Stable ergodicity and physical measures for weakly hyperbolic dynamical systems

Davi dos Anjos Obata

NNT : 2019SACLS488

THÈSE DE DOCTORAT

de

L'Université Paris-Saclay

École doctorale de mathématiques Hadamard (EDMH, ED 574)

Établissement d'inscription : Université Paris-Sud

Laboratoire d'accueil : Laboratoire de mathématiques d'Orsay, UMR 8628 CNRS

Spécialité de doctorat : Mathématiques fondamentales

Davi Joel DOS ANJOS OBATA

Ergodicité stable et mesures physiques pour des systèmes dynamiques faiblement hyperboliques

Date de soutenance : 17 Décembre 2019

Après avis des rapporteurs : François LEDRAPPIER (Sorbonne Université)
Federico RODRIGUEZ HERTZ (Pennsylvania State University)

Jury de soutenance :

Alexander ARBIETO	(UFRJ) Examinateur
Pierre BERGER	(Sorbonne Université) Examinateur
Jérôme BUZZI	(Université Paris-Sud) Président du jury
Sylvain CROVISIER	(Université Paris-Sud) Directeur de thèse
François LEDRAPPIER	(Sorbonne Université) Rapporteur

Contents

Acknowledgements

This thesis represents the end of another important phase of my life. Many people helped and influenced me, allowing me to conclude it. Without them, I believe I would not have made it.

First of all, I am very grateful to my Heavenly Father for helping me to go much further than I would have been able to go on my own. For placing so many wonderful people in my life that played a key role in my development.

I am extremely grateful for my best friend, my wife, my eternal companion, and the love of my life Tamy. She has given me unconditional support in everything. She has sacrificed a lot so that I would be able to do this thesis. I have no words to describe how her support has given me the stability and the energy to face with greater optimism this journey.

Eu sou muito grato a minha família que sempre me apoiou. Meus pais sempre me encorajaram e me ensinaram o valor dos estudos. Mesmo com quatro filhos, sempre se sacrificaram para que todos nós tivessemos uma boa educação. Durante a minha graduação, mestrado e doutorado, sempre deram bastante apoio para que eu pudesse me concentrar ao máximo em aprender. Sou muito grato por todo esse suporte, apoio e exemplo.

Um agradecimento especial também para os meus sogros, Eliane e Vitório. Sempre deram bastante apoio nessa jornada e o mais importante, confiando em mim com a sua filha. Agradeço a Carol, Welington, Bruno, Anninha, André, Kyoko, Filipe, Bella, Yuri, Thomas, May, Matheus, Mary, Yuri, Yves, Melissa, Marina e Emma, pelo apoio ao longo dos anos, e por fazerem da minha vida muito mais feliz.

I am very grateful for my advisor Sylvain Crovisier. I have been learning, and have learned, a lot from him. He is not only a brilliant mathematician, but he has always been very kind and patient in listening to me, and giving me ideas and suggestions. His way of doing mathematics has taught me a lot. He has also been a great mentor in teaching me several important things for my career as a mathematician. I have a lot of respect and admiration for the mathematician and person that he is.

Alexander Arbieto has been teaching me dynamics (and mathematics in general) since my third semester as an undergraduate student at UFRJ. His energy and love for mathematics has pushed me to be constantly seeking to learn new things (in all areas of mathematics). This allowed me to develop a good basis so that I could learn several different topics, and start to do research. I am thankful for his friendship and mentoring. It is also because of him that it was possible for me to come to France to work with Sylvain. I am very thankful for all of that.

I also thank Todd Fisher. He has always mentored me in several things. Not only he has given me great advises for my career, but has always given me encouragement and helped me with my applications. He has definitely made a lot of effort to help me in many things and I am very grateful for that.

I thank Bruno Santiago, who has always supported me and helped me a lot. We are very good friends and I always learn a lot from him not only about math, but also about many things in life.

I also thank Mauricio Poletti, who has also taught me a lot this last year and a half. We developed a good friendship and during this time, he has taught me many things that appear in this thesis. I enjoyed a lot talking about math in general with him.

I thank Federico Rodriguez Hertz, Aaron Brown, Pierre Berger, Pablo Carrasco, Ali Tahzibi, for many useful conversations that we had.

My time in France has been very enjoyable because of many good friends that I have made. I thank Frank, Martin L., Martin A., Thibault, Tiago, Anita, Danilo, Trang, Santiago, Jinhua, Yuntao, Minsung, Gabriel, the people from the football team in Orsay, and many many friends from church. To all of you I just want to say Thank you very much! I thank specially Thibault and Arnaud for reading the french introduction of this thesis and helping me to correct many things.

Since my undergrad I have had many friends that impacted me and helped me in my studies. I thank Gabriel, Aloizio, Nicolau, Braga, Paloma, Claudio, Hugo, João, Welington, Bernardo, Diego, Daniel, Jennyffer, Freddy, Bruno, o pessoal do bandejão, Vinicius, Leandro, Rodrigo, Gladshow, and others. Thank you for everything! I also thank Rafael Ribeiro and Pedro Terencio for their friendship of more than 14 years.

I am grateful for all the administrative staff at Orsay. I am also grateful to Frédéric Paulin and Stéphane Nonnenmacher that have always been very helpful with many things during my Ph.D.

I thank Federico Rodriguez Hertz and François Ledrappier for having accepted the invitation to be the rapporteurs of this thesis. I also thank Jérôme Buzzi, Alexander Arbieto, Pierre Berger and François Ledrappier for having accepted to be part of the Jury of my defense.

This work was supported by ERC 692925 NUHGD.

Introduction (Français)

Un objectif général des systèmes dynamiques est de comprendre le comportement à long terme des orbites en fonction d'une loi d'évolution. Ils apparaissent dans toutes les branches de la science, par exemple physique, biologie, économie, chimie, météorologie et autres. Normalement, il est difficile de décrire le comportement asymptotique des orbites d'un système, même pour les systèmes ayant des expressions simples.

C'est le cas, par exemple, du fameux problème des trois corps en mécanique céleste. Donnons-en une brève description. Supposons qu'il n'y ait que trois corps dans l'univers entier, et rien d'autre, par exemple le soleil, la lune et la terre, interagissant par la loi gravitationnelle. Supposons aussi qu'à un moment donné vous connaissiez la position exacte et la vitesse de chacun de ces corps par rapport à un système référentiel de coordonnées. Pouvons-nous dire exactement où se trouveront chacun de ces corps à un moment donné dans l'avenir ? En général, non. Parce que pour ce système d'équations, il n'existe pas de solution analytique générale.

Considérant le problème des trois corps, Poincaré en 1890 ([Po90]) a proposé une étude qualitative de ce système, au lieu d'une étude quantitative, où on essaierait d'obtenir la description de l'orbite de nombreux points sans réellement trouver des solutions explicites du système.

Une direction importante dans cette étude qualitative est d'essayer de comprendre un système dynamique avec un point de vue probabiliste. Ce domaine des mathématiques est appelé théorie ergodique. Un point central de la théorie ergodique est d'essayer de comprendre le comportement statistique des orbites.

La majorité de cette thèse est consacrée à l'étude des propriétés ergodiques des systèmes dynamiques. En particulier, pour certains systèmes dynamiques donnés, on cherche à décrire le comportement statistique de l'orbite de points «typiques».

Cette thèse aborde les sujets suivants (que nous décrirons plus en détail dans cette introduction) :

- stabilité ergodique pour les difféomorphismes préservant le volume (chapitres 1 et 2) ;
- la généricité de l'existence d'exposants de Lyapunov positifs pour des produits aléatoires de difféomorphismes de surface préservant le volume (chapitre 3) ;
- rigidité des mesures u-Gibbs pour des systèmes partiellement hyperboliques (chapitre 5) ;
- transitivité robuste (chapitre 4).

Dans ce qui suit, nous décrirons une partie de l'histoire de chacun de ces points et les résultats obtenus.

1. Stabilité ergodique

Nos résultats présentés dans cette partie sont contenus dans les chapitres 1 et 2 de cette thèse.

1.1. Ergodicité pour les dynamiques conservatives. Soit $f : M \to M$ un C^r-difféomorphisme d'une variété riemannienne compacte, connexe, orientée M, qui préserve une mesure de probabilité lisse m. Nous appelons un tel système un difféomorphisme conservatif (ou qui préserve le volume). Les systèmes conservatifs apparaissent naturellement en physique. En effet, chaque flot hamiltonien (qui décrit l'évolution dans le temps

des particules modélisées par un système hamiltonien) préserve une mesure lisse (appelée la mesure de Liouville).

Une caractéristique basique qu'un système conservatif peut avoir est **l'ergodicité**, ce qui signifie que du point de vue probabiliste, la dynamique ne peut pas être décomposée en pièces invariantes. Soyons plus précis, (f, m) est ergodique si et seulement si un ensemble mesurable et invariant $\Lambda \subset M$ a m-measure 0 ou 1.

Suite au travail de Birkhoff en 1931 ([Bi31]), avec son célèbre théorème ergodique, l'ergodicité de (f, m) est équivalente à ce qui suit : pour toute fonction m-integrable $\varphi : M \to \mathbb{R}$, il existe un ensemble de m-mesure pleine Λ_φ tel que pour chaque $x \in \Lambda_\varphi$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) = \int_M \varphi dm. \tag{1}$$

Donnons une interprétation de cette expression. Supposons que nous voulions mesurer la fréquence à laquelle l'orbite d'un point visite une région mesurable B dans l'espace. Considérons la fonction caractéristique de B, et dénotons-la par χ_B. Observons que la fonction χ_B est m-integrable. L'expression (3) indique que la proportion de temps que l'orbite de x passe dans B (côté gauche de (3)) coïncide avec la mesure de B (côté droit de (3)).

Birkhoff et Hopf ont conjecturé qu'un système dynamique conservatif «typique» devrait être ergodique. Soit $\mathrm{Homeo}_m(M)$ l'ensemble des homéomorphismes conservatifs de M. En 1941, Oxtoby et Ulam ont montré ([OU41]) qu'un système «typique» dans $\mathrm{Homeo}_m(M)$ est ergodique. Dans ce cas, «typique» signifie que cet homéomorphisme fait partie d'un sous-ensemble G_δ dense de $\mathrm{Homeo}_m(M)$.

Soit $\mathrm{Diff}^r_m(M)$ l'espace des C^r-diffeomorphismes de M, pour un certain $r \geq 1$. Après les travaux importants de Kolmogorov en 1954 [Ko54-1], Moser en 1962 [Mo62], et Arnold en 1963 [Ar63], la conjecture proposée par Birkhoff et Hopf est fausse en grande régularité. En effet, pour M de dimension au moins deux, il existe des ensembles ouverts dans $\mathrm{Diff}^\infty_m(M)$ de difféomorphismes non ergodiques. Aujourd'hui, leur résultat est connu comme le théorème KAM (Kolmogorov-Arnold-Moser). Il donne des conditions pour l'existence, et la persistance, d'une certaine région dans la variété avec mesure positive constituée de tores invariants, en particulier ceci est une obstruction pour l'ergodicité.

Durant l'ICM de 1954, Kolmogorov a déclaré ce qui suit (voir [Ko54-2], page 326) :«$\cdots$ il est extrêmement probable que, pour s arbitraires, il existe des exemples de systèmes canoniques avec s degrés de liberté et avec transitivité stable (c.-à-d. ergodicité) et mélangeant $\cdots$ j'ai à l'esprit le mouvement au long des géodésiques sur une variété compacte à courbure négative constante$\cdots$ ». En d'autres termes, Kolmogorov croyait déjà que tout flot conservatif suffisamment proche du flot géodésique d'une variété compacte à courbure négative constante était ergodique. Aujourd'hui, on appelle cela la stabilité ergodique.

Définition 1.1 (Stabilité ergodique)**.** *Soit $r > 1$, et $s \in [1, r]$. Un difféomorphisme $f \in \mathrm{Diff}^r_m(M)$ est C^s-**stablement ergodique** s'il existe un C^s-voisinage $\mathcal{U}$ de f tel que tout difféomorphisme $g \in \mathcal{U} \cap \mathrm{Diff}^r_m(M)$ est ergodique.*

Bien avant l'exposé de Kolmogorov, Hopf en 1939 avait déjà prouvé que le flot géodésique mentionné ci-dessus est ergodique par rapport à la mesure de Liouville, voir [Ho39]. Dans sa preuve, il a introduit un argument qui est actuellement appelé **l'argument de Hopf**. Ce flot a une caractéristique importante appelée hyperbolicité uniforme.

Un difféomorphisme f est **uniformément hyperbolique** (aussi appelé difféomorphisme d'Anosov), s'il y a une décomposition invariante du fibré tangent $TM = E^s \oplus E^u$ telle que E^s est contracté de façon uniforme dans le futur et E^u est contracté de façon uniforme dans le passé.

En 1967, Anosov [Ano67] a utilisé l'argument de Hopf pour démontrer l'ergodicité du flot géodésique des variétés compactes dont la courbure sectionnelle est strictement négative (non constante). Sa preuve a également donné que tout difféomorphisme C^2 conservatif

uniformément hyperbolique est ergodique. Comme l'hyperbolicité uniforme est une propriété C^1-ouverte, cela implique que les difféomorphismes C^2 d'Anosov sont C^1-stablement ergodiques.

Dans la suite, nous discuterons le problème de la stabilité ergodique en dehors du cadre uniformément hyperbolique. Nous mentionnons également que toutes les preuves connues de la stabilité ergodique utilisent des généralisations de l'argument de Hopf.

1.2. Ergodicité stable pour les difféomorphismes partiellement hyperboliques. *Nos résultats présentés dans cette partie sont contenus dans le chapitre 1 de cette thèse.*

Depuis près de trois décennies, les systèmes conservatifs uniformément hyperboliques ont été les seuls exemples connus de difféomorphismes stablement ergodiques. Ce n'est qu'en 1994 que Grayson-Pugh-Shub ont obtenu un exemple non Anosov ([GPS94]). Décrivons mieux leur exemple. Soit (S, g) une surface compacte avec une courbure négative constante, et soit $(\varphi_t)_{t\in\mathbb{R}}$ le flot géodésique défini dans le fibré tangent unitaire de S, que nous noterons T^1S. Dénotons par m la mesure de Liouville normalisée sur T^1S, qui est φ_t-invariante. Grayson-Pugh-Shub ont prouvé que si on considère le difféomorphisme $f = \varphi_1 \in \mathrm{Diff}^2_m(T^1S)$, alors f est C^2-stablement ergodique. Il s'agit d'un résultat non trivial et utilise une généralisation non triviale de l'argument de Hopf. Le difféomorphisme f a une forme plus faible d'hyperbolicité appelée hyperbolicité partielle, que nous définissons ci-dessous.

Un difféomorphisme f est **partiellement hyperbolique** si le fibré tangent admet une décomposition Df-invariante de la forme $TM = E^s \oplus E^c \oplus E^u$ telle que E^s est contracté de façon uniforme, E^u est dilaté de façon uniforme, et il existe $N \in \mathbb{N}$ telle que pour tout point $x \in M$ nous avons

$$\max\left\{\|Df^N(x)|_{E^c}\|.\|Df^{-N}(f^N(x))|_{E^u}\|, \|Df^N(x)|_{E^s}\|.\|Df^{-N}(f^N(x))|_{E^c}\|\right\} < \frac{1}{2}.$$

Il est bien connu que les sous-fibrés E^s et E^u sont uniquement intégrables, c'est-à-dire qu'il existe un unique feuilletage invariante, $\mathcal{F}^*$ qui est tangent à E^*, pour $* = s, u$. Nous remarquons que l'hyperbolicité partielle est une propriété C^1-ouverte parmi les difféomorphismes.

Une propriété clé dans la preuve de la stabilité ergodique de Grayson-Pugh-Shub ([GPS94]) est l'accessibilité. Un système partiellement hyperbolique est **accessible** si deux points quelconques de la variété peuvent être joints par une courbe qui est une concaténation de plusieurs courbes, chacune d'elles étant contenue dans une feuille stable ou instable. Nous disons qu'un difféomorphisme est **stablement accessible** si tout difféomorphisme dans un voisinage C^1 de celui-ci est accessible.

En se basant sur leur travail [GPS94], Pugh et Shub ont fait la conjecture suivante :

Conjecture (Conjecture 1 de Pugh-Shub, [PS97])**.** *La stabilité ergodique C^1 est C^r-dense parmi les difféomorphismes C^r conservatifs et partiellement hyperboliques, pour tout $r > 1$.*

Dans [PS97], Pugh et Shub proposent également un programme pour résoudre cette conjecture. Il comporte les deux conjectures suivantes.

Conjecture (Conjecture 2 de Pugh-Shub, [PS97])**.** *L'accessibilité implique l'ergodicité pour un difféomorphisme C^r, conservatif et partiellement hyperbolique, pour $r > 1$.*

Conjecture (Conjecture 3 de Pugh-Shub, [PS97])**.** *L'accessibilité stable est C^r-dense parmi les difféomorphismes C^r partiellement hyperboliques (conservatifs, ou non).*

Depuis lors, le thème de la stabilité ergodique pour les systèmes partiellement hyperboliques est devenu un sujet de recherche très actif. En particulier, de nombreux travaux ont été réalisés visant ces conjectures. Dans ce qui suit, nous allons faire une liste (incomplète) de certains des travaux liés à ces conjectures.

- Dans [HHU08], Hertz-Hertz-Ures ont montré que la conjecture 1 est vraie parmi les difféomorphismes C^r partiellement hyperboliques avec une direction centrale unidimensionnelle.
- Dans [HHTU11], Hertz-Hertz-Tahzibi-Ures ont montré que parmi les diffémorphismes C^r partiellement hyperboliques avec direction centrale bidimensionnelle, la stabilité ergodique est C^1-dense. Plus récemment, Avila-Crovisier-Wilkinson ont montré, dans [ACW17], que le même résultat est vrai sans l'hypothèse de centrale bidimensionnelle.
- Burns-Dolgopyat-Pesin ont montré, dans [BDP02], qu'un difféomorphisme C^r, conservatif et partiellement hyperbolique avec les conditions suivantes est ergodique : il est accessible ; et il a un ensemble de mesure positive dont les points ont tous leurs exposants de Lyapunov central (voir la définition ci-dessous) non nul et avec le même signe. Ce résultat est valable avec une propriété plus faible appelée accessibilité essentielle.
- Burns-Wilkinson ont montré, dans [BW10], que pour un difféomorphisme C^2, conservatif et partiellement hyperboliques, accessibilité (ou accessibilité essentielle), et une condition technique appelée "center bunching" implique l'ergodicité. C'est une généralisation non trivial d'un résultat précédent de Pugh-Shub dans [PS00].

Nous remarquons que tous ces travaux utilisent la propriété d'accessibilité (ou d'accessibilité essentielle). Si l'on donne un exemple explicite, l'accessibilité est généralement une propriété difficile à vérifier. L'un des objectifs de ce travail est de faire face au problème suivant :

Problème 1.2. *Prouver la stabilité ergodique des systèmes partiellement hyperboliques sans utiliser l'accessibilité (ou accessibilité essentielle) ?*

Comme nous allons voir dans la section suivante, il existe des exemples de difféomorphismes stablement ergodiques en dehors du cadre partiellement hyperbolique, voir la section 1.3. Une approche naturelle pour répondre à cette question est d'utiliser la théorie de Pesin pour les systèmes non uniformément hyperboliques.

Pour un difféomorphisme de regularité C^1, f, et une mesure invariante ν, pour ν-presque tout point $p \in M$ et pour chaque $v \in T_pM - \{0\}$ la limite suivante existe

$$\lambda(p, v) = \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n(p).v\|.$$

Le théorème d'Oseledets dit que $\lambda(p, .)$ peut avoir au maximum $\dim(M)$ différentes valeurs. Ces nombres sont appelés exposants Lyapunov. Une mesure invariante ν est non uniformément hyperbolique pour f si pour ν-presque tout point, chaque exposant de Lyapunov est non nul. Nous disons qu'un difféomorphisme conservatif f est non uniformément hyperbolique si la mesure lisse invariante m est non uniformément hyperbolique pour f.

Dans [Pes77], Pesin a adapté l'argument de Hopf pour le cas non-uniformément hyperbolique, et il a montré qu'une mesure lisse non uniformément hyperbolique a au maximum une quantité dénombrable de composantes ergodiques différentes. Nous remarquons que la seule théorie de Pesin ne donne pas l'ergodicité du système.

Berger et Carrasco ont introduit dans [BC14] un exemple de difféomorphisme partiellement hyperbolique, qui préserve le volume et est non uniformément hyperbolique. Cet exemple a un sous-fibré central bidimensionnel qui n'admet aucune autre décomposition dominée et Lebesgue presque tout point a un exposant de Lyapunov positif et un négatif dans la direction centrale. De plus, les propriétés de cet exemple sont C^2-robuste. On ne sait pas si cet exemple est accessible ou non.

Le chapitre 1 est dédié à prouver la stabilité ergodique C^2 de l'exemple de Berger-Carrasco. Avant d'introduire l'exemple et de donner l'énoncé précis de notre résultat, soulignons deux caractéristiques de notre travail qui le distinguent du reste des travaux précédents sur la stabilité ergodique des difféomorphismes partiellement hyperboliques :

- La stabilité ergodique d'un système avec un comportement mixte [1] et aucune décomposition dominée de la direction centrale (comme un renforcement du [BC14]) ;
- Une preuve de stabilité ergodique qui n'utilise pas l'accessibilité (ou l'accessibilité essentielle).

L'exemple de Berger-Carrasco. Pour $N \in \mathbb{N}$ nous notons par $s_N(x,y) = (2x - y + N\sin(x), x)$ l'application standard sur $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. Pour chaque N, l'application s_N préserve la mesure de Lebesgue induite par la métrique habituelle de $\mathbb{T}^2$. Cette application est liée à plusieurs problèmes de la physique, voir par exemple [Ch79, Iz80, SS95].

Il est conjecturé que pour $N \neq 0$ grand, l'application s_N a entropie positive pour la mesure de Lebesgue (voir [Si94], page 144). Selon la formule d'entropie de Pesin ([Pes77], Theorem 5.1), cela est équivalent à l'existence d'un ensemble de mesure de Lebesgue positive et dont les points ont un exposant de Lyapunov positif. L'existence de ces ensembles n'est connue pour aucune valeur de N. Nous renvoyons le lecteur à [BXY17, Du94, Go12] pour quelques résultats sur cette conjecture.

Soit $A \in SL(2,\mathbb{Z})$ une matrice hyperbolique qui définit un difféomorphisme Anosov sur $\mathbb{T}^2$, et soit $P_x : \mathbb{T}^2 \to \mathbb{T}^2$ la projection sur la première coordonnée de $\mathbb{T}^2$. Cette projection est induite par l'application linéaire de $\mathbb{R}^2$, que nous dénoterons également par P_x, définie par $P_x(a,b) = (a,0)$.

Considérons le tore $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$, et représentons-le en utilisant les coordonnées (x,y,z,w), où $x,y,z,w \in [0,2\pi)$. On peut naturellement identifier un point (z,w) sur le second tore avec un point (x,y) sur le premier tore en définissant $x = z$ et $y = w$. Pour chaque $N \in \mathbb{N}$ définissons

$$\begin{array}{rccc} f_N & : & \mathbb{T}^2 \times \mathbb{T}^2 & \longrightarrow & \mathbb{T}^2 \times \mathbb{T}^2 \\ & & (x,y,z,w) & \mapsto & (s_N(x,y) + P_x \circ A^N(z,w), A^{2N}(z,w)), \end{array}$$

où le point $A^N(z,w)$ sur le second tore est identifié avec le même point dans le premier tore comme décrit précédemment.

Ce difféomorphisme préserve la mesure de Lebesgue, que nous dénoterons par *Leb*. Pour N assez grand c'est un difféomorphisme partiellement hyperbolique, avec une direction centrale bidimensionnelle donnée par $E^c = \mathbb{R}^2 \times \{0\}$. Ce type de système a été considéré par Berger et Carrasco dans [BC14], où ils ont prouvé le théorème suivant.

Théorème 1.1 ([BC14], Théorème 1)**.** *Il existe $N_0 > 0$ et $c > 0$ tels que pour chaque $N \geq N_0$, pour Lebesgue presque tout point p et pour chaque $v \in \mathbb{R}^4$*

$$\lim_{n\to\infty} \left| \frac{1}{n} \log \|Df_N^n(p).v\| \right| > c \log N.$$

De plus, la même propriété s'applique à tout difféomorphisme conservatif dans un voisinage C^2 de f_N.

Nous remarquons que Viana a construit dans le théorème B de [Vi97], un exemple de difféomorphisme non conservatif partiellement hyperbolique avec les propriétés similaires à l'exemple de Berger-Carrasco : Lebesgue presque tout point a un exposant positif et un exposant négatif dans la direction centrale et il n'y a pas de décomposition dominée du centre, mais le système est dissipatif.

Définition 1.3. *Soit ν une mesure de probabilité invariante pour f. Nous disons que (f,ν) est Bernoulli s'il est mesurablement conjugué à un décalage de Bernoulli. Pour les difféomorphismes conservatifs, on dit que f est Bernoulli si (f, Leb) est Bernoulli.*

Nous remarquons que la propriété Bernoulli est plus forte que l'ergodicité.

1. Ceci signifie que presque tout point a un exposant de Lyapunov central positif et un négatif

Théorème A. Pour N suffisamment grand, f_N est C^2-stablement ergodique (en fait, Bernoulli). De plus, tout difféomorphisme préservant le volume dans un voisinage C^2 de f_N est Bernoulli.

Afin de montrer ce théorème, nous aurons besoin d'obtenir des estimations précises sur la taille des variétés invariantes dans la direction centrale pour certains points. Pour cela, nous avons besoin d'une meilleure estimation des exposants centraux, donnée par la proposition suivante.

Proposition 1.4. *Pour chaque $\delta \in (0,1)$, il existe $N_0 = N_0(\delta)$ tel que pour chaque $N \geq N_0$ il y a un voisinage C^2, $\mathcal{U}_N$, de f_N dans $\mathrm{Diff}^2_{Leb}(\mathbb{T}^4)$ avec les propriétés suivantes. Si $g \in \mathcal{U}_N$, alors Lebesgue presque tout point a un exposant de Lyapunov positif et négatif dans la direction centrale dont la valeur absolue est supérieure à $(1-\delta)\log N$.*

On peut montrer que f_N est C^2-approximé par des difféomorphismes stablement ergodiques avec une autre approche. Cette approche utilise l'accessibilité, qui peut être obtenue en utilisant les résultats de [HS17], et les critères d'ergodicité de [BW10]. Une telle approche n'utilise pas l'hyperbolicité non uniforme du système.

Plus tard, nous présenterons d'autres résultats ergodiques et topologiques obtenus dans cette thèse concernant l'exemple de Berger-Carrasco.

Autres remarques et questions. Les techniques utilisées pour montrer le théorème A sont basées sur des estimations précises de la taille et de la «géométrie» des variétés stables et instables de Pesin. Il y a une idée générale derrière cela : pour toute composante ergodique non uniformément hyperbolique de «grands» exposants de Lyapunov impliquent l'existence de «grandes» variétés stables/instables sur un ensemble de mesure grande (pour la composante). Avec un contrôle supplémentaire de la géométrie, nous sommes capables de trouver des intersections transverses entre les variétés stables/instables de points typiques pour deux composantes ergodiques quelconques, ce qui, selon l'argument de Hopf, implique l'ergodicité. Jusqu'où peut-on pousser ces techniques pour étudier l'ergodicité d'un système ? En particulier, nous pensons qu'elles pourraient être utile pour donner des réponses partielles à la conjecture 2 de Pugh-Shub.

Question 1.5. *Soit f un difféomorphisme C^2, conservatif partiellement hyperbolique avec une direction centrale bidimensionnelle. Supposons que f soit accessible, non uniformément hyperbolique avec des exposants centraux «grands» pour presque tout point. Est-ce que f est ergodique ?*

Remarquons que pour la question ci-dessus, si les exposants centraux ont le même signe, alors f est en fait stablement ergodique [BDP02]. Le scénario à considérer est donc celui où f a un comportement mixte le long du centre, tout comme dans l'exemple de Berger-Carrasco.

Dans «[ABW09], Avila-Bochi-Wilkinson» étudient les implications ergodiques d'une propriété appelée **center bunching non-uniforme**. En particulier, dans le corollaire C de cet article, ils donnent un critère d'ergodicité pour le systèmes C^2, conservatif, non uniformément «center bunching», avec un type plus fort d'accessibilité. Cette propriété de «center bunching» non-uniforme est présente, par exemple, dans les systèmes avec de «petits» exposants centraux.

Un autre cas qui n'est pas mentionné ci-dessus est celui où il existe un ensemble de mesure positive de points ayant à la fois un «petit» et un «grand» exposant central. Pour éviter ce cas, on peut considérer des systèmes avec certaines relations de symétrie pour les exposants. C'est le cas, par exemple, des symplectomorphismes.

Question 1.6. *Peut-on utiliser une combinaison des techniques mentionnées ci-dessus pour montrer qu'un symplectomorphisme C^2, partiellement hyperbolique, accessible, avec*

direction centrale bidimensionnelle a un nombre fini des composantes ergodiques ? Peut-on prouver l'ergodicité ?

1.3. Stabilité ergodique au-delà de l'hyperbolicité partielle. *Nos résultats présentés dans cette partie sont contenus dans le chapitre 2 de cette thèse.*

La plupart des travaux sur la stabilité ergodique ont été faits dans le cadre partiellement hyperbolique, et peu de choses ont été faites en dehors de ce cadre.

On sait que les difféomorphismes C^1-stablement ergodiques doivent avoir une forme plus faible d'hyperbolicité [AM07], appelée décomposition dominée. Nous disons qu'un difféomorphisme f admet une **décomposition dominée** s'il y a une décomposition du fibré tangent, $TM = E \oplus F$, en deux sous-fibrés non triviaux qui sont Df-invariants, de sorte que pour certain $N \geq 1$, tout vecteurs unitaires $v \in E(x)$ et $u \in F(x)$ vérifient

$$\|Df^N(x)v\| < \frac{1}{2}\|Df^N(x)u\|.$$

Nous définissons également une forme faible d'hyperbolicité partielle. Un difféomorphisme est **faiblement partiellement hyperbolique** s'il admet une décomposition dominée de la forme $TM = E \oplus E^u$, de sorte que le sous-fibré E^u est dilaté exponentiellement sous l'action de Df.

Problème 1.7. *Obtenir des critères de stabilité ergodique pour les systèmes avec une décomposition dominée, ou pour les systèmes qui sont faiblement partiellement hyperboliques.*

Bonatti et Viana en 2000 ([BV00]) ont obtenu un exemple de difféomorphisme faiblement partiellement hyperbolique qui n'est pas partiellement hyperbolique (c'est-à-dire qu'il existe seulement une direction instable forte mais pas de direction stable forte) et qui est C^1-stablement ergodique.

Dans le même article, Bonatti et Viana ont présenté un exemple de difféomorphisme qui préserve le volume et qui est C^1-robustement transitif [2] et a une décomposition dominée sans direction uniformément hyperbolique. À la fin de leur article, ils ont posé la question de savoir si ce système était C^1-stablement ergodique.

En 2004, Tahzibi dans sa thèse a donné une réponse positive à la question de Bonatti-Viana, obtenant le premier exemple d'un difféomorphisme C^1-stablement ergodique qui n'a aucune direction hyperbolique (voir [Tah04]). Depuis lors, il n'y a pas eu d'autres travaux sur la stabilité ergodique en dehors du scénario partiellement hyperbolique.

Puisque l'ergodicité est une caractéristique globale, il est naturel de rechercher des propriétés globales qui pourraient aider à obtenir l'ergodicité, ou la stabilité ergodique. Dans le cadre partiellement hyperbolique, comme nous l'avons expliqué précédemment, la propriété globale clé qui a été utilisée est l'accessibilité. Rappelons que m représente une mesure lisse sur la variété M.

Un des objectifs du chapitre 2 de cette thèse est de trouver de nouveaux critères de stabilité ergodique, en fait de stabilité Bernoulli, hors du scénario partiellement hyperbolique. En particulier, nous étudions les conséquences données par une propriété appelée **hyperbolicité par chaîne**, pour la définition précise voir définition 2.3 au chapitre 2. L'hyperbolicité par chaîne, elle a été définie et utilisée précédemment dans [Cro11, CP15]. Elle peut être vue comme une sorte d'hyperbolicité topologique demandant que f «contracte» topologiquement le long de la direction E, jusqu'à une certaine «échelle», et f^{-1} «contracte» topologiquement le long de la direction F, jusqu'à une certaine «échelle». En utilisant cette propriété globale pour étudier la stabilité ergodique des difféomorphismes ayant une décomposition dominée, nous obtenons le théorème suivant.

2. Un difféomorphisme est robustement transitif si dans un voisinage C^1, tout difféomorphisme possède orbite dense.

Théorème B. Soit $f \in \mathrm{Diff}^1_m(M)$. Si f est un difféomorphisme hyperbolique par chaîne pour une décomposition dominée $TM = E \oplus F$ et vérifiant

$$\int_M \log \|Df|_E\| dm < 0 \text{ et } \int_M \log \|Df^{-1}|_F\| dm < 0, \tag{2}$$

alors il existe un voisinage C^1 de $f, \mathcal{U}$, de sorte que tout difféomorphisme $g \in \mathcal{U} \cap \mathrm{Diff}^2_m(M)$ est ergodique, et même Bernoulli. En particulier, un tel difféomorphisme g est stablement Bernoulli.

Dans le cadre du théorème B, comme conséquence de (2) et de l'ergodicité, nous obtenons en fait que m-presque tout point a tous ses exposants de Lyapunov négatifs le long de E et tous ses exposants positifs le long de F.

Comme application du théorème B, nous obtenons le critère suivant de stabilité Bernoulli pour les systèmes faiblement partiellement hyperboliques.

Théorème C. Soit $f \in \mathrm{Diff}^2_m(M)$. Supposons que f est faiblement partiellement hyperbolique avec une décomposition dominée $TM = E \oplus E^u$ et hyperbolique par chaîne par rapport à la même décomposition. Si f a tous ses exposants de Lyapunov négatifs dans la direction E sur un ensemble de m-mesure positive, alors f est stablement ergodique, en fait stablement Bernoulli.

Ce théorème peut être vu comme une version du théorème 4 dans [BDP02] pour les difféomorphismes faiblement partiellement hyperboliques. Nous remarquons également que si $f \in \mathrm{Diff}^2_m(M)$ vérifie l'hypothèse du théorème B et que la direction F est uniformément hyperbolique, ce qui signifie $F = E^u$, alors (2) implique que f vérifie l'hypothèse du théorème C. Par contre, un difféomorphisme qui vérifie l'hypothèse du théorème C, ne vérifie pas nécessairement l'hypothèse du théorème B, a priori.

Le théorème B donne plus de flexibilité dans la construction de l'exemple considéré par Tahzibi dans [Tah04]. Pour construire l'exemple on fait une déformation supportée dans un nombre fini de petites boules autour de points fixes hyperboliques, en particulier, les déformations sont locales. Le théorème B s'applique à cet exemple et quantifie, d'une certaine manière, la taille des perturbations autorisées. En particulier, les déformations ne doivent pas nécessairement être locales. Dans la section 6 du chapitre 2 nous expliquons la construction d'un tel exemple non local. Nous remarquons que notre preuve est différente de celle de Tahzibi dans [Tah04].

Comme autre application du théorème B, et d'autres résultats, nous pouvons montrer la densité C^1 de difféomorphismes stablement Bernoulli parmi une certaine classe des difféomorphismes faiblement partiellement hyperboliques. Définissons cette classe.

Soit $\mathcal{D} \subset \mathrm{Diff}^2_m(M)$ le sous-ensemble des difféomorphismes f qui vérifient les propriétés suivantes :

- f est faiblement partiellement hyperbolique, avec une décomposition dominée $TM = E \oplus E^u$ et $dim(E) = 2$;
- f est hyperbolique par chaîne pour la décomposition $TM = E \oplus E^u$.

Définissons $\mathcal{WCH}^2_m(M)$ comme étant l'intérieur C^1 de $\mathcal{D}$ pour la topologie relative. Pour le tore de dimension d, cet ensemble est non vide, pour $d \geq 3$. Les exemples faiblement hyperboliques de Bonatti-Viana [BV00] appartiennent à cet ensemble. Nous avons le théorème suivant.

Théorème D. Stablement Bernoulli est C^1-dense dans $\mathcal{WCH}^2_m(M)$.

Nous remarquons que tous nos résultats restent vrais pour les diféomorphismes $C^{1+\alpha}$.

Autres remarques et questions. Nous terminons cette partie de l'introduction par quelques questions et commentaires.

Question 1.8. *Quels autres critères de stabilité ergodique, ou de stabilité Bernoulli, peut-on obtenir en utilisant l'hyperbolicité par chaîne ?*

Nous remarquons que l'exemple considéré par Tahzibi dans [Tah04] est isotope à un difféomorphisme d'Anosov linéaire.

Question 1.9. *Existe-t-il un difféomorphisme qui vérifie l'hypothèse du théorème B, ou du théorème C, qui n'est pas isotope à un difféomorphisme d'Anosov ?*

Potrie obtient une réponse négative pour cette question en dimension 3 sous certaines hypothèses, voir [Pot15].

Après notre travail, Núñez-Hertz dans [NH19] ont également obtenu un résultat de stabilité ergodique (en fait de stabilité Bernoulli) en dehors du cadre partiellement hyperbolique. Ils considèrent un difféomorphisme faiblement partiellement hyperbolique sur des variétés tridimensionnelles. La propriété globale (au lieu de l'hyperbolicité par chaîne) qu'ils utilisent pour étudier la stabilité ergodique est la minimalité de feuilletage instable fort. En particulier, ils obtiennent qu'un difféomorphisme C^1-générique [3] qui est faiblement partiellement hyperbolique et dont le feuilletage instable fort est minimal, est stablement Bernoulli. Même s'il y a quelques similitudes entre les deux preuves, elles utilisent des propriétés globales différentes, de sorte qu'elles peuvent être considérées comme complémentaires l'une à l'autre.

2. Généricité de l'existence d'exposants de Lyapunov positifs

Nos résultats présentés dans cette section sont d'un travail en collaboration avec Mauricio Poletti [4], et ils sont contenus dans le chapitre 3 de cette thèse.

Dans les années 60, Smale avait obtenu plusieurs résultats sur les conséquences dynamiques de l'hyperbolicité uniforme (voir [Sm67]). Depuis lors, la dynamique hyperbolique uniforme a été très bien comprise. Parmi les propriétés des ensembles transitifs hyperboliques, nous pouvons mentionner l'existence d'une dynamique symbolique, l'existence de points periodiques et de fers à cheval, l'entropie positive, etc. Même si l'hyperbolicité uniforme est une propriété C^1-ouverte, ce n'est pas une propriété C^1-dense.

Pour des mesures invariantes lisses, Pesin a proposé dans [Pes77] une notion plus faible d'hyperbolicité, appelée hyperbolicité non uniforme que nous avons définie dans la section précédente. Il s'avère que l'hyperbolicité non uniforme implique également plusieurs caractéristiques intéressantes de la dynamique, telles que l'existence d'orbites périodiques et de fers à cheval [Ka80], un nombre au plus dénombrable de composantes ergodiques pour les mesures physiques [Pes77], etc. Contrairement à l'hyperbolicité uniforme, on peut s'attendre a ce que l'hyperbolicité non uniforme soit vérifiée pour une grande classe de systèmes. Étant donnée une variété riemanniene compacte et lisse M, nous rappelons que m est une mesure lisse sur M.

Problème 2.1. *Quelle est la fréquence de l'hyperbolicité non uniforme dans* $\mathrm{Diff}^r_m(M)$ *?*

2.1. Hyperbolicité non uniforme dans la topologie C^1. Si M est une surface, alors un résultat remarquable de Mañé [Ma96] et Bochi [Bo02] nous dit que C^1-génériquement dans $\mathrm{Diff}^1_m(M)$ soit le difféomorphisme est Anosov, soit tous ses exposants Lyapunov sont nuls pour presque tout point.

Ce résultat a été récemment généralisé à toutes les dimensions par Avila-Crovisier-Wilkinson en 2016 ([ACW16]) : ils ont prouvé que pour une variété compacte de n'importe quelle dimension M, C^1-génériquement dans $\mathrm{Diff}^1_m(M)$, soit tous les exposants de Lyapunov sont nuls pour m-presque tout point, soit le système est non uniformément Anosov (ce

3. C^1-générique signifie qu'il appartient à un sous-ensemble G_δ dense de $\mathrm{Diff}^1_m(M)$.

4. CNRS-Laboratoire de Mathématiques d'Orsay, UMR 8628, Université Paris-Sud 11, Orsay Cedex 91405, France

qui signifie qu'il est non uniformément hyperbolique et la décomposition d'Oseledets est dominée).

Un comportement différent se produit localement et C^1-densément. Liang-Yang ont montré dans [LY17] que pour tout $r \geq 1$, il existe un sous-ensemble C^1-dense de $\mathrm{Diff}^r_m(M)$ de difféomorphismes ayant un ensemble de mesure m-positive dont les points ont tous leurs exposants Lyapunov non nuls. Ce résultat est valable pour les variétés dont la dimension est au moins égale à deux.

Au lieu d'un difféomorphisme de surface, considérons maintenant deux ou plusieurs difféomorphismes de surface préservant le volume et itérons les de façon aléatoire.

Question 2.2. *L'hyperbolicité non uniforme est-elle valide pour les produits aléatoires «typiques» de difféomorphismes de surface préservant le volume ?*

Fixons $d \in \mathbb{N}$ tel que $d \geq 2$, fixons des nombres réels positifs $p_1, \ldots, p_d$ tels que $p_1 + \cdots + p_d = 1$, et notons p la mesure de probabilité sur l'ensemble $\{1, \cdots, d\}$ définie par les poids p_i. Étant donnés les difféomorphismes $(f_1, \cdots, f_d) \in \mathrm{Diff}^r_m(S)^d$, nous considérons le produit aléatoire qu'ils génèrent : à chaque instant, la probabilité du difféomorphisme f_i à agir sur S est p_i. Formellement, le produit aléatoire est un produit tordu au dessus du décalage gauche en $\Sigma = \{1, \cdots, d\}^{\mathbb{Z}}$ et la mesure de Bernoulli donnée par $P = p^{\mathbb{Z}}$.

Pour P-presque toute suite $\tilde{x} = (x_i)_{i \in \mathbb{Z}}$ et m-presque tout point $p \in S$, les deux limites suivantes existent

$$\lambda^+(\tilde{x}, p) = \lim_{n \to +\infty} \frac{1}{n} \|Df^n_{\tilde{x}}(p)\| \text{ et } \lambda^-(\tilde{x}, p) = -\lim_{n \to +\infty} \frac{1}{n} \left\|Df^{-n}_{\tilde{x}}(p)\right\|,$$

où $f^n_{\tilde{x}} = f_{x_{n-1}} \circ \cdots \circ f_{x_0}$. Ces nombres sont appelés exposants de Lyapunov le long des fibres (ou exposants centraux). Nous pouvons également considérer l'exposant de Lyapunov central intégré, qui est donné par

$$L(f_1, \cdots, f_d) = \int_{\Sigma \times S} \lambda^+(\tilde{x}, p) dP \times m(\tilde{x}, p).$$

Dans cette introduction, nous avons choisi d'énoncer d'abord le théorème G avant les théorèmes E et F, car il illustre de manière plus simple le type de résultats que nous obtenons et comment il se compare avec le théorème de Bochi-Mañé. Notre résultat est le suivant.

Théorème G. Fixons $d \in \mathbb{N}$, un nombre réel $r \geq 1$ et fixons une mesure de probabilité p sur $\{1, \ldots, d\}$ telle que $p(\{i\}) < 1$, pour $i = 1, \cdots, d$. Alors il existe un sous-ensemble de $\mathrm{Diff}^r_m(S)^d$, C^1-ouvert et C^1-dense, de sorte que si $(f_1, \cdots, f_d)$ appartient à cet ensemble son produit aléatoire, ses exposants de Lyapunov centraux intégrés sont positifs.

Observons que ce résultat produit un ensemble C^1-ouvert dans $\mathrm{Diff}_m(S)^d$ pour lequel les exposants sont positifs dans certaines régions. En particulier, le résultat de Bochi-Mañé ne se vérifie pas dans le scénario des produits aléatoires. Nos résultats s'appliquent en fait à des produits tordu plus généraux. Décrivons maintenant les autres résultats que nous avons obtenus.

Soit $\tilde{M}$ une variété lisse, compacte, connexe et orientée et S une surface lisse, compacte et connexe. Considérons un espace fibré M sur $\tilde{M}$, défini par une projection lisse $\pi : M \to \tilde{M}$, avec fibres difféomorphes à S. Pour un point $x \in M$, nous écrivons S_x la fibre qui contient le point x. On dit qu'un difféomorphisme $f : M \to M$ préserve les fibres si pour n'importe quel $x \in M$ il vérifie $S_{f(x)} = f(S_x)$.

Pour l'espace fibré M un difféomorphisme $f : M \to M$ est un **produit tordu partiellement hyperbolique** si ce qui suit est verifié :

- f envoie fibre sur fibre ;
- f est un difféomorphisme partiellement hyperbolique, avec décomposition $TM = E^s \oplus E^c \oplus E^u$, tel que $E^c = \ker D\pi$.

Soit m la mesure de Lebesgue normalisée sur M et définissons $SP^r_m(M)$ comme étant l'ensemble des produits tordus C^r partiellement hyperboliques qui préservent la mesure de Lebesgue. Dans l'espace $SP^r_m(M)$ on peut considérer la topologie C^s, pour tout $s \in [0, r]$.

Pour m-presque tout point, considérons les plus grands et les plus petits exposants de Lyapunov dans la direction centrale, définis respectivement par

$$\lambda_c^+(x) = \lim_{n\to+\infty} \frac{1}{n}\|Df^n(x)|_{E^c_x}\| \text{ et } \lambda_c^-(x) = -\lim_{n\to+\infty} \frac{1}{n}\|Df^{-n}(x)|_{E^c_x}\|.$$

Dans ce cas $\det Df(x)|_{E^c_x} = 1$ (voir le chapitre 3). Cela implique que pour presque chaque point $x \in M$ nous avons que $\lambda_c^-(x) = -\lambda_c^+(x)$. Nous définissons l'exposant **de Lyapunov integré dans la direction centrale** par

$$L(f) = \int_M \lambda_c^+(x) dm(x).$$

Dans le résultat suivant, nous utilisons la notion de «center bunching», qui est une condition technique que nous ne définirons pas ici. Nous renvoyons le lecteur au chapitre 3 pour la définition précise.

Théorème E. Pour tout $r > 1$, parmi les produits tordus C^r, partiellement hyperboliques, qui préservent le volume et qui sont «center bunched», il existe un sous-ensemble C^1-dense et C^r-ouvert de difféomorphismes f vérifiant $L(f) > 0$.

Par [HS17], on sait que l'ergodicité est C^1-ouverte et C^r-dense dans le cadre du théorème précédent. Le résultat suivant découle immédiatement du théorème E.

Corollaire 2.3. *Dans le contexte du théorème E, il existe un sous-ensemble C^1-ouvert et C^1-dense tel que tout difféomorphisme dans ce sous-ensemble est non uniformément hyperbolique : pour m-presque tout point, tous les exposants de Lyapunov sont non nuls.*

Un autre scénario pour lequel nous obtenons des résultats est celui des produits tordus plus généraux. Soit Σ un espace métrique compact sans points isolés, soit $\sigma : \Sigma \to \Sigma$ un homéomorphisme hyperbolique (voir le chapitre 3 pour une définition précise) et $\tilde{\mu}$ une mesure σ-invariante qui a une propriété appelée structure de produit local (voir le chapitre 3 pour une définition détaillée). Cette propriété est valide pour des mesures importantes telles que les états d'équilibre de potentiels Hölder (voir [Bow75]).

Fixons $\alpha > 0$. Étant donnée une application (C, α)-Hölder de Σ dans $\mathrm{Diff}^r_m(S)$, $\tilde{x} \mapsto f_{\tilde{x}}$, on définit le produit tordu

$$\begin{array}{rcl} f : \Sigma \times S & \to & \Sigma \times S \\ (\tilde{x}, t) & \mapsto & f(\tilde{x}, t) = (\sigma(\tilde{x}), f_{\tilde{x}}(t)), \end{array}$$

où nous voulons dire par (C, α)-Hölder que

$$\mathrm{d}_{C^r}(f_{\tilde{x}}, f_{\tilde{y}}) \leq C\, \mathrm{d}_\Sigma(\tilde{x}, \tilde{y})^\alpha.$$

Observons que ce produit tordu préserve la mesure $\mu := \tilde{\mu} \times m$. Une telle application est appellée **$C^{r,\alpha}$-produit tordu** sur σ préservant μ.

A partir de maintenant nous fixons $C > 0$. Pour $\alpha > 0$ et $r \geq 1 + \alpha$, nous définissons $SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$ comme étant l'espace des $C^{r,\alpha}$-produits tordus sur σ, tels que l'application $\tilde{x} \mapsto f_{\tilde{x}}$ est (C, α)-Hölder. Dans cet espace, nous considérons la topologie C^s, pour tout $s \leq r$ définie comme suit : pour deux $C^{r,\alpha}$-produits tordus $f, g \in SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$, la distance C^s entre f et g est

$$\mathrm{d}_{C^s}(f, g) = \sup_{\tilde{x}\in\Sigma} \mathrm{d}_{C^s}(f_{\tilde{x}}, g_{\tilde{x}}),$$

où $\mathrm{d}_{C^s,\tilde{x}}(f_{\tilde{x}}, g_{\tilde{x}})$ est la distance C^s sur $\mathrm{Diff}^r_m(S)$. Nous rappelons que σ est toujours fixe.

Comme précédemment, nous pouvons définir les exposants dans les fibres comme suit

$$\lambda^+(\tilde{x}, t) = \lim_{n\to+\infty} \frac{1}{n}\|Df^n_{\tilde{x}}(t)\| \text{ et } \lambda^-(\tilde{x}, t) = -\lim_{n\to+\infty} \frac{1}{n}\|Df^{-n}_{\tilde{x}}(t)\|,$$

où $f^n_{\tilde{x}} = f_{\sigma^{n-1}(\tilde{x})} \circ \cdots \circ f_{\tilde{x}}$. Ceci est défini μ-presque partout.

Comme pour la notion de «center bunching», il existe une notion de «fiber bunching»qui garantit l'existence d'holonomies linéaires, voir le chapitre 3 pour des définitions précises.

Théorème F. Soit σ un homéomorphisme hyperbolique et $\tilde{\mu}$ une mesure invariante pour σ avec une structure de produit local. Pour tout $r > 1$ et $\alpha > 0$, il existe un sous-ensemble C^1-dense et C^r-ouvert de $SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$ de difféomorphismes f vérifiant $L(f) > 0$.

L'un des ingrédients clé de notre preuve est une condition que nous appelons «**pinching**» (voir le chapitre 3 pour la définition). Cette condition dit qu'il existe des fibres fixes (ou périodiques), de sorte que l'action du produit tordu sur cette fibre a des exposants positifs dans une certain régions de volume positif.

2.2. L'hyperbolicité non uniforme en topologie C^r. Pour les résultats ci-dessus, nous ne pouvons obtenir que la densité C^1 car nous utilisons que la condition de «pinching» est C^1-dense, après le résultat de [LY17]. La densité C^r de la condition de «pinching», en général, n'est pas connue pour $r > 1$.

Avec quelques informations sur les points périodiques d'un difféomorphisme, la condition de «pinching»peut être obtenue dans une classe de régularité plus élevée, et nous avons le résultat suivant.

Théorème H. Soit f comme dans les Théorèmes E, F ou G, et supposons de plus qu'il existe une fibre périodique $S_{\tilde{p}}$ telle que $f_{\tilde{p}} : S_{\tilde{p}} \to S_{\tilde{p}}$ a un point périodique elliptique. Alors f est C^r-accumulé par des ensembles C^r-ouverts des systèmes ayant des exposants Lyapunov intégrés positifs. De plus, dans le cas d'un produit aléatoire, ces ensembles sont C^1-ouverts.

Nous remarquons que Marin dans [Mar16] a prouvé qu'un symplectomorphisme C^r, partiellement hyperbolique avec un centre bidimensionnel, qui est accessible, qui vérifie un type de condition de «center bunching», et a un point périodique satisfaisant une condition (qu'elle appelle «pinching»), peut être C^r approximé par symplectomorphismes non uniformément hyperboliques ergodiques. Dans son argument, l'accessibilité et la préservation d'une forme de volume (induite par la forme symplectique) sont des propriétés cruciales, car elle utilise les résultats de [ASV13]. Nous remarquons que son résultat ne se limite pas simplement aux produits tordus. Il a été encore amélioré par Liang-Marin-Yang, dans [LMY18], qui montrent que dans un voisinage C^r d'un symplectomorphisme vérifiant les conditions ci-dessus, il existe un sous ensemble C^r-ouvert et C^r-dense des symplectomorphismes ergodiques non uniformément hyperboliques. Nous soulignons que, contrairement au résultat de Marin, nos arguments ne sont pas basés sur les résultats de [ASV13].

Après la conclusion de notre travail, Barrientos et Malicet [BM18] nous ont envoyé une prépublication d'un résultat similaire pour le produit aléatoire des diffeomorphismes. Ils montrent qu'après avoir fixé $k-1$ difféomorphismes de surface préservant le volume, $f_1, \ldots, f_{k-1}$, engendrant une action ergodique, alors pour tout f_k dans un sous-ensemble C^r-dense et C^1-ouvert de difféomorphismes préservant le volume, le produit aléatoire des $f_1, \ldots, f_k$ a des exposants positifs. Ils utilisent une approche différente qui ne nécessite pas de condition de «pinching», mais l'ergodicité des premiers $k-1$ difféomorphismes est essentielle dans leur argument.

Autres remarques et questions. Comme nous l'avons mentionné précédemment, nous ne savons pas si la condition de «pinching» est C^r-dense. C'est la seule chose qui nous empêche d'avoir la densité C^r dans les énoncés des Théorèmes E et F.

Question 2.4. *La condition de «pinching» est-elle C^r-dense (pour tout $r \geq 1$) dans* $\mathrm{Diff}^r_m(S)$ *?*

Pour les produits aléatoires, une autre question intéressante est la suivante :

Question 2.5. *Pour tout $d \in \mathbb{N}$ avec $d \geq 2$, est-ce que l'ergodicité du produit aléatoire est C^r générique dans* $\mathrm{Diff}^r_m(S)^d$ *?*

Si la réponse à cette question est positive, après notre résultat, on pourrait également conclure que génériquement, le produit aléatoire est non uniformément hyperbolique.

Rappelons que P est la mesure de Benoulli sur l'espace de décalage Σ_d. Une propriété plus faible que l'ergodicité est la **transitivité métrique**, qui est définie comme suit. Un produit aléatoire est métriquement transitif si pour P-presque toute suite $\tilde{x} = (x_i)_{i\in\mathbb{Z}}$, pour m-presque tout point $p \in S$, l'orbite $f^n_{\tilde{x}}(p)$ est dense dans S. En d'autres termes, une trajectoire typique est dense dans S. Une étape intermédiaire possible pour répondre à la question 2.5 est la question suivante.

Question 2.6. *Pour tout $d \in \mathbb{N}$ avec $d \geq 2$, est-ce que la transitivité métrique du produit aléatoire est C^r-générique dans* $\mathrm{Diff}^r_m(S)^d$ *?*

Nous remarquons que Koropecki et Nassiri ([KN10]) ont prouvé que pour tout $r \in \mathbb{N} \cup \{\infty\}$, il existe un sous-ensemble résiduel de $\mathrm{Diff}^r_m(S)^2$ tel que pour toute paire (f, g) dans ce sous-ensemble résiduel, l'action sur S induite par le demi-groupe généré par f et g est transitive.

3. Perturbations dissipatives de l'exemple de Berger-Carrasco

Nos résultats présentés dans cette partie sont contenus dans les chapitres 4 et 5 de cette thèse.

Nous avons défini dans la section 1 l'exemple introduit par Berger et Carrasco dans [BC14]. Nous avons considéré le tore de dimension deux $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, et pour chaque $N \in \mathbb{N}$ nous avons considéré l'application standard $s_N(x, y) = (2x - y + N\sin(x), x)$. Soit $A \in SL(2, \mathbb{Z})$ une matrice hyperbolique qui définit un difféomorphisme Anosov sur $\mathbb{T}^2$, et soit $P_x : \mathbb{T}^2 \to \mathbb{T}^2$ la projection sur la première coordonnée de $\mathbb{T}^2$.

Considérons le tore $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ et représentons-le en utilisant les coordonnées (x, y, z, w), où $x, y, z, w \in [0, 2\pi)$. Pour chaque $N \in \mathbb{N}$, nous définissons

$$\begin{array}{rccc} f_N & : & \mathbb{T}^2 \times \mathbb{T}^2 & \longrightarrow & \mathbb{T}^2 \times \mathbb{T}^2 \\ & & (x, y, z, w) & \mapsto & (s_N(x, y) + P_x \circ A^N(z, w), A^{2N}(z, w)). \end{array}$$

Dans le théorème A, nous n'avons considéré que des petites perturbations C^2 de f_N préservant le volume. Dans cette section, nous considérons également les systèmes non conservatifs et nous décrivons notre travail en relation avec le problème suivant.

Problème 3.1. *Quelles propriétés ergodiques et topologiques peut-on obtenir pour des perturbations dissipatives de f_N ?*

3.1. Transitivité robuste.

Le résultat que nous présentons dans cette partie est un travail commun avec Pablo Carrasco[5] *et il se trouve au chapitre 4 de cette thèse.*

Parmi les propriétés robustes qu'un système dynamique peut avoir, la transitivité a été l'une des plus étudiées. Rappelons qu'un difféomorphisme f est **transitif** si pour deux ensembles ouverts non vides U et V, il existe un entier $n \in \mathbb{N}$ tel que $f^n(U) \cap V \neq \emptyset$. La transitivité signifie que du point de vue topologique, le système ne peut pas être séparé en parties invariantes disjointes. Un difféomorphisme est **robustement transitif**, si chaque difféomorphisme dans un voisinage C^1 est transitif. Ceci peut être considéré comme une version topologique de la stabilité ergodique.

Les premiers exemples connus de difféomorphismes robustement transitifs sont donnés par les applications d'Anosov : si $f \in \mathrm{Diff}^1(M)$ est transitif et uniformément hyperbolique, alors il est C^1-robustement transitif. Il s'avère qu'un certain degré d'hyperbolique est nécessaire pour avoir la transitivité robuste. En effet, si $f \in \mathrm{Diff}^1(M)$ est robustement transitif et $dim M \leq 3$ alors f est hyperbolique/faiblement partiellement hyperbolique [Ma78, BDU99].

5. ICEx-UFMG, Avda. Presidente Antônio Carlos 6627, Belo Horizonte-MG,BR 31270-901.

En général, f admet une décomposition dominée du fibré tangent, $TM = E \oplus F$, tel que $det(Df^{-n_0}|E), det(Df^{n_0}|D) \leq 1/2$ pour un $n_0 \geq 1$ uniforme [BDP03]. Il est bon de souligner que les sous-fibrés E, F ci-dessus ne sont pas nécessairement uniformément expansifs, voir [BV00].

Quant aux exemples non hyperboliques, plusieurs sont connus. La liste ci-dessous donne une image générale (incomplète) des arguments utilisés pour établir une transitivité robuste pour les systèmes non hyperboliques.

- *Déformations des systèmes d'Anosov.* Le premier exemple concret d'application robustement transitif qui n'est pas uniformément hyperbolique a été donné par Shub dans [Sh71] ; plus tard dans [Ma78] Mañé a donné un type de construction similaire sur $\mathbb{T}^3$. Ils sont tous les deux partiellement hyperboliques et homotopes à un système Anosov. L'exemple donné dans [BV00] est aussi une déformation d'un difféomorphisme d'Anosov, et bien qu'il ne soit pas partiellement hyperbolique, il admet une décomposition dominée cohérente avec sa partie Anosov (comme les deux exemples précédents). Plus récemment, Potrie ([Po12] page 152) a donné un exemple de ce type, mais à la différence qu'il admet une décomposition dominée qui n'est pas cohérente avec sa partie hyperbolique. Dans ces cas, la preuve de la transitivité robuste est fondée sur le fait qu'ils ont un comportement hyperbolique dans une grande partie de l'espace.

- *Mélangeurs.* Ce mécanisme puissant a été introduit dans [BD96] par Bonatti-Díaz. Avec lui, les auteurs ont pu montrer que certaines perturbations des applications temps-t d'un flot mélangeant hyperbolique et du produit d'une application d'Anosov par l'identité (disons, sur $\mathbb{T}^3$), sont robustement transitives. Le même outil a été utilisé par Cheng-Gan-Shi dans [CGS18] pour présenter un produit tordu robustement transitif qui a quelques propriétés ergodiques intéressantes (leur type d'exemple est connu sous le nom de type Kan).

- *Minimalité du feuilletage stable/instable.* Il est facile de voir que si $f \in \mathrm{Diff}^1(M)$ admet un feuilletage minimal invariant expansif, alors f est transitif. Les conditions qui garantissent la persistance de ces types de feuilletage sont donc pertinentes pour montrer la transitivité robuste. Parmi ces conditions, la propriété SH introduite par Pujals-Sambarino [PS06] est particulièrement simple à vérifier, et peut être appliquée pour établir la transitivité robuste des systèmes transitifs partiellement hyperboliques où on a un certain contrôle sur le comportement des feuilletages stable/instable. Les exemples de Shub et Mañé cités précédemment entrent dans cette catégorie.

- *Expansion non-uniforme le long du centre.* Dans un travail récent [Ya16], Yang considère les systèmes partiellement hyperboliques avec un comportement central en expansion non uniforme, et montre que tout système conservatif et ergodique avec centre unidimensionnel est robustement transitif. L'auteur utilise le comportement d'expansion non uniforme du centre en remplacement de l'hyperbolicité, en utilisant des méthodes de la théorie ergodique lisse. Ces techniques semblent toutefois ne s'appliquer qu'aux systèmes avec centre unidimensionnel.

Nous ajoutons un nouvel exemple dans la liste ci-dessus. C'est ce qui est donné par l'exemple de Berger-Carrasco. Rappelons qu'il s'agit d'un système partiellement hyperbolique, avec une direction centrale bidimensionnelle, où il a à la fois expansion et contraction le long du centre, et il n'admet aucune autre décomposition dominée. Nous remarquons que la stabilité ergodique n'implique pas la transitivité robuste, voir [Sh14]. On obtient le résultat suivant.

Théorème I. Il existe $N_0 \in \mathbb{N}$ tel que pour tout $N \geq N_0$ le difféomorphisme f_N est C^1-robustement transitif (en fait, C^1-robustement topologiquement mélangeant).

Remarque 3.2. *Topologiquement mélangeant est une propriété plus forte que la transitivité : f est topologiquement mélangeant si pour deux ensembles ouverts U et V, il existe $n_0 \in \mathbb{N}$ tel que pour tout $n \geq n_0$ nous avons $f^n(U) \cap V \neq \emptyset$.*

Les preuves de transitivité robuste pour les difféomorphismes qui sont des déformations des systèmes d'Anosov, mentionnés ci-dessus, utilisent des informations sur un certain type de minimalité (ou ε-minimalité) des variétés stables/instables. Remarquons que notre exemple a un comportement de type hyperbolique dans une grande partie de la variété, comme dans les exemples qui sont des déformations des systèmes Anosov. Par contre, une différence importante dans notre preuve est que nous n'utilisons aucune information sur la minimalité (ou ε-minimalité) des feuilletages stable/instable.

3.2. Rigidité des mesures u-Gibbs. *Nos résultats présentés dans cette partie sont contenus dans le chapitre 5 de cette thèse.*

En dynamique, nous essayons généralement de comprendre le comportement asymptotique de l'orbite de nombreux points. Dans cette direction, il est naturel d'essayer de comprendre les propriétés et l'existence de certaines mesures invariantes qui capturent le comportement statistique d'un ensemble de points qui est relevants pour la mesure de Lebesgue. Précisons les choses plus en détails.

Soit f un difféomorphisme d'une variété fermée, compacte, connexe, orientable M. Étant donné une mesure de probabilité ergodique invariante μ, son **bassin** est défini par la formule suivante

$$B(\mu) = \left\{ p \in M : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(p)} \xrightarrow{n \to +\infty} \mu \right\},$$

où δ_p est la mesure de dirac sur p et la convergence est pour la topologie faible étoile. La mesure μ est **physique** si son bassin a une mesure de Lebesgue positive. En d'autres termes, les mesures physiques sont les mesures qui capturent le comportement asymptotique de beaucoup de points dans le point de vue de Lebesgue.

Dans les années 1970, Sinai, Ruelle et Bowen [Si72, Ru76, Bow75] ont montré que les systèmes uniformément hyperboliques de régularité $C^{1+\alpha}$ ont un nombre fini de mesures physiques qui décrit le comportement statistique de Lebesgue presque tout point. Aujourd'hui, les mesures qu'ils ont construites s'appellent **mesures SRB** (SRB pour Sinai-Ruelle-Bowen). Ces mesures ont une propriété géométrique importante : elles admettent des mesures conditionnelles le long de variétés instables qui sont absolument continues par rapport au volume des variétés instables. Après les travaux de Ledrappier dans [Le84], il existe une théorie ergodique bien développée pour ces mesures. Les mesures SRB hyperboliques forment une classe importante de mesures physiques.

Nous remarquons que dans le cadre hyperbolique, il y a une expansion/contraction uniforme et un décomposition dominée (ce qui implique que l'angle entre les directions d'expansion/contraction est uniformément loin de zéro). Ces deux points sont importants pour porter les constructions de telles mesures. Un problème général dans la théorie est le suivant.

Problème 3.3. *Quand existe-t-il des mesures SRB hyperboliques ?*

Il existe plusieurs travaux qui étudient des conditions qui garantissent l'existence de mesures SRB hyperboliques en dehors du cadre uniformément hyperbolique, voir par exemple [Yo98, BV00, ABV00, CDP16, CLP19, BO19]. Nous renvoyons également au récent article [CLP17] pour une discussion sur les différentes méthodes de construction de telles mesures (avec une emphase sur la méthode géomètrique). Généralement, il est difficile de démontrer l'existence de mesures SRB hyperboliques en dehors du cadre uniformément hyperbolique.

Dans le but d'étudier l'existence et unicité des mesures SRB hyperboliques pour les systèmes partiellement hyperboliques, une autre mesure invariante importante est la mesure dite **u-Gibbs**. Ce sont des mesures invariantes qui vérifient également une certaine

propriété géométrique. Dans le cadre partiellement hyperbolique, chaque mesure SRB hyperbolique est une mesure u-Gibbs.

Le chapitre 5 de cette thèse est un pas vers la connaissance de l'existence et unicité des mesures SRB hyperboliques pour les perturbations dissipatives de l'exemple de Berger-Carrasco. Comme nous l'avons déjà mentionné, il est important de comprendre les mesures u-Gibbs dans cette tâche. Nous obtenons un résultat de rigidité pour les mesures u-Gibbs dans un voisinage de cet exemple. En particulier, nous classifions toutes les mesures u-Gibbs possibles qui peuvent apparaître. Précisons les choses plus en détail.

Dans $\mathrm{Diff}^r(\mathbb{T}^4)$, on peut considérer le sous-espace $SP^r(\mathbb{T}^2 \times \mathbb{T}^2)$ de produits tordus, qui est l'ensemble des C^r-diffeomorphismes g de la forme suivante

$$g(x, y, z, w) = (g_1(x, y, z, w), g_2(z, w)),$$

où $g_2(.,.)$ est un difféomorphisme C^r de $\mathbb{T}^2$, et pour chaque $(z, w) \in \mathbb{T}^2$, $g_1(., ., z, w)$ est un difféomorphisme C^r de $\mathbb{T}^2$ également. Observons que $f_N \in SP^2(\mathbb{T}^2 \times \mathbb{T}^2)$. Nous remarquons aussi que pour N assez grand, si g est un produit tordu C^1 proche de f_N, alors g_2 est un difféomorphisme d'Anosov, et g est partiellement hyperbolique.

Nous rappelons que pour une application g, une mesure invariante μ est **Bernoulli** si le système (g, μ) est conjugué à un décalage de Bernoulli. Pour un produit tordu g comme ci-dessus, nous pouvons regarder les mesures conditionnelles de μ par rapport au feuilletage central. Si ces mesures conditionnelles sont atomiques, nous disons que μ a **désintégration atomique le long du feuilletage central**. Le principal résultat du chapitre 5 est le suivant :

Théorème J. Soit $\alpha \in (0, 1)$. Pour N suffisamment grand, il existe $\mathcal{U}_N^{sp}$ un voisinage C^2 de f_N qui est contenu dans $SP^{2+\alpha}(\mathbb{T}^2 \times \mathbb{T}^2)$, de sorte que si $g \in \mathcal{U}_N^{sp}$, pour toute mesure ergodique $\mu \in \mathrm{Gibbs}^u(g)$ seulement un des énoncés suivants est vérifié :

(1) μ est l'unique mesure SRB. C'est Bernoulli et $\mathrm{supp}(\mu) = \mathbb{T}^4$;

(2) μ a une désintégration atomique le long du feuilletage central, dont les mesures conditionnelles ont un nombre fini d'atomes.

La preuve du théorème J est une conséquence immédiate des théorèmes K et L ci-dessous.

Théorème K. Soit $\alpha \in (0, 1)$. Pour N suffisamment grand, il existe $\mathcal{U}_N^{sp}$ un voisinage C^2 de f_N qui est contenu dans $SP^{2+\alpha}(\mathbb{T}^2 \times \mathbb{T}^2)$, de sorte que si $g \in \mathcal{U}_N^{sp}$, pour toute mesure ergodique $\mu \in \mathrm{Gibbs}^u(g)$ seulement un des énoncés suivants est vérifié :

(1) μ est une mesure SRB hyperbolique ;

(2) μ a une désintégration atomique le long du feuilletage central, dont les mesures conditionnelles ont un nombre fini d'atomes.

Disons quelques mots sur le α qui apparaît dans les énoncés de théorèmes J et K. Ceci est dû à la preuve de Théorème K. Cette preuve utilise un résultat récent par Brown-Rodriguez Hertz dans [BRH17]. Dans leur article, ils classifient toutes les mesures stationnaires ergodiques et hyperboliques pour les produits aléatoires des difféomorphismes de surface C^2. Pour montrer leur résultat, ils prouvent en fait un théorème plus général, ce qui est valable pour les produits tordus abstraits plus généraux avec une surface donnée comme fibre (voir chapitre 5 pour plus de détails).

Le α qui apparaît dans les énoncés des théorèmes J et K, n'apparaît qu'à cause de la régularité C^2 des difféomorphismes considérés dans le résultat principal de Brown-Hertz dans [BRH17]. Si on obtient une version de leur résultat pour les difféomorphismes $C^{1+\beta}$, alors on pourrait supprimer le α de l'énoncé des nos théorèmes.

Théorème L. Pour N suffisamment grand, il existe $\mathcal{U}_N$ un voisinage C^2 de f_N dans $\mathrm{Diff}^2(\mathbb{T}^4)$ tel que si $g \in \mathcal{U}_N$, alors g a au maximum une mesure SRB. En plus, si μ_g est une mesure SRB pour g, alors $\mathrm{supp}(\mu_g) = \mathbb{T}^4$, c'est Bernoulli et hyperbolique.

Remarque 3.4. *Les théorèmes J et K sont pour un voisinage de f_N parmi les difféomorphismes qui sont produits tordus, $SP^2(\mathbb{T}^2 \times \mathbb{T}^2)$. Théorème L est le théorème d'unicité pour les mesures SRB, et il tient dans un voisinage de f_N dans* $\mathrm{Diff}^2(\mathbb{T}^4)$..

Autres remarques et questions. Résumons pourquoi dans les Théorèmes J et K nous avons la condition que les systèmes soient produits tordus pour $\mathbb{T}^2 \times \mathbb{T}^2$. Nous utilisons cette condition pour obtenir que le geuilletage central soit lisse. Ceci est utilisé pour prouver la proposition 2.23, qui déclare que nous pouvons utiliser le principe d'invariance (voir aussi corollaire 2.25). Une question intéressante est de savoir s'il existe un tel résultat de rigidité de mesure pour les systèmes qui ne sont pas des produits tordus pour $\mathbb{T}^2 \times \mathbb{T}^2$. Une première pas naturelle est donnée dans la quesiton suivante :

Question 3.5. *Y a-t-il un résultat de rigidité de mesure similaire pour les mesures u-Gibbs des difféomorphismes dans un voisinage de f_N dans* $\mathrm{Diff}^2(\mathbb{T}^4)$ *?*

Nous croyons que la condition (2) dans le Théorème J ne se réalise généralement pas. Comme il existe de bonnes informations hyperboliques pour les mesures u-Gibbs dans un voisinage de f_N, nous croyons aussi que l'existence générique d'une mesure SRB dans un voisinage de f_N devrait impliquer l'existence d'une mesure SRB pour tout système dans un voisinage de f_N. Nous le précisons dans la conjecture suivante :

Conjecture. *Tout difféomorphisme dans $\mathcal{U}_N^{sp}$ a une mesure SRB.*

Une stratégie intéressante pour prouver l'existence d'une mesure SRB dans un voisinage de f_N dans $\mathrm{Diff}^2(\mathbb{T}^4)$ est d'utiliser les résultats de [CDP16]. Pour ce faire, il faut montrer que la condition appelée **hyperbolicité effective** est vérifiée (voir section 1.2 dans [CDP16]). Cette condition paraît difficile à montrer, mais elle pourrait donner l'existence de mesures SRB en dehors du cas des produits tordus (ou fibrés).

Question 3.6. *Pour N suffisamment grand, pour tout difféomorphisme g qui est suffisamment C^2 proche de f_N, tient-il que g est effectivement hyperbolique ?*

4. Organisation de cette thèse

Cette thèse se divise en deux parties. La première partie (chapitres 1, 2 et 3) contient tous les résultats mentionnés ci-dessus qui sont liés à la dynamique conservative, et la seconde partie (chapitres 4 et 5) contient les résultats qui sont liés à la dynamique dissipative. Les chapitres 1, et 2 concernent la stabilité ergodique. Le chapitre 3 contient les résultats de l'auteur avec Mauricio Poletti sur la généricité de l'existence d'exposants positifs pour certains produits tordus. Le chapitre 4 contient le résultat de l'auteur avec Pablo Carrasco sur la transitivité robuste de l'exemple de Berger-Carrasco. Enfin, le chapitre 5 présente les résultats sur la rigidité des mesures u-Gibbs pour les perturbations dissipatives du même exemple.

Travaux contenus dans cette thèse et d'autres travaux

Cette thèse contient les travaux suivants :

- On the stable ergodicity of Berger-Carrasco's example. *Ergodic Theory and Dynamical Systems*, online version, 2018.
- On the stable ergodicity of diffeomorphisms with dominated splitting. *Nonlinearity*, 32 :445–463, 2019.
- On the genericity of positive exponents of conservative skew products with two-dimensional fibers. Joint work with Mauricio Poletti. Submited. *Preprint on arxiv :-1809.03874*, 2018.
- A new example of robustly transitive diffeomorphism. Joint work with Pablo Carrasco. Submited. *Preprint on arxiv :1904.11788*, 2019.

- Rigidity of u-Gibbs measures for certain partially hyperbolic skew products. 2019.

En 2015, j'avais commencé un doctorat à l'Universidade Federal do Rio de Janeiro (UFRJ) au Brésil, sous la supervision d'Alexander Arbieto. Quelque temps après, la possibilité s'est présentée de commencer un nouveau doctorat à l'Université Paris-Sud (Orsay), indépendamment de mon doctorat au Brésil, sous la direction de Sylvain Crovisier. Depuis janvier 2017, je travaille simultanément sur ces deux thèses, qui sont indépendantes l'une de l'autre. Ma thèse à l'UFRJ, que j'ai soutenu en juillet 2019, porte sur l'étude de différents types de centralisateurs pour les champs de vecteurs. Je mentionne également ici les travaux obtenus pour ma thèse au Brésil :

- On the centralizer of vector fields : Criteria of triviality and genericity results. Joint work with Martin Leguil and Bruno Santiago. Submited. *Preprint arXiv :1810.05085*, 2018.
- Symmetries of vector fields : The diffeomorphism centralizer. Submited. *Preprint arXiv :1903.05883*, 2019.

Introduction (English)

A general goal of dynamical systems is to understand the long term behavior of orbits given an evolutionary law. It appears in all branches of science, for example, physics, biology, economics, chemistry, meteorology, and others. Usually it is a hard problem to describe the asymptotic behavior of the orbits of a system, even for systems with simple expressions.

This is the case, for instance, with the famous three body problem in celestial mechanics. Let us give a brief description of it. Suppose there were only three bodies in the entire universe, and nothing else, for example the sun, the moon, and the earth, interacting by the gravitational law. Suppose also that at some moment you knew the exact position, and velocity of each of these bodies with respect to some referential system of coordinates. Can we say exactly where each of these bodies will be in any future moment? In general no. Because for this system of equations there is no general analytic solution.

Considering the three body problem, Poincaré in 1890 ([Po90]) proposed a qualitative study of this system, instead of a quantitative one, where one would try to obtain the description of the orbit of many points without actually finding explicit solutions of the system.

An important direction in this qualitative study is to try to understand a dynamical system with a probabilistic point of view. This area of mathematics is called ergodic theory. A central point in ergodic theory is to try to understand the statistical behavior of the orbits.

The majority of this thesis is dedicated to the study of ergodic properties of dynamical systems. In particular, for some given dynamical systems, we are interested in describing the statistical behavior of the orbit of "typical" points.

This thesis studies the following topics (which we will describe in more details throughout this introduction):

- stable ergodicity of volume preserving diffeomorphisms (chapters 1 and 2);
- genericity of the existence of positive Lyapunov exponents for random products of volume preserving surface diffeomorphisms (chapter 3);
- the existence and uniqueness of hyperbolic SRB measures (chapter 5);
- robust transitivity (chapter 4).

In what follows we will describe some of the history of each of these points and the results we obtained.

5. Stable ergodicity

Our results stated in this part are contained in chapters 1 and 2 of this thesis.

5.1. Ergodicity in conservative dynamics. Let $f : M \to M$ be a C^r-diffeomorphism of a compact, connected, oriented riemannian manifold M, which preserves some smooth probability measure m. We call such system a conservative diffeomorphism (or volume preserving). Conservative systems appears naturally in physics. Indeed, every hamiltonian flow (which describes the evolution with time of particles modeled by some hamiltonian system) preserves a smooth measure (called the Liouville measure).

A basic feature that a conservative system may have is **ergodicity**, which means that from the probabilistic point of view the dynamics cannot be decomposed into invariant

pieces. Let us be more precise, (f, m) is ergodic if and only if any f-invariant measurable set $\Lambda \subset M$ has m-measure 0 or 1.

After the work of Birkhoff in 1931 ([Bi31]), with his famous ergodic theorem, ergodicity of (f, m) is equivalent to the following: for any m-integrable function $\varphi : M \to \mathbb{R}$, there exists a set of full m-measure Λ_φ such that for each $x \in \Lambda_\varphi$, it is verified

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) = \int_M \varphi dm. \tag{3}$$

Let us give an interpretation of this expression. Suppose that we want to measure how often the orbit of a point visits some measurable region B in the space. Consider the characteristic function of B, and denote it by χ_B. Observe that the function χ_B is m-integrable. Expression (3) then states that the proportion of time that the orbit of x spends on B (left hand side of (3)) coincides with the measure of B (right hand side of (3)).

Birkhoff and Hopf conjectured that a "typical" conservative dynamical system should be ergodic. Let $\mathrm{Homeo}_m(M)$ be the set of conservative homeomorphisms of M. In 1941, Oxtoby-Ulam proved ([OU41]) that a "typical" system in $\mathrm{Homeo}_m(M)$ is ergodic. In this case, "typical" means a homeomorphism belonging to a dense G_δ subset of $\mathrm{Homeo}_m(M)$.

Let $\mathrm{Diff}^r_m(M)$ be the space of C^r-diffeomorphisms of M, for some $r \geq 1$. After the important works of Kolmogorov in 1954 [Ko54-1], Moser in 1962 [Mo62], and Arnold in 1963 [Ar63], the conjecture proposed by Birkhoff and Hopf, turned out to be false in higher regularity. Indeed, for M with dimension at least two, there exist open sets in $\mathrm{Diff}^\infty_m(M)$ of non ergodic diffeomorphisms. Nowadays, their result is known as KAM theorem (Kolmogorov-Arnold-Moser). It gives conditions for the existence, and persistence, of some region in the manifold with positive measure consisting of invariant tori, in particular this is an obstruction for ergodicity.

In Kolmogorov's ICM lecture in 1954 (see [Ko54-2], page 326), he stated the following: "... it is extremely likely that, for arbitrary s, there are examples of canonical systems with s degrees of freedom and with stable transitiveness (i.e. ergodicity) and mixing... I have in mind motion along geodesics on compact manifold of constant negative curvature..." In other words, Kolmogorov already believed that any conservative flow sufficiently close to the geodesic flow of compact manifolds with constant negative curvature was ergodic. Nowadays, this is known as stable ergodicity.

Definition 5.1 (Stable ergodicity)**.** *Let $r > 1$, and $s \in [1, r]$. A diffeomorphism $f \in \mathrm{Diff}^r_m(M)$ is C^s-**stably ergodic** if there exists a C^s-neighborhood $\mathcal{U}$ of f such that any diffeomorphism $g \in \mathcal{U} \cap \mathrm{Diff}^r_m(M)$ is ergodic.*

Many years before Kolmogorov's lecture, Hopf in 1939 had already proved that the geodesic flow mentioned above is ergodic with respect to the Liouville measure, see [Ho39]. In his proof he introduced an argument which is nowadays called the **Hopf argument**. This flow has an important feature called uniform hyperbolicity.

A diffeomorphism f is **uniformly hyperbolic** (also called Anosov diffeomorphisms), if there is a Df-invariant decomposition of the tangent bundle $TM = E^s \oplus E^u$ such that E^s contracts uniformly for the future, and E^u contracts uniformly for the past.

In 1967, Anosov [Ano67] used Hopf argument to prove the ergodicity of the geodesic flow of compact manifolds with strictly negative (non constant) sectional curvature. His proof also gave that any conservative uniformly hyperbolic C^2-diffeomorphism is ergodic. Since uniform hyperbolicity is a C^1-open property, this implies that C^2-Anosov diffeomorphisms are C^1-stably ergodic.

In what follows, we will discuss stable ergodicity outside the uniformly hyperbolic setting. We also mention that, every known proof of stable ergodicity uses some generalization of Hopf argument.

5.2. Stable ergodicity for partially hyperbolic diffeomorphisms. Our results stated in this part are contained in chapter 1 of this thesis.

For almost three decades conservative uniformly hyperbolic systems were the only known examples of stably ergodic diffeomorphisms. It was only in 1994 that Grayson-Pugh-Shub obtained a non Anosov example ([GPS94]). Let us describe better their example. Let (S, g) be a compact surface with constant negative curvature, and let $(\varphi_t)_{t\in\mathbb{R}}$ be the geodesic flow defined on the unit tangent bundle of S, which we denote by T^1S. Denote by m the normalized Liouville measure on T^1S, which is φ_t-invariant. Grayson-Pugh-Shub then proved that if one considers the diffeomorphism $f = \varphi_1 \in \mathrm{Diff}^2_m(T^1S)$, then f is C^2-stably ergodic. This is a non-trivial result, and uses a non-trivial generalization of Hopf argument. The diffeomorphism f has a weaker form of hyperbolicity called partial hyperbolicity, which we define below.

A diffeomorphism f is **partially hyperbolic** if the tangent bundle admits a Df-invariant decomposition of the form $TM = E^s \oplus E^c \oplus E^u$ such that E^s contracts uniformly, E^u expands uniformly, and there exists $N \in \mathbb{N}$ such that for any point $x \in M$ we have

$$\max\left\{\|Df^N(x)|_{E^c}\|.\|Df^{-N}(f^N(x))|_{E^u}\|, \|Df^N(x)|_{E^s}\|.\|Df^{-N}(f^N(x))|_{E^c}\|\right\} < \frac{1}{2}.$$

It is well known that the subbundles E^s and E^u are uniquely integrable, that is, there is an unique invariant foliation $\mathcal{F}^*$ which is tangent to E^*, for $* = s, u$. We remark that partial hyperbolicity is a C^1-open property among the diffeomorphisms.

A key property in the proof of stable ergodicity by Grayson-Pugh-Shub ([GPS94]) is the accessibility. A partially hyperbolic system is **accessible** if any two points in the manifold can be joined by a curve which is a concatenation of finitely many curves, each of them being contained in a stable or an unstable leaf. We say that a diffeomorphism is **stably accessible** if any diffeomorphism in a C^1-neighborhood of it is accessible.

Based on their work [GPS94], Pugh-Shub made the following conjecture:

Conjecture (Pugh-Shub conjecture 1, [PS97]). *C^1-stable ergodicity is C^r-dense among the C^r-partially hyperbolic conservative diffeomorphisms on a compact connected manifold, for any $r > 1$.*

In [PS97], Pugh-Shub also propose a program to solve this conjecture. Their program proposes the following two conjectures.

Conjecture (Pugh-Shub conjecture 2, [PS97]). *Accessibility implies ergodicity for a conservative partially hyperbolic C^r-diffeomorphism.*

Conjecture (Pugh-Shub conjecture 3, [PS97]). *Stable accessibility is C^r-dense among the partially hyperbolic C^r-diffeomorphisms (volume preserving, or not).*

Since then, the topic of stable ergodicity for partially hyperbolic systems has become a very active topic of research. In particular, many works have been done aiming these conjectures. In what follows we will make an (incomplete) list of some of the works related to these conjectures.

- In [HHU08], Hertz-Hertz-Ures proved that conjecture 1 is true among the C^r-partially hyperbolic diffeomorphisms with one dimensional center.
- In [HHTU11], Hertz-Hertz-Tahzibi-Ures proved that among C^r-partially hyperbolic diffemorphisms with two dimensional center, stable ergodicity is C^1-dense. More recently, Avila-Crovisier-Wilkinson proved, in [ACW17], that the same result holds without the assumption of two dimensional center.
- Burns-Dolgopyat-Pesin proved, in [BDP02], that a volume preserving partially hyperbolic C^r-diffeomorphism with the following conditions is ergodic: it is accessible; and it has a set of positive measure whose points have all center Lyapunov exponents (see the definition below) non zero and with the same sign. This result holds with a weaker assumption called essential accessibility.

- Burns-Wilkinson proved, in [BW10], that for volume preserving C^2-partially hyperbolic diffeomorphisms, accessibility (or essential accessibility), and a technical condition called center bunching implies ergodicity. This is a non-trivial strengthening of a previous result by Pugh-Shub in [PS00].

We remark that all these works use the accessibility (or essential accessibility) property. Given some explicit example, accessibility is usually a hard property to check. One of the goals of this work is to address the following problem:

Problem 5.2. *Prove the stable ergodicity for partially hyperbolic systems without using accessibility (or essential accessibility)?*

As we will see in the next section, there are examples of stably ergodic diffeomorphisms outside the partially hyperbolic setting, see section 5.3. A natural approach to answer this question is by using Pesin's theory for non-uniformly hyperbolic systems.

For a C^1-diffeomorphism f and an invariant measure ν, for ν-almost every point $p \in M$ and for every $v \in T_pM - \{0\}$ the following limit exists

$$\lambda(p, v) = \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n(p).v\|.$$

Oseledets' theorem states that $\lambda(p, .)$ can have at most $\dim(M)$ different values. Such numbers are called Lyapunov exponents. An f-invariant measure ν is non-uniformly hyperbolic for f if for ν-almost every point, every Lyapunov exponent is non zero. We say that a conservative diffeomorphism f is non-uniformly hyperbolic if the invariant smooth measure m is non-uniformly hyperbolic for f.

In [Pes77], Pesin adapted Hopf argument for the non-uniformly hyperbolic setting, and proved that a non-uniformly hyperbolic smooth measure has at most countably many different ergodic components. We remark that just Pesin's theory does not give ergodicity.

Berger and Carrasco introduced in [BC14] an example of a volume-preserving, partially hyperbolic diffeomorphism which is non-uniformly hyperbolic. This example has a two dimensional center bundle which does not admit any further dominated decomposition, and Lebesgue almost every point has both a positive and a negative Lyapunov exponent in the center direction. Furthermore, the properties of this example are C^2-robust. It is not known if this example is accessible or not.

