families and integration

$(HK)\int_a^b f$	Henstock-Kurzweil integral of f on $[a,b]$,	p. 25
$[\mathcal{P}]$	body of the family $\mathcal{P}$,	p. 25
$\delta(x)$	gauge,	p. 25
$\sigma(f,\mathcal{P})$	Riemann sum of f on the tagged family $\mathcal{P}$,	p. 25
$(HKP)\int_T f$	HKP integral of f on the current T,	p. 47
$\sigma(f,\mathcal{P})$	Riemann sum of f over the tagged family $\mathcal{P}$,	p. 47
$[\mathcal{P}]$	Body of the family $\mathcal{P}$,	p. 79
$\delta(x)$	gauge,	p. 81
I(f,T)	Integral of f on the current T,	p. 119
$(\mathfrak{P})\int f$	Pfeffer integral of f on the current T,	p. 121

Integral Currents

$\mathbb{M}$	Mass of a current,	p. 144
$\mathbb{F}(T)$	Flat norm of T,	p. 145
$\mathfrak{F}_{X,m}$	Moonens topology on $\mathbb{N}_m(X)$,	p. 145
$\mathbb{N}(T)$	Normal mass of a T,	p. 145
$\mathbb{N}_m(\mathbb{R}^n)$	Normal currents of dimension m in $\mathbb{R}^n$,	p. 145
∂T	Boundary of the current T,	p. 145
$\overrightarrow{T}$	m-vector field tangent to T,	p. 145
$\|T\|$	Carrying measure of T,	p. 145
$\mathcal{R}_m(\mathbb{R}^n)$	Rectifiable currents,	p. 146
$\mathbb{I}_m(\mathbb{R}^n)$	Integral currents of dimension m in $\mathbb{R}^n$,	p. 146
$\phi_\# T$	Pushforward of T by ϕ,	p. 146
$\langle T, f, y\rangle$	Slice of T by f at y,	p. 148
$\mathbf{E}^m \llcorner A$	Integral current associated to the BV set $A \subseteq \mathbb{R}^m$,	p. 150

Sets of finite perimeter

$\mathcal{BV}(A)$	Bounded sets of finite perimeter contained in A,	p. 149

$\mathrm{P}(A)$	Perimeter of A,	p. 149
$\mathrm{cl}_e\, A$	Essential closure of A,	p. 150
$\partial_e A$	Essential boundary of A,	p. 150

Pieces, subcurrents and functions

$\mathcal{S}_{\leqslant}(T)$	space of pieces of T,	p. 36
$\mathfrak{D}_T\, F(x)$	Derivate of F along T at x,	p. 39
$\mathcal{S}_{\leqslant}(T, x, \delta)$	δ fine pieces of T at x,	p. 39
$S \sqsubset T$	S is a subcurrent of T,	p. 58
$\mathbf{CH}^m(X)$	m-charges in the set X,	p. 70
$\phi^{\#} F$	Pull-back of F by ϕ,	p. 73
$\mathcal{S}(T, x, \eta, \delta)$	η-regular subcurrents of T touching x with diameter $< \delta$,	p. 74
$\mathfrak{D}_T\, F(x)$	Derivate of F along T at x,	p. 76

Chapter 2

One dimensional integration

2.1 The integral of Kurzweil and Henstock

2.1.1 Definition and classical properties

A nonnegative function defined on a set $E \subseteq \mathbb{R}$ is called a **gauge** if its zero set is **thin** , that is has σ-finite Hausdorff measure of dimension 0. (This boils down to being countable.) We will distinguish between gauges and positive gauges. In the classical definition of Henstock-Kurzweil integral gauges are always positive, but for our purpose it makes sense to allow the gauge to take the value zero in a thin set. The definition of a gauge changes in higher dimension, but we fix it for this chapter. A **tagged family** in $[a, b]$ is a finite collection of pairs $([a_j, b_j], x_j)_{j=1,2,\ldots,p}$ where one has $a \leqslant a_1 < b_1 \leqslant a_2 < \cdots \leqslant a_p < b_p \leqslant b$ and for all j, $x_j \in [a_{j-1}, a_j]$ The **body** of a family $\mathcal{P}$ is the union denoted by $[\mathcal{P}]$ of all the intervals in $\mathcal{P}$. A **tagged partition** in $[a, b]$ is a tagged family whose body is $[a, b]$. If δ is a gauge on $[a, b]$, we say that a tagged family (or a tagged partition) is δ-fine, when for all j, $b_j - a_j < \delta(x_j)$.

Definition 2.1.1. A function f defined almost everywhere in an interval $[a, b]$ is **Henstock-Kurzweil integrable on** $[a, b]$ if there exists a real number denoted by $(HK)\int_a^b f$ such that for all $\varepsilon > 0$, there exists a positive gauge δ on $[a, b]$ such that for each δ-fine tagged partition $\{([a = a_0, a_1], x_1), \ldots ([a_{p-1}, a_p = b], x_p)\}$, there holds:

$$\left| (HK)\int_a^b f - \sum_{j=1}^{p} f(x_j)(a_j - a_{j-1}) \right| < \varepsilon.$$

In the following, we will write $\sigma(f, \mathcal{P})$ for the sum on the left hand side, whenever $\mathcal{P}$ is a tagged family. This definition is well posed as a consequence of the following result.

Lemma 2.1.2 (Cousin)**.** *If I is a closed bounded interval and δ is a positive gauge on I, then a δ-fine tagged partition of I exists.*

Proof. Suppose no δ-fine tagged partition of I exists. Consider the two halves of I: I_1 and I_2. Either I_1 or I_2 does not admit a δ-fine tagged partition. By successive divisions, we can find a decreasing sequence of closed intervals of the form $I^p = I_{j_1,j_2,\ldots,j_p}$ where $j_k \in \{1, 2\}$ and $I_{j_1,\ldots,j_p,1}$

and $I_{j_1,\dots,j_p,2}$ are the two halves of $I_{j_1,\dots,j_p}$. We can choose the intervals I^p for $p = 1, 2, \dots$ so that none of them admits a δ-fine tagged partition. There exists $x \in I \cap \bigcap_{p=1}^{\infty} I^p$. Since δ is positive on I, $\delta(x) > 0$ and as $\operatorname{diam}(I^p) = 2^{-p} \operatorname{diam} I$, there exists p such that $\operatorname{diam} I^p < \delta(x)$. This implies that $((I^p, x))$ is a δ-fine tagged partition of I^p, a contradiction. □

The following function is the classical motivation for the introduction of the Henstock Kurzweil Integral.

Example 2.1.3. Consider the function F defined on $[-1, 1]$ by

$$F : x \mapsto \begin{cases} x^2 \sin x^{-2} \text{ if } x \neq 0, \\ 0 \text{ if } x = 0. \end{cases}$$

F continuous on $[-1, 1]$ and differentiable everywhere in $(-1, 1)$. However F' is not Lebesgue integrable. To see this, for $k = 1, 2, \dots$, consider the intervals

$$J_k := \left[\frac{1}{\sqrt{\frac{\pi}{2} + 2k\pi}}, \frac{1}{\sqrt{2k\pi}} \right].$$

On each of these intervals, F' is nonnegative, it is also bounded and thus Lebesgue integrable with

$$(\mathcal{L}) \int_{J_k} F' = F\left(\frac{1}{\sqrt{2k\pi}}\right) - F\left(\frac{1}{\sqrt{\frac{\pi}{2} + 2k\pi}}\right) = -\frac{1}{2k\pi + \frac{\pi}{2}}.$$

If F' (and thus $|F'|$) were Lebesgue integrable on $[0, 1]$, we would have

$$(\mathcal{L}) \int_0^1 |F'| \geqslant \sum_{k=1}^{\infty} (\mathcal{L}) \int_{J_k} |F'| = \sum_{k=1}^{\infty} \frac{1}{2k\pi + \frac{\pi}{2}} \geqslant \sum_{k=1}^{\infty} \frac{1}{2(k+1)\pi}.$$

As the last series diverges, there is a contradiction and F' is not Lebesgue integrable.

However, F' is Henstock Kurzweil integrable on every subinterval of $[-1, 1]$ and its HK integral on an interval $[a, b]$ is equal to $F(b) - F(a)$. This is a consequence of the fundamental Theorem 2.1.7

We list here the main direct consequences of the definition.

Proposition 2.1.4 (Cauchy Criterion for integrability)**.** *f is HK integrable on the interval $[a, b]$, if and only if for each $\varepsilon > 0$ there exists a positive gauge δ on $[a, b]$ such that whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are δ-fine tagged partitions of $[a, b]$, there holds*

$$|\sigma(f, \mathcal{P}_1) - \sigma(f, \mathcal{P}_2)| < \varepsilon.$$

Proposition 2.1.5. *Let f be a Henstock Kurzweil integrable function on the interval $[a, b]$:*

(1) If g is HK integrable on $[a, b]$ and λ is a real number, then $f + \lambda g$ is HK integrable and $(HK) \int_a^b (f + \lambda g) = (HK) \int_a^b f + \lambda \left((HK) \int_a^b g \right)$.

(2) If a function g is equal to f almost everywhere on $[a, b]$, then g is also HK integrable and has the same integral.

(3) If g is Lebesgue integrable, it is also HK integrable and the two integrals coincide.

(4) The restriction of f to a subinterval $[c,d] \subseteq [a,b]$ is Henstock Kurzweil integrable on $[c,d]$.

(5) The function $F : [a,b] \to \mathbb{R}; x \mapsto (HK)\int_a^x f$ is continuous it is called the **indefinite HK integral of** *F. Also, if f is nonnegative, F is nondecreasing.*

(6) (Saks-Henstock Lemma) For $\varepsilon > 0$ and δ a positive gauge corresponding to ε in the definition of integrability of f, given any tagged family $(([a_j,b_j],x_j))_{j=1}^p$ in $[a,b]$ there holds

$$\sum_{j=1}^{p} \left| f(x_j)(b_j - a_j) - (HK)\int_{a_j}^{b_j} f \right| < 2\varepsilon.$$

(7) The function F above is differentiable almost everywhere with derivative equal to f.

(8) f is Lebesgue measurable.

(9) f is Lebesgue integrable if and only if f and $|f|$ are Henstock Kurzweil integrable.

The proofs of these results can be found in any treaty on Henstock Kurzweil Integration (see chapter 9 in [31], the concise appendix H to [11], or the book [53] - in French). In section 2.3 and chapter 5 we prove results comparable to proposition 2.1.4 and proposition 2.1.5 (1) to (6) for the HKP integral on integral currents of dimension 1 Pfeffer integral on integral currents. The methods are very similar. We only outline the proofs which have not direct equivalent in Pfeffer integration on currents.

Proof of proposition 2.1.5 (7), (8) and (9). To prove (7), treat separately the upper and lower, left and right derivates of F and prove that they are equal almost everywhere to f. Consider the extended real values function:

$$\mathrm{D}^+ F(x) := \limsup_{t \to x^+} \frac{F(t) - F(x)}{t - x}$$

For $\alpha > 0$, let E_α be the subset of $[a,b)$ consisting of the points at which $\mathrm{D}^+ F > f(x) + \alpha$. Suppose E_α has positive Lebesgue measure λ. For each $x \in E_\alpha$, consider the nondegenerate intervals $[x,t]$ corresponding to $(F(t) - F(x))/(t - x) \geqslant f(x) + \alpha$. The collection $\mathcal{C}_\alpha$ of those intervals forms a Vitali cover of E_α (see section 6.2 of [11]). More importantly, if one defines a positive gauge δ corresponding to the integrability of f for $\varepsilon = \alpha\lambda/8$, the collection C_α^δ of those intervals $[x,t]$ in $\mathcal{C}_\alpha$ such that $t - x < \delta(x)$ is still a Vitali cover of E_α. By the Vitali Covering theorem (6.2.1 in [11]), there exists a finite disjoint subcollection $\mathcal{C}'$ of $\mathcal{C}_\alpha^\delta$ which covers a Borel subset of E_α of Lebesgue measure at least $\lambda/2$.

Notice that the tagged family $\{([x,t],x), [x,t] \in \mathcal{C}'\}$ is a δ fine tagged family in $[a,b]$. By the Saks Henstock Lemma,

$$\sum_{[x,t] \in \mathcal{C}'} |F(t) - F(x) - f(x)(t - x)| < 2\varepsilon.$$

However, by definition of $\mathcal{C}'$ and $\mathcal{C}_\alpha$ for all $[x,t] \in \mathcal{C}'$, $F(t) - F(x) - f(x)(t - x) \geqslant \alpha(t - x)$, hence

$$\sum_{[x,t] \in \mathcal{C}'} |F(t) - F(x) - f(x)(t - x)| \geqslant \alpha \sum_{[x,t] \in \mathcal{C}'} (t - x) \geqslant \alpha\lambda/2.$$

And we get $\alpha\lambda/4 = 2\varepsilon > \alpha\lambda/2$, a contradiction. Thus E_α has zero Lebesgue measure. Since this is true for all $\alpha > 0$, $\mathrm{D}^+ F(x) = f(x)$ for Lebesgue almost all x in $[a, b)$. Proceed in the same way for $\mathrm{D}_+ F$, $\mathrm{D}^- F$ and $\mathrm{D}_- F$ to prove that F is differentiable almost everywhere in $[a, b]$.

To prove (8) notice that for Lebesgue almost all x,

$$f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}.$$

Thus f is equal almost everywhere to the pointwise limit of a sequence of continuous functions. Such a limit is Borel measurable and f is thus Lebesgue measurable on $[a, b]$. Finally to prove (9), suppose that f and $|f|$ are HK integrable on $[a, b]$. It is sufficient to prove that $|f|$ is Lebesgue integrable, so we can suppose that f is nonnegative. Let F be the indefinite HK integral of f on $[a, b]$. By item (5), F is non decreasing and continuous, thus it is a function of bounded variation on $[a, b]$ and its derivative F' is Lebesgue integrable. As F' is equal to f almost everywhere, f is also Lebesgue integrable. □

Finally, we state three important convergence properties in the space of Henstock Kurzweil integrable functions:

Proposition 2.1.6. *Let $(f_n)_n$ be a sequence of HK integrable functions on the interval $[a, b]$. Suppose that $f_n \to f$ pointwise almost everywhere if either of the following conditions holds, then f is HK integrable and $(HK) \int f = \lim_n (HK) \int f_n$:*

(i) For almost all x, for all n, $f_n(x) \leqslant f_{n+1}(x)$ and $\sup_n (HK) \int f_n < +\infty$. (Monotone convergence Theorem)

(ii) There exist HK integrable functions g and h such that for all n, $g \leqslant f_n \leqslant h$ almost everywhere. (Dominated convergence Theorem)

(iii) $(f_n(x))_n$ is bounded for almost all $x \in [a, b]$ and for all $\varepsilon > 0$ there exists a positive gauge on $[a, b]$ such that for all n, for all δ-fine tagged partition $\mathcal{P}$ of $[a, b]$:

$$\left| \sigma(\mathcal{P}, f_n) - (HK) \int_a^b f_n \right| < \varepsilon.$$

(In the latter case, known as the Controlled Convergence Theorem, the sequence $(f_n)_n$ is called HK equiintegrable.)

Proof. The two first results can be proved using only the Saks Henstock Lemma and "purely Henstock Kurzweil techniques", we will give such a proof for the Monotone Convergence Theorem of HKP integration (see theorem 2.3.16). However, when possible, it is quicker to rely on Lebesgue integration results: the first statement can be proved using the Monotone Convergence Theorem for Lebesgue integration. Indeed, if $f_1 \leqslant f_n$ and both functions are HK integrable, then $f_n - f_1$ is nonnegative and HK, thus Lebesgue integrable. To conclude, it suffices to apply Lebesgue's Monotone Convergence Theorem to the sequence $(f_n - f_1)$. Similarly, to prove the second result, consider the sequence $f_n - g$ and use the Lebesgue Dominated Convergence Theorem using $h - g$ as an upper bound.

The third statement has no equivalent in Lebesgue integral, and relies strongly on the gauge. First redefine the f_n and f so that $f_n \to f$ everywhere and $(f_n(x))_n$ is bounded for all $x \in [a,b]$, this will not change the statement since the HK integral does not depend on the value of the function on a Lebesgue null set. Now, for $\varepsilon > 0$, choose δ as in the definition of the equiintegrability of the f_n. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two δ-fine tagged partitions of $[a,b]$, for all n, using the integrability of f_n yields

$$\begin{aligned}
&|\sigma(f,\mathcal{P}_1) - \sigma(f,\mathcal{P}_2)| \leqslant \\
&\qquad |\sigma(f,\mathcal{P}_1) - \sigma(f_n,\mathcal{P}_1)| + |\sigma(f_n,\mathcal{P}_1) - \sigma(f_n,\mathcal{P}_2)| + |\sigma(f_n,\mathcal{P}_2) - \sigma(f,\mathcal{P}_2)| \\
&\qquad\qquad \leqslant |\sigma(f,\mathcal{P}_1) - \sigma(f_n,\mathcal{P}_1)| + 2\varepsilon + |\sigma(f_n,\mathcal{P}_2) - \sigma(f,\mathcal{P}_2)| \\
&\qquad\qquad\qquad \leqslant \sum_{(x,I)\in\mathcal{P}_1} |f(x) - f_n(x)||I| + 2\varepsilon + \sum_{(x,I)\in\mathcal{P}_2} |f(x) - f_n(x)||I|.
\end{aligned}$$

Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are finite families, and $f_n \to f$ pointwise everywhere, for n large enough we have

$$|\sigma(f,\mathcal{P}_1) - \sigma(f,\mathcal{P}_2)| < 3\varepsilon$$

and by the Cauchy criterion for HK integrability (see proposition 2.1.4), f is Henstock-Kurzweil integrable on $[a,b]$. To see that the integral of f is the limit of the integrals of the f_n, consider δ adapted to ε for the integrability of the f_n and for the integrability of f. Fix a δ-fine tagged partition $\mathcal{P}$, we have for n large enough

$$\begin{aligned}
&\left|(HK)\int f - (HK)\int f_n\right| \leqslant \\
&\qquad \left|(HK)\int f - \sigma(f,\mathcal{P})\right| + \left|(HK)\int f_n - \sigma(f_n,\mathcal{P})\right| + |\sigma(f_n,\mathcal{P}) - \sigma(f,\mathcal{P})| < 3\varepsilon.
\end{aligned}$$

□

2.1.2 AC_* functions and the Fundamental Theorem of HK Integration

The next paragraph is mostly a copy of paragraphs 1.9-1.11 in [18].

Theorem 2.1.7. *If F is continuous on $[a,b]$ and differentiable at all but countably many points, then F' is HK integrable on $[a,b]$ and F is the indefinite integral of F'.*

Proof. Define f to be equal to F' wherever F is differentiable and to 0 where it is not. Since f is equal to F' almost everywhere, F' is HK integrable if and only f is. Fix $\varepsilon > 0$, let $y_1, y_2, \ldots$ be the points at which F is not differentiable. For $x \in [a,b]\backslash\{y_1, y_2, \ldots\}$, using the differentiability of F at x, choose a positive $\delta(x)$ such that for all $y \in [x-\delta(x), x+\delta(x)]$

$$|F(y) - F(x) - F'(x)(y-x)| < \frac{\varepsilon}{2}\frac{|y-x|}{b-a}.$$

For $j = 1, 2, \ldots$, using the continuity of F at y_j, choose $\delta(y_j)$ so that whenever $[c,d]$ is an interval

in $[a,b]$ containing y_j with $d-c<\delta(y_j)$, there holds

$$|F(d)-F(c)|<\frac{\varepsilon}{2^{j+2}}.$$

Suppose $\mathcal{P}=(([c_k,d_k],x_k))_{k=1,\dots,p}$ is a δ-fine tagged family in $[a,b]$ and for $k\in\{1,\dots,q\}$, with $q\leqslant p$ F is differentiable at x_k, whereas for $k\in\{q+1,\dots,p\}$, there exists j such that $x_k=y_j$ and a given j corresponds to at most two k since no more than two nonoverlapping intervals can contain the same point.

$$\begin{aligned}\sum_{k=1}^{p}|F(d_k)-F(c_k)-f(x_k)(d_k-c_k)|\leqslant&\\ \sum_{k=1}^{q}|F(d_k)-F(c_k)-F'(x_k)(d_k-c_k)|+\sum_{k=q+1}^{p}&|F(d_k)-F(c_k)|\\ <\frac{\varepsilon}{2}\frac{\sum_{k=1}^{q}(d_k-x+x-c_k)}{b-a}+2\sum_{j=1}^{\infty}\frac{\varepsilon}{2^{j+2}}&<\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, apply this to the case where $\mathcal{P}$ is a partition to show that f and F' are HK integrable in $[a,b]$. As this is true for any tagged family, this shows that F is the indefinite integral of f and F'. □

We now generalize this result to less regular functions F. This requires a new notion. A function F defined on $[a,b]$ is AC_* if for every set $D\subseteq[a,b]$ of zero Lebesgue measure and every $\varepsilon>0$, there exists a positive gauge δ on D such that whenever $\mathcal{P}$ is a δ-fine tagged family in $[a,b]$ and for all $(I,x)\in\mathcal{P}$, x is in D

$$\sum_{([c,d],x)\in\mathcal{P}}|F(d)-F(c)|<\varepsilon. \tag{2.1}$$

In particular, an AC_* function is continuous. If f is HK integrable, then its indefinite integral F is AC_*, indeed if D is a Lebesgue null set, we can consider the function $f_{D^c}:=f\mathbb{1}_{D^c}$.As HK integration is insensitive to modifications on Lebesgue null sets, F is also the primitive of f_D, so for $\varepsilon>0$, we can apply the Saks-Henstock Lemma (6) and find a gauge δ corresponding to $\varepsilon/2$ on $[a,b]$. Considering the gauge $\delta_D=\delta|_D$ by the Saks-Henstock Lemma for any δ_D fine tagged family $\mathcal{P}$ in $[a,b]$, since f_D is equal to zero on D, (2.1) holds. The following converse statement holds:

Proposition 2.1.8. *If F is AC_* and almost everywhere differentiable in $[a,b]$ then F' is HK integrable and*

$$F(b)-F(a)=(HK)\int_a^b F'.$$

Proof. Note that F is necessarily continuous. For $\varepsilon>0$, define δ first on the set of differentiability points as in the previous proof and define δ on the null set of non differentiability points as a gauge adapted to $\varepsilon/2$ in the definition of AC_* functions. □

In order to get a general condition which ensures that a function is AC_* and almost everywhere differentiable, we recall Stepanoff's Theorem. A function F defined on an interval I is **pointwise**

Lipschitz at the point $x \in I$ if

$$\mathrm{Lip}_x F := \limsup_{y \to x y \in I} \frac{|F(y) - F(x)|}{|y - x|} < +\infty$$

Theorem 2.1.9 (Stepanoff)**.** *If F pointwise Lipschitz at all points of some set $E \subseteq I$, then F is differentiable almost everywhere in E.*

For a proof of this result, see for instance [26, 3.1.9].

Proposition 2.1.10. *A continous function F which is pointwise Lipschitz at all but countably many points is AC_*.*

Proof. Let D be a null set in $[a, b]$ let $E := \{y_1, y_2, \ldots\}$ be the subset of D at which F is not pointwise Lipschitz. Fix $\varepsilon > 0$. Fix k in $\{1, 2, \ldots\}$; as F is continuous at y_k, there exists $\delta_k > 0$ such that whenever x, y are points in $(y - \delta_k, y + \delta_k) \cap [a, b]$,

$$F(y) - F(x) < 2^{-k-2}\varepsilon.$$

Now for $x \in D$, let

$$\delta(x) := \begin{cases} \dfrac{\varepsilon}{2(b-a)(1 + \mathrm{Lip}_x F)} & \text{if } x \in D \backslash E, \\ \delta_k \text{ if } x = y_k \text{ for some } k \in \{1, 2, \ldots\}. \end{cases}$$

The end of the proof is similar to that of theorem 2.1.7. □

Combining proposition 2.1.10, theorem 2.1.9 and proposition 2.1.8 yields

Theorem 2.1.11 (Fundamental Theorem of Henstock Kurzweil Integration)**.** *Let F be a continuous function on $[a, b]$. Suppose that F is pointwise Lipschitz at all but countably many points. Then F is differentiable almost everywhere and F' is Henstock Kurzweil integrable on $[a, b]$ with indefinite integral F.*

2.1.3 An equivalent definition of the HK integral

Remark 2.1.12 (Extension to Lipschitz curves)*.* All the above properties of the Henstock Kurzweil Integral can be extended to the case where the interval $[a, b]$ is replaced by a simple Lipschitz curve $\Gamma \subseteq \mathbb{R}^n$ (closed or not). Indeed, one can consider an arc-length parameterization γ of Γ and work on $f \circ \gamma$. If f is pointwise Lipschitz at $\gamma(x)$ along Γ, $f \circ \gamma$ is pointwise Lipschitz at x. The only thing that is not straighforward is relating differentiation in the ambient space $\mathbb{R}^n$ with differentiation along the curve. However, a Lipschitz curve has a tangent line at almost all points. In the next section, we consider countable sums of simple Lipschitz curves to develop Henstock-Kurzweil integration on Integral currents of dimension 1. The sum of curves can often be decomposed in several ways and example 2.1.13 shows that the choice of the decomposition can have an effect on the integral, hence the need for a definition of integrability that does not depend on the decomposition.

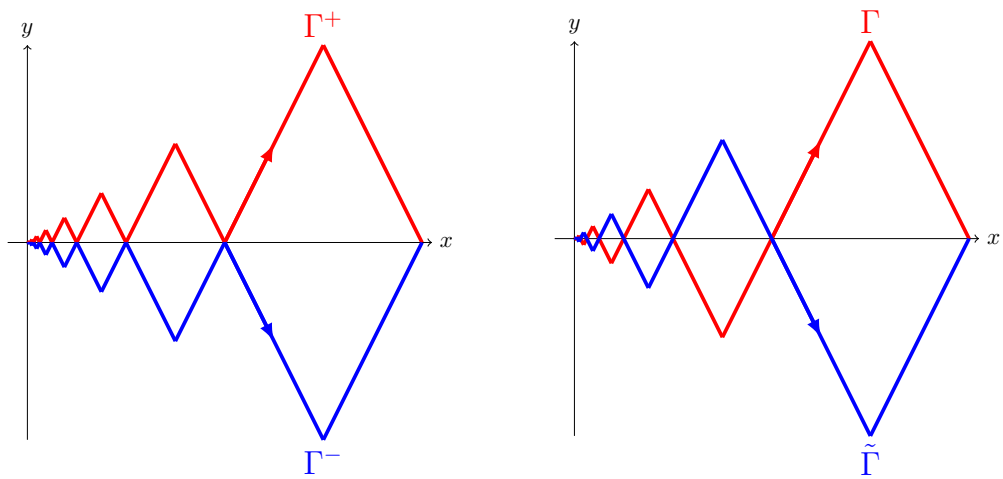

Figure 2.1: u is HK integrable on Γ^+ and Γ^- but not on Γ or $\tilde{\Gamma}$.

Example 2.1.13. In $\mathbb{R}^2$, consider the curve Γ^+ corresponding to the graph in $(0,1]$ of $x \mapsto f(x) := \operatorname{dist}(x, \{t \in (0,1], 2t\sin(t^{-2}) - 2t^{-1}\cos(t^{-2})\} = 0)$. Γ^+ is a Lipschitz curve and has length $\sqrt{2}$, orient Γ^+ towards the positive first coordinate. Let Γ^- be the the reflection of Γ^+ across the horizontal axis. The union of curves Γ^+ and Γ^- can also be seen as the (closure of) the union of the graphs on $(0,1]$ of $x \mapsto \pm\operatorname{sgn}(x\sin(x^{-2}) - 2x^{-1}\cos(x^{-2}))f(x)$. Let Γ and $\tilde{\Gamma}$ be the corresponding curves. Let u be the function defined in $\mathbb{R}^2$ by

$$(x_1, x_2) \mapsto \begin{cases} 2\operatorname{sgn}(x_2)\left(x_1\sin(x_1^{-2}) - 2x_1^{-1}\cos(x_1^{-2})\right) & \text{if } x_1 > 0, x_2 \neq 0, \\ 0 \text{ otherwise.} \end{cases}$$

If γ^+, γ^-, γ and $\tilde{\gamma}$ are respective arclength parametrizations of the curves above, the functions $u\circ\gamma^+$ and $u \circ \gamma^-$ are HK integrable on $[0, \sqrt{2}]$ with respective indefinite integrals $x \mapsto \pm\sqrt{2}x^2\sin(x^{-2})$, see example 2.1.3. However, the functions $u \circ \gamma$ and $u \circ \tilde{\gamma}$ are equal respectively to $\pm|(u \circ \gamma^+)'|$ which are not HK integrable. These curves are plotted in figure 2.1

In order to generalize the Henstock Kurzweil integral to other settings, it is necessary to use more flexible tools. In particular we need to remove the dependency on the parameters and allow for families instead of partitions (so that some "small" piece of the domain can be left out).

This will be formalized in the next section, but first state an equivalent definition of HK integrability on an interval. In order to define what "small" is we will consider functions F on the space of finite unions of disjoint intervals in $[a,b]$. Such a function is **subadditive** if given two families, $\mathcal{U}$ and $\mathcal{U}'$, of closed intervals of $[a,b]$ there holds

$$|F([\mathcal{U}] \cup [\mathcal{U}'])| \leqslant |F([\mathcal{U}])| + |F([\mathcal{U}'])|.$$

F is **additive** if for $\mathcal{U}$ and $\mathcal{U}'$ as above with $\mathcal{L}^1([\mathcal{U}] \cap [\mathcal{U}']) = 0$, there holds

$$F([\mathcal{U}] \cup [\mathcal{U}']) = F([\mathcal{U}]) + F([\mathcal{U}']).$$

F is **continuous** on the space of finite unions of intervals if given a sequence $\mathcal{U}_j$ of families of

intervals with $\#\mathcal{U}_j < C$ and $\mathcal{L}^1([\mathcal{U}_j]) \to 0$, there holds $F([\mathcal{U}_j]) \to 0$. In particular, it is equivalent to consider a continuous function $F : [a,b] \to \mathbb{R}$ and a continuous function on the space of finite unions of intervals of $[a,b]$, indeed a continuous function on a closed interval is uniformly continuous.

This definition seems impractical but we will see in the following section that it can be easily generalized to other supports and also to higher dimensions, in chapter 3. Indeed while intervals are not well suited to algebraic operations, they can be seen as currents of dimension 1 in $\mathbb{R}$, using their canonical orientation.

The following property is a reformulation of HK integrability in the language of Pfeffer integration (see Theorem 6.7.5 in [60]). We remind the reader that a gauge on an interval is a nonnegative function whose zero set is at most countable.

Proposition 2.1.14 (Equivalent integrability condition)**.** *A function f defined almost everywhere on $[a,b]$ is Henstock-Kurzweil integrable if and only if there exists a nonnegative subadditive continuous function G on the space of finite unions of intervals in $[a,b]$ and a real number I with the property that for all $\varepsilon > 0$, there exists a gauge δ - not necessarily positive everywhere - and a positive number τ such that whenever $\mathcal{P}$ is a δ-fine tagged family in $[a,b]$ with*

$$G([a,b]\backslash[\mathcal{P}]) < \tau,$$

there holds $|I - \sigma(f,\mathcal{P})| < \varepsilon$.

Before proving this, it makes sense to check that a tagged family satisfying the above contraints exists, it is a sort of generalization of Cousin's Lemma 2.1.2, where we consider families instead of partitions.

Lemma 2.1.15 (Howard-Cousin)**.** *Let δ be a gauge on $[a,b]$, not necessarily positive everywhere. Let G be a nonnegative subadditive continuous function on the space of finite union of closed intervals in $[a,b]$. For every $\tau > 0$, there exists a δ-fine tagged family $\mathcal{P}$ in $[a,b]$ with*

$$G([a,b]\backslash[\mathcal{P}]) < \tau. \tag{2.2}$$

Proof. We define a positive gauge δ^* on $[a,b]$ and use Cousin's Lemma to get a δ^*-fine tagged partition $\mathcal{P}^*$ of $[a,b]$ we then consider the subfamily $\mathcal{P}$ of $\mathcal{P}^*$ consisting of the pairs (x,I) where $\delta(x) > 0$. $\mathcal{P}$ is clearly a δ-fine tagged family, but it is necessary to check that (2.2) holds. This is where the choice of δ^* is critical. It relies heavily on the continuity of G: For every $\varepsilon > 0$, there exists $\eta > 0$ such that whenever I is a closed interval in $[a,b]$ with $b-a < \eta$, $G(I) < \varepsilon$. To see this, suppose the contrary, there exists a sequence $(I_j)_j$ of closed intervals in $[a,b]$ with $\mathcal{L}^1(I_j) < 1/j$ and $G(I_j) \geqslant \varepsilon$ for all j. This contradicts the definition of continuity of G.

For $j = 1,2,\dots$, choose η_j so that $G(I) < 2^{-(j+1)}\tau$ whenever I is an interval in $[a,b]$ with length less than η_j. Let $\delta^*(x_j) = \eta_j$ for $j = 1,2,\dots$ and for $x \in [a,b]\backslash E$, let $\delta^*(x) := \delta(x)$. δ^* is a positive gauge, so there exists a δ^*-fine tagged partition $\mathcal{P}^*$ of $[a,b]$. Clearly the tagged family $\mathcal{P} := \{(I,x) \in \mathcal{P}^*, x \in [a,b]\backslash E\}$ is δ-fine, and furthermore, as $\mathcal{Q} := \mathcal{P}^*\backslash\mathcal{P}$ is finite and G is

subadditive, there holds

$$G([\,a,b\,]\backslash[\mathcal{P}]) = G\left([\mathcal{P}^*\backslash\mathcal{P}]\right) \leqslant \sum_{(I,x)\in\mathcal{Q}} G(I).$$

Now for each $(I,x) \in \mathcal{Q}$, there exists a j such that $x = x_j$ and for all $j \in \{1,2,\ldots\}$, there are at most two such (I,x) as the intervals in $\mathcal{Q}$ are nonoverlapping and $x = x_j \in I$, since for such an $x = x_j$ there holds $\mathcal{L}^1(I) < \delta^*(x_j) = \eta_j$, we have

$$G([\,a,b\,]\backslash[\mathcal{P}]) < 2\sum_{j=1}^{\infty} 2^{-(j+1)}\tau \leqslant \tau.$$

□

Proof of proposition 2.1.14. First suppose that f is HK integrable on $[\,a,b\,]$. Let F be the indefinite integral of f. F can be identified to an additive and continuous function on the space of finite unions of intervals in $[\,a,b\,]$. Let G be the absolute value of F, it is subadditive, nonnegative and continuous. Fix $\varepsilon > 0$ and choose δ as in the definition of HK integrability for f. By the Saks Henstock Lemma, if $\mathcal{P}$ is a δ-fine tagged family in $[\,a,b\,]$ there holds

$$|\sigma(f,\mathcal{P}) - F([\mathcal{P}])| < 2\varepsilon$$

and by additivity of F

$$|F([\,a,b\,]) - F([\mathcal{P}])| \leqslant G([\,a,b\,]\backslash\mathcal{P}),$$

so if $\mathcal{P}$ is such that

$$G([\,a,b\,]\backslash[\mathcal{P}]) < \varepsilon,$$

we have

$$|\sigma(f,\mathcal{P}) - F([\mathcal{P}])| \leqslant |\sigma(f,\mathcal{P}) - F([\mathcal{P}])| + |F([\,a,b\,]) - F([\mathcal{P}])| \leqslant 3\varepsilon.$$

This proves that f satisfies the condition of the statement with $I = (HK)\int f = F([\,a,b\,])$.

The converses uses results of HKP integration, defined in the next section. Suppose that the condition in the statement holds. f is in fact HKP integrable on the current $\mathbf{E}^1 \llcorner [\,a,b\,] = \mathcal{L}^1 \llcorner [\,a,b\,] \wedge \mathbf{e}_x = [\![a,b]\!]$,as defined in section 2.3. This implies that G can be replaced by $|F|$, where F is the primitive of f for HKP integration. Thus F is additive and continuous (we use the correspondence between functions on $\mathcal{S}_{\leqslant}(\mathbf{E}_1 \llcorner [\,a,b\,])$ and the continuous functions on the space of intervals in $[\,a,b\,]$. This is proved in the same way as proposition 2.3.9. For $\varepsilon > 0$, choose a gauge δ on $[\,a,b\,]$ according to the statement. Let $\{x_1, x_2, \ldots\}$ be the zero set of δ. We define a positive gauge δ^* combining the condition of the proof of the Howard Cousin Lemma 2.1.15 (with parameter τ) with the choice $\delta(x_p) < (\max\{|f(x_p)|, 1\})^{-1}\varepsilon 2^{p+1}$.

For a δ^*-fine tagged partition, $\mathcal{P}^*$ of $[\,a,b\,]$, if $\mathcal{P}^* = \mathcal{P} \sqcup \mathcal{Q}$ there holds

$$|\sigma(f,\mathcal{P}^*) - F([\,a,b\,])| \leqslant |\sigma(f,\mathcal{P}^*) - F([\,a,b\,])| + |\sigma(f,\mathcal{Q})|.$$

Since $\mathcal{P}$ is δ-fine and $|F([\,a,b\,])| < \tau$ the first term is smaller than $\varepsilon/2$. The second term is controlled

by:

$$|\sigma(f,\Omega)| < \sum_{j=1}^{\infty} |f(x_j)|\delta(x_j) < \frac{\varepsilon}{2},$$

which concludes the proof. □

2.2 Current of dimension 1, decompositions, pieces

For basic definitions and statements on integral currents we refer to Appendix B, and for more details to [26, Chapter 4].

There is a notion of decomposition of currents. An integral current T is **decomposable** if there exists two non trivial integral currents Q and R with $Q+R=T$ and $\mathbb{M}(T)=\mathbb{M}(Q)+\mathbb{M}(R)$, $\mathbb{M}(\partial T)=\mathbb{M}(\partial Q)+\mathbb{M}(\partial R)$. If such a pair does not exist, T is called indecomposable.

The dimension one case is well understood: a current $T \in \mathbb{I}_1(\mathbb{R}^n)$ is indecomposable if and only if it is associated with an oriented simple Lipschitz curve with multiplicity 1. If the curve is closed, the current is a cycle, otherwise it has boundary mass 2. Such a curve Γ can be parameterized by arc length by $\gamma : [\,0,\mathbb{M}(T)\,] \to \mathbb{R}^n$, where γ Lipschitz and injective on $[\,0,\mathbb{M}(T))$ with $\gamma(0)=\gamma(\mathbb{M}(T))$ if and only if T is a cycle. Indecomposable currents of dimension 1 are thus of the form

$$T = \gamma_{\#}\mathbf{E}^1 \llcorner [0,\mathbb{M}(T)] = \mathcal{H}^1 \llcorner \Gamma \wedge \gamma'(\gamma^{-1}).$$

Proposition 2.2.1 (Decomposition of integral currents of dimension 1)**.** *A current $T \in \mathbb{I}_1(\mathbb{R}^n)$ can be written as a countable sum of indecomposable currents $T_j \in \mathbb{I}_1(\mathbb{R}^n)$, for $j \in J$, where J is either of the form $\{1,2,\ldots,q\}$ or of the form $\{1,2,\ldots\}$, and the T_j are cycles for j larger than some integer p. There holds*

$$\mathbb{M}(T) = \sum_{j\in J} \mathbb{M}(T_j) \text{ and } \mathbb{M}(\partial T) = \sum_{j\in J} \mathbb{M}(T_j).$$

Such a sequence $(T_j)_{j\in J}$ is called a decomposition of T, there may be infinitely many non equivalent decompositions of T (two decompositions are equivalent if they are equal up to a bijection in the indices in J). In the following, whenever a (finite or infinite) decomposition $T_1, T_2, \ldots$ of a current T will be mentioned we will implicitely associate to each T_j the Lipschitz function $\gamma_j : [\,0,\mathbb{M}(T_j)\,] \to \mathbb{R}^n$ with $|\gamma'| = 1$ almost everywhere and $T_j = \gamma_{j\#}[\![0,\mathbb{M}(T_j)]\!]$.

To relate this section with the previous one, notice that an integral current of dimension 1 in $\mathbb{R}^1$ is associated to a finite sum of disjoint oriented intervals with integer multiplicity. An indecomposable integral current of dimension 1 in $\mathbb{R}$ is an interval. In particular, there are no non trivial cycles in $\mathbb{R}^1$. When integrating on an interval, we cut the interval into little pieces. The canonical orientation of $\mathbb{R}$ and the fact that the intervals have multiplicity 1, allowed us to identify an interval both as a closed set in the topological sense, and as a piece of our current in the measure sense, while preserving the global orientation. However, general currents of dimension 1 in $\mathbb{R}^n$ for $n \geqslant 2$ require a different approach. We now turn to defining the "pieces" of the currents we are going to use in the Riemann sums, in order to integrate on a current.

2.2.1 Subcurrents, pieces of current

Let T be an integral current, an integral current S is a **piece of** T if

$$\|S\| \leqslant \|T\| \text{ and } \|T - S\| \leqslant \|T\|.$$

Remark 2.2.2. This notion differs from that of subcurrent defined in in any dimension where the condition is $\|S\| \perp \|T - S\|$, see chapter 3.

Example 2.2.3. Consider the current $S \in \mathbb{I}_1(\mathbb{R}^1)$ given by $T = 2\mathbf{E}^1 \llcorner [0,2]$, then

- The currents $\mathbf{E}^1 \llcorner [0,2]$ and $2\mathbf{E}^1 \llcorner [0,4/3]$ are pieces of T,
- $3\mathbf{E}^1 \llcorner [0,2]$, $3^{-1}\mathbf{E}^1 \llcorner [0,2]$ and $-\mathbf{E}^1 \llcorner [0,2]$ are not pieces of T.

Morally, removing a piece of T should decrease the mass of T locally, but not take more than what T contains.

Denote by $\mathcal{S}_{\leqslant}(T)$ the collection of all pieces of T. It is related to the space of subcurrents of T $\mathcal{S}(T)$ defined in chapter 3, in particular subcurrents of T are pieces of T.

Proposition 2.2.4. *$S \in \mathbb{I}_1(\mathbb{R}^n)$ is in $\mathcal{S}_{\leqslant}(T)$ if and only if there exists a $\|T\|$ measurable $g : \mathbb{R}^n \to [0,1]$ such that $S = T \llcorner g$.*

Proof. Suppose $S = T \llcorner g$, then $\|S\| = \|T\| \llcorner g \leqslant \|T\|$ and $\|T - S\| = \|T\| \llcorner (1-g) \leqslant \|T\|$.

Conversely, suppose S is in $\mathcal{S}_{\leqslant}(T)$. Then S can be written $\mathcal{H}^1 \llcorner (\theta_S \mathbb{1}_{M_S}) \wedge \overrightarrow{S}$ and $T = \mathcal{H}^1 \llcorner (\theta_T \mathbb{1}_{M_T}) \wedge \overrightarrow{T}$, where θ_S and θ_T are supposed non negative, respectively $\mathcal{H}^1 \llcorner M_S$ and $\mathcal{H}^1 \llcorner M_T$ almost everywhere. The hypotheses on S imply respectively that

$$\|S\| \leqslant \|T\| \Rightarrow \mathcal{H}^1(M_S \backslash M_T) = 0 \text{ and } \theta_S \leqslant \theta_T,\ \mathcal{H}^1 \llcorner M_T \text{ almost everywhere}$$

and that

$$\|T - S\| \leqslant \|T\| \Rightarrow |\theta_T \overrightarrow{T} - \theta_S \overrightarrow{S}| \leqslant \theta_T,\ \mathcal{H}^1 \llcorner M_T \text{ almost everywhere.}$$

This in turn implies that $\overrightarrow{T} = \overrightarrow{S}$ at $\mathcal{H}^1$ almost all points where θ_S is positive. Define the functions g by

$$g(x) = \begin{cases} 0 \text{ if } x \notin M_T, \text{ or } \theta_T(x) = 0, \\ \theta_S(x)/\theta_T(x) \text{ otherwise.} \end{cases}$$

Clearly $g(x) \in [0,1]$ for all $x \in \mathbb{R}^n$ and $S = T \llcorner g$. □

In particular elements of a decomposition of T are pieces of T, however an indecomposable piece of T may not be a piece of any element of decomposition of T (see figure 2.2).

2.2.2 Continuous function on the space of pieces of T

Definition 2.2.5. A function F on $\mathcal{S}_{\leqslant}(T)$ is **continuous** , if given a sequence $(S_j)_j$ in $\mathcal{S}_{\leqslant}(T)$ that converges to 0 in the flat norm with $\sup_j \mathbb{M}(\partial S_j) < +\infty$, we have $F(S_j) \to 0$. F is **additive** if

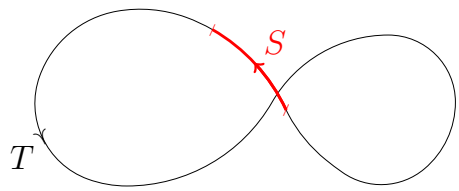

Figure 2.2: S is not a piece of an indecomposable element of T.

whenever S_1 and S_2 are in $\mathcal{S}_{\leqslant}(T)$ with $S_1+S_2 \in \mathcal{S}_{\leqslant}(T)$ (which is equivalent to $\|S_1\|+\|S_2\| \leqslant \|T\|$), there holds $F(S_1+S_2) = F(S_1)+F(S_2)$. F is **subadditive**, if instead for each S_1, S_2 as above, we have $F(S_1+S_2) \leqslant F(S_1)+F(S_2)$.

Examples of continuous additive functions on $\mathcal{S}_{\leqslant}(T)$ include the restriction of 1-charges as defined in [19]. 1-charges include representatives of continuous functions f and continuous differential 1-forms ω on $\operatorname{spt} T$ defined respectively as

$$\Theta_f : S \mapsto \partial S(f)$$

and

$$\Lambda_\omega : S \mapsto \int \langle \omega, \overrightarrow{S} \rangle \, \mathrm{d}\|S\|.$$

Furthermore, the mass function $S \mapsto \mathbb{M}(S)$ is continuous on $\mathcal{S}_{\leqslant}(T)$:

Proposition 2.2.6. *For every $T \in \mathbb{I}_1(\mathbb{R}^n)$, the function $S \mapsto \mathbb{M}(S)$ is continuous and additive on $\mathcal{S}_{\leqslant}(T)$.*

Proof. Additivity is clear. For the continuity, let $(S_j)_j$ be a sequence in $\mathcal{S}_{\leqslant}(T)$ converging in the flat norm to $S \in \mathcal{S}_{\leqslant}(T)$ with $\sup_j \mathbb{M}(\partial S_j) < +\infty$. First notice that $\mathbb{M}(S) \leqslant \liminf_j \mathbb{M}(S_j)$ by lower semi-continuity of mass in the flat norm topology. So all we have to show is that $\liminf_j \mathbb{M}(S_j) \geqslant \mathbb{M}(S)$. In order to do this, for $\varepsilon > 0$ define a smooth 1-form $\tilde{\omega}$ in $\mathbb{R}^n$ such that $|\tilde{\omega}(x)| \leqslant 1$ for all x and $R(\tilde{\omega}) \geqslant \mathbb{M}(R) - \varepsilon$ for each $R \in \mathcal{S}_{\leqslant}(T)$. To prove that such a form exist, consider the $\|T\|$ measurable function $\overrightarrow{T}$. Recall that $\|T\|$ is a Radon measure. Fix $\varepsilon > 0$, by Lusin's Theorem (see theorem A.4 or [11, 1.2, Theorem 2]), there exists a compact set $E \subseteq \mathbb{R}^n$ such that $\overrightarrow{T}$ is continuous on E and $\|T\|(\mathbb{R}^n \backslash E) < \varepsilon/2$. Consider the continuous differential form of degree 1,

$$\omega : E \to \Lambda^1(\mathbb{R}^n), x \mapsto \overrightarrow{T}^*(x).$$

where $\overrightarrow{T}^*(x)$ is the 1 covector associated to $\overrightarrow{T}$ by duality for the euclidian inner product. Extend ω continuously to the whole of $\mathbb{R}^n$ so that $|\omega(x)| \leqslant 1$ everywhere. ω can be approximated uniformly by a smooth differential form of degree 1, $\tilde{\omega}$ defined on $\mathbb{R}^n$ with $|\tilde{\omega}(x)| \leqslant 1$. Say, $|\tilde{\omega} - \omega|_\infty < \varepsilon/(2\,\mathbb{M}(T))$. Let R be a piece of T, there holds

$$\begin{aligned} R(\tilde{\omega}) = \int \langle \tilde{\omega}(x), \overrightarrow{T}(x) \rangle \, \mathrm{d}\|R\|(x) \\ > \int \langle \omega(x), \overrightarrow{T}(x) \rangle \, \mathrm{d}\|R\|(x) - \mathbb{M}(R)\varepsilon/(2\,\mathbb{M}(T)) \\ \geqslant \int \langle \overrightarrow{T}^*(x), \overrightarrow{T}(x) \rangle \, \mathrm{d}(\|R\| \llcorner E)(x) - |\omega|_\infty \|R\|(\mathbb{R}^n \backslash E) - \varepsilon/2 > \mathbb{M}(R) - \varepsilon. \end{aligned}$$

By definition of the flat convergence, $S_j(\tilde{\omega}) \to S(\tilde{\omega})$ which implies that $\mathbb{M}(S_j) \leqslant \mathbb{M}(S) - \varepsilon - \varepsilon$, for all large enough j. Since ε is arbitary, $\mathbb{M}(S_j) \to \mathbb{M}(S)$. □

As a consequence, to a $\|T\|$-Lebesgue integrable function f in $\mathbb{R}^n$, one can associate the continuous additive function on $\mathcal{S}_{\leqslant}(T)$

$$\tilde{\Lambda}_f : S \mapsto \int f \, \mathrm{d}\|S\|.$$

In the definition of Θ_f, one can ask whether the continuity assumption of f on $\operatorname{spt} T$ can be relaxed, for instance if f is continuous on $\operatorname{set}_1 \|T\|$, is that sufficient for Θ_f to be continuous. Clearly, if T is indecomposable, $\operatorname{set}_1 \|T\| = \operatorname{spt} T$, but if one considers a current that has a countable decomposition, things are different:

Proposition 2.2.7. *There exists an integral current T of dimension 1 in $\mathbb{R}^2$ along with a bounded function f continuous on $\operatorname{set}_1 \|T\|$, but not on $\operatorname{spt} T$ such that the function on $\mathcal{S}_{\leqslant}(T)$ associated to the variation of f*

$$\Theta_f : S \mapsto \partial S(f)$$

is not continuous.

Proof. Consider a union of disjoint circles $\bigcup_{j=1}^{\infty} C_j$. Where for $j = 1, 2, \ldots,$ C_j is centered at $(a_j, 0) = (2^{-j}, 0)$ and has radius $r_j := 3^{-(j+1)}$. Define the function f piecewise on each C_j so that $f = 1$ at the top of each circle, and $f = -1$ at the bottom of each circle and f is smooth. A good choice is $f(x_1, x_2) = r_j^{-1} y$ if $(x_1, x_2) \in C_j$. Let $\overrightarrow{T}$ be a field of tangent unit vectors to the circles, oriented positively and

$$T := (\mathcal{H}^2 \llcorner \bigcup_j C_j) \wedge \overrightarrow{T}.$$

Clearly $\operatorname{spt} T = \bigcup_j C_j \cup \{(0,0)\}$. Let us check that $\operatorname{set}_1 \|T\| = \bigcup_j C_j$: for $r > 0$ if $2^{1-j_0} \leqslant r \leqslant 2^{-j_0}$,

$$\|T\|(\mathrm{U}(0,r)) \leqslant \sum_{j \geqslant j_0} 2\pi r_j \leqslant 3^{j_0} \pi.$$

Thus $\Theta^{1*}(\|T\|, 0) = 0$ and $0 \notin \operatorname{set}_1 \|T\|$. Consider the sequence of pieces $S_j \in \mathcal{S}_{\leqslant}(T)$ corresponding to the half circles: $S_j = T \llcorner \{(x_1, x_2), 2^{-j} \leqslant x_1 \leqslant 2^{-j} + 3^{-j-1}\}$ (see figure 2.3). S_j tends to 0 in mass and for all j, $\mathbb{M}(\partial S_j) = 2$. However, $\partial S_j(f) = 2 \nrightarrow 0$. Therefore $S \mapsto \partial S(f)$ is not continuous on $\mathcal{S}_{\leqslant}(T)$. □

Note this never happens for an indecomposable current because of the following fact:

Proposition 2.2.8. *$T \in \mathbb{I}_1(\mathbb{R}^n)$ is indecomposable, $\operatorname{set}_1 \|T\| = \operatorname{spt} T$ and if $x \in \operatorname{spt} T \setminus \operatorname{spt} \partial T$, there holds $\Theta^1_*(\|T\|, x) \geqslant 1$.*

Proof. Rescaling, we can suppose that $\mathbb{M}(T) = 1$. There exists a Lipschitz curve: $\gamma : [0,1] \to \mathbb{R}^n$ with $|\gamma'| = 1$ Lebesgue almost everywhere in $[0,1]$ and $T = \gamma_{\#}(\mathbf{E}^1 \llcorner [0,1])$. such that either $\gamma(0) = \gamma(1)$ and γ is injective on $[0,1[$, or γ is bijective on its image. In any case $\gamma([0,1])$ is compact. Therefore $\gamma([0,1]) = \operatorname{spt} T$

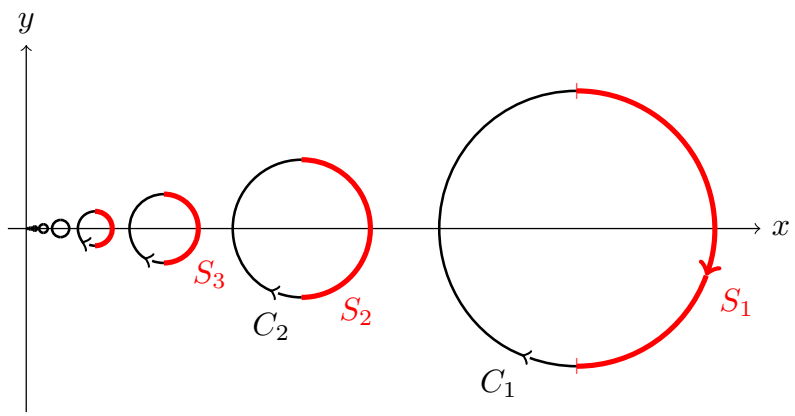

Figure 2.3: The current T and the sequence $(S_j)_j$ of pieces

Fix $x_0 \in \operatorname{spt} T$, there exists $t \in [0,1]$ such that $\gamma(t) = x$, thus for $r \in (0, 1/2)$

$$\|T\|(\mathrm{U}(x,r)) \geqslant \mathcal{L}^1([0,1] \cap [t-r, t+r]) \geqslant r.$$

This implies that $\Theta^1_*(\|T\|, x) \geqslant 1/2$ and $x \in \operatorname{set}_1 \|T\|$ and $\operatorname{spt} T \subseteq \operatorname{set}_1 \|T\|$. As the converse inclusion is clear, we are done. If x is not in $\operatorname{spt} \partial T$, for r small enough, we can replace r by $2r$ on the right hand side above and this implies $\Theta^1_*(\|T\|, x) \geqslant 1$. □

2.2.3 Derivation

For function on $\mathcal{S}_{\leqslant}(T)$ there is a notion of derivation along T, similar to the differentiation of measures in Radon-Nikodym Theory:

Definition 2.2.9. For x in $\operatorname{spt} T$ and $\delta > 0$, consider the subset $\mathcal{S}_{\leqslant}(T, x, \delta)$ of $\mathcal{S}_{\leqslant}(T)$ consisting of all pieces S of T such that

1. $x \in \operatorname{spt} S$,
2. S is indecomposable,
3. $\operatorname{diam} \operatorname{spt} S < \delta$.

If $\mathcal{S}_{\leqslant}(T, x, \delta)$ is not empty for some positive δ, x is called **good in** T. In this case, the **upper and lower derivatess** of F along T at x are respectively

$$\overline{\mathfrak{D}}_T F(x) := \inf_{\delta > 0} \sup_{\mathcal{S}_{\leqslant}(T,x,\delta)} \frac{F(S)}{\mathbb{M}(S)} \text{ and } \underline{\mathfrak{D}}_T F(x) := \sup_{\delta > 0} \inf_{\mathcal{S}_{\leqslant}(T,x,\delta)} \frac{F(S)}{\mathbb{M}(S)}.$$

F is **derivable along** T **at** $x \in \operatorname{set}_1 \|T\|$ if the upper and lower **derivates** of F at x along T coincide, the corresponding derivate is denoted $\mathfrak{D}_T F(x)$.

A related notion we will use is that of almost derivability: a function F on $\mathcal{S}_{\leqslant}(T)$ is **almost derivable** at $x \in \operatorname{set}_1 \|T\|$ if the upper and lower derivates of F along T at x are finite.

Remark 2.2.10. For general currents of dimension 1: There exist points $x \in \operatorname{set}_1 \|T\|$ and some $\delta > 0$, such that $\mathcal{S}_{\leqslant}(T, x, \delta) = \emptyset$. Such a point x cannot be in the support of any indecomposable piece of T, therefore the set of all these points is $\mathcal{H}^1$ null.

However, this set could be uncountable. An idea to define such a set would be to consider a fat Cantor set $[\,0,1\,]$. For instance, one could let C be the set obtained by removing iteratively the middle intervals of length 4^{-k} for $k = 1, 2, \ldots$ from $[\,0,1\,]$. C is a compact totally disconnected set with $\mathcal{L}^1(C) = 1/2 > 0$.

For each $k = 1, 2, \ldots$ there are 2^{k-1} segments of lenght 4^{-k} in the complement of C, denote them by S_k^j for $j = 1, 2, \ldots, 2^{k-1}$. In $\mathbb{R}^2$ let R_k^j be the rectangle $S_k^j \times [\,0, h_k\,]$ where the h_k form a summable sequence of real numbers with $\sum_{k=1}^{\infty} 2^k h_k < +\infty$.

We can consider the current $T \in \mathbb{I}_1(\mathbb{R}^2)$ defined by

$$T := \sum_{k,j} \partial(\mathbf{E}^2 \llcorner R_k^j).$$

(see figure 2.4) T is a cycle which has finite mass by the choice of h_k. Clearly $\operatorname{spt} T = C \cup \bigcup_{k,j} \operatorname{bdry} R_k^j \supseteq [\,0,1\,]$. The question is how to characterize $\operatorname{set}_1 \|T\|$ and whether there exists points of $C \cap \operatorname{set}_1 \|T\|$ such that there is no indecomposable piece S of T with $x \in \operatorname{spt} S$.

Claim 1. Suppose S is an indecomposable piece of T, then S is a piece of $\partial(\mathbf{E}^2 R_k^j)$ for some $k \in \{1, 2, \ldots\}$ and $j \in \{1, \ldots, 2^{k-1}\}$.

