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Sorbonne Université

École doctorale de sciences mathématiques de Paris centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Jesua Israel Epequin Chavez

Extremal representations for the finite Howe correspondence

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Life is long enough for the highest achievements.

Seneca, *On the Shortness of Life*

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Abstract

We study the Howe correspondence $\Theta : \mathscr{R}(G) \to \mathscr{R}(G')$, for irreducible dual pairs $(G, G') = (\mathrm{U}_m(\mathbb{F}_q), \mathrm{U}_{m'}(\mathbb{F}_q))$ and $(G, G') = (\mathrm{Sp}_{2m}(\mathbb{F}_q), \mathrm{O}^{\epsilon}_{2m'}(\mathbb{F}_q))$, where $\mathbb{F}_q$ denotes a finite field with q elements (q odd) and $\epsilon = \pm 1$. We establish the compatibility between the Howe correspondence and arbitrary Harish-Chandra series. We define and prove the existence of extremal (i.e. minimal and maximal) irreducible subrepresentations from the image $\Theta(\pi)$ of irreducible unipotent representations π of G. Finally, we prove how the study of the Howe correspondence between arbitrary Harish-Chandra series can be brought to the study of unipotent series, and use this to extend our results on extremal representations to arbitrary (i.e. not necessarily unipotent) irreducible representations π of G.

Keywords

Howe correspondence, Harish-Chandra series, Lusztig series.

Représentations extrémales pour la correspondance de Howe sur des corps finis

Résumé

On étudie la correspondance de Howe $\Theta : \mathscr{R}(G) \to \mathscr{R}(G')$, pour des paires duales irréductibles $(G, G') = (\mathrm{U}_m(\mathbb{F}_q), \mathrm{U}_{m'}(\mathbb{F}_q))$ et $(G, G') = (\mathrm{Sp}_{2m}(\mathbb{F}_q), \mathrm{O}^{\epsilon}_{2m'}(\mathbb{F}_q))$, où $\mathbb{F}_q$ désigne le corps fini à q éléments (q impair). On établit la compatibilité entre la correspondance de Howe et les séries arbitraires de Harish-Chandra. On définit et montre l'existence de sous-représentations extrémales (i.e. minimales et maximales) de l'image $\Theta(\pi)$ d'une représentation irréductible unipotente de π de G. Finalement, on démontre comment l'étude de la correspondance de Howe entre séries d'Harish-Chandra arbitraires peut être ramenée à l'étude des séries unipotentes, et on utilise ceci pour étendre nos résultats sur les représentations extrémales aux représentations irréductibles arbitraires (i.e. pas forcément unipotentes) π de G.

Mots-clés

Théorie de représentations, paires duales, séries d'Harish-Chandra, représentations de Weil, séries de Lusztig, correspondance de Howe.

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Chapter 1

Introduction

Let $\mathbb{F}_q$ be a finite field with q elements and odd characteristic. Denote the symplectic group $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ by $\mathrm{Sp}_{2n}(q)$. A pair $(G_m, G'_{m'})$ of reductive subgroups of $\mathrm{Sp}_{2n}(q)$, where each one is the centralizer of the other, is called a *reductive dual pair* (the indices m and m' refer to the *Witt indices* defined in Section 3.5). Roger Howe introduced in [20] a correspondence $\Theta_{m,m'} : \mathscr{R}(G_m) \to \mathscr{R}(G'_{m'})$ between the category of complex representations of these subgroups. It is obtained from a particular representation ω of $\mathrm{Sp}_{2n}(q)$, called the *Weil representation* (see Section 3.2). Indeed, the restriction $\omega_{m,m'}$ of ω to $G_m \cdot G'_{m'}$ decomposes as

$$\omega_{m,m'} = \sum \pi \otimes \Theta_{m,m'}(\pi),$$

where the sum is over the set of irreducible representations π of G_m. Extending by linearity to $\mathscr{R}(G_m)$, we obtain the so called *Howe correspondence*.

Our interest is in *irreducible dual pairs* (cf. [23]), because these pairs are the building blocks of all the others. Such a pair $(G_m, G'_{m'})$ in $\mathrm{Sp}_{2n}(q)$ can be either symplectic-orthogonal $(\mathrm{Sp}_{2m}(q), \mathrm{O}_{m'}(q))$, unitary $(\mathrm{U}_m(q), \mathrm{U}_{m'}(q))$ or linear $(\mathrm{GL}_m(q), \mathrm{GL}_{m'}(q))$, with $n = mm'$ in all cases.

For a fixed irreducible representation π of G_m, our main goal is to find certain extremal (i.e. minimal and maximal) representations in the set of irreducible components of $\Theta_{m,m'}(\pi)$, for unitary and symplectic-orthogonal pairs. The definition of extremal representation is canonical for unitary pairs, for symplectic-orthogonal pairs it is defined by means of the *Springer correspondence*. Our results generalize those found by Aubert, Kraśkiewicz, and Przebinda in [4]. We also explicit the effect of the Howe correspondence on Harish-Chandra series, this was done for series of unipotent representations in [5].

Chapter 2

In this chapter we define two cornerstone notions : *cuspidal* and *unipotent* representations. These are representations of the group of Frobenius-fixed (or *rational*) points of a

group $\mathbf{G}$ defined over $\mathbf{F}_q$. This group is denoted either G or $\mathbf{G}^F$, where F is the Frobenius morphism.

For a cuspidal representation ρ of $\mathbf{M}$ (see Definition 2.9), we define a *Harish-Chandra series* $\mathrm{Irr}(G, \rho)$, as the set of irreducible representations appearing in the parabolic induced representation $R_{\mathbf{M}}^{\mathbf{G}}(\rho)$. Harish-Chandra series, for different G-conjugacy classes of cuspidal pairs $(\mathbf{M}, \rho)$ of $\mathbf{G}$, provide a partition of the set of irreducible representations $\mathrm{Irr}(G)$ of the group G. Moreover, there is a one-to-one correspondence, that we call *Howlett-Lehrer bijection*, between a given Harish-Chandra series $\mathrm{Irr}(G, \rho)$ and the set of irreducible representations of a certain Weyl group $W_{\mathbf{G}}(\rho)$ (see Theorem 2.15). This bijection is crucial to our work.

The final section of this chapter introduces the *Deligne-Lusztig characters* $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, where $\mathbf{T}$ is a Frobenius-stable (or *rational*) maximal torus of $\mathbf{G}$, and θ is an irreducible representation of T. These furnish a partition of $\mathrm{Irr}(G)$, indexed by geometric conjugacy classes (s) in the Langlands dual $\mathbf{G}^*$ of $\mathbf{G}$. The blocks in this partition are called *Lusztig series* and denoted by $\mathscr{E}(G, (s))$. The chapter ends by showing how the trivial conjugacy class, $(s) = 1$ provides a "prototype" for all other Lusztig series, representations inside this particular series are called *unipotent*.

Results in this chapter point out to the importance of studying representations both unipotent and cuspidal.

Chapter 3

This chapter introduces all the ingredients necessary to define the Howe correspondence, it starts by defining and then classifying all *irreducible dual pairs* over finite fields. Then, we define the *Heisenberg group* (see Definition 3.2) H and endow it with an action of the symplectic group $\mathrm{Sp}_{2n}(q)$. By means of the Stone-von-Neumann theorem (see Theorem 3.3), this action yields an important representation of $\mathrm{Sp}_{2n}(q)$, called the *Weil representation* and denoted by ω. As mentioned above, the restriction of this representation to $G_m \cdot G'_{m'}$ allows us to define the *Howe correspondence* $\Theta_{m,m'}$.

The Stone-von-Neumann theorem (see Theorem 3.3) provides a representation, unique up to isomorphism, called *the Heisenberg representation*. This unicity implies that the interesting aspect of the representation theory of the Heisenberg group is not the set of isomorphism classes of irreducible representations, but rather the various objects in one such class and the isomorphisms among them. These objects are called *models* of the Heisenberg representation. We finish this chapter presenting the *mixed Schrödinger model* (Section 3.3).

Chapter 4

In [15], Gérardin introduced Weil representations $\omega^\flat$ of linear, symplectic and unitary groups over finite fields. For symplectic-orthogonal dual pairs, ω and $\omega^\flat$ have the same

restriction to $G_m \cdot G'_{m'}$; for linear and unitary pairs the restriction of $\omega^\flat$ to $G_m \cdot G'_{m'}$ is obtained from the restriction of ω multiplied by a certain representation of $G_m \cdot G'_{m'}$ with values in $\{\pm 1\}$. The study of the Howe correspondence can, therefore, be replaced by the study of the correspondence $\Theta^\flat_{m,m'} : \mathscr{R}(G_m) \to \mathscr{R}(G'_{m'})$ induced from $\omega^\flat$.

The main point in introducing the correspondence $\Theta^\flat_{m,m'}$ is that it respects unipotent representations (see Theorem 4.2) : if π belongs to $\mathscr{E}(G_m, (1))$ then $\Theta^\flat_{m,m'}(\pi)$ decomposes as sum of representations in $\mathscr{E}(G'_{m'}, (1))$.

Another important fact about the Howe correspondence is the compatibility with cuspidal representations (see Theorem 4.1) : for a cuspidal irreducible representation π of G_m there is a minimal m' such that $\Theta_{m,m'}(\pi)$ is not zero. Moreover, this representation is (irreducible and) cuspidal. The integer m' is known as the *first occurrence index*.

We are therefore brought to the study of representations both cuspidal and unipotent. Not many classical groups have such a representation, between those appearing in dual pairs we find the groups $\mathrm{GL}_1(q)$, $\mathrm{Sp}_{2k(k+1)}(q)$, $\mathrm{O}^\epsilon_{2k^2}(q)$ and $\mathrm{U}_{k(k+1)/2}(q)$ for a positive integer k and $\epsilon = \mathrm{sgn}(-1)^k$; these groups have only one cuspidal unipotent representation (trivial for linear groups) except for the orthogonal groups, which have two. Thus, we can associate to each cuspidal unipotent representation an integer k. This allows to see the correspondence $\Theta^\flat$ for unipotent cuspidal representations, as a function $\theta : \mathbb{N} \to \mathbb{N}$ (see Theorem 4.4).

Harish-Chandra series of cuspidal unipotent representations are therefore equal to $\mathrm{Irr}(G_m, \lambda \otimes 1)$, where λ is a cuspidal unipotent representation of a smaller group of same type as G_m. In [5] Aubert, Michel and Rouquier showed that $\Theta^\flat_{m,m'}$ maps this series into the set $\mathscr{R}(G'_{m'}, \lambda' \otimes 1)$ of representations spanned by $\mathrm{Irr}(G'_{m'}, \lambda' \otimes 1)$, where λ' is the first occurrence of λ. In other words, the Howe correspondence is compatible with Harish-Chandra series of unipotent representations.

Our first main result is a generalization of this fact. The starting point is the computation of coinvariants for the Weil representation (see Proposition 4.6):

Proposition. *Let $(G_m, G'_{m'})$ be a symplectic-orthogonal or unitary dual pair, M_k denote the Levi subgroup $\mathrm{GL}_k \times G_{m-k}$, and ${}^*R_k$ be the parabolic restriction functor from $G_m \times G'_{m'}$ to $M_k \times G'_{m'}$.*

There exists a $M_k \times G'_{m'}$ invariant filtration $0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_{r+1} = {}^*R_k(\omega^\flat_{m,m'})$, where $r = min\{k, m'\}$. Its successive quotients τ_{i+1}/τ_i verify

$$\tau_{i+1}/\tau_i \simeq \mathrm{Ind}^{M_k \times G'_{m'}}_{Q_{k-i} G_{m-k} \times P'_i} \mathbf{R}^{\mathrm{GL}_i} \otimes \omega^\flat_{m-k,m'-i},$$

where Q_{m-k} is the parabolic subgroup of GL_k consisting of upper triangular matrices $\left(\begin{smallmatrix} a_j & * \\ & a_{k-j} \end{smallmatrix}\right)$, and $\mathbf{R}^{\mathrm{GL}_i}$ is the natural representation of $\mathrm{GL}_i \times \mathrm{GL}_i$ on the space of complex functions defined on GL_i.

The proof of this result lies heavily on the mixed Schrödinger model. A direct conse-

quence is the following isomorphism :

$$^*R_k(\omega^\flat_{m,m'}) \simeq \bigoplus_{i=0}^{\min\{k,m'\}} \mathrm{Ind}_{Q_{k-i}G_{m-k}\times P'_i}^{M_k\times G'_{m'}} \mathbf{R}^{\mathrm{GL}_i} \otimes \omega^\flat_{m-k,m'-i}.$$

For a cuspidal representation ρ of G, let $\mathscr{R}(G,\rho)$ denote the subcategory of $\mathscr{R}(G)$ spanned by $\mathrm{Irr}(G,\rho)$. The compatibility between the Howe correspondence and arbitrary Harish-Chandra series follows from the previous proposition (see Theorem 4.15) :

Theorem A. *Let φ be an irreducible cuspidal representation of G_l. Suppose that its first occurrence index l' is not greater than m', and let φ' be the corresponding cuspidal irreducible representation of $G'_{l'}$. Then, the Howe correspondence $\Theta^\flat_{m,m'}$ sends $\mathrm{Irr}(G_m,\boldsymbol{\sigma}\otimes\varphi)$ to $\mathscr{R}(G'_{m'},\boldsymbol{\sigma}'\otimes\varphi')$, where $\boldsymbol{\sigma}'=\boldsymbol{\sigma}\otimes 1$ if $m'\geq l'+|\mathbf{s}|$, and $\boldsymbol{\sigma}=\boldsymbol{\sigma}'\otimes 1$ otherwise.*

Chapter 5

For type I dual pairs $(\mathrm{Sp}_{2m}(q),\mathrm{O}^\pm_{2m'}(q))$ and $(\mathrm{U}_m(q),\mathrm{U}_{m'}(q))$, the Howe correspondence between pairs of Harish-Chandra series of cuspidal unipotent representations can be seen as a correspondence between pairs of type $\mathbf{B}$ Weyl groups : $(\mathbf{B}_{m-k(k+1)},\mathbf{B}_{m'-\theta(k)(\theta(k)+1)})$ for symplectic-orthogonal pairs, and $(\mathbf{B}_{(1/2)(m-k(k+1)/2)},\mathbf{B}_{(1/2)(m'-\theta(k)(\theta(k)+1)/2)})$ for unitary pairs.

Call $(W_r,W_{r'})$ one of these pairs. In [5], Aubert, Michel and Rouquier find explicit representations $\Omega_{r,r'}$ (see Section 5.1) of $W_r\times W_{r'}$, that yield (conjecturally for symplectic orthogonal pairs) the Howe correspondence for these pairs of Weyl groups.

Take $\mathbf{G}$ to be a linear or unitary group. Let $C_{\mathbf{G}}(s)$ be the centralizer of the semisimple element s in $\mathbf{G}$ (see Proposition 5.4). This group decomposes as the direct product of certain reductive groups $\mathbf{G}_\lambda(s)$, for different eigenvalues λ of s. The group $\mathbf{G}_1(s)$ is a smaller group of the same kind as $\mathbf{G}$. Moreover, if $\mathbf{G}$ is linear (resp. unitary), then the group $\mathbf{G}_\#(s)=\prod_{\lambda\neq 1}\mathbf{G}_\lambda(s)$ is a product of linear (resp. linear or unitary) groups.

The "Jordan decomposition" theorem (see Theorem 2.29), together with this direct product decomposition of the centralizer $C_{\mathbf{G}}(s)$, yields a bijection $\Xi^{\mathbf{G}}_s$, between $\mathscr{E}(\mathbf{G}^F,(s))$ and $\mathscr{E}(\mathbf{G}_\#(s)^F,(1))\times\mathscr{E}(\mathbf{G}_1(s)^F,(1))$. We denote by $\pi_\#\otimes\pi_{(1)}$ the image of π under $\Xi^{\mathbf{G}}_s$.

Let $(G_m,G'_{m'})$ be an unitary dual pair and s_m be a semisimple rational element of $\mathbf{G}^*_m$. There is a semisimple geometric conjugacy class $(s'_{m'})$ in $\mathbf{G}'^*_{m'}$, such that the Howe correspondence sends representations in $\mathscr{E}(G_m,(s_m))$ to representations in $\mathscr{R}(G'_{m'},(s'_{m'}))$, the subcategory of $\mathscr{R}(G'_{m'})$ spanned by the Lusztig series $\mathrm{Irr}(G'_{m'},(s'_{m'})$. Moreover, in this situation s_m and $s'_{m'}$ can be obtained from "adding ones" to a common semisimple rational element s, whose eigenvalues are all different from 1, and belonging to a smaller group $\mathbf{G}$ (see Theorem 5.6).

The groups $(\mathbf{G}_m)_\#(s_m)$ and $(\mathbf{G}'_{m'})_\#(s'_{m'})$ are isomorphic to $\mathbf{G}_\#(s)$. Moreover, if the latter has rank l then the groups $(\mathbf{G}_m)_{(1)}(s_m)$ and $(\mathbf{G}'_{m'})_{(1)}(s'_{m'})$ are isomorphic to $\mathbf{G}_{m-l}$

and $\mathbf{G}'_{m'-l}$ respectively.

In Theorem 5.10, we use the Lusztig bijection to describe the Howe correspondence for the pair $(G_m, G'_{m'})$ in terms of the correspondence for a smaller dual pair (see Theorem 5.10):

Theorem B. *The following diagram is commutative :*

$$\begin{array}{ccc} \mathcal{E}(G_m,(s_m)) & \xrightarrow[\sim]{\Xi_{s_m}^{G_m}} & \mathcal{E}(\mathbf{G}_\#(s),(1)) \times \mathcal{E}(_{m-l},(1)) \\ \Big\downarrow \Theta_{m,m'} & & \Big\downarrow \mathrm{Id} \otimes \Theta_{m-l,m'-l} \\ \mathscr{R}(G'_{m'},(s'_{m'})) & \xrightarrow[\sim]{\Xi_{s'_{m'}}^{G'_{m'}}} & \mathscr{R}(\mathbf{G}_\#(s),(1)) \otimes \mathscr{R}(G'_{m'-l},(1)). \end{array}$$

.

Theorem A tells us that for a cuspidal pair $(\mathbf{L}, \rho)$ of $\mathbf{G}_m$, there is a unique cuspidal pair $(\mathbf{L}', \rho')$ of $\mathbf{G}'_{m'}$ such that $\Theta_{m,m'}$ send the series $\mathrm{Irr}(G_m, \rho)$ to $\mathscr{R}(G'_{m'}, \rho')$. In this chapter we prove that the cuspidal pairs $(\mathbf{L}_\#(s), \rho_\#)$ and $(\mathbf{L}'_\#(s'), \rho'_\#)$ are isomorphic. Moreover, if (s_m) and $(s'_{m'})$ are the geometric conjugacy classes of ρ and ρ' in $\mathbf{L}^*$ and $\mathbf{L}'^*$, the following holds (see Theorem 5.16) :

Theorem C. *The following diagram is commutative :*

$$\begin{array}{ccc} \mathrm{Irr}(G_m,\rho) & \xrightarrow[\Xi_s^{G_m}]{\sim} & \mathrm{Irr}(\mathbf{G}_\#(s),\rho_\#) \times \mathrm{Irr}(G_{m-l},\rho_{(1)}) \\ \Big\downarrow \Theta_{m,m'} & & \Big\downarrow \mathrm{Id} \otimes \Theta_{m-l,m'-l} \\ \mathscr{R}(G'_{m'},\rho') & \xrightarrow[\Xi_{s'}^{G_{m'}}]{\sim} & \mathscr{R}(\mathbf{G}_\#(s),\rho_\#) \otimes \mathscr{R}(G'_{m'-l},\rho'_{(1)}). \end{array}$$

Let $\mathbf{R}^G$ the natural representation of $G \times G$ on $\mathscr{L}(G)$ (see Section 2.1) and $\mathrm{pr}_\delta(\mathbf{R}^G)$ its projection onto $\mathrm{Irr}(G \times G, \delta \otimes \delta)$. These notations allow us to restate Theorem C (see Theorem 5.15) :

Theorem C'. *The projection $\omega_{m,m',\rho}$ of $\omega_{m,m'}$ onto $\mathscr{R}(G_m, \rho) \otimes \mathscr{R}(G'_{m'}, \rho')$ is identified with the representation $\mathrm{pr}_{\rho_\#}(\mathbf{R}^{G_\#(s)^F}) \otimes \omega_{m-l,m'-l,\rho_{(1)}}$ via the bijection*

$$\mathrm{Irr}(G_m \times G'_{m'}, \rho \otimes \rho') \simeq \mathrm{Irr}(C_{G_m}(s_m) \times C_{G'_{m'}}(s'_{m'}), \rho_u \otimes \rho'_u).$$

The Howlett-Lehrer bijection yields

$$\mathrm{Irr}(W_{\mathbf{G}_m}(\rho) \times W_{\mathbf{G}'_{m'}}(\rho')) \simeq \mathrm{Irr}(G_m \times G'_{m'}, \rho \otimes \rho').$$

Therefore, the representation $\omega_{m,m',\rho}$ induces a representation $\Omega_{m,m',\rho}$ of the direct product of Weyl groups $W_{\mathbf{G}_m}(\rho) \times W_{\mathbf{G}'_{m'}}(\rho')$. The main result of this chapter describes this representation in terms of a representation of a smaller pair of type B Weyl groups (see Theorem 5.17) :

Theorem D. *There is an bijection*

$$\mathrm{Irr}(W_{\mathbf{G}_m}(\rho) \times W_{\mathbf{G}'_{m'}}(\rho')) \simeq \mathrm{Irr}(W_{\mathbf{G}_{\#}(s)}(\rho_{\#}) \times W_{\mathbf{G}_{\#}(s)}(\rho_{\#})) \times \mathrm{Irr}(W_r, W_{r'}).$$

It identifies the representation $\Omega_{m,m',\rho}$ *with* $\mathbf{R}^{W_{\mathbf{G}_{\#}(s)}(\rho_{\#})} \otimes \Omega_{r,r'}$. *Moreover, there are isomorphisms of Weyl groups* $W_{\mathbf{G}_m}(\rho) \simeq W_{\mathbf{G}_{\#}(s)}(\rho_{\#}) \times W_r$, *and* $W_{\mathbf{G}'_{m'}}(\rho') \simeq W_{\mathbf{G}_{\#}(s)}(\rho_{\#}) \times W_{r'}$, *compatible with this bijection.*

This is an extension, to arbitrary Harish-Chandra series, of the main result in [5]. In this paper Aubert, Michel and Rouquier treated the case of unipotent Harish-Chandra series.

Chapter 6

This chapter presents the extremal representations for the Howe correspondence. We deal first with unipotent representations. In this case, the existence of such extremal representations is established by elementary calculations. Thanks to results in the previous chapter, the general case follows from the unipotent case.

Extremal unipotent representations. Let $(W_r, W_{r'})$ be a pair of type $\mathbf{B}$ Weyl groups as in the previous chapter. Irreducible representations of the Weyl group $\mathbf{B}_n$ are known to be parametrized by bipartitions of n (see Section 6.1), $\chi_{\xi,\eta}$ will denote the irreducible representation of W_n corresponding to the bipartition (ξ, η) of n. For a bipartition (ξ, η) of r, denote by $\Theta(\xi, \eta)$ the set of bipartitions of r' such that $\chi_{\xi,\eta} \otimes \chi_{\xi',\eta'}$ appears in $\Omega_{r,r'}$.

Symplectic-orthogonal pairs : Extremal representations for symplectic-orthogonal pairs are found by means of the *Springer correspondence.* This relates an irreducible representation of the Weyl group W_r of $\mathrm{Sp}_{2r}(\overline{\mathbb{F}}_q)$ to a pair $(\mathcal{O}, \rho)$ consisting of a unipotent conjugacy orbit $\mathcal{O}$ in $\mathrm{Sp}_{2r}(\overline{\mathbb{F}}_q)$ and an irreducible representation ρ of the group of connected components of the centralizer of any $u \in \mathcal{O}$. Unipotent conjugacy classes in $\mathrm{Sp}_{2r}(\overline{\mathbb{F}}_q)$ are parametrized by *symplectic* partitions of $2r$. Moreover, this parametrization identifies the *closure order* on unipotent conjugacy classes with the *natural order* on partitions (see Remark 6.5). Therefore, restricting our attention to unipotent orbits, we can obtain a map sending bipartitions (ξ, η) of r to symplectic partitions $\lambda(\xi, \eta)$ of $2r$.

For a fixed bipartition (ξ', η') of r', we define the *minimal* (resp. *maximal*) representation $(\xi_{\min}, \eta_{\min})$ (resp. $(\xi_{\max}, \eta_{\max})$) in $\Theta(\xi', \eta')$ as the one verifying $\lambda(\xi_{\min}, \eta_{\min}) \leq \lambda(\xi, \eta)$ (resp. $\lambda(\xi, \eta) \leq \lambda(\xi_{\max}, \eta_{\max})$) for all (ξ, η) in $\Theta(\xi', \eta')$. These amounts to ask for the minimal (resp. maximal) unipotent orbit given by the Springer correspondence. We refer to these representations as *extremal.*

Unitary pairs : Let λ_k be the unique cuspidal unipotent representation of $\mathrm{U}_{k(k+1)/2}(q)$. Representations in the Harish-Chandra series $\mathrm{Irr}(U_m(q), \lambda_k \otimes 1)$ are the virtual characters $R_\nu^{\mathrm{U}_m(q)}$ [12, Appendix, Proposition p.224] up to a sign. The partitions ν parametrizing

these characters are those having the partition $(k\ k-1\cdots 1)$ as 2-core and having as 2-quotients (of parameter 1) the irreducible representations of $\mathbf{B}_{(1/2)(m-k(k+1)/2)}$ obtained by the Howlett-Lehrer bijection between $\mathrm{Irr}(U_m(q), \lambda_k \otimes 1)$ and $\mathrm{Irr}(\mathbf{B}_{(1/2)(m-k(k+1)/2)})$ (cf. [13]). Denote by $\nu(\xi,\eta)$ the partition of m corresponding to the bipartition (ξ,η) of $r = \frac{1}{2}(m-k(k+1)/2)$.

For a fixed bipartition (ξ',η') of r', we define the *minimal* (resp. *maximal*) representation $(\xi'_{\min},\eta'_{\min})$ (resp. $(\xi'_{\max},\eta'_{\max})$) in $\Theta(\xi',\eta')$ as the one verifying $\nu(\xi'_{\min},\eta'_{\min}) \leq \nu(\xi,\eta)$ (resp. $\nu(\xi,\eta) \leq \nu(\xi'_{\max},\eta'_{\max})$) for all (ξ,η) in $\Theta(\xi',\eta')$.

In Theorems 6.9 to 6.21 we establish the following.

Theorem E. *Extremal representations exist in* $\Theta(\xi',\eta')$.

Extremal representations. Let π be an irreducible representation of G_m. The reduction to unipotent representations found in the previous chapter provides a bijection

$$\Theta_{m,m'}(\pi) \simeq \{\pi_\#\} \times \Theta_{m-l,m'-l}(\pi_{(1)}).$$

Hence, we can define the *minimal* (resp *maximal*) representation in $\Theta_{m,m'}(\pi)$ as the one corresponding to the minimal (resp. maximal) representation in $\Theta_{m-l,m'-l}(\pi_{(1)})$. Indeed, the latter is a series of unipotent representations (see Theorem 6.24).

Theorem F. *Let π be an irreducible representation of G_m. There exists a minimal (resp. maximal) irreducible representation $\pi_{\min}$ (resp. $\pi_{\max}$) in $\Theta_{m,m'}(\pi)$. It verifies $(\pi_{\min})_{(1)} = (\pi_{(1)})_{\min}$ (resp. $(\pi_{\max})_{(1)} = (\pi_{(1)})_{\max}$), where $(\pi_{(1)})_{\min}$ (resp. $(\pi_{(1)})_{\max}$) is the minimal (resp. maximal) representation in $\Theta_{m-l,m'-l}(\pi_{(1)})$.*

Chapter 7

In this chapter we discuss how our work could be extended to pairs $(\mathrm{Sp}_{2m}(q), \mathrm{O}_{2n+1}(q))$, containing odd orthogonal groups. Also, we point out to how it could be related to recent work of Gurevich and Howe [17], and of Atobe and Gan [3].

Chapter 2

Lusztig and Harish-Chandra theories

2.1 Some results on representation theory

In this section we state results to be used in the rest of this composition. Through all the statements G and N denote arbitrary finite groups and H denotes a subgroup of G. We will also denote by $\mathscr{R}(G)$ the category of complex representations of G. The proofs of results in this section are straightforward and therefore omitted.

Let G act on a set X and (φ, V) be a representation of G. Take $x \in X$, denote by $G \cdot x$ its orbit, by $\mathrm{St}(x)$ its stabilizer and by $\mathscr{S}(Z, V)$ the vector space of functions defined on $Z \subset X$ with values in V.