Chapter 1 is dedicated to prove the C^2-stable ergodicity of Berger-Carrasco's example. Before we introduce the example and give the precise statement of our result, let us stress two features of our work that distinguishes it from the rest of the previous works about stable ergodicity of partially hyperbolic diffeomorphisms:

- The stable ergodicity for a system with mixed behaviour [6], and no dominated splitting of the center direction (as a strengthening of [BC14]);
- A proof of stable ergodicity that does not uses accessibility (or essential accessibility).

Berger-Carrasco's example. For $N \in \mathbb{N}$ we denote by $s_N(x, y) = (2x - y + N \sin(x), x)$ the standard map on $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. For every N the map s_N preserves the Lebesgue measure induced by the usual metric of $\mathbb{T}^2$. This map is related to several physical problems, see for instance [Ch79, Iz80, SS95].

It is conjectured that for $N \neq 0$ large, the map s_N has positive entropy for the Lebesgue measure (see [Si94], page 144). By Pesin's entropy formula ([Pes77], Theorem 5.1), this is equivalent to the existence of a set of positive Lebesgue measure and whose points have a positive Lyapunov exponent. The existence of those sets is not known for any value of N. We refer the reader to [BXY17, Du94, Go12] for some results related to this conjecture.

Let $A \in SL(2, \mathbb{Z})$ be a hyperbolic matrix which defines an Anosov diffeomorphism on $\mathbb{T}^2$, and let $P_x : \mathbb{T}^2 \to \mathbb{T}^2$ be the projection on the first coordinate of $\mathbb{T}^2$. This

6. It means that almost every point has both a positive and a negative center Lyapunov exponent along the center direction

projection is induced by the linear map of $\mathbb{R}^2$, which we will also denote by P_x, defined by $P_x(a,b) = (a,0)$.

Consider the torus $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$, and represent it using the coordinates (x,y,z,w), where $x,y,z,w \in [0,2\pi)$. We may naturally identify a point (z,w) on the second torus with a point (x,y) on the first torus by setting $x = z$ and $y = w$. For each $N \in \mathbb{N}$ define

$$\begin{array}{rccc} f_N & : & \mathbb{T}^2 \times \mathbb{T}^2 & \longrightarrow & \mathbb{T}^2 \times \mathbb{T}^2 \\ & & (x,y,z,w) & \mapsto & (s_N(x,y) + P_x \circ A^N(z,w), A^{2N}(z,w)), \end{array}$$

where the point $A^N(z,w)$ on the second torus is being identified with the same point in the first torus as described previously.

This diffeomorphism preserves the Lebesgue measure, which we will denote by *Leb*. For N large enough it is a partially hyperbolic diffeomorphism, with a two dimensional center direction given by $E^c = \mathbb{R}^2 \times \{0\}$. This type of system was considered by Berger and Carrasco in [BC14], where they proved the following theorem.

Theorem 5.3 ([BC14], Theorem 1)**.** *There exist $N_0 > 0$ and $c > 0$ such that for every $N \geq N_0$, for Lebesgue almost every point m and for every $v \in \mathbb{R}^4$*

$$\lim_{n\to\infty} \left| \frac{1}{n} \log \|Df_N^n(m).v\| \right| > c \log N.$$

Moreover, the same property holds for any volume-preserving diffeomorphism in a C^2-neighborhood of f_N.

We remark that Viana constructed in theorem B of [Vi97], an example of a non-conservative partially hyperbolic diffeomorphism with similar properties as Berger-Carrasco's example: Lebesgue almost every point has a positive and a negative exponent in the center direction and there is no dominated splitting of the center, but the system is dissipative.

Definition 5.4. *Let ν be an invariant probability measure for f. We say that (f,ν) is Bernoulli if it is measurably conjugated to a Bernoulli shift. For volume-preserving diffeomorphisms, we say that f is Bernoulli if (f, Leb) is Bernoulli.*

We remark that the Bernoulli property is stronger than ergodicity.

Theorem A. For N large enough f_N is C^2-stably ergodic. Moreover, any volume-preserving diffeomorphism in a C^2-neighborhood of f_N is Bernoulli.

In order to prove this theorem we will need to obtain precise estimates on the size of the invariant manifolds in the center direction for certain points. For that we need a better estimate of the center exponents, given by the following proposition.

Proposition 5.5. *For every $\delta \in (0,1)$, there exists $N_0 = N_0(\delta)$ such that for every $N \geq N_0$ there is a C^2-neighborhood $\mathcal{U}_N$ of f_N in $\mathrm{Diff}^2_{Leb}(\mathbb{T}^4)$ with the following property. If $g \in \mathcal{U}_N$, then Lebesgue almost every point has a positive and a negative Lyapunov exponent in the center direction whose absolute value are greater than $(1-\delta)\log N$.*

One can show that f_N is C^2-approximated by stably ergodic diffeomorphisms with another approach. This approach uses accessibility, which can be obtained using the results in [HS17], and the criteria of ergodicity in [BW10]. Such approach does not use the non-uniform hyperbolicity of the system.

Later we will state more ergodic, and topological results obtained in this thesis regarding Berger-Carrasco's example.

Further remarks and questions. The techniques used to prove Theorem A are based on finding precise estimates on the size and "geometry" of stable and unstable Pesin's manifolds. There is a general idea behind it, which is that for any ergodic component, non-uniform hyperbolicity with "large" Lyapunov exponents implies the existence of "large"

stable/unstable manifolds in a set of large measure (for the component). With some aditional control on the geometry, we are able to find transverse intersections between stable/unstable manifolds of typical points for any two ergodic components, which by Hopf argument will imply ergodicity. How much further can these techniques be pushed to study the ergodicity of a system? In particular, we think that it might help to give partial answers to Pugh-Shub's conjecture 2.

Question 5.6. *Let f be a conservative partially hyperbolic C^2-diffeomorphism with two dimensional center. Suppose that f is accessible, non-uniformly hyperbolic with "large" center Lyapunov exponents for almost every point. Is f ergodic?*

Observe that for the question above, if the center Lyapunov exponents have the same sign, then f is actually stably ergodic [BDP02]. So the scenario to be considered is when f has mixed behavior along the center, just as in Berger-Carrasco's example.

In [ABW09], Avila-Bochi-Wilkinson study ergodic implications of a property called **non-uniform center bunching**. In particular, in corollary C of that paper, they give a criterion of ergodicity for volume preserving, non-uniformly center bunching C^2-systems, with a stronger type of accessibility property. This non-uniformly center bunching property is implied, for example, by systems with "small" center Lyapunov exponents.

Another case that is not mentioned above is when you have a set of positive measure of points having both one "small" and one "large" center exponents. To avoid this case, one may consider systems with some symmetry relations for the exponents. This happens, for instance, for symplectomorphisms.

Question 5.7. *Can one use a combination of the techniques mentioned above to prove that a partially hyperbolic, accessible, C^2-symplectomorphism with two dimensional center has finitely many ergodic components? Can one prove ergodicity?*

5.3. Stable ergodicity beyond partial hyperbolicity. *Our results stated in this part are contained in chapter 2 of this thesis.*

Most works done about stable ergodicity have been done in the partially hyperbolic setting, and not much has been done outside this setting.

It is known that C^1-stably ergodic diffeomorphisms must have some weaker form of hyperbolicity [AM07], called dominated splitting. We say that a diffeomorphism f admits a **dominated splitting** if there is a decomposition of the tangent bundle, $TM = E \oplus F$, into two non-trivial subbundles which are Df-invariant, such that for some $N \geq 1$, any unit vectors $v \in E(x)$ and $u \in F(x)$ verify

$$\|Df^N(x)v\| < \frac{1}{2}\|Df^N(x)u\|.$$

We also define a weark form of partial hyperbolicity. A diffeomorphism is **weakly partially hyperbolic** if it admits a dominated splitting of the form $TM = E \oplus E^u$, such that the subbundle E^u expands exponentially fast under the action of Df.

Problem 5.8. *Obtain stable ergodicity criteria for systems with a dominated decomposition, or for systems that are weakly partially hyperbolic?*

Bonatti-Viana in 2000 ([BV00]) obtained an example of a weakly partially hyperbolic diffeomorphism which is not partially hyperbolic (meaning there is only the strong unstable direction, and no strong stable direction), and which is C^1-stably ergodic.

In the same paper, Bonatti-Viana introduced an example of a volume preserving diffeomorphism, which is C^1-robustly transitive [7] and has a dominated decomposition without any uniformly hyperbolic direction. At the end of their paper, they asked if this system was C^1-stably ergodic.

7. A diffeomorphism is robustly transitive if in a C^1-neighborhood of it, any diffeomorphism has the property that it exists a point with dense orbit.

In 2004, Tahzibi in his thesis gave a positive answer to Bonatti-Viana's question, obtaining the first example of a C^1-stably ergodic diffeomorphism which does not have any hyperbolic direction (see [Tah04]). Since then there were no other work on stable ergodicity outside the partially hyperbolic scenario.

Since ergodicity is a global feature, it is natural to look for global properties that could help to obtain ergodicity, or stable ergodicity. In the partially hyperbolic setting, as we explained before, the key global property that has been used is accessibility. We recall that m represents some smooth measure on the manifold M.

One of the goals of chapter 2 of this thesis is to find new criteria of stable ergodicity, actually of stable Bernoulli, outside the partially hyperbolic scenario. In particular, we study the consequences given by a property called **chain-hyperbolicity**, for the precise definition see definition 2.3 in chapter 2. Chain-hyperbolicity has been defined and used before in [Cro11, CP15]. It can be seen as some type of topological hyperbolicity saying that f "contracts" topologically along the direction E, up to a certain "scale", and f^{-1} "contracts" topologically along the direction F, up to a certain "scale". Using this as the global property to study stable ergodicity for diffeomorphisms with a dominated splitting, we have the following theorem.

Theorem B. Let $f \in \mathrm{Diff}^1_m(M)$. If f is a chain-hyperbolic diffeomorphism for a dominated splitting $TM = E \oplus F$ and verifies

$$\int_M \log \|Df|_E\| dm < 0 \text{ and } \int_M \log \|Df^{-1}|_F\| dm < 0, \tag{4}$$

then there exists a C^1-neighborhood $\mathcal{U}$ of f, such that any diffeomorphism $g \in \mathcal{U} \cap \mathrm{Diff}^2_m(M)$ is ergodic, in fact Bernoulli. In particular, any such diffeomorphism g is stably Bernoulli.

In the setting of Theorem B, as a consequence of (4) and ergodicity, we actually obtain that m-almost every point has all Lyapunov exponents negative along E and all positive along F.

As one application of Theorem B, we obtain the following criterion of stable Bernoulli for weakly partially hyperbolic systems.

Theorem C. Let $f \in \mathrm{Diff}^2_m(M)$. Suppose that f is weakly partially hyperbolic with dominated splitting $TM = E \oplus E^u$ and chain-hyperbolic with respect to the same splitting. If f has all Lyapunov exponents negative along the direction E on a set of positive m-measure, then f is stably ergodic, in fact stably Bernoulli.

This theorem can be seen as a version of theorem 4 in [BDP02] for weakly partially hyperbolic diffeomorphisms. We also remark that if $f \in \mathrm{Diff}^2_m(M)$ verifies the hypothesis of Theorem B and the direction F is uniformly hyperbolic, meaning $F = E^u$, then (4) implies that f verifies the hypothesis of Theorem C. However, a diffeomorphism which verifies the hypothesis of Theorem C, does not necessarily verify the hypothesis of Theorem B, a priori.

Theorem B gives more flexibility in the construction of the example considered by Tahzibi in [Tah04]. To construct the example one makes a deformation supported in a finite number of small balls around hyperbolic fixed points, in particular, the deformations are local. Theorem B applies to this example and quantifies, in a certain way, how much one can make such a deformation, in particular, the deformations do not have to be local. In section 6 of chapter 2 we explain the construction of such an example in this non local way. We remark that our proof is different from the proof of Tahzibi in [Tah04].

As another application of Theorem B, and some others results, we can prove the C^1-density of stably Bernoulli diffeomorphisms among a certain class of weakly partially hyperbolic diffeomorphisms. Let us define this class.

Let $\mathcal{D} \subset \mathrm{Diff}^2_m(M)$ be the subset of diffeomorphisms f that verifies the following properties:

- f is weakly partially hyperbolic, with dominated splitting $TM = E \oplus E^u$ and $dim(E) = 2$;
- f is chain-hyperbolic for the splitting $TM = E \oplus E^u$.

Define $\mathcal{WCH}^2_m(M)$ to be the C^1-interior of $\mathcal{D}$ for the relative topology. For the d-torus this set is non empty, with $d \geq 3$. The weak partially hyperbolic examples in Bonatti-Viana [BV00] belong to this set. We have the following theorem.

Theorem D. Stable Bernoulli is C^1-dense on $\mathcal{WCH}^2_m(M)$.

We remark that all our results remain true for $C^{1+\alpha}$-diffeomorphisms.

Further remarks and questions. We finish this part of the introduction with some questions and comments.

Question 5.9. *What others criteria for stable ergodicity, or stable Bernoulli, can one obtain using chain-hyperbolicity?*

We point out that the example considered by Tahzibi in [Tah04] is isotopic to a linear Anosov diffeomorphism.

Question 5.10. *Is there a diffeomorphism that verifies the hypothesis of Theorem B, or Theorem C, which is not isotopic to an Anosov diffeomorphism?*

Rafael Potrie obtains a negative answer for this question in dimension 3 under some assumptions, see [Pot15].

After our work, Núñez-Hertz in [NH19] also obtained a result of stable ergodicity (indeed stable Bernoulli) outside the partially hyperbolic setting. They consider weakly partially hyperbolic diffeomorphism on three dimensional manifolds. The global property (instead of the chain hyperbolicity) they use to study stable ergodicity is the minimality of the strong unstable foliation. In particular, they obtain in dimension three that a C^1-generic [8] diffeomorphism, which is weakly partially hyperbolic and whose strong unstable foliation is minimal, is stably Bernoulli (meaning that any C^2-diffeomorphism C^1-close to it is Bernoulli). Even though there are some similarities in both proofs, they are use different "global" properties, so they can be seen as complementary to each other.

6. Genericity of the existence of positive Lyapunov exponents

Our results stated in this section are a joint work with Mauricio Poletti [9], and they are contained in chapter 3 of this thesis.

In the 60's, Smale had obtained several results about dynamical consequences of uniform hyperbolicity (see [Sm67]). Since then, uniform hyperbolic dynamics has been very well understood. For instance, hyperbolic transitive sets have several features, such as a symbolic dynamics associated to it, existence of periodic points and horseshoes, positive entropy, etc. Even though, uniform hyperbolicity is a C^1-open property, it is not a C^1-dense property.

For smooth invariant measures, Pesin proposed in [Pes77] a weaker notion of hyperbolicity, called non-uniform hyperbolicity which we defined in the previous section. It turns out that non-uniform hyperbolicity also implies several interesting features of the dynamics, such as existence of periodic orbits and horseshoes [Ka80], countably many ergodic components [Pes77], etc. Contrary to uniform hyperbolicity, one can expect non-uniform hyperbolicity to hold for a large class of systems.

Given a smooth compact riemannian manifold M, recall that m is a smooth measure on M.

8. C^1-generic means that it belongs to a dense G_δ subset of $\mathrm{Diff}^1_m(M)$.

9. CNRS-Laboratoire de Mathématiques d'Orsay, UMR 8628, Université Paris-Sud 11, Orsay Cedex 91405, France.

Problem 6.1. *How frequent is non-uniform hyperbolicity in* $\mathrm{Diff}^r_m(M)$*?*

6.1. Non-uniform hyperbolicity in the C^1-topology. If M is a surface, then a remarkable result by Mañé [Ma96] and Bochi [Bo02] states that C^1-generically in $\mathrm{Diff}^1_m(M)$ either the diffeomorphism is Anosov, or all its Lyapunov exponents are zero for almost every point.

This result has been recently generalized to any dimension by Avila-Crovisier-Wilkinson in 2016 ([ACW16]), where they proved that for a compact manifold of any dimension M, C^1-generically in $\mathrm{Diff}^1_m(M)$, either all the Lyapunov exponents are zero for m-almost every point, or the system is non-uniformly Anosov which means that it is non-uniformly hyperbolic and the Oseledets' splitting is dominated.

A different behavior happens locally, and C^1-densely. Liang-Yang proved in [LY17] that for any $r \geq 1$, there exists a C^1-dense subset of $\mathrm{Diff}^r_m(M)$ of diffeomorphisms having a set of positive m-measure whose points have its Lyapunov exponents all non zero. This result holds for manifolds with dimension at least two.

Now suppose that instead of one surface diffeomorphism, we considered two or more volume preserving surface diffeomorphisms iterating them in a random way.

Question 6.2. *Does non-uniform hyperbolicity hold for "typical" random products of volume preserving surface diffeomorphisms?*

Fix $d \in \mathbb{N}$ such that $d \geq 2$, fix positive real numbers $p_1, \ldots, p_d$ such that $p_1 + \cdots + p_d = 1$, and let p be the probability measure on the set $\{1, \cdots, d\}$ given by the numbers p_i. Given d-diffeomorphisms $(f_1, \ldots, f_d) \in \mathrm{Diff}^r_m(S)^d$, we consider the random product generated by them, where in each moment, the probability of the diffeomorphism f_i to act on S is p_i. Formally, the random product is an skew product over the left shift map in $\Sigma = \{1, \cdots, d\}^{\mathbb{Z}}$ and Bernoulli measure given by $P = p^{\mathbb{Z}}$.

For P-almost every sequence $\tilde{x} = (x_i)_{i \in \mathbb{Z}}$ and m-almost every point $p \in S$, the following two limits exist

$$\lambda^+(\tilde{x}, p) = \lim_{n \to +\infty} \frac{1}{n} \|Df^n_{\tilde{x}}(p)\| \text{ and } \lambda^-(\tilde{x}, p) = -\lim_{n \to +\infty} \frac{1}{n} \left\|Df^{-n}_{\tilde{x}}(p)\right\|,$$

where $f^n_{\tilde{x}} = f_{x_{n-1}} \circ \cdots \circ f_{x_0}$. These numbers are called the fiberwise (or center) Lyapunov exponents. We may also consider the integrated fiberwise Lyapunov exponent, which is given by

$$L(f_1, \cdots, f_d) = \int_{\Sigma \times S} \lambda^+(\tilde{x}, p) dP \times m(\tilde{x}, p).$$

In this introduction, we chose to state first Theorem G before Theorems E and F, since it illustrates in a simpler way the type of results we obtain, and how it compares with Bochi-Mañé's theorem. Our result is the following.

Theorem G. Fix $d \in \mathbb{N}$, a real number $r \geq 1$, and fix some probability measure p on $\{1, \ldots, d\}$ such that $p(\{i\}) < 1$, for $i = 1, \cdots, d$. Then there exists a C^1-open and C^1-dense subset of $\mathrm{Diff}^r_m(S)^d$ such that if $(f_1, \ldots, f_d)$ belongs to this set its random product has positive integrated Lyapunov exponents.

Observe that this result gives a C^1-open set in $\mathrm{Diff}_m(S)^d$ with positive exponents, in some region. In particular, an interpretation of this is that Bochi-Mañé's result does not hold in the random product scenario. Our results actually hold for more general skew products. Let us now describe the other results we obtain.

Let $\tilde{M}$ be a smooth, compact, connected and oriented manifold and S be a smooth, compact and connected surface. Consider a fiber bundle M over $\tilde{M}$, defined by a smooth projection $\pi : M \to \tilde{M}$, with fibers diffeomorphic to S. For a point $x \in M$, we write S_x the fiber that contains the point x. We say that a diffeomorphism $f : M \to M$ preserves fibers if for any $x \in M$ it holds $S_{f(x)} = f(S_x)$.

For the fiber bundle M a diffeomorphism $f : M \to M$ is a **partially hyperbolic skew product** if the following holds:

- f sends fiber to fiber;
- f is a partially hyperbolic diffeomorphism, with splitting $TM = E^s \oplus E^c \oplus E^u$, such that $E^c = \ker D\pi$.

Let m be the normalized Lebesgue measure on M and define $SP^r_m(M)$ to be the set of C^r-partially hyperbolic skew products that preserve the Lebesgue measure. In the space $SP^r_m(M)$ we may consider the C^s-topology, for any $s \in [0, r]$.

For m-almost every point, consider the greatest and smallest Lyapunov exponents along the center direction, defined respectively by

$$\lambda^+_c(x) = \lim_{n\to+\infty} \frac{1}{n}\left\|Df^n(x)|_{E^c_x}\right\| \text{ and } \lambda^-_c(x) = -\lim_{n\to+\infty} \frac{1}{n}\left\|Df^{-n}(x)|_{E^c_x}\right\|.$$

In this setting, we actually have that $\det Df(x)|_{E^c_x} = 1$ (see chapter 3). This implies that for almost every point $x \in M$ it is verified that $\lambda^-_c(x) = -\lambda^+_c(x)$. We define the **integrated Lyapunov exponent along the center direction** by

$$L(f) = \int_M \lambda^+_c(x) dm(x).$$

In the following result, we use the notion of center bunching, which is a technical condition that we will not define it here, we refer the reader to chapter 3 for the precise definition.

Theorem E. For any $r > 1$, among the volume preserving, C^r-partially hyperbolic skew products that are center bunched, there exists a C^1-dense and C^r-open subset of diffeomorphisms verifying the following: if f belongs to this subset, then $L(f) > 0$.

From [HS17], it is known that ergodicity is C^1-open and C^r-dense in the setting of the previous theorem. The next result follows immediately from Theorem E.

Corollary 6.3. *In the same setting of Theorem E, there exists a C^1-dense and C^r-open subset such that any diffeomorphism in this subset is non-uniformly hyperbolic, that is, m-almost every point has all its Lyapunov exponents non zero.*

Another scenario in which we obtain results is for more general skew products. Let Σ be a compact metric space with no isolated points, let $\sigma : \Sigma \to \Sigma$ be a hyperbolic homeomorphism (see chapter 3 for a precise definition), and $\tilde{\mu}$ be a σ-invariant measure that has a property called local product structure (we refer the reader to chapter 3 for precise definitions). This property holds for important measures such as the equilibrium states of Hölder potentials (see [Bow75]).

Fix $\alpha > 0$. Given a (C, α)-Hölder map from Σ to $\mathrm{Diff}^r_m(S)$, $\tilde{x} \mapsto f_{\tilde{x}}$, we define the skew product

$$\begin{array}{rcl} f : \Sigma \times S & \to & \Sigma \times S \\ (\tilde{x}, t) & \mapsto & f(\tilde{x}, t) = (\sigma(\tilde{x}), f_{\tilde{x}}(t)), \end{array}$$

where by (C, α)-Hölder we mean that

$$\mathrm{d}_{C^r}(f_{\tilde{x}}, f_{\tilde{y}}) \leq C\, \mathrm{d}_\Sigma(\tilde{x}, \tilde{y})^\alpha.$$

Observe that such skew product preserves the measure $\mu := \tilde{\mu} \times m$. Such map is called **$C^{r,\alpha}$-skew product** over σ that preserves μ.

From now on we fix $C > 0$. For $\alpha > 0$ and $r \geq 1 + \alpha$, we define $SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$ to be the space of $C^{r,\alpha}$ skew products over σ, such that the map $\tilde{x} \mapsto f_{\tilde{x}}$ is (C, α)-Hölder. In this space we consider the C^s-topology, for any $s \leq r$ defined as follows: for any two $C^{r,\alpha}$-skew products $f, g \in SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$, the C^s distance between f and g is

$$\mathrm{d}_{C^s}(f, g) = \sup_{\tilde{x}\in\Sigma} \mathrm{d}_{C^s}(f_{\tilde{x}}, g_{\tilde{x}}),$$

where $\mathrm{d}_{C^s,\tilde{x}}(f_{\tilde{x}}, g_{\tilde{x}})$ is the C^s distance on $\mathrm{Diff}^r_m(S)$. Keep in mind that σ is always fixed.

As before, we can define the fiberwise Lyapunov exponents as

$$\lambda^+(\tilde{x},t)=\lim_{n\to+\infty}\frac{1}{n}\|Df^n_{\tilde{x}}(t)\| \text{ and } \lambda^-(\tilde{x},t)=-\lim_{n\to+\infty}\frac{1}{n}\|Df^{-n}_{\tilde{x}}(t)\|,$$

where $f^n_{\tilde{x}}=f_{\sigma^{n-1}(\tilde{x})}\circ\cdots\circ f_{\tilde{x}}$. This is defined μ-almost everywhere.

Similar to the notion of center bunching, there is a notion of fiber bunching which guarantees the existence of linear holonomies, see chapter 3 for precise definitions.

Theorem F. Let σ be a hyperbolic homeomorphism and let $\tilde{\mu}$ be a σ-invariant measure with local product structure. For any $r>1$ and $\alpha>0$, there exists a C^1-dense and C^r-open subset of $SP^{r,\alpha}_{\sigma,\mu}(\Sigma\times S)$ verifying the following: if f belongs to this subset, then $L(f)>0$.

One of the key ingredients in our proof is a condition that we call **pinching** (see chapter 3 for the definition). This condition states that there exist some fixed (or periodic) fiber, such that the action of the skew product on this fiber has positive Lyapunov exponents in some region with positive volume.

6.2. Non-uniform hyperbolicity in the C^r-topology. In our results above, we can only get C^1 density because we use that the pinching condition is C^1-dense, after the result of [LY17]. The C^r-density of the pinching condition, in general, is not known for $r>1$.

With some information on the periodic points of a diffeomorphism, the pinching condition can be found in higher regularity, and we have the following result.

Theorem H. Let f be as in Theorem E, F or G, and suppose that there exist some periodic fiber $S_{\tilde{p}}$ such that $f_{\tilde{p}}:S_{\tilde{p}}\to S_{\tilde{p}}$ has an elliptic periodic point. Then f is C^r-accumulated by C^r-open sets with positive integrated Lyapunov exponents. Moreover, in the random product case these sets are C^1 open.

We remark that Marin in [Mar16] proved that a partially hyperbolic C^r-symplectomorphism with two dimensional center, which is accessible, verifies some center bunching condition, and has a periodic point satisfying some condition (which she calls pinching), can be C^r-approximated by ergodic non-uniformly hyperbolic symplectomorphisms. In her argument accessibility and the fact that it preserves some volume form (induced by the symplectic form) are crucial properties because she uses the results of [ASV13]. We remark that her result is not restricted to the skew product setting. It was furher improved by Liang-Marin-Yang, in [LMY18], that proved that in a C^r-neighborhood of a symplectomorphism verifying the conditions above, there is a C^r-open and C^r-dense subset of ergodic non-uniformly hyperbolic symplectomorphisms. We point out that differently from Marin's result, our arguments are not based on the results from [ASV13].

After the conclusion of our work, Barrientos and Malicet [BM18] sent us a preprint of a similar result for the random product of diffeomorphisms. They prove that after fixing $k-1$ volume preserving surface diffeomorphisms, $f_1,\dots,f_{k-1}$, such that the action generated by them is ergodic, then for any f_k inside a C^r dense and C^1 open subset of volume preserving diffeomorphisms, the random product of $f_1,\dots,f_k$ has positive exponents. They use a different approach that does not require some pinching condition, but ergodicity of the $k-1$-first diffeomorphisms is essential in their argument.

Further remarks and questions. As we mentioned before, we do not know if the pinching condition is C^r-dense. This is the only thing that stop us from having C^r-density in the statements of the Theorems E and F.

Question 6.4. *Is the pinching condition C^r-dense (for any $r\geq 1$) in* $\mathrm{Diff}^r_m(S)$*?*

For random products, another interesting question is the following:

Question 6.5. *For any $d \in \mathbb{N}$ with $d \geq 2$, does ergodicity for the random product holds C^r generically in $\mathrm{Diff}^r_m(S)^d$?*

If the answer to this question is positive, after our result, one would also conclude that generically the random product is non-uniformly hyperbolic.

Recall that P is the Benoulli measure on the shift space Σ_d. A property weaker than ergodicity is **metric transitivity**, which is defined as follows. A random product is metric transitive if for P-almost every sequence $\tilde{x} = (x_i)_{i\in\mathbb{Z}}$, for m-almost every point $p \in S$, the orbit $f^n_{\tilde{x}}(p)$ is dense in S. In other words, a typical trajectory is dense in S. A possible intermediate step in answering question 6.5 is the following question.

Question 6.6. *For any $d \in \mathbb{N}$ with $d \geq 2$, does metric transitivity for the random product holds C^r-generically in $\mathrm{Diff}^r_m(S)^d$?*

We remark that Koropecki-Nassiri ([KN10]) proved that for any $r \in \mathbb{N} \cup \{\infty\}$, there exists a C^r-residual subset of $\mathrm{Diff}^r_m(S)^2$ such that for any pair (f, g) in this residual subset, the action on S induced by the semigroup generated by f and g is transitive.

7. Dissipative perturbations of Berger-Carrasco's example

Our results stated in this section are contained in chapters 4 and 5 of this thesis.

We had defined in section 5 the example introduced by Berger and Carrasco in [BC14]. We considered the two torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, and for $N \in \mathbb{N}$ we considered the standard map $s_N(x, y) = (2x - y + N\sin(x), x)$. Let $A \in SL(2, \mathbb{Z})$ be a hyperbolic matrix which defines an Anosov diffeomorphism on $\mathbb{T}^2$, and let $P_x : \mathbb{T}^2 \to \mathbb{T}^2$ be the projection on the first coordinate of $\mathbb{T}^2$.

Consider the torus $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$, and represent it using the coordinates (x, y, z, w), where $x, y, z, w \in [0, 2\pi)$. For each $N \in \mathbb{N}$ we defined

$$\begin{array}{rccc} f_N & : & \mathbb{T}^2 \times \mathbb{T}^2 & \longrightarrow & \mathbb{T}^2 \times \mathbb{T}^2 \\ & & (x, y, z, w) & \mapsto & (s_N(x, y) + P_x \circ A^N(z, w), A^{2N}(z, w)). \end{array}$$

In Theorem A, we considered only C^2-small volume preserving perturbations of f_N. In this section we also consider non conservative systems and we describe our work related with the following problem.

Problem 7.1. *What ergodic, and topological properties can we obtain for dissipative perturbations of f_N?*

7.1. Robust transitivity. *Our result stated in this part is from a joint work with Pablo Carrasco*[10]*, and it is contained in chapter 4 of this thesis.*

Among the robust properties that a dynamical system may have, transitivity has been one of the most extensively researched. Recall that a diffeomorphism f is *transitive* if for any two non-empty open sets U and V, there is an integer $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. Transitivity means that from the topological point of view the system cannot be separated into disjoint invariant parts. A diffeomorphism is **robustly transitive**, if every diffeomorphism in a C^1-neighborhood of it is transitive. This can be seen as a topological version of stable ergodicity.

The first known examples of robustly transitive diffeomorphisms are given by Anosov maps: if $f \in \mathrm{Diff}^1(M)$ is transitive and uniformly hyperbolic, then it is C^1-robustly transitive. It turns out that certain degree of hyperbolicity is required in order to have robust transitivity. Indeed, if $f \in \mathrm{Diff}^1(M)$ is robustly transitive and $dim M \leq 3$ then f is hyperbolic/weakly partially hyperbolic [Ma78, BDU99]. In general, f admits a dominated decomposition of the tangent bundle, $TM = E \oplus F$, such that $det(Df^{-n_0}|E), det(Df^{n_0}|D) \leq 1/2$ for some uniform $n_0 \geq 1$ [BDP03]. It is worth to point out that the bundles E, F above are not necessarily uniformly expanding, see [BV00].

10. ICEx-UFMG, Avda. Presidente Antônio Carlos 6627, Belo Horizonte-MG,BR 31270-901.

As for non-hyperbolic examples, there are several known. The list below gives a rough (incomplete) picture of the arguments used to establish robust transitivity for non hyperbolic systems.

- *Deformations from Anosov systems.* The first concrete example of non-uniformly hyperbolic robustly transitive map was given by Shub in [Sh71]; later in [Ma78] Mañé gave a similar type of construction on $\mathbb{T}^3$. They are both partially hyperbolic and homotopic to an Anosov system. The example given in [BV00] is also a deformation of an Anosov diffeomorphism, and although it is not partially hyperbolic, it does admit a dominated splitting coherent with its Anosov part (as the previous two examples). More recently, Potrie ([Po12] page 152) gave an example of this type, but with the difference that it admits a dominated splitting which is not coherent with its hyperbolic part. In these cases, the proof of robust transitivity is founded in that they have hyperbolic-type behavior in a large part of the space.
- *Blenders.* This powerful mechanism was introduced in [BD96] by Bonatti-Díaz. With it the authors were able to prove that some perturbations of time-t maps of mixing hyperbolic flows, and of the product of an Anosov map times the identity (say, on $\mathbb{T}^3$), are robustly transitive. The same tool was used by Cheng-Gan-Shi in [CGS18] to present a robustly transitive skew-product which has some interesting ergodic properties (their type of example is known as Kan's type).
- *Minimality of the stable/unstable foliation.* It is easy to see that if $f \in \mathrm{Diff}^1(M)$ admits an invariant expanding minimal foliation, then f is transitive. Conditions that guarantee the persistence of these types of foliations are thus relevant for robust transitivity. Among these conditions, the property SH introduced by Pujals-Sambarino [PS06] is particularly simple to check, and can be applied to establish robust transitivity of transitive partially hyperbolic systems where one has some control on the behavior of the stable/unstable foliations. Shub and Mañé's examples cited before fall into this category.
- *Non-uniform expansion along the center.* In a recent work [Ya16], Yang considers partially hyperbolic systems with non-uniformly expanding center behavior, and shows that any conservative ergodic of such systems with one-dimensional center is robustly transitive. The author uses the non-uniform expanding character of the center as a replacement for hyperbolicity, employing methods of smooth ergodic theory. These techniques however seem to be applicable only for systems with one-dimensional center.

We add a new example in the list above. This is given by the Berger-Carrasco example. Recall that it is a partially hyperbolic system, with two dimensional center, where it has both expansion and contraction along the center, and it does not admit any further dominated decomposition. We remark that stable ergodicity does not imply robust transitivity, see [Sh14]. We obtain the following result.

Theorem I. There exists $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$ the diffeomorphism f_N is C^1-robustly transitive (in fact, C^1-robustly topologically mixing).

Remark 7.2. *Topologically mixing is a stronger property than transitivity: f is topologically mixing if for any two open sets U and V, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have $f^n(U) \cap V \neq \emptyset$.*

The proofs of robust transitivity for the diffeomorphisms which are deformations of Anosov systems, mentioned above, use information about some type of minimality (or ε-minimality) of stable/unstable manifolds. Observe that, our example has a hyperbolic-type behavior in a large part of the manifold, as in the examples which are deformations of Anosov systems. However, an important difference in our proof is that we do not use any information on the minimality (or ε-minimality) of stable/unstable foliations.

7.2. Rigidity of u-Gibbs measures. *Our results stated in this part are contained in chapter 5 of this thesis.*

In dynamics one usually tries to understand the asymptotic behavior of the orbit of many points. In this direction, it is natural to try to understand properties, and the existence, of certain invariant measures that capture the statistical behavior of a set of points that is relevant for the Lebesgue measure. Let us make this more precise.

Let f be a diffeomorphism of a closed, compact, connected, orientable manifold M. Given an invariant ergodic probability measure μ, its **basin** is defined as

$$B(\mu) = \left\{ p \in M : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(p)} \xrightarrow{n\to+\infty} \mu \right\},$$

where δ_p is the dirac measure on p and the convergence is for the weak*-topology. The measure μ is **physical** if its basin has positive Lebesgue measure. In other words, physical measures are the measures that capture the asymptotic behavior of many points in the Lebesgue point of view.

In the 1970s, Sinai, Ruelle and Bowen [Si72, Ru76, Bow75] proved that $C^{1+\alpha}$ uniformly hyperbolic systems have finitely many physical measures that describes the statistical behavior of Lebesgue almost every point. Nowadays, the measures they constructed are called **SRB measures** (SRB for Sinai-Ruelle-Bowen). These measures have an important geometrical property: they admit conditional measures along unstable manifolds which are absolutely continuous with respect to the volume of the unstable manifolds. After the work of Ledrappier in [Le84], there is a well developed ergodic theory for these measures. The hyperbolic SRB measures form an important class of physical measures.

We remark that in the hyperbolic setting there are uniform expansion/contraction, and a dominated splitting (which implies that the angle between the expanding/contracting directions is uniformly bounded from below). These two points are important to carry the constructions of such measures.

Problem 7.3. *When does it exist hyperbolic SRB measures?*

There are many works that study conditions that guarantee the existence of hyperbolic SRB measures outside the uniformly hyperbolic setting, see for instance [Yo98, BV00, ABV00, CDP16, CLP19, Ov19]. We also refer the reader to the recent survey [CLP17] for a discussion on the different methods of construction of such measures (with a focus on the geometrical method). Usually it is a hard problem to show the existence of hyperbolic SRB measures outside the uniformly hyperbolic setting.

In the task of studying the existence and uniqueness of hyperbolic SRB measures for partially hyperbolic systems, another important type of invariant measure are the so-called **u-Gibbs measure**, see definition 2.17. They are invariant measures that also verify some geometric property. In the partially hyperbolic setting, every hyperbolic SRB measure is an u-Gibbs measure.

Chapter 5 of this thesis is a step towards understanding the existence and uniqueness of hyperbolic SRB measures for dissipative perturbations of the Berger-Carrasco's example. As we mentioned before, understanding the u-Gibbs measures is important in this task. We obtain a rigidity result for u-Gibbs measures in a neighborhood of this example. In particular, we classify all the possible u-Gibbs measures that may appear. Let us make this more precise.

Inside $\mathrm{Diff}^r(\mathbb{T}^4)$, we may consider the subspace $SP^r(\mathbb{T}^2 \times \mathbb{T}^2)$ of skew products, which is the set of C^r-diffeomorphisms g of the form

$$g(x, y, z, w) = (g_1(x, y, z, w), g_2(z, w)),$$

where $g_2(.,.)$ is a C^r-diffeomorphism of $\mathbb{T}^2$, and for each $(z, w) \in \mathbb{T}^2$, $g_1(., ., z, w)$ is a C^r-diffeomorphism of $\mathbb{T}^2$ as well. Observe that $f_N \in SP^2(\mathbb{T}^2 \times \mathbb{T}^2)$. We also remark that for N large enough, if g is a skew product C^1-close enough to f_N, then g_2 is an Anosov diffeomorphism, and g is partially hyperbolic.

We recall that for a map g, a g-invariant measure μ is **Bernoulli** if the system (g, μ) is measurably conjugated to a Bernoulli shift. For a skew product g as above, one may look at the conditional measures of μ with respect to the center foliation. If these conditional measures are atomic, we say that μ has **atomic disintegration along the center foliation**, see section 2.3 for a precise definition. The main result in chapter 5 is the following:

Theorem J. Let $\alpha \in (0, 1)$. For N large enough, there exists $\mathcal{U}_N^{sp}$ a C^2-neighborhood of f_N contained in $SP^{2+\alpha}(\mathbb{T}^2 \times \mathbb{T}^2)$ such that for $g \in \mathcal{U}_N^{sp}$, and μ an ergodic u-Gibbs measure for g, one of the following holds true:

(1) μ is the unique SRB measure. It is Bernoulli and $\text{supp}(\mu) = \mathbb{T}^4$;

(2) μ has atomic disintegration along the center foliation, whose conditional measures has finitely many atoms.

The proof of Theorem J is an immediate consequence of Theorems K and L below.

Theorem K. Let $\alpha \in (0, 1)$. For N large enough, there exists $\mathcal{U}_N^{sp}$ a C^2-neighborhood of f_N contained in $SP^{2+\alpha}(\mathbb{T}^2 \times \mathbb{T}^2)$, such that for $g \in \mathcal{U}_N^{sp}$, for any ergodic measure $\mu \in \text{Gibbs}^u(g)$ one of the following holds:

(1) μ is a hyperbolic SRB measure, or

(2) μ has atomic disintegration along the center foliation, whose conditional measures has finitely many atoms.

Let us say a few words on the α that appears in the statement of Theorem J, and K. This is due to the proof of Theorem K. This proof uses a recent result by Brown-Rodriguez Hertz in [BRH17]. In their paper they classify all the ergodic, hyperbolic stationary measures for random products of surface C^2-diffeomorphisms. To prove their result, they actually prove a more general theorem, which holds for more general abstract skew products with a given surface as fiber (see chapter 5 for more details).

The α that appears in the statements of Theorems J and K, only appears because of the C^2-regularity of the diffeomorphisms considered in Brown-Rodriguez Hertz's main result in [BRH17]. If one obtains a version of their result for $C^{1+\beta}$-diffeomorphisms, then one could remove the α from the statement of our theorems.

Theorem L. For N large enough, there exists $\mathcal{U}_N$ a C^2-neighborhood of f_N in $\text{Diff}^2(\mathbb{T}^4)$ such that if $g \in \mathcal{U}_N$, then g has at most one SRB measure. Moreover, if μ_g is an SRB measure for g, then $\text{supp}(\mu_g) = \mathbb{T}^4$, it is Bernoulli and hyperbolic.

Remark 7.4. *Theorems J and K hold for a neighborhood of f_N inside the skew product diffeomorphisms, $SP^2(\mathbb{T}^2 \times \mathbb{T}^2)$. Theorem L is the uniqueness theorem for SRB measures, and it holds in a neighborhood of f_N inside* $\text{Diff}^2(\mathbb{T}^4)$.

Further remarks and questions. Let us summarize why in Theorems J and K we have the condition that the systems are skew products for $\mathbb{T}^2 \times \mathbb{T}^2$. We use this condition to obtain the smoothness of the center foliation. This is used to prove proposition 2.23, which states that we may use the invariance principle (see also corollary 2.25). An interesting question is to know if there exists such a measure rigidity result for systems which are not skew products for $\mathbb{T}^2 \times \mathbb{T}^2$. A first natural step is given in the following quesiton:

Question 7.5. *Is there a similar measure rigidity result for u-Gibbs measures of diffeomorphism in a neighborhood of f_N inside* $\text{Diff}^2(\mathbb{T}^4)$*?*

We believe that condition (2) in Theorem J usually does not happen. Since there are good hyperbolic information for u-Gibbs measures in a neighborhood of f_N. We also

believe that the generic existence of an SRB measure in a neighborhood of f_N should imply the existence of an SRB measure for any system in an entire neighborhood of f_N. We precise this in the following conjecture:

Conjecture. *Every diffeomorphism in $\mathcal{U}_N^{sp}$ has an SRB measure.*

An interesting strategy to prove the existence of an SRB measure in a neighborhood of f_N inside $\mathrm{Diff}^2(\mathbb{T}^4)$ is to use the results from [CDP16]. In order to do that, one needs to prove that the condition called **effective hyperbolicity** is verified (see section 1.2 in [CDP16]). This condition seems hard to prove, however it could give the existence of SRB measures outside the fibered case.

Question 7.6. *For N large enough, for any diffeomorphism g which is sufficiently C^2-close to f_N, does it hold that g is effective hyperbolic?*

8. Organization of this thesis

This thesis has two parts. The first part (chapters 1, 2, and 3) has all the results mentioned above which are related with conservative dynamics, and the second part (chapters 4, and 5) has the results which are related with dissipative dynamics. Chapters 1, and 2 are about stable ergodicity. Chapter 3 has the results of the author with Mauricio Poletti on the genericity of the existence of positive Lyapunov exponents for certain skew products. Chapter 4 has the result of the author with Pablo Carrasco about the robust transitivity of Berger-Carrasco's example. At last, chapter 5 has the results on the measure rigidity of u-Gibbs measures for dissipative perturbations of the same example.

Works contained in this thesis and other works

This thesis contains the following works:

- On the stable ergodicity of Berger-Carrasco's example. *Ergodic Theory and Dynamical Systems*, online version, 2018.
- On the stable ergodicity of diffeomorphisms with dominated splitting. *Nonlinearity*, 32:445–463, 2019.
- On the genericity of positive exponents of conservative skew products with two-dimensional fibers. Joint work with Mauricio Poletti. Submited. *Preprint on arxiv:1809.03874*, 2018.
- A new example of robustly transitive diffeomorphism. Joint work with Pablo Carrasco. Submited. *Preprint on arxiv:1904.11788*, 2019.
- Rigidity of u-Gibbs measures for certain partially hyperbolic skew products. 2019.

In 2015, I had started a Ph.D. at Universidade Federal do Rio de Janeiro (UFRJ) in Brazil, under the supervision of Alexander Arbieto. Some time after that, the opportunity appeared to start a new (and independent) Ph.D. at Université Paris-Sud (Orsay), under the supervision of Sylvain Crovisier. Since January 2017, I have been working on these two thesis simultaneously, which are independent of each other. My thesis at UFRJ, which I will defend it in July 2019, is focused on the study of different types centralizers for vector fields. I also mention here the works obtained for my thesis in Brazil:

- On the centralizer of vector fields: Criteria of triviality and genericity results. Joint work with Martin Leguil and Bruno Santiago. Submited. *Preprint arXiv:1810.05085*, 2018.
- Symmetries of vector fields: The diffeomorphism centralizer. Submited. *Preprint arXiv:1903.05883*, 2019.

Bibliography

[ABV00] J. Alves, C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Inventiones mathematicae*, 140:351–398, 2000.

[Ano67] D. Anosov. Geodesic flows on closed Riemannian manifolds of negative curvature. *Trudy Mat. Inst. Steklov.*, 90, 1967.

[AM07] A. Arbieto and C. Matheus. A pasting lemma and some applications for conservative systems. *Ergodic Theory Dynam. Systems*, 27:1399–1417, 2007.

[Ar63] V. Arnold. Proof of a theorem of A. N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian. *Uspehi Mat. Nauk.*, 18:13–40, 1963.

[ABW09] A. Avila, J. Bochi, and A. Wilkinson. Nonuniform center bunching and the genericity of ergodicity among C^1 partially hyperbolic symplectomorphisms. *Ann. Sci. Éc. Norm. Supér.*, 42:931–979, 2009.

[ACW17] A. Avila, S. Crovisier, and A. Wilkinson. C^1 density of stable ergodicity. to appear in *Advances in Math.*, preprint on arXiv:1709.04983, 2017.

[ACW16] A. Avila, S. Crovisier, and A. Wilkinson. Diffeomorphisms with positive metric entropy. *Publ. Math. IHES.*, 124:319–347, 2016.

[ASV13] A. Avila, J. Santamaria, and M. Viana. Holonomy invariance: rough regularity and applications to Lyapunov exponents. *Astérisque*, 358:13–74, 2013.

[AV10] A. Avila, and M. Viana. Extremal Lyapunov exponents: an invariance principle and applications. *Invent. Math.*, 181:115–189, 2010.

[BM18] P. G. Barrientos and D. Malicet. Extremal exponents of random products of conservative diffeomorphisms. Preprint on arXiv:1809.08619, 2018.

[BO19] S. Ben Ovadia. Generalized SRB measures, physical properties, and thermodynamic formalism of smooth hyperbolic systems. *Preprint arXiv:1904.10074*, 2019.

[BC14] P. Berger, and P. Carrasco. Non-uniformly hyperbolic diffeomorphisms derived from the standard map. *Comm. Math. Phys.*, 329:239–262, 2014.

[BT17] P. Berger and D. Turaev. On Herman's positive entropy conjecture. *to appear in Advances in Math.*, 2017.

[Bi31] G. Birkhoff. Proof of the ergodic theorem. *Proc. Natl. Acad. Sci.*, 17:656–660, 1931.

[BXY17] A. Blumenthal, J. Xue, and L.S. Young. Lyapunov exponents for random perturbations of some area-preserving maps including the standard map. *Ann. of Math.*, 185:285–310, 2017.

[Bo02] J. Bochi. Genericity of zero Lyapunov exponents. *Ergodic Theory Dynam. Systems*, 22:1667–1696, 2002.

[BD96] C. Bonatti, and L. Díaz. Persistent nonhyperbolic transitive diffeomorphisms. *Ann. of Math.*, 143:357–396, 1996.

[BDP03] C. Bonatti, L. Díaz, and E. Pujals. A $\mathcal{C}^1$-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources. *Ann. of Math.*, 158:355–418, 2003.

[BV00] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.

[Bow75] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lect. Notes in Math.* Springer Verlag, 1975.

[BRH17] A. Brown, and F. Rodriguez Hertz. Measure rigidity for random dynamics on surfaces and related skew products. *J. Amer. Math. Soc.*, 30:1055–1132, 2017.

[BDP02] K. Burns, D. Dolgopyat, and Y. Pesin. Partial hyperbolicity, Lyapunov exponents and stable ergodicity. *J. Statist. Phys.*, 108:927–942, 2002.

[BW10] K. Burns, and A. Wilkinson. On the ergodicity of partially hyperbolic systems. *Ann. of Math.*, 171:451–489, 2010.

[CGS18] C. Cheng, S. Gan, and Y. Shi. A robustly transitive diffeomorphism of Kan's type. *Disc. and Cont. Dyn. Sys.*, 38:867-888, 2018.

[Ch79] B. Chirikov. A universal instability of many-dimensional oscillator systems. *Phys. Rep.*, 52:264–379, 1979.

[CDP16] V. Climenhaga, D. Dolgopyat, and Y. Pesin. Non-stationary non-uniform hyperbolicity: SRB measures for dissipative maps. *Comm. Math. Phys.*, 346:553–602, 2016.

[CLP19] V. Climenhaga, S. Luzzatto, and Y. Pesin. SRB measures and Young towers for surface diffeomorphisms. *Preprint arXiv:1904.00034*, 2019.

[CLP17] V. Climenhaga, S. Luzzatto, and Y. Pesin. The geometric approach for constructing Sinai-Ruelle-Bowen measures. *J. Stat. Phys.*, 166:467–493, 2017.

[Cro11] S. Crovisier. Partial hyperbolicity far from homoclinic bifurcations. *Advances in Mathematics*, 226:673–726, 2011.

[CP15] S. Crovisier and E. Pujals. Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms. *Invent. Math.*, 201:385–517, 2015.

[BDU99] L. Díaz, E. Pujals, and R. Ures. Partial hyperbolicity and robust transitivity. *Acta Mathematica*, 183:1–43, 1999.

[Du94] P. Duarte. Plenty of elliptic islands for the standard family of area preserving maps. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 11:359–409, 1994.

[GPS94] M. Grayson, C. Pugh, and M. Shub. Stably ergodic diffeomorphisms. *Ann. of Math.*, 140:295–329, 1994.

[Go12] A. Gorodetski. On stochastic sea of the standard map. *Comm. Math. Phys.*, 309:155–192, 2012.

[HS17] V. Horita, and M. Sambarino. Stable ergodicity and accessibility for certain partially hyperbolic diffeomorphisms with bidimensional center leaves. *Comment. Math. Helv.*, 92:467–512, 2017.

[Ho39] E. Hopf. Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung. *Ber. Verh. Sächs. Akad. Wiss. Leipzig*, 91:216–304, 1939.

[Iz80] F. Izraelev. Nearly linear mappings and their applications. *Physica D: Nonlinear Phenomena*, 1:243–266, 1980.

[Ka80] A. Katok. Lyapunov exponents, entropy and periodic points of diffeomorphisms. *Publ. Math. IHES*, 51:137–173, 1980.

[Ko54-1] A. Kolmogorov. On conservation of conditionally periodic motions for a small change in Hamilton's function. *Dokl. Akad. Nauk SSSR (N.S.)*, 98:527–530, 1954.

[Ko54-2] A. Kolmogorov. Théorie générale des systèmes dynamiques et mécanique classique. *Proceedings of the International Congress of Mathematicians (Amsterdam 1954)*, 1:315–333.

[KN10] A. Koropecki, and M. Nassiri. Transitivity of generic semigroups of area-preserving surface diffeomorphisms. *Math. Z.*, 266:707–718, 2010.

[Le84] F. Ledrappier. Propriétés ergodiques des mesures de Sinaï. *Inst. Hautes Études Sci. Publ. Math.*, 59:163–188, 1984.

[LMY18] C. Liang, K. Marin, and J. Yang. Lyapunov exponents of partially hyperbolic volume-preserving maps with 2-dimensional center bundle. *Ann. Inst. H. Poincaré Anal. Non Linéaire.*, 35:1687–1706, 2018.

[LY17] C. Liang and Y. Yang. the C^1 density of nonuniform hyperbolicity in C^r conservative diffeomorphisms. *Proc. Amer. Math. Soc.*, 145:1539-1552, 2017.

[Ma78] R. Mañé. Contributions to the stability conjecture. *Topology*, 17:383–396, 1978.

[Ma96] R. Mañé. The Lyapunov exponents of generic area preserving diffeomorphisms. In *International Conference on Dynamical Systems (Montevideo, 1995)*, pages 110–119.

Longman, 1996.
[Mar16] K. Marin. C^r-density of (non-uniform) hyperbolicity in partially hyperbolic symplectic diffeomorphisms. *Comment. Math. Helv.*, 91:357–396, 2016.
[Mo62] J. Moser. On invariant curves of area-preserving mappings of an annulus. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl.*, 1962:1–20, 1962.
[NH19] G. Núñez, and J. Rodriguez Hertz. Minimality and stable bernouliness in dimension 3. *Preprint on arXiv:1905.04414*, 2019.
[OU41] J.Oxtoby, and S. Ulam. Measure-preserving homeomorphisms and metrical transitivity. *Ann. of Math.*, 42:874–920, 1941.
[Pes77] Y. Pesin. Characteristic Ljapunov exponents, and smooth ergodic theory. *Uspehi Mat. Nauk*, 32:55–112, 1977.
[Po90] H. Poincaré. Sur le problème des trois corps et les équations de la dynamique. *Acta math.*, 13:1–270, 1890.
[Po12] R. Potrie. Partial hyperbolicity and attracting regions in 3-dimensional manifolds. *Thesis. Available in http://www.cmat.edu.uy/ rpotrie/*, 2012.
[Pot15] R. Potrie. Partially hyperbolic diffeomorphisms with a trapping property. *Discrete Contin. Dyn. Syst.*, 35:5037–5054, 2015.
[PS00] C. Pugh, and M. Shub. Stable ergodicity and julienne quasi-conformality. *J. Eur. Math. Soc.*, 2:1–52, 2000.
[PS97] C. Pugh and M. Shub. Stable ergodicity and stable accessibility. *Differential equations and applications*, 258–268, 1997.
[PS06] E. Pujals, and M. Sambarino. A sufficient condition for robustly minimal foliations. *Ergodic Theory Dynam. Systems* 26:281-289, 2006.
[HHU08] F. Rodriguez Hertz, J. Rodriguez Hertz, and R. Ures. Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1D-center bundle. *Invent. Math.*, 172:353–381, 2008.
[HHTU11] F. Rodriguez Hertz, J. Rodriguez Hertz, A. Tahzibi, and R. Ures. New criteria for ergodicity and nonuniform hyperbolicity. *Duke Math. J.*, 160:599–629, 2011.
[Ru76] D. Ruelle. A measure associated with Axiom A attractors. *Amer. J. Math.*, 98:619–654, 1976.
[SS95] D. Shepelyansky, and A. Stone. Chaotic landau level mixing in classical and quantum wells. *Physical review letters*, 74, 1995.
[Sh14] Y. Shi. Perturbation of partially hyperbolic automorphisms on Heisenberg nilmanifolds and holonomy maps. *Thesis. Available in www.nuxeo.u-bourgogne.fr/nuxeo/site/esupversions/7adf4309-f3e7-4b82-841a-5279c2270f17*, 2014.
[Sh71] M. Shub. Topological transitive diffeomorphism on $\mathbb{T}^4$. *Lecture notes in Mathematics 206*. Springer, 1971.
[Si72] J. Sinaĭ. Gibbs measures in ergodic theory. *Uspehi Mat. Nauk*, 27:21–64, 1972.
[Si94] J. Sinaĭ. *Topics in ergodic theory*, volume 44 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1994.
[Sm67] S. Smale. Differentiable dynamical systems. *Bull. Am. Math. Soc.*, 73:747–817, 1967.
[Tah04] A. Tahzibi. Stably ergodic diffeomorphisms which are not partially hyperbolic. *Israel J. Math.*, 142:315–344, 2004.
[Vi97] M. Viana. Multidimensional nonhyperbolic attractors. *Inst. Hautes Études Sci. Publ. Math.*, 85:63–96, 1997.
[Ya16] J. Yang. Entropy along expanding foliations. *Preprint on arxiv:1601.05504*, 2016.
[Yo98] L. S. Young. Statistical properties of dynamical systems with some hyperbolicity. *Ann. of Math.*, 147:585–650, 1998.

Part I

Conservative dynamics: stable ergodicity and genericity of positive Lyapunov exponents

CHAPTER 1

On the stable ergodicity of Berger-Carrasco's example

1. Introduction

Let M be a smooth compact riemannian manifold and let ν be a Borel probability measure on M. Given a measurable transformation $f : M \to M$ that preserves ν, we say that f is ergodic with respect to ν if every invariant measurable set has either zero or full measure. Ergodicity means that from the probabilistic point of view the system cannot be decomposed into invariant smaller parts. In our scenario, f is ergodic if and only if for every continuous function $\varphi : M \to M$, for ν-almost every point $p \in M$ it is verified

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(p) = \int_M \varphi d\nu.$$

In 1939, Hopf introduced in [Ho39] an argument to prove that the geodesic flow on compact surfaces with constant negative curvature is ergodic with respect to the Liouville measure. Many years later, Anosov [Ano67], Anosov and Sinai [AS67] used the Hopf argument to prove ergodicity of hyperbolic systems that preserve a smooth measure. A diffeomorphism is hyperbolic, or Anosov, if its tangent bundle decomposes into two invariant subbundles, one is contracted and the other one is expanded exponentially fast by the action of the derivative. Hyperbolicity was the key property that allowed them to use the Hopf argument in these settings.

Since then several works extended the Hopf argument to more general settings, namely non-uniformly hyperbolic and partially hyperbolic systems.

For a C^1-diffeomorphism f and an invariant measure ν, Kingman's ergodic theorem implies that for ν-almost every point $p \in M$ and for every $v \in T_pM - \{0\}$ the following limit exists

$$\lambda(p, v) = \lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(p).v\|. \tag{5}$$

Oseledets' theorem states that $\lambda(p, .)$ can take at most $\dim(M)$ different values. Such numbers are called Lyapunov exponents. A f-invariant measure ν is non-uniformly hyperbolic for f if for ν-almost every point, every Lyapunov exponent is non zero.

In [Pes77], Pesin uses the Hopf argument to prove that if ν is a smooth, non-uniformly hyperbolic measure and f is a $C^{1+\alpha}$-diffeomorphism then ν has at most countably many ergodic components.

A diffeomorphism f is partially hyperbolic if there is a Df-invariant decomposition $TM = E^{ss} \oplus E^c \oplus E^{uu}$, such that $Df|_{E^{ss}}$ contracts, $Df|_{E^{uu}}$ expands and the behaviour of $Df|_{E^c}$ is bounded by the contraction of E^{ss} and the expansion of E^{uu}. See section 2 for a precise definition.

A key property for discussing the ergodicity of partially hyperbolic systems is the accessibility. A partially hyperbolic system is accessible if any two points can be joined by a curve which is a concatenation of finitely many curves, each of them being contained in a stable or an unstable leaf.

There are several works that use accessibility to extend the Hopf argument and prove ergodicity, see for instance [BP74, GPS94, PS00, BDP02, BW10, HHTU11]. Most proofs of the ergodicity for partially hyperbolic systems uses accessibility. Several of the extensions of the Hopf argument for accessible partially hyperbolic diffeormorphisms allow vanishing Lyapunov exponents along the center direction.

Berger and Carrasco introduced in [BC14] an example of a volume-preserving, partially hyperbolic diffeomorphism which is non-uniformly hyperbolic. This example has a two dimensional center bundle and Lebesgue almost every point has both positive and negative Lyapunov exponent in the center direction. Furthermore, the properties of this example are C^2-robust. It is not known if this example is accessible or not.

Definition 1.1. *A volume-preserving diffeomorphism f is* **C^2-stably ergodic** *if it admits a C^2-neighborhood such that any volume-preserving diffeomorphism inside this neighborhood is ergodic.*

In this paper we prove the following theorem.

Theorem A. *The Berger-Carrasco's example is C^2-stably ergodic.*

We stress two features of our work that distinguishes it from the rest of the previous works about ergodicity of partially hyperbolic diffeomorphisms:

- The stable ergodicity with mixed behaviour along the center direction and that does not admit a dominated splitting of the center direction (as a strengthening of [BC14]);
- A proof of stable ergodicity that does not uses accessibility.

We explain a couple points on why on definition 1.1 we use a C^2-neighborhood instead of a C^1-neighborhood, which is the one usually used to define stable ergodicity, see for instance [HHTU11]. First, the techniques we use depend on the uniform control of C^2-norms in a neighborhood. Second, it is not possible to have the mixed behaviour along the center for every volume-preserving, C^2-diffeomorphism in a C^1-neighborhood of Berger-Carrasco's example. This is due to theorem A' in [ACW17], which implies that arbitrarily C^1-close to Berger-Carrasco's example there is a volume-preserving, C^2-diffeomorphism which is stably ergodic and whose Lyapunov exponents along the center have the same sign.

From now on we will denote the normalized Lebesgue measure of a manifold by *Leb* and by $\mathrm{Diff}^r_{Leb}(M)$ the set of C^r-diffeomorphisms that preserve the Lebesgue measure.

Berger-Carrasco's example and the precise statement of the main theorem. For $N \in \mathbb{R}$ we denote by $s_N(x,y) = (2x - y + N\sin(x), x)$ the standard map on $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. For every N the map s_N preserves the Lebesgue measure induced by the usual metric of $\mathbb{T}^2$.

This map is related to several physical problems, see for instance [Ch79], [Iz80] and [SS95].

It is conjectured that for $N \neq 0$ the map s_N has positive entropy for the Lebesgue measure, see [Si94] page 144. By Pesin's entropy formula, see [Pes77] Theorem 5.1, this is equivalent to the existence of a set of positive Lebesgue measure and whose points have a positive Lyapunov exponent. The existence of those sets is not known for any value of N. See [BXY17], [Du94] and [Go12] for some results related to this conjecture.

Let $A \in SL(2,\mathbb{Z})$ be a hyperbolic matrix which defines an Anosov diffeomorphism on $\mathbb{T}^2$, let $P_x : \mathbb{T}^2 \to \mathbb{T}^2$ be the projection on the first coordinate of $\mathbb{T}^2$, this projection is induced by the linear map of $\mathbb{R}^2$, which we will also denote by P_x, given by $P_x(a,b) = (a,0)$. In a similar way define $P_y : \mathbb{T}^2 \to \mathbb{T}^2$ the projection on the second coordinate of the torus.

Consider the torus $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ and represent it using the coordinates (x,y,z,w), where $x,y,z,w \in [0,2\pi)$. We may naturally identify a point (z,w) on the second torus with a point (x,y) on the first torus by taking $x = z$ and $y = w$. For each $N \geq 0$ define

$$\begin{array}{rccc} f_N & : & \mathbb{T}^2 \times \mathbb{T}^2 & \longrightarrow & \mathbb{T}^2 \times \mathbb{T}^2 \\ & & (x,y,z,w) & \mapsto & (s_N(x,y) + P_x \circ A^N(z,w), A^{2N}(z,w)), \end{array}$$

where the point $A^N(z,w)$ on the second torus is being identified with the same point in the first torus as described previously.

This diffeomorphism preserves the Lebesgue measure. For N large enough it is a partially hyperbolic diffeomorphism, with two dimensional center direction given by $E^c = \mathbb{R}^2 \times \{0\}$. This type of system was considered by Berger and Carrasco in [BC14], where they proved the following theorem.

Theorem 1.2 ([BC14], Theorem 1)**.** *There exist $N_0 > 0$ and $c > 0$ such that for every $N \geq N_0$, for Lebesgue almost every point m and for every $v \in \mathbb{R}^4$*

$$\lim_{n\to\infty} \left| \frac{1}{n} \log \|Df_N^n(m).v\| \right| > c \log N.$$

Moreover, the same holds for any volume-preserving diffeomorphism in a C^2-neighborhood of f_N.

This theorem says that for N large enough the system f_N is non-uniformly hyperbolic. Indeed, along the center direction there is one positive and one negative Lyapunov exponent for Lebesgue almost every point.

We remark that Viana constructed in theorem B of [Vi97], an example of a non-conservative partially hyperbolic diffeomorphism with similar properties as in Berger and Carrasco's example, meaning Lebesgue almost every point has a positive and a negative exponent in the center direction and there is no dominated splitting of the center, but in the dissipative case. The approach used by Berger and Carrasco has some similarities with Viana's approach, which is to consider "unstable" curves and use combinatorial arguments to estimate the exponents over such a curve.

Definition 1.3. *Let ν be an invariant probability measure for f. We say that (f, ν) is Bernoulli if it is measurably conjugated to a Bernoulli shift. For volume-preserving diffeomorphisms, we say that f is Bernoulli if (f, Leb) is Bernoulli.*

The Bernoulli property is stronger than ergodicity. We can now give the precise statement of Theorem A.

Theorem A Restated. *For N large enough f_N is C^2-stably ergodic. Moreover, any volume-preserving diffeomorphism in a C^2-neighborhood of f_N is Bernoulli.*

In order to prove this theorem we will need to obtain precise estimates on the size of the invariant manifolds in the center direction for certain points. For that we will need a better estimate of the center exponents, given by the following proposition.

Proposition 1.4. *For every $\delta \in (0, 1)$, there exists $N_0 = N_0(\delta)$ such that for every $N \geq N_0$ there is a C^2-neighborhood $\mathcal{U}_N$ of f_N in $\mathrm{Diff}^2_{Leb}(\mathbb{T}^4)$ with the following property. If $g \in \mathcal{U}_N$, then Lebesgue almost every point has a positive and a negative Lyapunov exponent in the center direction whose absolute value are greater than $(1 - \delta) \log N$.*

We remark that one can show that f_N is C^2-approximated by stably ergodic diffeomorphisms with another approach. This approach uses accessibility, which can be obtained using the results in [HS17], and the criteria of ergodicity in [BW10]. Such approach does not use the non-uniform hyperbolicity of the system.

Strategy of the proof. The strategy of the proof has two parts. The first part is the construction of stable and unstable manifolds inside center leaves with precise estimates on its length and "geometry". The second part is the global strategy to obtain the ergodicity.

For the first part, the main tool is to use the construction of stable manifolds for surface diffeomorphisms, given by Crovisier and Pujals in theorem 5 of [CP18]. In order to do that two ingredients are needed. The first is a good control of the Lyapunov exponents along the center direction so it verifies some inequality, see the beginning of section 3.3 for a discussion. The second is to find sets with positive measure of points with good contraction and expansion for the Oseledecs splitting, for any ergodic component.

Proposition 1.4 gives the control needed of the Lyapunov exponents. To prove proposition 1.4, we follow the proof of theorem 1.2, given by Berger and Carrasco in [BC14], with the necessary adaptations to obtain a precise estimate of the Lyapunov exponents along the center. For the second ingredient, we use a version of the Pliss lemma, lemma 3.4. Following the construction of Crovisier and Pujals in [CP18], we obtain precise estimates of the length and the "geometry" of stable and unstable curves inside center leaves, given by propositions 3.11 and 5.6. So far what is obtained with this construction is that any ergodic component of the Lebesgue measure has a set of points with positive measure having stable and unstable curves in the center leaves of uniform size and controlled "geometry". That alone guarantees that there are at most finitely many ergodic components.

For the global strategy there are also two ingredients, the estimate on the measure of points with good expansion and contraction, given by Pliss lemma, and the density of the orbit of almost every center leaf among the center leaves.

The estimate on the measure given by Pliss lemma is used to obtain points that spend a long time inside a region with good hyperbolicity. This together with the control on the length and "geometry" of the stable and unstable curves inside the center leaves allows us to obtain points whose such curves are very large inside the center direction. The density of the orbit of almost every center leaf together with these large stable and unstable manifolds is then used to apply the Hopf argument and conclude the ergodicity.

We remark that in this proof we use the Hopf argument for non-uniformly hyperbolic systems and not the version usually used for partially hyperbolic diffeomorphisms, see for instance [BW10].

Organization of this chapter. In section 2 we will introduce several tools that will be used in the proof. We will assume that proposition 1.4 holds throughout sections 3, 4, 1.1 and 6, which are dedicated to prove the *main theorem*. The proof of proposition 1.4 is then given in Section 7.

2. Preliminaries

2.1. General theory and results.

Partial hyperbolicity and foliations. A C^r-diffeomorphism f, with $r \geq 1$, is **partially hyperbolic** if the tangent bundle has a decomposition $TM = E^{ss} \oplus E^c \oplus E^{uu}$, there is a riemannian metric on M and continuous functions $\chi^*_-, \chi^*_+ : M \to \mathbb{R}$, for $* = ss, c, uu$, with such that for any $m \in M$

$$\chi^{ss}_+(m) < 1 < \chi^{uu}_-(m) \text{ and } \chi^{ss}_+(m) < \chi^c_-(m) \leq \chi^c_+(m) < \chi^{uu}_-(m),$$

it also holds

$$\chi^{ss}_-(m) \leq m(Df(m)|_{E^{ss}_m}) \leq \|Df(m)|_{E^{ss}_m}\| \leq \chi^{ss}_+(m);$$

$$\chi^{c}_-(m) \leq m(Df(m)|_{E^{c}_m}) \leq \|Df(m)|_{E^{c}_m}\| \leq \chi^{c}_+(m);$$

$$\chi^{uu}_-(m) \leq m(Df(m)|_{E^{uu}_m}) \leq \|Df(m)|_{E^{uu}_m}\| \leq \chi^{uu}_+(m),$$

where $m(Df(m)_{E^*_m}) = \|(Df(m)|_{E^*_m})^{-1}\|^{-1}$ is the co-norm of $Df(m)|_{E^*_m}$, for $* = ss, c, uu$. If the functions in the definition of partial hyperbolicity can be taken constant, we say that f is *absolutely partially hyperbolic*.

It is well known that the distributions E^{ss} and E^{uu} are uniquely integrable, that is, there are two unique foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$, with C^r-leaves, that are tangent to E^{ss} and E^{uu} respectively. For a point $p \in M$ we will denote by $W^{ss}(p)$ a leaf of the foliation $\mathcal{F}^{ss}$, we will call such leaf the strong stable manifold of p. Similarly we define the strong unstable manifold of p and denote it by $W^{uu}(p)$.

Definition 2.1. *A partially hyperbolic diffeomorphism is* **center bunched** *if*

$$\chi^{ss}_+(m) < \frac{\chi^c_-(m)}{\chi^c_+(m)} \text{ and } \frac{\chi^c_+(m)}{\chi^c_-(m)} < \chi^{uu}_-(m), \text{ for every } m \in M.$$

We denote $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$.

Definition 2.2. *A partially hyperbolic diffeomorphism f is* **dynamically coherent** *if there are two invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$, with C^1-leaves, tangent to E^{cs} and E^{cu} respectively. From those two foliations one obtains another invariant foliation $\mathcal{F}^c = \mathcal{F}^{cs} \cap \mathcal{F}^{cu}$ that is tangent to E^c. We call those foliations the center-stable, center-unstable and center foliation.*

For any $R > 0$ we write $W^*_R(p)$ to be the disc of size R centered on p, for the Riemannian metric induced by the metric on M, contained in the leaf $W^*(p)$, for $* = ss, c, uu$.

The definition below allows one to obtain higher regularity of the leaves of such foliations.

Definition 2.3. *We say that a partially hyperbolic diffeomorphism f is* **r-normally hyperbolic** *if for any $m \in M$*

$$\chi^{ss}_+(m) < (\chi^c_-(m))^r \text{ and } (\chi^c_+(m))^r < \chi^{uu}_-(m).$$

Definition 2.4. *Let f and g be partially hyperbolic diffeomorphisms of M that are dynamically coherent, denote by $\mathcal{F}^c_f$ and $\mathcal{F}^c_g$ the center foliations. We say that f and g are* **leaf conjugated** *if there is a homeomorphism $h : M \to M$ that takes leaves of $\mathcal{F}^c_f$ to leaves of $\mathcal{F}^c_g$ and such that for any $L \in \mathcal{F}^c_f$ it is verified*

$$h(f(L)) = g(h(L)).$$

One may study the stability of partially hyperbolic systems up to leaf conjugacy. Related to this there is a technical notion called **plaque expansivity** which we will not define here, see chapter 7 of [HPS77] for the definition. The next theorem is important for the theory of stability of partially hyperbolic systems.

Theorem 2.5 ([HPS77], Theorem 7.4)**.** *Let $f : M \to M$ be a C^r-partially hyperbolic and dynamically coherent diffeomorphism. If f is r-normally hyperbolic and plaque expansive then any $g : M \to M$ in a C^r-neighborhood of f is partially hyperbolic and dynamically coherent. Moreover, g is leaf conjugated to f and the center leaves of g are C^r-immersed manifolds.*

Remark 2.6. *In the proof of the previous theorem, it is obtained for a fixed $R > 0$, if f satisfies the hypothesis of the theorem, then for g sufficiently C^r-close to f, for any $m \in M$, $W^c_{f,R}(m)$ is C^r-close to $W^c_{g,R}(m)$. In particular, if the center foliation is uniformly compact then for every g sufficiently C^r-close to f, for any $m \in M$, $W^c_f(m)$ is C^r-close to $W^c_g(m)$.*

It might be hard to check the condition of plaque expansiviness, but this is not the case when the center foliation of a dynamically coherent, partially hyperbolic diffeomorphism is at least C^1, see Theorem 7.4 of [HPS77]. Usually the invariant foliations that appear in dynamics are only Hölder.

We can also obtain a better regularity for the center direction given by the following theorem, see section 4 of [PSW12] for a discussion on this topic.

Theorem 2.7. *Let f be a C^2-partially hyperbolic diffeomorphism and let $\alpha > 0$ be a number such that for every $m \in M$ it is verified*

$$\chi^{ss}_+(m) < \chi^c_-(m)(\chi^{ss}_-(m))^\alpha \text{ and } \chi^c_+(m)(\chi^{uu}_+(m))^\alpha < \chi^{uu}_-(m),$$

then E^c is α-Hölder.

Pesin's theory. Let f be a C^1-diffeomorphism, for a number $\lambda \in \mathbb{R}$ define E^λ_p to be the subspace of the vector zero united with all vectors $v \in T_pM - \{0\}$ such that the number $\lambda(p, v) = \lambda$, where $\lambda(p, v)$ is the number defined in (5).

We say that a set R has full probability if for any f-invariant probability measure ν it is verified that $\nu(R) = 1$. The following theorem is known as the Oseledets theorem.

Theorem 2.8 ([BP02], Theorems 2.1.1 and 2.1.2)**.** *For any C^1-diffeomorphism f, there is a set $\mathcal{R}$ of full probability, such that for every $\varepsilon > 0$ it exists a measurable function $C_\varepsilon : \mathcal{R} \to (1, +\infty)$ with the following properties:*

(1) for any $p \in \mathcal{R}$ there are numbers $s(p) \in \mathbb{N}$, $\lambda_1(p) < \cdots < \lambda_{s(p)}(p)$ and a decomposition $T_pM = E_p^1 \oplus \cdots \oplus E_p^{s(p)}$;

(2) $s(f(p)) = s(p)$, $\lambda_i(f(p)) = \lambda_i(p)$ and $Df(p).E_p^i = E_{f(p)}^i$, for every $i = 1, \cdots, s(p)$;

(3) for every $v \in E_p^i - \{0\}$ and $n \in \mathbb{Z}$

$$C_\varepsilon(p)^{-1} e^{n.(\lambda_i(p)-\varepsilon)} \leq \frac{\|Df^n(p).v\|}{\|v\|} \leq C_\varepsilon(p) e^{n.(\lambda_i(p)+\varepsilon)} \text{ and } \lambda(p, v) = \lambda_i(p);$$

(4) the angle between E_p^i and E_p^j is greater than $C_\varepsilon(p)^{-1}$, if $i \neq j$;

(5) $C_\varepsilon(f(p)) \leq e^\varepsilon C_\varepsilon(p)$.

We call the set $\mathcal{R}$ the set of regular points. For a fixed $\varepsilon > 0$ and each $l \in \mathbb{N}$ we define the **Pesin block**

$$\mathcal{R}_{\varepsilon,l} = \{p \in \mathcal{R} : C_\varepsilon(p) \leq l\}. \tag{6}$$

We have the following decomposition

$$\mathcal{R} = \bigcup_{l \in \mathbb{N}} \mathcal{R}_{\varepsilon,l}. \tag{7}$$

A point $p \in \mathcal{R}$ has k negative Lyapunov exponents if

$$\sum_{i:\lambda_i(p)<0} dim(E_p^i) = k.$$

Similarly for positive or zero Lyapunov exponents. From now on, we assume that ν is a f-invariant measure, not necessarily ergodic, and there are numbers k and l such that ν-almost every point $p \in \mathcal{R}$ has k negative and l positive Lyapunov exponents.

For a regular point we write

$$E_p^s = \bigoplus_{i:\lambda_i(p)<0} E_p^i \text{ and } E_p^u = \bigoplus_{i:\lambda_i(p)>0} E_p^i. \tag{8}$$

Definition 2.9. *For f a C^2 diffeomorphism the* **stable Pesin manifold** *of the point $p \in \mathcal{R}$ is*

$$W^s(p) = \{q \in M : \limsup_{n \to +\infty} \frac{1}{n} \log d(f^n(p), f^n(q)) < 0\}.$$

Similarly one defines the **unstable Pesin manifold** *as*

$$W^u(p) = \{q \in M : \limsup_{n \to +\infty} \frac{1}{n} \log d(f^{-n}(p), f^{-n}(q)) < 0\}.$$

Remark 2.10. *If f is also partially hyperbolic, with $TM = E^{ss} \oplus E^c \oplus E^{uu}$ then the Oseledets splitting refines the partial hyperbolic splitting. This means that for a regular point $p \in \mathcal{R}$, there are numbers $1 \leq l_1 < l_2 < s(p)$ such that*

$$E_p^{ss} = \bigoplus_{i=1}^{l_1} E_p^i,\ E_p^c = \bigoplus_{i=l_1+1}^{l_2} E_p^i \text{ and } E_p^{uu} = \bigoplus_{i=l_2+1}^{s(p)} E_p^i.$$

This follows from a standard argument similar to the proof of the unicity of dominated splittings, see section B.1.2 from [BDV05]. It also holds that for any regular point p, $E_p^{ss} \subset E_p^s$ and $E_p^{uu} \subset E_p^u$.

Pesin's manifolds are immersed submanifolds, see section 4 of [Pes77]. A difficulty that appears is that such submanifolds in general do not vary continuously with the point, but they vary continuously on Pesin blocks. Let us make this more precise. For $p \in \mathcal{R}_{\varepsilon,l}$, define $W^s_{loc}(p)$ to be the connected component $D^s(p)$ of $W^s(p) \cap B(p,r)$ containing p, such that $\partial D^s(p) \subset \partial B(p,r)$ and $r > 0$ is a small fixed number depending only on $\varepsilon > 0$ and $l \in \mathbb{N}$.

Theorem 2.11 ([Pes77], Theorems 4.1 and 4.2)**.** *Let $f : M \to M$ be a C^2-diffeomorphism preserving a smooth measure ν and suppose that ν-almost every regular point p has the same number of negative and positive Lyapunov exponents. For each $l > 1$, $\varepsilon > 0$ small and $p \in \mathcal{R}_{\varepsilon,l}$, it is verified:*

(1) $W^s_{loc}(p)$ contains a disc centered at p and tangent to E^s_p;

(2) $p \mapsto W^s_{loc}(p)$ varies continuously in the C^1-topology over $\mathcal{R}_{\varepsilon,l}$.

A partition ξ of M is measurable with respect to a probability measure ν, if up to a set of ν-zero measure, the quotient M/ξ is separated by a countable number of measurable sets. Denote by $\hat{\nu}$ the quotient measure in M/ξ.

By Rokhlin's desintegration theorem [Ro52], for a measurable partition ξ, there is set of conditional measures $\{\nu^\xi_D : D \in \xi\}$ such that for $\hat{\nu}$-almost every $D \in \xi$ the measure ν^ξ_D is a probability measure supported on D, for each measurable set $B \subset M$ the application $D \mapsto \nu^\xi_D(B)$ is measurable and it holds

$$\nu(B) = \int_{M/\xi} \nu^\xi_D(B) d\hat{\nu}(D). \tag{9}$$

Fix $\mathcal{R}_{\varepsilon,l}$ a Pesin block. For $p \in \mathcal{R}_{\varepsilon,l}$ and for $\rho > 0$ small, define $B_s(p,\rho)$ as the union of the local stable pesin manifolds of the points $y \in B(p,\rho) \cap \mathcal{R}_{\varepsilon,l}$. Consider the measure $\nu_{p,\rho} = \nu|_{B_s(p,\rho)}$ and the measurable partition ξ_s given by the partition of $B_s(p,\rho)$ by local stable Pesin manifolds. For such a partition let $\{\nu^{\xi_s}_{p,\rho,D} : D \in \xi_s\}$ be the set of conditional measures of the desintegration of $\nu_{p,\rho}$ with respect to ξ_s.

Definition 2.12. *The measure ν has* **absolute continuous conditional measures on stable manifolds** *if for every Pesin block $\mathcal{R}_{\varepsilon,l}$, every $\rho > 0$ small enough, for $\hat{\nu}_{p,\rho}$-almost every $D \in \xi_s$, the measure $\nu^{\xi_s}_{p,\rho,D}$ is equivalent to the Lebesgue measure of a local stable Pesin manifold.*

We will also need the following definition.

Definition 2.13. *Take $p \in \mathcal{R}$ and let T_1 and T_2 be two disks transverse to $W^s(p)$ close to p. We define the* ***holonomy map*** *related to these disks as the map H defined on a subset of $T_1 \cap \mathcal{R}$, consisting of the points q such that $W^s_{loc}(q)$ intersects transversely T_2.*

Recall that we are assuming that the number of negative and positive Lyapunov exponents are the same ν-almost everywhere.

Definition 2.14. *We say that the stable partition is* **absolutely continuous** *if all holonomy maps are measurable and take sets with zero Lebesgue measure of T_1 to into sets of zero Lebesgue measure of T_2.*

Analogously we define all the above for the unstable partition.

Theorem 2.15 ([Pes77], Theorem 4.4)**.** *Let f be a C^2-diffeomorphism preserving a smooth measure ν and non-uniformly hyperbolic, then the stable and unstable partitions are absolutely continuous.*

Remark 2.16. *This theorem implies that ν has absolute continuous conditional measures with respect to the stable, or unstable, manifolds, see theorem* 5.11 *in [BP02]. In particular, a Fubini-like formula (9) holds locally.*

The notion of absolute continuity also makes sense for foliations, but for the holonomy maps of the foliation. The strong stable foliation $\mathcal{F}^{ss}$ of a C^2-partially hyperbolic diffeomorphism is absolutely continuous, see [Ano67].

Usually the partition by strong stable leaves, given by the foliation $\mathcal{F}^{ss}$, is not measurable. In a foliated chart U, one may consider the restricted foliation $\mathcal{F}^{ss}|_U$ and the partition by strong stable leaves forms a measurable partition of U. Thus one can disintegrate a smooth measure locally along such foliation. The absolute continuity of the strong stable foliation implies that the conditional measures of this disintegration are equivalent to the Lebesgue measure of these manifolds, in particular a Fubini-like formula also holds, see [PVW07] for a discussion.

Recall that a f-invariant measure ν is non-uniformly hyperbolic if for ν-almost every point all Lyapunov exponents are non-zero.

Theorem 2.17 ([Pes77], Theorems 7.2 and 8.1)**.** *Let f be a C^2-diffeomorphism preserving a smooth measure ν. If ν is non-uniformly hyperbolic then there are at most countably many ergodic components of ν, that is,*

$$\nu = \sum_{i\in\mathbb{N}} c_i \nu_i,$$

where $c_i \geq 0$, $\sum_{i\in\mathbb{N}} c_i = 1$, each ν_i is a f-invariant ergodic probability measure and if $i \neq j$ then $\nu_i \neq \nu_j$. Moreover, for each $i \in \mathbb{N}$, there is $k_i \in \mathbb{N}$ such that

$$\nu_i = \frac{1}{k_i} \sum_{j=1}^{k_i} \nu_{i,j},$$

where each $\nu_{i,j}$ is a f^{k_i}-invariant probability measure, the system $(f^{k_i}, \nu_{i,j})$ is Bernoulli and $\nu_{i,j} \neq \nu_{i,j}$ if $j \neq l$. Furthermore, f permutes the measures $\nu_{i,j}$, that is, $f_*(\nu_{i,j}) = \nu_{i,j+1}$ for $j = 1, \cdots, k_i - 1$ and $f_*(\nu_{i,k_i}) = \nu_{i,1}$, where $f_*(\nu)$ denotes the pushforward of a measure ν by f.

All the results for Pesin's theory were stated for C^2-diffeomorphisms, but they hold for $C^{1+\alpha}$-diffeomorphisms.

2.2. The strong stable and strong unstable holonomies. Let f be a partially hyperbolic, dynamically coherent diffeomorphism. Each leaf of the foliation $\mathcal{F}^{cs}$ is foliated by strong stable manifolds. For a point $p \in M$ and $q \in W_1^{ss}(p)$, where $W_1^{ss}(p)$ is the strong stable manifold of size 1, we can define the stable holonomy map restricted to the center-stable manifold, between center manifolds. Let us be more precise. We can choose two small numbers $R_1, R_2 > 0$, with the property that for any $z \in W_{R_1}^c(p)$, there is only one point in the intersection $W_2^{ss}(z) \cap W_{R_2}^c(q)$. We define $H_{p,q}^s(z) = W_2^{ss}(z) \cap W_{R_2}^c(q)$. With this construction we obtain a map $H_{p,q}^s : W_{R_1}^c(p) \to W_{R_2}^c(q)$. By the compactness of M we can take the numbers R_1 and R_2 to be constants, independent of p and q.

We can define analogously the unstable holonomy map, for $p \in M$ and $q \in W_1^{uu}(p)$, which we will denote by $H_{p,q}^u : W_{R_1}^c(p) \to W_{R_2}^c(q)$.

In [PSW97] and [PSW00], the authors prove that the map $H_{p,q}^s$ is C^1 if f is a partially hyperbolic, center bunched and dynamically coherent C^2-diffeomorphism. Indeed, the authors prove that the strong stable foliation is C^1 when restricted to a center-stable leaf. Consider the family of C^1-maps $\{H_{p,q}^s\}_{p\in M, q\in W_1^{ss}(p)}$.

Theorem 2.18. *Let f be an absolutely partially hyperbolic, dynamically coherent diffeomorphism. Suppose also that f verifies:*

(1) $\chi^c_- < 1$ *and* $\chi^c_+ > 1$;

(2) there exists $\theta \in (0,1)$*, such that*

$$(\chi^s_+)^\theta < \frac{\chi^c_-}{\chi^c_+}, \tag{10}$$

and

$$\chi^s_+ < \chi^c_-(\chi^s_-)^\theta \text{ and } \chi^c_+(\chi^u_+)^\theta < \chi^u_-. \tag{11}$$

Then the family $\{H^s_{p,q}\}_{p\in M, q\in W^s_1(p)}$ *is a family of* C^1*-maps depending continuously in the* C^1*-topology with the choices of the points* p *and* q*.*

Proof. We follow the approach found in [Br16], which is an approximation of the strong stable holonomies argument. In [Br16], the author proves that such holonomies between center manifolds is C^1 if f is $C^{1+\text{Hölder}}$ and verifies a few (stronger) bunching conditions, see section 2 of [Br16] for precise statements. For a detailed proof in our setting we refer the reader to [Ob18].

We remark that condition (10)is sometimes called **strong bunching** condition. By theorem 2.7, condition (11) implies that the center bundle is θ-Hölder.

We may fix a local approximation of the holonomy H^s_*, which we will denote by π^s_*, that verifies the following: there exists a constant $C > 0$ such that for any $p \in M$ and $q \in W^{ss}_1(p)$, there exists a $C^{1+\theta}$-map, which is a diffeomorphism onto its image, $\pi^s_{p,q} : W^c_{R_1}(p) \to W^c(q)$ that verifies

(1) $d(\pi^s_{p,q}(p), q) \le Cd(p,q)$;

(2) $d(D\pi^s_{p,q}(p).v, v) \le Cd(p,q)^\theta$, where $v \in SE^c_p$ and SE^c_p is the unit sphere on E^c_p;

(3) if $p' \in W^c_{loc}(p)$ and $q' \in W^{ss}_1(p') \cap W^c_{loc}(q)$, then $\pi^s_{p,q}$ coincides with $\pi^s_{p',q'}$ on $W^c_{loc}(p) \cap W^c_{loc}(p')$.