Proof. By contradiction, let $S \in \mathcal{S}_{\leqslant}(T)$ be indecomposable and fix $x \in \operatorname{spt} S \cap \operatorname{bdry} R_k j$ and $x' \in \operatorname{spt} S \cap \operatorname{bdry} R_{k'}^{j'}$ with $(k, j) \neq (k', j')$. Without loss of generality (taking an indecomposable piece of S), we can suppose that $\partial S = \delta_{x'} - \delta_x$. We can also suppose that $x = (x_1, 0)$ and $x' = (x_1', 0)$ with $x_1 < x_1'$ and $x_1 = \max(t \in S_k^j)$, $x' = \min(t, t \in S_{k'}^{j'}$. As S is indecomposable and the differential form $(z_1, z_2) \mapsto \mathbf{e}_1^*$ is the differential of $(z_1, z_2) \mapsto z_1$, there holds

$$\int \langle \mathbf{e}_1^*, \overrightarrow{S} \rangle \, \mathrm{d}\|S\| = x_1' - x_1.$$

However, since S is supported inside $[\,x_1, x_1'\,] \times \mathbb{R}$, and $\overrightarrow{S} = \overrightarrow{T} \in \{\mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2\}$, $\|S\|$ almost everywhere, there holds

$$\begin{aligned}\int \langle \mathbf{e}_1^*, \overrightarrow{S} \rangle \, \mathrm{d}\|S\| &\leqslant \|T\|(([\,x_1, x_1'\,] \times \mathbb{R}) \cap \{(z_1, z_2), \overrightarrow{T}(z_1, z_2) = \mathbf{e}_1\}) \\ &\leqslant \mathcal{L}^1(C^c \cap [\,x_1, x_1'\,]) < x_1' - x_1,\end{aligned}$$

where we used the fact that $C \cap [\,x_1, x_1'\,]$ contains a fat Cantor subset of C, which has positive Lebesgue measure. This is a contradiction. ■

The above claim implies that for all $x \in C \setminus \bigcup_{k,j} \operatorname{cl}(S_k^j)$, x is not in the support of any indecomposable piece of T. There remains to prove that $C \cap \operatorname{set}_1 \|T\| \setminus \bigcup_{k,j} \operatorname{cl}(S_k^j)$ is uncountable. For $x \in (0,1)$, $\Theta^{1*}(\|T\|, x) \geqslant \Theta^{*1}(\mathcal{H}^1 \llcorner E^c, x) = 1 - \Theta_*^1(\mathcal{H}^1 \llcorner C, x)$, so we only need to prove that $C \cap \{x, \Theta_*^1(\mathcal{H}^1 \llcorner C, x) < 1\}$ is uncountable.

In [9, Theorem 1], Buczolich proved that the set of points of a nowhere dense perfect set $P \subseteq \mathbb{R}$ where P has lower density larger than γ for any $\gamma > 0.5$ is always of first category in P. This

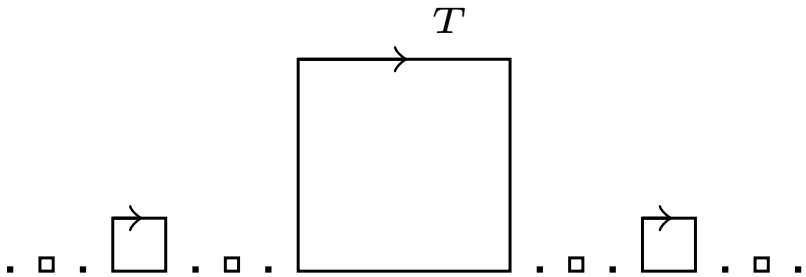

Figure 2.4: An integral current T defined using the complementary intervals to a Cantor set.

implies that the set of points of density less than 1 is of second category in P, which in turn implies that it is uncountable (P is a Baire space with the topology inherited from $\mathbb{R}$, see for instance [57, Chapter 9]). Note that there are more precise ways to characterize the points of a Cantor set with given densities, see for instance the paper by Besicovitch [6].

Example 2.2.11. Let Λ_f be the function on $\mathcal{S}_{\leqslant}(T)$ be associated to a Lebesgue $\|T\|$ integrable function f defined almost everywhere on $\operatorname{set}_1 \|T\|$ by

$$\Lambda_f : S \mapsto \int f \,\mathrm{d}\|S\|.$$

If f is continuous at $x \in \operatorname{set}_1 \|T\|$ and x is good in T, then Λ_f is derivable at x along T with derivate $\mathfrak{D}_T F(x) = f(x)$.

For a good point $x \in \operatorname{spt} T$, $\varepsilon > 0$, choose $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for all $y \in \mathrm{U}(x, \delta)$. For $S \in \mathcal{S}_{\leqslant}(T, x, \delta)$

$$|\Lambda_f(S) - f(x)\,\mathbb{M}(S)| \leqslant \int |f(y) - f(x)| \,\mathrm{d}\|S\|(y) \leqslant \varepsilon\,\mathbb{M}(S).$$

Letting ε go to zero, we can conclude.

Question 2.2.12. *If F is a continuous function defined on $\mathcal{S}_{\leqslant}(T)$, are the extended real valued functions $\underline{\mathfrak{D}}_T F$, $\overline{\mathfrak{D}}_T F$ and $\mathfrak{D}_T F$ are $\|T\|$ measurable? Borel measurable?.*

For Henstock-Kurzweil Integration in 1 dimension and for Pfeffer Integration on sets of finite perimeter, such results rely on the Vitali covering theorem and a derivation operation. A "covering" theorem using pieces of T would be useful. An alternative would be to study a suitable decomposition of T, but this approach is made difficult by the fact that there can be pieces of T which do not belong to any decomposition of T, as fig. 2.2 shows.

Definition 2.2.13. Let T be an integral current of dimension 1 in $\mathbb{R}^n$ and let u be a function defined on $\operatorname{set}_1 \|T\|$. Fix a point $x \in \operatorname{set}_1 \|T\|$ good in T. The function u is **differentiable along** T at x if there exists a linear form $\mathrm{D}u(x)$ on $\mathbb{R}^n$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $y \in \operatorname{set}_1 \|T\| \cap \mathrm{U}(x, \delta)$ and there is an $S \in \mathcal{S}_{\leqslant}(T, x, 3\delta)$ with $y \in \operatorname{set}_1 \|S\|$, there holds

$$|u(y) - u(x) - \mathrm{D}u(x) \cdot (y - x)| \leqslant \varepsilon |y - x|.$$

Note that if u is differentiable in $\mathbb{R}^n$ or differentiable on $\operatorname{spt} T$ in the sense of Whitney [72] then u is differentiable along T with the same differential.

Theorem 2.2.14. *Suppose that u is a continuous function on $\operatorname{spt} T$ for some $T \in \mathbb{I}_1(\mathbb{R}^n)$. Fix $x \in \operatorname{set}_1 \|T\|$ such that $\mathcal{S}_{\leqslant}(T, x, \delta) \neq \emptyset$ for some $\delta > 0$, then the following three statements hold*

(i) If u has pointwise Lipschitz constant 0 at x, then Θ_u is derivable at x along T and $\mathfrak{D}_T\, \Theta_u(x) = 0$.

(ii) If u is pointwise Lipschitz at x, then Θ_u is almost derivable at x with $-\operatorname{Lip}_x u \leqslant \underline{\mathfrak{D}}_T \Theta_u \leqslant \overline{\mathfrak{D}}_T \Theta_u \leqslant Lip_x u$.

(iii) If $\overrightarrow{T}$ has a $\|T\|$ approximately continuous representative at x (which we still denote by $\overrightarrow{T}$), $\|T\|$ has finite upper density at x and u is differentiable at x along T, then Θ_u is derivable at x along T, with $\mathfrak{D}_T\, \Theta_u(x) = \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle$.

Remark 2.2.15. The assumption that $\overrightarrow{T}$ has a $\|T\|$ approximately continuous representative at x is satisfied for $\|T\|$ almost all x. (See claim 1 in the proof of proposition 2.3.3.)

Proof. Let us start with (i) and (ii). For $\varepsilon > 0$, there exists δ such that whenever $y \in \operatorname{spt} T$ with $|y - x| < \delta$,

$$|u(y) - u(x)| < (M + \varepsilon)|y - x|,$$

with $M = \operatorname{Lip}_x u$. Given an indecomposable $S \in \mathcal{S}_{\leqslant}(T)$, with $x \in \operatorname{spt} S$ and $\operatorname{diam} \operatorname{spt} S < \delta$, S is of the form $\gamma_{\#}[\![0, \mathbb{M}(S)]\!]$ with $\gamma(0) = y_-$ and $\gamma(\mathbb{M}(S)) = y_+$. Since $|y_+ - x| + |x - y_-| \leqslant \mathbb{M}(S)$, we get

$$|\Theta_u(S)| = |u(y_+) - u(y_-)| \leqslant |u(y_+) - u(x)| + |u(x) - u(y_-)| \leqslant (M + \varepsilon)\, \mathbb{M}(S),$$

as ε is arbitrary, this is enough to prove (ii), and (i), as in the last case, $M = 0$.

We turn to (iii). If $\mathrm{D}u(x) = 0$, refer to (i), thus we can suppose $\mathrm{D}u(x) \neq 0$. Fix $\varepsilon > 0$. There exists $\delta_1 > 0$ such that for any $r \in (0, \delta_1)$,

$$\frac{\|T\|(\mathrm{B}(x, r))}{2r} \leqslant 2\theta, \tag{2.3}$$

with $\theta := \Theta^{1*}(\|T\|, x) \in (0, +\infty)$. Replace $\overrightarrow{T}$ with its $\|T\|$ approximately continuous representative at x. Denote by $E_{x,\varepsilon}$ the set

$$E_{x,\varepsilon} := \operatorname{set}_1 \|T\| \cap \left\{ y, |\overrightarrow{T}(y) - \overrightarrow{T}(x)| > \frac{\varepsilon}{2|\,\mathrm{D}u(x)|} \right\}$$

There exists $\delta_2 > 0$ which we can suppose less or equal to δ_1 such that whenever $r \in (0, \delta_2)$,

$$\frac{\|T\|(\mathrm{B}(x, r) \cap E_{x,\varepsilon})}{\|T\|(\mathrm{B}(x, r))} < \frac{\varepsilon}{4\theta|\,\mathrm{D}u(x)|} \tag{2.4}$$

For $S \in \mathcal{S}_{\leqslant}(T, x, \delta_2)$, the field $\overrightarrow{S}$ is equal $\|S\|$ almost everywhere to $\overrightarrow{T}$ and if S is "a curve joining x and y, with $\partial S = \delta_y - \delta_x$. As for $j = 1, \ldots, n$ the 1 form $z \mapsto \mathbf{e}_j^*$ is the differential of the 0-form

$z \mapsto z_j$, We can write:

$$y - x = (y_1 - x_1)\mathbf{e}_1 + \cdots + (y_n - x_n)\mathbf{e}_n = \sum_{j=1}^{n} \partial S(z \mapsto z_j)\mathbf{e}_j$$
$$= \sum_{j=1}^{n} S(z \mapsto \mathbf{e}_j^*)\mathbf{e}_j = \sum_{j=1}^{n} \int \langle \mathbf{e}_j^*, \overrightarrow{T} \rangle \, \mathrm{d}\|S\| \mathbf{e}_j = \int \overrightarrow{T} \, \mathrm{d}\|S\|.$$

The same identity with opposite sign is true if $\partial S = \delta_x - \delta_y$ instead. Denote by d_S the diameter of $\operatorname{spt} S$. By (2.3) and (2.4),

$$\begin{aligned}
|y - x - \mathbb{M}(S)\overrightarrow{T}(x)| &\leqslant \int |\overrightarrow{T}(x') - \overrightarrow{T}(x)| \, \mathrm{d}\|S\|(x') \\
&\leqslant 2\|S\| \left(E_{x,\varepsilon} \cap \mathrm{B}(x, d_S)\right) + \frac{\varepsilon}{2|\,\mathrm{D}u(x)|} \mathbb{M}(S) \\
&\leqslant \frac{\varepsilon \|T\|(\mathrm{B}(x, d_s))}{2\theta|\,\mathrm{D}u(x)|} + \frac{\varepsilon}{2|\,\mathrm{D}u(x)|} \mathbb{M}(S) \\
&\leqslant \frac{2\varepsilon\theta d_S}{\theta|\,\mathrm{D}u(x)|} + \frac{\varepsilon}{2|\,\mathrm{D}u(x)|} \mathbb{M}(S) \leqslant \frac{5\varepsilon}{2|\,\mathrm{D}u(x)|} \mathbb{M}(S), \qquad (2.5)
\end{aligned}$$

where in the second inequality, we used the fact that $|\overrightarrow{T}(x') - \overrightarrow{T}(x)| \leqslant 2$ for $\|T\|$ almost all x', in particular in the exceptionnal set $E_{x,\varepsilon}$, in the third inequality we used the fact that $\|S\| \leqslant \|T\|$ and (2.4) and in the last inequality, we used the fact that since S is indecomposable, $d_S \leqslant \mathbb{M}(S)$.

By differentiability of u along T at x, there exists $\delta_3 > 0$, such that for $y \in \mathrm{U}(0, \delta_3) \cap \operatorname{set}_1 \|T\|$ such that there exists $S \in \mathcal{S}_{\leqslant}(T, x, \delta_3)$ with $y \in \operatorname{spt} S$,

$$|u(y) - u(x) - \langle \mathrm{D}u(x), y - x \rangle| < \varepsilon |y - x|.$$

Let $\delta := \min\{\delta_1, \delta_2, \delta_3\}$ and choose $S \in \mathcal{S}_{\leqslant}(T, x, \delta)$. We can write S as $S^+ + S^-$ where S^+ and S^- are indecomposable, $\partial S^+ = \delta_{y^+} - \delta_x$ and $\partial S^- = \delta_x - \delta_{y^-}$, with $\mathbb{M}(S) = \mathbb{M}(S^+) + \mathbb{M}(S^-)$ and we have

$$\Theta_u(S) = \Theta_u(S^+) + \Theta_u(S^-) = u(y^+) - u(x) + u(x) - u(y^-).$$

Thus we can write

$$\begin{aligned}
&|\Theta_u(S) - \langle \mathrm{D}u(x), \overrightarrow{T}(x) \rangle \, \mathbb{M}(S)| \\
&\qquad \leqslant |u(y^+) - u(x) - \langle \mathrm{D}u(x), \overrightarrow{T}(x) \rangle \, \mathbb{M}(S^+)| + |u(x) - u(y^-) - \langle \mathrm{D}u(x), \overrightarrow{T}(x) \rangle \, \mathbb{M}(S^-)|
\end{aligned}$$

and study only the first term of the right hand side. We have

$$\begin{aligned}
&|u(y^+) - u(x) - \langle \mathrm{D}u(x), \overrightarrow{T}(x) \rangle \, \mathbb{M}(S^+)| \leqslant \\
&\qquad |u(y^+) - u(x) - \langle \mathrm{D}u(x), y - x \rangle| + |\langle \mathrm{D}u(x), y^+ - x \rangle - \langle \mathrm{D}u(x), \overrightarrow{T}(x) \rangle \, \mathbb{M}(S^+)| \\
&\qquad\qquad \leqslant \varepsilon |y^+ - x| + |\,\mathrm{D}u(x)| |y^+ - x - \mathbb{M}(S^+)\overrightarrow{T}(x)| \\
&\qquad\qquad\qquad \leqslant 4\varepsilon \, \mathbb{M}(S^+),
\end{aligned}$$

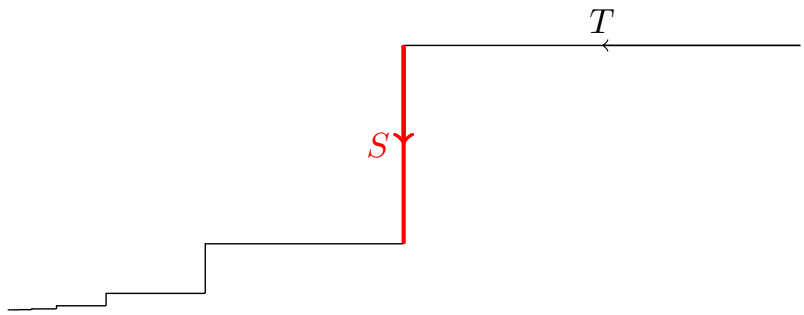

Figure 2.5: The piece S is not suitable for a differentiation basis of T at 0

by (2.5) applied to S^+. Doing the same with S^- and summing concludes the proof: there exists $\delta > 0$ such that for all $S \in \mathcal{S}_{\leqslant}(T, x, \delta)$,

$$|\Theta_u(S) - \mathbb{M}(S)\langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle| \leqslant \varepsilon\, \mathbb{M}(S)$$

and Θ_u is thus differentiable along T at x. □

If one assumes only approximate continuity of the tangent - as we just did - the assumption that the currents S used in the derivation are indecomposable is necessary:

Example 2.2.16. Consider the function $h : (x, y) \mapsto y$ and the current T associated to an infinite staircase with steps indexed by j, with height (y length) 3^{-j} and length (x-length) 2^{-j} symmetric in the x direction, converging at $(0, 0)$ (see figure 2.5). If one considers a sequence of subcurrents S_j composed of a very small "interval" (length 4^{-j}) around 0 and a vertical part of the step, there holds

$$\Theta_h(S_j) = 3^{-j}C + o(3^{-j}).$$

Thus $\lim_j \Theta_h(S_j)/\,\mathbb{M}(S_j) = C > 0$. However if one considers a sequence of indecomposable currents R_j touching 0 with mass going to 0, we will get by the above Theorem $\Theta_h(R_j)/\,\mathbb{M}(R_j) \to 0$.

An alternative restriction would be to bound the *regularity* of the pieces. This is actually how we proceed in higher dimension (see chapters 3,4 and 5) as indecomposability is not a practical tool in that setting.

2.3 Integration on currents of dimension 1

We first need an analogue to Cousin's Lemma in order to decompose a current of dimension 1 into small pieces.

2.3.1 Howard Cousin Lemma in dimension 1

Given a current $T \in \mathbb{I}_1(\mathbb{R}^n)$ and a gauge on $\mathrm{set}_1 \|T\|$, a **tagged family in** T is a finite collection $\mathcal{P}$ of pairs (S_j, x_j) for $j = 1, \ldots, p$, where

$$S_j \in \mathcal{S}_{\leqslant}(T),$$

$$x_j \in \operatorname{set}_1 \|T\| \cap \operatorname{spt} S_j,$$
$$S_j \text{ is indecomposable and}$$
$$\sum_{j=1}^{p} \|S_j\| \leqslant \|T\|.$$

The last condition allows the pieces to overlap in the case where T has multiplicity higher than 1. If T has multiplicity 1 almost everywhere, it imposes that the pieces be nonoverlapping. This differs from the subcurrents in higher dimension (see chapter 3). Such a tagged family is **subordinate to a decomposition** $T_1, T_2, \dots$ of T if there exists a partition of $\mathcal{P}$ indexed by k into families $\mathcal{P}_k$ in T_k.

A **gauge** on a set E is a nonnegative function δ such that $\{x \in E, \delta(x) = 0\}$ is countable. If δ is a gauge on a set $E \subseteq \operatorname{set}_1 \|T\|$, a **$\delta$-fine** tagged family in T is a tagged family as above satisfying

$$\forall (S, x) \in \mathcal{P},\ x \in E \text{ and } \operatorname{diam} \operatorname{spt} S < \delta(x).$$

Furthermore, given a nonnegative subadditive function G on $\mathcal{S}_{\leqslant}(T)$, and a positive real number τ, a tagged family $\mathcal{P}$ is (G, τ) **full** if $G(T - [\mathcal{P}]) < \tau$.

Lemma 2.3.1 (Howard-Cousin Lemma)**.** *Let T be an integral current of dimension 1 in $\mathbb{R}^n$. Let F be a subadditive continuous function on $\mathcal{S}_{\leqslant}(T)$. Given $\varepsilon > 0$ and δ a gauge on $\operatorname{set}_1 \|T\|$, for any decomposition $T_1, T_2, \dots,$ there exists a $(|F|, \varepsilon)$ full, δ-fine tagged family subordinate to this decomposition.*

Proof. Fix a decomposition of T. For each k choose $\gamma_k : [\,0, \mathbb{M}(T_k)\,] \to \mathbb{R}^n$ to parameterize T_k by arc-length, so that $T_k = \gamma_{k\#}[\![0, \mathbb{M}(T_k)]\!]$. Let $\delta_k := \delta \circ \gamma_k$, it is a gauge on $I_k := [\,0, \mathbb{M}(T_k)\,]$.

Since T is integral, there exists k_0 such that for all $k > k_0$, T_k is a cycle. Also $\mathbb{M}(T_k) \to 0$ as $k \to \infty$. Since F is continuous and subadditive, there exists k_ε such that

$$\left| F\left(T - \sum_{k=1}^{k_\varepsilon} T_k \right) \right| < \frac{\varepsilon}{2}.$$

For $k = 1, 2, \dots, k_0$, consider the interval $I_k[\,0, \mathbb{M}(T_k)\,]$, along with the gauge δ_k and the continuous additive function $\gamma_k^{\#} F$ on $\mathcal{S}_{\leqslant}(\mathbf{E}^1 \llcorner I_k)$ defined by $\gamma_k^{\#} F(\mathbf{E}^1 \llcorner [\,a, b\,]) = F(\gamma_{k\#}(\mathbf{E}^1 \llcorner [\,a, b\,])$, for $0 \leqslant a < b \leqslant \mathbb{M}(T)$. Note that it is enough to define $\gamma_k^{\#} F$ on indecomposable pieces of $\mathbf{E}^1 \llcorner I_k$ as all pieces are in this case a finite sum of disjoint indecomposable pieces. Apply lemma 2.1.15 to I_k, $\delta \circ \gamma_k$, $|\gamma_k^{\#} F|$ and $\varepsilon/(2k_0)$ to get a $\delta \circ \gamma_k$ fine $(\gamma_k^{\#}, \varepsilon/(2k_0))$ full tagged family $\mathcal{P}_k$ in I_k.

The collection $\gamma_{\#}\mathcal{P}_k$ defined by $\{(\gamma_{k\#} S, \gamma_k(x)), (S, x) \in \mathcal{P}_k\}$ is a δ fine tagged family in T_k (as γ_k has Lipschitz constant 1), which satisfies

$$|F(T_k - [\gamma_{\#}\mathcal{P}_k])| = (\gamma_k^{\#} F)(\mathbf{E}^1 \llcorner I_k - [\mathcal{P}_k])| < \frac{\varepsilon}{2k_0}.$$

Summing this inequality over $k = 1, 2, \dots, k_0$ yields

$$\left| F\left(\sum_{k=1}^{k_0} T_k - \left[\bigcup_{k=1}^{k_0} \gamma_{k,\#} \mathcal{P}_k \right] \right) \right| < \frac{\varepsilon}{2}.$$

And the collection $\mathcal{P} := \bigcup_{k=1}^{k_0} \gamma_k^{\#} \mathcal{P}_k$ is a therefore a tagged family in T which is δ fine and (F, ε) full. $\square$

2.3.2 AC_* functions on $\mathcal{S}_{\leqslant}(T)$

A function F on $\mathcal{S}_{\leqslant}(T)$ is AC_* if given a $\|T\|$ null set $E \subset \operatorname{set}_1 \|T\|$, for every $\varepsilon > 0$, there exists a gauge δ on E with

$$|F([\mathcal{P}])| < \varepsilon,$$

whenever $\mathcal{P}$ is a δ-fine tagged family in T. We say that a tagged family is **anchored** in a set E if for all (S, x) in this tagged family, $x \in E$. As the gage δ is defined only on E, here $\mathcal{P}$ is automatically anchored in E. The next two propositions are adapted from of [61, Theorems 3.6.6. and 3.6.7].

Proposition 2.3.2. *If F is a continuous additive function on $\mathcal{S}_{\leqslant}(T)$ which is AC_* and such that $\underline{\mathfrak{D}}_T F(x) \geqslant 0$ almost everywhere, then F is nonnegative, i.e. for all $S \in \mathcal{S}_{\leqslant}(T)$, $F(S) \geqslant 0$.*

Proof. It is enough to prove that $F(T) \geqslant 0$, indeed if T' is in $\mathcal{S}_{\leqslant}(T)$ the restriction of F to $\mathcal{S}_{\leqslant}(T')$ satisfies the hypothesis of the proposition. Let N be the set of points x such that $\underline{\mathfrak{D}}_T F(x) < 0$. For $\varepsilon > 0$, there exists a gauge δ_N on N such that $|F([\mathcal{P}])| < \varepsilon$ whenever $\mathcal{P}$ is a δ_N fine tagged family anchored in N. For each x at which $\underline{\mathfrak{D}}_T F(x) \geqslant 0$, there exists Δ_x such that for all $S \in \mathcal{S}_{\leqslant}(T, x, \Delta_x)$, $F(S) \geqslant -\varepsilon\, \mathbb{M}(S)/\, \mathbb{M}(T)$. Define a gauge δ on $\operatorname{set}_1 \|T\|$ by letting

$$\delta(x) = \begin{cases} \delta_N(x) \text{ if } x \in N, \\ \Delta_x \text{ otherwise.} \end{cases}$$

Using lemma 2.3.1, find a δ fine tagged family $\mathcal{P}$ in T with $|F(T - [\mathcal{P}])| < \varepsilon$. Let $\mathcal{P}_N$ be the subfamily of $\mathcal{P}$ consisting of all the elements anchored in N. Denoting $\mathcal{P}^*$ the complement of $\mathcal{P}$ yields:

$$F(T) \geqslant F([\mathcal{P}]) - F(T - [\mathcal{P}]) \geqslant F([\mathcal{P}^*]) + F([\mathcal{P}_N]) - \varepsilon \geqslant -3\varepsilon.$$

Since ε is arbitrary, $F(T) \geqslant 0$. $\square$

Proposition 2.3.3. *If a continuous additive function F is almost derivable everywhere in $\operatorname{set}_1 \|T\|$ except in a countable set E_T, then F is AC_*.*

Proof. Let N be a $\|T\|$ null set. For $\varepsilon > 0$, and $k = 1, 2, \ldots$, let U_k be a neighbourhood of N with $\|T\|(U_k) < 2^{-k}\varepsilon/k$. For $x \in N \backslash E_T$, choose a positive integer k_x and a positive Δ_x such that $\mathrm{U}(x, \Delta_x) \subseteq U_{k_x}$ and for all $S \in \mathcal{S}_{\leqslant}(T, x, \Delta_x)$, $|F(S)| \leqslant k_x\, \mathbb{M}(S)$. k_x and Δ_x exist by almost derivability of F at x. Define a gauge δ on N by

$$\delta(x) = \begin{cases} 0 \text{ if } x \in E_T, \\ \Delta_x \text{ if } x \in N \backslash E_T. \end{cases}$$

Given a δ fine tagged family $\mathcal{P}$ anchored in N, partition $\mathcal{P}$ into families $\mathcal{P}_k$ for $k = 1, 2, \ldots$ such that $(S, x) \in \mathcal{P}_k$ if and only if $k_x = k$ all but finitely many of these families are empty, there holds

$$|F([\mathcal{P}])| \leqslant \sum_{k=1}^{\infty} \sum_{(S,x) \in \mathcal{P}_k} |F(S)| \leqslant \sum_{k=1}^{\infty} k \sum_{(S,x) \in \mathcal{P}_k} \mathbb{M}(S) \leqslant \sum_{k=1}^{\infty} k \|T\|(U_k) < \varepsilon.$$

□

2.3.3 The HKP Integral on integral currents of dimension 1.

Definition 2.3.4. A function f defined $\|T\|$ almost everywhere on $\mathrm{set}_1 \|T\|$, is **Pfeffer 1 integrable** or HKP **integrable on** T if there exists a continuous additive function F on $\mathcal{S}_{\leqslant}(T)$ and for every $\varepsilon > 0$, there exists a gauge δ and a positive number τ such that whenever $\mathcal{P}$ is a δ-fine tagged family in T with $|F(T - [\mathcal{P}])| < \tau$, there holds:

$$|F(T) - \sigma(f, \mathcal{P})| < \varepsilon. \tag{2.6}$$

(Where $\sigma(f, \mathcal{P})$ denotes the Riemann sum $\sum_{(x,S)\in\mathcal{P}} f(x)\,\mathbb{M}(S)$.)

$F(T)$ is also the HKP integral of f on T and we sometimes denote it $(HKP)\int_T f$.

Question 2.3.5. *Is it equivalent to ask that each families be surbordinate to some decomposition? This is not clear because a piece of T can very well not be a piece of any decomposition (see figure 2.2).*

According to example 2.1.13, it is not sufficient to be integrable on all elements of one given decomposition to be integrable on the whole current. However, suppose f is integrable on each piece for two decompositions, is the integral the same?

We list the main basic properties of the integral, when the proofs are not given, they are similar to the ones in chapter 5 section 5.1 and the first section of this chapter:

Proposition 2.3.6 (Space of Pfeffer 1 integrable functions on T)**.** *The space of Pfeffer 1 integrable functions on T is a linear space and the integral: $f \mapsto I(f, T)$ is a linear operator. Furthermore, if $f \leqslant g$ and f and g are HKP integrable on T, then $(HKP)\int_T f \leqslant (HKP)\int_T g$.*

Proposition 2.3.7 (Cauchy criterion)**.** *f is HKP integrable on T if and only if there is a continuous nonnegative subadditive function G on $\mathcal{S}_{\leqslant}(T)$ and for every $\varepsilon > 0$, there exists a gauge δ and a positive τ such that for any two δ-fine (G, τ) full families $\mathcal{P}_1$ and $\mathcal{P}_2$,*

$$|\sigma(f, \mathcal{P}_1) - \sigma(f, \mathcal{P}_2)| < \varepsilon. \tag{2.7}$$

Proposition 2.3.8. *Let f be HKP integrable on the current $T \in \mathbb{I}_1(\mathbb{R}^n)$. For all $S \in \mathcal{S}_{\leqslant}(T)$, f is HKP integrable on S and $T - S$ and $I(f, S) + I(f, T - S) = I(f, T)$.*

Proof. Let G be a continuous nonnegative subadditive function on $\mathcal{S}_{\leqslant}(T)$ associated to the integrability of f on T. Fix $S \in \mathcal{S}_{\leqslant}(T)$, notice first that $G \llcorner \mathcal{S}_{\leqslant}(S)$ and $G \llcorner \mathcal{S}_{\leqslant}(T-S)$ are also nonnegative continuous and subadditive. Given $\varepsilon > 0$. Choose a gauge δ on $\mathrm{set}_1 \|T\|$ and a positive τ associated to $\varepsilon/2$ in the definition of integrability of f. $\delta \llcorner \mathrm{set}_1 \|T - S\|$ is a gauge on $\mathrm{set}_1 \|T - S\|$, so by lemma 2.3.1, there exists a δ fine $(G \llcorner \mathcal{S}_{\leqslant}(T - S), \tau/2)$ full tagged family $\mathcal{P}$ in $T - S$. Now given two δ fine $(G \llcorner \mathcal{S}_{\leqslant}(S), \tau/2)$ full families in S: $\mathcal{P}_1$ and $\mathcal{P}_2$, we define the concatenations $\mathcal{P} \cup \mathcal{P}_1$ and $\mathcal{P} \cup \mathcal{P}_2$. Since $[\mathcal{P}] \in \mathcal{S}_{\leqslant}(T - S)$ and $[\mathcal{P}_1], [\mathcal{P}_2] \in \mathcal{S}_{\leqslant}(S)$, we have $[\mathcal{P} \cup \mathcal{P}_1], [\mathcal{P} \cup \mathcal{P}_2] \in \mathcal{S}_{\leqslant}(T)$ so the concatenations are families in $\mathcal{S}_{\leqslant}(T)$. They are also δ fine and for $j = 1, 2$,

$$G(T - [\mathcal{P} \cup \mathcal{P}_j]) = G(T - S - [\mathcal{P}] + S - \mathcal{P}_j]) < G(T - S - [\mathcal{P}]) + G(S - \mathcal{P}_j) < \tau$$

by subadditivity of G and definition of $\mathcal{P}$ an $\mathcal{P}_j$. Therefore, by proposition 2.3.7

$$|\sigma(f, \mathcal{P} \cup \mathcal{P}_1) - \sigma(f, \mathcal{P} \cup \mathcal{P}_2)| = |\sigma(f, \mathcal{P}_1) - \sigma(f, \mathcal{P}_2)| < \varepsilon.$$

Thus, since ε, $\mathcal{P}_1$ and $\mathcal{P}_2$ are arbitrary one can apply the Cauchy Criterion proposition 2.3.7 to S, this proves that f is HKP integrable on S. By a similar argument f is HKP integrable on $T - S$. Therefore for $\varepsilon > 0$, choosing a gauge δ and a positive τ adapted to the integrability of f on T, S and $T - S$ at the same time, yields for δ-fine $(G, T - S, \tau/2)$ and $(G, S, \tau/2)$ full families $\mathcal{P}$ and $\mathcal{P}'$ in $T - S$ and S respectively

$$\begin{aligned} &|I(f, T) - (I(f, T - S) + I(f, S))| \\ &\qquad \leqslant |I(f, T) - \sigma(f, \mathcal{P} \cup \mathcal{P}')| + |I(f, T - S) - \sigma(f, \mathcal{P})| + |I(f, S) - \sigma(f, \mathcal{P}')| \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad < 3\varepsilon, \end{aligned}$$

because $\mathcal{P} \cup \mathcal{P}'$ is a δ fine (G, T, τ) full tagged family in T. As ε is as small as we want, this concludes the proof. □

This allows us to define a function F on $\mathcal{S}_{\leqslant}(T)$ by $S \mapsto I(f, S)$, called the **indefinite integral of f (on T)**.

Proposition 2.3.9. *The indefinite integral F of f defined above is additive and continuous on $\mathcal{S}_{\leqslant}(T)$.*

Proof. For the additivity: Let S_1 and S_2 be two pieces of T such that $S_1 + S_2 \in \mathcal{S}_{\leqslant}(T)$. Clearly S_1 and S_2 are pieces of $S_1 + S_2$, so it suffices to apply proposition 2.3.8 to see that $F(S_1) + F(S_2) = F(S_1 + S_2)$.

For the continuity: If $(S_j)_j$ is a sequence of pieces of T converging to $0 \in \mathcal{S}_w(T)$ with $\sup_j \mathbb{M}(\partial S_j) < \infty$. We want to show that $F(S_j) = I(f, S_j) \to 0$ as j tends to infinity. By additivity, it is equivalent to show that $I(f, T - S_j) \to I(f, T)$. For $\varepsilon > 0$ choose a gauge δ and a positive τ associated to the integrability of f on T. As seen above, for all j, δ and $\tau/2$, are associated to 2ε for the integrability of f on $T - S_j$. Let $\mathcal{P}$ be a δ fine $(G, T - S_j, \tau/2)$ full tagged family in $T - S_j$, it satisfies

$$|\sigma(f, \mathcal{P}) - F(T - S_j)| < 2\varepsilon.$$

By continuity of G, if j is large enough, we can suppose $G(T - [\mathcal{P}]) \leqslant G(T - S_j) + G(S_j - [\mathcal{P}]) < \tau/2 + \tau/2$, so $\mathcal{P}$ is (G, T, τ) full and

$$|\sigma(f, \mathcal{P}) - F(T)| < \varepsilon.$$

Therefore, for large enough j, $|F(T) - F(T - S_j)| < 3\varepsilon$ and we conclude that $F(S_j) \to 0$ as j tends to infinity. This proves that F is continuous on $\mathcal{S}_{\leqslant}(T)$. □

Theorem 2.3.10 (Saks-Henstock Lemma)**.** *f is HKP integrable on T if and only if there exists a continuous additive function F on $\mathcal{S}_{\leqslant}(T)$ satisfying: For all $\varepsilon > 0$, there exists a gauge δ on*

$\mathrm{set}_1 \|T\|$ *such that whenever* $\mathcal{P}$ *is a* δ*-fine tagged family in* T*:*

$$\sum_{(S,x)\in\mathcal{P}} |F(S) - f(x)\,\mathbb{M}(S)| < \varepsilon. \tag{2.8}$$

Proof. If the second condition in the statement is satisfied, it is straightforward to prove that f is HKP integrable on T, with integral $I(f,T) = F(T)$ and the "control function" $G = |F|$, indeed for $\varepsilon > 0$, if δ is a gauge on T associated to $\varepsilon/2$ in the statement of the theorem and $\mathcal{P}$ is a δ-fine, $(G, \varepsilon/2)$ full tagged family in T

$$|F(T) - \sigma(f,\mathcal{P})| \leqslant \left| F(T) - \sum_{(S,x)\in\mathcal{P}} F(S) \right| + \sum_{(S,x)\in\mathcal{P}} |F(S) - f(x)\,\mathbb{M}(S)| < \varepsilon.$$

Similarly, one proves that F is the indefinite integral of f.

Conversely, suppose f is Pfeffer 1 integrable on T. The proof is very similar to the case of Henstock Kurzweil integration. Suppose that f is HKP integrable on T and for $\varepsilon > 0$, fix a positive number $\tau < \varepsilon/4$ and a gauge δ on $\mathrm{set}_1 \|T\|$ such that whenever $\mathcal{P}$ is a δ fine $(|F|, \tau)$ full tagged family in T,

$$|\sigma(f,\mathcal{P}) - F(T)| < \frac{\varepsilon}{4}.$$

Let $\mathcal{P}$ be a δ fine tagged family in T, without any hypothesis on $|F(T - [\mathcal{P}])|$. Notice first that since $T - [\mathcal{P}]$ is an integral current, there exists a δ fine, $(|F|, \tau)$ full tagged family $\mathcal{Q}$ in $T - [\mathcal{P}]$, which implies that $\mathcal{P} \cup \mathcal{Q}$ is a δ fine $(|F|, \tau/2)$ full tagged family in T and

$$\sum_{(S,x)\in\mathcal{P}} |F(S) - f(x)\,\mathbb{M}(S)| \leqslant \sum_{(S,x)\in\mathcal{P}\cup\mathcal{Q}} |F(S) - f(x)\,\mathbb{M}(S)|. \tag{2.9}$$

Therefore, it is enough to prove that (2.8) holds for $(|F|, \tau)$ full families in T and we suppose that $\mathcal{P}$ is $(|F|, \tau)$ full. We can write $\mathcal{P} = \{(S_1, x_1), \ldots, (S_p, x_p)\}$ and, reordering, assume that for some $k_0 \leqslant p$, if $1 \leqslant j \leqslant k_0$, $|F(S_j) - f(x_j)\,\mathbb{M}(S_j)| \geqslant 0$ whereas for $k_0 + 1 \leqslant j \leqslant p$, $||F(S_j) - f(x_j)\,\mathbb{M}(S_j)| < 0$. For $j = 1, \ldots, p$ use the HKP integrability of f on S_j to define a δ fine, $(F \llcorner S_j, \tau/p)$ full tagged family $\mathcal{P}_j$ such that $|\sigma(f, \mathcal{P}_j) - F(S_j)| < \varepsilon/(2p)$. Consider the families

$$\begin{aligned} \mathcal{P}^+ &:= \{(S_1, x_1), \ldots, (S_{k_0}, x_{k_0})\} \cup \mathcal{P}_{k_0+1} \cup \cdots \cup \mathcal{P}_p, \text{ and} \\ \mathcal{P}^- &:= \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{k_0} \cup \{(S_1, x_1), \ldots, (S_{k_0}, x_{k_0})\}. \end{aligned}$$

$\mathcal{P}^+$ and $\mathcal{P}^-$ are both δ fine, $(|F|, \tau)$ full families in T, therefore (2.9) holds for both, furthermore there holds

$$\begin{aligned} \sum_{j=1}^{k_0} |F(S_j) - f(x_j)\,\mathbb{M}(S_j)| &= \left| \sum_{j=1}^{k_0} F(S_j) - f(x_j)\,\mathbb{M}(S_j) \right| \\ &\leqslant \left|\sigma(f,\mathcal{P}^+) - F(T)\right| + \sum_{j=k_0+1}^{p} |\sigma(f,\mathcal{P}_j) - F(S_j)| \leqslant \frac{\varepsilon}{4} + \frac{(p-k_0)\varepsilon}{2p} \end{aligned}$$

and symmetrically

$$\sum_{j=k_0+1}^{p} |F(S_j) - f(x_j)\,\mathbb{M}(S_j)| = \left| \sum_{j=k_0+1}^{p} F(S_j) - f(x_j)\,\mathbb{M}(S_j)) \right|$$
$$\leqslant \big|\sigma(f,\mathcal{P}^-) - F(T)\big| + \sum_{j=1}^{k_0} |\sigma(f,\mathcal{P}_j) - F(S_j)| \leqslant \frac{\varepsilon}{4} + \frac{k_0\varepsilon}{2p}.$$

Combining the two inequalities above yields

$$\sum_{j=1}^{p} |F(S_j) - f(x_j)\,\mathbb{M}(S_j)| < \varepsilon.$$

□

Proposition 2.3.11. *If f is HKP integrable on T, then given any decomposition $T = T_1 + T_2 + \dots$, f is HKP integrable on T_j for all j with $I(f,T) = \sum_j I(f,T_j)$. In fact $f \circ \gamma_j$ is HK integrable on $[\,0, \mathbb{M}(T_j)\,]$.*

Proof. The first part of the statement is clear. For the second part, it suffices to notice that $\sum_{j=1}^{k} T_j \to T$ as k goes to infinity with $\sup_k \mathbb{M}(\partial(\sum_{j=1}^{k} T_j)) \leqslant \mathbb{M}(\partial T)$ for all k. By continuity of the indefinite integral F of f on T, $\sum_{j=1}^{k} F(T_j) \to \sum_{j=1}^{\infty} F(T_j) = F(T)$. □

Proposition 2.3.12. *If f is defined almost everywhere in $\operatorname{set}_1 \|T\|$ and Lebesgue integrable with respect to $\|T\|$, then f is Pfeffer 1 integrable on T. As a consequence, the integral of a HKP integrable function does not depend on its values on a $\|T\|$ null set.*

Proof. Let f be Lebesgue integrable with respect to $\|T\|$, extend f by 0 so that it is defined everywhere in $\operatorname{spt} T$. Fix $\varepsilon > 0$. By the Vitali Caratheodory Theorem (see [63, 2.24], there exists two functions g and h with $g \leqslant f \leqslant h$ almost everywhere, $(\mathcal{L}) \int (h-g)\,\mathrm{d}\|T\| < \varepsilon$ and g and h are respectively upper and lower semi-continuous. By upper (respectively lower) semi continuity of g (respectively h), for each $x \in \operatorname{spt} T$, there exists $\delta(x) > 0$ such that whenever $y \in \operatorname{spt} T \cap \mathrm{U}(x, \delta(x))$,

$$g(y) \leqslant f(x) - \varepsilon \text{ (and respectively } h(y) \geqslant f(x) - \varepsilon).$$

(Note that $\delta(x)$ can be chosen for g and h at the same time for each x.) Suppose that $\mathcal{P}$ is a δ fine tagged family in T, with $\mathbb{M}(T - [\mathcal{P}]) < \varepsilon$,

$$(\mathcal{L}) \int g\,\mathrm{d}\|[\mathcal{P}]\| - \varepsilon\,\mathbb{M}([\mathcal{P}]) \leqslant \sigma(f,\mathcal{P}) \leqslant (\mathcal{L}) \int g\,\mathrm{d}\|[\mathcal{P}]\| + \varepsilon\,\mathbb{M}([\mathcal{P}]).$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two such families, there holds

$$|\sigma(f,\mathcal{P}_1) - \sigma(f,\mathcal{P}_2)| \leqslant (\mathcal{L}) \int (h-g)\,\mathrm{d}\|T\| + 2\varepsilon\,\mathbb{M}(T).$$

As ε is arbitrary, we can use proposition 2.3.7 to prove that f is Pfeffer 1 integrable. The HKP integral of f coincides with its Lebesgue integral. Indeed, choose a sequence $(\mathcal{P}_j)_j$ of δ fine families

in T with $\mathbb{M}(T - [\mathcal{P}_j]) \to 0$,

$$(\mathcal{L}) \int g \, \mathrm{d}\|[\mathcal{P}_j]\| \to (\mathcal{L}) \int g \, \mathrm{d}\|T\|$$

and the same holds for h.

In particular, if f is Pfeffer 1 integrable on T and g is equal to f, $\|T\|$ almost everywhere, then $g - f$ is equal to zero $\|T\|$ almost everywhere and is therefore Lebesgue integrable with respect to $\|T\|$, thus Pfeffer 1 integrable on T and $g = (g - f) + f$ is also Pfeffer 1 integrable with the same integral (and indefinite integral) as f. □

Proposition 2.3.13. *If f is Pfeffer 1 integrable on T, then its indefinite integral F is AC_*.*

Proof. Let N be a $\|T\|$ null set. By the Saks Henstock Lemma, for $\varepsilon > 0$, there exists a gauge δ on $\mathrm{set}_1 \|T\|$ such that

$$\sum_{(S,x)\in\mathcal{P}} |F(S) - f(x)\,\mathbb{M}(S)| < \varepsilon,$$

for every δ fine tagged family $\mathcal{P}$ in T. As F does not depend on the value of f on N, we can suppose that $f(x) = 0$ for all $x \in N$. If $\mathcal{P}$ is anchored in N, we have

$$|F([\mathcal{P}])| \leqslant \sum_{(S,x)\in\mathcal{P}} |F(S)| < \varepsilon,$$

which proves that F is AC_* on T. □

Proposition 2.3.14. *If f is HKP integrable, then it is $\|T\|$ measurable.*

Proof. Consider a decomposition of T: $T_1, T_2, \dots$ and a representative of f. f is HKP integrable on each $T_k =: \gamma_{k\#}(\mathbf{E}^1 \llcorner [0, \mathbb{M}(T_k)])$ and therefore, $f \circ \gamma_k$ is HK integrable on $[0, \mathbb{M}(T_k)]$ and thus Lebesgue measurable. Thus f is $\|T_k\|$ measurable, and also, $f_k := f \llcorner \mathrm{spt}\,\|T_k\|$ is $\|T\|$ measurable. Consider the function $\tilde{f} x \mapsto \sup_k f_k(x)$. $\tilde{f}$ is $\|T\|$ measurable as a pointwise supremum of measurable functions. The function $f - \tilde{f}$ is equal to zero at each point of $\mathrm{spt}\, T_1 \cup \mathrm{spt}\, T_2 \cup \dots \subseteq \mathrm{set}_1 \|T\|$. By definition of decomposition of currents, $\|T\| = \sum_{k=1}^{\infty} \|T_k\|$ and as indecomposable currents correspond to simple Lipschitz curve with integral multiplicity, for all k, $\mathrm{spt}\, T_k = \mathrm{set}_1 \|T_k\|$, therefore

$$\|T\|(\mathbb{R}^n \setminus \bigcup_{k=1}^{\infty} \mathrm{spt}\, T_k) = 0,$$

thus $f = \tilde{f}$, $\|T\|$ almost everywhere. This proves that f is $\|T\|$ measurable. □

Proposition 2.3.15. *Conversely to proposition 2.3.12, a function f is Lebesgue integrable with respect to $\|T\|$ if and only if f and $|f|$ are HKP integrable on T.*

Proof. Without loss of generality, we can suppose that f is nonnegative and HKP integrable, we also fix a representative of f with respect to $\|T\|$. It suffices to show that f is Lebesgue integrable with respect to $\|T\|$. For $k = 1, 2, \dots$, consider the function $f_k := f \mathbb{1}_{\{x, f(x) \leqslant k\}}$. Since f is $\|T\|$ measurable by proposition 2.3.14, f_k is $\|T\|$ measurable and bounded and thus Lebesgue integrable with respect to $\|T\|$ (which is a finite measure). the sequence f_k is nondecreasing and converges pointwise to f. Furthermore the sequence $\left((\mathcal{L}) \int f_k \, \mathrm{d}\|T\|\right)_k = \left((HKP) \int_T f_k\right)_k$ is bounded from

above by $(HKP)\int_T f$. By the Lebesgue Monotone Convergence Theorem, f is Lebesgue integrable with respect to $\|T\|$. $\square$

Theorem 2.3.16 (Monotone Convergence Theorem for the HKP integral.)**.** *Suppose that $(f_k)_{k=1,2,\dots}$ is a $\|T\|$ almost everywhere nondecreasing sequence of HKP integrable functions on T. If there exists $f : \mathrm{set}_1 \|T\| \to \mathbb{R}$ such that $f_k(x)$ converges to $f(x)$ $\|T\|$ almost everywhere and if furthermore, the sequence of integral: $(HKP)\int_T f_k$ for $k = 1, 2, \dots$ is bounded from above. Then f is HKP integrable on T with integral the limit of this sequence.*

We give a proof which does not rely on the measurability of f or on Lebesgue integration results, but relies only on gauge integration techniques.

Proof. Since the HKP integral of a function does not depend on its values in a $\|T\|$ null set, we can suppose that f_k converges pointwise to f everywhere and that for all $x \in \mathrm{set}_1 \|T\|$, the sequence $(f_k(x))_k$ is nondecreasing. Up to substracting f_1, we can also suppose that all the f_k are nonnegative (by linearity of the integral). For $k = 1, 2, \dots$, let F_k be the indefinite HKP integral of f_k on T it is nonnegative. Notice also that for all $S \in \mathcal{S}_{\leqslant}(T)$, and for $k \leqslant k'$, $F_k(S) \leqslant F_{k'}(S)$ by the last part of proposition 2.3.6. Since $F_k(T)$ is bounded from above, it converges to a limit $F(T)$, similarly we can define $F(S)$ for any $S \in \mathcal{S}_{\leqslant}(T)$ as both $(F_k(T - S))_k$ and $(F_k(S))$ are nondecreasing sequences bounded from above by $F(T) \geqslant F_k(S) + F_k(T - S)$. F is nonnegative. The function F on $\mathcal{S}_{\leqslant}(T)$ is also additive, indeed, suppose S, S' and $S + S'$ are in $\mathcal{S}_{\leqslant}(T)$, we have

$$F(S + S') = \lim_{k\to\infty} F_k(S + S') = \lim_{k\to\infty} (F_k(S) + F_k(S')) = F(S) + F(S').$$

Let us now prove that F is continuous. Fix sequence $(S_j)_j$ in $\mathcal{S}_{\leqslant}(T)$ with $\sup_j \mathbb{M}(\partial S_j) < \infty$ and $\mathbb{F}(S_j) \to 0$. For each k, the sequence $(F_k(S_j))_j$ goes to 0 as j goes to ∞ and similarly $F_k(T - S_j) \to F_k(T)$ as $j \to \infty$. Thus, since for all k and j, $F(T) \geqslant F(T - S_j) \geqslant F_k(T - S_j)$, given $\varepsilon > 0$ there exists k_0 such that for all $k \geqslant k_0$, $F_k(T) \geqslant F(T) - \varepsilon/2$.

There exists also j_0 such that for all $j \geqslant j_0$, $F_{k_0}(T - S_j) \geqslant F_{k_0}(T) - \varepsilon/2$. This implies that for all $j \geqslant j_0$ and all $k \geqslant k_0$,

$$F(T) \geqslant F(T - S_j) \geqslant F_k(T - S_j) \geqslant F_{k_0}(T - S_j) \geqslant F_{k_0}(T) - \frac{\varepsilon}{2} \geqslant F(T) - \varepsilon.$$

Thus F is nonnegative, additive and continuous on $\mathcal{S}_{\leqslant}(T)$. Since $F(S) \geqslant F_k(S)$ for all k, if $\mathcal{P}$ is an (F, τ) full tagged family in T for some $\tau > 0$, $\mathcal{P}$ is also (F_k, τ) full for all k.

From now on the argument follows the method of [53, 4.42]. Fix $\varepsilon > 0$, there exists l such that for all $k \geqslant l$, $F(T) - F_k(T) < \varepsilon/4$. For each $k \geqslant l$, fix a gauge δ'_k on $\mathrm{set}_1 \|T\|$ such that for all δ'_k fine, $(|F_k|, \varepsilon/4)$ full families $\mathcal{P}$ int T,

$$\sum_{(x,S)\in\mathcal{P}} |F_k(S) - f_k(x)\,\mathbb{M}(S)| < \frac{\varepsilon}{4^{k+2}}.$$

Define a new series of gauges $(\delta_k)_k$ such that for $x \in \mathrm{set}_1 \|T\|$,

$$\delta_k(x) := \min_{1\leqslant j\leqslant k} \delta'_k(x).$$

Note that δ_k is indeed a gauge, as a finite union of countable sets is countable. For each $x \in \operatorname{set}_1 \|T\|$, fix $l(x) \geqslant l$ so that $0 \leqslant f(x) - f_k(x) < \varepsilon/(4\,\mathbb{M}(T))$ whenever $k \geqslant l(x)$. And let $\delta(x) := \delta_{l(x)}(x)$ be a gauge on $\operatorname{set}_1 \|T\|$. To check that the zero set of δ is countable, notice that it is contained in the countable union of the zero sets of the gauges δ'_k.

Let $\mathcal{P}$ be a δ fine, $(F, \varepsilon/4)$ full tagged family in T. It is also $(F_k, \varepsilon/4)$ full, as we said above. Let l' be the maximum of the indices $l(x)$ over $(x, S) \in \mathcal{P}$. For $l \leqslant k \leqslant l'$ let $\mathcal{P}_k$ be the subfamily of $\mathcal{P}$ consisting of all the $(x, S) \in \mathcal{P}$ with $l(x) = k$. We can write

$$\begin{aligned}\sigma(f, \mathcal{P}) - F(T) &= \sum_{k=l}^{l'} \sigma(f, \mathcal{P}_k) - F(T) \\ &= \sum_{k=l}^{l'} (\sigma(f, \mathcal{P}_k) - \sigma(f_j, \mathcal{P}_k)) + \sum_{k=l}^{l'} (\sigma(f_k, \mathcal{P}_k) - F_k([\mathcal{P}_k])) + \sum_{k=l}^{l'} F_k([\mathcal{P}_k]) - F(T).\end{aligned}$$

To control the first term, by the choice of $l(x)$, for all k we have

$$0 \leqslant \sigma(f, \mathcal{P}_k) - \sigma(f_k \mathcal{P}_k) < \frac{\mathbb{M}([\mathcal{P}_k])}{\mathbb{M}(T)} \frac{\varepsilon}{4}.$$

Sum over $k = l, \ldots, l'$ to obtain

$$0 \leqslant \sum_{k=l}^{l'} \sigma(f, \mathcal{P}_k) - \sigma(f_k \mathcal{P}_k) < \frac{\mathbb{M}([\mathcal{P}])}{\mathbb{M}(T)} \frac{\varepsilon}{4} \leqslant \frac{\varepsilon}{4}.$$

For the second term, for any k by the Saks Henstock Lemma applied to f_k and $\mathcal{P}_k$ we have

$$|\sigma(f_k, \mathcal{P}_k) - F_k([\mathcal{P}_k])| \leqslant \frac{\varepsilon}{4^{k+2}}.$$

Which can be summed to get

$$\sum_{k=l}^{l'} |\sigma(f_k, \mathcal{P}_k) - F_k([\mathcal{P}_k])| \leqslant \frac{\varepsilon}{4}.$$

Finally, for the third term, notice that for all $k \geqslant l$

$$F_k([\mathcal{P}_k]) \geqslant F_l([\mathcal{P}_k]).$$

Summing over k yields

$$F(T) \geqslant F([\mathcal{P}]) \geqslant \sum_{k=l}^{l'} F_k([\mathcal{P}_k]) \geqslant F_l([\mathcal{P}]) \geqslant F_l(T) - \frac{\varepsilon}{4} \geqslant F(T) - \varepsilon/2,$$

as $\mathcal{P}$ is $(F_l, \varepsilon/4)$ full in T. Combining the three above estimates we get

$$|\sigma(f, \mathcal{P}) - F(T)| < \varepsilon,$$

which proves that f has HKP integral $F(T)$ on T. By the same reasonning one can prove that

f is HKP integrable on $S \in \mathcal{S}_{\leqslant}(T)$ with integral $F(S)$, thus F is the indefinite integral of f on T. □

2.3.4 Fundamental Theorem of Integration

Proposition 2.3.17. *If F is a continuous additive function on $\mathcal{S}_{\leqslant}(T)$ which is AC_* and derivable $\|T\|$ almost everywhere, then $x \mapsto \mathfrak{D}_T F(x)$ is HKP integrable on T with indefinite integral F.*

Proof. Let N be the set of non derivability points of F in $\operatorname{set}_1 \|T\|$. Let f be the function defined on $\operatorname{set}_1 \|T\|$ by $f(x) = 0$ if $x \in N$ and $f(x) = \mathfrak{D}_T F(x)$ otherwise. For $\varepsilon > 0$, let δ be a gauge on $\operatorname{set}_1 \|T\|$ such that whenever $\mathcal{P}$ is a δ-fine tagged family in T anchored in N, $|F([\mathcal{P}])| < \varepsilon$ and for all $x \in \operatorname{set}_1 \|T\| \backslash N$, $\delta(x)$ is a positive number such that for all $S \in \mathcal{S}_{\leqslant}(T, x, \delta(x))$

$$|F(S) - f(x)\,\mathbb{M}(S)| < \varepsilon\,\mathbb{M}(S).$$

If $\mathcal{P}$ is a δ-fine tagged family in T with $|F(T - [\mathcal{P}])| < \varepsilon$, let $\mathcal{P}_N$ be the subfamily of $\mathcal{P}$ containing all the pairs $(S, x) \in \mathcal{P}$ with $x \in N$. There holds

$$|F(T) - \sigma(f, \mathcal{P})| \leqslant |F(T - [\mathcal{P}])| + |F([\mathcal{P}_N])| + \sum_{(S,x)\in\mathcal{P}, x\notin N} |F(S) - f(x)\,\mathbb{M}(S)| < 3\varepsilon$$

Thus f is HKP integrable in T with $I(f, T) = F(T)$. Since $F|_{\mathcal{S}_{\leqslant}(S)}$ satisfies the hypothesis of the theorem for any $S \in \mathcal{S}_{\leqslant}(T)$, $I(f, S) = F(S)$ and F is the indefinite integral of $\mathfrak{D}_T F$ on T. □

Proposition 2.3.18. *If u is a continuous function on $\operatorname{spt} T$ which is differentiable $\|T\|$ almost everywhere and Θ_u is AC_*, then $x \mapsto \mathfrak{D}_T \Theta_u(x) = \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle$ is HKP integrable on T with indefinite integral Θ_u.*

Proof. Using proposition 2.3.17 it suffices to prove that the set

$$\{x \in \operatorname{set}_1 \|T\|, \Theta_u \text{ is not derivable at } x, \text{ or } \mathfrak{D}_T \Theta_u(x) \neq \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle\}$$

is $\|T\|$ negligible. As u is differentiable $\|T\|$ almost everywhere, by theorem 2.2.14 (iii) it suffices to prove that

$$N := \{x \in \operatorname{set}_1 \|T\|, \overrightarrow{T} \text{ has a } \|T\| \text{ approximately continuous representative at } x\}$$

is $\|T\|$ negligible.

Claim 1. The function $x \mapsto \overrightarrow{T}$ is $\|T\|$ approximately continuous $\|T\|$ almost everywhere, i.e. for $\|T\|$ almost every x, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Theta^{m*}(\|T\| \llcorner \{y, |\overrightarrow{T}(x) - \overrightarrow{T}(y)| \geqslant \delta\}, x) < \varepsilon.$$

Proof. The measure $\|T\|$ in $\mathbb{R}^n$ is finite and Borel regular, therefore the Besicovitch Covering Theorem (see [50, 2.7]) holds for $\|T\|$ (or in the words of [26], $\mathbb{R}^n$ is directionally limited (see [26, 2.8.9]) and by [26, 2.8.18], the pairs $(x, \mathrm{U}(x, r))$ for $x \in \mathbb{R}^n$, $r > 0$ form a Vitali relation for $\|T\|$).

Furthermore, the function $\overrightarrow{T} : \mathrm{set}_1 \|T\| \to \Lambda_1(\mathbb{R}^n)$ is $\|T\|$ measurable. By [26, 2.9.13], $\overrightarrow{T}$ is $\|T\|$ approximately continuous $\|T\|$ almost everywhere. ■

□

Denote by $good(T)$ the set of points in $\mathrm{set}_1 \|T\|$ which belong to the support of an indecomposable piece of T, i.e. the set of points $x \in \mathrm{set}_1 \|T\|$ such that there exists $\delta > 0$ (depending on x) such that $\mathcal{S}_{\leqslant}(T, x, \delta) \neq \emptyset$.

Theorem 2.3.19 (Fundamental theorem of integration)**.** *Let T be a fixed integral current of dimension 1 in $\mathbb{R}^n$, u a continuous function on $\mathrm{spt}\, T$. Suppose that u is pointwise Lipschitz at all but countably many points in $\mathrm{set}_1 \|T\| \cap good(T)$ and that u is differentiable $\|T\|$ almost everywhere, then $x \mapsto \langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle$ is HKP integrable on T and*

$$(\partial T)(u) = (HKP) \int_T \langle \mathrm{D}u, \overrightarrow{T}\rangle.$$

Proof. Let Θ_u be the function on $\mathcal{S}_{\leqslant}(T)$ associated to the variations of u. By theorem 2.2.14(ii), Θ_u is almost derivable at all points of $\mathrm{set}_1 \|T\|$ except for a countable set. By proposition 2.3.3, Θ_u is AC_*. By theorem 2.2.14 (iii), Θ_u is derivable $\|T\|$ almost everywhere along T with derivative equal to $\langle \mathrm{D}u(x), \overrightarrow{T}(x)\rangle$. Use propositions 2.3.17 and 2.3.18 to conclude. □

Chapter 3

Subcurrents and derivation.

