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Rubén MARTOS

The Baum-Connes conjecture for Quantum Groups

Stability properties and K-theory computations

Directeur : **Pierre FIMA**

Soutenue le *6 Septembre 2018* devant le jury composé de :

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Résumé

Cette thèse porte sur la *conjecture de Baum-Connes pour les groupes quantiques*. Le but principal de ce travail est l'étude de la stabilité de la conjecture de Baum-Connes par certaines constructions de groupes quantiques discrets.

Dans un premier temps, nous réalisons une étude détaillé et approfondie de la reformulation catégorielle de la conjecture de Baum-Connes d'après les travaux de R. Meyer et R. Nest [132]. Ensuite, nous appliquons ces techniques au cas concret des *groupes quantiques discrets sans torsion*.

Nous réalisons une étude exhaustive des produits croisés afin de pouvoir les manipuler aisément en connexion avec la conjecture de Baum-Connes. Notamment nous donnons une preuve de la propriété universelle d'un produit croisé réduit par un groupe quantique discret. Nous analysons également quelques propriétés d'importance pour le contexte de cette thèse. Mentionnons particulièrement la propriété d'associativité du produit croisé par rapport à un produit semi-direct.

En s'inspirant des travaux pionniers de J. Chabert [34] nous menons une généralisation pour les groupes quantiques discrets de la stabilité de la conjecture de Baum-Connes par rapport à un produit semi-direct. Deux propriétés d'invariance d'intérêt indépendant sont également étudiées, à savoir le phénomène de torsion et la K-moyennabilité. Nous observons que l'hypothèse *sans torsion* force un biproduit crosié compact à être un produit semi-direct quantique sans torsion. Ainsi, la conjecture de Baum-Connes correspondante ne fournit pas d'information remarquable dans ce cas.

La stratégie générale pour mener à bien une telle généralisation consiste à définir un foncteur de "*décomposition*" entre les catégories de Kasparov suivant l'opération de produit semi-direct. Nous observons que cette stratégie peut être extrapolée à d'autres constructions de groupes quantiques. Notamment un produit direct de groupe quantiques. Dans ce cas, nous établissons une connexion avec la formule de Künneth de manière analogue à ce qui a été démontré dans [37] par J. Chabert, S. Echterhoff et H. Oyono-Oyono pour les groupes localement compacts classiques. Les propriétés de torsion et de K-moyennabilité ont également été étudiées.

Nous savons, grâce à R. Vergnioux and C. Voigt [208], que la conjecture de Baum-Connes *forte* est préservée par le passage aux sous-groupes quantiques discrets *divisibles*. Le même résultat est vrai pour la propriété de torsion *forte*, grâce à Y. Arano et K. De Commer [3]. Dans ce travail nous montrons que la conjecture de Baum-Connes usuelle est préservée par le passage aux sous-groupes quantiques discrets *divisibles* sous l'hypothèse sans torsion. La propriété de K-moyennabilité a également été étudiée.

Une notable propriété de permanence inclue dans cette thèse est la stabilité de la conjecture de Baum-Connes *forte* par produit en couronne libre. Pour cela, nous réalisons une complète classification des actions de torsion pour un produit libre quantique, ce qui permet de donner une formulation adéquate de la conjecture de Baum-Connes *forte* pour un produit en couronne libre inspirés par le travail pionnier de C. Voigt [212]. Une application majeure est un calcul explicite de

K-théorie, dans trois situations pertinentes, pour le groupe quantique compact de Lemeux-Tarrago qui est monoïdallement équivalent à un produit en couronne libre [120]. Cette propriété de stabilité pour un produit en couronne libre ainsi que les calculs de K-théorie s'intègrent dans un travail en collaboration avec A. Freslon [127].

Pour conclure, nous nous questionnons sur les résultats obtenus afin de proposer une liste de questions, problems et objectifs que l'auteur a rencontré durant l'intégralité de la période de recherche de cette thèse et qui rassemblent quelques unes des lignes de travail pour ses projets futures de recherche.