This can be done in the following way: Consider a smooth subbundle $\widetilde{E}$ inside a cone close to the subbundle perpendicular to the subbundle E^c, with dimension $\dim(M) - \dim(E^c)$. Since E^c is θ-Hölder, the center manifolds are $C^{1+\theta}$. Hence, the restriction of $\widetilde{E}$ to any center manifold is a $C^{1+\theta}$-bundle. For each point $q \in M$ and $\rho > 0$, consider $L_{q,\rho} := \exp_q(\widetilde{E}(q,\rho))$ to be the projection of the ball of radius ρ by the exponential map over q. By the uniform transversality and the compactness of M, there exists a constant ρ_0 such that for any center leaf $W^c_{R_1}(p)$, the set $\{L_{q,\rho}\}_{q\in W^c_{R_1}(p)}$ forms an uniform foliated neighborhood of $W^c_{R_1}(p)$ (or a tubular neighborhood). Let $\pi^s_{p,q}$ be the holonomy defined by this local foliation, up to rescaling of the metric we may assume that it is well defined for $p \in M$ and $q \in W^{ss}_1(p)$. By the compactness of M we obtain the constant $C > 0$ above. Observe also that since the center leaves vary continuously in the C^1-topology, we obtain that the map $\pi^s_{p,q}$ varies continuously in the C^1-topology with the points p and q.

For any $p, q \in M$, with $q \in W^{ss}_1(p)$, and each $n \in \mathbb{N}$, write $p_n = f^n(p)$ and $q_n = f^n(q)$. We define

$$H^s_{p,q,n} = f^{-n} \circ \pi^s_{p_n,q_n} \circ f^n.$$

Since we are assuming that f is absolutely partially hyperbolic, only for this proof, write its partially hyperbolic constants as $\chi_s = \chi^{ss}_+(p)$, $\chi_c = \chi^c_-(p)$ and $\widehat{\chi}_c = (\chi^c_+(p))^{-1}$. Also only for this proof, for a diffeomorphism $g : N_1 \to N_2$, between manifolds N_1 and N_2, we will write $g_* : SN_1 \to SN_2$, the action induced by the derivative on the unitary bundles of N_1 and N_2.

Observe that the Lipschitz norm of f^{-1}_* restricted to a fiber S_xE^c is $(\chi_c\widehat{\chi}_c)^{-1}$. Also since f is a C^2-diffeomorphism, then f^{-1}_* is a C^1-diffeomorphism of SM, let $C_1 > 0$ be the C^1-norm of f^{-1} on M and C_2 to be the C^1-norm of f^{-1}_* on SM. For any two points $\xi = (x,v), \zeta = (y,u) \in SM$, we write

$$\xi_k = f^k_*(x,v) = (x_k, v_k) \text{ and } \zeta_k = f^k_*(y,u) = (y_k, u_k), \text{ for } k \in \mathbb{Z}.$$

In the setting that f is $C^{1+\text{Hölder}}$ and some bunching conditions, Brown proves in [Br16] that $(H^s_{p,q,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in the C^1-topology. Furthermore, this sequence converges exponentially fast to $H^s_{p,q}$.

These bunching conditions mentioned above are used to prove lemma 3.1 in [Br16]. In our C^2 scenario, we can obtain a similar lemma, using that f verifies (10) and (11).

Lemma 2.19. *There are constants $\delta, \alpha \in (0,1)$, that verify the following: If $\xi = (x,v)$, $\zeta = (y,u) \in SW^c(p)$, $K > 0$ and $n \geq 0$ verify $d(x_n, y_n) < K\chi_s^n$, $d(\xi_n, \zeta_n) \leq K\chi_s^{n\theta}$ and for every $0 \leq k \leq n$,*

$$d(x_k, y_k) \leq \delta.$$

Then, for all $0 \leq k \leq n$,

$$d(x_k, y_k) \leq K\chi_s^n.\chi_c^{-(n-k)} \text{ and } d(\xi_k, \zeta_k) \leq K\chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-(n-k)(1+\alpha)}.$$

In particular,

$$d(\xi, \zeta) \leq K\chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-n(1+\alpha)}.$$

Furthermore, θ and α can be chosen such that

$$\chi_s^{\theta}.(\widehat{\chi}_c\chi_c)^{-(1+\alpha)} < 1.$$

Proof. The proof is by backward induction in k. We will first denote by α and δ quantities that will be fixed later. Suppose that what we want holds for some $k \in \{1, \cdots n\}$, we will prove that it holds for $k-1$. Since x_k and y_k belongs to the same center manifold, we obtain

$$d(x_{k-1}, y_{k-1}) \leq \chi_c^{-1} d(x_k, y_k) \leq K\chi_s^n.\chi_c^{-n+k+1}.$$

For any $\beta \in (0,1)$, and since $d(x_k, y_k) \leq \delta$, we have

$$\begin{aligned}
d(f_*^{-1}(x_k, v_k), f_*^{-1}(y_k, u_k)) &\leq d(f_*^{-1}(x_k, v_k), f_*^{-1}(x_k, u_k)) + d(f_*^{-1}(x_k, u_k), f_*^{-1}(y_k, u_k)) \\
&\leq (\chi_c\widehat{\chi}_c)^{-1} d(v_k, u_k) + C_2 d(x_k, y_k). \\
&\leq (\chi_c\widehat{\chi}_c)^{-1}[1 + C_2.(\chi_c\widehat{\chi}_c) d(x_k, y_k)^{1-\beta}]. \max\{d(x_k, y_k)^{\beta}, d(v_k, u_k)\} \\
&\leq (\chi_c\widehat{\chi}_c)^{-1}[1 + C_2.(\chi_c\widehat{\chi}_c)\delta^{1-\beta}] \\
&\quad .K \max\{\chi_s^{n\beta}.\chi_c^{-(n-k)\beta}, \chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-(n-k)(1+\alpha)}\}.
\end{aligned}$$

We claim that we can choose α and β such that for any $n \in \mathbb{N}$ and $0 \leq k \leq n$ it holds

$$\chi_s^{n\beta}.\chi_c^{-(n-k)\beta} \leq \chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-(n-k)(1+\alpha)}.$$

This inequality is equivalent to

$$1 \leq \chi_s^{n(\theta-\beta)}.(\chi_c^{(\beta-1-\alpha)}\widehat{\chi}_c^{-(1+\alpha)})^{(n-k)}. \tag{12}$$

Since $\widehat{\chi}_c^{-1} > 1$, we can fix $\beta > \theta$ close enough to 1 such that $1 < \chi_c^{(\beta-1-\alpha)}\widehat{\chi}_c^{-(1+\alpha)}$. Let us explain. Observe that $(\chi_c)^{-\alpha} > 1$, for any $\alpha > 0$. Hence,

$$\chi_c^{\beta-1}(\hat{\chi}_c\chi_c)^{-\alpha}\hat{\chi}_c^{-1} > \chi_c^{\beta-1}\hat{\chi}_c^{-1}.$$

From this, one can see that if β is sufficiently close to 1, we have that $1 < \chi_c^{(\beta-1-\alpha)}\widehat{\chi}_c^{-(1+\alpha)}$. Since $\beta > \theta$, and hence $\theta - \beta$ is negative, we conclude (12).

We also need that

$$\chi_s^{\theta}.(\widehat{\chi}_c\chi_c)^{-(1+\alpha)} < 1. \tag{13}$$

By the strong center bunching condition (10), the inequality above holds if α is sufficiently close to 0. Fix $\alpha > 0$ that verifies (13).

Now fix $\delta > 0$ small enough such that

$$[1 + C_2.(\chi_c\widehat{\chi}_c)\delta^{1-\beta}] \leq (\chi_c\widehat{\chi}_c)^{-\alpha}.$$

We conclude,

$$d(f_*^{-1}(\xi^k), f_*^{-1}(\zeta^k)) \leq (\chi_c\widehat{\chi}_c)^{-(1+\alpha)}.K\chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-(n-k)(1+\alpha)}$$
$$= K\chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-(n-k-1)(1+\alpha)}$$

□

This lemma is specifically used to prove that the sequence $((H^s_{p,q,n})_*)_{n\in\mathbb{N}}$ is Cauchy. We can follow similar calculations as in [Br16] to conclude that for every $p \in M$ and $q \in W_1^{ss}(p)$ the sequence $(H^s_{p,q,n})_{n\in\mathbb{N}}$ is a Cauchy sequence that converges exponentially fast in the C^1-topology to $H^s_{p,q}$. The rate of convergence depends only on χ_s, χ_c and $\widehat{\chi}_c$. In particular, it is independent on the choices of the points p and q.

The family $\{\pi^s_{p,q}\}_{p\in M,q\in W_1^{ss}(p)}$ is a family of C^1-maps depending continuously in the C^1-topology with the choices of points p and q. For each $n \in \mathbb{N}$, consider the family $\{f^{-n} \circ \pi^s_{p_n,q_n} \circ f^n\}_{p\in M,q\in W_1^{ss}(p)}$ and observe that, since f is C^2, this is a family of C^2-maps depending continuously in the C^1-topology with the choices of the points p and q.

Since the rate of convergence does not depend on the choices of the points p and q, we conclude that the sequence of families $\left(\{f^{-n} \circ \pi^s_{p_n,q_n} \circ f^n\}_{p\in M,q\in W_1^{ss}(p)}\right)_{n\in\mathbb{N}}$ converges uniformly in the C^1-topology to the family $\{H^s_{p,q}\}_{p\in M,q\in W_1^{ss}(p)}$. Thus, the family $\{H^s_{p,q}\}_{p\in M,q\in W_1^{ss}(p)}$ is a family of C^1-maps depending continuously in the C^1-topology with the choices of p and q. □

2.3. Berger-Carrasco's example. Recall that for each $N \geq 0$ and $m = (x, y, z, w) \in \mathbb{T}^4$ we defined in section 1 the diffeomorphism

$$f_N(m) = (s_N(x, y) + P_x \circ A^N(z, w), A^{2N}(z, w)).$$

Observe that

$$Df_N(m) = \begin{pmatrix} Ds_N(x,y) & P_x \circ A^N \\ 0 & A^{2N} \end{pmatrix}.$$

It is useful to introduce $\Omega(x, y) = N \cos x + 2$, so that

$$Ds_N(x,y) = \begin{pmatrix} \Omega(x,y) & -1 \\ 1 & 0 \end{pmatrix}.$$

For a point $m = (x, y, z, w) \in \mathbb{T}^4$, we will write $\Omega(m) = \Omega(x, y)$ and $Ds_N(m) = Ds_N(x, y)$. Observe that

$$\frac{1}{2N} \leq \|Ds_N\| \leq 2N \text{ and } \|D^2 s_N\| \leq N. \tag{14}$$

Let $A \in SL(2, \mathbb{Z})$ be the linear Anosov matrix considered in the definition of the map f_N. Denote by $0 < \lambda < 1 < \mu = \lambda^{-1}$ the eigenvalues of A. Let e^s and e^u be unit eigenvectors of A for λ and μ, respectively.

Consider the involution $I(x, y, z, w) = (y, x, z, w)$ for $(x, y, z, w) \in \mathbb{T}^2$. An important feature of the map f_N is given by the following lemma.

Lemma 2.20 ([BC14], Lemma 1)**.** *The map f_N^{-1} is conjugated to the map*

$$(x, y, z, w) \mapsto (s_N(x, y) + P_x \circ A^{-N}(z, w), A^{-2N}(z, w)),$$

by the involution I.

This lemma allows us to prove certain properties for f_N and f_N^{-1} only by considering the map f_N, since the involution tell us that f_N and f_N^{-1} behave in the same way up to exchange the x and y coordinates. This will be used several times throughout paper.

Recall that $E^c = \mathbb{R}^2 \times \{0\}$ and that the system f_N is dynamically coherent.

Proposition 2.21. *Fix $\varepsilon > 0$ small, for N large enough there is a C^2-neighborhood $\mathcal{U}_N$ of f_N, such that if $g \in \mathcal{U}_N$ then g is dynamically coherent, its center leaves are C^2-submanifolds, g is leaf conjugated to f_N and for every $m \in \mathbb{T}^4$ the C^2-distance between $W^c_g(m)$ and $W^c_f(m)$ is smaller than ε.*

Proof. Take N large enough such that

$$\lambda^{2N} < (2N)^{-4}.$$

This inequality implies that f_N is 2-normally hyperbolic. Since its center foliation is smooth, by theorem 7.4 of [HPS77], f_N is plaque expansive. By theorem 2.3, for every g sufficiently C^2-close to f_N, g is dynamically coherent, leaf conjugated to f_N and its center leaves are C^2-submanifolds. Since the center foliation of f_N is uniformly compact, from remark 2.4, if $\mathcal{U}_N$ is small enough then for every $g \in \mathcal{U}_N$ and $m \in \mathbb{T}^4$ the center leaves $W^c_g(m)$ and $W^c_f(m)$ are ε-close in the C^2-topology. □

Remark 2.22. *Let $\theta \in (0,1)$. For N large enough we have*

$$\left(\lambda^{2N}\right)^{\theta} < (4N^2)^{-1} \text{ and } \left(\lambda^{2N}\right)^{1-\theta} < (2N)^{-1}.$$

This is exactly the conditions (10) and (11) from theorem 2.18.

Define $\pi_1(x,y,z,w) = (x,y) \in \mathbb{T}^2$ and $\pi_2(x,y,z,w) = (z,w) \in \mathbb{T}^2$. For convenience, a vector $(u,v) \in \mathbb{R}^2$ will be often identified with $(u,v,0,0) \in \mathbb{R}^4$, so that $Df_N(m).(u,v) = Df_N(m).(u,v,0,0)$. For a vector $v \in T_m\mathbb{T}^4$ we will write $v_1 = D\pi_1(m).v$.

3. The size of the invariant manifolds and cone estimates

In this section we obtain the main estimates to prove the ergodicity of f_N. Assuming proposition 1.4 and fixing a small $\delta > 0$, we prove:

Proposition 3.1. *For N large enough, for each ergodic component of the volume, for f_N, there exists a set with measure larger than $\frac{1-7\delta}{1+7\delta}$, such that:*

For any x in that set, there exist a stable and an unstable curves inside $W^c(x)$, with length bounded from below by N^{-7}. Moreover, the stable curve is transverse, inside $W^c(x)$, to the horizontal direction and the unstable curve is transverse to the vertical direction.

See lemma 3.5 and proposition 3.11 for precise statements.

Remark 3.2. *From now on the norm $\|.\|$ will be the norm induced by the usual metric of $\mathbb{T}^2$ or $\mathbb{T}^4$. We will omit the dependence of N by writing $f = f_N$.*

We fix two scales $\theta_1 = N^{-\frac{2}{5}}$ and $\theta_2 = N^{-\frac{3}{5}}$.

3.1. Points with good contraction and expansion. Since f is non-uniformly hyperbolic, by theorem 2.17, there are at most countably many ergodic components. Therefore $Leb = \sum_{i\in\mathbb{N}} c_i\nu_i$, where $c_i \geq 0$ and for every $i \in \mathbb{N}$ the probability measure ν_i is f-invariant and ergodic. As a consequence of Birkhoff's theorem, for each measure ν_i there exists a set Λ_i with full ν_i-measure such that for every $m \in \Lambda_i$

$$\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(m)} \xrightarrow[n\to+\infty]{} \nu_i \text{ and } \frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^{-j}(m)} \xrightarrow[n\to+\infty]{} \nu_i, \text{ in the } weak^*\text{-topology.} \tag{15}$$

Where δ_p is the dirac mass on the point p. If $\nu_i \neq \nu_j$ then $\Lambda_i \cap \Lambda_j = \emptyset$. Define

$$\Lambda = \bigcup_{i\in\mathbb{N}} \Lambda_i. \tag{16}$$

Recall that $\mathcal{R}$ is the set of regular points given by Oseledets theorem. By remark 2.10, the center direction is decomposed by the Oseledets splitting for almost every point,

that is, for $m \in \mathcal{R}$ there is a decomposition $E^c_m = E^-_m \oplus E^+_m$, where E^-_m is the Oseledets direction related to the negative center exponent and E^+_m is the direction related to the positive exponent.

For each $i \in \mathbb{N}$ define the sets

$$Z_i^- = \left\{ m \in \mathcal{R} \cap \Lambda_i : \forall n \geq 0 \text{ it holds } \left\| Df^n(m)|_{E^-_m} \right\| < \left(N^{-\frac{4}{5}} \right)^n \right\};$$

$$Z_i^+ = \left\{ m \in \mathcal{R} \cap \Lambda_i : \forall n \geq 0 \text{ it holds } \left\| Df^{-n}(m)|_{E^+_m} \right\| < \left(N^{-\frac{4}{5}} \right)^n \right\};$$

$$Z_i = f(Z_i^-) \cap f^{-1}(Z_i^+).$$

Define also

$$Z = \bigcup_{i \in \mathbb{N}} Z_i. \tag{17}$$

Remark 3.3. *For each $i \in \mathbb{N}$, by the definition of Z_i, $f^{-1}(Z_i) \subset Z_i^-$. Observe that*

$$1 \leq \left\| Df(f^{-1}(m))|_{E^-_{f^{-1}(m)}} \right\| . \left\| Df^{-1}(m)|_{E^-_m} \right\| \leq N^{-\frac{4}{5}} \left\| Df^{-1}(m)|_{E^-_m} \right\|$$

We conclude that $\left\| Df^{-1}(m)|_{E^-_m} \right\| \geq N^{\frac{4}{5}}$. Similarly $\left\| Df(m)|_{E^+_m} \right\| \geq N^{\frac{4}{5}}$.

We will need the following version of the Pliss lemma.

Lemma 3.4 ([CP18], Lemma 3.1)**.** *For any $\varepsilon > 0$, $\alpha_1 < \alpha_2$ and any sequence $(a_i) \in (\alpha_1, +\infty)^{\mathbb{N}}$ satisfying*

$$\limsup_{n \to +\infty} \frac{a_0 + \cdots + a_{n-1}}{n} \leq \alpha_2,$$

there exists a sequence of integers $0 < n_1 < n_2 < \cdots$ such that

(1) for any $k \geq 1$ and $n > n_k$, one has $\dfrac{a_{n_k} + \cdots + a_{n-1}}{(n - n_k)} \leq \alpha_2 + \varepsilon$;

(2) the upper density $\limsup \dfrac{k}{n_k}$ is larger than $\dfrac{\varepsilon}{\alpha_2 + \varepsilon - \alpha_1}$.

Using this lemma we prove the following.

Lemma 3.5. *Fix $\delta > 0$ small and assume that N is large enough such that proposition 1.4 holds for $f = f_N$. Then, it is verified $\nu_i(Z_i) \geq \frac{1-7\delta}{1+7\delta}$ and $Leb(Z) \geq \frac{1-7\delta}{1+7\delta}$.*

Proof. Since N is large enough, by proposition 1.4, for every $m \in \mathcal{R} \cap \Lambda_i$, and since $E^-(m)$ is one dimensional, we obtain

$$\lim_{n \to +\infty} \frac{1}{n} \log \|Df^n(m)|_{E^-_m}\| = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(m))|_{E^-_{f^j(m)}}\| \leq -(1-\delta) \log N.$$

Take $\varepsilon = \dfrac{1}{6} \log N$, $\alpha_1 = -\log N - \log 2$, $\alpha_2 = -(1-\delta) \log N$ and consider the sequence $\left(\log \|Df(f^j(m))|_{E^-_{f^j(m)}}\| \right)_{j \in \mathbb{N}}$. Applying Pliss lemma 3.4 for those quantities we obtain a sequence of integers $(n_k)_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$ and $n > n_k$

$$\frac{1}{n - n_k} \sum_{j=n_k}^{n-1} \log \|Df(f^j(m))|_{E^-_{f^j(m)}}\| \leq -(1-\delta) \log N + \frac{1}{6} \log N = \log N^{-\frac{5}{6}+\delta} < \log N^{-\frac{4}{5}}.$$

From this we conclude

$$\|Df^n(f^{n_k}(m))|_{E^-_{f^{n_k}(m)}}\| < \left(N^{-\frac{4}{5}} \right)^n, \ \forall n \geq 0.$$

Thus for every $k \in \mathbb{N}$ we have $f^{n_k}(m) \in Z_i^-$. Since $m \in \Lambda_i$, by Birkhoff's theorem and the second point in Pliss lemma

$$\begin{aligned}\nu_i(Z_i^-) &\geq \limsup_{k\to+\infty} \frac{k}{n_k} \\ &\geq \frac{\varepsilon}{-(1-\delta)\log N + \varepsilon + \log N + \log 2} \\ &= \frac{1}{(1+6\delta) + \frac{6\log 2}{\log N}} \geq \frac{1}{1+7\delta}.\end{aligned}$$

Similarly, $\nu_i(Z_i^+) \geq \frac{1}{1+7\delta}$. This implies that

$$\nu_i(\mathbb{T}^4 - Z_i^*) \leq \frac{7\delta}{1+7\delta}, \text{ for } * = -, +.$$

By choosing $\delta > 0$ small enough, the measure of these sets can be taken close to 1. From the definition of Z_i we conclude that

$$\nu_i(Z_i) = 1 - \nu_i(\mathbb{T}^4 - Z_i) \geq 1 - \frac{14\delta}{1+7\delta} = \frac{1-7\delta}{1+7\delta}.$$

Since $Z = \bigcup_{i\in\mathbb{N}} Z_i$ and the previous estimate is valid for every $i \in \mathbb{N}$, then

$$Leb(Z) \geq \frac{1-7\delta}{1+7\delta}. \qquad \square$$

Let $T = \left[\frac{1+7\delta}{28\delta}\right]$, we may assume that $\delta > 0$ is small enough such that $T > 20$, define

$$X = \bigcap_{k=-T+1}^{T-1} f^k(Z). \tag{18}$$

Lemma 3.6. *For N large enough, if ν_i is an ergodic component of the Lebesgue measure then*

$$\nu_i(X) > 0.$$

Proof. Recall that $\nu_i(Z_i) \geq \frac{1-7\delta}{1+7\delta}$, for N large enough, this implies that

$$\nu_i(\mathbb{T}^4 - Z_i) \leq \frac{14\delta}{1+7\delta}.$$

Therefore

$$\begin{aligned}\nu_i(X) = 1 - \nu_i(X^c) &\geq 1 - \sum_{j=-T+1}^{T-1} \nu_i(f^k(\mathbb{T}^4 - H)) \\ &\geq 1 - \left(2\left[\tfrac{(1+7\delta)}{28\delta}\right] - 2\right) \cdot \tfrac{14\delta}{1+7\delta} > 0.\end{aligned}$$

$\square$

3.2. Cone estimates. Let $V \subset \mathbb{R}^2$ be a one dimensional vector subspace inside $\mathbb{R}^2$ and let $V^\perp$ be the one dimensional subspace perpendicular to V. For any vector $w \in \mathbb{R}^2$ we can write $w = w_V + w_{V^\perp}$, the decomposition of w in V and $V^\perp$ coordinates. For $\theta > 0$ define

$$\mathscr{C}_\theta(V) = \{w \in \mathbb{R}^2 : \theta\|w_V\| \geq \|w_{V^\perp}\|\},$$

the cone inside $\mathbb{R}^2$ around V of size θ. For simplicity if $V = \mathbb{R}.(1,0)$ then we just write $\mathscr{C}_\theta^{hor} = \mathscr{C}_\theta(V)$ and $\mathscr{C}_\theta^{ver} = \mathscr{C}_\theta(V^\perp)$, we will call them the horizontal and vertical cones respectively. Throughout this paper, for a direction V, we will write

$$\mathscr{C}_\theta(V, m) = \mathscr{C}_\theta(V) \times \{0\} \subset T_m\mathbb{T}^4 = \mathbb{R}^2 \times \mathbb{R}^2.$$

Recall that $\theta_1 = N^{-\frac{2}{5}}$.

Lemma 3.7. *For N large enough, for every $m \in Z$ we have that $E^+_m \subset \mathscr{C}^{hor}_{\theta_1^{-1}}(m)$, with $\theta_1 = N^{-\frac{2}{5}}$. Furthermore, $\mathscr{C}_{\frac{\theta_1}{2}}(E^+_m, m) \subset \mathscr{C}^{hor}_{\frac{4}{\theta_1}}(m)$.The same is valid for the E^-_m direction and the vertical cone.*

Proof. From remark 3.3, we know that $\|Df(m)|_{E^+_m}\| \geq N^{\frac{4}{5}}$, for $m \in Z$. Take a vector of the form $(u, 1)$, with $|u| \leq N^{-\frac{2}{5}}$, then for N large enough

$$\begin{aligned} \|Df(m).(u,1)\| = \|(u\Omega(m) - 1, u)\| &\leq |u||\Omega(m)| + 1 + |u| \\ &\leq |u|(N+2) + 1 + |u| \leq N^{-\frac{2}{5}}.N^{1+\frac{1}{200}} + 1 \\ &\leq N^{\frac{3}{5}+\frac{1}{200}} + 1 \qquad\quad \leq N^{\frac{3}{5}+\frac{1}{100}} < N^{\frac{4}{5}}. \end{aligned}$$

Hence, if $m \in Z$ then $E^+_m \subset \mathscr{C}^{hor}_{\theta_1^{-1}}(m)$.

We want to determine $\theta > 0$ such that the cone $\mathscr{C}^{hor}_{\theta}(m)$ contains the cone $\mathscr{C}_{\frac{\theta_1}{2}}(E^+_m, m)$. For this purpose we will consider a cone $\mathscr{C}_{\frac{\theta_1}{2}}(V, m)$, where the direction V belongs to the boundary of the cone $\mathscr{C}^{hor}_{\theta_1^{-1}}(m)$.

Suppose V is generated by the unit vector $(x, \frac{x}{\theta_1})$, with $x > 0$. Observe that $V^\perp$ is generated by $(-\frac{x}{\theta_1}, x)$. One of the boundaries of the cone $\mathscr{C}^{hor}_{\theta}(m)$ we are looking for is generated by the vector $\frac{\theta_1}{2}(-\frac{x}{\theta_1}, x) + (x, \frac{x}{\theta_1})$.

The size of the cone θ is given by

$$\theta = \frac{2.[x(\theta_1^2+2)]}{2x\theta_1} = \frac{\theta_1^2+2}{\theta_1} < \frac{4}{\theta_1}.$$

Since the horizontal cones are symmetric with respect to the horizontal direction, we conclude that

$$\mathscr{C}_{\frac{\theta_1}{2}}(E^+_m, m) \subset \mathscr{C}^{hor}_{\theta}(m) \subsetneq \mathscr{C}^{hor}_{\frac{4}{\theta_1}}(m).$$

By the symmetry of f, given by lemma 2.20, the same holds of the stable direction but using vertical cones. □

We define some critical regions. For that, define $I_1 = I_1(N) = (-2N^{-\frac{3}{10}}, 2N^{-\frac{3}{10}})$, $I_2 = I_2(N) = \frac{I_1}{2}$, write $C_1 = \{\frac{\pi}{2} + I_1\} \cup \{\frac{3\pi}{2} + I_1\}$ and $C_2 = \{\frac{\pi}{2} + I_2\} \cup \{\frac{3\pi}{2} + I_2\}$. Consider the regions

$$\begin{aligned} Crit_1 &= \{C_1 \times S^1 \times \mathbb{T}^2\} \cup \{S^1 \times C_1 \times \mathbb{T}^2\} \\ Crit_2 &= \{C_2 \times S^1 \times \mathbb{T}^2\} \cup \{S^1 \times C_2 \times \mathbb{T}^2\}. \end{aligned}$$

Write $G_* = (Crit_*)^c$, for $* = 1, 2$ and observe that $G_1 \subset G_2$. Observe also that each G_* has four connected components, $\{G_{*,j}\}_{j=1}^4$. Each $G_{*,j}$ is a square and we can choose the index j such that $G_{1,j} \subset G_{2,j}$.

Remark 3.8. *The distance between the boundaries of these two sets is*

$$d(\partial G_{1,j}, \partial G_{2,j}) = N^{-\frac{3}{10}} > N^{-7}, \text{ for } 1 \leq j \leq 4.$$

Recall that $\theta_2 = N^{-\frac{3}{5}}$.

Lemma 3.9. *If N is large enough then*

(1) $Z \subset G_1 \subset G_2$;

(2) If $m \in G_2$ then $Df(m).(\mathscr{C}^{hor}_{\frac{4}{\theta_1}}(m)) \subset \mathscr{C}^{hor}_{\theta_2}(f(m))$;

(3) If γ is a C^1-curve inside a center leaf, with length $l(\gamma) \geq N^{-\frac{3}{10}}$, such that $\gamma \subset G_2$ and is tangent to $\mathscr{C}^{hor}_{\theta_2}$ then $l(f(\gamma)) > 4\pi$.

Similar statements hold for the vertical cone and f^{-1}.

PROOF. 1. If $m \notin G_1$ then for N large enough, $|\cos x| < 4N^{-\frac{3}{10}}$, in particular

$$\|Df(m)|_{E^c_m}\| \leq N|\cos x| + 4 < 4N^{\frac{7}{10}-\frac{1}{200}} + 4 < N^{\frac{7}{10}-\frac{1}{100}} < N^{\frac{4}{5}}.$$

Using the symmetry given by lemma 2.20 and since for $m \notin G_1$ we have $|\cos y| < 4N^{-\frac{3}{10}}$, a similar calculation gives

$$\|Df^{-1}(m)|_{E^c_m}\| < N^{\frac{4}{5}}.$$

Thus $Z \subset G_1 \subset G_2$.

(2) For any $m \in G_2$, $(u,v) \in \mathscr{C}^{hor}_{\frac{4}{\theta_1}}(m)$ we have

$$\theta_2(|\Omega(m)||u| - |v|) \geq \theta_2|u|\left(\frac{1}{2}.N^{\frac{7}{10}} - 2 - 4N^{\frac{3}{5}}\right) = |u|\left(\frac{1}{2}N^{\frac{1}{10}} - 2N^{-\frac{3}{5}} - 4\right) > |u|.$$

(3) For any $m \in G_2$ observe that

$$|\cos x| \geq \frac{N^{-\frac{3}{10}}}{2}. \tag{19}$$

For $(u,v) \in \mathscr{C}^{hor}_{\theta_2}(m)$ an unit vector, we must have

$$\begin{aligned}
\|Df(m).(u,v)\| \geq |\Omega(m)||u| - |v| &\geq |u|(|\Omega(m)| - \theta_2) \\
&\geq \tfrac{\|(u,v)\|}{1+\theta_2}(|\Omega(m)| - \theta_2) \geq \tfrac{1}{2}(N|\cos x| - 2 - \theta_2) \\
&\geq \tfrac{N^{\frac{7}{10}}}{4} - 1 - \tfrac{\theta_2}{2} > N^{\frac{1}{2}}.
\end{aligned}$$

Thus we have

$$l(f(\gamma)) \geq N^{\frac{1}{2}}.N^{-\frac{3}{10}} = N^{\frac{2}{10}} > 4\pi. \qquad \square$$

Remark 3.10. *Observe that the condition $\gamma \subset G_2$ in the previous lemma can be replaced by $P_x(\pi_1(\gamma)) \subset P_x(\pi_1(G_2))$. The same holds for the past changing P_x by P_y and horizontal to vertical cones.*

3.3. A lower bound on the size of the invariant manifolds. Let $(S_n)_{n=0}^{+\infty}$ be a sequence of surfaces, such that each surface has a metric that induces a distance $d_n(.,.)$ and let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of diffeomorphisms $\psi_n : S_{n-1} \to S_n$. A curve $\gamma \subset S_0$ is a stable manifold for the sequence $(\psi_n)_{n\in\mathbb{N}}$ if any two points x and y on γ verifies that $d_n(\psi_n \circ \cdots \circ \psi_1(x), \psi_n \circ \cdots \circ \psi_1(y))$ converges to zero exponentially fast. We say that γ has size bounded from below by $r > 0$, if $l_0(\gamma) \geq r$, where $l_0(.)$ is the length of γ inside S_0.

The next proposition gives us the existence of stable and unstable curves tangent to the center direction, with good estimates on its sizes and its tangent directions. The proof of this proposition follows the exact same steps as theorem 5 in [CP18], but with the changes necessary to get the estimates we need.

Theorem 5 in [CP18] proves the existence of stable manifolds with uniform size and "geometry" in the following scenario. Let $g : S \to S$ be a C^2-diffeomorphism of a compact surface and let $\sigma, \tilde{\sigma}, \rho, \tilde{\rho} \in (0,1)$ be constants such that

$$\frac{\tilde{\sigma}\tilde{\rho}}{\sigma\rho} > \sigma. \tag{20}$$

For any point $x \in S$ having a direction $E \subset T_xS$ such that for all $n \geq 0$

$$\tilde{\sigma}^n \leq \|Dg^n(x)|_E\| \leq \sigma^n \text{ and } \tilde{\rho}^n \leq \frac{\|Dg^n(x)|_E\|^2}{|\det Dg^n(x)|} \leq \rho^n$$

they obtain stable manifolds for such points. Inequality (20) is important in the construction. That is why we need a good control on the Lyapunov exponent along the center, given by proposition 1.4.

Proposition 3.11. *For N large enough, for each $m \in Z$, there are two C^1-curves $W^*(m)$ contained in $W^c(m)$, tangent to E^*_m and with length bounded from below by $r_0 = N^{-7}$, for $* = -, +$. Those curves are C^1-stable and unstable manifolds for f, respectively. Moreover, $T_pW^+_{r_0}(m) \subset \mathscr{C}^{hor}_{\frac{4}{\theta_1}}(p)$ and $T_qW^-_{r_0}(m) \subset \mathscr{C}^{ver}_{\frac{4}{\theta_1}}(q)$, for every $p \in W^+_{r_0}(m)$ and $q \in W^-_{r_0}(m)$.*

Proof. We use some of the notation of the proof of Theorem 5 in [CP18]. If $m \in Z$, by the definition of Z, $m \in Z_i$ for some $i \in \mathbb{N}$. Since $Z_i = f(Z_i^-) \cap f^{-1}(Z_i^+)$ we have that $f^{-1}(m) \in Z_i^-$, for this point it holds that

$$(2N)^{-n} \leq \left\| Df^n(f^{-1}(m))|_{E^-_{f^{-1}(m)}} \right\| < \left(N^{-\frac{4}{5}}\right)^n, \ \forall n \geq 0.$$

Since $\left|\det Df(p)|_{E^c_p}\right| = |\det Ds_N(p)| = 1$ for every $p \in \mathbb{T}^4$, it also holds

$$(2N)^{-2n} \leq \frac{\left\| Df^n(f^{-1}(m))|_{E^-_{f^{-1}(m)}} \right\|^2}{\left|\det Df^n(f^{-1}(m))|_{E^c_{f^{-1}(m)}}\right|} < \left(N^{-2.(\frac{4}{5})}\right)^n, \ \forall n \geq 0.$$

For each $n \in \mathbb{N}$ consider $\psi_n : V_n \to T_{f^n(m)}\mathbb{T}^2$ to be the lifted dynamics by the exponential map of the diffeomorphism $f|_{W^c(f^{n-1}(m))}$ along the orbit of m, that goes from a neighborhood V_n of 0 in $T_{f^{n-1}(m)}\mathbb{T}^2$ to a neighborhood of 0 in $T_{f^n(m)}\mathbb{T}^2$. Since the center leaves are C^2, we have that $f|_{W^c(f^{n-1}(m))}$ is a C^2-diffeomorphism, this implies that ψ_n is a C^2-diffeomorphism into its image.

Take $\sigma = N^{-\frac{4}{5}}$, $\tilde{\sigma} = (2N)^{-1}$, $\rho = \sigma^2$ and $\tilde{\rho} = \tilde{\sigma}^2$. Consider

$$\lambda_1 = 2N^{-\frac{4}{5}} = 2\sigma \text{ and } \lambda_2 = \frac{1}{2.(2N)^2} = \frac{\tilde{\rho}}{2},$$

and take

$$C_0 = 3 > \sum_{k\geq 0}\left(\frac{\sigma}{\lambda_1}\right)^k = 2 = \sum_{k\geq 0}\left(\frac{\lambda_2}{\tilde{\rho}}\right)^k.$$

Let $E_n = E^-_{f^{n-1}(m)}$ and $F_n = E_n^\perp$ and use the basis $E_n \oplus F_n$. We define

$$m_n = \left\| Df^n(f^{-1}(m))|_{E^-_{f^{-1}(m))}} \right\| \text{ and } M_n = \frac{|\det Df^n|_{E^c}(f^{-1}(m))|}{m_n} = \frac{1}{m_n}.$$

Using this notation it is also defined

$$A_n = \sum_{k\geq 0} \lambda_1^{-k} m_{n+k}/m_n,$$

$$B_n = \sum_{k=0}^{n} \lambda_2^{k-n} \frac{M_k/M_n}{m_k/m_n}.$$

The proof of theorem 5 in [CP18] gives

$$A_n \leq C_0\left(\frac{\lambda_1}{\tilde{\sigma}}\right)^n \text{ and } B_n \leq C_0\left(\frac{\rho}{\lambda_2}\right)^n. \tag{21}$$

Define the change of coordinates in $T_{f^{n-1}(m)}\mathbb{T}^2$ given by $\Delta_n = Diag(A_n, A_nB_n)$, where the map Δ_n is defined using the coordinates $E_n \oplus F_n$. Observe that A_n and B_n are larger or equal to 1, in particular, $\|\Delta_n\| = A_nB_n$ and $\|\Delta_n^{-1}\| = A_n^{-1} < 1$.

Write $h_n = \Delta_{n+1} \circ \psi_n \circ \Delta_n^{-1}$ and $H_n = \Delta_{n+1} \circ D\psi_n(0) \circ \Delta_n^{-1}$. We have

$$H_n = \begin{pmatrix} a & d \\ 0 & c \end{pmatrix} \text{ and } H_n^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{d}{ca} \\ 0 & \frac{1}{c} \end{pmatrix}.$$

From the proof of theorem 5 in [CP18], we obtain

$$(\|Df|_{E^c}\|.\|Df^{-1}|_{E^c}\|^2)^{-1} \leq |a| < \lambda_1 \tag{22}$$

$$|a|\lambda_2^{-1} \leq |c| \leq \lambda_1\lambda_2^{-1}\|Df|_{E^c}\|.\|Df^{-1}|_{E^c}\| + \lambda_1\|Df^{-1}|_{E^c}\|^2 \tag{23}$$

$$|d| \leq \|Df|_{E^c}\|.\|Df^{-1}|_{E^c}\||a|. \tag{24}$$

Using inequalities (23) and (24), we have

$$\left|\frac{d}{c}\right| \leq \frac{\|Df|_{E^c}\|.\|Df^{-1}|_{E^c}\||a|}{|a|\lambda_2^{-1}} < \frac{(2N)^2}{2.(2N)^2} = \frac{1}{2}.$$

Let us set $\xi = \frac{\tilde{\sigma}\lambda_2}{\lambda_1^2\rho}$ and observe that for N large enough $\xi > 4$. For $\eta \leq \frac{1}{2}$ we will consider $\widetilde{\mathscr{C}}_{(\eta,n)} = \mathscr{C}_\eta(E_n)$ the cone of size η around the direction E_n. If $(u,v) \in \widetilde{\mathscr{C}}_{(\eta,n+1)}$, using (22) and the estimate on $\left|\frac{d}{c}\right|$, we have

$$\begin{aligned}\|H_n^{-1}.(u,v)\| &\geq \left|\tfrac{u}{a}\right| - \left|\tfrac{dv}{ca}\right| \qquad \geq \left|\tfrac{u}{a}\right|\left(1 - \left|\tfrac{d\eta}{c}\right|\right) \\ &\geq \tfrac{\|(u,v)\|}{(1+\eta)\lambda_1}\left(1-\tfrac{\eta}{2}\right) \geq \tfrac{\|(u,v)\|}{\frac{3}{2}\lambda_1}.\tfrac{1}{2}.\tfrac{3}{2} = \tfrac{\|(u,v)\|}{2\lambda_1} > \tfrac{\|(u,v)\|}{\xi\lambda_1}.\end{aligned}$$

We conclude that the vectors of the cone $\widetilde{\mathscr{C}}_{(\eta,n+1)}$ are expanded by $\frac{1}{2\lambda_1}$ by H_n^{-1}. Observe that if a linear map is $\frac{\eta}{6}$-close to H_n^{-1} then the vectors inside $\widetilde{\mathscr{C}}_{\eta,n+1}$ are expanded by at least $(4\lambda_1)^{-1} > (\xi\lambda_1)^{-1}$. It is easy to see that $L\left(\widetilde{\mathscr{C}}_{(\eta,n+1)}\right) \subset \widetilde{\mathscr{C}}_{(\eta,n)}$ for any linear map L which is $\frac{\eta}{6}$-close to H_n^{-1}.

Recall that $\|Df|_E^c\| \leq 2N$ and $\|D^2f^{-1}|_{W^c}\| \leq N$. Since $\|\Delta_{n+1}^{-1}\| < 1$, we obtain

$$\|Dh_n^{-1}(0) - Dh_n^{-1}(y)\| \leq \|\Delta_n\|.\|\Delta_{n+1}^{-1}\|.\|D^2f^{-1}|_{W^c}\|.\|\Delta_{n+1}^{-1}\|.\|y\| \leq NA_nB_n\|y\|.$$

Using (21), we have that $Dh_n^{-1}(y)$ is $\frac{\eta}{4|a|}$-close to H_n^{-1} in a ball of radius

$$\tilde{r}_{n+1} = \frac{\eta}{6NA_nB_n} > \frac{\eta}{6NC_0^2}\left(\frac{\tilde{\sigma}\lambda_2}{\lambda_1\rho}\right)^n > \frac{\eta}{54N}.(4\lambda_1)^n.$$

Since Dh_n^{-1} expands the vectors inside the cone $\widetilde{\mathscr{C}}_{\eta,n+1}$ by at least $(4\lambda_1)^{-1} > (\xi\lambda)^{-1}$, we can take

$$\tilde{r}_0 = \frac{\eta}{54N}.\frac{1}{4\lambda_1} = \frac{\eta}{216N\lambda_1}.$$

The proof of theorem 5 in [CP18] gives us a C^1-curve inside $T_{f^{-1}(m)}\mathbb{T}^2$ tangent to the cone $\widetilde{\mathscr{C}}_{\eta,0}$, of size $\tilde{r}_0$, which is a stable manifold for the sequence $(h_n)_{n\in\mathbb{N}}$.

To obtain a stable manifold for the sequence $(\psi_n)_{n\in\mathbb{N}}$ we need to apply Δ_0 to this curve. Recall that $\Delta_0 = Diag(A_0, A_0)$, in particular it preserves the size and direction of a cone. Thus, we obtain that $\Delta_0(\widetilde{\mathscr{C}}_{(\eta,0)}) = \mathscr{C}_\eta(E^-_{f^{-1}(m)})$.

To obtain a stable manifold for f, instead of the sequence $(\psi_n)_{n\in\mathbb{N}}$, we must project this curve by the exponential map, this projection will be denoted by $W^-(f^{-1}(m))$. Since $\mathbb{T}^2$ is the flat torus, the derivative of the exponential map is the identity. We conclude that the stable manifold for f at the point $f^{-1}(m)$ is tangent to $\mathscr{C}_\eta(E^-_{f^{-1}(m)})$.

Now we estimate the size of the cones in the proposition at the point m. So far, the only restriction we have is $\eta \leq \frac{1}{2}$. Since $\|Df^{-1}|_{E^c}\|$ and $\|Df|_{E^c}\|$ are bounded from above by $2N$,

$$Df(f^{-1}(m)).\mathscr{C}_\eta(E^-_{f^{-1}(m)}, f^{-1}(m)) \subset \mathscr{C}_{4N^2\eta}(E^-_m, m).$$

Using the estimates from lemma 3.7, we want $4N^2\eta \leq \frac{\theta_1}{2} = \left(2N^{\frac{2}{5}}\right)^{-1}$, therefore, the additional restriction we put now is $\eta < \left(8N^{2+\frac{2}{5}}\right)^{-1}$. Since N is large, we can take $\eta = N^{-3}$, for instance. By lemma 3.7, we have $E^-_m \subset \mathscr{C}^{ver}_{\theta_1^{-1}}$ and $\mathscr{C}_{4N^2\eta}(E^-_m) \subset \mathscr{C}^{ver}_{\frac{4}{\theta_1}}$. This proves the estimate on the cones of the proposition.

With this restriction, now we estimate the size of the stable manifold at the point m. For $\eta = N^{-3}$ and since $\lambda_1 = 2N^{-\frac{4}{5}}$, we obtain for N large enough,

$$\tilde{r}_0 = \frac{\eta}{216N\lambda_1} = \frac{1}{532.N^{4-\frac{4}{5}}} > \frac{1}{N^5}.$$

From this one can conclude that the stable manifold at the point $f^{-1}(m)$ has size bounded below by N^{-5}, this implies that at the point m the stable manifold has size bounded by $(2N)^{-1}.N^{-5} > N^{-7} = r_0$, which concludes the proof for $W^-_{r_0}(m)$. The proof for the unstable manifold is analogous. □

Remark 3.12. *From item* 1 *of lemma 3.9 and Remark 3.8, if $m \in Z$ then $W^*_{r_0}(m) \subset G_2$, for $* = -, +$.*

4. Ergodicity of the system f_N

In this section assuming proposition 1.4, we prove:

Theorem 4.1. *For N large enough f_N is ergodic.*

The proof is by contradiction. Suppose that $f = f_N$ is not ergodic, then there are at least two different ergodic components, ν_1 and ν_2. Let $\varphi : \mathbb{T}^4 \to \mathbb{R}$ be a continuous function such that

$$\int \varphi d\nu_1 \neq \int \varphi d\nu_2.$$

Consider the forward and backward Birkhoff's average

$$\varphi^+(m) = \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j(m) \text{ and } \varphi^-(m) = \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^{-j}(m).$$

Recall that we defined at the beginning of section 3, the set Λ_i as the set of points such that any $m_i \in \Lambda_i$, it holds that $\varphi^+(m_i) = \varphi^-(m_i) = \int \varphi d\nu_i$, for $i = 1, 2$ and any continuous function $\varphi : \mathbb{T}^4 \to \mathbb{R}$.

First we remark that for almost every $m \in \mathbb{T}^4$ the stable part of the Oseledets decomposition, defined in (41), is given by $E^s_m = E^{ss}_m \oplus E^-_m$. By theorem 2.11 there is a C^1 stable Pesin manifold, $W^s(m)$, such that $T_m W^s(m) = E^{ss}_m \oplus E^-_m$, analogously for the unstable direction. Recall that the stable Pesin manifold has a topological characterization given by

$$W^s(m) = \{y \in \mathbb{T}^4 : \limsup_{n\to+\infty} \frac{1}{n} \log d(f^n(m), f^n(y)) < 0\}.$$

The set Z was defined in (17). For $m \in Z$ consider

$$\widehat{W}^s(m) = \bigcup_{y\in W^-_{r_0}(m)} W^{ss}(y),$$

where $W^-_{r_0}(m)$ is the stable manifold constructed in proposition 3.11 and $r_0 = N^{-7}$.

Remark 4.2. *By the topological characterization of the stable Pesin manifold we conclude that $\widehat{W}^s(m) \subset W^s(m)$. Observe that the strong stable manifold subfoliates the Pesin stable manifold, in particular $\widehat{W}^s(m)$ is open inside the Pesin manifold. We conclude that $\widehat{W}^s(m)$ is a C^1-submanifold and for every $m \in Z$ the stable Pesin manifold contain a disc of size r_0. Analogously for the unstable manifold.*

Since φ is continuous, for every $z \in W^s(m)$ and $w \in W^u(m)$, with $m \in \Lambda$, we obtain $\varphi^+(m) = \varphi^+(z)$ and $\varphi^-(w) = \varphi^-(m)$, where Λ was defined in (16) and has full Lebesgue measure.

Claim 1. *There exists an invariant set B of full Lebesgue measure, such that for every $m \in B$ and for Lebesgue almost every point $z \in W^u(m)$ it is verified $\varphi^-(z) = \varphi^+(z)$.*

PROOF. Let Λ be as before. By theorem 2.15, the unstable partition is absolutely continuous, in particular, a Fubini-like formula holds. Since the set Λ has full Lebesgue measure, this implies that it exists a set of full Lebesgue measure $B_0 \subset \Lambda$ such that for any $m \in B_0$, the set $W^u(m) \cap \Lambda$ has full Lebesgue measure inside $W^u(m)$. Consider

$$B = \bigcap_{j\in\mathbb{Z}} f^j(B_0).$$

This set is f-invariant, it has full Lebesgue measure and verifies the conclusion of the claim. □

Recall that we defined $X = \bigcap_{k=-T+1}^{T-1} f^k(Z)$ and $\theta_2 = N^{-\frac{3}{5}}$. Recall also that we defined in section 3.2 the sets G_1 and G_2.

Lemma 4.3. *For N large enough and $n \geq 15$, for every $m \in X$ there are two curves $\gamma^-_{-n}(m) \subset f^{-n}(W^-_{r_0}(m))$ and $\gamma^+_n(m) \subset f^n(W^+_{r_0}(m))$ with length greater than 4π. The tangent vectors of each of those curves are contained in the cone $\mathscr{C}^{ver}_{\theta_2}$ and $\mathscr{C}^{hor}_{\theta_2}$, respectively.*

PROOF. If $m \in X$ then

$$\{f^{-T+1}(m), \cdots, f^{T-1}(m)\} \subset Z \subset G_1 \subset G_2, \text{ where } T = \left[\frac{1+7\delta}{28\delta}\right] > 20.$$

Define $W^+_k(m) = f^k(W^+_{r_0}(m))$ and observe that for every $z \in W^+_k(m)$, if $z \in G_2$ and $T_z W^+_k(m) \subset \mathscr{C}^{hor}_{\theta_2}$ then $T_{f(z)} W^+_{k+1}(m) \subset \mathscr{C}^{hor}_{\theta_2}$.

By Proposition 3.11, $TW^+_0(m) \subset \mathscr{C}^{hor}_{\frac{4}{\theta_1}}$. Since $m \in Z \subset G_1$, by remark 3.8 we conclude that $W^+_0(m) \subset G_2$. Item 2 of lemma 3.9 implies that $TW^+_1(m) \subset \mathscr{C}^{hor}_{\theta_2}$.

If $p \in G_2$ and $(u, v) \in \mathscr{C}^{hor}_{\theta_2}(p)$ is an unit vector, then $\|Df(p).(u, v)\| > N^{\frac{1}{2}}$. For a C^1-curve γ containing m with length N^{-7}, such that $\gamma \subset G_2$ and $T\gamma \subset \mathscr{C}^{hor}_{\theta_2}$, let $k \in \mathbb{N}$ be the largest number such that $f^j(\gamma) \subset G_2$, for every $j = 1, \cdots, k$. Since the vectors inside $\mathscr{C}^{hor}_{\theta_2}$ are expanded by at least $N^{\frac{1}{2}}$ and the cone $\mathscr{C}^{hor}_{\theta_2}$ is preserved by the derivative of the points in G_2, we conclude that $k \leq 14$.

Let $k^+_0 \in \mathbb{N}$ be the smallest number such that $W^+_{k^+_0}(m) \cap \partial G_2 \neq \emptyset$. Recall that if $p \in G_2$ and $(u, v) \in \mathscr{C}^{hor}_{\frac{4}{\theta_1}}$ is a unit vector, then by (19), $\|Df(p).(u, v)\| > 1$. Since $r_0 = N^{-7}$, we obtain that the curve $W^+_1(m) \subset \mathscr{C}^{hor}_{\theta_2}$ has length at least N^{-7} and is tangent to $\mathscr{C}^{hor}_{\theta_2}$, by the previous paragraph $k^+_0 \leq 15$.

If $m \in X$, the connected component of $W^+_{k^+_0}(m) \cap G_2$ containing $f^{k^+_0}(m)$, which we will denote by $\widehat{W}^+_{k^+_0}(m)$, intersects the boundary of G_2 and $T\widehat{W}^+_{k^+_0}(m) \subset \mathscr{C}^{hor}_{\theta_2}$. Since $k^+_0 < T$, we know that $f^{k^+_0}(m) \in Z \subset G_1 \subset G_2$. We conclude that $\widehat{W}^+_{k^+_0}(m)$ also intersects the boundary of G_1.

Let $\gamma^+_{k^+_0}$ be a connected component of $\widehat{W}^+_{k^+_0}(m) \cap (G_2 - G_1)$, such that $\gamma^+_{k^+_0} \cap \partial G_1 \neq \emptyset$ and $\gamma^+_{k^+_0} \cap \partial G_2 \neq \emptyset$, see figure 1. The curve $\gamma^+_{k^+_0}$ is a C^1-curve that verifies the hypothesis of item 3 from lemma 3.9. Thus $l(f(\gamma^+_{k^+_0})) > 4\pi$, $Tf(\gamma^+_{k^+_0}) \subset \mathscr{C}^{hor}_{\theta_2}$ and by definition $f(\gamma^+_{k^+_0}) \subset W^+_{k^+_0+1}(m)$. Define $\gamma_{k^+_0+1}(m) = f(\gamma^+_{k^+_0})$.

Let

$$\widetilde{G} = \left\{(x, y, z, w) \in \mathbb{T}^4 : N^{-\frac{3}{10}} \leq |x - \frac{\pi}{2}| \leq 2N^{-\frac{3}{10}} \text{ or } N^{-\frac{3}{10}} \leq |x - \frac{3\pi}{2}| \leq 2N^{-\frac{3}{10}}\right\}.$$

It is easy to see that $\widetilde{G}$ has four connected components, each connected component having two boundaries. Since the critical region only depends on the coordinate x, for any point $p \in \widetilde{G}$, the derivative $Df(p)$ expands any vector inside $\mathscr{C}^{hor}_{\theta_2}$ by at least $N^{\frac{1}{2}}$.

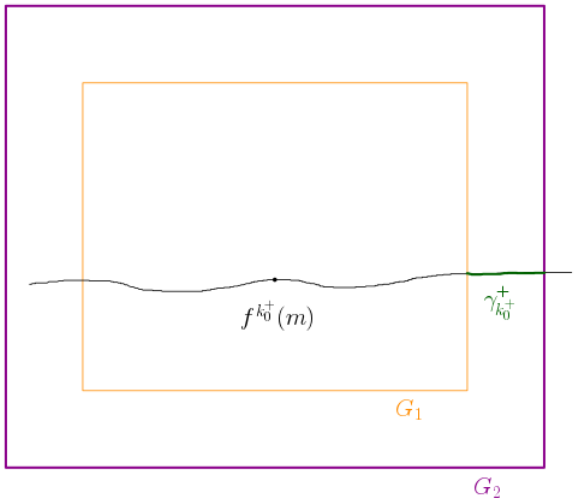

FIGURE 1. The curve $\gamma^+_{k_0^+}$

We build $\gamma^+_n \subset f(\gamma^+_{n-1})$ inductively for $n > k_0^+ + 1$. Let us build it for $n = k_0^+ + 2$. Observe that $P_x(\pi_1(\gamma^+_{k_0^+ +1})) = S^1$. Consider then $\widetilde{\gamma}^+_{k_0^+ +1}$ to be a connected component of $\gamma^+_{k_0^+ +1}(m) \cap \widetilde{G}$ that intersects the two boundaries of a connected component of $\widetilde{G}$. Define $\gamma^+_{k_0^+ +2}(m) = f(\tilde{\gamma}^+_{k_0^+ +1})$, observe that $l(\gamma^+_{k_0^+ +2}(m)) > 4\pi$ and $Tf(\gamma_{k_0^+ +2}(m)) \subset \mathscr{C}^{hor}_{\theta_2}$. In this way we can build inductively the curves $\gamma^+_n(m)$ that satisfy the conclusions of the lemma. In a similar way we construct the curves $\gamma^-_{-n}(m)$. Since $k_0^+ \leq 15$ and $k_0^- \leq 15$, then this certainly holds for $n > 15$. □

For each $R > 0$, $n \geq 15$ and $m \in X$, define

$$W^s_{R,-n}(m) = \bigcup_{q \in \gamma^-_{-n}(m)} W^{ss}_R(q), \tag{25}$$

where the curve $\gamma^-_{-n}(m)$ is the curve given by lemma 4.3. Define in a similar way the set $W^u_{R,n}(m)$. For the same reason as we explained in remark 4.2, we obtain that $W^s_{R,-n}(m)$ and $W^u_{R,n}(m)$ are C^1-submanifolds.

Lemma 4.4. *Fix $\theta_3 > 0$ such that $\theta_3 > \theta_2$ and that satisfies $\mathscr{C}^{hor}_{\theta_3} \cap \mathscr{C}^{ver}_{\theta_3} = \{0\}$. There exists $0 < R < 1$ such that if $n \geq 15$, $m \in X$ and $m^- \in W^s_{R,-n}(m)$, then*

$$T(W^s_{2,-n}(m) \cap W^c(m^-)) \subset \mathscr{C}^{ver}_{\theta_3}.$$

A similar result holds for $W^u_{R,n}(m)$.

PROOF. For any $p \in \mathbb{T}^4$, it holds that $\pi_2(W^{ss}(p)) = W^{ss}_A(\pi_2(p))$, where $W^{ss}_A(\pi_2(p))$ is the stable manifold of the point $\pi_2(p)$ for the linear Anosov system. Thus, given any point $q \in W^{ss}_1(p)$, for every $b \in W^c(p)$ there is only one point in $W^{ss}(b) \cap W^c(q)$. We define the stable holonomy map

$$\begin{array}{rccc} H^s_{p,q}: & W^c(p) & \longrightarrow & W^c(q) \\ & b & \mapsto & W^{ss}(b) \cap W^c(q). \end{array}$$

Locally this map is given by the holonomy map defined in section 2. This is a C^1-diffeomorphism and we can naturally write $DH^s_{p,q}(p) : \mathbb{R}^2 \to \mathbb{R}^2$.

By remark 2.22, f_N verifies the conditions of theorem 2.18. As a consequence of theorem 2.18, we obtain that the family of maps above vary continuously in the C^1-topology with the points (p, q). Since $DH^s_{p,p} = Id$, by the compactness of $\mathbb{T}^4$, there is $R \in (0, 1)$ such that for any $q \in W^{ss}_R(p)$ it holds $DH^s_{p,q}(p).(\mathscr{C}^{ver}_{\theta_2}) \subset \mathscr{C}^{ver}_{\theta_3}$.

Observe that $W^s_{2,-n}(m)$ is contained inside a center-stable leaf, which is subfoliated by strong stable leaves. For this subfoliation, restricted to a center-stable leaf, the center manifolds are transversals. Thus for $m^- \in W^s_{R,-n}(m)$, the $W^s_{2,-n}(m) \cap W^c(m^-)$ is given by $H^s_{m,m^-}(\gamma^-_{-n}(m))$. By our choice of R and since $T\gamma^-_{-n}(m) \subset \mathscr{C}^{ver}_{\theta_2}$ the conclusion of the lemma follows. □

Lemma 4.5. *There is a set of full measure $D \subset \mathbb{T}^4$ such that for every $p \in D$ the orbit of $W^c(p)$ is dense among the center leaves.*

Proof. For the linear Anosov A^{2N}, there is a set D_A of full measure, with the property that every point in D_A has dense orbit. This follows from the ergodicity of A^{2N} for the Lebesgue measure.

Since the Lebesgue measure of $\mathbb{T}^4$ is the product measure of the Lebesgue measure of each $\mathbb{T}^2$, take $D = \pi_2^{-1}(D_A)$. For any $p \in \mathbb{T}^4$ it holds that

$$\pi_2(f(W^c(p)) = A^{2N}(\pi_2(p)).$$

For any $q \in \mathbb{T}^2$, $\pi_2^{-1}(q)$ is a center leaf. Thus the dynamics among the center leaves is conjugated to A^{2N} by π_2. Therefore, for any $p \in D$, since $\pi_2(p) \in D_A$ we conclude that the orbit of $W^c(p)$ is dense among the center leaves. □

Take $m_1 \in X \cap D \cap B \cap \Lambda_1$ and $m_2 \in X \cap D \cap B \cap \Lambda_2$. From the definition of Λ_1 and Λ_2, for these two points

$$\varphi^-(m_1) = \int \varphi d\nu_1 \text{ and } \varphi^+(m_2) = \int \varphi d\nu_2.$$

Fix a center leaf $W^c(q)$. Since $m_1, m_2 \in D$, there are two sequences $n_k \to +\infty$ and $l_j \to +\infty$, such that

$$f^{n_k}(W^c(m_1)) \to W^c(q) \text{ and } f^{-l_j}(W^c(m_2)) \to W^c(q).$$

By lemma 4.3, there are curves $\gamma^+_{n_k}(m_1)$ and $\gamma^-_{-l_j}(m_2)$ with length bigger that 4π and contained in the cone $\mathscr{C}^{hor}_{\theta_2}$ and $\mathscr{C}^{ver}_{\theta_2}$, respectively. Take R given by lemma 4.4 and consider the sets

$$L^u_k(m_1) = \bigcup_{z \in \gamma^+_{n_k}(m_1)} W^{uu}_R(z) \subset W^u(f^{n_k}(m_1))$$
$$L^s_j(m_2) = \bigcup_{z \in \gamma^-_{-l_j}(m_2)} W^{ss}_R(z) \subset W^s(f^{-l_j}(m_2)).$$

For k and j large enough, $f^{n_k}(W^c(m_1))$ and $f^{-l_j}(W^c(m_2))$ are very close to the leaf $W^c(q)$. Thus by the control on the angles that we obtained in lemma 4.4, there is a transversal intersection between $L^u_k(m_1)$ and $L^s_j(m_2)$. In particular $W^u(f^{n_k}(m_1))$ and $W^s(f^{-l_j}(m_2))$ intersects transversely. Before we continue with the proof we make the following remark.

Remark 4.6. *This transverse intersection between stable and unstable manifolds is the key property to obtain ergodicity. We will see that the rest of the proof is a standard application of Hopf argument in the non-uniformly hyperbolic scenario. Three properties imply this transverse intersection:*

(1) For any point inside a certain set with full measure for any ergodic component, there exists a stable curve inside the center manifold, with large size and controlled

geometry. Similarly the existence of such a set but with the existence of an unstable curve. This is given by lemma 4.3. Indeed, we can take the sets

$$X^s = \bigcup_{n\geq 15} f^{-n}(X) \text{ and } X^u = \bigcup_{n\geq 15} f^n(X);$$

(2) The control of the holonomies, which will give a control on the tangent space of Pesin's manifolds considered in (25). This is given by lemma 4.4;

(3) The density of the orbit of almost every center leaf, which is given by lemma 4.5.

Now we continue with the proof. Fix $\varepsilon > 0$ small and $l \in \mathbb{N}$ large enough such that the Pesin block $\mathcal{R}_{\varepsilon,l}$ has positive ν_2 measure. By theorem 2.11, there is a number $\varepsilon_1 > 0$ such that every point $q \in \mathcal{R}_{\varepsilon,l}$ has a disc contained in $W^s(q)$ of size ε_1, which we will denote it by $W^s_{loc}(q)$. Furthermore, those discs vary C^1-continuously with the point $q \in \mathcal{R}_{\varepsilon.l}$.

Let p be a point of transversal intersection between $L^u_k(m_1)$ and $L^s_j(m_2)$. Take $M > 0$ large enough such that $f^{M-l_j}(m_2) \in \mathcal{R}_{\varepsilon,l}$ and $d(f^{M-l_j}(m_2), f^M(p)) << \varepsilon_1$, such M exists since m_2 is a typical point for ν_2 and the set $\mathcal{R}_{\varepsilon,l}$ has positive ν_2-measure. We may assume that $f^{M-l_j}(m_2)$ is a density point of $\mathcal{R}_{\varepsilon,l} \cap \Lambda_2$. Fix a disc T transverse to $W^s_{loc}(f^{M-l_j}(m_2))$ such that $\mathcal{R}_{\varepsilon,l} \cap \Lambda_2 \cap T$ has positive measure inside T.

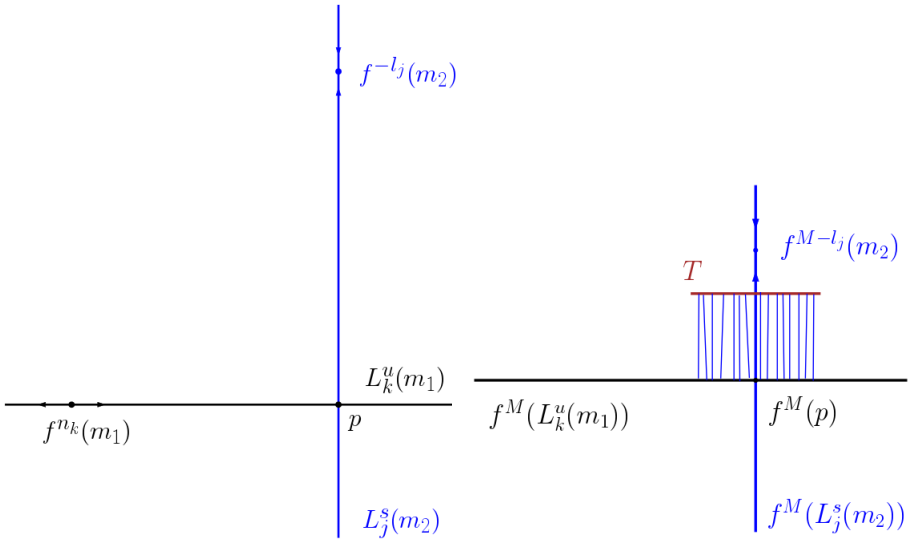

FIGURE 2. The transverse intersection and the holonomy

Consider a disc $D^u \subset f^M(L^u_k(m_1))$ centered in $f^M(p)$ and observe that this disc is transverse to $W^s_{loc}(f^{M-l_j}(m_2))$. By the absolute continuity of the Pesin manifolds, we conclude that the set $A = \{W^s_{loc}(z) \cap D^u : z \in \mathcal{R}_{\varepsilon,l} \cap \Lambda_2 \cap T\}$ has positive measure inside $W^u(f^{M+n_k}(m_1))$.

By the invariance of B, we know that $f^{M+n_k}(m_1) \in B$ and for almost every point $q \in W^u(f^{M+n_k}(m_1))$, it holds that $\varphi^+(q) = \varphi^-(q)$. Fix $\hat{z} \in A$ such that $\varphi^+(\hat{z}) = \varphi^-(\hat{z})$ and let $z \in \mathcal{R}_{\varepsilon,l} \cap \Lambda_2 \cap T$ be the point with $\hat{z} \in W^s_{loc}(z)$.

Since $z \in \Lambda_2$ and $\hat{z} \in W^s(z)$, we know that $\varphi^+(m_2) = \varphi^+(z) = \varphi^+(\hat{z})$. On the other hand, $\hat{z} \in W^u(f^{M+n_k}(m_1))$ implies that $\varphi^-(\hat{z}) = \varphi^-(m_1)$. Thus,

$$\int \varphi d\nu_1 = \varphi^-(m_1) = \varphi^-(\hat{z}) = \varphi^+(\hat{z}) = \varphi^+(z) = \varphi^+(m_2) = \int \varphi d\nu_2.$$

This is a contradiction since we assumed that $\int \varphi d\nu_1 \neq \int \varphi d\nu_2$. We conclude that there is only one ergodic component, in particular, the Lebesgue measure is ergodic. Thus we have proved that for N large enough, $f_N = f$ is ergodic.

5. Stable ergodicity of the system f_N

In this section we show how to adapt the proof of the ergodicity of f_N to obtain C^2-stable ergodicity. Recall that for a vector $v \in T_m\mathbb{T}^4$, we defined $v_1 = D\pi_1(m).v$. For a direction $E \subset T_m\mathbb{T}^4$ we will write $(E)_1 = D\pi_1(m).E$. For this section we fix $0 < \delta << 1$ small and we are assuming that N is large and $\mathcal{U}_N$ is small enough such that proposition 1.4 holds. Using proposition 2.9 and the estimates in (14), one easily obtains the following lemma.

Lemma 5.1. *For each $\beta > 0$, if N is large and $\mathcal{U}_N$ is small enough, for $g \in \mathcal{U}_N$ it holds*

(1) g is partially hyperbolic, with a decomposition $TM = E^{ss}_g \oplus E^c_g \oplus E^{uu}_g$;

(2) g is dynamically coherent and leaf conjugated to f by a homeomorphism $h_g : \mathbb{T}^4 \to \mathbb{T}^4$;

(3) $d_{C^2}(W^c_g(m), W^c_f(m)) \leq \beta$;

(4) $\|Dg(m)|_{E^c_{g,m}}\| \in (e^{-\beta}\|Df(m)|_{E^c_{f,m}}\|, e^{\beta}\|Df(m)|_{E^c_{f,m}}\|)$;

(5) $|\det Dg(m)|_{E^c_{g,m}}| \in (e^{-\beta}, e^{\beta})$;

(6) $\|D^2g(m)|_{W^c_g(m)}\| \leq 2N$;

(7) $\max\{\|Dg(m)|_{E^c_{g,m}}\|, \|Dg^{-1}(m)|_{E^c_{g,m}}\|\} \leq 2N$;

(8) $\min\{m(Dg(m)|_{E^c_{g,m}}), m(Dg^{-1}(m)|_{E^c_{g,m}})\} \geq (2N)^{-1}$;

(9) $\|Dg(m).v^c\| \in (e^{-\beta}\|Dg(m).v^c_1\|, e^{\beta}\|Dg(m).v^c_1\|)$, where $v^c \in E^c_{g,m}$ and $v^c_1 = D\pi_1(m).v^c$;

(10) for points $p \in \mathbb{T}^4$ and $q \in W^c_g(p)$, let $exp^c_q : T_qW^c_g(p) \to W^c_g(p)$ be the exponential map of the center leaf. For any C^1-curve $\gamma \subset B(0, \frac{1}{2}) \subset T_qW^c_g(p)$, it holds $l_q(\gamma) \in (e^{-\beta}l(exp^c_q(\gamma)), e^{\beta}l(exp^c_q(\gamma)))$, where $l_q(\gamma)$ is the length of the curve with respect to the inner product $< ., . >_q$ on $T_qW^c_g(p)$, the usual metric of $\mathbb{T}^4$ at the point q.

From now on we fix $0 < \beta << 1$. By proposition 1.4, every diffeomorphism $g \in \mathcal{U}_N$ is non-uniformly hyperbolic. Using theorem 2.17, we obtain the ergodic decomposition $Leb = \sum_{i\in\mathbb{N}} c_i\nu_{g,i}$. We define similarly as in section 3 the sets $\{\Lambda_{g,i}\}_{i\in\mathbb{N}}$. Let $\mathcal{R}_g$ be the set of regular points for g. For a regular point $p \in \mathcal{R}_g$, let $E^-_{g,p}$ and $E^+_{g,p}$ be the directions of the Oseledets splitting. It holds that $E^c_{g,p} = E^-_{g,p} \oplus E^+_{g,p}$.

We define the sets

$$Z^-_{g,i} = \left\{m \in \mathcal{R}_g \cap \Lambda_{g,i} : \forall n \geq 0 \text{ it holds } \left\|Dg^n(m)|_{E^-_{g,m}}\right\| < \left(N^{-\frac{4}{5}}\right)^n\right\};$$

$$Z^+_{g,i} = \left\{m \in \mathcal{R}_g \cap \Lambda_{g,i} : \forall n \geq 0 \text{ it holds } \left\|Dg^{-n}(m)|_{E^+_{g,m}}\right\| < \left(N^{-\frac{4}{5}}\right)^n\right\};$$

$$Z_{g,i} = g(Z^-_{g,i}) \cap g^{-1}(Z^+_{g,i});$$

$$Z_g = \bigcup_{i\in\mathbb{N}} Z_{g,i}.$$

Lemma 5.2. *For every $g \in \mathcal{U}_N$, it holds that $\nu_{g,i}(Z_{g,i}) \geq \frac{1-7\delta}{1+7\delta}$ and $Leb(Z_g) \geq \frac{1-7\delta}{1+7\delta}$.*

The proof is analogous to the proof of lemma 3.5. Let $T = \left[\frac{1+7\delta}{28\delta}\right]$ and define

$$X_g = \bigcap_{k=-T+1}^{T-1} g^k(Z_g). \tag{26}$$

The proof of the next lemma is the same as the proof of lemma 3.6.

Lemma 5.3. *For N large and $\mathcal{U}_N$ small enough, if $\nu_{g,i}$ is an ergodic component of the Lebesgue measure then*

$$\nu_{g,i}(X_g) > 0.$$

Now we make a few estimates on the cones. Recall that $\theta_1 = N^{-\frac{2}{5}}$.

Lemma 5.4. *If N is large and $\mathcal{U}_N$ is small enough then for each $g \in \mathcal{U}_N$, for every $m \in Z_g$, it is verified that $(E^+_{g,m})_1 \subset \mathscr{C}^{hor}_{\theta_1^{-1}}(m)$. Furthermore, $\mathscr{C}_{\frac{\theta_1}{2}}((E^+_{g,m})_1, m) \subset \mathscr{C}^{hor}_{\frac{4}{\theta_1}}(m)$. The same holds for the $E^-_{g,m}$ and the vertical cone.*

Proof. For $m \in Z_g$, it holds that $\|Dg(m)|_{E^+_{g,m}}\| \geq N^{\frac{4}{5}}$. Take a vector of the form $(u, 1)$, identifying $(u, 1) = (u, 1, 0, 0)$, with $|u| \leq N^{-\frac{2}{5}}$. For N large enough and from the calculations made in the proof of lemma 3.7, which for this part does not use that $m \in Z_g$, we obtain

$$\|Dg(m).(u, 1)\| \leq e^{\beta}\|Df(m).(u, 1)\| \leq e^{\beta} N^{\frac{3}{5}+\frac{1}{100}} < N^{\frac{3}{5}+\frac{1}{50}}.$$

Suppose that such $(u, 1)$ generates $(E^+_{g,m})_1$, then

$$\|Dg(m)|_{E^+_{g,m}}\| \leq e^{\beta} \frac{\|Dg(m).(u, 1)\|}{\|(u, 1)\|} \leq N^{\frac{3}{5}+\frac{1}{25}} < N^{\frac{4}{5}},$$

which is a contradiction since $m \in Z_g$. The proof of the second part of the lemma is exactly the same as in lemma 3.7. □

Recall that we defined in section 3.2 the sets $Crit_1$, $Crit_2$, G_1 and G_2. Also recall that $\theta_2 = N^{-\frac{3}{5}}$. We obtain the following lemma, by continuity and lemma 3.9.

Lemma 5.5. *For N large, $\mathcal{U}_N$ small enough and $g \in \mathcal{U}_N$, it holds that*

(1) $Z_g \subset G_1 \subset G_2$;

(2) If $m \in G_2$ then $\left(Dg(m).\mathscr{C}^{hor}_{\frac{4}{\theta_1}}(m)\right)_1 \subset \mathscr{C}^{hor}_{\theta_2}(g(m))$;

(3) If $\gamma \subset G_2$ is a C^1-curve inside a center leaf such that the curve $\pi_1(\gamma)$ is tangent to $\mathscr{C}^{hor}_{\theta_2}$ and has length $l(\pi_1(\gamma)) \geq N^{-\frac{3}{10}}$ then $l(g(\gamma)) > 4\pi$.

Similar statements hold for the vertical cone and g^{-1}.

Proof. 1. For $m \notin G_1$, by item 4 of lemma 5.1, it holds

$$\|Dg(m)|_{E^c_{g,m}}\| \leq e^{\beta}\|Df(m)|_{E^c_{f,m}}\| < e^{\beta} N^{\frac{7}{10}-\frac{1}{100}} < N^{\frac{4}{5}}.$$

(2) The proof of item 2 of lemma 3.9 actually gives that for $m \in G_2$, it holds

$$Df(m).(\mathscr{C}^{hor}_{\frac{4}{\theta_1}}(m)) \subset \mathscr{C}^{hor}_{\frac{\theta_2}{K}}(f(m)),$$

where $K = \frac{1}{2}N^{\frac{1}{10}} - 2N^{-\frac{3}{5}} - 4$. In particular, the inclusion of item 2 of lemma 3.9 is uniformly strict. Thus, if $\mathcal{U}_N$ is small enough the conclusion follows.

(3) From the estimates made in the proof of item 3 of lemma 3.9 and by items 4 and 9 of lemma 5.1, it follows that

$$l(g(\gamma)) \geq l(g(\pi_1(\gamma))) > e^{-\beta} N^{\frac{1}{2}-\frac{3}{10}} > 4\pi.$$ □

Now we estimate the size of the stable and unstable manifolds analogous to proposition 3.11.

Proposition 5.6. *Let N be large and $\mathcal{U}_N$ be small enough. For $g \in \mathcal{U}_N$ and $m \in Z_g$, there are two C^1-curves, $W^*_g(m)$, contained in $W^c_g(m)$, tangent to $E^*_{g,m}$ and with length bounded from below by $r_0 = N^{-7}$, for $* = -, +$. Those curves are C^1-stable and unstable manifolds*

for g, respectively. Moreover, $\left(T_pW^+_{g,r_0}(m)\right)_1 \subset \mathscr{C}^{hor}_{\frac{4}{\theta_1}}(p)$ and $\left(T_qW^-_{g,r_0}(m)\right)_1 \subset \mathscr{C}^{ver}_{\frac{4}{\theta_1}}(q)$, for every $p \in W^+_{g,r_0}(m)$ and $q \in W^-_{g,r_0}(m)$.

PROOF. The main difference in the proof is that we have to project by $D\pi_1$ the tangent directions of the curves constructed. By lemma 5.1 we will have good control of what happens after this projection, obtaining the desired estimates.

Using item 5 of lemma 5.1, for $m \in Z_g$, it holds that

$$(2N)^{-n} \leq \left\| Dg^n(g^{-1}(m))|_{E^-_{g,g^{-1}(m)}} \right\| < \left(N^{-\frac{4}{5}}\right)^n,$$

and

$$(2N)^{-2n}e^{-n\beta} \leq \frac{\left\| Dg^n(g^{-1}(m))|_{E^-_{g,g^{-1}(m)}} \right\|^2}{\left|\det Dg(g^{-1}(m))|_{E^c_{g,g^{-1}(m)}}\right|} < \left(e^{\beta}N^{-2.\left(\frac{4}{5}\right)}\right)^n.$$

In the same way as in the proof of proposition 3.11, consider the lifted dynamics $\psi_n : V_n \to T_{g^n(m)}W^c_g(g^n(m))$ of the diffeomorphism $g|_{W^c_g(g^{n-1}(m))}$, that goes from a neighborhood V_n of 0 in $T_{g^{n-1}(m)}W^c_g(g^{n-1}(m))$ onto a neighborhood of 0 in $T_{g^n(m)}W^c_g(g^n(m))$. Since the center leaves are C^2, we have that $g|_{W^c_g(g^{n-1}(m)}$ is a C^2-diffeomorphism, which implies that ψ_n is a C^2-diffeomorphisms into its image.

Take $\sigma = N^{-\frac{4}{5}}$, $\lambda_1 = 2\sigma$, $\tilde{\sigma} = (2N)^{-1}$, $\rho = e^{\beta}\sigma^2$, $\tilde{\rho} = e^{-\beta}\tilde{\sigma}^2$, $\lambda_2 = \frac{\tilde{\rho}}{2}$ and $C_0 = 3$. Let $\xi = \frac{\tilde{\sigma}\lambda_2}{\lambda_1^2\rho}$ and observe that for N large enough

$$\xi = \frac{\tilde{\sigma}\lambda_2}{\lambda_1^2\rho} = 2^{-6}e^{-2\beta}N^{\frac{1}{5}} > 4.$$

Following the same construction as in proposition 3.11, one obtains the maps Δ_n, h_n and H_n. Recall that

$$H_n = \begin{pmatrix} a & d \\ 0 & c \end{pmatrix} \text{ and } H_n^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{d}{ca} \\ 0 & \frac{1}{c} \end{pmatrix}.$$

It also holds that

$$(\|Dg|_{E^c_g}\|.\|Dg^{-1}|_{E^c_g}\|^2)^{-1} \leq |a| < \lambda_1 \tag{27}$$

$$|a|\lambda_2^{-1} \leq |c| \leq \lambda_1\lambda_2^{-1}\|Dg|_{E^c_g}\|.\|Dg^{-1}|_{E^c_g}\| + \lambda_1\|Dg^{-1}|_{E^c_g}\|^2 \tag{28}$$

$$|d| \leq \|Dg|_{E^c_g}\|.\|Dg^{-1}|_{E^c_g}\||a|. \tag{29}$$

By item 4 of lemma 5.1 and using the previous inequalities

$$\left|\frac{d}{c}\right| \leq \frac{\|Dg|_{E^c_g}\|.\|Dg^{-1}|_{E^c_g}\||a|}{|a|\lambda_2^{-1}} < \frac{e^{2\beta}(2N)^2}{2e^{\beta}.(2N)^2} = \frac{e^{\beta}}{2}.$$

For $\eta \leq \frac{1}{2}$ define the cone $\widetilde{\mathscr{C}}_{(\eta,n)} = \mathscr{C}_{\eta}(E_n)$, with cone size η around the direction E_n inside $T_{g^{n-1}(m)}W^c_g(g^{n-1}(m))$. Using the estimate on $\left|\frac{d}{c}\right|$, following the same steps as in the proof of proposition 3.11, we obtain that any linear map $\frac{\eta}{6}$-close to H_n^{-1} contracts the cone $\widetilde{\mathscr{C}}_{(\eta,n+1)}$ and expands any vector inside $\widetilde{\mathscr{C}}_{(\eta,n+1)}$ by at least $\frac{1}{4\lambda_1}$.

By item 6 of lemma 5.1, for any point $q \in \mathbb{T}^4$, it holds that $\|D^2g(q)|_{W^c_g(q)}\| \leq 2N$. Thus $(Dh_n(y))^{-1}$ is $\frac{\eta}{6}$-close to H_n^{-1} in the ball of radius

$$\tilde{r}_{n+1} = \frac{\eta}{12N\|\Delta_n\|} > \frac{\eta}{108N}(4\lambda_1)^n.$$

Arguing similarly as in the proof of proposition 3.11, we can take

$$\tilde{r}_0 = \frac{\eta}{432N\lambda_1}.$$

Also by similar reasons as in the proof of proposition 3.11, taking $\eta = N^{-3}$ we obtain a stable manifold for the sequence $(\psi_n)_{n\in\mathbb{N}}$ with size bounded from below by $\tilde{r}_0 > N^{-4+\frac{2}{5}}$, for N large enough. The projection of this stable manifold by the exponential map gives the stable manifold $W_g^-(g^{-1}(m))$ for g at the point $g^{-1}(m)$. By item 10 of lemma 5.1, this stable manifold has size bounded from below by $e^{-\beta}.N^{-4+\frac{2}{5}} > N^{-5}$. Thus $W_g^-(m) = g(W_g^-(g^{-1}(m)))$ has size bounded from below by $r_0 = N^{-7}$.

The stable manifold for the sequence (ψ_n) is tangent to the cone $\widetilde{\mathscr{C}}_{(\eta,0)}$ and at the origin is tangent to the direction $E^-_{g,m}$. By items 3, 7 and 8 of lemma 5.1, for any $q \in \mathbb{T}^4$

$$\left(Dg(q).(\widetilde{\mathscr{C}}_{2\eta,0})_1\right)_1 \subset \mathscr{C}_{e^{2\beta}8N^2\eta}((E^-_{g,m})_1, m), \tag{30}$$

where $(\widetilde{\mathscr{C}}_{2\eta,0})_1$ is identified with $(\widetilde{\mathscr{C}}_{2\eta,0})_1 \times \{0\}$.

The stable manifold $W_g^-(g^{-1}(m))$ at the point q is tangent to $Dexp^c_m((exp^c_m)^{-1}(q)).\widetilde{\mathscr{C}}_{(\eta,0)}$. If $\beta > 0$ is small enough, then $Dexp^c_m(p)$ is close to be the identity, for any $p \in B(0, \frac{1}{2})$. Thus $\left(T_qW_g^-(g^{-1}(m))\right)_1 \subset \left(\widetilde{\mathscr{C}}_{2\eta,0}\right)_1$. By (30), we obtain

$$\left(T_qW^-_{g,r_0}(m)\right)_1 \subset \mathscr{C}_{e^{2\beta}8N^2\eta}((E^-_{g,m})_1, q).$$

By lemma 5.4 and our choice of η, we conclude that

$$\left(T_qW^-_{g,r_0}(m)\right)_1 \subset \mathscr{C}^{ver}_{\frac{4}{\theta_1}}(q). \qquad \square$$

So far we have obtained the results analogous to section 3. Now we will obtain the results analogous to the results used in section 4 to obtain the ergodicity of f. The following is analogous to lemma 4.3.

Lemma 5.7. *For N large, $\mathcal{U}_N$ small and $n > 15$, if $\nu_{g,i}$ is an ergodic component of the Lebesgue measure, then for every $m \in X_g$ there are two curves $\gamma^-_{g,-n}(m) \subset g^{-n}(W^-_{g,r_0}(m))$ and $\gamma^+_{g,n}(m) \subset g^n(W^+_{g,r_0}(m))$ with length greater than 4π, such that $\left(T\gamma^-_{g,-n}(m)\right)_1 \subset \mathscr{C}^{ver}_{\theta_2}$ and $\left(T\gamma^+_{g,n}(m)\right)_1 \subset \mathscr{C}^{hor}_{\theta_2}$.*

Proof. The difference from the fibered case is the need to consider the projection π_1. For $m \in X_g$, it holds that $W^+_{g,r_0}(m) \subset G_2$. Define $W^+_{k,g}(m) = g^k(W^+_{g,r_0}(m))$. By lemma 5.6, $\left(TW^+_{g,r_0}(m)\right)_1 \subset \mathscr{C}^{hor}_{\frac{4}{\theta_1}}$ and by lemma 5.5, $\left(TW^+_{1,g}(m)\right)_1 \subset \mathscr{C}^{hor}_{\theta_2}$.

Construct in a similar way as in the proof of lemma 4.3 the number $k_0^+ \in \mathbb{N}$ and the curve $\gamma^+_{k_0^+,g}$. Since this curve must intersect ∂G_1 and ∂G_2, it has length $l(\pi_1(\gamma^+_{k_0^+,g})) \geq N^{-\frac{3}{10}}$ and $\pi_1(\gamma^+_{k_0^+,g})$ is tangent to $\mathscr{C}^{hor}_{\theta_2}$. By lemma 5.5, $l(g(\gamma^+_{k_0^+,g})) > 4\pi$ and $\pi_1(g(\gamma^+_{k_0^+,g}))$ is tangent to $\mathscr{C}^{hor}_{\theta_2}$. The rest of the proof is the same as the proof of lemma 4.3. $\square$

For $R > 0$, let

$$W^s_{g,R,-n}(m) = \bigcup_{q\in\gamma^-_{g,-n}(m)} W^{ss}_{g,R}(q),$$

where the curve $\gamma^-_{g,-n}(m)$ is the curve given by the previous lemma. Define similarly $W^u_{g,R,n}(m)$. For the same reason as we explained in remark 4.2, we obtain that $W^s_{g,R,-n}(m)$ and $W^u_{g,R,n}(m)$ are C^1-submanifolds. The next lemma is similar to lemma 4.4.

Lemma 5.8. *Fix $\theta_3 > 0$ such that $\theta_3 > \theta_2$ and satisfies $\mathscr{C}^{hor}_{\theta_3} \cap \mathscr{C}^{ver}_{\theta_3} = \{0\}$. For $g \in \mathcal{U}_N$, there exists $0 < R < 1$ such that if $n \geq 15$, $m \in X_g$ and $m^- \in W^s_{g,R,-n}(m) \subset W^s_{g,2,-n}(m)$, then*

$$\left(T(W^s_{g,2,-n}(m) \cap W^c_g(m^-))\right)_1 \subset \mathscr{C}^{ver}_{\theta_3}.$$

A similar result holds for $W^u_{g,R,n}(m)$.

The main difference for the non fibered case is given in the following proposition.

Proposition 5.9. *For N large and $\mathcal{U}_N$ small enough, if $g \in \mathcal{U}_N$ then for Lebesgue almost every point $m \in \mathbb{T}^4$ its central leaf $W^c_g(m)$ has dense orbit among the center leaves.*

PROOF. For $\mathcal{U}_N$ small enough, for every $g \in \mathcal{U}_N$ there is a homeomorphism $h_g : \mathbb{T}^4 \to \mathbb{T}^4$, that takes center leaves of f_N to center leaves of g, such that for every $m \in \mathbb{T}^4$ it is verified

$$g \circ h_g(W^c_f(m)) = h_g \circ f(W^c_f(m))$$

Consider the quotients $M_f = \mathbb{T}^4/\sim^c_f$ and $M_g = \mathbb{T}^4/\sim^c_g$, where $p \sim^c_* q$ if and only if $q \in W^c_*(p)$ for $* = f, g$. We denote $\pi_f : \mathbb{T}^4 \to M_f$ and $\pi_g : \mathbb{T}^4 \to M_g$ the respective projections. Observe that $M_f = \mathbb{T}^2$ and that the induced dynamics $\tilde{f} : M_f \to M_f$ of f is given by A^{2N}. Endow M_g with the distance d_g given by the Hausdorff distance on the center leaves, that is,

$$d_g(L, W) = d_{\text{Haus}}(\pi_g^{-1}(L), \pi_g^{-1}(W)).$$

By the leaf conjugacy equation, the induced dynamics $\tilde{g} : M_g \to M_g$ of g is conjugated to the linear Anosov A^{2N} on $\mathbb{T}^2$ by the homeomorphism induced by h_g, which we will denote by $\tilde{h}_g$. Denote by $W^s_{A^{2N}}(.)$ the stable manifold of A^{2N} on $\mathbb{T}^2$ and let

$$W^s_{\tilde{g}}(L) = \{W \in M_g : \lim_{n \to +\infty} d_g(\tilde{g}^n(L), \tilde{g}^n(W)) = 0\},$$

be the stable set of L.