3.1 Currents and subcurrents

This section is devoted to defining the elements which we will use in Riemann sums later: subcurrents. These are comparable to the subsets of finite perimeter used in the Pfeffer Integral to decompose bounded sets of finite perimeter. Here we want to decompose an integral current instead. We outline in particular the key properties of sets of finite perimeter which are not shared by subcurrents. We refer the reader to Appendix B for the basic definitions and notations concerning integral currents.

3.1.1 Subcurrents: definition and properties

To "cut" an integral current into "pieces" which are still current is not trivial: indeed the restriction of an integral current to a measurable set is not necessarily an integral current. In the same way as intersecting a set of finite perimeter with a measurable set. However "almost all restrictions" are good and yield a space of objects which we call subcurrents. We make this precise in this subsection.

Proposition 3.1.1 (Restrictions of rectifiable currents to sets)**.** *For a rectifiable current $T \in \mathcal{R}_m(\mathbb{R}^n)$ and $\|T\|$ measurable sets A and B in $\mathbb{R}^n$,*

(i) $T \llcorner A = \|T\| \llcorner A \wedge \overrightarrow{T}$ *is in* $\mathcal{R}_m(\mathbb{R}^n)$ *and if* $S := T \llcorner A$ *then* $\|S\| = \|T\| \llcorner A$.

(ii) $(T \llcorner A) \llcorner B = T \llcorner (A \cap B)$.

(iii) $T \llcorner (A \cup B)$ *and* $T \llcorner (A \cap B)$ *are also in* $\mathcal{R}_m(\mathbb{R}^n)$ *and*

$$T \llcorner (A \cup B) = T \llcorner A + T \llcorner B - T \llcorner (A \cap B).$$

Proof. The statements in (i) follow from the definition of the operator $\llcorner$. To prove (ii), notice that $(T \llcorner A) \llcorner B = (T \llcorner \mathbb{1}_A) \llcorner \mathbb{1}_B = T \llcorner (\mathbb{1}_A \mathbb{1}_B)$ and that $\mathbb{1}_A \mathbb{1}_B = \mathbb{1}_{A \cap B}$. For (iii) use the identity:

$$\mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A \cap B}.$$

□

Definition 3.1.2. Let $T \in \mathbb{I}_m(\mathbb{R}^n)$. A current $S \in \mathbb{I}_m(\mathbb{R}^n)$ is a **subcurrent of** T if there exists a Borel set A such that $S = T \llcorner A$. We denote by $S \sqsubset T$ the relation S **is a subcurrent of** T. Two subcurrents of T are said to be **non-overlapping** if their carrying measures are mutually singular.

Proposition 3.1.3. *The following statement concerning subcurrents hold*

(i) *If $S \sqsubset T$, then $\mathbb{M}(S) \leqslant \mathbb{M}(T)$, the inequality is strict if $S \neq T$.*

(ii) *The relation $\sqsubset$ is transitive.*

(iii) *If $S \sqsubset T$, then $\operatorname{spt} S \subseteq \operatorname{spt} T$.*

(iv) *If $S \sqsubset T$ and $A \subseteq \mathbb{R}^n$ is $\|T\|$ measurable with $T \llcorner A \in \mathbb{I}_m(\mathbb{R}^n)$ ($T \llcorner A \sqsubset T$), then $S \llcorner A \sqsubset T \llcorner A$.*

(v) *If $S \sqsubset T$ with $S = T \llcorner B$ and $B \subseteq A$, where $A \subseteq \mathbb{R}^n$ is $\|T\|$ measurable and $T \llcorner A \sqsubset T$ then $S \sqsubset T \llcorner A$.*

(vi) *An integral m-current S is a subcurrent of T if and only if there holds*

$$\|S\| \perp \|T - S\|. \tag{3.1}$$

(vii) *If S and S' are nonoverlapping subcurrents of T, $S + S' \sqsubset T$.*

(viii) *If $T \in \mathbb{I}_m(\mathbb{R}^n)$ and $T' \in \mathbb{I}_m(\mathbb{R}^{n'})$ are integral currents and ϕ is lipschitz and bijective from $\operatorname{spt} T$ to its image with $T' = \phi_\# T$, then if A is $\|T\|$ measurable and $S := T \llcorner A$ is a subcurrent of T, then $S' := \phi_\# S$ is a subcurrent of T' with $S' = T' \llcorner \phi(A \cap \operatorname{spt} T)$.*

(ix) *If $T' \sqsubset T$ then $\operatorname{set}_m \|T'\| \subseteq \operatorname{set}_m \|T\|$, furthermore, if S and S' are subcurrents of T with $S + S' \sqsubset T$, then $\operatorname{set}_m \|S + S'\| = \operatorname{set}_m \|S\| \cup \operatorname{set}_m \|S'\|$.*

(x) *(Slicing) If $S \sqsubset T$ and $f : \mathbb{R}^n \to \mathbb{R}^k$, with $k \leqslant m$ is Lipschitz. Then for $\mathcal{L}^k$ almost all $y \in \mathbb{R}^k$*

$$\langle S, f, y\rangle \sqsubset \langle T, f, y\rangle$$

Proof. To prove (i) $\mathbb{M}(S) = \|T \llcorner A\|(\mathbb{R}^n) = \|T\|(A) \leqslant \|T\|(\mathbb{R}^n) = \mathbb{M}(T)$ and if $S = T \llcorner A \neq T$ where A is $\|T\|$ measurable, then $T - S = T \llcorner A^c \neq 0$ thus $\|T\|(A^c) > 0$ and $\mathbb{M}(T) = \|T\|(\mathbb{R}^n) > \|T\|(A) = \mathbb{M}(S)$. To prove (ii), suppose that $T, T', T" \in \mathbb{I}_m(\mathbb{R}^n)$ with $T' \sqsubset T$ and $T" \sqsubset T'$, then there exists a $\|T\|$ measurable A and B in $\mathbb{R}^n$ such that $T' = T \llcorner A$ and $T" = T' \llcorner B$. Since $(T \llcorner A) \llcorner B = T \llcorner (A \cap B)$, $T"$ is a subcurrent of T and (ii) holds. For (iii) There exists a $\|T\|$ measurable $A \subseteq \mathbb{R}^n$ with $\operatorname{spt} S = \operatorname{spt}(T \llcorner \mathbb{1}_A) \subseteq \operatorname{spt} T \cap \operatorname{cl}(\{x, |\mathbb{1}_{\operatorname{cl} A}(x)| > 0\})$. If T, S and A are as in (iv), then there exists B measurable such that $S = T \llcorner B$, thus $S \llcorner A = (T \llcorner B) \llcorner A = (T \llcorner A) \llcorner B$ and conclude using the fact that $T \llcorner A$ are integral. (v) holds since in that setting $S = S \llcorner A$ is integral. To prove (vi), suppose first that S is a subcurrent of T, S is integral and there exists a $\|T\|$ measurable $A \subseteq \mathbb{R}^n$ such that $S = T \llcorner A$. Then $\|S\| = \|T\| \llcorner A$ (by proposition 3.1.1) and since $T - S = T \llcorner A^c$, $\|T - S\| = \|T\| \llcorner A^c$ and (3.1) holds. Conversely, if S is integral and (3.1) holds, let us show that there exists a measurable A such that $S = T \llcorner A$. First since T and

S is in $\mathbb{I}_m(\mathbb{R}^n)$, there exists θ, M and $\overrightarrow{T}$ as well θ', M' and $\overrightarrow{S}$ as in (2) with $T = \theta\mathcal{H}^m \llcorner M \wedge \overrightarrow{S}$ and $S = \theta'\mathcal{H}^m \llcorner M' \wedge \overrightarrow{S}$. Thus

$$\begin{aligned} \|S\| &= \theta'\mathcal{H}^m \llcorner M' \\ \|T - S\| &= \theta\mathcal{H}^m \llcorner (M\backslash M') + \theta'\mathcal{H}^m \llcorner (M'\backslash M) + |\theta\overrightarrow{T} - \theta'\overrightarrow{S}|\mathcal{H}^m \llcorner (M \cap M'). \end{aligned}$$

Applying (3.1) and noting that θ' is positive $\mathcal{H}^m \llcorner M'$ almost everywhere yields

$$\begin{aligned} \theta' &= 0, \quad \mathcal{H}^m \text{ almost everywhere on } M'\backslash M \text{ and} \\ |\theta\overrightarrow{T} - \theta'\overrightarrow{S}| &= 0, \quad \mathcal{H}^m \text{ almost everywhere on } M \cap M'. \end{aligned}$$

This implies that $\theta\overrightarrow{T} = \theta'\overrightarrow{S}$, $\mathcal{H}^m \llcorner (M \cap M')$ almost everywhere and since θ and θ' are positive and $\overrightarrow{T}$ and $\overrightarrow{T}'$ have unit length, $\theta = \theta'$ and $\overrightarrow{T} = \overrightarrow{S}$ $\mathcal{H}^m \llcorner (M \cap M')$ almost everywhere. Thus, up to redefinition on a set of zero $\mathcal{H}^m$ measure, $M' \subseteq M$ and $S = \theta\mathcal{H}^m \llcorner M' \wedge \overrightarrow{T}$. This translates as $S = T \llcorner M'$ and since S is integral, $S \sqsubset T$. For (vii), let $S := T \llcorner A$ and $S' := T \llcorner A'$. Since S and S' are non overlapping, there holds $\|T \llcorner A\| \perp \|T \llcorner A'\|$, we can therefore suppose $A \cap A' = \emptyset$. This implies $S + S' = T \llcorner A + T \llcorner A' = T \llcorner (A \cup A')$, and $S + S'$ is therefore a subcurrent of T.

To prove (viii), remark that ϕ is Lipschitz on $\operatorname{spt} T$ and therefore also on $\operatorname{spt} S$, thus $S' := \phi_\# S$ is also an integral current of dimension m supported in $\phi(\operatorname{spt} S)$ by [26, 4.1.14]. By the last paragraph of the same section

$$\|\phi_\#(T - S)\| \leqslant (\operatorname{Lip}\phi)^m \, \phi_*\|T - S\| \text{ and } \|\phi_\# S\| \leqslant (\operatorname{Lip}\phi)^m \, \phi_*\|S\|, \tag{3.2}$$

where for a measure μ, $\phi_*\mu$ denotes the image measure of μ by ϕ. Using the fact that ϕ is bijective, we get for all $\|T\|$ measurable E in $\phi(\operatorname{spt} T)$:

$$\phi_*\|S\|(\phi(E)) = \|S\|(E) \text{ and } \phi_*\|T - S\|(\phi(E)) = \|T - S\|(E).$$

As $\|S\| \perp \|T - S\|$, there exists a $\|T\|$ measurable A, such that $\|S\|(A) = \|S\|(\mathbb{R}^n)$ and $\|T - S\|(A) = 0$ and similarly $\|T - S\|(A^c) = \|T - S\|(\mathbb{R}^n)$ and $\|S\|(A^c) = 0$. This implies

$$\begin{aligned} \phi_*\|S\|(\phi(A)) = \|S\|(A) = \|S\|(\mathbb{R}^n) &= \phi_*\|S\|(\mathbb{R}^{n'}), \\ &\text{and} \\ \phi_*\|T - S\|(\phi(A)) &= 0 \end{aligned}$$

As well as

$$\begin{aligned} \phi_*\|T - S\|(\phi(A^c)) = \|T - S\|(A^c) = \|T - S\|(\mathbb{R}^n) &= \phi_*\|T - S\|(\mathbb{R}^{n'}), \\ &\text{and} \\ \phi_*\|S\|(\phi(A^c)) &= 0, \end{aligned}$$

thus $\phi_*\|T - S\| \perp \phi_*\|S\|$, which combined with (3.2) implies

$$\|T' - S'\| = \|\phi_\#(T - S)\| \perp \|S'\|$$

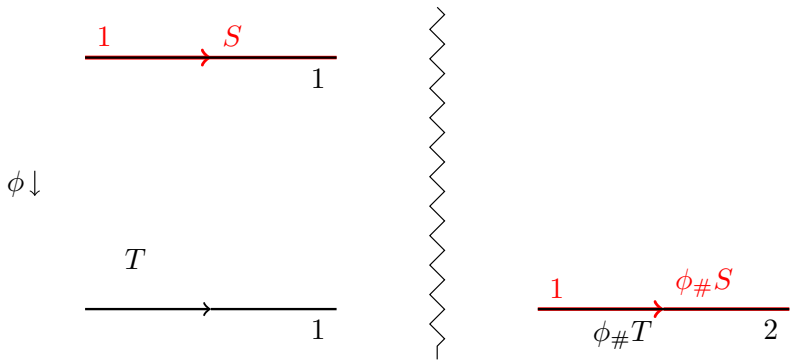

Figure 3.1: S is a subcurrent of T but $\phi_\# S$ is not a subcurrent of $\phi_\# T$.

since T' and S' are integral currents of dimension m in $\mathbb{R}^{n'}$, we get $S' \sqsubset T'$. For (ix), let $x \in \operatorname{set}_m T'$, with $T' = T \llcorner A$. $\|T\| \llcorner A$ has positive upper m-density at x and $\|T\|(B) \geqslant \|T\|(A \cap B) = \|T \llcorner A\|(B)$ for each $\|T\|$ measurable B so $x \in \operatorname{set}_m T$. Finally to prove (x), let $T = \theta \mathcal{H}^m \llcorner M \wedge \overrightarrow{T}$, S and f be as in the statement. Notice that $S = T \llcorner \mu$, where μ is $\|T\|$ measurable and takes values in $\{0, 1\}$. By B.5 for almost all $y \in \mathbb{R}^k$ there holds at the same time

$$\begin{aligned} \|\langle T - S, f, y\rangle\| = &\quad \theta(1-\mu)\mathcal{H}^{m-k} \llcorner (M \cap f^{-1}(\{y\})) \\ &\text{and} \\ \|\langle S, f, y\rangle\| = &\quad \theta\mu\mathcal{H}^{m-k} \llcorner (M \cap f^{-1}(\{y\})). \end{aligned}$$

And these two currents are integral. Thus $\|\langle T - S, f, y\rangle\| \perp \|\langle S, f, y\rangle\|$. Since $\langle T - S, f, y\rangle = \langle T, f, y\rangle - \langle S, f, y\rangle$, we have indeed $\langle S, f, y\rangle \sqsubset \langle T, f, y\rangle$ for almost all $y \in \mathbb{R}^k$. $\square$

Remark 3.1.4. In (viii), the assumption that ϕ is bijective is necessary. Indeed, consider the integral current T of dimension 1 in $\mathbb{R}^2$ given by the sum of the currents associated to the segments $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ with multiplicity 1 and oriented towards positive first coordinate and the map $\phi : \mathbb{R}^2 \to \mathbb{R}^2, (x_1, x_2) \mapsto (x_1, 0)$ (ϕ is Lipschitz and proper when restricted to $\operatorname{spt} T$). Let $S = T \llcorner ([0,1] \times \{0\})$, S is a subcurrent of T, but we have $\phi_\# S = S$ and $\phi_\# T = 2S$, so $\phi_\# S$ is not a subcurrent of $\phi_\# T$ (see figure 3.1). This is also a problem when considering pieces, for instance if one considers two segments with opposite orientation, which cancel out when pushed forward by ϕ.

Proposition 3.1.5. *Given a sequence $(S_j)_j$ of m-dimensional integral currents converging to $S \in \mathbb{I}_m(\mathbb{R}^n)$ in the flat norm, given $x \in \mathbb{R}^n$, for $\mathcal{L}^1$ almost all $r > 0$, there exists a subsequence $(S_{j_{k,r}})_k$ with $S_{j_{k,r}} \llcorner \mathrm{U}(x,r) \to S \llcorner \mathrm{U}(x,r)$ in the flat norm as $k \to +\infty$.*

Proof. Without loss of generality, we can suppose that $S = 0$. There exists two sequences of integral currents $(R_j)_j$ in $\mathbb{I}_m(\mathbb{R}^n)$ and $(Q_j)_j$ in $\mathbb{I}_{m+1}(\mathbb{R}^n)$ (take $Q_j = 0$ if $m = n$) with $\mathbb{M}(R_j) + \mathbb{M}(Q_j) \to 0$ and for all j

$$S_j = R_j + \partial Q_j.$$

Restricting to $\mathrm{U}(x,r)$ yields

$$S_j \llcorner \mathrm{U}(x,r) = R_j \llcorner \mathrm{U}(x,r) + (\partial Q_j) \llcorner \mathrm{U}(x,r).$$

For all j, for $\mathcal{L}^1$ almost all $r \in]0, +\infty[$, $\langle Q_j, \mathrm{dist}(x, \cdot), r\rangle$ is integral and:

$$S_j \llcorner \mathrm{U}(x,r) = R_j \llcorner \mathrm{U}(x,r) - \langle Q_j, \mathrm{dist}(x,\cdot), r\rangle + \partial(Q_j \llcorner \mathrm{U}(x,r)),$$

The first two terms on the right hand side go to zero as j goes to ∞. Furthermore, there holds

$$\int_0^{+\infty} \mathbb{M}(\langle Q_j, \mathrm{dist}(x,\cdot), r\rangle)\,\mathrm{d}r \leqslant \mathbb{M}(Q_j)$$

and since $\mathbb{M}(Q_j)$ goes to 0 as j goes to ∞, for almost all $r > 0$, $\mathbb{M}(\langle Q_j, \mathrm{dist}(x,\cdot), r\rangle)$ converges to 0 up to extraction of a subsequence. The subsequence choice depends on r. The corresponding subsequence $S_{j_{k,r}} \llcorner \mathrm{U}(x,r)$ thus converges to zero in the flat norm. □

3.1.2 Topology on the space of subcurrents of T

Recall that in [19], the authors define a topology $\mathfrak{F}_{X,m}$ on the space $\mathbb{N}_m(X)$ of normal currents of dimension m supported in a set $X \subset \mathbb{R}^n$. This topology is not metrizable, though it is hereditarily sequential and it coincides with the flat norm topology on every space of the form:

$$\mathbb{N}_{m,c}(X) := \mathbb{N}_m(X) \cap \{S, \mathbb{N}(S) < c\}.$$

Consider the space $\mathcal{S}(T)$ of subcurrents of a given integral current $T \in \mathbb{I}_m(\mathbb{R}^n)$. Clearly all the currents $S \in \mathcal{S}(T)$ are in $\mathbb{N}_m(\mathrm{spt}\,T)$.

Theorem 3.1.6. *For $c > 0$, the space $\mathcal{S}(T) \cap \mathbb{N}_{m,c}(\mathrm{spt}\,T)$ is compact in $(\mathbb{N}_m(\mathrm{spt}\,T), \mathfrak{F}_{\mathrm{spt}\,T,m})$. Furthermore, the mass operator $\mathbb{M}$ is continuous on $\mathcal{S}(T)$ and convergence in the flat norm is equivalent to convergence in mass.*

Since $\mathbb{N}_m(\mathrm{spt}\,T)$ is sequential and $\mathrm{spt}\,T$ is compact, for the first claim, it suffices to prove that any sequence in $\mathcal{S}(T)$ with bounded normal mass has a subsequence which converges in $\mathcal{S}(T)$.

Proof. Let us start with currents of dimension 0. $T \in \mathbb{I}_0(\mathbb{R}^n)$ is of the form $T = \sum_{s=1}^p \theta_s \delta_{x_s}$, where the x_s are points in $\mathbb{R}^n$ and the θ_s are integers. The space of subcurrents of T is identified with $\mathcal{P}(\{1, \dots, s\})$ which is finite. Therefore any sequence in $\mathcal{S}(T)$ has limit points in $\mathcal{S}(T)$. For the convergence in mass, note that any converging sequence is eventually constant.

In the general case, using the Compactness Theorem [26, 4.2.17], one gets that $S_j \to S$ in the flat norm for some $S \in \mathbb{I}_m(\mathbb{R}^n)$. A slightly stronger result [19, Theorem 4.2] states that if $K \subseteq \mathbb{R}^n$ is compact, the space $(\mathbb{N}_{m,c}(K), \mathfrak{F})$ is compact. Using this result with $K = \mathrm{spt}\,T$, one can assume that $(S_j)_j$ converges in the flat norm to $S \in \mathbb{I}_m(\mathrm{spt}\,T)$.

By proposition 3.1.5, for all $x \in \mathbb{R}^n$ for almost all $r > 0$, there exists a subsequence $(S_{j_{k,r}})_k$, with $S_{j_{k,r}} \llcorner \mathrm{U}(x,r) \to S \llcorner \mathrm{U}(x,r)$ in the flat norm as k goes to ∞. By lower semi-continuity of the mass with respect to flat convergence, for all $x \in \mathbb{R}^m$ and almost all $r > 0$,

$$\frac{\|S\|(\mathrm{U}(x,r))}{\boldsymbol{\alpha}(m)r^m} \leqslant \liminf_{k\to\infty} \frac{\|S_{j_{k,r}}\|(\mathrm{U}(x,r))}{\boldsymbol{\alpha}(m)r^m} \leqslant \frac{\|T\|(\mathrm{U}(x,r))}{\boldsymbol{\alpha}(m)r^m}$$

and letting r go to 0,

$$\liminf_{r\to 0} \frac{\|S\|(\mathrm{U}(x,r))}{\boldsymbol{\alpha}(m)r^m} \leqslant \liminf_{r\to 0} \frac{\|T\|(\mathrm{U}(x,r))}{\boldsymbol{\alpha}(m)r^m}.$$

Now, $\|S\|$ (respectively $\|S_j\|$ for all j) is an inner regular measure, thus for each $r > 0$, and each $\varepsilon > 0$, there exists $r' \in]0, r[$ such that $\|S\|(\mathrm{U}(x,r) \setminus \mathrm{B}(x,r')) \leqslant \varepsilon r^m$. This implies that:

$$\liminf_{r\to 0} \frac{\|S\|(\mathrm{U}(x,r))}{\boldsymbol{\alpha}(m)r^m} = \liminf_{r\to 0} \frac{\|S\|(\mathrm{B}(x,r))}{\boldsymbol{\alpha}(m)r^m}$$

and the same identity holds for T, yielding

$$\Theta^m_*(\|S\|, x) \leqslant \Theta^m_*(\|T\|, x). \tag{3.3}$$

By proposition 3.1.3 (ix), the points of positive higher density of $\|S_j\|$ are points of positive higher density of $\|T\|$. Thus all the points where $\|S\|$ has a positive and defined density are points of positive density of $\|T\|$. Since T is an integral current of dimension m, we can write $\|T\| = \theta_T \mathcal{H}^m \llcorner M_T$, where M_T is $(\mathcal{H}^m, m)$ rectifiable and contains $\mathrm{set}_m \|T\|$ as $\mathcal{H}^m(\mathrm{set}_m \|T\| \Delta M_T) = 0$. On the other hand, $\|S\| = \theta_S \mathcal{H}^m \llcorner M_S$, where M_S is $(\mathcal{H}^m, m)$ rectifiable and $\|S\|$ has density higher than 1 $\mathcal{H}^m \llcorner M_S$ almost everywhere thus $\mathcal{H}^m(M_S \setminus M_T) = 0$. This implies that $\|S\| = \theta_S \mathcal{H}^m \llcorner M_T$ where we extend θ_S by 0 in $M_T \setminus M_S$. And since S is integral, $\overrightarrow{S}$ is tangent to M_T, $\|S\|$ almost everywhere. This implies that $\overrightarrow{S} = \pm \overrightarrow{T}$, $\|S\|$ almost everywhere, we can redefine θ_S, $\|T\|$ almost everywhere, allowing it to take negative values in order to have $S = \theta_S \|T\| \wedge \overrightarrow{T}$. Note that by (3.3), $|\theta_S| \leqslant \theta_T$, $\|T\|$ almost everywhere.

Symmetrically, consider the sequence $T - S_j$, it converges in the flat norm to $T - S$ and the same reasonning as above applies, which yields

$$T - S = (\theta_T - \theta_S)\mathcal{H}^m \llcorner M_T \wedge \overrightarrow{T}.$$

In particular, $\theta_T - \theta_S \leqslant \theta_T$, which implies that θ_S is nonnegative $\|T\|$ almost everywhere and

$$\mathbb{M}(T) = \mathbb{M}(T - S) + \mathbb{M}(S) = \int \theta_S/\theta_T \,\mathrm{d}\|T\| + \int (\theta_T - \theta_S)/\theta_T \,\mathrm{d}\|T\|.$$

Since $\mathbb{M}(T) = \mathbb{M}(T - S_j) + \mathbb{M}(S_j)$ for all j by proposition 3.1.3 (vi) and mass is lower semi-continous with respect to the flat norm convergence, there holds

$$\mathbb{M}(S_j) \to \mathbb{M}(S)$$

and mass is continous for the convergence in $\mathfrak{F} \llcorner \mathcal{S}(T)$. for all $c > 0$. If $\mathbb{M}(S) = 0$, $S = 0$ is a subcurrent and S_j converges to S in mass and we are done. On the contrary, suppose that $\mathbb{M}(S) > 0$. We will show that θ_S takes the values 0 or θ_T, $\|T\|$ almost everywhere, making S a subcurrent of T.

Let $A := \{x, \theta_S(x) = \theta_T(x)\}$. A is $\|T\|$ measurable and therefore $S \llcorner A = T \llcorner A$ is a rectifiable current. Consider the sequence $S_j - S \llcorner A$, it converges in the flat norm to $S - S \llcorner A$, therefore for $f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ Lipschitz, for almost all $y \in \mathbb{R}^{n-m}$ there holds

$$\langle S_j - S \llcorner A, f, y\rangle \to \langle S - S \llcorner A, f, y\rangle$$

in the flat norm. Since for almost all y, $\langle S, f, y\rangle$ is a subcurrent of $\langle T, f, y\rangle$, therefore for such a y,

for all x in the support of $\langle S, f, y\rangle$,

$$\theta_S(x) = \theta_{\langle S,f,y\rangle}(x) = \theta_{\langle T,f,y\rangle}(x) = \theta_T(x).$$

This implies that $\langle S, f, y\rangle = \langle S \llcorner A, f, y\rangle$ and therefore, for almost all $y \in \mathbb{R}^{n-m}$, $\langle S - S \llcorner A, f, y\rangle = 0$. In particular, choosing f as in B.6, we get : $S = S \llcorner A = T \llcorner A$ and the proof is complete.

□

3.1.3 The special case of codimension 0: sets of finite perimeter

Some definitions and classical results on sets of finite perimeter are contained in Appendix C. In this paragraph, we outline the main properties of currents representing sets of finite perimeter which are not shared by general integral currents.

Let us first point out that the isoperimetric inequality doesn't hold for general integral currents, as can be shown by considering the two dimensional currents associated to spheres of arbitrary diameter in $\mathbb{R}^3$. This example also shows that the flat norm and the mass norm are not equivalent in positive codimension.

Proposition 3.1.7. *Let A be a set of finite perimeter, then the following statements hold:*

(1) If B is an $\mathcal{L}^m$ measurable subset of $\mathbb{R}^m$, then $[\![A]\!] \llcorner B$ is a subcurrent of $[\![A]\!]$ if and only if $A \cap B$ is a set of finite perimeter. In particular, $[\![A]\!] \llcorner B \in \mathcal{S}([\![A]\!])$ whenever B has finite perimeter.

(2) If B and B' are $\mathcal{L}^m$ measurable sets, with $[\![A]\!] \llcorner B$ and $[\![A]\!] \llcorner B'$ subcurrents of $[\![A]\!]$, then $[\![A]\!] \llcorner (B \cap B') \in \mathcal{S}([\![A]\!])$ as well.

3.1.4 Intersection of subcurrents

It is not true in general that if $T \llcorner A$ and $T \llcorner B$ are subcurrents of T the current given by $T \llcorner (A \cap B)$ is as subcurrent of T as well (we call this current the **intersection** of $T \llcorner A$ and $T \llcorner B$). Two counterexamples are given below. This is true however for currents of codimension 0, as will be shown in the next section.

Example 3.1.8. Consider the set $2^{-\mathbb{N}} := \{1, 1/2, 1/4, \ldots\}$ and $E := \bigcup_{k=1}^{\infty} [\,2^{-2k}, 2^{-2k-1}\,]$ in $\mathbb{R}$. Define the function f on $[\,0,1\,]$: $x \mapsto \mathrm{dist}(x, 2^{-\mathbb{N}})$ And denote by Γ its graph in $\mathbb{R}^2$. Note that f is a Lispchitz function, thus Γ is a Lipschitz curve with finite length. We also look at the function $g : x \mapsto \mathrm{dist}(x, 2^{-\mathbb{N}})\mathbb{1}_{E^c}(x)$ and call its graph Γ'.

Let S be the integral current of dimension 1 in $\mathbb{R}^2$ associated to the curve Γ oriented towards positive first coordinate and $R := [\![(0,0),(1,0)]\!]$ the current associated to the segment $[\,0,1\,] \times \{0\}$ oriented in the same direction as S. The sum $T := S + R$ is in $\mathbb{I}_1(\mathbb{R}^2)$. And $T \llcorner \Gamma$ and $T \llcorner \Gamma'$ are subcurrents of T. $T \llcorner (\Gamma \cap \Gamma')$ is an infinite sum of currents associated to disjoint intervals and therefore has infinite boundary mass: it is not a subcurrent of T. See figure 3.2.

One can ask whether such a situation can also occurs if T is an indecomposable current. In dimension 1 will certainly not be the case as an indecomposable integral current of dimension 1 is either a simple Lipschitz curve or a cycle. It is not hard, however, to define an indecomposable

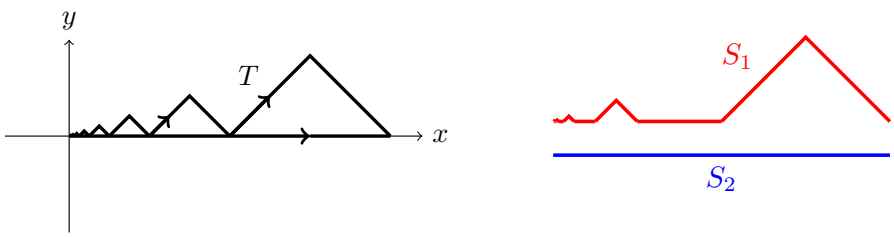

Figure 3.2: The intersection of S_1 and S_2 is not a subcurrent of T.

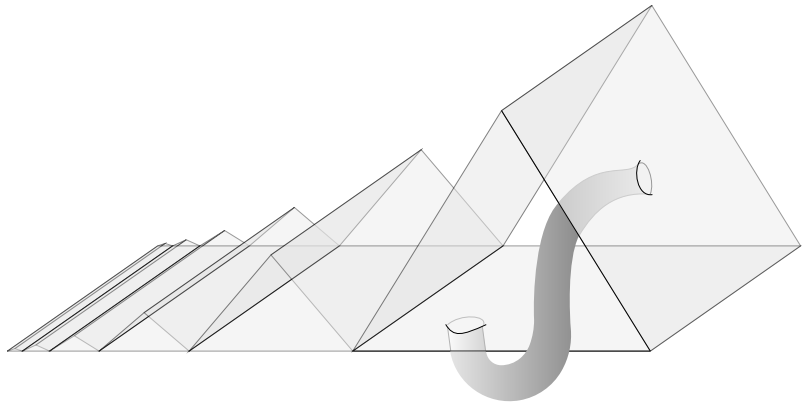

Figure 3.3: An indecomposable current which does not have the Intersection Property.

integral current in $\mathbb{I}_2(\mathbb{R}^3)$ which does not have the weak intersection property. This is done by "tensorizing" T above with an interval in the third dimension and then connecting the two elements of decomposition of T by cutting out two disks from the two components and joining them by a tube in such a way as to preserve the orientation without adding any boundary.

Now that we have this example in mind, we can study currents which behave better with respect to the intersection of their subcurrents. A current $T \in \mathbb{I}_m(\mathbb{R}^n)$ has the **weak intersection property** whenever the intersection of two of its subcurrents is also a subcurrent, i.e. if $S := T \llcorner A$ and $S' := T \llcorner A'$ are subcurrents of T, then $S \sqcap S' := T \llcorner (A \cap A')$ is also a subcurrent of T.

A stronger property might be more useful, as it is quantitative T has the **strong intersection property** : whenever there exists a constant $C_T^a > 0$ such that for S, S' and $S \sqcap S'$ as above:

$$\mathbb{M}(\partial(S \sqcap S')) \leqslant C_T^a + \mathbb{M}(\partial S) + \mathbb{M}(\partial S').$$

Clearly the strong intersection propertyimpliesthe weak intersection property. Note that as the intersection of two sets of finite perimeter is a set of finite perimeter, an integral current representing bounded set of finite perimeter has the strong intersection property.

Consider the 1-dimensional current from example 3.1.8 restricted to $\{(x, y) \in \mathbb{R}^2, x \geqslant (k\pi)^{-1}\}$ for some $k \geqslant 1$, call it T_k. T_k is integral and has the strong intersection property with the constant $C_{T_k}^a = k-1$.It is not clear whether an integral current may have only the weak intersection property but not the strong one.

3.1.5 Inner approximation of currents

It is essential, when working with sets of finite perimeter or integral currents, to distinguish the set of density points from the support. However, in the case of sets of finite perimeter a result of Giacomelli and Tamanini [68] states that a bounded set of finite perimeter can be approximated from the inside by an essentially closed set, that is a set whose set of density points is closed, with little variation in the perimeter. The authors obtain two different approximation results. We reformulate them in the language of currents:

Definition 3.1.9 (Inner approximation properties)**.** A current T in $\mathbb{I}_m(\mathbb{R}^n)$ has:

- The **inner approximation property** if there exists $C > 0$ such that given $\varepsilon > 0$, there exists a subcurrent $T_\varepsilon \in \mathcal{S}(T)$ such that:
  - (i) $\operatorname{set}_m \|T_\varepsilon\| = \operatorname{spt} T_\varepsilon$,
  - (ii) $\mathbb{M}(T - T_\varepsilon) < \varepsilon$ and
  - (iii) $\mathbb{M}(\partial T_\varepsilon) \leqslant C$
- The **strong inner approximation property** if given $\varepsilon > 0$, there exists a subcurrent $T_\varepsilon \in \mathcal{S}(T)$ such that:
  - (i) $\operatorname{set}_m \|T_\varepsilon\| = \operatorname{spt} T_\varepsilon$,
  - (ii) $\mathbb{N}(T - T_\varepsilon) < \varepsilon$.

Note that these two properties are not a priori equivalent, though the second one implies the first one. However, the following is true:

Theorem 3.1.10 (Giacomelli-Tamanini, [68])**.** *If A is a bounded set of finite perimeter in $\mathbb{R}^m$, then $\mathbf{E}^m \llcorner A$ has both the strong and weak inner approximation properties.*

The proof of this theorem rely on the minimization of functionals on the family of subsets of A with finite perimeter and on the isoperimetric inequality to get a quantitative estimate on the density of the subset at boundary points. This method will therefore not work for integral currents of positive codimension. The following example provides candidates for integral currents which do not have the Strong or the Weak Inner Approximation Property.

Example 3.1.11. We consider an integral current of dimension 2 in $\mathbb{R}^3$, which is a cycle and *probably* does not have the Strong (respectively the Weak) Approximation Property. These currents are sums of *tentacles* which split in a fractal way to cover a four corners Cantor set C.This is inspired by a construction attributed to Nöbeling of a bounded set of finite perimeter in $\mathbb{R}^3$ whose topological boundary contains a Cantor set of dimension 2.

The set $C \times \{z_\infty\}$ is in $\operatorname{spt} T \setminus \operatorname{set}_2 \|T\|$ and the subcurrents $T \llcorner \{x, y, z, z < z_k\}$ are arbitrarily close to T in mass, but their boundary mass is positive and constant (respectively goes to infinity). This shows that they are not good inner approximations. We conjecture that this example does not have the Weak (respectively Strong) Inner Approximation Properties and give a criterion which could be used to prove it. Consider the "four corner Cantor set" C built from the square $C_0 := [0, 1]^2$ in $\mathbb{R}^2$, by iterated contractions of ratio $1/4 \leqslant \alpha < 1/2$ centered at the corners of the

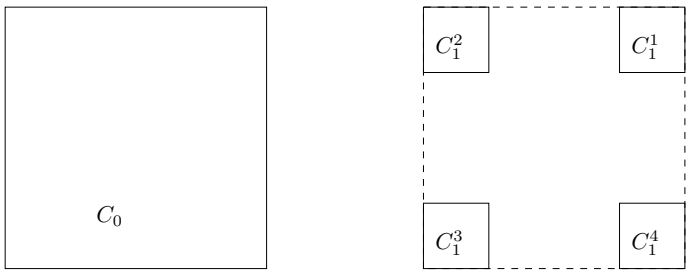

Figure 3.4: Labeling of the squares for the construction of the Cantor set.

squares. For $k = \{0, 1, 2, \ldots\}$, call C_k the k-th step of the construction of the Cantor set: C_k is a union of 4^k disjoint closed squares of side lengths α^k and we have $C := \bigcap_{k>0} C_k$. We number the vertices of each square by $1, 2, 3, 4$ in the counter-clockwise direction in a consistent fashion as in figure 3.4 and call the triangles in C_k: C_k^j for $j \in \{1, 2, 3, 4\}^k$ where for $l \in \{1, 2, 3, 4\}$, $C_{k+1}^{(j,l)}$ is the descendant of C_k^j corresponding to its l-th vertex.

Choose $r \in (1/4, \min\{\alpha, 1/(2\sqrt{2})\})$. For $j \in \{1, \ldots, 4^k\}$ we call D_k^j the closed disk with radius r^k centered at the center of C_k^j (at $k = 0$, we simply call the corresponding disk D_0). Note that D_k^j is contained in C_k^j as $r^k \leqslant \alpha^k$. In each disk D_k^j, place four disjoint disks of radius r^{k+1}, not touching the boundary of D_k^j and with their centers situated at the middle points of the segments linking the center of C_k^j to its vertices. Call these disks $\tilde{D}_k^{(j,l)}$ according to the number $l \in \{1, 2, 3, 4\}$ of the closest vertex of C_k^j. The distance between the centers of $\tilde{D}_k^{(j,l)}$ and $D_{k+1}^{(j,l)}$ is

$$d_{k+1} := \frac{\sqrt{2}}{2}\alpha^k(1-\alpha) - r^{-k}(1-r) \lesssim \alpha^k,$$

where $\lesssim$ stands for less or equal up to multiplication by a positive constant which does not depend on k. Let $h < 1$, $z_0 := 0$, for $k \in \{1, 2, \ldots\}$ let $z_k := \sum_{k'=1}^{k} h^{k'}$ and $z_\infty := \sum_{k=1}^{\infty} h^k$. For $k \in \{1, 2, \ldots\}$ and $j \in \{1, \ldots, 3\}^k$ and $l \in \{1, 2, 3, 4\}$ call $S_k^{j,l}$ the tilted hollow cylinder whose bottom basis is $\tilde{D}_{k-1}^j \times \{z_{k-1}\}$ and whose top basis is $D_k^j \times \{z_k\}$. For $k \in \{1, 2, \ldots\}$ and $j \in \{1, 2, 3\}^k$, the area of S_k^j is equal to $2\pi 4^{-k}\sqrt{h^{2k} + d_k^2}$. For convenience, we choose the set S_k^j so that it does not contain the bottom and top circles. (See figure 3.5.) Let

$$M := \left(\bigcup_{l\in\{1,2,3,4\}} \tilde{D}_0^l\right) \times \{z_0\} \cup \bigcup_{\substack{k\in\{1,2,\ldots\}\\ j\in\{1,2,3,4\}^k}} \left(\left(D_k^j \setminus \left(\bigcup_{l\in\{1,2,3,4\}} \operatorname{int} \tilde{D}_k^{(j,l)}\right)\right) \times \{z_k\} \cup S_k^j\right).$$

Note that it is possible to modify M to get a single connected rectifiable set.

Claim 1. M is countably $\mathcal{H}^2$ rectifiable and

$$\mathcal{H}^2(M) = \frac{\pi}{4} + \sum_{k\in\{1,2,\ldots\}} 4^k \left(\left(r^{2k}\pi(1-4r^2)\right) + 2\pi r^k\sqrt{h^{2k} + d_k^2}\right) < +\infty.$$

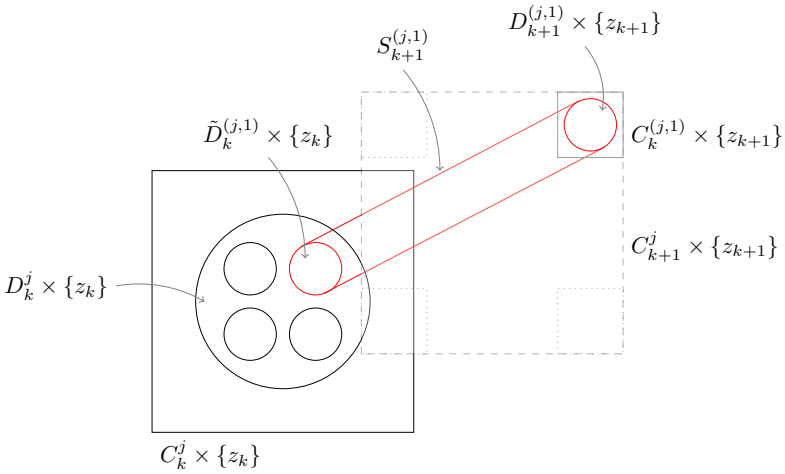

Figure 3.5: The cylinder $S_{k+1}^{(j,1)}$ connects the disks D_k^j and $D_{k+1}^{(j,1)}$.

In particular, for $k_0 \in \{0, 1, 2, \ldots\}$,

$$\mathcal{H}^2(M \cap \{(x, y, z), z > z_{k_0}\}) = \sum_{k > k_0} 4^k \left(\left(r^{2k}\pi(1 - 4r^2)\right) + 2\pi r^k \sqrt{h^{2k} + d_k^2} \right)$$
$$\lesssim 4^{-k_0} r^{2k_0} + (4rh)^{k_0} + (4r\alpha)^{k_0}.$$

Let e_x, e_y, e_z be the orthonormal basis of $\mathbb{R}^3$ in our coordinates (x, y, z) and let $\overrightarrow{R}$ be a 2-vector field defined $\mathcal{H}^2(M)$ almost everywhere as:

- $R(x, y, z) = -e_x \wedge e_y$ if $(x, y, z) \in \bigcup_{l \in \{1,2,3,4\}} \operatorname{int} \tilde{D}_0^l \times \{0\}$,
- $R(x, y, z) = e_x \wedge e_y$ if $(x, y, z) \in D_k^j \times \{z_0\}$ for some $k \in \{1, 2, \ldots\}$ and $j \in \{1, 2, 3, 4\}^k$.
- $R(x, y, z)$ is the tangent unit simple 2-vector to S_k^j pointing outwards if $(x, y, z) \in S_k^j$ for some k, j.

Claim 2. The current $T := \mathcal{H}^2 \llcorner M \wedge \overrightarrow{R}$ is integral and is a cycle ($\partial T = 0$).

Proof. M is rectifiable and has finite mass and $\overrightarrow{R}$ is tangent to M $\mathcal{H}^2 \llcorner M$ almost everywhere, thus T is a rectifiable current. To see that T is a cycle (and thus an integral current) we show that it is a limit of cycles in the flat norm. For $k \geqslant 1$, let

$$M_k := \left(\bigcup_{l \in \{1,2,3,4\}} \tilde{D}_0^l \right) \times \{z_0\}$$
$$\cup \bigcup_{\substack{k' \in \{1,2,\ldots,k-1\} \\ j \in \{1,2,3,4\}^{k'}}} \left(\left(D_{k'}^j \setminus \left(\bigcup_{l \in \{1,2,3,4\}} \operatorname{int} \tilde{D}_{k'}^{(j,l)} \right) \right) \times \{z_{k'}\} \cup S_{k'}^j \right)$$
$$\cup \bigcup_{j \in \{1,2,3,4\}^k} D_k^j \times \{z_k\}$$

and let $T_k := \mathcal{H}^2 \llcorner M_k \wedge \overrightarrow{R}$. T_k is clearly integral and a cycle. There holds:

$$\mathbb{M}(T - T_k) = \sum_{j \in \{1,2,3,4\}^k} \mathcal{H}^2(D_k^j) + \mathcal{H}^2(M \cap \{(x,y,z), z > z_k\}).$$

This goes to 0 and the claim is proved. ■

Claim 3. Under some conditions on α and h, $\operatorname{set}_2 \|T\| = M$. Furthermore, $\operatorname{spt} T \setminus \operatorname{set}_2 T$ is equal to the Cantor set $C = \bigcap_{k \in \{1,2,\ldots,\}} C_k$.

Proof. Let us characterize $\operatorname{set}_2 \|T\|$. Let $x \in M$, x is either contained in a disk with 4 cicular holes or in a tilted cylinder contained in M so $x \in \operatorname{set}_2 \|T\|$, $M \subseteq \operatorname{set}_2 \|T\|$. Conversely, $\operatorname{spt} T = \operatorname{cl}(M) = M \cup C$, so $\operatorname{set}_2 \|T\| \subseteq M \cup C$. (Incidentally, this proves that $\operatorname{spt} T \setminus \operatorname{set}_2 \|T\| \subseteq C$.) Let us show that if $x \in C$, x is not a point of density of $\|T\| = \mathcal{H}^2 \llcorner M$. Let $\min(1, z_\infty) > t > 0$, there exists $k \in \{1, 2, \ldots\}$ such that $\alpha^{-k} \leqslant t < \alpha^{-k-1}$ and $k_0 \in \{0, 1, 2, \ldots\}$ such that $z_{k_0} < z_\infty - t \leqslant z_{k_0+1}$. this implies that $h^{k_0} \lesssim t$ Suppose without loss of generality that $x \in C_k^1$, by the choice of k and k_0,

$$\mathrm{U}(x,t) \subseteq C_k^1 \times \{z > z_{k_0}\}.$$

By self similarity (in the (x,y) directions) of the construction of M, we infer:

$$\|T\|(\mathrm{U}(x,t)) \leqslant 4^{-k} \mathcal{H}^2(M \cap \{z > z_{k_0}\}) \lesssim 4^{-k}((4r^2)^{k_0} + (4rh)^{k_0} + (4r\alpha)^{k_0}),$$

and

$$\frac{\|T\|(\mathrm{U}(x,t))}{t^2} \lesssim t^{-2} 4^{-k}((4r^2)^{k_0} + (4rh)^{k_0} + (4r\alpha)^{k_0})) \\ \lesssim h^{-2k_0} 4^{-k}((4r^2)^{k_0} + (4rh)^{k_0} + (4r\alpha)^{k_0}).$$

where we used the fact that $t \gtrsim \alpha^k$ and $t \gtrsim \alpha^{k_0}$. Choosing $\alpha = h$, we get $k = k_0$ and

$$\frac{\|T\|(\mathrm{U}(x,t))}{t^2} \lesssim \alpha^{-2k} 4^{-k}((4r^2)^k + 2(4r\alpha)^k) \lesssim (r^2/\alpha^2)^k + (r\alpha)^k.$$

This goes to 0 as $k \to \infty$ if $h = \alpha > r$. If we only want to prove that T does not have the Strong Inner Approximation Property, it suffices to choose $\alpha = h = 1/3$ and $r = 1/4$. To prove that T does not have the Weak Inner Approximation Property, we can choose $r = 3/10$ instead. Both choices, ensure that $C \times \{z_\infty\} \cap \operatorname{set}_m \|T\| = \emptyset$. ■

For $k = 1, 2, \ldots$, consider the current $T_k := T \llcorner \{(x,y,z), z < z_k\}$. By construction, $\mathbb{M}(T_k) \to \mathbb{M}(T)$ as k goes to ∞. However, the boundary of $T - T_k$ is a sum of 4^{k-1} oriented circles of radius r^{k-1}. Therefore $\mathbb{M}(\partial(T - T_k)) = 4^{k-1}(2\pi r^{k-1})$ which does not go to 0 provided $r \geqslant 1/4$ and goes to infinity provided $r > 1/4$ for instance if $r = 3/10$. This tends to show that T cannot be approximated from the inside. However it is not a proof.

The question is thus: how could one prove that T does not have one of the approximation properties? The next Lemma gives a way to do it. It uses the generalized Stokes' Theorem which we prove in chapter 4 on the approximating currents.

Claim 4. **If** there exists a differential form ω of degree 1 continuous on M and such that

1. There exists a constant $L > 0$ such that $\|\omega\| < L$ uniformly on M,
2. ω is pointwise differentiable along M at each point where M is smooth and
3. $x \mapsto \langle \mathrm{d}\omega(x), \overrightarrow{T}(x)\rangle$ is Lebesgue integrable with respect to $\|T\|$ with integral 0.

Then T does not have the Strong Inner Approximation Property.

If we replace condition 3 by

3' $x \mapsto \langle \mathrm{d}\omega(x), \overrightarrow{T}(x)\rangle$ is uniformly bounded from below, locally Lebesgue integrable with respect to $\|T\|$ in the open set $\mathbb{R}^3 \cap \{(x, y, z), z < z_\infty\}$ but satisfies $\int_M \langle \mathrm{d}\omega, \overrightarrow{T}\rangle = +\infty$,

then T does not have the Weak Inner Approximation Property.

Proof. We only prove the second part of the statement, the first part is easier by definition of integrability and the same use of the generalized Stokes' Theorem.

Suppose such a differential form exists. For each positive integer p, there exists a compact set K_p such that $\|\langle \mathrm{d}\omega, \overrightarrow{T}\rangle\|_{\mathbb{L}^1(\|T \llcorner K_p\|)} \geqslant p$ and $\mathbb{M}(T \llcorner K_p^c) < 1/p$. Choose a positive integer p, by properties of the Lebesgue integral, there exists $\eta > 0$ such that if A is a $\|T\|$ measurable set with $\|T \llcorner K_p\|(K_p \backslash A) < \eta$,

$$\int_A \langle \mathrm{d}\omega, \overrightarrow{T}\rangle \, \mathrm{d}\|T\| \geqslant p - 1/p.$$

We can also suppose that $\eta < 1/p$. Now fix $T' \sqsubset T$ and suppose that $\operatorname{spt} T' = \operatorname{set}_2 \|T'\|$ and $\mathbb{M}(T - T') < \eta$, then $T' = T \llcorner \operatorname{spt} T'$. There holds $\|T\|(K \backslash \operatorname{spt} T') \leqslant \|T\|(M \backslash - \operatorname{spt} T') < \eta$. As the integrand is nonnegative, we have

$$\int \langle \mathrm{d}\omega, \overrightarrow{T}\rangle \, \mathrm{d}\|T'\| \geqslant \int_{K_p \backslash \operatorname{spt} T'} \langle \mathrm{d}\omega, \overrightarrow{T}\rangle \, \mathrm{d}\|T\| \geqslant p - 1/p.$$

As $\operatorname{spt} T' = \operatorname{set}_2 T' \subseteq \operatorname{set}_2 T$, we have necessarily $\operatorname{spt} T' \cap C = \emptyset$ and $\operatorname{spt} T' \subset\subset M$. Therefore $\operatorname{spt} T'$ is contained in a finite union of compact submanifolds intersecting only at the boundary, in particular, T has C^1 BV charts except on a disposable set (in the language of theorem 4.2.10), as $\operatorname{spt} T'$ is compactly supported in M, theorem 4.3.1 applies to ω and T' with

$$\int \langle \omega, \overrightarrow{\partial T'}\rangle \, \mathrm{d}\|\partial T'\| = \int \langle \mathrm{d}\omega, \overrightarrow{T}\rangle \, \mathrm{d}\|T'\| \geqslant p - 1/p.$$

Since $\|\omega(x)\| \leqslant 2$ for all $x \in M$, we have

$$\mathbb{M}(\partial T') \geqslant \frac{p - 1/p}{2}.$$

This proves that for each constant $C > 0$, there exists $\eta > 0$ such that if T' is a subcurrent of T with support contained in M and $\mathbb{M}(T - T') < \eta$, then $\mathbb{M}(\partial T') > C$, which implies that T does not have the weak inner approximation property. ■

All there remains to do is prove that such a form exists. We have not been able to do this for the moment.

One can ask whether the approximation property is hereditary, in other words, if T has the strong/weak approximation property, do his subcurrents have it as well?

Example 3.1.12. If the example above truly does not have the Weak Approximation Property, there exists a integral current of dimension 2 in $\mathbb{R}^3$ which has the Strong Approximation Property, while not all his subcurrents have the Weak Approximation Property.

For sequences $k_1, k_2, \dots$ and $\varepsilon_1, \varepsilon_2, \dots$ with $\varepsilon_s << h^{k_s}$, add to T cycles representing closed hollow cylinders of the form

$$(\tilde{D}^j_{k_s} \times \{z_{k_s}, z_{k_s} + \varepsilon_s\}) \cup (\partial \tilde{D}^j_{k_s} \times [z_{k_s}, z_{k_s} + \varepsilon_s]).$$

This yields an integral current $\tilde{T}$. If k_s goes to ∞ quickly enough, $\mathrm{set}_2 \tilde{T}$ will be equal to the union of M and the cylinders and therefore not intersect C. The currents $T_s := \tilde{T} \llcorner \{(x, y, z), z \leqslant z_{k_s}\}$ are subcurrents of $\tilde{T}$ and cycles which approximate $\tilde{T}$ in mass. They also are essentially closed. This shows that $\tilde{T}$ has the Weak Approximation Property. However, T is a subcurrent of T' and still does not have the Weak Approximation Property.

3.2 Functions on the space of subcurrents.

3.2.1 Continuous functions on $\mathcal{S}(T)$, m-charges and flat cochains

Definition 3.2.1. Given a current $T \in \mathbb{I}_m(\mathbb{R}^n)$, a function F on $\mathcal{S}(T)$ is a **continuous function on** $\mathcal{S}(T)$ if:

1. F is additive: if S and S' in $\mathcal{S}(T)$ are non-overlapping, $F(S + S') = F(S) + F(S')$.
2. If $(S_k)_k$ is a sequence in $\mathcal{S}(T)$ with $\sup_k \mathbb{N}(S_k) < \infty$ and $\mathbb{F}(S_k) \to 0$, then $F(S_k) \to 0$ as k goes to ∞.

Proposition 3.2.2. *If T has the strong intersection property, and F is subadditive and continuous on T, then given a sequence $(S_k)_k$ in $\mathcal{S}(T)$ with $\sup_k \mathbb{N}(S_k) < \infty$ converging in the flat norm to $S \in \mathcal{S}(T)$, then $F(S_k) \to F(S)$.*

Proof. To see this, apply the definition of continuity to the sequence of symmetric differences of S_j and S, then use the additivity. More precisely if $S_j = T \llcorner A_j$ and $S = T \llcorner A$, consider the sequences $R_j := T \llcorner (A_j \backslash A) = T \llcorner (A_j \cap A^c))$ and $R'_j := T \llcorner (A \backslash A_j)$. By the Strong Intersection Property and the assumptions on $\mathbb{M}(\partial S_j)$, $\sup_j \mathbb{M}(\partial R_j) < +\infty$. Also, by theorem 3.1.6 the S_j converge to S in mass, which implies that $\mathbb{M}(R_j)$ goes to 0 and the R_j thus tend to 0 in the flat norm, thus $F(R_j) \to 0$. The same holds for the R'_j. Since $S_j = S - R'_j + R_j$ and F is subadditive, one can show that $|F(S_j) - F(S)| \leqslant |F(R'_j)| + |F(R_j)|$, it suffices to pass to the limit to conclude. □

It is not clear whether this is also true when T only has the weak intersection property, as the normal masses of the symetric differences can go to ∞.

Proposition 3.2.3. *The space of continuous additive functions on $\mathcal{S}(T)$, is a vector space. It contains the restrictions to $\mathcal{S}(T)$ of m-charges on $\operatorname{spt} T$ in the sense of Moonens: $\mathbf{CH}^m(\operatorname{spt} T)$.*

Proof. The first part of the statement is clear. To prove the second, consider an element of $\mathbf{CH}^m(\operatorname{spt} T)$. It is a continuous linear form on the space $\mathbb{N}(\operatorname{spt} T)$ which contains $\mathcal{S}(T)$. □

It is crucial to distinguish the charges on a current defined here from m-charges in the sense of [19] as the inclusion above is usually strict. See in particular example 3.2.8. However, m-charges are of particular interest for us, in particular the following two types:

Example 3.2.4. Let $T \in \mathbb{I}_m(\mathbb{R}^n)$, the following functions from $\mathcal{S}(T)$ to $\mathbb{R}$ are charges on T:

- if ω is a continuous differential form of degree m on $\operatorname{spt} T$, the function

$$\Lambda_\omega : S \mapsto \int \langle \omega(x); \overrightarrow{S}(x)\rangle \, \mathrm{d}\|S\|(x). \tag{3.4}$$

- if ζ is a continuous differential form of degree $m-1$ on $\operatorname{spt} T$, we call **circulation of** ζ the function:

$$\Theta_\zeta : S \mapsto \int \langle \zeta(x); \overrightarrow{\partial S}(x)\rangle \, \mathrm{d}\|\partial S\|(x). \tag{3.5}$$

These two functions are the canonical examples of m-charges. Indeed, any m-charge on $\operatorname{spt} T$ can be expressed as $\Theta_\zeta + \Lambda_\omega$ for a pair (ω, ζ) as above.

Proposition 3.2.5. *If $u \in \mathbb{L}^1(\|T\|) \cap \mathbb{L}^\infty(\|T\|)$, the function*

$$\begin{aligned} F_u : \mathcal{S}(T) &\to \mathbb{R} \\ S &\mapsto \int u \, \mathrm{d}\|S\| \end{aligned}$$

is a continuous additive function on $\mathcal{S}(T)$.