Mots Clés. groupes quantique, Baum-Connes, K-théorie, catégorie triangulée, catégorie de Kasparov, C^*-catégorie (tensorielle), produit semi-direct, produit en couronne libre, produit direct, torsion quantique, K-moyennabilité.

Abstract

The present dissertation is focused on the *Baum-Connes conjecture for quantum groups*. The main purpose of this work is the study of the Baum-Connes conjecture stability under some constructions of discrete quantum groups.

In a first phase, we carry out a detailed and extensive study about the categorical reformulation of the Baum-Connes conjecture according to the results of R. Meyer and R. Nest [132]. Next, we apply these techniques to the specific case of *torsion-free discrete quantum groups*.

We carry out an exhaustive study of crossed products in order to handle them comfortably in connexion with the Baum-Connes conjecture. Notably, we give a proof of the universal property satisfied by a reduced crossed product by a discrete quantum group. We analyze as well some important properties for this dissertation. Let us mention in particular the associativity property of the crossed product with respect to a semi-direct product.

Being inspired by the pionneer work of J. Chabert [34], we perform a generalization for discrete quantum groups of the invariance property of the Baum-Connes conjecture under the semi-direct product construction. Two permanence properties of own interest are studied as well. Namely, the torsion-freeness and the K-amenability. We observe that the torsion-freeness assumption forces a compact bicrossed product to be a torsion-free quantum semi-direct product, so that the corresponding Baum-Connes conjecture does not give any relevant information in this case.

The general strategy used to accomplish such a generalization consists in defining a *"decomposition"* functor between the corresponding Kasparov categories in accordance with the semi-direct product operation. Thus, we observe that this strategy can be extrapolate to other (quantum) group constructions. Namely, to a a quantum direct product. In this case, we state a connexion with the Künneth formula as pointed out in [37] by J. Chabert, S. Echterhoff and H. Oyono-Oyono for classical locally compact groups. The properties of torsion-frenness and K-amenability are also analyzed.

It is known, thanks to R. Vergnioux and C. Voigt [208], that the *strong* Baum-Connes conjecture is preserved by *divisible* discrete quantum subgroups. The same is true for the *strong* torsion-freeness property, thanks to Y. Arano and K. De Commer [3]. Here we show that the usual Baum-Connes conjecture is preserved by *divisible* discrete quantum subgroups under torsion-freeness assumption. The K-amenability property is analyzed too.

A notably permanence property included in this dissertation is the invariance of the *strong* Baum-Connes conjecture under the free wreath product construction. For this, we carry out a complete classification of torsion actions of a quantum free product, which allows to give an appropriated formulation of the *strong* Baum-Connes conjecture for a free wreath product inspired by the pioneer work of C. Voigt [212]. A major application is an explicit K-theory computation, in three relevant situations, for the Lemeux-Tarrago's compact quantum group which is monoidally

equivalent to a free wreath product [120]. Both this stability property for a free wreath product and the K-theory computations are part of a collaboration work with A. Freslon [127].

To conclude, we question ourselves about the results obtained in order to suggest a list of questions, problems and goals that the author has encountered during the whole research period of the present dissertation and that are part of his future research projects.

Keywords. quantum group, Baum-Connes, K-theory, triangulated category, Kasparov category, C^*(-tensor) category, semi-direct product, free wreath product, direct product, quantum torsion, K-amenability.