Claim 2. *For every $m \in \mathbb{T}^4$, for every $q \in W^c_g(m)$, it is verified that*

$$\pi_g(W^{ss}_g(q)) = W^s_{\tilde{g}}(\pi_g(m)) = \tilde{h}_g(W^s_{A^{2N}}(\pi_f(h_g^{-1}(m)))),$$

and π_g is a bijection from $W^{ss}_g(q)$ to $W^s_{\tilde{g}}(\pi_g(m))$.

PROOF. The leaf conjugacy equation implies that $W^s_{\tilde{g}}(\pi_g(m)) = \tilde{h}_g(W^s_{A^{2N}}(\pi_f(h_g^{-1}(m))))$, in particular, $W^s_{\tilde{g}}(\pi_g(m))$ is a continuous curve homeomorphic to a line.

It is immediate to see that $\pi_g(W^{ss}_g(q)) \subset W^s_{\tilde{g}}(\pi_g(m))$. We also have that $W^{ss}_g(q) \cap W^c_g(q) = \{q\}$. Indeed, since the angle between E^c_g and E^{ss}_g is uniformly bounded away from zero and the center foliation is uniformly compact, the map $\pi_g|_{W^{ss}_{g,loc}(z)}$ is injective, for every $z \in \mathbb{T}^4$ and for some small uniform size of stable leaf which we write $W^{ss}_{loc}(z)$. If there were two points $\{p, q\} \subset W^{ss}_g(q) \cap W^c_g(q)$ then for n large enough $\{g^n(p), g^n(q)\} \subset W^{ss}_{g,loc}(g^n(q)) \cap W^c_g(g^n(q))$, which contradicts the fact that $\pi_g|_{W^{ss}_{g,loc}(q)}$ is injective. It remains to show the surjectivity.

We work inside $W^{cs}(m)$, which is foliated by strong stable manifolds. Take $P \in W^s_{\tilde{g}}(\pi_g(m))$ and consider its central leaf $F = \pi_g^{-1}(P)$. This is a transversal section of the C^1 foliation by strong stable manifolds inside the manifold $W^{cs}_g(m)$. Consider the set $L_{m,F} = \{z \in W^c_g(m) : W^{ss}_g(z) \cap F \neq \emptyset\}$.

Fix a small $\varepsilon > 0$. Since the angle between E^{ss}_g and E^c is uniformly bounded away from zero and the center foliation is uniformly compact, for any point $p \in \mathbb{T}^4$, it holds that

$$\mathcal{V}^s_g(p) := \bigcup_{q \in W^c_g(p)} W^{ss}_{g,\varepsilon}(q),$$

contains a neighborhood of $W^c_g(p)$ inside $W^{cs}_g(p)$ of uniform size, independent of p.

Since $P \in W^s_{\tilde{g}}(\pi_g(m))$, take n large enough such that $\pi_g^{-1}(\tilde{g}^n(P)) \cap \mathcal{V}^s_g(g^n(m)) \neq \emptyset$. Thus, there exists some $q_n \in W^c_g(g^n(m))$ such that $W^{ss}_{g,\varepsilon}(q_n) \cap \pi_g^{-1}(\tilde{g}^n(P)) \neq \emptyset$. We conclude that $W^{ss}_g(g^{-n}(q_n)) \cap F \neq \emptyset$, in particular, $L_{m,F} \neq \emptyset$.

If $\hat{p} \in L_{m,F}$ let $\gamma_{\hat{p},F}$ be a simple C^1 curve contained in $W^{ss}_g(\hat{p})$ connecting $\hat{p}$ and F, there is a foliated chart containing $\gamma_{\hat{p},F}$. Since F is transversal to the foliation, we have that there is an open neighborhood of $\hat{p}$ inside $W^c_g(m)$ such that the strong stable manifold of every point in this neighborhood intersects F, thus $L_{m,F}$ is open.

Since $W^c_g(m)$ and F are compact the distance, inside $W^{cs}_g(m)$, between them is smaller than a constant $\tilde{R} > 0$. Observe that the tangent spaces of stable manifolds are contained inside a cone, transverse to the central direction in $W^{cs}_g(m)$. Thus, for $\hat{p} \in L_{m,F}$, the length of the piece of $W^{ss}_g(\hat{p})$ starting in $\hat{p}$ and ending in F is bounded by a constant $C > 0$.

Let $(p_n)_{n\in\mathbb{N}} \subset L_{m,F}$ be a sequence such that $p_n \to p \in W^c_g(m)$. Consider $W^{ss}_{g,2C}(p)$ the strong stable manifold of size $2C$. Since compact parts of the strong stable manifold vary continuously with the point, $W^{ss}_{g,2C}(p_n)$ converges in the C^2-topology to $W^{ss}_{g,2C}(p)$. Take the sequence of points $(q_n)_{n\in\mathbb{N}}$ defined as $q_n \in W^{ss}_{g,2C}(p_n) \cap F$. Thus, $q_n \to q \in W^{ss}_{g,2C}(p)$ and since F is closed, $q \in F$. Therefore $q \in W^{ss}_{g,2C}(p) \cap F$ and $L_{m,F}$ is closed. Since $W^c_g(m)$ is connected, it follows that $L_{m,F} = W^c_g(m)$. □

For the linear Anosov A^{2N} the stable foliation is minimal. Let m be a generic point of an ergodic component $\nu_{g,i}$ of the Lebesgue measure for g, suppose also that m is a density point for the set $\Lambda_{g,i}$ defined at the beginning of this section. By absolute continuity of the strong stable foliation almost every point inside $W^{ss}_g(q)$ is in the ergodic component of m, for $q \in \Lambda_{g,i}$. Using the minimality of the stable foliation of the linear Anosov and by the leaf conjugacy $W^s_{\tilde{g}}(\pi_g(m))$ is dense in M_g.

Take U a small open set in M_g. Since the center foliation is uniformly compact, $\hat{U} = \pi_g^{-1}(U)$ is a saturated open set such that any two center leaves in $\hat{U}$ are C^2-close to each other. By the previous claim $W^{ss}_g(m) \cap \hat{U} \neq \emptyset$.

Let $B(m, \varepsilon)$ be a small ball around m such that $Leb(B(m, \varepsilon) \cap \Lambda_{g,i})$ has almost full measure inside $B(m, \varepsilon)$. By absolute continuity

$$Leb(W^{ss}_g(B(m, \delta) \cap \Lambda_{g,i}) \cap \hat{U} \cap \Lambda_{g,i}) > 0.$$

In particular $\nu_{g,i}(\Lambda_{g,i} \cap \hat{U}) > 0$. Since m is a generic point for $\nu_{g,i}$, its future orbit visits $\hat{U}$ infinitely many times. This is true for any open set U inside M_g, which concludes the proof of the proposition. □

Now let N be large and $\mathcal{U}_N$ be small enough such that lemmas 5.7, 5.8 and proposition 5.9 hold. For $g \in \mathcal{U}_N$, if g is not ergodic, we can follow the exact same steps as in the proof of ergodicity of f and find a contradiction. We conclude that every $g \in \mathcal{U}_N$ is ergodic.

6. The Bernoulli property

In this section we explain how to adapt the proof of ergodicity to obtain the Bernoulli property. Let $f = f_N$ for N large enough. By theorem 2.17, since the Lebesgue measure is ergodic for f, there exists $k \in \mathbb{N}$ and probability measures $\nu_1, \cdots, \nu_k$, which are f^k-invariant, such that

$$Leb = \frac{1}{k} \sum_{j=1}^{k} \nu_i,$$

where each (f^k, ν_i) is Bernoulli. Suppose $k > 1$. The measures $\{\nu_i\}_{i=1}^k$ form the ergodic decomposition of the Lebesgue measure for f^k. As we stated in remark 4.6, three properties imply the existence of transverse intersections between Pesin's manifolds of points in different ergodic components.

Observe that $f^{-k}(X^s) \subset X^s$, where we defined the set X^s in item 1 of remark 4.6. Similarly $f^k(X^u) \subset X^u$. Thus, item 1 of remark 4.6 is valid for f^k.

Once we have the curves obtained in item 1 of remark 4.6 and since a stable manifold for f is a stable manifold for f^k, using the control on the holonomies given by lemma 4.4 we obtain item 2 of remark 4.6.

To obtain item 3 of remark 4.6 we need the following lemma.

Lemma 6.1. *There is a set of full measure $D \subset \mathbb{T}^4$ such that for every $p \in D$ the f^k-orbit of $W^c(p)$ is dense among the center leaves.*

PROOF. The linear Anosov A^{2N} is totally ergodic, that is, for any $j \in \mathbb{N}$, A^{2Nj} is ergodic. In particular A^{2Nk} is ergodic. The proof is the analogous to the proof of lemma 4.5. □

Following the same steps of the proof of ergodicity for f, which is just Hopf argument in the non-uniformly hyperbolic scenario, we conclude that f^k is ergodic. This is a contradiction, since the ergodic decomposition of the Lebesgue measure is given by the measures $\{\nu_i\}_{i=1}^k$ and $k > 1$. Thus $k = 1$. In particular f is Bernoulli.

For $g \in \mathcal{U}_N$ to prove that g is Bernoulli one follows the same steps as in the proof that f is Bernoulli. Observe that the stable and unstable foliations of A^{2Nj} are minimal, for any $j \in \mathbb{N}$. With this observation one easily proves a lemma analogous to lemma 5.9.

7. Estimative of center Lyapunov exponents

The goal of this section is to prove proposition 1.4. To prove this proposition, we follow and adapt the proof of theorem 1.2 given by Berger and Carrasco in [BC14] with the necessary changes. For a C^1-curve γ and a measurable set $A \subset \gamma$, write $Leb(A)$ the measure of A with respect to the Lebesgue measure in γ induced by the metric of $\mathbb{T}^4$. Also denote $f = f_N$. In this section we will refer to the strong unstable manifold by unstable manifold.

7.1. The estimate for f_N. The goal of this section is to prove the estimate given by proposition 1.4 for f.

Recall that we denoted $e^u = (e_1^u, e_2^u) \in \mathbb{R}^2$ an unit eigenvector of A for the eigenvalue $1 < \mu = \lambda^{-1}$, where $\lambda \in (0,1)$ is the eigenvalue for the contractive direction of A. Recall also that we defined the linear map $P_x : \mathbb{R}^2 \to \mathbb{R}^2$ given by $P_x(a,b) = (a,0)$.

Lemma 7.1 ([BC14], Proposition 1)**.** *There is a differentiable function $\alpha : \mathbb{T}^4 \to \mathbb{R}^2$ such that the unstable direction of f is generated by the vector field $(\alpha(m), e^u)$, where*

$$Df(m).(\alpha(m), e^u) = \mu^{2N}(\alpha(f(m)), e^u) \text{ and } \|\alpha(m) - \lambda^N P_x(e^u)\| \leq \lambda^{2N}.$$

Definition 7.2. *A u-curve is a C^1-curve $\gamma : [0, 2\pi] \to M$ such that $\dfrac{d\gamma}{dt}(t) = \dfrac{(\alpha(\gamma(t)), e^u)}{\lambda^N \|P_x(e^u)\|}$, for every $t \in [0, 2\pi]$.*

Observe that for a u-curve γ

$$\frac{df^k \circ \gamma}{dt}(t) = \frac{\mu^{2Nk}(\alpha(f^k(\gamma(t))), e^u)}{\lambda^N \|P_x(e^u)\|}, \ \forall t \in [0, 2\pi] \text{ and } \forall k \geq 0. \tag{31}$$

The u-curves will play a fundamental role in the proof. The key property of a u-curve is that $\|\alpha(\gamma(t)).(\lambda^N \|P_x(e^u)\|)^{-1} - (1,0)\| \leq \lambda^{2N}$. This will allow us to control the amount of time that a u-curve spend in a critical region, which is a region on $\mathbb{T}^4$ that only depends on the x coordinate.

Since we are interested in Lyapunov exponents along the center direction we will introduce certain types of vector fields along u-curves that will be useful in this task. After that we will be ready to give a criteria to obtain large positive Lyapunov exponents along the center direction for almost every point in $\mathbb{T}^4$.

Definition 7.3. *An adapted field (γ, X) over a u-curve γ is an unitary vector field X such that*

(1) X is tangent to the center direction;

(2) X is $(C_X, 1/2)$-Hölder along γ, that is

$$\|X_m - X_{m'}\| \leq C_X d_\gamma(m, m')^{\frac{1}{2}}, \ \forall m, m' \in \gamma,$$

where $C_X < 30N^2\lambda^N$ and d_γ is the distance measured along γ.

Remark 7.4. *The estimate on the Hölder constant used in [BC14] is $20N^2\lambda^N$, instead of $30N^2\lambda^N$ as above. This is due to the fact that the parametrization of the torus $\mathbb{T}^4$ is by intervals of length 2π instead of 1 in the proof of lemma 2 in [BC14]. However, this change on the estimate of the Hölder constant does not affect the rest of the proof.*

Berger and Carrasco proved that for N large enough and for every (γ, X) adapted field

$$\|X_m - X_{m'}\| \leq \lambda^{N/3}, \ \forall m, m' \in \gamma.$$

Fix an adapted field X and denote by $X^k = \frac{(f^k)_* X}{\|(f^k)_* X\|}$, where

$$\left((f^k)_* X\right)_m = Df^k(f^{-k}(m)).X_{f^{-k}(m)}.$$

Lemma 7.5 ([BC14], Lemma 2)**.** *For N large enough, for every adapted field (γ, X), for every $k \geq 0$ and every $1 \leq j \leq [\mu^{2Nk}]$, the pair $(\gamma_j^k, X^k|_{\gamma_j^k})$ is an adapted field.*

We prove this lemma here, just to justify the change on the Hölder constant mentioned before in remark 7.4.

Proof. Let (γ, X) be an adapted field. We have

$$\begin{aligned}&\|Df(m)X_m - Df(m')X_{m'}\| \leq \\ &\|Df(m)X_m - Df(m)X_{m'}\| + \|Df(m)X_{m'} - Df(m')X_{m'}\| = I + II.\end{aligned}$$

Since X is $(C_X, \frac{1}{2})$-Hölder, we obtain

$$I \leq 2N\|X_m - X_{m'}\| \leq 2NC_X d(m, m')^{\frac{1}{2}}.$$

If $d(m, m') \leq 1$, then

$$II \leq Nd(m, m') < Nd(m, m')^{\frac{1}{2}} < 7Nd(m, m')^{\frac{1}{2}}.$$

Observe that $d(m, m') \leq 2\pi < 7$, for any two points $m, m' \in \mathbb{T}^4$. If $d(m, m') > 1$, then

$$II \leq Nd(m, m') \leq 7N < 7Nd(m, m')^{\frac{1}{2}}.$$

We conclude that

$$\|Df(m)X_m - Df(m')X_{m'}\| < (7N + 2NC_X)d(m, m')^{\frac{1}{2}}. \tag{32}$$

Also,

$$\begin{aligned}\|(X^1)_m - (X^1)_{m'}\| &= \frac{1}{\|f_*X_m\|\|f_*X_{m'}\|}\,\|\|f_*X_{m'}\|f_*X_m - \|f_*X_m\|f_*X_{m'}\| \\ &\leq \frac{1}{\|f_*X_m\|\|f_*X_{m'}\|}\,(\|\|f_*X_{m'}\|f_*X_m - \|f_*X_{m'}\|f_*X_{m'}\| \\ &\quad + \|\|f_*X_{m'}\|f_*X_{m'} - \|f_*X_m\|f_*X_{m'}\|) \\ &\leq \frac{2}{\|f_*X_m\|}\|f_*X_m - f_*X_{m'}\| \\ &= \frac{2}{\|f_*X_m\|}\|Df(f^{-1}(m))X_{f^{-1}(m)} - Df(f^{-1}(m'))X_{f^{-1}(m')}\|.\end{aligned}$$

Using (91) for the points $f^{-1}(m)$ and $f^{-1}(m')$, we have

$$\|Df(f^{-1}(m))X_{f^{-1}(m)} - Df(f^{-1}(m'))X_{f^{-1}(m')}\| \leq N\lambda^N(1 + \lambda^N)(7 + 2C_X)d(m, m')^{\frac{1}{2}}.$$

Recall that $\|Df|_{E_f^c}\| \geq (2N)^{-1}$, hence

$$\|X^1(m) - X^1(m')\| \leq \frac{2N\lambda^N(1 + \lambda^N)(7 + 2C_X)d(m, m')^{\frac{1}{2}}}{\|f_*X(m)\|} \leq 4(1+\lambda^N)N^2\lambda^N(7+2C_X)d(m, m')^{\frac{1}{2}}.$$

Observe that $C_{X^1} := 4(1+\lambda^N)N^2\lambda^N(7+2C_X)$ estimates the Hölder constant of X^1. Since $C_X < 30N^2\lambda^N$, for N large enough

$$C_{X^1} = 4(1+\lambda^N)N^2\lambda^N(7+2C_0) \leq 4(1+\lambda^N)N^2\lambda^N(7.1) < 30N^2\lambda^N.$$

□

Denote by $d\gamma$ the Lebesgue measure induced on γ and by $|\gamma|$ the length of γ. Define

$$I_n^{\gamma,X} := \frac{1}{|\gamma|}\int_\gamma \log\|Df^n.X\|d\gamma.$$

Now we prove the following criteria to obtain positive Lyapunov exponents along the center direction.

Proposition 7.6. *Suppose that there exists $C > 0$ such that for every u-curve γ there is an adapted vector field X which satisfies for n large enough*

$$\frac{I_n^{\gamma,X}}{n} > C.$$

Then Lebesgue almost every point in $\mathbb{T}^4$ has a Lyapunov exponent along the central direction which is larger than $(1-2\lambda^{2N})C$.

Proof. We will prove that for every $\rho > 0$, for almost every point there is a Lyapunov exponent greater than $(1-2\lambda^{2N}-\rho)C$ in the center direction. Suppose not, then there is a set with positive measure B such that every point in this set does not have a Lyapunov exponent greater than $(1-2\lambda^{2N}-\rho)C$. Since the unstable foliation is absolutely continous there is an unstable manifold L^u that intersects B in a subset with positive Lebesgue measure inside L^u. Let $q \in L^u$ be a Lebesgue density point of $L^u \cap B$.

Let $r_k = 2\pi\lambda^{2Nk}$ and let $\gamma_{r_k} : [-r_k, r_k] \to M$ to be a piece of u-curve such that $\gamma_{r_k}(0) = q$, since q is a density point then

$$\frac{Leb(\gamma_{r_k} \cap B)}{Leb(\gamma_{r_k})} \to 1.$$

Take $\beta < \rho$ and let k be large enough such that $Leb(\gamma_{r_k} \cap B^c) < \frac{\beta C}{\log 2N} Leb(\gamma_{r_k})$. Observe that $f^k \circ \gamma_{r_k}$ is a u-curve, let X_{r_k} be the vector field over γ_{r_k}, such that $(f^k \circ \gamma_{r_k}, (f^k)_* X_{r_k})$ satisfies the hypothesis of the lemma. Let

$$\chi(m) = \limsup_{n\to\infty} \frac{\log\|Df^n(m).X_{r_k}\|}{n},$$

thus for every $m \in B$, $\chi(m) < (1-2\lambda^{2N}-\rho)C$. From (31) and lemma 3.3, for N large enough we obtain

$$\frac{1}{\left\|\frac{d(f^k\circ\gamma_{r_k})}{dt}\right\|} \geq \frac{1-2\lambda^{2N}}{\mu^{2Nk}}.$$

In particular,

$$\begin{aligned}
\int_{\gamma_{r_k}} \chi d\gamma_{r_k} &= \int_{f^k\circ\gamma_{r_k}} \chi\circ f^{-k} \frac{1}{\left\|\frac{d(f^k\circ\gamma_{r_k})}{dt}\right\|} d(f^k\circ\gamma_{r_k}) \\
&\geq \frac{1-2\lambda^{2N}}{\mu^{2Nk}} \int_{f^k\circ\gamma_{r_k}} \chi\circ f^{-k} d(f^k\circ\gamma_{r_k}) \\
&= \lambda^{2Nk}(1-2\lambda^{2N}) \limsup_{n\to+\infty} \int_{f^k\circ\gamma_{r_k}} \frac{\log\|Df^n(m).(f^k)_*X_{r_k}\|}{n} d(f^k\circ\gamma_{r_k}) \\
&\geq \lambda^{2Nk}(1-2\lambda^{2N})|f^k\circ\gamma_{r_k}|C > (1-2\lambda^{2N})C|\gamma_{r_k}|.
\end{aligned}$$

On the other hand

$$\begin{aligned}\int_{\gamma_{r_k}} \chi d\gamma_{r_k} &= \int_{\gamma_{r_k}\cap B} \chi d\gamma_{r_k} + \int_{\gamma_{r_k}\cap B^c} \chi d\gamma_{r_k} \\ &\leq (1-2\lambda^{2N}-\rho)C|\gamma_{r_k}| + \log 2N.C.\beta(\log 2N)^{-1}|\gamma_{r_k}| \\ &= (1-2\lambda^{2N}-\rho+\beta)C|\gamma_{r_k}| < (1-2\lambda^{2N})C|\gamma_{r_k}|\end{aligned}$$

which is a contradiction. Since it holds for every $\rho > 0$, one concludes the proof of the proposition. □

We can represent the curve $f^k\circ\gamma$ as the concatenation $f^k\circ\gamma = \gamma_1^k * \cdots * \gamma_{[\mu^{2Nk}]}^k * \gamma_{[\mu^{2Nk}]+1}^k$, where γ_i^k is a u-curve for every $1 \leq i \leq [\mu^{2Nk}]$, $\gamma_{[\mu^{2Nk}]+1}^k$ is a piece of a u-curve, $[.]$ denotes the integer part of a number and $*$ denotes the concatenation between the curves. Berger and Carrasco proved the following formula, see section 3 of [BC14].

Lemma 7.7. *For every adapted field (γ, X) and $n \in \mathbb{N}$, for each $k = 0, \cdots, n-1$ there exists a number $\beta_k \in [-2\lambda^{2N}, 2\lambda^{2N}]$ such that*

$$\begin{aligned}I_n^{\gamma,X} &= \frac{1}{|\gamma|}\int_\gamma \log\|Df^n.X\|d\gamma \\ &= \sum_{k=0}^{n-1}\frac{1+\beta_k}{\mu^{2Nk}|\gamma|}\left(\sum_{j=1}^{[\mu^{2Nk}]}\int_{\gamma_j^k}\log\|Df.X^k\|d\gamma_j^k + \int_{\gamma_{[\mu^{2Nk}]+1}^k}\log\|Df.X^k\|d\gamma_{[\mu^{2Nk}]+1}\right),\end{aligned}$$

where $\beta_k \in [-2\lambda^{2N}, 2\lambda^{2N}]$.

This formula will allow us to study the growth of $I_n^{\gamma,X}$ by studying the pieces $\int_{\gamma_j^k}\log\|Df.X^k\|d\gamma_j^k$. In order to analyze these pieces we will define the notion of "good" and "bad" pieces. The estimate on the growth of $I_n^{\gamma,X}$ will come from an induction on n and a combinatorial argument, to estimate the number of "good" and "bad" pieces that appears in this formula.

Fix $\tilde{\delta} > 0$ small, the number N will be chosen after in function of $\tilde{\delta}$. Let

$$E(\gamma, X) = I_1^{\gamma,X} = \frac{1}{|\gamma|}\int_\gamma \log\|Df(m).X_m\|d\gamma.$$

Recall that for $m = (x, y, z, w) \in \mathbb{T}^4$, we defined $\Omega(m) = N\cos(x) + 2$. Define $v_m = (1, \Omega(m))$ and $u_m = (\Omega(m), -1)$. They form an orthogonal basis of the center direction. Let X be an unit vector field tangent to the center direction, thus using this basis we have

$$X_m = \frac{\cos(\theta_X(m))}{\sqrt{1+\Omega(m)^2}}v_m + \frac{\sin(\theta_X(m))}{\sqrt{1+\Omega(m)^2}}u_m.$$

Where $\theta_X(m)$ is the angle that X_m makes with v_m. Using the basis (v_m, u_m) the derivative can be written as

$$Df(m).X_m = \left(\sin(\theta_X(m)).\sqrt{1+\Omega(m)^2}, \frac{\cos(\theta_X(m)) + \sin(\theta_X(m)).\Omega(m)}{\sqrt{1+(\Omega(m))^2}}\right),$$

then

$$\|Df(m).X_m\| \geq |\sin(\theta_X(m))|.\sqrt{1+\Omega(m)^2} \geq |\sin(\theta_X(m))|.|\Omega(m)|.$$

If N is large enough and if $|x-\pi/2| > 2.N^{-\tilde{\delta}}$ and $|x-3\pi/2| \geq 2.N^{-\tilde{\delta}}$ then $|\cos(x)| \geq N^{-\tilde{\delta}}$.

Define the critical strip as

$$Crit = \left\{(x,y,z,w) \in \mathbb{T}^4 : |x-\pi/2| < 2.N^{-\tilde{\delta}} \text{ or } |x-3\pi/2| < 2.N^{-\tilde{\delta}}\right\},$$

thus the length of the projection of the critical strip on the first coordinate is $l(Crit) < 8.N^{-\tilde{\delta}}$, which converges to zero as N goes to infitiny.

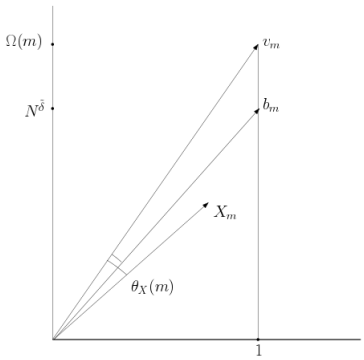

FIGURE 3. The triangle formed by 0, b_m and v_m

Lemma 7.8. *For N large enough, if $m \notin Crit$ then $|\Omega(m)| \geq N^{1-2\tilde\delta}$ and $\|Df(m).X_m\| \geq N^{1-2\tilde\delta}.|\sin(\theta_X(m))|$.*

The proof is straightforward with the fact that if $m \notin Crit$ then $|\cos(x)| \geq N^{-\tilde\delta}$.

Definition 7.9. *Consider the cone $\Delta_{\tilde\delta} = \{(u,v) \in \mathbb{R}^2 : N^{\tilde\delta}|u| \geq |v|\}$. If an adapted vector field (γ, X) is tangent to this cone we say that it is a $\tilde\delta$-**good** adapted vector field. Otherwise we say that it is $\tilde\delta$-**bad**.*

Lemma 7.10. *For N sufficiently large and for every $\tilde\delta$-good adapted vector field (γ, X)*

$$|\sin(\theta_X(m))| > N^{-4\tilde\delta} \quad \forall m \notin Crit.$$

Furthermore, for a $\tilde\delta$-good adapted field (γ, X), if $m \notin Crit$ then $\|Df(m).X_m\| \geq N^{1-6\tilde\delta}$

PROOF. Recall that $v_m = (1, \Omega(m))$ and suppose that $\Omega(m) > 0$. Let $b_m = (1, N^{\tilde\delta})$ and consider the triangle formed by the points 0, b_m and v_m, see figure 3. Denote by $\measuredangle(u,v)$ the angle between two vectors $u, v \in \mathbb{R}^2$. By the law of sines

$$\frac{\sin(\measuredangle(v_m, b_m))}{\|v_m - b_m\|} = \frac{\sin(\measuredangle(v_m - b_m, b_m))}{\|v_m\|}. \tag{33}$$

For a good adapted field (γ, X), it holds $|\sin(\theta_X(m))| \geq |\sin(\measuredangle(v_m, b_m))|$. Recall that $m \notin Crit$, by lemma 7.8 we have $N^{1-2\tilde\delta} \leq |\Omega(m)| \leq N$. Observe that

$$\sin(\measuredangle(v_m - b_m, b_m)) = \frac{1}{\|b_m\|}. \tag{34}$$

By (33) and (34), for N large enough we obtain

$$|\sin(\theta_X(m))| \geq \frac{|\Omega(m)| - N^{\tilde\delta}}{\sqrt{1+N^{2\tilde\delta}}.\sqrt{1+\Omega(m)^2}} \geq N^{-4\tilde\delta}.$$

It follows from this inequality and lemma 7.8, that for a $\tilde\delta$-good adapted field (γ, X), if $m \notin Crit$ then $\|Df(m).X_m\| \geq N^{1-6\tilde\delta}$. If $\Omega(m) < 0$ we can obtain the same estimate taking $b_m = (1, -N^{\tilde\delta})$. □

Proposition 7.11. *For N sufficiently large if (γ, X) is a $\tilde\delta$-good adapted vector field then $E(\gamma, X) \geq (1 - 7\tilde\delta)\log N$.*

PROOF. Recall that for a u-curve $\frac{d\gamma}{dt}(t) = \frac{(\alpha(\gamma(t)), e^u)}{\lambda^N \|P_x(e^u)\|}$ and $\|\alpha(\gamma(t)).(\lambda^N \|P_x(e^u)\|)^{-1} - (1,0)\| \leq \lambda^{2N}$. In particular, using that $l(Crit) \leq 8N^{-\tilde\delta}$, for N large enough the measure of $\gamma \cap Crit$ is smaller than $10N^{-\tilde\delta}|\gamma|$.

The previous lemma give us an estimate for points outside the critical strip. For points inside the critical strips we use that $\|Df|_{E^c}\| \geq (2N)^{-1}$. Thus for N large enough we get

$$\begin{aligned} |\gamma| E(\gamma, X) &= \int_{\gamma \cap Crit} \log \|Df(m).X_m\| d\gamma + \int_{\gamma \cap Crit^c} \log \|Df(m).X_m\| d\gamma \\ &\geq \left(1 - \tfrac{10}{N^{\tilde{\delta}}}\right).(1 - 6\tilde{\delta}) \log N|\gamma| - \left(\tfrac{10}{N^{\tilde{\delta}}}\right). \log 2N|\gamma| \geq (1 - 7\tilde{\delta}) \log N|\gamma|. \end{aligned}$$

□

Recall that $f^k \circ \gamma = \gamma_1^k * \cdots * \gamma_{[\mu^{2Nk}]}^k * \gamma_{[\mu^{2Nk}]+1}^k$ and define

$$G_k = G_k(\gamma, X) = \left\{ 1 \leq j \leq [\mu^{2Nk}] : \left(\gamma_j^k, \frac{f_*^k X}{\|f_*^k X\|}\right) \text{ is } \tilde{\delta}\text{-good.} \right\}, \tag{35}$$

$$B_k = B_k(\gamma, X) = \left\{ 1 \leq j \leq [\mu^{2Nk}] : \left(\gamma_j^k, \frac{f_*^k X}{\|f_*^k X\|}\right) \text{ is } \tilde{\delta}\text{-bad.} \right\}. \tag{36}$$

Lemma 7.12. *For N sufficiently large, if (γ, X) is a $\tilde{\delta}$-good adapted field and $f^{-1}(\gamma_j^1) \cap Crit = \emptyset$, then $(\gamma_j^1, \frac{f_* X}{\|f_* X\|})$ is also $\tilde{\delta}$-good.*

Proof. Let $m \notin Crit$ and $v \in (-N^{\tilde{\delta}}, N^{\tilde{\delta}})$. It is verified

$$Df(m).(1, v) = (\Omega(m) - v, 1).$$

By lemma 7.10

$$|\Omega(m) - v| \geq |\Omega(m)| - |v| \geq N^{1-2\tilde{\delta}} - N^{\tilde{\delta}},$$

which is arbitrarily large as N grows. This implies that the vector $(\Omega(m) - v, 1)$ is inside the cone $\Delta_{\tilde{\delta}}$, because it will be very close to the x axis. □

The next lemma is the same as lemma 6 in [BC14].

Lemma 7.13. *For N sufficiently large, for every $\tilde{\delta}$-bad adapted vector field, there is a strip S_X of length π such that if $f^{-1}(\gamma_j^1) \subset S_X$ then $(\gamma_j^1, \frac{f_* X}{\|f_* X\|})$ is $\tilde{\delta}$-good.*

Let $\eta_N = \frac{5}{\pi N^{\tilde{\delta}}}$. The following proposition is analoguous to proposition 4 in [BC14].

Proposition 7.14. *For N large enough, for every $\tilde{\delta}$-bad adapted field*

$$\#G_1 \geq \tfrac{1}{3}\mu^{2N} \quad \text{and} \quad \#B_1 \leq \tfrac{2}{3}\mu^{2N}.$$

For every $\tilde{\delta}$-good adapted field

$$\#G_1 \geq (1 - \eta_N)\, \mu^{2N} \quad \text{and} \quad \#B_1 \leq (\eta_N)\, \mu^{2N}.$$

Proof. Using lemma 7.13 there is a strip S_X of length π such that if $f^{-1}(\gamma_j^1) \subset S_X$, this represents almost half of the pieces γ_j^1, for N large enough we conclude the first part of the proposition. For the second part we use lemma 7.12 and the fact that $l(Crit) \leq 8N^{-\tilde{\delta}}$ and by a similar argument, for N large, it holds the second part of the proposition. □

Now in general for any $k \in \mathbb{N}$,

$$\begin{aligned} \#G_{k+1} &\geq (1 - \eta_N)\mu^{2N} \#G_k + \tfrac{1}{3}\mu^{2N} \#B_k \\ \#B_{k+1} &\leq \eta_N \mu^{2N} \#G_k + \tfrac{2}{3}\mu^{2N} \#B_k \end{aligned}$$

Lemma 7.15. *For any $K \geq 1$, if N is large enough then for any $k \geq 0$ and any $\tilde{\delta}$-good adapted vector field (γ, X), it is verified $\#G_k \geq K.\#B_k$.*

PROOF. Since (γ, X) is $\tilde{\delta}$-good then $B_0 = 0$ and $\#G_0 = 1 > K.\#B_0$. By our previous remark if N is large enough then it is also valid for $k = 1$, let us suppose that it is valid for k and prove it for $k+1$.

$$\frac{\#B_{k+1}}{\#G_{k+1}} \leq \frac{\eta_N \mu^{2N} \#G_k + \frac{2}{3}\mu^{2N}\#B_k}{(1-\eta_N)\mu^{2N}\#G_k + \frac{1}{3}\mu^{2N}\#B_k} \leq \frac{\eta_N \mu^{2N}\#G_k + \frac{2}{3}\mu^{2N}K^{-1}\#G_k}{(1-\eta_N)\mu^{2N}\#G_k}$$
$$= \frac{\eta_N + \frac{2}{3}K^{-1}}{1-\eta_N} < \frac{3}{4K}.$$

Where the last inequality holds for N large. Thus $\#G_{k+1} > \frac{4K}{3}\#B_{k+1} > K\#B_{k+1}$. □

Now we can get the estimate on the Lyapunov exponent that we wanted.

Lemma 7.16. *For N large enough and for every $\tilde{\delta}$-good adapted vector field (γ, X) and for every large enough $n \geq 1$ we have*

$$\frac{I_n^{\gamma,X}}{n} \geq (1 - 10\tilde{\delta}) \log N.$$

PROOF. Fix $K > 0$ large enough such that $K^{-1} < \tilde{\delta}$. Let (γ, X) be a $\tilde{\delta}$-good adapted vector field, by the previous lemma $\#G_k > \frac{1}{1+K^{-1}}\mu^{2Nk}$. Using the formula given by lemma 7.7, the estimate obtained for $\tilde{\delta}$-good adapted vector field in proposition 7.11 and for every $\tilde{\delta}$-bad adapted vector field using that $\|Df|_{E^c}\| \geq (2N)^{-1}$, we conclude

$$\begin{aligned}
\tfrac{I_n^{\gamma,X}}{n} &\geq \frac{1}{n}\sum_{k=0}^{n-1} \frac{(1-2\lambda^{2N})}{\mu^{2Nk}} \left(\#G_k.(1-7\tilde{\delta})\log N - (\#B_k + 1)\log 2N \right) \\
&\geq \frac{1}{n}\sum_{k=0}^{n-1} (1-2\lambda^{2N}) \left(\frac{1}{1+K^{-1}}.(1-7\tilde{\delta})\log N - K^{-1}\log 2 - K^{-1}\log N - \frac{\log 2N}{\mu^{2Nk}} \right) \\
&\geq (1 - 10\tilde{\delta}) \log N
\end{aligned}$$

For N large enough. □

With this lemma we can prove the estimate of proposition 1.4 for f_N.

Corollary 7.17. *For $\delta > 0$, if N is large enough then almost every point has a Lyapunov exponent on the center direction greater than $(1-\delta)\log N$ for f_N.*

PROOF. Take $\tilde{\delta} = \frac{\delta}{30}$ and let N be large enough such that the previous lemma holds. Thus we can take $C = (1 - 10\tilde{\delta})\log N = (1 - \frac{\delta}{3})\log N$, where C is the constant from proposition 7.6. Assume that N is large enough such that $(1-2\lambda^{2N})(1-\frac{\delta}{3}) > (1-\delta)$. The result follows from proposition 7.6. □

7.2. Robustness of the estimate. In this section we prove proposition 1.4. For a C^1-curve γ we will denote by Leb_γ the Lebesgue measure induced by the Riemaniann metric in the curve. Recall that for each $N \in \mathbb{N}$ we denote by $\mathcal{U}_N \subset \mathrm{Diff}^2_{Leb}(\mathbb{T}^4)$ a C^2-neighborhood of f_N.

Lemma 7.18. *For $\varepsilon_1 > 0$ small, if N is large and $\mathcal{U}_N$ is small enough then for every $g \in U$ and for all unit vectors $v^s \in E^{ss}_g$, $v^c \in E^c_g$ and $v^u \in E^{uu}_g$, the following holds:*

(1) $e^{-\varepsilon_1}\lambda^{2N} \leq \|Dg(v^s)\| \leq e^{\varepsilon_1}\lambda^{2N}$;
(2) $e^{-\varepsilon_1}\mu^{2N} \leq \|Dg(v^u)\| \leq e^{\varepsilon_1}\mu^{2N}$;
(3) $\frac{1}{2N} \leq \|Dg(v^c)\| \leq 2N$;
(4) $\|D^2 g^{-1}\| \leq 2N$ *and* $\|D^2 g\| \leq 2N$;

(5) E^c_g *is* $\frac{1}{2}$*-Hölder.*

PROOF. The only statement that does not follow directly from C^2-continuity for N large enough is (5). Observe that

$$e^{\varepsilon_1}\lambda^{2N} < (2N)^{-1}e^{-\frac{\varepsilon_1}{2}}\lambda^N.$$

Hence, by theorem 2.7 it follows that E^c_g is $\frac{1}{2}$-Hölder. □

Definition 7.19. *A u-curve for g is a* C^1*-curve* $\gamma = (\gamma_x, \gamma_y, \gamma_z, \gamma_w) : [0, 2\pi] \to M$ *tangent to* E^u_g *and such that* $\left|\frac{d\gamma_x}{dt}(t)\right| = 1$, $\forall t \in [0, 2\pi]$. *For every* $k \geq 0$ *there exists an integer* $N_k = N_k(\gamma) \in \left[[(e^{-\varepsilon_1}\mu^{2N})^k], [(e^{-\varepsilon_1}\mu^{2N})^k]\right]$ *such that the curve* $g^k \circ \gamma$ *can be writen as*

$$g^k \circ \gamma = \gamma^k_1 * \cdots * \gamma_{N_k} * \gamma^k_{N_k+1}$$

where γ^k_j *for* $j = 1, \cdots, N_k$*, are u-curves and* $\gamma^k_{N_k+1}$ *is a segment of u-curve.*

Observe that this definition of an u-curve is different from the one given in definition 7.2. The advantage of definition 7.2 is that during the calculations we do not have to deal with bounded distortion estimates. Since for the general case it is natural to appear bounded distortion estimates, see lemma 3.8, we just normalize the curve on the x-direction in the previous definition.

Lemma 7.20 ([BC14], Corollary 5)**.** *For* $\varepsilon_2 > 0$ *small, if* N *is large and* $\mathcal{U}_N$ *is small enough then for every* $g \in \mathcal{U}_N$ *and any unit vector* $v^u \in E^{uu}_{g,m}$*, it holds that*

$$|P_x(D\pi_1.v^u)| \in [(\lambda^N(\|P_x(e^u) - 3\lambda^N\|), (\lambda^N(\|P_x(e^u) + 3\lambda^N\|)].$$

In particular, any two u-curves (γ, γ') *satisfy:*

$$e^{-\varepsilon_2}l(\gamma) \leq l(\gamma') \leq e^{\varepsilon_2}l(\gamma).$$

Define similarly as in definition 7.3 an adapted field (γ, X). Also define the unstable jacobian of g^k as

$$J^{uu}_{g^k}(m) = |\det Dg^k(m)|_{E^{uu}_g}|, \ \forall m \in \mathbb{T}^4.$$

By item 2 of lemma 7.18, for $g \in \mathcal{U}_N$ and for every $m \in \mathbb{T}^4$

$$e^{-\varepsilon_1}\lambda^{2N} \leq J^{uu}_{g^{-1}}(m) \leq e^{\varepsilon_1}\lambda^{2N}.$$

The proof of the next lemma is classical and can be found in [BC14], lemma 8.

Lemma 7.21 (Bounded distortion)**.** *For* $\varepsilon_3 > 0$ *small, if* N *is large and* $\mathcal{U}_N$ *is small enough, for every* $g \in \mathcal{U}_N$ *and any u-curve* γ *for* g*, for every* $k \geq 0$*, it holds*

$$\forall m, m' \in \gamma, \ e^{-\varepsilon_3} \leq \frac{J^{uu}_{g^{-k}}(m)}{J^{uu}_{g^{-k}}(m')} \leq e^{\varepsilon_3}.$$

This lemma implies that for $g \in \mathcal{U}_N$ and for any u-curve γ for g, if $A \subset \gamma$ is any measurable set, for every $k \geq 0$, it holds

$$e^{-\varepsilon_3}\frac{Leb(A)}{Leb(\gamma)} \geq \frac{Leb(g^{-k}(A))}{Leb(g^{-k}(\gamma))} \leq e^{\varepsilon_3}\frac{Leb(A)}{Leb(\gamma)}.$$

Let (γ, X) be an adapted field, define

$$I^{\gamma,X}_n = \frac{1}{|\gamma|}\int_\gamma \log\|Dg^n.X\|d\gamma.$$

For the fibered case, proposition 7.6 gives us precise estimates for the Lyapunov exponent along the center direction. In the general case we have the following proposition.

Proposition 7.22. *Suppose that there exists $C > 0$ with the following property: for every u-curve γ there exists an adapted vector field (γ, X) for g and for all $n > 0$ large enough*

$$\frac{I_n^{\gamma,X}}{n} > C.$$

Then the map g has a positive exponent in the center direction greater than $e^{-2\varepsilon_3}C$ for Leb-almost every point.

Proof. The new ingredient in the proof is the bounded distortion estimates. Suppose not, then there exists a measurable set B with positive measure such that every point in B has exponents in the center direction strictly smaller than $e^{-2\varepsilon_3}C$. By the absolute continuity of the unstable foliation, there is an unstable manifold γ that intersects B on a set of positive measure, for the Lebesgue measure of γ. Let $b \in \gamma \cap B$ be a density point and take $\gamma_k = g^{-k} \circ \beta_k$, where β_k is a u-curve with $\beta_k(0) = g^k(b)$. We have that $l(\gamma_k) \to 0$ and by bounded distortion, lemma 3.8

$$\frac{Leb(\gamma_k \cap B)}{Leb(\gamma_k)} \longrightarrow 1.$$

Take k large enough such that

$$\frac{Leb(\gamma_k \cap B^c)}{Leb(\gamma_k)} < \frac{e^{-2\varepsilon_3}(e^{\varepsilon_3} - 1)C}{2 \log 2N}.$$

Using bounded distortion again, for any $m^k \in g^k(\gamma_k)$

$$J^{uu}_{g^{-k}}(m^k) \geq \frac{Leb(\gamma_k)}{Leb(g^k(\gamma_k))} e^{-\varepsilon_3}.$$

Define $\chi_k(m) = \limsup_{n \to +\infty} \frac{1}{n} \log \|Dg^n(g^k(m)).X_{g^k(m)}\|$ for all $m \in \gamma_k$, where X is the vector field such that (β_k, X) verifies the hypothesis of the lemma.

$$\begin{aligned}\int_{\gamma_k} \chi_k d\gamma_k &= \int_{g^k(\gamma_k)} \chi_k \circ g^{-k} J^{uu}_{g^{-k}} d(g^k(\gamma_k)) \\ &\geq e^{-\varepsilon_3} \frac{Leb(\gamma_k)}{Leb(g^k(\gamma_k))} \int_{g^k(\gamma_k)} \chi_k \circ g^{-k} d(g^k(\gamma_k)) \geq e^{-\varepsilon_3} C Leb(\gamma_k).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\gamma_k} \chi_k d\gamma_k &= \int_{\gamma_k \cap B} \chi_k d\gamma_k + \int_{\gamma_k \cap B^c} \chi_k d\gamma_k \\ &\leq e^{-2\varepsilon_3} C Leb(\gamma_k) + \frac{\log 2N e^{-2\varepsilon_3}(e^{\varepsilon_3} - 1) C Leb(\gamma_k)}{2 \log 2N} \\ &< e^{-\varepsilon_3} C Leb(\gamma_k)\end{aligned}$$

which is a contradiction. □

Denote by

$$E(\gamma, X) = \frac{1}{|\gamma|} \int_\gamma \log \|Dg(m).X_m\| d\gamma(m),$$

where (γ, X) is an adapted field. For X a vector field on γ define

$$\widetilde{X}(m) = \frac{\pi_1(X(m))}{\|\pi_1(X(m))\|}.$$

Definition 7.23. *An adapted field (γ, X) is $\tilde{\delta}$-good if for every $m \in \gamma$, $\widetilde{X}(m) \in \Delta_{\tilde{\delta}}$.*

If $\mathcal{U}_N$ is small enough then the center leaves are very close to the horizontal tori, very similar to the proof of proposition 7.11 we obtain:

Proposition 7.24. *For N large and $\mathcal{U}_N$ small enough, for all $g \in \mathcal{U}_N$ and (γ, X) an $\tilde{\delta}$-good adapted field for g, it is verified that $E(\gamma, X) \geq (1 - 8\tilde{\delta}) \log N$.*

Recall that for $k \geq 0$ and a u-curve γ the number $N_k = N_k(\gamma)$ was the maximum number of u curves that subdivide $g^k \circ \gamma$. For an adapted field (γ, X) define $Y^k = \frac{g_*^k X}{\|g_*^k X\|}$. The following lemma is the analogous to lemma 7.5.

Lemma 7.25 ([BC14], Lemma 9)**.** *For N large and $\mathcal{U}_N$ small enough, let $g \in \mathcal{U}_N$ and (γ, X) be an adapted field for g. For $k \geq 0$, every possible pair $(\gamma_j^k, Y^k|_{\gamma_j^k})$, with $1 \leq j \leq N_k(\gamma)$ is an adapted field.*

Similar to lemma 7.7, Berger and Carrasco proved the following formula, see section 6 of [BC14].

Lemma 7.26. *For every adapted field (γ, X) and any $n \in \mathbb{N}$*

$$I_n^{\gamma,X} = \sum_{k=0}^{n-1} \left(R_k + \sum_{j=0}^{N_k} \frac{1}{|\gamma|} \int_{\gamma_j^k} \log \|Dg(m).Y_m^k\| J_{g^{-k}}^{uu} d\gamma_j^k \right),$$

where $R_k = \frac{1}{|\gamma|} \int_{\gamma_{N_k+1}^k} \log \|Dg(m).Y_m^k\| J_{g^{-k}}^{uu} d\gamma_{N_k+1}^k$.

We remark that this formula and the formula obtained in lemma 7.7 are obtained in the same way, just by using the change of variables formula multiple times. The difference in this one is that we keep the unstable jacobian in the formula. As a consequence of this formula we obtain

$$I_n^{\gamma,X} \geq \sum_{k=0}^{n-1} \left(R_k + e^{-\varepsilon_2} \sum_{j=0}^{N_k} (\min_{\gamma_j^k} J_{g^{-k}}^{uu}) E(\gamma_j^k, Y^k) \right), \tag{37}$$

where the term $e^{-\varepsilon_2}$ comes from $\frac{|\gamma_j^k|}{|\gamma|}$, which is bigger than $e^{-\varepsilon_2}$.

Observe that

$$|R_k| \leq \frac{(e^{-\varepsilon_1}\lambda)^{2Nk} \log 2N}{\lambda^N (1 - 2\lambda^N) \|P_x(e^u)\|} \xrightarrow{k \to +\infty} 0.$$

Hence

$$\frac{1}{n} \sum_{k=0}^{n-1} |R_k| \longrightarrow 0.$$

For (γ, X) an adapted field we define similarly as in the previous section the sets $G_k = G_k(\gamma, X)$ and $B_k = B_k(\gamma, X)$. The key lemma is the next one which is the analog of lemma 7.15.

Lemma 7.27. *For $K \geq 1$, for N large and $\mathcal{U}_N$ small enough, for every $g \in \mathcal{U}_N$ and every (γ, X) a $\tilde{\delta}$-good adapted field it holds that*

$$\sum_{j \in G_k} \min_{\gamma_j^k} J_{g^{-k}}^{uu} \geq K \sum_{j \in B_k} \max_{\gamma_j^k} J_{g^{-k}}^{uu}.$$

The proof uses the next lemma, which is the analog of lemmas 7.12 and 7.13.

Lemma 7.28. *For N large and U small enough, for every $g \in \mathcal{U}_N$, every adapted field (γ, X)*

(1) If (γ, X) is a $\tilde{\delta}$-good adapted field and if j is so that $g^{-1}\gamma_j^1$ does not intersect the strip Crit, then the field $(\gamma_j^1, \frac{g_* X}{\|g_* X\|})$ is $\tilde{\delta}$-good.

(2) *If (γ, X) is $\tilde{\delta}$-bad, there exists a strip S of length π such that for every j satisfying $g^{-1}\gamma_j^1 \subset S$, the field $(\gamma_1^j, \frac{g_*X}{\|g_*X\|})$ is $\tilde{\delta}$-good.*

The proof of this lemma is similar to the proof of lemma 12 in [BC14] and uses the estimate obtained in lemma 7.10.

Proof of lemma 7.27. We follow exactly Berger-Carrasco's proof of lemma 10 in [BC14] with the constants we chose and taking $\eta_N = \frac{5}{\pi N^\delta}$. The proof goes by induction, it is valid for $k = 0$ and suppose it is true for k. Using lemmas 7.20 and 7.28, following exactly the same proof of Berger and Carrasco, we obtain

$$\sum_{j\in G_{k+1}} \min_{\gamma_j^k} J^{uu}_{g^{-k-1}} \geq e^{-(\varepsilon_2+\varepsilon_3)}(1-\eta_N) \sum_{j\in G_k} \min_{\gamma_j^k} J^{uu}_{g^{-k}}.$$

It is also obtained

$$\sum_{j\in B_{k+1}} \max_{\gamma_j^k} J^{uu}_{g^{-k-1}} \leq \left(e^{\varepsilon_2+2\varepsilon_3}\eta_N + \frac{2.2}{3.K}e^{\varepsilon_3}\right) . \left(\sum_{j\in G_k} \min_{\gamma_j^k} J^{uu}_{g^{-k}}\right) + \lambda^{\frac{N}{2}} e^{\varepsilon_3}.$$

Thus

$$\frac{\sum_{j\in B_{k+1}} \max_{\gamma_j^k} J^{uu}_{g^{-k-1}}}{\sum_{j\in G_{k+1}} \min_{\gamma_j^k} J^{uu}_{g^{-k-1}}} \leq \frac{e^{\varepsilon_2+2\varepsilon_3}\eta_N + \frac{2.2}{3.K}e^{\varepsilon_3}}{e^{-(\varepsilon_2+\varepsilon_3)}(1-\eta_N)} + \frac{\lambda^{\frac{N}{2}}}{e^{-(\varepsilon_2+2\varepsilon_3)}(1-\eta_N)} < \frac{1}{K},$$

since we fixed ε_2 and ε_3 very small, for N large enough we obtain the last inequality. □

From now on we fix $K > (\tilde{\delta})^{-1}$ and assume that N is large and $\mathcal{U}_N$ is small enough such that lemma 7.27 holds.

Lemma 7.29. *For N large and $\mathcal{U}_N$ small enough, for every $g \in \mathcal{U}_N$, every adapted field (γ, X) and $k \geq 0$, it holds*

$$e^{-(\varepsilon_2+\varepsilon_3)} \leq \sum_{j\in G_k} \min_{\gamma_j^k} J^{uu}_{g^{-k}} + \sum_{j\in B_k} \max_{\gamma_j^k} J^{uu}_{g^{-k}} \leq e^{2(\varepsilon_2+\varepsilon_3)}.$$

Proof. Of course the lemma is true for $k = 0$. Following the same steps as the proof of lemma 11 in [BC14], one obtains

$$\begin{aligned}
1 &= \frac{1}{|\gamma|}\int_\gamma d\gamma = \frac{1}{|\gamma|}\sum_{j=1}^{N_k+1}\int_{\gamma_j^k} J^{uu}_{g^{-k}} d\gamma_j^k \\
&\geq \sum_{j\in G_k} \frac{|\gamma_j^k|}{|\gamma|} \min_{\gamma_j^k} J^{uu}_{g^{-k}} + e^{-\varepsilon_3}\sum_{j\in B_k} \frac{|\gamma_j^k|}{|\gamma|} \max_{\gamma_j^k} J^{uu}_{g^{-k}} - \int_{\gamma^k_{N_k+1}} \max_{\gamma^k_{N_k+1}} J^{uu}_{g^{-k}} d\gamma^k_{N_k+1} \\
\Rightarrow 1 &\geq e^{-(\varepsilon_2+\varepsilon_3)} \left(\left(\sum_{j\in G_k} \min_{\gamma_j^k} J^{uu}_{g^{-k}}\right) + \left(\sum_{j\in B_k} \max_{\gamma_j^k} J^{uu}_{g^{-k}}\right) - \frac{(e^{-\varepsilon_1}.\mu)^{-2Nk}}{\lambda^N(1-2\lambda^N)\|P_x(e^u)\|}\right).
\end{aligned}$$

For N large enough

$$1 + \frac{e^{-(\varepsilon_2+\varepsilon_3)}(e^{-\varepsilon_1}.\mu)^{-2Nk}}{\lambda^N(1-2\lambda^N)\|P_x(e^u)\|} < e^{\varepsilon_2+\varepsilon_3}.$$

Hence

$$\left(\sum_{j\in G_k} \min_{\gamma_j^k} J^{uu}_{g^{-k}}\right) + \left(\sum_{j\in B_k} \max_{\gamma_j^k} J^{uu}_{g^{-k}}\right) \leq e^{2(\varepsilon_2+\varepsilon_3)}.$$

Similarly one obtains the other inequality. □

We remark that this lemma for the fibered case is immediate, since in this case $\#G_k + \#B_k = [\mu^{2Nk}]$ and by the way we parametrize u-curves for the fibered case, $J^{uu}_{f^{-k}} = \mu^{-2Nk}$. Since the calculations for the fibered case are more direct, the application of this lemma is hidden inside the proof of lemma 7.16. For the general case we use this lemma to obtain inequality (38) below. This is done in the following way. By lemmas 7.27 and 7.29,

$$e^{-2(\varepsilon_2+\varepsilon_3)} \leq (1+K^{-1}) \sum_{j\in G_k} \min_{\gamma_j^k} J^{uu}_{g^{-k}},$$

which implies that

$$\frac{e^{-2(\varepsilon_2+\varepsilon_3)}}{1+K^{-1}} \leq \sum_{j\in G_k} \min_{\gamma_j^k} J^{uu}_{g^{-k}}. \tag{38}$$

Proposition 7.30. *For N large and $\mathcal{U}_N$ small enough, for every $g \in \mathcal{U}_N$, any $\tilde{\delta}$-good adapted field (γ, X) and every $k \geq 0$, it holds*

$$e^{-\varepsilon_2} \sum_{j=0}^{N_k} (\min_{\gamma_j^k} J^{uu}_{g^{-k}}) E(\gamma_j^k, Y^k) \geq (1-12\tilde{\delta}) \log N.$$

Proof. We have

$$\sum_{j=0}^{N_k} (\min_{\gamma_j^k} J^{uu}_{g^{-k}}) E(\gamma_j^k, Y^k) = \sum_{j\in G_k} (\min_{\gamma_j^k} J^{uu}_{g^{-k}}) E(\gamma_j^k, Y^k) + \sum_{j\in B_k} (\min_{\gamma_j^k} J^{uu}_{g^{-k}}) E(\gamma_j^k, Y^k)$$

By lemmas 7.27 and 7.29 and proposition 7.24 we obtain

$$\begin{aligned} e^{-\varepsilon_2} \sum_{j=0}^{N_k} (\min_{\gamma_j^k} J^{uu}_{g^{-k}}) E(\gamma_j^k, Y^k) &\geq e^{-\varepsilon_2}(1-8\tilde{\delta}) \log N \sum_{j\in G_k} \min_{\gamma_j^k} J^{uu}_{g^{-k}} - e^{-\varepsilon_2} \log 2N \sum_{j\in B_k} \min_{\gamma_j^k} J^{uu}_{g^{-k}} \\ &\geq e^{\varepsilon_2} \left((1-8\tilde{\delta}) - \frac{\log 2N}{K} \right) \sum_{j\in G_k} \min_{\gamma_j^k} J^{uu}_{g^{-k}} \\ &\geq \frac{e^{-2(2\varepsilon_2+\varepsilon_3)}(1-10\tilde{\delta}) \log N}{1+K^{-1}} > (1-12\tilde{\delta}) \log N. \end{aligned}$$

□

Proof of Proposition 1.4. Take $\tilde{\delta} = \frac{\delta}{15}$. By proposition 7.30, for N large and $\mathcal{U}_N$ small enough, for $g \in \mathcal{U}_N$ and any $\tilde{\delta}$-good adapted field (γ, X), for g, it holds that

$$\sum_{j=0}^{N_k} (\min_{\gamma_j^k} J^{uu}_{g^{-k}}) E(\gamma_j^k, Y^k) \geq (1-12\tilde{\delta}) \log N.$$

Using inequality (37), for n large enough

$$\frac{I_n^{\gamma,X}}{n} \geq (1-14\tilde{\delta}) \log N.$$

Since we could have chosen $\varepsilon_3 > 0$ small enough such that $e^{-\varepsilon_3}(1-14\tilde{\delta}) \geq (1-15\tilde{\delta})$ by proposition 7.22, almost every point has a Lyapunov exponent for g in the center direction larger than

$$(1-15\tilde{\delta}) \log N = (1-\delta) \log N.$$

All we did is also valid for g^{-1}, if $\mathcal{U}_N$ is small enough, thus almost every point has a negative Lyapunov exponent in the center direction smaller than $-(1-\delta)\log N$. □

Bibliography

[Ano67] D. Anosov. Geodesic flows on closed Riemannian manifolds of negative curvature. *Trudy Mat. Inst. Steklov.*, 90, 1967.

[AS67] D. Anosov, and Y. Sinaĭ. Certain smooth ergodic systems. *Uspehi Mat. Nauk*, 22:107–172, 1967.

[ACW17] A. Avila, S. Crovisier, and A. Wilkinson. C^1 density of stable ergodicity. to appear in Advances in Math., 2017.

[BP02] L. Barreira, and Y. Pesin. *Lyapunov exponents and smooth ergodic theory*, volume 23 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2002.

[BC14] P. Berger, and P. Carrasco. Non-uniformly hyperbolic diffeomorphisms derived from the standard map. *Comm. Math. Phys.*, 329:239–262, 2014.

[BXY17] A. Blumenthal, J. Xue, and L.S. Young. Lyapunov exponents for random perturbations of some area-preserving maps including the standard map. *Ann. of Math.*, 185:285–310, 2017.

[BDV05] C. Bonatti, L. Díaz, and M. Viana. *Dynamics beyond uniform hyperbolicity*, volume 102 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005.

[BP74] M. Brin, and Y. Pesin. Partially hyperbolic dynamical systems. *Izv. Akad. Nauk SSSR Ser. Mat.*, 38:170–212, 1974.

[Br16] A. Brown. Smoothness of stable holonomies inside center-stable manifolds and the C^2 hypothesis in Pugh-Shub and Ledrappier-Young theory. *Preprint arXiv:1608.05886*, 2016.

[BDP02] K. Burns, D. Dolgopyat, and Y. Pesin. Partial hyperbolicity, Lyapunov exponents and stable ergodicity. 108:927–942, 2002.

[BW10] K. Burns, and A. Wilkinson. On the ergodicity of partially hyperbolic systems. *Ann. of Math.*, 171:451–489, 2010.

[Ch79] B. Chirikov. A universal instability of many-dimensional oscillator systems. *Phys. Rep.*, 52:264–379, 1979.

[CP18] S. Crovisier, and E. Pujals. Strongly dissipative surface diffeomorphisms. *Comment. Math. Helv.*, 93:377–400, 2018.

[Du94] P. Duarte. Plenty of elliptic islands for the standard family of area preserving maps. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 11:359–409, 1994.

[Go12] A. Gorodetski. On stochastic sea of the standard map. *Comm. Math. Phys.*, 309:155–192, 2012.

[GPS94] M. Grayson, C. Pugh, and M. Shub. Stably ergodic diffeomorphisms. *Ann. of Math.*, 140:295–329, 1994.

[HHTU11] F. Rodriguez Hertz, J. Rodriguez Hertz, A. Tahzibi, and R. Ures. New criteria for ergodicity and nonuniform hyperbolicity. *Duke Math. J.*, 160:599–629, 2011.

[HPS77] M. Hirsch, C. Pugh, and M. Shub. *Invariant manifolds*. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977.

[Ho39] E. Hopf. *Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung*, 1939.

[HS17] V. Horita, and M. Sambarino. Stable ergodicity and accessibility for certain partially hyperbolic diffeomorphisms with bidimensional center leaves. *Comment. Math. Helv.*, 92:467–512, 2017.

[Iz80] F. Izraelev. Nearly linear mappings and their applications. *Physica D: Nonlinear Phenomena*, 1:243–266, 1980.

[Ob18] D. Obata. On the holonomies of strong stable foliations. *Notes on D. Obata's personal webpage*, 2018.

[Pes77] Y. Pesin. Characteristic Ljapunov exponents, and smooth ergodic theory. *Uspehi Mat. Nauk*, 32:55–112, 1977.

[PS00] C. Pugh, and M. Shub. Stable ergodicity and julienne quasi-conformality. *J. Eur. Math. Soc.*, 2:1–52, 2000.

[PSW97] C. Pugh, M. Shub, and A. Wilkinson. Hölder foliations. *Duke Math. J.*, 86:517–546, 1997.

[PSW00] C. Pugh, M. Shub, and A. Wilkinson. Correction to: "Hölder foliations". *Duke Math. J.*, 105:105–106, 2000.

[PSW12] C. Pugh, M. Shub, and A. Wilkinson. Hölder foliations, revisited. *J. Mod. Dyn.*, 6:79–120, 2012.

[PVW07] C. Pugh, M. Viana, and A. Wilkinson. Absolute continuity of foliations. *Preprint in M. Viana's personal webpage*, 2007.

[Ro52] V. Rohlin. On the fundamental ideas of measure theory. *Amer. Math. Soc. Translation*, 1952.

[SS95] D. Shepelyansky, and A. Stone. Chaotic landau level mixing in classical and quantum wells. *Physical review letters*, 74, 1995.

[Si94] Y. Sinaĭ. *Topics in ergodic theory*, volume 44 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1994.

[Vi97] M. Viana. Multidimensional nonhyperbolic attractors. *Inst. Hautes Études Sci. Publ. Math.*, 85:63–96, 1997.

CHAPTER 2

On the stable ergodicity of diffeomorphisms with dominated splitting

1. Introduction

Conservative dynamics appears naturally in several different areas of mathematics and physics. By a conservative dynamical system we mean a diffeomorphism of a smooth compact connected riemannian manifold M, that preserves a volume form ω. A key property that a conservative system may have is ergodicity.

Let m be the probability measure induced by ω. We say that the conservative system f is *ergodic* if every measurable f-invariant set has either full or zero m-measure. From the probabilistic point of view, ergodicity means that the system cannot be decomposed into smaller f-invariant parts. A key characterization of ergodicity is given as a consequence of the well known Birkhoff's ergodic theorem. In our setting, this can be stated as follows: f is ergodic if and only if for every continuous function $\varphi : M \to R$, for m-almost every point $x \in M$, it holds

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) = \int_M \varphi dm. \tag{39}$$

An important problem in the theory of dynamical systems is to know when a conservative system is ergodic. Another important question is to know when ergodicity is a robust property. Let $\mathrm{Diff}^r_\omega(M)$ be the space of C^r-diffeomorphisms of M that preserves the volume form ω.

Definition 1.1. *A diffeomorphism $f \in \mathrm{Diff}^2_\omega(M)$ is stably ergodic if there exists a C^1-neighborhood $\mathcal{U}$ of f, such that any diffeomorphism $g \in \mathcal{U} \cap \mathrm{Diff}^2_\omega(M)$ is ergodic.*

Hopf introduced an argument to prove that the geodesic flow on compact surfaces of constant negative curvature is ergodic for the Liouville measure, see [Hop39]. Anosov [Ano67], Anosov and Sinai [AS67] used Hopf argument to prove that every C^2 hyperbolic diffeomorphism is ergodic, see [Mn87] for the definition of hyperbolic, or Anosov, diffeomorphism. Since hyperbolicity is a C^1-open property, we conclude that conservative hyperbolic diffeomorphisms are stably ergodic. What about outside the hyperbolic world?

Since ergodicity is a global feature, it is natural to look for global properties of a diffeomorphism that could help to obtain ergodicity, or stable ergodicity. Among the partially hyperbolic diffeomorphisms the key global property used is the *accessibility*: any two points in the manifold can be connected by a path contained in finitely many stable and unstable manifolds, see [BW10] for precise definitions of partial hyperbolicity and accessibility. Pugh and Shub conjectured in [PS97] that accessibility implies ergodicity. This conjecture remains open and one usually needs some extra assumption to conclude ergodicity.

Most works done about the problem of stable ergodicity consider partially hyperbolic systems, see for instance [BP74], [GPS94], [PS00], [BDP02], [AMP04] and [BW10]. Not much has been done outside the partially hyperbolic scenario.

It is known that stably ergodic diffeomorphisms must have some weaker form of hyperbolicity [AM07], called dominated splitting. We say that a diffeomorphism f admits a *dominated splitting* if there is a decomposition of the tangent bundle, $TM = E \oplus F$, into two non-trivial subbundles which are Df-invariant, such that for some $N \geq 1$, any unit

vectors $v \in E(x)$ and $u \in F(x)$ verify

$$\|Df^N(x)v\| < \frac{1}{2}\|Df^N(x)u\|.$$

Tahzibi in [Tah04] constructs an example of a stably ergodic diffeomorphism which is not partially hyperbolic. Another important property in ergodic theory is the following.

Definition 1.2. *Let ν be an invariant measure for f. We say that (f, ν) is Bernoulli if it is measurably conjugated to a Bernoulli shift. If f is a diffeomorphism that preserves a smooth measure m, we say that f is Bernoulli if (f, m) is Bernoulli.*

We remark that the Bernoulli property is stronger than ergodicity. Switching ergodicity by the Bernoulli property in definition 1.1, we obtain the definition of *stably Bernoulli*.

One of the goals of this work is to find new criteria of stable ergodicity, actually of stable Bernoulli, outside the partially hyperbolic scenario. In particular, we study the consequences given by a property called *chain-hyperbolicity*, see definition 2.3. Chain-hyperbolicity has been defined and used before in [Cro11], [CP15]. It can be seen as some type of topological hyperbolicity saying that f "contracts" topologically along the direction E, up to a certain "scale", and f^{-1} "contracts" topologically along the direction F, up to a certain "scale". Using this as the global property to study stable ergodicity, we have the following theorem.

Theorem B. *Let $f \in \mathrm{Diff}^1_\omega(M)$. If f is a chain-hyperbolic diffeomorphism for a dominated splitting $TM = E \oplus F$ and verifies*

$$\int_M \log \|Df|_E\| dm < 0 \text{ and } \int_M \log \|Df^{-1}|_F\| dm < 0, \tag{40}$$

then there exists a C^1-neighborhood $\mathcal{U}$ of f, such that any diffeomorphism $g \in \mathcal{U} \cap \mathrm{Diff}^2_\omega(M)$ is ergodic, in fact Bernoulli. In particular, any such diffeomorphism g is stably Bernoulli.

In the setting of theorem B, as a consequence of (40) and ergodicity, we actually obtain that m-almost every point has all Lyapunov exponents negative along E and all positive along F, see section 2 for the definition of Lyapunov exponent.

A diffeomorphism is weakly partially hyperbolic if it admits a dominated splitting of the form $TM = E \oplus E^{uu}$, such that the subbundle E^{uu} expands exponentially fast under the action of Df. As one application of theorem B, we obtain the following criterion of stable Bernoulli for weakly partially hyperbolic systems.

Theorem C. *Let $f \in \mathrm{Diff}^2_\omega(M)$. Suppose that f is weakly partially hyperbolic with dominated splitting $TM = E \oplus E^{uu}$ and chain-hyperbolic with respect to the same splitting. If f has all Lyapunov exponents negative along the direction E on a set of positive m-measure, then f is stably ergodic, in fact stably Bernoulli.*

This theorem can be seen as a version of theorem 4 in [BDP02] for weakly partially hyperbolic diffeomorphisms. We also remark that if $f \in \mathrm{Diff}^2_\omega(M)$ verifies the hypothesis of theorem B and the direction F is hyperbolic, meaning $F = E^{uu}$, then (40) implies that f verifies the hypothesis of theorem C. However, a diffeomorphism which verifies the hypothesis of theorem C, does not necessarily verify the hypothesis of theorem B, a priori.

Theorem B gives more flexibility in the construction of the example considered by Tahzibi. To construct the example one makes a deformation supported in a finite number of small balls around hyperbolic fixed points, in particular, the deformations are local. Theorem B applies to this example and quantifies, in a certain way, how much one can make such a deformation, in particular, the deformations do not have to be local. In section 6 we will explain the construction of such an example in this non local way. We remark that our proof is different from the proof of Tahzibi in [Tah04].

Pugh and Shub conjectured in [PS97], that stable ergodicity is C^r-dense among the partially hyperbolic conservative C^r-diffeomorphisms. A remarkable result by Avila, Crovisier

and Wilkinson states that stable ergodicity is C^1-dense among the partially hyperbolic conservative C^r-diffeomorphisms, indeed they obtain stable Bernoulli, see theorem A' in [ACW17].

As another application of theorem B and some others results, we can prove the C^1-density of stably Bernoulli diffeomorphisms among a certain class of weakly partially hyperbolic diffeomorphisms.

Let $\mathcal{D} \subset \mathrm{Diff}^2_\omega(M)$ be the subset of diffeomorphisms f that verifies the following properties:

- f is weakly partially hyperbolic, with dominated splitting $TM = E \oplus E^{uu}$ and $dim(E) = 2$;
- f is chain-hyperbolic for the splitting $TM = E \oplus E^{uu}$.

Define $\mathcal{WCH}^2_\omega(M)$ to be the C^1-interior of $\mathcal{D}$ for the relative topology. For the d-torus this set is non empty, with $d \geq 3$. The examples of Bonatti-Viana, see section 6.2 of [BV00], belong to this set. Indeed, the arguments in section 6 also apply to such examples, justifying that they belong to $\mathcal{WCH}^2_\omega(\mathbb{T}^d)$. We have the following theorem.

Theorem D. *Stable Bernoulli is C^1-dense on $\mathcal{WCH}^2_\omega(M)$.*

We remark that all our results remain true for $C^{1+\alpha}$-diffeomorphisms. We conclude this introduction with some questions.

Question 1.3. *What others criteria for stable ergodicity, or stable Bernoulli, can one obtain using chain-hyperbolicity?*

The example considered by Tahzibi in [Tah04] is isotopic to a linear Anosov diffeomorphism. This particular example was first considered by Bonatti and Viana in [BV00] where they proved robust transitivity. This type of construction allow us to obtain diffeomorphisms that verify the hypothesis of theorem B and which are not partially hyperbolic. For more details on the construction see section 6. We conclude the introduction with the following question.

Question 1.4. *Is there a diffeomorphism that verifies the hypothesis of theorem B, or theorem C, which is not isotopic to an Anosov diffeomorphism?*

Rafael Potrie obtains a positive answer for this question under some assumptions, see [Pot15].

Organization of this chapter. In section 2 we introduce all the tools needed in our proofs, in particular from the chain-hyperbolicity property and Pesin's theory. Theorems A, B and C are proved in sections 3, 4 and 5, respectively. In section 6 we explain Tahzibi's construction in a non local way to obtain examples of diffeomorphisms that verify the hypothesis of theorem A and are not partially hyperbolic.

2. Preliminaries

2.1. Dominated splittings and chain hyperbolic homoclinic classes. Let $f \in \mathrm{Diff}^1(M)$ be a C^1-diffeomorphism of M that admits a dominated splitting $TM = E \oplus F$. It is well known that dominated splitting is a C^1-open property, meaning that if g is sufficiently C^1-close to f, then g admits a dominated splitting $T_M = E_g \oplus F_g$, where $\dim(E_g) = \dim(E)$ and $\dim(F_g) = \dim(F)$. It is also well known that the maps $g \mapsto E_g$ and $g \mapsto F_g$ are continuous in a C^1-neighborhood of f. We call E_g and F_g the continuations of the subbbundles E and F. In particular, this implies that the maps

$$g \mapsto \int_M \log \|Dg|_{E_g}\| dm \text{ and } g \mapsto \int_M \log \|Dg^{-1}|_{F_g}\| dm$$

are continuous in a C^1-neighborhood of f.

Definition 2.1 (Plaque Family). *A plaque family tangent to E is a continuous map $\mathcal{W}$ from E into M that verifies:*

- *For each $x \in M$, the induced map $\mathcal{W}_x : E(x) \to M$ is a C^1-embedding which satisfies $\mathcal{W}_x(0) = x$ and whose image is tangent to $E(x)$ at x;*
- *$(\mathcal{W}_x)_{x\in M}$ is a continuous family of C^1-embeddings.*

A plaque family is locally invariant if there exists $\rho > 0$ such that for each $x \in M$, the image $f(\mathcal{W}_x(B(0,\rho)))$ is contained in the plaque $\mathcal{W}_{f(x)}$.

The condition of dominated splitting alone cannot guarantee that the subbundles E or F are integrable. In [HPS77], the authors proved that for diffeomorphisms with dominated splitting $TM = E \oplus F$, there are always locally invariant plaque families for f tangent to the direction E. Similarly, there are always locally invariant plaque families for f^{-1} tangent to the direction F.

Definition 2.2 (Trapped plaques). *A plaque family $(\mathcal{W}_x)_{x\in M}$ is trapped for f if for each $x \in M$, it holds*

$$f(\overline{\mathcal{W}_x}) \subset \mathcal{W}_{f(x)},$$

where $\overline{\mathcal{W}_x}$ denotes the closure of $\mathcal{W}_x$.

Let $\mathrm{Per}(f)$ be the set of all periodic points of f. For $p \in \mathrm{Per}(f)$, we write $O_f(p)$ the orbit of p for f. If it is clear that we are considering the orbit for f we will just write $O(p)$. A periodic point p is hyperbolic if there is a dominated splitting over $O(p)$, $T_{O(p)}M = E^{ss} \oplus E^{uu}$, such that E^{ss} is contracted and E^{uu} is expanded exponentially fast under the action of Df. It is well known that if f is a C^r-diffeomorphism, for any hyperbolic periodic point p of f, there is an immersed C^r-submanifold $W^{ss}(p,f)$, called the stable manifold of p, which is tangent to $E^{ss}(p)$ at p. Similarly, there is an immersed C^r-submanifold called the unstable manifold, which we will denote it by $W^{uu}(p,f)$.

Let $\pi(p) \in \mathbb{N}$ be the period of the periodic point p and write

$$W^{ss}(O(p),f) = \bigcup_{j=0}^{\pi(p)-1} W^{ss}(f^j(p),f),$$

to be the *stable manifold* of the orbit of p. Analogously, we define $W^{uu}(O(p),f)$.

Given two immersed submanifolds S_1 and S_2 of M, we say that a point x is a point of *transverse intersection* between S_1 and S_2 if $x \in S_1 \cap S_2$ and $T_xM = T_xS_1 + T_xS_2$. We denote the set of points of transverse intersection between S_1 and S_2 by $S_1 \pitchfork S_2$.

Given two hyperbolic periodic points $p, q \in \mathrm{Per}(f)$, we say that they are *homoclinically related* if $W^{ss}(O(p),f) \pitchfork W^{uu}(O(q),f) \neq \emptyset$ and $W^{uu}(O(p),f) \pitchfork W^{ss}(O(q),f) \neq \emptyset$. We write $p \sim q$ if p is homoclinically related to q.

The *homoclinic class* of p is defined as

$$H(p,f) = \overline{\{q \in \mathrm{Per}(f) : p \sim q\}}.$$

If it is clear that we are referring to f we will just write $H(p)$ as the homoclinic class of p for f.

Definition 2.3 (Chain-hyperbolicity). *We say that $f \in \mathrm{Diff}^r(M)$ is chain-hyperbolic if:*

(1) there exists a periodic hyperbolic point p such that $H(p) = M$;

(2) there is a dominated splitting $TM = E \oplus F$;

(3) there is a plaque family $(\mathcal{W}^E_x)_{x\in M}$ tangent to E which is trapped by f. There is also another plaque family $(\mathcal{W}^F_x)_{x\in M}$ tangent to F which is trapped by f^{-1};

(4) there are two periodic hyperbolic points, q_s and q_u, homoclinically related to p such that the stable manifold of q_s contains the plaque $\mathcal{W}^E_{q_s}$ and the unstable manifold of q_u contains the plaque $\mathcal{W}^F_{q_u}$.

Let f be a weakly partially hyperbolic diffeomorphism, with dominated splitting $TM = E \oplus E^{uu}$. It is well known that the subbundle E^{uu} is uniquely integrable, that is, there exists an unique foliation $\mathcal{W}^{uu}$, called unstable foliation, that is tangent to E^{uu}. If f is also chain-hyperbolic with respect to the same dominated splitting, then the plaque family for the direction E^{uu} can be taken as a family of discs inside the unstable foliation $\mathcal{W}^{uu}$.

A key consequence of chain-hyperbolicity for us is given in the following lemma.

Lemma 2.4 ([CP15], Lemma 3.2)**.** *If $f \in \mathrm{Diff}^1(M)$ is chain-hyperbolic, there exists a dense set $\mathcal{P} \subset M$ of hyperbolic periodic points homoclinically related to p, such that for any point $q \in \mathcal{P}$, the plaques $\mathcal{W}^E_q$ and $\mathcal{W}^F_q$ are respectively contained in the stable and in the unstable manifolds of q.*

Given $0 < \theta \leq 1$, we define the cone of size θ around the direction E as

$$\mathcal{C}^E_\theta = \{(v_E, v_F) \in E \oplus F : \theta\|v_E\| \geq \|v_F\|\}.$$

Remark 2.5. *Since both plaque families are continuous, by compactness, there exists $r > 0$ such that for every $x \in M$ the plaque $\mathcal{W}^E_x$ contains a C^1-disc of radius r, centered in x and tangent to $E(x)$. Furthermore, by domination, for some small $\theta > 0$, we can assume that these discs are tangent to $\mathcal{C}^E_\theta$. An analogous result holds for the plaque family $\{\mathcal{W}^F_x\}_{x\in M}$. Thus, lemma 2.4 states that densely there are periodic points homoclinically related to p whose stable and unstable manifolds have size bounded from below by r and "good" geometry, meaning controlled angles.*

If f is chain-hyperbolic, it is easy to see that for every $N \in \mathbb{N}$, properties 2 through 4 in definition 2.3 remain valid for f^N. On the other hand, it is not so immediate that property 1 holds for f^N. It could happen that the entire manifold is no longer a homoclinic class and it could be divided into finitely many distinct homoclinic classes. As a consequence of lemma 2.4, we obtain that this is not the case.

Lemma 2.6. *Let M be connected. If f is chain-hyperbolic, then for every $N \in \mathbb{N}$ it holds that f^N is chain-hyperbolic.*

Proof. Let p be the hyperbolic periodic point for f in the definition of chain-hyperbolicity and fix $N \in \mathbb{N}$. It suffices to prove that $H(p, f^N) = M$.

Let $\mathcal{P} \subset M$ be the set of hyperbolic periodic points given by lemma 2.4 for f. Notice that for f^N, the set $\mathcal{P}$ is also formed by hyperbolic periodic points with stable and unstable manifolds of uniform size, given by the plaques $\mathcal{W}^E$ and $\mathcal{W}^F$.

Let $\varepsilon > 0$ be small enough such that any two points in $q', q'' \in \mathcal{P}$ that are ε-close to each other verify

$$W^E_{q'} \pitchfork W^F_{q''} \neq \emptyset \text{ and } W^E_{q''} \pitchfork W^F_{q'} \neq \emptyset.$$

In particular q' and q'' are homoclinically related for f^n, for any $n \in \mathbb{N}$. The existence of ε is a consequence of remark 2.5.

For any two points $q', q'' \in \mathcal{P}$ we can take a finite set of points $\{q_0, \cdots, q_k\} \subset \mathcal{P}$, such that $q_0 = q'$, $q_k = q''$ and for every $i = 0, \cdots, k-1$ it holds $d(q_i, q_{i+1}) < \varepsilon$. This implies that any two points $q', q'' \in \mathcal{P}$ are homoclinically related for f^N. By the density of $\mathcal{P}$ there exists a point $q \in \mathcal{P}$ such that q is homoclinically related with p for f^N. We conclude that $H(p, f^N) = M$, which finishes the proof. □

Remark 2.7. *In the setting of lemma 2.6, from its proof and using the inclination lemma, see lemma 7.1 in [PDM82], we obtain the following: for any $\varepsilon > 0$ small enough, there exists $\{q_0, \cdots, q_k\} \subset \mathcal{P}$ which is ε-dense, such that $W^{ss}(q_i, f) \pitchfork W^{uu}(p, f) \neq \emptyset$ and $W^{uu}(q_i, f) \pitchfork W^{ss}(p, f) \neq \emptyset$ for $i = 0, \cdots, k$. We remark that here we consider the stable and unstable manifold of the point and not of the orbit. This property holds in a C^1-neighborhood of f.*

One defines a *chain-hyperbolic homoclinic class* as a homoclinic class $H(p)$ that verifies conditions 2 through 4 in definition 2.3. We remark that the same argument as in the proof of lemma 2.6 implies that if $H(p)$ is a connected, chain-hyperbolic homoclinic class for f, then for every $N \in \mathbb{N}$ it holds that $H(p)$ is a chain-hyperbolic homoclinic class for f^N.

2.2. Pesin's theory and criterion of ergodicity. Let $f \in \mathrm{Diff}^1(M)$ be a C^1-diffeomorphism. A number $\lambda \in \mathbb{R}$ is a Lyapunov exponent of f at x if there exists a nonzero vector $v \in T_xM$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Df^n(x)v\| = \lambda.$$

For a point x and a vector $v \in T_xM$, write

$$\lambda(x, v) := \lim_{n\to+\infty} \frac{1}{n} \log \|Df^n(x)v\|.$$

A key theorem in smooth ergodic theory is the Oseledet's theorem.

Theorem 2.8 ([BP02], Theorems 2.1.1 and 2.1.2)**.** *Let $f \in \mathrm{Diff}^1_\omega(M)$. There exists a set $\mathcal{R}_f$ of full Lebesgue measure, such that for any $x \in \mathcal{R}_f$ there is a number $1 \leq l(x) \leq \dim(M)$ and there are $l(x)$ Lyapunov exponents $\lambda_1(x) < \cdots < \lambda_{l(x)}(x)$. For this point $x \in \mathcal{R}_f$, there is a decomposition of the tangent space over x*

$$T_xM = E_1(x) \oplus \cdots \oplus E_{l(x)}(x),$$

which is Df-invariant. This decomposition varies measurably with $x \in \mathcal{R}_f$ and for every $v_i \in E_i(x) - \{0\}$, it holds that $\lambda(x, v_i) = \lambda_i(x)$, for $i = 1, \cdots, l(x)$.

A point of the set $\mathcal{R}_f$, given by the previous theorem, is called a *regular point*. A f-invariant measure μ is *non-uniformly hyperbolic* if for μ-almost every point all its Lyapunov exponents are non-zero.

For a regular point $x \in \mathcal{R}_f$, we write

$$E^s(x) = \bigoplus_{i:\lambda_i(p)<0} E_i(p) \text{ and } E^u(p) = \bigoplus_{i:\lambda_i(p)>0} E_i(p). \tag{41}$$

Definition 2.9. *For a C^2-diffeomorphism $f \in \mathrm{Diff}^2_\omega(M)$, the stable Pesin manifold of the point $x \in \mathcal{R}_f$ is*

$$W^s(x, f) = \{y \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0\}.$$

Similarly one defines the unstable Pesin manifold as

$$W^u(x, f) = \{y \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0\}.$$

Let $f \in \mathrm{Diff}^2_\omega(M)$, for Lebesgue almost every point $x \in \mathcal{R}_f$, the Pesin's manifolds are immersed C^1-submanifolds, see section 4 of [Pes77]. Let $p \in \mathrm{Per}(f)$ be a hyperbolic periodic point. Define the following sets:

$$\begin{aligned} H^s(O(p)) &= \{x \in \mathcal{R}_f : W^s(x, f) \pitchfork W^{uu}(O(p), f) \neq \emptyset\}, \\ H^u(O(p)) &= \{x \in \mathcal{R}_f : W^u(x, f) \pitchfork W^{ss}(O(p), f) \neq \emptyset\}. \end{aligned}$$

Define the *ergodic homoclinic class* of p by

$$H_{erg}(O(p)) = H^s(O(p)) \cap H^u(O(p)).$$

It is easy to see that $H_{erg}(p)$ is f-invariant. Given two measurable sets $A, B \subset M$ we write $A \stackrel{\circ}{=} B$, if A only differs from B in a set of zero Lebesgue-measure. Given a measurable set

Λ with positive m-measure, we define m_Λ to be the normalized restriction of the measure m to the set Λ, that is, for any measurable set A,

$$m_\Lambda(A) = \frac{m(A \cap \Lambda)}{m(\Lambda)}.$$

The following theorem will give us a criterion for ergodicity.

Theorem 2.10 ([HHTU11], Theorem A)**.** *Let $f \in \mathrm{Diff}^2_\omega(M)$. For a hyperbolic periodic point $p \in Per(f)$, if $m(H^s(O(p))) > 0$ and $m(H^u(O(p))) > 0$, then*

$$H_{erg}(O(p)) \stackrel{\circ}{=} H^s(O(p)) \stackrel{\circ}{=} H^u(O(p)).$$

Moreover $f|_{H_{erg}(O(p))}$ is ergodic and non-uniformly hyperbolic, with respect to the measure $m_{H_{erg}(O(p))}$. In particular, if $m(H_{erg}(O(p))) = 1$ then f is ergodic.

We will also need the following result by Pesin.

Theorem 2.11 ([Pes77], Theorem 8.1)**.** *Let f be a C^2-diffeomorphism preserving a smooth measure m. Suppose that f is non-uniformly hyperbolic and ergodic for the measure m. Then there exist $K \in \mathbb{N}$ and measurable sets with positive m-measure $\Lambda_1, \cdots \Lambda_K$ which are pairwise disjoints, such that $f(\Lambda_i) = \Lambda_{i+1}$ for $i = 1, \cdots, K-1$, $f(\Lambda_K) = \Lambda_1$ and for each $j = 1, \cdots, K$, the system $(f^K, m|_{\Lambda_j})$ is Bernoulli. In particular, if $K = 1$ then (f, m) is Bernoulli.*

For a hyperbolic periodic point p, define $h^s(p) = \{x \in \mathcal{R}_f : W^s(x, f) \pitchfork W^{uu}(p, f) \neq \emptyset\}$, notice that in this definition we are taking the unstable manifold of the point p and not the unstable manifold of the orbit of p. Analogously, we define the set $h^u(p)$. We define the *pointwise ergodic homoclinic class* as

$$h_{ber}(p) = h^s(p) \cap h^u(p). \tag{42}$$

As a consequence of theorems 2.10 and 2.11, we obtain the following corollary.