Proof. If u is identically zero, the result is clear, suppose u is non trivial. By definition, u is $\|T\|$ measurable. Let $\overrightarrow{T}^*$ be the $\|T\|$-almost everywhere defined dual m-covector field to $\overrightarrow{T}$ so that $\langle \overrightarrow{T}^*(x); \overrightarrow{T}(x)\rangle = 1 = |T^*(x)|$, $\|T\|$-almost everywhere. Let $\omega := u\overrightarrow{T}^*$, ω is a $\|T\|$ measurable m-form in $\mathbb{R}^n$. Find a representative of ω defined everywhere in $\operatorname{set}_m(\|T\|)$ (still denoted by ω). Extending it by 0 in $\mathbb{R}^n \backslash (\operatorname{set}_m \|T\|)$, view ω as a function from $\mathbb{R}^n$ to the normed finite dimensional vector space $\Lambda^m(\mathbb{R}^n)$. Recall that $\|T\|(\mathbb{R}^n) < \infty$ and that $\|T\|$ is a Radon measure in $\mathbb{R}^n$, thus a regular measure, so by Lusin's Theorem (see [23, 1.2 Theorem 2]) applied to $\Lambda_m(\mathbb{R}^n)$ (which is a finite dimensional vector space). given $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq \operatorname{set}_m(\|T\|)$ such that $\|T\|(\operatorname{set}_m \|T\| \backslash K_\varepsilon) < \varepsilon |u|_\infty^{-1}$ and ω is continuous on K_ε. The map $\omega|_{K_\varepsilon}$ is uniformly continuous, extend it to the whole of $\mathbb{R}^n$ to get a continuous map ω_ε:

$$\begin{aligned} \omega_\varepsilon &\in \mathcal{C}(\mathbb{R}^n, \Lambda^m(\mathbb{R}^n)) \\ \omega_\varepsilon &= \omega \text{ on } K_\varepsilon \\ |\omega_\varepsilon|_\infty &\leqslant |u|_\infty. \end{aligned}$$

There exists a smooth form of degree m in $\mathbb{R}^n$, $\tilde{\omega}_\varepsilon \in \mathcal{D}^m(\mathbb{R}^n)$ such that

$$\begin{aligned} |\tilde{\omega}_\varepsilon|_\infty &\leqslant |\omega_\varepsilon|_\infty \\ |\tilde{\omega}_\varepsilon - \omega_\varepsilon|_\infty &\leqslant \varepsilon \, \mathbb{M}(T)^{-1} \end{aligned}$$

Given a subcurrent S of T, with $S = Q + \partial R$ and $\mathbb{F}(S) = \mathbb{M}(Q) + \mathbb{M}(R)$, there holds $\mathbb{M}(S - S \llcorner K_\varepsilon) \leqslant$

$\varepsilon|u|_\infty^{-1}$ and

$$\begin{aligned}
|F(S)| = \left|\int u \,\mathrm{d}\|S\|\right| &= \left|\int u(x)\langle \overrightarrow{T}^*(x); \overrightarrow{T}(x)\rangle \,\mathrm{d}\|S\|(x)\right| \\
&= \left|\int \langle \omega(x); \overrightarrow{T}(x)\rangle \,\mathrm{d}\|S\|(x)\right| \\
&\leqslant \left|\int \langle \omega; \overrightarrow{T}\rangle \,\mathrm{d}\|S \llcorner K\|\right| + |\omega|_\infty \|S\|(\mathbb{R}^n \backslash K_\varepsilon) \\
&\leqslant \left|\int \langle \omega_\varepsilon; \overrightarrow{T}\rangle \,\mathrm{d}\|S\|\right| + |\omega_\varepsilon|_\infty \|S\|(\mathbb{R}^n \backslash K_\varepsilon) + |\omega|_\infty \|S\|(\mathbb{R}^n \backslash K_\varepsilon) \\
&\leqslant \left|\int \langle \tilde{\omega}_\varepsilon; \overrightarrow{S}\rangle \,\mathrm{d}\|S\|\right| + |\tilde{\omega}_\varepsilon - \omega_\varepsilon|_\infty \,\mathbb{M}(S) + 2|\omega|_\infty \|S\|(\mathbb{R}^n \backslash K_\varepsilon) \\
&\leqslant |\langle \tilde{\omega}_\varepsilon; Q\rangle| + |\langle \mathrm{d}\tilde{\omega}_\varepsilon; R\rangle| + \varepsilon \frac{\mathbb{M}(S)}{\mathbb{M}(T)} + 2\varepsilon \\
&\leqslant \max(|\tilde{\omega}_\varepsilon|_\infty, |\,\mathrm{d}\tilde{\omega}_\varepsilon|_\infty)\,\mathbb{F}(S) + 3\varepsilon.
\end{aligned}$$

Thus there exists a constant C_ε (depending on ε) such that for $S \in \mathcal{S}(T)$:

$$\left|\int u \,\mathrm{d}\|S\|\right| \leqslant C_\varepsilon \,\mathbb{F}(S) + 3\varepsilon.$$

This implies that for every $\varepsilon > 0$, given a sequence $(S_k)_k$ in $\mathcal{S}(T)$ with $\mathbb{F}(S_k) \to 0$, $|F(S_k)|$ is smaller than 4ε for a large enough k. Thus this sequence goes to 0 as k goes to ∞. □

Corollary 3.2.6. *The mass operator $\mathbb{M}$ is continuous with respect to the flat norm $\mathbb{F}$ on the space $\mathcal{S}(T)$. Furthermore, the hypothesis in proposition 3.2.5 that the function u is in $\mathbb{L}^\infty(\|T\|)$ can be lifted, so that each $\|T\|$ integrable function is naturally associated to a continuous additive function on $\mathcal{S}(T)$.*

Proof. Take $u(x) = 1$ in proposition 3.2.5 for $\varepsilon > 0$ there exists $\theta > 0$ such that for $S \sqsubset T$:

$$\mathbb{M}(S) = \int u \,\mathrm{d}\|S\| \leqslant \theta\,\mathbb{F}(S) + \varepsilon.$$

Thus, given a sequence S_k converging to 0 in the flat norm, for all $\varepsilon > 0$,

$$\limsup_{k\to\infty} \mathbb{M}(S_k) \leqslant \varepsilon,$$

So $\mathbb{M}(S_k) \to 0$ as k goes to ∞.

Let $u \in \mathbb{L}^1(\|T\|)$ and let $(S_k)_k$ be a sequence of subcurrents of T. For all k, there exists A_k, $\|T\|$ measurable, such that $S_k = T \llcorner A_k$. Suppose that $\mathbb{F}(S_k)$ goes to 0, this implies that $\mathbb{M}(S_k) = \|T\|(A_k) \to 0$. Consider the sequence of $\|T\|$-integrable functions $(u\mathbb{1}_{A_k})_k$, it converges $\|T\|$-almost everywhere to 0. The dominated convergence theorem implies that $\int u \,\mathrm{d}\|S_k\| = \int u \,\mathrm{d}\|T \llcorner A_k\| \to 0$. Thus $S \mapsto \int u \,\mathrm{d}\|S\|$ is continuous. □

Remark 3.2.7. Notice that, as the example below shows:

1. There is no a priori control on the rate of this convergence. This is a consequence of the absence of an isoperimetric inequality on the space of subcurrents.

2. $\mathbb{M}$ is not an m-charge in the sense of [19].

Example 3.2.8. For $k = 1, 2, \ldots$: Define $x_k := (2^{1-k}, 0) \in \mathbb{R}^2$ and $r_k := 3^{-k} > 0$. Denote by C_k the integral current of dimension 1 associated to the oriented circle in $\mathbb{R}^2$ of center x_k and radius r_k. One has $\mathbb{M}(C_k) = 2\pi 3^{-k}$ and $\mathbb{F}(C_k) = 3^{-2k}\pi$.

The C_k have pairwise disjoint supports. Let $C_\infty := \sum_{k=1}^{\infty} C_k$. C_∞ is an integral current of dimension 1 without boundary and each C_k is a subcurrent of C_∞ with $C_k = C_\infty \llcorner \operatorname{spt} C_k$. Consider the sequence $(C_k)_k$ in $\mathcal{S}(C_\infty)$. There holds

$$\frac{\mathbb{M}(C_k)}{\mathbb{F}(C_k)} = 2\ 3^k,$$

which accounts for the first part of the remark. For $k = 1, 2$, consider the current $3^k C_k$ in $\mathbb{I}_1(\operatorname{spt} C_\infty)$. The elements of the sequence $(3^k C_k)_k$ has constant mass and normal mass equal to 2π but their flat norms go to 0. Thus $\mathbb{M}$ is not in $\mathbf{CH}^1(\operatorname{spt} C_\infty)$.

The notion of flat cochains is due to Whitney [73], we give here the definition of Federer [26].

Definition 3.2.9. A flat cochain of dimension m in an open set $U \subseteq \mathbb{R}^n$ is a linear functional α on the space of m-dimensional flat chains supported in U such that for every compact subset K of U, there exists $\theta > 0$ such that if S is a flat chain of dimension m supported in K

$$\alpha(S) \leqslant \theta\, \mathbb{F}(S).$$

This definition implies that the restriction of a flat chain α of dimension m to the space of subcurrents of an integral current T of dimension m is a continuous additive function on $\mathcal{S}(T)$. Indeed, subcurrents of T are flat chains all supported in $\operatorname{spt} T$ which is compact, subadditivity comes from the linearity of α and the continuity with respect to the flat norm of α guaranties the continuity of the induced function on $\mathcal{S}(T)$. One can also notice that the restriction of an m-dimensional flat chain to the space of normal currents of dimension m is an m-charge in the sense of [19] and that we showed above that the restrictions of m-charges on $\operatorname{spt} T$ to $\mathcal{S}(T)$ are continuous additive functions on $\mathcal{S}(T)$. Conversely, by example 3.2.8, there exists a current $C_\infty \in \mathbb{I}_1(\mathbb{R}^2)$ and a continuous additive function on $\mathcal{S}(C_\infty)$ ($S \mapsto \mathbb{M}(S)$ restricted to $\mathcal{S}(C_\infty)$) which is not in $\mathbf{CH}^1(\operatorname{spt} C_\infty)$ this function cannot be the restriction of a flat chain. This example depends strongly on the positive codimension and one could wonder if such is the case in codimension 0, however flat cochains of codimension 0 in $\mathbb{R}^n$ correspond to elements of $\mathbb{L}^\infty(\mathbb{R}^n)$ and we proved in corollary 3.2.6 that for any current $T \in \mathbb{I}_n(\mathbb{R}^n)$ all $\|T\|$-integrable functions represent continuous additive functions on $\mathcal{S}(T)$, including unbounded functions which do not correspond to any flat cochain.

We also define the pull-back operation on continuous additive functions, as a dual to the push-forward operation of subcurrents. Given $T \in \mathbb{I}_m(\mathbb{R}^n)$ and $\phi : \operatorname{spt} T \to \mathbb{R}^{n'}$ lipschitz and bijective from $\operatorname{spt} T$ to its image. Consider the current $T' = \phi_\# T$. If F is a continuous function on $\mathcal{S}(T')$, then the we call **pull-back of F by ϕ** the function $\phi^\# F$ on $\mathcal{S}(T)$ given by $\phi^\# F(S) = F(\phi_\# S)$ for any $S \in \mathcal{S}(T)$.

Proposition 3.2.10. *$\phi^\# F$ is continuous (respectively additive, subadditive and nonnegative) whenever F is.*

Proof. The nonnegativity is clear. The additivity and subadditivity are straightforward as well: if S_1 and S_2 are nonoverlapping subcurrents of T, then $S_1 + S_2$ is a subcurrent of T by proposition 3.1.3(vii) and $\phi_\#(S_1 + S_2) = \phi_\# S_1 + \phi_\# S_2$ by linearity of the pushforward operator $\phi_\#$ on $\mathbb{I}_m(\mathbb{R}^n)$. This implies $\phi^\# F(S_1 + S_2) = \phi^\# F(S_1) + \phi^\# F(S_2)$ if F is additive (and $\leqslant$ if F is subadditive).

For the continuity, suppose that S_j is a sequence in $\mathcal{S}(T)$ which converges to 0 in the flat norm with $\sup_j \mathbb{M}(\partial S_j) < +\infty$. Then by the lipschitz regularity of ϕ, $\phi_\# S_j$ converges to 0 in the flat norm with boundary mass uniformly bounded. This implies that $(\phi^\#(S_j) = F(\phi_\# S_j) \to 0$ and F is therefore continuous on $\mathcal{S}(T)$. □

3.2.2 Derivation of a function on a current

The notions used in this section are adapted from [61]. Let $T \in \mathbb{I}_m(\mathbb{R}^n)$, given $\eta > 0$, we say that a subcurrent $S \in \mathcal{S}(T)$ is **η-regular** if

$$\operatorname{reg}(S) := \frac{\mathbb{M}(S)}{\mathbb{M}(\partial S)\operatorname{diam}\operatorname{spt} S} > \eta.$$

(if $\mathbb{M}(\partial S) = 0$, let $\operatorname{reg}(S) = +\infty$.) For $x \in \operatorname{set}_m \|T\|$, $\delta > 0$ and $\eta > 0$, denote by $\mathcal{S}(T, x, \eta, \delta)$ the family of currents S in $\mathcal{S}(T)$ such that

1. $x \in \operatorname{spt} S$,
2. $\operatorname{diam}\operatorname{spt} S < \delta$,
3. S is η-regular.

These families are our derivation bases. If given $\eta > 0$ and $x \in \operatorname{set}_m \|T\|$, for every $\delta > 0$, $\mathcal{S}(T, x, \eta, \delta)$ is not empty, we say that T is **η-good** at x. In the following, we show that a point x in the set $\operatorname{set}_m \|T\| \setminus \operatorname{set}_{m-1} \|\partial T\|$. This raises the question:

Question 3.2.11. *For which current T is $\|\partial T\|(\operatorname{set}_m \|T\| \setminus \operatorname{set}_{m-1} \|\partial T\|$ large?*

If T represents a set of finite perimeter $A \in \mathbb{R}^m$, there holds

$$\operatorname{set}_m \|T\| \cap \operatorname{set}_{m-1} \|\partial T\| = \partial_e A \cup (\operatorname{int}_e A \setminus \operatorname{int}_c A),$$

where $\operatorname{int}_c A$ is the **critical interior of** A, as defined in Appendix C. As $\mathcal{H}^{m-1}(\partial_e A) = \mathrm{P}(A) < +\infty$ and $\mathcal{H}^{m-1}(\operatorname{int}_e A \setminus \operatorname{int}_c A) = 0$, the set of bad points in $\operatorname{set}_m \|T\|$ is thin. If T is the bilipschitz pushforward of a bounded set of finite perimeter, the $\mathcal{H}^{m-1}$ measure of this set is therefore equal to $\mathbb{M}(\partial T)$.

Note, that for any integral current $T \in \mathbb{I}_m(\mathbb{R}^n)$, as ∂T has finite mass, $\operatorname{set}_{m-1} \|\partial T\|$ has finite $\mathcal{H}^{m-1}$ measure and is thus m-thin.

Proposition 3.2.12. *If $x \in \operatorname{set}_m \|T\| \setminus \operatorname{set}_{m-1} \|\partial T\|$, and $\theta^* := \Theta^{*m}(\|T\|, x) < +\infty$. Then T is 2^{-2m-3}-good at x.*

Proof. Let x be as above, for $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$(1-\varepsilon)\theta^* < \frac{\|T\|(\mathrm{B}(x, r_\varepsilon/2))}{\boldsymbol{\alpha}(m)(r_\varepsilon/2)^m}$$
$$\forall r \in (r_\varepsilon/2, r_\varepsilon), \frac{\|T\|(\mathrm{B}(x, r_\varepsilon))}{\boldsymbol{\alpha}(m) r^m} \leqslant (1+\varepsilon)\theta^*$$

and

$$\forall r \in (0, r_\varepsilon), \frac{\|\partial T\|(\mathrm{B}(x, r))}{\boldsymbol{\alpha}(m-1) r^{m-1}} < \varepsilon.$$

Recall from proposition B.3 that for almost all $r \in (0, r_0)$, $T \llcorner \mathrm{B}(x, r) \sqsubset T$ and from slicing theory

$$\int_0^{r_\varepsilon} \mathbb{M}(\langle T, \mathrm{dist}(\cdot, x), r\rangle)\, \mathrm{d}r \leqslant \mathbb{M}(T \llcorner \mathrm{B}(x, r_0)).$$

This implies that there exists $r \in (r_\varepsilon/2, r_\varepsilon)$ such that

$$2 r_\varepsilon^{-1}\, \mathbb{M}(T \llcorner \mathrm{B}(x, r_\varepsilon)) > \mathbb{M}(\langle T, \mathrm{dist}(\cdot, x), r\rangle) \geqslant \mathbb{M}(\partial(T \llcorner (\mathrm{B}(x, r))) - \|\partial T\|(\mathrm{B}(x, r)).$$

By the choice of r_ε, we get

$$\begin{aligned}\mathbb{M}(\partial(T \llcorner \mathrm{B}(x, r))) &\leqslant 2 r_\varepsilon^{-1}\, \mathbb{M}(T \llcorner \mathrm{B}(x, r_\varepsilon)) + \|\partial T\|(\mathrm{B}(x, r)) \\ &< 2(1+\varepsilon)\theta^* \boldsymbol{\alpha}(m) r_\varepsilon^{m-1} + \varepsilon \boldsymbol{\alpha}(m-1) r^{m-1}\end{aligned}$$

and

$$\mathbb{M}(T \llcorner \mathrm{B}(x, r)) \geqslant \mathbb{M}(T \llcorner \mathrm{B}(x, r_\varepsilon/2)) \geqslant (1-\varepsilon)\theta^* \boldsymbol{\alpha}(m) r_\varepsilon^m \geqslant 2^{-m}(1-\varepsilon)\theta^* \boldsymbol{\alpha}(m) r^m.$$

Finally, as $\mathrm{diam}(\mathrm{spt}\, T \llcorner \mathrm{B}(x, r)) \leqslant 2r$,

$$\begin{aligned}\mathrm{reg}(T \llcorner \mathrm{B}(x, r)) &\geqslant \frac{2^{-m}(1-\varepsilon)\theta^* \boldsymbol{\alpha}(m) r^m}{2r(2(1+\varepsilon)\theta^* \boldsymbol{\alpha}(m) r_\varepsilon^{m-1} + 2\varepsilon \boldsymbol{\alpha}(m-1) r^{m-1})} \\ &> \frac{(1-\varepsilon)\theta^* \boldsymbol{\alpha}(m)}{2^{2m+2}((1+\varepsilon)\theta^* \boldsymbol{\alpha}(m) + 2^{m+1}\varepsilon \boldsymbol{\alpha}(m-1))}.\end{aligned}$$

For ε small enough, there exists $r > 0$ as small as desired such that $\mathrm{reg}(T \llcorner \mathrm{B}(x, r)) > 2^{-(2m-3)}$.

□

Definition 3.2.13 (Derivate)**.** Given an function F on $\mathcal{S}(T)$ for $T \in \mathbb{I}_m(\mathbb{R}^n)$, $x \in \mathrm{set}_m \|T\|$ and $\eta > 0$, we let

$$\underline{\mathfrak{D}}_T^\eta F(x) := \sup_{\delta > 0} \inf_{S \in \mathcal{S}(T, x, \eta, \delta)} \frac{F(S)}{\mathbb{M}(S)} \quad \text{and} \quad \overline{\mathfrak{D}}_T^\eta F(x) := \inf_{\delta > 0} \sup_{\mathcal{S}(T, x, \eta, \delta)} \frac{F(S)}{\mathbb{M}(S)}.$$

and the lower and upper derivates of F at x along T:

$$\underline{\mathfrak{D}}_T F(x) := \inf_{\eta > 0} \underline{\mathfrak{D}}_T^\eta F(x) \quad \text{and} \quad \overline{\mathfrak{D}}_T F(x) := \sup_{\eta > 0} \overline{\mathfrak{D}}_T^\eta F(x)$$

If $\overline{\mathfrak{D}}_T F(x)$ and $\underline{\mathfrak{D}}_T F(x)$ coincide and are finite, we denote this number by $\mathfrak{D}_T\, F(x)$ and call it the **derivate of F along T at x**, F is then **derivable at x along T**.

In particular, the mass operator, seen as a function on $\mathcal{S}(T)$ is derivable at a point $x \in \operatorname{set}_m \|T\|$, provided T is η-good at x for some $\eta > 0$. We also prove t

The following example (adapted from [61, 2.3.2]) is the main motivation for the above definition:

Example 3.2.14. Given

(i) $T \in \mathbb{I}_m(\mathbb{R}^n)$ and $x \in \operatorname{set}_m \|T\|$ such that for some $r > 0$

$$T \llcorner \mathrm{U}(x, r) = \theta \phi_{\#}(\mathbf{E}^m \llcorner A),$$

with A a set of finite perimeter in $\mathbb{R}^m$, $\phi : \operatorname{cl} A \to \mathbb{R}^n$ a map bilipschitz on its image and θ a positive integer.

(ii) ζ a $\|T\|$ measurable differential form of degree $m-1$ defined on $\operatorname{set}_m \|T\|$ and Θ_ζ the associated continuous function on $\mathcal{S}(T)$ as in example 3.2.4 (3.5).

If ζ is differentiable at x and ϕ is C^1 in a neighbourhood of $\phi^{-1}(x)$ then Θ_ζ is derivable at x along T with

$$\mathfrak{D}_T\, \Theta_\zeta(x) = \langle \mathrm{d}\zeta(x); \mathrm{D}\phi(\phi^{-1}(x)\rangle.$$

The claim in this example is proved in the following Lemma, which is inspired from Proposition 2.8 in [54], as is the definition of derivation.

Lemma 3.2.15. *Let T be an integral current of dimension m in $\mathbb{R}^n$, let $x \in \operatorname{set}_m T$ and let ω be a $\|T\|$ measurable $(m-1)$-form in the neighbourhood of x. Suppose that*

(i) For some $r_0 > 0$, the tangent m-vector field to T $\overrightarrow{T}$ has a representative in $\operatorname{set}_m \|T\| \cap \mathrm{B}(x, r_0)$ which is continuous at x,

(ii) $\|T\|$ has finite upper density at x.

(iii) ζ is differentiable at x

If additionnaly x is η-good at x in T for some $\eta > 0$, then the function Θ_ζ associated to ζ on T is derivable at x and its derivate is:

$$\mathfrak{D}_T\, \Theta_\zeta(x) = \langle \mathrm{d}\zeta(x), \overrightarrow{T}(x)\rangle.$$

Proof. Fix $\varepsilon > 0$ and $\eta > 0$ such that T is η-good at x. Since ζ is differentiable at x, for every positive ε_1, there exists $\delta_1 > 0$ such that for $y \in \mathrm{U}(x, \delta_1)$

$$|\zeta(y) - \zeta(x) - (y - x) \lrcorner\, \mathrm{d}\zeta(x)| \leqslant \varepsilon_1 |y - x|,$$

where $\lrcorner$ is the inner product in the notation of [26]. Notice that $y \mapsto \zeta(x)$ is a closed and smooth $(m-1)$-form and that $y \mapsto (y - x) \lrcorner\, \mathrm{d}\zeta(x)$ is a smooth $(m-1)$-form in $\mathbb{R}^m$ whose differential

is the constant form $y \mapsto \mathrm{d}\zeta(x)$. Furthermore, since $\overrightarrow{T}$ is continuous, for every positive ε_2, there exists $\delta_2 > 0$ such that for $\|T\|$-almost all $y \in \mathrm{U}(x, \delta_2)$, $\overrightarrow{T}(y)$ is defined,

$$|\overrightarrow{T}(y) - \overrightarrow{T}(x)| < \varepsilon_2.$$

Let $\Delta := \min\{\delta_1, \delta_2\}$ and suppose S is in $\mathcal{S}(T)$, by the smooth Stokes Theorem, and since the form $y \mapsto \zeta(x)$ is constant and therefore closed, there holds:

$$\Theta_\zeta(S) = \int \langle \zeta(y), \overrightarrow{\partial S}(y)\rangle \,\mathrm{d}\|\partial S\| = \int \langle \zeta(y) - \zeta(x), \overrightarrow{\partial S}(y)\rangle \,\mathrm{d}\|\partial S\|$$

and $\|S\|$-almost everywhere $\overrightarrow{S} = \overrightarrow{T}$ thus

$$\begin{aligned}
\langle \mathrm{d}\zeta(x), \overrightarrow{T}(x)\rangle \,\mathbb{M}(S) &= \int \langle \mathrm{d}\zeta(x), \overrightarrow{T}(x)\rangle \,\mathrm{d}\|S\|(y) \\
&= \int \langle \mathrm{d}\zeta(x), \overrightarrow{S}(y)\rangle \,\mathrm{d}\|S\|(y) - \int \langle \mathrm{d}\zeta(x), \overrightarrow{T}(y) - \overrightarrow{T}(x)\rangle \,\mathrm{d}\|S\|(y) \\
&= \int \langle (y-x) \lrcorner \,\mathrm{d}\zeta(x), \overrightarrow{\partial S}(y)\rangle \,\mathrm{d}\|\partial S\|(y) - \int \langle \mathrm{d}\zeta(x), \overrightarrow{T}(y) - \overrightarrow{T}(x)\rangle \,\mathrm{d}\|S\|(y).
\end{aligned}$$

Thus one has the following estimate:

$$\begin{aligned}
\Big|\Theta_\zeta(S) - \langle \mathrm{d}\zeta(x), \overrightarrow{T}(x)\rangle \,\mathbb{M}(S)\Big| &\leqslant \\
&\Big|\int \langle \zeta(y) - \zeta(x) - (y-x) \lrcorner \,\mathrm{d}\zeta(x), \overrightarrow{\partial S}(y)\rangle \,\mathrm{d}\|\partial S\|(y)\Big| + \varepsilon_2 |\,\mathrm{d}\zeta(x)|\,\mathbb{M}(S) \\
&\leqslant \varepsilon_1 \operatorname{diam}(\operatorname{spt} S)\,\mathbb{M}(\partial S) + \varepsilon_2 |\,\mathrm{d}\zeta(x)|\,\mathbb{M}(S).
\end{aligned}$$

Letting $\varepsilon_1 := \eta\varepsilon/2$ and $\varepsilon_2 := \varepsilon|\,\mathrm{d}\zeta(x)|^{-1}/2$ (or $\varepsilon_2 := 0$ if $\mathrm{d}\zeta(x) = 0$) yields

$$\forall S \in \mathcal{S}(T, x, \eta, \Delta),\ \left|\frac{\Theta_\zeta(S)}{\mathbb{M}(S)} - \langle \mathrm{d}\zeta(x), \overrightarrow{T}(x)\rangle\right| < \varepsilon.$$

Letting Δ go to 0, yields:

$$\langle \mathrm{d}\zeta(x), \overrightarrow{T}(x)\rangle - \varepsilon \leqslant \underline{\mathfrak{D}}_T^\eta \Theta_\zeta(x) \leqslant \overline{\mathfrak{D}}_T^\eta \Theta_\zeta(x) < \langle \mathrm{d}\zeta(x), \overrightarrow{T}(x)\rangle + \varepsilon.$$

We can now let ε go down to zero to get the derivate of Θ_ζ as claimed. □

The above proof can be modified to obtain the following:

Proposition 3.2.16. *If ζ is differentiable with differential 0 at $x \in \operatorname{set}_m T$ and the only assumption on T is that it is η-good at x for some $\eta > 0$, there holds*

$$\mathfrak{D}_T\, \Theta_\zeta(x) = 0.$$

Conjecture 3.2.17. *The condition above that the tangent vector be continuous at x is necessary. Indeed there exists a current $T \in \mathbb{I}_2(\mathbb{R}^3)$, and a smooth differential form ζ of degree 1 in $\mathbb{R}^3$ such that*

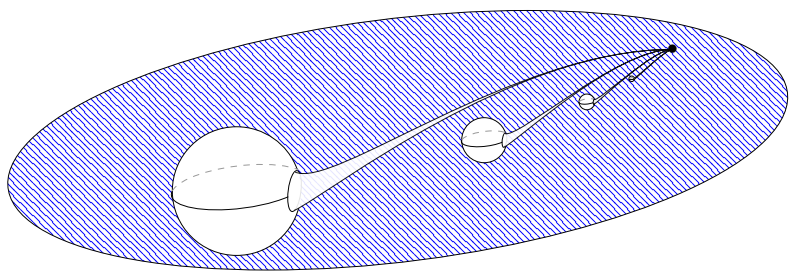

Figure 3.6: The pointed spheres accumulate at the point 0 with zero density.

(i) T is continuous on its definition set, except at the point $0 \in \mathbb{R}^3 \operatorname{set}_2 \|T\|$, where it is only approximately continuous.

(ii) T has density 1 everywhere.

(iii) The function on T: Θ_ζ is not derivable at 0.

Idea for a proof. Let $\chi : \mathbb{R}^3 \to [0,1]$ be a smooth cutoff function equal to 1 in $\mathrm{B}(0,1)$ and to 0 outside of $\mathrm{B}(0,2)$. Let ζ be the differential form in $\mathbb{R}^3$ defined by $\zeta : (x,y,z) \mapsto \chi(x,y,z)(-y\mathbf{e}_1^* + x\mathbf{e}_2^*)$.

Consider the two dimensional disk in $\mathbb{R}^3$ given by $D := \mathrm{B}^2(0,1) \times \{0\}$. Consider also three decreasing sequences $(d_j)_j$, $(R_j)_j$ and $(r_j)_j$ such that $x_j \to 0$, and for all j, $r_j < R_j < (d_j - d_{j+1})/10$. Consider a sequence of disjoint spheres of radius R_j centered at a point at distance d_j from 0 and connected "smoothly" to 0 by a cusp glued to the sphere in place of a disk of radius r_j. (See figure 3.6.)

Suppose we can construct these pointy spheres $(S_j)_j$ so that they do not intersect nor touch the plane $\{(x,y,z), z = 0\}$, have summable $\mathcal{H}^2$ measure, and such that their union has total $\mathcal{H}^2$-density 0 at 0. define a continuous unit simple tangent 2-vector field to each S_j: $\overrightarrow{S}_j$. Note that for all j, the currents $\mathcal{H}^2 \llcorner S_j \wedge \overrightarrow{T}_j$ is integral and a cycle. We let

$$T := \mathcal{H}^2 \llcorner D \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 + \sum_j \mathcal{H}^2 \llcorner S_j \wedge \overrightarrow{S}_j.$$

T is in $\mathbb{I}_2(\mathbb{R}^3)$, and its tangent vector $\overrightarrow{T}$ is continuous everywhere except at 0, where it is approximately continuous since the density of the union of the S_j is 0 at 0. We consider two sequences of subcurrents, for $j = 1, 2, \ldots$ $T_j := T \llcorner (\mathrm{B}^2(0,1/j) \times \{0\})$ and $Q_j := T \llcorner S_j$.

We have $\operatorname{reg}(T_j) = 1/2$ and $\operatorname{reg}(Q_j) = +\infty$, furthermore 0 is both in $\operatorname{spt} T_j$ and in $\operatorname{spt} Q_j$ for all j. However, there holds: $\Theta_\zeta(Q_j) = 0$ as Q_j is a cycle, and $\Theta_\zeta(T_j) = 2\pi j^{-2}$. Therefore, $\lim_j \Theta_\zeta(T_j)/\mathbb{M}(T_j) \to 2$. This proves that Θ_ζ is not derivable along T at 0. $\square$

Chapter 4

The Howard-Cousin Property and the Stokes-Cartan Theorem

In order to use gauge integration on an integral current T, T needs to have a good decomposition property we call Howard-Cousin Property. This property is modelled on the Howard-Cousin Lemma, which applies to bounded sets of finite perimeter. We start by stating and proving the Howard-Cousin Lemma, before defining the Howard-Cousin Property. Afterwards we give examples of classes of integral currents which have the Howard-Cousin Property, using a simple criterion on the set of points where the currents are not bilipschitz images of sets of finite perimeter. We then apply the Howard-Cousin Property on a class of currents to prove a generalized version of Stokes' Theorem on those. Finally we give examples of currents on which Stoke's Theorem does not hold, implying that they do not have the Howard-Cousin Property.

4.1 The Howard-Cousin Lemma in codimension 0

In order to prove the Howard-Cousin Lemma, we need a few classical results of Measure Theory, which are stated in Appendix A, as well as some facts related to covers by dyadic cubes, which we state and prove in the following subsection. We work on bounded subsets of $\mathbb{R}^m$, we say that two sets are **nonoverlapping** when the Lebesgue measure of their intersection is zero. This notion can be passed on to equivalence classes with respect to Lebesgue null sets. If $\mathcal{P}$ is a collection of subsets (respectively of tagged subsets (P_i, x_i)) of $\mathbb{R}^m$, we denote by $[\mathcal{P}]$ the union of all sets in $\mathcal{P}$ (respectively $\bigcup_{(P_i,x_i)\in\mathcal{P}} P_i$). $[\mathcal{P}]$ is called the **body** of $\mathcal{P}$. If $\mathcal{P}_0, \mathcal{P}_1, ...$ are collections, we denote by $\bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ the collection of sets that are in at least one of the collections $\mathcal{P}_n$. The restriction of a function or measure f, to a set A will be denoted by $f \llcorner A$. A set $E \subseteq \mathbb{R}^n$ is **m-thin** if it has $\mathcal{H}^{m-1}$ σ-finite measure. Most of the time, we will say only **thin** when the dimension m of the domain: current or set of finite perimeter, will be clear.

4.1.1 Cubes

Definition 4.1.1. A semi-cube is a set of the form $C = [a_1, a_1 + c) \times ... \times [a_m, a_m + c)$ where $(a_1, ..., a_m) \in \mathbb{R}^m$ and $c > 0$. For $n \in \mathbb{N}$ let $\mathcal{K}_n$ be the **n-th family of dyadic cubes with respect**

to C, that is the collection of 2^{mn} disjoint semi-cubes contained in C with diameter $2^{-n}\operatorname{diam}(C)$. A **dyadic cube with respect to** C is a member of the family $\mathcal{K} := \bigcup_{n\in\mathbb{N}} \mathcal{K}_n$.

When the largest cube C is not mentionned it is fixed, but arbitrary for now.

Lemma 4.1.2. *Given $\mathcal{R}$ a subcollection of $\mathcal{K}$, There exists a unique nonoverlapping subcollection $\mathcal{Q} \subseteq \mathcal{R}$ such that $[\mathcal{Q}] = [\mathcal{R}]$ and if $Q \in \mathcal{R}$ contains $Q' \in \mathcal{Q}$, then, $Q = Q'$. We call $\mathcal{Q}$ the* **maximal subfamily** *of $\mathcal{R}$.*

Proof. For $x \in [\mathcal{R}]$ define the collection $\mathcal{R}_x \subseteq \mathcal{R}$ of cubes K such that $x \in K$. $\mathcal{R}_x$ is well ordered by inclusion and bounded from above by $C \in \mathcal{K}_0$, it thus admits a maximum K_x. Let $\mathcal{Q} := \{K_x, x \in [\mathcal{R}]\}$. There holds $[\mathcal{Q}] = [\mathcal{R}]$ and if L' and L'' are two cubes in $\mathcal{Q}$ and they intersect then we can suppose $L' \subseteq L''$. There exists $x \in L'$ such that $L' = K_x$ and $x \in L'' \in \mathcal{Q}_x$ so by maximality $L' = L''$.

To prove that $\mathcal{Q}$ is unique, suppose that a family $\tilde{\mathcal{Q}}$ satisfies the hypotheses of the Lemma. Choose $x \in [\mathcal{K}] = [\mathcal{Q}] = [\tilde{\mathcal{Q}}]$ there exists $Q \in \mathcal{Q}$ and $\tilde{Q} \in \tilde{\mathcal{Q}}$ with $x \in Q$ and $x \in \tilde{Q}$. Since Q is maximal in $\{Q' \in \mathcal{K} | x \in Q'\}$, $\tilde{Q} \subseteq Q \in \mathcal{K}$. Thus, by construction, $\tilde{Q} = Q$. Since this argument works for every x in $[\mathcal{K}]$ and since every $\tilde{Q} \in \tilde{\mathcal{Q}}$ contains one such x, which is in turn contained in some $Q \in \mathcal{Q}$, this proves that $\mathcal{Q} = \mathcal{Q}'$. □

Lemma 4.1.3 (Number of parallel cubes around the same point)**.** *In $\mathbb{R}^m$, given a nonoverlapping family $\mathcal{Q}$ of parallel cubes, if a point x lies in the closure of every cube in $\mathcal{Q}$, then the cardinality of the family is less than 2^m.*

Proof. Without loss of generality, suppose $x = 0$. Define π_k the projection on the k-th coordinate in $\mathbb{R}^m$. For each $Q \in \mathcal{Q}$, and each $k \in \{1, ..., m\}$, $\pi_k(Q) = [a_k^Q, b_k^Q)$ with $a_k^Q < b_k^Q$ and $a_k^Q \leqslant 0 \leqslant b_k^Q$, so we can write : $Q = \cap_{k=1}^m (\mathbb{R}^{k-1} \times [a_k^Q, b_k^Q) \times \mathbb{R}^{m-k})$. Define the application $\phi : \mathcal{Q} \to \{0,1\}^m$ as follows:

$$\phi_k(Q) := \begin{cases} 0 \text{ if } b_k^Q = 0, \\ 1 \text{ otherwise.} \end{cases} \tag{4.1}$$

We claim that ϕ is injective. Indeed, suppose $\phi(Q) = \phi(R)$ for $Q, R \in \mathcal{Q}$. Since $Q \cap R = \cap_{k=1}^m \mathbb{R}^{k-1} \times [a_k^Q, b_k^Q) \cap [a_k^R, b_k^R) \times \mathbb{R}^{m-k}$,the two cubes intersect. Therefore $Q = R$. This implies that $\#\mathcal{Q} \leqslant \#\{0,1\}^m = 2^m$. □

4.1.2 Continuous functions on sets of finite perimeter

Fix a bounded set of finite perimeter A. An application G from $\mathcal{BV}(A)$ to $\mathbb{R}$ is called:

- **Subadditive**, if for all $A, B \in \mathcal{BV}$, then $|G(A \cup B)| \leqslant |G(A)| + |G(B)|$.
- **$\mathcal{T}$-continuous**, if G is continuous from $(\mathcal{BV}(A), \mathcal{T})$ to $\mathbb{R}$ with the usual topology.

Remark 4.1.4. Additive $\mathcal{F}$-continuous functions in $\mathrm{BV}(A)$are called **charges in** A, they form a linear space denoted by $\mathbf{CH}^m(A)$.

Lemma 4.1.5 (Equivalent definition with sequences)**.** *An application $G : \mathcal{BV} \to \mathbb{R}$ which is subadditive on disjoint sets is a subadditive $\mathcal{T}$-continuous function if and only if it satisfies one of the following equivalent conditions.*

(i) Given $\varepsilon > 0$ and n a positive integer, there exists $\theta > 0$ such that for $B \in \mathcal{BV}_n$,

$$|G(A)| \leqslant \theta|B| + \varepsilon\|B\|.$$

(ii) Given a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{BV}$, such that there exists $r > 0$, $M > 0$, such that for any $n \in \mathbb{N}$, $A_n \subseteq B(0,r)$, and $\|A_n\| < M$, if $|A_n| \to 0$ as $n \to \infty$, then $F(A_n) \to 0$ as well.

Proof. It is clear that $\mathcal{T}$-continuity and (i) each imply (ii). Let us show that $\mathcal{T}$-continuity implies (i). To show that (ii) implies $\mathcal{T}$-continuity, it suffices to show that $(\mathcal{BV}, \mathcal{T})$ is sequential. □

4.1.3 Gauges and families in a set of finite perimeter

A function $\delta : A \subseteq \mathbb{R}^m \to [0, +\infty)$ is called a **gauge in** A if its zero set $\delta^{-1}(0) \subseteq A$ is **thin** , that is, has σ-finite $\mathcal{H}^{m-1}$ measure. A **family** in A is a finite collection of non overlapping sets of finite perimeter $B_1, \ldots, B_p$, where each B_j is contained in A and $|B_j \cap B_{j'}| = 0$ whenever $j \neq j'$, together with **tags** $x_1, \ldots, x_p$, where $x_j \in \operatorname{cl}_e A \cap \operatorname{cl} B_j$ for all j. If δ is a gauge on $\operatorname{cl}_e A$, a family $((B_1, x_1), \ldots, (B_p, x_p))$ in A is **δ-fine** whenever $\operatorname{diam}(B_j) < \delta(x_j)$ for all j.

Given a positive η, a bounded set of finite perimeter B is called **η-regular** if

$$\frac{|B|}{\|B\| \operatorname{diam} B} > \eta.$$

The above ratio is the **regularity of** B. The regularity of a cube in $\mathbb{R}^m$ is given by $\eta_m := 1/(2m\sqrt{m})$.

If $\eta \in (0, \eta_m)$ is fixed, and $\mathcal{P} := ((B_1, x_1), \ldots, (B_p, x_p))$ is a family in A, we say that $\mathcal{P}$ is η-regular, provided each B_j is η-regular. If instead, η is a gauge on $\operatorname{cl}_e A$ (called regularity gauge to distinguish it from the gauge used in the fineness condition), we say that $\mathcal{P}$ is η-regular, provided each B_j is $\eta(x_j)$-regular, with $\eta(x_j) > 0$.

Definition 4.1.6. For $x \in \mathbb{R}^m$ and $\eta > 0$, a sequence $(C_j)_j$ in $\mathrm{BV}(\mathbb{R}^m)$ is said to **η-converge to** x whenever the C_j are eventually η-regular and $\operatorname{diam}(C_j \cup \{x\}) \to 0$ as j goes to infinity.

The following result is classical for sets of finite perimeter. It will be adapted to work with currents, see proposition 3.2.12 in the next section.

Lemma 4.1.7 ([61] lemma 2.5.2)**.** *Let A have finite perimeter in $\mathbb{R}^m$, $x \in \operatorname{int}_c A$ and $\eta > 0$. Consider a sequence $(C_j)_j$ of non trivial sets of finite perimeters in $\mathbb{R}^m$ which η-converges to x and with $x \in \operatorname{cl}(C_j)$ for all j, then*

$$\lim_j \frac{|C_j \cap A|}{|C_j|} = 1,$$

and the sequence $(C_j \cap A)_j$ θ-converges to x for each $\theta \in (0, m)$. In particular if $(C_j)_j$ is a sequence of cubes containing x, then $(C_j \cap A)_j$ is eventually θ-regular for all $\theta \in (0, 1/(2^m\sqrt{m}))$.

Note that the condition that x be in the critical interior of A cannot be lifted.

Proof. The last part of the statement is a direct consequence of the second part. Indeed, all cubes (of dimension m) in $\mathbb{R}^m$ have regularity $1/(2^m\sqrt{m})$.

To prove the first part of the statement, for $j = 0, 1, 2, \dots$, let $d_j := \operatorname{diam}(C_j \cup \{x\})$. By the isoperimetric inequality applied to C_j and the fact that all C_j are η-regular, for all j

$$\frac{|C_j|}{d_j^m} \geqslant \frac{|C_j|}{d_j^m}\frac{m^m \boldsymbol{\alpha}(m)}{2^m}\frac{|C_j|^{m-1}}{\|C_j\|^m} \geqslant \frac{m^m \boldsymbol{\alpha}(m)}{2^m}\left(\frac{|C_j|}{\|C_j\| d_j}\right)^m \geqslant \frac{m^m \boldsymbol{\alpha}(m)\eta^m}{2^m}. \tag{4.2}$$

As $x \in \operatorname{int}_c A \subseteq \operatorname{int}_e A$, there holds $\Theta^m(\mathcal{L}^m \llcorner A, x) = 1$ and we claim

$$\lim_j \frac{|C_j \cap A|}{|C_j|} = 1.$$

Indeed, suppose that there exists an increasing sequence of indices j_k starting a j_0 and an $\varepsilon > 0$ such that for all $k \geqslant 0$

$$|C_{j_k} \cap A| \leqslant (1-\varepsilon)|C_{j_k}|.$$

Passing to the larger balls, $\mathrm{B}(x, d_{j_k})$ we get

$$|\,\mathrm{B}(x, d_j) \cap A| \leqslant |\,\mathrm{B}(x, d_{j_k})| - \varepsilon |C_{j_k}|,$$

whereas by the assumption that x is a point of density 1 of A, for k large enough we have

$$|\,\mathrm{B}(x, d_j) \cap A| \geqslant \left(1 - \frac{\varepsilon(\eta m)^m}{10}\right)|\,\mathrm{B}(x, d_{j_k})|.$$

Combining the two inequalities yields

$$\left(1 - \frac{\varepsilon(\eta m)^m}{10}\right)|\,\mathrm{B}(x, d_{j_k})| \leqslant |\,\mathrm{B}(x, d_{j_k})| - \varepsilon |C_{j_k}|.$$

Dividing by $|\,\mathrm{B}(x, d_{j_k})| = \boldsymbol{\alpha}(m) 2^{-m} d_{j_k}^m$ and then using (4.2), we get

$$1 - \frac{\varepsilon(\eta m)^m}{10} \leqslant 1 - \varepsilon \frac{|C_{j_k}|}{\boldsymbol{\alpha}(m) d_{j_k}^m} \leqslant 1 - \varepsilon(\eta m)^m,$$

a contradiction.

Now, notice that

$$\|A \cap C_j\| \leqslant \|C_j\| + \mathcal{H}^{m-1}(\partial_* A \cap \mathrm{B}(x, d_j))$$

and compute

$$\frac{d_j \|A \cap C_j\|}{|C_j|} \leqslant \frac{d_j \|C_j\|}{|C_j|} + \frac{\mathcal{H}^{m-1}(\partial_* A \cap \mathrm{B}(x, d_j))}{d_j^{m-1}}\frac{d_j^m}{|C_j|} < \frac{1}{\eta} + C\eta^{-m}\beta_j,$$

where $\beta_j := d_j^{-(m-1)}\mathcal{H}^{m-1}(\partial_* A \cap \mathrm{B}(x, d_j)) \to 0$ as $x \in \operatorname{int}_c A$. Fix $\theta \in (0, \eta)$, the sets $A \cap C_j \cup \{x\}$ are eventually θ-regular. As $\operatorname{diam}(C_j \cup \{x\}) \to 0$ implies $\operatorname{diam}(A \cap C_j \cup \{x\}) \to 0$. This concludes the proof. □

In the following, we prove that we can partition a bounded set of finite perimeter with respect to a gauge δ in the body a δ-fine, regular family plus an error set which is controlled in the topology $\mathfrak{T}$. We start with the easier case of a positive gauge and a cube, then remove the positivity constraint,

before extending the result to a bounded essentially closed set of finite perimeter and then to the general case.

4.1.4 Statement and sketch of proof

This Lemma was first proved by Howard (Lemma 5 in [40]) with a constant regularity constraint on the subsets. Here, we state it with a (large enough) **regularity gauge** instead. This modification does not require any real change in the proof and will enable us to use the lemma on currents whose local regularity is not uniform.

Lemma 4.1.8 (Howard-Cousin)**.** *Let A be a bounded BV set in $\mathbb{R}^m$, G be a nonnegative subadditive $\mathfrak{T}$-continuous function on $\mathcal{BV}(A)$, let δ be a gauge on $\mathrm{cl}_e\, A$ and η be another gauge on $\mathrm{cl}_e\, A$, which we call* **regularity gauge***, it must satisfy $\eta(x) < 1/(2m\sqrt{m})$ for all $x \in \mathrm{cl}_e\, A$. Fix $\varepsilon > 0$. There exists a δ-fine, η-regular family $\mathcal{P} = \{(P_1, x_1), ..., (P_q, x_q)\}$ in A such that*

$$|G(A - [\mathcal{P}])| < \varepsilon. \tag{4.3}$$

The proof of Lemma 4.1.8 has three main steps. One starts by proving by contradiction a covering result (Lemma 4.1.10) for a positive gauge δ^* on a cube.

In the following step, one deals with the zero set T of the gauge Δ in a closed cube (Lemma 4.1.12). It requires a good cover of T by cubes, controlling their boundary to ensure that their union is ε-small for the subadditive $\mathfrak{T}$-continuous function G (Lemma 4.1.12). This is done by a careful application of Lemma A.3 on the equivalence between Hausdorff measure and net measure. The last step consists in defining a good positive gauge on a cube that contains A, in order to be able to apply Lemma 4.1.12. One defines a first intermediate gauge on $\mathrm{cl}_e\, A$ to ensure that each cube in the desired family will have a regular intersection with A. This can be done by forcing the tag points to be in the critical interior of A and taking a small enough (but positive) value of the gauge at each point of $\mathrm{int}_c\, A$ (depending on the point x and on $\eta(x)$).

We then extend the gauge to take the value 0 on $(\mathrm{cl}_e\, A \backslash \mathrm{int}_c\, A)$ which is thin by Lemma C.2. In a second step, one defines a gauge on a closed cube C containing $\mathrm{cl}_e\, A$ in order to apply Lemma 4.1.12 and so that any cube intersecting A is tagged inside $\mathrm{int}_c\, A$. In order to do this, one need to approximate A by an essentially closed BV subset A_ε such that $|G(A \backslash A_\varepsilon)| < \varepsilon/2$ which can be done using Lemma C.3 (the weak inner approximation property of bounded sets of finite perimeter). Finally one applies Lemma 4.1.12 to C with the new gauge, $\varepsilon/2$ and the restriction of G to $\mathcal{BV}(A_\varepsilon)$ which is still a subadditive $\mathfrak{T}$-continuous function.

4.1.5 Proof of Howard-Cousin Lemma

Definition 4.1.9. Fix a positive integer p, a cube C, a gauge δ on $\mathrm{cl}(C)$ and a family $\mathcal{P} = \{(x_1, Q_1), \ldots, (x_m, Q_p)\}$ where $\{Q_1, \ldots, Q_p\}$ is a nonoverlapping family of dyadic cubes with respect to C such that for each j, $x_j \in \mathrm{cl}(Q_j)$. We say that $\mathcal{P}$ is a δ-fine tagged **partition** of C in case $C = \bigcup_j Q_j$ and for each j, $\mathrm{diam}(Q_j) < \delta(x_j)$.

Lemma 4.1.10 (Cousin)**.** *Given a cube C and positive gauge δ on $\mathrm{cl}(C)$, there exists a δ-fine tagged partition of C.*

Proof. Suppose that no partition of C in dyadic cubes is δ-fine. This implies that at least one of the 2^m cubes in the family $\mathcal{K}_1$, say for instance C_1, admits no $(\delta \llcorner C_1)$-fine partition. We thus define inductively a decreasing sequence of cubes $(C_j)_{j\geqslant 1}$ with $C_j \in \mathcal{K}_j$ and there is no $(\delta \llcorner C_j)$-fine partition of C_j, for all $j \geqslant 1$. Let $x \in \mathbb{R}^m$ be such that $\{x\} = \bigcap_{j\geqslant 1} \mathrm{cl}(C_i) \subseteq \mathrm{cl}(C)$, by definition, $\delta(x) > 0$ so there exists $j \geqslant 1$ such that $\mathrm{diam}(C_j) < \delta(x)$. Hence $((C_j, x))$ is a δ-fine partition of C_j which is a contradiction. □

Lemma 4.1.11 (Thin set covering)**.** *Given a subadditive $\mathfrak{T}$-continuous function in $\mathcal{BV}(\mathbb{R}^m)$, G, a cube $C \subseteq \mathbb{R}^m$, a thin set $E \subseteq \mathrm{cl}(C)$ and $\varepsilon > 0$. There exists a nonoverlapping locally finite family $\mathcal{T}$ of dyadic cubes with respect to C, the union of which is a neighbourhood of E, such that for any finite subfamily $\mathcal{D} \subseteq \mathcal{T}$, $|G([\mathcal{D}])| < \varepsilon$. Furthermore, any $x \in E$ lies in the interior of the union of a finite subfamily of $\mathcal{T}$.*

Proof. By continuity of G, given a positive integer n, there exists a constant β_n such that for a BV subset B of $\mathbb{R}^m$ included in C with

$$|B| < 3^m 2^{m^2} \beta_n, \tag{4.4}$$

$$\|B\| < 2m 3^m 2^{m^2} \tag{4.5}$$

there holds $|G(B)| < \varepsilon/2^n$. Write E as the union of sets E_n with $\mathcal{H}^{m-1}(E_n) < 1$ for all $n > 1$, this is possible thanks to Lemma A.2. For each n, there exists $\delta_n < \beta_n$ such that $\mathcal{H}^{m-1}_{\delta_n}(E_n) < 1$. Following Lemma A.3, cover E_n with a family $\mathcal{T}_n$ of dyadic cubes of diameter less than δ_n, which we can suppose to be non overlapping, and such that $\sum_{K\in\mathcal{T}_n} \mathrm{diam}(K)^{m-1} < 3^m 2^{m^2}$. In particular, since $\|K\| \leqslant 2m\,\mathrm{diam}(K)^{m-1}$, and $|K| \leqslant \mathrm{diam}(K)^{m-1}\beta_n$, there holds:

$$\sum_{K\in\mathcal{T}_n} |K| < 3^m 2^{m^2} \beta_n,$$

$$\sum_{K\in\mathcal{T}_n} \|K\| < 2m 3^m 2^{m^2}.$$

This implies that the union of **any** subfamily of $\mathcal{T}_n$ is a subset of C with finite perimeter satisfying the conditions 4.4 and 4.5 for this particular n. Let $\mathcal{T}$ be the maximal nonoverlapping subfamily of the union of the families $\mathcal{T}_n$, it is a covering of a neigbourhood of E and satisfies the conditions of the lemma. Indeed, take any finite subfamily $\mathcal{E}$ and write it as a disjoint union of a finite number of subfamilies $\mathcal{E}_n \subseteq \mathcal{T}_n$, where $n \in \{n_1, ..., n_k\}$. There holds:

$$|G([\mathcal{E}])| \leqslant \sum_{i=1}^{k} |G([\mathcal{E}_{n_i}])| < \sum_{n\geqslant 1} \varepsilon/2^n. \tag{4.6}$$

Furthermore by Lemma A.3, each $x \in E$ is contained in the interior of the union of a finite family of $\mathcal{T}_n$, therefore, since we chose maximal cubes in $\mathcal{T}$, x it is contained in the interior of a finite union of cubes of $\mathcal{T}$ □

Lemma 4.1.12 (Howard-Cousin in the cube)**.** *Let G be a subadditive $\mathfrak{T}$-continuous function, C a semi-cube, Δ a positive gauge on $\mathrm{cl}(C)$, and $\varepsilon > 0$. There exists a Δ-fine family of cubes $\mathcal{P} = \{(P_1, x_1), ..., (P_q, x_q)\}$ in C such that $|G(C - [\mathcal{P}])| < \varepsilon$.*

Proof. Let $E = \{\Delta = 0\}$, cover a neigbourhood of E with locally finite families $\mathcal{T}$ and $\mathcal{T}_n$ of cubes as in Lemma 4.1.11 (each being nonoverlapping) and define a gauge Δ_+ on $\mathrm{cl}(C)$ as follows:

$$\Delta_+(x) = \begin{cases} \Delta(x) \text{ if } x \notin E, \\ \min\{\mathrm{diam}(K), x \in \mathrm{cl}\, K, K \in \mathcal{T}\} \text{ if } x \in E. \end{cases}$$

Note that the min is indeed taken over a finite family of cubes by Lemma 4.1.3. By Lemma 4.1.10 there exists a Δ_+-fine partition of C composed of dyadic cubes :

$$\mathcal{Q} = \{(P_1, x_1), ..., (P_q, x_q), (P_{q+1}, x_{q+1}), ..., (P_p, x_p)\},$$

which we can suppose to be ordered so that $j > q$ if and only if there exists $K \in \mathcal{T}$ such that $P_j \subseteq K$. Let $\mathcal{D} \subseteq \mathcal{T}$ be the collection of the cubes that contain some P_j and $D = [\mathcal{D}]$. Since $\mathcal{P}$ is finite the diameter of the P_i is bounded from below, and since $\mathcal{T}$ is nonoverlapping, $\mathcal{D}$ must be finite. For $j \in \{1, ..., q\}$, P_j does not overlap $\mathcal{D}$ (otherwise it would either contain a cube in $\mathcal{D}$ and thus a P_l for $l > q$ or be contained in a cube in $\mathcal{D} \subseteq \mathcal{T}$. Furthermore, by definition of Δ_+, $x_j \notin T$. Indeed, suppose the contrary, then $\mathrm{diam}\, P_j = \Delta_+(x_j)$ since $[\mathcal{T}]$ contains a neighbourhood of $x_j \in T$ (by lemma 4.1.11), since all cubes are dyadic, there exists $K \in \mathcal{T}$ with either $P_j \subseteq K$ (but then K would be in $\mathcal{D}$) or $K \subsetneq P_j$, which means $\mathrm{diam}\, K < \mathrm{diam}\, P_j$, which would be a contradiction. Thus $\mathcal{P} = \{(P_1, x_1), ..., (P_q, x_q)\}$ is the desired family: since $D = \cup\mathcal{D}$ is a finite union of cubes in $\mathcal{T}$, Lemma 4.1.11 ensures that $|G(D)| < \varepsilon$. □

The following lemma is a consequence of lemma 4.1.7:

Lemma 4.1.13 (Regularity of $Q \cap A$)**.** *Let $A \in \mathcal{BV}$ and $x \in \mathrm{int}_c\, A$ with $\eta(x) \in (0, m^{-3/2}/2)$. Then there exists $\sigma_x > 0$ such that if Q is a cube with $\mathrm{diam}\, Q < \sigma_x$ and $x \in \mathrm{cl}(Q)$, then $Q \cap A$ is $\eta(x)$-regular.*

Corollary 4.1.14 (Existence of a good positive gauge)**.** *Let $A \in \mathcal{BV}$, there exists σ, a positive gauge on $\mathrm{cl_e}\, A$ such that if $\{(P_1, x_1), ..., (P_q, x_q)\}$ is a σ-fine family with $\forall i, x_i \in \mathrm{cl_e}\, A$, $\{(P_1 \cap A, x_1), ..., (P_q \cap A, x_q)\}$ is a δ-fine, η-regular family in A.*

Proof. Define σ as follows:

$$\sigma(x) := \begin{cases} \sigma_x \text{ if } x \in \mathrm{int}_c\, A \text{ as in Lemma 4.1.13}, \\ 0 \text{ if } x \in (\mathrm{cl_e}\, A \setminus \mathrm{int}_c\, A). \end{cases}$$

□

Proof of Howard-Cousin's Lemma. Thanks to Lemma C.3 and the continuity property of subadditive $\mathfrak{T}$-continuous functions, there exists $A_\varepsilon \subseteq A$ essentially closed such that $|G(A - A_\varepsilon)| < \varepsilon/2$. Choose a gauge σ on $\mathrm{cl_e}\, A_\varepsilon$ as in Corollary 4.1.14. Let C be a cube containing $\mathrm{cl_e}\, A_\varepsilon$. define a positive gauge Δ on $\mathrm{cl}(C)$:

$$\Delta(x) := \begin{cases} \min\{\delta(x), \sigma(x)\} \text{ if } x \in \mathrm{cl_e}\, A_\varepsilon \\ \mathrm{dist}(x, \mathrm{cl_e}\, A_\varepsilon) \text{ if } x \in (\mathrm{cl}(C) \setminus \mathrm{cl_e}\, A). \end{cases}$$

For $B \in \mathcal{BV}$, define $G_\varepsilon(B) := G(A_\varepsilon \cap B)$. One can show that the application G is a subadditive $\mathfrak{T}$-continuous function in $\mathbb{R}^m$: The subadditivity commes from that of F and the continuity from that of G and from the fact that $\|B \cap A_\varepsilon\| \leqslant \|B\| + \|A_\varepsilon\|$. Use Lemma 4.1.12 with $\varepsilon/2$ to define a Δ-fine family of cubes in C and restrict it to A_ε to get a regular δ-fine family thanks to Lemma 4.1.13. $\square$

4.2 Howard-Cousin Property for currents.

4.2.1 The property

We now adapt the notions used above to the case of integral currents of dimension m in $\mathbb{R}^n$. A **gauge** on a set A is a nonnegative function on A whose zero set is $\mathcal{H}^{m-1}$ σ-finite. A **family in** T is a collection $\mathcal{P} := ((S_1, x_1), \ldots, (S_p, x_p))$ where for all j, S_j is a subcurrent of T ($S_j \sqsubset T$) and $x_j \in \operatorname{spt} S_j \cap \operatorname{set}_m \|T\|$ and for all $j \neq j'$, $\|S_j\| \perp \|S_{j'}\|$ (S_j and $S_{j'}$ are called **nonoverlapping** if the last condition holds).

If δ and η are gauges in $A \subseteq \operatorname{set}_m \|T\|$, the family $\mathcal{P}$ is δ-fine and η-regular, provided for $j = 1, \ldots, p$:

$$\operatorname{diam} \operatorname{spt} S_j < \delta(x_j)$$

and

$$\frac{\mathbb{M}(S_j)}{\mathbb{M}(\partial S_j) \operatorname{diam}(\operatorname{spt} S_j)} > \eta(x_j) > 0.$$

We will call η a **regularity gauge on** T to distinguish its role from that of δ.

Definition 4.2.1. An integral current $T \in \mathbb{I}_m(\mathbb{R}^n)$ has the **Howard-Cousin property with regularity gauge** η_T **and exceptional set** E_T, if $E_T \subseteq \operatorname{spt} T$, η_T is a gauge on $\operatorname{set}_m \|T\| \backslash E_T$ such that given a continuous subadditive function G on $\mathcal{S}(T)$, two real numbers $\tau \in (0, 1)$, a gauge δ on $\operatorname{set}_m \|T\| \backslash E_T$ and a gauge η on $\operatorname{set}_m \|T\| \backslash E_T$ with $0 \leqslant \eta \leqslant \eta_T$, there exists a δ-fine, η-regular family $\mathcal{P} = ((S_1, x_1), \ldots, (S_p, x_p))$ in T such that

$$\left| G\left(T - \sum_{j=1}^{p} S_j\right) \right| < \tau. \tag{4.7}$$

(A family in T is called (G, τ)**-full** if (4.7) holds.) T has the **hereditary Howard-Cousin Property** if for all $S \in \mathcal{S}(T)$, S has the Howard-Cousin Property with exceptional set $E_S \subseteq E_T \cap \operatorname{spt} S$ and regularity gauge $\eta_S = \eta_T|_{\operatorname{set}_m \|S\| \backslash E_S}$.