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Contents

Introduction **1**

1 Background **15**
1.1 Conventions and notations . . . 16
1.2 Triangulated categories . . . 21
1.2.1 Elementary facts . . . 22
1.2.2 Meyer-Nest's homological algebra . . . 35
1.2.3 Reformulation of the Baum-Connes conjecture . . . 43
1.2.4 Meyer-Nest's homological algebra *revisited* . . . 55
1.3 Compact Quantum Groups . . . 61
1.3.1 Woronowicz's theory . . . 64
1.3.2 Locally compact case . . . 86
1.4 Actions of Quantum Groups . . . 89
1.4.1 Actions of Discrete Quantum Groups . . . 90
1.4.2 Spectral theory for Compact Quantum Groups . . . 102
1.4.3 Induced actions from Discrete Quantum Subgroups . . . 114
1.5 Crossed Products by Discrete (Quantum) Groups . . . 121
1.5.1 Classical crossed products . . . 121
1.5.2 Quantum crossed products . . . 128
1.5.3 Further properties . . . 135
1.6 Torsion phenomena in the quantum setting . . . 143
1.6.1 Torsion *à la Meyer-Nest* . . . 143
1.6.2 Torsion *à l'Arano-De Commer* . . . 152
1.7 KK-theory in the quantum setting . . . 158
1.7.1 Quantum Kasparov's theory and Baaj-Skandalis Duality . . . 158
1.7.2 Quantum Baum-Connes conjecture . . . 168

2 Construction of Compact Quantum Groups **175**
2.1 Typical examples . . . 176
2.2 Quantum direct product . . . 184
2.3 Quantum semi-direct product . . . 186
2.4 Compact bicrossed product . . . 206
2.5 Quantum free product . . . 208
2.6 Free wreath product . . . 211

3 Stability properties for the QBCc **215**
3.1 The Baum-Connes property for *compact* quantum groups . . . 217
3.2 The Baum-Connes property for a quantum subgroup . . . 219
3.2.1 Torsion property . . . 220
3.2.2 The Baum-Connes property . . . 220
3.2.3 K-amenability property . . . 224
3.3 The Baum-Connes property for a quantum direct product . . . 225
3.3.1 Torsion property . . . 226
3.3.2 The Baum-Connes property . . . 227
3.3.3 K-amenability property . . . 235
3.4 The Baum-Connes property for a quantum semi-direct product . . . 238
3.4.1 Torsion property . . . 238
3.4.2 The Baum-Connes property . . . 239
3.4.3 K-amenability property . . . 249
3.5 The Baum-Connes property for a compact bicrossed product . . . 250
3.5.1 K-amenability property . . . 251
3.6 The Baum-Connes property for a quantum free product . . . 253
3.6.1 Torsion property . . . 253
3.6.2 The Baum-Connes property . . . 258
3.6.3 K-amenability property . . . 260
3.7 The Baum-Connes property for a free wreath product . . . 260
3.7.1 Torsion property . . . 261
3.7.2 The Baum-Connes property . . . 263
3.7.3 K-amenability property . . . 265

4 An application: the K-theory for the *Lemeux-Tarrago's* $\widehat{\mathbb{H}}_q < \mathbb{G} * \widehat{SU_q}(2)$ **267**
4.1 Strategies for K-theory computations . . . 268
4.1.1 Torsion-free discrete quantum group case . . . 268
4.1.2 Torsion discrete quantum group case . . . 270
4.2 The *Lemeux-Tarrago's* $\widehat{\mathbb{H}}_q < \mathbb{G} * \widehat{SU_q}(2)$ . . . 273
4.2.1 Preliminary computations . . . 276
4.2.2 $\mathbb{G} := O^+(n)$ is a free orthogonal quantum group . . . 281
4.2.3 $\mathbb{G} := U_1^+ * \ldots * U_k^+ * O_1^+ * \ldots * O_l^+$ is a free quantum group . . . 285
4.2.4 $\mathbb{G} := \mathbb{F}_n$ is the classical free group on n generators . . . 286

5 Conclusion: open questions and possible lines of attack **289**
5.1 Stability of the Baum-Connes property . . . 290
5.2 Maximal torus strategy . . . 292
5.3 K-theory computations . . . 294
5.4 Formulation of the Baum-Connes property for arbitrary quantum groups . . . 296

Appendices **298**

A Generalities **301**
A.1 Elements of C^*-algebras 301
A.2 Elements of von Neumann algebras 308
A.3 Elements of Hilbert modules 309
A.4 Elements of multiplier algebras 317