Corollary 2.12. *Let $f \in \mathrm{Diff}^2_\omega(M)$. For a hyperbolic periodic point $p \in Per(f)$, with period $\pi(p)$, if $m(h^s(p)) > 0$ and $m(h^u(p)) > 0$, then*

$$h_{ber}(p) \stackrel{\circ}{=} h^s(p) \stackrel{\circ}{=} h^u(p).$$

Moreover $f^{\pi(p)}|_{h_{ber}(p)}$ is Bernoulli and non-uniformly hyperbolic, with respect to the measure $m_{h_{ber}(p)}$. In particular, if $m(h_{ber}(p)) = 1$ then (f, m) is Bernoulli.

Proof. Apply theorem 2.10 for $f^{\pi(p)}$ and conclude that $h_{ber}(p) \stackrel{\circ}{=} h^s(p) \stackrel{\circ}{=} h^u(p)$. Applying theorem 2.10 for f, we obtain that $H_{erg}(O(p))$ is a non-uniformly hyperbolic ergodic component of f. Using theorem 2.11 and the fact that $f^{\pi(p)}(h_{ber}(p)) = h_{ber}(p)$ we conclude that $(f^{\pi(p)}|_{h_{ber}(p)}, m|_{h_{ber}(p)})$ is Bernoulli. Again by theorem 2.11, if $m(h_{ber}(p)) = 1$ then (f, m) is Bernoulli. □

3. Proof of Theorem B

Fix $N \in \mathbb{N}$, such that for any $x \in M$ it holds

$$\|Df^N|_{E(x)}\| . \|Df^{-N}|_{F(f^N(x))}\| < \frac{1}{2}.$$

Let $\mathcal{U}_1$ be a C^1-neighborhood of f such that for any $g \in \mathcal{U}_1$ it is verified

$$\|Dg^N|_{E_g(x)}\| . \|Dg^{-N}|_{F_g(g^N(x))}\| < \frac{1}{2},$$

where $E_g(.)$ and $F_g(.)$ are the continuations of the subbundles E and F, which we defined at the beginning of section 2. For each $g \in \mathcal{U}_1$, define the auxiliary functions

$$\varphi_g(x) = \log \|Dg^N|_{E_g(x)}\| \text{ and } \psi_g(x) = \log \|Dg^{-N}|_{F_g(x)}\|. \tag{43}$$

By our assumption (40), take a constant $\beta > 0$ such that

$$\int_M \log \|Df|_E\| dm < -2\beta \text{ and } \int_M \log \|Df^{-1}|_F\| dm < -2\beta.$$

By our discussion at the beginning of section 2, we can assume that $\mathcal{U}_1$ is small enough such that for any $g \in \mathcal{U}_1$, it is verified that

$$\int_M \log \|Dg|_{E_g}\| dm < -\beta \text{ and } \int_M \log \|Dg^{-1}|_{F_g}\| dm < -\beta.$$

Let $\sigma = \min\{\frac{\log 2}{2}, \beta\}$ and observe that for every $g \in \mathcal{U}_1$, it holds that

$$\int_M \varphi_g dm < -\sigma \text{ and } \int_M \psi_g dm < -\sigma. \tag{44}$$

We remark that the information that $\sigma \leq \frac{\log 2}{2}$ will be used later in the proof of lemma 3.2. For each $g \in \mathcal{U}_1$, define the sets

$$A_g = \left\{ x \in M : \limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Dg^N|_{E_g(g^{jN}(x))}\| \leq -\sigma \right\};$$

$$B_g = \left\{ x \in M : \limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Dg^{-N}|_{F_g(g^{-jN}(x))}\| \leq -\sigma \right\}.$$

We have the following lemma.

Lemma 3.1. *For every $g \in \mathcal{U}_1 \cap \mathrm{Diff}^2_\omega(M)$, both A_g and B_g have positive m-measure.*

Proof. Let us prove that A_g has positive measure, the proof is analogous for B_g. From (44), we have

$$\int_M \varphi_g dm < -\sigma.$$

Consider the Birkhoff average

$$\widetilde{\varphi}_g(.) := \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_g \circ g^{jN}(.).$$

By Birkhoff's theorem, $\widetilde{\varphi}_g$ is defined for almost every point and

$$\int_M \widetilde{\varphi}_g dm = \int_M \varphi_g dm < -\sigma. \tag{45}$$

Observe that the set $A = \{x \in M : \widetilde{\varphi}_g(x) < -\sigma\}$ is contained in A_g. From (45), we conclude that A has positive measure, which implies that A_g has positive measure as well. □

This lemma will allow us to verify the conditions for theorem 2.10 to hold. Using the domination we can prove the following lemma.

Lemma 3.2. *For every $g \in \mathcal{U}_1 \cap \mathrm{Diff}^2_{Leb}(\mathbb{T}^4)$ it holds that $m(A_g \cup B_g) = 1$.*

Proof. Let $g \in \mathcal{U}_1$ and μ be a g^N-invariant ergodic measure. Suppose that $\mu(A_g) = 0$. The domination implies that for every $x \in M$

$$\varphi_g(x) + \psi_g \circ g^N(x) < -\log 2.$$

Since μ is ergodic, for μ-almost every point $x \in M - A_g$ it holds that

$$\int_M \varphi_g d\mu = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_g \circ g^{jN}(x) > -\sigma \geq -\frac{\log 2}{2}.$$

Thus, by domination

$$\int_M \psi_g \circ g^N d\mu = \int_M \psi_g d\mu < -\log 2 + \frac{\log 2}{2} = -\frac{\log 2}{2} \leq -\sigma.$$

Since μ is ergodic, for μ-almost every point x, it holds

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi_g \circ g^{-jN}(x) = \int_M \psi_g d\mu < -\sigma.$$

In particular, $\mu(B_g) = 1$. Since the sets A_g and B_g are invariant, we obtain that for any ergodic measure μ it holds that $\mu(A_g \cup B_g) = 1$. Using the ergodic decomposition theorem, see theorem 6.4 in [Mn87], we conclude that $m(A_g \cup B_g) = 1$.

□

For $g \in \mathcal{U}_1 \cap \mathrm{Diff}^2_\omega(M)$, recall that $\mathcal{R}_g$ is the set of regular points for g. For a regular point $x \in A_g \cap \mathcal{R}_g$ all the Lyapunov exponents for g^N on $E_g(x)$ are negative. Indeed,

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Dg^{nN}|_{E_g(x)}\| \leq \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Dg^N|_{E_g(g^{jN}(x))}\| < -\sigma.$$

For $x \in \mathcal{R}_g$, consider the stable Pesin manifold $W^s(x, g)$ for g^N and for g. Similarly we define those sets for the unstable Pesin manifold and we denote it by $W^u(x, g)$.

Lemma 3.3. *There are a C^1-neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of f and two constants $r_0, \theta_0 > 0$ that verify the following: For $g \in \mathcal{U}_2 \cap \mathrm{Diff}^2_\omega(M)$ and for any $x \in A_g \cap \mathcal{R}_g$ there exists $n \geq 0$, such that $W^s(g^{-nN}(x), g)$ contains a C^1-disc of radius r_0, centered in $g^{-nN}(x)$ and tangent to $\mathcal{C}^E_{\theta_0}$.*

The proof of the existence of $r_0 > 0$ can be found in lemma 2 of [BDP02]. The proof uses the notion of hyperbolic times and ideas from [ABV00]. The existence of θ_0 follows from domination. A similar result holds for B_g and we can suppose that r_0 and θ_0 are the same for both sets, A_g and B_g.

Remark 3.4. *In theorem 3.11 in [ABC11], the authors prove the existence of Pesin manifolds for C^1-diffeomorphisms with a dominated splitting. From this result, we conclude that the conclusion of lemma 3.3 also holds for $g \in \mathcal{U}_2 \cap \mathrm{Diff}^1_\omega(M)$.*

We remark that in the proof of lemmas 3.1, 3.2 and 3.3, we do not use the chain-hyperbolicity condition. These lemmas are true for any C^1-diffeomorphism that preserves volume and that admits a dominated splitting $TM = E \oplus F$ which verifies (40).

By hypothesis f is chain-hyperbolic. Let $p \in Per(f)$ be the hyperbolic point in the definition of chain-hyperbolicity such that $H(p) = M$, see definition 2.3. We may assume that $\mathcal{U}_2$ is small enough such that for any $g \in \mathcal{U}_2$, there exists a hyperbolic periodic point $p_g \in Per(g)$, which is the continuation of the periodic point p.

Let $\mathcal{P}$ be the dense set of hyperbolic periodic points given by lemma 2.4. By remark 2.5, there exist two constants $r_1, \theta_1 > 0$ and a dense set of hyperbolic periodic points $\mathcal{P}$ homoclinically related with p, such that for any $q \in \mathcal{P}$, the stable manifold of q contains a C^1-disc centered in q, with radius r_1 and tangent to $\mathcal{C}^E_{\theta_1}$ and the unstable manifold of q contains a C^1-disc centered in q of radius r_1 and tangent to $\mathcal{C}^F_{\theta_1}$.

This is the main property that we use from chain-hyperbolicity. As a consequence of that, we obtain the following proposition.

Proposition 3.5. *There exists a C^1-neighborhood $\mathcal{U}_3 \subset \mathcal{U}_2$ of f, such that for any $g \in \mathcal{U}_3 \cap \mathrm{Diff}^2_\omega(M)$ and for any $x \in A_g \cap \mathcal{R}_g$, it is verified*

$$W^s(x, g) \pitchfork W^{uu}(p_g, g) \neq \emptyset.$$

Similarly, for any $y \in B_g$, it holds that $W^u(y, g) \pitchfork W^{ss}(p_g, g) \neq \emptyset$.

PROOF. Take $r = \frac{\min\{r_0, r_1\}}{2}$ and $\theta = 2\max\{\theta_0, \theta_1\}$. It is easy to see that there is $\varepsilon > 0$ such that any two points x and y with $d(x, y) < \varepsilon$, verify the following: any two C^1-discs D_1 and D_2, centered in x and y, respectively, with radius r and such that D_1 is tangent to $\mathcal{C}^E_\theta$ and D_2 is tangent to $\mathcal{C}^F_\theta$, have a transverse intersection. Fix such $\varepsilon > 0$.

By remark 2.7, fix a finite set of hyperbolic periodic points for f, $\{q_0, \cdots, q_k\} \subset \mathcal{P}$, which is $\frac{\varepsilon}{2}$-dense on M, such that

$$W^{ss}(q_i, f) \pitchfork W^{uu}(p, f) \neq \emptyset \text{ and } W^{uu}(q_i, f) \pitchfork W^{ss}(p, f) \neq \emptyset. \tag{46}$$

Consider a C^1-neighborhood $\mathcal{U}_3 \subset \mathcal{U}_2$ of f, small enough, such that for any $g \in \mathcal{U}_3 \cap \mathrm{Diff}^2_\omega(M)$ the following properties are verified:

- For any $i = 0 \cdots, k$, the continuation $q_{i,g}$ is defined and (46) holds for $q_{i,g}$ and p_g;
- the set $\{q_{0,g}, \cdots, q_{k,g}\}$ is ε-dense on M;
- the stable manifold of $q_{i,g}$ contains a C^1-disc centered in $q_{i,g}$, of radius r and tangent to $\mathcal{C}^E_\theta$, for every $i = 1, \cdots, m$. Similarly, the unstable manifold of $q_{i,g}$ contains a C^1-disc centered in $q_{i,g}$, of radius r and tangent to $\mathcal{C}^F_\theta$.

Let $x \in A_g \cap \mathcal{R}_g$. By lemma 3.3 and by our choice of r and θ, there exists some $n \geq 0$ such that $W^s(g^{-nN}(x))$ contains a C^1-disc of radius r and tangent to $\mathcal{C}^E_\theta$. There is some hyperbolic periodic point $q_{i,g}$ which is ε-close to $g^{-nN}(x)$, thus

$$W^s(g^{-nN}(x), g) \pitchfork W^{uu}(q_{i,g}, g) \neq \emptyset,$$

By (46) for $q_{i,g}$ and p_g and by the inclination lemma, we conclude that $W^s(x, g) \pitchfork W^{uu}(p_g, g) \neq \emptyset$. The argument is analogous for $x \in B_g \cap \mathcal{R}_g$. □

We remark that a homoclinic class with dominated splitting has a dense set of periodic points such that each of these points has an iterate with either the stable or unstable manifold of uniform size. Without the chain-hyperbolicity condition, we cannot guarantee the existence of a dense set of periodic points whose both stable and unstable manifolds have uniform size, this property was crucial in the proof of proposition 3.5.

Let us prove that any $g \in \mathcal{U}_3 \cap \mathrm{Diff}^2_\omega(M)$ is Bernoulli. Recall that we defined in section 2 the sets $h^s(p_g)$ and $h^u(p_g)$. By proposition 3.5, we have $(A_g \cap \mathcal{R}_g) \subset h^s(p_g)$ and $(B_g \cap \mathcal{R}_g) \subset h^u(p_g)$. Since the set of regular points $\mathcal{R}_g$ has full measure, by lemma 3.1 we conclude that

$$m(h^s(p_g)) > 0 \text{ and } m(h^u(p_g)) > 0.$$

Corollary 2.12 implies that $h_{ber}(p_g) \stackrel{\circ}{=} h^s(p_g) \stackrel{\circ}{=} h^u(p_g)$ and $(g^{\pi(p_g)}|_{h_{ber}(p_g)}, m|_{h_{ber}(p)})$ is Bernoulli. By lemma 3.2, we obtain that $m(h_{ber}(p_g)) = 1$, which implies that g is Bernoulli. This concludes the proof of theorem B.

Remark 3.6. *Lemmas 3.1, 3.2 and proposition 3.5 also hold for diffeomorphisms $g \in \mathcal{U}_i \cap \mathrm{Diff}^1_\omega(M)$, for $i = 1, 2$.*

4. Proof of Theorem C

Recall that if f is a weakly partially hyperbolic diffeomorphism with dominated splitting $TM = E \oplus E^{uu}$, then the unstable direction is uniquely integrable by a foliation $\mathcal{W}^{uu}$. For a point $x \in M$, let $W^{uu}(x)$ be the leaf that contains the point x.

Definition 4.1. *The unstable foliation of a weakly partially hyperbolic diffeomorphisms f is dynamically minimal if for any point $x \in M$, the set*

$$W^{uu}(O(x)) := \bigcup_{n \in \mathbb{Z}} W^{uu}(f^n(x), f)$$

is dense on the manifold.

The key property in the proof of theorem C is given in the following proposition, which is a consequence of chain-hyperbolicity.

Proposition 4.2. *Let f be a C^1-diffeomorphism, which does not have to preserve a volume form. If f is weakly partially hyperbolic with dominated splitting $TM = E \oplus E^{uu}$ and is chain-hyperbolic with respect to the same splitting, then the unstable foliation $\mathcal{W}^{uu}$ is dynamically minimal.*

PROOF. Let $p \in M$ be a hyperbolic periodic point such that $H(p) = M$, given in the definition of chain-hyperbolicity. By the definition of homoclinic class it is immediate that $\mathcal{W}^{uu}_f(O(p))$ is dense on M. By lemma 2.4, there is a dense set $\mathcal{P}$ of periodic points homoclinically related to p, such that for any $q \in \mathcal{P}$ its stable manifold contains the plaque $\mathcal{W}^E_q$. In particular every $q \in \mathcal{P}$ also verifies that $W^{uu}(O(q))$ is dense on M.

For any point $x \in M$, there exists a periodic point $q \in \mathcal{P}$ such that $W^{uu}(x) \pitchfork \mathcal{W}^E_q \neq \emptyset$. This is an immediate consequence of the density of the set $\mathcal{P}$, the uniform size of the plaques $\mathcal{W}^E_q$ and the fact that such plaques are tangent to a cone $\mathcal{C}^E_\theta$ for some small θ. The proposition then follows by the inclination lemma. □

Remark 4.3. *With an adaptation of this argument one can actually obtain the minimality of the unstable foliation, i.e., every unstable leaf is dense. The adaptation of the argument will appear in a work of Gabriel Nuñez and Jana Rodriguez Hertz, where they study minimal foliations and stable ergodicity for weakly partially hyperbolic systems.*

Let $f \in \mathrm{Diff}^2_\omega(M)$ be a weakly partially hyperbolic diffeomorphism. For any $x \in M$ consider two small discs T_1 and T_2 close to x and transverse to $W^{uu}_{loc}(x)$. The unstable holonomy between T_1 and T_2 is the map $H : T_1 \to T_2$ defined as $H(q) = W^{uu}_{loc}(q) \cap T_2$, this map is well defined by the transversality of the discs T_1 and T_2 and the fact that $W^{uu}_{loc}(q)$ vary continuously with the choice of q. Since f is C^2, it is well known that the unstable foliation is *absolutely continuous*, that is, the map H takes sets of zero Lebesgue measure inside T_1 into sets of zero measure inside T_2.

In the C^2-scenario we obtain the following lemma, which is an adaptation of an argument due to Brin in [Br75], for the weakly partially hyperbolic scenario.

Lemma 4.4. *Let $f \in \mathrm{Diff}^2_\omega(M)$ be a weakly partially hyperbolic diffeomorphism with dominated splitting $TM = E \oplus E^{uu}$. If f is chain-hyperbolic with respect to the same splitting, then m-almost every point has dense orbit.*

PROOF. First observe that it is enough to prove that for any open set $U \subset M$, the set of points whose orbit intersects U has full m-measure. A point $x \in M$ is backwards recurrent if it is an accumulation point of the sequence $(f^{-n}(x))_{n\in\mathbb{N}}$. Let $R \subset M$ the set of backwards recurrent points, by Poincaré recurrence theorem this set has full m-measure. It is a classical consequence of the absolute continuity of the unstable foliation that there exists a set $\Lambda \subset M$ of full m-measure such that for any point $x \in \Lambda$, the set $W^{uu}(x) \cap R$ has full Lebesgue measure inside the submanifold $W^{uu}(x)$, see for instance lemma 5 in [BDP02]. Observe that we can suppose that the same holds for $f^n(x)$, for any $n \in \mathbb{Z}$.

Fix an open set U and take $x \in \Lambda$. By proposition 4.2, there exists $k \in \mathbb{Z}$ such that $W^{uu}(f^k(x)) \cap U \neq \emptyset$. Since the set U is open, the set $W^{uu}(f^k(x)) \cap U$ has positive Lebesgue measure inside $W^{uu}(f^k(x))$. In particular, there exits a point $y \in W^{uu}(f^k(x)) \cap U$ which is backwards recurrent. Since the unstable manifold contracts for backwards iterates and by the backwards recurrence of y, there exists $n \in \mathbb{N}$ such that $f^{k-n}(x) \in U$. This concludes the proof of the lemma. □

We now proceed to the proof of theorem C. Let f be a diffeomorphism verifying the hypothesis of theorem C. Let $A \subset M$ be the f-invariant set of points such that all the Lyapunov exponents along the direction E are negative. By hypothesis, the set A has positive measure. Since the direction E^{uu} is uniformly hyperbolic, a standard argument using Birkhoff's ergodic theorem and the absolute continuity of the Pesin manifolds and the strong unstable foliation, implies that every ergodic component of $f|_A$ coincides with

an open set (mod 0), see for instance the proof of theorem 1 in [BDP02]. By lemma 4.4, m-almost every point has dense orbit. We can easily conclude that f is ergodic.

By ergodicity, m-almost every point has all its exponents negative along the direction E. This implies that for $N \in \mathbb{N}$ large enough

$$\int \log \|Df^N|_E\| dm < 0.$$

By lemma 2.6, the diffeomorphism f^N is chain-hyperbolic. Theorem B implies that f^N is stably Bernoulli. Let $p \in Per(f)$ be the hyperbolic periodic point in the definition of chain hyperbolicity. From the proof of theorem B, for any $g \in \mathrm{Diff}^2_\omega(M)$ in a C^1-neighborhood of f, it holds that $m(h_{ber}(g)) = 1$, which implies that f is stably Bernoulli.

5. Proof of Theorem D

Let $f \in \mathcal{WCH}^2_\omega(M)$. There exists a dominated splitting $TM = E \oplus E^{uu}$ such that $dim(E) = 2$. We separate the proof of theorem D in two cases. The first case is when there exists a sequence $(g_n)_{n\in\mathbb{N}}\mathcal{WCH}^2_\omega(M)$ converging to f in the C^1-topology, such that for each g_n the subbundle E_{g_n} admits a dominated splitting into two one dimensional bundles $E = E^1_{g_n} \oplus E^2_{g_n}$. The second case is when C^1-robustly inside $\mathcal{WCH}^2_\omega(M)$ the center direction does not admit any further dominated decomposition.

Case 1: In this case we have that arbitrarily C^1-close to f, there exists a diffeomorphism $g_n \in \mathcal{WCH}^2_\omega(M)$ such that $TM = E^1_{g_n} \oplus E^2_{g_n} \oplus E^{uu}$, with $dim(E^i_{g_n}) = 1$, for $i = 1, 2$. Since $E^1_{g_n}$ is one dimensional and g_n preserves volume, it follows that $E^1_{g_n}$ is uniformly contracted, see proposition 0.5 in [BDP03]. Hence, we have a partially hyperbolic diffeomorphisms with one dimensional center. By theorem A' in [ACW17], we have that g_n is C^1-approximated by a stably Bernoulli diffeomorphism.

Case 2: In this case, using theorem A from [ACW16], we take a diffeomorphism $g \in \mathrm{Diff}^1_\omega(M)$ arbitrarily C^1-close to f, which is non-uniformly hyperbolic and has negative exponent in the direction E_g. Thus, for $N \in \mathbb{N}$ large enough

$$\int_M \log \|Dg^N|_{E_g}\| dm < 0. \tag{47}$$

Recall that condition (47) is C^1-open. By theorem 1 from [Avi10], we can take a diffeomorphism $\tilde{g} \in \mathrm{Diff}^2_\omega(M)$ arbitrarily C^1-close to g such that $\tilde{g}$ verifies (47). By the definition of $\mathcal{WCH}^2_\omega(M)$, we can assume that $\tilde{g} \in \mathcal{WCH}^2_\omega(M)$. By lemma 2.6, $\tilde{g}^N$ is chain-hyperbolic. Using theorem B, we conclude that $\tilde{g}^N$ is stably Bernoulli. By similar reason as in the end of the proof of theorem C, we conclude that $\tilde{g}$ is stably Bernoulli.

6. The example

In theorem C of [BV00], the authors give the first example of a robustly transitive diffeomorphism having no invariant hyperbolic subbundle. Tahzibi proved in [Tah04] the stable ergodicity of this example. The construction is made by deforming an Anosov diffeomorphism inside small balls. In this section we explain the construction of the example in a not so local way so that the hypothesis of theorem B holds. In a certain way theorem B quantifies how much the Anosov diffeomorphism can be deformed and keep the stable ergodicity.

Let $A \in SL(4, \mathbb{Z})$ be a hyperbolic matrix with four distinct eigenvalues $0 < \lambda_{ss} < \lambda_s < 1 < \lambda_u < \lambda_{uu}$, with unit eigenvectors e_{ss}, e_s, e_u and e_{uu}. On $\mathbb{R}^4$ consider the coordinate system formed by the basis $\{e_{ss}, e_s, e_u, e_{uu}\}$. We write A^s the restriction of A to the stable directions and A^u the restriction of A to the unstable directions.

Consider the Anosov diffeomorphisms $f_A : \mathbb{T}^4 \to \mathbb{T}^4$ induced by A, with hyperbolic splitting $T\mathbb{T}^4 = E^{ss} \oplus E^s \oplus E^u \oplus E^{uu}$. Let p_1 and p_2 be two fixed points of f_A.

For each $a, b \in (0, 1)$, let $U^{a,b}_1$ and $U^{a,b}_2$ be neighborhoods of p_1 and p_2, respectively, defined as follows: let $\exp_{p_1} : T_{p_1}\mathbb{T}^4 \to \mathbb{T}^4$ be the exponential map on the point p_1 and define $U^{a,b}_1 = \exp_{p_1}(D^2_a \times D^2_b)$, where $D^2_a \times D^2_b$ is the product of two discs of radius a and

b, respectively, and D_a^2 is contained in the subspace generated by $\{e_{ss}, e_s\}$ and D_b^2 on the subspace generated by $\{e_u, e_{uu}\}$. Similarly we define $U_2^{a,b} = \exp_{p_2}(D_a^2 \times D_b^2)$. Observe that the exponential map, $\exp_{p_i}(.)$, sends sets of the form $\{x\} \times D_b^2$ on unstable manifolds of f_A inside $U_i^{a,b}$, for $i = 1, 2$ and $x \in D_a^2$. Similarly it sends sets of the form $D_a^2 \times \{y\}$ on stable manifolds of f_A.

Fix $R > 0$ such that for a and b sufficiently small, $U_1^{a,R} \cap U_2^{R,b} = \emptyset$. Write $U_1^a = U_1^{a,R}$ and $U_2^b = U_2^{R,b}$. We will describe the construction in U_1^a, the construction in U_2^b is analogous.

Let $g : D_1^2 \times D_R^2 \to D_1^2$, be a smooth map with the following properties:

(1) for each $y \in D_R^2$, $g(., y)$ is a diffeomorphism of D_1^2, which is the identity in a neighborhood of the boundary of D_1^2 and preserves area;

(2) $g(., y)$ is the identity if y belongs to a neighborhood of the boundary of D_R^2;

(3) $\|D_x g\| < \lambda^u$;

(4) for $y = 0$, the composition $g_0(A^s(x)) = g(A^s(x), 0)$ has three fixed points on D_1^2, one saddle and two sinks, where one of the sinks has a complex eigenvalue.

Such map can be obtained using Hamiltonian flows, see section 6 of [BV00]. For each $a \in (0, 1)$, consider the diffeomorphism

$$\begin{array}{rccc} \tilde{g}_a : D_a^2 \times D_R^2 & \longrightarrow & D_a^2 \times D_R^2 \\ (x, y) & \mapsto & (ag(a^{-1}x, y), y). \end{array}$$

By properties 1 and 2 of the map g, using the exponential chart $\exp_{p_1}$, we extend the diffeomorphism $\tilde{g}_a$ to a diffeomorphism G_a of $\mathbb{T}^4$, such that $G_a(q) = \exp_{p_1} \circ \tilde{g}_a \circ \exp_{p_1}^{-1}(q)$, if $q \in U_1^a$ and $G_a(q) = q$ otherwise. By item 1, we have that G_a preserves volume. For each $a \in (0, 1)$, consider the diffeomorphism $f_a = G_a \circ f_A$ of $\mathbb{T}^4$. Property 4 of g implies that f_a has a fixed point of index 1 and another fixed point of index 2 with complex stable eigenvalue.

Observe that if $f_A(x, y) \in U_1^a$, then using the coordinates $(E^{ss} \oplus E^s) \oplus (E^u \oplus E^{uu})$ we obtain

$$Df_a(x, y) = \begin{pmatrix} D_x g(a^{-1}A^s(x), A^u(y))A^s & aD_y g(a^{-1}A^s(x), A^u(y))A^u \\ 0 & A^u \end{pmatrix}.$$

Property 3 of the map g implies that $\|D_x g\|.\|A^s\| < \lambda^u$. Thus, if a is small enough Df_a expands vectors uniformly inside a thin cone $\mathcal{C}^u$ around the directions $E^u \oplus E^{uu}$, there is a dominated splitting $T\mathbb{T}^4 = E_a^{cs} \oplus E_a^u$ such that $dim(E_a^{cs}) = 2$ and E_a^{cs} does not admit any further decomposition.

By a similar construction, exchanging U_1^a by U_2^b and exchanging the roles of the stable and unstable directions, for each $b \in (0, 1)$, we obtain a volume preserving diffeomorphism H_b of $\mathbb{T}^4$. We consider the two parameter family of conservative diffeomorphisms $f_{a,b} = H_b \circ G_a \circ f_A$. We now describe a few properties that $f_{a,b}$ has for a and b small.

(1) $f_{a,b}$ admits a dominated splitting of the form $T\mathbb{T}^4 = E_{ab}^{cs} \oplus E_{ab}^{cu}$, where $dim(E_{ab}^{cs}) = 2$. It does not admit any further dominated decomposition. We also have that E_{ab}^{cs} converges to $E^{ss} \oplus E^s$ and E_{ab}^{cu} converges to $E^u \oplus E^{uu}$ when a, b goes to zero.

(2) $f_{a,b}$ has one periodic point of index 1 and one periodic point of index 3.

(3) There is a thin cone $\mathcal{C}^s$ around the direction E_{ab}^{cs}, such that if $x \notin U_2^b$ then $Df_{a,b}^{-1}(x)$ expands vectors uniformly in $\mathcal{C}^s$. Similarly, there is a thin cone $\mathcal{C}^u$ around the direction E_{ab}^{cu}, such that if $x \notin U_1^a$ then $Df_{a,b}(x)$ expands vectors uniformly inside $\mathcal{C}^u$.

(4) It holds that

$$\int_{\mathbb{T}^4} \log \|Df_{a,b}|_{E_{ab}^{cs}}\| dm < 0 \text{ and } \int_{\mathbb{T}^4} \log \|Df_{a,b}^{-1}|_{E_{ab}^{cu}}\| dm < 0. \tag{48}$$

Let us explain why this property holds. Notice that there exists a constant C_1 such that $m(U_1^a) < C_1 a^2$ and $m(U_1^b) < C_1 b^2$. There exists $C_2 > 1$ such that

$C_2^{-1} < \|(Df_{a,b})^{-1}\|^{-1} < \|Df_{a,b}\| < C_2$, for any a and b small enough. Using property (c) of $f_{a,b}$ we can easily conclude (48).

Observe that such properties are C^1-open. We fix a periodic point q, whose orbit remains outside $U_1^a \cup U_2^b$. Take $f \in \mathrm{Diff}^1_\omega(\mathbb{T}^4)$ a diffeomorphisms sufficiently C^1-close to $f_{a,b}$ such that the homoclinic class $H(q_f)$ is the entire manifold $\mathbb{T}^4$, where q_f is the continuation of the hyperbolic periodic point q. This is possible since C^1-generically in $\mathrm{Diff}^1_\omega(\mathbb{T}^4)$ the entire manifold is the homoclinic class of any periodic point, see theorem 1.3 in [BC03].

We now explain how to obtain trapped plaque families, as in the definition of chain-hyperbolicity. Let C_2 be the constant that appeared in the explanation of property (d) of $f_{a,b}$. Fix $r > 0$ such that for any $p \in \mathbb{T}^4$, any disc $D(x)$ with radius r, center x and tangent to $\mathcal{C}^s$, the set $D(x) \cap U_1^a$ has at most one connected component. Let $\rho = \frac{r}{C_2}$.

Lemma 6.1. *If a is small enough, for any $x \in \mathbb{T}^4$, any disc $D(x)$ tangent to $\mathcal{C}^s$ with radius ρ and centered in x, it holds that $f^{-1}(D(x))$ strictly contains a disc of radius ρ, centered in $f^{-1}(x)$ and tangent to $\mathcal{C}^s$.*

Proof. Let $D(x)$ be such a disc and let $\mu > 1$ be a constant such that for any unit vector $v \in \mathcal{C}^s(x)$ and $x \notin U_1^a$, it holds that $\|Df^{-1}(x)v\| > \mu$. This comes from property (c) of $f_{a,b}$ and the fact that f is C^1-close to $f_{a,b}$.

Domination implies that $f^{-1}(D(x))$ is tangent to $\mathcal{C}^s$. By our choice of ρ, we have that $f^{-1}(D(x)) \cap U_1^a$ has at most one connected component. Since $f^{-1}(D(x))$ is tangent to the cone $\mathcal{C}^s$, there exists a constant $C_3 > 1$ such that the diameter of the set $f^{-1}(D(x)) \cap U_1^a$ is bounded from above by $C_3 a$.

Let $\gamma \subset f^{-1}(D(x))$ be a curve minimizing distance between $f^{-1}(x)$ and $\partial f^{-1}(D(x))$. It holds that $l(\gamma) > \rho \|Df^{-1}\|^{-1} > \frac{\rho}{C_2}$, where $l(\gamma)$ is the length of the curve γ. We split γ in two parts: $\gamma_1 = \gamma \cap U_1^a$ and $\gamma_2 = \gamma - \gamma_1$. Observe that $l(\gamma_1) < C_3 a$. Thus

$$\frac{l(\gamma_1)}{l(\gamma)} < \frac{C_2 C_3 a}{\rho} = Ka.$$

Thus,

$$\begin{aligned} l(f(\gamma)) &= l(f(\gamma_1)) + l(f(\gamma_2)) < C_2 l(\gamma_1) + \mu^{-1} l(\gamma_2) \\ &< C_2 K a l(\gamma) + \mu^{-1} l(\gamma) = (C_2 K a + \mu^{-1}) l(\gamma) < l(\gamma), \end{aligned}$$

where the last inequality holds for $a > 0$ small enough. Observe that $f(\gamma)$ is a curve connecting x to $\partial D(x)$, we conclude that $d(x, \partial D(x)) < d(f^{-1}(x), \partial f^{-1}(D(x)))$, so $f^{-1}(D(x))$ contains a disc centered in $f^{-1}(x)$ with radius ρ. □

Following the proof of theorem 3.1 in [BF13], using lemma 6.1 in the place of claim 3.2 in the same paper, a construction using graph transforms allows us to obtain a plaque family $(\mathcal{W}^{cs}_x)_{x \in \mathbb{T}^4}$ which is trapped for f, such that any plaque $\mathcal{W}^{cs}_x$ is a disc of center x, radius ρ and is tangent to $\mathcal{C}^s$. Similarly, for b small enough we obtain a plaque family $(\mathcal{W}^{cu}_x)_{x \in \mathbb{T}^4}$, which is trapped for f^{-1}.

By taking a, b small enough, we can also suppose that

$$\bigcup_{q_f \in O_f(q_f)} (\mathcal{W}^{cs}_{q_f} \cup \mathcal{W}^{cu}_{q_f}) \cap (U_1^a \cup U_2^b) = \emptyset. \tag{49}$$

A standard argument known as the coherence argument (see for instance the argument used in step 2 in the proof of theorem 3.1 in [BF13]), implies that the plaques $\mathcal{W}^{cs}_{q_f}$ and $\mathcal{W}^{cu}_{q_f}$ are contained in the stable and unstable manifolds of q_f, respectively. Thus, conditions 3 and 4 in definition 2.3 hold.

Property 1 of $f_{a,b}$ implies that condition 2 in definition 2.3 is verified. Since $\mathbb{T}^4$ is the homoclinic class of q_f, we conclude that condition 1 in definition 2.3 is verified. Therefore, f is chain-hyperbolic.

We conclude that all the conditions in the hypothesis of theorem B are verified, thus there is a C^1-neighborhood $\mathcal{U}$ of f such that any diffeormorphism $g \in \mathcal{U} \cap \mathrm{Diff}^2_\omega(\mathbb{T}^4)$ is Bernoulli.

We remark that this construction is not local because the cilinders U_1^a and U_2^b, where we made the changes, have a fix size R in one of the directions, either E^{cs} or E^{cu}.

Bibliography

[ABC11] F. Abdenur, C. Bonatti, and S. Crovisier. Nonuniform hyperbolicity for C^1-generic diffeomorphisms. *Israel J. Math.*, 183:1–60, 2011.

[ABV00] J. Alves, C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Inventiones mathematicae*, 140:351–398, 2000.

[Ano67] D. Anosov. Geodesic flows on closed Riemannian manifolds of negative curvature. *Trudy Mat. Inst. Steklov.*, 90, 1967.

[AS67] D. Anosov and Y. Sinaĭ. Certain smooth ergodic systems. *Uspehi Mat. Nauk*, 22:107–172, 1967.

[AM07] A. Arbieto and C. Matheus. A pasting lemma and some applications for conservative systems. *Ergodic Theory and Dynamical Systems*, 27:1399–1417, 2007.

[AMP04] A. Arbieto, C. Matheus and M.J. Pacífico. The Bernoulli property for weakly hyperbolic systems. *J. Statist. Phys.*, 117:243–260, 2004.

[Avi10] A. Avila. On the regularization of conservative maps. *Acta Math*, 205:5–18, 2010.

[ACW17] A. Avila, S. Crovisier and A. Wilkinson. C^1 density of stable ergodicity. *Preprint: arXiv:1709.04983*, 2017.

[ACW16] A. Avila, S. Crovisier and A. Wilkinson. Diffeomorphisms with positive metric entropy. *Publ. Math. Inst. Hautes Études Sci.*, 124:319–347, 2016.

[BP02] L. Barreira and Y. Pesin. Lyapunov exponents and smooth ergodic theory, volume 23 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2002.

[BC03] C. Bonatti and S. Crovisier. Récurrence and généricité. *C. R. Math. Acad. Sci. Paris*, 336:839–844, 2003.

[BDP03] C. Bonatti, L. Díaz and E. Pujals. A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. *Ann. of Math.*, 158:355–418, 2003.

[BV00] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.

[Br75] M. Brin Topological transitivity of a certain class of dynamical systems, and flows of frames on manifolds of negative curvature. *Funkcional. Anal. i Priložen.*, 1:9–19, 1975.

[BP74] M. Brin and Y. Pesin. Partially hyperbolic dynamical systems. *Izv. Akad. Nauk SSSR Ser. Mat.*, 38:170–212, 1974.

[BDP02] K. Burns, D. Dolgopyat, and Y. Pesin. Partial hyperbolicity, Lyapunov exponents and stable ergodicity. *Journal of Statistical Physics*, 108:927–942, 2002.

[BW10] K. Burns and A. Wilkinson. On the ergodicity of partially hyperbolic systems. *Ann. of Math.*, 171:451–489, 2010.

[BF13] J. Buzzi and T. Fisher Entropic stability beyond partial hyperbolicity. *J. Mod. Dyn.*, 4:527–552.

[Cro11] S. Crovisier. Partial hyperbolicity far from homoclinic bifurcations. *Advances in Mathematics*, 226:673–726, 2011.

[CP15] S. Crovisier and E. Pujals. Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms. *Inventiones mathematicae*, 201:385–517, 2015.

[GPS94] M. Grayson, C. Pugh, and M. Shub. Stably ergodic diffeomorphisms. *Ann. of Math.*, 140:295–329, 1994.

[HHTU11] F.R. Hertz, M.R. Hertz, A. Tahzibi, and R. Ures. New criteria for ergodicity and nonuniform hyperbolicity. *Duke Math. J.*, 160:599–629, 2011.

[HPS77] M. Hirsch, C. Pugh, and M. Shub. *Invariant manifolds.* Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977.

[Hop39] E. Hopf. *Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung.* 1939.

[Mn87] R. Mañé. *Ergodic theory and differentiable dynamics*, volume 8 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1987.

[PDM82] J. Palis and W. De Melo. Geometric theory of dynamical systems. Springer-Verlag, New York-Berlin, 1982.

[Pes77] Y. Pesin. Characteristic Ljapunov exponents, and smooth ergodic theory. *Uspehi Mat. Nauk*, 32:55–112, 1977.

[Pot15] R. Potrie. Partially hyperbolic diffeomorphisms with a trapping property. *Discrete Contin. Dyn. Syst.*, 35:5037–5054, 2015.

[PS97] C. Pugh and M. Shub. Stable ergodicity and stable accessibility. In *Differential equations and applications (Hangzhou, 1996)*, pages 258–268. Int. Press, Cambridge, MA, 1997.

[PS00] C. Pugh and M. Shub. Stable ergodicity and julienne quasi-conformality. *J. Eur. Math. Soc.*, 2:1–52, 2000.

[Tah04] A. Tahzibi. Stably ergodic diffeomorphisms which are not partially hyperbolic. *Israel J. Math.*, 142:315–344, 2004.

CHAPTER 3

On the genericity of positive exponents of conservative skew products with two-dimensional fibers

This chapter is based on a joint work with Mauricio Poletti. [1]

1. Introduction

In the 60's, Smale had obtained several results about dynamical implications of uniform hyperbolicity (see [Sm67]). Since then, uniform hyperbolic dynamics have been very well understood. For instance, hyperbolic transitive sets have several features, such as a symbolic dynamics associated to it, existence of periodic points and horseshoes, positive entropy, ergodicity (in the volume preserving scenario), etc. Even though, uniform hyperbolicity is a C^1-open property, it is not a C^1-dense property.

For invariant measures, Pesin proposed in [Pe77] a weaker notion of hyperbolicity, called non-uniform hyperbolicity. A diffeomorphism f that preserves a probability measure μ is called non-uniformly hyperbolic if all its Lyapunov exponents are non-zero (see [BP01] for precise definitions). It turns out that non-uniform hyperbolicity also imply several interesting features of the dynamics, such as existence of periodic orbits and horseshoes [Ka80], countably many ergodic components for smooth measures [Pe77], etc.

Given a probability measure μ on a compact Riemannian manifold M, one can consider the space of C^r-diffeomorphisms that preserves this measure $\mathrm{Diff}^r_\mu(M)$. A natural question is to know how frequent is non-uniform hyperbolicity in $\mathrm{Diff}^r_\mu(M)$?

There are several results related to this question, most of them for the case when μ is the Lebesgue measure. For instance, a remarkable result by Mañé [Ma96] and Bochi [Bo02] proved that on surfaces any area preserving diffeomorphism which is not Anosov can be C^1-approximated by an area preserving diffeomorphism with some zero Lyapunov exponent (see section 2 for the definition of Anosov diffeomorphism).

In this paper we address this question for some skew products over hyperbolic maps. Let us define the scenarios we will be working with.

Let $\tilde{M}$ be a smooth, compact, connected and oriented manifold and S be a smooth, compact and connected surface. Consider a fiber bundle M over $\tilde{M}$, defined by a smooth projection $\pi : M \to \tilde{M}$, with fibers diffeomorphic to S. For a point $x \in M$, we write S_x the fiber that contains the point x. We say that a diffeomorphism $f : M \to M$ *preserves fibers* if for any $x \in M$ it holds $S_{f(x)} = f(S_x)$.

A diffeomorphism f is *partially hyperbolic* if there is a Df-invariant decomposition of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, such that $Df|_{E^s}$ contracts, $Df|_{E^u}$ expands and the behavior of $Df|_{E^c}$ is bounded by the contraction along E^s and the expansion along E^u, see section 2.2 for a precise definition.

For the fiber bundle M a diffeomorphism $f : M \to M$ is a *partially hyperbolic skew product* if the following holds:

- f preserves fibers;
- f is a partially hyperbolic diffeomorphism, with splitting $TM = E^s \oplus E^c \oplus E^u$, such that $E^c = \ker D\pi$.

1. CNRS-Laboratoire de Mathématiques d'Orsay, UMR 8628, Université Paris-Sud 11, Orsay Cedex 91405, France.

Let leb be the normalized Lebesgue measure on M and define $SP^r_{\text{leb}}(M)$ to be the set of C^r-partially hyperbolic skew products that preserve the Lebesgue measure. In the space $SP^r_{\text{leb}}(M)$ we consider the C^s-topology, for any $s \leq r$.

By Oseledets' theorem (see for instance [BP01]), for Lebesgue almost every point $x \in M$, the greatest and smallest Lyapunov exponents along the center direction, defined respectively by

$$\lambda_c^+(x) = \lim_{n\to+\infty} \frac{1}{n}\left\|Df^n(x)|_{E^c_x}\right\| \text{ and } \lambda_c^-(x) = -\lim_{n\to+\infty} \frac{1}{n}\left\|Df^{-n}(x)|_{E^c_x}\right\|,$$

exist. The conditions in the definition of partially hyperbolic skew product implies that $\det Df(x)|_{E^c_x} = 1$ (see section 2). This implies that for almost every point $x \in M$ it is verified that $\lambda_c^-(x) = -\lambda_c^+(x)$. We define the *integrated Lyapunov exponent along the center direction* by

$$L(f) = \int_M \lambda_c^+(x) d\text{leb}(x).$$

An important notion in the study of Lyapunov exponents for partially hyperbolic diffeomorphisms is the notion of center bunching, this is used to obtain the existence of linear holonomies, see section 2 for a precise definition. In this paper we prove the following theorem.

Theorem E. *For any $r > 1$, among the volume preserving, C^r-partially hyperbolic skew products that are center bunched, there exists a C^1-dense and C^r-open subset of diffeomorphisms verifying the following: if f belongs to this subset, then $L(f) > 0$.*

We say that $f \in SP^r_{\text{leb}}(M)$ is *non-uniformly hyperbolic* if for Lebesgue almost every point it holds that $\lambda^+(x) > 0$. From [AV] (or [HS17]), it is known that ergodicity is C^1-open and C^r-dense in the setting of the previous theorem. The next result follows immediately from theorem E.

Corollary 1.1. *In the same setting of theorem E, there exists a C^1-dense and C^r-open subset such that any diffeomorphism in this subset is non-uniformly hyperbolic.*

Marin in [Mar16] proved that a partially hyperbolic C^r-symplectomorphism with two dimensional center, which is accessible, verifies some center bunching condition, and has a periodic point satisfying some condition (which she calls pinching), can be C^r-approximated by ergodic non-uniformly hyperbolic symplectomorphisms. In her argument accessibility and the fact that it preserves some volume form (induced by the symplectic form) are crucial properties because she uses the results of [ASV13]. We remark that her result is not restricted to the skew product setting. It was furher improved by Liang-Marin-Yang, in [LMY18-2], that proved that in a C^r-neighborhood of a symplectomorphism verifying the conditions above, there is a C^r-open and C^r-dense subset of ergodic non-uniformly hyperbolic symplectomorphisms. We point out that differently from Marin's result, our arguments are not based on the results from [ASV13].

Let Σ be a compact metric space with no isolated points, let $\sigma : \Sigma \to \Sigma$ be a hyperbolic homeomorphism and $\tilde{\mu}$ be a σ-invariant measure that has a property called *local product structure* (see section 2.3 for precise definitions). This property holds for important measures such as the equilibrium states of Hölder potentials (see [Bow75]).

Let S be a compact, oriented C^r-surface. Fix some $\alpha > 0$, by abuse of notation let leb be the normalized Lebesgue measure on S and $\text{Diff}^r_{\text{leb}}(S)$ be the space of C^r diffeomorphisms that preserves leb. Given a (C, α)-Hölder map from Σ to $\text{Diff}^r_{\text{leb}}(S)$, $\tilde{x} \mapsto f_{\tilde{x}}$, by this we mean that

$$\text{d}_{C^r}(f_{\tilde{x}}, f_{\tilde{y}}) \leq C\, \text{d}_\Sigma(\tilde{x}, \tilde{y})^\alpha.$$

We define the skew product

$$\begin{array}{rcl} f : \Sigma \times S & \to & \Sigma \times S \\ (\tilde{x}, t) & \mapsto & f(\tilde{x}, t) = (\sigma(\tilde{x}), f_{\tilde{x}}(t)). \end{array}$$

Observe that such a skew product preserves the measure $\mu := \tilde{\mu} \times \text{leb}$. Such a map is called $C^{r,\alpha}$-*skew product* over σ that preserves μ.

From now on we fix $C > 0$. For $\alpha > 0$ and $r \geq 1 + \alpha$, we define $SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$ to be the space of $C^{r,\alpha}$ skew products over σ, such that the map $\tilde{x} \mapsto f_{\tilde{x}}$ is (C, α)-Hölder. In this space we consider the C^s-topology, for any $s \leq r$ defined as follows: for any two $C^{r,\alpha}$-skew products $f, g \in SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$, the C^s distance between f and g is

$$\mathrm{d}_{C^s}(f, g) = \sup_{\tilde{x} \in \Sigma} \mathrm{d}_{C^s}(f_{\tilde{x}}, g_{\tilde{x}}), \tag{50}$$

where $\mathrm{d}_{C^s}(f_{\tilde{x}}, g_{\tilde{x}})$ is the C^s distance on $\mathrm{Diff}^r_{\text{leb}}(S)$. Keep in mind that σ is always fixed.

As our map f is smooth on the fiber direction, we can define the center Lyapunov exponents as

$$\lambda^+(\tilde{x}, t) = \lim_{n \to +\infty} \frac{1}{n} \|Df^n_{\tilde{x}}(t)\| \text{ and } \lambda^-(\tilde{x}, t) = -\lim_{n \to +\infty} \frac{1}{n} \left\|Df^{-n}_{\tilde{x}}(t)\right\|,$$

where $f^n_{\tilde{x}} = f_{\sigma^{n-1}(\tilde{x})} \circ \cdots \circ f_{\tilde{x}}$. This is defined μ-almost everywhere.

Similar to the notion of center bunching, there is a notion of fiber bunching which guarantees the existence of linear holonomies, see section 2 for precise definitions.

Theorem F. *Let σ be a hyperbolic homeomorphism and let $\tilde{\mu}$ be a σ-invariant measure with local product structure. For any $r > 1$ and $\alpha > 0$, there exists a C^1-dense and C^r-open subset of $SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$ verifying the following: if f belongs to this subset, then $L(f) > 0$.*

In Theorem F we need to fix the Hölder constant because, as opposed to the C^r distance in the setting of Theorem E, the distance defined in (50) does not take in account the relation between the Hölder norm of $\tilde{x} \mapsto f_{\tilde{x}}$ and $\tilde{x} \mapsto g_{\tilde{x}}$. This hypothesis can be replaced by a finer C^r topology that takes into account the Hölder distance between the maps defining f and g.

Remark 1.2. *Actually in theorems E and F the open sets are C^1 open sets on subsets of C^r with bounded C^r norm if $r < 2$ and bounded C^2 norm if $r \geq 2$.*

A particular case very studied in the literature is called *random product of diffeomorphisms*, see for instance [KN10], [BRH17]. This is defined by a set of C^r-diffeomorphisms $f_1, \ldots, f_d \in \mathrm{Diff}^r_{\text{leb}}(S)$ and $p_1, \ldots, p_d$ positive real numbers such that $p_1 + \cdots + p_d = 1$ where the probability p of the diffeomorphism f_i to act on S is p_i. Formally this is a skew product of over the shift map in $\{1, \ldots, d\}^{\mathbb{Z}}$ with Bernoulli probability $P = p^{\mathbb{Z}}$. With our techniques, we obtain the following theorem.

Theorem G. *Fix $d \in \mathbb{N}$, a real number $r \geq 1$, and fix some probability measure p on $\{1, \ldots, d\}$ such that $p(\{i\}) < 1$, for $i = 1, \cdots, d$. Then there exists a C^1-open and C^1-dense subset of $\mathrm{Diff}^r_{\text{leb}}(S)^d$ such that if $(f_1, \ldots, f_d)$ belongs to this set its random product has positive integrated Lyapunov exponents.*

Observe that in this case we actually get a C^1 open set.

We can only get C^1 density because we use the result of [LY17] where they prove the C^1 density of volume preserving diffeomorphisms with positive Lyapunov exponents to find what we call *pinching* points (see definition 3.7), this result is not known in the C^r topology.

With some information on the periodic diffeomorphism, pinching can be found in some higher regularity, as we explain in section 9.1 we have the following result.

Theorem H. *Let f be as in theorem E, F or G and there exist some periodic fiber $S_{\tilde{p}}$ such that $f_{\tilde{p}} : S_{\tilde{p}} \to S_{\tilde{p}}$ has an elliptic periodic point. Then f is C^r-accumulated by open sets with positive integrated Lyapunov exponents. Moreover, in the random product case this sets are C^1 open.*

After the conclusion of this work, Barrientos and Malicet [BM18] send us a preprint of a similar result for the random product of diffeomorphisms. They prove that after fixing $k-1$ volume preserving surface diffeomorphisms, $f_1, \dots, f_{k-1}$, such that the action generated by them is ergodic then for any f_k inside a C^r dense and C^1 open subset of volume preserving diffeomorphisms, the random product of $f_1, \dots, f_k$ has positive exponents. They use a different approach that do not require some pinching condition, but ergodicity of the $k-1$-first diffeomorphisms is essential in their argument.

Idea of the proof. Our proof is based on the *invariance principle* (see Ledrappier [Le86], Avila-Viana [AV10]) which states that if a cocycle has only one exponent then the invariant measures for the projective cocycles have some extra invariance.

The bunching conditions guaranties the existence of a family of linear maps called stable and unstable holonomies (this is explained in section 3). The linear cocycle given by $(f, Df\,|_S)$ induce a cocycle F in the projective tangent bundle. By the invariance principle we get that if $\lambda_c^+ = \lambda_c^- = 0$ then the disintegration of every F-invariant measure that projects to μ is invariant by the stable and unstable holonomies μ-almost everywhere.

We use the properties of μ to prove that this su-invariance almost everywhere of the disintegration can be extended to leb-almost every point in every fiber $\tilde{x} \times S$. Then we perturb f such that we break the the su-invariance in a set of positive Lebesgue measure inside an f-periodic fiber. By the previous comments, F does not admit any su-invariant measure thus (f, Df_S) should have positive exponents.

To prove that the positive exponents are C^r stable we prove that the limit su-invariant measures is su-invariant.

This kind of strategy of breaking the invariance principle to prove genericity of positive exponents was used for linear cocycles over hyperbolic maps and measures with product strucure [Vi08], over partially hyperbolic accessible volume preserving maps (for linear cocycles [ASV13] and for the derivative cocycle in the center direction [Mar16]). In these cases the properties of the dynamics and the invariant measure allows to extend the su-invariant disintegration in a subset of full measure to a continuous disintegration everywhere, then the su-invariance can be broken by just breaking it in one point.

The main difficulty in our case is that we can not extend the disintegration to a continuous disintegration everywhere. A similar argument was used by the second author in [Pol18] for linear symplectic cocycles (independent of the base map) over partially hyperbolic skew products.

2. Preliminaries and precise statements

In this section we recall some definitions and give the precise statements of the main theorems.

2.1. Disintegration of measures. Let M be a fiber bundle over $\tilde{M}$, μ be a probability measure on M and $\tilde{\mu} = \pi_*\mu$, we say that a family of probability measures $\tilde{x} \mapsto \mu_{\tilde{x}}$, defined $\tilde{\mu}$-almost everywhere, is a disintegration of μ with respect to the fibers if

- for every $A \subset M$, $\tilde{x} \mapsto \mu_{\tilde{x}}(A)$ is measurable,
- $\mu(A) = \int \mu_{\tilde{x}}(A) d\tilde{\mu}(\tilde{x})$,
- $\mu_{\tilde{x}}(\pi^{-1}(\tilde{x})) = 1$.

The partition into fibers verifies a measurability condition and by Rokhlin disintegration theorem (see [VO15] chapter 5 for details) for any probability measure μ there exists a disintegration of μ with respect to the fibers. Moreover this disintegration is unique almost everywhere.

2.2. Partial hyperbolicity and restatement of theorem E. A C^r-diffeomorphism $f : M \to M$ is partially hyperbolic if the tangent bundle has a Df-invariant decomposition $TM = E^s \oplus E^c \oplus E^u$ and there is a riemannian metric such that for any $x \in M$ it holds

$$\begin{aligned} &\left\|Df(x)|_{E^s_x}\right\| < 1 < m(Df(x)|_{E^u_x}), \\ &\left\|Df(x)|_{E^s_x}\right\| < m(Df(x)|_{E^c_x}) \leq \left\|Df(x)|_{E^c_x}\right\| < m(Df(x)|_{E^u_x}), \end{aligned}$$

where $m(Df(x)) = \|Df(x)^{-1}\|^{-1}$ is the co-norm. It is well known that the distribution E^s is uniquely integrable, that is, it exists a unique foliation $\mathcal{F}^s$ tangent to E^s, whose leaves are C^r immersed submanifolds. For a point $x \in M$ we denote the leaf of such foliation that contains x by $W^{ss}(x)$ and we call it the *strong stable manifold* of x. Similarly, we define the strong unstable manifold of x and denote it by $W^{uu}(x)$. If the subbundle E^c is trivial, then we say that f is *Anosov*.

Recall that $SP^r_{\text{leb}}(M)$ is the space of partially hyperbolic skew products, for the fiber bundle M, with fibers S and base $\tilde{M}$. Let $f \in SP^r_{\text{leb}}(M)$, the invariance of the fibers implies that there exists some C^r-diffeomorphism $\tilde{f} : \tilde{M} \to \tilde{M}$ such that $\tilde{f} \circ \pi = \pi \circ f$. Since π is a smooth map, $\tilde{\mu} = \pi_* \text{leb}$ is a smooth volume measure in $\tilde{M}$.

Let $x \to \mu^c_x$ be the disintegration of leb on the fibers S_x. Since M is a smooth fiber bundle, the measure μ^c_x is a smooth volume measure on S_x and $x \to \mu^c_x$ is continuous in the weak* topology. Observe that the invariance of leb implies that for $\tilde{\mu}$-almost every $\tilde{x} \in \tilde{M}$, $f_{\tilde{x}*}\mu^c_{\tilde{x}} = \mu^c_{\tilde{f}(\tilde{x})}$, then by the continuity of the disintegration, this is actually true for every $\tilde{x} \in \tilde{M}$.

Lemma 2.1. *$\tilde{f} : \tilde{M} \to \tilde{M}$ is an Anosov diffeomorphism.*

Proof. Since the directions E^c and E^s have angle bounded away from 0, there exists a constant $C > 0$ such that for every $v \in E^s_x$, it holds

$$\frac{1}{C}\|v\| \leq \|D\pi(x)v\| \leq C\|v\|.$$

Let $\tilde{x} = \pi(x)$ and define $\tilde{E}^s_{\tilde{x}} = D\pi(x)E^s_x$. Since $D\pi(f(x))Df(x) = D\tilde{f}(\tilde{x})D\pi(x)$, for $\tilde{v} \in \tilde{E}^s_{\tilde{x}}$ such that $\tilde{v} = D\pi(x)v$, with $v \in E^s_x$, for every $n \in \mathbb{Z}$ we have

$$\begin{aligned}\left\|D\tilde{f}^n(\tilde{x})\tilde{v}\right\| &= \|D\pi(f^n(x))Df^n(x)v\| \\ &\leq C\|Df^n(x)v\| \\ &\leq C^2\left\|Df^n(x)\mid_{E^s_x}\right\|\|\tilde{v}\|.\end{aligned}$$

Analogously, $C^{-2}\left\|Df^n(x)\mid_{E^s_x}{}^{-1}\right\|^{-1}\|\tilde{v}\| \leq \left\|D\tilde{f}^n(\tilde{x})\tilde{v}\right\|$, hence, any $\tilde{v} \in \tilde{E}^s_{\tilde{x}}$ is contracted exponentially fast when $n \to \infty$ and expanded exponentially fast when $n \to -\infty$. We can also define $\tilde{E}^u_{\tilde{x}} = D\pi(x)E^u_x$, this subspace is contracted when $n \to -\infty$ and expanded when $n \to \infty$.

Since $T_xM = E^s_x \oplus E^u_x \oplus E^c_x$ and $E^c_x = \ker D\pi(x)$, we conclude that $T_{\tilde{x}}\tilde{M} = E^s_{\tilde{x}} \oplus E^s_{\tilde{x}}$. The exponential expansion and contraction of these directions implies that this subspaces are uniquely defined, meaning they do not depend on $x \in \pi^{-1}(\tilde{x})$. We can define such a splitting for any $\tilde{x} \in \tilde{M}$ and conclude that $D\tilde{f}(\tilde{x})\tilde{E}^*_{\tilde{x}} = \tilde{E}^*_{\tilde{f}(\tilde{x})}$ for $* = u, s$, therefore $\tilde{f}$ is Anosov. □

Definition 2.2 (α-center bunching)**.** *A C^1-partially hyperbolic diffeomorphism f is α-center bunched if for every $x \in M$ it holds*

$$\frac{\|Df(x)|_{E^c_x}\|}{m(Df(x)|_{E^c_x})}\|Df(x)|_{E^s_x}\|^\alpha < 1 \text{ and } 1 < \frac{m(Df(x)|_{E^c_x})}{\|Df(x)|_{E^c_x}\|}m(Df(x)|_{E^u_x})^\alpha.$$

Observe that this condition is C^1-open. For $\alpha > 0$, we define $CB^{r,\alpha}_{\text{leb}}(M)$ as the C^1-open set of α-center bunched, C^r-diffeomorphisms inside $SP^r_{\text{leb}}(M)$. If $\alpha = 1$ we just write $CB^r_{\text{leb}}(M) := CB^{r,1}_{\text{leb}}(M)$. We now state the precise statement of theorem E

Theorem E. [Restatement of theorem E] If $r = 1 + \alpha$, for $\alpha \in (0, 1)$, it exists a C^1-dense and C^r-open subset of $CB^{r,\alpha}_{\text{leb}}(M)$ such that for any diffeomorphism f in this subset it holds that $L(f) > 0$. If $r \geq 2$, then the same result holds inside $CB^r_{\text{leb}}(M)$.

2.3. Hyperbolic homeomorphisms and restatement of theorem F. Let Σ be a compact metric space with no isolated points. A homeomorphism $\sigma : \Sigma \to \Sigma$ is called *hyperbolic* if for some $\epsilon > 0$, for any $\tilde{x} \in \Sigma$, there exist *local stable* and *unstable sets* of $\tilde{x}$ with respect to σ defined by

$$W^s_{loc}(\tilde{y}) = \{\tilde{x}, \mathrm{d}(\sigma^k(\tilde{x}), \sigma^k(\tilde{y})) < \epsilon \text{ for every } k \geq 0\} \text{ and}$$
$$W^u_{loc}(\tilde{y}) = \{\tilde{x} : \mathrm{d}(\sigma^k(\tilde{x}), \sigma^k(\tilde{y})) < \epsilon \text{ for every } k \leq 0\}.$$

such that there exist $0 < \lambda < 1$ and $\tau > 0$ with the properties

(i) $\mathrm{d}(\sigma^n(\tilde{y}_1), \sigma^n(\tilde{y}_2)) \leq \lambda^n \mathrm{d}(\tilde{y}_1, \tilde{y}_2)$ for any $\tilde{y} \in \Sigma$, $\tilde{y}_1, \tilde{y}_2 \in W^s_{loc}(\tilde{y})$ and $n \geq 0$;

(ii) $\mathrm{d}(\sigma^{-n}(\tilde{y}_1), \sigma^{-n}(\tilde{y}_2)) \leq \lambda^n \mathrm{d}(\tilde{y}_1, \tilde{y}_2)$ for any $\tilde{y} \in \Sigma$, $\tilde{y}_1, \tilde{y}_2 \in W^u_{loc}(\tilde{y})$ and $n \geq 0$;

(iii) if $\mathrm{d}(\tilde{x}, \tilde{y}) \leq \tau$, then $W^u_{loc}(\tilde{x})$ and $W^s_{loc}(\tilde{y})$ intersect in a unique point, which is denoted by $[\tilde{x}, \tilde{y}]$ and depends continuously on $\tilde{x}$ and $\tilde{y}$.

Anosov diffeomorphisms, Markovian shifts, non trivial hyperbolic atractors and horseshoes are examples of hypebolic homeomorphisms. Property (iii) defines a local product structure of Σ, this means for every $\tilde{x} \in \Sigma$ there exists a neighborhood $\tilde{x} \in \mathcal{V} \subset \Sigma$ such that $[\cdot, \cdot] : W^s_{loc}(\tilde{x}) \times W^u_{loc}(\tilde{x}) \to \mathcal{V}$, $(\tilde{y}, \tilde{z}) \mapsto [\tilde{y}, \tilde{z}]$ is a homeomorphism.

Definition 2.3. *Let $\hat{\mu}$ be a σ-invariant probability measure, we say that $\hat{\mu}$ has* product structure *if locally in the product coordinates we can write $\hat{\mu} = \rho\mu^s \times \mu^u$, where μ^s is a measure on $W^s_{loc}(\tilde{x})$, μ^u is a measure on $W^u_{loc}(\tilde{x})$ and ρ is a positive measurable function. We also assume that $\hat{\mu}$ is fully supported.*

The local product structure property is verified by equilibrium states of Hölder potentials, also for the Lebesgue measure for Anosov diffeomorphisms. Consider the normalized Lebesgue measure on S, which by abuse of notation we write leb, and define $\mu = \hat{\mu} \times \text{leb}$. Observe that this is a f-invariant probability measure for every $f \in SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$.

Take $\beta > 0$, we say that f is β-*fiber bunched* if for any $\tilde{x} \in \Sigma$ it holds

$$\frac{\|Df_{\tilde{x}}\|}{m(Df_{\tilde{x}})}\lambda^\beta < 1 \text{ and } \frac{\|Df^{-1}_{\tilde{x}}\|}{m(Df^{-1}_{\tilde{x}})}\lambda^\beta < 1, \tag{51}$$

where $\|Df_{\tilde{x}}\| = \sup_{t \in S_{\tilde{x}}}\{\|Df_{\tilde{x}}(t)\|\}$ and $m(Df_{\tilde{x}}) = \sup_{t \in S_{\tilde{x}}}\{\|Df_{\tilde{x}}(t)^{-1}\|^{-1}\}$. Observe that this condition is C^1-open. Define $\mathcal{FB}^{r,\beta}_{\text{leb}}(\Sigma \times S)$ as the set of β-fiber bunched skew products inside $SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$.

Theorem F. [Restatement of theorem F] Let $\sigma : \Sigma \to \Sigma$ be a hyperbolic homeomorphism and $\hat{\mu}$ a σ-invariant probability measure with product structure let $\mu = \tilde{\mu} \times leb$. For $r > 1$ and $\alpha > 0$, there exists a C^r-open and C^1-dense subset of $\mathcal{FB}^{r,\alpha^2}_{\text{leb}}(\Sigma \times S) \subset SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$ such that any f belonging to this subset verifies $L(f, \mu) > 0$.

3. Holonomies and the invariance principle

In this section we are going to define the key concepts of holonomies that we are going to use. We will also introduce the invariance principle, which has been used many times in the study of Lyapunov exponents for cocycles.

3.1. Holonomies. Recall that by lemma 2.1, $\tilde{f} : \tilde{M} \to \tilde{M}$ is an Anosov map, in particular it is a hyperbolic homeomorphism. Therefore, we will define the following concepts in the topological setting.

For $\tilde{x} \in \Sigma$, we denote $S_{\tilde{x}}$ the fiber over $\tilde{x}$. In the topological case, as the space is a product the sub-index is not important but we use it just to stress that the definitions work for smooth fiber bundles.

We say that f admits α-*Hölder stable holonomies* if for every $\tilde{y} \in W^s_{loc}(\tilde{x})$ there exist functions $h^s_{\tilde{x}\tilde{y}} : S_{\tilde{x}} \to S_{\tilde{y}}$ such that

(a) $h^s_{\sigma^j(\tilde{x}),\sigma^j(\tilde{y})} = f^j_{\tilde{y}} \circ h^s_{\tilde{x},\tilde{y}} \circ (f^j_{\tilde{x}})^{-1}$ for every $j \geq 1$;

(b) $h^s_{\tilde{x},\tilde{x}} = \mathrm{id}$ and $h^s_{\tilde{x},\tilde{y}} = h^s_{\tilde{z},\tilde{y}} \circ h^s_{\tilde{x},\tilde{z}}$, for any $z \in W^s_{loc}(\tilde{x})$;

(c) there exists $L > 0$ such that $\mathrm{d}_{\Sigma\times S}(h^s_{\tilde{x},\tilde{y}}(t), t) \leq L\,\mathrm{d}_\Sigma(\tilde{x},\tilde{y})^\alpha$ for every $t \in S_{\tilde{x}}$;

(d) $\tilde{x},\tilde{y} \mapsto h^s_{\tilde{x},\tilde{y}}$ is uniformly continuous in $\{\tilde{x},\tilde{y} \in \Sigma : \tilde{y} \in W^s_{loc}(\tilde{x})\}$.

For $p \in \Sigma \times S$ define $Df_c(p) := Df_{\pi(p)}(p)$, where $\pi : \Sigma \times S \to \Sigma$ is the natural projection. We define the local strong stable set of $x \in \Sigma \times S$ as the set

$$W^{ss}_{loc}(x) = \{h^s_{\tilde{y},\tilde{x}}(x), \text{ where } \tilde{x} = \pi(x) \text{ and } \tilde{y} \in W^s_{loc}(\tilde{x})\}. \tag{52}$$

We remark that while the set $W^s_{loc}(\tilde{x})$ is the strong stable set for σ contained in the basis Σ, the set $W^{ss}_{loc}(x)$ is the strong stable set for the skew product f, which is contained in $\Sigma \times S$, see figure 1. We also remark that $\pi(W^{ss}_{loc}(x)) = W^s_{loc}(\tilde{x})$.

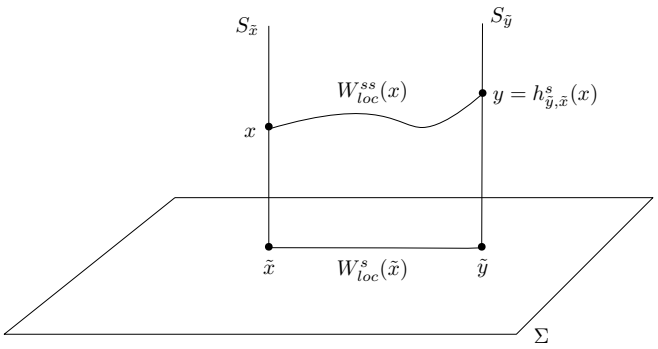

FIGURE 1. The strong stable set

Let $TS = \{(\tilde{x},t,v); (\tilde{x},t) \in \Sigma \times S, v \in T_tS_{\tilde{x}}\}$ be the fiber bundle tangent to S. For $y \in W^{ss}_{loc}(x)$, we say that f admits *α-Hölder linear stable holonomies*, if there exist linear maps $H^s_{xy} : T_xS_{\tilde{x}} \to T_yS_{\tilde{y}}$ such that:

(a) $H^s_{f^j(p),f^j(q)} = Df^j_c(q) \circ H^s_{p,q} \circ (Df^j_c(p))^{-1}$, for every $j \geq 1$;

(b) $H^s_{p,p} = \mathrm{id}$ and $H^s_{p,q} = H^s_{z,q} \circ H^s_{p,z}$, for any $z \in W^{ss}_{loc}(p)$;

(c) there exists $L > 0$ such that $\mathrm{d}_{TS}(H^s_{p,q}(v), v) \leq L\,\mathrm{d}(p,q)^\alpha$;

(d) $p,q \mapsto H^s_{p,q}$ is uniformly continuous in $\{p,q \in \Sigma \times S; q \in W^{ss}_{loc}(p)\}$.

We are going to prove the existence of the holonomies for fiber-bunched skew products of $SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$.

Proposition 3.1. *If f is α-fiber bunched, then for every $\tilde{x} \in W^s_{loc}(\tilde{y})$ the limit*

$$\lim_{n\to\infty} (f^n_{\tilde{y}})^{-1} \circ f^n_{\tilde{x}},$$

exists and defines an α-Hölder stable holonomy.

PROOF. For n sufficiently large we can identify $S_{\sigma^n(\tilde{x})}$ with $S_{\sigma^n(\tilde{y})}$ using local charts. Define $h^n = (f^n_{\tilde{y}})^{-1} \circ f^n_{\tilde{x}}$. We are going to prove that the maps $h^n : S \to S$ form a Cauchy sequence in the $C^0(S)$-topology. Using (51) and that $f_{\tilde{x}}$ varies α-Hölder continuously with the base point, we obtain the following estimate

$$\begin{aligned}
\mathrm{d}(h^{n+1}(t), h^n(t)) &= \mathrm{d}\left((f^n_{\tilde{y}})^{-1} \circ f^{-1}_{\tilde{f}^n(\tilde{y})} \circ f_{\tilde{f}^n(\tilde{x})} \circ f^n_{\tilde{x}}(t), (f^n_{\tilde{y}})^{-1} \circ f^n_{\tilde{x}}(t)\right) \\
&\leq \sup_{t\in S} \left\|(Df^n_{\tilde{y}}(t))^{-1}\right\| \mathrm{d}_{C^0(S)}\left(f^{-1}_{\tilde{f}^n(\tilde{y})} \circ f_{\tilde{f}^n(\tilde{x})}, \mathrm{id}\right) \\
&\leq \sup_{t\in S} \left\|(Df^n_{\tilde{y}}(t))^{-1}\right\| C\lambda^{n\alpha}\,\mathrm{d}(\tilde{x},\tilde{y})^\alpha \\
&\leq C\theta^n\,\mathrm{d}(\tilde{x},\tilde{y})^\alpha,
\end{aligned}$$

for some $\theta \in (0,1)$. Hence, $(h^n)_{n\in\mathbb{N}}$ is a Cauchy sequence and converges uniformly in the C^0-topology. Thus, we define $h^s_{\tilde{x},\tilde{y}} = \lim_{n\to\infty} h^n$. The properties of the holonomy follow directly from the definition of the limit. □

The hypothesis in proposition 3.1 could be weakened. Indeed, we do not need the α-fiber bunching condition, but something weaker that can be seen as a type of "dominated splitting" condition (the contraction on the basis is stronger than the contractions on the fibers). Similar considerations also hold for the unstable holonomies.

Remark 3.2. *For the smooth case the holonomy can also be defined by the strong stable foliation restricted to a center stable manifold, specifically* $h_{\tilde{x}\tilde{y}}(t) = W^{ss}_{loc}(t) \cap S_{\tilde{y}}$.

Let f be α-fiber bunched. To define the α-Hölder linear stable holonomy we first need to find the contraction rate of f in the strong stable set.

Take $(\tilde{x}, t)$ and $(\tilde{y}, h^s_{\tilde{x}\tilde{y}}(t))$ in the same strong stable set, then

$$\begin{aligned} \mathrm{d}_S(f^n(h^s_{\tilde{x},\tilde{y}}(t)), f^n(t)) &= \mathrm{d}_S(h^s_{\tilde{f}^n(\tilde{x})\tilde{f}^n(\tilde{y})} f^n(t), f^n(t)) \\ &\leq L\, \mathrm{d}_\Sigma(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y}))^\alpha \\ &\leq L\lambda^{n\alpha}\, \mathrm{d}_\Sigma(\tilde{x}, \tilde{y})^\alpha \\ &\leq L\lambda^{n\alpha}\, \mathrm{d}_{\Sigma\times S}((\tilde{x}, t), (\tilde{y}, h^s_{\tilde{x}\tilde{y}}(t)))^\alpha \end{aligned} \tag{53}$$

so the contraction rate is at least λ^α.

Proposition 3.3. *If f is α^2-fiber bunched, then for every $x \in W^{ss}_{loc}(y)$ the limit*

$$\lim_{n\to\infty} (Df^n_c(y))^{-1} \circ Df^n_c(x),$$

exists and defines an α-Hölder linear stable holonomy.

Proof. For n sufficiently large we can identify $T_{f^n(x)}S_{\sigma^n(\tilde{x})}$ with $T_{f^n(y)}S_{\sigma^n(\tilde{y})}$ using local charts, also observe that as $r - 1 \geq \alpha$ there exists $C > 0$ such that

$$\begin{aligned} \left\|(Df^n_{\tilde{y}}(t)) - (Df^n_{\tilde{x}}(t'))\right\| &\leq H\, \mathrm{d}_\Sigma(\tilde{x}, \tilde{y})^\alpha + \|f\|_{C^{\min(r,2)}}\, \mathrm{d}_S(t, t')^{\min(r-1,1)} \\ &\leq C\, \mathrm{d}_{\Sigma\times S}((\tilde{y}, t)(\tilde{x}, t'))^\alpha. \end{aligned}$$

Define $H^n = (Df^n_c(y))^{-1} \circ Df^n_c(x)$. We are going to prove that $(H^n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

$$\begin{aligned} \left\|H^{n+1} - H^n\right\| &= \left\|(Df^{n+1}_c(y))^{-1} \circ Df^{n+1}_c(x) - (Df^n_x(y))^{-1} \circ Df^n_c(y)\right\| \\ &\leq \left\|(Df^n_c(y))^{-1}\right\| \left\|(Df_c(f^n(y)))^{-1} \circ Df_c(f^n(x)) - \mathrm{id}\right\| \|(Df^n_c(x))\| \\ &\leq C\left\|(Df^n_c(y))^{-1}\right\| \|(Df^n_c(x))\|\, \mathrm{d}(f^n(x), f^n(y))^\alpha \\ &\leq_{by(53)} C\left\|(Df^n_c(y))^{-1}\right\| \|(Df^n_c(x))\| (\lambda^\alpha\, \mathrm{d}(f^n(x), f^n(y)))^\alpha \\ &\leq \left\|(Df^n_c(y))^{-1}\right\| \|(Df^n_c(x))\| \lambda^{\alpha^2 n}\, \mathrm{d}(x, y)^\alpha. \end{aligned}$$

Observe that the α^2-fiber bunching condition is open, in particular, there exist $\theta \in (0, 1)$ and $\delta > 0$ such that if $\mathrm{d}(x', y') < \delta$, then

$$\left\|(Df_c(y'))^{-1}\right\| \left\|(Df_c(x'))\right\| \lambda^{\alpha^2} < \theta.$$

Therefore, for j sufficiently large

$$\left\|Df_c(f^j(y))^{-1}\right\| \left\|Df_c(f^j(x))\right\| \lambda^{\alpha^2} < \theta.$$

Thus,

$$\left\|H^{n+1} - H^n\right\| \leq C\theta^n\, \mathrm{d}(x, y)^\alpha.$$

The sequence $(H^n)_{n\in\mathbb{N}}$ is a Cauchy sequence and converges uniformly. Define $H^s_{x,y} = \lim_{n\to\infty} H^n$. The properties of the holonomy follow directly from the definition of the limit. □

Remark 3.4. *For the $C^{1+\alpha}$-diffeomorphism case as the α-center bunching condition takes in account the contraction over the strong stable direction the argument works with α-center bunching instead of α^2-fiber bunched.*

Remark 3.5. *In the $C^{1+\alpha}$-diffeomorphism case, the existence of the linear holonomies H^s, given by proposition 3.3, does not imply that the holonomy maps h^s, obtained in proposition 3.1, are C^1 maps. Indeed, to conclude that the maps h^s are C^1, one needs to prove that the sequence $(h^n)_{n\in\mathbb{N}}$ is Cauchy for the C^1-topology, where this sequence was defined in the proof of proposition 3.1. The calculation becomes more delicate and in fact one needs a stronger bunching condition, see [Br16].*

Remark 3.6. *Observe that the convergence in propositions 3.1 and 3.3 is uniform in sets with bounded Hölder constant.*

This implies that the holonomies h^s and h^u vary continuously with f with respect to the C^0 topology in subsets with bounded Hölder constant, and the linear holonomies H^s and H^u vary continuously with f with respect to the C^1 topology on subsets on subsets with bounded Hölder constant of Df_c.

In particular in the random product case the holomies varies continuously with $f_1, \cdots, f_k$ in the C^1 topology.

Analogously we define the unstable and linear unstable holonomies as the stable and linear stable holonomies for f^{-1}.

3.2. The invariance principle and criterion for the existence of positive exponents. Let $f \in \mathcal{FB}^{r,\alpha^2}_{leb}(\Sigma \times S)$ be a skew product over a hyperbolic homeomorphism $\sigma : \Sigma \to \Sigma$. In [Bow70], Bowen proved that a hyperbolic homeomorphism is semi-conjugated to a subshift of finite type. In particular, the set of periodic points of σ is non empty. Also using the symbolic dynamics associated to σ, it holds that for any periodic point $\tilde{p}$ it exists $\tilde{z} \in \Sigma$ such that $\tilde{z} \in W^u_{loc}(\tilde{p}) \cap \sigma^{-i}(W^s_{loc}(\tilde{p})) - \{\tilde{p}\}$, for some $i \in \mathbb{N}$.

Definition 3.7 (Pinching)**.** *We say that the cocycle (f, Df_c) is* pinching *if there exists a periodic point $\tilde{p} \in \Sigma$, for σ, such that the cocycle $(f^\kappa_{\tilde{p}}, Df^\kappa_{\tilde{p}})$ verifies*

$$\int_{S_{\tilde{p}}} \lambda^+(Df^\kappa_{\tilde{p}}, t) d\mu^c_{\tilde{p}}(t) > 0,$$

where κ is the period of $\tilde{p}$.

Let $\tilde{p}$ be a periodic point for σ of period κ. Consider $NUH_{\tilde{p}} \subset S_{\tilde{p}}$ to be the set of points t inside $S_{\tilde{p}}$ such that $\lambda^+(Df^\kappa_{\tilde{p}}, t) > 0$. Let $PS_{\tilde{p}}$ be the projectivization of the tangent bundle $TS_{\tilde{p}}$. By Oseledets theorem, on $NUH_{\tilde{p}}$ there exists a measurable function $t \mapsto (e^u(t), e^s(t)) \in PS_{\tilde{p}} \times PS_{\tilde{p}}$, where $e^u(t)$ is the Oseledets space corresponding to the positive Lyapunov exponent and $e^s(t)$ is the Oseledets space corresponding to the negative exponent.

Definition 3.8 (Twisting)**.** *Let (f, Df_c) be a pinching cocycle for the periodic point $\tilde{p} \in \Sigma$. We say that the cocycle is* twisting *(see figure 2) if there exist $j \in \mathbb{N}$, $\tilde{z} \in \tilde{W}^u_{loc}(\tilde{p}) \cap \left(\tilde{f}^{-i}(\tilde{W}^s_{loc}(\tilde{p})) - \{\tilde{p}\}\right)$ and a set $K \subset NUH_{\tilde{p}}$ such that $\mu^c_{\tilde{p}}(K) > 0$ and K verifies the following:*

$$(H_t)^j_* \left(\{e^+(t), e^-(t)\}\right) \cap \{e^+(h^j(t)), e^-(h^j(t))\} = \emptyset,$$

where

- *$h : S_{\tilde{p}} \to S_{\tilde{p}}$ is the composition $h^s_{\sigma^i(\tilde{z}),\tilde{p}} \circ f^i_{\tilde{z}} \circ h^u_{\tilde{p},\tilde{z}}$;*
- *$H_t : T_t S_{\tilde{p}} \to T_{h(t)} S_{\tilde{p}}$ is the composition $H^s_{f^i(t_{\tilde{z}}),h(t)} \circ Df^i_{\tilde{z}}(t_{\tilde{z}}) \circ H^u_{t,t_{\tilde{z}}}$, where $t_{\tilde{z}} = h^u_{\tilde{p},\tilde{z}}(t)$;*
- *$(H_t)^j = H_{h^{j-1}(t)} \circ \cdots \circ H_t$;*
- *$(H_t)^j_*$ is the action induced by H^j_t on the projective bundle $PS_{\tilde{p}}$.*

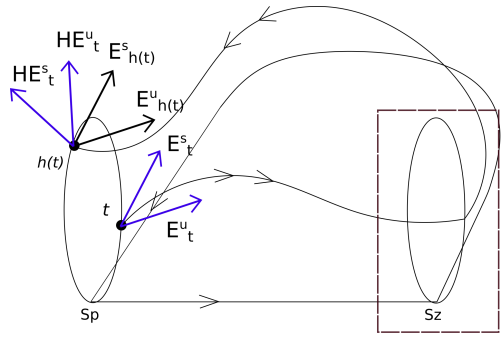

FIGURE 2. Twisting

These two definitions combined with the invariance principle gives us a criterion (theorem 3.11) for the existence of positive exponents for a open set of $SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$. Let us explain how this follows.

Recall that $TS = \{(\tilde{x}, t, v); (\tilde{x}, t) \in \Sigma \times S, v \in T_t S_{\tilde{x}}\}$ is the fiber bundle tangent to S, the derivative cocycle $Df_c : TS \to TS$ induce an action on the projective fiber bundle PS, which we will denote it by $F : PS \to PS$.

For a probability measure m on PS, that projects on μ, we write m_x the disintegration of m with respect to the projective fibers $P_x S_{\tilde{x}}$.

A F-invariant measure m is an *u-state* if there exists a set of full μ-measure, $M' \subset M$, such that for every $x, y \in M$, with $y \in W^{uu}(x)$, we have that $m_y = H^u_{x,y_*} m_x$. One defines a *s*-state analogously, replacing the roles of unstable by stable holonomies. If m is both s and u state, we call it a *su*-state. Using results from [Pol18], we have the following proposition.

Proposition 3.9. *If (f, Df_c) pinching and twisting then it does not admit a su-state projecting on μ.*

PROOF. Suppose that there exists m a F-invariant measure that is a *su*-state. Observe that the holonomies h preserves volume on S.

By Proposition 7.1 of [Pol18] there exist a disintegration that is *su/c*-invariant, this means that for every $\tilde{x}, \tilde{y} \in \Sigma$ in the same stable set for $\mu^c_{\tilde{x}}$-almost every $t \in S_{\tilde{x}}$

$$(H^s_{t,h^s_{\tilde{x},\tilde{y}}(t)})_* m_t = m_{h^s_{\tilde{x},\tilde{y}}(t)} \tag{54}$$

and the same property changing stable by unstable.

The pinching condition implies that for $\mu^c_{\tilde{p}}$-almost every $t \in NUH_{\tilde{p}}$, $m_t = a(t)\delta_{e^u(t)} + b(t)\delta_{e^s(t)}$, where $a(t), b(t) \in \mathbb{R}^+$ with $a(t) + b(t) = 1$. Now (54) implies that for $\mu^c_{\tilde{p}}$-almost every $t \in NUH_{\tilde{p}}$ and for every $j \in \mathbb{Z}$

$$a(t)\delta_{H^j_t e^u(t)} + b(t)\delta_{H^j_t e^s(t)} = a(h^j(t))\delta_{e^u(h^j(t))} + b(h^j(t))\delta_{e^s(h^j(t))}$$

this contradicts the twisting condition. □

Lemma 3.10. *If m^k are u-states for F_k, that projects to μ such that $F_k \to F$ and $m^k \to m$ in the weak*-topology, then m is an u-state.*

The proof of this lemma is very technical and also of independent interest. We prove it in more generality in section section 8, see theorem 8.3.

Theorem 3.11. *Let $\sigma : \Sigma \to \Sigma$ be a hyperbolic homeomorphism, $\hat{\mu}$ be a σ-invariant measure with product structure and let $f \in \mathcal{FB}^{1+\alpha,\alpha^2}_{leb}(\Sigma \times S)$. If (f, Df_c) is pinching and twisting then $L(f, \mu) > 0$, for $\mu = \hat{\mu} \times$ leb. Moreover, there exist a neighborhood of f in the C^r-topology with positive integrated Lyapunov exponents.*

PROOF. By [ASV13] if $L(f) = 0$ every F-invariant measure that projects to μ is an su-state, so using proposition 3.9 we prove that $L(f, \mu) > 0$.

For the second part assume that there exist $f_k \to f$ with $L(f_k, \mu) = 0$, then every F_k-invariant measure m^k is an su-state. Take a sub-sequence of m^k that converges to some m. By Lemma 3.10 m is a su-state, which is a contradiction. □

We remark that if $r \geq 2$ we can take $\alpha = 1$ in the previous statement. We also remark that the same theorem is true if $f \in CB^{r,\alpha}_{\text{leb}}(M)$.

4. Proof of Theorem E

From now on, we assume that f is α-fiber bunched. By proposition 3.1 and remark 3.4 there exists α-Hölder (linear) holonomies.

Let M be a fiber bundle over $\tilde{M}$, with fiber S, and take $f \in CB^r_{\text{leb}}(M)$. By lemma 2.1, f induces a diffeomorphism $\tilde{f} : \tilde{M} \to \tilde{M}$ which is Anosov. Moreover, $\tilde{f}$ preserves $\tilde{\mu} = \pi_* leb$. As in the previous section, we write $Df_c = Df|_{E^c}$ and recall that $\mu^c_{\tilde{x}}$ is the disintegration of the Lebesgue measure on the fiber $S_{\tilde{x}}$. Since M is a smooth bundle, the measure $\mu^c_{\tilde{x}}$ is just the Lebesgue measure on $S_{\tilde{x}}$.

In the following lemma we show how to perturb the cocycle (f, Df_c), which is pinching, to obtain a cocycle that is also twisting.

Lemma 4.1. *Let (f, Df_c) be a pinching cocycle for the periodic point $\tilde{p}$. Then there exists g arbitrarily C^r-close to f such that the cocycle (g, Dg_c) is pinching and twisting. Moreover, $g^\kappa|_{S_{\tilde{p}}} = f^\kappa|_{S_{\tilde{p}}}$, where $\kappa \in \mathbb{N}$ is the period of the periodic point $\tilde{p}$ for $\tilde{f}$.*

PROOF. Let $K \subset S_{\tilde{p}}$ be a compact set with the following properties:

- $\mu^c_{\tilde{p}}(K) > 0$;
- every point $t \in K$ has one positive and one negative Lyapunov exponent for the cocycle (f^κ, Df_c);
- the map $t \mapsto (e^u(t), e^s(t))$ is continuous on K, where $e^u(t)$ and $e^s(t)$ are the Osedelets spaces.