Lemma 2.1. *Keep the above notations. The representation φ and the action of G on X induce a linear G action on $\mathscr{S}(G \cdot x, V)$. Moreover, we have a G-isomorphism,*

$$\mathscr{S}(G \cdot x, V) \simeq \mathrm{Ind}_{\mathrm{St}(x)}^{G} \varphi.$$

Suppose now that G acts on N. Both groups are embedded in the semidirect product $G \ltimes N$. An element $g \in G$ (resp. $n \in N$) will be denoted by the same letter when regarded as contained in $G \ltimes N$.

Lemma 2.2. *Let φ be a representation of $G \ltimes N$. Then,*

$$\varphi(g)\varphi(n)\varphi(g^{-1}) = \varphi(g \cdot n).$$

In other words, giving a representation of a semidirect product $G \ltimes N$ amounts to give two representations, ω of G and ρ of N, such that

$$\omega(g)\rho(n)\omega(g^{-1}) = \rho(g \cdot n).$$

Lemma 2.3. *Let φ be a representation of $H \ltimes N$.*

a) The formula $\phi(f)(g,n) = \varphi(g \cdot n)f(g)$ defines a G-isomorphism,

$$\phi : \mathrm{Ind}_H^G \varphi|_H \simeq (\mathrm{Ind}_{H\ltimes N}^{G\ltimes N} \varphi)|_G.$$

b) The restriction induces a G-isomorphism,

$$(\mathrm{Ind}_{H\ltimes N}^{G\ltimes N} \varphi)_N \simeq \mathrm{Ind}_H^G(\varphi)_N.$$

Lemma 2.4. *Let χ be a character of G. We have a G-isomorphism*

$$\chi \otimes \mathrm{Ind}_H^G \varphi \simeq \mathrm{Ind}_H^G(\chi|_H \otimes \varphi)$$

We denote by $\mathbf{R}^G$, the natural representation of the group $G \times G$ on the space $\mathscr{L}(G)$, of functions defined on G with values in $\mathbb{C}$. That is, for f belonging to $\mathscr{L}(G)$:

$$(\mathbf{R}^G(x,z)f)(y) = f(xyz^{-1}),\ \ x,y,z \in G.$$

For representations φ and π of G,

$$\mathrm{Hom}_{G\times G}(\mathbf{R}^G, \varphi \otimes \rho) \simeq \mathrm{Hom}_G(\varphi, \rho).$$

The representation $\mathbf{R}^G$ is characterized by the above bijection. Indeed, it implies that

$$\mathbf{R}^G = \sum_{\pi \in \mathrm{Irr}(G)} \pi \otimes \pi.$$

Finally, for two representations φ and ρ of G, denote

$$\langle \varphi, \rho \rangle_G = \dim \mathrm{Hom}_G(\varphi, \rho).$$

2.2 Harish-Chandra theory

In this section $\mathbf{G}$ will denote a reductive algebraic group defined over $\mathbb{F}_q$ with Frobenius morphism F. The (finite) group $\mathbf{G}^F$ of elements of $\mathbf{G}$, fixed by the Frobenius, is also denoted by G, and its element are called *rational points* of $\mathbf{G}$. Finite groups arising in this manner are known as *finite groups of Lie type*. A subgroup $\mathbf{H}$ of $\mathbf{G}$ stable by F will be called *rational*, and its group of rational points denoted by $\mathbf{H}^F$ or H. In this section we deal with both connected and disconnected groups.

Harish-Chandra theory provides a way to further our understanding of the irreducible representations of finite groups of Lie type. Because all the groups appearing in this section are rational, we will omit this assumption.

Parabolic and Levi subgroups for connected groups are well known. For disconnected groups we have the following definition.

Definition 2.5. *A* parabolic subgroup $\mathbf{P}$ *of* $\mathbf{G}$ *is defined as the normalizer in* $\mathbf{G}$ *of a parabolic subgroup* $\mathbf{P}^\circ$ *of the connected component* $\mathbf{G}^\circ$ *of* $\mathbf{G}$. *A* Levi subgroup *of* $\mathbf{P}$ *is the normalizer in* $\mathbf{G}$ *of the couple* $\mathbf{M}^\circ \subset \mathbf{P}^\circ$ *where* $\mathbf{M}^\circ$ *is a Levi subgroup of* $\mathbf{P}^\circ$.

The following proposition applies to both connected and disconnected groups.

Proposition 2.6. *[10, Proposition 4.1]*

1. *Let* $\mathbf{Q} \subset \mathbf{P}$ *be two parabolic groups. If* $\mathbf{M}$ *is a Levi subgroup of* $\mathbf{Q}$, *there exists a unique Levi subgroup* $\mathbf{L}$ *of* $\mathbf{P}$ *containing it.*

2. *For* $\mathbf{L}$ *and* $\mathbf{P}$ *before, the following are equivalent :*

   a) $\mathbf{M}$ *is a Levi subgroup of a parabolic subgroup of* $\mathbf{L}$.

   b) $\mathbf{M}$ *is a Levi subgroup of a parabolic subgroup of* $\mathbf{G}$, *and* $\mathbf{M} \subset \mathbf{L}$.

If $\mathbf{L}$ is a Levi subgroup of a parabolic subgroup of the group $\mathbf{G}$, and we do not need to specify what the parabolic is, then we will say that $\mathbf{L}$ is a Levi of $\mathbf{G}$.

Let $\mathbf{M}$ be a Levi subgroup and $\mathbf{N}$ the unipotent radical of the parabolic $\mathbf{P}$ of the group $\mathbf{G}$. Let δ be a representation of the group M of rational points of $\mathbf{M}$. The canonical isomorphism $\mathbf{P}/\mathbf{N} = \mathbf{M}$ allows us to lift δ to a representation of P, that we denote by the same letter. Setting

$$R_{\mathbf{M}}^{\mathbf{G}}(\delta) := \mathrm{Ind}_P^G \, \delta,$$

where Ind denotes classical induction, we obtain a representation of G which is known as the *parabolic induction* of δ.

Let now (π, V) be a representation of G and $V(N)$ the subspace generated by $\pi(n)v - v$, with $n \in N$ and $v \in V$. The fact that M normalizes N, implies that $V(N)$ is stable by the former. We obtain in this way a representation of M in $V/V(N)$, known as the *parabolic restriction* of π, and denoted by ${}^*R_{\mathbf{M}}^{\mathbf{G}}(\pi)$.

The relation between the two functors just defined is similar to that between classic induction and restriction. For example, we have the Frobenius reciprocity :

$$\mathrm{Hom}_M({}^*R_{\mathbf{M}}^{\mathbf{G}}(\pi), \delta) = \mathrm{Hom}_G(\pi, R_{\mathbf{M}}^{\mathbf{G}}(\delta)).$$

The parabolic subgroup used in the constructions above does not appear in the notation since these functors do not depend upon the choice of a parabolic containing our Levi. This is a consequence of the Mackey formula whose proof can be found in [10, Proposition 6.1].

We next talk about transitivity. this is a crucial property of parabolic induction and restriction.

Proposition 2.7. *[10, Proposition 4.7] Let* $\mathbf{Q} \subset \mathbf{P}$ *two parabolic subgroups of* $\mathbf{G}$ *and* $\mathbf{M} \subset \mathbf{L}$ *two corresponding Levi subgroups. Then*

$$R_{\mathbf{L}}^{\mathbf{G}} \circ R_{\mathbf{M}}^{\mathbf{L}} = R_{\mathbf{M}}^{\mathbf{G}}$$

Using Frobenius reciprocity, we conclude that a similar result holds for parabolic restriction.

We note that thanks to Proposition 2.6, $\mathbf{M}$ is actually a Levi subgroup of $\mathbf{L}$ so that it is congruent to speak of parabolic induction from M to L.

Proposition 2.8. *[9, Proposition 3.8] Let* χ *be a linear character of* G*, trivial on unipotent elements, and* φ *be a representation of* G*. Then*

$${}^*R_{\mathbf{M}}^{\mathbf{G}}(\chi \otimes \varphi) \simeq \chi|_M \otimes {}^*R_{\mathbf{M}}^{\mathbf{G}}\varphi.$$

Definition 2.9. *Let* $\mathbf{L}$ *be a Levi subgroup of the group* $\mathbf{G}$ *and* δ *an irreducible representation of* L*. We say that* δ *is* cuspidal *(or that the pair* $(\mathbf{L}, \delta)$ *is cuspidal) if* ${}^*R_{\mathbf{M}}^{\mathbf{L}}(\delta) = 0$ *for all proper Levi subgroups* $\mathbf{M}$ *of* $\mathbf{L}$*.*

Frobenius reciprocity implies that δ is cuspidal if and only if for all Levi subgroups $\mathbf{M}$ of $\mathbf{L}$ and all irreducible representations ρ of $\mathbf{M}$,

$$\mathrm{Hom}_L(\delta, R_{\mathbf{M}}^{\mathbf{L}}(\rho)) = 0.$$

Let x be an element of $\mathbf{G}$. Denote by ${}^xM$ the Levi subgroup xMx^{-1} and by ${}^x\delta$ the representation of ${}^xM$ defined by ${}^x\delta(m) = \delta(x^{-1}mx)$. It is not hard to see that, if two pairs $(\mathbf{M}, \delta)$ and $(\mathbf{M}', \delta')$ are G-conjugated (that is, if there exists $x \in G$ such that $M' = {}^xM$ and $\delta' = {}^x\delta$), then $R_{\mathbf{M}}^{\mathbf{G}}(\delta)$ and $R_{\mathbf{M}'}^{\mathbf{G}}(\delta')$ are isomorphic representations.

Theorem 2.10. *Let* χ *be an irreducible representation of* G*. Then, up to rational conjugation, there exists a unique cuspidal pair* $(\mathbf{L}, \delta)$ *for which* $\mathrm{Hom}_G(\chi, R_{\mathbf{L}}^{\mathbf{G}}(\delta))$ *is not trivial.*

This theorem gives us a partition of $\mathrm{Irr}(G)$ in series parametrized by cuspidal pairs $(\mathbf{M}, \delta)$. The set of irreducible representations of G appearing in $R_{\mathbf{M}}^{\mathbf{G}}(\delta)$ is called the *Harish-Chandra series* of $(\mathbf{M}, \delta)$, and is denoted by $\mathrm{Irr}(\mathbf{G}, \delta)$. The set of representations of G spanned by this series will be denoted by $\mathscr{R}(G, \delta)$.

Let ${}^Fx$ denote the image of $x \in \mathbf{G}$ by the Frobenius morphism.

Proposition 2.11. *[31] Let* $\mathbf{G}$ *be a connected group defined over* $\mathbb{F}_q$*, and let* F *be its Frobenius morphism. The map* $\mathscr{L}$*, sending* $x \in \mathbf{G}$ *to* $x^{-1}{}^Fx \in \mathbf{G}$*, is surjective.*

This map is known as the *Lang map*, and the result actually holds if we replace the Frobenius by any surjective endomorphism of $\mathbf{G}$ with a finite number of fixed points.

Let again $\mathbf{P}$ denote a parabolic subgroup of $\mathbf{G}$. Using Proposition 2.11 one can show that $\mathbf{P}$ contains a rational Borel subgroup of $\mathbf{G}$. Since all rational Borel subgroups in $\mathbf{G}$ are conjugated by rational elements, we have the following result.

Proposition 2.12. *Let $\mathbf{B}$ be a fixed rational Borel subgroup of $\mathbf{G}$. Then every rational parabolic subgroup is conjugated, by a rational element of $\mathbf{G}$, to a parabolic containing $\mathbf{B}$.*

For classical groups, we usually fix the Borel subgroup of upper triangular matrices and call *standard parabolics* all the parabolic subgroups containing it. Rational Levi subgroups of standard parabolic are called *standard Levi*.

Corollary 2.13. *Harish-Chandra series of a classical groups can be indexed by classes of cuspidal pairs $(\mathbf{M}, \rho)$ where $\mathbf{M}$ is a standard Levi.*

Example 2.14. *Let $\mathbf{G}$ be equal to $\mathbf{GL}_n$, and fix the Borel group consisting of upper diagonal matrices. Standard parabolic are indexed by partitions $\mathbf{t} = (t_1, \ldots, t_k)$ of n. Thus, we can take the standard Levi $\mathbf{M}$ to be $\mathbf{M} = \mathbf{GL}_{t_1} \times \ldots \times \mathbf{GL}_{t_k}$, for different partitions $\mathbf{t}$ of n. The situation is similar for the other classical groups.*

Harish-Chandra series are in turn parametrized by irreducible representations of certain Hecke algebras. For a cuspidal representation δ of M we put

$$W_{\mathbf{G}}(\delta) = \{x \in N_G(\mathbf{M})/M : {}^x\delta = \delta\}.$$

Theorem 2.15. *[21, Corollary 5.4] There is an isomorphism*

$$\mathrm{End}_G(R_{\mathbf{M}}^{\mathbf{G}}(\delta)) \simeq \mathbb{C}[W_{\mathbf{G}}(\delta)].$$

In particular, the set of irreducible components of $R_{\mathbf{M}}^{\mathbf{G}}(\delta)$ is in bijective correspondence with the irreducible representations of $W_{\mathbf{G}}(\delta)$.

In other words, the irreducible representations in the Harish-Chandra series of $(\mathbf{M}, \delta)$ are parametrized by characters of $W_{\mathbf{G}}(\delta)$. We will sometimes refer to this parametrization as the *Howlett-Lehrer bijection*.

Definitions and results above extend to disconnected groups (keeping the same statements). The relation between parabolic induction and restriction for connected and disconnected groups is presented in the following lemma.

Lemma 2.16. *[6, Proposition 10.10] Let $\mathbf{G}$ be a disconnected group, $\mathbf{G}^\circ$ its identity component, a Levi $\mathbf{M}^\circ \subset \mathbf{P}^\circ$ of $\mathbf{G}^\circ$ and $\mathbf{M} \subset \mathbf{P}$ the corresponding Levi of $\mathbf{G}$. Then*

$$\begin{aligned} {}^*R_{\mathbf{M}^\circ}^{\mathbf{G}^\circ} \circ \mathrm{Res}_{G^\circ}^{G} &= \mathrm{Res}_{M^\circ}^{M} \circ {}^*R_{\mathbf{M}}^{\mathbf{G}} \\ {}^*R_{\mathbf{M}}^{\mathbf{G}} \circ \mathrm{Ind}_{G^\circ}^{G} &= \mathrm{Ind}_{M^\circ}^{M} \circ {}^*R_{\mathbf{M}^\circ}^{\mathbf{G}^\circ}. \end{aligned}$$

We end this section showing the relation between the cuspidal representations of a disconnected group and those of its identity component. It is an easy corollary of the previous lemma.

Proposition 2.17. *An irreducible representation ψ of G is cuspidal if and only if it is an irreducible component of* $\mathrm{Ind}_{G^\circ}^{G}\,\psi^\circ$ *for a certain cuspidal irreducible representation ψ° of G°.*

2.3 Deligne-Lusztig theory

The definition of parabolic induction involves a rational Levi contained in a rational parabolic. Deligne and Lusztig extended this construction to the case where the rational Levi is not contained in any rational parabolic. This construction, when specialized to maximal tori will give us a decomposition of the category of irreducible representations of G, similar to that obtained from parabolic induction.

Let $\mathbf{M}$ be a rational Levi of a parabolic $\mathbf{P}$, and let $\mathbf{N}$ be the unipotent radical of $\mathbf{P}$. The group of fixed points G acts on the left on $\mathscr{L}^{-1}(\mathbf{N})$ while (as $\mathbf{M}$ normalizes $\mathbf{N}$) M acts on the right. This induces, for all integers k, a G-module-M structure on the vector spaces $H_c^k(\mathscr{L}^{-1}(\mathbf{N}))$ of l-adic cohomology with compact support (cf. [10, Chapter 10]). In this way see the *virtual vector space* $H_c^*(\mathscr{L}^{-1}(\mathbf{N})) = \sum_k (-1)^k H_c^k(\mathscr{L}^{-1}(\mathbf{N}))$ as a G-module-M.

Definition 2.18. *The functor $R_{\mathbf{M}}^{\mathbf{G}} : E \to H_c^*(\mathscr{L}^{-1}(\mathbf{N})) \otimes_{\mathbb{C}[\mathbf{M}]} E$ going from the category of M-modules to that of G-modules is know as the* Lusztig induction.

If $\mathbf{P}$ is rational, then Lusztig induction becomes parabolic induction. That is why we use the same notation for both inductions. Indeed, in this case the unipotent radical $\mathbf{N}$ is also rational, and the fact that x and ${}^F x$ have same class modulo $\mathbf{N}$ for any $x \in \mathscr{L}^{-1}(\mathbf{N})$, implies that the mapping $\mathscr{L}^{-1}(\mathbf{N}) \to \mathbf{G}/\mathbf{N}$ has image $(\mathbf{G}/\mathbf{U})^F \simeq G/N$. This morphism has fibers isomorphic to $\mathbf{N}$. Since G/N is finite, its cohomology groups are trivial except in degree zero, for which $H_c^0(G/N) \simeq \overline{\mathbb{Q}}_l[G/N]$. This implies [10, p. 81] that $H_c^*(\mathscr{L}^{-1}(\mathbf{N})) \simeq \overline{\mathbb{Q}}_l[G/N]$ as G-modules-L, whence the result.

For a representation ρ of M, the Lusztig induction $R_{\mathbf{M}}^{\mathbf{G}}(\rho)$ does not provide necessarily a representation of G. In fact, its associated character decomposes as sum of irreducible characters with coefficients in $\mathbb{Z}$, not necessarily positive. This kind of class functions are called *virtual representations*. A simple example will be provided below.

In order to get a partition of the set of irreducible representations of G, we need the Mackey formula to hold. It does if we restrict ourselves to maximal rational tori [10, Theorem 11.13]. Let $\mathbf{T}$ and $\mathbf{T}'$ be such tori. For two one-dimensional representations θ and θ' of T and T' :

$$\mathrm{Hom}_G(R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')) = 0$$

whenever $(\mathbf{T}, \theta)$ and $(\mathbf{T}', \theta')$ are not G-conjugated. Moreover, the virtual characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ are isomorphic whenever $(\mathbf{T}, \theta)$ and $(\mathbf{T}', \theta')$ are conjugated under G. The virtual characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ are called *Deligne-Lusztig characters*. These provide a set of orthogonal class functions, indexed by G-conjugacy classes of pairs $(\mathbf{T}, \theta)$. We are interested in increasing our understanding of these classes.

For a maximal torus $\mathbf{T}$, we define its *Weyl group* $W(\mathbf{T})$ by $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. Two elements x, y in W are said to be *F-conjugated* if there exists z in W, such that $x = zy^F z^{-1}$, this defines an equivalence relation in W. Classes for this relation are called *F-conjugacy classes.*

Any two rational maximal tori in $\mathbf{G}$ are conjugated in $\mathbf{G}$ but not necessarily by a rational element. Thus, in general, there can be multiple G-conjugacy classes of such tori. Consider a fixed maximal rational torus $\mathbf{T}$ and denote by $W(\mathbf{T})$ its Weyl group. Any other maximal torus of $\mathbf{G}$ will then have the form ${}^g\mathbf{T}$ for some $g \in \mathbf{G}$. The following proposition furnishes a parametrization of the set of classes of maximal tori.

Proposition 2.19. *[8, Section 3.3] With the notations above.*

a) The torus ${}^g\mathbf{T}$ is rational if and only if $g^{-1F}g$ belongs to the normalizer of $\mathbf{T}$ in $\mathbf{G}$.

b) If two maximal rational torus ${}^g\mathbf{T}$, and ${}^{g'}\mathbf{T}$ are G-conjugated then the $g^{-1F}g$ and $g'^{-1F}g'$ are F-conjugated in the Weyl group $W(\mathbf{T})$.

c) The map sending ${}^g\mathbf{T}$ to $g^{-1F}g$ defines a bijection between the G-classes of rational maximal tori of $\mathbf{G}$ and the F-conjugacy classes of $W(\mathbf{T})$.

Tori in a G-conjugacy class corresponding to the F-conjugacy class of $w \in W(\mathbf{T})$ will be said to be of *type w*.

Consider the case where $\mathbf{G}$ is $\mathbf{GL}_n$ and $\mathbf{T}$ is the torus (rational) of diagonal matrices. Its Weyl group W can be identified with the symmetric group $\mathfrak{S}_n$. Thus, the action of the standard Frobenius F (defined by $F(x) = x^q$) on W is trivial, and classes of rational tori are parametrized by conjugacy classes of $\mathfrak{S}_n$. For any $\sigma \in \mathfrak{S}_n$, let T_σ be a maximal torus of type σ with respect to T.

Characters of irreducible representations of GL_n in the Harish-Chandra series of $(\mathbf{T}, 1)$ are called *unipotent* (we provide a general definition below). These are in bijection with the irreducible characters of the symmetric group $\mathfrak{S}_n$, hence in bijection with the partitions of n. Let μ be a partition of n and denote the irreducible representation of $\mathfrak{S}_n$ by the same letter.

Proposition 2.20. *[10, Section 15.4]*

a) Let μ denote a partition of n. The characters of unipotent irreducible representations are given by

$$R_\mu = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mu(\sigma) R_{\mathbf{T}_\sigma}^{\mathbf{GL}_n}(1).$$

c) $R_{(n)} = 1_{\mathrm{GL}_n}$ *and* $R_{(1^n)} = \mathrm{St}_{\mathbf{GL}_n}$.

Example 2.21. *For* $\mathbf{G} = \mathbf{GL}_2$*, consider again the torus* $\mathbf{T}$ *of diagonal matrices. Its Weyl group* W *has two elements* $\{1, s\}$*, where* s *is the transposition* $(1\,2)$*. Thus, there are two classes of tori in* $\mathbf{GL}_2$*. From Proposition 2.20, the corresponding characters are* $R_{\mathbf{T}}^{\mathbf{GL}_2}(1) = 1_{\mathrm{GL}_2} + \mathrm{St}_{\mathbf{GL}_2}$ *and* $R_{\mathbf{T}_s}^{\mathbf{GL}_2}(1) = 1_{\mathrm{GL}_2} - \mathrm{St}_{\mathbf{GL}_2}$.

The previous example shows that two orthogonal Deligne-Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ may have common constituents. Thus, the partition of $\mathrm{Irr}(G)$ we intend to obtain cannot be indexed by the set of G-conjugacy classes of pairs $(\mathbf{T}, \theta)$. In order to get a disjoint union we need the weaker notion of geometric conjugacy class, it basically tells us that two pairs are conjugate up to scalar extension.

Let $ad(g)$ denote conjugation by g and $N_{F^n/F} : \mathbf{T}^{F^n} \to \mathbf{T}^F$ be the map given by $t \mapsto t^F t \cdots {}^{F^{n-1}}t$. This last generalizes the Frobenius morphism.

Definition 2.22. *Let* $\mathbf{T}$ *and* $\mathbf{T}'$ *be two rational maximal tori, and let* θ *and* θ' *be characters respectively of* T *and* T'*. We say that the pairs* $(\mathbf{T}, \theta)$ *and* $(\mathbf{T}', \theta')$ *are* geometrically conjugated *if there exists a positive integer* n *and* $g \in \mathbf{G}^{F^n}$ *such that* $\mathbf{T}' = {}^g\mathbf{T}$ *and* $\theta' \circ N_{F^n/F} = \theta \circ N_{F^n/F} \circ ad(g)$

Proposition 2.23. *[10, Proposition 13.3] Let* $\mathbf{T}$ *and* $\mathbf{T}'$ *be two rational maximal tori, and let* θ *and* θ' *be characters respectively of* T *and* T'*. If* $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ *and* $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ *share an irreducible constituent, then the pairs* $(\mathbf{T}, \theta)$ *and* $(\mathbf{T}', \theta')$ *are geometrically conjugated.*

We still need to show that any irreducible character of G appears in the induced representation $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ of some pair $(\mathbf{T}, \theta)$. We first give a definition.

Definition 2.24. *We call* uniform functions *the class functions of* G *that are linear combinations of Deligne-Lusztig characters.*

The character of the regular representation of G is a uniform function [10, Corollary 12.14]. Since every irreducible representation χ appears on the regular representation, there exists a pair $(\mathbf{T}, \theta)$ whose Deligne-Lusztig character $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ contains χ.

The previous observations and Proposition 2.23 tell us that, the set of Deligne-Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ indexed by $(\mathbf{T}, \theta)$ belonging to the same geometric conjugacy class, provide a partition of the set of irreducible representations of G.

The following proposition gives a parametrization of the set of geometric conjugacy classes.

Proposition 2.25. *[10, Proposition 13.12] Let* $(\mathbf{G}, F)$ *and* $(\mathbf{G}^*, F^*)$ *be dual to each other. Geometric conjugacy classes of pairs* $(\mathbf{T}, \theta)$ *in* $\mathbf{G}$ *are in bijection with* F^**-stable conjugacy classes of semi-simple elements of* $\mathbf{G}^*$.

We shall be concerned with the following classical groups $\mathbf{G}$ and their Frobenius morphisms F :

a) $\mathbf{G} = \mathbf{Sp}_{2n}$, $F = F_q$,

b) $\mathbf{G} = \mathbf{SO}_{2n}^{\pm}$, $F = F_q$,

c) $\mathbf{G} = \mathbf{SO}_{2n+1}$, $F = F_q$,

d) $\mathbf{G} = \mathbf{GL}_n$, $F = F_q$ or $F = {}^{\mathrm{t}}F_q{}^{-1}$

Their dual groups $\mathbf{G}^*$ and the isogenies F^* may be idenfitied respectively with :

a) $\mathbf{G} = \mathbf{SO}_{2n+1}$, $F = F_q$,

b) $\mathbf{G} = \mathbf{SO}_{2n}^{\pm}$, $F = F_q$,

c) $\mathbf{G} = \mathbf{Sp}_{2n}$, $F = F_q$,

d) $\mathbf{G} = \mathbf{GL}_n$, $F = F_q$ or $F = {}^{\mathrm{t}}F_q{}^{-1}$

Definition 2.26. *A* Lusztig series $\mathscr{E}(G,(s))$ *corresponding to the geometric conjugacy class* (s) *of a semi-simple rational element* $s \in \mathbf{G}^*$*, is the set of irreducible representations of* G *appearing in* $R_{\mathbf{T}}(\theta)$ *for* $(\mathbf{T},\theta)$ *belonging to the geometric conjugacy class associated to* (s) *by proposition 2.25.*

The paragraph preceding proposition 2.25 can be rewriten in terms of the last definition.

Proposition 2.27. *Lusztig series associated to different geometric conjugacy classes of rational semi-simple* $(s) \in \mathbf{G}^*$ *form a partition of the set of irreducible representations of* G*.*

We have seen that the $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ are parametrized by G-conjucagy classes of pairs $(\mathbf{T},\theta)$. Using the dual group, we can give another parametrization.

Proposition 2.28. *The* G*-conjugacy classes of pairs* $(\mathbf{T},\theta)$ *where* $\mathbf{T}$ *is a rational maximal torus of* $\mathbf{G}$ *and* θ *belongs to* $\mathrm{Irr}(T)$ *are in one-to-one correspondence with the* $\mathbf{G}^{*F^*}$*-conjugacy classes of pairs* $(\mathbf{T}^*,s)$ *where* s *is a semi-simple element of* $\mathbf{G}^{*F^*}$ *and* $\mathbf{T}^*$ *is a rational maximal torus containing* s*.*

Using this proposition, we will sometimes use the notation $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ for $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. Also, we put $\epsilon_{\mathbf{G}} = (-1)^{\mathbb{F}_q-\mathrm{rank}(\mathbf{G})}$ (cf. [10, Definition 8.3]).

The representations corresponding to the series $\mathscr{E}(G,1)$ of the trivial element in G^* are called *unipotent representations*. These irreducible representations are a prototype for the other ones, this is the main result of Lusztig's classification of characters of finite groups of Lie type (cf. [27]).

Theorem 2.29. *Let* $\mathbf{G}$ *be a connected reductive group with connected center, and let* s *be a semisimple rational element of* $\mathbf{G}^*$*. There is a bijection, denoted by* $\mathfrak{L}_s^G$*, from* $\mathscr{E}(G,(s))$ *to* $\mathscr{E}(C_{\mathbf{G}^*}(s)^{F^*},(1))$ *such that, extended by linearity to virtual characters, it sends* $\epsilon_{\mathbf{G}} R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ *to* $\epsilon_{C_{\mathbf{G}^*}(s)} R_{\mathbf{T}^*}^{C_{\mathbf{G}^*}(s)}(1)$*, for any maximal rational torus* $\mathbf{T}^*$ *of* $C_{\mathbf{G}^*}(s)$*.*

We ask for the center of $\mathbf{G}$ to be connected, because in that case all centralizers $C_{\mathbf{G}^*}(s)$ are connected (cf. [10, Lemma 13.14]). The bijection in the previous theorem is usually called *Lusztig bijection*. We will sometimes let π_u denote the image of π by $\mathfrak{L}_s^G$.