We will sometimes say that (T, E_T, η_T) has the (hereditary) Howard-Cousin Property, to be more concise. A first remark is that if $T \in \mathbb{I}_m(\mathbb{R}^n)$ has the Howard-Cousin Property and θ is an integer, then θT has the Howard-Cousin Property with the same exceptional set and regularity gauge. In the light of the previous section, the following proposition is natural:

Proposition 4.2.2. *If $A \subset \mathbb{R}^m$ is a bounded set of finite perimeter, then the integral current $\mathbf{E}^m \llcorner A$ has the Howard-Cousin Property with empty exceptional set and constant regularity gauge taking the value $1/(2m\sqrt{m})$.*

Proof. Fix $\varepsilon > 0$, a gauge δ on $\operatorname{set}_m \|T\|$ and a nonnegative subadditive function G on $\mathcal{S}(T)$. Notice that since $\operatorname{set}_m \|T\| = \operatorname{cl}_e A$, δ is also a gauge on $\operatorname{cl}_e A$ and since $\mathcal{S}(T) = \{S = T \llcorner B, B \in \mathrm{BV}(A)\}$, G is naturally associated to a subadditive nonnegative function on $\mathrm{BV}(A)$, which we denote by $\tilde{G}$, there holds $\tilde{G}(B) = G(T \llcorner B)$.

Fix $\eta \in (0, 1/(2m\sqrt{m}))$ and apply the Howard-Cousin Lemma to the set of finite perimeter A, the gauge δ and the nonnegative subadditive function $\tilde{G}$ to get an η regular, δ-fine family $((B_1, x_1), \dots, (B_p, x_p))$ in A with $\tilde{G}(A \backslash (B_1 \cup \cdots \cup B_p)) < \varepsilon$. The family $((T_1 := T \llcorner B_1, x_1), \dots, (T_p := T \llcorner B_p, x_p))$ is δ-fine and η-regular and by definition of $\tilde{G}$, we have $G(T - (T_1 + \cdots + T_p)) = \tilde{G}(A \backslash (B_1 \cup B_p)) < \varepsilon$. □

Proposition 4.2.3. *Given $T \in \mathbb{I}_m(\mathbb{R}^n)$ has the Howard-Cousin Property with exceptional set E_T and regularity gauge η_T and a current $T' \in \mathbb{I}_m(\mathbb{R}^{n'})$ is of the form $T' = \phi_\# T$, where ϕ is a bilipschitz map from $\operatorname{spt} T$ to its image in $\mathbb{R}^{n'}$, then T' has the Howard-Cousin Property with exceptional set $E_{T'} = \phi(E_T)$ and regularity gauge $\eta_{T'} := (\operatorname{Lip} \phi^{-1})^{-m} (\operatorname{Lip} \phi)^{-m} \eta_T \circ \phi$.*

Proof. As ϕ is bilipschitz, thus it sends m-thin sets into m thin sets. Furthermore, by proposition B.2, $\phi(\operatorname{set}_m \|T\|) = \operatorname{set}_m \|T'\|$, $\eta_{T'}$ defined in the statement is a gauge on $\operatorname{set}_m \|T'\| \backslash \phi(E_T)$. Fix $\varepsilon > 0$, a continuous function G on $\mathcal{S}(T')$ and gauges δ' and η' on $\operatorname{set}_m \|T'\| \backslash \phi(E_T)$ with $0 \leqslant \eta \leqslant \eta_{T'}$. Consider the pull-back $\phi^\# G$ of G by ϕ on $\mathcal{S}(T)$. By proposition 3.2.10, $\phi^\# G$ is nonnegative continuous and subadditive on $\mathcal{S}(T)$. By the same argument as above, the function $\tilde{\delta}$ defined for all x in $\operatorname{set}_m \|T\| \backslash E_T$ by $\tilde{\delta}(x) := \delta'(\phi(x)) / \operatorname{Lip} \phi$ is a gauge on $\operatorname{set}_m \|T\| \backslash E_T$. Similarly, the function

$$\tilde{\eta} := (\operatorname{Lip} \phi^{-1})^m (\operatorname{Lip} \phi)^m \eta' \circ \phi$$

is a gauge on $\operatorname{set}_m \|T\| \backslash E_T$ and satisfies $0 \leqslant \tilde{\eta} \leqslant (\operatorname{Lip} \phi^{-1})^m (\operatorname{Lip} \phi)^m \eta_{T'} \circ \phi = \eta_T$. By the Howard-Cousin Property of (T, E_T, η_T), there exits a $\tilde{\delta}$-fine $\tilde{\eta}$-regular tagged family $\mathcal{P} = ((S_1, x_1), \dots, (S_p, x_p))$ in T with $G(T - [\mathcal{P}]) < \varepsilon$. Consider the collection

$$\mathcal{P}' := ((\phi_\# S_1, \phi(x_1)), \dots, (\phi_\# (S_p), \phi(x_p))).$$

First, notice that $\mathcal{P}'$ is a tagged family in T'. This is due to the bilipschitz property of ϕ: ϕ sends a point of positive upper density of T to a point of positive upper density of T' and a point of the support of a subcurrent S to a point of $\operatorname{spt} \phi_\# S$. $\mathcal{P}'$ is also non overlapping by the bijectivity of ϕ (we used proposition B.2 again).

By definition of the pull-back of G, there holds

$$G(T' - [\mathcal{P}']) = G(\phi_\# (T - \sum_{j=1}^{p} S_j)) = \phi^\# G(T - \sum_{j=1}^{p} S_j) < \varepsilon.$$

We also see that $\mathcal{P}'$ is δ-fine: indeed

$$\operatorname{diam} \operatorname{spt} \phi_\# S_j \leqslant (\operatorname{Lip} \phi)(\operatorname{diam} \operatorname{spt} S_j) < (\operatorname{Lip} \phi) \tilde{\delta}(x_j) \leqslant \delta'(\phi(x_j)).$$

We turn to the regularity of $\mathcal{P}'$. By the Lipschitz property of ϕ there holds for any $S \sqsubset T$,

$$\mathbb{M}(\partial(\phi_\# S)) \leqslant (\operatorname{Lip} \phi)^{m-1} \mathbb{M}(\partial S)$$

and since ϕ^{-1} is also Lipschitz, and for such S, $\phi_\#^{-1}(\phi_\# S) = S$, we have

$$\mathbb{M}(\phi_\# S) \geqslant (\operatorname{Lip}\phi^{-1})^{-m}\,\mathbb{M}(\phi_\# S)$$

which implies for all j, as S_j is $\tilde{\eta}(x_j)$ regular

$$\frac{\mathbb{M}(\phi_\# S_j)}{\mathbb{M}(\partial(\phi_\# S_j))\operatorname{diam}\operatorname{spt}(\phi_\# S_j)} \geqslant \frac{1}{(\operatorname{Lip}\phi^{-1})^m(\operatorname{Lip}\phi)^m}\frac{\mathbb{M}(S_j)}{\mathbb{M}(\partial(S_j))\operatorname{diam}\operatorname{spt}(S_j)} > \frac{1}{(\operatorname{Lip}\phi^{-1})^m(\operatorname{Lip}\phi)^m}\tilde{\eta}(x_j) = \eta'(\phi(x_j)).$$

As η' is an arbitrary gauge with $0 \leqslant \eta' \leqslant \eta_{T'}$, and δ', G and ε are arbitrary as well, the current $T' = \phi_\# T$ has the Howard-Cousin Property with exceptional set $\phi(E_T)$ and regularity gauge $\eta_{T'}$ (defined in the statement). □

Using these results and triangulation argument, one could prove that a current associated to a compact C^1 submanifold of $\mathbb{R}^n$ with boundary has the Howard-Cousin property with empty exceptional set and constant regularity gauge. Similarly, one can also consider a bounded "finite perimeter subset" of a C^1 submanifold with boundary in $\mathbb{R}^n$.

However, we do not need to use difficult topological arguments and we give a more general criterion for a current to have the Howard-Cousin property in the next section. This argument will cover the C^1 manifold case.

Remark 4.2.4. Given a current $T \in \mathbb{I}_m(\mathbb{R}^n)$ and a set $E \subseteq \operatorname{spt} T$. Suppose two regularity gauges, η and η' are given on $\operatorname{set}_m \|T\| \backslash E$ such that (T, E, η) and (T, E, η') have the Howard-Cousin Property. The gauge $\min(\eta, \eta')$ is also a suitable regularity gauge for the Howard-Cousin property of T. However, it is not clear that $\max(\eta, \eta')$ is also suitable. This may have important implications on the definition of the Pfeffer Integral on the current T which we will study in chapter 5.

Remark 4.2.5. In section 4.4.3, we exhibit an integral current S which does not have the Howard-Cousin Property. We also build another integral current T which has the Howard-Cousin Property, but such that $S \sqsubset T$. This implies that T does not have the hereditary Howard-Cousin Property.

Proposition 4.2.6. *The Howard-Cousin Property is stable by "disjoint sum", more precisely, Given non overlapping integral currents of dimension m in $\mathbb{R}^n$, T and T', sets $E \subseteq \operatorname{spt} T$ and $E' \subseteq \operatorname{spt} T'$ and gauges η on $\operatorname{set}_m \|T\| \backslash E$ and η' on $\operatorname{set}_m \|T'\| \backslash E'$, such that (T, E, η_T) and $(T', E', \eta_{T'})$ have the Howard-Cousin Property, then $T + T'$ has the Howard-Cousin Property with exceptional set $\tilde{E} := (E \backslash \operatorname{set}_m \|T'\|) \cup (E' \backslash \operatorname{set}_m \|T\|) \cup (E \cap E')$ and regularity gauge $\tilde{\eta}$, where for $x \in \operatorname{set}_m \|T + T'\| \backslash \tilde{E}$,*

$$\tilde{\eta}(x) := \begin{cases} \eta_T(x), \text{ if } x \in (\operatorname{set}_m \|T\| \backslash E) \backslash (\operatorname{set}_m \|T'\| \backslash E'), \\ \eta_{T'}(x), \text{ if } x \in (\operatorname{set}_m \|T'\| \backslash E') \backslash (\operatorname{set}_m \|T\| \backslash E), \\ \min(\eta_T(x), \eta_{T'}(x)), \text{ if } x \in (\operatorname{set}_m \|T\| \backslash E) \cap (\operatorname{set}_m \|T'\| \backslash E'). \end{cases}$$

Proof. Let us first prove that $\tilde{\eta}$ is well defined. Without loss of generality, we can suppose that $E \subseteq \operatorname{set}_m \|T\|$ and $E' \subseteq \operatorname{set}_m \|T'\|$. Note that $\operatorname{set}_m \|T + T'\| = \operatorname{set}_m \|T\| \cup \operatorname{set}_m \|T'\|$. Indeed,

if $T := (\theta\mathcal{H}^m \llcorner M) \wedge \overrightarrow{T}$ and $T' := (\theta'\mathcal{H}^m \llcorner M') \wedge \overrightarrow{T}'$ are nonoverlapping, then $\theta\mathcal{H}^m \llcorner M$ and $\theta'^m\mathcal{H}^m \llcorner M'$ are orthogonal, thus since θ and θ' are positive $\mathcal{H}^m$ almost everywhere on M and M' respectively, $\mathcal{H}^m(M \cap M') = 0$ and $T + T' = \mathcal{H}^m \llcorner (M \cup M') \wedge (\theta \mathbb{1}_M \overrightarrow{T} + \theta' \mathbb{1}_{M'} \overrightarrow{T}')$, thus

$$\|T + T'\| = \|T\| + \|T'\|,$$

leading for all $x \in \mathbb{R}^n$ to

$$\Theta^{m*}(\|T + T'\|, x) \geqslant \max\{\Theta^{m*}(\|T\|, x), \Theta^{m*}(\|T'\|, x)\}.$$

Now, we have defined $\tilde{\eta}$ on $(\mathrm{set}_m \|T\| \backslash E) \cup (\mathrm{set}_m \|T'\| \backslash E')$ and there holds

$$\begin{aligned}(\mathrm{set}_m \|T\| \backslash E) \cup (\mathrm{set}_m \|T'\| \backslash E') = \\ (\mathrm{set}_m \|T\| \cup \mathrm{set}_m \|T'\|) \backslash ((E \cap E') \cup (E \backslash \mathrm{set}_m \|T'\|) \cup (E' \backslash \mathrm{set}_m \|T\|)) = \tilde{E}.\end{aligned}$$

and $\tilde{\eta}$ is defined on all of $\tilde{E}$. Fix a $\mathcal{F}$-continuous subadditive function G on $\mathcal{S}(T + T')$, two gauges δ and η on $\mathrm{set}_m \|T + T'\| \backslash \tilde{E}$ with $\eta \leqslant \tilde{\eta}$ and a positive real number ε. Notice that $\mathcal{S}(T) \subseteq \mathcal{S}(T + T')$ and that $G \llcorner T := G|_{\mathcal{S}(T)}$ is a continuous subadditive function on $\mathcal{S}(T)$. Apply the Howard-Cousin Property of (T, E, η_T) for the function $G \llcorner T$, the gauge $\delta|_{\mathrm{set}_m \|T\| \backslash E}$, positive constant $\varepsilon/2$ and regularity gauge $\eta|_{\mathrm{set}_m \|T\| \backslash E} \leqslant \eta_T$ to get a tagged family $\mathcal{P}$ in T, which is δ-fine, η-regular and $(G \llcorner T, \varepsilon/2)$-full. Do the same for T' to get a tagged family $\mathcal{P}'$ in T' which is δ-fine η regular and $(G \llcorner T', \varepsilon/2)$-full. Both $\mathcal{P}$ and $\mathcal{P}'$ are tagged families in $T + T'$, and they are non overlapping, as T and T' are . Therefore the concatenation of $\mathcal{P}$ and $\mathcal{P}'$ is a δ-fine, $\tau\tilde{\eta}$-regular, (G, ε)-full, tagged family in $T + T'$. This proves that $T + T'$ has the Howard-Cousin Property with exceptional set $\tilde{E}$ and regularity gauge $\tilde{\eta}$. □

4.2.2 Disposable sets for a current

In the following, we will study currents which are well behaved outside of an exceptional set. It will be necessary to *remove* these sets, by looking at subcurrents whose support do not intersect the exceptionnal set.

Definition 4.2.7. A set $E \subseteq \mathbb{R}^n$ is called **disposable for** T if there exists $C > 0$ such that for any ε, there exists a subcurrent $T_\varepsilon \sqsubset T$ such that $\operatorname{spt} T_\varepsilon \cap \operatorname{cl} E = \emptyset$ and

$$\begin{aligned}\mathbb{N}(T - T_\varepsilon) &< C, \\ \mathbb{M}(T - T_\varepsilon) &< \varepsilon.\end{aligned}$$

If E is disposable for T and all of its subcurrents, we call E **hereditarily disposable in** T.

This definition implies in particular that if E is disposable in T, and G is a continuous function on $\mathcal{S}(T)$, for every $\varepsilon > 0$, there exists $T_\varepsilon \sqsubset T$ with $\operatorname{spt} T_\varepsilon \cap \operatorname{cl} E = \emptyset$ and $|G(T - T_\varepsilon)| < \varepsilon$.

Proposition 4.2.8. *If T and E are such that*

$$\lim_{r\to 0} \|T\|(\mathrm{U}(E,r)) = 0$$

and

$$l := \liminf_{r\to 0} \mathbb{M}(\partial(T \llcorner \mathrm{U}(E,r))) < +\infty,$$

then E is disposable for T.

Furthermore, we can strenghten the last condition to: "There exists $l > 0$ and for all $r_0 > 0$ there exists a subset of $(0, r_0)$ of positive Lebesgue measure, such that for all r in this subset $\mathbb{M}(\partial(T \llcorner \mathrm{U}(E,r))) \leqslant l$."
In this case E is hereditarily disposable for T.

The last condition is satisfied in particular if we replace the lim inf by a lim sup in the previous condition.

Proof. First, notice that for any $r > 0$, $\operatorname{spt} T \llcorner (\mathrm{U}(E,r)^c) \cap \operatorname{cl} E = \emptyset$. Given $\varepsilon > 0$, one can clearly choose $r > 0$ small enough to have at once

$$\mathbb{M}(T - T \llcorner (\mathrm{U}(E,r)^c)) = \mathbb{M}(T \llcorner \mathrm{U}(E,r)) < \varepsilon$$

and

$$\mathbb{N}(T - T \llcorner (\mathrm{U}(E,r)^c)) = \mathbb{N}(T \llcorner \mathrm{U}(E,r)) \leqslant \varepsilon + l,$$

Let $C := l + 1$ and $T_\varepsilon := T \llcorner (\mathrm{U}(E,r))^c$.

Suppose now that the hypothesis holds with the last specified above. Let S be a subcurrent of T. Clearly for all $r > 0$, $\mathbb{M}(S \llcorner \mathrm{U}(E,r)) \leqslant \mathbb{M}(T \llcorner \mathrm{U}(E,r))$. Furthermore, since the function $f : x \mapsto \operatorname{dist}(E,x)$ is Lipschitz, there holds for almost all $r > 0$:

$$\langle S, f, r\rangle = \partial(S \llcorner \mathrm{U}(E,r)) - (\partial S) \llcorner \mathrm{U}(E,r).$$

Since S is a subcurrent of T, by proposition 3.1.3(x), for almost all $r > 0$, $\langle S, f, r\rangle \sqsubset \langle T, f, r\rangle$ and therefore $\mathbb{M}(\langle S, f, r\rangle) \leqslant \mathbb{M}(\langle T, f, r\rangle)$. Then, there exists $r > 0$, arbitrarily small, such that:

$$\begin{aligned}
\mathbb{M}(\partial(S \llcorner \mathrm{U}(E,r))) &\leqslant \mathbb{M}((\partial S) \llcorner \mathrm{U}(E,r)) + \mathbb{M}(\langle S, f, r\rangle) \\
&\leqslant \mathbb{M}(\partial S) + \mathbb{M}(\langle T, f, r\rangle) \\
&\leqslant \mathbb{M}(\partial S) + \mathbb{M}(\partial(T \llcorner \mathrm{U}(E,r))) + \mathbb{M}((\partial T) \llcorner \mathrm{U}(E,r)) \\
&\leqslant \mathbb{M}(\partial S) + l + \mathbb{M}(\partial T).
\end{aligned}$$

E is therefore disposable for S. □

Finally, note that the Strong Intersection Property implies "hereditariness of disposability":

Proposition 4.2.9. *Suppose that E is disposable in T and that T has the Strong Intersection Property as defined in section 3.1.4. Then E is hereditarily disposable in T.*

Proof. Let $C > 0$ be the constant corresponding to the disposability of E. Let S be a subcurrent of T. For $\varepsilon > 0$ let $T_\varepsilon \sqsubset T$ be such that $\mathbb{N}(T - T_\varepsilon) < C$ and $\mathbb{M}(T - T_\varepsilon) < \varepsilon$. By the strong

intersection property for T, there exists $C_T > 0$ such that $\mathbb{N}(T_\varepsilon \sqcap S) \leqslant \mathbb{N}(T_\varepsilon) + \mathbb{N}(S) + C_T$. Then $T_\varepsilon \sqcap S \sqsubset S$, $T_\varepsilon \sqcap S \sqsubset T_\varepsilon$ and $E \cap \operatorname{spt}(T_\varepsilon \sqcap S) \subseteq E \cap T_\varepsilon = \emptyset$, and furthermore

$$\begin{aligned}\mathbb{N}(S - (S \sqcap T_\varepsilon)) &\leqslant \mathbb{N}(S) + \mathbb{N}(S \sqcap T_\varepsilon)\\ &\leqslant 2\,\mathbb{N}(S) + \mathbb{N}(T_\varepsilon) + C_T \leqslant 2\,\mathbb{N}(S) + \mathbb{N}(T) + C + C_T\end{aligned}$$

and

$$\mathbb{M}(S - (S \sqcap T_\varepsilon)) \leqslant \mathbb{M}(T - T_\varepsilon) < \varepsilon.$$

So $S \sqcap T_\varepsilon$ is the required subcurrent. As ε is arbitrary, the proof is complete. □

Disposable sets are key to proving the Howard-Cousin Property of currents. The example mentionned above of a current which has the Howard-Cousin Property, but not hereditarily relies on a set which is disposable but not hereditarily disposable for this current (see section 4.4.3).

4.2.3 A criterion to have the Howard-Cousin Property

The next statement gives a general criterion for a current to have the Howard-Cousin Property. In the following, we give two examples of classes which satisfy this criterion.

Theorem 4.2.10. *Consider a current $T \in \mathbb{I}_m(\mathbb{R}^n)$ and a set $E \subseteq \mathbb{R}^n$ such that*

(i) E is disposable in T

(ii) For all $x \in \operatorname{spt} T \backslash E$, there exists a neighbourhood V_x of x in $\mathbb{R}^n$ along with a bounded BV set $A_x \subseteq \mathbb{R}^m$, a positive integer θ_x and a map $\phi_x : \operatorname{cl} A_x \to \mathbb{R}^n$ bilipschitz on its image with

$$T \llcorner V_x = \theta_x \phi_{x\#}(\mathbf{E}^m \llcorner A_x).$$

Then

1. *There exists a regularity gauge η on $\operatorname{set}_m \|T\| \backslash E$ such that T has the Howard-Cousin property with exceptional set E and regularity gauge η.*

2. *One can choose the tagged family $\mathcal{P}$ in the Howard-Cousin Property to consist of pairs (x, S) where the currents S are bilipschitz pushforwards of currents of the form $\theta_S \mathbf{E}^m \llcorner B_S$, where B_S is a bounded set of finite perimeter in $\mathbb{R}^m$.*

3. *If all the ϕ_x are also C^1, the previous statement holds with "C^1, bilipschitz pushforwards" instead of only bilipschitz pushforwards.*

4. *If E is hereditarily disposable, then (T, E, η) has the Hereditary Howard-Cousin Property.*

We call the data of $(V_x, A_x, \theta_x, \phi_x)$ for some x as above a **Bilipschitz-BV chart of T near** x, if ϕ_x is also C^1, we then speak of a **C^1-BV chart**. A point $x \in \operatorname{spt} T$ such that T has a Bilipschitz-BV chart in the neighbourhood of x is a **regular point** of T. The complement in $\operatorname{spt} T$ of the set of regular points is called the **nonsmooth set** of T.

Proof. First we need to define a regularity gauge η_T on $\mathrm{set}_m \|T\|$. For each $x \in \operatorname{spt} T \backslash E$, V_x is open, so there exists $R(x) > 0$ with $\mathrm{U}(x,r) \subseteq V_x$ for all $r \in (0, R(x)\,]$. Since T is an integral current, by a standard slicing argument, we can choose $R(x)$ a little smaller so that $T \llcorner \mathrm{U}(x, R(x))$ is also an integral current and a subcurrent of $T \llcorner V_x$. Note that $(\phi_x^{-1})_\#(T \llcorner \mathrm{U}(x,R(x)) \sqsubset (\phi_x^{-1})_\#(T \llcorner V_x) = \theta_x \mathbf{E}^m \llcorner A_x$, so there exists $A'_x \subseteq A_x$ of finite perimeter in $\mathbb{R}^m$ with $T \llcorner \mathrm{U}(x,R(x)) = \theta \phi_{x\#}(\mathbf{E}^m \llcorner A'_x)$. Without loss of generality, we can thus suppose that $V_x = \mathrm{U}(x,R(x))$ and $A_x = A'_x$.

Consider the cover of $\operatorname{spt} T \backslash E$ consisting of the open balls $\mathrm{U}(x,R(x))$ with $x \in \operatorname{spt} T \backslash E$. Since $\operatorname{spt} T \backslash E \subseteq \mathbb{R}^m$ is paracompact, there exists a locally finite subcover $\mathcal{U} := \{U_\alpha = \mathrm{U}(x_\alpha, R(x_\alpha)), \alpha \in I\}$. Relabel accordingly $\phi_\alpha := \phi_{x_\alpha}$ and similarly, $A_\alpha := A_{x_\alpha}$, $\theta_\alpha := \theta_{x_\alpha}$ and $r_\alpha := R(x_\alpha)$. We define a new cover of $\operatorname{spt} T \backslash E$ by Bilipschitz BV charts, by choosing for each $x \in \operatorname{spt} T \backslash E$ the *least regular chart of* T *near* x in our locally finite collection of charts. Forgetting the original labels, for $x \in \operatorname{spt} T \backslash E$ let

$$\begin{aligned}
\alpha(x) &:= \operatorname{argmin}\{(\operatorname{Lip}\phi_\alpha)^{-m}(\operatorname{Lip}\phi_\alpha^{-1})^{-m}, \alpha, x \in \mathrm{U}(x_\alpha, r_\alpha)\},\\
\theta_x &:= \theta_{\alpha(x)},\\
r_x &:= \sup\{r > 0, \mathrm{U}(x,r) \subseteq U_{\alpha(x)}\},\\
A_x &:= \phi_{\alpha(x)}^{-1}\left(\phi_{\alpha(x)}(A_{\alpha(x)}) \cap B(x, r_x)\right),\\
\phi_x &:= \phi_{\alpha(x)}|_{A_x}.
\end{aligned}$$

Note that for all $x \in \operatorname{spt} T \backslash E$, the new $(\mathrm{U}(x,r_x), A_x, \theta_x, \phi_x)$ is still a Bilipschitz-BV chart of T near x. Indeed, A_x is the intersection of two bounded sets of finite perimeter, thus it has finite perimeter. ϕ_x remains Bilipschitz on its image and

$$T \llcorner \mathrm{U}(x,r_x) = \theta_x \phi_{x\#}(\mathbf{E}^m \llcorner A_x)$$

is an integral current. Lastly define

$$\eta_T(x) := (\operatorname{Lip}\phi_{\alpha(x)})^{-m}(\operatorname{Lip}\phi_{\alpha(x)}^{-1})^{-m} m^{-3/2}/2.$$

Let $\varepsilon > 0$ and G be a $\mathcal{F}$-continuous subadditive function on $\mathbb{N}_m(\operatorname{spt} T)$. By the definition of an disposable set, there exists $C > 0$ such that for each $\nu > 0$, there exists $T_\nu \sqsubset T$ such that $\operatorname{spt} T_\nu \cap E = \emptyset$ and

$$\begin{aligned}
\mathbb{N}(T - T_\nu) &< C,\\
\mathbb{M}(T - T_\nu) &< \nu.
\end{aligned}$$

In particular, as G is $\mathcal{F}$-continuous, there exists ν such that $G(T - T_\nu) \leqslant \varepsilon/2$. For such a ν, let $T' := T_\nu$.

Now, $\operatorname{spt} T'$ is closed and for each $x \in \operatorname{spt} T'$, by the hypothesis of the theorem, T admits a Bilipschitz-BV chart at x: $(\mathrm{U}(x,r_x), A_x, \theta_x, \phi_x)$. The collection $(\mathrm{U}(x, r_x/2))_{x \in \operatorname{spt} T'}$ forms an open cover of $\operatorname{spt} T'$, which is compact, thus there exists a finite subcover indexed by $(x_1, \ldots, x_p)$. Label the corresponding bilipschitz-BV charts $(\mathrm{U}(x_j, r_j), A_j, \theta_j, \phi_j) := (\mathrm{U}(x, r_{x_j}), A_{x_j}, \theta_{x_j}, \phi_{x_j})$.

From this finite cover, we now construct a partition of T' into bilipschitz-BV charts. First,

since T' is integral, by proposition B.3 there exists $s_1 \in [r_1/2, r_1[$ such that $T' \llcorner \mathrm{U}(x_1, r_1)$ is integral. Let $T_1 := T' \llcorner \mathrm{U}(x_1, s_1)$. Recursively, for $j \in \{2, \dots, p-1\}$ choose $s_j \in [r_j/2, r_j[$ such that $T_j := (T' - \sum_{k=1}^{j-1} T_k) \llcorner \mathrm{U}(x_j, s_j)$ is an integral current. Lastly, let $T_p := T' - \sum_{j=1}^{p-1} T_j$.

Claim 1. The next two statements hold

1. The T_j, for $j \in \{1, \dots, p\}$ are nonoverlapping subcurrents of T' and $T' = \sum_{j=1}^{p} T_j$.
2. For $j \in \{1, \dots, p\}$, there exists a set of finite perimeter $B_j \subseteq A_j$ such that $T_j = \theta_j \phi_{j\#}(\mathbf{E}^m \llcorner B_j)$.

Proof of claim 1. We start by the first part of the claim. By construction, the T_j are integral currents for $j = 1, \dots, p-1$, T_p is thus also an integral current and $T' = \sum_{j=1}^{p} T_j$. We prove recursively that the T_j are nonoverlapping subcurrents of T' for $j = 1, \dots, p$. $T_1 = T' \llcorner \mathrm{U}(x_1, s_1)$ is integral by proposition 3.1.3. Suppose that $T_1, \dots, T_{j-1}$ are nonoverlapping subcurrents of T', this implies by recursive use of proposition 3.1.3 (vii) that $\sum_{k=1}^{j-1} T_k$ is a subcurrent of T'. By definition, T_j is a subcurrent of $T' - \sum_{k=1}^{j-1} T_j$, thus

$$\|T_j\| << \left\| T' - \sum_{k=1}^{j-1} T_k \right\|$$

and

$$\|T_j\| \perp \left\| T' - \sum_{k=1}^{j-1} T_k - T_j \right\|.$$

We can write

$$\|T' - T_j\| = \left\| T' - \sum_{k=1}^{j-1} T_k - T_j \right\| + \left\| \sum_{k=1}^{j-1} T_k \right\| \perp \|T_j\|,$$

where we used the fact that for three measures μ, ν and ξ, $\mu << \nu$ and $\nu \perp \xi$ implies $\mu \perp \xi$ which implies that T_j does not overlap with each T_k for $k < j$ (nor with their sum). By proposition 3.1.3 (vi), T_j is therefore also a subcurrent of T'. Finally, $\sum_{j=1}^{p-1} T_j$ is a subcurrent of T' and by definition of subcurrents, T_p is also a subcurrent of T' which does not overlap with the T_j for j from 1 to $p-1$.

For the second part of the claim, by definition, each T_j for $j \leqslant p-1$ is supported inside $\mathrm{U}(x_j, r_j)$ and $T' - \sum_{k=1}^{j} T_k$ is supported in the complement of $\bigcup_{k=1}^{j} \mathrm{U}(x_k, s_k)$. For the p-th step, since $\operatorname{spt} T' \subseteq \bigcup_{k=1}^{p} \mathrm{U}(x_k, r_k/2)$, this implies that $T_p = T - \sum_{j=1}^{p-1} T_j$ is supported in $\operatorname{spt} T' \setminus \bigcup_{j=1}^{p-1} \mathrm{U}(x_k, r_k) \subseteq \mathrm{U}(x_p, r_p)$.

Thus for all $j \in \{1, \dots, p\}$, $T_j \sqsubset T \llcorner \mathrm{U}(x_j, r_j)$ and by the transitivity of the relation $\sqsubset$ (see proposition 3.1.3(ii)),

$$T_j = T_j \llcorner \mathrm{U}(x_j, r_j) \sqsubset T' \llcorner \mathrm{U}(x_j, r_j) \sqsubset T \llcorner \mathrm{U}(x_j, r_j).$$

This implies $T_j \sqsubset \theta_j \phi_{j\#}(\mathbf{E}^m \llcorner A_j)$. By proposition 3.1.3(viii), $\phi_{j\#}^{-1} T_j \sqsubset \theta_j(\mathbf{E}^m \llcorner A_j)$ and there exists $B_j \subseteq A_j$ such that $T_j = \theta_j \phi_{j\#}(\mathbf{E}^m \llcorner B_j)$. ■

The end of the proof consists in applying lemma 4.1.8 to each B_j. Fix $j \in \{1, \dots, p\}$. For $y \in \operatorname{cl}_e B_j$, let

$$\eta_j(y) := \varepsilon \eta_T(\phi_j(y)) (\operatorname{Lip} \phi_j^{-1})^m (\operatorname{Lip} \phi_j)^m.$$

Note that by the choice of η_T at $\phi_j(y)$, and the fact that $\phi_j(y)$ belongs to the set $\mathrm{U}(x_j, s_j)$, there holds

$$\eta_j(y) \leqslant \varepsilon \frac{(\operatorname{Lip}\phi_j^{-1}\operatorname{Lip}\phi_j)^m}{(\operatorname{Lip}\phi_{\alpha(\phi_j(x))}^{-1}\operatorname{Lip}\phi_{\alpha(\phi_j(x))})^m} m^{-3/2}/2 < m^{-3/2}/2,$$

ensuring that we can apply lemma 4.1.8 to B_j. Let δ and η be gauges on $\operatorname{set}_m \|T\|\backslash E$ with $\eta \leqslant \eta_T$. For $j = 1, \ldots, p$, $\operatorname{set}_m \|T_j\| \subseteq \operatorname{set}_m \|T\|$ and since ϕ_j is bilipschitz from $\operatorname{cl} B_j$ into its image in $\operatorname{spt} T$, $\phi_j(\operatorname{set}_m \|\mathbf{E}^m B_j\|) \subseteq \operatorname{set}_m \|T_j\|$ and furthermore, ϕ_j^{-1} maps an m-thin set into a m-thin set. Thus the function δ_j defined for y in $\operatorname{cl}_e B_j = \phi_j^{-1}(\operatorname{set}_m \|T_j\|)$ by

$$\delta_j(y) := \frac{\delta(\phi_j(y))}{\operatorname{Lip}\phi_j}.$$

is a gauge on B_j. Apply the Howard-Cousin Lemma to B_j with the subadditive $\mathcal{F}$-continuous function $\phi_j^{\#}G$ defined at proposition 3.2.10, the gauge δ_j, the regularity gauge η_j and the error term $\varepsilon/(2p\theta_j)$, to get a δ_j-fine, η_j-regular tagged family $\mathcal{C}^j := ((C_1^j, y_1^j), \ldots, (C_{q_j}^j, y_{q_j}^j))$ in B_j such that:

$$\left|\phi_j^{\#}G\left(\mathbf{E}^m \llcorner B_j - \sum_{k=1}^{q_j}(\mathbf{E}^m \llcorner C_k^j)\right)\right| < \varepsilon_j.$$

From which we directly infer

$$\left|G\left(T_j - \sum_{k=1}^{q_j}\theta_j\phi_{j\#}(\mathbf{E}^m \llcorner C_k^j)\right)\right| < \theta_j\varepsilon_j.$$

Furthermore, $\mathcal{C}^j$ being δ_j-fine and η_j regular implies that for all $k \in \{1, \ldots, q_j\}$ there holds:

$$\begin{aligned}
\phi_j(y_k^j) &\in \operatorname{spt}(\phi_{j\#}(\mathbf{E}^m \llcorner C_k^j)),\\
\operatorname{diam}\operatorname{spt}(\theta_j\phi_{j\#}(\mathbf{E}^m \llcorner C_k^j)) \leqslant (\operatorname{Lip}\phi_j)\operatorname{diam} C_k^j &< (\operatorname{Lip}\phi_j)\delta_j(y_k^j) < \delta(\phi_j(y_k^j)),\\
\mathbb{M}(\phi_{j\#}(\mathbf{E}^m \llcorner C_k^j)) &\geqslant \frac{|C_k^j|}{(\operatorname{Lip}\phi_j^{-1})^m},\\
\mathbb{M}(\partial\phi_{j\#}(\mathbf{E}^m \llcorner C_k^j)) &\leqslant (\operatorname{Lip}\phi_j)^{m-1}\|C_k^j\|.
\end{aligned}$$

In particular:

$$\begin{aligned}
&\frac{\mathbb{M}(\theta_j\phi_{j\#}(\mathbf{E}^m \llcorner C_k^j))}{\mathbb{M}(\theta_j\partial\phi_{j\#}(\mathbf{E}^m \llcorner C_k^j))\operatorname{diam}\operatorname{spt}(\phi_{j\#}(\mathbf{E}^m \llcorner C_k^j))}\\
&\qquad \geqslant \frac{(\operatorname{Lip}\phi_j^{-1})^{-m}|C_k^j|}{(\operatorname{Lip}\phi_j)^m\|C_k^j\|\operatorname{diam} C_k^j} > \frac{\eta_j(y_k^j)}{(\operatorname{Lip}\phi_j^{-1})^m(\operatorname{Lip}\phi_j)^m} = \eta_T(\phi_j(y_k^j)) \geqslant \eta(\phi_j(y_k^j)).
\end{aligned}$$

and the family $\mathcal{D}^j := ((\theta_j\phi_{j\#}(\mathbf{E}^m \llcorner C_1^j), \phi_j(y_1^j)), \ldots, (\theta_j\phi_{j\#}(\mathbf{E}^m \llcorner C_{q_j}^j), \phi_j(y_{q_j}^j)))$ is δ-fine and η-regular in T'. Finally let $\mathcal{D}$ be the concatenation of the families $\mathcal{D}^j$ for $j = 1, \ldots, p$, $\mathcal{D}$ is δ-fine, and η regular and by subadditivity of G, there holds:

$$|G(T - [\mathcal{D}])| \leqslant |G(T - T')| + \sum_{j=1}^{p}\left|G\left(T_j - \mathcal{D}^j\right)\right| < \frac{\varepsilon}{2} + \sum_{j=1}^{p}\frac{\varepsilon}{2p} = \varepsilon.$$

As δ, ε and η are arbitraty, statement 1 is proved. By construction 2 holds and if the ϕ_x are chosen C^1, there is nothing to change in the above proof to show that 3 holds.

Let us finally prove 4. Let S be a subcurrent of T, since E is hereditarily disposable for T, it is disposable for S. Therefore, the only thing to do is prove that the second hypothesis of the Theorem applies to S as well and be careful in order to obtain the a regularity gauge on $\mathrm{set}_m \|S\|\backslash E$ compatible with η_T. For the second point, we can suppose that the process of redefining the Bilip-BV charts using a locally finite cover of $\operatorname{spt} T\backslash E$ by charts. This is done independently on S and we get $\eta_S := \eta_T|_{\mathrm{set}_m \|S\|\backslash E}$. Fix $x \in \operatorname{spt} S\backslash E$, as $\operatorname{spt} S \subseteq \operatorname{spt} T$, there exists a bilipschitz-BV chart $(V_x, A_x, \theta_x, \phi_x)$ for T such that

$$T \llcorner V_x = \theta_x \phi_{x\#}(\mathbf{E}^m \llcorner A_x).$$

Up to replacing V_x by a ball $\mathrm{U}(x, r_x)$ for some small enough r_x, we get

$$S \llcorner \mathrm{U}(x, r_x) \sqsubset T \llcorner \mathrm{U}(x, r_x) = \theta_x \phi_{x\#}(\mathbf{E}^m \llcorner (\phi^{-1}(\mathrm{U}(x, r_x) \cap \phi(A_x)) \cap A_x))$$

Therefore, there exists a bounded set of finite perimeter $B_x \subseteq A_x$ such that $(\mathrm{U}(x, r_x), B_x, \theta_x, \phi_x)$ is a bilipschitz-BV chart for S. Thus we can apply statement 1 to S to prove that S has the Howard-Cousin Property with exceptional set E and a regularity gauge $\eta_S = \eta_T|_{\mathrm{set}_m \|S\|\backslash E}$. □

Corollary 4.2.11. *Currents associated to compact oriented C^1 submanifolds with boundary have the hereditary Howard-Cousin Property with empty exceptional set and constant regularity gauge.*

Proof. Let $T \in \mathbb{I}_m(\mathbb{R}^n)$ be associated with M, a C^1 compact submanifold with boundary of dimension m in $\mathbb{R}^n$. Let $x \in M$, then there exists an open neighbourhood V_x of x in $\mathbb{R}^n$, along with a ball or a half ball A_x in $\mathbb{R}^m$ and a C^1 diffeomorphism ϕ_x from $\operatorname{cl} A_x$ to $\operatorname{cl}(V_x) \cap M$. Since a ball and a half ball have finite perimeter and are bounded, $(V_x, A_x, \phi_x, 1)$ is a C^1-BV chart on T near x up to a change of orientation of ϕ_x. As we can take a finite subcoverof M by the V_x, we can choose η_T as the minimum of a finite family, uniformly. □

We now state a result about chains definable in o-minimal structures. These are a type of integral currents, their precise definition and properties are in chapter 6

Proposition 4.2.12. *Definable chains in an o-minimal structure have the hereditary Howard-Cousin Property.*

Proof. In corollary 6.3.3, we prove that the singular set of an definable chain is hereditary disposable. This is sufficient to apply theorem 4.2.10. □

In the next paragraph, we give another sufficient condition for the set of nonsmooth points of a current to be disposable and apply it to mass minimizing currents.

4.2.4 Minkowski content and disposability

We recall the notion of Minkowski content. Given a set $E \subseteq \mathbb{R}^n$ and a positive number r, the **r-neighbourhood of E** is the open set

$$\mathrm{U}(E, r) := \{x \in \mathbb{R}^n, \operatorname{dist}(x, E) < r\}.$$

The **upper and lower Minkowski content** of E in dimension $m > 0$ are then defined respectively as

$$\mathcal{M}^{*m}(E) := \limsup_{r>0} \frac{\mathcal{L}^n(\mathrm{U}(E,r))}{\boldsymbol{\alpha}(n-m)r^{n-m}} \text{ and } \mathcal{M}^m_*(E) := \liminf_{r>0} \frac{\mathcal{L}^n(\mathrm{U}(E,r))}{\boldsymbol{\alpha}(n-m)r^{n-m}}.$$

When those two numbers coincide, they are called the **m-dimensional Minkowski content of** E, denoted by $\mathcal{M}^m(E)$. Notice that the Minkowski content of an unbounded set in any dimension is always infinite. We recall an important property of the Minkowski content: If $E \subseteq \mathbb{R}^m$ is m rectifiable and closed, then $\mathcal{H}^m(E) = \mathcal{M}^m(E)$ (see [26, 3.2.39]). Notice that the closedness hypothesis is important as, by definition, the the Minkowski content of a set E is always equal to that of $\operatorname{cl} E$, whereas we can have $\mathcal{H}^m(E) < \mathcal{H}^m(\operatorname{cl} E)$. The rectifiability condition is important as well, as a $(\mathcal{H}^2, 2)$ rectifiable compact set of finite $\mathcal{H}^2$ measure in $\mathbb{R}^3$ may have infinite Minkowski content, see the last paragraph of [26, 3.2.40]. See [50, 5.3-5.6] for more details on Minkowski dimension and content.

Similarly, given a measure μ, define the **m-dimensional μ-Minkowski content**, replacing the Lebesgue measure by μ in the above definition. In the following we are interested in the $m-1$-dimensional $\|T\|$-Minkowski content of the set of nonsmooth points of a current T. We will show that if the $m-1$-dimensional $\|T\|$ Minkowski content of the set of nonsmooth points of T is finite, then this singular set is disposable for T.

Furthermore, given a current $T \in \mathbb{I}_m(\mathbb{R}^n)$, if $E \subseteq \mathbb{R}^n$ has finite $m-1$ dimensional $\|T\|$-Minkowski content and $S \sqsubset T$, then E has finite $m-1$ dimensional $\|S\|$ Minkowski content. This is an easy consequence of the two following facts.

- If S is a subcurrent of T, then $\|S\| \leqslant \|T\|$.
- If S is a subcurrent of T, then the set of nonsmooth points of S is contained in the setof nonsmooth points of T.

A first consequence of this, is that if the set of nonsmooth points of T has with finite $m-1$ dimensional $\|T\|$-Minkowski content, then it is hereditarily disposable. Therefore such a T has the hereditary Howard-Cousin Property. In fact, having small $\|T\|$-Minkowski content also implies the Strong Intersection Property.

Proposition 4.2.13. *If the set E_T of nonsmooth points of T has finite $\|T\|$-Minkowski content of dimension $m-1$, then*

1. *E_T is hereditarily disposable (which implies T has the hereditary Howard-Cousin Property as in theorem 4.2.10)*
2. *and T has the Strong Intersection Property.*

Proof. For the first part, it suffices to prove that T and E_T satisfy the conditions of proposition 4.2.8. Let C denote the Minkowski content of the singular set E_T of T. There exists $r_0 > 0$ such that for all $r \in (0, r_0]$

$$\|T\|(\mathrm{U}(E_T, r)) < 2Cr^{m-(m-1)} = 2Cr.$$

Fix $\varepsilon > 0$ and choose r_0 small enough so that for all $r \in (0, r_0)$ one has also $\|T\|(\mathrm{U}(E_T, r)) < \varepsilon$. Since T is an integral current, for r in a set of positive measure in $(r_0/2, r_0)$, $\langle T, \operatorname{dist}(E_t, \cdot), r\rangle$ is

an integral current and

$$\mathbb{M}(\langle T, \operatorname{dist}(E_T, \cdot), r\rangle) \leqslant \frac{2}{r_0}\|T\|(\mathrm{U}(E_T, r_0)) \leqslant 4C.$$

From this we infer

$$\begin{aligned}\mathbb{M}(\partial(T \llcorner (\mathrm{U}(E_T, r)))) &\leqslant \mathbb{M}((\partial T) \llcorner \mathrm{U}(E_T, r)) + \mathbb{M}(< T, f, r >)\\ &\leqslant \mathbb{M}(\partial T) + 4C.\end{aligned}$$

This proves that for every $\varepsilon > 0$ and $r_1 > 0$, there exists a set of positive measure in $(0, r_1)$ such that

$$\begin{aligned}&\|T\|(\mathrm{U}(E_T, r)) < \varepsilon,\\ &\mathbb{M}(\partial(T \llcorner (\mathrm{U}(E_T, r)))) \leqslant 4C + \mathbb{M}(\partial T),\end{aligned}$$

therefore proposition 4.2.8 applies, E_T is hereditarily disposable in T and by theorem 4.2.10 T has the Hereditary Howard-Cousin Property.

To prove that T has the Strong Intersection Property, let $S_1 := T \llcorner A_1$ and $S_2 := T \llcorner A_2$ be subcurrents of T, let $R := T \llcorner (A_1 \cap A_2)$, we will show that R is integral and therefore a subcurrent of T. Choose $\varepsilon > 0$ there exists $r_0 > 0$ such that for all $r \in (0, r_0)$

$$\max(\mathbb{M}(S_1 \llcorner \mathrm{B}(E_T, r)), \mathbb{M}(S_2 \llcorner \mathrm{B}(E_T, r)), \mathbb{M}(T \llcorner \mathrm{B}(\operatorname{cl} E_T, r))) < \varepsilon.$$

By the assumption on the small Minkowski content, one can in fact choose r_0 so small that for all $r \in (0, r_0)$,

$$\|T\|(\mathrm{B}(\operatorname{cl} E_T, r)) < (C + \varepsilon)r.$$

By slicing theory, since T, S_1 and S_2 are integral currents, for r in a set of positive measure in $(r_0/2, r_0)$, for $j = 1, 2$, there holds

$$\begin{aligned}\mathbb{M}(\partial(S_j \llcorner (\mathrm{U}(\operatorname{cl} E_T, r))^c)) &\leqslant \mathbb{M}(\partial S_j) + \mathbb{M}(< S_j, f, r >)\\ &\leqslant \mathbb{M}(\partial S_j) + \mathbb{M}(< T, f, r >)\\ &\leqslant \mathbb{M}(\partial S_j) + \frac{\|T\|(\mathrm{B}(\operatorname{cl} E_T, r))}{r_0/2}\\ &\leqslant \mathbb{M}(\partial S_j) + 4(C + \varepsilon),\end{aligned}$$

where f is the functions : $x \mapsto \operatorname{dist}(x, \operatorname{cl} E_T)$ and we used the fact that for almost all r, $< S_j, f, r >$ is a subcurrent of $< T, f, r >$, by proposition 3.1.3 (x). The set of positive measure in $(r_0/2, r_0)$ is chosen from a set of full measure which depends on j, but the subsequent choice is made with respect to T and therefore does not depend on j, up to a null set. Choose an r in this set.

Let $T' := T \llcorner (\mathrm{U}(\operatorname{cl} E_T, r))^c)$ and $S'_j := \partial(S_j \llcorner (\mathrm{U}(\operatorname{cl} E_T, r))^c)$, for $j = 1, 2$. $S'_j \sqsubset T'$. We will show that the intersection of S'_1 and S'_2 is a subcurrent of T' with boundary mass less than $\mathbb{M}(\partial S'_1) + \mathbb{M}(\partial S'_2)$. If $S'_j = T' \llcorner A_j$, let $R' := T' \llcorner (A_1 \cap A_2)$. As $\operatorname{spt} T'$ is compact and the U_x are open and cover $\operatorname{spt} T'$, we can extract a finite subcover $(U_k)_{k=1}^p$. A first consequence is that

$\partial R'$ has finite mass, furthermore, one can define a partition of unity subordinate to this cover : $\psi_1, \ldots, \psi_p$ with $\sum_{k=1}^{p} \psi_k = 1$ in a neigbourhood of $\operatorname{spt} T'$. Clearly

$$\|\partial R'\| \llcorner \psi_k \leqslant \|\partial S_1'\| \llcorner \psi_k + \|\partial S_2'\| \llcorner \psi_k$$

Let $\phi \in \mathcal{D}^{m-1}(\mathbb{R}^n)$ and notice that necessarily, $\operatorname{spt} \partial R' \subseteq \operatorname{spt} T'$, thus

$$\partial R'(\phi) = \partial R' \left(\left(\sum_{k=1}^{p} \psi_k \right) \phi \right) = \sum_{k=1}^{p} \partial R'(\psi_k \phi).$$

In particular,

$$|\partial R'(\phi)| \leqslant \sum_{k=1}^{p} \|\partial S_1'\|(\psi_k \phi) + \|\partial S_2'\|(\psi_k \phi) = \|\partial S_1'\|(\phi) + \|\partial S_2'\|(\phi),$$

So $\mathbb{M}(\partial R') \leqslant \mathbb{M}(\partial S_1') + \mathbb{M}(\partial S_2')$.

We have thus proved that R' is a subcurrent of T' (and therefore of T) with

$$\mathbb{M}(R - R') = \mathbb{M}(R \llcorner \mathrm{B}(E_T, r)) \leqslant \mathbb{M}(T \llcorner \mathrm{B}(E_T, r)) < \varepsilon.$$

and

$$\mathbb{M}(\partial R') \leqslant \mathbb{M}(\partial S_1) + \mathbb{M}(\partial S_2) + 8(C + \varepsilon).$$

Choosing a sequence ε_n going to 0 and an R_n' for each ε_n, one obtains a sequence converging to R in mass. Thus $\partial R_n'$ converges to ∂R in the flat norm. By the lower semi continuity of mass with respect to the flat norm, we get $\mathbb{M}(\partial R) \leqslant \mathbb{M}(\partial S_1) + \mathbb{M}(\partial S_2) + 8\mathcal{M}(E_T)$. □

4.2.5 Mass minimizing currents and stationary varifolds

We state here a series of result which combined imply that codimension 1 area minimizing minimal currents have a disposable exceptional set. Let us start with the interior regularity results. A general survey is contained in [12], and the proofs can be found in section 37 of [66] and in [67].

Theorem 4.2.14 (Classical regularity for mass minimizing currents of dimension $n-1$ in $\mathbb{R}^n$)**.** *If T is an integral current of dimension $n-1$ in $\mathbb{R}^n$, which minimises $\mathbb{M}$ among all integral currents with boundary ∂T, then there exists a set $Sing(T) \subseteq \operatorname{spt} T$ called the singular set of T which has Hausdorff dimension at most $n-8$ and such that at all points of $\operatorname{spt} T \backslash (Sing(T) \cup \operatorname{spt} \partial T)$, T is locally an integer multiple of the oriented graph of an analytic function. Furthermore, $Sing(T)$ is countably $(\mathcal{H}^{n-8}, n-8)$ rectifiable.*

This result relies in particular the following fact:

Lemma 4.2.15. *With the conditions of theorem 4.2.14, if $x \in \operatorname{spt} T \backslash \operatorname{spt} \partial T$ and T has a flat tangent cone at x, then $x \notin Sing(T)$.*

Concerning the boundary points in $\operatorname{spt} T$, if the boundary is nice enough, the next classical result states that they are well enough behaved for our purpose.

Theorem 4.2.16 (Boundary Regularity of Hardt and Simon [34]). *Suppose that T is an $n-1$ dimensional integral current in $\mathbb{R}^n$ such that*

1. *∂T represents a connected oriented $C^{1,\alpha}$ submanifold of $\mathbb{R}^n$ (with $\alpha \in (0,1)$ and*
2. *T is a mass minimizer in the class $\{S \in \mathbb{I}_{n-1}(\mathbb{R}^n), \partial S = T\}$.*

Then there exists a neighbourhood V of $\operatorname{spt}\partial T$ such that $V \cap \operatorname{spt} T$ is a $C^{1,\alpha}$ submanifold.

A recent improvement to the interior regularity theory was obtained by Cheeger and Naber in [10, Theorem 5.8], where they control the $\|T\|$ measure of an r-neighbourhood of the singular set:

Theorem 4.2.17 (Cheeger and Naber). *Suppose $T \in \mathbb{I}_{n-1}(\mathbb{R}^n)$ satisfies the conditions of theorem 4.2.14, then for $x \in \operatorname{spt} T \setminus \operatorname{spt}\partial T$, $R > 0$ such that $\mathrm{U}(x,R) \cap \operatorname{spt}\partial T = \emptyset$ and $\eta > 0$, there exists $C > 0$ such that for $r \in (0,R)$:*

$$\|T\|(\mathrm{U}(x,R) \cap \mathrm{U}(Sing(T),r)) \leqslant Cr^{7-\eta}.$$

(Note that the results of Cheeger and Naber, and the more recent results of Naber and Valtorta are much stronger - and more quantitative - than what we need.) From this result, we directly infer that $Sing(T)$ has locally null $n-2$ dimensional $\|T\|$ Minkowski content:

$$r^{-1}\|T\|(\mathrm{U}(x,R) \cap \mathrm{U}(Sing(T),r)) \to 0, \text{ as } r \to 0^+.$$

This implies in particular that $Sing(T)$ is hereditarily disposable in T and allows us to apply the results of the previous sections to prove:

Proposition 4.2.18. *Suppose T is an area minimizing current of dimension $n-1$ in $\mathbb{R}^n$ and ∂T represents a closed oriented $C^{1,\alpha}$ submanifold of $\mathbb{R}^n$ for some $\alpha > 0$ (with multiplicity one). Then the singularity set of T has null $m-1$ dimensional T-Minkowski content and T has the hereditary Howard-Cousin Property with exceptional set $\operatorname{spt}\partial T \cup Sing(T)$.*

Proof. Notice that the boundary regularity theorem 4.2.16 does not imply that T has C^1-BV charts at a point of $\operatorname{spt}\partial T$. Indeed, there could be points of the boundary where T has a jump in multiplicity when crossing $\operatorname{spt}\partial T$. There are two ways to go around this problem: one can either generalize the notion of bilipschitz-BV chart in order to include the sum of two charts based on the same open set and same map ϕ, but with different sets of finite perimeter and multiplicities, or as we choose to do, prove that $\operatorname{spt}\partial T$ is hereditarily disposable.

To do this, we want to control the $n-2$ dimensional $\|T\|$ Minkowski content of $\operatorname{spt}\partial T$. We start to prove that the multiplicity of T is bounded from above in a neighbourhood of $\operatorname{spt}\partial T$. Fix a point $x \in \operatorname{spt}\partial T$. There exists $r > 0$ such that $\operatorname{spt} T \cap \mathrm{U}(x,r)$ is a $C^{1,\alpha}$ submanifold of $\mathbb{R}^n$ of dimension $n-1$. Choosing r smaller, we can also suppose that $\operatorname{spt}\partial T$ separates $\operatorname{spt} T \cap \mathrm{U}(x,r)$ into two connected relatively open subsets. Using a parameterization of $\operatorname{spt} T \cap \mathrm{U}(x,r)$ on a connected open set of $\mathbb{R}^{n-1}$, we can therefore apply the constancy theorem for currents (see section 4.1.7 in [26]) to prove that the multiplicity of T takes only one value in each connected component of $(\operatorname{spt} T \cap \mathrm{U}(x,r)) \setminus \operatorname{spt}\partial T$. Covering $\operatorname{spt}\partial T$ with finitely many such balls (by compactness), we get an upper bound θ_{max} for the density of $\|T\|$ in a neighbourhood.

There exists a neighbourhood U of $\operatorname{spt}\partial T$ such that $\operatorname{spt}\partial T$ is a closed $C^{1,\alpha}$ submanifold of dimension $n-2$ of $\operatorname{spt} T \cap U$, which is itself a $C^{1,\alpha}$ submanifold of dimension $n-1$ of $\mathbb{R}^n$. By standard differential geometry, there exists a $C > 0$ such that for $r > 0$ small enough,

$$\mathcal{H}^{n-1}(\operatorname{spt} T \cap \mathrm{U}(\operatorname{spt}\partial T, r)) \leqslant C\mathcal{H}^{n-2}(\operatorname{spt}\partial T)\ r.$$

Thus for $r > 0$ small enough,

$$\|T\| \llcorner \mathrm{U}(\operatorname{spt}\partial T, r) \leqslant C\theta_{max}\mathcal{H}^{n-2}(\operatorname{spt}\partial T)\ r,$$

and $\limsup_{r\to 0} \|T\| \llcorner \mathrm{U}(\operatorname{spt}\partial T, r)/r < +\infty$ and $\operatorname{spt}\partial T$ has finite $n-2$ $\|T\|$ Minkowski content. By definition, Minkowski content additive on disjoint compact sets, therefore $\operatorname{spt}\partial T \cup Sing(T)$ has also finite $n-2$ $\|T\|$ Minkowski content, making it hereditarily disposable in T by proposition 4.2.13. Outside of this set, T represents locally analytic hypersurfaces of $\mathbb{R}^n$ and therefore has C^1 BV charts. Applying theorem 4.2.10, we find that T has the hereditary Howard-Cousin Property with exceptional set $\operatorname{spt}\partial T \cup Sing(T)$. □

If T is an area minimizing currents of higher codimension, there is no equivalent of lemma 4.2.15 as there can be *flat singular points* (or *double points*) see [12] for an introduction, the regularity theory in [3] or the recent simplifications [16, 14, 15]). Therefore, even though boundary regularity results exist as announced in [13] and Minkowski content estimates are valid for the singular strata (see [56]), the singular set of the current can be larger than the strata and no bound is know. However, assuming some control on the singularities on the boundary and the absence of double points, the singular set of T should be disposable.

Furthermore for stationary varifolds (see [66] and [2]) the singular strata have controlled classical Minkowski content, see [56]. Thus, if T is a current supported on a stationary varifold, assuming there are no double points and we control the singularities at the boundary, the singular set of T should be disposable as well, and T should have the hereditary Howard-Cousin Property with exceptional set $Sing(T) \cup \operatorname{spt}\partial T$.

4.3 Stokes-Cartan Theorem for the Lebesgue Integral

Using the results of chapter 3 and the previous sections, we obtain a general version of Stokes' Theorem on a family of integral currents.

4.3.1 Outline and statement

In this section, we consider a current $T \in \mathbb{I}_m(\mathbb{R}^n)$ which has C^1-BV charts at all points of its support except for a disposable set E_T. By the previous sections, T has the Howard-Cousin Property with exceptional set E_T and a regularity gauge η_T.