B Categories **321**
B.1 Generalities 321
B.2 Abelian categories 328
B.3 C^*-tensor categories. Categorical picture of Quantum Groups 332

Bibliography **343**

Introduction

La realité qui nous entoure est purement géométrique et pour maîtriser une telle géométrie nous avons utilisé, depuis l'époque d'Euclide, des outils et des postulats naturellement construits de l'intuition élaborée à partir de nos observations immédiates. Cependant, tout au long du XXème siècle la physique a fournit des nouveaux paradigmes pour décrire la *réalité*.

D'une part, la théorie de la relativité d'A. Einstein a donné une vision plus mathématique de l'Univers en l'encadrant dans une géométrie beaucoup plus souple que la géométrie euclidienne. Il a mis en relief le caractère local de la réalité directe que nous apercevons. D'autre part, la physique quantique stimulée par W. Heisenberg a donné une vision plus changeante de l'Univers en l'encadrant dans ce cas dans une mer de possibilités, chacune d'elles étant dépendantes des observations faites.

Dû précisément à ce caractère changeant, la notion classique de *position* doit être remplacée par la notion d'*opérateur*. Par conséquent, l'espace géométrique où la réalité quantique se développe n'est plus un espace usuel au sens de Descartes, mais un espace où la mesure de ses propriétés n'est pas une opération commutative (*principe d'incertitude*).

Ce changement de point de vu s'illustre en mathématiques par la naissance de la *Géométrie non Commutative*. Son but fondateur était donc d'allier sous un même cadre conceptuel l'aspect non commutatif ou opérationnel de la physique quantique avec l'aspect purement géométrique de la théorie de la relativité.

Depuis qu'A. Connes a commencé à développer, avec beaucoup de succès, cette théorie en 1979, la Géométrie non Commutative et l'Algèbre d'Opérateurs plus généralement ont expérimenté un élargissement remarquable grâce aux travaux de mathématiciens de très grande envergure.

Rappelons le célèbre théorème de *dualité de Gelfand*: *"la catégorie des espaces topologiques Hausdorff localement compacts est équivalente à la catégorie des C^*-algèbres commutatives"*. Ce théorème représente le point de départ du changement de philosophie pour la géométrie non commutative d'A. Connes [41] et il contient également l'idée essentielle pour aboutir à la notion de *groupe quantique*. La théorie des groupes s'est développée sous des perspectives différentes et variées depuis l'époque d'E. Galois et la moderne théorie des groupes quantiques a essayé de créer d'énigmatiques reflets dans le cadre de la géométrie non commutative.

Il est compliqué de donner une description exhaustive de l'évolution historique de ces objects car la notion de groupe quantique apparaît aussi bien dans la physique que dans les mathématiques et avec des significations différentes *a priori*. Également, les contributions au développement de ces objets ont été très éparses dans le panorama mathématique entre les années 60 et 80, ce qui contribuait à une vision assez hétérogène de la théorie des groupes quantiques. À l'heure actuelle,

même s'il existent encore des débats concernant certaines terminologies ou notations, nous pouvons dire que les groupes quantiques ont réussi à consolider ses bases en créant ainsi une théorie bien comprise qui continue sa croissance en élargissant ses domaines d'action et d'application.

Afin de donner un aperçu sur cette théorie qui permette d'expliquer aussi bien leurs origines que leur état actuel, nous pouvons distinguer trois axes principaux autour desquels se développent les travaux sur les groupes quantiques : *i*) *analyse harmonique et théorie des représentations*, *ii*) *espaces homogènes et groupes de transformations* et *iii*) *combinatoire et géométrie.*

i) *Analyse harmonique et théorie des représentations.* L'analyse harmonique classique étudie la représentation de fonctions (définies sur $\mathbb{R}$) comme une superposition de fonctions élémentaires trigonométriques par le biais de la *transformée de Fourier.* À la moitié du XXème siècle on a observé que certaines propriétés de $\mathbb{R}$ et de la transformée de Fourier classique peuvent être généralisées pour n'importe quel groupe localement compact en donnant lieu à ce que l'on appel l'*analyse harmonique abstraite.*