Remark 2.30. *Let $\mathscr{R}(G,(s))$ be the set of representations spanned by $\mathscr{E}(G,(s))$. Lusztig's bijection can be extended by linearity to this set. Moreover, this extension becomes an isometry, i.e.*

$$\langle \varphi, \rho \rangle_G = \langle \mathfrak{L}_s^G(\varphi), \mathfrak{L}_s^G(\rho) \rangle_{C_{\mathbf{G}^*}(s)^{F^*}},$$

for φ and ρ in $\mathscr{R}(G,(s))$.

We need to extend Theorem 2.29 to disconnected groups. Indeed, we will apply it to orthogonal groups $\mathrm{O}_{2n}(q)$. In order to do this we need first to extend previous definitions to this groups.

If $\mathbf{G}$ is a reductive disconnected group and $\mathbf{T}$ is a maximal rational torus (necessarily contained in $\mathbf{G}^\circ$), we set $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = \mathrm{Ind}_{G^\circ}^G R_{\mathbf{T}}^{\mathbf{G}^\circ}(\theta)$. This definition allows us to extend the notion of unipotent representation and uniform function (keeping the same definitions).

Proposition 2.31. *[5, Proposition 1.7] Theorem 2.29 holds for the groups $\mathbf{G} = \mathbf{G}^* = \mathrm{O}_{2l}$, $\mathbf{G}^F = \mathbf{G}^{*F^*} = \mathrm{O}_{2l}^\epsilon(q)$.*

Let $\mathbf{M}$ be a rational Levi contained in a rational parabolic. Let ρ be a cuspidal representation of the group of rational points M, (s) be the corresponding geometric conjugacy class in $\mathbf{M}^*$, and denote by ρ_u the image of ρ by $\mathfrak{L}_s^M$. Lusztig's bijection $\mathfrak{L}_s^G$ induces, by restriction, a one-to-one correspondence between $\mathrm{Irr}(G,\rho)$ and $\mathrm{Irr}(C_{\mathbf{G}^*}(s)^{F^*}, \rho_u)$. Thanks to Theorem 2.15, this yields a one-to-one correspondence between $\mathrm{Irr}(W_{\mathbf{G}}(\rho))$ and $\mathrm{Irr}(W_{C_{\mathbf{G}^*}(s)}(\rho_u))$.

Proposition 2.32. *[27, Chapter 8] Keep the notations as above. There is an isomorphism $W_{\mathbf{G}}(\rho) \simeq W_{C_{\mathbf{G}^*}(s)}(\rho_u)$, compatible with the one-to-one correspondence between $\mathrm{Irr}(W_{\mathbf{G}}(\rho))$ and $\mathrm{Irr}(W_{C_{\mathbf{G}^*}(s)}(\rho_u))$, induced by Lusztig's bijection.*

We finish this chapter with the following proposition. It concerns the $\mathbb{F}_q$-rank of rational Levi contained in rational parabolic. We provide the proof for lack of reference.

Proposition 2.33. *If $\mathbf{M}$ is a rational Levi contained in a rational parabolic subgroup of $\mathbf{G}$, then $\epsilon_{\mathbf{M}}$ is equal to $\epsilon_{\mathbf{G}}$.*

Proof. By definition, we need to prove the equality between the $\mathbb{F}_q$-rank of $\mathbf{M}$ and $\mathbf{G}$ (cf. [10, Definition 8.3]). This is defined as the $\mathbb{F}_q$-rank of a rational maximal torus contained in a rational Borel. Let $\mathbf{P}$ be the rational parabolic containing $\mathbf{M}$, and consider a rational maximal torus $\mathbf{T}$ of $\mathbf{M}$. We can choose a rational Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ contained in $\mathbf{P}$ and containing $\mathbf{T}$. Since it is also contained in the rational Borel subgroup $\mathbf{B} \cap \mathbf{M}$ of $\mathbf{M}$, the $\mathbb{F}_q$-rank of $\mathbf{M}$ and $\mathbf{G}$ are equal. $\square$

Chapter 3

Howe correspondence

3.1 Dual pairs

In this section we will present the reductive dual pairs over finite fields $\mathbb{F}_q$. We will suppose the characteristic p of the field to be odd.

Let W be a symplectic vector space over $\mathbb{F}_q$. The group of isometries for the non degenerate symplectic form over W is denoted by $\mathrm{Sp}(W)$, and called *symplectic group*. By choosing a suitable base we can consider the symplectic group as a group of matrices, in this situation we will also denote it by $\mathrm{Sp}_{2n}(q)$, where $\dim W = 2n$.

For a group G and a subgroup H, we let

$$C_G(H) = \{x \in G \mid xz = zx \text{ for all } z \in H\}$$

Definition 3.1. *A* reductive dual pair (G, G') *in* $\mathrm{Sp}(W)$ *is a pair of reductive subgroups* G *and* G' *of* $\mathrm{Sp}(W)$ *such that*

$$C_{\mathrm{Sp}(W)}(G) = G', \text{ and } C_{\mathrm{Sp}(W)}(G') = G.$$

We will usually omit the word reductive and call (G, G') *a dual pair.*

If $W = W_1 \oplus W_2$ is an orthogonal sum decomposition, and if (G_1, G_1') and (G_2, G_2') are dual pairs in $\mathrm{Sp}(W_1)$ and $\mathrm{Sp}(W_2)$ respectively, then $(G, G') = (G_1 \times G_2, G_1' \times G_2')$ is a dual pair in $\mathrm{Sp}(W)$. Such a pair is said to be *reducible*. A dual pair (G, G') which does not arise in this way is said to be *irreducible*. For example, if W is irreducible for the action of $G \cdot G'$, then the dual pair (G, G') is irreducible. Every dual pair can be written as a product of irreducible dual pairs. We present now the classification of irreducible dual pairs over finite fields.

(1) Let V_1 and V_2 be vector spaces over $\mathbb{F}_q$. Suppose V_1 has a symplectic form $\langle\ ,\ \rangle_1$ and V_2 has a quadratic form $\langle\ ,\ \rangle_2$. The group of isometries of the latter is called *orthogonal group* and denoted by $\mathrm{O}(V_2)$.

The $\mathbb{F}_q$ vector space $W = V_1 \otimes_{\mathbb{F}_q} V_2$ has a symplectic form defined by

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle = \langle u_1, v_1 \rangle_1 \langle u_2, v_2 \rangle_2.$$

We can see $\mathrm{Sp}(V_1)$ and $\mathrm{O}(V_2)$ as subgroups of $\mathrm{Sp}(W)$ via the natural map $\mathrm{Sp}(V_1) \times \mathrm{O}(V_2) \to \mathrm{Sp}(W)$. The pair $(\mathrm{Sp}(V_1), \mathrm{O}(V_2))$ so obtained is an irreducible dual pair, it is called *symplectic-orthogonal*.

(2) Consider the quadratic extension $\mathbb{F}_{q^2}$ of $\mathbb{F}_q$ and let F denote its Frobenius morphism. Let V_1 be a vector space over $\mathbb{F}_{q^2}$ with a non-degenerate skew-Hermitian form $\langle\ ,\ \rangle_1$, i.e. such that

$$\langle \alpha u, \beta v \rangle_1 = \alpha^F \beta \langle u, v \rangle_1, \text{ and } {}^F\langle u, v \rangle_1 = -\langle v, u \rangle_1.$$

The group $\mathrm{U}(V_1)$ of isometries of this form is called *unitary group*. Similarly, let V_2 be a $\mathbb{F}_{q^2}$-vector space with a Hermitian form $\langle\ ,\ \rangle_2$, i.e. such that

$$\langle \alpha u, \beta v \rangle_2 = \alpha^F \beta \langle u, v \rangle_2, \text{ and } {}^F\langle u, v \rangle_2 = \langle v, u \rangle_2,$$

and let $\mathrm{U}(V_2)$ be the corresponding unitary group.

Denote by W the $\mathbb{F}_q$-vector space underlying the $\mathbb{F}_{q^2}$-vector space $V_1 \otimes_{\mathbb{F}_{q^2}} V_2$. It has a symplectic form defined by

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle = \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\langle u_1, v_1 \rangle_1 {}^F\langle u_2, v_2 \rangle_2).$$

Again, via the natural map $\mathrm{U}(V_1) \times \mathrm{U}(V_2) \to \mathrm{Sp}(W)$, we can see $\mathrm{U}(V_1)$ and $\mathrm{U}(V_2)$ as subgroups of $\mathrm{Sp}(W)$. The irreducible dual pair $(\mathrm{U}(V_1), \mathrm{U}(V_2))$ obtained this way is called *unitary*.

Unitary and symplectic-orthogonal dual pairs are said to be of *type I*.

(3) Let V_1 and V_2 be vector spaces over $\mathbb{F}_q$. As for type I dual pairs we have a natural action of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ on $V = V_1 \otimes V_2$, and therefore an induced action on its dual V^*. By considering the diagonal action we get a map

$$\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \to \mathrm{GL}(W)$$

where $W = V \oplus V^*$. This last vector space can be given a symplectic form

$$\langle x + x^*, y + y^* \rangle = y^*(x) - x^*(y),$$

that makes $\mathrm{GL}(V_1)$ and $\mathrm{GL}(V_2)$ subgroups of $\mathrm{Sp}(W)$. Dual pairs $(\mathrm{GL}(V_1), \mathrm{GL}(V_2))$ arising this way are called *linear*. They are also said to be of *type II*.

It might seem contradictory that in (2) above, we called "unitary group" a group arising from a Hermitian or a skew-Hermitian form (a priori these need not be isomorphic). But

in fact, it is easy to provide an isomorphism between them.

3.2 Howe Correspondence

The Howe correspondence relates representations of the groups belonging to a dual pair. In order to introduce it we need to study the representation theory of the *Heisenberg group*.

Definition 3.2. *The* Heisenberg group *is the group with underlying set* $H(W) = \{(w,t) : w \in W, t \in \mathbb{F}_q\}$ *and product*

$$(w,t)\cdot(w',t') = (w+w', t+t'+\frac{1}{2}\langle w,w'\rangle)$$

The representation theory of the Heisenberg group is simple. Let us take an irreducible representation ρ of $H(W)$, Schur's lemma implies that its restriction to the center $Z \simeq \mathbb{F}_q$ of $H(W)$ equals $\psi_\rho \cdot 1$, where ψ_ρ is a character of $\mathbb{F}_q$.

If $\psi_\rho = 1$ then ρ factors to $H(W)/Z \simeq W$ which is abelian, so ρ is itself a character (has dimension one). The case of non-trivial central character is described by the following celebrated theorem [29]:

Theorem 3.3 (Stone-von-Neumann)**.** *For any non-trivial character ψ of Z there exists (up to equivalence) a unique irreducible representation ρ of $H(W)$ such that $\psi_\rho = \psi$.*

The representation ρ_ψ appearing in this theorem is known as the *Heisenberg representation*. It depends on ψ, so we denote it by ρ_ψ. The unicity up to isomorphim means that there are multiple realizations (or *models*) of the Heisenberg representation, below we present one of these.

The action of $\mathrm{Sp}(W)$ on W lifts to an action on $H(W)$ and hence to one on the set $\mathrm{Irr}(H)$ of equivalence classes of irreducible representations. Fix a character ψ of the $\mathbb{F}_q$. The action of $\mathrm{Sp}(W)$ on $H(W)$ fixes the elements of its center. Thus, ρ_ψ and $x\cdot\rho_\psi$ agree on Z, for any $x \in \mathrm{Sp}(W)$, and any irreducible representation ρ_ψ of $H(W)$. The unicity part in Theorem 3.3 implies that there is an operator $\omega_\psi(x)$ verifying

$$\rho_\psi(x\cdot w,t) = \omega_\psi(x)\rho_\psi(w,t)\omega_\psi(x)^{-1}.$$

Schur's lemma shows that ω_ψ is a projective representation of $\mathrm{Sp}(W)$, that is

$$\omega_\psi(xy) = \alpha(x,y)\omega_\psi(x)\omega_\psi(y),$$

for a certain complex two cocycle $\alpha(x,y)$. Since $H^2(\mathrm{Sp}(W),\mathbb{C}^\times) = 0$, this cocycle is a coboundary, so that $\alpha(x,y) = f(x)f(y)f(xy)^{-1}$ for a certain complex function f on $\mathrm{Sp}(W)$. Scaling ω_ψ by f gives us a true representation of $\mathrm{Sp}(W)$ (that we denote by the same letter). We call ω_ψ the *Weil representation* of $\mathrm{Sp}(W)$.

This representation depends on the character ψ first defined, but this dependence is weak. Indeed, there are only two possible oscillator representations. Let a belong to $\mathbb{F}_q^*$, and denote by ψ_a the character $\psi_a(z) = \psi(az)$.

Proposition 3.4. *[17, Proposition 2.3.3] Two Weil representations ω_ψ and ω'_ψ are isomorphic if and only if $\psi' = \psi_{s^2}$ for a certain $s \in \mathbb{F}_q^*$*

For an irreducible dual pair (G, G'), we introduced a natural map from $G \times G'$ to $\mathrm{Sp}(W)$. Pulling back the Weil representation by this map we get a representation $\omega_{G \cdot G'}$ of $G \times G'$. This representation decomposes as a sum :

$$\omega_{G \cdot G'} = \sum m_{\pi,\pi'} \pi \otimes \pi',$$

where the sum is over the set of irreducible representations π and π' of G and G' respectively. We can rearrange this sum in order to get

$$\omega_{G \cdot G'} = \sum_{\pi \in \mathrm{Irr}(G)} \pi \otimes \Theta(\pi),$$

where $\Theta(\pi) = \sum m_{\pi,\pi'} \pi'$ is a (not necessarily irreducible) representation of G. We obtain in this way a map Θ from the set of irreducible representations of G to the set of representations of G'. It is called the *Howe correspondence*.

3.3 Mixed Schrödinger model

In order to keep track of dimension, in this section we modify slightly the notations from previous sections.

Let W_n denote the symplectic space of dimension $2n$. Fix a character ψ of $\mathbb{F}_q$; from the Stone-von-Neumann theorem we deduced the existence of the Heisenberg representation ρ_n of the Heisenberg group $H(W_n)$, and the existence of the Weil representation ω_n (both depending on ψ) of the symplectic group $\mathrm{Sp}(W_n)$. They verified the intertwining relation

$$\omega_n(x)\rho_n(h)\omega_n(x^{-1}) = \rho_n(x \cdot h),$$

for $x \in \mathrm{Sp}(W_n)$ and $h \in H(W_n)$. This, according to Lemma 2.2, is the same as giving a representation of the semidirect product $\mathrm{Sp}(W_n) \ltimes H(W_n)$, whose restriction to $\mathrm{Sp}(W_n)$ and $H(W_n)$ is ω_n and ρ_n respectively.

If $W_n = V_1 \oplus V_2$ is an orthogonal sum of symplectic vector spaces, then there is a short exact sequence

$$1 \to \mathbb{F}_q \xrightarrow{i} H(V_1) \times H(V_2) \xrightarrow{j} H(W_n) \to 1,$$

where the arrows are defined by $i(t) = (t, -t)$ and $j((v_1, t_1), (v_2, t_2)) = (v_1 + v_2, t_1 + t_2)$. Let (ρ_1, V_1) and (ρ_2, V_2) be models of the Heisenberg representations of $H(V_1)$ and $H(V_2)$

respectively. The previous exact sequence allows us to show that $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$ is a model for the Heisenberg representation of $H(W_n)$. Moreover, the Weil representation ω_n of $\mathrm{Sp}(W_n)$ coincides with $\omega_1 \otimes \omega_2$ on its subgroup $\mathrm{Sp}(V_1) \times \mathrm{Sp}(V_2)$.

Let $\{e_1, \ldots, e_n, e'_n, \ldots, e'_1\}$ be a symplectic base of W_n. Let $k \leq n$ and X_k (resp. Y_k) be a totally isotropic subspace spanned by the k first (resp. last) vectors in this base. The non-degenerate pairing $X_k \times Y_k \to \mathbb{C}$ induced by the symplectic form, allows identifying Y_k with the dual space $\check{X}_k$ of X_k. Therefore, there is an *Witt decomposition* :

$$W_n \simeq (X_k \oplus \check{X}_k) \textcircled{\perp} W_{n-k}.$$

The Heisenberg representation of $H(X_k \oplus \check{X}_k)$ can be realized in the space $\mathscr{S}(\check{X}_k)$ of complex functions defined in $\check{X}_k$. Let (ρ_{n-k}, S_{n-k}) be a model of the Heisenberg representation of $H(W_{n-k})$. From the previous paragraph, we deduce that the tensor product $\mathscr{S}(\check{X}_k) \otimes S_{n-k}$ provides a model for the Heisenberg representation of $H(W_n)$, called *mixed Schrödinger model*. Moreover, this space can be identified with $\mathscr{S}(\check{X}_k, S_{n-k})$.

For convenience, an element $w \in W_n$ (resp. $t \in \mathbb{F}_q$) will be denoted by the same letter when regarded as an element of $H(W_n)$.

Proposition 3.5. *With the above notation, the action of the Heisenberg representation ρ_n in $\mathscr{S}(\check{X}_k, S_{n-k})$ is given by,*

$$\rho_n(w,t)f(\check{y}) = \psi(\langle \check{y}, x\rangle + \langle \check{x}, x\rangle/2 + t)\rho_{n-k}(w_{n-k})f(\check{x} + \check{y}),$$

where $w = x + w_{n-k} + \check{x}$, $x \in X_k$, $w_{n-k} \in W_{n-k}$ and $\check{x}, \check{y} \in \check{X}_k$.

The proof of this result is an easy consequence of Example I.4 in [30]. A straightforward corollary is the following.

Corollary 3.6. *If $x \in X_k$ then,*

$$\rho_n(x)f(\check{y}) = \psi(\langle \check{y}, x\rangle)f(\check{y}).$$

Explicit formulas for the action of the Weil representation ω_n on the vector space $\mathscr{S}(\check{X}_k, S_{n-k})$ are also known. Let P_k be the stabilizer of X_k in $\mathrm{Sp}(W_n)$. It is a maximal parabolic group with a Levi decomposition $P_k = M_k \ltimes N_k$. Its Levi subgroup M_k consist of matrices $m(a,u) = \mathrm{diag}(a, u, {}^{\mathrm{t}}a^{-1})$ for $a \in \mathrm{GL}_k$ and $u \in \mathrm{Sp}(W_{n-k})$. The unipotent radical N_k is the group of matrices

$$n(c,d) = \begin{pmatrix} 1 & c & d - c\,{}^{\mathrm{t}}c/2 \\ & 1 & -{}^{\mathrm{t}}c \\ & & 1 \end{pmatrix},$$

where d is a symmetric matrix.

Consider the representation χ of $P_k \ltimes H(X_k \oplus W_{n-k})$, trivial on its unipotent radical N_k, and defined by

$$\chi(m) = \det(a)^{(q-1)/2}\omega_{n-k}(u).$$

on elements $m = m(a, u)$ of the M_k, and by

$$\chi(x + w_0, t) = \rho_{n-k}(w_0, t)$$

on elements of the Heisenberg group. Theorem 2.4 of [15] implies that there is a unique representation of $\mathrm{Sp}(W_n) \ltimes H(W_n)$ whose restriction to $P_k \ltimes H(W_n)$ agrees with the induced representation of χ to this group. The space $\mathscr{S}(\check{X}_k, S_{n-k})$ can be identified with the vector space of this induced representation. The latter consists of functions $f : P_k \ltimes H(W_n) \to S_{n-k}$ verifying $f(\delta\gamma) = \chi(\delta)f(\gamma)$ for δ in $P(X_k) \ltimes H(X_k \otimes W_{n-k})$ and γ in $H(W_n)$. The group P_k acts by right translations on this function space, and so it also acts on $\mathscr{S}(\check{X}_k, S_{n-k})$. A straightforward computation shows that the former action is given by

$$\omega_n(m)f(\check{x}) = \det(a)^{(q-1)/2}\omega_{n-k}(u)f({}^{\mathrm{t}}a\check{x})$$

for elements $m = m(a, u)$ of the Levi subgroup M_k, and by

$$\omega_n(n)f(\check{x}) = \psi(\langle d\check{x}, \check{x}\rangle/2)\rho_{n-k}({}^{\mathrm{t}}c\check{x})f(\check{x}),$$

for elements $n = n(c, d)$ of the unipotent radical N_k.

3.4 Weil representations for unitary groups

Let ψ be a fixed nontrivial character of $\mathbb{F}_q$. For a dual pair (G, G'), the Howe correspondence, as it has been defined, arises from the restriction to $G \cdot G'$ of the Weil representation ω_ψ of the symplectic group $\mathrm{Sp}_{2n}(q)$. For unitary pairs, we will see (Section 4.2) that the Howe correspondence is not compatible with unipotent representations. This shortcoming can be fixed by studying the correspondence coming from a Weil representation introduced by Gérardin in [15].

Let V be a vector space over $\mathbb{F}_{q^2}$, of dimension m, and provided with a skew-hermitian form $[\ ,\ ]$. The underlying $\mathbb{F}_q$-vector space of V is denoted by W. It has a non-degenerate symplectic form $\langle\ ,\ \rangle = \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}[\ ,\ ]$. The unitary group $U(V)$ is canonically embedded in the symplectic group $\mathrm{Sp}(W)$.

Let $H(V)$ be the set of elements (w, t), with $w \in W$, $t \in \mathbb{F}_{q^2}$ such that $t - t^q = \langle w, w\rangle$. As before, it is called *the Heisenberg group*.

Lemma 3.7. *[15, Lemma 3.1] Let the notations be as above. Then,*

a) $H(V)$ is a group for the law given by :

$$(w,t)\cdot(w',t') = (w+w', t+t'+[w,w']).$$

Moreover, it has center isomorphic to $\mathbb{F}_q$.

b) The mapping sending (w,z) to $(w, z+[w,w]/2)$ defines an isomorphism of $H(V)$ onto $H(W)$.

Gérardin shows that, up to isomorphism, there is a unique representation $\rho^\flat_\psi$ of $H(V)$ given by ψ in the center $\mathbb{F}_q$ (cf. [15, Section 3.3]). As before, this is known as the *Heisenberg representation*.

Theorem 3.8. *[15, Theorem 3.3] Keep the above notations. Then :*

a) Any realisation $\rho^\flat_\psi$ of the Heisenberg representation can be extended to a representation of $U(V)\ltimes H(V)$. The representations $\omega^\flat_\psi$ of $U(V)$ so obtained are isomorphic.

b) Under the isomorphism of $H(V)$ and $H(W)$ given in Lemma 3.7,

$$\omega^\flat_\psi = \nu_m \otimes \omega_\psi, \quad \text{on } U(V).$$

where ν_m is the unique nontrivial one-dimensional real representation of $U(V)$, it is defined by $\nu_m(u) = (\det u)^{(q+1)/2}$ for all $u\in U(V)$.

For unitary pairs, we will use correspondence obtained, not from the Weil representation ω_ψ, but from the *Weil representation* $\omega^\flat_\psi$ introduced in the last theorem.

Let $(U(V_1), U(V_2))$ be such a pair, and let m_1 (resp. m_2) denote the dimension of V_1 (resp. V_2). In Section 3.1, we embedded these unitary groups in $\mathrm{Sp}(W)$, where W is the $\mathbb{F}_q$-vector space underlying the $\mathbb{F}_{q^2}$-vector space $V = V_1\otimes_{\mathbb{F}_{q^2}} V_2$. The latter can be equipped with the skew-hermitian form

$$\langle u_1\otimes u_2, v_1\otimes v_2\rangle = \langle u_1, v_1\rangle_1{}^F\langle u_2, v_2\rangle_2.$$

This form provides a unitary group $U(V)$, embedded in $\mathrm{Sp}(W)$. Moreover, the embedding of $U(V_1)$ and $U(V_2)$ in $\mathrm{Sp}(W)$ can be factored through $U(V)$. The identity

$$\det(x_1\otimes x_2) = (\det x_1)^{m_2}(\det x_2)^{m_1},\ \ x_1\in U(V_1),\ x_2\in U(V_2),$$

and item (b) in Theorem 3.8 imply that the restriction of $\omega^\flat_\psi$ to $U(V_1)\cdot U(V_2)$ equals the restriction of ω_ψ to $U(V_1)\cdot U(V_2)$ multiplied by $\nu_{m_1}^{m_2}\otimes\nu_{m_2}^{m_1}$. The standard Howe correspondence (i.e. the one coming from ω_ψ) can be, therefore, obtained from the *Howe correspondence* $\Theta^\flat$, induced by $\omega^\flat_\psi$.

3.5 Witt towers and stable range

A Witt towers

Some of the nicest properties of theta correspondence involve its compatibility with *Witt towers*. For groups belonging to type I dual pairs, these towers $\mathbf{T} = \{G_n\}_{n\in\mathbb{N}}$, are the following :

- For unitary groups there are two Witt towers, one whose groups are $G_n = U_{2n}(q)$ for $n \in \mathbb{N}$ and the other for groups $G_n = U_{2n+1}(q)$ for $n \in \mathbb{N}$. The first one will be denoted by $\mathbf{U}^+$ and the second one by $\mathbf{U}^-$.
- In the symplectic case there is only one Witt tower, formed by groups $G_n = \mathrm{Sp}_{2n}(q)$ for $n \in \mathbb{N}$. It will be denoted by $\mathbf{Sp}$.
- Even orthogonal groups provide two Witt towers whose groups are $G_n = \mathrm{O}^+_{2n}(q)$, and $G_n = \mathrm{O}^-_{2n}(q)$, for positive integers n. These will be denoted by $\mathbf{O}^+$ and $\mathbf{O}^-$ respectively.

Likewise, there are Witt towers of Hermitian, symplectic and quadratic spaces. The group G_m (resp. the space V_m) in the Witt tower $\mathbf{T}$ is said to have *Witt index* equal to m.

B Stable range

Now we discuss the behaviour of the theta correspondence as the groups $(G_m, G'_{m'})$ in a type I dual pair vary in their respective Witt towers.

Definition 3.9. *The pair $(G_m, G'_{m'})$ is in the* stable range (with G_m smaller) *if the defining module for $G'_{m'}$ has a totally isotropic subspace of the same dimension as the defining module of G_m.*

For instance, for pairs $(\mathrm{Sp}_{2m}(q), \mathrm{O}_{2m'}(q))$ the stable range condition (with $\mathrm{O}_{2m'}(q)$ smaller) means that $m > 2m'$.

Fix the group G_m and let $G'_{m'}$ vary on its Witt tower. The following proposition tells us that once we reach the stable range, every representation of G_m appears in the Howe correspondence.

Proposition 3.10. *[23, Propositions 4.3 and 4.5] If the dual pair $(G_m, G'_{m'})$ is in the stable range, then for every irreducible representation π of G_m, $\Theta(\pi) \neq 0$.*

Chapter 4

Correspondence between Harish-Chandra series

Throughout this chapter we fix a nontrivial character ψ of $\mathbb{F}_q$, and write ω (resp. $\omega^\flat$) instead of ω_ψ (resp. $\omega_p si^\flat$).

4.1 Compatibility with unipotent representations

The goal of this section is to show that the Howe correspondence behaves well with respect to Harish-Chandra series of cuspidal unipotent representations. In order for this to hold, we need to consider the correspondence coming, not from the Weil representation ω, but from the representations $\omega^\flat$ defined in [15]. In this paper Gérardin introduced Weil representations for symplectic, unitary and linear groups. In the previous chapter we studied the unitary case.

Let $(G_m, G'_{m'})$ be a type I dual pair. For symplectic-orthogonal pairs, ω and $\omega^\flat$ have same restriction to $G_m \cdot G'_{m'}$. For unitary pairs, we saw in Section 3.4 that these restrictions differ by multiplication by $\nu_m^{m'} \otimes \nu_{m'}^m$. The restriction of ω (resp. $\omega^\flat$) to $G_m \cdot G'_{m'}$ defines, by inflation, a representation of $G_m \times G'_{m'}$, that we denote by $\omega_{m,m'}$ (resp. $\omega^\flat_{m,m'}$). The theta correspondence obtained from this representation will be denoted by $\Theta_{m,m'}$ (resp. $\Theta^\flat_{m,m'}$).

Let $\mathbf{T}$ and $\mathbf{T}'$ be two Witt towers such that pairs $(G_m, G'_{m'})$ with $G_m \in \mathbf{T}$ and $G'_{m'} \in \mathbf{T}'$ form a dual pair of type I. That is, let $(\mathbf{T}, \mathbf{T}')$ be either $(\mathbf{Sp}, \mathbf{O}^\epsilon)$ or $(\mathbf{U}^\epsilon, \mathbf{U}^{\epsilon'})$, for ϵ and ϵ' equal to ± 1.

Theorem 4.1. *[2, Theorem 2.2] Suppose π is an irreducible cuspidal representation of G_m and*

1. *$\Theta_{m,m'}(\pi) = \pi'$ in the correspondence for $(G_m, G'_{m'})$*
2. *π does not occur in the correspondence for (G_m, G'_k) for $k < m'$, that is $\Theta_{m,k}(\pi) = 0$.*

Then π' is an irreducible cuspidal representation of G_m. The same holds for $\Theta^\flat_{m,m'}$.