Such compact sets always exist, see section 4.2.1 in [BP01]. Fix $\tilde{z} \in W^u_{loc}(\tilde{p})$ such that $\tilde{f}^i(\tilde{z}) \in W^s_{loc}(\tilde{p})$, with $i > 0$. Write $h = h^s_{\tilde{f}^i(\tilde{z}),\tilde{p}} \circ f^i_{\tilde{z}} \circ h^u_{\tilde{p},\tilde{z}}$. By the formula given in proposition 3.1, we have that $h^u_{\tilde{p},\tilde{z}}$ only depends on $f^n(\tilde{z})$ and $f^n(\tilde{p})$, for $n \leq 0$, and $h^s_{\tilde{f}^i(\tilde{z}),\tilde{p}}$ only depends on $f^{n+i}(\tilde{z})$ and $f^n(\tilde{p})$, for $n \geq 0$.

Take t a density point of K and $j_t \in \mathbb{N}$ such that $h^{j_t}(t)$ is also a density point of K, this is can be done because h preserves $\mu^c_{\tilde{p}}$ and as a consequence of Poincaré recurrence theorem (Referencia). We can take j_t to be the smallest natural number that verifies this condition for t. Let $t_{\tilde{z}} = h^u_{\tilde{p},\tilde{z}}(t)$ and let $H_t = H^s_{f^i(t_{\tilde{z}}),h(t)} \circ Df^i_{\tilde{z}}(t_{\tilde{z}}) \circ H^u_{t,t_{\tilde{z}}}$. For H_t, the map $H^u_{t,t_{\tilde{z}}}$ only depends on $f^n(t)$ and $f^n(\tilde{z})$, for $n \leq 0$, and $H^s_{f^i(t_{\tilde{z}}),h(t)}$ only depends on $f^n(h(t))$ and $f^n(f^i(\tilde{z}))$, for $n \geq 0$.

If the twisting condition did not hold, then for almost every $t' \in K$ we would have

$$(H_{t'})^{j_{t'}} \left(\{e^u(t'), e^s(t')\}\right) \cap \{e^u(h^{j_{t'}}(t')), e^s(h^{j_{t'}}(t'))\} \neq \emptyset, \tag{55}$$

in particular, we can assume that this holds for t.

Take $V \subset S_{\tilde{z}}$ a neighborhood of $t_{\tilde{z}}$ in $S_{\tilde{z}}$ small enough such that the sets V, $\cdots$, $h^u_{\tilde{p},\tilde{z}} \circ h^{j-1}(V)$ are pairwise disjoint. Also fix a small open neighborhood of $\tilde{z}$, $\tilde{U} \subset \tilde{M}$, such that the sets $\tilde{U}$, $\cdots$, $\tilde{f}^{i-1}(\tilde{U})$ are pairwise disjoint. Consider the neighborhood of $\tilde{z}$ given by $\tilde{U} \times V$.

Let $D \subset \mathbb{R}^2$ be the unitary disc and let $U = B^{d-2}(0,1) \subset \mathbb{R}^{d-2}$, where d is the dimension of M and $B^{d-2}(0,1)$ is the unitary ball on $\mathbb{R}^{d-2}$. Take a smooth parametrization $\Phi : U \times D \to M$ such that $\Phi(U \times D) \subset \tilde{U} \times V$ and for each $y \in U$ we have that $\Phi(\{y\} \times D) \subset \pi^{-1}(\pi(\Phi(\{y\} \times D))$. We also take Φ verifying $\Phi(0) = t_{\tilde{z}}$ and $\Phi_*^{-1}\mu\,|_{V \times \tilde{U}}$ is the standard volume in $U \times D$.

Using polar coordinates on D, we consider the vector field $X(r',\theta) = r'\frac{\partial}{\partial\theta}$. Let $\rho : [0,1] \to [0,1]$ be a smooth bump function that verifies, $\rho(r') = 1$ if $r' \in [0, \frac{1}{3}]$, and $\rho(r') = 0$ if $r' \in [\frac{2}{3}, 1]$. Using coordinates (y, r', θ) on $U \times D$, where $y \in U$, we define the vector field $\hat{X}$ on $U \times D$ by

$$\hat{X}(y, r', \theta) = \left(0, \rho(\|y\|)\rho(r')r'\frac{\partial}{\partial\theta}\right). \tag{56}$$

For $T \in \mathbb{R}$, let ϕ_T be the flow generated by $\hat{X}$. Observe that $\phi_T = \mathrm{id}$ in a neighborhood of the boundary of $U \times D$, also that $\phi_T(0) = 0$ and $D\phi_T(0) = id \times R_T$, where R_T is the rotation counterclockwise of angle T. Using the parametrization Φ we define a flow $\phi'_T : M \to M$ as follows: if $q \notin \tilde{U} \times V$, then $\phi'_T(q) = \mathrm{id}$. If $q \in V \times \tilde{U}$, then

$$\phi'_T(q) = \Phi \circ \phi_T \circ \Phi^{-1}(q).$$

Take $g_T = f \circ \phi'_T$. Our perturbation does not affect the orbit of $t_{\tilde{z}}$, this implies that $h^i_T(t)$ is not affected for $i = 0, \dots, j_t$, in particular, $h^{j_t}_T(t) = h^{j_t}(t)$ and the linear holonomy is given by

$$H^T_t = H^s_{f^i(t_{\tilde{z}}),h(t)} \circ Df^i_{\tilde{z}}(s) \circ D\Phi(0) \circ R_t \circ (D\Phi(0))^{-1} \circ H^u_{t,t_{tz}}.$$

Furthermore, $\left(H^T_{h(t)}\right)^{j_t-1} = \left(H_{h(t)}\right)^{j_t-1}$.

For $T > 0$ small, g_T is C^r-close to f and it holds that

$$\left(H^T_t\right)^{j_t}\left(\{e^u(t), e^s(t)\}\right) \cap \{e^u(h^{j_t}(t)), e^s(h^{j_t}(t))\} = \emptyset.$$

Observe that $g^\kappa_T|_{S_{\tilde{p}}} = f^\kappa|_{S_{\tilde{p}}}$. Since $h^{j_t}(t)$ is a density point of K and for the points of K the Oseledets splitting varies continuously, we conclude that g_T is twisting. □

As a consequence of this lemma and theorem 3.11, we have the following theorem.

Theorem 4.2. *Let $f \in CB^r_{\mathrm{leb}}(M)$ be a diffeomorphism such that (f, Df_c) is pinching. Then arbitrarily C^r-close to f, there exist C^r-open sets, inside $CB^r_{\mathrm{leb}}(M)$, of diffeomorphisms with positive integrated center Lyapunov exponent.*

Proof. By lemma 4.1, arbitrarily C^r-close to f there exists a diffeomorphism g which is pinching and twisting. By theorem 3.11, we conclude that $L(g, \mu) > 0$. By lemma 3.10, we obtain that $L(., \mu)$ is positive in a C^r-neighborhood of g, since otherwise g would be accumulated by diffeomorphisms admiting a *su*-state, which would imply that g has a *su*-state. □

Remark 4.3. *Using theorem 8.3, we can conclude that the C^r-open sets in the statement of theorem 4.2 is among the partially hyperbolic volume preserving C^r-diffeomorphisms, not necessarily skew products.*

We will need the following theorem.

Theorem 4.4 (Theorem 1.2 in [LY17])**.** *Let f be a volume preserving, C^r-diffeomorphism of a compact manifold M of dimension $d \geq 2$. Then arbitrarily C^1-close to f there exists a volume preserving C^r-diffeomorphism g without zero exponents on a set of positive measure.*

In our scenario, we want to use a fibered version of theorem 4.4. For that we need the following lemma.

Lemma 4.5. *Let $\mathrm{Diff}^r_{\mathrm{leb}}(S)$ be the space of C^r-diffeomorphisms of a surface S that preserve the Lebesgue measure. If $g \in \mathrm{Diff}^r_{\mathrm{leb}}(S)$ is sufficiently C^1-close to the identity, then there exists a smooth path $(g_t)_{t\in[0,1]} \subset \mathrm{Diff}^r_{\mathrm{leb}}(S)$ connecting g to the identity such that for any $t \in [0,1]$, g_t is C^1-close to the identity.*

PROOF. This proof uses several basic tools from symplectic geometry which we will not define, but we refer the reader to [Can01], chapters 3 and 9, for all the definitions and results that we use here.

Let ω be the volume form that generates leb on S. Since S is a surface, we have that (S, ω) is a symplectic manifold and the volume preserving diffeomorphisms are the symplectomorphisms of this manifold.

We consider two other symplectic manifolds $(S \times S, \pi_1^*\omega - \pi_2^*\omega)$, where π_i is the projection on the i-th coordinate, and the cotangent bundle (T^*S, ω_0) with the canonical symplectic form. Let Δ be the diagonal on $S \times S$. It is well known that Δ is a Lagrangian submanifold of $S \times S$ and that the zero section is a Lagrangian submanifold in T^*S. By Weinstein's tubular neighborhood theorem (chapter 9 of [Can01]), there exists a smooth symplectomorphism $\varphi : U_1 \to U_2$ taking Δ to the zero section, where U_1 is a neighborhood of Δ and U_2 is a neighborhood of the zero section.

If g is a C^r-symplectomorphism sufficiently C^1-close to the identity, then $\text{graph}(g)$ is contained in U_1. Furthermore, its graph is a C^r-Lagrangian submanifold. Using φ, $\text{graph}(g)$ is identified with a C^r-Lagrangian submanifold Γ_g of U_2. Since the zero section is transverse to the fibers in T^*S and Γ_g is C^1-close to the zero section, we have that Γ_g is also transverse to the fibers in T^*S. Thus, it exists a 1-form ν with regularity C^r such that $\Gamma_g = \text{graph}(\nu)$.

It is true that the graph of a 1-form ν is Lagrangian if and only if $d\nu = 0$. Consider the smooth family of 1-forms $\nu_t = (1-t)\nu$, for $t \in [0,1]$. Observe that $d\nu_t = 0$ and that ν_t is a path connecting ν to the zero form. Hence, the family $(\text{graph}(\nu_t))_{t\in[0,1]}$ is a smooth family of C^r-Lagrangian submanifolds which are C^1-close to the zero section. Using φ and a transversality argument again, we obtain that each of these graphs defines a C^r-diffeomorphism $g_t : S \to S$ which is C^1-close to the identity. Furthermore, since its graphs are Lagrangian submanifolds, it holds that g_t is a symplectomorphism for each $t \in [0,1]$. Thus, we have obtained a smooth family of symplectomorphisms $(g_t)_{t\in[0,1]}$, verifying the conditions of the lemma. □

CONCLUSION OF THE PROOF OF THEOREM E. We claim that the pinching condition is C^1-dense inside $CB^{r,\alpha}_{\text{leb}}(M)$. Indeed, let $f \in CB^{r,\alpha}_{\text{leb}}(M)$ and let $\tilde{p} \in \tilde{M}$ be a periodic point for $\tilde{f}$. For simplicity, suppose that $\tilde{p}$ is a fixed point and fix a small open neighborhood $\tilde{U} \subset \tilde{M}$ of $\tilde{p}$ such that in a trivialization chart of the bundle M, a neighborhood of $S_{\tilde{p}}$ is given by $\tilde{U} \times S$.

Consider $f_{\tilde{p}} = f|_{S_{\tilde{p}}}$. By theorem 4.4, there exists $g : S \to S$ volume preserving which is C^1-close to $f_{\tilde{p}}$. Take $G = f_{\tilde{p}}^{-1} \circ g$ and observe that G is C^1-close to the identity. Let $(G_t)_{t\in[0,1]}$ be the smooth path connecting h to the identity given by lemma 4.5. Fix $\rho > 0$ small enough such that the ball, $\tilde{B}(\tilde{p}, \rho)$, of radius ρ and center $\tilde{p}$ is contained in $\tilde{U}$. We define the fibered diffeomorphism

$$\begin{array}{rcl} \mathcal{G} : \tilde{B}(\tilde{p},\rho) \times S & \to & \tilde{B}(\tilde{p},\rho) \times S \\ (\tilde{q}, t) & \mapsto & (\tilde{q}, G_{\frac{d(\tilde{p},\tilde{q})}{\rho}}(t)). \end{array}$$

We can extend $\mathcal{G}$ to a diffemorphism which is the identity outside $\tilde{U} \times S$ and inside this neighborhood coincides with $\mathcal{G}$. We also denote this diffeomorphism by $\mathcal{G}$. Since G_t is C^1-close to the identity, we have that $\mathcal{G}$ is also C^1-close to the identity. Also observe that $\mathcal{G} \circ f|_{S_{\tilde{p}}} = g$, which has positive integrated Lyapunov exponent. Hence, $\mathcal{G} \circ f$ is pinching, which proves the claim.

Since the pinching condition is C^1-dense, using theorem 4.2 we finish the proof of theorem E. □

5. Proof of Theorem F

Let $\sigma : \Sigma \to \Sigma$ be a hyperbolic homeomorphism and $\mathcal{FB}^{r,\alpha^2}_{\text{leb}}(\Sigma \times S)$ be the set of fiber bunched skew products defined in section 2. By the results in section 3 the linear and

non-linear holonomies are well defined inside $\mathcal{FB}^{r,\alpha^2}_{\text{leb}}(\Sigma \times S)$. Similar to lemma 4.1, but in the scenario of theorem B, we have the following lemma.

Lemma 5.1. *If $f \in \mathcal{FB}^{r,\alpha^2}_{\text{leb}}(\Sigma \times S)$ is a fiber-bunched cocycle such that (f, Df_c) is pinching for the periodic point $\tilde{p}$, then there exists $g \in \mathcal{FB}^{r,\alpha^2}_{\text{leb}}(\Sigma \times S)$ arbitrarily C^r-close to f such that the cocycle (g, Dg_c) is pinching and twisting. Moreover, $g^\kappa_{\tilde{p}} = f^\kappa_{\tilde{p}}$, where $\kappa \in \mathbb{N}$ is the period of the periodic point $\tilde{p}$ for σ.*

Proof. The proof follows the same steps as the proof of lemma 4.1, the only difference is in the construction of the vector field $\hat{X}$ defined in (56). Let us explain how to adapt the construction of such vector field. Let $K \subset \{\tilde{p}\} \times S$, $t \in K$, $\tilde{z} \in W^s_{loc}(\tilde{p})$, $t_{\tilde{z}} = h^u_{\tilde{p},\tilde{z}}(t)$, $V \subset \{\tilde{z}\} \times S$ and $\tilde{U} \subset \Sigma$ be as in the proof of lemma 4.1.

Write $t_{\tilde{z}} = (t_1, t_2)$. Consider $V' \subset V$ a small neighborhood of t_2 inside V and $\varphi : D \to V'$ be a C^r-parametrization from the unit disc D onto V' such that $\varphi(0) = t_2$. Let $U = B(t_1, \delta)$ be the ball inside Σ centered in t_1 with radius $\delta > 0$ small enough such that $U \subset \tilde{U}$. Consider $\rho_1 : U \to [0,1]$ defined by $\rho_1(\tilde{x}) = \frac{d(\tilde{x}, \partial U)}{\delta}$, where ∂U is the boundary of U. We remark that the function ρ_1 is Lipschitz continuous. Take $\rho_2 : [0,1] \to [0,1]$ a smooth bump function as in the proof of lemma 4.1 and consider $X(r', \theta) = r' \frac{\partial}{\partial \theta}$ to be the vector field on D using polar coordinates. On $U \times D$ consider the vector field defined on each fiber by

$$\hat{X}(y, r', \theta) = \rho_1(y)\rho_2(r')X(r', \theta) \in \{y\} \times T_{(r',\theta)}D.$$

The vector field $\hat{X}$ induces a fibered flow, which we denote it by ϕ_t. Let $\Phi : U \times D \to U \times V'$ be defined by $\Phi = Id \times \varphi$ and for each $t \in \mathbb{R}$ we define $\phi'_t = \Phi \circ \phi_t \circ \Phi^{-1}$. In the same way as in the proof of lemma 4.1 we have that ϕ'_t extends to a homeomorphism $g_t : \Sigma \times S \to \Sigma \times S$, which is C^r on the fibers. Furthermore, $Dg_t(t_{\tilde{z}}) = D\varphi(0)R_t(D\varphi(0))^{-1}$. The rest of the proof is the same as before. □

Using lemma 5.1, we can prove in the same way as in theorem 4.2 the following theorem.

Theorem 5.2. *Let $f \in \mathcal{FB}^{r,\alpha^2}_{\text{leb}}(\Sigma \times S)$ be a skew product such that (f, Df_c) is pinching. Then arbitrarily C^r-close to f, there exist C^r-open sets of skew products with positive integrated center Lyapunov exponent.*

Using theorems 4.4, 5.2 and lemma 4.5, and following the same constructions in the conclusion of the proof of theorem E, we can conclude the proof of theorem F.

6. Proof of Theorem G

Recall that p is a probability measure on $\{1, \cdots, d\}$, for some $d \in \mathbb{N}$, and given $(f_1, \cdots, f_d) \in \text{Diff}^r_{\text{leb}}(S)^d$ we are interested in studying the random product generated by this set with distribution p. This random product can be seen as the skew product

$$\begin{array}{rcl} f : \{1, \cdots, d\}^{\mathbb{Z}} \times S & \to & \{1, \cdots, d\}^{\mathbb{Z}} \times S \\ ((x_j)_{j\in\mathbb{Z}}, t) & \mapsto & ((x_{j+1})_{j\in\mathbb{Z}}, f_{x_0}(t)), \end{array}$$

with the measure $\mu = p^{\mathbb{Z}} \times \text{leb}$. The proof of theorem G follows the same lines as theorems E and F, but the perturbations are simpler.

Observe that in this scenario, the local linear and non-linear holonomies are just the identity maps. This follows because the skew-product is constant on open sets of $\{1, \cdots, d\}^{\mathbb{Z}}$.

We say that a random product is *pinching* if there exists $i \in \{1, \cdots, d\}$ such that f_i has positive Lyapunov exponent on a set of positive Lebesgue measure. The definition of twisting remains the same as definition 3.8, but considering the fixed point given by the constant sequence formed by i.

Lemma 6.1. *Consider the random product formed by $(f_1, \cdots, f_d) \in \mathrm{Diff}^r_{\mathrm{leb}}(S)^d$ and probability p. If (f, Df_c) is pinching, then there exists $(g_1, \cdots, g_d) \in \mathrm{Diff}^r_{\mathrm{leb}}(S)^d$ arbitrarily C^r-close to $(f_1, \cdots, f_d)$ such that (g, Dg_c) is twisting.*

Proof. The only difference from the proof of this lemma and lemma 5.1 is that we want to produce the twisting property just by perturbing one of the diffeomorphisms considered in the random product. Since (f, Df_c) is pinching, there exists $i \in \{1, \cdots, d\}$ such that f_i has positive Lyapunov exponent on a set of positive measure. Consider the fixed point $\tilde{p} = (\tilde{p}_j)_{j\in\mathbb{Z}}$, where $\tilde{p}_j = i$ for every $j \in \mathbb{Z}$. Fix $l \in \{1, \cdots, d\}$ such that $l \neq i$ and consider the point $\tilde{z} = (\tilde{z}_j)_{j\in\mathbb{Z}}$, where $\tilde{z}_1 = l$ and $\tilde{z}_j = i$ for $j \neq 1$. Observe that $\tilde{z}$ is a homoclinic point of $\tilde{p}$.

Take $K \subset \{\tilde{p}\} \times S$, $t \in K$, $t_{\tilde{z}} = h^u_{\tilde{p},\tilde{z}}(t)$ and $V \subset \{\tilde{z}\} \times S$ be as in the proof of lemma 4.1. Notice that since the holonomies are the identity, $t_{\tilde{z}} = t$. Let $V' \subset V$ and $\varphi : D \to V'$ be a C^r-diffeomorphisms such that $\varphi(0) = t$. Using polar coordinates, consider $X(r', \theta) = r' \frac{\partial}{\partial\theta}$. Take $\rho : [0, 1] \to [0, 1]$ a bump function as in the proof of lemma 4.1 and consider the vector field $X'(r', \theta) = \rho(r')X(r', \theta)$. For each $t \in \mathbb{R}$, let ϕ_t be the time t of the flow on D generated by X'. Define a flow on V' by

$$\Phi_t = \varphi \circ \phi_t \circ \varphi^{-1}.$$

Since Φ_t is the identity on a neighborhood of the boundary of V', we can extend it to a flow on S, which is the identity outside V'. We also denote this flow by Φ_t. For each $t \in \mathbb{R}$ consider the diffeomorphism $g^t_l = f_l \circ \Phi_t$. For t small, g^t_l is C^r-close to f_l. By similar arguments as in the proof of lemma 4.1, we can find $t \in \mathbb{R}$ arbitrarily small such that the random product formed by $(g_1 \cdots, g_d)$, with $g_j = f_j$ if $j \neq l$ and $g_l = g^t_l$, is twisting. □

By theorem 4.4, the pinching condition is C^1-dense in $\mathrm{Diff}^r_{\mathrm{leb}}(S)^d$. Using theorem 5.2 and following the same steps in the conclusion of the proof of theorem E we conclude the proof of theorem G.

7. Proof of Theorem H

Notice that in the proof of theorems E, F, and G, the only place where we use C^1-perturbations is to use theorem 4.4, which states that by arbitrarily small C^1-perturbation one can get the pinching condition. All the other perturbations that we make are C^r. In a recent work, Berger and Turaev proved the following interesting result.

Theorem 7.1 (Theorem A in [BT17])**.** *For any surface S, if a diffeomorphism $f \in \mathrm{Diff}^\infty_{\mathrm{leb}}(S)$ has a periodic point which is not hyperbolic, then there is an arbitrarily small C^∞-perturbation of f such that the perturbed map $g \in \mathrm{Diff}^\infty_{\mathrm{leb}}(S)$ has a set of positive measure with positive Lyapunov exponents.*

In the setting of theorem H, if $f_{\tilde{p}}$ has an elliptic periodic point, since elliptic periodic points are robust and using that $\mathrm{Diff}^\infty_{\mathrm{leb}}(S)$ is C^r-dense in $\mathrm{Diff}^r_{\mathrm{leb}}(S)$ (see [Av10]), then C^r-close to $f_{\tilde{p}}$ there is a diffeomorphism with positive exponents in a set of positive measure. With the same constructions as before we obtain that in this scenario, arbitrarily C^r-close to f there is a map which is pinching and twisting, which concludes the proof of theorem H.

8. u-states are closed

This section is devoted to prove the u-states converge to u-states. This result maybe of independent interest we prove a more general version here. During the preparation of this paper a proof of this fact was given by Liang, Marin and Yang [LMY18-1] for the volume preserving case using recent results about entropy [TY18]. Our proof is more direct and does not use entropy.

Let $f_k : M \to M$ be sequence of partially hyperbolic maps admitting linear holonomies converging C^1 to f and E be a fiber bundle over M with smooth fibers $\mathcal{E}$. We say that a

sequence of cocycles $F^k : E \mapsto E$ over f_k converges to $F : E \mapsto E$ if F^k converges C^0 to F and the holonomies converges uniformly in local unstable manifolds.

By remark 3.6 if $f_k \in SP^r_{\text{leb}}(M)$, $f_k \to f$ in C^1 and have the $C^{1+\alpha}$ norm uniformly bounded then $F_k \to F$.

This kind of results was proved for other particular cases, for example [TY18] where they proved it when f_k are equal to a fixed Anosov map, [Pol18] where this is proved for f_k a fixed partially hyperbolic map. The principal dificulty here is that the base map is no longer fixed.

We want to prove that if probability measures m^k on E are F^k-invariant u-states and $m^k \to m$ then m is an u-state for F.

First let us focus on the smooth case. We actually prove a more general version than we need here, we prove that the u-states are closed for every volume preserving partially hyperbolic map (not necessarily a skew product).

The proof will be an adaptation of the proof of [Pol18, Theorem A.1] where the result was proved when the base map is fixed and the cocycle is independent of the base map.

Take $H = [-1,1]^{d_u}$ and $V = [-1,1]^{d_{cs}}$. Lets write $(x,y) \in [-1,1]^d$ where $x \in H$ and $y \in V$. For each $p \in M$ and f C^1 partially hyperbolic diffeomorphism we call a Hölder continuous map $\Theta_f : [-1,1]^d \to \mathcal{V} \subset M$ *foliated chart* centered at p, if

- $\Theta_f(H \times \{y\}) \subset W^{uu}_{loc}(\Theta_f(0,y))$,
- $f \mapsto \Theta_f$ is continuous in the $C^0([-1,1]^d)$ topology in a neighborhood of f,
- for each $y \in V$, $\Theta_{f,y} := \Theta_f \mid_{H\times\{y\}}$ is C^1 and $f \mapsto \Theta_{f,y}$ is continuous in the $C^r(H)$ topology in a neighborhood of f,
- $\Theta_f \mid_{\{0\}\times V}$ is C^1, moreover we can take $\Theta_f(\{0\} \times V) = \Gamma$ constant in a neighborhood of f.

Lemma 8.1. *Let $f : M \to M$ be partially hyperbolic $C^{1+\alpha}$, $\alpha > 0$, volume preserving diffeomorphism and $\Theta : H \times V \to \mathcal{V}$ be a foliated chart, then the pull back of the Lebesgue measure $\eta = \Theta_*^{-1} \operatorname{vol} \mid_{\mathcal{V}}$ has the form $\rho_f dxdy$, where $dxdy$ is the usual volume form in $H \times V$ and ρ_f varies continuously with f with respect to the C^1 norm in subsets of $C^{1+\alpha}$ bounded norm.*

Proof. Let $\nu = \operatorname{vol} \mid_{\mathcal{V}}$ and take the disintegration of ν into local unstable manifolds $\nu = \nu_{y'} d\nu_\Gamma$ where $y' \in \Gamma = \Theta(\{0\} \times V)$, by the absolute continuity of the unstable foliation ν_Γ is the (normalized) lebesgue measure on Γ and $\nu_{y'} = \varrho_f vol_{W^u(y')}$ where ϱ_f is defined up to normalization by

$$\frac{\varrho_f(x')}{\varrho_f(y')} = \lim_{n\to-\infty} \frac{\det Df^n \mid_{W^u(x')}}{\det Df^n \mid_{W^u(y')}} \tag{57}$$

moreover the limit in (57) is uniform in subsets of bounded $C^{1+\alpha}$ norm. So, normalizing as $\nu_{y'}(H \times y') = 1$, ϱ_f depends continuously on f with respect to the C^1 topology in subsets of bounded $C^{1+\alpha}$ norm.

Disintegrating η in horizontals we have $\eta = \eta_y d\eta_V$, where $\eta_y = \Theta_*^{-1}\nu_{\Theta(y)}$ and $\eta_V = \Theta_*^{-1}\nu_\Sigma$.

Let $\varphi : H \to \mathbb{R}$ be a continuous function, then

$$\begin{aligned}
\int \varphi d\eta_y &= \int \varphi \circ \Theta^{-1}(t) \nu_{\Theta(y)}(t) \\
&= \int \varphi \circ \Theta^{-1}(t) \varrho_f(t) d \operatorname{vol}_{W^u(\Theta(y))}(t). \\
&= \int \varphi |\det \Theta_y(t)| \varrho_f(\Theta(t)) dx
\end{aligned}$$

So $\eta_y = \rho'_f(\cdot, y)dx$ where ρ'_f is continuous and depends continuously on f. Analogously $\eta_V = \rho''_f dy$, then by Fubini $\eta = \rho_f dxdy$, with ρ_f continuous on f. □

Now lets define our foliated charts for the skew products over hyperbolic homeomorphisms.

Fix a local chart given by the product structure

$$\theta : W^s_{loc}(\tilde{x}_0) \times W^u_{loc}(\tilde{x}_0) \to \mathcal{V} \subset \tilde{M},\ \theta(x,y) = [x,y].$$

Let $x^u \in W^u_{loc}(\tilde{x}_0)$ be fixed, for any $\tilde{x} = (x^-, x^+) \in W^s_{loc}(\tilde{x}_0) \times W^u_{loc}(\tilde{x}_0)$, let $\phi(\tilde{x}) = [x^-, x^u]$ and then define

$$\Theta(\tilde{x}, t) = \big(\tilde{x}, h^u_{\varphi(\tilde{x}),\tilde{x}}(t)\big).$$

This local chart sends $\{x^-\} \times W^u_{loc}(\tilde{x}_0) \times \{t\}$ to $W^{uu}_{loc}(x^-, x^u, t)$. Take $H = W^u_{loc}(\tilde{x}_0)$ and $V = W^s_{loc}(\tilde{x}_0) \times S$, then the foliated chart is given by $\Theta : H \times V \mapsto \mathcal{V}$.

As $h^u_{\varphi(\tilde{x}),\tilde{x}}$ preserves the volume on S the next lemma follows

Lemma 8.2. *Let $\sigma : \Sigma \to \Sigma$ be a hyperbolic homeomorphism with invariant measure $\hat{\mu}$ and $f \in SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$, then $\Theta^*\tilde{\mu} \times \text{leb} = \tilde{\mu} \times \text{leb}$.*

Now we can prove that u-states converges to u-states in both cases. Let f_k be a partially hypebolic volume preserving map or $f_k \in SP^{r,\alpha}_{\sigma,\mu}(\Sigma \times S)$, that converges to f in the corresponding topology. We have our theorem

Theorem 8.3. *Let $F_k \to F$ and m^k be u-states converging weakly to m, then m is an u-state for F.*

Proof. Let μ be the lebesgue measure if we are in the partially hyperbolic volume preserving case or $\hat{\mu} \times \text{leb}$ in the other case.

Denote by $\Theta_k : H \times V \to \mathcal{V}$ the foliated chart for f_k and let $\eta^k = \Theta_k^* \mu \mid_{\mathcal{V}}$. By lemmas 8.1 and 8.2 we can write $\eta^k = \rho_k \eta$ for some $\rho_k : H \times V \to \mathbb{R}$, where $\eta = \Theta^* \mu \mid_{\mathcal{V}}$, moreover there exist $0 < c < C$ such that $c \le \rho_k \le C$.

Observe that it is sufficient to prove the result locally, so we can suppose that the projective tangent bundle is locally a product $\mathcal{V} \times \mathcal{E}$.

Define $\mathcal{H}_k : H \times V \times \mathcal{E} \to H \times V \times \mathcal{E}$ as

$$(x, y, v) \to (x, y, H^{u,k}_{\Theta_k(x,y),\Theta_k(0,y)}(v))$$

Let $\hat{m}^k = \big(\Theta_k^{-1} \times \text{id}\big) * m^k \mid_{\mathcal{V}\times\mathcal{E}}$, define $\tilde{m}^k = \mathcal{H}_{k*}\hat{m}^k$. Observe that m^k is a u-state if and only if the disintegration of $\tilde{m}^k$ with respect to the projective fibers is such that

$$\tilde{m}^k_{(x,y)} = \tilde{m}^k_{(x',y)} \text{ for } \eta\text{-almost every } (x,y), (x',y) \in H \times V. \tag{58}$$

Denote by $\mathbf{B}_0$ the sigma algebra of sets $H \times \{y\}$, $y \in V$, then (58) is equivalent to $p \mapsto \tilde{m}^k_p$ being $\mathbf{B}_0$ measurable.

So we are left to prove that $m^k \to m$ implies that $p \mapsto \tilde{m}_p$ is also $\mathbf{B}_0$ measurable.

We follow the proof of [Pol18, Proposition A.7] where this was proved for cocycles with a fixed base map. In order to do this we first need to adapt Lemma A.3 and Lemma A.4 of [Pol18] to our scenario.

Lemma 8.4 (Addaptation of Lemma A.3)**.** *Let $\phi : H \times V \times \mathcal{E} \to \mathbb{R}$ be a measurable bounded function such that for every $p \in H \times V$, $v \mapsto \phi(p, v)$ is continuous, then $\int \phi d\hat{m}^k \to \int \phi d\hat{m}$.*

Proof. To simplify the notation let $\hat{\Theta}_k := \Theta_k \times \text{id}$. Recall that ${\Theta_k^{-1}}_* \mu_k \mid_{\mathcal{V}} = \rho_k \eta$ is the projection of $\hat{m}^k = (\hat{\Theta}_k)_* m^k$.

We have

$$\left| \int \phi \circ \hat{\Theta}_k^{-1} dm^k - \int \phi \circ \hat{\Theta}^{-1} dm \right| \le$$

$$\left| \int \phi \circ \hat{\Theta}_k^{-1} - \phi \circ \hat{\Theta}^{-1} dm^k \right| + \left| \int \phi \circ \hat{\Theta}^{-1} dm^k - \int \phi \circ \hat{\Theta}^{-1} dm \right|.$$

As m^k and m projects to the same measure and $\phi \circ \hat{\Theta}^{-1}(p, \cdot)$ is continuous for every p, a simple argument shows that the second term goes to zero when $k \to \infty$ (for the details see [Pol18, lemma A.3]), so we focus on the first term.

Fix $\varepsilon > 0$ and take a compact set $K \subset H \times V$ such that $\eta(H \times V \setminus K) < \frac{\varepsilon}{C\|\phi\|}$ and ϕ is continuous in $K \times \mathcal{E}$. Take $\phi' : H \times V \times \mathcal{E} \to \mathbb{R}$ a continuous function such that $\phi(p, v) = \phi'(p, v)$ for every $p \in K$, $v \in \mathcal{E}$ and $\|\phi'\| \leq \|\phi\|$, then

$$\left| \int \phi \circ \hat{\Theta}_k^{-1} - \phi \circ \hat{\Theta}^{-1} dm^k \right| \leq$$
$$\int \left| \phi' \circ \hat{\Theta}_k^{-1} - \phi' \circ \hat{\Theta}^{-1} \right| dm^k + \int |\phi' - \phi| d(\hat{\Theta}_k^{-1})_* m^k + \int |\phi' - \phi| d\hat{\Theta}_*^{-1} m^k.$$

The first term goes to zero by continuity of ϕ' and the second and last term are bounded by $\sup\{\rho_k\}\eta(H \times V \setminus K) < \epsilon$. □

Lemma 8.5 (Addaptation of Lemma A.4)**.** *If $m^k \to m$ then $\tilde{m}^k \to \tilde{m}$.*

Proof. Take $\phi : H \times V \times \mathcal{E} \to \mathbb{R}$ uniformly continuous, then

$$\left| \int \phi d\tilde{m}^k - \int \phi d\tilde{m} \right| \leq$$
$$\left| \int \phi \circ \mathcal{H} d\hat{m}^k - \int \phi \circ \mathcal{H} d\hat{m} \right| + \int |\phi \circ \mathcal{H}_k - \phi \circ \mathcal{H}| d\hat{m}^k$$

The first term goes to zero by lemma 8.4 and the second one by uniform convergence of the holonomies. □

Now the [Pol18, Proposition A.7] can be directly addapted. So we have that $p \mapsto \tilde{m}_p$ is $\mathbf{B}_0$ measurable concluding the proof. □

Remark 8.6. *Observe that Theorem 8.3 can be addapted to a non conservative setting, when μ_k are f_k-invariant probabilities converging to an f-invariant probability μ such that $\eta^k = \Theta_k^* \mu_k \mid_{\mathcal{V}}$ has the form $\eta^k = \rho_k \eta$ for some $\rho_k : H \times V \to \mathbb{R}$ and there exist $0 < c < C$ such that $c \leq \rho_k \leq C$.*

9. Further applications and questions

In this section we explain some further applications of our techniques and some conjectures.

9.1. C^r density. As we mentioned in section 7, the only place where we use C^1-perturbations is for theorem 4.4, which in the setting of theorem H is replaced by theorem 7.1. Both theorems are used to obtain the pinching property. So to prove C^r density of positive exponents using our techniques we have to answer the following question: *Is the pinching property C^r dense? Or is it C^r dense in some sub-class of diffeomorphisms?*

9.2. Higher dimension. In the whole work we assume that S is a 2-dimensional manifold. First because in the two dimensional case positive exponents in a positive measures set implies non uniformly hyperbolicity in this set. In higher dimension using our techniques we can only hope to have at least one positive exponents.

For a technicall reason, we use that S has dimension 2 in the proof of Lemma 4.1 to do the perturbation. We belive that this can be addapted to a higher dimensional case (see [Pol18] where a higher dimensional case is treated for linear cocycles). Also we use the 2-dimensional assumption in Lemma 4.5, we do not know if this lemma is true in the volume preserving case for higher dimensional manifolds.

In the case of random product, as the perturbations are locally constant we do not need the homotopy constructed in Lemma 4.5, so we conjecture that

Conjecture. *Let S be a manifold of dimension greater than 2, then there exist a C^1 dense and C^r open set of volume preserving diffeomorphisms such that the random product has at least one positive exponent.*

Bibliography

[Av10] A. Avila. On the regularization of conservative maps. *Acta Mathematica*, 205:5–18, 2010.

[ASV13] A. Avila, J. Santamaria, and M. Viana. Holonomy invariance: rough regularity and applications to Lyapunov exponents. *Astérisque*, 358:13–74, 2013.

[AV] A. Avila and M. Viana. On stable accessibility with 2-dimensional center. Preprint in M. Viana's personal webpage.

[AV10] A. Avila and M. Viana. Extremal Lyapunov exponents: an invariance principle and applications. *Invent. Math.*, 181(1):115–189, 2010.

[BP01] L. Barreira and Ya. Pesin. Lectures on Lyapunov exponents and smooth ergodic theory. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, volume 69 of *Proc. Sympos. Pure Math.*, pages 3–106. Amer. Math. Soc., 2001. Appendix A by M. Brin and Appendix B by D. Dolgopyat, H. Hu and Ya. Pesin.

[BM18] P. G. Barrientos and D. Malicet. Extremal exponents of random products of conservative diffeomorphisms. Preprint on arXiv:1809.08619, 2018.

[BT17] P. Berger and D. Turaev. On Herman's positive entropy conjecture. *to appear in Advances in Math.*, 2017.

[Bo02] J. Bochi. Genericity of zero Lyapunov exponents. *Ergod. Th. & Dynam. Sys.*, 22:1667–1696, 2002.

[Bow70] R. Bowen. Markov partitions for Axiom A diffeomorphisms. *Amer. J. Math.*, 92:725–747, 1970.

[Bow75] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lect. Notes in Math.* Springer Verlag, 1975.

[Br16] A. Brown. Smoothness of stable holonomies inside center-stable manifolds and the C^2 hypothesis in pugh-shub and ledrappier-young theory. Preprint arXiv:1608.05886, 2016.

[BRH17] A. Brown and F. Hertz. Measure rigidity for random dynamics on surfaces and related skew products. *Journal of the American Mathematical Society*, 30(4):1055–1132, 2017.

[BW10] K. Burns and A. Wilkinson. On the ergodicity of partially hyperbolic systems. *Annals of Math.*, 171:451–489, 2010.

[Can01] A. Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[HS17] V. Horita and M. Sambarino. Stable ergodicity and accessibility for certain partially hyperbolic diffeomorphisms with bidimensional center leaves. *Comment. Math. Helv.*, 92(3):467–512, 2017.

[Ka80] A. Katok. Lyapunov exponents, entropy and periodic points of diffeomorphisms. *Publ. Math. IHES*, 51:137–173, 1980.

[KN10] A. Koropecki and M. Nassiri. Transitivity of generic semigroups of area-preserving surface diffeomorphisms. *Mathematische Zeitschrift*, 266(3):707–718, 2010.

[Le86] F. Ledrappier. Positivity of the exponent for stationary sequences of matrices. In *Lyapunov exponents (Bremen, 1984)*, volume 1186 of *Lect. Notes Math.*, pages 56–73. Springer-Verlag, 1986.

[LMY18-1] C. Liang, K. Marin, and J. Yang. C^1-openness of non-uniform hyperbolic diffeomorphisms with bounded C^2 norm. Preprint arXiv:1808.04003, 2018.

[LMY18-2] C. Liang, K. Marin, and J. Yang. Lyapunov exponents of partially hyperbolic volume-preserving maps with 2-dimensional center bundle. *Ann. Inst. H. Poincaré*

Anal. Non Linéaire., 35:1687–1706, 2018.

[LY17] C. Liang and Y. Yang. the c^1 density of nonuniform hyperbolicity in c^r conservative diffeomorphisms. *Proc. Amer. Math. Soc. 145 (2017), 1539-1552*, 2017.

[Ma96] R. Mañé. The Lyapunov exponents of generic area preserving diffeomorphisms. In *International Conference on Dynamical Systems (Montevideo, 1995)*, pages 110–119. Longman, 1996.

[Mar16] K. Marin. C^r-density of (non-uniform) hyperbolicity in partially hyperbolic symplectic diffeomorphisms. *Comment. Math. Helv.*, 91(2):357–396, 2016.

[Pe77] Ya. B. Pesin. Characteristic Lyapunov exponents and smooth ergodic theory. *Russian Math. Surveys*, 324:55–114, 1977.

[Pol18] M. Poletti. Stably positive lyapunov exponents for symplectic linear cocycles over partially hyperbolic diffeomorphisms. *Discrete & Contin. Dyn. Syst.*, 38, 2018.

[Sm67] S. Smale. Differentiable dynamical systems. *Bull. Am. Math. Soc.*, 73:747–817, 1967.

[TY18] A. Tahzibi and J. Yang. Invariance principle and rigidity of high entropy measures. *To appear in Transactions of American Mathematica Society*, 2018.

[Vi08] M. Viana. Almost all cocycles over any hyperbolic system have nonvanishing Lyapunov exponents. *Ann. of Math.*, 167:643–680, 2008.

[VO15] M. Viana and K. Oliveira. *Foundations of Ergodic Theory.* Cambridge University Press, 2015.

Part II

Dissipative dynamics: Robust transitivity and physical measures

CHAPTER 4

A new example of robustly transitive diffeomorphism

This chapter is based on a joint work with Pablo D. Carrasco. [1]

1. Introduction and Main Theorem

Let M be a closed Riemannian manifold and denote by $\mathrm{Diff}(M)$ the space of $\mathcal{C}^1$ diffeomorphisms in M, equipped with the $\mathcal{C}^1$ topology. In trying to understand $\mathrm{Diff}(M)$, properties that are stable under perturbations play a central role in the study. This is true not only from a theoretical point of view (i.e. understanding open sets in $\mathrm{Diff}(M)$), but also from an applied one, as it is desirable to maintain the same qualitative conclusions even in presence of small errors. Following common use, we will say that $f \in \mathrm{Diff}(M)$ has the property $\mathcal{P}$ *robustly* if $\mathcal{P}$ is also valid in an open set $\mathcal{U}_f \subset \mathrm{Diff}(M)$ containing f.

Among the robust properties that have been studied, transitivity has been one of the most extensively researched. Recall that a diffeomorphism f is *transitive* if for any two non-empty open sets U and V, there is an integer $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. The first known examples of robustly transitive diffeomorphisms are given by Anosov maps [**?**]: if $f \in \mathrm{Diff}(M)$ is transitive and uniformly hyperbolic, then it is robustly transitive. It turns out that certain degree of hyperbolicity is required in order to have robust transitivity. Indeed, if $f \in \mathrm{Diff}(M)$ is robustly transitive and $dim M \leq 3$ then f is hyperbolic/partially hyperbolic [Ma78, BDU99]; in general, f admits Df-invariant bundles E, F such that $det(Df^{-n_0}|E), det(Df^{n_0}|D) \leq 1/2$ for some uniform $n_0 \geq 1$ [BDP03]. It is worth to point out that the bundles E, F above are not necessarily uniformly expanding [BV00].

As for non-hyperbolic examples, there are several known. The list below gives a rough (and necessarily, incomplete) picture of the arguments used to establish robust transitivity for non hyperbolic systems.

- *Deformations from Anosov systems.* The first concrete example of non-uniformly hyperbolic robustly transitive map was given by M. Shub in [Sh71]; later in [Ma78] R. Mañe gave a similar type of construction on $\mathbb{T}^3$. They are both partially hyperbolic (see next section) and homotopic to an Anosov system, in particular with hyperbolic action on homology. The example given in [BV00] is also a deformation of an Anosov diffeomorphism, and although it is not partially hyperbolic, it does admit a dominated splitting [2] coherent with its Anosov part (as the previous two examples). More recently, R. Potrie ([Po12] page 152) gave an example of this type, but with the difference that it admits a dominated splitting which is not coherent with its hyperbolic part. In these cases, the proof of robust transitivity is founded in that they have hyperbolic-type behavior in a large part of the space.
- *Blenders.* This powerful mechanism was introduced in [BD96] by C. Bonatti and L. Díaz. With it the authors were able to prove that some perturbations of time-t maps of mixing hyperbolic flows, and of the product of an Anosov map times the identity (say, on $\mathbb{T}^3$), are robustly transitive. Note that in the first case the examples are homotopic to the identity, while in the second the action on homology on the fiber direction is trivial. The same tool was used by C. Cheng, S. Gan and Y. Shi in [CGS18] to present a robustly transitive skew-product, but

1. ICEx-UFMG, Avda. Presidente Antônio Carlos 6627, Belo Horizonte-MG,BR 31270-901.

2. An f invariant closed set Λ admits an dominated splitting if $T_\Lambda M = E \oplus F$ is an Df-invariant decomposition satisfying $\|Df^{n_0}(m)|E(m)\| \cdot \|Df^{-n_0}|F(f^{n_0}m)\| \leq \frac{1}{2}$ for some uniform $n_0 \geq 1$.

where the fiber action in homology is given by minus the identity. We also point out that the example in [CGS18] has some interesting ergodic properties.

- *Minimality of the stable/unstable foliation.* It is easy to see that if $f \in \mathrm{Diff}(M)$ admits an invariant expanding minimal foliation, then f is transitive. Conditions that guarantee the persistence of these types of foliations are thus relevant for robust transitivity. Among this, the property SH introduced by E. Pujals and M. Sambarino [PS06] is particularly simple to check, and can be applied to establish robust transitivity of transitive partially hyperbolic systems where one has some control on the behavior of the stable/unstable foliations. Shub and Mañe's examples cited before fall into this category.
- *Non-uniform expansion along the center.* In a recent work [Ya18], J. Yang considers partially hyperbolic systems with non-uniformly expanding center behavior, and shows that any conservative ergodic of such systems with one-dimensional center is robustly transitive. The author uses the non-uniform expanding character of the center as a replacement for hyperbolicity, employing methods of smooth ergodic theory. These techniques however seem to be applicable only for systems with one-dimensional center.

In this note we add a different type of example to the previous list. We present a diffeomorphism that is again a partially hyperbolic skew-product on $\mathbb{T}^4$, but with non-hyperbolic action on homology. More importantly, the tangent bundle of the fiber does not admit any one-dimensional invariant direction, nor does it have a non-uniform expanding/contracting behavior.

Let $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ and for each $N > 0$ we consider the *standard map* given by $s_N(x, y) = (2N \sin x + 2x - y, x)$. Fix $A \in \mathrm{SL}(2, \mathbb{Z})$ a hyperbolic matrix. On $\mathbb{T}^4$ we use coordinates (x, y, z, w), and for each N we consider the skew product $f_N : \mathbb{T}^2 \times \mathbb{T}^2 \longrightarrow \mathbb{T}^2 \times \mathbb{T}^2$ given by [3]

$$f_N(x, y, z, w) = (s_N(x, y) + P_x \circ A^{[N]}(z, w), A^{[2N]}(z, w)), \text{ where } P_x(x, y) = (x, 0).$$

This diffeomorphism was introduced in [BC14] where it is proven that for large N it is non-uniformly hyperbolic (i.e. all its Lyapunov exponents are Lebesgue almost everywhere different from zero), and remains so by $\mathcal{C}^2$ conservative perturbations: these maps are in fact ergodic with respect to the Lebesgue measure [Ob18]. It is direct to verify that the action on homology of f_N is not hyperbolic, and that its fiber direction does not admit a dominated splitting (since $Df_N|\mathbb{R}^2 \times \{0\} = Ds_N$). Finally, the example does not satisfy the SH property, and the question whether its stable/unstable foliations are (robustly) minimal seems to be outside the reach of current technology.

Here we establish the following.

Theorem I. *There exists $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$ the diffeomorphism f_N is robustly transitive (in fact, robustly topologically mixing).*

Remark 1.1. *Topologically mixing is a stronger property than transitivity: f is topologically mixing if for any two open sets U and V, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have $f^n(U) \cap V \neq \emptyset$.*

The proofs of robust transitivity for the diffeomorphisms which are deformations of Anosov systems, mentioned above, use information about some type of minimality (or ε-minimality) of stable/unstable manifolds. Observe that, our example has a hyperbolic-type behavior in a large part of the manifold, as in the examples which are deformations of Anosov systems. However, an important difference in our proof is that we do not use any information on the minimality (or ε-minimality) of stable/unstable foliations.

3. Here $[N]$ denotes the integer part of N.

2. Preliminaries

In this section we present the tools we will use. We first state some general facts about partially hyperbolic diffeormorphisms and then some facts about the example we are studying.

2.1. Partial hyperbolicity and foliations. A diffeomorphism $f \in \mathrm{Diff}(M)$ is *partially hyperbolic* if there exist a Df-invariant decomposition $TM = E^{ss}_f \oplus E^c_f \oplus E^{uu}_f$ and a Riemannian metric on M such that for any $m \in M$

$$\|Df(m)|_{E^{ss}_f(m)}\| < 1 < \|(Df(m)|_{E^{uu}_f(m)})^{-1}\|^{-1}$$

$$\|Df(m)|_{E^{ss}_f(m)}\| < \|(Df(m)|_{E^{c}_f(m)})^{-1}\|^{-1} \leq \|Df(m)|_{E^{c}_f(m)}\| < \|(Df(m)|_{E^{uu}_f(m)})^{-1}\|^{-1}.$$

The set $PH(M)$ of partially hyperbolic diffeomorphisms is an open subset of $\mathrm{Diff}(M)$. It is well known that the distributions E^{ss}_f and E^{uu}_f are uniquely integrable [HPS77], that is, there are two unique foliations $\mathcal{F}^{ss}_f$ and $\mathcal{F}^{uu}_f$, with C^1-leaves, that are tangent to E^{ss}_f and E^{uu}_f respectively. For a point $m \in M$ we will denote by $W^{ss}_f(m)$ a leaf of the foliation $\mathcal{F}^{ss}$, we will call such leaf the strong stable manifold of m. Similarly, we define the strong unstable manifold of m and denote it by $W^{uu}_f(m)$. We denote $E^{cs}_f = E^s_f \oplus E^c_f$ and $E^{cu}_f = E^c_f \oplus E^u_f$.

Definition 2.1. *A partially hyperbolic diffeomorphism f is dynamically coherent if there are two invariant foliations $\mathcal{F}^{cs}_f$ and $\mathcal{F}^{cu}_f$, with C^1-leaves, tangent to E^{cs}_f and E^{cu}_f respectively. From those two foliations one obtains another invariant foliation $\mathcal{F}^c_f = \mathcal{F}^{cs}_f \cap \mathcal{F}^{cu}_f$ with $\mathcal{C}^1$ leaves that is tangent to E^c_f. We call these foliations the center-stable, center-unstable and center foliation.*

For $R > 0$ we denote by $W^*_f(m; R)$ the disc of size R centered on m, measured by the intrinsic metric in $W^*_f(m)$, for $* = ss, c, uu$.

Definition 2.2. *Let $f, g \in PH(M)$ dynamically coherent. We say that f and g are leaf conjugated if there is a homeomorphism (called a leaf conjugacy) $h : M \to M$ that sends leaves of $\mathcal{F}^c_f$ to leaves of $\mathcal{F}^c_g$ and such that for any $L \in \mathcal{F}^c_f$ it is verified*

$$h(f(L)) = g(h(L)).$$

One may study the stability of partially hyperbolic systems up to leaf conjugacies. The next theorem is a good representative of this situation.

Theorem 2.3 ([HPS77], Theorem 7.4)**.** *Consider $f \in PH(M)$ having a differentiable*[4] *center foliation. Then there exists an open neighborhood $U_f \subset \mathrm{Diff}(M)$ of f such that any $g \in U_f$ is partially hyperbolic, dynamically coherent, and leaf conjugate to f. The corresponding leaf conjugacy between g and f depends continuously on g.*

Let V and N be compact manifolds. We define a partially hyperbolic skew-product as a diffeomorphism $f \in PH(V \times N)$ of the form

$$f(p, q) = (F_q(p), A(q)) \quad (p, q) \in V \times N,$$

where $A : N \to N$ is a hyperbolic diffeomorphism, for each $q \in N$, the map $F_q : V \to V$ is a C^1-diffeomorphism and depends continuously with the choice of the point q, and

$$\|A|_{E^{ss}(q)}\| < \|(DF_q(p))^{-1}\|^{-1} \leq \|DF_q(p)\| < \|(A|_{E^{uu}(q)})^{-1}\|^{-1}, \forall (p, q) \in V \times N.$$

In this case f is dynamically coherent with center foliation $\mathcal{F}^c = \{V \times \{z\} : z \in N\}$. The example f_N that we are considering is of this type.

4. More generally, the differentiability condition can be replaced by plaque expansivity. See Chapter 7 of [HPS77]

Remark 2.4. *Using theorem 2.3 one checks that if f is a partially hyperbolic skew-product, then any diffeomorphism g sufficiently close to f is also partially hyperbolic, and has a center foliation $\mathcal{F}^c_g$ given by a trivial fibration with leaves diffeomorphic to V. These leaves approach (in the Hausdorff metric) the horizontal foliation $\{V \times \{z\} : z \in N\}$ as $g \longrightarrow f$.*

2.2. Some estimates for the example. Recall that for each $N \geq 0$ and $m = (x, y, z, w) \in \mathbb{T}^4$ we defined the diffeomorphism

$$f_N(x, y, z, w) = (s_N(x, y) + P_x \circ A^{[N]}(z, w), A^{[2N]}(z, w)), \text{ where } P_x(x, y) = (x, 0).$$

Its derivative can be computed in block form

$$Df_N(m) = \begin{pmatrix} Ds_N(x, y) & P_x \circ A^{[N]} \\ 0 & A^{[2N]} \end{pmatrix}, \tag{59}$$

where

$$Ds_N(x, y) = \begin{pmatrix} N \cos x + 2 & -1 \\ 1 & 0 \end{pmatrix}. \tag{60}$$

For a point $m = (x, y, z, w) \in \mathbb{T}^4$ we will write $Ds_N(m) = Ds_N(x, y)$. Observe that

$$\frac{1}{2N} \leq \|Ds_N\| \leq 2N \text{ and } \|D^2 s_N\| \leq N.$$

Denote by $0 < \lambda < 1 < \mu = \lambda^{-1}$ the eigenvalues of A, and let e^s, e^u be unit eigenvectors of for λ and μ, respectively. Consider the involution $\mathcal{I}(x, y, z, w) = (y, x, z, w)$ for $(x, y, z, w) \in \mathbb{T}^2$. An important feature of the map f_N is given by the following lemma.

Lemma 2.5 ([BC14], Lemma 1)**.** *The map f_N^{-1} is conjugated to the map*

$$(x, y, z, w) \mapsto (s_N(x, y) + P_x \circ A^{-[N]}(z, w), A^{-[2N]}(z, w)),$$

by the involution $\mathcal{I}$.

This lemma allows us to prove certain properties for f_N and f_N^{-1} only by considering the map f_N, since the involution tell us that f_N and f_N^{-1} behave in the same way up to exchanging the x and y coordinates.

Lemma 2.6 ([BC14], Corollary 5)**.** *For N sufficiently large, there exists a C^1-neighborhood $\mathcal{U}_N$ of f_N such that for any $g \in \mathcal{U}_N$, for any point $m \in \mathbb{T}^4$ and for any unit vector $v = (v_x, v_y, v_z, v_w)$ in $E^{uu}_g(m)$, we have*

$$\lambda^N \left(\|P_x(e^u)\| - 3\lambda^N \right) \leq |v_x| \leq \lambda^N \left(\|P_x(e^u)\| + 3\lambda^N \right).$$

By lemma 2.5, similar statement holds for the strong stable direction, but projecting on the y direction.

For $m \in \mathbb{T}^4$, we identify $T_m\mathbb{T}^4 = \mathbb{R}^4$; since the center bundle E^c of f_N is tangent to the horizontal fibers, by an abuse of notation we write $E^c = \mathbb{R}^2 \times \{0\} = \mathbb{R}^2$ (the first two coordinates). We define $\pi_h : \mathbb{T}^2 \times \mathbb{T}^2 \longrightarrow \mathbb{T}^2, proj_h : \mathbb{R}^4 \longrightarrow E^c$ to be the corresponding projections. Similarly, since the hyperbolic directions E^s_A and E^u_A of A on $\mathbb{T}^2$ are constant, by the same abuse of notation we will write $E^*_A \subset \mathbb{R}^4, * = s, u$ for the directions that determine on $\mathbb{R}^4$. If $v \in \mathbb{R}^4$ we write $v = (v_c, v_s, v_u)$ using the decomposition

$$\mathbb{R}^4 = E^c \oplus E^s_A \oplus E^u_A \tag{61}$$

For $\alpha > 0$ we define the stable cone of size α over m by

$$\mathcal{C}^s_\alpha(m) = \{v \in T_m\mathbb{T}^4 : v = (v_c, v_s, v_u) : \|v_c + v_u\| < \alpha \|v_s\|\}.$$

Note that $\mathcal{C}^s_\alpha = \{\mathcal{C}^s_\alpha(m)\}_{m \in \mathbb{T}^4}$ is a continuous cone field over M. Analogously, we define the unstable cone field $\mathcal{C}^u_\alpha$ of size α.

Lemma 2.7. *Fix $\alpha > 0$. If N is sufficiently large there exists an open neighborhood $\mathcal{U}_N$ of f_N in $\mathrm{Diff}(M)$ such that for every $g \in \mathcal{U}_N$, the strong stable direction E^{ss}_g of g is contained in $\mathcal{C}^s_\alpha$. Similarly, the strong unstable direction E^{uu}_g of g is contained in $\mathcal{C}^u_\alpha$.*

Proof. By (59) we deduce

$$\frac{1}{2N} \leq m(Ds_N) \leq \|Ds_N\| \leq 2N$$
$$\lambda^N \leq m(P_x \circ A^N) \leq \|P_x \circ A^N\| \leq \lambda^{-N}.$$

On the other hand, the strength of the expansion (or contraction) of A^{2N} is λ^{-2N} (respectively λ^{2N}), which is exponentially bigger than the estimates above. Therefore, a simple calculation for N sufficiently large concludes the proof of the lemma for the case $g = f_N$. Noting that all bounds are stable by $\mathcal{C}^1$ perturbations we finish the proof. □

Lemma 2.7 states that for N large enough, the strong stable direction is close to the stable direction of the linear Anosov A. Similarly, the strong unstable direction is close to the unstable direction of the linear Anosov A.

Define $I = I(N) = (-2N^{-\frac{3}{10}}, 2N^{-\frac{3}{10}})$ and write $C = \{\frac{\pi}{2} + I\} \cup \{\frac{3\pi}{2} + I\}$. Consider the regions

$$Crit^u = C \times S^1 \times \mathbb{T}^2 \tag{62}$$
$$Crit^s = S^1 \times C \times \mathbb{T}^2. \tag{63}$$

We define the *good regions* as the sets $G^* = (Crit_*)^c$, for $* = s, u$. For each $\theta > 0$, we define the horizontal cone of size θ along the center, as

$$\mathscr{C}^{hor}_\theta := \{v = (v_x, v_y) \in E^c : \|v_y\| < \theta \|v_x\|\}.$$

We define similarly the vertical cone, but exchanging the roles of v_x and v_y in the definition. Fix $\theta = N^{-\frac{3}{5}}$.

Lemma 2.8. *For every N sufficiently large there exists an open neighborhood $\mathcal{U}_N$ of f_N with the following property: if $g \in \mathcal{U}_N, m \in G^u$ then*

$$v \in \mathscr{C}^{hor}_\theta \Rightarrow proj_h(Dg(m)v) \subset \mathscr{C}^{hor}_\theta \text{ and } \|Dg(m)v\| > N^{\frac{1}{2}}\|V\|.$$

Furthermore, if γ is a C^1-curve contained in a center leaf satisfying

- $proj_h\left(\frac{d\gamma}{dt}(t)\right) \in \mathscr{C}^{hor}_\theta \ \forall t$, *and*
- *it has length greater than $N^{-\frac{3}{10}}$,*

then the curve $g \circ \gamma$ has length greater than 4π and its horizontal projection is tangent to $\mathscr{C}^{hor}_\theta$.

Proof. The proof follows from the proof of lemma 3.9 and 5.5 in [Ob18] □

As an easy consequence of Remark 2.4, we have the following proposition:

Proposition 2.9. *Fix $\varepsilon > 0$ small, for N large enough there is a C^1-neighborhood $\mathcal{U}_N$ of f_N, such that if $g \in \mathcal{U}_N$ then g is dynamically coherent, its center leaves are C^1-submanifolds, g is leaf conjugated to f_N and for every $m \in \mathbb{T}^4$ the C^1-distance between $W^c_g(m)$ and $W^c_f(m)$ is smaller than ε.*

3. Topologically mixing: proof of the Theorem I

Lemma 3.1. *For every N large enough, there exists a C^1-neighborhood $\mathcal{U}_N$ of f_N such that any $g \in \mathcal{U}_N$ verifies the following properties:*

(1) If $\gamma^u \subset \mathcal{F}^{uu}_g$ is non trivial curve then there exists a point $m \in \gamma^u$ and a number $n_u \geq 0$ such that $g^n(m) \in G^u$ for every $n \geq n_u$.

(2) *If $\gamma^s \subset \mathcal{F}^{ss}_g$ is non trivial curve then there exists a point $m \in \gamma^s$ and a number $n_s \geq 0$ such that $g^{-n}(m) \in G^s$ for every $n \geq n_s$.*

PROOF. Suppose that N is large enough and $\mathcal{U}_N$ is the C^1-open set given by lemma 2.6. Let $g \in \mathcal{U}_N$ and γ^u be a non trivial curve contained in a strong unstable manifold of g. Take $n_u \geq 0$ to be the smallest integer such that $\gamma^u_{n_u} := g^{n_u}(\gamma^u)$ has length greater than $\lambda^{-N}\left(\|P_x(e^u)\| + 3\lambda^N\right)^{-1}$.

By lemma 2.6, we have that $\frac{l(\gamma^u_{n_u} \cap G^u)}{l(\gamma^u_{n_u})} > 1 - 10N^{-\frac{3}{10}}$. This implies that we may take $\gamma^u_{n_u+1} \subset \gamma^u_{n_u}$ a compact connected curve contained in G^u with length greater than $\frac{1}{2}$.

It is easy to see that for N large enough, for any curve γ contained in a strong unstable manifold with length greater than $\frac{1}{2}$, then $g(\gamma)$ has length greater than $\lambda^{-N}\left(\|P_x(e^u)\| + 3\lambda^N\right)^{-1}$. Therefore, the length of $g(\gamma^u_{n_u+1})$ is greater than $\lambda^{-N}\left(\|P_x(e^u)\| + 3\lambda^N\right)^{-1}$.

Repeating this argument, we find a decreasing sequence of compact sets

$$\gamma^u_{n_u+1} \supset \gamma^u_{n_u+2} \supset \cdots,$$

with the property that $g^j(\gamma^u_{n_u+n}) \subset G^u$, for $j = 0, \cdots, n-1$. Take

$$m^u \in \bigcap_{n\in\mathbb{N}} \gamma^u_{n_u+n}.$$

By construction, the point $m = g^{-n_u}(m^u)$ verifies the conclusion of our lemma. The argument for the stable curves is the same, working with backward iterates.

□

Using the skew product structure of f_N, we can prove the following lemma:

Lemma 3.2. *There exists a constant $R > 0$ with the following property: for N sufficiently large, there exists a C^1-neighborhood $\mathcal{U}$ of f_N such that for any $g \in \mathcal{U}_N$ and any two points $p, q \in \mathbb{T}^4$ we have that for any $m_p \in W^c_g(p)$ there exists $m_q \in W^c_g(q)$ such that $W^{uu}_g(m_p; R) \cap W^{ss}_g(m_q; R) \neq \emptyset$.*

PROOF. First let us prove that if N is sufficiently large, we have the conclusion of the lemma for f_N. The robustness of this property will then follow by a transversality argument.

We consider $\pi_v : \mathbb{T}^4 \to \mathbb{T}^2$ the projection on the last two coordinates, and start by noticing that (due to minimality of the foliations $\mathcal{F}^{ss}_A, \mathcal{F}^{uu}_A$ in $\mathbb{T}^2$) there is a number $R_1 > 0$ with the property that for any $p, q \in \mathbb{T}^2$ the disc $W^s_A(p;, R_1)$ intersects transversely $W^u_A(q; R_1)$. By lemma 2.7, there exists a constant $R_2 > R_1$ such that for any point $m \in \mathbb{T}^4$ we have $\pi_v(W^{ss}_{f_N}(m; R_2)) \supset W^s_A(\pi_v(m); R_1)$.

For any $m \in \mathbb{T}^4$ we consider the C^1-submanifolds

$$W^{cs}_{f_N}(m; R_2) = \bigcup_{p\in W^c_{f_N}(m)} W^{ss}_{f_N}(p;, R_2)$$

$$W^{cu}_{f_N}(m; R_2) = \bigcup_{p\in W^c_{f_N}(m)} W^{uu}_{f_N}(p;, R_2)$$

By our choice of R_2, R_1, for any $m_1, m_2 \in \mathbb{T}^4$ the sets $W^{cs}_{f_N,R_2}(m_1)$ and $W^{cu}_{f_N,R_2}(m_2)$ intersect transversely; indeed their intersection is a center leaf, which shows that the conclusion of the lemma holds for f_N. Since the manifolds $W^{cs}_g(\cdot; R_2), W^{cu}_g(\cdot; R_2)$ depend continuously on g ([HPS77] chapter 5), the lemma folows. □

We fix R as in lemma above and recall that $\theta = N^{-\frac{3}{5}}$.

Proposition 3.3. *If N is sufficiently large there exists $\mathcal{U}_N \subset \mathrm{Diff}(M)$ neighborhood of f_N such that for any $g \in \mathcal{U}_N$ and any open set $U \subset \mathbb{T}^4$, there exists a number $n'_u \geq 0$ with the following property: for every $n \geq n'_u$ there exists a C^1 curve $\gamma^+_n \subset g^n(U)$ satisfying:*

- γ_n^+ *is contained in a center leaf.*
- $\pi_h(\gamma_n^+)$ *is tangent to* $\mathscr{C}_\theta^{hor}$.
- γ_n^+ *has length greater than* 4π.
- $\bigcup_{q\in\gamma_n^+} W_g^{uu}(q;R) \subset g^n(U)$.

Similarly, there exists $n_s' \geq 0$ *such that for any* $n \geq n_s'$, *there exists a* C^1 *curve* $\gamma_n^- \subset g^{-n}(U)$ *satisfying*

- γ_n^- *is contained in a center leaf.*
- $\pi_h(\gamma_n^-)$ *is tangent to* $\mathscr{C}_\theta^{ver}$.
- γ_n^- *has length greater than* 4π
- $\bigcup_{q\in\gamma_n^-} W_g^{ss}(q;R) \subset g^{-n}(U)$.

Proof. Choose $N and \mathcal{U}_N$ so that the conclusions of lemmas 3.1 and 3.2 hold. Fix $g \in \mathcal{U}_N$ and also fix two open sets $U, V \subset \mathbb{T}^4$. Take a small unstable curve $\gamma^u \subset U$ and consider $n_u \geq 0, m^u \in \gamma^u$ given in lemma 3.1 (i.e. $g^n(m^u) \in G^u \ \forall n \geq n_u$). Set $m^+ := g^{n_u}(m^u)$.

Since $m^+ \in g^{n_u}(U)$ and the set $g^{n_u}(U)$ is open, we may take a curve

$$\gamma^+ \subset \left(W^c(m^+) \cap g^{n_u}(U) \cap G^u\right)$$

centered in m^+, such that $\pi_h(\gamma^+)$ is a horizontal segment on the torus $\mathbb{T}^2$. By lemma 2.8, the image $g(\gamma^+)$ projects to a curve tangent to $\mathscr{C}_\theta^{hor}$ and verifies $length(g(\gamma^+)) > N^{\frac{1}{2}} length(\gamma^+)$. The same argument as in the proof of lemmas 4.3 and 5.7 in [Ob18] implies that there exists $n_1 \in \mathbb{N}$ such that for any $n \geq n_1$, there is a C^1 curve $\gamma_n^+ \subset g^n(\gamma^+)$ with length greater than 4π and $\pi_h(\gamma_n^+)$ is tangent to $\mathscr{C}_\theta^{hor}$.

Take $r > 0$ so that

$$\bigcup_{q\in\gamma^+} W_g^{uu}(q;r) \subset g^{n_u}(U).$$

Fix $n_2 \in \mathbb{N}$ such that for any $q \in \gamma^+$ and $n \geq n_2$, we have that $g^n(W_g^{uu}(q;r)) \supset W_g^{uu}(g^n(q;R))$. Finally, take $n_u' = \max\{n_1, n_2\}$. It follows directly that for any $n \geq n_u'$ we have

$$\bigcup_{q\in\gamma_n^+} W_g^{uu}(q;R) \subset g^n(U),$$

which finishes the proof of the first part. A similar argument for g^{-1} completes the proof of the proposition. □

Consider the vertical foliation $\mathcal{F}_{ver} = \{\{z\} \times \mathbb{T}^2 : z \in \mathbb{T}^2\}$. Observe that if N is sufficiently large and g sufficiently C^1-close to f_N, we have that $W_g^c(m)$ intersects each vertical torus $\{z\} \times \mathbb{T}^2$ in exactly one point, for any $m \in \mathbb{T}^4$. Hence, for any two points $m_1, m_2 \in \mathbb{T}^4$, the map from $W_g^c(m_1)$ to $W_g^c(m_2)$ defined by $h_{m_1,m_2}^g(p) = W_g^c(m_2) \cap \mathcal{F}_{ver}(p)$ is well defined. Note that $h_{m_1,m_2}^{f_N}$ is just the identity map, independently of the points m_1, m_2.

We recall also the notion of holonomy. For $p, q \in M$ with $q \in W_{f_N}^{ss}(p)$ define $H_{p,q}^{s,f_N} : W_{f_N}^c(p) \longrightarrow W_{f_N}^c(q)$, the stable holonomy between p and q, by

$$H_{p,q}^{s,f_N}(w) = W_{f_N}^{ss}(w) \cap W_{f_N}^c(q), \text{ for } w \in W_{f_N}^c(p).$$

It is easy to see that this is a well defined map. Analogously we define $H_{p,q}^{s,g}$ for g close to f_N. Similarly, we define the unstable holonomy map $H_{p,q}^{u,g}$ using $\mathcal{F}_g^{uu}$ instead of $\mathcal{F}_g^{ss}$.

Let $R > 0$ be the constant given by lemma 3.2.

Lemma 3.4. *For every $\varepsilon > 0$, there exists $N_0 := N_0(\varepsilon)$ with the following property: for $N \geq N_0$ there exists a C^1-neighborhood $\mathcal{U}_N$ of f_N such that if $g \in \mathcal{U}_N, p \in \mathbb{T}^4$ and $q \in W_g^{ss}(p; R)$ then $d_{C^0}(h_{p,q}^g, H_{p,q}^{s,g}) < \varepsilon$. Analogous result holds for the unstable holonomy.*

Proof. Fix $\varepsilon > 0$. Let us first prove that the conclusion holds for f_N, for N large. Using the coordinate system we defined in (61), we consider the constant vector field $X^s = \{0\} \times e^s$, where e^s is the unitary vector that generates the stable direction for the linear Anosov A chosen at the beginning. Let $\{X_t^s\}_t$ be the flow generated by X^s. As mentioned, since the system is a skew product, any stable manifold of f_N projects to a stable manifold of A. In particular, for $p \in \mathbb{T}^4$ and $q \in W_{f_N}^{ss}(p; R)$ there exists an unique number $T(q) \in \mathbb{R}$ such that $X_{T(q)}^s(.)$ is a diffeomorphism between $W_{f_N}^c(p)$ and $W_{f_N}^c(q)$. It is easy to see that $h_{p,q}^{f_N}(m) = X_{T(q)}^s$ for $m \in W_{f_N}^c(p)$.

By lemma 2.7, after fixing $\alpha \ll \frac{\varepsilon}{R}$, for N large enough $E_{f_N}^{ss}$ belongs to the cone of size α around the direction $\{0\} \times E_A^s$. This implies that for any point $m \in \mathbb{T}^4$, the Hausdorff distance between the strong stable manifold $W_{f_N}^{ss}(m; R)$ and the piece of X^s-orbit $X_{[-R,R]}^s(m)$ is less than ε. By the definition of $H_{p,q}^{s,f_N}$, we conclude that $d_{C^0}(h_{p,q}^{f_N}, H_{p,q}^{s,f_N}) < \varepsilon$.

Since the center leaves and compact parts of strong stable leaves vary C^1-continuously with the choice of a diffeomorphism g in a neighborhood of f_N, we conclude that for any g sufficiently C^1-close to f_N, $p \in \mathbb{T}^4$ and $q \in W_g^{ss}(p; R)$ we have $d_{C^0}(h_{p,q}^g, H_{p,q}^{s,g}) < \varepsilon$. □

Proof of Theorem I. Fix $\varepsilon > 0$ small and let N be large enough with corresponding neighborhood $\mathcal{U}_N$ small enough such that the conclusions of proposition 3.3 and lemmas 3.2 and 3.4 hold. Fix $g \in \mathcal{U}_N$ and let $U, V \subset \mathbb{T}^4$ be any two open sets.

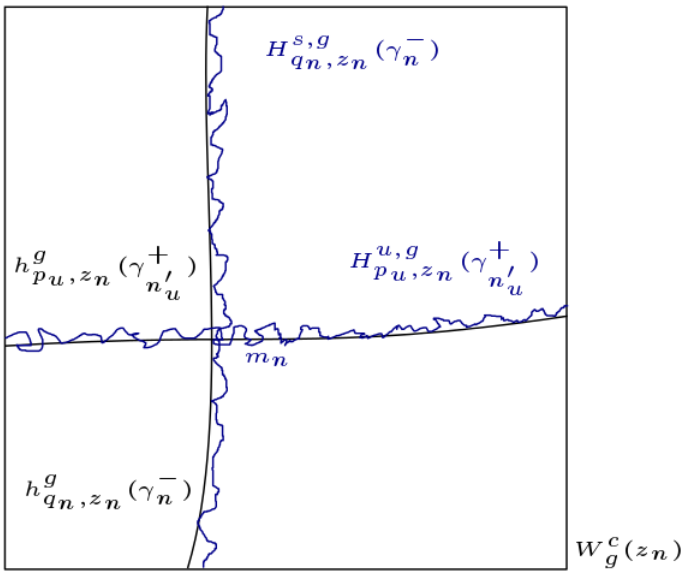

Figure 1. Intersection between $H_{q_n,z_n}^{s,g}(\gamma_n^-)$ and $H_{p_u,z_n}^{u,g}(\gamma_{n'_u}^+)$

By proposition 3.3 applied to U for the future and V for the past, we obtain two numbers $n'_u, n'_s \geq 0$ that verify the conclusion of the proposition. For $n \geq n'_s$ consider the curve $\gamma_n^- \subset W_g^c(q_n) \cap g^{-n}(V)$ that is almost vertical, and $\gamma_{n'_u}^+ \subset W_g^c(p_u) \cap g^{n'_u}(U)$ be the almost horizontal curve given by the proposition.

Applying lemma 3.2 we deduce the existence of a point $z_n \in \mathbb{T}^4$ such that $W_g^{cs}(q_n; R) \cap W_g^{cu}(p_u; R) = W_g^c(z_n)$. Observe that the image of $h_{q_n,z_n}^g(\gamma_n^-)$ is a curve C^0-close to a vertical curve of length 4π. By lemma 3.4, the curve $H_{q_n,z_n}^{s,g}(\gamma_n^-)$ is also C^0-close to a vertical curve of length 4π. Similarly, $H_{p_u,z_n}^{u,g}(\gamma_{n'_u}^+)$ is a curve C^0-close to a horizontal curve of length 4π. Therefore, the curves $H_{q_n,z_n}^{s,g}(\gamma_n^-)$ and $H_{p_u,z_n}^{u,g}(\gamma_{n'_u}^+)$ must intersect at some point $m_n \in W_g^c(z_n)$ (see figure 1).

By proposition 3.3, the point m_n belongs to $g^{n'_u}(U) \cap g^{-n}(V)$. In particular, $g^{n'_u+n}(U) \cap V \neq \emptyset$. Hence, for any $n \geq n'_u + n'_s$ we have that $g^n(U) \cap V \neq \emptyset$ and g is topologically mixing. This concludes the proof of the Theorem I.

□

Bibliography

[BC14] P. Berger, and P. Carrasco. Non-uniformly hyperbolic diffeomorphisms derived from the standard map. *Comm. Math. Phys.*, 329:239–262, 2014.

[BD96] C. Bonatti, and L. Díaz. Persistent nonhyperbolic transitive diffeomorphisms. *Annals of Mathematics*, 143(2):357, 1996.

[BDP03] C. Bonatti, L. Díaz, and E. Pujals. A $\mathcal{C}^1$-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources. *Annals of Mathematics*, 158(2):355–418, 2003.

[BV00] C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel Journal of Mathematics*, 115(1):157–193, 2000.

[CGS18] C. Cheng, S. Gan, and Y. Shi. A robustly transitive diffeomorphism of Kan's type. *Disc. and Cont. Dyn. Sys.*, 38(2):867-888, 2018.

[BDU99] L. Díaz, E. Pujals, and R. Ures. Partial hyperbolicity and robust transitivity. *Acta Mathematica*, 183(1):1–43, 1999.

[HPS77] M. Hirsch, C. Pugh, and M. Shub. Invariant manifolds. *Lecture Notes in Mathematics*, Vol. 583. Springer-Verlag, Berlin-New York, 1977

[Ma78] R. Mañé. Contributions to the stability conjecture. *Topology*, 17(4):383–396, 1978.

[Ob18] D. Obata. On the stable ergodicity of Berger-Carrasco's example. *Ergodic Theory and Dynamical Systems*, Online version, 2018.

[Po12] R. Potrie. Partial hyperbolicity and attracting regions in 3-dimensional manifolds. *Thesis. Available in http://www.cmat.edu.uy/ rpotrie/*, 2012.

[PS06] E. Pujals, and M. Sambarino. A sufficient condition for robustly minimal foliations. *Ergodic Theory and Dynamical Systems* 26(1), 281-289, 2006.

[Sh71] M. Shub. Topological transitive diffeomorphism on $\mathbb{T}^4$. *Lecture notes in Mathematics 206*. Springer, 1971.

[Sm67] S. Smale. Differentiable Dynamical Systems. *Bull. Amer. Math. Soc.*, 73(73):747–817, 1967.

[Ya18] J. Yang. Entropy along expanding foliations. *Preprint on arxiv:1601.05504*, 2016.

CHAPTER 5

Rigidity of u-Gibbs measures for certain partially hyperbolic skew products

1. Introduction

In dynamics one usually tries to understand the asymptotic behavior of the orbit of many points. In this direction, it is natural to try to understand properties, and the existence, of certain invariant measures that capture the statistical behavior of a set of points that is relevant for the Lebesgue measure. Let us make this more precise.

Let f be a diffeomorphism of a closed, compact, connected, orientable manifold M. Given an invariant ergodic [1] probability measure μ, its **basin** is defined as

$$B(\mu) = \left\{ p \in M : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(p)} \xrightarrow{n \to +\infty} \mu \right\},$$

where δ_p is the dirac measure on p and the convergence is for the weak*-topology. The measure μ is **physical** if its basin has positive Lebesgue measure. In other words, physical measures are the measures that capture the asymptotic behavior of many points in the Lebesgue point of view.

In the 1970s, Sinai, Ruelle and Bowen [Si72, Ru76, Bow75] proved that $C^{1+\alpha}$ uniformly hyperbolic systems have finitely many physical measures that describes the statistical behavior of Lebesgue almost every point. Nowadays, the measures they constructed are called **SRB measures** (SRB for Sinai-Ruelle-Bowen), see definition 2.13. These measures have an important geometrical property: they admit conditional measures along unstable manifolds which are absolutely continuous with respect to the volume of the unstable manifolds. After the work of Ledrappier in [Le84], there is a well developed ergodic theory for these measures. The hyperbolic [2] SRB measures form an important class of physical measures.

We remark that in the hyperbolic setting there are uniform expansion/contraction, and a dominated splitting [3] (which implies that the angle between the expanding/contracting directions is uniformly bounded from below). These two points are important to carry the constructions of such measures.

There are many works that study conditions that guarantee the existence of hyperbolic SRB measures outside the uniformly hyperbolic setting, see for instance [Yo98, BV00, ABV00, CDP16, CLP19, Ov19]. We also refer the reader to the recent survey [CLP17] for a discussion on the different methods of construction of such measures (with a focus on the geometrical method). Usually it is a hard problem to show the existence of hyperbolic SRB measures outside the uniformly hyperbolic setting.

In the task of studying the existence and uniqueness of hyperbolic SRB measures for partially hyperbolic systems [4], another important type of invariant measure are the so-called **u-Gibbs measure**, see definition 2.17. They are invariant measures that also verify some geometric property. In the partially hyperbolic setting, every hyperbolic SRB measure is an u-Gibbs measure.

1. Recall that μ is ergodic if and only if any f-invariant measurable set Λ has measure 0 or 1.
2. See section 2.2 for the definition of hyperbolic measure.
3. An invariant set Λ admits a dominated splitting if it admits an invariant splitting $T_\Lambda M = E \oplus F$, such that $\exists N \in \mathbb{N}$ that verifies $\|Df^N(p)|_{E_p}\| \|Df^{-N}(f^N(p))|_{F_{f^N(p)}}\| < \frac{1}{2}$.
4. See section 2 for the definition.

Berger and Carrasco introduced in [BC14] an example of a partially hyperbolic system, with two dimensional center, and such that among the volume preserving systems it is robustly non-uniformly hyperbolic with both expansion/contraction along the center and it does not admit a decomposition of the center in dominated directions.

In [Ob18-2], the author proves that the Berger-Carrasco's example and any C^2-small volume preserving perturbation of it is ergodic. The main result in this chapter is a step towards understanding the existence and uniqueness of hyperbolic SRB measures for dissipative perturbations of this example. As we mentioned before, understanding the u-Gibbs measures is important in this task. We obtain a rigidity result for u-Gibbs measures in a neighborhood of this example. In particular, we classify all the possible u-Gibbs measures that may appear.

The example and precise statement of the results. For $N \in \mathbb{R}$ we denote by $s_N(x,y) = (2x - y + N\sin(x), x)$ the standard map on $\mathbb{T}^2 := \mathbb{R}^2/2\pi\mathbb{Z}^2$. For every N the map s_N preserves the Lebesgue measure induced by the usual metric of $\mathbb{T}^2$. This map is related to several physical problems, see for instance [Ch79], [Iz80] and [SS95].

It is conjectured that for $N \neq 0$ the map s_N has positive entropy for the Lebesgue measure, see [Si94] page 144. By Pesin's entropy formula, see [Pe77] Theorem 5.1, this is equivalent to the existence of a set of positive Lebesgue measure, whose points have a positive Lyapunov exponent. The existence of those sets is not known for any value of N. See [BXY17], [Du94] and [Go12] for some results related to this conjecture.

Let $A \in SL(2,\mathbb{Z})$ be a hyperbolic matrix which defines an Anosov diffeomorphism on $\mathbb{T}^2$, let $P_x : \mathbb{T}^2 \to \mathbb{T}^2$ be the projection on the first coordinate of $\mathbb{T}^2$, this projection is induced by the linear map of $\mathbb{R}^2$, which we will also write P_x, given by $P_x(a,b) = (a,0)$.

Consider the torus $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ and represent it using the coordinates (x,y,z,w), where $x,y,z,w \in [0,2\pi)$. We may naturally identify a point (z,w) on the second torus with a point (x,y) on the first torus by taking $x = z$ and $y = w$. For each $N \in \mathbb{N}$ define

$$\begin{array}{rccc} f_N & : & \mathbb{T}^2 \times \mathbb{T}^2 & \longrightarrow & \mathbb{T}^2 \times \mathbb{T}^2 \\ & & (x,y,z,w) & \mapsto & (s_N(x,y) + P_x \circ A^N(z,w), A^{2N}(z,w)). \end{array}$$

This diffeomorphism preserves the Lebesgue measure. For N large enough it is a partially hyperbolic diffeomorphism, with two dimensional center direction given by $E^c = \mathbb{R}^2 \times \{0\}$. This type of system was considered by Berger-Carrasco in [BC14], where they proved that for N large enough f_N is C^2-robustly non-uniformly hyperbolic among the volume preserving diffeomorphisms.

For $r \geq 1$ we consider $\mathrm{Diff}^r(\mathbb{T}^4)$ to be the set of C^r-diffeomorphisms of $\mathbb{T}^4$. Inside Diff^r, we may consider the subspace $SP^r(\mathbb{T}^2 \times \mathbb{T}^2)$ of skew products, which is the set of C^r-diffeomorphisms g of the form

$$g(x,y,z,w) = (g_1(x,y,z,w), g_2(z,w)),$$

where $g_2(.,.)$ is a C^r-diffeomorphism of $\mathbb{T}^2$, and for each $(z,w) \in \mathbb{T}^2$, $g_1(.,.,z,w)$ is a C^r-diffeomorphism of $\mathbb{T}^2$ as well. Observe that $f_N \in SP^2(\mathbb{T}^2 \times \mathbb{T}^2)$. We also remark that for N large enough, if g is a skew product C^1-close enough to f_N, then g_2 is an Anosov diffeomorphism, and g is partially hyperbolic.

We recall that for a map g, a g-invariant measure μ is **Bernoulli** if the system (g,μ) is measurably conjugated to a Bernoulli shift. For a skew product g as above, one may look at the conditional measures of μ with respect to the center foliation. If these conditional measures are atomic, we say that μ has **atomic disintegration along the center foliation**, see section 2.3 for a precise definition. Our main result in this chapter is the following:

Theorem J. *Let $\alpha \in (0,1)$. For N large enough, there exists $\mathcal{U}_N^{sp}$ a C^2-neighborhood of f_N contained in $SP^{2+\alpha}(\mathbb{T}^2 \times \mathbb{T}^2)$ such that for $g \in \mathcal{U}_N^{sp}$, and μ an ergodic u-Gibbs measure for g, one of the following holds true:*

(1) μ is the unique SRB measure. It is Bernoulli and $\mathrm{supp}(\mu) = \mathbb{T}^4$*;*

(2) *μ has atomic disintegration along the center foliation, whose conditional measures has finitely many atoms.*

The proof of Theorem J is an immediate consequence of Theorems K and L below.

Theorem K. *Let $\alpha \in (0,1)$. For N large enough, there exists $\mathcal{U}_N^{sp}$ a C^2-neighborhood of f_N contained in $SP^{2+\alpha}(\mathbb{T}^2 \times \mathbb{T}^2)$, such that for $g \in \mathcal{U}_N^{sp}$, for any ergodic measure $\mu \in \text{Gibbs}^u(g)$ one of the following holds:*

(1) *μ is a hyperbolic SRB measure, or*

(2) *μ has atomic disintegration along the center foliation, whose conditional measures has finitely many atoms.*

The proof of Theorem K uses a recent result by Brown-Rodriguez Hertz in [BRH17]. In their paper they classify all the ergodic, hyperbolic stationary measures for random products of surface C^2-diffeomorphisms. To prove their result, they actually prove a more general theorem, which holds for more general abstract skew products with a given surface as fiber (see section 5 for more details).

We remark that the α that appears in the statements of the Theorems J and K, only appears because in the statement of Brown-Rodriguez Hertz's main result in [BRH17], the surface diffeomorphisms they consider have regularity C^2. If one obtains a version of their result for $C^{1+\beta}$-diffeomorphisms, then one could remove the α from the statement (see section 5).