   Le point de départ est l'existence d'une mesure appropriée sur tout groupe localement compact, que l'on appelle *measure de Haar* (en honneur de A. Haar), qui est unique à multiplication d'une constante positive près et qui est notée μ. Cette mesure joue le rôle de la mesure de Lebesgue sur $\mathbb{R}$. L'autre ingrédient essentiel pour développer cette théorie est la notion de *caractère* qui joue, quant à lui, le rôle des fonctions élémentaires trigonométriques dans une décomposition de type Fourier classique. Plus précisément, si G est un groupe *abélien* localement compact, alors $\widehat{G}$ dénote le groupe (abélien) des caractères de G, ce soit, l'ensemble des homomorphismes de groupes entre G et S^1. Il s'appelle *groupe dual de* G. La théorie générale affirme que $\widehat{G}$ est toujours un groupe localement compact. Remarquons tout de même que l'hypothèse "G abélien" est nécessaire pour achever cette conclusion puisque l'on utilise la dualité de Gelfand pour identifier $\widehat{G}$ avec le spectre de $L^1(G,\mu)$. Dans ce contexte, les propriétés de la transformée de Fourier classique trouvent une raison conceptuelle en vertu de ce que l'on appelle la *dualité de Pontryagin* [159] (en honneur de L. Pontryagin) et qui assure que $G \cong \widehat{\widehat{G}}$, pour tout groupe abélien localement compact G. En outre, si G est compact, alors son dual de Pontryagin est un groupe discret.

   Observons le fait suivant : S^1 peut être vu comme le groupe des opérateurs unitaires de $\mathbb{C}$, c'est à dire, le groupe d'automorphismes de $\mathbb{C}$ en tant qu'espace de Hilbert. Par conséquent, un caractère de G peut être vu comme une *représentation unitaire de dimension* 1 *de* G et réciproquement. Autrement dit, il existe une correspondance bijective entre les caractères d'un groupe abélien localement compact et ses représentations unitaires de dimension 1.

   Ainsi, la dualité de Pontryagin répond à une question de *reconstruction* : *pouvons-nous reconstruire un groupe à partir de la catégorie de ses représentations ?*

   Si nous éliminons l'hypothèse de commutativité de G, alors il faut trouver un remplaçant convenable pour $\widehat{G}$. Ceci sera l'*ensemble des représentations unitaires et irréductibles de* G. Observons en revanche que cet ensemble ne sera plus un groupe car le produit tensoriel de représentations irréductibles n'est pas forcément une représentation irréductible (pour cette raison nous seront intéressés à étudier ce que l'on appelle les *règles de fusion*, c'est à dire, les règles de décomposition en irréductibles d'un produit tensoriel de représentations irréductibles). Or, nous pouvons munir cet ensemble d'une structure de C^**-catégorie tensorielle.*

   La question de reconstruction précédente devient ainsi une question plus profonde, à savoir : *quelles sont les catégories équivalentes à une catégorie de représentations d'un groupe?*

Cette problématique de reconstruction dans le cadre non commutatif est connue sous le nom de *dualité de Tannaka-Krein* et elle est le point de départ de l'approche mathématique pour la naissance des groupes quantiques. Les premières tentatives de résoudre ce problème ont été menées par T. Tannaka [185] et par M. G. Krein [109], [110] en utilisant tout de même la théorie de Peter-Weyl pour les groupes compacts [149].