The occurrence of a representation π of G_m in the Howe correspondence for $(G_m, G'_{m'})$ with m' minimal, that is $\Theta_{m,k}(\pi) = 0$ (or equivalently $\Theta^\flat_{m,k}(\pi) = 0$) for $k < m'$, is referred to as the *first occurrence*. The integer m' will be called *first occurrence index* for π. It is important to stress that, in this case, the representation $\Theta_{m,m'}(\pi)$ (resp. $\Theta_{m,m'}(\pi)^\flat$) is irreducible. As mentioned before, this is not always the case. For the first occurrence we denote $\Theta_{m,m'}(\pi)$ (resp. $\Theta_{m,m'}(\pi)^\flat$) by $\theta(\pi)$ (resp. $\theta^\flat(\pi)$).

Theorem 4.2. *[5, Proposition 2.3] Under the notation of Theorem 4.1, let π and π' be irreducible representations of G_m and $G'_{m'}$ respectively. Suppose the representation $\pi \otimes \pi'$ appears in the Weil representation $\omega^\flat_{m,m'}$ of the pair $(G_m, G'_{m'})$. Then π is unipotent if and only if π' is unipotent.*

Theorems 4.1 and 4.2 together imply that, in the case of first occurrence, for dual pairs of type I, the Howe correspondence $\omega^\flat_{m,m'}$ takes cuspidal unipotent representations to cuspidal unipotent representations.

There are very few groups, belonging to a dual pair, that have unipotent cuspidal representations. For instance, the only type II group having a cuspidal unipotent representation is $\mathrm{GL}_1(q)$, moreover this representation is the trivial one. Between type I groups we have the following result.

Theorem 4.3. *[25, Theorem 3.22] The following groups:*

1. $\mathrm{Sp}_{2n}(q)$, $n = k(k+1)$,
2. $\mathrm{SO}^\epsilon_{2n}(q)$, $n = k^2$, $\epsilon = \mathrm{sgn}(-1)^k$,
3. $\mathrm{U}_n(q)$, $n = (k^2+k)/2$

are the only groups in their respective Witt towers having a cuspidal unipotent representation. Moreover, in each case the group possesses a unique cuspidal unipotent representation denoted by λ_k.

Let ϵ be equal to ± 1. Thanks to Proposition 2.17, irreducible cuspidal (resp. unipotent) representations of $O^\epsilon_n(q)$ appear as constituents of $\mathrm{Ind}_{\mathrm{SO}^\epsilon_n(q)}^{O^\epsilon_n(q)}(\pi)$ where π is a cuspidal (resp. unipotent) representation of $\mathrm{SO}^\epsilon_n(q)$.

The representation $\mathrm{Ind}_{\mathrm{SO}^\epsilon_{2k^2}(q)}^{O^\epsilon_{2k^2}(q)}(\lambda_k)$ is not irreducible. Moreover, the unicity of λ_k implies that it decomposes as :

$$\mathrm{Ind}_{\mathrm{SO}^\epsilon_{2k^2}(q)}^{O^\epsilon_{2k^2}(q)}(\lambda_k) = \lambda_k^I \oplus \lambda_k^{II}$$

where λ_k^I and λ_k^{II} are the irreducible cuspidal unipotent representations of $\mathrm{O}^\epsilon_{2k^2}(q)$. Furthermore, these representations differ by tensoring with the sgn character of $\mathrm{O}^\epsilon_{2k^2}(q)$.

Theorem 4.3 gives us restrictions on the dimensions of type I groups which have cuspidal unipotent representations. The following theorem gives us the corresponding first occurrence indices.

Theorem 4.4. *[2, Theorems 4.1 and 5.2] The Howe correspondence $\Theta^\flat_{m,m'}$ for dual pairs $(\mathrm{U}_n(q), \mathrm{U}_m(q))$ and $(\mathrm{Sp}_{2m}(q), \mathrm{O}^\epsilon_{2n}(q))$ takes cuspidal unipotent representations to cuspidal unipotent representations as follows :*

- *For towers $(\mathbf{Sp}, \mathbf{O}^\epsilon)$, λ_k corresponds to λ^{II}_k if ϵ is the sign of $(-1)^k$ and to λ^I_{k+1} otherwise.*
- *For towers $(\mathbf{U}^\epsilon, \mathbf{U}^{\epsilon'})$, λ_k corresponds to $\lambda_{k'}$ where $k' = k+1$ or $k' = k-1$. We take k so that ϵ is the sign of $(-1)^{k(k+1)/2}$, and we choose k' such that $\epsilon' = (-1)^{k'(k'+1)/2}$.*

Moreover, these cases give the first occurrence $\theta^\flat(\lambda_k)$ of λ_k.

This theorem allows us to write the Howe correspondence $\Theta^\flat_{m,m'}$ between cuspidal unipotent representations as a function on natural integers $\theta : \mathbb{N} \to \mathbb{N}$, defined by $\theta(\lambda_k) = \lambda_{\theta(k)}$. In fact, it shows that for symplectic-orthogonal pairs, $\theta(k)$ is either k or $k+1$; whereas for unitary pairs, $\theta(k) = k \pm 1$.

Let $\mathbf{T} = \{G_n\}_{n\in\mathbb{N}}$ be a Witt tower. The maximal torus of G_k consisting of diagonal matrices will be denoted by T_k. Given two groups G_l and G_m in $\mathbf{T}$, such that $l < m$, we can include G_l inside the Levi subgroup $G_l \times T_{m-l}$ of G_m. Let λ be a cuspidal representation of G_l, and $\mathscr{R}(G_m, \lambda\otimes 1)$ the subset of $\mathscr{R}(G_m)$ whose elements are spanned by $\mathrm{Irr}(G_m, \lambda\otimes 1)$.

Let $\mathbf{T}$ and $\mathbf{T}'$ be two Witt towers such that pairs $(G_m, G'_{m'})$ with $G_m \in \mathbf{T}$ and $G'_{m'} \in \mathbf{T}'$ form a dual pairs of type I.

Theorem 4.5. *[5, Théorème 3.7] Let λ be a cuspidal representation of $G_l \in \mathbf{T}$, let l' be its first occurrence index and $\lambda' = \theta^\flat(\lambda)$ the corresponding cuspidal representation of $G'_{l'} \in \mathbf{T}'$. For $\gamma \in \mathrm{Irr}(G_m, \lambda\otimes 1)$, $\Theta^\flat_{m,m'}(\gamma) = 0$ whenever $m' < l'$ and $\Theta^\flat_{m,m'}(\gamma) \in \mathscr{R}(G'_{m'}, \lambda'\otimes 1)$ otherwise. Moreover, the representation λ of G_l is unipotent if and only if λ' is a unipotent representation of $G'_{l'}$.*

4.2 Howe correspondence and cuspidal support

Theorem 4.5 shows the behaviour of the Howe correspondence $\Theta^\flat_{m,m'}$ with respect to Harish-Chandra series of unipotent representations. We aim at generalizing this result to arbitrary Harish-Chandra series. In order to do so, we first construct a filtration of a certain space of coinvariants of the Weil representation $\omega_{m,m'}$. Next we will describe the successive quotients of this filtration. These coinvariant calculation relies strongly on the Schrödinger mixed model.

A Weil representation coinvariants

Let $\mathbf{T}$ and $\mathbf{T}'$ be two Witt towers such that pairs $(G_m, G'_{m'})$ with $G_m \in \mathbf{T}$ and $G'_{m'} \in \mathbf{T}'$ form a dual pairs of type I. Let D be a field equal to $\mathbb{F}_q$ when the pair is symplectic-orthogonal, and equal to $\mathbb{F}_{q^2}$ when the pair is unitary. Denote by N the norm of D over $\mathbb{F}_q$, it is the identity when $D = \mathbb{F}_q$ and it is defined by $N(x) = x^{q+1}$ when $D = \mathbb{F}_{q^2}$. Let W_m (resp. $W'_{m'}$) be the D-vector space of Witt index m (resp. m') on which G_m (resp. $G'_{m'}$) acts; denote by n (resp. n') its dimension.

As in Section 3.3, let X_k denote the totally isotropic subspace of W_m, spanned by the k first vectors of a hermitian base, $k \leq m$. Let P_k be the stabilizer of X_k in G_m. Denote by N_k its unipotent radical, $\mathrm{GL}_k = \mathrm{GL}(X_k)$ and $M_k = \mathrm{GL}_k \times G_{m-k}$ a Levi subgroup of P_k. Let Q_j be the stabilizer of X_j in GL_k, it is a parabolic group whose elements are matrices $\left(\begin{smallmatrix} a_j & * \\ & a_{k-j} \end{smallmatrix}\right)$. Finally, denote by $X'_{k'}$, $\mathrm{GL}'_{k'}$, $P'_{k'}$, $N'_{k'}$, $M'_{k'}$ and $Q'_{j'}$the analogous groups for $G'_{m'}$. It is important to stress that all these groups are groups of matrices with coefficients in $\mathbb{F}_q$ when the dual pair is symplectic-orthogonal, and with coefficients in $\mathbb{F}_{q^2}$ when the dual pair is unitary

Let $\xi_k = (\mathrm{N} \circ \det)^{(q-1)/2}$ be the unique non-trivial real one-dimensional representation of the general linear group GL_k with coefficients in D, that is $\xi_k = \det^{(q-1)/2}$ if $D = \mathbb{F}_q$, and $\xi_k = \det^{(q^2-1)/2}$ if $D = \mathbb{F}_{q^2}$. Finally, let $\mathbf{R}^G$ denote the natural representation of $G \times G$ on the space $\mathscr{S}(G)$ of complex functions defined on the group G.

Proposition 4.6. *Let ${}^*R_k$ and ${}^*R'_{k'}$ be the parabolic restriction functor from $G_m \times G'_{m'}$ to $M_k \times G'_{m'}$ and $G_m \times M'_{k'}$ respectively.*

a) *There exists a $M_k \times G'_{m'}$ invariant filtration $0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_{r+1} = {}^*R_k(\omega_{m,m'})$, where $r = min\{k, m'\}$. Its successive quotients τ_{i+1}/τ_i verify*

$$\tau_{i+1}/\tau_i \simeq \mathrm{Ind}_{Q_{k-i}G_{m-k} \times P'_i}^{M_k \times G'_{m'}} \xi_{k,i} \mathbf{R}^{\mathrm{GL}_i} \otimes \omega_{m-k,m'-i},$$

where $\xi_{k,i}$ is the character $\xi_{k-i}^{n'} \otimes \xi_i^n \otimes \xi_i^{n'}$ of $\mathrm{GL}_{k-i} \times \mathrm{GL}_i \times \mathrm{GL}'_i \subset Q_{k-i} \times P'_i$.

b) *Likewise, there is a $G_m \times M'_{k'}$ invariant filtration $0 = \tau'_0 \subset \tau'_1 \subset \cdots \subset \tau'_{r'+1} = {}^*R'_{k'}(\omega_{m,m'})$, where $r' = min\{k', m\}$. Its successive quotients τ'_{i+1}/τ'_i verify*

$$\tau'_{i+1}/\tau'_i \simeq \mathrm{Ind}_{P_i \times Q'_{k'-i}G'_{m'-k'}}^{G_m \times M'_{k'}} \xi'_{k',i} \mathbf{R}^{\mathrm{GL}_i} \otimes \omega_{m-i,m'-k'},$$

where $\xi'_{k',i}$ is the character $\xi_i^n \otimes \xi_{k'-i}^n \otimes \xi_i^{n'}$ of $\mathrm{GL}_i \times \mathrm{GL}'_{k'-i} \times \mathrm{GL}'_i \subset P_i \times Q'_{k'-i}$.

The proof of the previous proposition is long, so it is presented in a separate section. An easy consequence of it is the following.

Corollary 4.7. *a) The parabolic restriction* ${}^*R_k(\omega_{m,m'})$ *verifies :*

$$ {}^*R_k(\omega_{m,m'}) \simeq \bigoplus_{i=0}^{\min\{k,m'\}} \mathrm{Ind}_{Q_{k-i}G_{m-k}\times P'_i}^{M_k\times G'_{m'}} \xi_{k,i}\mathbf{R}^{\mathrm{GL}_i}\otimes\omega_{m-k,m'-i}. $$

b) Likewise, the parabolic restriction ${}^*R'_{k'}(\omega_{m,m'})$ *verifies*

$$ {}^*R'_{k'}(\omega_{m,m'}) \simeq \bigoplus_{i=0}^{\min\{k',m\}} \mathrm{Ind}_{P_i\times Q'_{k'-i}G'_{m'-k'}}^{G_m\times M'_{k'}} \xi'_{k',i}\mathbf{R}^{\mathrm{GL}_i}\otimes\omega_{m-i,m'-k'}. $$

B Proof of Proposition 4.6

We start by calculating the coinvariants relative to the group $N_{k,1}$ consisting of matrices $n(0,d)$. Then, we compute the coinvariants of the group $N_{k,2}$ consisting of matrices $n(c,0)$ (see Section 3.3 for notation). Due to the short exact sequence

$$ 1\to N_{k,1}\to N_k\to N_{k,2}\to 1, $$

performing these two computations is equivalent to calculating the coinvariants of N_k. In each calculation, we will use a mixed Schrödinger model because it allows us to express the Jacquet functor in terms of restriction of functions.

Coinvariants relative to $N_{k,1}$: Consider the Witt decomposition $W_m\simeq X_k\oplus W_{m-k}\oplus\check{X}_k$, where $\check{X}_k$ denotes the dual space of X_k. Tensoring by $W'_{m'}$ produces

$$ W_m\otimes W'_{m'}\simeq (X_k\otimes W'_{m'}\oplus\check{X}_k\otimes W'_{m'})\,\textcircled{\perp}\,W_{m-k}\otimes W'_{m'} $$

The mixed Schödinger model corresponding to this orthogonal sum makes $\mathrm{Sp}(W_m\otimes W'_{m'})$ act on $\mathscr{S}(\check{X}_k\otimes W'_{m'},S_{m-k,m'})$ where $S_{m-k,m'}$ is a model of the Heisenberg representation $\rho_{m-k,m'}$ of $H(W_{m-k}\otimes W'_{m'})$. The Weil representation makes the parabolic $P_k\times G'_{m'}$ of $G_m\times G'_{m'}$ act on $\mathscr{S}(\check{X}_k\otimes W'_{m'},S_{m-k,m'})$. For $f\in\mathscr{S}(\check{X}_k\otimes W'_{m'},S_{m-k,m'})$ and $\check{x}$ in $\check{X}_k\otimes W'_{m'}\simeq\mathrm{Hom}(X_k,W'_{m'})$:

∘ If $m=m(a,u)\in M_k$ and $g'\in G'_{m'}$.

$$ \omega(m,g')f(\check{x})=\xi_k(a)^{n'}\omega_{m-k,m'}(u,g')f(g'\check{x}a^{-1}). $$

∘ If $n=n(c,d)\in N_k$,

$$ \omega(n)f(\check{x})=\psi(\langle d\check{x},\check{x}\rangle/2)\rho_{m-k,m'}({}^{\mathrm{t}}c\check{x})f(\check{x}), $$

where $\langle\cdot,\cdot\rangle$ is the natural paring between $\check{X}_k\otimes W'_{m'}$ and its dual.

Therefore, the action of $n(0,d)\in N_{k,1}$ is given by

$$\omega(n(0,d))f(\check{x}) = \psi(\langle d\check{x}, \check{x}\rangle/2)f(\check{x}),$$

so the $f \in \mathscr{S}(\check{X}_k \otimes W'_{m'}, S_{m-k,m'})$ fixed by $N_{k,1}$ are those whose support is contained in $\mathscr{S}(Z, S_{m-k,m'})$ where Z is the subspace consisting of those $x \in \mathrm{Hom}(X_k, W'_{m'})$ whose image is totally isotropic. This implies the following.

Proposition 4.8. *Restriction to Z defines a $P_k \times G'_{m'}$ isomorphism between $\mathscr{S}(Z, S_{m-k,m'})$ and the space of coinvariants relative to $N_{k,1}$.*

The subspace Z is invariant under the action of $P_k \times G'_{m'}$. The orbits of this action are $Z_i = \{\check{x} \in Z | \,\mathrm{rk}(\check{x}) = i\}$ for $i = 0, \ldots, \min\{k, m'\}$. We denote $\min\{k, m'\}$ by r for short.

This orbit decomposition $Z = \cup Z_i$ provides the following finite $P_k \times G'_{m'}$ invariant filtration

$$\{0\} = \mathscr{S}_0 \subset \mathscr{S}_1 \subset \cdots \subset \mathscr{S}_{r+1} = \mathscr{S}(Z, S_{m-k,m'}),$$

where $\mathscr{S}_i = \{f \in \mathscr{S}(Z, S_{m-k,m'}) | f(\check{x})$ is trivial when $\mathrm{rk}(\check{x}) \geq i\}$. The subquotients of this filtration are $\mathscr{S}_{i+1}/\mathscr{S}_i \simeq \mathscr{S}(Z_i, S_{m-k,m'})$, for $i = 0, \ldots, r$.

For each i take $z_i \in Z_i$ and consider the representation $(\varphi_i, S_{m-k,m'})$ of $P_k \times G'_{m'}$ defined by :

$$\varphi_i(mn, g') = \xi_k(a)^{n'} \omega_{m-k,m'}(u, g') \rho_{m-k,m'}(\check{c}z_i)$$

for $g' \in G'_{m'}$, $n = n(c,d) \in N_k$ and $m = m(a,u) \in M_k$. The action of $M_k \times G'_{m'}$ via this representation and the natural action of $M_k \times G'_{m'}$ on Z_i induce a representation of $M_k \times G'_{m'}$ on $\mathscr{S}(Z_i, S_{m-k,m'})$. This representation is equal to the restriction of $\mathscr{S}_{i+1}/\mathscr{S}_i$ to $M_k \times G'_{m'}$. Hence, Lemma 2.1 provides a $M_k \times G'_{m'}$ isomorphism,

$$\mathscr{S}_{i+1}/\mathscr{S}_i \simeq \mathrm{Ind}_{\mathrm{St}(z_i)G_{m-k}}^{M_k \times G'_{m'}} \varphi_i.$$

where $\mathrm{St}(z_i)$ is the stabilizer of z_i in $\mathrm{GL}_k \times G'_{m'}$.

The group $P_k \times G'_{m'}$ can be written as a semidirect product $(M_k \times G'_{m'}) \ltimes N_k$. The stabilizer H_i of z_i inside of $P_k \times G'_{m'}$ can be also expressed as a semidirect product $H_i = \mathrm{St}(z_i)G_{m-k} \ltimes N_k$ so that Lemma 2.3 provides an isomorphism of $M_k \times G'_{m'}$-modules

$$\phi : \mathscr{S}_{i+1}/\mathscr{S}_i \simeq \mathrm{Ind}_{H_i}^{P_k \times G'_{m'}} \varphi_i.$$

That lemma also gives an explicit formula for ϕ which in this case becomes

$$\phi(f)(mn, g') = \varphi_i(mn, g') f(g' z_i a^{-1}),$$

where $m = m(a,u) \in M_k$, $g' \in G'_{m'}$ and $n \in N_k$. A direct calculation shows that ϕ is à fortiori a N_k-morphism and hence an isomorphism of $P_k \times G'_{m'}$ modules.

Finally, Lemma 2.3 gives us a $M_k \times G'_{m'}$ isomorphism

$$(\mathscr{S}_{i+1}/\mathscr{S}_i)_{N_k} \simeq \mathrm{Ind}_{\mathrm{St}(z_i)G_{m-k}}^{M_k \times G'_{m'}}(\varphi_i)_{N_k}. \tag{4.1}$$

To continue the proof, we need to compute the coinvariants of φ_i relative to N_k. Due to the fact that $N_{k,1}$ already acts trivially, this computation comes down to that of the coinvariants relative to $N_{k,2}$.

Coinvariants relative to $N_{k,2}$: Let X'_i be the image of z_i. Consider the Witt decomposition $W'_{m'} \simeq X'_i \oplus W'_{m'-i} \oplus \check{X}'_i$. Tensoring by W_{m-k} we obtain

$$W_{m-k} \otimes W'_{m'} \simeq (X'_i \otimes W_{m-k} \oplus \check{X}'_i \otimes W_{m-k}) \textcircled{\scriptsize$\perp$} W_{m-k} \otimes W'_{m'-i}. \tag{4.2}$$

The related mixed model yields a realization of φ_i in the vector space $S_{m-k,m'} \simeq \mathscr{S}(X'_i \otimes W_{m-k}, S_{m-k,m'-i})$, where $S_{m-k,m'-i}$ is a model of the Heisenberg representation of $H(W_{m-k} \otimes W'_{m'-i})$. The explicit action of N_k on this space is obtained thanks to Corollary 3.6 :

$$\varphi_i(n)f(\check{x}) = \rho_{m-k,m'}(\check{c}z_i)f(\check{x}) = \psi(\langle \check{c}z_i, \check{x}\rangle)f(\check{x}),$$

where $n = n(c,d) \in N_k$ and $\check{x} \in X'_i \otimes W_{m-k}$.

We conclude that the $f \in \mathscr{S}(X'_i \otimes W_{m-k}, S_{m-k,m'-i})$ invariant by N_k are those trivial on $X'_i \otimes W_{m-k} \setminus \{0\}$. This provides the following

Proposition 4.9. *The restriction to zero provides an isomorphism of $M_k \times G'_{m'}$ modules between $\mathscr{S}(0, S_{m-k,m'-i}) \simeq S_{m-k,m'-i}$ and the space $(S_{m-k,m'})_{N_k}$ of coinvariants of φ_i relative to N_k.*

It is easy to see that if (a, g') belongs $\mathrm{St}(z_i)$ then g' stabilizes X'_i. Hence, $\mathrm{St}(z_i)G_{m-k}$ is contained in $M_k \times P'_i$, where P'_i is the stabilizer of $X'_i \subset W'_{m'}$. The mixed model formulas for the orthogonal sum decomposition (4.2) allow us to compute the action of $G_{m-k} \times P'_i$ by $\omega_{m-k,m'}$ on $\mathscr{S}(W_{m-k} \otimes X'_i, S_{m-k,m'-i})$. From this we deduce the action of $\mathrm{St}(z_i)G_{m-k} \subset M_k \times P'_i$ by the representation φ_i on $S_{m-k,m'-i}$:

$$\varphi_i(m, m'n') = \xi_k(a)^{n'}\xi_i(a')^n\omega_{m-k,m'-i}(u, u'). \tag{4.3}$$

for $m = m(a,u) \in M_k$, $m' = m'(a', u') \in M'_i$ and $n' \in N'_i$. We remark that the action of the unipotent radical N'_i is trivial.

To finish our computation we need to explicit the elements of the stabilizer $\mathrm{St}(z_i)$ of z_i in $\mathrm{GL}_k \times G'_{m'}$. If $(a, g') \in \mathrm{St}(z_i)$ then a preserves the kernel X_{k-i} of z_i. Hence, $\mathrm{St}(z_i)$ is contained in the parabolic $Q_{k-i} \times P'_i$ of $\mathrm{GL}_k \times G'_{m'}$. Moreover, for the given basis,

$z_i = \left(\begin{smallmatrix} 0_{k-i,i} & 1_{i,i} \\ 0 & 0 \end{smallmatrix}\right)$, so (a, g') belongs to $\mathrm{St}(z_i)$, if and only if

$$a = \begin{pmatrix} a_{k-i} & * \\ & a_i \end{pmatrix}, \; g' = \begin{pmatrix} a_i & * & * \\ & u' & * \\ & & \check{a}_i^{-1} \end{pmatrix}.$$

Considering $a \in Q_{k-i}$ and $g' = m'n' \in P_i'$ (where $m' = m'(a_i, u')$) as in the previous line, the action (4.3) becomes

$$\varphi_i(m, m'n') = \xi_{k-i}(a_{k-i})^{n'} \xi_i(a_i)^{n'+n} \omega_{m-k,m'-i}(u, u'),$$

denoting $(\mathscr{S}_i)_{N_k}$ by τ_i, the isomorphism (4.1) turns into

$$\begin{aligned} \tau_{i+1}/\tau_i &\simeq \mathrm{Ind}_{\mathrm{St}(z_i)G_{m-k}}^{M_k \times G'_{m'}} \xi_{k-i}^{n'} \xi_i^{n'+n} \otimes \omega_{m-k,m'-i} \\ &= \mathrm{Ind}_{Q_{k-i}G_{m-k} \times P_i'}^{M_k \times G'_{m'}} \xi_{k,i} \mathbf{R}^{\mathrm{GL}_i} \otimes \omega_{m-k,m'-i}. \end{aligned}$$

The last equality above was obtained by transitivity of induction. Performing this induction comes down to induce $\xi_i^{n'+n}$ from GL_i to $\mathrm{GL}_i \times \mathrm{GL}_i$ (the inclusion being diagonal). Thanks to Lemma 2.4, the resulting representation is $\chi_i \mathbf{R}^{\mathrm{GL}_i}$, where χ_i is any character whose restriction to GL_i equals $\xi_i^{n'+n}$. We can choose χ_i to be equal to $\xi_i^n \otimes \xi_i^{n'}$. This finishes the proof of part (a) of the theorem. Part (b) has an analogous proof.

C The main theorem

In this section we state and prove the theorem showing the behaviour of the Howe correspondence with respect to Harish-Chandra series.

Let G_m be a symplectic, orthogonal or unitary group of Witt index m. The set of standard parabolic subgroups of G_m can be parametrized by sequences $\mathbf{t} = (t_1, \ldots, t_r)$, such that $|\mathbf{t}| = \sum_{i=1}^r t_i$ is not greater than m. The corresponding parabolic, denoted by $P_{\mathbf{t}}$, is the stabilizer of an isotropic flag

$$V_{t_1} \subset V_{t_1+t_2} \subset \cdots \subset V_{\|\mathbf{t}\|},$$

and has a Levi decomposition $P_{\mathbf{t}} = L_{\mathbf{t}} \ltimes U_{\mathbf{t}}$ where $L_{\mathbf{t}}$ is equal to $\mathrm{GL}_{t_1} \times \cdots \times \mathrm{GL}_{t_r} \times G_{m-|\mathbf{t}|}$. For this standard Levi, we denote parabolic induction by $R_{\mathbf{t}}$ and parabolic restriction by ${}^*R_{\mathbf{t}}$.

Let π be an irreducible representation of G_m. There exists a set $\mathbf{t} = (t_1, \ldots, t_r)$ of positive integers such that $|\mathbf{t}| \leq m$, and cuspidal irreducible representations σ_i of GL_{t_i} and φ of $G_{m-|\mathbf{t}|}$ such that

$$\mathrm{Hom}_{G_m}(\pi, R_{\mathbf{t}}(\boldsymbol{\sigma} \otimes \varphi)) \neq 0.$$

where $\boldsymbol{\sigma} = \sigma_1 \otimes \ldots \otimes \sigma_r$. We write

$$[\pi] = [\sigma_1, \ldots, \sigma_r, \varphi],$$

and call it *cuspidal support* of π.

For the rest of this chapter we denote by $\Theta_{m,m'}(\pi)$ the set of irreducible representations π' of $G'_{m'}$, such that

$$\mathrm{Hom}_{G_m \times G'_{m'}}(\omega_{m,m'}, \pi \otimes \pi') \neq 0.$$

Likewise, for an irreducible representation π' of $G'_{m'}$, denote by $\Theta_{m',m}(\pi')$ the set of irreducible representations π of G_m verifying the same inequality.

Theorem 4.10. *Let $\pi \in \mathrm{Irr}(G_m)$ with cuspidal support $[\pi] = [\sigma_1, \ldots, \sigma_r, \varphi]$ and let $\pi' \in \Theta_{m,m'}(\pi)$. Denote by $m'(\varphi)$ the first occurrence index of φ.*

a) *If $m' \geq m'(\varphi) + |\mathbf{t}|$ then*

$$[\pi'] = [\xi_{t_1}^{n+n'}\sigma_1, \ldots, \xi_{t_r}^{n+n'}\sigma_r, \xi_1^n, \ldots, \xi_1^n, \theta(\varphi)].$$

b) *When $m' < m'(\varphi)+|\mathbf{t}|$ there exists a sequence $i_1 \leq \cdots \leq i_d$, with $d = m'(\varphi)+|\mathbf{t}|-m'$, such that $\sigma_{i_k} = \xi_1^{n'}$ and*

$$[\pi'] = [\xi_{t_1}^{n+n'}\sigma_1, \ldots, \widehat{\xi_{t_{i_1}}^{n+n'}\sigma_{i_1}} \ldots \widehat{\xi_{t_{i_d}}^{n+n'}\sigma_{i_d}} \ldots, \xi_{t_r}^{n+n'}\sigma_r, \theta(\varphi)].$$

Proof. First, take π cuspidal and assume that $m' > m'(\pi)$. Let $[\pi'] = [\sigma'_1, \ldots, \sigma'_r, \varphi']$ and let σ' be an irreducible representation of $\mathrm{GL}_{|\mathbf{t}'|}$ with cuspidal support $[\sigma'] = [\sigma'_1, \ldots, \sigma'_r]$ such that π' is an irreducible component of $R'_{|\mathbf{t}'|}(\sigma' \otimes \varphi')$.