Given a continuous differential form ω on $\operatorname{spt} T$, we associate to it its circulation, which is a continuous and additive function on $\mathcal{S}(T)$:

$$\Theta_\omega : S \mapsto \int \langle \omega, \overrightarrow{S} \rangle \, \mathrm{d}\|S\|.$$

Assuming that ω is differentiable on $\operatorname{set}_m \|T\| \backslash (E_T \cup E_\omega)$ we use the good properties of C^1-BV charts to prove that Θ_ω is derivable at all points of $\operatorname{set}_m \|T\| \backslash (E_T \cup E_\omega \cup \operatorname{set}_{m-1} \|\partial T\|)$ with derivative

$$\mathfrak{D}_T \, \Theta_\omega : x \mapsto \langle \mathrm{d}\omega, \overrightarrow{\partial S} \rangle.$$

Supposing that $\mathfrak{D}_T \, \Theta_\omega$ is Lebesgue integrable with respect to $\|T\|$, we prove that its integral can be approximated by Riemann sums over certain tagged families in T. On the other hand, Riemann sums over well chosen tagged families in T approximate $\Theta_\omega(T)$. Combining the two conditions, we use the Howard-Cousin Property of T to prove that

$$\Theta_\omega(T) = \int \mathfrak{D}_T \, \Theta_\omega \, \mathrm{d}\|T\| = \int \langle \mathrm{d}\omega, \overrightarrow{T} \rangle \, \mathrm{d}\|T\|.$$

With an abuse of notation, this boils down to $\partial T(\omega) = T(\mathrm{d}\omega)$, which is a classical formulation of Stokes' Theorem for currents. Let us state our result more precisely:

Theorem 4.3.1. *Given a current $T \in \mathbb{I}_m(\mathbb{R}^n)$ and a differential form $\omega : \mathbb{R}^n \to \Lambda^{m-1}\mathbb{R}^n$ of degree $m-1$, if*

(i) T has C^1-BV charts at all points of $\operatorname{spt} T$ except on a disposable set E_T,

(ii) ω is continuous on $\operatorname{spt} T$

(iii) ω is differentiable at each point of $\operatorname{set}_m \|T\| \backslash E_\omega$, where E_ω has σ-finite $\mathcal{H}^{m-1}$ measure.

(iv) $x \mapsto \langle \mathrm{d}\omega(x), \overrightarrow{T}(x) \rangle$ is Lebesgue integrable with respect to $\|T\|$,

then there holds

$$\int \langle \mathrm{d}\omega(x), \overrightarrow{T}(x) \rangle \, \mathrm{d}\|T\|(x) = \int \langle \omega(y), \overrightarrow{\partial T}(y) \rangle \, \mathrm{d}\|\partial T\|(y). \tag{4.8}$$

4.3.2 Proof

To prove this theorem, we start with a Lemma on the derivation of Θ_ω at points in the C^1-BV charts.

Lemma 4.3.2. *With the hypotheses of theorem 4.3.1, fix $x \in \operatorname{set}_m \|T\| \backslash (E_T \cup E_\omega \cup \operatorname{set}_{m-1} \|\partial T\|$. Then Θ_ω is derivable at x with derivative*

$$\mathfrak{D}_T \, \Theta_\omega(x) = \langle \mathrm{d}\omega(x), \overrightarrow{\partial S}(x) \rangle.$$

Proof. If we can apply proposition 3.2.12 and lemma 3.2.15, we are done. Thus it suffices to prove that $\Theta^{m*}(\|T\|, x) < +\infty$, $\overrightarrow{T}$ has a continuous representative at x.

There exists $r > 0$, a non-zero integer θ, a bounded set of finite perimeter $A \subseteq \mathbb{R}^m$ and a C^1 bilipschitz map ϕ from $\operatorname{cl} A$ to its image in $\mathbb{R}^n$ such that

$$T \llcorner \mathrm{U}(x, r) = \theta \phi_\#(\mathbf{E}^m \llcorner A).$$

We can also suppose that $x = \phi(0)$. This implies that $\Theta^{m*}(\|T\|, x) \leqslant (\operatorname{Lip} \phi)^m \Theta^{m*}(\mathcal{L}^m, 0) = (\operatorname{Lip} \phi)^m$ and proves that $\|T\|$ has finite upper m density at x. Thus proposition 3.2.12 applies, and T is 2^{-2m-3} good at x.

Furthermore ϕ is C^1, so by [26, (4.1.30)], $\|T \llcorner V\|$ almost everywhere, we have

$$\overrightarrow{T} = \mathrm{J}^m \phi(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_m)/|\mathrm{J}^m \phi|.$$

Thus $\overrightarrow{T}$ has a continuous representative in a neighbourhood of x. And we can write $\overrightarrow{T}(x)$ without ambiguity. We apply lemma 3.2.15 to conclude. □

Proof of theorem 4.3.1. The proof relies on approximations of the two sides of (4.8) by Riemann sums over tagged families. The existence of such families is a consequence of the Howard-Cousin Property of T with exceptional set E_T and regularity gauge η_T, which is itself a consequence of theorem 4.2.10. Without loss of generality, we can suppose that $\eta_T \leqslant 2^{-2m-3}$. Let us fix $\varepsilon > 0$.

We start by giving a criterion for approximating the right hand side of (4.8).:

Claim 1. There exists a gauge δ_0 on $\mathrm{set}_m \|T\| \backslash E_T$ such that whenever $\mathcal{P}$ is a δ_0-fine η_T-regular tagged family in T with $|\Theta_\omega(T - [\mathcal{P}]) < \varepsilon/4$,

$$|\Theta_\omega(T) - \sigma(\mathfrak{D}_T\, \Theta_\omega, \mathcal{P})| < \varepsilon/2.$$

Proof of claim 1. For $x \in \mathrm{set}_m \|T\| \backslash E_T$, if $x \in E_\omega \cup \mathrm{set}_{m-1} \|\partial T\|$ let $\delta_0(x) = 0$. Otherwise, according to lemma 4.3.2 let $\delta_0(x)$ be a positive number such that for all $S \in \mathcal{S}(T, x, \delta_0(x), \min(2^{-2m-3}, \eta_T(x)))$,

$$|\Theta_\omega(S) - \mathfrak{D}_T\, \Theta_\omega(x)\, \mathbb{M}(S)| < \frac{\varepsilon\, \mathbb{M}(S)}{4\, \mathbb{M}(T)}.$$

Note that δ_0 is a gauge on $\mathrm{set}_m \|T\| \backslash E_T$, as E_ω and $\mathrm{set}_{m-1} \|\partial T\|$ are m-thin. Given a δ_0-fine $\min(2^{-2m-3}, \eta_T)$-regular, $(|\Theta_\omega|, \varepsilon/4)$-full tagged family $\mathcal{P}$ in T, we have

$$\begin{aligned} |\sigma(\mathfrak{D}_T\, \Theta_\omega, \mathcal{P}) - \Theta_\omega(T)| &\leqslant |\sigma(\mathfrak{D}_T\, \Theta_\omega, \mathcal{P}) - \Theta_\omega([\mathcal{P}])| + |\Theta_\omega(T - [\mathcal{P}])| \\ &\leqslant \sum_{(x,S)\in\mathcal{P}} |\mathfrak{D}_T\, \Theta_\omega(x)\, \mathbb{M}(S) - \Theta_\omega(S)| + \frac{\varepsilon}{4} < \frac{\varepsilon}{4} \frac{\sum_{(x,S)\in\mathcal{P}} \mathbb{M}(S)}{\mathbb{M}(T)} + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}. \end{aligned}$$

■

To estimate the right hand side, a similar condition exists.

Claim 2. There exists a positive gauge δ_1 on $\mathrm{set}_m \|T\|$ and a positive real number τ, such that if $\mathcal{P}$ is a δ-fine tagged family in T with $\mathbb{M}(T - [\mathcal{P}]) < \tau$,

$$\left| \int \langle \mathrm{d}\omega, \overrightarrow{T} \rangle \, \mathrm{d}\|T\| - \sigma(\mathfrak{D}_T\, \Theta_\omega, \mathcal{P}) \right| < \varepsilon/2.$$

Proof of claim claim 2. The measure $\|T\|$ is a finite Borel regular measure and $f : x \mapsto \langle \mathrm{d}\omega(x), \overrightarrow{T}(x) \rangle$ is Lebesgue integrable on $\mathrm{set}_m \|T\| \backslash (E_\omega \cup E_T \cup \mathrm{set}_{m-1} \|\partial T\|)$, with respect to $\|T\|$. Therefore by the Vitali Caratheodory Theorem (see theorem A.5) there exist extended-real valued functions g and h defined on $\mathrm{set}_m \|T\| \backslash (E_T \cup E_\omega \cup \mathrm{set}_{m-1} \|\partial T\|)$, respectively upper and lower semi-continuous, such that $g \leqslant f \leqslant h$ and $\int (h - g)\, \mathrm{d}\|T\| < \varepsilon/4$. By upper semi-continuity of g and symmetrically, by lower semi-continuity of h, for each $x \in \mathrm{set}_m \|T\| \backslash (E_T \cup E_\omega \cup \mathrm{set}_{m-1} \|\partial T\|)$, there exists a

positive $\delta_1(x)$ such that for all $y \in \mathrm{B}(x, \delta(x))$

$$g(y) - \frac{\varepsilon}{8\,\mathbb{M}(T)} \leqslant g(x) \leqslant f(x) \leqslant h(x) \leqslant h(y) + \frac{\varepsilon}{8\,\mathbb{M}(T)}.$$

For $x \in (\mathrm{set}_m \|T\| \backslash E_T) \cap (E_\omega \cup \mathrm{set}_{m-1} \|\partial T\|)$, let $\delta_1(x) = 0$ and note that as E_ω and $\mathrm{set}_{m-1} \|\partial T\|$ are m-thin, δ_1 is a gauge on $\mathrm{set}_m \|T\| \backslash E_T$. If $\mathcal{P}$ is a δ_1-fine tagged family in T, for $(x, S) \in \mathcal{P}$

$$\int g \,\mathrm{d}\|S\| - \frac{\varepsilon\,\mathbb{M}(S)}{8\,\mathbb{M}(T)} \leqslant g(x)\,\mathbb{M}(S) \leqslant f(x)\,\mathbb{M}(S) \leqslant \int h \,\mathrm{d}\|S\| + \frac{\varepsilon\,\mathbb{M}(S)}{8\,\mathbb{M}(T)}.$$

Summing over (x, S) in $\mathcal{P}$ and recalling that $\mathbb{M}([\mathcal{P}]) \leqslant \mathbb{M}(T)$ yields

$$\int g \,\mathrm{d}\|[\mathcal{P}]\| - \frac{\varepsilon}{8} \leqslant \sigma(f, \mathcal{P}) \leqslant \int h \,\mathrm{d}\|[\mathcal{P}]\| + \frac{\varepsilon}{8}.$$

As $h - g$ and $f - g$ are nonnegative and integrable with $f - g \leqslant h - g$, we have for all $Q \sqsubset T$

$$0 \leqslant \int f \,\mathrm{d}\|Q\| - \int g \,\mathrm{d}\|Q\| \leqslant \int (h - g) \,\mathrm{d}\|Q\| \leqslant \varepsilon/4$$

And similarly

$$0 \leqslant \int h \,\mathrm{d}\|Q\| - \int f \,\mathrm{d}\|Q\| \leqslant \varepsilon/4$$

This implies that

$$\left| \int f \,\mathrm{d}\|\mathcal{P}\| - \sigma(f, \mathcal{P}) \right| \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{8}.$$

Furthermore, as f is Lebesgue integrable, there exists τ such that if $Q \sqsubset T$ and $\mathbb{M}(T - Q) < \tau$

$$\left| \int f \,\mathrm{d}\|T\| - \int f \,\mathrm{d}\|Q\| \right| < \varepsilon/4.$$

Therefore, if $\mathcal{P}$ is a δ_1-fine, $(\mathbb{M}, \tau)$-full tagged family in T,

$$\left| \int f \,\mathrm{d}\|T\| - \sigma(f, \mathcal{P}) \right| < \frac{\varepsilon}{2}.$$

■

Fix δ_0, δ_1 and τ, according to the two claims above. The function δ defined on $\mathrm{set}_m \|T\| \backslash E_T$ by $\min(\delta_0, \delta_1)$ is a gauge. The function on $\mathcal{S}(T)$ defined by $G : S \mapsto \max(\mathbb{M}(S), |\Theta_\omega(S)|)$ is continuous, nonnegative and subadditive. We can therefore apply the Howard-Cousin Property of (T, E_T, η_T) to obtain a δ-fine, η_T-regular tagged family $\mathcal{P}$ in T with $G(T - [\mathcal{P}]) < \min(\varepsilon/4, \tau)$. Applying the two claims above, we get

$$\left| \Theta_\omega(T)| - \int \mathfrak{D}_T\, \Theta_\omega \,\mathrm{d}\|T\| \right| \leqslant |\sigma(\mathfrak{D}_T\, \Theta_\omega, \mathcal{P}) - \Theta_\omega(T)| + \left| \int \mathfrak{D}_T\, \Theta_\omega \,\mathrm{d}\|T\| - \sigma(\mathfrak{D}_T\, \Theta_\omega, \mathcal{P}) \right| < \varepsilon.$$

As ε is arbitrary, the term on the left hand side is equal to zero and the proof is complete. □

4.4 Counter-examples

In this section, we give examples of currents on which the Stokes Theorem in such a generality does not hold. A consequence is that these currents do not have the Howard Cousin Property with exceptional set contained in their nonsmooth set. Furthermore, we give an example of a current which has the Howard-Cousin Property with a given exceptional set, but not hereditarily, as it has a subcurrent which is the first counterexample of the section.

Theorem 4.4.1. *There exists an integral current S of dimension 2 in $\mathbb{R}^3$ such that*

1. $\operatorname{spt} S = \operatorname{cl} M$, *where M is a C^∞ submanifold of $\mathbb{R}^3$.*
2. $M = \Psi(R)$, *where $\Psi : R := [\,0,\pi\,] \times [\,0, y_\infty[\to M$ is a C^∞ diffeomorphism and Ψ can be extended to a homeomorphism $\bar{\Psi} : \operatorname{cl} R \to \operatorname{cl} M$.*
3. $\Psi = (\mathrm{id}_{R^2}, \psi)$, *where ψ is C^∞ on R and can be extended to $\operatorname{cl}(R)$ by continuity.*
4. $\Psi \llcorner \partial R$ *is bilipschitz and* $\partial S = \Psi_\#(\partial(\mathbf{E}^2 \llcorner R))$.
5. *For all* $t \in [\,0, y_\infty[\;\; S \llcorner \mathbb{R} \times [0,t\,] \times \mathbb{R} = \Psi_\#(\mathbf{E}^2 \llcorner ([\,0,\pi\,] \times [\,0,t\,]))$.

And a continuous form ω of degree 1 in $\mathbb{R}^3$ which is C^1 on $\mathbb{R}^3 \backslash E$ where $E := [\,0,\pi\,] \times \{y_\infty\} \times [\,-1,1\,]$ and such that:

$$\langle \mathrm{d}\omega(x), \overrightarrow{T}(x)\rangle = 0 \text{ for all } x \in M \backslash E, \text{ but } \int \langle \omega, \overrightarrow{\partial S}\rangle \, \mathrm{d}\|\partial S\| \neq 0.$$

Remark 4.4.2. This implies that the current S does not have the weak Stokes-Cartan property, even though it has a smooth chart at all points of its support which are not in the support of the boundary. Therefore, it does not have the Howard Cousin Property.

Given a smooth enough surface M associated to an integral current S, and a subset M' of M associated to a subcurrent S' of S, call **circulation** of ω around M' the quantity:

$$\int_{\partial M'} \omega = \Theta_\omega(S').$$

The idea behind the construction that follows is to find a form ω defined on a surface M and a small subset E of the surface (here the segment) which is not removable for the circulation of the form. That is, there is no way to remove a small neighbourhood of E from the surface without changing the circulation of the form by a large amount: if M' is nice, closed, contains most of M but is separated from E, then

$$\left| \int_{\partial M'} \omega - \int_{\partial M} \omega \right| > C.$$

Morally, crucial things happen to the circulation in any neighbourhood of E, even if it cannot be seen on E only. Thus removing a neighbourhood of E has a lot of impact on the circulation of the form. To achieve this, E which is rather well behaved (as segments usually are) can be seen as a limit of a badly behaved sequence of curves Γ_k converging to E in Hausdorff distance. Though each of these curves is smooth and they converge to a segment, the curves oscillate a lot with a higher and higher frequency but a lower and lower amplitude. This allows a lot of length to be packed in

a small neighbourhood of E. As the curves tend to E in the Hausdorff distance, their lengths go to infinity. Thus most of the length "vanishes" at the limit. Interpolating correctly between those curves yields a surface S which has E as a piece of its boundary and is smooth everywhere except at the points of E. Here we define E as the segment $[0,\pi]\times\{y_\infty\}\times\{0\}\subset\mathbb{R}^3$ and the curves Γ_k for $k\in\mathbb{N}^*$ as the graphs in $\mathbb{R}^3$ of the functions:

$$\begin{aligned} f_k : [0,\pi]\times\{y_k\}\subset\mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto h^k \sin\left(\frac{x}{\lambda^k}\right), \end{aligned}$$

where $h<1$ and $\lambda^{-1}\in\mathbb{N}^*$, with $h\lambda^{-1}<1$ and $y_k\nearrow y_\infty$ as $k\to\infty$.

The form ω should be defined on S as the differential of a function u which ensures that it will be locally closed. To make use of the "vanishing" of the infinite length, one can define ω in such a way that for all k:

$$\int_{\Gamma_k}\omega(\tau_{\Gamma_k})\,\mathrm{d}l = 1,$$

where τ_{Γ_k} is the tangent vector to Γ_k. Furthermore one can ask that $\omega(\tau_{\Gamma_k})$ be constant along the curve Γ_k, which implies that $\omega(\tau_{\Gamma_k})$ converges uniformly to 0 as k tends to ∞, that is as one gets closer to E, yielding:

$$\int_E \omega(\tau_E)\,\mathrm{d}l = 0.$$

Some of the circulation has "vanished" along with the infinite lengths of the curves. The continuity of ω allows us to consider the operator "circulation of ω" as a continuous function on $\mathcal{S}(S)$ (see chapter 3 for definitions). In the following, this construction is made rigorous.

4.4.1 The main example

Let a,h,λ be positive real parameters with $a,h,\lambda<1$ with $\lambda^{-1}\in\mathbb{N}$. Let also $\phi:[0,1]\to[0,1]$ be a C^∞ non-decreasing function with derivative everywhere lower than 2, which is constant equal to 0 in a neighbourhood of 0 and constant equal to 1 in a neigbourhood of 1. Let $y_0:=0$, for $k=1,2,\dots$ define

$$y_k := \sum_{j=1}^{k} a^j$$

and $y_\infty := \sum_{j=1}^{\infty} a^j$.

$$\begin{aligned} \phi_k : [y_k, y_{k+1}[&\to [0,1] \\ y &\mapsto \phi\left(\frac{y-y_k}{a^{k+1}}\right). \end{aligned}$$

Let alos

$$\begin{aligned} f_0 : [0,\pi] &\to \mathbb{R},\ x\mapsto 0 \\ \text{and for } k>0,\ f_k : [0,\pi] &\to \mathbb{R},\ x\mapsto h^k\sin\left(\frac{x}{\lambda^k}\right). \end{aligned}$$

Interpolate between the f_k as follows

$$\begin{aligned} \psi_k : [0,\pi] \times [y_k, y_{k+1}[\quad &\to \quad \mathbb{R} \\ (x,y) \mapsto (1-\phi_k(y))\, f_k(x) \quad &+ \quad \phi_k(y) f_{k+1}(x). \end{aligned}$$

Consider the following function obtained by patching together the ψ_k:

$$\begin{aligned} \psi : R := [0,\pi] \times [0, y_\infty] \quad &\to \quad \mathbb{R} \\ (x,y) \quad &\mapsto \quad \begin{cases} \psi_k(x,y) & \text{if } y \in [y_k, y_{k+1}]\,, \\ 0 & \text{if } y = y_\infty. \end{cases} \end{aligned}$$

Claim 1. Each ψ_k is Lipschitz on its domain and can be extended to a Lipschitz function on the closure of it's domain. ψ is continuous on its domain and C^∞ in the interior of its domain. Furthermore ψ is identically 0 on the boundary of its domain.

Proof. The first and the last statement hold by definition. For the smoothness: inside each strip, ψ is smooth. At the junction between two strips, it is constant in the y direction and smooth in the x direction, thus smooth. The continuity only needs to be proven when $y \to y_\infty$. Since h is positive and strictly smaller than 1 and $\|f_k\|_\infty \leqslant h^k$, for all k there holds $|\psi_k(x,y)| \leqslant 2h^k$ on its domain. Thus for $y > y_k$ and $x \in [\,0,\pi\,]$, we have $|\psi(x,y)| < 2h^k$ and $\psi(x,y)$ tends to 0 (uniformly) as y tends to y_∞. This proves that ψ is continuous on R. ■

For convenience define

$$\Psi : [0,\pi] \times [0, y_\infty] \to \mathbb{R}^3, (x,y) \mapsto (x, y, \psi(x,y)$$

In $\mathbb{R}^3$, let M be the graph of ψ over $]0,\pi[\times]0, y_\infty[$, (see figure 4.4.1). M is countably 2-rectifiable. Since segments are 2-rectifiable, *cl* M is also countably 2-rectifiable. The oriented surface in the Theorem is the current supported on $\operatorname{cl} M$ with multiplicity 1 and a normal pointing towards the positive z direction (3rd coordinate, we will say "upwards" sometimes). To make this construction rigorous, we construct such a current and prove that it is integral. For $k = 1, 2, \ldots$, let $S_k := \Psi_\#(\mathbf{E}^2 \llcorner [\,0,\pi\,] \times [0, y_{k+1}])$. S_k is an integral current, as Ψ coincides with Ψ_k on $[0,\pi] \times [\,0, y_{k+1}\,]$ and Ψ_k is Lipschitz. We now consider the sequence of integral currents $(\sum_{j=1}^k S_j)$ and prove that it has a limit under some conditions on a, h, λ, which we call S.

Claim 2. If $ha\lambda^{-1} < 1$, then S is an integral current whose boundary is that of the rectangle $[0,\pi] \times [0, y_\infty] \times \{0\}$. The mass of this boundary is $2\pi + 2y_\infty$.

Proof. The mass of S_k is the area (or $\mathcal{H}^2$ measure) of the graph of ψ_k:

$$\mathbb{M}(S_k) =: A_k = \int_0^\pi \int_{y_k}^{y_{k+1}} \sqrt{1 + (\partial_x \psi_k)^2 + (\partial_y \psi_k)^2}\, \mathrm{d}y\, \mathrm{d}x.$$

It is controlled as follows:

$$A_k \leqslant \pi a^k \sqrt{1 + 16 h^{2k} \lambda^{-2k} + 4 h^{2k} a^{-2k}},$$

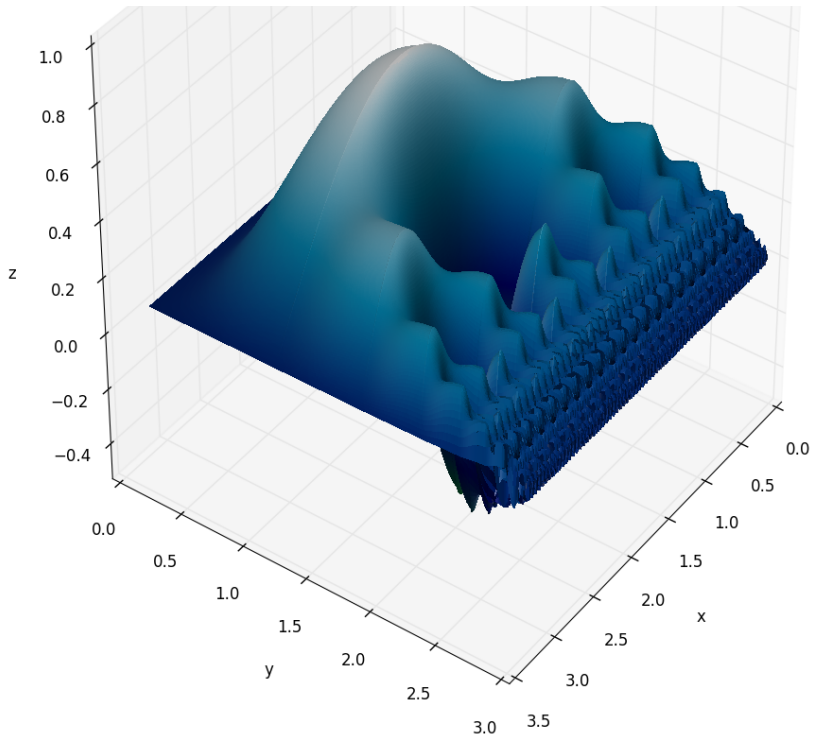

Figure 4.1: Example of surface with non-removable singular set

which is summable in k provided $a < 1$, $h < 1$ and $ha\lambda^{-1} < 1$. **These conditions will be supposed to hold from now on**. They imply that $(\sum_{j=0}^{k} S_j)_k$ is a Cauchy sequence in the mass topology. By [26, 4.1.24], the class of integer multiplicity rectifiable currents is complete in the mass topology and therefore the sequence $(\sum_{j=0}^{k} S_j)_k$ tends to a rectifiable current S. (This is not developped much in Federer's book, but it is a simple application of the definition of integral multiplicity rectifiable currents by pushforwards of polyhedral chains.) Denoting by τ the vector field equal to the upper normal to M we get

$$S = (\mathcal{H}^2 \llcorner M) \wedge \tau.$$

To show that S is an integral current, consider the sequence

$$(T_k)_k := (\partial(\sum_{j=0}^{k} S_j))_k = (\sum_{j=0}^{k} \partial S_j)_k.$$

Let T be the integral 1 dimensional current associated to the boundary of $\Psi(\operatorname{bdry} R)$ with the same orientation as the T_k for $k = 1, 2, \ldots$ and multiplicity 1. For $k = 1, 2, \ldots$, $T - T_k$ is the boundary of a current R_k with

$$\|R_k\| \leqslant \mathcal{H}^2 \llcorner \left([0, \pi] \times \{y_{k+1} \times [-h^{k+1}, h^{k+1}] \cup [0, \pi] \times [y_{k+1}, y_\infty] \times \{0\}\right).$$

Thus $\mathbb{F}(T - T_k) \leqslant (h^{k+1} + y_\infty - y_{k+1})\pi \to 0$ and $\partial S_k \to \partial S = T$ in the flat norm. As T has finite mass (equal to $2\pi + 2y_\infty$), S is an integral current. ■

For $\tilde{y}$ in $[0, y_\infty]$, denote by $L(\tilde{y})$ the length of the section of M by the plane $\{(x, y, z) \in \mathbb{R}^3, y = \tilde{y}\}$, notice that there holds:

$$L(\tilde{y}) = \int_0^\pi \sqrt{1 + (\partial_x \psi)^2(x, \tilde{y})}\, \mathrm{d}x.$$

Claim 3. If $h/\lambda > 1$, then $L(\tilde{y})$ tends to infinity as $\tilde{y}$ tends to y_∞ from below, whereas $L(y_\infty) = \pi$.

Proof. To get a first idea, notice that for $k \in \mathbb{N}^*$

$$L(y_k) \geqslant \frac{\pi h^k}{2\lambda^k}.$$

Indeed:

$$\begin{aligned} L(y_k) = \int_0^\pi \sqrt{1 + \frac{h^{2k}}{\lambda^{2k}} \cos^2(x\lambda^{-k})}\, \mathrm{d}x \\ \geqslant \int_0^\pi \frac{h^k}{\lambda^k} |\cos(x\lambda^{-k})|\, \mathrm{d}x \\ \geqslant \frac{h^k}{\lambda^k} \int_0^\pi |\cos(x\lambda^{-k})|\, \mathrm{d}x \geqslant \frac{2h^k}{\lambda^k}. \end{aligned}$$

Let $y \in [y_k, y_{k+1}[$ and $\theta := \phi_k(y)$, there holds:

$$\begin{aligned} L(y) &= \int_0^\pi \sqrt{1 + \left((1-\theta)\frac{h^k}{\lambda^k}\cos(x\lambda^{-k}) + \theta\frac{h^{k+1}}{\lambda^{k+1}}\cos(x\lambda^{-k-1})\right)^2}\, \mathrm{d}x \\ &\geqslant \int_0^\pi \left|(1-\theta)\frac{h^k}{\lambda^k}\cos(x\lambda^{-k}) + \theta\frac{h^{k+1}}{\lambda^{k+1}}\cos(x\lambda^{-k-1})\right| \mathrm{d}x \\ &\geqslant \frac{h^k}{\lambda^k}\int_0^\pi \left|(1-\theta)\cos(x\lambda^{-k}) + \theta\frac{h}{\lambda}\cos(x\lambda^{-k-1})\right| \mathrm{d}x \\ &\geqslant \frac{h^k}{\lambda^k}\int_0^\pi \left|(1-\theta)\cos(t) + \theta\frac{h}{\lambda}\cos(t\lambda^{-1})\right| \mathrm{d}t, \end{aligned}$$

where the last inequality is obtained by noticing that the integrand is $\pi\lambda^{-k}$ periodic in x and using the change of variables $t := x\lambda^{-k}$.

Let $\alpha(\theta)$ be the value of the last integral. α does not depend on k, is positive for each value of θ. $\theta \mapsto \alpha(\theta)$ is also continuous. Therefore it has a positive minimum. Let C be this minimum, we get

$$\forall k \in \mathbb{N}^*, \forall y \in [y_k, y_k + 1[,\ L(y) \geqslant C\frac{h^k}{\lambda^k},$$

which proves the claim. ■

Define the functions L on $[0, \pi] \times [0, y_\infty]$ and u on $\operatorname{cl} M$ as

$$L(x, y) \quad := \quad \int_0^x \sqrt{1 + (\partial_x \psi)^2(t, y)}\, \mathrm{d}t.$$

Note that $L(\pi, y) = L(y)$ hence the notation is non-ambiguous. Also, ψ is constant on $\{y = y_\infty\}$ so $\partial_x \psi$ exists and is zero on this segment. $u(x, y) := L(x, y)/L(y)$ corresponds to the portion of the

length $L(y)$ which one has to walk on M to get from the point $(0, y, 0)$ to $(x, y, \psi(x, y))$ staying in the same y-coordinate plane. u is as smooth as ψ and $\mathrm{d}u$ is therefore a closed 1 form on R which is equal to

$$\begin{aligned}\mathrm{d}u(x,y) &= \partial_x u\, \mathbf{e}_x^* + \partial_y u\, \mathbf{e}_y^* \\ &= \frac{\sqrt{1+(\partial_x\psi)^2(x,y)}}{L(y)}\mathbf{e}_x^* + \frac{L(\pi,y)\partial_y L(x,y) - L(x,y)\partial_y L(\pi,y)}{L(y)^2}\mathbf{e}_y^*. \quad (4.9)\end{aligned}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ form the canonical basis of $\mathbb{R}^2$ and $\mathbf{e}_x^*$ and $\mathbf{e}_y^*$ is the corresponding 1-covector base. It is now possible to define a 1-form ω_M on M as the pullback of $\mathrm{d}u$ by the projection of M on R $\pi := (x, y, z) \mapsto (x, y)$.

$$\omega_M := \pi^* \,\mathrm{d}u.$$

The 1-form ω_M acts on a tangent vector $\mathbf{v}$ to M by

$$\langle \omega_M(\Psi(x,y)), \mathbf{v}\rangle = \langle \mathrm{d}u(x,y), \mathrm{D}\pi(\Psi(x,y))(\mathbf{v})\rangle. \quad (4.10)$$

Claim 4. The form ω_M is C^1 and closed on M, it can be continuously extended to a form $\omega : \mathbb{R}^3 \to \Lambda^1(\mathbb{R}^3)$ with value 0 on cl $M \cap \{y = y_\infty\}$, and in such a way that it is C^1 in $\mathbb{R}^3 \backslash M \cap \{y = y_\infty\}$.

Proof. ω_M is clearly C^1 on M. It is closed as the pullback of the closed form $\mathrm{d}u$. We now consider ω_M as a map from M to $\Lambda^1(\mathbb{R}^3)$ which we will extend to the whole of $\mathbb{R}^3$. Ψ, u and π can be extended in a C^1 fashion to $\mathrm{cl}\, M \backslash \{(x, y, z), y = y_\infty\}$ this implies that ω_M can be extended in a continuous way to $\mathrm{cl}\, M \backslash \{(x, y, z), y = y_\infty\}$. We now extend ω_M by zero at the points of $\{(x, y, z), y = y_\infty\}$. To do this we consider a tangent basis to M.

Denote by $(\tau_1, \tau_2, \tau_3)(x_0, y_0, z_0)$ the direct orthonormal basis at $(x_0, y_0, z_0) \in M$ with τ_3 the normal vector to S pointing towards positive z and τ_1 tangent to S and the plane $\{(x, y_0, z)\}$. Letting (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$, remark that (τ_1, τ_2, τ_3) is the Gram-Schmidt orthonormalization of $(\mathrm{D}\Psi \cdot e_1, \mathrm{D}\Psi \cdot e_2, e_3)$. In the canonical coordinates for all points in R

$$\tau_1 \circ \Psi = \frac{1}{\sqrt{1+(\partial_x\psi)^2}}\begin{pmatrix}1\\0\\\partial_x\psi\end{pmatrix},$$

$$\tau_2 \circ \Psi = \frac{1}{\sqrt{1+(\partial_x\psi)^2}\sqrt{1+(\partial_x\psi)^2+(\partial_y\psi)^2}}\begin{pmatrix}-\partial_x\psi\partial_y\psi\\1+(\partial_x\psi)^2\\\partial_y\psi\end{pmatrix}$$

and

$$\tau_3 \circ \Psi = \frac{1}{\sqrt{1+(\partial_x\psi)^2+(\partial_y\psi)^2}}\begin{pmatrix}-\partial_x\psi\\-\partial_y\psi\\1\end{pmatrix}$$

In particular,

$$\mathrm{D}\pi(\tau_1) = \frac{1}{\sqrt{1+(\partial_x\psi)^2}}\mathbf{e}_x \quad (4.11)$$

and

$$\mathrm{D}\pi(\tau_2) = \frac{1}{\sqrt{1+(\partial_x\psi)^2}\sqrt{1+(\partial_x\psi)^2+(\partial_y\psi)^2}}\left(-\partial_x\psi\partial_y\psi\mathbf{e}_x + (1+(\partial_x\psi)^2)\mathbf{e}_y\right) \tag{4.12}$$

Choose a tangent vector $\mathbf{v}$ to M, it is of the form $\mathbf{v} = l_1\tau_1 + l_2\tau_2$, suppose l_1 and l_2 are in $(-1,1)$. Combining (4.9), (4.10), (4.11) and (4.12) yields

$$\begin{aligned}
\langle\omega,\mathbf{v}\rangle = \\
&\frac{\sqrt{1+(\partial_x\psi))^2}}{L(y)}\left(l_1\frac{1}{\sqrt{1+(\partial_x\psi)^2}} - l_2\frac{\partial_x\psi\partial_y\psi}{\sqrt{1+(\partial_x\psi)^2}\sqrt{1+(\partial_x\psi)^2+(\partial_y\psi)^2}}\right) \\
&+\frac{L(y)\partial_y L(x,y) - L(x,y)\partial_y L(y)}{L(y)^2}\frac{\sqrt{1+(\partial_x\psi)^2}\; l_2}{\sqrt{1+(\partial_x\psi)^2+(\partial_y\psi)^2}} \\
&= \frac{l_1}{L(y)} + \\
&\frac{l_2\sqrt{1+(\partial_x\psi)^2}}{\sqrt{1+(\partial_x\psi)^2+(\partial_y\psi)^2}}\frac{L(y)\partial_y L(x,y) - L(x,y)\partial_y L(y)}{L(y)^2} \\
&- \frac{l_2}{\sqrt{1+(\partial_x\psi)^2+(\partial_y\psi)^2}}\frac{\partial_x\psi\partial_y\psi}{L(y)}.
\end{aligned}$$

To show that ω goes to 0 as y tends to ∞ we only have to prove that the coefficients in front of l_1 and l_2 in the three terms of the above sum go to zero uniformly in y. For the first term, it follows from claim 3. For the third term, notice that $|\partial_y\psi| \leqslant Ch^k a^{-k}$ whenever $y \in [\,y_k, y_{k+1}[$ and that for all $\alpha > 0$, $\sqrt{(1+\alpha^2)} \geqslant \alpha$. Thus, the third term is controlled by $(h/a)^k/(h/\lambda)^k$, which goes to 0 as y tends to y_∞, provided $\lambda < a$. Let us control the second term, it is sufficient to prove that

$$\frac{L(y)\partial_y L(x,y) - L(x,y)\partial_y L(y)}{L(y)^2} \to 0 \text{ as } y \to y_\infty$$

uniformly in x. For $k = 1, 2, \ldots$, consider the k-th strip: $y \in [\,y_k, y_{k+1}[$. In this strip, ψ can be extended to be $\lambda^k x$ periodic in the x direction. Therefore, we can rewrite

$$L(x,y) = \left\lfloor\frac{x}{\lambda^k\pi}\right\rfloor L(y) + \int_0^{x-\pi\lambda^k\left\lfloor\frac{x}{\lambda^k\pi}\right\rfloor}\sqrt{1+(\partial_x\psi)^2}\,\mathrm{d}x$$

and its derivative in the y direction satisfies

$$\partial_y L(x,y) = \left\lfloor\frac{x}{\lambda^k\pi}\right\rfloor \partial_y L(y) + \int_0^{x-\pi\lambda^k\left\lfloor\frac{x}{\lambda^k\pi}\right\rfloor}\frac{\partial_{yx}\psi\partial_x\psi}{\sqrt{1+(\partial_x\psi)^2}}\,\mathrm{d}x.$$

Thus, there holds

$$\begin{aligned}
L(y)\partial_y L(x,y) - L(x,y)\partial_y L(y) = \\
&L(y)\left(\left\lfloor\frac{x}{\lambda^k\pi}\right\rfloor \partial_y L(y) + \int_0^{x-\pi\lambda^k\left\lfloor\frac{x}{\lambda^k\pi}\right\rfloor}\frac{\partial_{yx}\psi\partial_x\psi}{\sqrt{1+(\partial_x\psi)^2}}\,\mathrm{d}x\right)
\end{aligned}$$

$$- \partial_y L(y) \left(\left\lfloor \frac{x}{\lambda^k \pi} \right\rfloor L(y) + \int_0^{x-\pi\lambda^k \left\lfloor \frac{x}{\lambda^k \pi} \right\rfloor} \sqrt{1+(\partial_x \psi)^2}\, \mathrm{d}x \right).$$

The first terms in each parenthesis cancel out and dividing by $L(y)^2$ yields:

$$\frac{L(y)\partial_y L(x,y) - L(x,y)\partial_y L(y)}{L(y)^2} = \frac{\displaystyle\int_0^{x-\pi\lambda^k \left\lfloor \frac{x}{\lambda^k \pi} \right\rfloor} \frac{\partial_{yx}\psi \partial_x \psi}{\sqrt{1+(\partial_x\psi)^2}}\, \mathrm{d}x}{L(y)} - \frac{\partial_y L(y) \displaystyle\int_0^{x-\pi\lambda^k \left\lfloor \frac{x}{\lambda^k \pi} \right\rfloor} \sqrt{1+(\partial_x\psi)^2}\, \mathrm{d}x}{L(y)^2}.$$

This can be controlled by noticing that the length of the interval over which the integrals are calculated is less than $2\lambda^k \pi$ and using the following straightforward estimates:

$$\begin{aligned} |\partial_x \psi| &\leqslant \sqrt{1+(\partial_x\psi)^2} \lesssim h^k \lambda^{-k}, \\ |\partial_{xy}\psi| &\lesssim h^k \lambda^{-k} a^{-k}, \end{aligned}$$

thanks to which one obtains

$$|\partial_y L(y)| \lesssim h^k \lambda^{-k} a^{-k}$$

and using the fact proven above that $L(y) \gtrsim h^k \lambda^{-k}$ yields

$$\frac{L(y)\partial_y L(x,y) - L(x,y)\partial_y L(y)}{L(y)^2} \lesssim \lambda^k a^{-k}.$$

This tends to 0 provided $a > \lambda$, which proves that $\omega_M \to 0$ uniformly as $y \to y_\infty$. Consider the form $\omega : \mathrm{cl}\ M \to \Lambda^1(\mathbb{R}^3)$ obtained by extending ω_M continuously on cl M and in a C^1 way to cl $M \cap \{(x,y,z), y < y_\infty\}$.

The circulation of ω around S is:

$$\int \langle \omega(x), \overrightarrow{\partial S} \rangle\, \mathrm{d}\|\partial S\| = \int_0^\pi 1\, \mathrm{d}x + 0 + 0 + \int_0^\pi 0\, \mathrm{d}x = 1.$$

To extend ω_M to the whole of $\mathbb{R}^3$, start by extending ψ, π-periodically to $\mathbb{R} \times [\,0, y_\infty\,]$ and let $\chi : \mathbb{R} \to [\,0,1\,]$ be a C^∞ function, equal to 1 in $[\,-1,1\,]$ and to 0 outside of $[\,-2,2\,]$. For $y \in [\,0, y_\infty\,]$, $x \in [\,-\pi, 2\pi\,]$ let

$$\omega(x,y,\psi(x,y)) := \chi((2x-\pi)/\pi) \begin{cases} \omega(x-\pi, y, \phi(x-\pi,y)) \text{ if } x \geqslant \pi, \\ \omega(x+\pi, y, \phi(x+\pi,y)) \text{ if } x \leqslant 0. \end{cases}$$

For $(x,y,z) \in \mathbb{R} \times \mathbb{R}^- \times \mathbb{R}$, let $\omega(x,y,z) := \chi(y)\omega(x,y,z)$ and for $y > y_\infty$ let $\omega(x,y,z) = 0$. So constructed, ω is continous in $\mathbb{R}^3$ and smooth in $\mathbb{R}^3 \backslash [0,\pi] \times \{y_\infty\} \times [\,-1,1\,]$. ■

Claim 5. There is a choice of parameters a, h, λ which is compatible with all the above conditions.

Proof. There must hold:

$$\begin{aligned} 0 < a, h, \lambda < 1, \\ ha < \lambda, \\ h > \lambda, \\ a > \lambda. \end{aligned}$$

A possible choice would be $\lambda = 1/4$, $a = h = 1/3$. □

Remark 4.4.3. $E = [0, \pi] \times \{y_\infty\} \times \{0\}$ is not removable with respect to $\int_{\partial \cdot} \omega$ in the sense that for all ε letting $M_\varepsilon := M \cap \{y \leqslant y_\infty - \varepsilon\}$, M' is a smooth manifold, which can be made "very close" to M, but since ω is smooth and closed on M_ε, there holds:

$$\int_{\partial M_\varepsilon} \omega = 0,$$

whereas

$$\int_{\partial M} \omega = 1.$$

This is made possible because E is not disposable in S (see chapter 5)

Note also that the current S is indecomposable. This is a consequence of the following lemma and of the fact that $\mathbf{E}^2 \llcorner ([\,0, \pi\,] \times [\,0, y_\infty\,])$ is indecomposable (this is true of all currents associated to bounded convex sets of finite perimeter).

Lemma 4.4.4. *Let T be an integral current of dimension m in $\mathbb{R}^{n-1}$ which is indecomposable and has compact support. Let $\xi : \operatorname{spt} T \to \mathbb{R}$ be a continuous function, which is differentiable at $\|T\|$ almost every point at which T has a tangent m-plane. Let $\Xi(x) := (x, \xi(x))$. Define the pushforward R of T by Φ, by associating to $\omega \in \mathcal{D}^m(\mathbb{R}^n)$:*

$$R := \int \langle \omega(\Xi(x)), \mathrm{D}\Xi_x \tau_T(x) \rangle \, \mathrm{J}_m \, \Xi(x) \, \mathrm{d}\|T\|(x)$$

and suppose it is an integral current (of dimension m in $\mathbb{R}^n$). Then R is indecomposable.

Proof. The projection $\pi : \operatorname{spt} R \to \mathbb{R}^m, (x, \xi(x)) \mapsto x$ is lipschitz and proper. Thus the pushforward $\pi_\# R$ is well defined. There holds: $\pi_\# R = T$. Suppose R_1 and R_2 are non zero and decompose R. Then $\pi_\# R_1$ and $\pi_\# R_2$ decompose T. One of them has to be zero. Suppose $\Xi_\# R_1$ is zero. However, since π is linear (thus smooth) from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ and bijective from $\operatorname{spt} R$ to $\operatorname{spt} T$, we have by [26, 4.1.30]

$$\pi_\# R_1 = \mathcal{H}^2 \llcorner \eta,$$

where for $\mathcal{H}^m$ y in $\mathbb{R}^{n-1}$,

$$\eta(y) = \sum_{x \in \pi^{-1}(y)} \frac{\theta_{R_1}(x)}{\mathrm{J}_m \, \pi} \, [\Lambda_m \pi(x)] \, \overrightarrow{R_1}(x) = \theta_{R_1}(y, \Xi(y)) \, [\Lambda_m \pi(y, \Xi(y))] \, \overrightarrow{R_1}(y, \Xi(y)).$$

The assumption that $\pi_\# R_1 = 0$ implies that

$$\theta_{R_1}(x) = 0,$$
$$\text{or}$$
$$[\Lambda_m \pi(x)]\, \overrightarrow{R_1}(x) = 0$$

for $\|R_1\|$ almost all x.

Recall that by definition of R and R_1 and the assumption on Ξ, for $\|R_1\|$-almost all $x \in \mathbb{R}^n$, $\overrightarrow{R_1}(x) = \overrightarrow{R}(x) = \mathrm{D}\Xi_{\pi(x)} \overrightarrow{T}(\pi(x))$ and $\overrightarrow{T}(\pi(x)) \neq 0$. Thus for those x, we have

$$[\Lambda_m \pi(x)]\, \overrightarrow{R_1}(x) = [\Lambda_m \pi(x)]\, \mathrm{D}\Xi_{\pi(x)} \overrightarrow{T}(\pi(x)) = \overrightarrow{T}(\pi(x) \neq 0,$$

which implies that θ_{R_1} must be zero $\|R_1\|$ almost everywhere and $R_1 = 0$. So (R_1, R_2) is not a proper decomposition of R. R is therefore indecomposable. □

4.4.2 A variation with only one singular point

Theorem 4.4.5. *There exists a compact surface $\tilde{M}$ in $\mathbb{R}^3$, such that $\tilde{M}\backslash\{0\}$ is a C^2 submanifold with boundary the horizontal circle of radius 1, z-coordinate 0 and center 0. To $\tilde{M}$ we can associate an integral current $\tilde{S}$ of dimension 2 whose boundary is the corresponding positively oriented circle with multiplicity 1. Together with a form $\tilde{\omega}$ which is continuous on $\mathbb{R}^3$ and differentiable on $\mathbb{R}^3\backslash(\{(0,0)\} \times [-1,1])$ such that $\langle \mathrm{d}\tilde{\omega}(x), \overrightarrow{\tilde{S}}(x)\rangle = 0$ for $x \in \tilde{M}\backslash\{0\}$, $\int \langle \tilde{\omega}, \overrightarrow{\partial \tilde{S}}\rangle \, \mathrm{d}\|\partial \tilde{S}\| = 1$.*

Before constructing the surface and form, note that a simple pushforward argument: collapsing the singular set of the previous example onto one point does not work, as the conditions on the parameters do not allow continuity of the form at the singular point. However, working in cylindrical coordinates and constructing a surface on concentric crowns, we get a different set of conditions on the parameters, thanks to the different area element ($r\,\mathrm{d}r\,\mathrm{d}\theta$ instead of $\mathrm{d}x\,\mathrm{d}y$) and we can define a continuous form according to the specifications of the statement.

There is probably a way to work with pushforward operations, but one would have to work with a map which is not Lipschitz globally, rather Lipschitz on each crown. We prefer to start afresh with the experience from the previous example.

Proof. The construction is a translation of the previous one, albeit in cylindrical coordinates. Let a, h, λ be parameters in $(0,1)$ with $\lambda^{-1} \in \{2, 3, \ldots\}$.

Let $r_0 := 1$, and for $k = 1, 2, \ldots$, let

$$r_k := 1 - \frac{\sum_{j=0}^{k-1} a^j}{\sum_{j=1}^{+\infty} a^j} = a^k.$$

For $k \in \{0, 1, 2, \ldots\}$, let $\tilde{f}_k : [0, 2\pi] \to \mathbb{R}$ be a function with for all $\theta \in [0, 2\pi]$

$$\tilde{f}_0(\theta) = 0 \text{ and}$$
$$\tilde{f}_k(\theta) = h^k \sin(\lambda^{-k}\theta) \text{ for } k \geqslant 1.$$

Consider the closed unit disk D in $\mathbb{R}^2$, and for $(r,\theta) \in]r_{k+1}, r_k] \times [0, 2\pi]$, define

$$\tilde{\psi}_k(r,\theta) := \phi\left(\frac{r - r_{k+1}}{a^k}\right) \tilde{f}_k(\theta) + \left(1 - \phi\left(\frac{r - r_{k+1}}{a^k}\right)\right) \tilde{f}_{k+1}(\theta),$$

where ϕ is the smooth transition function defined in the previous section. Let

$$\tilde{\psi} : D \to \mathbb{R}; (r,\theta) \mapsto \begin{cases} 0 \text{ if } r = 0, \\ \tilde{\psi}_k(r,\theta) \text{ if } r \in]r_{k-1}, r_k] \end{cases}$$

and the function $\tilde{\Psi} : D \to R^3; (r,\theta) \mapsto (r,\theta,z)$ (in cylindrical coordinates). Define also the length at radius r as:

$$\tilde{L}(r) := \int_0^{2\pi} \sqrt{1 + \frac{(\partial_\theta \tilde{\psi}(r,\theta))^2}{r^2}}\, r \,\mathrm{d}\theta.$$

By the same argument as previously $\tilde{L}(r) \to \infty$ as $r \to 0$, provided $h > \lambda$. The area of the graph of $\tilde{\Psi}$ which is situated between r_k and r_{k+1} is:

$$\tilde{A}_k := \int_0^{2\pi} \int_{r_{k+1}}^{r_k} \sqrt{1 + (\partial_r \tilde{\psi}_k(r,\theta))^2 + \frac{(\partial_\theta \psi_k'(r,\theta))^2}{r^2}}\, r \,\mathrm{d}r \,\mathrm{d}\theta$$

and we can write

$$\begin{aligned} \tilde{A}_k &\leqslant 2\pi \int_{r_{k+1}}^{r_k} \sqrt{1 + 16h^{2k}a^{-2k} + 4h^{2k}\lambda^{-2k}r^{-2}}\, r \,\mathrm{d}r \\ &\leqslant 2\pi a^k (1-a) \sqrt{a^{2k} + 16h^{2k}a^{-2k} + 4h^{2k}\lambda^{-2k}}\, a^k \end{aligned}$$

since by definition, $r_k = a^k$. Thus $(A_k)_{k \geqslant 0}$ is summable provided $ha^2/\lambda < 1$. And we can define the current $\tilde{S}$ carried by $\tilde{M} := \tilde{\Psi}(\operatorname{int} D \backslash \{0\})$. As above, $\tilde{S}$ is integral. And we have $\partial \tilde{S} = \tilde{\Psi}_\# \partial(\mathbf{E}^2 \llcorner D)$.

Define the function $\tilde{u} : D \to \mathbb{S}^1 \simeq \mathbb{R}/\{0 \sim 1\}; (r,\theta) \mapsto (\tilde{L}(r,\theta)/\tilde{L}(r))/\{0 \sim 1\}$, where

$$\tilde{L}(r,\theta) := \int_0^{\theta} \sqrt{1 + \frac{(\partial_\theta \tilde{\psi}(r,\theta))^2}{r^2}}\, r \,\mathrm{d}\theta.$$

We get for $(r,\theta) \in D \backslash \{0\}$:

$$\begin{aligned} \mathrm{d}\tilde{u}(r,\theta) &= r^{-1}\partial_\theta \tilde{u} \mathbf{e}_\theta^* + \partial_r \tilde{u} \mathbf{e}_r^* \\ &= \frac{\sqrt{1 + r^{-2}(\partial_\theta \tilde{\psi})^2(r,\theta)}}{\tilde{L}(r)} \mathbf{e}_\theta^* + \frac{\tilde{L}(r)\partial_r \tilde{L}(r,\theta) - \tilde{L}(r,\theta)\partial_r \tilde{L}(r)}{\tilde{L}(r)^2} \mathbf{e}_r^*. \end{aligned}$$

Pullback $\mathrm{d}\tilde{u}$ by the projection on the horizontal plane: $\pi : (r,\theta,z) \mapsto (r,\theta)$ to get a C^1 1-form $\tilde{\omega}$ defined on $\tilde{M}$. $\tilde{\omega}$ is closed, as $\mathrm{d}\tilde{u}$ also is.

Let us show that we can extend $\tilde{\omega}$ by 0 at 0. Consider the cylindrical coordinates orthonormal basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ in $\mathbb{R}^3 \backslash (\{(0,0)\} \times \mathbb{R})$. In these coordinates, we express the vectors of a tangent

basis to $\tilde{M}$ at each point of $\tilde{M}$:

$$\tilde{\tau}_1 \circ \tilde{\Psi} = \frac{1}{\sqrt{1 + r^{-2}(\partial_\theta \tilde{\psi})^2}} \begin{pmatrix} 1 \\ 0 \\ r^{-2}\partial_\theta \tilde{\psi} \end{pmatrix},$$

$$\tilde{\tau}_2 \circ \tilde{\Psi} = \frac{1}{\sqrt{1 + r^{-2}(\partial_\theta \tilde{\psi})^2}\sqrt{1 + r^{-2}(\partial_\theta \tilde{\psi})^2 + (\partial_r \tilde{\psi})^2}} \begin{pmatrix} -r^{-1}\partial_\theta \tilde{\psi}\partial_r \tilde{\psi} \\ 1 + r^{-2}(\partial_\theta \tilde{\psi})^2 \\ \partial_r \tilde{\psi} \end{pmatrix}$$

and the normal vector $\tilde{\tau}_3$ to $\tilde{M}$ making $(\tilde{\tau}_3, \tilde{\tau}_2, \tilde{\tau}_3)$ into a direct orthonormal basis of $\mathbb{R}^3$.

$$\tilde{\tau}_3 \circ \tilde{\Psi} = \frac{1}{\sqrt{1 + r^{-2}(\partial_\theta \tilde{\psi})^2 + (\partial_r \tilde{\psi})^2}} \begin{pmatrix} -r^{-1}\partial_\theta \tilde{\psi} \\ -\partial_r \tilde{\psi} \\ 1 \end{pmatrix}.$$

We thus have

$$\pi(\tilde{\tau}_1) = \frac{1}{\sqrt{1 + r^{-2}(\partial_\theta \tilde{\psi})^2}} \mathbf{e}_\theta \tag{4.13}$$

and

$$\pi(\tilde{\tau}_2) = \frac{\left(-r^{-1}\partial_\theta \tilde{\psi}\partial_r \tilde{\psi}\mathbf{e}_\theta + (1 + r^{-2}(\partial_\theta \tilde{\psi})^2)\mathbf{e}_r\right)}{\sqrt{1 + r^{-2}(\partial_\theta \tilde{\psi})^2}\sqrt{1 + r^{-2}(\partial_\theta \tilde{\psi})^2 + (\partial_r \tilde{\psi})^2}} \tag{4.14}$$

For a vector $\mathbf{v} = l_1\tilde{\tau}_1 + l_2\tilde{\tau}_2$ tangent to $\tilde{M}$ at the point $\tilde{\Psi}(r, \theta)$, we can write:

$$\langle \tilde{\omega}(\tilde{\Psi}(r, \theta)), \mathbf{v} \rangle = \langle \mathrm{d}\tilde{u}(r, \theta), \pi\mathbf{v} \rangle$$

and we get

$$\begin{aligned} \langle \tilde{\omega}(\tilde{\Psi}(r, \theta)), \mathbf{v} \rangle = \frac{l_1}{\tilde{L}(r)} + & \\ & \frac{l_2\sqrt{1 + r^{-2}(\partial_\theta \tilde{\psi})^2}}{\sqrt{1 + r^{-2}(\partial_\theta \tilde{\psi})^2 + (\partial_r \tilde{\psi})^2}} \frac{\tilde{L}(r)\partial_r \tilde{L}(r, \theta) - \tilde{L}(r, \theta)\partial_r \tilde{L}(r)}{\tilde{L}(r)^2} \\ & - \frac{l_2}{\sqrt{1 + r^{-2}(\partial_\theta \tilde{\psi})^2 + (\partial_r \tilde{\psi})^2}} \frac{r^{-1}\partial_\theta \tilde{\psi}\partial_r \tilde{\psi}}{\tilde{L}(r)}. \end{aligned}$$

Finally for $r \in (r_{k+1}, r_k\,]$

$$\begin{aligned} |\tilde{\omega}(r, \theta)| \lesssim & \\ & \frac{1}{\tilde{L}(r)} \max\left\{1, \frac{\tilde{L}(r)\partial_r \tilde{L}(r, \theta) - \tilde{L}(r, \theta)\partial_r \tilde{L}(r)}{\tilde{L}(r)}, \frac{r^{-1}\partial_\theta \tilde{\psi}\partial_r \psi}{\sqrt{1 + r^{-2}(\partial_\theta \tilde{\psi})^2 + (\partial_r \tilde{\psi})^2}}\right\} \\ & \lesssim \max\left\{\lambda^k h^{-k}, \lambda^k a^{-k}, \lambda^k a^{-k}\right\}, \end{aligned}$$

Where we used the same periodicity argument as in the previous case. Thus if $\lambda < \min\{h, a\}$, $\tilde{\omega}$ can be extended to 0 by continuity at 0. A good choice of parameters is $\lambda = 1/4$, and $h = a = 1/3$.

To extend $\tilde{\omega}$ to the whole of $\mathbb{R}^3$: let $\tilde{\omega}(0,0,z) := 0$ for all z and for $(r,\theta,z) \in (0,1] \times [0,2\pi] \times \mathbb{R}$, let

$$\tilde{\omega}(r,\theta,z) := \chi(z - \tilde{\psi}(r,\theta))\tilde{\omega}(r,\theta,\tilde{\psi}(r,\theta)),$$

where $\chi : \mathbb{R} \to [0,1]$ is smooth compactly supported and equal to 1 in $[-1,1]$. For $r > 1$, and all θ, z, let $\tilde{\omega}(r,\theta,z) := \chi(z)\tilde{\omega}(1,\theta,z)$. Thus ω is a continuous 1 form in $\mathbb{R}^3$, compactly supported and C^1 in $\mathbb{R}^3 \backslash \{(0,0)\} \times [-1,1]$. □

4.4.3 An example related to the hereditary Howard-Cousin Property

To conclude on the current S, as announced in chapter 5, we present a current T which has the Howard Cousin Property and which admits S as a subcurrent:

$$T = \left(\mathcal{H}^2 \llcorner \left(M \cup R \cup \bigcup_{k=1}^{\infty} (H_k \cup D_k)\right)\right) \wedge \overrightarrow{T},$$

where R is the intersection of the topological boundary of the convex enveloppe of M with the half space $z \leqslant 0$. In particular R is the graph of a the function ψ_R defined on $[0,\pi] \times [0, y_\infty]$. ψ_R is the minimum of a convex continuous function and the function $(x,y) \mapsto 0$ and we have $\psi_R \leqslant \psi$. We also consider the function $\Psi_R = (\mathrm{id}_{\mathbb{R}^2}, \psi_R))$.

- for all k, H_k and D_k are very close to

$$\{(x, y_k, z), x \in [0,\pi], z \in [\psi_R(x,y_k), \psi(x,y_k)]\},$$

- H_k is moved slightly in the negative y direction and D_k slightly in the positive y direction, except for the top and bottom boundaries, so that

$$\begin{aligned} H_k \cap R = D_k \cap R &= \Psi_R([0,\pi] \times \{y_k\}) \\ H_k \cap M = D_k \cap M &= \Psi([0,\pi] \times \{y_k\}) \\ H_k \cap D_k &= \Psi_R([0,\pi] \times \{y_k\}) \cup \Psi([0,\pi] \times \{y_k\}) \end{aligned}$$

The simple unit 2-vector field $\overrightarrow{T}$ is chosen so that

- $\overrightarrow{T} = \overrightarrow{S}$ $\mathcal{H}^2$ almost everywhere on M.
- $T \llcorner (M \cup R)$ is a cycle.
- $T \llcorner (H_k \cup D_k)$ is a cycle for $k = 1, 2, \ldots$.
- For all $k \geqslant 1$, $Q_k := T \llcorner \{(x,y,z) \in \mathbb{R}^3, y > y_k\}$ is a cycle.