Theorem L. *For N large enough, there exists $\mathcal{U}_N$ a C^2-neighborhood of f_N in $\text{Diff}^2(\mathbb{T}^4)$ such that if $g \in \mathcal{U}_N$, then g has at most one SRB measure. Moreover, if μ is an SRB measure for g, then $\text{supp}(\mu) = \mathbb{T}^4$, it is Bernoulli and hyperbolic.*

Remark 1.1. *Theorems J and K hold for a neighborhood of f_N inside the skew product diffeomorphisms, $SP^2(\mathbb{T}^2 \times \mathbb{T}^2)$. Theorem L is the uniqueness theorem for SRB measures, and it holds in a neighborhood of f_N inside $\text{Diff}^2(\mathbb{T}^4)$.*

Strategy of the proofs of our theorems. As we mentioned before, Theorem J is an immediate consequence of Theorems K and L. Using the calculations to prove non-uniform hyperbolicity of f_N from [BC14], and the adaptations made in [Ob18-2], we prove that in a neighborhood of f_N in $\text{Diff}^2(\mathbb{T}^4)$, every u-Gibbs measure is hyperbolic with both a positive and a negative Lyapunov exponent along the center.

To prove Theorem K we do the following. For a skew product g sufficiently close to f_N and an ergodic u-Gibbs measure μ, we prove that after a measurable change of coordinates using the unstable holonomies, we are in the setting of [BRH17]. To justify that the change of coordinates mentioned above take us to the setting of Brown-Rodriguez Hertz's rigidity result, we use the version of the invariance principle by Tahzibi-Yang in [TY19]. The translation of their rigidity result to our scenario is given by Theorem 5.3, in particular, there are only three possibilities for an ergodic u-Gibbs measure: either it is SRB; or it has atomic disintegrations along the center foliation; or the Oseledecs direction for the negative center Lyapunov exponent is invariant by the derivative of unstable holonomies. Using some estimates from [BC14], we prove that the third case never happens (see proposition 6.1). In the case that a u-Gibbs measure has atomic disintegration along the center foliation, we prove that it only has finitely many atoms concluding the proof of Theorem K.

The proof of Theorem L is based on the techniques developed by the author in [Ob18-2]. Using such techniques we can prove that any u-Gibbs measure has a set of large measure, whose points have "large" stable and unstable manifolds. Furthermore, we can obtain precise control on the "geometry" of these invariant manifolds. This allows us to prove that any two u-Gibbs measures are homoclinically related (see definition 2.15 and Theorem 4.1). Hence, we conclude that in a neighborhood of f_N (inside $\text{Diff}^2(\mathbb{T}^4)$) there exists at

most one SRB measure. The techniques will also allow us to conclude that such a measure is Bernoulli. Using some arguments from the recent work [CO19] of the author with P. Carrasco, we prove that if there exists an SRB measure then it has full support.

Further remarks and questions. Let us summarize why in Theorems J and K we have the condition that the systems are skew products for $\mathbb{T}^2 \times \mathbb{T}^2$. We use this condition to obtain the smoothness of the center foliation. This is used to prove proposition 2.23, which states that we may use the invariance principle (see also corollary 2.25). An interesting question is to know if there exists such a measure rigidity result for systems which are not skew products for $\mathbb{T}^2 \times \mathbb{T}^2$. A first natural step is given in the following quesiton:

Question 1.2. *Is there a similar measure rigidity result for u-Gibbs measures of diffeomorphism in a neighborhood of f_N inside $\mathrm{Diff}^2(\mathbb{T}^4)$?*

We believe that condition (2) in Theorem J usually does not happen. Since there are good hyperbolic information for u-Gibbs measures in a neighborhood of f_N. We also believe that the generic existence of an SRB measure in a neighborhood of f_N should imply the existence of an SRB measure for any system in an entire neighborhood of f_N. We precise this in the following conjecture:

Conjecture. *Every diffeomorphism in $\mathcal{U}_N^{sp}$ has an SRB measure.*

An interesting strategy to prove the existence of an SRB measure in a neighborhood of f_N inside $\mathrm{Diff}^2(\mathbb{T}^4)$ is to use the results from [CDP16]. In order to do that, one needs to prove that the condition called **effective hyperbolicity** is verified (see section 1.2 in [CDP16]). This condition seems hard to prove, however it could give the existence of SRB measures outside the fibered case.

Question 1.3. *For N large enough, for any diffeomorphism g which is sufficiently C^2-close to f_N, does it hold that g is effective hyperbolic?*

Organization of the chapter. In section 2, we review several tools that we will use in this work. In particular, results on partially hyperbolic systems and accessibility classes, u-Gibbs and SRB measures, and the invariance principle. Sections 3 and 4 are dedicated to prove Theorem L. In these sections we show how the techniques from [Ob18-2], and [BC14], are used to obtain precise control on the center Lyapunov exponents of u-Gibbs measures, and how to obtain the uniqueness of the SRB measure.

In section 5 we state the main result of Brown-Rodriguez Hertz in [BRH17], and we show how after a measurable change of coordinates of our systems we are in the setting of their result. Sections 6 and 7 are dedicated to prove Theorem K. In the appendix we prove that with some stronger bunching condition the strong unstable holonomy between center manifolds has regularity C^2, this is used in the proof of Theorem K.

2. Preliminaries

2.1. Partial hyperbolicity, holonomies and accessibility classes.

Partial hyperbolicity and foliations. A C^r-diffeomorphism f, with $r \geq 1$, is **partially hyperbolic** if the tangent bundle has a decomposition $TM = E^{ss} \oplus E^c \oplus E^{uu}$, there is a riemannian metric on M and continuous functions $\chi^{ss}, \chi^{uu}, \chi^c_-, \chi^c_+ : M \to \mathbb{R}$, such that for any $m \in M$

$$\chi^{ss}(m) < 1 < \chi^{uu}(m) \text{ and } \chi^{ss}(m) < \chi^c_-(m) \leq \chi^c_+(m) < \chi^{uu}(m),$$

it also holds

$$\chi^c_-(m) \leq m(Df(m)|_{E^c_m}) \leq \|Df(m)|_{E^c_m}\| \leq \chi^c_+(m);$$

$$\|Df(m)|_{E^{ss}_m}\| \leq \chi^{ss}(m) \text{ and } \chi^{uu}(m) \leq m(Df(m)|_{E^{uu}_m}),$$

where $m(Df(m)_{E^*_m}) = \|(Df(m)|_{E^*_m})^{-1}\|^{-1}$ is the co-norm of $Df(m)|_{E^*_m}$, for $* = c, uu$. If the functions in the definition of partial hyperbolicity can be taken constant, we say that f is **absolutely partially hyperbolic**.

It is well known that the distributions E^{ss} and E^{uu} are uniquely integrable, that is, there are two unique foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$, with C^r-leaves, that are tangent to E^{ss} and E^{uu} respectively. For a point $p \in M$ we will denote by $W^{ss}(p)$ a leaf of the foliation $\mathcal{F}^{ss}$, we will call such leaf the strong stable manifold of p. Similarly we define the strong unstable manifold of p and denote it by $W^{uu}(p)$.

Definition 2.1. *A partially hyperbolic diffeomorphism is* **center bunched** *if*

$$\chi^{ss}(m) < \frac{\chi^c_-(m)}{\chi^c_+(m)} \text{ and } \frac{\chi^c_+(m)}{\chi^c_-(m)} < \chi^{uu}(m), \text{ for every } m \in M.$$

We denote $E^{cs} = E^{ss} \oplus E^c$ and $E^{cu} = E^c \oplus E^{uu}$.

Definition 2.2. *A partially hyperbolic diffeomorphism f is* **dynamically coherent** *if there are two invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$, with C^1-leaves, tangent to E^{cs} and E^{cu} respectively. From those two foliations one obtains another invariant foliation $\mathcal{F}^c = \mathcal{F}^{cs} \cap \mathcal{F}^{cu}$ that is tangent to E^c. We call those foliations the center-stable, center-unstable and center foliation.*

For any $R > 0$ we write $W^*_R(p)$ to be the disc of size R centered on p, for the Riemannian metric induced by the metric on M, contained in the leaf $W^*(p)$, for $* = ss, c, uu$.

The definition below allows one to obtain higher regularity of the leaves of such foliations.

Definition 2.3. *We say that a partially hyperbolic diffeomorphism f is* **r-normally hyperbolic** *if for any $m \in M$*

$$\chi^{ss}(m) < (\chi^c_-(m))^r \text{ and } (\chi^c_+(m))^r < \chi^{uu}(m).$$

Definition 2.4. *Let f and g be partially hyperbolic diffeomorphisms of M that are dynamically coherent. Denote by $\mathcal{F}^c_f$ and $\mathcal{F}^c_g$ the center foliations. We say that f and g are* **leaf conjugated** *if there is a homeomorphism $h : M \to M$ that takes leaves of $\mathcal{F}^c_f$ to leaves of $\mathcal{F}^c_g$ and such that for any $L \in \mathcal{F}^c_f$ it is verified*

$$h(f(L)) = g(h(L)).$$

One may study the stability of partially hyperbolic systems up to leaf conjugacy. Related to this there is a technical notion called **plaque expansivity** which we will not define here, see chapter 7 of [HPS77] for the definition. The next theorem is important for the theory of stability of partially hyperbolic systems.

Theorem 2.5 ([HPS77], Theorem 7.4)**.** *Let $f : M \to M$ be a C^r-partially hyperbolic and dynamically coherent diffeomorphism. If f is r-normally hyperbolic and plaque expansive then any $g : M \to M$ in a C^r-neighborhood of f is partially hyperbolic and dynamically coherent. Moreover, g is leaf conjugated to f and the center leaves of g are C^r-immersed manifolds.*

Remark 2.6. *Fix $R > 0$, and let f be a diffeomorphism that satisfies the hypothesis of the previous theorem. The proof of this theorem implies that for g sufficiently C^r-close to f, for any $m \in M$ we have that $W^c_{f,R}(m)$ is C^r-close to $W^c_{g,R}(m)$. In particular, if the center foliation is uniformly compact then for every g sufficiently C^r-close to f, for any $m \in M$, $W^c_f(m)$ is C^r-close to $W^c_g(m)$.*

It might be hard to check the condition of plaque expansiviness, but this is not the case when the center foliation of a dynamically coherent, partially hyperbolic diffeomorphism

is at least C^1, see Theorem 7.4 of [HPS77]. Usually the invariant foliations that appear in dynamics are only Hölder.

We can also obtain a better regularity for the center direction given by the following theorem, see section 4 of [PSW12] for a discussion on this topic.

Theorem 2.7. *Let f be a C^2-partially hyperbolic diffeomorphism and let $\theta > 0$ be a number such that for every $m \in M$ it is verified*

$$\chi^{ss}(m) < \chi^c_-(m)m(Df(m)|_{E^{ss}})^\theta \text{ and } \chi^c_+(m)\|Df(m)|_{E^{uu}}\|^\theta < \chi^{uu}(m),$$

then E^c is θ-Hölder.

Unstable holonomies. Let f be a partially hyperbolic, dynamically coherent diffeomorphism. Each leaf of the foliation $\mathcal{F}^{cs}$ is foliated by strong stable manifolds. For a point $p \in M$ and $q \in W^{ss}_1(p)$, where $W^{ss}_1(p)$ is the strong stable manifold of size 1, we can define the stable holonomy map restricted to the center-stable manifold, between center manifolds. Let us be more precise. We can choose two small numbers $R_1, R_2 > 0$, with the property that for any $z \in W^c_{R_1}(p)$, there is only one point in the intersection $W^{ss}_2(z) \cap W^c_{R_2}(q)$. We define $H^s_{p,q}(z) = W^{ss}_2(z) \cap W^c_{R_2}(q)$. With this construction we obtain a map $H^s_{p,q} : W^c_{R_1}(p) \to W^c_{R_2}(q)$. By the compactness of M we can take the numbers R_1 and R_2 to be constants, independent of p and q.

We can define analogously the unstable holonomy map, for $p \in M$ and $q \in W^{uu}_1(p)$, which we will denote by $H^u_{p,q} : W^c_{R_1}(p) \to W^c_{R_2}(q)$.

In [PSW97] and [PSW00], the authors prove that the map $H^s_{p,q}$ is C^1 if f is a partially hyperbolic, center bunched and dynamically coherent C^2-diffeomorphism. Indeed, the authors prove that the strong stable foliation is C^1 when restricted to a center-stable leaf. Consider the family of C^1-maps $\{H^s_{p,q}\}_{p\in M, q\in W^{ss}_1(p)}$.

Theorem 2.8. *Let f be an absolutely partially hyperbolic, dynamically coherent diffeomorphism with regularity C^2. Suppose also that f verifies:*

(1) $\chi^c_- < 1$ and $\chi^c_+ > 1$;

(2) there exists $\theta \in (0,1)$, such that

$$(\chi^{ss})^\theta < \frac{\chi^c_-}{\chi^c_+} \text{ and } \frac{\chi^c_+}{\chi^c_-} < (\chi^{uu})^\theta; \tag{64}$$

and also

$$\chi^{ss} < \chi^c_- m(Df|_{E^{ss}})^\theta \text{ and } \chi^c_+\|Df|_{E^{uu}}\|^\theta < \chi^{uu}. \tag{65}$$

Then the family $\{H^s_{p,q}\}_{p\in M, q\in W^{ss}_1(p)}$ is a family of C^1-maps depending continuously in the C^1-topology with the choices of the points p and q. Furthermore, there exists a constant $C > 0$ such that for any $p \in M$, $q \in W^{ss}_1(p)$, and any unit vector $v \in E^c_p$, it is verified

$$d\left(\frac{H^s_{p,q}(p)v}{\|H^s_{p,q}(p)v\|}, v\right) < Cd(p,q)^\theta. \tag{66}$$

Similar results holds for the family of unstable holonomies $\{H^u_{p,q}\}_{p\in M, q\in W^{uu}_1(p)}$.

The proof of this theorem can be found in [Ob18-1], which is an adaptation of the arguments from [Br16] by Brown. In what follows, we give the main points of this proof mostly to justify (66). For all the details, we refer the reader to [Ob18-1].

Sketch of the proof. By Theorem 2.7, condition (65) implies that the center bundle E^c is θ-Hölder (see section 4 in [PSW12]). The condition (64) is sometimes called the **strong bunching** condition.

We may fix a local approximation of the holonomy H^s_*, which we will denote by π^s_*, that verifies the following: there exists a constant $\tilde{C} > 0$ such that for any $p \in M$ and $q \in W^{ss}_1(p)$, there exists a $C^{1+\theta}$-map, which is a diffeomorphism onto its image, $\pi^s_{p,q} : W^c_{R_1}(p) \to W^c(q)$ that verifies

(1) $d(\pi^s_{p,q}(p), q) \leq \tilde{C} d(p, q)$;

(2) $d(D\pi^s_{p,q}(p).v, v) \leq \tilde{C} d(p, q)^\theta$, where $v \in SE^c_p$, and SE^c_p is the unit sphere on E^c_p;

(3) if $p' \in W^c_{loc}(p)$ and $q' \in W^{ss}_1(p') \cap W^c_{loc}(q)$, then $\pi^s_{p,q}$ coincides with $\pi^s_{p',q'}$ on $W^c_{loc}(p) \cap W^c_{loc}(p')$.

This can be done in the following way: Consider a smooth subbundle $\widetilde{E}$ inside a cone which is close to the direction perpendicular to the subbundle E^c, with dimension $\dim(M) - \dim(E^c)$. Since E^c is θ-Hölder, the center manifolds are $C^{1+\theta}$. Hence, the restriction of $\widetilde{E}$ to any center manifold is a $C^{1+\theta}$-bundle. For each point $q \in M$ and $\rho > 0$, consider $L_{q,\rho} := \exp_q(\widetilde{E}(q, \rho))$ to be the projection of the ball of radius ρ by the exponential map over q. By the uniform transversality and the compactness of M, there exists a constant ρ_0 such that for any center leaf $W^c_{R_1}(p)$, the set $\{L_{q,\rho}\}_{q \in W^c_{R_1}(p)}$ forms an uniform foliated neighborhood of $W^c_{R_1}(p)$ (or a tubular neighborhood). Let $\pi^s_{p,q}$ be the holonomy defined by this local foliation, up to rescaling of the metric we may assume that it is well defined for $p \in M$ and $q \in W^{ss}_1(p)$. By the compactness of M we obtain the constant $\tilde{C} > 0$ above. Observe also that since the center leaves vary continuously in the C^1-topology, we obtain that the map $\pi^s_{p,q}$ varies continuously in the C^1-topology with the points p and q.

For any $p, q \in M$ and each $n \in \mathbb{N}$, write $p_n = f^n(p)$ and $q_n = f^n(q)$. We define

$$H^s_{p,q,n} = f^{-n} \circ \pi^s_{p_n,q_n} \circ f^n.$$

If it is clear that we are talking about two points p and $q \in W^{ss}_1(p)$ we will only write $H^s_n = H^s_{p,q,n}$ and similarly $\pi^s_n = \pi^s_{p_n,q_n}$.

Since we are assuming that f is absolutely partially hyperbolic, only for this proof, we write its partially hyperbolic constants as $\chi_s = \chi^{ss}$, $\chi_c = \chi^c_-$ and $\widehat{\chi}_c = (\chi^c_+)^{-1}$. Also, for a diffeomorphism $g : N_1 \to N_2$, between manifolds N_1 and N_2, we will write $g_* : SN_1 \to SN_2$, the action induced by the derivative on the unitary bundles of N_1 and N_2.

The proof of Theorem 2.8 follows the steps in [Br16]. The first step is to prove that $(H^s_n)_{n\in\mathbb{N}}$ is uniformly Cauchy in the C^0-topology. The second step is to prove that the sequence $((H^s_n)_*)_{n\in\mathbb{N}}$ is uniformly Cauchy. The third step is to prove that for any vector $v \in E^c_p$, the sequence $(\|DH^s_n(p)v\|)_{n\in\mathbb{N}}$ is also uniformly Cauchy. In all these three steps it is obtained that the rate of convergence of these sequence does not depend on the choices of the points p and q. The uniform convergence in the C^1-topology of the sequence $(H^s_n)_{n\in\mathbb{N}}$ then follows from these three steps. In this paper, we only describe in more details step two, for the details of the other two steps we refer the reader to [Ob18-1].

Observe that the Lipschitz norm of f^{-1}_* restricted to a fiber S_xE^c is $(\chi_c\widehat{\chi}_c)^{-1}$. Since f is a C^2-diffeomorphism, then f^{-1}_* is a C^1-diffeomorphism of SM, let $C_1 > 0$ be the C^1-norm of f^{-1} on M and C_2 to be the C^1-norm of f^{-1}_* on SM. For $\xi = (x, v) \in S_xM$, write $\xi_k = f^k_*(x, v) = (x_k, v_k)$, with $k \in \mathbb{Z}$.

In [Br16], the author uses the strong bunching condition (64) above, but he also uses another type of bunching (see Theorem 4.1 in [Br16]). In the proof, this different type of bunching is only used to obtain a version of lemma 2.9 below. In our setting, instead of asking for this other type of bunching, we ask that $\chi_c < 1$ and $\hat{\chi}_c < 1$. We obtain the following lemma.

Lemma 2.9. *There are constants $\delta, \alpha \in (0, 1)$, that verify the following: if $\xi = (x, v)$, $\zeta = (y, u) \in SW^c(p)$, $K > 0$ and $n \geq 0$ verify $d(x_n, y_n) < K\chi^n_s$, $d(\xi_n, \zeta_n) \leq K\chi^{n\theta}_s$, and for every $0 \leq k \leq n$,*

$$d(x_k, y_k) \leq \delta.$$

Then, for all $0 \leq k \leq n$,

$$d(x_k, y_k) \leq K\chi^n_s.\chi_c^{-(n-k)} \text{ and } d(\xi_k, \zeta_k) \leq K\chi^{n\theta}_s.(\chi_c\widehat{\chi}_c)^{-(n-k)(1+\alpha)}.$$

In particular,

$$d(\xi, \zeta) \leq K\chi^{n\theta}_s.(\chi_c\widehat{\chi}_c)^{-n(1+\alpha)}.$$

Furthermore, α can be chosen such that

$$\chi_s^{\theta}.(\widehat{\chi}_c\chi_c)^{-(1+\alpha)} < 1.$$

PROOF. The proof is by backward induction in k. We will first denote by α and δ quantities that will be fixed later. Since x_k and y_k belongs to the same center manifold, we obtain

$$d(x_{k-1}, y_{k-1}) \leq \chi_c^{-1} d(x_k, y_k) \leq K\chi_s^n.\chi_c^{-n+k+1}.$$

For any $\beta \in (0,1)$, and since $d(x_k, y_k) \leq \delta$, we have

$$\begin{aligned} d(f_*^{-1}(x_k, v_k), f_*^{-1}(y_k, u_k)) &\leq d(f_*^{-1}(x_k, v_k), f_*^{-1}(x_k, u_k)) + d(f_*^{-1}(x_k, u_k), f_*^{-1}(y_k, u_k)) \\ &\leq (\chi_c\widehat{\chi}_c)^{-1} d(v_k, u_k) + C_2 d(x_k, y_k). \\ &\leq (\chi_c\widehat{\chi}_c)^{-1}[1 + C_2.(\chi_c\widehat{\chi}_c)d(x_k, y_k)^{1-\beta}].\max\{d(x_k, y_k)^{\beta}, d(v_k, u_k)\} \\ &\leq (\chi_c\widehat{\chi}_c)^{-1}[1 + C_2.(\chi_c\widehat{\chi}_c)\delta^{1-\beta}] \\ &\quad .K\max\{\chi_s^{n\beta}.\chi_c^{-(n-k)\beta}, \chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-(n-k)(1+\alpha)}\}. \end{aligned}$$

We claim that we can choose α and β such that for any $n \in \mathbb{N}$ and $0 \leq k \leq n$ it holds

$$\chi_s^{n\beta}.\chi_c^{-(n-k)\beta} \leq \chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-(n-k)(1+\alpha)}.$$

This inequality is equivalent to

$$1 \leq \chi_s^{n(\theta-\beta)}.(\chi_c^{(\beta-1-\alpha)}\widehat{\chi}_c^{-(1+\alpha)})^{(n-k)}. \tag{67}$$

Since $\widehat{\chi}_c^{-1} > 1$, we can fix $\beta > \theta$ close enough to 1 such that $1 < \chi_c^{(\beta-1-\alpha)}\widehat{\chi}_c^{-(1+\alpha)}$. Let us explain. Observe that $(\chi_c)^{-\alpha} > 1$, for any $\alpha > 0$. Hence,

$$\chi_c^{\beta-1}(\hat{\chi}_c\chi_c)^{-\alpha}\hat{\chi}_c^{-1} > \chi_c^{\beta-1}\hat{\chi}_c^{-1}.$$

From this, one can see that if β is sufficiently close to 1, we have that $1 < \chi_c^{(\beta-1-\alpha)}\widehat{\chi}_c^{-(1+\alpha)}$. Since $\beta > \theta$, and hence $\theta - \beta$ is negative, we conclude (67).

We also need that

$$\chi_s^{\theta}.(\widehat{\chi}_c\chi_c)^{-(1+\alpha)} < 1. \tag{68}$$

By the strong center bunching condition (64), the inequality above holds if α is sufficiently close to 0. Fix $\alpha > 0$ that verifies (68).

Now fix $\delta > 0$ small enough such that

$$[1 + C_2.(\chi_c\widehat{\chi}_c)\delta^{1-\beta}] \leq (\chi_c\widehat{\chi}_c)^{-\alpha}.$$

We conclude,

$$\begin{aligned} d(f_*^{-1}(\xi^k), f_*^{-1}(\zeta^k)) &\leq (\chi_c\widehat{\chi}_c)^{-(1+\alpha)}.K\chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-(n-k)(1+\alpha)} \\ &= K\chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-(n-k-1)(1+\alpha)} \end{aligned}$$

□

Fix $\xi = (z, l) \in SW^c_{R_1}(p)$. Write $\zeta^n := (H_n^s)_*(\xi)$ and $\zeta_j^n := f_*^j(\zeta^n)$, for any $j \in \mathbb{Z}$. We also write $w = H^s_{p,q}(z)$, $\zeta^n = (H_n^s)_*(\xi) = (x, v)$ and $\zeta^{n+1} = (H_{n+1}^s)_*(\xi) = (y, u)$. Observe that $\zeta_n^n = (\pi_n^s)_*(\xi_n)$ and $\zeta_n^{n+1} = f_*^{-1}((\pi_{n+1}^s)_*(\xi_{n+1}))$. First we have

$$\begin{aligned} d(\pi_n^s(z_n), f^{-1}(\pi_{n+1}^s(z_{n+1}))) &\leq d(z_n, \pi_n^s(z_n)) + d(f^{-1}(z_{n+1}), f^{-1}(\pi_n^s(z_{n+1})) \\ &\leq \tilde{C}\chi_s^n d(z, w) + C_1\tilde{C}\chi_s^{n+1} d(z, w) \\ &\leq 2\tilde{C}C_1\chi_s^n d(z, w). \end{aligned}$$

The previous estimate shows that $d(x_n, y_n) \leq 2\tilde{C}C_1 d(z,w)\chi_s^n$. Also it is verified for any $0 \leq k \leq n$

$$d(x_k, y_k) \leq 2\tilde{C}C_1 d(z,w)\chi_s^n \chi_c^{-(n-k)}. \tag{69}$$

Let δ be the constant given by lemma 2.9. By domination, if n is large enough, we conclude that $d(x_k, y_k) < \delta$. This n can be taken uniform, independently of p and q.

Also, using that $f_*^{-1}(\xi_{n+1}) = \xi_n$, we obtain

$$\begin{aligned} d(\zeta_n^n, \zeta_n^{n+1}) &= d((\pi_n^s)_*(\xi_n), f_*^{-1}(\pi_{n+1}^s)_*(\xi_{n+1})) \\ &\leq d(\xi_n, (\pi_n^s)_*(\xi_n)) + d(f_*^{-1}(\xi_{n+1}), f_*^{-1}(\pi_{n+1}^s)_*(\xi_{n+1})). \end{aligned}$$

By property 2 of π_*^s, we have $d(\xi_n, (\pi_n^s)_*(\xi_n)) \leq \tilde{C}d(z,w)^\theta \chi_s^{n\theta}$. For the second term in the inequality we have

$$\begin{aligned} d(f_*^{-1}(\xi_{n+1}), f_*^{-1}(\pi_{n+1}^s)_*(\xi_{n+1})) &= d(f_*^{-1}(z_{n+1}, l_{n+1}), f_*^{-1}(y_{n+1}, u_{n+1})) \\ &\leq d(f_*^{-1}(z_{n+1}, l_{n+1}), f_*^{-1}(z_{n+1}, u_{n+1})) \\ &\quad + d(f_*^{-1}(z_{n+1}, u_{n+1}), f_*^{-1}(y_{n+1}, u_{n+1})) \\ &\leq C_2 d(l_{n+1}, u_{n+1}) + C_2 d(z_{n+1}, y_{n+1}) \\ &\leq \tilde{C}C_2 d(z,w)^\theta \chi_s^{(n+1)\theta} + \tilde{C}C_2 d(z,w)\chi_s^{n+1} \\ &\leq (\tilde{C}C_2 + \tilde{C}C_2 d(z,w)^{1-\theta}\chi_s^{(n+1)(1-\theta)}) d(z,w)^\theta \chi_s^{(n+1)\theta} \\ &\leq (\tilde{C}C_2 + \tilde{C}C_2 d(z,w)^{1-\theta}) d(z,w)^\theta \chi_s^{(n+1)\theta}. \end{aligned}$$

Thus,

$$d(\zeta_n^n, \zeta_n^{n+1}) \leq [\tilde{C} + (\tilde{C}C_2 + \tilde{C}C_2 d(z,w)^{1-\theta})] d(z,w)^\theta \chi_s^{n\theta}.$$

By compactness, $d(z,w)$ is bounded from above independently of p and q. Hence, take a constant C_3 such that $d(\zeta_n^n, \zeta_n^{n+1}) \leq C_3 d(z,w)^\theta \chi_s^{n\theta}$. Fix $K_1 = \max\{2\tilde{C}C_1, C_3\}$, and observe that we are in the setting of lemma 2.9, for $K = K(z,w) := K_1 d(z,w)^\theta$. Let α be the constant given by the same lemma. We conclude that

$$d(\zeta^n, \zeta^{n+1}) \leq K\chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-n(1+\alpha)} = K_1\chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-n(1+\alpha)} d(z,w)^\theta, \text{ for } n \text{ large enough.}$$

In particular, the sequence $(\zeta^n)_{n\in\mathbb{N}}$ is Cauchy. Since this holds uniformly for any ξ, we obtain that $((H_n^s)_*)_{n\in\mathbb{N}}$ is a Cauchy sequence whose speed of convergence does not depend on the choices of the the points p and q.

If $d(p,q) \leq \delta$ then for any $n \geq 0$ it holds that

$$d((H_n^s)_*, (H_{n+1}^s)_*) \leq K_1\chi_s^{n\theta}.(\chi_c\widehat{\chi}_c)^{-n(1+\alpha)} d(p,q)^\theta.$$

Write $(H_{p,q}^s)_* = \lim_{n\to+\infty} (H_n^s)_*$. Hence, there exists a constant $K_2 > 0$ such that for $p, q \in M$ with $d(p,q) < \delta$, we have

$$d(Id_*, (H_{p,q}^s)_*) \leq d(Id_*, (\pi^s)_*) + \sum_{j=0}^{+\infty} d((H_j^s)_*, (H_{j+1}^s)_*) \leq K_2 d(p,q)^\theta.$$

Since $\delta > 0$ is a constant, there is a maximum number $T = [\frac{1}{\delta}]$ such that there are at most $T+1$ points, $\{x_1, \cdots, x_{T+1}\} \subset W_1^s(p)$ verifying $x_1 = p$, $x_{T+1} = q$ and $d(x_i, x_{i+1}) < \delta$. Since $H_{p,q}^s(.) = H_{x_T, x_{T+1}}^s \circ \cdots \circ H_{x_1,x_2}^s(.)$, we conclude that there exists a constant $C > 0$ such that

$$d(Id_*, (H_{p,q}^s)_*) \leq Cd(p,q)^\theta. \tag{70}$$

This concludes the proof of the second step that we mentioned above. In particular, it also proves the conclusion (66) in the statement of this theorem. □

Suppose that f is a partially hyperbolic, center bunched skew product on $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$, with the Anosov map on the base $f_2 : \mathbb{T}^2 \to \mathbb{T}^2$. Observe that for any $p \in \mathbb{T}^4$, its unstable manifold $W^{uu}(p)$ projects to the unstable manifold of $\pi_2(p)$ of f_2. In particular, for each $p \in \mathbb{T}^4$ and $q \in W^{uu}(p)$ and since the center leaves are uniformly compact (indeed they

are just the fibers), the unstable holonomy map can be defined on the entire center leaf $H^u_{p,q} : W^c(p) \to W^c(q)$. By Theorem 2.5, this property is C^1-open.

Using the f-invariance of the center and strong unstable foliations, it is easy to see that for any $n \in \mathbb{Z}$, for each p, q as above, we have

$$H^u_{f^n(p),f^n(q)} \circ f^n = f^n \circ H^u_{p,q}.$$

We remark that in the skew product case, we may also use the notation $H^u_{p_2,q_2}$ to denote the unstable holonomy between $\pi_2^{-1}(p_2)$ and $\pi_2^{-1}(q_2)$, for p_2 and q_2 belonging to the same unstable manifold of f_2. Sometimes we will use this notation.

Higher regularity of unstable holonomies. Let f be a $C^{2+\alpha}$ absolutely partially hyperbolic skew product of $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ and let $\chi^{ss}, \chi^c_-, \chi^c_+, \chi^{uu}$ be the partially hyperbolic constants of f. We say that f verifies the $(2,\alpha)$**-center unstable bunching condition** if

$$\left(\frac{\chi^c_+}{\chi^c_-}\right)^2 < \chi^{uu} \text{ and } \frac{\chi^c_+}{(\chi^c_-)^2} < (\chi^{uu})^\alpha. \tag{71}$$

Similarly, f verifies the $(2,\alpha)$**-center stable bunching condition** if

$$\chi^{ss} < \left(\frac{\chi^c_-}{\chi^c_+}\right)^2 \text{ and } (\chi^{ss})^\alpha < \frac{\chi^c_-}{(\chi^c_+)^2}. \tag{72}$$

If f verifies condition (71) and (72) then we say that f is $(2,\alpha)$**-center bunched**.

We use the $(2,\alpha)$-center bunching condition to obtain C^2-regularity of the unstable holonomy inside a center unstable leaf. This is given in the following theorem.

Theorem 2.10. *Let f be a $C^{2+\alpha}$ absolutely partially hyperbolic skew product of $\mathbb{T}^4$, and fix $R > 0$. If f is $(2,\alpha)$-unstable center bunched, then $\{H^u_{p,q}\}_{p\in\mathbb{T}^4, q\in W^{uu}_R(p)}$ is a family of C^2-diffeomorphisms of $\mathbb{T}^2$ whose C^2-norm varies continuously with the choices of p and q.*

This theorem is proved in the appendix (see section 8).

2.2. Pesin's theory and SRB measures. Let f be a C^1-diffeomorphism. A number $\lambda \in \mathbb{R}$ is a **Lyapunov exponent** if there exists a point $p \in M$ and a non zero vector $v \in T_pM$ such that $\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(p)v\| = \lambda$. We write $\lambda(p,v) := \lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(p)v\|$.

We say that a set R has full probability if for any f-invariant probability measure ν it is verified that $\nu(R) = 1$. The following theorem is known as the Oseledets theorem.

Theorem 2.11 ([BP02], Theorems 2.1.1 and 2.1.2)**.** *For any C^1-diffeomorphism f, there is a set $\mathcal{R}$ of full probability, such that for every $\varepsilon > 0$ it exists a measurable function $C_\varepsilon : \mathcal{R} \to (1, +\infty)$ with the following properties:*

(1) for any $p \in \mathcal{R}$ there are numbers $s(p) \in \mathbb{N}$, $\lambda_1(p) < \cdots < \lambda_{s(p)}(p)$ and a decomposition $T_pM = E^1_p \oplus \cdots \oplus E^{s(p)}_p$;

(2) $s(f(p)) = s(p)$, $\lambda_i(f(p)) = \lambda_i(p)$ and $Df(p).E^i_p = E^i_{f(p)}$, for every $i = 1, \cdots, s(p)$;

(3) for every $v \in E^i_p - \{0\}$, $\lambda(p,v) = \lambda^i(p)$.

We call the set $\mathcal{R}$ the set of **regular points**. A point $p \in \mathcal{R}$ has k negative Lyapunov exponents if

$$\sum_{i:\lambda_i(p)<0} dim(E^i_p) = k.$$

Similarly for positive or zero Lyapunov exponents. From now on, we assume that ν is a f-invariant measure, not necessarily ergodic, and there are numbers k and l such that ν-almost every point $p \in \mathcal{R}$ has k negative and l positive Lyapunov exponents.

For a regular point we write

$$E^s_p = \bigoplus_{i:\lambda_i(p)<0} E^i_p \text{ and } E^u_p = \bigoplus_{i:\lambda_i(p)>0} E^i_p. \tag{73}$$

It is well known that for a C^2-diffeomorphism f and an invariant measure ν, then for ν-almost every p, the set defined by

$$W^s(p) = \{q \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f^n(p), f^n(q)) < 0\}$$

is an immersed submanifold such that $T_pW^s(p) = E^s_p$ (see section 4 of [Pe77]). We call $W^s(p)$ the **stable Pesin manifold** of the point p. Similarly, the set defined by

$$W^u(p) = \{q \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f^{-n}(p), f^{-n}(q)) < 0\}$$

is an immersed submanifold such that $T_pW^u(p) = E^u_p$. We call $W^u(p)$ the **unstable Pesin manifold** of the point p. Since these manifolds exist for ν-almost every point, the unstable manifolds $\{W^u(p)\}_{p\in\mathcal{R}}$ form a partition of a ν-full measure subset of M.

Remark 2.12. *If f is also partially hyperbolic, with $TM = E^{ss} \oplus E^c \oplus E^{uu}$ then the Oseledets splitting refines the partially hyperbolic splitting. This means that for a regular point $p \in \mathcal{R}$, there are numbers $1 \le l_1 < l_2 < s(p)$ such that*

$$E^{ss}_p = \bigoplus_{i=1}^{l_1} E^i_p,\ E^c_p = \bigoplus_{i=l_1+1}^{l_2} E^i_p \text{ and } E^{uu}_p = \bigoplus_{i=l_2+1}^{s(p)} E^i_p.$$

This follows from a standard argument similar to the proof of the uniqueness of dominated splittings, see section B.1.2 from [BDV05]. It also holds that for any regular point p, $E^{ss}_p \subset E^s_p$ and $E^{uu}_p \subset E^u_p$.

A partition ξ of M is **measurable** with respect to a probability measure ν, if up to a set of ν-zero measure, the quotient M/ξ is separated by a countable number of measurable sets. Denote by $\hat{\nu}$ the quotient measure in M/ξ.

By Rokhlin's disintegration theorem [Ro52], for a measurable partition ξ, there is set of conditional measures $\{\nu^\xi_D : D \in \xi\}$ such that for $\hat{\nu}$-almost every $D \in \xi$ the measure ν^ξ_D is a probability measure supported on D, for each measurable set $B \subset M$ the application $D \mapsto \nu^\xi_D(B)$ is measurable and

$$\nu(B) = \int_{M/\xi} \nu^\xi_D(B) d\hat{\nu}(D). \tag{74}$$

From now on we suppose that f is a C^2-diffeomorphism and ν has no zero Lyapunov exponents. We call such a measure **hyperbolic**. We remark that usually the unstable partition $\{W^u(p)\}_{p\in\mathcal{R}}$ is not a measurable partition. We say that a ν-measurable partition ξ^u is **u-subordinated** if for for ν-almost every p, the following conditions are verified:

- $\xi^u(p) \subset W^u(p)$;
- $\xi^u(p)$ contains an open neighborhood of p inside $W^u(p)$.

Definition 2.13 (SRB measure)**.** *A measure ν is **SRB** if for any u-subordinated measurable partition ξ^u, for ν-almost every p, the conditional measure $\nu^u_{\xi^u(p)}$ is absolutely continuous with respect to the riemannian volume of $W^u(p)$.*

There is a well developed ergodic theory for hyperbolic SRB measures. We now state some results obtained by Ledrappier in [Le84].

Theorem 2.14 ([Le84], Corollary 4.10 and Theorem 5.10.)**.** *Let f be a C^2-diffeomorphism and ν a hyperbolic SRB measure. Then there are at most countably many ergodic components of ν, that is,*

$$\nu = \sum_{i\in\mathbb{N}} c_i \nu_i,$$

where $c_i \geq 0$, $\sum_{i\in\mathbb{N}} c_i = 1$, each ν_i is an f-invariant ergodic SRB measure such that if $i \neq j$, and $c_i, c_j > 0$ then $\nu_i \neq \nu_j$. Moreover, for each $i \in \mathbb{N}$ such that $c_i > 0$, there exists $k_i \in \mathbb{N}$ such that

$$\nu_i = \frac{1}{k_i} \sum_{j=1}^{k_i} \nu_{i,j},$$

where each $\nu_{i,j}$ is an f^{k_i}-invariant probability measure, the system $(f^{k_i}, \nu_{i,j})$ is Bernoulli and $\nu_{i,j} \neq \nu_{i,l}$ if $j \neq l$. Furthermore, f permutes the measures $\nu_{i,j}$, that is, $f_*(\nu_{i,j}) = \nu_{i,j+1}$ for $j = 1, \cdots, k_i - 1$ and $f_*(\nu_{i,k_i}) = \nu_{i,1}$, where $f_*(\nu)$ denotes the pushforward of a measure ν by f.

Now given two hyperbolic ergodic measure, μ and ν, we say that stable manifolds of μ intersects transversely unstable manifolds of ν if the following holds: there exist a set Λ^s with positive μ-measure and a set Λ^u with positive ν-measure, such that for each $p \in \Lambda^s$ and $q \in \Lambda^u$, there exists $n_1, n_2 \in \mathbb{Z}$ with

$$W^s(f^{n_1}(p)) \pitchfork W^u(f^{n_2}(q)) \neq \emptyset.$$

In this case we write $\mu \pitchfork_{su} \nu$.

Definition 2.15. *For μ and ν hyperbolic ergodic measures, we say that μ is **homoclinically related** with ν, if $\mu \pitchfork_{su} \nu$ and $\nu \pitchfork_{su} \mu$. We write $\mu \sim_{hom} \nu$.*

In the case that μ and ν are ergodic SRB measures, homoclinic relation actually implies that they are the same.

Theorem 2.16. *Let μ and ν be two hyperbolic, ergodic SRB measures. If $\mu \sim_{hom} \nu$ then $\mu = \nu$.*

The proof of Theorem 2.16 is a consequence of Hopf's argument adapted to the non-uniformly hyperbolic scenario. This type of argument has been done in many places, see for instance Lemma 3.2 in [HS16].

We remark that all the results stated in this section were stated for C^2-diffeomorphisms, but they hold for $C^{1+\alpha}$-diffeomorphisms.

2.3. u-Gibbs measures and the invariance principle.

u-Gibbs measures. Let f be a C^2-partially hyperbolic diffeomorphism and let μ be an f-invariant measure. We say that a μ-measurable partition ξ^{uu} is subordinated to the foliation $\mathcal{F}^{uu}$, if for μ-almost every p, $\xi^{uu}(p) \subset W^{uu}(p)$ and $\xi^{uu}(p)$ contains an open neighborhood of p inside $W^{uu}(p)$. For simplicity, we will write the conditional measure $\mu^{uu}_{\xi^{uu}(p)}$ by μ^{uu}_p.

Definition 2.17 (u-Gibbs)**.** *An f-invariant measure μ is **u-Gibbs** if for any μ-measurable partition ξ^{uu} subordinated to $\mathcal{F}^{uu}$, for μ-almost every point p, the conditional measure μ^{uu}_p is absolutely continuous with respect to the Lebesgue measure of $W^{uu}(p)$. We denote the set of u-Gibbs measures of f by $Gibbs^u(f)$.*

We remark that for $C^{1+\alpha}$-partially hyperbolic diffeomorphisms these measures always exist, and they capture all the possible statistical behavior of Lebesgue almost every point. That is, for Lebesgue almost every p any accumulation point of the sequence $\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(p)}$ is an u-Gibbs measure (see Theorem 11.16 in [BDV05]).

Let us consider the strong unstable foliation $\mathcal{F}^{uu}$ and μ an f-invariant measure. We say that a μ-measurable partition ξ^{uu} subordinated to $\mathcal{F}^{uu}$ is **increasing** if for μ-almost every p, we have $\xi^{uu}(f(p)) \subset f(\xi^{uu}(p))$. We define the **$\mu$-partial entropy along $\mathcal{F}^{uu}$** by

$$h_\mu(f, \mathcal{F}^{uu}) = H_\mu(f^{-1}\xi^{uu}|\xi^{uu}) := -\int_M \log \mu_p^{uu}(f^{-1}\xi^{uu}(p))\mu(p), \tag{75}$$

where $f^{-1}\xi^{uu}(p)$ is the element of the partition $f^{-1}\xi^{uu}$ containing p. The definition above does not depend on the choice of the μ-measurable partition ξ^{uu}. The notion of partial entropy along expanding foliations has been introduced in [VY17] and [Ya16] (see also [LY85-1]).

Let $Jac^{uu}(p) = |\det(Df(p)|_{E^{uu}})|$. In the case that E^{uu} has dimension one, for any ergodic f-invariant measure, we write λ_μ^{uu} to be the Lyapunov exponent of the strong unstable direction. The following result can be found in [Ya16] and [Le84].

Proposition 2.18 ([Ya16], Proposition 5.2, and [Le84], Theorem 3.4)**.** *Let μ be an u-Gibbs measure. Then*

$$h_\mu(f, \mathcal{F}^{uu}) = \int_M \log Jac^{uu}(p) d\mu(p).$$

In particular, if E^{uu} is one dimensional and μ is ergodic then $h_\mu(f, \mathcal{F}^{uu}) = \lambda_\mu^{uu}$.

The invariance principle. An important tool in this work is the invariance principle which was first developed by Furstenberg in [Fu63] and by Ledrappier in [Le86]. We also mention the work of Avila-Viana in [AV10]. In this work we use the version of the invariance principle given by Tahzibi-Yang in [TY19], which we describe in this section. This relates entropy along strong unstable foliations with the so called u-invariance of certain measures. Their results hold for large classes of partially hyperbolic skew products, however, we will state them for skew products on $\mathbb{T}^2 \times \mathbb{T}^2$.

Let f be a C^2-partially hyperbolic center bunched skew product and let f_2 be the Anosov diffeomorphism on the base. We remark that on $\mathbb{T}^2$, every Anosov diffeomorphism is transitive. Fix a f_2-invariant measure ν. Let ξ_2^{uu} be a ν-measurable partition of $\mathbb{T}^2$ which is subordinated to the foliation $\mathcal{F}_2^{uu}$ (the unstable foliation of f_2 on $\mathbb{T}^2$), and consider the μ-measurable partition ξ^{uu} of $\mathbb{T}^4$ subordinated to $\mathcal{F}^{uu}$ which refines the partition $\pi_2^{-1}(\xi_2^{uu})$ with the property that for μ-almost every p, $\pi_2(\xi^{uu}(p)) = \xi_2^{uu}(\pi_2(p))$.

Definition 2.19. *We say that an f-invariant measure μ is an **u-state projecting on ν**, if $(\pi_2)_* = \nu$ and for μ-almost every p,*

$$(\pi_2)_*\mu_p^{uu} = \nu_{\pi_2(p)}^{uu}. \tag{76}$$

We denote the set of u-state measures projecting on ν by $\mathrm{State}_\nu^u(f)$.

Remark 2.20. *In [TY19], the authors call the measures from definition 2.19 u-Gibbs measures projecting on ν. Since we already use the name u-Gibbs for the measures from definition 2.17, we changed the name in our paper. Even though later we will see that in our setting both definitions coincide once the measure ν is an SRB-measure for the Anosov diffeomorphism on the basis (see proposition 2.23).*

The following result is a characterization using entropy for a measure to belong to $\mathrm{State}_\nu^u(f)$.

Theorem 2.21 ([TY19], Theorem A)**.** *Let f be a C^2-partially hyperbolic skew product as above and let ν be an f_2-invariant measure. Suppose that μ is an f-invariant measure such that $(\pi_2)_*\mu = \nu$. Then $h_\mu(f, \mathcal{F}^{uu}) \leq h_\nu(f_2)$ and the equality holds if and only if $\mu \in \mathrm{State}_\nu^u(f)$.*

Let μ be an f-invariant probability measure, and recall that the partition by center leaves form a μ-measurable partition of $\mathbb{T}^4$. Consider $\{\mu_{p_2}^c\}$ be the disintegration of μ with

respect to the center leaves. We say that μ has **atomic disintegration along the center foliation** if for μ-almost every p the measure $\mu^c_{\pi_2(p)}$ is an atomic.

Proposition 2.22 ([TY19], Proposition 5.4)**.** *A measure μ is an u-state projecting on ν if and only if there exists a set $X \subset \mathbb{T}^2$ of full ν-measure such that for any two points $p_2, q_2 \in X$ in the same unstable leaf, we have that*

$$\mu^c_{q_2} = (H^u_{p_2,q_2})_* \mu^c_{p_2}. \tag{77}$$

The property described by (77) is called **u-invariance of the conditional measures** $\{\mu^c_{p_2}\}_{p_2 \in \mathbb{T}^2}$.

Since f_2 is a transitive C^2-Anosov diffeomorphism, it is well known that it admits an unique SRB measure ν, see [Bow75, Ru76, Si68]. Consider now the set $\text{State}^u_\nu(f)$. In what follows we will show that $\text{Gibbs}^u(f) = \text{State}^u_\nu(f)$.

Proposition 2.23. *For f and ν as above,* $\text{Gibbs}^u(f) = \text{State}^u_\nu(f)$.

To prove this proposition, we will need the following lemma.

Lemma 2.24. *Let $\mu \in \text{Gibbs}^u(f)$, then $(\pi_2)_*\mu = \nu$.*

Proof. It is enough to prove that $\tilde{\nu} := (\pi_2)_*\mu$ is an SRB measure for f_2. Since f_2 admits only one SRB measure, it follows that $\tilde{\nu} = \nu$.

Let ξ^{uu}_2 be a $\tilde{\nu}$-measurable partition subordinated to $\mathcal{F}^{uu}_2$. Observe that the partition $\xi^{cu} = \pi_2^{-1}(\xi^{uu})$ is μ-measurable and denote by μ^{cu}_p the conditional measures of μ with respect to this partition. The partition ξ^{cu} is refined by the μ-measurable partition ξ^{uu} which is subordinated to $\mathcal{F}^{uu}$ and such that for μ-almost every p, we have $\pi_2(\xi^{uu}(p)) = \xi^{uu}_2(\pi_2(p))$.

Take a $\tilde{\nu}$-generic point $p_2 \in \mathbb{T}^2$ and let $B \subset \xi^{uu}_2(p_2)$ be a set of zero Lebesgue measure inside the unstable manifold of p_2. Since the foliation by center fibers is smooth (because we are in the skew product setting), and the strong unstable manifolds of f are uniformly transverse to the center direction inside the cu-leaves, we have that for $\mu^{cu}_{p_2}$-almost every q the set $\xi^{uu}(q) \cap \pi_2^{-1}(B)$ has zero Lebesgue measure inside $W^{uu}_f(q)$. In particular, the u-Gibbs property of μ implies that $\mu^{uu}_q(\pi_2^{-1}(B)) = 0$. We conclude

$$\tilde{\nu}^{uu}_{p_2}(B) = \int_M \mu^{uu}_q(\pi_2^{-1}(B)) d\mu^{cu}(q) = 0.$$

This is true for any set B of zero Lebesgue measure. This implies that $\tilde{\nu}^{uu}_{p_2}$ is absolutely continuous with respect to the Lebesgue measure of $W^{uu}_{f_2}(p_2)$ and the measure $\tilde{\nu}$ is SRB. □

Proof of 2.23. From (76) and the fact that the foliation by horizontal fiber is smooth, it is immediate that $\text{State}^u_\nu(f) \subset \text{Gibbs}^u(f)$.

Since the strong unstable direction is uniformly transverse to the center fibers inside the cu-leaves and it projects to $E^{uu}_{f_2}$, and since the center direction is orthogonal to the base, there exists a constant $C \geq 1$ such that for any $p \in \mathbb{T}^4$ and any $v^{uu} \in E^{uu}_p$ we have

$$\frac{1}{C}\|v^{uu}\| \leq \|D\pi_2(p)v^{uu}\| \leq \|v^{uu}\|.$$

Suppose that $\mu \in \text{Gibbs}^u(f)$ is an ergodic measure. Let p be a generic point for μ and let $v^{uu} \in E^{uu}_p$ be an unit vector. Observe that for any $n \in \mathbb{N}$ we have

$$\frac{1}{C}\|Df^n(p)v^{uu}\| \leq \|D\pi_2(f^n(p))Df^n(p)v^{uu}\| \leq \|Df^n(p)v^{uu}\|.$$

Since f is a skew product and $\pi_2 \circ f = f \circ \pi_2$, we obtain that $D\pi_2(f^n(p))Df^n(p)v^{uu} = Df^n_2(\pi_2(p))D\pi_2(p)v^{uu}$. By lemma 2.24, we may assume that $\pi_2(p)$ is a generic point for

ν. We conclude that

$$\lambda_\mu^{uu} = \lim_{n\to+\infty} \frac{1}{n} \log \|Df^n(p)v^{uu}\| = \lim_{n\to+\infty} \frac{1}{n} \log \|Df_2^n(\pi_2(p))D\pi_2(p)v^{uu}\| = \lambda_\nu^{uu},$$

where λ_μ^{uu} is the Lyapunov exponent of f for μ along the strong unstable direction and λ_ν^{uu} is the Lyapunov exponent of f_2 for ν along the unstable direction.

It is well known that the measure ν verifies Pesin's formula (since it is also an SRB measure for f_2), see [Le84], and hence $h_\nu(f_2) = \lambda_\nu^{uu}$. By proposition 2.18, we have that $h_\mu(f, \mathcal{F}^{uu}) = \lambda_\mu^{uu}$. We conclude that $h_\mu(f, \mathcal{F}^{uu}) = h_\nu(f_2)$. By Theorem 2.21 we obtain that $\mu \in \text{State}_\nu^u(f)$. □

The main conclusion of proposition 2.23 is the following corollary.

Corollary 2.25. *For f as above, any u-Gibbs measure μ has u-invariant center conditional measures.*

3. Center Lyapunov exponents for u-Gibbs measures

In this section we explain how the techniques developed by Berger-Carrasco in [BC14], and the adaptations of their techniques made by the author in [Ob18-2], actually give estimates for the Lyapunov exponents for any u-Gibbs measure. We prove the following theorem:

Theorem 3.1. *For every $\delta \in (0,1)$, there exists $N_0 = N_0(\delta)$ such that for every $N \geq N_0$, there exists $\mathcal{U}_N$ a C^2-neighborhood of f_N inside $\text{Diff}^2(\mathbb{T}^4)$ with the following property. If $g \in \mathcal{U}_N$ and μ is an u-Gibbs measure, then μ-almost every point has a positive and a negative Lyapunov exponent along the center whose absolute value is greater than $(1-\delta)\log N$. In particular, in a neighborhood of f_N any u-Gibbs measure is hyperbolic.*

Remark 3.2. *Even though the results from [BC14, Ob18-2] are in the volume preserving scenario, several of the lemmas and propositions still valid for dissipative perturbations. In what follows, we will use several results from these works. The only point in this section that will need an adaptation for u-Gibbs measures is given in proposition 3.11.*

Let $A \in SL(2,\mathbb{Z})$ be the linear Anosov matrix considered in the definition of the map f_N. Denote by $0 < \lambda < 1 < \tilde{\mu} = \lambda^{-1}$ the eigenvalues of A. Let e^s and e^u be unit eigenvectors of A for λ and $\tilde{\mu}$, respectively. Recall that we defined the linear map $P_x : \mathbb{R}^2 \to \mathbb{R}^2$ given by $P_x(a,b) = (a,0)$.

Lemma 3.3 ([BC14], Proposition 1)**.** *There is a differentiable function $\alpha : \mathbb{T}^4 \to \mathbb{R}^2$ such that the unstable direction of f_N is generated by the vector field $(\alpha(m), e^u)$, where*

$$Df_N(m).(\alpha(m), e^u) = \tilde{\mu}^{2N}(\alpha(f_N(m)), e^u) \text{ and } \|\alpha(m) - \lambda^N P_x(e^u)\| \leq \lambda^{2N}.$$

Observe that $|\det Df_N|_{E^c_{f_N}}| = 1$.

Lemma 3.4 ([Ob18-2], Lemma 7.17)**.** *For $\varepsilon_1 > 0$ and $\beta > 0$ small, if N is large and $\mathcal{U}_N$ is small enough then for every $g \in \mathcal{U}_N$ and for all unit vectors $v^s \in E^{ss}_g$, $v^c \in E^c_g$ and $v^u \in E^{uu}_g$, the following holds:*

(1) $e^{-\varepsilon_1}\tilde{\mu}^{2N} \leq \|Dg(v^u)\| \leq e^{\varepsilon_1}\tilde{\mu}^{2N}$;

(2) $\frac{1}{2N} \leq \|Dg(v^c)\| \leq 2N$;

(3) $\|D^2 g^{-1}\| \leq 2N$ *and* $\|D^2 g\| \leq 2N$;

(4) $|\det Dg|_{E^c_g}| \in (e^{-\beta}, e^{\beta})$;

(5) E^c_g *is* $\frac{1}{2}$*-Hölder.*

A key element in Berger-Carrasco's proof is to consider center vector fields over certain pieces of strong unstable curve. Consider $g \in \mathcal{U}_N$.

Definition 3.5 ([BC14], Definition 7.18)**.** *An u-curve for g is a C^1-curve $\gamma = (\gamma_x, \gamma_y, \gamma_z, \gamma_w) : [0, 2\pi] \to M$ tangent to E^{uu}_g and such that $\left|\frac{d\gamma_x}{dt}(t)\right| = 1$, $\forall t \in [0, 2\pi]$. For every $k \geq 0$ there exists an integer $N_k = N_k(\gamma) \in \left[(e^{-\varepsilon_1}\tilde{\mu}^{2N})^k, (e^{\varepsilon_1}\tilde{\mu}^{2N})^k\right]$ such that the curve $g^k \circ \gamma$ can be writen as*

$$g^k \circ \gamma = \gamma_1^k * \cdots * \gamma_{N_k} * \gamma_{N_k+1}^k$$

where γ_j^k for $j = 1, \cdots, N_k$, are u-curves and $\gamma_{N_k+1}^k$ is a segment of u-curve.

The following lemma controls the length of u-curves.

Lemma 3.6 ([BC14], Corollary 5)**.** *For N is large and $\mathcal{U}_N$ small enough then for every $g \in \mathcal{U}_N$ and any unit vector $v^u \in E^{uu}_{g,m}$, it holds that*

$$|P_x(D\pi_1.v^u)| \in [(\lambda^N(\|P_x(e^u)\| - 3\lambda^N), (\lambda^N(\|P_x(e^u)\| + 3\lambda^N)].$$

An easy consequence of this lemma is the following.

Corollary 3.7. *For any $\varepsilon_2 > 0$, if N is large and $\mathcal{U}_N$ is small enough, then any two u-curves (γ, γ') satisfy:*

$$e^{-\varepsilon_2}|\gamma| \leq |\gamma'| \leq e^{\varepsilon_2} l|\gamma|, \tag{78}$$

where $|\gamma|$ denotes the length of the curve γ.

We define the unstable jacobian of g^k as

$$J^{uu}_{g^k}(m) = |\det Dg^k(m)|_{E^{uu}_g}|, \ \forall m \in \mathbb{T}^4. \tag{79}$$

By item 2 of lemma 3.4, for $g \in \mathcal{U}_N$ and for every $m \in \mathbb{T}^4$

$$e^{-\varepsilon_1}\lambda^{2N} \leq J^{uu}_{g^{-1}}(m) \leq e^{\varepsilon_1}\lambda^{2N}. \tag{80}$$

Lemma 3.8 ([Ob18-2], Lemma 7.20)**.** *For $\varepsilon_3 > 0$ small, if N is large and $\mathcal{U}_N$ is small enough, for every $g \in \mathcal{U}_N$ and any u-curve γ for g, for every $k \geq 0$, we have*

$$\forall m, m' \in \gamma, \ e^{-\varepsilon_3} \leq \frac{J^{uu}_{g^{-k}}(m)}{J^{uu}_{g^{-k}}(m')} \leq e^{\varepsilon_3}.$$

This lemma implies that for $g \in \mathcal{U}_N$ and for any u-curve γ for g, if $A \subset \gamma$ is any measurable set, for every $k \geq 0$, it holds

$$e^{-\varepsilon_3}\frac{Leb(A)}{Leb(\gamma)} \leq \frac{Leb(g^{-k}(A))}{Leb(g^{-k}(\gamma))} \leq e^{\varepsilon_3}\frac{Leb(A)}{Leb(\gamma)}.$$

Definition 3.9. *An adapted field (γ, X) over an u-curve γ is an unitary vector field X such that*

(1) X is tangent to the center direction;

(2) X is $(C_X, 1/2)$-Hölder along γ, that is

$$\|X_m - X_{m'}\| \leq C_X d_\gamma(m, m')^{\frac{1}{2}}, \ \forall m, m' \in \gamma,$$

where $C_X < 30N^2\lambda^N$ and d_γ is the distance measured along γ.

Remark 3.10. *The estimate on the Hölder constant used in [BC14, Ob18-2] is $20N^2\lambda^N$, instead of $30N^2\lambda^N$ as above. This is due to the fact that the parametrization of the torus $\mathbb{T}^4$ is by intervals of length 2π instead of 1 in the proof of lemma 2 in [BC14]. However, this change on the estimate of the Hölder constant does not affect the rest of the proof.*

Let (γ, X) be an adapted field, and define

$$I_n^{\gamma,X} = \frac{1}{|\gamma|}\int_\gamma \log\|Dg^n.X\|d\gamma.$$

Proposition 3.11. *Suppose that there exists $C > 0$ with the following property: for every u-curve γ there exists an adapted vector field (γ, X) for g and for all $n > 0$ large enough*

$$\frac{I_n^{\gamma,X}}{n} > C.$$

Then any u-Gibbs measure μ for g has a positive Lyapunov exponent along the center direction greater than $e^{-2\varepsilon_3}C$.

Proof. Suppose not, then there exist an u-Gibbs measure μ and a measurable set B with positive μ-measure such that every point in B has exponents in the center direction strictly smaller than $e^{-2\varepsilon_3}C$. Since μ has disintegration along unstable leaves equivalent to the Lebesgue measure along the leaves, there is an unstable manifold γ that intersects B on a set of positive measure for the Lebesgue measure of γ. Let $b \in \gamma \cap B$ be a density point and take $\gamma_k = g^{-k} \circ \beta_k$, where β_k is a u-curve with $\beta_k(0) = g^k(b)$. We have that $l(\gamma_k) \to 0$ and by bounded distortion (lemma 3.8)

$$\frac{Leb(\gamma_k \cap B)}{Leb(\gamma_k)} \longrightarrow 1.$$

Take k large enough such that

$$\frac{Leb(\gamma_k \cap B^c)}{Leb(\gamma_k)} < \frac{e^{-2\varepsilon_3}(e^{\varepsilon_3} - 1)C}{2\log 2N}.$$

Using bounded distortion again, for any $m^k \in g^k(\gamma_k)$

$$J^{uu}_{g^{-k}}(m^k) \geq \frac{Leb(\gamma_k)}{Leb(g^k(\gamma_k))} e^{-\varepsilon_3}.$$

Define $\chi_k(m) = \limsup_{n\to+\infty} \frac{1}{n} \log \|Dg^n(g^k(m)).X^k_{g^k(m)}\|$ for all $m \in \gamma_k$, where X^k is the vector field such that (β_k, X^k) verifies the hypothesis of the proposition. Since for μ-almost every point the Lyapunov exponents exist, using the dominated convergence theorem, we have

$$\begin{aligned}
\int_{\gamma_k} \chi_k d\gamma_k &= \int_{\beta_k} \chi_k \circ g^{-k} J^{uu}_{g^{-k}} d\beta_k \\
&\geq e^{-\varepsilon_3} \frac{Leb(\gamma_k)}{Leb(\beta_k)} \int_{\beta_k} \chi_k \circ g^{-k} d\beta_k \\
&= e^{-\varepsilon_3} \frac{Leb(\gamma_k)}{Leb(\beta_k)} \limsup_{n\to+\infty} \frac{I_n^{\beta_k, X^k}}{n} . Leb(\beta_k) \geq e^{-\varepsilon_3} C Leb(\gamma_k).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{\gamma_k} \chi_k d\gamma_k &= \int_{\gamma_k \cap B} \chi_k d\gamma_k + \int_{\gamma_k \cap B^c} \chi_k d\gamma_k \\
&\leq e^{-2\varepsilon_3} C Leb(\gamma_k) + \frac{\log 2N e^{-2\varepsilon_3}(e^{\varepsilon_3} - 1) C Leb(\gamma_k)}{2\log 2N} \\
&< e^{-\varepsilon_3} C Leb(\gamma_k)
\end{aligned}$$

which is a contradiction. □

Write

$$E(\gamma, X) = \frac{1}{|\gamma|} \int_\gamma \log \|Dg(m).X_m\| d\gamma(m),$$

where (γ, X) is an adapted field. Let $\pi_1 : \mathbb{T}^4 \to \mathbb{T}^2$ be the projection defined by $\pi_1(x, y, z, w) = (x, y)$. For X a vector field on γ define

$$\widetilde{X}_m = \frac{D\pi_1(X_m)}{\|D\pi_1(X_m)\|}.$$

In what follows, we let $\tilde{\delta} > 0$ to be a positive constant that we will fix later.

Definition 3.12. *Consider the cone $\Delta_{\tilde{\delta}} = \{(u,v) \in \mathbb{R}^2 : N^{\tilde{\delta}}|u| \geq |v|\}$. Let (γ, X) be an adapted vector field. If for every $m \in \gamma$ we have that $\tilde{X}(m) \in \Delta_{\tilde{\delta}}$ then we say that (γ, X) is a $\tilde{\delta}$-**good** adapted vector field. Otherwise we say that it is $\tilde{\delta}$-**bad**.*

Recall that for $k \geq 0$ and an u-curve γ the number $N_k = N_k(\gamma)$ denotes the maximum number of u curves that subdivide $g^k \circ \gamma$. For an adapted field (γ, X) define the unit vector field over $g^k(\gamma)$, $Y^k = \frac{g^k_* X}{\|g^k_* X\|}$, where $g^k_* X_m = Dg^k(g^{-k}(m))X_{g^{-k}(m)}$.

Lemma 3.13 ([BC14], Lemma 9)**.** *For N large and $\mathcal{U}_N$ small enough, let $g \in \mathcal{U}_N$ and (γ, X) be an adapted field for g. For $k \geq 0$, every possible pair $(\gamma^k_j, Y^k|_{\gamma^k_j})$, with $1 \leq j \leq N_k(\gamma)$ is an adapted field.*

The following formula is proved in section 6 of [BC14].

Lemma 3.14. *For every adapted field (γ, X) and any $n \in \mathbb{N}$*

$$I^{\gamma,X}_n = \sum_{k=0}^{n-1} \left(R_k + \sum_{j=0}^{N_k} \frac{1}{|\gamma|} \int_{\gamma^k_j} \log \|Dg(m).Y^k_m\| J^{uu}_{g^{-k}} d\gamma^k_j \right),$$

where $R_k = \frac{1}{|\gamma|} \int_{\gamma^k_{N_k+1}} \log \|Dg(m).Y^k_m\| J^{uu}_{g^{-k}} d\gamma^k_{N_k+1}$.

As a consequence of lemma 3.14, and using (78), we obtain

$$I^{\gamma,X}_n \geq \sum_{k=0}^{n-1} \left(R_k + e^{-\varepsilon_2} \sum_{j=0}^{N_k} (\min_{\gamma^k_j} J^{uu}_{g^{-k}}) E(\gamma^k_j, Y^k) \right). \tag{81}$$

Since $\gamma^k_{N_k+1}$ is a piece of an u-curve, then

$$\frac{|\gamma^k_{N_k+1}|}{|\gamma|} < 2.$$

By (80), we have

$$\begin{aligned} |R_k| = \frac{1}{|\gamma|} \int_{\gamma^k_{N_k+1}} \log \|Dg(m).Y^k_m\| J^{uu}_{g^{-k}} d\gamma^k_{N_k+1} &< \frac{|\gamma^k_{N_k+1}|}{|\gamma|} (e^{\varepsilon_1}\lambda)^{2Nk} \log 2N \\ &< 2(e^{\varepsilon_1}\lambda)^{2Nk} \log 2N \xrightarrow{k\to+\infty} 0. \end{aligned}$$

Hence,

$$\frac{1}{n} \sum_{k=0}^{n-1} |R_k| \longrightarrow 0.$$

The following is the key proposition that will give us the estimate that we need.

Proposition 3.15 ([Ob18-2], Proposition 7.29)**.** *For N large and $\mathcal{U}_N$ small enough, for every $g \in \mathcal{U}_N$, any $\tilde{\delta}$-good adapted field (γ, X) and every $k \geq 0$, we have*

$$e^{-\varepsilon_2} \sum_{j=0}^{N_k} (\min_{\gamma^k_j} J^{uu}_{g^{-k}}) E(\gamma^k_j, Y^k) \geq (1 - 12\tilde{\delta}) \log N.$$

Remark 3.16. *In [Ob18-2], the term $e^{-\varepsilon_2}$ on the right hand side of the equation (81) is missing. The same term is also missing in the statement of proposition 7.29 in [Ob18-2]. Since we can fix ε_2 arbitrarily close to 0, this does not affect the rest of the proof in [Ob18-2] to obtain the estimate of the center Lyapunov exponents.*

Now, we can proceed with the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. Take $\tilde{\delta} = \frac{2\delta}{15}$. Let N be large and let $\mathcal{U}_N$ be small enough such that it verifies proposition 3.15. Fix $g \in \mathcal{U}_N$ and let μ be an u-Gibbs measure for g. Consider any u-curve γ and any $\tilde{\delta}$-good vector field X on γ. By proposition 3.15, and using inequality (81), for n large enough

$$\frac{I_n^{\gamma,X}}{n} \geq (1 - 14\tilde{\delta}) \log N. \tag{82}$$

Since we could have chosen $\varepsilon_3 > 0$ small enough such that $e^{-\varepsilon_3}(1 - 14\tilde{\delta}) \geq (1 - 15\tilde{\delta})$ by proposition 3.11, μ-almost every point has a Lyapunov exponent for g in the center direction larger than

$$(1 - 15\tilde{\delta}) \log N = (1 - 2\delta) \log N.$$

By condition (4) in lemma 3.4, we have that for μ-almost every point m the sum of the center Lyapunov exponents belongs to the interval $(-\beta, \beta)$, that is, $-\beta < \lambda^-(m) + \lambda^+(m) < \beta$. By taking $\beta > 0$ small, after fixing δ, we conclude that

$$\lambda^-(m) < \beta - \lambda^+(m) < \beta - (1 - 2\delta) \log N < (1 - \delta) \log N.$$

Therefore, we obtain that for N large and $\mathcal{U}_N$ small enough, for $g \in \mathcal{U}_N$, any u-Gibbs measure $\mu \in \text{Gibbs}^u(g)$ verifies that μ-almost every point m has both a positive and a negative Lyapunov exponent on the center with absolute value larger than $(1-\delta) \log N$. □

4. Proof of Theorem L

Recall that in section 2 we defined the notion of homoclinically related measures (see definition 2.15). The goal of this section is to prove Theorem L. This is based in the techniques developed by the author in [Ob18-2]. We actually prove the following theorem, which is more general than Theorem L:

Theorem 4.1. *For N large and $\mathcal{U}_N$ small enough, for any $k \in \mathbb{N}$ the following holds: if $g \in \mathcal{U}_N$ and μ_1, μ_2 are two ergodic u-Gibbs measures for g^k, then μ_1 is homoclinically related to μ_2.*

For an SRB measure, we can also obtain the following proposition.

Proposition 4.2. *For N large and $\mathcal{U}_N$ small enough, let $g \in \mathcal{U}_N$ and let μ be an SRB measure for g. Then* $\text{supp}(\mu) = \mathbb{T}^4$.

PROOF OF THEOREM L ASSUMING THEOREM 4.1 AND PROPOSITION 4.2. Let N be large and $\mathcal{U}_N$ be small enough such that Theorem 4.1 holds and fix $g \in \mathcal{U}_N$. If μ_1 and μ_2 are two ergodic SRB measures for g, by Theorem 4.1, μ_1 is homoclinically related to μ_2. By Theorem 2.16, $\mu_1 = \mu_2$, and therefore g has at most one SRB measure.

Suppose that μ is an SRB measure for g. By Theorem 2.14, there exist $k \in \mathbb{N}$ and k measures which are g^k-invariant and SRB, $\mu_1, \cdots, \mu_k$, such that $\mu_i \neq \mu_j$ for $j \neq i$ and

$$\mu = \frac{1}{k} \sum_{j=1}^{k} \mu_j.$$

Moreover, $g_*(\mu_j) = \mu_{j+1}$, with the identification of $k + 1 = 1$, and (g^k, μ^k) is Bernoulli. Observe that if $k = 1$, then μ is Bernoulli for g.

Suppose $k > 1$, by Theorem 4.1, we have that for any $i, j \in \{1, \cdots, k\}$ with $i \neq j$, the measures μ_i and μ_j are homoclinically related. Since these measures are SRB, we obtain that $\mu_i = \mu_j$, which is a contradiction with the fact that $\mu_i \neq \mu_j$. Hence, $k = 1$ and the measure μ is Bernoulli for g. Proposition 4.2 states that if μ is SRB then it has full support. □

The rest of this section is mostly dedicated to prove Theorem 4.1. As we will see, the proof of this theorem is essentially contained in the proof of the stable ergodicity for the map f_N in [Ob18-2]. We will refer the reader to [Ob18-2] for the proofs of several of the lemmas and propositions that we will use in this section, and we remark that they are also valid outside the volume preserving setting. At the end of the section we explain how to obtain proposition 4.2. The argument involved in the proof of proposition 4.2 is a combination of some estimates obtained to prove Theorem 4.1 and arguments from [CO19].

4.1. Estimates for stable and unstable manifolds of u-Gibbs measures. For a vector $v \in T_m\mathbb{T}^4$, write $v_1 = D\pi_1(m).v$. For a direction $E \subset T_m\mathbb{T}^4$ we will write $(E)_1 = D\pi_1(m).E$. For this section we fix $0 < \delta << 1$ small and we are assuming that N is large and $\mathcal{U}_N$ is small enough such that Theorem 3.1 holds. For this subsection we fix two constants (depending on N), $\theta_1 := N^{-\frac{2}{5}}$ and $\theta_2 := N^{-\frac{3}{5}}$.

Let $g \in \mathcal{U}_N$. For each ergodic measure μ for g let Λ_μ be the set of points $m \in \mathbb{T}^4$ such that

$$\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(m)} \xrightarrow[n\to+\infty]{} \mu \text{ and } \frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^{-j}(m)} \xrightarrow[n\to+\infty]{} \mu, \text{ in the } weak^*\text{-topology.}$$

Where δ_p is the dirac mass on the point p. Birkhoff's theorem implies that $\mu(\Lambda_\mu) = 1$. Recall that $\mathcal{R}_g$ is the set of regular points for g. By Theorem 3.11, if μ is an u-Gibbs measure for g, then for each $m \in \mathcal{R}_g \cap \Lambda_\mu$ there are two directions $E^-_{g,m}$ and $E^+_{g,m}$ contained in $E^c_{g,m}$, which are the Oseledets' directions with respect to the negative and positive Lyapunov exponent, respectively.

For each $\mu \in \text{Gibbs}^u(g)$, we define the sets

$$Z^-_\mu = \left\{m \in \mathcal{R}_g \cap \Lambda_\mu : \forall n \geq 0 \text{ it holds } \left\|Dg^n(m)|_{E^-_{g,m}}\right\| < \left(N^{-\frac{4}{5}}\right)^n\right\};$$

$$Z^+_\mu = \left\{m \in \mathcal{R}_g \cap \Lambda_\mu : \forall n \geq 0 \text{ it holds } \left\|Dg^{-n}(m)|_{E^+_{g,m}}\right\| < \left(N^{-\frac{4}{5}}\right)^n\right\};$$

$$Z_\mu = g(Z^-_\mu) \cap g^{-1}(Z^+_\mu);$$

$$Z_g = \bigcup_{\mu \in \text{Gibbs}^u(g)} Z_\mu.$$

The proof of the following lemma is the same as lemma 5.2 in [Ob18-2]. It is an application of Pliss lemma.

Lemma 4.3 ([Ob18-2], lemma 5.2)**.** *Let $g \in \mathcal{U}_N$. If μ is an ergodic u-Gibbs measure for g, then $\mu(Z_g) \geq \frac{1-7\delta}{1+7\delta}$.*

Let $T = \left[\frac{1+7\delta}{28\delta}\right]$ and define

$$X_g = \bigcap_{k=-T+1}^{T-1} g^k(Z_g). \tag{83}$$

An easy consequence of the estimate in lemma 4.3 is given in the following lemma.

Lemma 4.4 ([Ob18-2], lemma 5.3)**.** *For N large and $\mathcal{U}_N$ small enough, if μ is an u-Gibbs measure for g then*

$$\mu(X_g) > 0.$$

For a vector $v \in \mathbb{R}^2$ we write $v = (v_h, v_v)$, where v_h and v_v are the coordinates of v with respect to the basis $(1, 0)$ and $(0, 1)$. For each $\theta > 0$ we consider the horizontal and vertical cones

$$\mathscr{C}^{hor}_\theta = \{v \in \mathbb{R}^2 : \theta\|v_h\| \geq \|v_v\|\} \text{ and } \mathscr{C}^{ver}_\theta = \{v \in \mathbb{R}^2 : \theta\|v_v\| \geq \|v_h\|\}.$$

One of the key ingredients in the proof of stable ergodicity of f_N is based in a version of the stable manifold theorem given by Crovisier-Pujals in [CP16]. Using their construction we can obtain precise estimates on the sizes of stable and unstable manifolds inside the center direction for u-Gibbs measures. This is given in the following proposition. Fix $\theta_1 = N^{-\frac{2}{5}}$.

Proposition 4.5 ([Ob18-2], Proposition 5.6)**.** *Let N be large and $\mathcal{U}_N$ be small enough. For $g \in \mathcal{U}_N$ and $m \in Z_g$, there are two C^1-curves, $W^*_g(m)$, contained in $W^c_g(m)$, tangent to $E^*_{g,m}$ and with length bounded from below by $r_0 = N^{-7}$, for $* = -, +$. Those curves are C^1-stable and unstable manifolds for g, respectively. Moreover, $\left(T_p W^+_{g,r_0}(m)\right)_1 \subset \mathscr{C}^{hor}_{\frac{4}{\theta_1}}(p)$ and $\left(T_q W^-_{g,r_0}(m)\right)_1 \subset \mathscr{C}^{ver}_{\frac{4}{\theta_1}}(q)$, for every $p \in W^+_{g,r_0}(m)$ and $q \in W^-_{g,r_0}(m)$.*

We remark that the proof of this proposition only uses the estimates for points in the set Z_g and estimates on the C^2-norm of g. The proof is exactly the same as the proof of proposition 5.6 in [Ob18-2]

Let $\theta_2 = N^{-\frac{3}{5}}$. Proposition 4.5 is one of the key ingredients to prove the following lemma.

Lemma 4.6 ([Ob18-2], Lemma 5.7)**.** *For N large, $\mathcal{U}_N$ small and $n > 15$, let $g \in \mathcal{U}_N$. Then for every $m \in X_g$ there are two curves $\gamma^-_{g,-n}(m) \subset g^{-n}(W^-_{g,r_0}(m))$ and $\gamma^+_{g,n}(m) \subset g^n(W^+_{g,r_0}(m))$ with length greater than 4π, such that $\left(T\gamma^-_{g,-n}(m)\right)_1 \subset \mathscr{C}^{ver}_{\theta_2}$ and $\left(T\gamma^+_{g,n}(m)\right)_1 \subset \mathscr{C}^{hor}_{\theta_2}$.*

We remark that the statement of lemma 5.7 from [Ob18-2], which is the equivalent of lemma 4.6 above, involves a measure $\nu_{g,i}$. However, the proof only uses the estimates of the points in the set Z_g and the definition of X_g. Therefore, the proof of lemma 4.6 is exactly the same as the proof of lemma 5.7 from [Ob18-2]

For $R > 0$, let

$$W^s_{g,R,-n}(m) = \bigcup_{q \in \gamma^-_{g,-n}(m)} W^{ss}_{g,R}(q),$$

where the curve $\gamma^-_{g,-n}(m)$ is the curve given by the previous lemma. Define similarly the set $W^u_{g,R,n}(m)$, but using the strong unstable manifolds.

Let $m \in X_g$ be a typical point for an u-Gibbs measure μ. Recall that the stable Pesin manifold is a C^1-immersed submanifold and it has a topological characterization given by

$$W^s(m) = \{y \in \mathbb{T}^4 : \limsup_{n \to +\infty} \frac{1}{n} \log d(f^n(m), f^n(y)) < 0\}.$$

By the topological characterization of the stable Pesin manifold and by the definition of $W^s_{g,R,-n}(m)$, it is easy to see that $W^s_{g,R,-n}(m) \subset g^{-n}(W^s(m))$. Observe that the strong stable manifolds subfoliate the Pesin stable manifold, in particular $W^s_{g,R,-n}(m)$ is open inside the Pesin manifold $g^{-n}(W^s(m))$. We conclude that $W^s_{g,R,-n}(m)$ is a C^1-submanifold. An analogous conclusion holds for unstable manifolds.

The next lemma allows us to control the tangent space of these stable and unstable manifolds inside the center direction.

Lemma 4.7 ([Ob18-2], Lemma 5.8)**.** *Fix $\theta_3 > 0$ such that $\theta_3 > \theta_2$ and satisfies $\mathscr{C}^{hor}_{\theta_3} \cap \mathscr{C}^{ver}_{\theta_3} = \{0\}$. For $g \in \mathcal{U}_N$, there exists $0 < R < 1$ such that if $n \geq 15$, $m \in X_g$ and $m^- \in W^s_{g,R,-n}(m) \subset W^s_{g,2,-n}(m)$, then*

$$\left(T(W^s_{g,2,-n}(m) \cap W^c_g(m^-))\right)_1 \subset \mathscr{C}^{ver}_{\theta_3}.$$

A similar result holds for $W^u_{g,R,n}(m)$.

The last ingredient for the proof we will need is the following proposition.

Proposition 4.8. *For N large and $\mathcal{U}_N$ small enough, if $g \in \mathcal{U}_N$ then for any ergodic u-Gibbs measure μ for g and for any $k \in \mathbb{N}$, the following property holds: for μ-almost every point $m \in \mathbb{T}^4$, its center leaf $W^c_g(m)$ has dense g^k-orbit in $\mathbb{T}^4$.*

The proof of this proposition is essentially the same as the proof of Proposition 5.9 in [Ob18-2]. For the sake of completeness we will include it here.

Proof. For $\mathcal{U}_N$ small enough, for every $g \in \mathcal{U}_N$ there is a homeomorphism $h_g : \mathbb{T}^4 \to \mathbb{T}^4$, that takes center leaves of f_N to center leaves of g, such that for every $m \in \mathbb{T}^4$ it is verified

$$g \circ h_g(W^c_f(m)) = h_g \circ f(W^c_f(m))$$

Consider the quotients $M_f = \mathbb{T}^4/\sim^c_f$ and $M_g = \mathbb{T}^4/\sim^c_g$, where $p \sim^c_* q$ if and only if $q \in W^c_*(p)$ for $* = f, g$. We denote $\pi_f : \mathbb{T}^4 \to M_f$ and $\pi_g : \mathbb{T}^4 \to M_g$ the respective projections. Observe that $M_f = \mathbb{T}^2$ and that the induced dynamics $\tilde{f} : M_f \to M_f$ of f is given by A^{2N}. Endow M_g with the distance d_g given by the Hausdorff distance on the center leaves, that is,

$$d_g(L, W) = d_{\text{Haus}}(\pi_g^{-1}(L), \pi_g^{-1}(W)).$$

By the leaf conjugacy equation, the induced dynamics $\tilde{g} : M_g \to M_g$ of g is conjugated to the linear Anosov A^{2N} on $\mathbb{T}^2$ by the homeomorphism induced by h_g, which we will denote by $\tilde{h}_g$. Denote by $W^{uu}_{A^{2N}}(.)$ the stable manifold of A^{2N} on $\mathbb{T}^2$ and let

$$W^{uu}_{\tilde{g}}(L) = \{W \in M_g : \lim_{n \to +\infty} d_g(\tilde{g}^{-n}(L), \tilde{g}^{-n}(W)) = 0\},$$

be the stable set of L.

Claim 3 (Claim 2 in the proof of Proposition 5.9 from [Ob18-2])**.** *For every $m \in \mathbb{T}^4$, for every $q \in W^c_g(m)$, it is verified that*

$$\pi_g(W^{uu}_g(q)) = W^{uu}_{\tilde{g}}(\pi_g(m)) = \tilde{h}_g(W^{uu}_{A^{2N}}(\pi_f(h_g^{-1}(m)))),$$

and π_g is a bijection from $W^{uu}_g(q)$ to $W^{uu}_{\tilde{g}}(\pi_g(m))$.

For the linear Anosov A^{2N} the unstable foliation is minimal, that is, every unstable manifold of A^{2N} is dense in $\mathbb{T}^2$. Let μ be an ergodic u-Gibbs measure for g and fix m a generic point for μ. Using the minimality of the unstable foliation of the linear Anosov and by the leaf conjugacy $W^{uu}_{\tilde{g}}(\pi_g(m))$ is dense in M_g.

Take U a small open set in M_g. Since the center foliation is uniformly compact, $\hat{U} = \pi_g^{-1}(U)$ is a saturated open set such that any two center leaves in $\hat{U}$ are close to each other. By the previous claim $W^{uu}_g(m) \cap \hat{U} \neq \emptyset$.

Since μ is an u-Gibbs measure, we have that $W^{uu}_g(m)$ is contained in the support of μ. Hence, $\text{supp}(\mu) \cap \hat{U} \neq \emptyset$. In particular, $\mu(\hat{U}) > 0$. Recall that m is a generic point for μ, therefore, its future orbit visits $\hat{U}$ infinitely many times. This is true for any open set U inside M_g, which concludes the proof of the proposition for $k = 1$.

For $k \in \mathbb{N}$, we remark that an unstable leaf for A^{2N} is an unstable leaf for A^{2Nk}, in particular, the unstable foliation of A^{2Nk} is minimal. The map g^k is leaf conjugated to A^{2Nk}. The same argument as above concludes the proof of the proposition. □

Proof of Theorem 4.1. Let N be large and $\mathcal{U}_N$ be small enough such that lemmas 4.6, 4.7 and proposition 4.8 hold. Fix $g \in \mathcal{U}_N$ and μ_1, μ_2 be two ergodic u-Gibbs measures for g.

Recall that we defined the set X_g in (83) and let Λ_{μ_i} be the set of typical points that we defined before for the measures μ_i, for $i = 1, 2$. Since $\mu_i(X_g) > 0$ and $\mu_i(\Lambda_{\mu_i}) = 1$, the set $X_i = X_g \cap \Lambda_i$ has positive μ_i measure as well, for $i = 1, 2$.

For any two points $m_1 \in X_1$ and $m_2 \in X_2$, we will prove that the stable manifold of m_1 has a transverse intersection with the unstable manifold of m_2. Fix a center leaf $W^c_g(q)$, the center leaf of some point $q \in \mathbb{T}^4$. By proposition 4.8 and remark **??**, the forward and past

iterates of $W^c_g(m_i)$ are dense in $\mathbb{T}^4$, for $i = 1, 2$. Hence, there are two sequences $n_k \to +\infty$ and $l_j \to +\infty$, such that

$$g^{n_k}(W^c_g(m_1)) \to W^c_g(q) \text{ and } g^{-l_j}(W^c_g(m_2)) \to W^c(q).$$

By lemma 4.6, there are curves $\gamma^+_{g,n_k}(m_1)$ and $\gamma^-_{g,-l_j}(m_2)$ with length bigger that 4π and contained in the cone $\mathscr{C}^{hor}_{\theta_2}$ and $\mathscr{C}^{ver}_{\theta_2}$, respectively. Take R given by lemma 4.7 and consider the sets

$$L^u_k(m_1) = \bigcup_{z \in \gamma^+_{g,n_k}(m_1)} W^{uu}_{g,R}(z) \subset W^u(g^{n_k}(m_1))$$
$$L^s_j(m_2) = \bigcup_{z \in \gamma^-_{g,-l_j}(m_2)} W^{ss}_{g,R}(z) \subset W^s(g^{-l_j}(m_2)).$$

For k and j large enough, $g^{n_k}(W^c_g(m_1))$ and $g^{-l_j}(W^c_g(m_2))$ are very close to the leaf $W^c_g(q)$. Thus by the control on the angles that we obtained in lemma 4.7, there is a transverse intersection between $L^u_k(m_1)$ and $L^s_j(m_2)$. In particular, $W^u_g(g^{n_k}(m_1))$ and $W^s_g(g^{-l_j}(m_2))$ intersect transversely. Since transverse intersections are invariant by iterates, we conclude that $W^u_g(m_1)$ and $W^s_g(m_2)$ have a transverse intersection.

Repeating this argument, exchanging the roles of m_1 and m_2, implies that $W^u_g(m_2)$ and $W^s_g(m_1)$ have a transverse intersection. Since the set X_i has positive μ_i measure, for $i = 1, 2$, we conclude that μ_1 is homoclinically related to μ_2. This finishes the proof of Theorem 4.1 for g, in the case $k = 1$.

Let $k \in \mathbb{N}$. Following the same steps as above, it is easy to prove that any two ergodic u-Gibbs measures for g^k, μ_1 and μ_2, are homoclinically related. □

4.2. Proof of proposition 4.2. We will need a few results from [CO19].

Lemma 4.9 ([CO19], Lemma 3.2)**.** *There exists a constant $R > 0$ with the following property: for N sufficiently large, there exists a C^1-neighborhood $\mathcal{U}$ of f_N such that for any $g \in \mathcal{U}_N$ and any two points $p, q \in \mathbb{T}^4$ we have that for any $m_p \in W^c_g(p)$ there exists $m_q \in W^c_g(q)$ such that $W^{uu}_{g,R}(m_p) \cap W^{ss}_{g,R}(m_q) \neq \emptyset$.*

Fix $\theta = N^{-\frac{3}{5}}$ and recall that in subsection 4.1, we defined the vertical cone $\mathscr{C}^{ver}_\theta$.