By Frobenius reciprocity and Corollary 4.7,

$$\begin{aligned}\langle \omega_{m,m'}, \pi \otimes \pi' \rangle \leq & \langle \omega_{m,m'}, \pi \otimes R'_{|\mathbf{t}'|}(\sigma' \otimes \varphi') \rangle = \langle {}^*R'_{|\mathbf{t}'|}(\omega), \pi \otimes \sigma' \otimes \varphi' \rangle \\ = & \sum_{i=0}^{\min\{m,|\mathbf{t}'|\}} \langle \tau'_{i+1}/\tau'_i, \pi \otimes \sigma' \otimes \varphi' \rangle.\end{aligned}$$

Due to the fact that π is cuspidal, only the term corresponding to $i = 0$ can contribute, and that term gives :

$$\langle \xi'_{|\mathbf{t}'|,0}\omega_{m,m'-|\mathbf{t}'|}, \pi \otimes \sigma' \otimes \varphi' \rangle = \langle \xi'_{|\mathbf{t}'|,0}, \sigma' \rangle \langle \omega_{m,m'-|\mathbf{t}'|}, \pi \otimes \varphi' \rangle,$$

so in this case we have $\sigma' = \xi'_{|\mathbf{t}'|,0}$, and φ' is the first occurrence $\theta(\pi)$ of π. The former has cuspidal support $[\sigma'] = [\xi_1^n, \ldots, \xi_1^n]$, hence

$$[\pi'] = [\xi_1^n, \ldots, \xi_1^n, \theta(\pi)].$$

This proves part (a) in the case when $r = 0$, i.e. when π is cuspidal.

Now consider π non-cuspidal with support $[\pi] = [\sigma_1, \ldots, \sigma_r, \varphi]$. There is an irreducible representation φ_1 of G_{m-t_1} with cuspidal support $[\varphi_1] = [\sigma_2, \ldots, \sigma_r, \varphi]$ such that π is an irreducible component of $R_{t_1}(\sigma_1 \otimes \varphi_1)$.

By Frobenius reciprocity and Corollary 4.7,

$$\begin{aligned}\langle \omega_{m,m'}, \pi \otimes \pi' \rangle \leq & \langle \omega_{m,m'}, R_{t_1}(\sigma_1 \otimes \varphi_1) \otimes \pi' \rangle = \langle {}^*R_{t_1}(\omega_{m,m'}), \sigma_1 \otimes \varphi_1 \otimes \pi' \rangle \\ = & \sum_{i=0}^{\min\{t_1, m'\}} \langle \tau_{i+1}/\tau_i, \sigma_1 \otimes \varphi_1 \otimes \pi' \rangle.\end{aligned}$$

Due to the fact that σ_1 is cuspidal only the terms corresponding to $i = 0$ and $i = t_1$ contribute to the sum.

◦ The term corresponding to $i = 0$ is :

$$\langle \xi_{t_1,0} \omega_{m-t_1,m'}, \sigma_1 \otimes \varphi_1 \otimes \pi' \rangle = \langle \xi_{t_1,0}, \sigma_1 \rangle \langle \omega_{m-t_1,m'}, \varphi_1 \otimes \pi' \rangle.$$

This is non-trivial if and only if $\sigma_1 = \xi_{t_1,0}$ and $\pi' = \theta(\varphi_1)$. The former equality implies that the character $\xi_{t_1,0}$ of GL_{t_1} is cuspidal so that $t_1 = 1$ and $\sigma_1 = \xi_1^{n'}$.

◦ When $i = t_1$ the corresponding term yields :

$$\langle \xi_{t_1,t_1} \mathbf{R}^{\mathrm{GL}_{t_1}} \otimes \omega_{m-t_1,m'-t_1}, \sigma_1 \otimes \varphi_1 \otimes {}^*R'_{t_1}(\pi') \rangle.$$

This is non zero only if there is a simple M'_{s_1} submodule $\sigma'_1 \otimes \varphi'_1$ of ${}^*R'_{t_1}(\pi')$ such that

$$\langle \xi_{t_1,t_1} \mathbf{R}^{\mathrm{GL}_{t_1}} \otimes \omega_{m-t_1,m'-t_1}, \sigma_1 \otimes \varphi_1 \otimes \sigma'_1 \otimes \varphi'_1 \rangle \neq 0.$$

This in turn equals $\langle \xi_{t_1}^n \sigma_1, \xi_{t_1}^{n'} \sigma'_1 \rangle \langle \omega_{m-t_1,m'-t_1}, \varphi_1 \otimes \varphi'_1 \rangle$. Hence, in this case we obtain $\sigma'_1 = \xi_{t_1}^{n+n'} \sigma_1$ and $\varphi'_1 = \theta(\varphi_1)$.

Suppose that $m' \geq |\mathbf{t}| + m'(\varphi)$. If the first case above holds, since $m' \geq |\mathbf{t}| - t_1 + m'(\varphi)$, we can apply an induction argument to $\pi' = \theta(\varphi_1)$ and φ_1 in order to obtain

$$[\pi'] = [\xi_{t_2}^{n+n'} \sigma_2, \ldots, \xi_{t_r}^{n+n'} \sigma_r, \xi_1^n, \ldots, \xi_1^n, \theta(\varphi)].$$

Performing a permutation yields (a), because in this case $\xi_1^{n+n'} \sigma_1 = \xi_1^n$. If the second case above holds, since $m' - t_1 \geq |\mathbf{t}| - t_1 + m'(\varphi)$, the inductive hypothesis applied to φ'_1 and φ_1 together with $\sigma'_1 = \xi_{t_1}^{n+n'} \sigma_1$ yield (a) of Theorem 4.10.

Now suppose that $m' < |\mathbf{t}| + m'(\varphi)$. In this situation the first case above must occur at least $d = |\mathbf{t}| + m'(\varphi) - m'$ times. This would yield a subsequence $\sigma_{i_i}, \ldots, \sigma_{i_d}$ is as in part (b) of the Theorem 4.10. □

Note that the "tails" to be added in one direction are just those which are canceled in

the other.

This theorem can be restated in terms of Harish-Chandra series. Let $\boldsymbol{\sigma}$ denote the cuspidal representation $\sigma_1 \otimes \ldots \otimes \sigma_r$ of $\mathrm{GL}_{\mathbf{t}} = \mathrm{GL}_{t_1} \times \ldots \mathrm{GL}_{t_r}$ and $\xi_{\mathbf{t}}$ be its character $\xi_{t_1} \otimes \ldots \otimes \xi_{t_r}$.

Theorem 4.11. *Let $\boldsymbol{T}$ and $\boldsymbol{T'}$ be two Witt towers such that pairs $(G_m, G'_{m'})$ with $G_m \in \boldsymbol{T}$ and $G'_{m'} \in \boldsymbol{T'}$ are of type I. Let φ be a cuspidal representation of G_l, l' be its first occurrence index, and φ' be the corresponding cuspidal representation $\theta(\varphi)$ of $G'_{l'}$. The Howe correspondence $\Theta_{m,m'}$ sends $\mathrm{Irr}(G_m, \boldsymbol{\sigma} \otimes \varphi)$ to $\mathscr{R}(G'_{m'}, \boldsymbol{\sigma'} \otimes \varphi')$, where $\boldsymbol{\sigma'} = \xi_{\mathbf{t}}^{n'+n} \boldsymbol{\sigma} \otimes \xi_1^n$, if $m' \geq l' + |\mathbf{t}|$, and $\boldsymbol{\sigma} = \xi_{\mathbf{t}}^{n+n'} \boldsymbol{\sigma'} \otimes \xi_1^{n'}$ otherwise.*

Suppose that the representation $\boldsymbol{\sigma} \otimes \varphi$ of $\mathrm{GL}_{\mathbf{t}} \times G_l$ is unipotent. In this situation, $\mathbf{t}$ becomes $\mathbf{t} = (1^{m-l})$, so that $\mathrm{GL}_{\mathbf{t}}$ becomes the torus T_{m-l} of diagonal matrices. Moreover, the representation $\boldsymbol{\sigma}$ becomes trivial. However, the representation $\boldsymbol{\sigma'}$ is not necessarily trivial. Indeed, for unitary dual pairs, the characters $\xi_{\mathbf{s}}^{n+n'}$ and $\xi_1^{n'}$ appearing in Theorem 4.11, need not be trivial (because neither n nor n' need be even). This means that the Howe correspondence Θ does not preserve unipotent representations.

4.3 Results for Gérardin's Weil representation

As proved in the first section, the Howe correspondence $\Theta^\flat$ is compatible with unipotent representations. In constrast, in the last section we saw that this is not necessarily true for Θ.

We now state analogues, for the Howe correspondence $\Theta^\flat$, of results obtained in the last section. These will provide a generalisation of Theorem 4.5. We keep notations from Section 4.1 : we denote by $\omega^\flat_{m,m'}$ the inflation to $G_m \times G'_{m'}$ of the restriction of $\omega^\flat$ to $G_m \cdot G'_{m'}$, and by $\Theta^\flat_{m,m'}$ the ensuing correspondence.

Proposition 4.12. *Let ${}^*R_k$ and ${}^*R'_{k'}$ be the parabolic restriction functor from $G_m \times G'_{m'}$ to $M_k \times G'_{m'}$ and $G_m \times M'_{k'}$ respectively.*

a) *There exists a $M_k \times G'_{m'}$ invariant filtration $0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_{r+1} = {}^*R_k(\omega^\flat_{m,m'})$, where $r = min\{k, m'\}$. Its successive quotients τ_{i+1}/τ_i verify*

$$\tau_{i+1}/\tau_i \simeq \mathrm{Ind}_{Q_{k-i} G_{m-k} \times P'_i}^{M_k \times G'_{m'}} \mathbf{R}^{\mathrm{GL}_i} \otimes \omega^\flat_{m-k,m'-i}.$$

b) *Likewise, there is a $G_m \times M'_{k'}$ invariant filtration $0 = \tau'_0 \subset \tau'_1 \subset \cdots \subset \tau'_{r'+1} = {}^*R'_{k'}(\omega^\flat_{m,m'})$, where $r' = min\{k', m\}$. Its successive quotients τ'_{i+1}/τ'_i verify*

$$\tau'_{i+1}/\tau'_i \simeq \mathrm{Ind}_{P_i \times Q'_{k'-i} G'_{m'-k'}}^{G_m \times M'_{k'}} \mathbf{R}^{\mathrm{GL}_i} \otimes \omega^\flat_{m-i,m'-k'}.$$

Proof. This is a consequence of Proposition 4.6.

For symplectic-orthogonal dual pairs $(\mathrm{Sp}_{2m}, \mathrm{O}_{2m'})$ there is nothing to be done. Indeed, in this case, the powers of the quadratic characters appearing in that proposition vanish because of the parity of the dimensions.

For unitary pairs it is enough to see that the restriction of $\nu_m^{m'} \otimes \nu_{m'}^m$ to $Q_{k-i}G_{m-k} \times P_i'$ is equal to $\nu_{m-k}^{m'-i} \otimes \nu_{m'-i}^{m-k}$ multiplied by $\xi_{k,i}$. Thus, (a) follows from Proposition 4.6. The proof of part (b) is similar. □

From this proposition we can obtain analogues to Corollary 4.7 and Theorem 4.10. The proofs are, mutatis mutandis, the same.

Corollary 4.13. *The parabolic restriction* ${}^*R_k(\omega^\flat_{m,m'})$ *verifies*

$$ {}^*R_k(\omega^\flat_{m,m'}) \simeq \bigoplus_{i=0}^{\min\{k,m'\}} \mathrm{Ind}_{Q_{k-i}G_{m-k}\times P_i'}^{M_k\times G'_{m'}} \mathbf{R}^{\mathrm{GL}_i} \otimes \omega^\flat_{m-k,m'-i}. $$

Similarly, the parabolic restriction ${}^*R'_{k'}(\omega^\flat_{m,m'})$ *verifies*

$$ {}^*R'_{k'}(\omega^\flat_{m,m'}) \simeq \bigoplus_{i=0}^{\min\{k',m\}} \mathrm{Ind}_{P_i\times Q'_{k'-i}G'_{m'-k'}}^{G_m\times M'_{k'}} \mathbf{R}^{\mathrm{GL}_i} \otimes \omega^\flat_{m-i,m'-k'}. $$

Theorem 4.14. *Let* $\pi \in \mathrm{Irr}(G_m)$ *with cuspidal support* $[\pi] = [\sigma_1, \dots, \sigma_r, \varphi]$ *and let* $\pi' \in \Theta^\flat_{m,m'}(\pi)$.

a) *If* $m' \geq m'(\varphi) + |\mathbf{t}|$ *then*

$$ [\pi'] = [\sigma_1, \dots, \sigma_r, 1, \dots, 1, \theta^\flat(\varphi)]. $$

b) *When* $m' < m'(\varphi)+|\mathbf{t}|$ *there exists a sequence* $i_1 \leq \cdots \leq i_d$, *with* $d = m'(\varphi)+|\mathbf{t}|-m'$, *such that* $\sigma_{i_k} = 1$ *and*

$$ [\pi'] = [\sigma_1, \dots, \widehat{\sigma_{i_1}}, \dots, \widehat{\sigma_{i_d}}, \dots, \sigma_r, \theta^\flat(\varphi)]. $$

Let $\mathbf{T}$ and $\mathbf{T}'$ be two Witt towers such that pairs $(G_m, G'_{m'})$ with $G_m \in \mathbf{T}$ and $G'_{m'} \in \mathbf{T}'$ are of type I.

Theorem 4.15. *Let* φ *be a cuspidal representation of* G_l, l' *be its first occurrence index, and* φ' *be the corresponding cuspidal representation* $\theta^\flat(\varphi)$ *of* $G'_{l'}$. *The Howe correspondence* $\Theta^\flat_{m,m'}$ *sends* $\mathrm{Irr}(G_m, \boldsymbol{\sigma} \otimes \varphi)$ *to* $\mathscr{R}(G'_{m'}, \boldsymbol{\sigma}' \otimes \varphi')$, *where* $\boldsymbol{\sigma}' = \boldsymbol{\sigma} \otimes 1$ *if* $m' \geq l' + |\mathbf{t}|$, *and* $\boldsymbol{\sigma} = \boldsymbol{\sigma}' \otimes 1$ *otherwise.*

This theorem generalizes Theorem 4.5. Indeed, if we ask for the representation $\boldsymbol{\sigma} \otimes \varphi$ of $\mathrm{GL}_{\mathbf{t}} \times G_l$ to be also unipotent, then $\mathbf{t}$ becomes $\mathbf{t} = (1^{m-l})$, the representation $\boldsymbol{\sigma}$ becomes the trivial representation of the torus T_{m-l} of diagonal matrices, and $\boldsymbol{\sigma}'$ becomes the trivial representation of the torus $T_{m'-l'}$.

Chapter 5

Correspondence for Weyl groups

5.1 A correspondence between Weyl groups

Let G_m belong to a Witt tower of symplectic, unitary or orthogonal groups. The standard Levi subgroups of G_m are $L_{\mathbf{t}} = \mathrm{GL}_{\mathbf{t}} \times G_{m-|\mathbf{t}|}$. A unipotent (resp. cuspidal) representation of this Levi is then given by $\boldsymbol{\sigma} \otimes \varphi$ where $\boldsymbol{\sigma}$ and φ are unipotent (resp. cuspidal) representations of $\mathrm{GL}_{\mathbf{t}}$ and $G_{m-|\mathbf{t}|}$ respectively. Therefore, thanks to Theorem 4.3, and the remark preceding it, the only Levi having cuspidal unipotent representations are $L_k = T_r \times G_{m(k)}$, where T_r is the torus of diagonal matrices of rang $r = (m - m(k))/2$ and $m(k)$ is equal to $k^2 + k$ for symplectic, $(k^2 + k)/2$ for unitary, and k^2 for orthogonal groups. For the first two kinds of groups, this representation is unique and equal to $1 \otimes \lambda_k$ where λ_k is the only cuspidal unipotent representation of $G_{m(k)}$, its Harish-Chandra series will be denoted by $\mathrm{Irr}(G_m)_k$, and the set of representations spanned by this series will be denoted by $\mathscr{R}(G_m)_k$. For orthogonal groups $\mathrm{O}^{\epsilon}_{2m}(q)$ and k verifying $\epsilon = (-1)^k$, we will have two unipotent cuspidal representations $1 \otimes \lambda_k^I$ and $1 \otimes \lambda_k^{II}$ coming from the two cuspidal unipotent representations λ_k^I and λ_k^{II} of $\mathrm{O}^{\epsilon}_{2k^2}(q)$. The corresponding Harish-Chandra series are denoted by $\mathrm{Irr}(\mathrm{O}^{\epsilon}_{2m}(q))^I_k$ and $\mathrm{Irr}(\mathrm{O}^{\epsilon}_{2m}(q))^{II}_k$, and the spanned sets by $\mathscr{R}(\mathrm{O}^{\epsilon}_{2m}(q))^I_k$ and $\mathscr{R}(\mathrm{O}^{\epsilon}_{2m}(q))^{II}_k$ respectively.

Theorem 2.15 tells us that representations in the Harish-Chandra series corresponding to a cuspidal unipotent representation, are parametrized by irreducible representations of a Weyl group. For symplectic or unitary groups, the series $\mathrm{Irr}(G_m)_k$ yields a group

$$W_{G_m}(1 \otimes \lambda_k) = \{x \in N_{G_m}(L_k)/L_k : {}^x\lambda_k = \lambda_k\}.$$

The unicity of λ_k implies that the condition on the elements of the previous group is trivial so $W_{\mathrm{Sp}_{2m}(q)}(\lambda_k)$ reduces to $N_{G_m}(L_k)/L_k$, which is a Weyl group of type $\mathbf{B}_{(m-m(k))/2}$. The same reasoning allows us to state that the series $\mathrm{Irr}(\mathrm{O}^{\epsilon}_{2m}(q))^I_k$ and $\mathrm{Irr}(\mathrm{O}^{\epsilon}_{2m}(q))^{II}_k$ are in bijection with the irreducible representations of a Weyl group of type $\mathbf{B}_{m-k^2}$.

These remarks, together with Theorems 4.4 and 4.5 imply that for the type I dual

pairs $(\mathrm{Sp}_{2m}(q), \mathrm{O}^{\epsilon}_{2m'}(q))$ and $(\mathrm{U}_m(q), \mathrm{U}_{m'}(q))$, the Howe correspondence between Harish-Chandra series of cuspidal unipotent representations leads to a correspondence between pairs of type **B** Weyl groups : $(\mathbf{B}_{m-k(k+1)}, \mathbf{B}_{m'-\theta(k)^2})$ for symplectic-orthogonal pairs, and $(\mathbf{B}_{\frac{1}{2}(m-k(k+1)/2)}, \mathbf{B}_{\frac{1}{2}(m'-\theta(k)(\theta(k)+1)/2)})$ for unitary pairs.

Let $(G_m, G'_{m'})$ be a type I dual pair and $(W_r, W_{r'})$ be one of the corresponding pairs of Weyl groups from the previous paragraph. Denote $\theta(k)$ by k', and denote the projection of $\omega_{m,m'}$ onto $\mathscr{R}(G_m)_k \otimes \mathscr{R}(G'_{m'})_{k'}$ by $\omega_{m,m',k}$. Finally let $\phi : W_n \to \{\pm 1\}$ denote the unique group homomorphism whose kernel is a group of type **D** [14, Proposition 1.4.10].

Conjecture 5.1. *[5, Conjecture 3.11] Let $(\mathrm{Sp}_{2m}(q), \mathrm{O}^{\epsilon}_{2m'}(q))$ be a symplectic-orthogonal dual pair. There is a bijection*

$$\mathrm{Irr}(\mathrm{Sp}_{2m}(q))_k \times \mathrm{Irr}(\mathrm{O}^{\epsilon}_{2m'}(q))^{\Gamma}_{k'} \simeq \mathrm{Irr}(W_r \times W_{r'}).$$

where $\Gamma = II$ if $\epsilon = (-1)^k$ and $\Gamma = I$ otherwise. Moreover, it identifies $\omega_{m,m',k}$ to the representation $\Omega_{r,r'}$ whose character is :

$$\sum_{l=0}^{\min(r,r')} \sum_{\chi \in \mathrm{Irr}(W_l)} (\mathrm{Ind}^{W_r}_{W_l \times W_{r-l}} \chi \otimes \phi) \otimes (\mathrm{Ind}^{W'_r}_{W_l \times W_{r'-l}} \chi \otimes \phi), \tag{5.1}$$

for $(\mathrm{Sp}_{2m}(q), \mathrm{O}^{\epsilon}_{2m'}(q))$ if $\epsilon = (-1)^k$; and

$$\sum_{l=0}^{\min(r,r')} \sum_{\chi \in \mathrm{Irr}(W_l)} (\mathrm{Ind}^{W_r}_{W_l \times W_{r-l}} \chi \otimes 1) \otimes (\mathrm{Ind}^{W'_r}_{W_l \times W_{r'-l}} \chi \otimes \phi), \tag{5.2}$$

otherwise.

Theorem 5.2. *[5, Theorem 3.10] Let $(U_m(q), U_{m'}(q))$ be a unitary dual pair. The bijection*

$$\mathrm{Irr}(U_m(q))_k \times \mathrm{Irr}(U_{m'}(q))_{k'} \simeq \mathrm{Irr}(W_r \times W_{r'}),$$

identifies the representation $\omega_{m,m',k}$ with the representation $\Omega_{r,r'}$ whose character is :

$$\sum_{l=0}^{\min(r,r')} \sum_{\chi \in \mathrm{Irr}(W_l)} (\mathrm{Ind}^{W_r}_{W_l \times W_{r-l}} \chi \otimes 1) \otimes (\mathrm{Ind}^{W'_r}_{W_l \times W_{r'-l}} \phi\chi \otimes 1), \tag{5.3}$$

for the pair $(\mathrm{U}_m(q), \mathrm{U}_{m'}(q))$ if k is odd or $k = k' = 0$;

$$\sum_{l=0}^{\min(r,r')} \sum_{\chi \in \mathrm{Irr}(W_l)} (\mathrm{Ind}^{W_r}_{W_l \times W_{r-l}} \chi \otimes \phi) \otimes (\mathrm{Ind}^{W'_r}_{W_l \times W_{r'-l}} \phi\chi \otimes 1), \tag{5.4}$$

for the pair $(\mathrm{U}_m(q), \mathrm{U}_{m'}(q))$ otherwise.

5.2 Reduction to unipotent representations

A Centralizers of rational semisimple elements

Let $\mathbf{G}$ be a linear or unitary group of Witt index m. As usual, linear groups arise as fixed points of the standard Frobenius $F = F_q$, while unitary groups come from the twisted Frobenius $F = {}^{w_0 \mathrm{t}}F_q{}^{-1}$, where w_0 denotes a permutation matrix with ones on the antidiagonal and zeroes elsewhere. The choice of these Frobenius allows the group of upper triangular matrices to be a rational Borel.

The pair $(\mathbf{G}, F)$ is autodual, this means that its Langlands dual $(\mathbf{G}^*, F^*)$ can be identified with $(\mathbf{G}, F)$. Therefore, for every rational semisimple s in $\mathbf{G}^* \simeq \mathbf{G}$, the Lusztig bijection (cf. Theorem 2.29) becomes

$$\mathfrak{L}_s^{\mathbf{G}} : \mathscr{E}(\mathbf{G}^F, (s)) \simeq \mathscr{E}(C_{\mathbf{G}}(s)^F, (1))$$

We usually denote by π_u the image of π by the above bijection. In order to reduce the study of the Howe correspondence to unipotent representations, we must therefore get a better understanding of the centralizers of rational semisimple elements.

Let $\mathbf{T}$ be the rational maximal torus of $\mathbf{G}$ consisting of diagonal matrices. By definition, a semisimple element of $\mathbf{G}$ is conjugated to an element of $\mathbf{T}$. Hence, in order to describe the centralizer of rational semisimple elements of $\mathbf{G}$, it is enough to do so for centralizers of diagonal matrices, that are conjugated to rational elements.

Let t be a diagonal matrix such that $s = {}^x t$ is rational for a certain $x \in \mathbf{G}$. Let w be the class of $x^{-1}F(x)$ in the Weyl group $W(\mathbf{T})$ of $\mathbf{G}$. The fact that s is rational is equivalent to the equality ${}^{wF}t = t$. Therefore, the centralizer of t in $\mathbf{G}$ is stable by wF. Moreover, the pair $(wF, C_{\mathbf{G}}(t))$ is conjugated by x to $(F, C_{\mathbf{G}}(s))$, so we can replace $C_{\mathbf{G}}(s)^F$ by $C_{\mathbf{G}}(t)^{wF}$ on the right side of the bijection $\mathfrak{L}_s^{\mathbf{G}}$.

Example 5.3. *Let $\mathbf{G}$ be a linear (resp. unitary) group of Witt index m, and $\mathrm{Fr} = F_q$ (resp. $\mathrm{Fr} = F_q^{-1}$). The diagonal matrix t with pairwise distinct eigenvalues $\lambda, {}^{\mathrm{Fr}}\lambda, \ldots, {}^{\mathrm{Fr}^{m-1}}\lambda$, is conjugated to a rational element of $\mathbf{G}$. Furthermore, the permutation matrix w corresponding to $(1\,2\ldots n)$ verifies the equality ${}^{wF}t = t$. The centralizer of t in $\mathbf{G}$ is $C_{\mathbf{G}}(t) = \mathbf{GL}_1^m$, and the group of its fixed points by wF is $\mathrm{GL}_1(q^m)$ (resp. $\mathrm{GL}_1(q^m)$ if m is even and $\mathbf{U}_1(q^m)$ if m is odd).*

Semisimple elements t in the previous paragraph are said to be *elliptic*. This example shows that elliptic semisimple elements have tori as centralizers. Moreover, the rational points of these centralizers are linear of unitary groups of rank 1 and coefficients in possibly some extension of $\mathbb{F}_q$.

Let t be a diagonal matrix of a linear (resp. unitary) group and $\nu_t(\lambda)$ be the algebraic multiplicity of the eigenvalue λ of t. The Frobenius F_q (resp. F_q^{-1}) acts on the set of

eigenvalues of t. We denote by $\mathbf{G}_\lambda(t)$ the product of $\mathbf{GL}_{\nu_t(\gamma)}$ for γ in the orbit of λ. As before w denotes an element of the Weyl group $W(\mathbf{T})$ of $\mathbf{G}$ such that ${}^{wF}t = t$.

Proposition 5.4. *Let $\mathbf{G}$ be a linear (resp. unitary) group. The centralizer of t in $\mathbf{G}$ can be expressed as a product of groups $\mathbf{G}_\lambda(t)$, indexed by F_q-orbits of eigenvalues λ of t. Each group $\mathbf{G}_\lambda(t)$ is stable by wF and, if m is the cardinal of the orbit of λ, then its group of fixed points by this morphism equals $GL_{\nu_t(\lambda)}(q^m)$ (resp. $\mathrm{GL}_{\nu_t(\lambda)}(q^m)$ when m is even, and $\mathbf{U}_{\nu_t(\lambda)}(q^m)$ when m is odd).*

Let t be diagonal matrix of a unitary (resp. linear) group. If t has an eigenvalue equal to 1, then the corresponding F_q-orbit is a singleton. Therefore, thanks to the previous proposition, $\mathbf{G}_1(t)$ becomes the unitary (resp. linear) group $\mathbf{G}_{\nu_t(1)}$ of rank $\nu_t(1)$.

Following the notation in [5], for a rational semisimple element s of $\mathbf{G}$, we call $\mathbf{G}_\#(s)$ the product of $\mathbf{G}_\lambda(s)$ for $\lambda \neq 1$, and $\mathbf{G}_{(1)}(s)$ the group $\mathbf{G}_{\nu_s(1)}$. As mentioned in the previous paragraph, the latter is a group of the same kind as $\mathbf{G}$, but of smaller rank.

Proposition 5.4 yields a natural bijection between $\mathscr{E}(C_{\mathbf{G}}(s)^F, (1))$ and $\mathscr{E}(\mathbf{G}_\#(s)^F, (1)) \times \mathscr{E}(\mathbf{G}_{(1)}(s)^F, (1))$. Composing with $\mathfrak{L}_s^{\mathbf{G}}$ we obtain a one-to-one correspondence :

$$\Xi_s^{\mathbf{G}} : \mathscr{E}(\mathbf{G}^F, (s)) \simeq \mathscr{E}(\mathbf{G}_\#(s)^F, (1)) \times \mathscr{E}(\mathbf{G}_{(1)}(s)^F, (1)). \tag{5.5}$$

We usually denote by $\pi_\# \otimes \pi_{(1)}$ the image of π by the above bijection. According to previous notation $\pi_u \simeq \pi_\# \otimes \pi_{(1)}$.

We can extend this bijection to proper standard Levi (indeed, these are products of linear and unitary groups). Therefore, in (5.5) we can take $\mathbf{G}$ to be a linear, unitary, or a standard Levi subgroup of these.