Proposition 4.4.6. *T is an integral current and has the Howard Cousin Property with exceptional set $E := [0,\pi] \times \{(y_\infty, 0)\}$ but not the hereditary Howard Cousin Property.*

Sketch of proof. For the first point, let us check that T has finite mass: it is a consequence of the fact that the mass of R is finite, and that $\mathbb{M}(D_k) \leqslant \mathbb{M}(G_k) \leqslant \pi(2h^k)$, making them summable. Since we are summing cycles, the limit current T is a cycle and therefore an integral current. Notice that S is a subcurrent of T, as one can write $S = T \llcorner M$.

The current S defined at the beggining of this section is a subcurrent of T and does not have the Howard-Cousin Property with exceptional set E. Thus T does not have the Hereditary Howard-Cousin Property with exceptional set E.

However, we can prove that T has the Howard Cousin Property with exceptional set E and a regularity gauge η_T, where η_T is piecewise constant in $\operatorname{spt} T \backslash E$ and defined using the charts, notice that E is disposable as one can remove a cycle $Q \sqsubset T$ of arbitrarily small mass such that $\operatorname{spt}(T - Q) \cap E = \emptyset$. For $\varepsilon > 0$, there exists $k > 0$ such that $Q := Q_k$ has mass smaller than ε, and zero boundary mass.

With this choice of Q, $T - Q$ is a finite sum of bilipschitz pushforwards of currents associated with compact BV sets and therefore has the Howard Cousin Property with empty exceptional set and gauge η_T defined above. Given a nonnegative subadditive continuous functions G on $\mathcal{S}(T)$, a real number $\varepsilon > 0$, a gauge δ on $\operatorname{set}_m \|T\| \backslash E$ and a regularity gauge $\eta \leqslant \eta_T$, we can thus find $Q \in \mathcal{S}$ such that $G(Q) < \varepsilon/2$ and $T - Q$ has the Howard-Cousin Property with empty singular set and regularity gauge $\eta_T|_{\operatorname{set}_m \|T-Q\|}$, thus there exists a tagged family $\mathcal{P}$ in Q, which is $\delta|_{\operatorname{set}_m \|T-Q\|}$-fine, $\eta|_{\operatorname{set}_m \|T-Q\|}$-regular and $(G, \varepsilon/2)$-full. $\mathcal{P}$ is thus a δ-fine, η-regular, (G, ε)-full tagged family in T. □

4.4.4 Remarks on other counter-examples

In looking for a counter-example to a generalized Stokes' Theorem, the example of 3.1.11 could be a good candidate in the case where T has the Weak Inner Approximation Property and $\operatorname{set}_2 \|T\|$ could be made into a smooth manifold. However, the singular set of these currents has Hausdorff dimension larger than 1 - as it contains a four corner Cantor set built with ratio 1/3. Nonetheless, the high dimensional part of the singular set is contained outside of $\operatorname{set}_2 \|T\|$, this is already interesting. If we could make T and ω smooth outside of the singular set of T, it would be a good counter example, as there are no singularities in $\operatorname{set}_2 \|T\|$. Note however that ω would have to be discontinuous, but could stay bounded.

Finally we would like to conclude by mentionning intermediate structures between the o-minimal world and the usual GMT world with its Cantor sets and oscillations. More precisely, in o-minimal geometry, by the Cell Decomposition Theorem, there is only one notion of dimension, or in other words, if an set is definable, its topological dimension coincides with its Hausdorff and Minkowski dimension (and all notions of metric dimension, as far as I know). In [39], P. Hieronymi and C. Miller have proved that a structure has this property (for closed sets) if and only if it does not define the set of relative integers.

An example of such a structure is $(\overline{\mathbb{R}}, 2^{-\mathbb{N}})$. In this structure, oscillation can happen (consider the function $f : \mathbb{R} \to \mathbb{R}; x \mapsto \operatorname{dist}(x, 2^{-\mathbb{N}})$), but it has to decay or explode quickly and cannot give rise to structures of intermediate density. In particular, countable sets of Minkowski dimension larger than 0 are not definable. It is therefore natural to ask:

Question 4.4.7 (Intermediate case)**.** *Is there an integral current T definable in $(\overline{\mathbb{R}}, 2^{-\mathbb{N}})$ which does not have the Stokes-Cartan Property? We could use for instance $x \mapsto \operatorname{dist}(x, 2^{-\mathbb{N}})$?*

Chapter 5

Integration on currents

5.1 Pfeffer Integration

In this section, $T \in \mathbb{I}_m(\mathbb{R}^n)$ has the Howard-Cousin Property with exceptional set E_T and regularity gauge η_T. Eventually we will assume that (T, E_T, η_T) has the hereditary Howard-Cousin Property.

Definition 5.1.1. A function f defined on $\operatorname{set}_m \|T\| \backslash E_T$ is $\mathfrak{P}_{T,E_T,\eta_T})$**-integrable** . if there exists a nonnegative subadditive continuous function G on $\mathcal{S}(T)$ and a real number $I(f,T)$ such that for every $\varepsilon \in (0,1)$ there exists $\tau > 0$, and a gauge δ on $\operatorname{set}_m \|T\| \backslash E$ such that for all δ-fine, $(\varepsilon\eta_T)$-regular tagged family $\mathcal{P}$ which is (G,τ)-full in T, i.e.

$$G\left(T - \sum_{j=1}^{p} S_j\right) < \tau, \tag{5.1}$$

there holds

$$|\sigma(f,\mathcal{P}) - I(f,T)| < \varepsilon, \tag{5.2}$$

where

$$\sigma(f,\mathcal{P}) := \sum_{j=1}^{p} \mathbb{M}(S_j) f(x_j).$$

Remark 5.1.2. Note that for this notion to be well defined, the Howard-Cousin property of (T, E_T, η_T) is necessary. Also it is not clear that if one uses two different regularity gauges η_T and η'_T on $\operatorname{set}_m \|T\| \backslash E_T$, such that (T, E_T, η_T) and (T, E_T, η'_T) have the Howard-Cousin Property, the integrals so defined coincide. It would be the case, provided T had the Howard-Cousin Property with the regularity gauge $\max(\eta, \eta')$ which is not clear. However, the integral so defined coincides with the Lebesgue integral for Lebesgue integrable functions as we show below, and this does not depend on the regularity gauge. In the following, η is fixed and the $\mathfrak{P}_{(T,E_T,\eta_T)}$-integral: $I(f,T)$ depends on it implicitely.

Proposition 5.1.3 (Space of Pfeffer integrable functions)**.** *The integral $I(f,T)$ is uniquely defined. The space of $\mathfrak{P}_\eta$-integrable functions is a real vector space and the integral is a linear operator. Furthermore, if $f \leqslant g$ on $\operatorname{set}_m \|T\|$ and both functions are $\mathfrak{P}_\eta$ integrable on T, then $I(f,T) \leqslant I(g,T)$.*

Proof. For the first point, suppose two different integrals exist I_1 and I_2 along with $G_{1,2}$ as in the definition of integrability. Let $G := \max(G_1, G_2)$ and $\varepsilon := |I_1 - I_2|/3 > 0$ and chose δ_1, δ_2 as in the definition. Let $\delta := \min\{\delta_1, \delta_2\}$. By the Howard-Cousin property, there exists a tagged family $\mathcal{P}$ which is δ-fine, $\varepsilon\eta$-regular, and such that (5.1) holds. But by the choice of δ and G, $\mathcal{P}$ is also δ_k-fine and satisfies (5.1) with G_k instead of G, for $k = 1, 2$, thus by the assumptions on I_1, I_2 and ε, there holds

$$|I_1 - I_2| \leqslant |I_1 - \sigma(f, \mathcal{P})| + |\sigma(f, \mathcal{P}) - I_2| \leqslant 2\varepsilon < |I_1 - I_2|,$$

a contradiction.

For the second and third point, consider two functions f and g, Pfeffer integrable on T and a real number λ. If $\lambda = 0$, it is clear that $\lambda f = 0$ is integrable with integral $I(f, T)$. For non zero λ, to prove that λf is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable on T with integral $\lambda I(f, T)$, choose the same G as for f. Then, given $\varepsilon' > 0$, choose a gauge δ corresponding to $\varepsilon := \varepsilon'/|\lambda|$ for the integrability of f and multiply (5.2) by $|\lambda|$ to get the result for any δ-fine $(\varepsilon\eta_T)$-regular tagged family such that (5.1).

To show that $f + g$ is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable with integral $I(f, T) + I(g, T)$, choose G_f and G_g corresponding to the integrability of f and g respectively and let $G := \max(G_f, G_g)$. Given $\varepsilon > 0$, choose gauges δ_f, δ_g and positive numbers τ_f, τ_g corresponding to the integrability of f and g respectively, but with $\varepsilon/2$ instead of ε. Let $\delta := \min\{\delta_1, \delta_2\}$ and $\tau := \min\{\tau_1, \tau_2\}$. Let $\mathcal{P}$ be a δ-fine $\varepsilon\eta_T$-regular tagged family in T such that (5.1) holds. There also holds

$$G_{f,g}\left(T - [\mathcal{P}]\right) \leqslant G\left(T - [\mathcal{P}]\right) < \tau \leqslant \tau_{f,g},$$

so by the integrability of f and g,

$$|\sigma(f + g, \mathcal{P}) - I(f, T) - I(g, T)| \leqslant |\sigma(f, \mathcal{P}) - I(f, T)| + |\sigma(g, \mathcal{P}) - I(g, T)| < \varepsilon,$$

which proves that $f + g$ is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable on T and $I(f + g, T) = I(f, T) + I(g, T)$.

For the last point, by linearity, it suffices to show that if f is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable on T and nonnegative, then $I(f, T)$ is nonnegative, this follows from the definition of integrability as necessarily in this case, all sums $\sigma(f, \mathcal{P})$ are nonnegative. □

Proposition 5.1.4 (Cauchy Criterion)**.** *If f and T are as above, then f is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable if and only there exists a function G subadditive and continuous on $\mathcal{S}(T)$ and for all $\varepsilon > 0$, there exists a gauge δ on $\operatorname{set}_m \|T\| \backslash E_T$ such that for any pair $(\mathcal{P}_1, \mathcal{P}_2)$ of δ-fine, $\varepsilon\eta_T$-regular tagged families in T satisfying* (5.1)*, there holds:*

$$|\sigma(f, \mathcal{P}_1) - \sigma(f, \mathcal{P}_2)| < \varepsilon. \tag{5.3}$$

Proof. If f is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable, apply the definition of integrability with $\varepsilon/2$ instead of ε to find δ and $\tau > 0$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two δ-fine, $(\varepsilon\eta)$-regular tagged families in T satisfying (5.1). There holds:

$$|\sigma(f, \mathcal{P}_1) - \sigma(f, \mathcal{P}_2)| \leqslant |\sigma(f, \mathcal{P}_1) - I(f, T)| + |I(f, T) - \sigma(f, \mathcal{P}_2)| < \varepsilon.$$

Conversely suppose (5.3) holds for all $\mathcal{P}_1$, $\mathcal{P}_2$ as above. Let $\varepsilon_k := 2^{-k}$ with $k = 1, 2, \ldots$ and

choose for each k, a positive numbers τ_k as well as a gauge δ_k as in the statement. Noting that the minimum of a finite family of gauges is still a gauge we can suppose that $(\delta_k)_k$ and $(\tau_k)_k$ are decreasing. Using the Howard-Cousin Property of T, for $k = 1, 2, \dots$, choose a tagged family in T: $\mathcal{P}_k$ which is δ_k-fine, satisfies

$$G\left(T - [\mathcal{P}_k]\right) < \tau_k,$$

and we can ask furthermore that $\mathcal{P}_1$ be $\eta/2$ regular and that for $k \geqslant 2$, $\mathcal{P}_k$ be $(\varepsilon_{k-1}\eta)$-regular. We can thus apply the hypothesis for ε_k to $\mathcal{P}_k$ and $\mathcal{P}_{k+1}$, yielding

$$|\sigma(f, \mathcal{P}_k) - \sigma(f, \mathcal{P}_{k+1})| < \varepsilon_k.$$

This shows that the sequence $\sigma(\mathcal{P}_k, f)$ converges to a real number I, we will show that I is the integral of f on T. For $\varepsilon > 0$, suppose that $\varepsilon/3 \in [\,\varepsilon_k, \varepsilon_{k-1})$. Let $\mathcal{P}$ be a δ_k-fine, $(\varepsilon\eta)$-regular tagged family such that

$$G\left(T - [\mathcal{P}]\right) < \tau_k.$$

As $\varepsilon_k < \varepsilon$, the family $\mathcal{P}$ is also $(\varepsilon_k\eta)$-regular, so by the choice of δ_k and τ_k, there holds

$$|\sigma(f, \mathcal{P}) - \sigma(f, \mathcal{P}_k)| < \varepsilon_k.$$

Thus since $\sum_{k' \geqslant k} |\sigma(f, \mathcal{P}) - \sigma(f, \mathcal{P}_k)| < 2\varepsilon_k$, we get

$$|\sigma(f, \mathcal{P}) - I| < 3\varepsilon_k \leqslant \varepsilon,$$

which concludes the proof. □

Now that the integral is well defined, we can also denote it $(\mathfrak{P}_{(T,E_T,\eta_T)}) \int f$.

Proposition 5.1.5 (Adapted from 3.1.6-3.1.9 in [61])**.** *A function which is Lebesgue integrable with respect to $\|T\|$ is also $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable on T and the Pfeffer integral of f on T coincides with the Lebesgue integral with respect to $\|T\|$.*

Proof. Let f be in $\mathbb{L}^1(\|T\|)$, extend f by zero to get a function $\bar{f}$ defined everywhere in $\operatorname{spt} T$. Remember that $\|T\|$ is a Radon measure. By the Vitali Caratheodory Theorem (see [63, 2.25]), given $\varepsilon > 0$, there exist extended-real valued functions g and h on $\operatorname{set}_m \|T\|$ respectively upper and lower semi-continuous, such that $g \leqslant \bar{f} \leqslant h$ almost everywhere and $\int (h - g)\, \mathrm{d}\|T\| < \varepsilon$. By upper semi-continuity of g and symmetrically, by lower semi-continuity of h, for each $x \in \operatorname{spt} T$, there exists a positive $\delta(x)$ such that for all $y \in \mathrm{B}(x, \delta(x))$

$$g(y) \leqslant \bar{f}(x) + \varepsilon \text{ and } h(y) \geqslant \bar{f}(x) - \varepsilon.$$

If $\mathcal{P} := ((S_1, x_1), \dots, (S_p, x_p))$ is a δ-fine tagged family in T, there holds

$$\int g\, \mathrm{d}\|[\mathcal{P}]\| - \varepsilon\, \mathbb{M}([\mathcal{P}]) \leqslant \sigma(\bar{f}, \mathcal{P}) \leqslant \int h\, \mathrm{d}\|[\mathcal{P}]\| + \varepsilon\, \mathbb{M}([\mathcal{P}]).$$

Since $\mathbb{M}([\mathcal{P}]) \leqslant \mathbb{M}(T)$, if $\mathcal{P}$ and $\mathcal{Q}$ are two such families, there holds:

$$|\sigma(\bar{f}, \mathcal{P}) - \sigma(\bar{f}, \mathcal{Q})| \leqslant (2\,\mathbb{M}(T) + 1)\varepsilon.$$

Applying proposition 5.1.4 to $\bar{f}$ we conclude that $\bar{f}$ is $\mathfrak{P}_{(T,E_T,\eta_T)}$-integrable on T. For $\mathbb{M}([\mathcal{P}])$ close to $\mathbb{M}(T)$ we have

$$\int g \,\mathrm{d}\|[\mathcal{P}]\| > \int g \,\mathrm{d}\|T\| - \varepsilon \geqslant \int \bar{f} \,\mathrm{d}\|T\| - 2\varepsilon$$

and similarly $\int h \,\mathrm{d}\|[\mathcal{P}]\| \leqslant \int \bar{f} \,\mathrm{d}\|T\| + 2\varepsilon$. This implies that $(\mathfrak{P}_{(T,E_T,\eta_T)}) \int f \,\mathrm{d}\|T\| = \mathcal{L} \int f \,\mathrm{d}\|T\|$. □

In particular, if f and g coincide except on a $\|T\|$ null set and f is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable, then $f - g$ is Lebesgue integrable with integral zero. Thus g is also $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable and $(\mathfrak{P}_{(T,E_T,\eta_T)}) \int f = (\mathfrak{P}_{(T,E_T,\eta_T)}) \int g$.

Proposition 5.1.6 (Restriction of the integral to a subcurrent)**.** *Suppose f is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable on T and $S \sqsubset T$ is such that S and $T - S$ both have the Howard-Cousin Property with exceptional sets $E_S, E_{T-S} \subseteq E_T$ and regularity gauges $\eta_S := \eta_T|_{\mathrm{set}_m \|S\| \setminus E_S}$ and $\eta_{(T-S)} := \eta_T|_{\mathrm{set}_m \|T-S\| \setminus E_{T-S}}$ respectively, then f is $\mathfrak{P}_{(S,E_S,\eta_S)}$ integrable on S.*

Proof. Fix $\varepsilon > 0$ and let G, δ and τ correspond to the integrability of f on T for ε. $\delta_S := \delta|_{\mathrm{set}_m \|S\| \setminus E_S}$ is a gauge. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two δ_S-fine $\varepsilon\eta_S$-regular tagged families in S such that for $j = 1, 2$

$$G(S - [\mathcal{P}_j]) < \tau/2.$$

Similarly, $\delta_{T-S} := \delta|_{\mathrm{set}_m \|T-S\| \setminus E_{T-S}}$ is a gauge and by the Howard-Cousin Property of $(T - S, E_{T-S}, \eta_{T-S})$, there exists a δ_{T-S} fine $\varepsilon\eta_{T-S}$ regular tagged family $\mathcal{P}'$ in $T - S$ with

$$G(T - S - [\mathcal{P}']) < \tau/2.$$

For $j = 1, 2$, let $\tilde{\mathcal{P}}_j$ be the reunion of $\mathcal{P}_j$ and $\mathcal{P}'$. $\tilde{\mathcal{P}}_j$ is a δ fine $\varepsilon\eta$ regular family in T and by the subadditivity of G

$$G(T - [\tilde{\mathcal{P}}_j]) < \tau.$$

Thus by the integrability of f on T, there holds

$$|\sigma(\tilde{\mathcal{P}}_1, f) - \sigma(\tilde{\mathcal{P}}_2, f)| < \varepsilon.$$

So decomposing $\sigma(\tilde{\mathcal{P}}_j, f)$ into $\sigma(\mathcal{P}_j, f) + \sigma(\mathcal{P}', f)$, we get:

$$|\sigma(\mathcal{P}_1, f) - \sigma(\mathcal{P}_2, f)| < \varepsilon$$

and by the Cauchy criterion for Pfeffer integrability, f is $\mathfrak{P}_{(S,E_S,\eta_S)}$ integrable on S. □

In the following we suppose that f is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable on T and that T has the hereditary Howard-Cousin Property, this allows us to consider the **indefinite integral of f on T**:

$$F : \mathcal{S}(T) \to \mathbb{R}; S \mapsto I(f, S) = (\mathfrak{P}_{(S,E_S,\eta_S)}) \int f.$$

Proposition 5.1.7. *If T has the hereditary Howard-Cousin Property, then the indefinite integral of f, F, is additive and continuous on $\mathcal{S}(T)$.*

Proof. Let S_1, S_2 be non overlapping subcurrents of T, $S := S_1 + S_2$ is a subcurrent of T as well and S_1 and S_2 are subcurrents of S. Choose $\varepsilon > 0$ and let δ be a gauge on S corresponding to the integrability of f on S, choose $\tau > 0$ accordingly. With the notations of proposition 5.1.6 and without loss of generality, we can suppose that δ_{S_1} and δ_{S_2} are adapted to the integrability of f on S_1 and S_2 respectively with $\tau/2$ and $\varepsilon/2$ as parameters. For $j = 1, 2$, let $\mathcal{P}_j$ be a δ_{S_j}-fine, $\varepsilon\eta_{S_j}$ regular family in S_j with

$$G(S_j - [\mathcal{P}_j]) < \tau/2.$$

By the integrability of f on S_j, we get $|\sigma(f, \mathcal{P}_j) - I(f, S_j)| < \varepsilon/2$. Notice that $\mathcal{P}_1 \cup \mathcal{P}_2$ is a δ_S-fine $\varepsilon\eta_S$-regular family in S, with

$$G(S - [\mathcal{P}_1 \cup \mathcal{P}_2]) < \tau.$$

so $|\sigma(f, \mathcal{P}_1 \cup \mathcal{P}_2) - I(f, S)| < \varepsilon$. And we conclude:

$$|I(f, S) - I(f, S_1) - I(f, S_2)| < 2\varepsilon.$$

By the arbitrariness of ε, we conclude that $I(f, S) = I(f, S_1) + I(f, S_2)$ and the indefinite integral of f is thus additive.

Let us now prove that $S \mapsto I(f, S)$ is $\mathcal{F}$-continuous. To do this, by the definition of $\mathcal{F}$-continuity, it suffices to show that if $(S_k)_k$ is a sequence in $\mathcal{S}(T)$ which goes to 0 in the flat norm with uniformly bounded boundary mass, then

$$I(f, S_k) \to 0, \text{ as } k \to \infty.$$

By additivity of the indefinite integral, it is equivalent to show that $I(f, T - S_k) \to I(f, T)$. Fix such a sequence $(S_k)_k$, let $\varepsilon > 0$ and choose δ and τ corresponding to the integrability of f on T. Now for k large enough we have $G(S_k) < \tau/2$, because G is continuous. Fix such a k (arbitrary, large enough). Let δ' and τ' be a gauge on $T - S_k$ corresponding to the integrability of f on $T - S_k$ for ε. By the Howard-Cousin property of $T - S_k$, there exists a $\min\{\delta, \delta'\}$-fine, $\varepsilon\eta_{S_k}$-regular family $\mathcal{P}$ in $T - S_k$ such that

$$G(T - S_k - [\mathcal{P}]) < \tau/2$$

This implies $|I(f, T - S_k) - \sigma(f, \mathcal{P})| < \varepsilon$. Since S_k and $[\mathcal{P}]$ are non overlapping subcurrents of T, then $S_k + [\mathcal{P}] \sqsubset T$ and we have

$$G(T - [\mathcal{P}]) \leqslant G(T - S_k - [\mathcal{P}]) + G(T - S_k) < \tau$$

and by the integrability of f on T we get

$$|I(f, T) - \sigma(f, \mathcal{P})| < \varepsilon.$$

Combining the two estimates yields $|I(f, S_k)| = |I(f, T) - I(f, T - S_k)| < 2\varepsilon$. This holds for all k large enough and by the arbitrariness of ε, we conclude that $I(f, S_k) \to 0$ as $k \to \infty$. Thus the

indefinite integral is continuous on $\mathcal{S}(T)$. $\square$

Proposition 5.1.8 (Saks-Henstock Lemma)**.** *If T has the hereditary Howard-Cousin Property with exceptional set E_T and regularity gauge η_T, f is Pfeffer integrable on T if and only if there exists a continuous additive function F on $\mathcal{S}(T)$ such that: for all $\varepsilon > 0$, there exists a gauge δ on T such that for all δ-fine $\varepsilon\eta_T$-regular tagged family $((S_1, x_1), \ldots, (S_p, x_p))$ in T we have*

$$\sum_{j=1}^{p} |F(S_j) - f(x_j)\,\mathbb{M}(S_j)| < \varepsilon. \tag{5.4}$$

This statement differs from the definition of integrability in particular by the place of the absolute value inside the sum and the fact that we do not ask that the family covers *most* of T.

Proof. Suppose such a function F exists. We show that f satisfies the conditions of definition 5.1.1 for with G replaced by $|F|$ and $I(f, T)$ by $F(T)$. Indeed, fix $\varepsilon > 0$ and choose δ a gauge on $\mathrm{set}_m \|T\|$ corresponding to $\varepsilon/2$ in the condition of the Saks-Henstock Lemma. If $\mathcal{P} = ((S_j, x_j))_{j=1}^{p}$ is a δ-fine $\varepsilon\eta$-regular family in T with

$$|F|(T - [\mathcal{P}]) < \varepsilon/2,$$

then

$$\begin{aligned} |F(T) - \sigma(f, \mathcal{P})| &\leqslant \left| \sum_{j=1}^{p} F(S_j) - f(x_j)\,\mathbb{M}(S_j) \right| + |F\,(T - [\mathcal{P}])| \\ &\leqslant \sum_{j=1}^{p} |F(S_j) - f(x_j)\,\mathbb{M}(S_j)| + |F|\,(T - [\mathcal{P}]) \\ &< \varepsilon. \end{aligned}$$

Conversely, if f is $\mathfrak{P}_{(T, E_T, \eta_T)}$-integrable, let F be its indefinite integral. First notice that we can replace G by $|F|$ and τ by $\varepsilon/2$ in the definition of the integral of f. Furthermore, by the Howard-Cousin Property, given $\varepsilon > 0$, $\tau > 0$ and a gauge δ on T, we can complete any δ-fine $\varepsilon\eta_T$-regular tagged family $\mathcal{P}$ in T into a δ-fine $\varepsilon\eta_T$-regular tagged family $\mathcal{Q}$ in T with $|F|(T - [\mathcal{Q}]) < \tau$. Therefore, it suffices to prove that (5.4) holds for such a tagged family.

Fix $\varepsilon > 0$, choose a gauge δ and corresponding to the integrability of f with parameter $\varepsilon/10$. Consider a tagged family $\mathcal{P} = ((S_j, x_j))_{j=1}^{p}$ in T which is δ-fine and $\varepsilon\eta_T$-regular, such that

$$|F(T) - \sigma(f, \mathcal{P})| < \varepsilon/10 \text{ and } |F|(T - [\mathcal{P}]) < \varepsilon/11.$$

For $j = 1, \ldots, p$, use the Howard-Cousin Property of S_j and the integrability of f on S_j to define a δ-fine $\varepsilon\eta_T$-regular tagged family $\mathcal{P}_j$ in S_j, such that

$$|\sigma(f, \mathcal{P}_j) - F(S_j)| < \frac{\varepsilon}{10p} \text{ and } |F|(S_j - [\mathcal{P}_j]) < \frac{\varepsilon}{11p}.$$

Reorder $\mathcal{P}$ so that

$$|F(S_j) - f(x_j)\,\mathbb{M}(S_j)| = \begin{cases} F(S_j) - f(x_j)\,\mathbb{M}(S_j), & \text{for } j \in \{1,\dots,k\} \text{ and} \\ f(x_j)\,\mathbb{M}(S_j) - F(S_j), & \text{for } j \in \{k+1,\dots,p\}. \end{cases}$$

If one can choose $k = p$ or $k = 0$, the proof is finished. Suppose this is not the case. Let $\mathcal{P}_+ := \mathcal{P}_1 \cup \dots \cup \mathcal{P}_k$ and $\mathcal{P}_- := \mathcal{P}_{k+1} \cup \dots \cup \mathcal{P}_p$. Consider the tagged family $((S_1, x_1), \dots, (S_k, x_k)) \cup \mathcal{P}_-$. It is δ-fine $\varepsilon\eta_T$-regular in T and

$$|F|(T - (S_1 + \dots + S_k + [\mathcal{P}_-])) \leqslant |F|(T - [\mathcal{P}]) + \sum_{j=k+1}^{p} |F|(S_j - [\mathcal{P}_j]) \leqslant \frac{2\varepsilon}{11}.$$

Thus, by the integrability of f on T, there holds:

$$\left| F(T) - \sum_{j=1}^{k} f(x_j)\,\mathbb{M}(S_j) - \sum_{j=k+1}^{p} \sigma(f, \mathcal{P}_j) \right| < \frac{\varepsilon}{10}.$$

So by the assumptions on the $\mathcal{P}_j$:

$$\begin{aligned} \left| \sum_{j=1}^{k} (F(S_j) - f(x_j)\,\mathbb{M}(S_j)) \right| &\leqslant \frac{\varepsilon}{10} + \left| F(T - (S_1 + \dots + S_k)) - \sum_{j=k+1}^{p} \sigma(f, \mathcal{P}_j) \right| \\ &\leqslant \frac{\varepsilon}{10} + \left| F(T - [\mathcal{P}]) + F\left(\sum_{j=k+1}^{p} S_j - [\mathcal{P}_-] \right) - \sum_{j=k+1}^{p} (F(S_j) - \sigma(f, \mathcal{P}_j)) \right| \\ &\leqslant \frac{\varepsilon}{10} + \frac{\varepsilon}{11} + \frac{\varepsilon}{11} + \frac{\varepsilon}{10} < \frac{\varepsilon}{2}. \end{aligned}$$

A similar estimates holds summing over $j = k+1, \dots, p$ and combining the two, we get

$$\begin{aligned} &\sum_{j=1}^{p} |F(S_j) - f(x_j)\,\mathbb{M}(S_j)| \\ &= \left| \sum_{j=1}^{k} (F(S_j) - f(x_j)\,\mathbb{M}(S_j)) \right| + \left| \sum_{j=k+1}^{p} (F(S_j) - f(x_j)\,\mathbb{M}(S_j)) \right| < \varepsilon. \end{aligned}$$

□

Conjecture 5.1.9 (Monotone convergence for the $\mathfrak{P}_{(T,E_T,\eta_T)}$ integral.)**.** *Let f_n be a sequence of $\mathfrak{P}_\eta$-integrable functions on T such that*

$$f_n(x) \leqslant f_{n+1}(x) \text{ for } \|T\| \text{ almost all } x \in \mathbb{R}^n, \tag{5.5}$$

$$\sup_n (\mathfrak{P}_{(T,E_T,\eta_T)}) \int f_n < +\infty. \tag{5.6}$$

Then if $(f_n)_n$ converges $\|T\|$ almost everywhere to a function f on $\operatorname{set}_m \|T\| \backslash E_T$, f is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable with $(\mathfrak{P}_{(T,E_T,\eta_T)}) \int f = \lim_{n\to+\infty} (\mathfrak{P}_{(T,E_T,\eta_T)}) \int f_n$.

Idea for a proof. The proof of theorem 2.3.16 can be translated directly to the higher dimensional

setting, adding the condition of $\varepsilon\eta_T$ regularity and using the Howard-Cousin Property whenever we choose a family. □

It would be useful to know that a $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable function on T is $\|T\|$ measurable. In the codimension zero case, this is usually proved using the fact that the function is almost everywhere the derivative of its indefinite integral followed by Vitali's Covering Theorem (see Proposition 2.3.5 in [61]). Here since we have not proved a "Vitali Covering Theorem" using subcurrents as "subsets", we have to reduce to the case where the current is locally parameterized by bilipschitz maps over sets of finite perimeter in $\mathbb{R}^m$.

Proposition 5.1.10. *If $T \in \mathbb{I}_m(\mathbb{R}^n)$ satisfies the conditions of theorem 4.2.10 and f is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable on T, then f is $\|T\|$ measurable.*

Proof. If E is the subset of $\operatorname{spt} T$ where T does not have Bilipschitz BV charts, recall that $\|T\|(E) = 0$ therefore we only need to consider $f|_{\operatorname{spt} T\setminus E}$.

Claim 1. f is $\|T\|$ measurable if and only if for $\|T\|$ almost all points x, x has a neighbourhood V_x such that $f|_{U_x}$ is $\|T\|$ measurable.

Proof. For the forward direction, taking a locally finite subcover of $(U_x)_x$ and using a partition of unity subordinate to this locally finite cover, we can express f as a countable sum of $\|T\|$ measurable functions and conclude. The converse is clear. ■

Choose a Bilipschitz-BV chart (V, A, ϕ, θ) of T and notice that since f is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable on $T \llcorner V$, the function $f \circ \phi$ is $\mathcal{R}$ integrable on A in the sense of Pfeffer (see Definition 5.5.3 in [61]). Therefore $f \circ \phi$ is Lebesgue measurable. Now, as $\|T \llcorner V\| = \theta\phi_*(\mathcal{L}^m \llcorner A)$, $f|_V$ is $\|T\| \llcorner V$ measurable, thus $\|T\|$ measurable. □

Corollary 5.1.11. *In the same setting, if f is a nonnegative function which is $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable on T, then f is Lebesgue integrable with respect to $\|T\|$. If follows that a function f is Lebesgue integrable with respect to $\|T\|$ if and only if f and $|f|$ are $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable on T.*

Proof. We follow proposition 2.3.15. If f is nonnegative and $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrable on T, then f is measurable and can be approximated from below by the sequence $f_k := f\mathbb{1}_{\{x\in \operatorname{set}_m \|T\|, f(x)<k\}}$ for $k = 1, 2, \ldots$. The f_k are in $\mathcal{L}^\infty(\|T\|)$ and their integrals coincides with the $\mathfrak{P}_{(T,E_T,\eta_T)}$ integrals of f_k on T. As the latter sequence is nondecreasing and bounded from above, the former is as well and we can use the Lebesgue Monotone convergence theorem to conclude.

The second part of the statement follows from the linearity of the $\mathfrak{P}_{(T,E_T,\eta_T)}$ and the Lebesgue integral, which allow us to split f into its nonnegative and nonpositive parts. □

5.2 Fundamental Theorem of Pfeffer integration and Stokes' Theorem

Theorem 5.2.1. *Let T be an integral current of dimension m in $\mathbb{R}^n$ and F a continuous additive function on $\mathcal{S}(T)$. Suppose T has the Hereditary Howard-Cousin Property with exceptional set E_T and regularity gauge η_T and F is derivable at each point of $\operatorname{set}_m \|T\|\setminus E_T$, except maybe on a set of σ-finite $\mathcal{H}^{m-1}$ measure. Then the function $x \mapsto \mathfrak{D}_T F(x)$ is $\mathfrak{P}_{(T,E_T,\eta_T)}$-integrable on T and its primitive is F. This does not depend on the regularity gauge η_T.*

Proof. It suffices to prove that $x \mapsto \mathfrak{D}_T F(x)$ and F satisfy the conditions of the Saks-Henstock Lemma 5.1.8. Fix $\varepsilon \in (0,1)$. Consider a point $x \in \operatorname{set}_m \|T\| \backslash E_T$ at which F is derivable and $\eta_T(x) > 0$. By derivability of F, there exists a real number $\delta(x) > 0$ such that if S is an $\varepsilon\eta_T(x)$ regular subcurrent of T with $x \in \operatorname{spt} S$ and $\operatorname{diam} \operatorname{spt} S < \delta(x)$, then

$$|F(S) - \mathfrak{D}_T F(x)\, \mathbb{M}(S)| < \frac{\varepsilon}{\mathbb{M}(T)}\, \mathbb{M}(S).$$

Fix $\delta(x)$ in this way for all x at which F is differentiable and $\eta(x) > 0$ and let $\delta(x) = 0$ on the rest of $\operatorname{set}_m \|T\| \backslash E_T$. The function δ is a gauge on $\operatorname{set}_m \|T\| \backslash E_T$.

Let $\mathcal{P} := ((S_1, x_1), \ldots, (S_p, x_p))$ be a δ-fine, $\varepsilon\eta$-regular family in T, then

$$\sum_{j=1}^{p} |F(S_j) - \mathfrak{D}_T F(x_j)\, \mathbb{M}(S_j)| < \frac{\varepsilon}{\mathbb{M}(T)} \sum_{j=1}^{p} \mathbb{M}(S_j) \leqslant \varepsilon.$$

By proposition 5.1.8, $x \mapsto \mathfrak{D}_T F(x)$ is $\mathfrak{P}_{(T, E_T, \eta_T)}$-integrable with indefinite integral F. Changing the regularity gauge does not change the result as F is already fixed. □

We now suppose that T satisfies the Hereditary Howard-Cousin Property with exceptional set E_T and regularity gauge η_T.

Definition 5.2.2. $T \in \mathbb{I}_m(\mathbb{R}^n)$ has the **Pfeffer-Stokes Property** if for any differential form ω of degree $m-1$ in $\mathbb{R}^n$ such that:

(i) ω is continuous on $\operatorname{spt} T$,

(ii) $\exists E \subseteq \operatorname{set}_m(\|T\|)$ of $\mathcal{H}^{m-1}$ σ-finite measure such that ω is differentiable at each point of $\operatorname{set}_m(\|T\|) \backslash E$,

The map $x \mapsto \langle \mathrm{d}\omega(x), \overrightarrow{T}(x) \rangle$ is Pfeffer integrable on T and there holds

$$(\mathfrak{P}_{(T, E_T, \eta_T)}) \int \langle \mathrm{d}\omega(x), \overrightarrow{T}(x) \rangle \, \mathrm{d}\|T\|(x) = \int \langle \omega(y), \overrightarrow{\partial T}(y) \rangle \, \mathrm{d}\|\partial T\|(y).$$

Alternatively, we say that T has the **weak Stokes-Cartan Property** if whenever ω verifies (i) and (ii), with

$$\langle \mathrm{d}\omega(x), \overrightarrow{T}(x) \rangle = 0, \text{ for } \|T\| \text{ almost all } x,$$

there holds $\int \langle \omega, \overrightarrow{\partial T} \rangle \, \mathrm{d}\|\partial T\| = 0$.

In the case of bounded sets of finite perimeter and C^1 submanifolds with boundary, W. Pfeffer has proved that the Pfeffer-Stokes Property is always verified (see [58]). In positive codimension however, there exist currents in $\mathbb{I}_m(\mathbb{R}^n)$ which do not have the weak Pfeffer-Stokes Property even though they are associated with a smooth submanifold (not "with boundary"). We provide such a counter-example in section 4.4, in this example, the form verifies $\langle \mathrm{d}\omega, \overrightarrow{T} \rangle = 0$ on $\operatorname{spt} T \backslash E$, where E is a set of finite $\mathcal{H}^{m-1}$ measure and is the exceptional set of T for the Howard-Cousin Property.

Theorem 5.2.3. *Let T be an integral current of dimension m in $\mathbb{R}^n$ having*

1. *Bilipschitz-BV charts in the neighbourhood of every point of* $\operatorname{spt} T \backslash E_T$ *where E_T is disposable and m-thin for T and*

2. *C^1-BV charts in the neighbourhood of every point of* $\operatorname{spt} T \backslash E'_T$*, where* E'_T *is* $\mathcal{H}^{m-1}$*-sigma finite.*

Then T has the Stokes-Cartan Property. If only the first condition holds, then T has the weak Stokes-Cartan Property.

Proof of theorem 5.2.3. Let T be as in the assumptions of theorem 5.2.3 and ω be an $(m-1)$-form on $\operatorname{spt} T$ such that

- ω is continuous on $\operatorname{spt} T$,
- ω is differentiable on $\operatorname{set}_m T \backslash E_\omega$ where $E_\omega \subseteq \operatorname{spt} T$ is m-thin,

Consider the function Θ_ω on $\mathcal{S}(T)$ associated to the circulation of ω: $S \mapsto \langle \partial S, \omega \rangle$. We will show that Θ_ω is derivable at all points of $\operatorname{set}_m \|T\|$ except on an m-thin set and that when it is the case, there holds

$$\mathfrak{D}_T(\Theta_\omega)(x) = \langle \mathrm{d}\omega(x), \overrightarrow{T}(x) \rangle.$$

It will indeed allow us to use theorem 5.2.1 and conclude.

By lemma 3.2.15, this is true at all points $x \in \operatorname{set}_m \|T\|$ such that

- ω is differentiable at x.
- T has bounded upper density at x.
- $\overrightarrow{T}$ has a a continuous representative in $\operatorname{set}_m \|T\|$ at x, still denoted by $x \mapsto \overrightarrow{T}(x)$.
- T is $\tilde{\eta}$ good at x.

The first point holds whenever $x \in \operatorname{set}_m \|T\| \backslash E_\omega$. We showed in the proof of lemma 4.3.2 the second and third points are verified whenever T has a C^1-BV chart in a neighbourhood of x and $x \in \operatorname{set}_m \|T\| \backslash \operatorname{set}_{m-1} \|\partial T\|$. Thus Θ_ω is derivable along T at each point of $\operatorname{set}_m \|T\| \backslash (E_T \cup E_\omega \cup \operatorname{set}_{m-1} \|\partial T\|$, since $E_\omega \cup \operatorname{set}_{m-1} \|\partial T\|$ is m-thin, we are done. $\square$

Chapter 6

Definable chains in o-minimal structures

O-minimal structures are families of sets which are stable under countably many operations and have good regularity properties. Let us start with a short presentation of these structures and of the o-minimal tools we use in this chapter.

6.1 O-minimality

The first part of the paper [45] is a good introduction to o-minimal geometry. A more complete study can be found in [21], which is written with geometric and topological applications in mind, and is therefore also accessible to analysts. Usually many results such as the Monotonicity Theorem and the Cell Decomposition Theorem are stated as decompositions into continuous pieces and later extended to C^1 pieces. Here we state directly the C^1 decomposition. See Chapter 3 of [21] for the continuous versions and Chapter 7 of the same book for the C^1 versions.

Definition 6.1.1. A set E in $\mathbb{R}^n$ is called **semi-algebraic** if it is a finite union of sets E_k there exists polynomials P_0^k and $Q_1^k, \dots, Q_{q_k}^k$ with n real variables such that

$$E_k = \{x \in \mathbb{R}^n, P_0^k(x) = 0, Q_1^k(x) > 0, \dots, Q_{q_k}^k(x) > 0\}.$$

In particular, the graphs and epigraphs of polynomials and rational fractions are semi algebraic sets.

Definition 6.1.2. A **structure** $\mathcal{M}$ over the field of real numbers $\mathbb{R}$ is the data for each natural number n of a collection $\mathcal{M}_n \subseteq \mathcal{P}(\mathbb{R}^n)$, satisfying the following five properties:

1. For every n, $\mathcal{M}_n$ is stable under finite union, intersection, and complement operations.

2. For every n, n', $\mathcal{M}_{n+n'} \supseteq \mathcal{M}_n \times \mathcal{M}_{n'} := \{A \times B, A \in \mathcal{M}_n, B \in \mathcal{M}_{n'}\}$.

3. For every $n > 0$, and every $A \in \mathcal{M}_n$, if π is the projection on the $n-1$ first coordinates of $\mathbb{R}^n$, $\pi(A) \in \mathcal{M}_{n-1}$.

4. For every $n > 0$, $\mathcal{M}^n$ contains all the semialgebraic sets in $\mathbb{R}^n$.

A set in one of the $\mathcal{M}_n$ is called **definable in** $\mathcal{M}$. To a structure, one can associate its definable functions and maps: Given, positive integers n and m, definable sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ in $\mathcal{M}$, a map f from A to B is **definable in** $\mathcal{M}$ if its graph $\{(x, y) \in A \times B | f(x) = y\}$ is a definable subset of $\mathbb{R}^{n+m}$.

An easy consequence of this definition is that given a set $A \in \mathcal{M}_n$ for $n > 1$, the sets

$$\{x \in \mathbb{R}^{n-1}, \exists x \in \mathbb{R}, (x, x') \in A\}, \text{ and } \{x \in \mathbb{R}^{n-1}, \forall x \in \mathbb{R}, (x, x') \in A\}$$

are in $\mathcal{M}_{n-1}$, for instance if A is the graph of a definable function f from a subset of $\mathbb{R}^{n-1}$ to $\mathbb{R}$, the first of theses two sets is the domain of f. In the same way, the image and preimage of a definable set by definable maps are definable. Also, a structure is stable by linear transformations, indeed the graphs of linear maps are semi-algebraic (in fact semi-linear) sets. Suppose $L \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $A \in \mathcal{M}_n$, then

$$L(A) = \{y \in \mathbb{R}^m, \exists x \in \mathbb{R}^n L(x) = y\} = \{y \in \mathbb{R}^m, \exists x \in \mathbb{R}^n, (x, y) \in \text{graph}(L)\}.$$

Given a structure $\mathcal{M}$, one can expand it by adding a collection of subsets of the $\mathbb{R}^n$ for $n = 1, 2, \ldots$ to it. The **expansion of** $\mathcal{M}$ by a set $A \subseteq \mathbb{R}^n$ is the smallest structure $\mathcal{M}'$ such that $\mathcal{M}'^n$ contains A. To prove that such a structure exists, notice that $(\mathcal{P}(\mathbb{R}^n))_n$ is a structure and contains all the additional sets, note also that structures are stable by intersection. Take the intersection of all the structures which contain $\mathcal{M}$ and our additional sets to get the smallest one.

Definition 6.1.3. An **o-minimal structure** $\mathcal{M}$ over $\mathbb{R}$ is a structure satisfying the additional property

(5) $\mathcal{M}_1$ is exactly the collection of all the finite unions of segments and points.

Notice that we cannot expand an o-minimal structure by any set and expect the expansions to be o-minimal. To see this, just add the set of natural numbers to $\mathcal{M}_1$. We will often call condition (5), as the **o-minimality condition**, or as the **tameness property**. Let us mention two equivalent definitions which are often used to describe o-minimal structures, as they give a more geometric intuition of their behavior.

Proposition 6.1.4. *Condition (5) above is equivalent to:*

(5') Every definable set in $\mathbb{R}^n$ has a finite number of connected components.

(5") In $\mathbb{R}^n$, given a definable curve and a definable set, the intersection of the curve and the set has a finite number of components. (A definable curve is the image of a definable map from an interval of R to $\mathbb{R}^n$.)

Proof. Notice at first that (5') $\Rightarrow$ (5"), indeed the intersection of a definable curve and a definable set is itself a definable set, which by (5') implies that it must have a finite number of components. For the second implication, notice that $\mathbb{R}$ is a definable curve in $\mathbb{R}$. As $\mathbb{R}$ is a definable curve in any structure satisfying (4), property (5") implies (5).

For the last implication, the Cell Decomposition Theorem, 6.2.2 follows from (5). It implies that a definable set is a finite union of cells, which are all connected. Thus (5) implies (5'). $\square$

In particular, any discrete definable set is finite. In this way, sets in an o-minimal structure inherit many of the good properties of (semi)algebraic sets. We would also like to point out that, connected components of definable sets are definable. The next statement gives a first glimpse of the consequences of definition 6.1.3.

Theorem 6.1.5 (C^1 Monotonicity Theorem)**.** *Let $I \subseteq \mathbb{R}$ be definable in an o-minimal structure, $\mathcal{M}$ and $f : I \to \mathbb{R}$ be definable in M. Then the derivative of f: f' is finite except at finitely many points and it is a definable function. Furthermore, there exists $p \in \mathbb{N}$ and $I_1, \ldots, I_p$ disjoints intervals contained in I (they can be open, closed, half-open or degenerate) such that*

- $\bigcup_j I_j = I$ *and*
- f *is C^1 and monotone on each I_j.*

Example 6.1.6. The following structures on $\mathbb{R}$ are o-minimal:

a) The structure (denoted by $\mathbb{R}$) of semi-algebraic sets, defined above.

b) The structure $\mathbb{R}_{an}$ of globally subanalytic sets. $A \subseteq \mathbb{R}^n$ is definable in $\mathbb{R}_{an}$ if at the neighbourhood of each of its points, A is a projection of the graph of an analytic function restricted to a compact set.

c) The structure $(\mathbb{R}, exp)$ (resp. $(\mathbb{R}_{an}, exp)$) generated by the semi-algebraic sets (resp. $\mathbb{R}_{an}$) expanded by the graph of the exponential function on $\mathbb{R}$. (see [74, 22])

d) The Pfaffian closure of any o-minimal structure. (see [75])

Definition 6.1.7. Given p and n integers, an index set $B \subseteq \mathbb{R}^p$ and for each $t \in B$ a set $A_t \subseteq \mathbb{R}^m$. The family $(A_t)_{t \in B}$ is called **definable** in the o-minimal structure $\mathcal{M}$ if

- B is definable and
- The set $\bigcup_{t \in B} A_t \times \{t\} \subseteq \mathbb{R}^p \times \mathbb{R}^n$ is definable.

In particular, for all t, A_t is definable in $\mathbb{R}^n$. The sets A_t are the **fibers** of the definable family.

Given a definable family $(A_r)_{r \in B}$ of subsets of $\mathbb{R}$, the map $r \mapsto \inf A_r$ from B to $\mathbb{R} \cup \{\pm\infty\}$ is definable. As a consequence the euclidian distance from a point or a definable set is also definable. This implies that metric properties and the topology induced by the metric can be used in the o-minimal setting: all open and closed balls are definable and if A is a definable set in $\mathbb{R}^n$, its closure $\operatorname{cl} A$, interior $\operatorname{int} A$, topological boundary $\operatorname{bdry} A$ and ε-neighbourhood : $\mathrm{U}(A, \varepsilon) := \{x \in \mathbb{R}^n, \operatorname{dist}(x, A) < \varepsilon\}$ are definable as well. For instance the interior is defined by the formula

$$\operatorname{int} A = \{x \in \mathbb{R}^n, \exists r > 0, \forall y \in \mathbb{R}^n, \operatorname{dist}(y, x) < r \Rightarrow y \in A\}$$

6.2 Properties of definable sets, definable families

In the following, we fix an o-minimal structure $\mathcal{M}$.

6.2.1 Cell decomposition

There is no a priori bound on the number of components of a definable set. It depends on the complexity of the formula used to define it. Similarly: given a polynomial, one can not bound the number of its roots, unless one has information on the degree. Since functions definable in an o-minimal structure can be much wilder than polynomials, we do not always have an information as strong as the degree. Nonetheless, given an definable set, one can decompose it in a finite number of **cells**. We use a C^1 version of Definition 2.3 in Chapter 3 of [21], see chapter 7 of the same book. In the following we will use "cell" for "C^1-cell".

Definition 6.2.1 (C^1 k-cell)**.** For a positive integer m, let $(i_1, \ldots, i_m)$ be a sequence of zeros and ones. An $(i_1, \ldots, i_m)$**-cell** is a subset of $\mathbb{R}^m$ obtained by induction as follows:

(i) A (0)-cell is a single point in $\mathbb{R}$. A (1) cell in is an open interval $(a, b) \subseteq \mathbb{R}$ where $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$.

(ii) If $(i_1, \ldots, i_{m-1})$ is an $(m-1)$-uple of zeros and ones an $(i_1, \ldots, i_{m-1}, 0)$-cell, C', is the graph in $\mathbb{R}^m$ of a C^1 definable function from an $(i_1, \ldots, i_{m-1})$ cell in $\mathbb{R}^{m-1}$ to $\mathbb{R}$.

$$C' = \{(x_1, \ldots, x_m) \in \mathbb{R}^m, (x_1, \ldots, x_{m-1}) \in C \text{ and } x_m = f(x_1, \ldots, x_{m-1})\}.$$

An $(i_1, \ldots, i_{m-1}, 1)$-cell, C", is the set which is contained between the graphs of two C^1 function, f and g, defined on the same $(i_1, \ldots, i_{m-1})$-cell: $C \subseteq R^{m-1}$, with $f < g$ on this cell.

$$C" = \{(x_1, \ldots, x_m) \in \mathbb{R}^m, (x_1, \ldots, x_{m-1}) \in C \text{ and } f(x_1, \ldots, x_{m-1}) < x_m < g(x_1, \ldots, x_{m-1})\}.$$

We say that the cell C is the **base** of the cells C' and C". For $k \in \{0, \ldots, m\}$, a k**-cell** in $\mathbb{R}^m$ is an $(i_1, \ldots, i_m)$-cell for a finite sequence $i_1, \ldots, i_m$ with $\sum_j i_j = k$.

Among the properties of cells, note that:

- Cells are definable sets.
- m-cells in $\mathbb{R}^m$ are open sets.
- An m-cell in $\mathbb{R}^n$ is a m-dimensional C^1 submanifold. (Which enables us to speak about C^1 functions on a cell.) More precisely, up to a permutation of the axes, an m cell in $\mathbb{R}^n$ is the graph of a C^1 from on an m-cell in $\mathbb{R}^m$ to $\mathbb{R}^{n-m}$.

Note also that cells can be unbounded and that no regularity of the functions f and g are required at the boundary. Indeed, consider $U_1 = \mathbb{R}^2 \cap \{x^2 + y^2 < 1\} \cap \{(x, y), y < 1\}$ and $f_1 : U_1 \to \mathbb{R}; (x, y) \mapsto (x^2 + y^2)^{1/4}$ (which corresponds to "half a cusp"), or $U_2 = \mathbb{R}^2 \cap \{(x, y), 0 < y < |x| < 1\}$ and $f_2 : U_2 \to \mathbb{R}; (x, y) \mapsto y/x$ (which we call a piece of "helix"). A **cell decomposition** of $\mathbb{R}$ is a finite family of points and interval partitioning $\mathbb{R}$. For $m > 1$, a cell decomposition of $\mathbb{R}^m$ is a partition of $\mathbb{R}^m$ into cells whose bases form a cell decomposition of $\mathbb{R}^{m-1}$.

Theorem 6.2.2 (C^1 cell decomposition)**.** *For $m \geqslant 0$.*

(I) If $A \subseteq \mathbb{R}^m$ is definable, there is a decomposition of $\mathbb{R}^m$ partitioning A. That is all cells of the decomposition are either contained in A or disjoint from it.

(II) Given a finite family of definable sets in $\mathbb{R}^m$, they can be partitionned in cells simultaneously. (i.e. they are partitionned by the same decomposition of $\mathbb{R}^m$.)

(III) Given a definable map $f : A \to B$ where $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ are definable sets, there is a cell decomposition of $\mathbb{R}^m$ which partitions A and such that f is C^1 on each cell of the cell decomposition.

Remark 6.2.3. (III) above is usually stated for functions instead of maps. However, applying (III) for functions successively to each component of f yields a finite collection of sets (the collection of all the cells). Apply (II) to this family of sets to get a cell decomposition which partitions all of these sets. In particular every component of f is C^1 on each of the new cells, thus f itself is C^1 on each of the new cell.

A conclusion of this theorem is that any set definable in an o-minimal structure has a well defined Hausdorff dimension, which is equal to the maximum of the dimensions of its cells and is m-rectifiable in this dimension. The proof of theorem 6.2.2 relies on the following result which is primordial to us. This is Lemma 2.13 in Chapter 3 of [21].

The Cell Decomposition Theorem can be applied to decompose elements of a definable family. Given a set $E \in \mathbb{R}^{m+n}$ and $r \in \mathbb{R}^m$, define the **fiber** of E at r by

$$E_r := \{y \in \mathbb{R}^n, (r, y) \in E\}.$$

The following is a reformulation of [21, (Proposition 3.5)].

Corollary 6.2.4. *Given a definable family $(A_r)_{r\in B}$ in $\mathbb{R}^n$ with B definable in $\mathbb{R}^m$, there exists a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{m+n}$ which partitions $\bigcup_{r\in B} A_r \times \{r\}$. And such that for all $r \in B$, the fiber $\mathcal{D}_r$ given by*

$$\mathcal{D}_r := \{C_r, C \in \mathcal{D}, C_r \neq \emptyset\}.$$

is a cell decomposition of $\mathbb{R}^n$ which partitions A_r.

Even without the cell decomposition theorem, one can prove the uniform finiteness property for definable families, which states that finite fibers are uniformly finite:

Lemma 6.2.5 (Uniform finiteness property)**.** *Let Y be a definable family in $\mathbb{R}^{m+1}$. For $r \in \mathbb{R}^m$, denote by Y_r the projection on the last coordinate of $Y \cap (\{r\} \times \mathbb{R})$. If for all $r \in \mathbb{R}^m$, Y_r is a finite set, then there exists a natural number M such that the cardinal of Y_r is smaller than M for all $r \in \mathbb{R}^m$.*

A more general version of this property, which is a consequence of the cell decomposition theorem is

Corollary 6.2.6. *(Chapter 3, paragraphs 3.6 and 3.7 in [21]) Let $(Y_r)_{r\in\mathbb{R}^m}$ be a definable family in $\mathbb{R}^n$. There exists M_Y such that for $r \in \mathbb{R}^m$, Y_r has less than M_Y components. In particular, if Y_r is a finite set, then it has cardinal at most M_Y.*

Finally, we state a crucial result concerning the topology of definable sets:

Theorem 6.2.7 (Triangulation Theorem)**.** *Every definable $S \subseteq \mathbb{R}^m$ is definably homeomorphic to a polyhedron $|K| \subseteq \mathbb{R}^m$. (Two sets are called* **definably homeomorphic** *if there is a definable homeomorphism between them.)*

Note that this result is topological in nature, as although definable homeomorphisms are much nicer than general homeomorphisms, they can exhibit singularities such as cusps. Thus, the metric properties of a definable set can a priori be very different from that of a polyhedron.

6.2.2 Control of Hausdorff measures

An important result, which is a consequence of Theorem 6.2.2 is the following:

Proposition 6.2.8. *A bounded definable set of dimension m in $\mathbb{R}^n$ has finite m-dimensional Hausdorff measure. In particular, a bounded definable set of dimension m in $\mathbb{R}^m$ has finite perimeter (it is a Caccioppoli set). Also the perimeter of this set is equal to the $\mathcal{H}^{m-1}$ measure of its topological boundary.*

(The last fact is a consequence of the fact that the topological boundary of an m cell is a finite union of disjoint $(m-1)$-cells which are $(m-1)$-rectifiable and of lower dimensional cells.) This control of the Hausdorff measure can be extended to definable families of sets. If m and n are integers, $m \leqslant n$, B a definable set in $\mathbb{R}^m$ and $(A_r)_{r\in B}$ a bounded definable family in $\mathbb{R}^n$, (i.e. $A := \bigcup_{r\in B} A_r \times \{r\} \subset \mathbb{R}^{n+m}$ with the A_r all contained in a same compact subset of $\mathbb{R}^n$). The following theorem was proved in [48], building on results of [26, 4.3.8] and [35].

Theorem 6.2.9 (Uniform Hausdorff finiteness)**.** *Let $k \leqslant n$, if for each $r \in B$, A_r is of dimension at most k, then there exists $C \in \mathbb{R}$ such that*

$$\forall r \in B,\ \mathcal{H}^k(A_r) \leqslant C.$$

Proof. Each A_r being k-rectifiable, by [26, 3.3.13]

$$\mathcal{H}^k(A_r) = \mathcal{I}_1^k(A_r),$$

where $\mathcal{I}_1^k$ is the (one of the) integral geometric measure. Which can be expressed as follows for a Borel set $E \subseteq \mathbb{R}^n$:

$$\mathcal{I}_1^k(E) := c(n,k) \int_{\mathrm{G}(n,k)} \int_{\mathbb{R}^k} \#\left(E \cap p^{-1}(y)\right) \mathrm{d}\mathcal{L}^k(y)\, \mathrm{d}\theta^{n,k}(p). \tag{6.1}$$

Here, $\mathrm{G}(n,k)$ is the Grassmannian of k planes W in $\mathbb{R}^n$ which we represent by the corresponding orthogonal projections $p : \mathbb{R}^n \to W \simeq \mathbb{R}^k$, writing $p \in \mathrm{G}(n,k)$ for convenience. $\theta^{n,k}$ is the unique probability measure on $\mathrm{G}(n,k)$ which is invariant by rotation and $c(n,k)$ is a positive constant. (See 2.10.5 and 2.10.15 [26] for the construction.) Whenever E is a k-rectifiable set, the cardinality in the integral is finite $(\mathrm{d}\theta^{n,k} \times \mathrm{d}\mathcal{L}^k)$-almost everywhere. The idea is to show that there is a uniform bound on this cardinality. To be able to use the above formula in an o-minimal setting as well as tools such as uniform finiteness, one has to make sense of "definability" for subsets of the Grassmannian. A reference for this is Chapter 10 in [21]: *Definable spaces and quotients.*

Definition 6.2.10. An equivalence relation $\sim$ in a definable set A is **definable** if its graph ($\{(x,y) \in A \times A | x \sim y\}$) is a definable set. A set B in $A/\sim$ is **definable** if it can be written as $\tilde{B}/\sim$ where $\tilde{B} \subseteq A$ is definable.

An oriented k dimensional linear subspace in $\mathbb{R}^n$ is represented by a family of k vectors, that is an element of $M(n,k)$, with rank k. Having rank k is an algebraic property (equivalent to the existence of a nonzero minor of size k so the set of such families is definable in $\mathbb{R}^{n\times k}$). Furthermore, there is an algebraic equivalence relation on the linearly independent rank k families of vectors in $\mathbb{R}^n$ such that two families are equivalent if and only if they generate the same linear subspace of $\mathbb{R}^n$. Indeed, one can consider the two families of rank k expressed in the same basis as an element A of $M(n,2k)$, the families span the same k vector space, if and only if A has rank k. Note that reordering the vectors or changing their sign has no effect on the above properties.