Lemma 4.10 ([CO19], Proposition 3.3)**.** *If N is sufficiently large there exists $\mathcal{U}_N \subset \mathrm{Diff}^2(\mathbb{T}^4)$ a C^1-neighborhood of f_N such that for any $g \in \mathcal{U}_N$ and any open set $U \subset \mathbb{T}^4$, there exists $n_s \geq 0$ such that for any $n \geq n_s$, there exists a C^1 curve $\gamma^-_n \subset g^{-n}(U)$ satisfying:*

- *γ^-_n is contained in a center leaf.*
- *$\pi_1(\gamma^-_n)$ is tangent to $\mathscr{C}^{ver}_\theta$.*
- *γ^-_n has length greater than 4π*
- *$\bigcup_{q \in \gamma^-_n} W^{ss}_{g,R}(q) \subset g^{-n}(U)$.*

Consider the vertical foliation $\mathcal{F}_{ver} = \{\{z\} \times \mathbb{T}^2 : z \in \mathbb{T}^2\}$. Observe that for any diffeomorphism g sufficiently C^1-close to f_N, we have that $W^c_g(m)$ intersects each vertical torus $\{z\} \times \mathbb{T}^2$ in exactly one point, for any $m \in \mathbb{T}^4$. Hence, for any two points $m_1, m_2 \in \mathbb{T}^4$, the map from $W^c_g(m_1)$ to $W^c_g(m_2)$ defined by $h^g_{m_1,m_2}(p) = W^c_g(m_2) \cap \mathcal{F}_{ver}(p)$ is well defined. Note that, after identifying all the horizontal tori with $\mathbb{T}^2$, the map $h^{f_N}_{m_1,m_2}$ is just the identity, independently of the points m_1, m_2.

Lemma 4.11 ([CO19], Lemma 3.4)**.** *For every $\varepsilon > 0$, there exists $N_0 := N_0(\varepsilon)$ with the following property: for $N \geq N_0$ there exists a C^1-neighborhood $\mathcal{U}_N$ of f_N such that if*

$g \in \mathcal{U}_N, p \in \mathbb{T}^4$ and $q \in W^{ss}_{g,R}(p)$ then $d_{C^0}(h^g_{p,q}, H^s_{p,q}) < \varepsilon$. Analogous result holds for the unstable holonomy.

Proof of proposition 4.2. Let N be large and $\mathcal{U}_N$ be small enough such that lemmas 4.6, 4.9, 4.10, and 4.11 hold. Let $g \in \mathcal{U}_N$ and suppose that μ is an SRB measure for g. Fix $U \subset \mathbb{T}^4$ an open set, we must prove that $\mathrm{supp}(\mu) \cap U \neq \emptyset$.

Since μ is SRB, its supports contains entire Pesin unstable manifolds. By lemma 4.6, we can take a μ-generic point m_u with the property that for n_u large enough there exists $\gamma^+_{g,n_u}(m) \subset g^{n_u}(W^+_{g,r_0}(m))$ a curve of length greater than 4π and whose projection by π_1 is tangent to $\mathscr{C}^{hor}_\theta$.

For n_s large enough, let $\gamma^-_{n_s}$ be the curve given by lemma 4.10 for U and g. As a consequence of lemmas 4.9, and 4.11, we conclude that

$$\left(\bigcup_{q \in \gamma^-_{n_s}} W^{ss}_{g,R}(q) \right) \cap \left(\bigcup_{p \in \gamma^+_{n_u}} W^{uu}_{g,R}(p) \right) \neq \emptyset. \tag{84}$$

We refer the reader to [CO19] for more details on this argument. By (84), we obtain that $g^{n_s+n_u}(W^u(m_u)) \cap U \neq \emptyset$, and since μ is SRB we conclude that $\mathrm{supp}(\mu) \cap U \neq \emptyset$. □

5. Rigidity of u-Gibbs measures

The main tool to study the existence of SRB measures that we will use is a recent result by Brown-Rodriguez Hertz on measure rigidity for random dynamics of surface diffeomorphisms. The goal of this section is to explain the statement of their result and how it can be applied to our scenario after a measurable change of coordinates using the unstable holonomies (see Theorem 5.3).

5.1. Measure rigidity for general skew products. Let $(\Omega, \mathcal{B}_\Omega, \nu)$ be a Polish probability space, that is, Ω has the topology of a complete separable metric space, $\mathcal{B}_\Omega$ is the Borel σ-algebra of Ω and ν is a Borel probability measure on Ω. Let $\theta : (\Omega, \mathcal{B}_\Omega, \nu) \to (\Omega, \mathcal{B}_\Omega, \nu)$ be an invertible, measure-preserving and ergodic transformation. Let S be a compact smooth surface and $\mathrm{Diff}^2(S)$ be the set of C^2-diffeomorphisms of S. We consider a measurable map that for each point $\xi \in \Omega$ associates a diffeomorphism $f_\xi \in \mathrm{Diff}^2(S)$. For each $n \in \mathbb{Z}$ we define

$$\begin{aligned} f^0_\xi &:= Id, \\ f^n_\xi &:= f_{\theta^{n-1}(\xi)} \circ \cdots \circ f_\xi \text{ for } n > 0, \\ f^n_\xi &:= (f_{\theta^n(\xi)})^{-1} \circ \cdots \circ (f_{\theta^{-1}(\xi)})^{-1} \text{ for } n < 0. \end{aligned}$$

We consider the skew product over θ given by the map $\xi \mapsto f_\xi$, which is defined by

$$\begin{array}{rccc} F : S \times \Omega & \longrightarrow & S \times \Omega \\ (x, \xi) & \mapsto & (f_\xi(x), \theta(\xi)). \end{array}$$

With the notation above, we may write $F^n(x, \xi) = (f^n_\xi(x), \theta^n(\xi))$. Write $X = S \times \Omega$ and let $\pi_2 : X \to \Omega$ be the natural projection on Ω.

Let μ be an F-ergodic probability measure, such that $(\pi_2)_* \mu = \nu$. Observe that the partition by the fibers S is measurable. Therefore, we have a family of conditional measures defined in a set D of full ν-measure $\{\mu_\xi\}_{\xi \in D}$ with respect to the partition induced by π_2. For ν-almost every ξ, the measure μ_ξ is supported on $S_\xi := S \times \{\xi\}$. There is a trivial identification of S_ξ with S, hence, by an abuse of notation we consider the map $\xi \mapsto \mu_\xi$ to be a ν-measurable map from Ω to the space of Borel probability measures of S.

To talk about SRB measures in this setting, we need to first talk about Lyapunov exponents and stable and unstable manifolds. Write $TX := TS \times \Omega$ and let $DF : TX \to TX$ to be the linear cocycle defined by

$$DF((x, v), \xi) = ((f_\xi(x), Df_\xi(x)v), \theta(\xi)).$$

Suppose that the following integrability condition holds

$$\int_\Omega \log^+(\|f_\xi\|_{C^2}) + \log^+(\|f_\xi^{-1}\|_{C^2}) d\nu(\xi) < \infty, \tag{85}$$

where $\log^+(.) = \max\{0, \log(.)\}$ and $\|f_\xi\|_{C^2}$ is the C^2-norm of f_ξ. Applying Oseledec's theorem for the linear cocycle DF, there is a μ-measurable decomposition $T_{(\xi,x)}X = \bigoplus_j E^j_{(x,\xi)}$ such that the space $E^j_{(x,\xi)}$ is the space corresponding to the Lyapunov exponent λ^j_μ, where $\{\lambda^j_\mu\}_j$ are the Lyapunov exponents of DF.

From now on, let us suppose that the measure μ is hyperbolic on the fibers, meaning, all the Lyapunov exponents are non zero. The integrability condition (85) is used to have Pesin's theory for fibered systems. In particular, for μ-almost every point there exists stable and unstable manifolds, which may possibly be just points in the case that all the exponents are negative or positive. We refer the reader to section 6 in [BRH17] for more details.

Suppose that μ has at least one positive Lyapunov exponent. The family of unstable manifolds $\{W^u(x,\xi)\}_{(x,\xi)\in X}$ forms a partition of a μ-full measure subset of X. Usually this partition is not measurable. In this context, we say that a measurable partition $\mathcal{P}$ is **u-subordinated** if for μ-almost every (x,ξ), there exists a positive number $r > 0$ such that $W^u_r(x,\xi) \subset \mathcal{P}(x,\xi) \subset W^u(x,\xi)$.

Definition 5.1 (Fiber-wise SRB)**.** *An F-invariant probability measure μ is fiber-wise SRB if for any u-subordinated measurable partition $\mathcal{P}$, for μ-almost every (x,ξ), the conditional measure $\mu^{\mathcal{P}}_{(x,\xi)}$ is absolutely continuous with respect to the riemannian volume on $W^u(x,\xi)$.*

Let $\mathcal{P}_\Omega$ be a measurable partition of Ω. We say that $\mathcal{P}_\Omega$ is **increasing** if for ν-almost every point ξ we have

$$\mathcal{P}_\Omega(\theta(\xi)) \subset \theta(\mathcal{P}_\Omega(\xi)).$$

Let $\hat{\mathcal{F}}(\mathcal{P}_\Omega) \subset \mathcal{B}_\Omega$ be the sub-σ-algebra generated by $\mathcal{P}_\Omega$. We say that $\hat{\mathcal{F}}(\mathcal{P}_\Omega)$ is an **increasing** sub-σ-algebra. We remark that in [BRH17], the authors call these partitions and sub-σ-algebra decreasing instead of increasing. We changed it here to be in harmony with the notion of increasing that we defined in section 2.

Let $\mathcal{F}(\mathcal{P}_\Omega)$ be the μ-completion of $\mathcal{B}_S \otimes \hat{\mathcal{F}}(\mathcal{P}_\Omega)$, where $\mathcal{B}_S$ is the Borel σ-algebra on S. For a hyperbolic measure μ, we may also look at the Oseledec's direction $E^s(x,\xi)$ as a measurable map of X that takes values on the projectivization of TX. We are now ready to state the main theorem in [BRH17].

Theorem 5.2 ([BRH17], Theorem 4.10)**.** *Let $F : X \to X$ be as above verifying the integrability condition (85), let $\mathcal{P}_\Omega$ be a measurable increasing partition of Ω and let μ be a hyperbolic F-invariant measure such that $(\pi_2)_*\mu = \nu$. Suppose that the family of conditional measures on the fibers $\{\mu_\xi\}$ are non-atomic almost surely. Furthermore, assume that*

(1) $\xi \mapsto f_\xi^{-1}$ is $\hat{\mathcal{F}}(\mathcal{P}_\Omega)$-measurable, and

(2) $\xi \mapsto \mu_\xi$ is $\hat{\mathcal{F}}(\mathcal{P}_\Omega)$-measurable.

Then either $(x,\xi) \mapsto E^s(x,\xi)$ is $\mathcal{F}(\mathcal{P}_\Omega)$-measurable of μ is fiber-wise SRB.

5.2. Change of coordinates. Fix $\alpha \in (0,1)$. In this section, we show how to use Theorem 5.2 to obtain the following theorem:

Theorem 5.3. *For N large enough, there exists $\mathcal{U}^{SP}_N$ a C^2-neighborhood of f_N in $SP^2(\mathbb{T}^2 \times \mathbb{T}^2)$ such that for $g \in \mathcal{U}^{SP}_N \cap \mathrm{Diff}^{2+\alpha}(\mathbb{T}^4)$, for any ergodic $\mu \in \mathrm{Gibbs}^u(g)$ one of the following holds:*

(1) for μ-almost every $p \in \mathbb{T}^4$ the measure μ^c_p is atomic;

(2) μ is SRB;

(3) for μ-almost every $p \in \mathbb{T}^4$, and for Lebesgue almost every point q in $W^{uu}_{loc}(p)$

$$E^-_{g,q} = DH^u_{p,q}(p)E^-_{g,p}.$$

To prove Theorem 5.3 we will define a measurable change of coordinates using the strong unstable holonomies, so that after this change of coordinates we are in the setting of Theorem 5.2.

Recall that $\lambda < 1$ is the rate of contraction of the linear Anosov A. Let N be large enough such that

$$(4N^2)^2 \left(\lambda^{2N}\right)^\alpha < 1.$$

In particular, if the C^2-neighborhood $\mathcal{U}^{SP}_N$ of f_N is sufficiently small, then for every $g \in \mathcal{U}^{SP}_N$ we have

$$\left(\frac{\|Dg|_{E^c_g}\|}{\|m(Dg|_{E^c_g})\|}\right)^2 (m(Dg|_{E^{uu}_g}))^{-\alpha} < 1. \tag{86}$$

Fix $g \in \mathcal{U}^{SP}_N \cap \mathrm{Diff}^{2+\alpha}(\mathbb{T}^4)$ and some $R > 1$. Condition (86) above is the $(2,\alpha)$-unstable center bunching condition defined in (71). By Theorem 2.10, for any $p \in \mathbb{T}^4$, $q \in W^{uu}_{g,R}(p)$ the unstable holonomy $H^u_{p,q} : W^c_g(p) \to W^c_g(q)$ is a C^2-diffeomorphism, whose C^2-norm varies continuously with the choices of p and q as above.

Since g is a partially hyperbolic skew product, we have that $g(p_1,p_2) = (g_{p_2}(p_1), g_2(p_2))$, where $g_2(p_2)$ is a $C^{2+\alpha}$-Anosov diffeomorphism of $\mathbb{T}^2$ which is topologically conjugated to A^{2N}. It is well known that a transitive $C^{1+\alpha}$-Anosov diffeomorphism has an unique ergodic u-Gibbs measure. Let ν be such a measure for g_2 on $\mathbb{T}^2$.

Fix $\mathcal{R} = \{R_1, \cdots, R_m\}$ a small Markov partition for A and observe that $\mathcal{R}$ is also a Markov partition for A^{2N} for every $N \in \mathbb{N}$. By taking N sufficiently large we may suppose that the transition matrix P_{2N} associated with $\mathcal{R}$ for A^{2N} verifies $(P_{2N})_{i,j} = 1$, for every $i,j = 1, \cdots m$. Let $\mathcal{R}_g$ be the image of $\mathcal{R}$ by the conjugacy map between A^{2N} and g_2. It is easy to see that $\mathcal{R}_g$ is a small Markov partition for g_2 and the conjugacy implies that it has the same transition matrix P_{2N}. Define $\Sigma := \{1, \cdots, m\}^{\mathbb{Z}}$ which is the shift space associated with $\mathcal{R}_g$ for g_2, let $\sigma : \Sigma \to \Sigma$ be the left shift map, and let $\Theta : \Sigma \to \mathbb{T}^2$ be the continuous surjection that defines the semi-conjugacy between σ and g_2.

Let us set some notations. Write $\Sigma^- = \{(\xi_i)_{i\leq 0} : \xi_i \in \{1, \cdots, m\}\}$ and $\Sigma^+ := \{(\xi_i)_{i>0} : \xi_i \in \{1, \cdots, m\}\}$. Let $\pi^- : \Sigma \to \Sigma^-$ and $\pi^+ : \Sigma \to \Sigma^+$ be the natural projections. For a point $\xi \in \Sigma$ we write $\xi^- := \pi^-(\xi)$ and $\xi^+ := \pi^+(\xi)$ and we use the notation $\xi = (\xi^-, \xi^+)$. The local unstable set of a point $\xi \in \Sigma$ is

$$\Sigma^u_{loc}(\xi) = \{\eta \in \Sigma : \eta^- = \xi^-\}.$$

Define $\nu_\sigma := \Theta_*\nu$, and observe that this is an ergodic, σ-invariant measure. The partition Σ^u_{loc} on local unstable sets forms a ν_σ-measurable partition of Σ. Let $\mathcal{P}^u$ be the ν-measurable u-subordinated partition given by the intersection of local unstable manifolds of g_2 with the rectangles from the Markov partition $\mathcal{R}_g$. Notice that $\mathcal{P}^u$ is equivalent (on a set of full ν-measure) to the partition Σ^u_{loc} (on a set of full ν_σ-measure).

It is easy to see that the partition Σ^u_{loc} is an increasing partition. Let $\mathcal{B}^u$ be the sub-σ-algebra generated by the partition on local unstable sets. This is an increasing sub-σ-algebra.

It is well known that Θ is bijective in a set of full ν_σ-measure, which we will denote by $\hat{D}$. We may further assume that $\hat{D}$ is σ-invariant. Let $D := \Theta(\hat{D})$ this is a g_2-invariant set of full ν-measure. Define $\Psi = Id \times \Theta^{-1}$, and notice that it is an isomorphism between $\mathbb{T}^2 \times D$ and $\mathbb{T}^2 \times \hat{D}$. Let $\pi'_2 : \mathbb{T}^2 \times \Sigma \to \Sigma$ be the natural projection on the second coordinate.

Let μ be an ergodic u-Gibbs measure for g. By lemma 2.24, $\nu = (\pi_2)_*\mu$. Consider the measure $\hat{\mu} := \Psi_*\mu$, and observe that it verifies $(\pi'_2)_*\hat{\mu} = \nu_\sigma$. We define the skew product on $\mathbb{T}^2 \times \hat{D}$ by $\hat{g}(x,\xi) = (\hat{g}_\xi(x), \sigma(\xi))$, where $\hat{g}_\xi := g_{\Theta(\xi)}$. We may extend $\hat{g}$ to $\mathbb{T}^2 \times \Sigma$ by setting $\hat{g}_\xi = Id$, for $\xi \notin \hat{D}$. Observe that (g,μ) is isomorphic (or measurably conjugated) to $(\hat{g},\hat{\mu})$ by the isomorphism Ψ. Since Ψ is just the identity in the first coordinate, it is

$$\begin{array}{ccc}
(\mathbb{T}^2\times\mathbb{T}^2,\mu) & \xrightarrow{g} & (\mathbb{T}^2\times\mathbb{T}^2,\mu)\\
\downarrow\Psi & & \downarrow\Psi\\
(\mathbb{T}^2\times\Sigma,\hat{\mu}) & \xrightarrow{\hat{g}} & (\mathbb{T}^2\times\Sigma,\hat{\mu})\\
\downarrow\Phi & & \downarrow\Phi\\
(\mathbb{T}^2\times\Sigma,\tilde{\mu}) & \xrightarrow{\tilde{g}} & (\mathbb{T}^2\times\Sigma,\tilde{\mu})
\end{array}$$

FIGURE 1. Changes of coordinates

immediate that the center Lyapunov exponents of μ are the same as the fiber Lyapunov exponents of $\hat{\mu}_\sigma$. Furthermore, μ is SRB if and only if $\hat{\mu}$ is fiber-wise SRB.

We now introduce a change of coordinate in the fibers for the skew product $\hat{g}$ in a way that the new skew product will verify the conditions to apply Theorem 5.2.

Fix $\eta^+\in\Sigma^+$ and define the function $\phi:\Sigma\to\Sigma$ by $\phi(\xi)=(\xi^-,\eta^+)$ for every $\xi\in\Sigma$. Observe that for each $\xi\in\Sigma$, $\phi(\xi)\in\Sigma^u_{loc}(\xi)$. In particular ϕ is $\mathcal{B}^u$-measurable.

For each $\xi\in\hat{D}$, since $\Theta(\xi)$ and $\Theta(\phi(\xi))$ belongs to the same local unstable manifold for g_2, we define

$$\begin{array}{rccc}
\Phi_\xi:\mathbb{T}^2 & \longrightarrow & \mathbb{T}^2\\
x & \mapsto & H^u_{\Theta(\xi),\Theta(\phi(\xi))}(x).
\end{array}$$

To simplify our notation, we write $H^u_{\xi,\phi(\xi)}:=H^u_{\Theta(\xi),\Theta(\phi(\xi))}$. We also define $\Phi:\mathbb{T}^2\times D\to\mathbb{T}^2\times D$ by $\Phi(x,\xi)=(\Phi_\xi(x),\xi)$. We can extend the definition of Φ to $\mathbb{T}^2\times\Sigma$ by setting $\Phi_\xi=Id$ for $\xi\notin\hat{D}$. We consider a skew product $\tilde{g}$ on $\mathbb{T}^2\times\Sigma$ defined by

$$\tilde{g}=\Phi\circ\hat{g}\circ\Phi^{-1}. \tag{87}$$

Consider the ergodic $\tilde{g}$-invariant measure $\tilde{\mu}=\Phi_*\hat{\mu}$ and observe that $(\pi'_2)_*\tilde{\mu}=\nu_\sigma$. The partition on the fibers $\mathbb{T}^2$ forms a measurable partition of $\mathbb{T}^2\times\Sigma$. Let $\{\tilde{\mu}_\xi\}_{\xi\in\Sigma}$ be the family of conditional measures with respect to the fibers. Figure 1 represents all these changes of coordinates that are conjugacies on subsets of full measure.

Lemma 5.4. *The maps $\xi\mapsto\tilde{g}_\xi^{-1}$ and $\xi\mapsto\tilde{\mu}_\xi$ are $\mathcal{B}^u$-measurable.*

PROOF. Recall that $\tilde{g}_\xi^{-1}=(\tilde{g}_{\sigma^{-1}(\xi)})^{-1}$. Since the unstable holonomy commutes with g, and by the definition of $\hat{g}$, in what follows we will use that $H^u_{\xi,\eta}\circ\hat{g}_{\sigma^{-1}(\xi)}=\hat{g}_{\sigma^{-1}(\eta)}\circ H^u_{\sigma^{-1}(\xi),\sigma^{-1}(\eta)}$. By (87), we have

$$\begin{array}{rcl}
\tilde{g}_{\sigma^{-1}(\xi)}(x) & = & H^u_{\xi,\phi(\xi)}\circ\hat{g}_{\sigma^{-1}(\xi)}\circ H^u_{\phi(\sigma^{-1}(\xi)),\sigma^{-1}(\xi)}(x)\\
& = & H^u_{\xi,\phi(\xi)}\circ H^u_{\sigma(\phi(\sigma^{-1}(\xi))),\xi}\circ\hat{g}_{\phi(\sigma^{-1}(\xi))}(x)=H^u_{\sigma(\phi(\sigma^{-1}(\xi))),\phi(\xi)}\circ\hat{g}_{\phi(\sigma^{-1}(\xi))}(x).
\end{array}$$

Notice that $\phi(\xi)$ and $\phi(\sigma^{-1}(\xi))$ depend only on ξ^-, in particular $\tilde{g}_{\sigma^{-1}(\xi)}$ depends only on ξ^-. If $\eta\in\Sigma^u_{loc}(\xi)$, which means that $\eta^-=\xi^-$, then $\tilde{g}_{\sigma^{-1}(\xi)}=\tilde{g}_{\sigma^{-1}(\eta)}$ and hence the map $\xi\mapsto\tilde{g}_\xi^{-1}$ is constant on local unstable sets and it is $\mathcal{B}^u$-measurable.

Since μ is an u-Gibbs measure, and it projects to ν, corollary 2.25 implies that for ν-almost every p_2, and for Lebesgue almost every $q_2\in W^{uu}_{g_2}(p_2)$ (for the riemannian volume of $W^{uu}_{g_2}(p_2)$), we have

$$\mu^c_{q_2}=(H^u_{p_2,q_2})_*\mu^c_{p_2}. \tag{88}$$

At first, the disintegration $\mu^c_{q_2}$ is defined for almost every point inside the unstable manifold of p_2. However, using (88), for any $q_2\in W^{uu}_{g_2}(p_2)$, we may consider the measure $\mu_{q_2}=(H^u_{p_2,q_2})_*\mu_{p_2}$. This defines a new disintegration that coincides with the original one in μ-almost every point with the advantage that for ν-almost every point the disintegration is defined along entire unstable manifolds.

Since Ψ is the identity on the fibers and a conjugation with the shift on the basis, for ν_σ-almost every ξ we obtain $\mu_{\Theta(\xi)}=\hat{\mu}_\xi$. Let us see the equivalent of property (88) for $\hat{\mu}$.

Consider the disintegration of ν_σ on the measurable partition Σ^u_{loc}. For ν_σ-almost every ξ, let ν^ξ_σ be the conditional measure on $\Sigma^u_{loc}(\xi)$. Hence, for ν^ξ_σ-almost every η, we have that $\hat{\mu}_\eta = (H^u_{\xi,\eta})_*\hat{\mu}_\xi$.

In an analogous way as we did for μ, we define the measure μ_η for every η in the local unstable set of ξ and this defines a new disintegration that coincides with the original disintegration on a set of full measure. By an abuse of notation we will use the notation $\hat{\mu}_\xi$ for the conditional measure of this new disintegration. We remark that this disintegration has the advantage of being defined along entire local unstable sets.

By the definition of Φ we see that for ν_σ-almost every ξ and for any $\eta \in \Sigma^u_{loc}(\xi)$ the measure $\tilde{\mu}_\eta = (H^u_{\eta,\phi(\xi)})_*\hat{\mu}_\eta = \hat{\mu}_{\phi(\xi)}$. In particular, the map $\xi \mapsto \tilde{\mu}_\xi$ is constant on local unstable sets and it is $\mathcal{B}^u$-measurable. □

Proof of Theorem 5.3. First, let us explain how the skew product $\tilde{g}$ verifies the hypothesis of Theorem 5.2. Since Σ^u_{loc} is a decreasing partition, we have that $\mathcal{B}^u$ is a decreasing sub-σ-algebra. Let $\mathcal{B}^*$ be the $\tilde{\mu}$-completion of $\mathcal{B}_{\mathbb{T}^2} \otimes \mathcal{B}^u$, where $\mathcal{B}_{\mathbb{T}^2}$ is the Borel σ-algebra on $\mathbb{T}^2$. Recall that

$$\tilde{g}_\xi = H^u_{\sigma(\phi(\xi)),\phi(\sigma(\xi))} \circ \hat{g}_{\phi(\xi)}.$$

We claim that there exists a constant $R > 0$ such that for any $\xi \in \Sigma$, we have $\Theta(\sigma(\phi(\xi))) \in W^{uu}_{g_2,R}(\Theta(\phi(\sigma(\xi))))$. Indeed, recall that we had fixed $\mathcal{R}_g = \{R_{g,1}, \cdots, R_{g,m}\}$ a small Markov partition for g_2. Since $\phi(\xi) \in \Sigma^u_{loc}(\xi)$, we obtain that $\Theta(\phi(\xi))$ and $\Theta(\xi)$ belongs to the same local unstable manifold intersected with some rectangle $R_{g,i}$. Since the expansion rate of unstable manifolds for g_2 is close to λ^{-2N}, which is a constant, there exists $R_1 > 0$ that verifies $\Theta(\sigma(\phi(\xi))) \in W^{uu}_{g_2,R_1}(\Theta(\sigma(\xi)))$, for any $\xi \in \Sigma$. To conclude, we observe that $\Theta(\phi(\sigma(\xi))) \in W^{uu}_{g_2,loc}(\Theta(\sigma(\xi)))$. Hence, by fixing R sufficiently large we conclude our claim.

Since g is $C^{2+\alpha}$, Theorem 2.10 in the appendix implies that for every $\xi \in \Sigma$, the holonomy $H^u_{\sigma(\phi(\xi)),\phi(\sigma(\xi))}$ is a C^2-diffeomorphism of $\mathbb{T}^2$ with uniformly bounded C^2-norm. Since $\hat{g}_\xi = g_{\Theta(\xi)}$, we also have that all the C^2-diffeomophisms $\hat{g}_\xi$ belong to a compact subset of $\mathrm{Diff}^2(\mathbb{T}^2)$. We conclude that for every ξ, the C^2-norm of $\tilde{g}_\xi$ is uniformly bounded. Similar conclusion holds for $\tilde{g}^{-1}_\xi$. In particular, the skew product $\tilde{g}$ verifies the integrability condition (85).

It is easy to see that the fiber-wise Lyapunov exponents of $(\tilde{g}, \tilde{\mu})$ are the same as the center Lyapunov exponents of (g, μ). In particular, $\tilde{\mu}$ is a hyperbolic measure with a positive and a negative fiber-wise Lyapunov exponent.

Lemma 5.4 states that $(\tilde{g}, \tilde{\mu})$ verifies the conditions (1) and (2) in the hypothesis of Theorem 5.2. Since the skew products $\tilde{g}$ fibers over the system (σ, ν_σ), which is ergodic, we conclude that either

(1) the measure $\tilde{\mu}_\xi$ is atomic for ν_σ-almost every ξ;

(2) $\tilde{\mu}$ is fiber-wise SRB;

(3) the stable distribution $(x, \xi) \mapsto E^-_{\tilde{g}}(x, \xi)$ is $\mathcal{B}^*$-measurable.

Notice that the composition $(\Phi \circ \Psi)$ takes fibers of $\mathbb{T}^2 \times \mathbb{T}^2$ into fibers of $\mathbb{T}^2 \times \Sigma$. Furthermore, it acts as a C^2-diffeomorphism on each fiber. Observe also that it measurably conjugates the dynamics of g and $\tilde{g}$ on a set of full μ-measure. In particular, for ν-almost every $p_2 \in \mathbb{T}^2$ we have

$$\mu^c_{p_2} = (\Phi \circ \Psi)^{-1}_* \tilde{\mu}_{\Theta^{-1}(p_2)}. \tag{89}$$

From (89) above, $\tilde{\mu}_\xi$ is atomic if and only if $\mu_{\Theta(\xi)}$ is atomic, for ν_σ-almost every ξ.

Since μ is an u-Gibbs measure, it will be an SRB measure if and only if it is fiber-wise SRB in the sense of definition 5.1. From (89), we conclude that $\tilde{\mu}$ is fiber-wise SRB for $\tilde{g}$ if and only if μ is fiber-wise SRB for g.

For the map $(x, \xi) \mapsto E^-_{\tilde{g}}(x, \xi)$ to be $\mathcal{B}^*$-measurable, it is equivalent to the following: for $\tilde{\mu}$-almost every (x, ξ) and for ν^ξ_σ-almost every $\eta \in \Sigma^u_{loc}(\xi)$, we have that $E^-_{\tilde{g}}(x, \xi) =$

$E^-_{\tilde{g}}(x,\eta)$. Observe that the points (x,ξ) and (x,η) belong to the same local unstable set for $\tilde{g}$. By the conjugacy $(\Psi\circ\Phi)$, we conclude that

$$E^-_{\tilde{g}}(x,\xi) = DH^u_{\Theta(\xi),\Theta(\phi(\xi))}(x)E^-_{g,(x,\Theta(\xi))}.$$

Since the measure is u-Gibbs, the third condition above is equivalent to for μ-almost every $p\in\mathbb{T}^4$, for Lebesgue almost every point $q\in W^{uu}_{loc}(p)$, we have $E^-_{g,q} = DH^u_{p,q}(p)E^-_{g,p}$.

All these conclusions hold for any $g\in \mathrm{Diff}^{2+\alpha}(\mathbb{T}^2)$ sufficiently C^2-close to f_N. This concludes the proof. □

6. The non invariance of stable directions by u-holonomies

In this section we fix N large and $\mathcal{U}_N$ small enough such that Theorem 3.1 holds for some small fixed $\delta>0$. In particular, if $g\in\mathcal{U}_N$ then any u-Gibbs measure for g has both a positive and a negative center Lyapunov exponent for μ almost every point. Since μ has absolutely continuous disintegration with respect to strong unstable manifolds, for μ-almost every point p, Lesbesgue almost every point $q\in W^{uu}_g(p)$ has a well defined Oseledec's stable and unstable directions in the center, where the Lebesgue measure we are considering is the measure restricted to the strong unstable manifold $W^{uu}_g(p)$.

Recall that for any $p\in\mathbb{T}^4$ and any $q\in W^{uu}_g(p)$, there is a well defined unstable holonomy map $H^u_{p,q}: W^c_g(p)\to W^c_g(q)$. Furthermore, this map is a C^1-diffeomorphism. The main result in this section is the following:

Proposition 6.1. *Let $g\in\mathcal{U}_N$ and let μ be an u-Gibbs measure for g. For any $\varepsilon>0$, the following property holds: for μ-almost every p, there exists a set D^u contained in $W^{uu}_{g,\varepsilon}(p)$ with positive Lebesgue measure (for the riemannian volume of $W^{uu}_{g,\varepsilon}(p)$) such that for any $q\in D^u$ it is verified that*

$$DH^u_{p,q}(p)E^-_{g,p}\neq E^-_{g,q}.$$

The rest of this section is dedicated to prove proposition 6.1.

Let $g\in\mathcal{U}_N$, for any $p\in\mathbb{T}^4$, for any piece of strong unstable manifold γ^u_p containing p and any unit vector $v\in E^c_{g,p}$, we define a unitary vector field over γ^u_p defined as follows: for any $q\in\gamma^u_p$ we write

$$PH^u_{p,q}(p)v = \frac{DH^u_{p,q}(p)v}{\|DH^u_{p,q}(p)v\|}, \tag{90}$$

and define $v'_q := PH^u_{p,q}(p)v$. First we study the regularity of the vector field v'.

Lemma 6.2. *Let $g\in\mathcal{U}_N$. There exists a constant $C>0$ that verifies the following: for any $p\in\mathbb{T}^4$, let $\gamma^u_p := W^{uu}_{g,1}(p)$ be the strong unstable manifold of size 1, for any unit vector $v\in E^c_{g,p}$, the vector field v^u defined above is $(C,\frac{1}{2})$-Hölder.*

Proof. Observe that, for N large enough, we have

$$\left(\lambda^{2N}\right)^{\frac{1}{2}} < (4N^2)^{-1} \text{ and } \left(\lambda^{2N}\right)^{\frac{1}{2}} < (2N)^{-1}.$$

This means that f_N verifies the conditions (64) and (65) from Theorem 2.8, for $\theta=\frac{1}{2}$. In particular, any g sufficiently C^1-close to f_N also verifies (64) and (65). Lemma 6.2 then follows from the conclusion (66), for unstable holonomies, of Theorem 2.8. □

Next, we will see how the center bunching condition "smoothes" a center vector field over a piece of strong unstable manifold. This is a crucial point for us, so that it will allow us to apply some of the techniques and estimates from section 3 to prove proposition 6.1.

Lemma 6.3. *Let $g\in\mathcal{U}_N$. For any piece of strong unstable curve γ^u and any X unitary vector field over γ^u tangent to E^c_g which is $(C_0,\frac{1}{2})$-Hölder, for some $C_0:=C_0(X)>0$, the following holds: there exists $n_0\in\mathbb{N}$, which depends only on C_0, such that for every $n\geq n_0$, the vector field $X_n := \frac{g^n_*(X)}{\|g^n_*(X)\|}$ over $g^n(\gamma^u)$ is $(C_n,\frac{1}{2})$-Hölder with $C_n<30N^2\lambda^N$.*

PROOF. The proof of this lemma is essentially contained in the proof of lemma 1 from [BC14]. However, we will repeat the main steps of the argument here. For simplicity we will prove the lemma for f_N, which we will denote by f. Using the estimates from lemma 3.4, one can adapt the calculations for any $g \in \mathcal{U}_N$.

Let us just review some estimates for f. Recall that

$$Df(x,y,z,w) = \begin{pmatrix} Ds_N(x,y) & P_x \circ A^N(z,w) \\ 0 & A^{2N}(z,w) \end{pmatrix}.$$

Hence, $\|Df(x,y,z,w)|_{E^c}\| = \|Ds_n(x,y)\| \leq 2N$. Since Ds_N is the only non linear term, $\|D^2 f\| = \|D^2 s_N\| \leq N$.

Let γ^u be a piece of a strong unstable manifold and X a $(C_0, \frac{1}{2})$-Hölder unitary vector field over γ^u. Let us estimate C_1, the Hölder constant of X_1 over $f(\gamma^u)$. First, for any $m, m' \in \gamma^u$, we have

$$\begin{aligned} &\|Df(m)X_m - Df(m')X_{m'}\| \leq \\ &\|Df(m)X_m - Df(m)X_{m'}\| + \|Df(m)X_{m'} - Df(m')X_{m'}\| = I + II. \end{aligned}$$

Since X is $(C_0, \frac{1}{2})$-Hölder, we obtain

$$I \leq 2N\|X_m - X_{m'}\| \leq 2NC_0 d(m,m')^{\frac{1}{2}}.$$

If $d(m,m') \leq 1$, then

$$II \leq Nd(m,m') < Nd(m,m')^{\frac{1}{2}} < 7Nd(m,m')^{\frac{1}{2}}.$$

Observe that $d(m,m') \leq 2\pi < 7$, for any two points $m, m' \in \mathbb{T}^4$. If $d(m,m') > 1$, then

$$II \leq Nd(m,m') \leq 7N < 7Nd(m,m')^{\frac{1}{2}}.$$

We conclude that

$$\|Df(m)X_m - Df(m')X_{m'}\| < (7N + 2NC_0)d(m,m')^{\frac{1}{2}}. \tag{91}$$

Also,

$$\begin{aligned} \|(X_1)_m - (X_1)_{m'}\| &= \frac{1}{\|f_*X_m\|\|f_*X_{m'}\|}\,\|\|f_*X_{m'}\|f_*X_m - \|f_*X_m\|f_*X_{m'}\| \\ &\leq \frac{1}{\|f_*X_m\|\|f_*X_{m'}\|}\,(\|\|f_*X_{m'}\|f_*X_m - \|f_*X_{m'}\|f_*X_{m'}\| \\ &\quad + \|\|f_*X_{m'}\|f_*X_{m'} - \|f_*X_m\|f_*X_{m'}\|) \\ &\leq \frac{2}{\|f_*X_m\|}\|f_*X_m - f_*X_{m'}\| \\ &= \frac{2}{\|f_*X_m\|}\|Df(f^{-1}(m))X_{f^{-1}(m)} - Df(f^{-1}(m'))X_{f^{-1}(m')}\|. \end{aligned}$$

Using (91) for the points $f^{-1}(m)$ and $f^{-1}(m')$, we have

$$\|Df(f^{-1}(m))X_{f^{-1}(m)} - Df(f^{-1}(m'))X_{f^{-1}(m')}\| \leq N\lambda^N(1+\lambda^N)(7+2C_0)d(m,m')^{\frac{1}{2}}.$$

Recall that $\|Df|_{E^c_f}\| \geq (2N)^{-1}$, hence

$$\|X_1(m) - X_1(m')\| \leq \frac{2N\lambda^N(1+\lambda^N)(7+2C_0)d(m,m')^{\frac{1}{2}}}{\|f_*X(m)\|} \leq 4(1+\lambda^N)N^2\lambda^N(7+2C_0)d(m,m')^{\frac{1}{2}}.$$

Observe that $4(1+\lambda^N)N^2\lambda^N(7+2C_0)$ estimates the Hölder constant of X_1. If $C_0 \leq \frac{1}{10}$, then for N large enough

$$4(1+\lambda^N)N^2\lambda^N(7+2C_0) \leq 4(1+\lambda^N)N^2\lambda^N(7.2) < 30N^2\lambda^N.$$

Hence, $C_1 < 30N^2\lambda^N$ and the same calculations imply that $C_n < 30N^2\lambda^N$, for every $n \geq 1$. Now suppose that $C_0 > \frac{1}{10}$. Then, for N large enough

$$\frac{4(1+\lambda^N)N^2\lambda^N(7+2C_0)}{C_0} = 4(1+\lambda^N)N^2\lambda^N\left(\frac{7}{C_0}+2\right) < 4(1+\lambda^N)N^2\lambda^N 72 < \frac{1}{2}.$$

This implies that $C_1 < \frac{C_0}{2}$. Therefore, there exists $\tilde{n} \in \mathbb{N}$ such that $C_{\tilde{n}} < \left(\frac{1}{2}\right)^{\tilde{n}} C_0 \leq \frac{1}{10}$. Take $n_0 = \tilde{n} + 1$. We conclude that for every $n \geq n_0$, $C_n < 30N^2\lambda^N$. □

Proof of proposition 6.1. If the conclusion of proposition 6.1 did not hold, there would exist a diffeomorphism $g \in \mathcal{U}_N$, an u-Gibbs measure μ and a measurable set D of positive μ-measure such that for any $p \in D$ and for Lebesgue almost every point $q \in W_g^{uu}(p)$ we would have $DH_{p,q}^u(E_{g,p}^-) = E_{g,q}^-$. Fix $p \in D$ and let $\gamma^u := W_{g,1}^{uu}(p)$. Consider v an unit vector on $E_{g,p}^-$ and let v' be the unit vector field over γ^u defined as in (90).

Let C be the constant given by lemma 6.2. Therefore, v' is a $(C, \frac{1}{2})$-Hölder vector field over γ^u. Let $n_0 \in \mathbb{N}$ be given by lemma 6.3. Hence, for $n \geq n_0$, the vector field $v'_n := \frac{g_*^n(v')}{\|g_*^n(v')\|}$ is $(C_n, \frac{1}{2})$-Hölder over $\gamma_n^u := g^n(\gamma^u)$, with $C_n < 30N^2\lambda^N$.

Suppose that n_0 is large enough such that $l(\gamma_{n0}^u) > 2\pi$. Hence, we may consider a C^1-curve $\tilde{\gamma} : [0, 2\pi] \to \mathbb{T}^4$ such that $\tilde{\gamma} = (\tilde{\gamma}_x, \tilde{\gamma}_y, \tilde{\gamma}_z, \tilde{\gamma}_w)$ with $\left|\frac{d\tilde{\gamma}_x}{dt}\right| = 1$, $\tilde{\gamma}([0, 2\pi]) \subset \gamma_{n_0}^u$, and define $\tilde{v} = v_{n_0}^u$. Following definition 3.9, the pair $(\tilde{\gamma}, \tilde{v})$ is an adapted field.

Recall that $\delta > 0$ is fixed and in section 3, on the proof of Theorem 3.1, we fixed $\tilde{\delta} = \frac{2\delta}{15}$. For each $k \geq 0$, we write $\tilde{v}_k = v'_{n_0+k}$ and recall that there exists $N_k \in \mathbb{N}$ such that

$$g^k \circ \tilde{\gamma} = \tilde{\gamma}_1^k * \cdots * \tilde{\gamma}_{N_k}^k * \tilde{\gamma}_{N_k+1}^k,$$

where $\tilde{\gamma}_j^k$ is an u-curve for $j = 1, \cdots, N_k$ and $\tilde{\gamma}_{N_k+1}^k$ is a segment of a u-curve. By lemma 3.13, every pair $(\tilde{\gamma}_j^k, \tilde{v}_k|_{\tilde{\gamma}_j^k})$ is an adapted field for $j = 1, \cdots, N_k$.

Recall that in section 3, we had defined the notion of $\tilde{\delta}$-good adapted field (see definition 3.12). We will need the following lemma.

Lemma 6.4 ([Ob18-2], Lemma 7.27)**.** *Let $g \in \mathcal{U}_N$, and let (γ, X) be a $\tilde{\delta}$-bad adapted field. Then there exists a strip S of length π such that for every j satisfying $g^{-1}\gamma_j^1 \subset S$, the field $(\gamma_1^j, \frac{g_*X}{\|g_*X\|})$ is $\tilde{\delta}$-good.*

Let $(\hat{\gamma}, \hat{v})$ be a $\tilde{\delta}$-good adapted field defined as follows: if $(\tilde{\gamma}, \tilde{v})$ is a $\tilde{\delta}$-good adapted field then $(\hat{\gamma}, \hat{v}) = (\tilde{\gamma}, \tilde{v})$. Otherwise, by the previous lemma, we may choose $j \in \{1, \cdots N_k\}$ such that $(\tilde{\gamma}_j^1, \tilde{v}_1|_{\tilde{\gamma}_j^1})$ is a $\tilde{\delta}$-good adapted field. In this case, we define $(\hat{\gamma}, \hat{v}) = (\tilde{\gamma}_j^1, \tilde{v}_1|_{\tilde{\gamma}_j^1})$.

Let $K \in \{n_0, n_0+1\}$ be such that $g^{-K}(\hat{\gamma}) \subset \gamma^u$ and write $\hat{\gamma}_{-K} := g^{-K}(\hat{\gamma})$. Recall that we had defined $J_{g^k}^{uu}(.) = |\det Dg^k(.)|_{E_g^{uu}}|$. For any $n \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{|\hat{\gamma}|}\int_{\hat{\gamma}_{-K}} \log\|Dg^{K+n}v'\| d\hat{\gamma}_{-K} &= \frac{1}{|\hat{\gamma}|}\left(\sum_{i=0}^{K-1}\int_{g^i\circ\hat{\gamma}_{-K}} \log\|Dgv'_i\| J_{g^{-i}}^{uu} d(g\circ\hat{\gamma}_{-K})\right.\\ &\quad \left. + \int_{\hat{\gamma}} \log\|Dg^n\hat{v}\| J_{g^{-K}}^{uu} d\hat{\gamma}\right)\\ &= M_K + \frac{1}{|\hat{\gamma}|}\int_{\hat{\gamma}} \log\|Dg^n\hat{v}\| J_{g^{-K}}^{uu} d\hat{\gamma} = M_k + I_n^{\hat{\gamma},\hat{v}},\end{aligned}$$

where M_K does not depend on n. Since $(\hat{\gamma}, \hat{v})$ is a $\tilde{\delta}$-good curve, by (82) in section 3, for n large enough we have

$$\frac{I_n^{\hat{\gamma},\hat{v}}}{n} \geq (1 - 14\tilde{\delta}) \log N.$$

Therefore,

$$\limsup_{n\to+\infty} \frac{1}{|\hat{\gamma}|}\int_{\hat{\gamma}_{-K}} \frac{\log\|Dg^{K+n}v^u\|}{n} d\hat{\gamma}_{-K} = \limsup_{n\to+\infty} \frac{M_K}{n} + \frac{I_n^{\hat{\gamma},\hat{v}}}{n} \geq (1 - 14\tilde{\delta}) \log N > 0. \quad (92)$$

However, by assumption, for Lebesgue almost every $q \in W_g^{uu}(p)$ the vector v_q^u belongs to $E_{g,q}^-$. In particular, there exists a number $\lambda^- < 0$ such that for Lebesgue almost every

$q \in W_g^{uu}(p)$

$$\lim_{n\to+\infty} \frac{\log \|Dg^{K+n}(q)v_q^u\|}{n} = \lambda^- < 0. \tag{93}$$

By (93) and applying the dominated convergence theorem, we obtain

$$\begin{aligned}\limsup_{n\to+\infty} \frac{1}{|\hat{\gamma}|}\int_{\hat{\gamma}_{-K}} \frac{\log \|Dg^{K+n}v^u\|}{n} d\hat{\gamma}_{-K} &= \frac{1}{|\hat{\gamma}|}\int_{\hat{\gamma}_{-K}} \limsup_{n\to+\infty} \frac{\log \|Dg^{K+n}v^u\|}{n} d\hat{\gamma}_{-K},\\ &= \frac{|\hat{\gamma}_{-K}|}{|\hat{\gamma}|}\lambda^- < 0.\end{aligned}$$

which is a contradiction with (92). □

7. Proof of Theorem K

In this section we conclude the proof of Theorem K. Let $\alpha \in (0,1)$ and take N large enough such that Theorem 5.3 holds and let $\mathcal{U}_N^{sp}$ be a small C^2-neighborhood of f_N is $SP^2(\mathbb{T}^2 \times \mathbb{T}^2)$. Take $g \in \mathcal{U}_N^{sp} \cap SP^{2+\alpha}(\mathbb{T}^4)$ and take an ergodic measure $\mu \in \text{Gibbs}^u(g)$. Recall that μ projects to the measure ν which is the unique u-Gibbs measure of g_2. By Theorem 5.3, there are three possibilities:

(1) for μ-almost every $p \in \mathbb{T}^4$, the measure μ_p^c is atomic;

(2) μ is SRB;

(3) for μ-almost every $p \in \mathbb{T}^4$, for Lebesgue almost every point q in $W_{loc}^{uu}(p)$, we have

$$E_{g,q}^- = DH_{p,q}^u(p)E_{g,p}^-.$$

Suppose that μ does not have atomic center disintegration. By Proposition 6.1, μ cannot verify item 3 above. Therefore, μ must be an SRB measure. The following lemma concludes the proof of Theorem K.

Lemma 7.1. *If μ has atomic disintegration along the center, then there exists $k \in \mathbb{N}$ such that for ν-almost every $p_2 \in \mathbb{T}^2$ the measure $\mu_{p_2}^c$ has k-atoms.*

Proof. We already know that the measure $\mu_{p_2}^c$ is atomic for ν-almost every $p_2 \in \mathbb{T}^2$. For each $n \in \mathbb{N}$ consider the set $B_n := \{p \in \mathbb{T}^4 : \mu_{\pi_2(p)}^c(\{p\}) > \frac{1}{n}\}$. It is easy to see that B_n is a g-invariant set for each $n \in \mathbb{N}$. For n sufficiently large $\mu(B_n) > 0$, and by ergodicity $\mu(B_n) = 1$. Hence, for n large enough, every atom of $\mu_{p_2}^c$ has measure larger than $\frac{1}{n}$, for ν-almost every $p_2 \in \mathbb{T}^2$, and therefore there are at most n atoms. Since the measure μ is f-invariant, for ν-almost every p_2 we have $\mu_{p_2}^c = (g^{-1})_*\mu_{g_2(p_2)}^c$. For each $l \in \mathbb{N}$, consider the set $F_l := \{p \in \mathbb{T}^4 : \mu_p^c \text{ has exactly } l\text{-atoms.}\}$. By the previous observation, F_l is an invariant set, and since for ν-almost every p_2 the measure $\mu_{p_2}^c$ has at most n-atoms, we find some $k \in \mathbb{N}$ such that $\mu(F_k) = 1$. □

8. Appendix: Regularity of unstable holonomies

In this appendix we prove Theorem 2.10. Let f be a $C^{2+\alpha}$ absolutely partially hyperbolic skew product of $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ and let $\chi^{ss}, \chi_-^c, \chi_+^c, \chi^{uu}$ be the partially hyperbolic constants of f. We say that f verifies the $(2,\alpha)$**-center unstable bunching condition** if

$$\left(\frac{\chi_+^c}{\chi_-^c}\right)^2 < \chi^{uu} \text{ and } \frac{\chi_+^c}{(\chi_-^c)^2} < (\chi^{uu})^\alpha. \tag{94}$$

Similarly, f verifies the $(2,\alpha)$**-center stable bunching condition** if

$$\chi^{ss} < \left(\frac{\chi_-^c}{\chi_+^c}\right)^2 \text{ and } (\chi^{ss})^\alpha < \frac{\chi_-^c}{(\chi_+^c)^2}. \tag{95}$$

If f verifies condition (94) and (95) then we say that f is $(2,\alpha)$**-center bunched**. In this section, for any point $p \in \mathbb{T}^4$ and any $n \in \mathbb{Z}$ we write $p_n := f^n(p)$.

In this appendix, we use the $(2,\alpha)$-center bunching condition to obtain C^2-regularity of the unstable holonomy inside a center unstable leaf. Recall that given p and q belonging to the same strong unstable leaf, then there exists a well defined strong unstable holonomy map $H^u_{p,q} : W^c(p) \to W^c(q)$. Since the center manifolds are $\mathbb{T}^2$, we have that each unstable holonomy is a diffeomorphism of $\mathbb{T}^2$. For each $R > 0$, we consider the family $\{H^u_{p,q}\}_{p\in\mathbb{T}^4, q\in W^{uu}_R(p)}$. The main theorem of the appendix is the following:

Theorem 8.1 (Theorem 2.10)**.** *Let f be a $C^{2+\alpha}$ absolutely partially hyperbolic skew product of $\mathbb{T}^4$, and fix $R > 0$. If f is $(2,\alpha)$-center unstable bunched, then $\{H^u_{p,q}\}_{p\in\mathbb{T}^4, q\in W^{uu}_R(p)}$ is a family of C^2-diffeomorphisms of $\mathbb{T}^2$ whose C^2-norm varies continuously with the choices of p and q.*

It is easy to see that this theorem follows from the case that $R = 1$. Observe that the $(2,\alpha)$-unstable center unstable bunching condition implies that

$$\frac{\chi^c_+}{\chi^c_-} < \chi^{uu}.$$

This condition is the regular bunching condition which is sufficient to prove that the unstable holonomy is a C^1-diffeomorphism. For each $n \in \mathbb{Z}$, for each $p \in \mathbb{T}^4$ and $q \in W^{uu}_1(p)$ we have

$$f^n \circ H^u_{p,q} = H^u_{p_n,q_n} \circ f^n \text{ and } Df^n(H^u_{p,q}(.))DH^u_{p,q}(.) = DH^u_{p_n,q_n}(f^n(.))Df^n(.). \tag{96}$$

Since the center leaves are $\mathbb{T}^2$, all its tangent spaces have a canonical identification with $\mathbb{R}^2$. In particular, we may consider $DH^u_{p,q}(.)$ to be a continuous map from $\mathbb{T}^2$ to $L(\mathbb{R}^2,\mathbb{R}^2)$, where $L(\mathbb{R}^2,\mathbb{R}^2)$ is the set of linear maps from R^2 to $\mathbb{R}^2$. Thus, the family $\{DH^u_{p,q}(.)\}_{p\in\mathbb{T}^4, q\in W^{uu}_1(p)}$ is a continuous family that takes value on $C^0(\mathbb{T}^2, L(\mathbb{R}^2,\mathbb{R}^2))$. Furthermore, there exists an uniform constant $C \geq 1$ such that

$$\|DH^u_{p,q}(.) - Id\| < Cd(p,q). \tag{97}$$

Fix some constant $K > C$ and let $\mathcal{L}$ be the set defined as follows: an element $\mathcal{L}$ is a of continuous family of maps $\{A_{p,q}\}_{p\in\mathbb{T}^4, q\in W^{uu}_1(p)}$ that takes value on $C^0(\mathbb{T}^2, L(\mathbb{R}^2,\mathbb{R}^2))$ such that $\|A_{p,q} - Id\| < Kd(p,q)$. For simplicity, we will denote a family $\{A_{p,q}\}_{p\in\mathbb{T}^4, q\in W^{uu}_1(p)}$ by $\mathcal{A}$, such that $\mathcal{A}_{p,q}(.) = A_{p,q}(.)$. We will also write the continuous family given the derivative of the unstable holonomy just by DH^u.

Observe that $\mathcal{L}$ has a natural distance defined by

$$d(\mathcal{A},\mathcal{B}) = \sup_{p\in\mathbb{T}^4, q\in W^{uu}_1(p)} \left\{ \sup_{x\in\mathbb{T}^2} \|\mathcal{A}_{p,q}(x) - \mathcal{B}_{p,q}(x)\| \right\}.$$

For each $n \in \mathbb{N}$ we define $\Gamma_n : \mathcal{L} \to \mathcal{L}$ in the following way: for each $p \in \mathbb{T}^4$ and $q \in W^{uu}_1(p)$, then

$$\Gamma_n(\mathcal{A})_{p,q}(.) = Df^n(H^u_{p_{-n},q_{-n}}(f^{-n}(.)))\mathcal{A}_{p_{-n},q_{-n}}(f^{-n}(.))Df^{-n}(.). \tag{98}$$

By (96), for any $n \in \mathbb{N}$ the derivative of the unstable holonomy DH^u is Γ_n-invariant, that is, $\Gamma_n(DH^u) = DH^u$. In the next lemma we prove that it is the only element of $\mathcal{L}$ that has this property.

Lemma 8.2. *For any $\mathcal{A} \in \mathcal{L}$, the limit $\lim_{n\to+\infty} \Gamma_n(\mathcal{A})$ exists and it is equal to DH^u. Moreover, DH^u is the only element of $\mathcal{L}$ which is Γ_n-invariant for every $n \in \mathbb{N}$.*

PROOF. Let $\mathcal{A} \in \mathcal{L}$. Fix $p \in \mathbb{T}^4$ and $q \in W_1^{uu}(p)$, and we will write $H^u_{-n}(.) = H^u_{p_{-n},q_{-n}}(.)$. We will use a similar notation for $\mathcal{A}_{p_{-n},q_{-n}}$. For any $x \in W^c(p)$, we have

$$\begin{aligned}\|\Gamma_n(\mathcal{A})_{p,q}(x) - DH^u_{p,q}(x)\| &= \|Df^n(H^u_{-n}(x_{-n}))\left(\mathcal{A}_{-n}(x_{-n}) - DH^u_{-n}(x_{-n})\right)Df^{-n}(x)\| \\ &\leq \left(\frac{\chi^c_+}{\chi^c_-}\right)^n \|\mathcal{A}_{-n}(x_{-n}) - DH^u_{-n}(x_{-n})\| \\ &\leq \left(\frac{\chi^c_+}{\chi^c_-}\right)^n \left(\|\mathcal{A}_{-n}(x_{-n}) - Id\| + \|DH^u_{-n}(x_{-n}) - Id\|\right) \\ &\leq \left(\frac{\chi^c_+}{\chi^c_-}\right)^n (K + C)(\chi^{uu})^{-n} d(p,q).\end{aligned}$$

The center bunching condition implies that

$$\frac{\chi^c_+}{\chi^c_-}(\chi^{uu})^{-1} < 1.$$

Hence, $\|\Gamma_n(\mathcal{A})_{p,q}(x) - DH^u_{p,q}(x)\|$ goes to zero uniformly as n goes to infinity. Since $d(p,q) \leq 1$, this estimate is independent of the points p, q and x. In other words, $\lim_{n\to+\infty} d(\Gamma_n(\mathcal{A}), DH^u) = 0$. Moreover, if $\mathcal{A}$ is Γ_n-invariant for every $n \in \mathbb{N}$, then $\lim_{n\to+\infty} d(\Gamma_n(\mathcal{A}), DH^u) = d(\mathcal{A}, DH^u) = 0$ and thus $\mathcal{A} = DH^u$. □

For a C^2-diffeomorphism $g : \mathbb{T}^2 \to \mathbb{T}^2$, we have that $Dg(.)$ is a map that belongs to $C^1(\mathbb{T}^2, L(\mathbb{R}^2, \mathbb{R}^2))$. In particular, $D^2g(.)$ is a map that belongs to $C^0(\mathbb{T}^2, L(\mathbb{R}^2, L(\mathbb{R}^2, \mathbb{R}^2)))$, where $L(\mathbb{R}^2, L(\mathbb{R}^2, \mathbb{R}^2))$ is the space of linear maps from $\mathbb{R}^2$ to $L(\mathbb{R}^2, \mathbb{R}^2)$. The space $L(\mathbb{R}^2, L(\mathbb{R}^2, \mathbb{R}^2))$ can be identified with the space $L^2(\mathbb{R}^2, \mathbb{R}^2)$, which is the space of bilinear maps of $\mathbb{R}^2$ taking values in $\mathbb{R}^2$. The space $L^2(\mathbb{R}^2, \mathbb{R}^2)$ has a norm given by

$$\|B\| = \sup\{\|B(u,v)\| : \|u\| = \|v\| = 1\}.$$

Using this norm, we can naturally define a C^0-metric in $C^0(\mathbb{T}^2, L^2(\mathbb{R}^2, \mathbb{R}^2)))$, which gives a C^1-metric in $C^1(\mathbb{T}^2, L(\mathbb{R}^2, \mathbb{R}^2))$ that we will denote it by $d^*_{C^1}(.,.)$. We remark that the space $C^1(\mathbb{T}^2, L(\mathbb{R}^2, \mathbb{R}^2))$ is complete with $d^*_{C^1}(.,.)$.

Consider the set $\mathcal{L}^1$ of the elements $\mathcal{A}$ of $\mathcal{L}$ such that for each $p \in \mathbb{T}^4$ and $q \in W_1^{uu}(p)$ we have $\mathcal{A}_{p,q}(.) \in C^1(\mathbb{T}^2, L(\mathbb{R}^2, \mathbb{R}^2))$ and it varies continuously in the C^1-topology with the choices of the points p and q. We define the C^1-distance on $\mathcal{L}^1$ by

$$d_{C^1}(\mathcal{A}, \mathcal{B}) = \sup_{p\in\mathbb{T}^4, q\in W_1^{uu}(p)} \left\{d^*_{C^1}(\mathcal{A}_{p,q}(.), \mathcal{B}_{p,q}(.))\right\}.$$

It is easy to see that $\mathcal{L}^1$ is closed for the metric d_{C^1}. The strategy to prove Theorem 2.10 is the following: we consider the family Id in $\mathcal{L}^1$ which is just the identity for any choices of $p \in \mathbb{T}^4$ and $q \in W_1^{uu}(p)$, next we consider the sequence $\{\Gamma_n(Id)\}_{n\in\mathbb{N}}$ and we prove that this sequence is Cauchy for the metric d_{C^1}. Lemma 8.2 implies that $\Gamma_n(Id)$ converges C^0 to DH^u. However, $\Gamma_n(Id)$ also converges C^1 and therefore $DH^u \in \mathcal{L}^1$, which implies that $\{H^u_{p,q}\}_{p\in\mathbb{T}^4, q\in W_1^{uu}(p)}$ is a continuous family of C^2-diffeomorphisms.

PROOF OF THEOREM 2.10. As we explained in the previous paragraph, to prove Theorem 2.10 it is enough to prove that the sequence $\{\Gamma_n(Id)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. We fix $p \in \mathbb{T}^4$, $q \in W_1^{uu}(p)$ and $x \in \mathbb{T}^2$. For each $n \in \mathbb{N}$, we define $H^u_{-n} := H^u_{p_{-n},q_{-n}}(x_-n)$ and $\Gamma_n := \Gamma_n(Id)_{p,q}(x)$. Observe that

$$\Gamma_n = Df^n(H^u_{-n})Df^{-n}(x) = Df^n(H^u_{-n})Df(x_{-n-1})Df^{-1}(x_{-n})Df^{-n}(x).$$

By (96), for each $j = 1, \cdots, n$, we have $f^j(H^u_{-n}) = H^u_{-n+j}$. Hence,

$$\Gamma_{n+1} = Df^n(H^u_{-n})Df(H^u_{-n-1})Df^{-1}(x_{-n})Df^{-n}(x).$$

We want to estimate $\|D\Gamma_{n+1} - D\Gamma_n\|$. First, let us evaluate $D\Gamma_{n+1}$ and $D\Gamma_n$. In what follows, for a diffeomorphism g, we will write $D^2g(y)[.,.]$ to represent the bilinear form of its

second derivative on the point y. By the chain rule and using that $Df(x_{-n-1})Df^{-1}(x_{-n}) = Id$, we obtain

$$\begin{aligned}
D\Gamma_n[.,.] &= D\left(Df^n(H^u_{-n})Df(x_{-n-1})Df^{-1}(x_{-n})Df^{-n}(x)\right)[.,.] \\
&= D^2f^n(H^u_{-n})\left[DH^u_{-n}Df^{-n}(x).,Df^{-n}(x).\right] && (\mathrm{I}_n) \\
&\quad + Df^n(H^u_{-n})D^2f(x_{-n-1})\left[Df^{-n-1}(x).,Df^{-n-1}(x).\right] && (\mathrm{II}_n) \\
&\quad + Df^n(H^u_{-n})Df(x_{-n-1})D^2f^{-1}(x_{-n})\left[Df^{-n}(x).,Df^{-n}(x).\right] && (\mathrm{III}_n) \\
&\quad + Df^n(H^u_{-n})D^2f^{-n}(x)[.,.] && (\mathrm{IV}_n) \\
&= \mathrm{I}_n + \mathrm{II}_n + \mathrm{III}_n + \mathrm{IV}_n.
\end{aligned}$$

Similarly,

$$\begin{aligned}
D\Gamma_{n+1}[.,.] &= D\left(Df^n(H^u_{-n})Df(H^u_{-n-1})Df^{-1}(x_{-n})Df^{-n}(x)\right)[.,.] \\
&= D^2f^n(H^u_{-n})\left[DH^u_{-n}Df^{-n}(x).,Df(H^u_{-n-1})Df^{-n-1}(x).\right] && (\mathrm{I}'_n) \\
&\quad + Df^n(H^u_{-n})D^2f(H^u_{-n-1})\left[DH^u_{-n-1}Df^{-n-1}(x).,Df^{-n-1}(x).\right] && (\mathrm{II}'_n) \\
&\quad + Df^n(H^u_{-n})Df(H^u_{-n-1})D^2f^{-1}(x_{-n})\left[Df^{-n}(x).,Df^{-n}(x).\right] && (\mathrm{III}'_n) \\
&\quad + Df^{n+1}(H^u_{-n-1})Df^{-1}(x_{-n})D^2f^{-n}(x)[.,.] && (\mathrm{IV}'_n) \\
&= \mathrm{I}'_n + \mathrm{II}'_n + \mathrm{III}'_n + \mathrm{IV}'_n.
\end{aligned}$$

To estimate $\|\Gamma_{n+1} - \Gamma_n\|$ we will separate it into four estimates.

The estimate for $\|\mathrm{I}'_n - \mathrm{I}_n\|$. Let us first write the expressions for I_n and I'_n. In what follows we use that $f^j(H^u_{-n}) = H^u_{-n+j}$, for any $j \in \mathbb{Z}$. Then,

$$\begin{aligned}
I_n &= D^2f(H^u_{-1})\left[Df^{n-1}(H^u_{-n})DH^u_{-n}Df^{-n}(x).,Df^{n-1}(H^u_{-n})Df^{-n}(x).\right] && (\tilde{\mathrm{I}}_{n,1}) \\
&\quad + Df(H^u_{-1})D^2f(H^u_{-2})\left[Df^{n-2}(H^u_{-n})DH^u_{-n}Df^{-n}(x).,Df^{n-2}(H^u_{-n})Df^{-n}(x).\right] && (\tilde{\mathrm{I}}_{n,2}) \\
&\qquad\qquad \vdots \\
&\quad + Df(H^u_{-1})\cdots Df(H^u_{-n+1})D^2f(H^u_{-n})\left[DH^u_{-n}Df^{-n}(x).,Df^{-n}(x).\right]. && (\tilde{\mathrm{I}}_{n,n})
\end{aligned}$$

We also have

$$\begin{aligned}
I'_n &= D^2f(H^u_{-1})\left[Df^{n-1}(H^u_{-n})DH^u_{-n}Df^{-n}(x).,Df^{n}(H^u_{-n-1})Df^{-n-1}(x).\right] && (\tilde{\mathrm{I}}'_{n,1}) \\
&\quad + Df(H^u_{-1})D^2f(H^u_{-2})\left[Df^{n-2}(H^u_{-n})DH^u_{-n}Df^{-n}(x).,Df^{n-1}(H^u_{-n-1})Df^{-n-1}(x).\right] && (\tilde{\mathrm{I}}'_{n,2}) \\
&\qquad\qquad \vdots \\
&\quad + Df(H^u_{-1})\cdots Df(H^u_{-n+1})D^2f(H^u_{-n})\left[DH^u_{-n}Df^{-n}(x).,Df^{-n-1}(x).\right]. && (\tilde{\mathrm{I}}'_{n,n})
\end{aligned}$$

Let $\tilde{C}_n = Df(H^u_{-n-1})Df^{-1}(x_{-n}) - Id$. For each $j = 1, \cdots n$, we obtain

$$\begin{aligned}
\|\tilde{\mathrm{I}}_{n,j} - \tilde{\mathrm{I}}_{n,j}\| &= \|Df(H^u_{-1}) \cdots Df(H^u_{-j+1}). \\
&\qquad D^2f(H^u_{-j}) \left[Df^{n-j}(H^u_{-n})DH^u_{-n}Df^{-n}(x)., Df^{n-j}(H^u_{-n})\tilde{C}_nDf^{-n}(x). \right] \| \\
&\leq \|Df^{j-1}|_{E^c}\| \|f\|_{C^2} \|Df^{n-j}|_{E^c}\|^2 \|DH^u_{-n}\| \|Df^{-n}|_{E^c}\|^2 \|\tilde{C}_n\| \\
&\leq \|f\|_{C^2} \frac{(\chi^c_+)^{j-1}.(\chi^c_+)^{2(n-j)}}{(\chi^c_-)^{2n}} \|DH^u_{-n}\| \|\tilde{C}_n\| \\
&= \|f\|_{C^2} \|DH^u_{-n}\| \left(\frac{\chi^c_+}{\chi^c_-} \right)^{2n} \|\tilde{C}_n\| (\chi^c_+)^{-j-1} < \|f\|_{C^2} \|DH^u_{-n}\| \left(\frac{\chi^c_+}{\chi^c_-} \right)^{2n} \|\tilde{C}_n\|.
\end{aligned}$$

We remark that in the last inequality we used that $\chi^c_+ > 1$. By (97), for every $n \in \mathbb{N}$, we have $\|DH^u_{-n}\| < K$, for some constant $K \geq 1$. Also

$$\begin{aligned}
\|\tilde{C}_n\| &= \|Df(H^u_{-n-1})Df^{-1}(x_{-n}) - Id\| \\
&= \| \left(Df(H^u_{-n-1}) - Df(x_{-n-1}) \right) Df^{-1}(x_{-n})\| \\
&\leq \frac{1}{\chi^c_-} \|f\|_{C^2} d(x_{-n-1}, H^u_{-n-1}) \\
&\leq \frac{1}{\chi^c_-} \|f\|_{C^2} (\chi^{uu})^{-n-1} d(p,q) \leq \frac{1}{\chi^c_-} \|f\|_{C^2} (\chi^{uu})^{-n-1}.
\end{aligned}$$

Hence,

$$\|\tilde{\mathrm{I}}_{n,j} - \tilde{\mathrm{I}}_{n,j}\| \leq \frac{\|f\|^2_{C^2} K}{\chi^{uu}\chi^c_+\chi^c_-} \left[\left(\frac{\chi^c_+}{\chi^c_-} \right)^2 (\chi^{uu})^{-1} \right]^n.$$

Take the constant

$$C_1 := \frac{\|f\|^2_{C^2} K}{\chi^{uu}\chi^c_+\chi^c_-}$$

and observe that

$$\|\mathrm{I}'_n - \mathrm{I}_n\| \leq \sum_{j=1}^{n} \|\tilde{\mathrm{I}}_{n,j} - \tilde{\mathrm{I}}_{n,j}\| \leq C_1 \left(\frac{(\chi^c_+)^2}{\chi^{uu}(\chi^c_-)^2} \right)^n. \tag{99}$$

This gives the estimate we need for $\|\mathrm{I}'_n - \mathrm{I}_n\|$.

The estimate for $\|\mathrm{II}'_n - \mathrm{II}_n\|$. This is the only part in the proof of Theorem 2.10 that we use that f is $C^{2+\alpha}$. Let

$$\tilde{\mathrm{II}}_n := D^2f(x_{-n-1}) \left[Df^{-n-1}(x)., Df^{-n-1}(x). \right] - D^2f(H^u_{-n-1}) \left[DH^u_{-n-1}Df^{-n-1}(x)., Df^{-n-1}(x). \right].$$

Notice that

$$\|\mathrm{II}'_n - \mathrm{II}_n\| = \|Df^n(H^u_{-n})\tilde{\mathrm{II}}_n\| \leq (\chi^c_+)^n \|\tilde{\mathrm{II}}_n\|.$$

By the triangular inequality,

$$\begin{aligned}
\|\tilde{\mathrm{II}}_n\| \leq &\|D^2f(x_{-n-1}) \left[Df^{-n-1}(x)., Df^{-n-1}(x). \right] \\
&- D^2f(x_{-n-1}) \left[DH^u_{-n-1}Df^{-n-1}(x)., Df^{-n-1}(x). \right] \| \\
&+ \|D^2f(x_{-n-1}) \left[DH^u_{-n-1}Df^{-n-1}(x)., Df^{-n-1}(x). \right] \\
&- D^2f(H^u_{-n-1}) \left[DH^u_{-n-1}Df^{-n-1}(x)., Df^{-n-1}(x). \right] \| \\
= &\|D_n\| + \|E_n\|.
\end{aligned}$$

Let us estimate each of these terms..

$$\begin{aligned}\|D_n\| &= \left\|D^2f(x_{-n-1})\left[\left(Id - DH^u_{-n-1}\right)Df^{-n-1}(x).,Df^{-n-1}(x).\right]\right\| \\ &\leq \|f\|_{C^2}\|Id - DH^u_{-n-1}\|\|Df^{-n-1}(x)|_{E^c}\|^2 \\ &\leq \|f\|_{C^2}\left(\frac{1}{(\chi^c_-)^2}\right)^{n+1} C(\chi^{uu})^{-n-1}d(p,q) \\ &\leq \|f\|_{C^2}C\left(\frac{1}{\chi^{uu}(\chi^c_-)^2}\right)^{n+1} \leq \|f\|_{C^2}C\left(\frac{1}{(\chi^{uu})^\alpha(\chi^c_-)^2}\right)^{n+1}.\end{aligned}$$

Since f is $C^{2+\alpha}$, There exists a constant $C_H \geq 1$ such that $\|D^2f(z)[.,.] - D^2f(w)[.,.]\| \leq C_H d(z,w)^\alpha$. Recall that $\|DH^u_{-j}\| < K$, for every $j \in \mathbb{N}$ and some constant $K \geq 1$. Therefore,

$$\begin{aligned}\|E_n\| &\leq C_H d(x_{-n-1}, H^u_{-n-1})^\alpha \|DH^u_{-n-1}\|\|Df^{-n-1}|_{E^c}\|^2 \\ &\leq C_H K \frac{1}{(\chi^c_-)^{2(n+1)}}(\chi^{uu})^{-\alpha(n+1)}d(p,q) \\ &\leq C_H K\left(\frac{1}{(\chi^{uu})^\alpha(\chi^c_-)^2}\right)^{n+1}.\end{aligned}$$

Take the constant

$$C_2 := \left(\|f\|_{C^2}C + C_H K\right)\frac{1}{(\chi^{uu})^\alpha(\chi^c_-)^2}.$$

We obtain

$$\|\mathrm{II}'_n - \mathrm{II}_n\| \leq C_2\left(\frac{\chi^c_+}{(\chi^{uu})^\alpha(\chi^c_-)^2}\right)^n. \tag{100}$$

The $(2,\alpha)$-center bunching condition implies that the right hand side of (100) goes exponentially fast to zero. This gives the estimate we need for $\|\mathrm{II}'_n - \mathrm{II}_n\|$.

The estimate for $\|\mathrm{III}'_n - \mathrm{III}_n\|$. Observe that

$$\begin{aligned}\|\mathrm{III}'_n - \mathrm{III}_n\| &= \|Df^n(H^u_{-n})\left(Df(H^u_{-n-1}) - Df(x_{-n-1})\right)D^2f^{-1}(x_{-n})\left[Df^{-n}(x).,Df^{-n}(x).\right]\| \\ &\leq (\chi^c_+)^n\|Df(H^u_{-n-1}) - Df(x_{-n-1})\|(\chi^c_-)^{-2n}.\end{aligned}$$

We have

$$\|Df(H^u_{-n-1}) - Df(x_{-n-1})\| \leq \|f\|_{C^2}(\chi^{uu})^{-n-1}.$$

By taking

$$C_3 := \frac{\|f\|^2_C}{\chi^{uu}},$$

we conclude that

$$\|\mathrm{III}'_n - \mathrm{III}_n\| \leq C_3\left(\frac{\chi^c_+}{\chi^{uu}(\chi^c_-)^2}\right)^n. \tag{101}$$

This concludes the estimate we need for $\|\mathrm{III}'_n - \mathrm{III}_n\|$.

The estimate for $\|\mathrm{IV}'_n - \mathrm{IV}_n\|$. Notice that

$$\begin{aligned}\|\mathrm{IV}'_n - \mathrm{IV}_n\| &= \|Df^n(H^u_{-n})\left(Df(H^u_{-n-1})Df^{-1}(x_{-n}) - Id\right)D^2f^{-n}(x)[.,.]\| \\ &\leq (\chi^c_+)^n\|\left(Df(H^u_{-n-1}) - Df(x_{-n-1})\right)Df^{-1}(x_{-n})\|\|D^2f^{-n}(x)\| \\ &\leq (\chi^c_+)^n\|f\|_{C^2}(\chi^{uu})^{-n-1}(\chi^c_-)^{-1}\|D^2f^{-n}(x)\|.\end{aligned}$$

Let us estimate $\|D^2 f^{-n}(x)\|$. First, observe that

$$\begin{aligned} D^2 f^{-n}(x)[.,.] &= D^2 f^{-1}(x_{-n+1})\left[Df^{-n+1}(x).,Df^{-n+1}(x).\right] \\ &\quad + Df^{-1}(x_{-n+1})D^2 f^{-1}(x_{-n+2})\left[Df^{-n+2}(x).,Df^{-n+2}(x).\right] \\ &\qquad\qquad \vdots \\ &\quad + Df^{-n+1}(x_{-1})D^2 f(x)[.,.]. \end{aligned}$$

Using that $\|D^2 f^{-1}(.)\| \leq \|f^{-1}\|_{C^2}$ and by the expression above, we obtain

$$\|D^2 f^{-n}(x)\| \leq \|f^{-1}\|_{C^2} \sum_{j=0}^{n-1} (\chi_-^c)^{-j}(\chi_-^c)^{-2n+2j} = \|f^{-1}\|_{C^2}(\chi_-^c)^{-2n} \sum_{j=0}^{n-1} (\chi_-^c)^j.$$

Since $\chi_-^c < 1$, the sum $\sum_{j\in\mathbb{N}}(\chi_-^c)^j$ converges. Define the constant C_4 as

$$C_4 := \frac{\|f\|_{C^2}\|f^{-1}\|_{C^2}\sum_{j\in\mathbb{N}}(\chi_-^c)^j}{\chi^{uu}\chi_-^c}.$$

We conclude that

$$\|\mathrm{IV}_n' - \mathrm{IV}_n\| \leq C_4 \left(\frac{\chi_+^c}{\chi^{uu}(\chi_-^c)^2}\right)^n. \tag{102}$$

Conclusion of the proof of Theorem 2.10. Take

$$\chi = \max\left\{\frac{(\chi_+^c)^2}{\chi^{uu}(\chi_-^c)^2}, \frac{\chi_+^c}{(\chi^{uu})^\alpha(\chi_-^c)^2}, \frac{\chi_+^c}{\chi^{uu}(\chi_-^c)^2}\right\},$$

and observe that by the $(2,\alpha)$-center bunching condition $\chi < 1$. Fix the constant $\hat{C} := C_1 + C_2 + C_3 + C_4$. By (99),(100), (101) and (102) we obtain that

$$\|\Gamma_{n+1} - \Gamma_n\| \leq \hat{C}\chi^n.$$

Therefore, $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence for the C^1-topology. Observe that all these estimates and constants are uniform with the choices of $p \in \mathbb{T}^4$, $q \in W_1^{uu}(p)$ and $x \in W^c(p)$. We conclude that $\{\Gamma_n(Id)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}^1$ for the C^1-topology. Since $\Gamma_n(Id)$ converges C^0 to DH^u, we conclude that DH^u is C^1. This implies that $\{H^u_{p,q}(.)\}_{p\in\mathbb{T}^4}, q \in W_1^{uu}(p)\}$ is a continuous family of C^2-diffeomorphisms whose C^2-norm varies continuously with the choices of p and q as above. □

Bibliography

[ABV00] J. Alves, C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.*, 140:351–398, 2000.

[AV10] A. Avila, and M. Viana. Extremal Lyapunov exponents: an invariance principle and applications. *Invent. Math.*, 181:115–189, 2010.

[BP02] L. Barreira, and Y. Pesin. *Lyapunov exponents and smooth ergodic theory*, volume 23 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2002.

[Ov19] S. Ben Ovadia. Generalized SRB measures, physical properties, and thermodynamic formalism of smooth hyperbolic systems. *Preprint arXiv:1904.10074*, 2019.

[BC14] P. Berger, and P. Carrasco. Non-uniformly hyperbolic diffeomorphisms derived from the standard map. *Comm. Math. Phys.*, 329:239–262, 2014.

[BXY17] A. Blumenthal, J. Xue, and L.S. Young. Lyapunov exponents for random perturbations of some area-preserving maps including the standard map. *Ann. of Math.*, 185:285–310, 2017.

[BDV05] C. Bonatti, L. Díaz, and M. Viana. *Dynamics beyond uniform hyperbolicity*, volume 102 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005.

[BV00] C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.

[Bow75] R. Bowen. Equilibrium states and the ergodic theory of Anosov diffeomorphisms. *Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York*, Volume 470, 1975.

[Br16] A. Brown. Smoothness of stable holonomies inside center-stable manifolds and the C^2 hypothesis in Pugh-Shub and Ledrappier-Young theory. *Preprint arXiv:1608.05886*, 2016.

[BRH17] A. Brown, and F. Rodriguez Hertz. Measure rigidity for random dynamics on surfaces and related skew products. *J. Amer. Math. Soc.*, 30(4):1055–1132, 2017.

[CO19] P. Carrasco, and D. Obata. A new example of robustly transitive diffeomorphism. *Preprint on arXiv:1904.11788*, 2019.

[Ch79] B. Chirikov. A universal instability of many-dimensional oscillator systems. *Phys. Rep.*, 52:264–379, 1979.

[CDP16] V. Climenhaga, D. Dolgopyat, and Y. Pesin. Non-stationary non-uniform hyperbolicity: SRB measures for dissipative maps. *Comm. Math. Phys.*, 346:553–602, 2016.

[CLP19] V. Climenhaga, S. Luzzatto, and Y. Pesin. SRB measures and Young towers for surface diffeomorphisms. *Preprint arXiv:1904.00034*, 2019.

[CLP17] V. Climenhaga, S. Luzzatto, and Y. Pesin. The geometric approach for constructing Sinai-Ruelle-Bowen measures. *J. Stat. Phys.*, 166:467–493, 2017.

[CP16] S. Crovisier, and E. Pujals. Strongly dissipative surface diffeomorphisms. *Preprint arXiv:1608.05999*, 2016.

[Du94] P. Duarte. Plenty of elliptic islands for the standard family of area preserving maps. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 11:359–409, 1994.

[Fu63] H. Furstenberg. Noncommuting random products. *Trans. Amer. Math. Soc.*, 108:377–428, 1963.

[Go12] A. Gorodetski. On stochastic sea of the standard map. *Comm. Math. Phys.*, 309:155–192, 2012.

[HS16] M. Hirayama, and N. Sumi. On the ergodicity of hyperbolic Sinaĭ-Ruelle-Bowen measures: the constant unstable dimension case. *Ergodic Theory Dynam. Systems*, 36:1494–1515, 2016.

[HPS77] M. Hirsch, C. Pugh, and M. Shub. *Invariant manifolds*. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977.

[Iz80] F. Izraelev. Nearly linear mappings and their applications. *Physica D: Nonlinear Phenomena*, 1:243–266, 1980.

[Le86] F. Ledrappier. Positivity of the exponent for stationary sequences of matrices, in Lyapunov exponents (Bremen, 1984). *Lecture Notes in Math.*, 1186:56–73, Springer, Berlin, 1986.

[Le84] F. Ledrappier. Propriétés ergodiques des mesures de Sinaï. *Inst. Hautes Études Sci. Publ. Math.*, 59:163–188, 1984.

[LY85-1] F. Ledrappier, and L.S. Young. The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin's entropy formula. *Ann. of Math.*, 122:509–539, 1985.

[Ob18-1] D. Obata. On the holonomies of strong stable foliations. *Notes on D. Obata's personal webpage*, 2018.

[Ob18-2] D. Obata. On the stable ergodicity of Berger-Carrasco's example. *Ergodic Theory and Dynamical Systems*, Online version, 2018.

[Pe77] Y. Pesin. Characteristic Ljapunov exponents, and smooth ergodic theory. *Uspehi Mat. Nauk*, 32:55–112, 1977.

[PSW97] C. Pugh, M. Shub, and A. Wilkinson. Hölder foliations. *Duke Math. J.*, 86:517–546, 1997.

[PSW00] C. Pugh, M. Shub, and A. Wilkinson. Correction to: "Hölder foliations". *Duke Math. J.*, 105:105–106, 2000.

[PSW12] C. Pugh, M. Shub, and A. Wilkinson. Hölder foliations, revisited. *J. Mod. Dyn.*, 6:79–120, 2012.

[Ro52] V. Rohlin. On the fundamental ideas of measure theory. *Amer. Math. Soc. Translation*, 1952.

[Ru76] D. Ruelle. A measure associated with Axiom A attractors. *Amer. J. Math.*, 98:619–654, 1976.

[SS95] D. Shepelyansky, and A. Stone. Chaotic landau level mixing in classical and quantum wells. *Physical review letters*, 1995.

[Si68] J. Sinaĭ. Markov partitions and Y-diffeomorphisms. *Funkcional. Anal. i Priložen*, 2:64–89, 1968.

[Si72] J. Sinaĭ. Gibbs measures in ergodic theory. *Uspehi Mat. Nauk*, 27:21–64, 1972.

[Si94] J. Sinaĭ. *Topics in ergodic theory*, volume 44 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1994.

[TY19] A. Tahzibi, and J. Yang. Invariance principle and rigidity of high entropy measures. *Trans. Amer. Math. Soc.* 371:1231–1251, 2019.

[VY17] M. Viana, and J. Yang. Measure-theoretical properties of center foliations. *Modern theory of dynamical systems, Contemp. Math.* 692:291–320, 2017.

[Ya16] J. Yang. Entropy along expanding foliations. *Preprint on arXiv:1601.05504*, 2016.

[Yo98] L.S. Young. Statistical properties of dynamical systems with some hyperbolicity. *Ann. of Math.*, 147:585–650, 1998.

université PARIS-SACLAY | École doctorale de mathématiques Hadamard (EDMH)

Titre : Ergodicité stable et mesures physiques pour des systèmes dynamiques faiblement hyperboliques.

Mots Clefs : stabilité ergodique, exposants de Lyapunov, mesures SRB hyperboliques, mesures u-Gibbs, mesures physiques, hiperbolicité partielle, décomposition dominée.

Résumé : Dans cette thèse, nous étudions les sujets suivants :

- la stabilité ergodique pour les systèmes conservatifs ;
- la généricité de l'existence d'exposants positifs pour certains produits tordus avec fibres de dimension deux ;
- rigidité des mesures u-Gibbs pour certains systèmes partiellement hyperboliques;
- la transitivité robuste.

Nous donnons une preuve de la stabilité ergodique pour certains systèmes partiellement hyperboliques sans utiliser l'accessibilité. Ces systèmes ont été introduits par Pierre Berger et Pablo Carrasco, et ils ont les propriétés suivantes : ils possèdent une direction centrale bidimensionnelle ; ils sont non-uniformément hyperboliques avec un exposant positif et un exposant négatif le long de la direction centrale pour presque tout point, et la décomposition d'Oseledets n'est pas dominée.
Dans un autre travail, nous donnons des critères de stabilité ergodique pour des systèmes ayant une décomposition dominée. En particulier, nous explorons la notion d'hyperbolicité par chaîne introduite par Sylvain Crovisier et Enrique Pujals. À l'aide de cette notion, nous donnons des critères explicites de stabilité ergodique et nous donnons quelques applications.
Dans un travail commun avec Mauricio Poletti, nous prouvons que le produit aléatoire de difféomorphismes de surface conservatifs possède génériquement une région avec des exposants positifs. Nos résultats s'appliquent également aux produits tordus plus généraux.
Nous étudions également les perturbations dissipatives de l'exemple de Berger-Carrasco. Nous classifions toutes les mesures u-Gibbs qui peuvent apparaître dans un voisinage de l'exemple. Dans ce voisinage, nous prouvons que toute mesure u-Gibbs est soit l'unique mesure SRB du système, soit la désintégration dans le feuilletage central est atomique.
Dans un travail commun avec Pablo Carrasco, nous prouvons que cet exemple est robustement transitif (en fait robustement topologiquement mélangeant).

université PARIS-SACLAY | École doctorale de mathématiques Hadamard (EDMH)

Title : Stable ergodicity and physical measures for weakly hyperbolic dynamical systems.

Keys words : stable ergodicity, Lyapunov exponents, hyperbolic SRB measures, u-Gibbs measures, physical measures, partial hyperbolicity, dominated splitting.

Abstract : In this thesis we study the following topics:

- stable ergodicity for conservative systems;
- genericity of the existence of positive exponents for some skew products with two dimensional fibers;
- rigidity of u-Gibbs measure for certain partially hyperbolic systems;
- robust transitivity.

We give a proof of stable ergodicity for a certain partially hyperbolic system without using accessibility. This system was introduced by Pierre Berger and Pablo Carrasco, and it has the following properties: it has a two dimensional center direction; it is non-uniformly hyperbolic having both a positive and a negative exponent along the center for almost every point, and the Oseledets decomposition is not dominated.
In a different work, we find criteria of stable ergodicity for systems with a dominated splitting. In particular, we explore the notion of chain-hyperbolicity introduced by Sylvain Crovisier and Enrique Pujals. With this notion we give explicit criteria of stable ergodicity, and we give some applications.
In a joint work with Mauricio Poletti, we prove that the random product of conservative surface diffeomorphisms generically has a region with positive exponents. Our results also hold for more general skew products.
We also study dissipative perturbations of the Berger-Carrasco example. We classify all the u-Gibbs measures that may appear inside a neighborhood of the example. In this neighborhood, we prove that any u-Gibbs measure is either the unique SRB measure of the system or it has atomic disintegration along the center foliation. In a joint work with Pablo Carrasco, we prove that this example is robustly transitive (indeed robustly topologically mixing).