Let $\mathbf{G}'$ be a group of same type as $\mathbf{G}$ and of Witt index m. Let $\mathscr{J}_{m,m'}$ denote the injective map sending $a \in \mathbf{G}$ to $\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) \in \mathbf{G}'$ for $m \leq m'$ and $\mathbf{T}_{l,0}$ be the subgroup of the l dimensional torus $\mathbf{T}_l$ whose diagonal matrices have eigenvalues different from 1. Finally, let $\mathbf{L}$ be a standard Levi subgroup of $\mathbf{G}$, and s be a rational semisimple element of $\mathbf{L}$.

Lemma 5.5. *The groups $C_{\mathbf{L}}(s)$, $\mathbf{L}_\#(s)$ and $\mathbf{L}_{(1)}(s)$ are Levi subgroups of $C_{\mathbf{G}_m}(s)$, $\mathbf{G}_\#(s)$ and $\mathbf{G}_{(1)}(s)$ respectively.*

Proof. Let $\mathbf{P} = \mathbf{LU}$ be a parabolic containing $\mathbf{L}$. Proposition 2.3 in [10] implies that $C_{\mathbf{P}}(s) = \mathbf{P} \cap C_{\mathbf{G}_m}(s)$ is a parabolic group of $C_{\mathbf{G}_m}(s)$. Indeed, the latter is a connected reductive group of maximal rank in $\mathbf{G}_m$. Moreover, according to the same proposition, this parabolic has a Levi $C_{\mathbf{L}}(s) = \mathbf{L} \cap C_{\mathbf{G}_m}(s)$ and unipotent radical $C_{\mathbf{U}}(s) = \mathbf{U} \cap C_{\mathbf{G}_m}(s)$.

Let $l \leq m$, and s_0 be an element of $\mathbf{T}_{l,0}$ such that $s = \mathscr{J}_{l,m}(s_0)$. A reasoning similar to the one in the previous paragraph yields the assertion for $\mathbf{L}_\#$. Indeed, it is the centralizer of s_0 in the Levi subgroup $\mathbf{L} \cap \mathbf{G}_{m-\nu_s(1)}$ of $\mathbf{G}_{m-\nu_s(1)}$, whereas $\mathbf{G}_\#(s)$ is the centralizer of s_0 in $\mathbf{G}_{m-\nu_s(1)}$. The assertion for $\mathbf{L}_{(1)}(s)$ has an analogous proof. $\square$

B Reduction to unipotent Lusztig series

Let $(G_m, G'_{m'})$ be a unitary or linear dual pair. The Howe correspondence for this pair behaves well vis-à-vis Lusztig series. In other words, for a geometric conjugacy class (s_m) in $\mathbf{G}^*_m$, there is a corresponding geometric conjugacy class $(s'_{m'})$ in $\mathbf{G}'^*_{m'}$, such that $\Theta_{m,m'}(\pi)$ belongs to $\mathscr{R}(G'_{m'}, (s'_{m'}))$, whenever π belongs to $\mathscr{E}(G_m, (s_m))$. The correspondence between semisimple classes is given in terms of the embedding $\mathscr{J}_{m,m'}$, introduced in the previous section.

Theorem 5.6. *Let π (resp. π') be an irreducible representation of G_m (resp. $G'_{m'}$) and (s_m) (resp. $(s'_{m'})$) be the geometric conjugacy class corresponding to π (resp. π'). If $\pi \otimes \pi'$ appears in $\omega_{m,m'}$, then we have $(s'_{m'}) = \mathscr{J}_{m,m'}(s_m)$ for $m < m'$, and $(s_m) = \mathscr{J}_{m',m}(s'_{m'})$ otherwise.*

We are interested in the relation between $(\mathbf{G}_m)_\#(s_m)$ and $(\mathbf{G}'_{m'})_\#(s'_{m'})$, as well as the one between $(\mathbf{G}_m)_{(1)}(s_m)$ and $(\mathbf{G}'_{m'})_{(1)}(s'_{m'})$. This is the content of the following lemma.

Lemma 5.7. *Let $(G_m, G'_{m'})$ be a unitary (resp. linear) dual pair. Suppose $m < m'$ and $(s') = \mathscr{J}_{m,m'}(s)$. The groups $(\mathbf{G}_m)_\#(s)$ and $(\mathbf{G}'_{m'})_\#(s')$ are isomorphic. Furthermore, $(\mathbf{G}_m)_{(1)}(s_m) \simeq \mathbf{G}_{m-l}$ and $(\mathbf{G}'_{m'})_{(1)}(s'_{m'}) \simeq \mathbf{G}_{m'-l}$ are unitary (resp. linear) groups.*

Proof. The isomorphism between $(\mathbf{G}_m)_\#(s)$ and $(\mathbf{G}'_{m'})_\#(s')$ comes from the fact that s and s' have the same eigenvalues different from 1 and with the same multiplicities. The other two isomorphisms (discussed in the previous section) are a consequence of Proposition 5.4. □

Let (s_m) and $(s'_{m'})$ be as in Theorem 5.6. According to the same, there exist $l \leq \min\{m, m'\}$, and $s \in \mathbf{T}_{l,0}$ such that $(s_m) = \mathscr{J}_{l,m}(s)$ and $(s'_{m'}) = \mathscr{J}_{l,m'}(s)$. Lemma 5.7 tells us that $((\mathbf{G}_m)_{(1)}(s_m), (\mathbf{G}'_{m'})_{(1)}(s'_{m'}))$ is a dual pair of the same kind as $(G_m, G'_{m'})$ but of smaller size. It also says that the groups $(\mathbf{G}_m)_\#(s_m)$ and $(\mathbf{G}'_{m'})_\#(s'_{m'})$ are isomorphic to $\mathbf{G}_\#(s) = \prod_{\lambda \neq 1} \mathbf{G}_\lambda(s)$.

Proposition 5.8. *[5, Proposition 2.4] Keep the notations above. Let $\omega_{m,m',s}$ denote the projection of the Weil representation $\omega_{m,m'}$ onto $\mathscr{R}(G_m, (s_m)) \otimes \mathscr{R}(G'_{m'}, (s_{m'}))$. We have :*

$$\omega_{m,m'} = \bigoplus_{l=0}^{\min(m,m')} \bigoplus_{s \in \mathbf{T}_{l,0}} \omega_{m,m',s}.$$

We state that $\omega_{m,m',s}$ can be described in terms of a correspondence between unipotent characters, defined either by the unipotent projection $\mathrm{pr}_{\mathrm{unip}}$ of $\mathbf{R}^{\mathbf{G}_\#(s)^F}$ (cf. Section 2.1), or by the unipotent projection of the Weil representation of a smaller dual pair.

Theorem 5.9. *Let s belong to $\mathbf{T}_{l,0}$. For linear or unitary pairs, the representation $\omega_{m,m',s}$, is the image by the Lusztig bijection*

$$\mathscr{E}(G_m \times G'_{m'}, (s_m) \times (s_{m'})) \simeq \mathscr{E}(C_{G_m}(s_m) \times C_{G'_{m'}}(s_{m'}), 1),$$

of the representation

$$\mathrm{pr}_{\mathrm{unip}}(\mathbf{R}^{\mathbf{G}_\#(s)^F}) \otimes \omega_{m-l,m'-l,1}.$$

For clarity, we express the last result as a commutative diagram.

Theorem 5.10. *The following diagram is commutative :*

$$\begin{array}{ccc}
\mathscr{E}(G_m, (s_m)) & \xrightarrow[\sim]{\Xi_{s_m}^{G_m}} & \mathscr{E}(\mathbf{G}_\#(s), (1)) \times \mathscr{E}(G_{m-l}, (1)) \\
\downarrow \Theta_{m,m'} & & \downarrow \mathrm{Id} \otimes \Theta_{m-l,m'-l} \\
\mathscr{R}(G'_{m'}, (s'_{m'})) & \xrightarrow[\sim]{\Xi_{s'_{m'}}^{G'_{m'}}} & \mathscr{R}(\mathbf{G}_\#(s), (1)) \otimes \mathscr{R}(G'_{m'-l}, (1)).
\end{array}$$

.

Proof. We need to show that if π' is an irreducible component of $\Theta_{m,m'}(\pi)$, then $\pi_\# \simeq \pi'_\#$ and $\pi'_{(1)}$ is an irreducible subrepresentation of $\Theta_{m-l,m'-l}(\pi_{(1)})$.

According to Theorem 5.9, $\pi_\# \otimes \pi'_\#$ appears in the (unipotent part) of the representation $\mathbf{R}^{\mathbf{G}_\#(s)^F}$ of $\mathbf{G}_\#(s)^F \times \mathbf{G}_\#(s)^F$. This implies that $\pi_\# \simeq \pi'_\#$. The same theorem tells us that $\pi_{(1)} \otimes \pi'_{(1)}$ is an irreducible constituent of the (unipotent part) of the Weil representation $\omega_{m-l,m'-l}$ associated to the dual pair $(G_{m-l}, G_{m'-l})$. Hence, $\pi'_{(1)}$ appears in $\Theta_{m-l,m'-l}(\pi_{(1)})$. □

C Reduction to unipotent Harish-Chandra series

Theorem 5.10 shows how the study of the theta correspondence can be brought to the study of a correspondence between unipotent representations of a smaller dual pair of the same kind. In this section we will use this result to show how to describe the theta correspondence between Harish-Chandra series in terms of series for cuspidal unipotent representations.

Let $\mathbf{L}$ be a standard rational Levi contained in a rational parabolic of a unitary group $\mathbf{G}$, and $L \subset G$ be its group of rational points, let s be a rational semisimple element of the dual group $\mathbf{L}^*$. In view of Lemma 5.5 it makes sense to consider parabolic inductions $R_{C_{\mathbf{L}^*}(s)}^{C_{\mathbf{G}^*}(s)}$, $R_{\mathbf{L}_\#(s)}^{\mathbf{G}_\#(s)}$ and $R_{\mathbf{L}_{(1)}(s)}^{\mathbf{G}_{(1)}(s)}$. The subsequent results show the effect of the Lusztig bijection on the parabolic induction from L to G.

Proposition 5.11. *Parabolic induction R_L^G sends the Lusztig series $\mathscr{E}(L, (s))$ to $\mathscr{R}(G, (s))$. Furthermore, the following diagram is commutative :*

$$\begin{array}{ccc} \mathscr{E}(L,(s)) & \xrightarrow[\mathfrak{L}_s^L]{\sim} & \mathscr{E}(C_{\mathbf{L}^*}(s)^F,(1)) \\ \Big\downarrow R_L^G & & \Big\downarrow R_{C_{\mathbf{L}^*}(s)}^{C_{\mathbf{G}^*}(s)} \\ \mathscr{R}(G,(s)) & \xrightarrow[\mathfrak{L}_s^G]{\sim} & \mathscr{R}(C_{\mathbf{G}^*}(s)^F,(1)). \end{array}$$

Proof. Let π be an irreducible representation in $\mathscr{E}(L,(s))$. Due to the fact that $\mathbf{L}$ is a product of linear and unitary groups, all central functions in L are uniform. Therefore we can express π as a linear combination with integral coefficients $\pi = \sum_{s\in\mathbf{T}^*} n_{\mathbf{T}} R_{\mathbf{T}^*}^{\mathbf{L}}(s)$. Transitivity of Lusztig induction implies that $R_{\mathbf{L}}^{\mathbf{G}}(\pi) = \sum_{s\in\mathbf{T}^*} n_{\mathbf{T}} R_{\mathbf{T}^*}^{\mathbf{G}}(s)$, this representation belongs to $\mathscr{R}(G,(s))$.

Applying the Lusztig bijection to the last sum in the previous paragraph, we obtain

$$\mathfrak{L}_s^{\mathbf{G}} R_{\mathbf{L}}^{\mathbf{G}}(\pi) = \epsilon_{\mathbf{G}}\epsilon_{C_{\mathbf{G}}^*(s)} \sum_{s\in\mathbf{T}^*} n_{\mathbf{T}} R_{\mathbf{T}^*}^{C_{\mathbf{G}^*}(s)}(s),$$

this representation belongs to $\mathscr{R}(C_{\mathbf{G}^*}(s)^F,(1))$. On the other side, inducing the representation $\mathfrak{L}_s^L(\pi) = \epsilon_{\mathbf{L}}\epsilon_{C_{\mathbf{L}^*}(s)} \sum_{s\in\mathbf{T}^*} n_{\mathbf{T}} R_{\mathbf{T}^*}^{C_{\mathbf{L}^*}(s)}(s)$ to $C_{\mathbf{G}^*}(s)^F$, we obtain

$$R_{C_{\mathbf{L}^*}(s)}^{C_{\mathbf{G}^*}(s)} \mathfrak{L}_s^L(\pi) = \epsilon_{\mathbf{L}}\epsilon_{C_{\mathbf{L}^*}(s)} \sum_{s\in\mathbf{T}^*} n_{\mathbf{T}} R_{\mathbf{T}}^{C_{\mathbf{G}^*}(s)}(s).$$

The theorem follows from Proposition 2.33 and these computations. Indeed, $\mathbf{L}$ (resp. $C_{\mathbf{L}^*}(s)$) is a rational Levi contained in a rational parabolic subgroup of $\mathbf{G}$ (resp. $C_{\mathbf{G}^*}(s)$). □

Corollary 5.12. *The following diagram is commutative :*

$$\begin{array}{ccc} \mathscr{E}(L,(s)) & \xrightarrow[\sim]{\Xi_s^L} & \mathscr{E}(\mathbf{L}_\#(s)^F,(1)) \times \mathscr{E}(\mathbf{L}_{(1)}(s)^F,(1)) \\ \Big\downarrow R_L^G & & \Big\downarrow R_{\mathbf{L}_\#(s)}^{\mathbf{G}_\#(s)} \times R_{\mathbf{L}_{(1)}(s)}^{\mathbf{G}_{(1)}(s)} \\ \mathscr{R}(G,(s)) & \xrightarrow[\sim]{\Xi_s^G} & \mathscr{R}(\mathbf{G}_\#(s)^F,(1)) \otimes \mathscr{R}(\mathbf{G}_{(1)}(s)^F,(1)). \end{array}$$

Let $(G_m, G'_{m'})$ be a unitary dual pair. Theorem 4.10 asserts that, for a cuspidal pair $(\mathbf{L},\rho)$ of $\mathbf{G}_m$, we can find a unique cuspidal pair $(\mathbf{L}',\rho')$ of $\mathbf{G}'_{m'}$ whose Harish-Chandra series are related by the theta correspondence. This means that $\Theta_{m,m'}$ sends the series $\mathrm{Irr}(G_m,\rho)$ to $\mathscr{R}(G'_{m'},\rho')$.

Proposition 5.13. *Let $\omega_{m,m',\rho}$ denote the projection of $\omega_{m,m'}$ onto $\mathscr{R}(G_m,\rho)\otimes\mathscr{R}(G'_{m'},\rho')$. Then*

$$\omega_{m,m'} = \bigoplus_{(\mathbf{L},\rho)} \omega_{m,m',\rho},$$

where the sum runs over all rational conjugacy classes of cuspidal pairs of G_m.

Consider the cuspidal pairs $(\mathbf{L},\rho)$ of $\mathbf{G}_m$ and $(\mathbf{L}',\rho')$ of $\mathbf{G}'_{m'}$ as above. Thanks to Corollary 2.13 we can take $\mathbf{L}$ and $\mathbf{L}'$ to be standard Levi. According to Theorem 4.15 we

can write $\mathbf{L} \simeq \mathbf{G}_l \times \mathbf{GL_t} \times \mathbf{T}_r$, $\mathbf{L}' \simeq \mathbf{G}'_{l'} \times \mathbf{GL_t} \times \mathbf{T}_{r'}$, $\rho \simeq \varphi \otimes \boldsymbol{\sigma} \otimes 1$ and $\rho' \simeq \theta(\varphi) \otimes \boldsymbol{\sigma} \otimes 1$, where $\boldsymbol{\sigma} = \sigma_1 \otimes \ldots \otimes \sigma_d$ is a product of non-trivial cuspidal representations.

We denote $\theta(\varphi)$ by φ'. Let s_φ, $s_{\varphi'}$ and $s_{\boldsymbol{\sigma}} = s_1 \times \cdots \times s_d$ be rational semisimple elements of $\mathbf{G}_l$, $\mathbf{G}'_{l'}$ and $\mathbf{GL_t}$, whose geometric conjugacy classes correspond to φ, φ' and $\boldsymbol{\sigma}$ respectively. The geometric conjugacy class corresponding to ρ (resp. ρ') in $\mathbf{L}^*$ (resp. $\mathbf{L}'^*$) is $(s_\varphi \times s_{\boldsymbol{\sigma}} \times 1)$ (resp. $(s_{\varphi'} \times s_{\boldsymbol{\sigma}} \times 1)$).

With notations from Section A of this chapter, $\mathbf{L}_\#(s) \simeq (\mathbf{G}_l)_\#(s_\varphi) \times (\mathbf{GL_t})_\#(s_{\boldsymbol{\sigma}})$ and $\mathbf{L}_{(1)}(s) \simeq (\mathbf{G}_l)_{(1)}(s_\varphi) \times (\mathbf{GL_t})_{(1)}(s_{\boldsymbol{\sigma}}) \times \mathbf{T}_r$. There are analogous isomorphisms for $\mathbf{L}'$.

Using these isomorphisms, we can define $\rho_\#$, $\rho'_\#$, $\rho_{(1)}$ and $\rho'_{(1)}$ in a canonical way. It is natural to ask what are the relations between these representations, the following proposition provides the answer. In its statement λ_k denotes the unique cuspidal unipotent representation of the unitary group $G_{k(k+1)/2}$.

Proposition 5.14. *The groups $\mathbf{L}_\#(s)$ and $\mathbf{L}'_\#(s')$ are isomorphic, $\mathbf{L}_{(1)}(s)$ is isomorphic to $\mathbf{G}_{k(k+1)/2} \times \mathbf{T}_r$, for a certain k, and $\mathbf{L}'_{(1)}(s')$ to $\mathbf{G}_{k'(k'+1)/2} \times \mathbf{T}_{r'}$, where $k' = \theta(k)$ is the first occurrence. Moreover, $\rho_\#$ is isomorphic to $\rho'_\#$, $\rho_{(1)} = \lambda_k \otimes 1$, and $\rho'_{(1)} = \lambda_{k'} \otimes 1$.*

Proof. Since $\boldsymbol{\sigma} = \sigma_1 \otimes \ldots \otimes \sigma_d$ is a product of non-trivial cuspidal representations, $s_{\boldsymbol{\sigma}} = s_1 \times \ldots \times s_d$ is a product of elliptic semisimple elements with eigenvalues different from 1. Therefore $C_{\mathbf{GL_t}}(s_{\boldsymbol{\sigma}})_\# = C_{\mathbf{GL_t}}(s_{\boldsymbol{\sigma}})$ and $C_{\mathbf{GL_t}}(s_{\boldsymbol{\sigma}})_{(1)} = 1$. Since $(\mathbf{G}_l)_\#$ is isomorphic to $(\mathbf{G}'_{l'})_\#$ (cf. Lemma 5.7), the groups $\mathbf{L}_\#(s)$ and $\mathbf{L}'_\#(s')$ are isomorphic.

Lusztig's bijection applied to the series $\mathscr{E}(\mathrm{GL}_{\mathbf{t}}, (s_{\boldsymbol{\sigma}}))$ sends the cuspidal representation $\boldsymbol{\sigma}$ to a cuspidal unipotent representation $\boldsymbol{\sigma}_u$ of the group of rational points of $C_{\mathbf{GL_t}}(s_{\boldsymbol{\sigma}})$. The condition on the semisimple element $s_{\boldsymbol{\sigma}}$ (mentioned in the previous paragraph), implies that this group is a torus, and $\boldsymbol{\sigma}_u$ is the trivial representation. Therefore $\rho_\# = \varphi_\# \otimes 1$ and $\rho'_\# = \varphi'_\# \otimes 1$. Furthermore, $\rho_{(1)} = \varphi_{(1)} \otimes 1_r$ and $\rho'_{(1)} = \varphi'_{(1)} \otimes 1_{r'}$.

Theorem 5.9 applied to the pair $(G_l, G_{l'})$ implies that $\varphi_\# \simeq \varphi'_\#$, and that $\varphi'_{(1)} = \theta(\varphi_{(1)})$. Since the latter is a first occurrence between unipotent cuspidal representations, there exists an integer k such that $(\mathbf{G}_l)_{(1)}(s_\varphi) \simeq U_{k(k+1)/2}$ and $(\mathbf{G}'_{l'})_{(1)}(s_{\varphi'}) \simeq U_{k'(k'+1)/2}$ where $k' = \theta(k)$ (cf. Theorem 4.4). This completes the proof. □

Let (s) be the geometric conjugacy class in $\mathbf{L}^*$ whose Lusztig series contains ρ, and denote by $\mathrm{pr}_{\rho_\#}(\mathbf{R}^{G_\#(s)^F})$ the projection of the representation $\mathbf{R}^{G_\#(s)^F}$ onto the Harish-Chandra series $\mathrm{Irr}(G_\#(s) \times G_\#(s), \rho_\# \otimes \rho_\#)$. The following is the main theorem of this section.

Theorem 5.15. *The representation $\omega_{m,m',\rho}$ is identified with $\mathrm{pr}_{\rho_\#}(\mathbf{R}^{G_\#(s)^F}) \otimes \omega_{m-l,m'-l,\rho_{(1)}}$ via the bijection*

$$\mathrm{Irr}(G_m \times G'_{m'}, \rho \otimes \rho') \simeq \mathrm{Irr}(C_{G_m}(s_m) \times C_{G'_{m'}}(s'_{m'}), \rho_u \otimes \rho'_u). \tag{5.6}$$

Proof. Proposition 5.11 tells us that the (parabolic) induced representation $R_{L\times L'}^{G_m\times G'_{m'}}(\rho\otimes\rho')$ belongs to $\mathscr{R}(G_m\times G'_{m'},(s_m)\times(s'_{m'}))$. Therefore the Harish-Chandra series $\mathrm{Irr}(G_m\times G'_{m'},\rho\otimes\rho')$ is contained in $\mathscr{E}(G_m\times G'_{m'},(s_m)\times(s'_{m'}))$.

Let π and π' be irreducible representations of G_m and $G'_{m'}$ respectively. Proposition 5.11 tells us that,

$$\langle \pi_u\otimes\pi'_u, R_{C_{\mathbf{L}^*}(s_m)}^{C_{\mathbf{G}^*}(s_m)}(\rho_u)\otimes R_{C_{\mathbf{L}'^*}(s'_{m'})}^{C_{\mathbf{G}'^*}(s'_{m'})}(\rho'_u)\rangle$$

is equal to

$$\langle \pi_u, (R_{C_{\mathbf{L}^*}(s_m)}^{C_{\mathbf{G}^*}(s_m)}\rho)_u\rangle\langle \pi'_u, (R_{C_{\mathbf{L}'^*}(s'_{m'})}^{C_{\mathbf{G}'^*}(s'_{m'})}\rho')_u\rangle,$$

which in turn equals (cf. Remark 2.30)

$$\langle \pi, R_{C_{\mathbf{L}^*}(s_m)}^{C_{\mathbf{G}^*}(s_m)}\rho\rangle\langle \pi', R_{C_{\mathbf{L}'^*}(s'_{m'})}^{C_{\mathbf{G}'^*}(s'_{m'})}\rho'\rangle.$$

Therefore, the Lusztig bijection in Theorem 5.9 restricts to (5.6). The statement about the representation $\omega_{m,m',\rho}$ also follows. □

An easy consequence is the following theorem. We omit the proof because it is the same as Theorem 5.10.

Theorem 5.16. *The following diagram is commutative :*

$$\begin{array}{ccc}
\mathrm{Irr}(G_m,\rho) & \xrightarrow[\Xi_s^{G_m}]{\sim} & \mathrm{Irr}(\mathbf{G}_\#(s),\rho_\#)\times\mathrm{Irr}(G_{m-l},\rho_{(1)}) \\
\downarrow{\scriptstyle \Theta_{m,m'}} & & \downarrow{\scriptstyle \mathrm{Id}\otimes\Theta_{m-l,m'-l}} \\
\mathscr{R}(G'_{m'},\rho') & \xrightarrow[\Xi_{s'}^{G_{m'}}]{\sim} & \mathscr{R}(\mathbf{G}_\#(s),\rho_\#)\otimes\mathscr{R}(G'_{m'-l},\rho'_{(1)}).
\end{array}$$

5.3 A generalized correspondence for Weyl groups

Let ρ and ρ' be two irreducible representations of G_m and $G'_{m'}$ respectively. By Theorem 2.15, we have a bijection

$$\mathrm{Irr}(W_{\mathbf{G}_m}(\rho)\times W_{\mathbf{G}'_{m'}}(\rho'))\simeq\mathrm{Irr}(G_m\times G'_{m'},\rho\otimes\rho'). \tag{5.7}$$

Therefore, the representation $\omega_{m,m',\rho}$ induces a representation $\Omega_{m,m',\rho}$ of the direct product of Weyl groups $W_{\mathbf{G}_m}(\rho)\times W_{\mathbf{G}'_{m'}}(\rho')$. We aim at describing this representation in terms of the representations introduced in Conjecture 5.1 and Theorem 5.2.

In Theorem 5.15 we showed that $\omega_{m,m',\rho}$ is identified with $\mathrm{pr}_{\rho_\#}(\mathbf{R}^{G_\#(s)^F})\otimes\omega_{m-l,m'-l,\rho_{(1)}}$ via the Lusztig bijection

$$\mathrm{Irr}(G_m\times G'_{m'},\rho\otimes\rho')\simeq\mathrm{Irr}(C_{\mathbf{G}_m}(s_m)\times C_{\mathbf{G}'_{m'}}(s'_{m'}),\rho_u\otimes\rho'_u). \tag{5.8}$$

Once again, Theorem 2.15 provides a bijection

$$\mathrm{Irr}(\mathbf{G}_{\#}(s) \times \mathbf{G}_{\#}(s), \rho_{\#} \otimes \rho_{\#}) \simeq \mathrm{Irr}(W_{\mathbf{G}_{\#}(s)}(\rho_{\#}) \times W_{\mathbf{G}_{\#}(s)}(\rho_{\#})). \tag{5.9}$$

Moreover, the representation $\mathrm{pr}_{\rho_{\#}}(\mathbf{R}^{\mathbf{G}_{\#}(s)^F})$ corresponds to $\mathbf{R}^{W_{\mathbf{G}_{\#}(s)}(\rho_{\#})}$ via this isomorphism.

Recall that $(\mathbf{G}_m)_{(1)}(s_m)$ and $(\mathbf{G}'_{m'})_{(1)}(s'_{m'})$ are isomorphic to the unitary groups $\mathbf{G}_{m-l}$ and $\mathbf{G}'_{m'-l}$ respectively (cf. Lemma 5.7). Proposition 5.14 provides isomorphisms $\rho_{(1)} \simeq \lambda_k \otimes 1_r$, and $\rho'_{(1)} \simeq \lambda_{k'} \otimes 1_{r'}$. This, together with Theorem 5.2 yields a bijection

$$\mathrm{Irr}((\mathbf{G}_m)_{(1)}(s_m) \times (\mathbf{G}'_{m'})_{(1)}(s'_{m'}), \rho_{(1)} \otimes \rho'_{(1)}) \simeq \mathrm{Irr}(W_r, W_{r'}), \tag{5.10}$$

where $r = \frac{1}{2}(m - l - k(k+1)/2)$ and $r' = \frac{1}{2}(m' - l - k'(k'+1)/2)$. Moreover, this isomorphism identifies $\omega_{m,m',\rho_{(1)}}$ with the representation $\Omega_{r,r'}$ defined in Section 5.1. We can now state the main result of this section.

Theorem 5.17. *Keep the notation above. There is an bijection*

$$\mathrm{Irr}(W_{\mathbf{G}_m}(\rho) \times W_{\mathbf{G}'_{m'}}(\rho')) \simeq \mathrm{Irr}(W_{\mathbf{G}_{\#}(s)}(\rho_{\#}) \times W_{\mathbf{G}_{\#}(s)}(\rho_{\#})) \times \mathrm{Irr}(W_r, W_{r'}).$$

It identifies the representation $\Omega_{m,m',\rho}$ with $\mathbf{R}^{W_{\mathbf{G}_{\#}(s)}(\rho_{\#})} \otimes \Omega_{r,r'}$. Moreover, there are isomorphisms of Weyl groups $W_{\mathbf{G}_m}(\rho) \simeq W_{\mathbf{G}_{\#}(s)}(\rho_{\#}) \times W_r$, and $W_{\mathbf{G}'_{m'}}(\rho') \simeq W_{\mathbf{G}_{\#}(s)}(\rho_{\#}) \times W_{r'}$, compatible with this bijection.

Proof. The bijection in the statement comes from composing bijections (5.7) to (5.10).