One can therefore construct $\mathrm{G}(n,k)$ as a definable quotient of a definable subset of $\mathbb{R}^{n\times k}$ and it makes sense to talk about definable subsets of $\mathrm{G}(n,k)$. One has

$$\forall r \in \mathbb{R}^m,\ \mathcal{H}^k(A_r) = \int_{\mathrm{G}(n,k)} \int_{\mathbb{R}^k} \#\left(A_r \cap p^{-1}(y)\right) \mathrm{d}\mathcal{L}^k(y)\, \mathrm{d}\theta^{n,k}(p).$$

By assumption, all the A_r are contained in the same $\mathrm{U}(0,R) \subseteq \mathbb{R}^n$ for some $R > 0$, so the above formula can be rewritten:

$$\mathcal{H}^k(A_r) = \int_{\mathrm{G}(n,k)} \int_{\mathrm{U}(0,R)\subseteq\mathbb{R}^k} \#\left(A_r \cap p^{-1}(y)\right) \mathrm{d}\mathcal{L}^k(y)\, \mathrm{d}\theta^{n,k}(p).$$

As $\mathcal{H}^k(A_r)$ is finite for all r, the cardinal above is bounded $\theta^{n,k} \times \mathcal{L}^k$-almost everywhere for each r. Thus if we can show that it is uniformly bounded, it implies a uniform bound on $\mathcal{H}^k(A_r)$ as desired. Consider the definable set:

$$B := \mathbb{R}^n \times \mathrm{G}(n,k) \times \mathbb{R}^k \times \mathbb{R}^m \cap \{(x,p,y,r) | x \in A_r, p(x) = y\},$$

one has

$$\#\left(A_r \cap p^{-1}(y)\right) = \#(B \cap \mathbb{R}^n \times \{p\} \times \{y\} \times \{r\}).$$

Consider the definable family $(B_{(p,y,r)})_{(p,y,r)\in\mathrm{G}(n,k)\times\mathbb{R}^k\times\mathbb{R}^m}$. By Corollary 6.2.6, the cardinal of its fibers is uniformly bounded on the set of parameters (p,y,r) where it is finite. Furthermore, as each A_r is k-rectifiable, for $\theta^{n,k} \times \mathcal{L}^k$ almost (p,y), the set $A_r \cap p^{-1}(y)$ is countable and as it is definable, it is finite. Since the sets A_r are uniformly bounded, for instance contained in some ball $\mathrm{U}(0,R)$, the set of (p,y) where $A_r \cap p^{-1}(y)$ is non empty has $\theta^{n,k} \times \mathcal{L}^k$ measure less than $\theta^{n,k}(\mathrm{G}(n,k))\boldsymbol{\alpha}(k)R^k$. The integrals of a family of almost everywhere uniformly bounded functions on a set of finite measure are uniformly bounded as well. This completes the proof. □

6.3 Definable chains

We define chains in an o-minimal structure using the formalism of integral currents/flat chains. Analytic chains have been studied extensively by Hardt in [35], following Federer in [26, 4.2.28] and [25]. Recently, Funk develloped a theory of semi algebraic chains in [30]. We merely extend

the basic results to the o-minimal case. Given integers $m \leqslant n$, a **definable chain** T of dimension m is defined by

- a bounded definable set $M \subseteq \mathbb{R}^m$ of dimension m,
- a definable integer valued function θ defined on M and
- a definable field of unit simple m-vectors $\overrightarrow{T}$, defined on each m-cell of M and tangent to m.

In the notation of [26]

$$T = (\mathcal{H}^m \llcorner M) \wedge (\theta \overrightarrow{T}).$$

This makes T into a current of dimension m in $\mathbb{R}^n$. Since M is necessarily m-rectifiable, T is in fact a rectifiable current with integer multiplicity in $\mathbb{R}^n$. The space of definable chains of dimension m in $\mathbb{R}^n$ is denoted by $\mathbb{I}_m^{Def}(\mathbb{R}^n)$. $T \in \mathbb{I}_m^{Def}(\mathbb{R}^n)$ is called **cellular** if M consists of a single m-cell, θ is constant equal to 1 on M and $\overrightarrow{T}$ is continuous on M. This definition is motivated by the next result:

Proposition 6.3.1. *A definable chain of dimension k is a finite sum of integer multiples of non overlapping cellular chains. (Two currents are* **nonoverlapping** *whenever their carrying measures are orthogonal.)*

Proof. The space of m-vectors in $\mathbb{R}^n$: $\Lambda_m(\mathbb{R}^n)$ is a definable quotient space, in the same way as $\mathrm{G}(n, m)$ above. Orientation is indeed an algebraic property, as the sign of a determinant. However, in order to use the Cell Decomposition Theorem, we prefer to see $\Lambda_m(\mathbb{R}^n)$ simply as a vector space.

By the Cell Decomposition Theorem 6.2.2 applied to M, θ and $\overrightarrow{T}$ seen as a function with values in $\Lambda_m(\mathbb{R}^n)$, there exists a decomposition of $\mathbb{R}^n$ such that $\overrightarrow{T}$ and θ are continous on each cell of dimension m. This is a **decomposition adapted to T**. θ is then constant on each cell of dimension m. Denote those cells by $C_1, \ldots, C_p$ and the corresponding values of θ by $\theta_1, \ldots, \theta_p$. As the union of the lower dimensional cells has $\mathcal{H}^m$ measure 0, we get

$$T = \sum_{j=1}^{p} \theta_j \left((\mathcal{H}^m \llcorner C_j) \wedge \overrightarrow{T} \right),$$

which concludes the proof. □

Theorem 6.3.2. *The family $I_m^{Def}(\mathbb{R}^n)$ is an abelian subgroup of $\mathbb{I}_m(\mathbb{R}^n)$. Furthermore, if $m \geqslant 1$, $\partial T \in I_{m-1}^{Def}(\mathbb{R}^n)$.*

Proof. Let us first prove that $I_m^{Def}(\mathbb{R}^n)$ is indeed an abelian group for the addition of currents; 0 is definable and each element has an inverse obtained by changing the sign of $\overrightarrow{T}$ (or of θ). Let T and T' be in $I_m^{Def}(\mathbb{R}^n)$, we can write

$$T = (\mathcal{H}^m \llcorner M) \wedge (\theta \overrightarrow{T}) \text{ and } T' = (\mathcal{H}^m \llcorner M') \wedge (\theta' \overrightarrow{T}').$$

Let $\mathcal{D}$ be a cell decomposition in $\mathbb{R}^n$ adapted to T and T'. In particular for each m-cell $C \in \mathcal{D}$, $M \cap C = C$ or $\emptyset$, θ is constant on C and $\overrightarrow{T}$ is continuous on C and the same holds for M', θ' and $\overrightarrow{T}'$.

Denote by $\mathcal{D}_T$ the subset of $\mathcal{D}$ consisting of all m-cells C such that $M \cap C \neq \emptyset$ and by $\mathcal{D}_T$, the corresponding subset of $\mathcal{D}$ for T'. We can write

$$T = \sum_{C \in \mathcal{D}_T} (\mathcal{H}^m \llcorner C) \wedge (\theta \overrightarrow{T}) \text{ and } T' = \sum_{C \in \mathcal{D}_{T'}} (\mathcal{H}^m \llcorner C) \wedge (\theta' \overrightarrow{T}').$$

We can thus write

$$T + T' = \sum_{C \in \mathcal{D}_{T'} \setminus \mathcal{D}_T} (\mathcal{H}^m \llcorner C) \wedge (\theta' \overrightarrow{T}') + \sum_{C \in \mathcal{D}_T \setminus \mathcal{D}_{T'}} (\mathcal{H}^m \llcorner C) \wedge (\theta \overrightarrow{T}) + \sum_{C \in \mathcal{D}_{T'} \cap \mathcal{D}_T} (\mathcal{H}^m \llcorner C) \wedge (\theta \overrightarrow{T} + \theta' \overrightarrow{T}')$$

Notice that at a point of an m-cell C, there are only two unit simple m-vectors tangent to C opposite to one another. Thus, we can partition $\mathcal{D}_T \cap \mathcal{D}_{T'}$ into the cells C where $\theta \overrightarrow{T} + \theta' \overrightarrow{T} = \alpha_C \overrightarrow{T}$ where α_C is a non zero integer and those where $\theta' T + \theta' \overrightarrow{T}' = 0$. Denoting by $\mathcal{D}^*_{T,T'}$ the first collection, we get:

$$T + T' = \sum_{C \in \mathcal{D}_{T'} \setminus \mathcal{D}_T} (\mathcal{H}^m \llcorner C) \wedge (\theta' \overrightarrow{T}') + \sum_{C \in \mathcal{D}_T \setminus \mathcal{D}_{T'}} (\mathcal{H}^m \llcorner C) \wedge (\theta \overrightarrow{T}) + \sum_{C \in \mathcal{D}_{T'} \cap \mathcal{D}_T} (\mathcal{H}^m \llcorner C) \wedge (\alpha_C \overrightarrow{T}).$$

Thus $T + T'$ is a finite sum of nonoverlapping cellular currents, and therefore a definable chain.

Let us now show that definable chains are integral currents. As $\mathbb{I}_m(\mathbb{R}^n)$ is stable by addition, it suffices to prove this for cellular chains. By [26, 4.2.16], all there is to prove is that ∂T has finite mass. Let $T = (\mathcal{H}^m \llcorner C) \wedge \overrightarrow{T}$ be a cellular chain. By definition of a cell (permuting the axes if necessary), C is the graph of a definable map $f \in C^1(U, \mathbb{R}^{n-m})$, where U is a definable m-dimensional cell in $\mathbb{R}^m$. Let $\phi = (id, f)$ be the associated map from $\mathbb{R}^m$ to $\mathbb{R}^n$. As C is bounded by definition of a cellular chain, U and ϕ are bounded as well. Being top dimensional and bounded, U has finite perimeter.

Notice that $\mathbf{E}^m \llcorner U$ is an integral current of dimension m in $\mathbb{R}^m$ and we would like to say, in the notation of [26] that T is the pushforward of $\mathbf{E}^m \llcorner U$ by ϕ. However, ϕ is not necessarily Lipschitz or even continuous on $\operatorname{cl} U$. We therefore approximate T from the inside. For $\varepsilon \geqslant 0$, consider $U_\varepsilon := U \cap \{x, \operatorname{dist}(x, \partial U) \geqslant \varepsilon\}$. For all ε, U_ε is bounded, open and definable, which implies that $\mathbf{E}^m \llcorner U_\varepsilon$ is integral. As ϕ is C^1 and proper (because it is a diffeomorphism) on $\operatorname{cl} U_\varepsilon = \operatorname{spt} \mathbf{E}^m \llcorner U_\varepsilon$, we can push this current forward by ϕ (see [26, 4.1.14]) and get $T_\varepsilon := \phi_\#(\mathbf{E}^m \llcorner U_\varepsilon) \in \mathbb{I}_m(\mathbb{R}^n)$. In fact, there holds :

$$T_\varepsilon = \mathcal{H}^m \llcorner \phi(U_\varepsilon) \wedge \mathrm{D}\phi(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_m).$$

and

$$\partial T_\varepsilon = \phi_\#(\partial(\mathbf{E}^m \llcorner U_\varepsilon)).$$

Since U_ε has finite perimeter, $\partial(\mathbf{E}^m \llcorner U_\varepsilon) = \mathcal{H}^{m-1} \llcorner \partial_* U_\varepsilon \wedge \tau_\varepsilon$, where $\partial_* U_\varepsilon$ is the reduced boundary of U_ε, which is $(\mathcal{H}^{m-1}, m-1)$ rectifiable and τ_ε is a simple unit $(m-1)$-vector field tangent to $\partial_* U_\varepsilon$ (the orientation is induced by that of $\mathbf{E}^m$). Furthermore as U_ε is definable, $\mathcal{H}^{m-1}(\operatorname{bdry} U_\varepsilon \setminus \partial_* U_\varepsilon) =$

0 by proposition 6.2.8. Thus

$$\partial T_\varepsilon = \mathcal{H}^{m-1} \llcorner \phi(\operatorname{bdry} U_\varepsilon) \wedge \mathrm{D}\phi(\tau_\varepsilon).$$

This implies that $\mathbb{M}(\partial T_\varepsilon) = \mathcal{H}^{m-1}(\phi(\operatorname{bdry} U_\varepsilon))$. $(U_\varepsilon)_{\varepsilon>0}$ is a definable family of sets and so are $(\operatorname{bdry} U_\varepsilon)_{\varepsilon>0}$ and $(\phi(\operatorname{bdry} U_\varepsilon))_{\varepsilon>0}$. Note also that the last family has dimension $m-1$ and is bounded (being contained in $\mathrm{cl}(\phi(U))$ which is compact). By theorem 6.2.9, there exists a positive M such that for all $\varepsilon > 0$, $\mathcal{H}^{m-1}(\phi(\operatorname{bdry} U_\varepsilon)) < M$. Thus $\mathbb{M}(\partial T_\varepsilon)$ is uniformly bounded.

On the other hand, $\bigcup_{\varepsilon>0} U_\varepsilon = U$ implies that we have $\mathcal{H}^m(\phi(U \backslash U_\varepsilon)) \to 0$ and as $T - T_\varepsilon$ is supported in $\phi(U \backslash U_\varepsilon)$, $\mathbb{M}(T - T_\varepsilon) \to 0$ as ε goes to 0. ∂T_ε converges to ∂T in the flat norm and by the control over $\mathbb{M}(\partial T_\varepsilon)$ above and semi-continuity of the boundary mass with respect to flat convergence, $\mathbb{M}(\partial T) \leqslant M$ and T is integral.

Let us now show that ∂T is a definable chain. We have $\operatorname{spt} \partial T \subseteq \mathrm{cl}(C) \backslash C =: \mathrm{fr}(C)$, which is a union of cells of dimension $m-1$ or less (we fix this decomposition). Let $C_1, \ldots, C_p$ be the $m-1$ dimensional cells of $\mathrm{fr}(C)$ and choose a simple unit $m-1$ vector field ζ tangent to each of the $m-1$ cells. As ∂T is integral an supported in a finite union of $(m-1)$-submanifolds, there exists an integer valued function θ defined on each C_i such that

$$\partial T = \theta \left(\mathcal{H}^m \llcorner \bigcup_{j=1}^{p} C_j \right) \wedge \zeta.$$

As $\partial(\partial T) = 0$, we can apply the following variant of the constancy theorem separately on each C_j:

Claim 1. Suppose M is a connected oriented C^1 submanifold of dimension k in $\mathbb{R}^n$ and $S \in \mathbb{F}_k(\mathbb{R}^n)$ is such that $\operatorname{spt} \partial S \subset \mathbb{R}^n \backslash M$. Then, denoting by ζ the oriented k vector field tangent to M, there exists $c \in \mathbb{R}$ such that:

$$S \llcorner M = c\, \mathcal{H}^k \llcorner M \wedge \zeta.$$

Proof of claim 1. Let x be a point in M, there exists a neighbourhood of x in M which is C^1 diffeomorphic to some open ball in $\mathbb{R}^k$. Restricting to a smaller neighbourhood $V_x \subseteq M$ of x and a smaller ball $\mathrm{U}(0,r)$ in $\mathbb{R}^k$, one can ask that the diffeorphism $\phi_x : \mathrm{U}(0,r) \to V_x$ be bilipshitz (thus proper). Push $S \llcorner V$ forward with ϕ^{-1} to get a k current in $\mathrm{U}(0,r)$, notice that $\partial(S \llcorner V)$ is supported in $M \backslash V$ and apply the Constancy Theorem ([26, 4.1.7]) to get $c_x \in \mathbb{R}$ such that

$$\phi_\#^{-1}(S \llcorner V_x) = c_x \mathbf{E}^k \llcorner \mathrm{U}(0,x).$$

Choosing the right orientation once and for all, we have $\phi_\#(\mathbf{E}^k \llcorner \mathrm{U}(0,r)) = \mathcal{H}^m \llcorner V \wedge \zeta$, where ζ is the tangent unit k vector field to M, which implies:

$$\phi_\#(\phi_\#^{-1}(S \llcorner V_x)) = (S \llcorner V_x) = c_x\, \mathcal{H}^k \llcorner V \wedge \zeta.$$

The first equality holds because we consider flat chains, and not any current, which implies that only the data of the pushforward map on the support of the current matter. As M is connected c_x should not depend on x, and the Proposition holds. ■

Using this separately on each C_j, we infer that θ must be piecewise constant, this yields

$$\partial T = \sum_{j=1}^{p} \theta_j \mathcal{H}^m \llcorner C_j \wedge \zeta.$$

As $x \mapsto \sum_j \mathbb{1}_{C_j}\theta_j$ and ζ are definable, ∂T is a definable chain. □

Using the argument of the above proof, we can show that a definable chain can be well approximated *from the inside* by a sum of multiples of disjoint C^1 manifolds with Lipschitz boundary.

Corollary 6.3.3. *The singular set of a definable chain is hereditarily disposable, i.e. if $T \in \mathbb{I}_m^{Def}(\mathbb{R}^n)$ and S is a subcurrent of T (see chapter 3, S does not have to be definable), there exists $C > 0$ such that for all υ, there exists a subcurrent $S_\upsilon \sqsubset S$ with*

$$S_\upsilon = \sum_j \theta_j \phi_{j\#}(\mathbf{E}^m \llcorner U_j),$$

where $\forall j, j' \in \{1, \ldots, p\}$ with $j \neq j'$,

(i) $\theta_j \in \{1, 2, \ldots\}$,

(ii) ϕ_j *is C^1 bilipshitz from* $\operatorname{cl} U_j$ *to its image,*

(iii) $\phi_j(\operatorname{cl} U_j) \cap \phi_{j'}(\operatorname{cl} U_{j'}) = \emptyset$

and

$$\begin{aligned} \mathbb{F}(S - S_\upsilon) &< \upsilon, \\ \mathbb{N}(S - S_\upsilon) &< C. \end{aligned}$$

Proof. We prove that the union of the lower dimensional cells of $\operatorname{spt} T$ (for some cell decomposition) is hereditarily disposable, it is enough to prove the claim as away from its boundary, a cell is a bilipschitz C^1 image of an open set of $\mathbb{R}^m$. To do so, we apply the second part of proposition 4.2.8.

Without loss of generality, we can suppose that T is supported on the closure of an m-cell D. As $\operatorname{bdry} D$ is $\|T\|$ null and $\|T\|$ is Borel Regular, we have $\|T\|(\mathrm{U}(\operatorname{bdry} D, r)) \to 0$ as $r \to 0$. and it suffices to prove that there exists $C > 0$ such that for $r > 0$,

$$\mathbb{M}(\partial(T \llcorner \mathrm{U}(\operatorname{bdry} D, r))) < C.$$

We proceed as in the previous proof to show that $(\operatorname{spt}(\partial(T \llcorner \mathrm{U}(\operatorname{bdry} D, r))))_{r \in (0,1)}$ is a bounded definable family in $\mathbb{R}^n$ with finite $\mathcal{H}^{m-1}$ measure for all r, and thus uniformly bounded $\mathcal{H}^{m-1}$ measure by theorem 6.2.9. As for all $r \in (0, 1)$, $\|\partial(T \llcorner \mathrm{U}(\operatorname{bdry} D, r))\|$ has multiplicity 1 almost everywhere, we have the same control on the boundary mass. □

Appendix

General notations and conventions

We denote the inclusion of sets by $\subseteq$. Union and intersection of sets are represented by the usual $\cup$ and $\cap$ and the **symmetric difference** of two sets is denoted by Δ. $\mathbb{R}$ is the set of real numbers and for $k = 2, 3, \ldots$, $\mathbb{R}^k$ is the k dimensional Euclidean space. For $x, y \in \mathbb{R}^k$, $x \cdot y$ is the usual scalar product of x and y and $|x|$ is the induced norm on x. $\mathrm{U}(x, r)$ and $\mathrm{B}(x, r)$ denote respectively the open and closed balls of center x and radius $r > 0$. If E is a subset of $\mathbb{R}^k$, and $x \in \mathbb{R}^k$, $\mathrm{dist}(x, E)$ denotes the Euclidian distance from x to E.

We work with the topology induced by the Euclidean norm and for $A \subseteq \mathbb{R}^k$, we denote by $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bdry} A$ the interior, closure and boundary of the set A.. If f is a function, or a map defined on the set X and $E \subseteq X$, $f|_E$ is the restriction of f to E.

Appendix A Measure Theory

In top dimension, we work with the Lebesgue measure $\mathcal{L}$. If A is a subset of $\mathbb{R}^m$, most of the time we denote its Lebesgue (exterior) measure by $|A|$. For $s \geqslant 0$ we define the s-dimensional Hausdorff measure: $\mathcal{H}^s$. We consider them as outer measures. One should not worry too much about non-measurable sets in our setting thanks to the following classical result:

Lemma A.1 (Theorem 1.6 in [24])**.** *For $E \subseteq \mathbb{R}^m$ there exists a G_δ set $B \subseteq \mathbb{R}^m$ with $E \subseteq B$ and $\mathcal{H}^s(E) = \mathcal{H}^s(B)$. In particular B is $\mathcal{H}^s$-measurable. This shows that $\mathcal{H}^s$ is a regular outer measure.*

For the next lemma, we will consider Σ the σ-algebra of $\mathcal{H}^s$-measurable sets in $\mathbb{R}^m$.

Lemma A.2. *For $s > 0$, $\mathcal{H}^s$ is atomless, that is, there is no $A \in \Sigma$ such that $0 < \mathcal{H}^s(A)$ and for any measurable subset B of A, $\mathcal{H}^s(B) < \mathcal{H}^s(A)$ implies $\mathcal{H}^s(B) = 0$. This implies in particular that given $t > 0$ and any set S with finite $\mathcal{H}^s$ (outer) measure, there exist an integer $p \geqslant 1$ and subsets $S_1, ..., S_p$ of S such that $\bigcup_{j=1}^p S_j = S$ and for each i, $\mathcal{H}^s(S_j) < t$.*

The following proof is very close to the one from [62].

Proof. To prove the first part of the statement, suppose $A \in \Sigma$ is an atom for $\mathcal{H}^s$. Let us divide $\mathbb{R}^m$ in semicubes of side length 1 : $C_1, C_2....$ Each C_j is $\mathcal{H}^s$-measurable and $\mathcal{H}^s(A \cap C_j) + \mathcal{H}^s(A - C_j) = \mathcal{H}^s(A)$. As $\mathcal{H}^s(A) = \sum_j \mathcal{H}^s(A \cap C_j)$ we can suppose that $\mathcal{H}^s(A \cap C_1) \neq 0$ which implies that $A \cap C_1$ is an atom of with $\mathcal{H}^s(A \cap C_1) = \mathcal{H}^s(A)$. We can thus suppose that $A = A \cap C_1$. Iterating the

process with the (finite) dyadic cube families related to C_1, one finds an infinite strictly decreasing sequence of dyadic cubes $(C_1^j)_j$ with $\mathcal{H}^s(A \cap C_1^j) = \mathcal{H}^s(A)$. Let x be the intersection of the closures, $\mathcal{H}^s(A) \leqslant \mathcal{H}^s(\cap_j C_1^j) \leqslant \mathcal{H}^s(\{x\}) = 0$. Thus there is no atom for $\mathcal{H}^s$.

Let us now prove the second assertion. Suppose a set S is measurable.

Claim 1. For any $A \in \Sigma$ of finite measure,there exists $B \subseteq A$ measurable with $0 < \mathcal{H}^s(B) < t$.

Proof of claim 1. Since A is not an atom, there exists $C \subseteq A$ measurable with $0 < \mathcal{H}^s(C) < \mathcal{H}^s(A)$, so either $\mathcal{H}^s(A - C) < t$ and we are done or $\mathcal{H}^s(A - C) \geqslant t$; in the latter case either $\mathcal{H}^s(A - C) \leqslant \mathcal{H}^s(A)/2$ or $\mathcal{H}^s(C) \leqslant \mathcal{H}^s(A)/2$. Pick one or the other accordingly and repeat the process k times until $\mathcal{H}^s(A)/2^k < t$, one of the two last sets can be chosen for B. ■

Now for $A \in \Sigma$, let $\eta(A) := \sup\{\mathcal{H}^s(B), B \in \Sigma, B \subseteq A, \mathcal{H}^s(B) < t\}$. If $0 < \mathcal{H}^s(A) < \infty$, $\eta(A) > 0$. Let us now cover S, if $\mathcal{H}^s(S) < t$, we are done. If not, let $B_1 \in \Sigma, B_1 \subseteq S$ with $\eta(S)/2 < \mathcal{H}^s(B_1) < t$. If $\mathcal{H}^s(S - B_1) < t$, we are done too. If not continue in the same way, defining $B_k \in \Sigma, B_k \subseteq (S - \bigcup_{j=1}^{k-1} B_j)$ with $\eta(S - \bigcup_{j=1}^{k-1} B_j)/2 < \mathcal{H}^s(B_k) < t$, if the process terminates we are done, if it doesn't then we have obtained a sequence $(B_k)_{k \geqslant 1}$ of disjoint measurable subsets of S thus $\sum_{j=1}^{\infty} \mathcal{H}^s(B_j) < \infty$ which implies that $\eta(S - \bigcup_{j=1}^{k} B_j)$ goes to zero as k tends to ∞.Let $B_0 := S - \bigcup_{j=1}^{\infty} B_j$, since η is decreasing, $\eta(B_0) \leqslant \eta(S - \bigcup_{j=1}^{k} B_j)$ for all k, thus $\mathcal{H}^s(B_0) = 0$. There exists $n \geqslant 1$ such that $\sum_{j=n}^{\infty} \mathcal{H}^s(B_j) < t$ so letting $S_j := B_j$ for $j < n$ and $S_n := B_0 \cup \bigcup_{j=n}^{\infty} B_j$ we obtain the desired cover.

If S is not measurable, consider a measurable $\tilde{S}$ that contains S and has the same measure, find a cover of $\tilde{S}$ by small measurable sets $\{\tilde{S}_i\}_{i=1,\dots,n}$ and define $S_i := \tilde{S}_i \cap S$. □

Lemma A.3. *There exists a constant $\kappa = \kappa(n) > 1$ such that for each closed cube $C \in \mathbb{R}^m$, $E \subseteq C$, $m > 0$ and $\delta > 0$, if $c > \mathcal{H}^m(E)$ there exists a countable family $\mathcal{Q}$ of nonoverlapping dyadic cubes (relative to C) satisfying* $\operatorname{diam}(Q) \leqslant \delta$ *for $Q \in \mathcal{Q}$, $\sum_{Q \in \mathcal{Q}} \operatorname{diam}(Q)^m \leqslant \kappa c$ and such that every point $x \in E$ belongs to* $\operatorname{int}^C[\mathcal{Q}_x]$ *where* int^C *stands for the topological interior with respect to C and $\mathcal{Q}_x := \{Q \in \mathcal{Q}, x \in \operatorname{cl} Q\}$ is a finite subfamily of $\mathcal{Q}$.*

Proof. In [24, Theorem 5.1], the author proves that there exists a constant $\kappa = \kappa(m)$ (one can take $\kappa = 3^m 2^{(m^2)}$) such that any set A with $\operatorname{diam}(A) < \delta$ is contained in κ non overlapping dyadic cubes of diameter $\delta_A < \operatorname{diam}(A)$.

Choose any δ-covering $\mathcal{U}$ of E by sets open in C. We can suppose that $\mathcal{U}$ is finite. Associate to each $U \in \mathcal{U}$ a covering $\mathcal{Q}_U$ of U by κ non overlapping dyadic cubes of diameter $\delta_U < \operatorname{diam}(U)$. For $x \in E$ and $U \in \mathcal{U}$ with $x \in U$ define $\mathcal{Q}_{U,x} := \{Q \in \mathcal{Q}_U, x \in \operatorname{cl} Q\}$; it is a finite collection of cubes. There holds $x \in \operatorname{int}^C[Q_{U,x}]$. E is contained in the countable collection $\mathcal{Q}_0 := \bigcup_{U \in \mathcal{U}} \mathcal{Q}_U$. Denote by $\mathcal{Q}$ the maximal subfamily of $\mathcal{Q}_0$ and compute

$$\sum_{Q \in \mathcal{Q}} d(Q)^m \leqslant \sum_{U \in \mathcal{U}} \sum_{Q \in \mathcal{Q}_U} d(Q)^m \leqslant \kappa \sum_{U \in \mathcal{U}} d(U)^m.$$

As $\mathcal{U}$ is an arbitrary δ-covering of E by open sets, one gets the desired upper bound for $\sum_{Q \in \mathcal{Q}} d(Q)^m$. Fix any $x \in E$. There is a subfamily $\mathcal{Q}_x := \{Q \in \mathcal{Q}, x \in \operatorname{cl} Q\} \subseteq \mathcal{Q}$ with $x \in \operatorname{int}_C \cup \mathcal{Q}_x$. In fact, $\mathcal{Q}_x$ is the maximal subfamily of the reunion of the families $\mathcal{Q}_{U,x}$, this implies that for $Q \in \mathcal{Q}_x$ and $U \in \mathcal{U}$ $\operatorname{diam}(Q) \geqslant \delta_U$. So $\mathcal{Q}_x$ is finite for all x. □

The next Theorem is used a few times, it can be found in [63, Theorem 2.24] or [26, 2.3.5].

Theorem A.4 (Lusin's Theorem)**.** *Given a metric space X and a Borel regular measure μ on X, if $A \subseteq X$ is μ-measurable and has finite μ measure, f is a μ-measurable function defined on X and $\varepsilon > 0$, then there exists a compact set $C \subseteq A$ such that f is continuous on C and $\mu(A \backslash C) < \varepsilon$.*

We use the next result to relate Lebesgue integrability to gauge integrability, it can be found in [63, Theorem 2.25].

Theorem A.5 (Vitali Caratheodory Theorem)**.** *If X is a locally compact Hausdorff space, μ is Borel regular and locally finite and f is μ integrable and $\varepsilon > 0$, then there exists two functions g and h with $g \leqslant f \leqslant h$, g is upper semi-continuous and h is lower semi-continuous and*

$$\int_X h - g \, \mathrm{d}\mu < \varepsilon.$$

Let us also state an easy property of Lebesgue integrable function which we use several times in the text.

Proposition A.6 (Continuity of the Lebesgue integral)**.** *If (X, μ) is a measure space and $f : X \to \mathbb{R}$ is Lebesgue integrable on X, μ, then given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever A is a subset of X with $\mu(X \backslash A) < \delta$, $|\int_X f - \int_A f| < \varepsilon$.*

(The proof is easy by approximation from below by a simple function if f is nonnegative.)

We conclude this section with a few definitions from Geometric Measure Theory: Given a measure μ in $\mathbb{R}^n$, the **support** of μ denoted by $\operatorname{spt} \mu$ is the smallest closed set such that $\mu(A) = 0$, whenever $A \cap \operatorname{spt} \mu = \emptyset$. Given a point $x \in \mathbb{R}^n$ and $m \in \{0, 1, \ldots, n\}$ and $\boldsymbol{\alpha}(m)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^m$, we define the **upper and lower m-dimensional densities of μ at x**:

$$\begin{aligned} \boldsymbol{\Theta}^{m*}(\mu, x) &:= \limsup_{r \to 0} \frac{\mu(\mathrm{U}(x, r))}{\boldsymbol{\alpha}(m) r^m} \text{ and} \\ \boldsymbol{\Theta}^{m}_{*}(\mu, x) &:= \liminf_{r \to 0} \frac{\mu(\mathrm{U}(x, r))}{\boldsymbol{\alpha}(m) r^m}. \end{aligned}$$

If $\boldsymbol{\Theta}^{m*}(\mu, x) = \boldsymbol{\Theta}^{m}_{*}(\mu, x)$, we call this number the **m-dimensional density of μ at x** and denote it simply by $\boldsymbol{\Theta}^m(\mu, x)$. Finally, we define $\operatorname{set}_m \mu$, the **m-density set of μ** as

$$\operatorname{set}_m \mu = \mathbb{R}^n \cap \{x, \boldsymbol{\Theta}^{m*}(\mu, x) > 0\}.$$

Appendix B Integral currents

B.1 Notations and classical definitions

We use mostly the notation of Federer [26]. Given integers $m \leqslant n$, denote by $\Lambda_m(\mathbb{R}^n)$ and $\Lambda^m(\mathbb{R}^n)$ respectively the spaces of m-vectors and m-covectors in $\mathbb{R}^n$ where the duality comes from the euclidean structure of $\mathbb{R}^n$. The inner product, denoted by $\langle \cdot ; \cdot \rangle$ is the corresponding bilinear map from $\Lambda^m(\mathbb{R}^n) \times \Lambda_m(\mathbb{R}^n)$ to $\mathbb{R}$.

The euclidian norm $|\cdot|$ and the **mass** (resp. **comass**) $\|\cdot\|$ of m-vectors (resp. m-covectors) are equivalent and coincide on simple m-vectors (resp. m-covectors) we will thus only consider the comass: for $\phi \in \Lambda^m(\mathbb{R}^n)$

$$\|\phi\| := \sup\{\langle \xi; \phi\rangle, \xi \in \Lambda_m(\mathbb{R}^n) \text{ is simple and } |\xi| \leqslant 1\}$$

and the mass: for $\xi \in \Lambda_m(\mathbb{R}^n)$

$$\|\xi\| := \sup\{\langle \xi; \phi\rangle, \phi \in \Lambda_m(\mathbb{R}^n) \text{ and } \|\phi\| \leqslant 1\}.$$

We will also use the exterior multiplication denoted by $\wedge$ as well as the interior multiplication operators, for $0 \leqslant p \leqslant m \leqslant n$:

$$\llcorner : \Lambda_m(\mathbb{R}^n) \times \Lambda^p(\mathbb{R}^n) \to \Lambda_{m-p}(\mathbb{R}^n),$$
$$\lrcorner : \Lambda_p(\mathbb{R}^n) \times \Lambda^m(\mathbb{R}^n) \to \Lambda^{m-p}(\mathbb{R}^n),$$

defined by the inner product identities

$$\langle \xi \llcorner \beta; \alpha\rangle = \langle \xi; \alpha \wedge \beta\rangle, \text{ when } \xi \in \Lambda_m(\mathbb{R}^n), \alpha \in \Lambda^{m-p}(\mathbb{R}^n), \beta \in \Lambda^p(\mathbb{R}^n),$$
$$\langle \xi; \zeta \lrcorner \phi\rangle = \langle \xi \wedge \zeta; \phi\rangle, \text{ when } \xi \in \Lambda_{m-p}(\mathbb{R}^n), \zeta \in \Lambda_p(\mathbb{R}^n), \phi \in \Lambda^m(\mathbb{R}^n).$$

In particular, if μ is a measure on a set X with values in some $\Lambda_m(\mathbb{R}^n)$ and α is a $\|\mu\|$-measurable function from X to $\Lambda^p(\mathbb{R}^n)$ with $0 \leqslant p \leqslant m$, we consider $\mu \llcorner \alpha$. In the case where A is a measurable subset of X, we will often write $\mu \llcorner A$ instead of $\mu \llcorner \mathbb{1}_A$ for the restriction of the measure μ to A.

$\mathcal{L}^m$ is the Lebesgue measure in $\mathbb{R}^m$, $\mathbf{E}^m$ stands for $\mathcal{L}^m \wedge \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_m$, where $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is the canonical basis of $\mathbb{R}^m$. For $s \geqslant 0$, $\mathcal{H}^s$ is the Hausdorff measure of dimension s in the ambient (euclidean) space considered at the moment. Finally, the notation spt stands for the support of a function or measure.

For $m \leqslant n$ We denote by $\mathcal{D}^m(\mathbb{R}^n)$ the space of C^∞ differential forms of degree m in $\mathbb{R}^n$ with compact support. We consider the norm $\|\cdot\|$, called **comass** defined for $\phi \in \mathcal{D}^m(\mathbb{R}^n)$ by

$$\|\phi\| := \sup_{x \in \mathbb{R}^n} \|\phi(x)\|.$$

B.2 Currents

The space $\mathcal{D}_m(\mathbb{R}^n)$ of currents of dimension m in $\mathbb{R}^n$ is the dual space of $\mathcal{D}^m(\mathbb{R}^n)$ in the sense that a linear form T on $\mathcal{D}^m(\mathbb{R}^n)$ is in $\mathcal{D}_m(\mathbb{R}^n)$ if and only if, for each compact $K \subseteq \mathbb{R}^n$, there exists $M > 0$ such that for all $\phi \in \mathcal{D}^m(\mathbb{R}^n)$ with $\operatorname{spt} \phi \subseteq K$

$$T(\phi) \leqslant M\|\phi\|.$$

The **mass** of a current T of dimension m, denoted by $\mathbb{M}(T)$ is given by:

$$\mathbb{M}(T) := \sup\{T(\phi), \phi \in \mathcal{D}^m(\mathbb{R}^n) \text{ and } \|\phi\| \leqslant 1\}.$$

If a current T has finite mass, then by [26, sections 4.1.5-7], it is representable by integration as follows:

$$T : \phi \mapsto \int \langle \overrightarrow{T}(x); \phi \rangle \, \mathrm{d}\|T\|(x),$$

where $\|T\|$ is a Radon measure in $\mathbb{R}^n$ with $\|T\|(\mathbb{R}^n) = \mathbb{M}(T)$ called the carrying measure of T and $\overrightarrow{T}$ is a unit simple m-vector field.

The differential operation on differential forms: $\mathrm{d}\colon \mathcal{D}^m(\mathbb{R}^n) \to \mathcal{D}^{m-1}(\mathbb{R}^n)$ induces a **boundary operator** ∂ on currents by:

$$\partial T(\phi) := T(\mathrm{d}\phi), \text{ for } T \in \mathcal{D}_m(\mathbb{R}^n) \text{ and } \phi \in \mathcal{D}^{m-1}(\mathbb{R}^n).$$

Of course ∂T need not be a current, however if it is, then $\partial(\partial T) = 0$ as $\mathrm{dd} = 0$. If T and ∂T are currents with finite mass and T *has compact support*, T is called **normal**. The space of normal currents of dimension m in $\mathbb{R}^n$ is denoted by $\mathbb{N}_m(\mathbb{R}^n)$ and for $T \in \mathbb{N}_m(\mathbb{R}^n)$, the number $\mathbb{N}(T) = \mathbb{M}(T) + \mathbb{M}(\partial T)$ is called the normal mass of T. For $X \subseteq \mathbb{R}^n$, we denote by $\mathbb{N}_m(X)$ the space of normal currents of dimension m in $\mathbb{R}^n$ whose support is contained in X. Lastly, the flat norm in $\mathbb{R}^n$ if T is a current of dimension m its flat norm if given by:

$$\mathbb{F}(T) := \inf\{\mathbb{M}(T - \partial S) + \mathbb{M}(S), \text{ where } S \in \mathbb{N}_{m+1}(\mathbb{R}^n)\}.$$

Given a compact set K, the space of **flat chains** of dimension m in K denoted by $\mathbb{F}_m(K)$ is the closure of $\mathbb{N}_m(K)$ with respect to the flat norm in $\mathbb{R}^n$. The space of m-dimensional flat chains in $\mathbb{R}^n$ is the reunion of the $\mathbb{F}_m(K)$ for all compact subsets K of $\mathbb{R}^n$. It is also the closure with respect to the flat norm of the space of polyhedral chains with real coefficients. An important property is that $\mathbb{M}$ and $\mathbb{N}$ are lower semi continuous with respect to the flat norm.

For a set $X \subseteq \mathbb{R}^n$ and $m \leqslant n$, using the flat norm and the normal mass, De Pauw, Moonens and Pfeffer defined in [19] a topology $\mathfrak{F}_{X,m}$ on the space $\mathbb{N}_m(X)$ with the following properties:

a) $(\mathbb{N}_m(X), \mathfrak{F}_{X,m})$ is locally convex, separable.

b) For every compact set $K \subset \mathbb{R}^n$ and every $c > 0$, the flat norm topology coincides with $\mathfrak{F}_{X,m}$ on the set

$$\mathbb{N}_{m,c}(K) := \{T \in \mathbb{N}_m(X \cap K) : \mathbb{N}(T) \leqslant c\}$$

in particular if X is compact, $\mathbb{N}_{m,c}(X)$ is compact in the topology $\mathfrak{F}_{X,m}$. This statement is not classical: in [26, 4.2.17], it is proved under the additional assumption that X is a Lipschitz neighbourhood retract. In [19, Theorem 4.2], a similar argument is used which is valid for any compact set X.

c) $(\mathbb{N}_m(\mathbb{R}^n), \mathfrak{F}_{\mathbb{R}^n})$ is not metrizable, however it is hereditary sequential and all sequences $(T_j)_j$ in $\mathbb{N}_m(X)$ such that

(i) The T_j are all supported in the same compact set K,

(ii) $\lim_j \mathbb{F}(T_j) = 0$ and

(iii) $\sup_j \mathbb{N}(T_j) < \infty$

converge in $\mathfrak{F}_{X,m}$.

A set $X \subseteq \mathbb{R}^n$ is $(\mathcal{H}^m, m)$-rectifiable (see [26, 3.2.14]) if $\mathcal{H}^m(X) < \infty$ and there exists a countable family of bounded subsets E_j of $\mathbb{R}^m$ and lipschitz functions $\phi_j : E_j \to \mathbb{R}^n$ such that

$$\mathcal{H}^m\left(X \setminus \bigcup_j f(E_j)\right) = 0.$$

A current $T \in \mathcal{D}_m(\mathbb{R}^n)$ is rectifiable, $(T \in \mathcal{R}_m(\mathbb{R}^n))$ if there exists a triple $(M, \theta, \overrightarrow{T})$, where

(i) $M \subseteq \mathbb{R}^n$ is bounded, $(\mathcal{H}^m, m)$-rectifiable and $\mathcal{H}^m$ measurable,

(ii) θ and $\overrightarrow{T}$ are $\mathcal{H}^m \llcorner M$ measurable maps,

(iii) θ is positive integer valued and $\mathcal{H}^m \llcorner M$ integrable

(iv) and $\overrightarrow{T}$ is a unit simple m-vector field tangent to M $\mathcal{H}^m \llcorner M$ almost everywhere,

such that T is representable by integration as:

$$T : \omega \mapsto \int_M \theta(x) \langle \omega(x), \overrightarrow{T}(x) \rangle \, \mathrm{d}\mathcal{H}^m(x). \tag{2}$$

In particular

- $\|T\| = \theta \mathcal{H}^m \llcorner M$
- the density of dimension m of $\|T\|$, $\Theta_m(\|T\|, \cdot)$ is $\|T\|$ almost everywhere defined and equal to θ.
- By definition, all rectifiable currents have finite mass.

This leads to the definition of our last fundamental family of currents: **integral currents** ; the space $\mathbb{I}_m(\mathbb{R}^n)$ of m-dimensional integral currents in $\mathbb{R}^n$ is given by

$$\mathbb{I}_m(\mathbb{R}^n) := \{T \in \mathcal{R}_m(\mathbb{R}^n), \partial T \in \mathcal{R}_{m-1}(\mathbb{R}^n)\}.$$

A crucial result is that a rectifiable current is an integral current if and only if its boundary has finite boundary mass (see [26, 4.2.16 (2)]).

Given $m \leqslant n$, another natural number n', $T \in \mathbb{F}_m(\mathbb{R}^n)$, an open set U containing $\operatorname{spt} T$ and a map $\phi : U \to \mathbb{R}^{n'}$, which is lipschitz and such that $\phi \llcorner \operatorname{spt} T$ is proper, we define: $\phi_\# T$, the **pushforward of** T **by** ϕ as the limit as j goes to infinity of the currents

$$(\phi_j)_\# T : \omega \in \mathcal{D}^m(\mathbb{R}^{n'}) \mapsto \langle (\phi_j)^\# \omega, T \rangle,$$

where the ϕ_j are maps defined on a neighbourhood U of $\operatorname{spt} T$ which are C^1 with $\operatorname{Lip} \phi_j$ uniformly bounded and $|\phi_j - \phi| < 1/i$ on $\operatorname{spt} T$ and $\phi_j^\# \omega$ is the classical pullback of a differential form by a smooth map. See [26, 4.1.14] for details on the construction and properties of the pushforward. The following will be useful:

Proposition B.1 (See 4.1.15 in [26]). *Let m, n, n', T and ϕ be as above. If ϕ' agrees with ϕ on $\operatorname{spt} T$, then $\phi_\# T = \phi'_\# T$.*

Note that if ϕ is a Lipschitz map from $\operatorname{spt} T$ to $\mathbb{R}^{n'}$, it can be extended to the whole of $\mathbb{R}^n$ in a Lipschitz way, not increasing the Lipschitz constant by more than a factor $\sqrt{n'}$. See for instance [23, 3.1 Theorem 1]. If ϕ is proper on $\operatorname{spt} T$, its extension will be as well. This together with the above proposition shows that we can push currents forward by maps which are only defined, Lipschitz and proper on $\operatorname{spt} T$.

Another useful result concerning the pushforward by a Lipschitz map is the following

Proposition B.2. *If T is a rectifiable current of dimension m in $\mathbb{R}^n$ and ϕ is Lipschitz from $\operatorname{spt} T$ to $\mathbb{R}^{n'}$, then $\mathbb{M}(\phi_\# T) \leqslant (\operatorname{Lip}\phi)^m\, \mathbb{M}(T)$ and as a corollary $\mathbb{F}(\phi_\# T) \leqslant \max\{(\operatorname{Lip}\phi)^{m+1}, (\operatorname{Lip}\phi)^m\}\, \mathbb{F}(T)$. Furthermore $\operatorname{set}_m \|\phi_\# T\| \subseteq \phi(\operatorname{set}_m \|T\|)$. In particular if ϕ is bilipschitz, the two sets are equal.*

Proof. The proof of the first statement and it corollary can be found in [26, 4.1.14]. For the second part of the statement, apply the first part to the rectifiable current $T \llcorner \mathrm{B}(x,r)$ for $x \in \operatorname{set}_m \|T\|$ and $r > 0$, then use the fact that $\phi(\mathrm{B}(x,r)) \subseteq \mathrm{B}(\phi(x), (\operatorname{Lip}\phi)r)$. □

B.3 Slicing

As a motivation, we start by giving a simple property concerning the restriction of an integral current to a ball:

Proposition B.3 (Restriction of integral currents to balls)**.** *Consider $T \in \mathbb{I}_m(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. For Lebesgue almost all $r \in]0,+\infty[$, $T \llcorner \mathrm{U}(x,r)$ is integral. This also holds when replacing $\mathrm{U}(x,r)$ by $\mathrm{B}(x,r)$*

Proof. Notice first that $T \llcorner \mathrm{U}(x,r)$ is a rectifiable current. Consider the function : $f :]0,+\infty[\to [0,+\infty[\,, r \mapsto \|T\|(\mathrm{U}(x,r))$. f is nondecreasing and therefore Borel measurable and differentiable Lebesgue almost everywhere in $]0,+\infty[$ (see for instance [11, Chapter 6.3]). Let r be a point of differentiability of f and for $h \in (0,r)$ let χ_h be a smooth nonnegative function such that $\mathbb{1}_{\mathrm{U}(x,r-h)} \leqslant \chi_h \leqslant \mathbb{1}_{\mathrm{U}(x,r)}$ and $\operatorname{Lip}\chi_h \leqslant 2h^{-1}$. As f is continuous at r, there holds

$$0 \leqslant \mathbb{M}(T \llcorner \mathrm{U}(x,r) - T \llcorner \chi_h) \leqslant \mathbb{M}(T \llcorner \mathrm{U}(x,r)) - \mathbb{M}(T \llcorner \mathrm{U}(x,r-h)) \to 0 \text{ as } h \to 0.$$

and $T \llcorner \chi_h$ converges to $T \llcorner \mathrm{U}(x,r)$ in the mass norm, when h goes to 0. This implies that $\partial(T \llcorner \chi_h)$ converges to $\partial(T \llcorner \mathrm{U}(x,r))$ in the flat norm as h goes to 0. Since the mass is lower-semi-continuous with respect to the flat norm, there holds:

$$\mathbb{M}(\partial(T \llcorner \mathrm{U}(x,r))) \leqslant \liminf_{h\to 0} \mathbb{M}(\partial(T \llcorner \chi_h)).$$

To conclude, for $h > 0$, χ_h is smooth with compact support and $\mathrm{d}\chi_h$ is a smooth 1 form, so

$$\partial(T \llcorner \chi_h) = (\partial T) \llcorner \chi_h + T \llcorner \mathrm{d}\chi_h$$

and since χ_h is supported in $\mathrm{B}(x,r) \setminus \mathrm{U}(x,r-h)$ and $\|T\|(\mathrm{B}(x,r)) = \|T\|(\mathrm{U}(x,r))$ by continuity of

f at r we get:

$$\begin{aligned}\mathbb{M}(\partial(T \llcorner \chi_h)) &\leqslant \mathbb{M}((\partial T) \llcorner \chi_h)) + \mathbb{M}(T \llcorner (\mathrm{d}\chi_h)) \\ &\leqslant \mathbb{M}(\partial T) + \frac{2}{h}\|T\|(\mathrm{B}(x,r) \setminus \mathrm{U}(x, r-h)) \\ &\leqslant \mathbb{M}(\partial T) + 2\frac{f(r) - f(r-h)}{h}.\end{aligned}$$

f having finite derivative at r allows us to conclude

$$\mathbb{M}(\partial(T \llcorner \mathrm{U}(x,r))) \leqslant \mathbb{M}(\partial T) + 2f'(r),$$

, and $T \llcorner \mathrm{U}(x,r) \in \mathbb{I}_m(\mathbb{R}^n)$. Since this is valid for Lebesgue almost all r in $]0,\infty[$, the proof is complete. □

In some sense, the above result shows that for almost every sphere $\mathbb{S}$, the "intersection" of T with $\mathbb{S}$ is an integral current. To generalize this notion, consider a function f which is Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}$. The **slice** of T by f at $r \in \mathbb{R}$ is defined by

$$\langle T, f, r\rangle := \partial(T \llcorner \{x, f(x) < r\}) - (\partial T) \llcorner \{x, f(x) < r\}.$$

More generally, if $T \in \mathbb{F}_m(\mathbb{R}^n)$, f is a Lipschitz map from $\mathbb{R}^n$ to $\mathbb{R}^k$ with $k \in \{1, \ldots, m\}$. The **slice of T by f at** $y \in \mathbb{R}^k$, denoted by $\langle T, f, y\rangle \in \mathbb{F}_{m-k}(\mathbb{R}^n)$, is defined for $\mathcal{L}^k$ almost all y in $\mathbb{R}^k$. See [26, 4.3.1-11] for the precise definition and an extensive study. For our purpose, we only list a few key properties of slices.

Proposition B.4 (Mass and flat norm control of slices, see 4.3.2 in [26])**.** *If T and f are as above,*

$$\int_{\mathbb{R}^k} \mathbb{F}(\langle T, f, y\rangle)\, \mathrm{d}\mathcal{L}^k(y) \leqslant (\mathrm{Lip}\, f)^k\, \mathbb{F}(T).$$

Similarly, if T has finite mass

$$\int_{\mathbb{R}^k} \mathbb{M}(\langle T, f, y\rangle)\, \mathrm{d}\mathcal{L}^k(y) \leqslant (\mathrm{Lip}\, f)^k\, \mathbb{M}(T),$$

in particular almost all slices of T have finite mass.

As a consequence, we infer that if T_j is a sequence in $\mathbb{F}_m(\mathbb{R}^n)$ which converges in the flat norm (respectively in the mass norm) to T, almost all the slices $\langle T_j, f, y\rangle$ converge inf the flat norm (respectively in the mass norm) to $\langle T, f, y\rangle$. To see this, just apply the previous proposition to $T - T_j$ and consider only the subset of $\mathbb{R}^k$ composed of all y such that $\langle T_j, f, y\rangle$ for all j and $\langle T, f, y\rangle$ have finite flat norm (respectively mass norm).

The last result we need is a combination of 4.3.6 and 4.3.8 in [26]:

Theorem B.5. *Suppose $T = \theta\mathcal{H}^m \llcorner M \wedge \overrightarrow{T}$ is a rectifiable current in $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^k$ with $k \leqslant m$ is Lipschitz, then for almost all $y \in \mathbb{R}^k$,*

$$\|\langle T, f, y\rangle\| = \theta\mathcal{H}^{m-k} \llcorner (M \cap f^{-1}(\{y\})).$$

In particular, almost all slices are rectifiable. Additionally, if T is integral, almost all its slices are integral.

Slicing has been used in particular to characterise rectifiable currents, see the works of Fleming [29] and White [71]. White uses flat chains of dimension 0 obtained by slicing a flat chain of dimension m by m-plane parallel to the coordinate axes. That is slicing by functions of the type $f : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_m)$. We conclude this introduction by noting that rectifiable currents of dimension 0 are particularly easy to study as they are finite sums of integer multiples of Dirac masses. The following theorem is a good illustration:

Theorem B.6 (White [71] 3.2)**.** *Let T be a flat chain of dimension m in $\mathbb{R}^n$. If for all m-uple $(j_1, \dots, j_m)$, for almost all $y \in \mathbb{R}^m$,*

$$\langle T, (x_1, \dots, x_n) \mapsto (x_{j_1}, \dots, x_{j_m}), y\rangle = 0,$$

then $T = 0$.

B.4 Facts on integral currents of dimension 1

A **rectifiable current of dimension** 1 , T is a vector valued measure supported on a $(\mathcal{H}^1, 1)$-rectifiable set M:

$$T = \|T\| \wedge \overrightarrow{T} = (\mathcal{H}^1 \llcorner \theta \mathbb{1}_M) \wedge \overrightarrow{T}),$$

where T is a unit lenght vector defined $\mathcal{H}^1 \llcorner M$ almost everywhere and θ is an integer valued $\mathcal{H}^1 \llcorner M$ integrable function. $\operatorname{spt} T$ is the support of the measure $\|T\|$ and $\operatorname{set}_1 \|T\|$ is the set of its points of positive upper density.

An **integral current of dimension** 1 is a rectifiable current T whose boundary ∂T is a finite sum of integer multiples of Dirac measures. In particular, given a C^∞ function f defined on a neighbourhood of $\operatorname{spt} T$, one has:

$$\int \langle \mathrm{D} f(x), \overrightarrow{T}(x)\rangle \, \mathrm{d}\|T\|(x) = \partial T(f) := \sum_{j=1}^{p} \alpha_j f(x_j).$$

To a current T we can associate its mass $\mathbb{M}(T) = \|T\|(\mathbb{R}^n)$ and its boundary mass $\mathbb{M}(\partial T) = \sum_{j=1}^{p} |\alpha_j|$. If $\partial T = 0$, T is called a **cycle**.

Appendix C Sets of finite perimeter

A set of finite perimeter (also called BV set) in $\mathbb{R}^m$ is a Lebesgue measurable set $A \subseteq \mathbb{R}^m$ such that the quantity:

$$\mathrm{P}(A) := \sup \left\{ \int_A \operatorname{div} \mathbf{v} \, \mathrm{d}\mathcal{L}^m \text{ for } \mathbf{v} \in \mathcal{C}_c^1(\mathbb{R}^m, \mathbb{R}^m) \text{ with } |\mathbf{v}|_\infty \leqslant 1 \right\}.$$

$\mathrm{P}(A)$ is the **perimeter** of A. Let $\mathcal{BV}$ be the space of bounded sets in $\mathbb{R}^m$ that have finite perimeter and $\mathcal{BV}(A) := \mathcal{BV} \cap \{B, B \subseteq A\}$, where we identify sets whose indicator functions are equal in L^1.

Let $A \in \mathcal{BV}$ define :

- The **essential closure** of A :

$$\mathrm{cl_e}\, A = \left\{ x \in \mathbb{R}^m, \limsup_{\varepsilon \to 0} \frac{|A \cap B(x,\varepsilon)|}{|B(x,\varepsilon)|} > 0 \right\}.$$

- The **essential interior** of A :

$$\mathrm{int_e}\, A = \left\{ x \in \mathbb{R}^m, \liminf_{\varepsilon \to 0} \frac{|A \cap B(x,\varepsilon)|}{|B(x,\varepsilon)|} = 1 \right\}.$$

- The **essential boundary** of A : $\partial_e A = \mathrm{cl_e}\, A - \mathrm{int_e}\, A$.

- The **reduced boundary** of A : $\partial_* A$ is defined as usual ($\partial_* A \subseteq \mathrm{bdry}\, A$).

- The **critical interior** of A :

$$\mathrm{int}_c\, A = \left\{ x \in \mathbb{R}^m, \limsup_{\varepsilon \to 0} \frac{\mathcal{H}^{m-1}(\mathrm{cl}_e\, A \cap B(x,\varepsilon))}{\varepsilon^{m-1}} = 0 \right\}.$$

Each bounded set of finite perimeter $A \subseteq \mathbb{R}^m$ is associated with an integral current $\mathbf{E}^m \llcorner A \mathcal{L}^m \llcorner A \wedge \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_m$ with

$$\mathbb{M}(\mathbf{E}^m \llcorner A) = |A| := \mathcal{L}^m(A)$$
$$\text{and}$$
$$\mathbb{M}(\partial(\mathbf{E}^m \llcorner A)) = \mathrm{P}(A).$$

A thorough study of top dimensional normal currents, including all sets of finite perimeter, is performed in [26, 4.5], a more specific study is in [4]

Proposition C.1. *For a bounded set A of finite perimeter in $\mathbb{R}^m$, we list here a few classical facts:*

(i) $\mathbb{F}(\mathbf{E}^m \llcorner A) = \mathbb{M}(\mathbf{E}^m \llcorner A) = |A|$.

(ii) If B is another bounded set of finite perimeter in $\mathbb{R}^m$, then $A \cap B$ has finite perimeter as well and $\mathrm{P}(A \cap B) \leqslant \mathrm{P}(A) + \mathrm{P}(B)$. The same holds for $A \cup B$ and $A \backslash B$ and we have the identity

$$\mathbf{E}^m \llcorner (A \cup B) = \mathbf{E}^m \llcorner A + \mathbf{E}^m \llcorner B - \mathbf{E}^m \llcorner (A \cap B).$$

(iii) $|A|^{\frac{m-1}{m}} \leqslant m^{-1}\alpha(m)^{-1/m}\, \mathrm{P}(A)$ *where $\alpha(m)$ is the euclidean volume of the unit ball in $\mathbb{R}^m$. This is known as the isoperimetric inequality*

(iv) $\mathrm{P}(A) = \mathcal{H}^{m-1}(\partial_e A) = \mathcal{H}^{m-1}(\partial_* A)$.

Lemma C.2. *If $A \in \mathrm{BV}(\mathbb{R}^m)$, the following hold :*

$$\mathcal{H}^{m-1}(\partial_e A) = \mathcal{H}^{m-1}(\partial_* A) = \mathrm{P}(A) \text{ and}$$
$$\mathcal{H}^{m-1}(\mathrm{int}_e\, A \backslash \mathrm{int}_c\, A) = 0.$$

(For a proof see Observation 2.5.1 in [61])

Lemma C.3 (Approximation by closed BV subsets)**.** *Let $A \in \mathcal{BV}$. There is a sequence $(A_i)_i \subseteq \mathcal{BV}$ with for all i $A_i \subseteq A$, $\mathrm{P}(A_i) \leqslant \mathrm{P}(A)$ and the sequence $|A - A_i|$ goes to zero.*

Definition C.4 (Topology on $\mathcal{BV}$)**.** For each positive integer n, let

$$\mathcal{BV}_n := \left\{ A \in \mathcal{BV}, A \subseteq \overline{B(0,n)} \text{ and } \mathrm{P}(A) \leqslant n \right\}.$$

Note that for all n, $\mathcal{BV}_n$ is compact in the L_1 topology. Denote by $\mathcal{T}$ the largest topology for which all the inclusions $(\mathcal{BV}_n, |.|) \hookrightarrow (\mathcal{BV}, \mathcal{T})$ are continuous.

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