The group $W_{\mathbf{G}_m}(\rho)$ is defined as $N_{G_m}(L, \rho)/L$. Moreover, according to Proposition 2.32, there is an isomorphism

$$N_{G_m}(L, \rho)/L \simeq N_{C_{G_m}(s)}(C_L(s_\rho), \rho_u)/C_L(s_\rho).$$

The group on the left side is $W_{\mathbf{G}_m}(\rho)$, whereas the group on the right side above is isomorphic to the direct product $N_{\mathbf{G}_{\#}(s)}(L_{\#}(s), \rho_{\#})/L_{\#}(s) \times N_{\mathbf{G}_{(1)}(s)}(L_{(1)}(s), \rho_{(1)})/L_{(1)}(s)$. By definition this last group is $W_{\mathbf{G}_{\#}(s)}(\rho_{\#}) \times W_r$. Also, the compatibility follows from Proposition 2.32. The assertion for $W_{\mathbf{G}'_{m'}}(\rho')$ has an analogous proof. $\square$

Chapter 6

Extremal representations

6.1 Extremal unipotent representations

We will deal with symplectic-orthogonal and unitary pairs separately because the definiton of "extremal" (i.e."minimal" and "maximal") representation changes from one pair to the other.

We call *partitions of* $n \in \mathbb{N}$ the non-increasing sequences of positive integers

$$\mu = (\mu_1, \ldots, \mu_k), \quad \text{such that} \sum_{1 \leq j \leq k} \mu_j = n.$$

Sometimes we will allow ourselves to add a certain number of zeroes at the end of a partition. The length $l(\mu)$ of μ is the number of μ_j different from zero and we will denote the above sum by $\|\mu\|$. We will denote by ${}^{\mathrm{t}}\mu$ the *dual partition* of μ, that is the partition such that ${}^{\mathrm{t}}\mu_i$ is the cardinal of the set $\{j | \mu_j \geq i\}$, for $1 \leq i \leq \mu_i$.

There is a natural order on the set of partitions of an integer $n \in \mathbb{N}$. Take μ and μ' two partitions of the same integer with length l and l'. Then $\mu \leq \mu'$ if and only if

$$\mu_1 + \ldots + \mu_k \leq \mu'_1 + \ldots + \mu'_k, \;\; 1 \leq k \leq l(\mu).$$

Following [5] we introduce another order between partitions.

Definition 6.1. *Let μ and μ' be two partitions (of possibly different integers). The* intersection $\mu \cap \mu'$ *of μ and μ' is the partition* $(\inf(\mu_1, \mu'_1), \ldots, \inf(\mu_l, \mu'_l))$*, where* $l = \inf\{l(\mu), l(\mu')\}$*. We say that μ is* contained *in μ' if $\mu \cap \mu' = \mu$. They are said to be* close *if $|\mu'_i - \mu_i| \leq 1$ for all i. Finally we say that μ* precedes *μ' and we denote it by $\mu \preceq \mu'$ if μ is contained in μ' and they are close.*

The relation $\preceq$ defines an order. It is important to stress that this order is stronger than the classical order between partitions, i.e. $\mu \preceq \mu'$ implies $\mu \leq \mu'$.

We call *bipartitions of* $n \in \mathbb{N}$ the set of pairs of partitions (λ, μ) such that $\|\lambda\|+\|\mu\| = n$. We denote by $\mathscr{P}_2(n)$ the set of bipartitions (λ, μ) of n.

Irreducible characters of a Weyl group W_n of type B or C are known to be parametrised by bipartitions of n [14, Theorem 5.5.6]. We denote $\chi_{\lambda,\mu}$ the irreducible representation of W_n corresponding to the bipartition (λ, μ) of n.

Proposition 6.2. *[14, Chapter 5] Let (λ, μ) be a bipartition of the integer r, then*

1. $\mathrm{Ind}_{W_r \times W_{l-r}}^{W_l} \chi_{\lambda,\mu} \otimes \phi = \sum_{{}^t\mu \preceq {}^t\mu'} \chi_{\lambda,\mu'}$.
2. $\mathrm{Ind}_{W_r \times W_{l-r}}^{W_l} \chi_{\lambda,\mu} \otimes 1 = \sum_{{}^t\lambda \preceq {}^t\lambda'} \chi_{\lambda',\mu}$.
3. $\phi \otimes \chi_{\lambda,\mu} = \chi_{\mu,\lambda}$

Achar and Henderson [1] introduced the following order between bipartitions

Definition 6.3. *For (ρ, σ), $(\mu, \nu) \in \mathscr{P}_2(n)$, we say that $(\rho, \sigma) \leq (\mu, \nu)$ if and only if the following inequalities hold for all $k \geq 0$:*

$$\rho_1 + \sigma_1 + \cdots + \rho_k + \sigma_k \leq \mu_1 + \nu_1 + \cdots + \mu_k + \nu_k, \text{ and}$$
$$\rho_1 + \sigma_1 + \cdots + \rho_k + \sigma_k + \rho_{k+1} \leq \mu_1 + \nu_1 + \cdots + \mu_k + \nu_k + \mu_{k+1}.$$

We will refer to this as the Achar-Henderson order.

This order has a geometric interpretation. Let V be a vector space of dimension n over an algebraically closed field. The $\mathrm{GL}(V)$-orbits on the *enhanced nilpotent cone* $V \times \mathcal{N}$ (where $\mathcal{N}$ is the variety of nilpotent endomorphisms of V) are parametrized by bipartitions of n (cf. [1, Proposition 2.3]). If we denote by $\mathcal{O}_{\mu,\nu}$ the orbit corresponding to the bipartition (μ, ν) of n, then $\mathcal{O}_{\rho,\sigma}$ is contained in $\overline{\mathcal{O}_{\mu,\nu}}$ (the closure of $\mathcal{O}_{\mu,\nu}$), if and only if $(\rho, \sigma) \leq (\mu, \nu)$ (cf. [1, Theorem 3.9]), the former defines an order called the *closure order*. The Achar-Henderson order on bipartitions is then compatible with this closure order.

A Symplectic-orthogonal pairs

Let $W_n = W(C_n)$ be the Weyl group of $\mathrm{Sp}_{2n}(\overline{\mathbb{F}}_q)$. In [26] Lusztig generalized the Springer correspondence introduced by the Springer in [38] for finite fields of large characteristic. This correspondence is an injective map from the set of irreducible representations of W_n into the set of pairs $(\mathcal{O}, \psi)$ where $\mathcal{O}$ is a unipotent conjugacy class of $\mathrm{Sp}_{2n}(\overline{\mathbb{F}}_q)$ and ψ is an irreducible character of the group $A(u)$ of connected components of the centraliser $C(u)$ of any $u \in \mathcal{O}$.

Recall that a partition is called *symplectic* if each odd part appears with even multiplicity. There is a bijection between symplectic partitions of $2n$ and unipotent conjugacy

classes of $\mathrm{Sp}_{2n}(\overline{\mathbb{F}}_q)$. We denote by $\mathcal{O}_\lambda$ the unipotent orbit associated to the symplectic partition λ.

Consider a symplectic partition of $2n$ by adding a zero if necessary we can suppose λ has an even number $2k$ of parts. We now define $\lambda^*_j = \lambda_{2k-j+1} + j - 1$ for $j = 1, \ldots, 2k$. We divide λ^* into its odd and even parts. Is has the same number of each. Let the odd parts be

$$2\xi^*_1 + 1 < 2\xi^*_2 + 1 < \ldots < 2\xi^*_k + 1$$

and the even parts be

$$2\eta^*_1 < 2\eta^*_2 < \ldots < 2\eta^*_k.$$

Then we have

$$\xi^*_1 < \xi^*_2 < \cdots < \xi^*_k$$
$$\eta^*_1 < \eta^*_2 < \cdots < \eta^*_k.$$

Next we define $\xi_i = \xi^*_{k-i+1} - (k-i)$ and $\eta_i = \eta^*_{k-i+1} - (k-i)$ for each i. We obtain in this way a bipartition (ξ, η) of n. The injective map ι, sending λ to (ξ, η), is closely related to the Springer correspondence.

Given a bipartition (ξ, η) of n, we ensure that ξ has one more part than η by adding zeroes to ξ if necessary, call k the number of parts of η. We associate to (ξ, η) the following *u-symbol*

$$\begin{pmatrix} \xi_{k+1} & \xi_k + 2 & \cdots & \xi_1 + 2k \\ & \eta_k + 1 & \cdots & \eta_1 + 2k - 1 \end{pmatrix}$$

The bipartition (ξ, η) is in the image of the above map if and only if its associated u-symbol is *distinguished*, that is

$$\xi_{k+1} \leq \eta_k + 1 \leq \xi_k + 2 \leq \eta_{k-1} + 3 \leq \cdots$$

In this situation the Springer map sends the representation $\chi_{\xi,\eta}$ of W_n to the pair $(\mathcal{O}_\lambda, 1)$ where λ is the symplectic partition such that $\iota(\lambda) = (\xi, \eta)$, and 1 is the trivial representation of $A(u)$.

The set of all u-symbols which share the same entries with the same multiplicities (in different arrangements) is called a *similarity class*. Each similarity class contains exactly one distinguished u-symbol.

Suppose the bipartition (ξ, η) is not in the image of the above map. If we call (ξ', η') the distinguished u-symbol similar to (ξ, η) and we let λ be the symplectic partition verifying $\iota(\lambda) = (\xi', \eta')$, then the Springer correspondence maps $\chi_{\xi,\eta}$ into the pair $(\mathcal{O}_\lambda, \psi)$ for some character ψ of $A(u)$.

Remark 6.4. *In order to stress the dependance of the symplectic partition λ on the bipartition (ξ,η), we will sometimes write $\lambda=\lambda(\xi,\eta)$. This defines a map sending bipartitions (ξ,η) of n to symplectic partitions $\lambda(\xi,\eta)$ of $2n$.*

Remark 6.5. *The* closure order *between unipotent conjugacy classes is defined by $\mathcal{O}\leq\mathcal{O}'$ if and only if $\mathcal{O}\subset\overline{\mathcal{O}'}$. We saw above that unipotent classes are indexed by symplectic partitions. This bijection is so that the closure order between unipotent classes corresponds to the classical order between the corresponding symplectic partitions. That is, $\mathcal{O}_\lambda\leq\mathcal{O}_{\lambda'}$ if and only if $\lambda\leq\lambda'$.*

Consider a symplectic-orthogonal pair $(\mathrm{Sp}_{2m}(q),\mathrm{O}^\epsilon_{2m'}(q))$. Theorems 4.4 and 4.5 show that the Howe correspondence relates the unipotent Harish-Chandra series $\mathrm{Irr}(\mathrm{Sp}_{2m}(q))_k$ to the series $\mathrm{Irr}(\mathrm{O}^\epsilon_{2m'}(q))^I_k$ if ϵ is the sign of $(-1)^k$, and to the series $\mathrm{Irr}(\mathrm{O}^\epsilon_{2m'}(q))^{II}_{k+1}$ otherwise. This in turn induces a correspondence for the pair of type **B** Weyl groups $(W_{m-k(k+1)},W_{m'-\theta(k)^2})$, where $\theta(k)$ is equal to k or $k+1$ (depending on the cases above).

Let $l=m-k(k+1)$ and $l'=m'-\theta(k)^2$. The correspondence between Weyl groups is conjectured to be given by a representation $\Omega_{l,l'}$, defined in the previous section. Fix a bipartition (ξ',η') of l', we denote by $\Theta(\xi',\eta')$ the set of all irreducible representations $\chi_{\xi,\eta}$ of $\mathrm{Irr}(W_l)$ such that $\chi_{\xi,\eta}\otimes\chi_{\xi',\eta'}$ is an irreducible component of $\Omega_{l,l'}$.

Definition 6.6. *The representation $(\xi_{\min},\eta_{\min})$ (resp. $(\xi_{\max},\eta_{\max})$) in $\Theta(\xi',\eta')$, verifying $\lambda(\xi_{\min},\eta_{\min})\leq\lambda(\xi,\eta)$ (resp. $\lambda(\xi,\eta)\leq\lambda(\xi_{\max},\eta_{\max})$) for all (ξ,η) in $\Theta(\xi',\eta')$, is called* minimal *(resp.* maximal*).*

Note that a priori, these extremal representations need not exist. Our goal is to prove the existence of such representations.

The explicit form of $\Omega_{l,l'}$ depends on the sign of ϵ and the parity of k. Aubert, Michel and Rouquier conjectured that it is given by the representations in (5.1) and (5.2) above. We study these two cases independently.

We suppose from now on that $l>2l'$. This condition is reminiscent of the stable range (see Section 3.5). Indeed, suppose k (and therefore $\theta(k)$) is zero (or equivalently, that the Harish-Chandra series are principal (cf. [10, Chapter 6])). In this case, $l=m$ and $l'=m'$, so that the condition $l>2l'$ actually means that the dual pair $(\mathrm{Sp}_{2m}(q),\mathrm{O}^\epsilon_{2m'}(q))$ is in the stable range (with $\mathrm{O}^\epsilon_{2m'}(q)$ smaller).

First case

Consider the pair $(\mathrm{Sp}_{2m}(q),\mathrm{O}^+_{2m'}(q))$ for k even, or $(\mathrm{Sp}_{2m}(q),\mathrm{O}^-_{2m'}(q))$ for k odd. In these cases, the representation $\Omega_{l,l'}$ is conjectured to be given by

$$\Omega_{l,l'}=\sum_{r=0}^{\min(l,l')}\sum_{\chi\in\mathrm{Irr}(W_r)}(\mathrm{Ind}^{W_l}_{W_r\times W_{l-r}}\chi\otimes\phi)\otimes(\mathrm{Ind}^{W'_l}_{W_r\times W_{l'-r}}\chi\otimes\phi). \tag{6.1}$$

Proposition 6.2 allows us to index the second sum above by bipartitions $(\xi, \zeta) \in \mathscr{P}_2(r)$, the representation then becomes

$$\Omega_{l,l'} = \sum_{r=0}^{\min(l,l')} \sum_{(\xi,\zeta)\in\mathscr{P}_2(r)} \sum_{\eta,\eta'} \chi_{\xi,\eta} \otimes \chi_{\xi,\eta'}, \tag{6.2}$$

where the third sum is over partitions η and η' of $l - |\xi|$ and $l' - |\xi|$ such that ${}^{\mathrm{t}}\zeta \preceq {}^{\mathrm{t}}\eta$ and ${}^{\mathrm{t}}\zeta \preceq {}^{\mathrm{t}}\eta'$.

Lemma 6.7. *Suppose $\Omega_{l,l'}$ is given by 6.1 and let (ξ', η') be a bipartition of l'.*

1. *The bipartition (ξ, η) belongs to $\Theta(\xi', \eta')$ if and only if ${}^{\mathrm{t}}\eta$ and ${}^{\mathrm{t}}\eta'$ have a common predecessor for the $\preceq$ order, and $\xi = \xi'$.*

2. *The smallest element of $\Theta(\xi', \eta')$ for the Achar-Henderson order corresponds to $(\xi', (l - l') \cup \eta')$*

3. *The largest element of $\Theta(\xi', \eta')$ for the Achar-Henderson order corresponds to $(\xi', (l - l' + \eta'_1 + \eta'_2, \eta'_3, \cdots, \eta'_r))$.*

Proof. Item 1 is an easy consequence of equality (6.2).

The representations belonging to $\Theta(\xi', \eta')$ correspond to bipartitions having ξ' as first component, and whose second component shares a common predecessor (for $\preceq$) with η'. Thus we need to prove that the smallest partition (for the natural order) having a common predecessor with η' is $(l - l') \cup \eta'$.

So let $\eta = (\eta_1, \ldots, \eta_r)$ and $\eta' = (\eta'_1, \ldots, \eta'_{r+1})$ have a common predecessor $\zeta = (\zeta_1, \ldots, \zeta_r)$ for the $\preceq$ order (we can suppose that $l(\zeta) = l(\eta') = l(\eta) - 1$ by adding zeros), this implies

$$0 \leq \zeta_r \leq \eta'_r \leq \cdots \leq \eta'_2 \leq \zeta_1 \leq \eta'_1$$
$$0 \leq \eta_{r+1} \leq \zeta_r \leq \eta_r \leq \cdots \leq \eta_2 \leq \zeta_1 \leq \eta_1,$$

which in turn imply

$$\eta'_k \geq \eta_{k+1} \text{ for } k = 1, \ldots, r. \tag{6.3}$$

As (ξ, η) and (ξ, η') are bipartitions of l and l' respectively, $|\eta| - |\eta'| = l - l'$, i.e.

$$\sum_{i=1}^{r+1} \eta_i = l - l' + \sum_{i=1}^{r} \eta'_i. \tag{6.4}$$

This equality and the inequalities in (6.3) provide

$$\sum_{i=1}^{k+1} \eta_i \geq l - l' + \sum_{i=1}^{k} \eta'_i,$$

for $k = 0, \ldots, r$, i.e. $\eta' \geq (l - l') \cup \eta$. This proves item 2.

The proof of item 3 is analogous to that of the previous. Indeed, we have

$$\eta_k \geq \eta'_{k+1} \text{ for } k = 1, \ldots, r-1.$$

These together with equality (6.4) imply

$$\sum_{i=1}^{k} \eta_i \leq l - l' + \eta'_1 + \sum_{i=1}^{k} \eta'_{i+1},$$

for $k = 1, \ldots, r+1$, where we set $\eta'_i = 0$ for $i > r$. This implies the assertion. □

In Remark 6.4 we defined a map sending bipartitions (ξ, η) of l to symplectic partitions $\lambda = \lambda(\xi, \eta)$ of $2l$, it was obtained from the Springer correspondence for $\mathrm{Sp}_{2l}(\overline{\mathbb{F}}_q)$. The following result states that this map is increasing when restricted to $\Theta(\xi', \eta')$.

Proposition 6.8. *Under the assumption that $\Omega_{l,l'}$ is given by 6.1, if $(\xi', \zeta) \leq (\xi', \zeta')$ belongs to $\Theta(\xi', \eta')$, then $\lambda(\xi', \zeta) \leq \lambda(\xi', \zeta')$.*

Proof. By adding zeros we can suppose that the ζ and ζ' have same number k of parts and that ξ' has one more part than both. Let

$$\begin{pmatrix} \xi'_{k+1} & \xi'_k + 2 & \cdots & \xi'_1 + 2k \\ & \zeta_k + 1 & \cdots & \zeta_1 + 2k - 1 \end{pmatrix} \text{ and } \begin{pmatrix} \xi'_{k+1} & \xi'_k + 2 & \cdots & \xi'_1 + 2k \\ & \zeta'_k + 1 & \cdots & \zeta'_1 + 2k - 1 \end{pmatrix}$$

be the u-symbols corresponding to (ξ', ζ) and (ξ', ζ') respectively. Let

$$\begin{pmatrix} \gamma_{2k+1} & \gamma_{2k-1} & \cdots & \gamma_1 \\ & \gamma_{2k} & \cdots & \gamma_2 \end{pmatrix} \text{ and } \begin{pmatrix} \gamma'_{2k+1} & \gamma'_{2k-1} & \cdots & \gamma'_1 \\ & \gamma'_{2k} & \cdots & \gamma'_2 \end{pmatrix}$$

their associated distinguished u-symbols. The bipartitions (α, β) and (α', β') corresponding to these (by the algorithm described at the beginning of this section) verify $(\alpha, \beta) \leq (\alpha', \beta')$ if and only if $\gamma \leq \gamma'$, where $\gamma = (\gamma_1, \ldots, \gamma_{2k+1})$ and $\gamma = (\gamma_1, \ldots, \gamma_{2k+1})$. This is in turn equivalent to $\lambda(\xi', \zeta) \leq \lambda(\xi', \zeta')$.

In remains to verify the inequality $\gamma \leq \gamma'$. Take $r \in \{1, \ldots, 2k+1\}$, we must show that

$$\gamma_1 + \cdots + \gamma_r \leq \gamma'_1 + \cdots + \gamma'_r.$$

The sum in the left can be expressed as

$$\gamma_1 + \cdots + \gamma_r = \sum_{i=1}^{t} \zeta_i + 2(k-i) + 1 + \sum_{i=1}^{s} \xi'_i + 2(k-i+1),$$

where $t + s = r$. Likewise,

$$\gamma'_1 + \cdots + \gamma'_r = \sum_{i=1}^{t'} \zeta'_i + 2(k-i) + 1 + \sum_{i=1}^{s'} \xi'_i + 2(k-i+1),$$

where $t' + s' = r$. Suppose that $s' \geq s$ (the case $s \geq s'$ has a similar proof). Distinguishedness implies

$$\xi'_{s+i} + 2(k+1-s-i) \geq \zeta'_{t'+i} + 2(k-t'-i) + 1$$

for all $i = 1, \ldots, s' - s$, so

$$\begin{aligned}
\sum_{i=1}^{t'} \zeta'_i + 2(k-i) + 1 + \sum_{i=1}^{s'} \xi_i + 2(k-i+1) \\
&\geq \sum_{i=1}^{t} \zeta'_i + 2(k-i) + 1 + \sum_{i=1}^{s} \xi_i + 2(k-i+1) \\
&\geq \sum_{i=1}^{t} \zeta_i + 2(k-i) + 1 + \sum_{i=1}^{s} \xi_i + 2(k-i+1),
\end{aligned}$$

the last inequality coming from $\zeta' \geq \zeta$. □

The main result of this section is a consequence of the previous proposition.

Theorem 6.9. *Suppose $\Omega_{l,l'}$ is given by 6.1. Let (ξ', η') be a bipartition of l'. There exists a minimal (resp. maximal) representation in $\Theta(\xi', \eta')$. It is given by the bipartition $(\xi', (l - l') \cup \eta')$ (resp. $(\xi', (l - l' + \eta'_1 + \eta'_2, \eta'_3, \cdots, \eta'_r))$).*

Second case

We now analyse pairs $(\mathrm{Sp}_{2m}(q), \mathrm{O}^+_{2m'}(q))$ for k odd and $(\mathrm{Sp}_{2m}(q), \mathrm{O}^-_{2m'}(q))$ for k even. In these cases the representation $\Omega_{l,l'}$ is conjectured to be given by

$$\Omega_{l,l'} = \sum_{r=0}^{\min(l,l')} \sum_{\chi \in \mathrm{Irr}(W_r)} (\mathrm{Ind}_{W_r \times W_{l-r}}^{W_l} \chi \otimes 1) \otimes (\mathrm{Ind}_{W_r \times W_{l'-r}}^{W'_l} \chi \otimes \phi). \tag{6.5}$$

Proposition 6.2 allows us to index the second sum above by bipartitions $(\xi, \zeta) \in \mathscr{P}_2(r)$, the representation then becomes

$$\Omega_{l,l'} = \sum_{r=0}^{\min(l,l')} \sum_{(\xi,\eta)\in\mathcal{P}_2(r)} \sum_{\xi',\eta'} \chi_{\xi',\eta} \otimes \chi_{\xi,\eta'}, \tag{6.6}$$

Where the sum is over partitions ξ' and η' of $l - |\eta|$ and $l' - |\xi|$ such that ${}^{\mathrm{t}}\xi \preceq {}^{\mathrm{t}}\xi'$ and ${}^{\mathrm{t}}\eta \preceq {}^{\mathrm{t}}\eta'$.

Lemma 6.10. *Suppose $\Omega_{l,l'}$ is given by 6.5, let (ξ', η') be a bipartition of l' and $\Theta(\xi', \eta')$ be the set of the representations $\chi_{\xi,\eta}$ of $\mathrm{Irr}(W_l)$ such that $\chi_{\xi,\eta} \otimes \chi_{\xi',\eta'}$ appears in $\Omega_{l,l'}$.*

1. *The representation $\chi_{\xi,\eta}$ belongs to $\Theta(\xi', \eta')$ if and only if ${}^{\mathrm{t}}\xi' \preceq {}^{\mathrm{t}}\xi$ and ${}^{\mathrm{t}}\eta \preceq {}^{\mathrm{t}}\eta'$.*

2. *Suppose the number of parts of η, η' and ξ' are the same and equal to an integer r, and that ξ has one more part than these three. Then for P, $Q \subset \{1, \ldots, r\}$ arbitrary,*

$$\xi_1 + \sum_P \xi_{i+1} + \sum_Q \eta_i \geq (l - l') + \sum_P \xi'_i + \sum_Q \eta'_i.$$

   In particular, the smallest element of $\Theta(\xi', \eta')$ for Achar-Henderson order is $((l - l') \cup \xi', \eta')$.

3. *Under the same asumptions of the previous item, for $P \subset \{1, \ldots, r+1\}$ and $Q \subset \{1, \ldots, r\}$*

$$\sum_P \xi_i + \sum_Q \eta_i \leq l - l' + \eta'_1 + \sum_P \xi'_i + \sum_Q \eta'_{i+1}.$$

   Thus, the largest element of $\Theta(\xi', \eta')$ for the Achar-Henderson order corresponds to the bipartition $((l - l' + \eta'_1 + \xi'_1, \xi'_2, \ldots, \xi'_r), (\eta'_2, \ldots, \eta'_r))$.

Proof. Item 1 is a straightforward consequence of equality (6.6).

Let $\chi_{\xi,\eta}$ belong to $\Theta(\xi', \eta')$ so that ${}^{\mathrm{t}}\xi' \preceq {}^{\mathrm{t}}\xi$ and ${}^{\mathrm{t}}\eta \preceq {}^{\mathrm{t}}\eta'$. These imply

$$\xi_{r+1} \leq \xi'_k \leq \xi_k \text{ and } \eta_k \leq \eta'_k \tag{6.7}$$

for all $k = 1, \ldots, r$. Rewriting

$$|\xi| + |\eta| - |\xi'| - |\eta'| = l - l'$$

as

$$\xi_1 - (l - l') + \sum_{i=1}^{r} \xi_{i+1} - \xi'_i + \sum_{i=1}^{r} \eta_i - \eta'_i = 0,$$

inequalities (6.7) imply

$$\xi_1 - (l - l') + \sum_P \xi_{i+1} - \xi'_i + \sum_Q \eta_i - \eta'_i \geq 0,$$

for P, $Q \subset \{1, \ldots, r\}$ arbitrary. It is simple to see this implies $(\xi, \eta) \geq ((l - l') \cup \xi', \eta')$.

The proof of item (3) is similar to that of (2). □

Unlike the previous section, in this case the map $\lambda = \lambda(\xi, \eta)$ is not increasing when restricted to $\Theta(\xi', \eta')$. However, we can prove that the representations $(\xi_{\min}, \eta_{\min})$ and $(\xi_{\max}, \eta_{\max})$, obtained in items (2) and (3) of the previous theorem, are indeed the minimal and maximal representations in $\Theta(\xi', \eta')$ (according to Definition 6.6).

Theorem 6.11. *Suppose $\Omega_{l,l'}$ is given by 6.5, and let (ξ', η') be a bipartition of l'. Then, the minimal (resp. maximal) representation in $\Theta(\xi', \eta')$ correspond to the bipartition $((l - l') \cup \xi', \eta')$ (resp. $((l - l' + \eta'_1 + \xi'_1, \xi'_2, \ldots, \xi'_r), (\eta'_2, \ldots, \eta'_r))$).*

Proof. As in the proof of Proposition 6.9 we can suppose that η, η' and ξ' have the same number k of parts and that ξ has $k + 1$ parts. Let

$$\begin{pmatrix} \xi_{k+1} & \xi_k + 2 & \cdots & \xi_1 + 2k \\ & \eta_k + 1 & \cdots & \eta_1 + 2k - 1 \end{pmatrix} \text{ and } \begin{pmatrix} \xi'_k & \xi'_{k-1} + 2 & \cdots & l - l' + 2k \\ & \eta'_k + 1 & \cdots & \eta'_1 + 2k - 1 \end{pmatrix}$$

be the u-symbols corresponding to (ξ, η) and $((l - l') \cup \xi', \eta')$ respectively. Let

$$\begin{pmatrix} \gamma_{2k+1} & \gamma_{2k-1} & \cdots & \gamma_1 \\ & \gamma_{2k} & \cdots & \gamma_2 \end{pmatrix} \text{ and } \begin{pmatrix} \gamma'_{2k+1} & \gamma'_{2k-1} & \cdots & \gamma'_1 \\ & \gamma'_{2k} & \cdots & \gamma'_2 \end{pmatrix}$$

be the corresponding distinguished u-symbols. Distinguishedness and the inequality $l > 2l'$ imply that $\gamma'_1 = l - l' + 2k$ and $\gamma_1 = \xi_1 + 2k$.

For $r \in \{1, \ldots, 2k + 1\}$, there exist P, $Q \subset \{1, \ldots, r\}$ such that

$$\gamma'_1 + \cdots + \gamma'_r = (l - l' + 2k) + \sum_P \xi'_i + 2(k - i) + \sum_Q \eta'_i + 2(k - i) + 1.$$

The right side in the above inequality is smaller than

$$(\xi_1 + 2k) + \sum_P \xi_{i+1} + 2(k - i - 1) + \sum_Q \eta_i + 2(k - i) + 1$$

by item 2 of Lemma 6.10. The last sum is smaller than $\gamma_1 + \cdots + \gamma_r$ (because we are dealing with a distinguished symbol). This means that $\gamma' \leq \gamma$, and as in proof of Proposition 6.9, it is equivalent to $\lambda((l - l') \cup \xi', \eta') \leq \lambda(\xi, \eta)$.

The assertion concerning the maximal representation has a similar proof. □

B Unitary pairs

Let $\mathbf{GL}_n$ be the general linear group of rank n with coefficients in $\overline{\mathbb{F}}_q$, and let w_0 be the permutation matrix with ones in the antidiagonal and zeroes elsewhere. Endow this group with a Frobenius endomorphism $F = {}^{w_0\mathrm{t}}F_q{}^{-1}$, where F_q denotes the standard Frobenius. Its group $\mathbf{GL}_n^F$ of rational points is the unitary group $\mathbf{U}_n(q)$. The choice of this Frobenius allows the group of diagonal matrices $\mathbf{T}$ to be a rational maximal torus. Denote its Weyl group $W(\mathbf{T})$ by W. It is isomorphic to the symmetric group $\mathfrak{S}_n$

As in Proposition 2.20, let μ be a partition of n, and denote by the same letter the corresponding irreducible character of $\mathfrak{S}_n$. Define

$$R_\mu = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \mu(\sigma) R_{\mathbf{T}_\sigma}^{\mathbf{U}_n}(1).$$

These central characters provide all the unipotent representations of $\mathbf{U}_n(q)$ up to the sign

$$\varepsilon_\mu = (-1)^{\sum_{i=1}^k \binom{\mu_i}{2} + \binom{m(m-1)}{2}}.$$

Proposition 6.12. *The characters of unipotent irreducible representations of $\boldsymbol{U}_n(q)$ are given by $\varepsilon_\mu R_\mu$, for different partitions μ of n.*

In what follows we need to use some combinatorics related to partitions. We start by defining the *Young diagram* associated to the partition μ of m as the subset of $\mathbb{N}\times\mathbb{N}$ defined by :

$$D(\mu) = \{(i,j)\in\mathbb{N}\times\mathbb{N} | 1\le i\le \mu_j\}.$$

We can visualize this set as a finite collection of squares with side length 1, in the north-east quadrant of the euclidean plane, arranged in left-justified rows, and such that the i-th row has μ_i boxes. We call *rim* of μ, the boxes belonging to $\{(i,j)\in D(\mu) | (i,j+1)\notin D(\mu) \text{ or } (i+1,j)\notin D(\mu)\}$. We call *rim 2-hook* of μ a pair $\{(i,j),(i,j+1)\}$ or $\{(i,j),(i+1,j)\}$ of elements of the rim of μ, such that we obtain the diagram of a partition after removing these elements from $D(\mu)$. Finally, the 2-*core* of μ is the partition obtained by removing as many rim 2-hooks as possible from the diagram of μ.

Let again $\mu = (\mu_1,\dots,\mu_r)$ be a partition of m, and let $t\ge r$ be an integer. We call *t-set of β-numbers* associated to μ, the set $\beta = \{\beta_i\}$ where $\beta_i = \mu_i + t - i$. Conversely, to each decreasing sequence $\beta = \{\beta_1,\dots,\beta_t\}$ of positive integers, we can associate a partition μ, defined by $\mu_i = \beta_i + t - i$.

If β denotes the t-set of β-numbers associated to μ, we set $\beta(0) = \{\beta_i/2 | \beta_i \text{ is even}\}$ and $\beta(1) = \{(\beta_i - 1)/2 | \beta_i \text{ is odd}\}$. Let $\mu(0)$ and $\mu(1)$ be the partitions associated to the sets of β-numbers $\beta(0)$ and $\beta(1)$. Then, the pair $(\mu(0),\mu(1))$ depends only on the congruence class $\bar{t}$ of t modulo 2 . We call $\mu(0)$ and $\mu(1)$ the 2-*quotients of parameter* $\bar{t}$ of μ .

As explained above, we have a bijection between the category $\mathscr{R}(\mathrm{U}_m(q))_k$, of com-

plex representations spanned by $\mathrm{Irr}(\mathrm{U}_m(q))_k$, and the set of irreducible representations of $W_{\frac{1}{2}(m-k(k+1)/2)}$. This bijection allows us to describe explicitely the characters in this Harish-Chandra series.

Proposition 6.13. *[12, Appendice, proposition p. 224]*

1. *The unique cuspidal unipotent representation λ_k of the unitary group $\mathrm{U}_{(k^2+k)/2}(q)$ is $\varepsilon_{\tau_k} R_{\tau_k}$ where τ_k is the k-th 2-core $\tau_k = (k, \ldots, 1)$*

2. *For $n \geq 1/2(k^2+k)$ the irreducible characters of $\mathscr{R}(\mathrm{U}_n(q))_k$ are $\varepsilon_\mu R_\mu$ where μ is a partition of n of 2-core τ_k. This character is related to the bipartition $(\mu(0), \mu(1))$ (where $\mu(0)$ and $\mu(1)$ are the 2-quotients of parameter 1 of μ) under the bijection given in Theorem 2.15.*

This theorem tell us that for a fixed k there is a bijection between the bipartitions of $\frac{1}{2}(n - k(k+1)/2)$, the representations in the Harish-Chandra series $\mathscr{R}(\mathrm{U}_n(q))_k$, and the partitions of n with 2-core τ_k.

The definition of 2-core is not useful to do computations. We need to express it otherwise.

Proposition 6.14. *[5, Lemma 5.8] If μ' is a partition obtained from μ by removing a 2-rimhook, then the β-set of μ' is $\{\beta_1, \ldots, \beta_{j_1}, \beta_j - 2, \beta_{j+1}, \ldots, \beta_t\}$, for a certain $j \leq t$. In particular, the β-sets of a partition and its 2-core have the same number of even (resp. odd) elements.*

An easy calculation shows that its β-set is $\beta_k = \{0, 2, \ldots, 2t_0 - 2, 1, 3, \ldots, 2t_1 - 1\}$ where $t_0 = |\beta_k(0)|$ and $t_1 = |\beta_k(1)|$. These two last numbers depend on the parity of k : if k is even then $t_0 = t + k + 1/2$ and $t_1 = t - k - 1/2$; and if k is odd then $t_0 = t - k/2$ and $t_1 = t + k/2$.

Proposition 6.15. *Let μ and μ' be two bipartitions of the same integer l and let β and β' be their β-sets repectively. Then μ and μ' have the same 2-core if and only if $|\beta(0)| = |\beta'(0)|$ and $|\beta(1)| = |\beta'(1)|$.*

Proof. Suppose that μ and μ' have the same 2-core. Corollary 6.14 says that the number of even (resp. odd) elements in the β-sets of μ and μ' equal the number of even (resp. odd) elements in the β-set of the common 2-core, so we have $|\beta(0)| = |\beta'(0)|$ and $|\beta(1)| = |\beta'(1)|$.

If μ and μ' have different 2-cores τ_k and $\tau_{k'}$, assuming that $k < k'$ we have 4 cases depending on the parity of k and k'. For instance, if they're both odd then $|\beta_k(1)| = t + k/2 < t + k'/2 = |\beta_{k'}(1)|$ and $|\beta_{k'}(0)| = t - k'/2 < t - k/2 = |\beta_k(0)|$, so by Corollary 6.14 $|\beta'(0)| < |\beta(0)|$ and $|\beta(1)| < |\beta'(1)|$. The other 3 cases are analogous. □

Consider a unitary pair $(\mathrm{U}_m(q), \mathrm{U}_{m'}(q))$. Theorems 4.4 and 4.5 show that the Howe correspondence relates the unipotent Harish-Chandra series $\mathrm{Irr}(U_m(q))_k$ to $\mathrm{Irr}(U_{m'}(q))_{\theta(k)}$.

This induces a correspondence for the pair $(W_{\frac{1}{2}(m-k(k+1)/2)}, W_{\frac{1}{2}(m'-\theta(k)(\theta(k)+1)/2)})$ of type **B** Weyl groups. The integer $\theta(k)$ is equal to $k-1$ or $k+1$, depending on the class of k modulo 4 (cf. Theorem 4.4).

Let $l = \frac{1}{2}(m - k(k+1)/2)$ and $l' = \frac{1}{2}(m' - \theta(k)(\theta(k)+1)/2)$. The correspondence for the pair $(W_l, W_{l'})$ is proven (and not conjectural as for symplectic-orthogonal pairs) to be given by the representation $\Omega_{l,l'}$ defined in Section 5.1. Fix a bipartition (λ, μ) of l, and denote by $\Theta(\lambda, \mu)$ the set of all irreducible representations of $W_{l'}$ such that $\chi_{\lambda,\mu} \otimes \chi_{\lambda',\mu'}$ appears in $\Omega_{l,l'}$.

Fix an integer k such that $k(k+1) \leq 2m$, and set $l = \frac{1}{2}(m - k(k+1)/2)$. From proposition 6.13 we have a bijection between $\mathrm{Irr}(W_l)$ and $\mathscr{R}(U_m(q))_k$, sending the irreducible representation $\chi_{\lambda,\mu}$ to the unipotent character $\varepsilon_\nu R_\nu$, where $\nu = \nu(\lambda, \mu)$ is the partition of n with τ_k as 2-core and (λ, μ) as 2-quotient of parameter 1.

We now introduce an order in the set $\Theta(\lambda, \mu)$.

Definition 6.16. *Let (λ', μ') and $(\tilde{\lambda}, \tilde{\mu})$ belong to $\Theta(\lambda, \mu)$. Then $(\lambda', \mu') \preceq (\tilde{\lambda}, \tilde{\mu})$ if and only if $\nu(\lambda', \mu') \leq \nu(\tilde{\lambda}, \tilde{\mu})$. This defines a partial order in $\Theta(\lambda, \mu)$.*

We show in the following sections that $\Theta(\lambda, \mu)$ admits a minimal (resp. maximal) representation, for the order just defined. We denote this representation by $(\lambda_{\min}, \mu_{\min})$ (resp. $(\lambda_{\max}, \mu_{\max})$)

The explicit form of $\Omega_{l,l'}$ depends on the parity of k, and is given by the representations (5.3) and (5.4) above. We study these two cases separately. As for symplectic-orthogonal pairs, we suppose that $l' > 2l$.

First case

Consider the pair $(\mathrm{U}_m(q), \mathrm{U}_{m'}(q))$ for k odd or $k = \theta(k) = 0$. In these cases, the Howe correspondence is given by the representation $\Omega_{l,l'}$ whose character is :

$$\Omega_{l,l'} = \sum_{r=0}^{\min(l,l')} \sum_{\chi \in \mathrm{Irr}(W_r)} (\mathrm{Ind}_{W_r \times W_{l-r}}^{W_l} \chi \otimes 1) \otimes (\mathrm{Ind}_{W_r \times W_{l'-r}}^{W_{l'}} \phi\chi \otimes 1),$$

Proposition 6.2 allows us to write the sum above as

$$\Omega_{l,l'} = \sum_{r=0}^{\min(l,l')} \sum_{(\lambda,\mu) \in \mathcal{P}_2(r)} \sum_{\lambda',\mu'} \chi_{\lambda',\mu} \otimes \chi_{\mu',\lambda},$$

the third sum being over partitions λ' and μ' of $l - |\mu|$ and $l' - |\lambda|$ such that ${}^{\mathrm{t}}\lambda \preceq {}^{\mathrm{t}}\lambda'$ and ${}^{\mathrm{t}}\mu \preceq {}^{\mathrm{t}}\mu'$. The following statement (and its proof) is similar to that of Lemma 6.7.

Lemma 6.17. *Let (λ, μ) be a bipartition of l.*

1. *Representations in* $\Theta(\lambda,\mu)$ *correspond to bipartitions* (μ',λ') *such that* ${}^{\mathrm{t}}\lambda' \preceq {}^{\mathrm{t}}\lambda$ *and* ${}^{\mathrm{t}}\mu \preceq {}^{\mathrm{t}}\mu'$.

2. *The smallest element of* $\Theta(\lambda,\mu)$ *for the Achar-Henderson order is* $((l'-l)\cup\mu,\lambda)$.

3. *The largest representation in* $\Theta(\lambda,\mu)$ *for the Achar-Henderson order is* $((l'-l+\lambda_1+\mu_1,\mu_2,\cdots,\mu_r),(\lambda_2,\cdots,\lambda_r))$

Theorem 6.18. *There exists a minimal (resp. maximal) representation in* $\Theta(\lambda,\mu)$ *(cf. Defiintion 6.16), it corresponds to the bipartition* $((l'-l)\cup\mu,\lambda)$ *(resp.* $((l'-l+\lambda_1+\mu_1,\mu_2,\cdots,\mu_r),(\lambda_2,\cdots,\lambda_r)))$ *of* l'.

Proof. Consider all the elements of $\Theta(\lambda,\mu)$ having the same second component, say λ'. As in Lemma 6.7 we can prove that the smallest (resp. largest) of these bipartitions for the Achar-Henderson order is $((a)\cup\mu,\lambda')$ (resp. $((a+\mu_1,\mu_2,\ldots,\mu_r),\lambda')$) with $a = l'-l+|\lambda|-|\lambda'|$.

Recall that representations $\epsilon_\nu R_\nu$ in $\mathscr{R}(U_{m'}(q))_{k'}$ have $\tau_{k'}$ as common 2-core and that this fixes the length of the partitions in the 2-quotient (see Proposition 6.15). We can use Theorem 6.21 to assert that the smallest (resp. largest) partition ν (for the classical order on partitions) having a 2-quotient with λ' as second component corresponds to the 2-quotient $((a)\cup\mu,\lambda')$ (resp. $((a+\mu_1,\mu_2,\ldots,\mu_r),\lambda')$).

We need still to compare partitions ν having 2-quotients of the form $((a')\cup\kappa,\lambda')$ for a fixed κ. Indeed, for λ' fixed, both the minimal and maximal 2-quotients are of this form (with $\kappa=\mu$ for the minimal and $\kappa=(\mu_2,\ldots,\mu_r)$ for the maximal). Let's consider two 2-quotients $((a')\cup\kappa,\lambda')$ and $((\tilde{a})\cup\kappa,\tilde{\lambda})$ such that

$$((a')\cup\kappa,\lambda') \leq ((\tilde{a})\cup\kappa,\tilde{\lambda}).$$

This amounts to the following inequalities

$$a'+\sum_{i=1}^{k}\lambda'_i \leq \tilde{a}+\sum_{i=1}^{k}\tilde{\lambda}_i,\ \ k=0,1,\ldots,t_1, \tag{6.8}$$

$\mathrm{f}t_1$ is the number of parts of λ' and $\tilde{\lambda}$ and $t_0 = l(\nu)+1$ (By Proposition 6.15, these lengths are fixed by the 2-core). The beta sets β' and $\tilde{\beta}$ corresponding to $((a')\cup\kappa,\lambda')$ and $((\tilde{a})\cup\kappa,\tilde{\lambda})$ respectively,

$$\beta' = \{2(a'+t_0-1),2(\kappa_{i-1}+t_0-i),2(\lambda'_j+t_1-j)+1\}$$
$$\tilde{\beta} = \{2(\tilde{a}+t_0-1),2(\kappa_{i-1}+t_0-i),2(\tilde{\lambda}_j+t_1-j)+1\},$$

where $2\leq i\leq t_0, 1\leq j\leq t_1$ in both sets.

Let $t = t_0 + t_1$ and $\beta'_1 > \cdots > \beta'_t$ (resp. $\tilde{\beta}_1 > \cdots > \tilde{\beta}_t$) denote the elements of β' (resp. $\tilde{\beta}$) after reordering. For $l = 1, \ldots, t$

$$\begin{aligned}\sum_{i=1}^{l} \beta'_i &= 2(a' + t_0 - 1) + \sum_{n=2}^{N} 2(\kappa_{n-1} + t_0 - n) + \sum_{m=1}^{M} 2(\lambda'_m + t_1 - 1) + 1 \\ &= 2(\tilde{a} + t_0 - 1) + \sum_{n=2}^{N} 2(\kappa_{n-1} + t_0 - n) + \sum_{m=1}^{M} 2(\tilde{\lambda}_m + t_1 - 1) + 1 \\ &\leq \sum_{i=1}^{l} \tilde{\beta}_i,\end{aligned}$$

the second equality coming from (6.8) and the last from the fact that the terms on the right are the l biggest in $\tilde{\beta}$. The result holds thanks to Lemma 6.17. □

Second case

Consider the pair $(\mathrm{U}_m(q), \mathrm{U}_{m'}(q))$ for k even and different from zero. In this case, the representation $\Omega_{l,l'}$ is given by

$$\Omega_{l,l'} = \sum_{r=0}^{\min(l,l')} \sum_{\chi \in \mathrm{Irr}(W_r)} (\mathrm{Ind}_{W_r \times W_{l-r}}^{W_l} \chi \otimes \phi) \otimes (\mathrm{Ind}_{W_r \times W_{l'-r}}^{W'_l} \phi\chi \otimes 1),$$

Once again, we use Proposition 6.2 to rewrite this sum as

$$\Omega_{l,l'} = \sum_{r=0}^{\min(l,l')} \sum_{(\lambda,\nu) \in \mathcal{P}_2(r)} \sum_{\mu,\mu'} \chi_{\lambda,\mu} \otimes \chi_{\mu',\lambda},$$

where the third sum is over partitions μ and μ' of $l - |\lambda|$ and $l' - |\lambda|$ such that ${}^{\mathrm{t}}\nu \preceq {}^{\mathrm{t}}\mu$ and ${}^{\mathrm{t}}\nu \preceq {}^{\mathrm{t}}\mu'$.

Lemma 6.19. *Let (λ, μ) be a bipartition of l.*

1. *An irreducible representation $\chi_{\mu',\lambda'}$ of $W_{l'}$ belongs to $\Theta(\lambda, \mu)$ if and only if ${}^{\mathrm{t}}\mu$ and ${}^{\mathrm{t}}\mu'$ have a common predecesor for the order $\preceq$ and that $\lambda' = \lambda$.*

2. *The smallest element of $\Theta(\lambda, \mu)$ for the Achar-Henderson order is $((l' - l) \cup \mu, \lambda)$.*

3. *The largest element of $\Theta(\lambda, \mu)$ for the Achar-Henderson order is $((l' - l + \mu_1 + \mu_2, \mu_3, \ldots, \mu_r), \lambda)$.*

Proposition 6.20. *Let ν and ν' denote two partitions of m. Suppose they have the same 2-core and that their 2-quotients are (μ, λ) and (μ', λ) respectively. If $(\mu, \lambda) \leq (\mu', \lambda)$ then $\nu \leq \nu'$.*

Proof. Call t_1 the number of parts of λ. As ν and ν' have the same 2-core we can suppose μ and μ' to have the same number of parts t_0 so that the β-sets of ν and ν' are

$$\beta = \{2(\mu_i + t - i), 2(\lambda_j + t - j) + 1 | 1 \leq i \leq t_0, 1 \leq j \leq t_1\}$$

and

$$\beta = \{2(\mu'_i + t - i), 2(\lambda_j + t - j) + 1 | 1 \leq i \leq t_0, 1 \leq j \leq t_1\}$$

respectively.

Suppose that, after ordering the elements of β are $\beta_1 > \cdots > \beta_t$, and those of β' are $\beta'_1 > \cdots > \beta'_t$ for $t = t_0 + t_1$. The hypothesis $(\mu, \lambda) \leq (\mu', \lambda)$ is equivalent to $\mu \leq \mu'$. For all k we can find non negatives integers r and s verifying $r + s = k$ such that

$$\begin{aligned}\sum_{i=1}^{k} \beta_i &= \sum_{i=1}^{r} 2(\mu_i + t - i) + \sum_{i=1}^{s} 2(\lambda_i + t - i) + 1 \\ &\leq \sum_{i=1}^{r} 2(\mu'_i + t - i) + \sum_{i=1}^{s} 2(\lambda_i + t - i) + 1 \leq \sum_{i=1}^{k} \beta'_i.\end{aligned}$$

The last inequality is true because the elements to its right are the k biggest elements of β'.

It is easy to see that the set of inequalities

$$\sum_{i=1}^{k} \beta_i \leq \sum_{i=1}^{k} \beta'_i$$

for $k = 1, \ldots, t$ is equivalent to $\nu \leq \nu'$. The last assertion is a consequence of Lemma 6.19. □

The bijection between $\mathrm{Irr}(W_l)$ and $\mathscr{R}(U_n(q))_k$, sending $\chi_{\lambda,\mu}$ to $\varepsilon_\nu R_\nu$, where ν is a partition of n with 2-core τ_k and 2-quotient (of parameter 1) (λ, μ) (Proposition 6.13), can be seen as one sending bipartitions (λ, μ) of l to partitions $\nu = \nu(\lambda, \mu)$ of n (with a fixed 2-core τ_k). Proposition 6.20 tells us that this map is increasing when restricted to $\Theta(\lambda, \mu)$. Indeed, all bipartitions in this set share the same second component. These remarks and Lemma 6.19 imply the following :

Theorem 6.21. *Let (λ, μ) be a bipartition of l. There exists a minimal (resp. maximal) representation in $\Theta(\lambda, \mu)$. It is given by the bipartition $((l' - l) \cup \mu, \lambda)$ (resp. $((l' - l + \mu_1 + \mu_2, \mu_3, \ldots, \mu_r), \lambda)$).*

6.2 Extremal representations

In Section 5.2, we saw how the Lusztig bijection allows the study of the Howe correspondence to be brought to the study of its effect on unipotent representations. Together with the results in the previous section, this enables us to find extremal representations in $\Theta(\pi)$ for π not necessarily unipotent.

Let $(G_m, G'_{m'})$ denote an unitary dual pair, and π be an irreducible representation of G_m. Once again we denote by $\Theta_{m,m'}(\pi)$ the set of irreducible representations π' of $G'_{m'}$, such that

$$\mathrm{Hom}_{G_m \times G'_{m'}}(\omega_{m,m'}, \pi \otimes \pi') \neq 0.$$

We now proceed to describe the set $\Theta_{m,m'}(\pi)$. Knowing the form of the representations it contains is the first step in finding the extremal representations.

Proposition 6.22. *The representation π' of $G'_{m'}$ belongs to $\Theta_{m,m'}(\pi)$, if and only if, $\pi'_{\#} = \pi_{\#}$ and $\pi'_{(1)}$ belongs to $\Theta_{m-l,m'-l}(\pi_{(1)})$. In particular, the map sending $\pi' \in \mathscr{R}(G'_{m'})$ to $\pi'_{(1)} \in \mathscr{R}(G'_{m'-l})$ defines a bijection between $\Theta_{m,m'}(\pi)$ and $\Theta_{m-l,m'-l}(\pi_{(1)})$.*

Proof. This is a restatement of Theorem 5.16. □

This proposition tells us, in other words, that the Lusztig bijection restricts to

$$\mathfrak{L}_s^{\mathbf{G}'_{m'}} : \Theta_{m,m'}(\pi) \simeq \{\pi_{\#}\} \times \Theta_{m-l,m'-l}(\pi_{(1)}).$$

In Definition 6.16 we introduced a partial order on the set $\Theta_{m,m'}(\pi)$, where π is a unipotent representation of G_m. Using Proposition 6.22, we can extend this order to arbitrary irreducible representations.

Definition 6.23. *Let π' and φ' belong to $\Theta_{m,m'}(\pi)$. Then $\pi' \leq \varphi'$ if and only if $\pi'_{(1)} \leq \varphi'_{(1)}$. This defines a partial order in $\Theta_{m,m'}(\pi)$.*

When π is a unipotent representation, the order defined on $\Theta_{m,m'}(\pi)$ is not necessarily total. However, we were able to find a minimal and a maximal representation for this order. The same is true for arbitrary irreducible representations.

Theorem 6.24. *Let π be an irreducible representation of G_m. There exists a minimal (resp. maximal) irreducible representation $\pi_{\min}$ (resp. $\pi_{\max}$) in $\Theta_{m,m'}(\pi)$. It is defined by $(\pi_{\min})_{(1)} = (\pi_{(1)})_{\min}$ (resp. $(\pi_{\max})_{(1)} = (\pi_{(1)})_{\max}$), where $(\pi_{(1)})_{\min}$ (resp. $(\pi_{(1)})_{\max}$) is the minimal (resp. maximal) representation in $\Theta_{m-l,m'-l}(\pi_{(1)})$.*

Proof. According to Theorems 6.18 and 6.21, there is an irreducible representation $(\pi_{(1)})_{\min}$ (resp. $(\pi_{(1)})_{\max}$) in $\Theta_{m-l,m'-l}(\pi_{(1)})$ verifying $(\pi_{(1)})_{\min} \leq \pi'_{(1)}$ (resp. $\pi'_{(1)} \leq (\pi_{(1)})_{\max}$) for all $\pi'_{(1)}$ in $\Theta_{m-l,m'-l}(\pi_{(1)})$. Thanks to Proposition 6.22, we see that $\pi_{\min}$ (resp. $\pi_{\max}$) verifying $(\pi_{\min})_{(1)} = (\pi_{(1)})_{\min}$ (resp. $(\pi_{\max})_{(1)} = (\pi_{(1)})_{\max}$) is the desired minimal (resp. maximal) representation. □

Chapter 7

Perspectives

In the study conducted in this manuscript we have excluded the pairs $(\mathrm{Sp}_{2m}(q), \mathrm{O}_{2n+1}(q))$. Indeed, we used the characters, found by Aubert, Michel and Rouquier in [5], expressing the Howe correspondence between unipotent Harish-Chandra series as a correspondence between Weyl groups of type **B** (see Conjecture 5.1). They, in turn, found these characters from the results in [39]. In this paper Srinivasan studied how the (unipotent part of) Weil representations decompose in terms of the Deligne-Lusztig virtual representations for all dual pairs (including linear pairs) but for pairs with odd orthogonal groups. It could be possible to extend our study to all dual pairs by using the results in a recently published paper by Pan (cf. [33]).

Recently, Gurevich and Howe [17, Theorem 3.3.3] have found a way to extract a "minimal" irreducible subrepresentation $\eta(\pi)$ of $\Theta(\pi)$ for symplectic-orthogonal dual pairs. Their definition of minimal differs from ours in that they use the notion or *rank* of a representation in its definition. It would be interesting to compare this representation with the extremal representations we have obtained. This could lead to solve Conjecture 3.3.4 in their paper.

Let K is a extension of degree 1 (resp. 2) over a fixed non archimedean local field F of characteristic 0, and G be either a symplectic or orthogonal (resp. unitary) group with coefficients in K. The *local Langlands correspondence* classifies irreducible representations π of G in terms of their L-parameters (ϕ, ε), where ϕ is a conjugate self-dual representation of the Weil-Deligne group of K and ε is an irreducible character of the component group associated to ϕ. In [3], Atobe and Gan express the local Howe correspondence as a correspondence between L-parameters for discrete (in fact tempered) representations.

With notation from Section 5.1, let $\chi_{\xi',\eta'}$ be an irreducible representation of $W_{l'}$ and $\chi_{\xi,\eta}$ denote one of the extremal representations obtained in Theorems 6.9 to 6.21. In our study we have excluded the representations appearing in the second component of the pairs obtained by the Springer correspondence. It should be possible, however, to obtain an analogue over finite fields to the results in [3] by including them. That is, by studying

the relation between the pairs $(\mathcal{O}', \psi')$ and $(\mathcal{O}, \psi)$ corresponding to $\chi_{\xi',\eta'}$ and $\chi_{\xi,\eta}$ via the Springer correspondence.

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List of symbols

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$R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, Deligne-Lusztig character, 25
$R_{\mathbf{T}^*}^{\mathbf{G}}(s)$, 27
$W(\mathbf{T})$, Weyl group of $\mathbf{T}$, 25
$W_{\mathbf{G}}(\delta)$, 23
W_n, Weyl group of type B or C, 60
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$\mathrm{Irr}(G_m)_k$, $\mathrm{Irr}(G_m)_k$, $\mathrm{Irr}(\mathrm{O}_{2m}^{\epsilon}(q))_k^{II}$, 49
$\mathrm{Irr}(\mathbf{G},\delta)$, Harish-Chandra series, 22
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$\mathrm{St}(x)$, stabilizer of x, 19
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$\Xi_s^{\mathbf{G}}$, 52
$\chi_{\lambda,\mu}$, 60
$\epsilon_{\mathbf{G}}$, 27
$\lambda(\xi,\eta)$, 62
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$\mathbb{F}_q$, finite field with q elements, 11
$\mathbf{G}^*$, dual group, 26
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$\mathbf{R}^G$, 20
$\mathfrak{L}_s^G$, Lusztig bijection, 27
$\mathscr{E}(G,(s))$, Lusztig series, 27
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$\mathscr{P}_2(n)$, bipartitions of n, 60
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$\mathscr{R}(G,\delta)$, representations spanned by $\mathrm{Irr}(\mathbf{G},\delta)$, 22
$\mathscr{S}(Z,V)$, 19
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ω_{ψ}, Weil representation, 31
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