Or, dans ces travaux le groupe et son dual sont des objets de nature différente et dans ce sens on obtient seulement une solution partielle par rapport au cas abélien. Il faut attendre G. I. Kac qui avait eu l'idée de décrire un groupe localement compact *et* son dual en termes d'algèbres de von Neumann munies d'une co-multiplication [91], [92], [93]. Ainsi la philosophie générale de la géométrie non commutative entre en jeu. Durant les années 70 la *théorie des algèbres de Kac* est développée de façon indépendante par G. V. Kac-L. I. Vainerman [198], [199] et par M. Enock-J-M. Schwartz [58], [59]. Ensuite, une version C^*-algébrique de la théorie des algèbres de Kac est aussi développée par J-M. Vallin [200] et par M. Enock et J-M. Vallin [60]. Ces deux approches achèvent certainement une théorie de dualité en termes de la problématique de reconstruction de Tannaka-Krein qui inclut *tous* les groupes localement compacts. Néanmoins, la théorie reste encore limitée à cause des nombreux axiomes dans la définition d'algèbre de Kac et de la manque d'exemples en dehors de ceux provenant des groupes classiques.

En 1987, S. L. Woronowicz [226] donne une première définition de *groupe quantique compact* $\mathbb{G}$ en termes de C^*-algèbres unifères munies d'une co-multiplication. Dans les années qui suivent, il améliore sa propre approche avec les articles [228], [229] et [231]. Il développe une théorie très riche et proche de celle des groupes compacts. D'une part, il démontre l'existence d'un état distingué, appelé *état de Haar*, qui joue le rôle de la measure de Haar sur un groupe localement compact suivant la dualité de Gelfand. D'autre part, il développe une théorie des représentations satisfaisante qui lui permet d'achever un analogue du théorème de Peter-Weyl dans le cadre non commutatif. Plus précisément, il obtient que toute représentation unitaire de $\mathbb{G}$ se décompose comme une somme directe de représentations irréductibles de dimension finie et, de plus, les coefficients de ces représentations sont liés par des relations d'orthogonalité par rapport à l'état de Haar. En ce qui concerne la théorie de dualité, les groupes quantiques compacts donnent une réponse affirmative. Nous pouvons construire un groupe quantique (non compact) $\widehat{\mathbb{G}}$ tel que $\mathbb{G} \cong \widehat{\widehat{\mathbb{G}}}$, ce qui s'appelle *dualité de Tannaka-Krein-Woronowicz*. En effet, $\widehat{\mathbb{G}}$ est complètement défini par la catégorie des représentations irréductibles de dimension finie de $\mathbb{G}$. En vertu de cette dualité, $\widehat{\mathbb{G}}$ est appelé *groupe quantique dual de* $\mathbb{G}$ et les duaux de groupes quantiques compacts sont appelés *groupes quantiques discrets*.

En outre, la théorie de Woronowicz est remplie d'exemples dont le plus important est $SU_q(2)$ [227]. En effet, les objets issus des travaux de V. G. Drinfeld [55] et de M. Jimbo [87] sur les déformations d'algèbres enveloppantes de groupes de Lie trouvent leur place dans la théorie de S. L. Woronowicz grâce aux travaux de M. Rosso [166]. Également, des groupes classiques comme le groupe unitaire, le groupe orthogonal ou le groupe de permutations trouvent une version quantique dans la théorie de S. L. Woronowicz notés respectivement $U^+(n)$, $O^+(n)$, S_N^+ et qui ont été introduits par S. Wang dans [215], [217].

Afin de donner une généralisation complète de l'analyse harmonique classique, le cas *localement compact* dans le cadre quantique a été abordé par différents auteurs sous différentes perspectives. Citons l'élégante approche des *unitaires multiplicatifs* de S. Baaj et G. Skandalis [7] ou l'approche purement algébrique dans le cadre des *algèbres de Hopf non unifères* de A. van Daele [201], [202]. Mais la définition de *groupe quantique localement compact* qui est à ce jour communément

acceptée est celle donnée par J. Kustermans et S. Vaes [112], [113] dans les années 2000 en termes d'algèbres de von Neumann et de C^*-algèbres. Dans ce cas, la théorie devient beaucoup plus technique et délicate que celle de Woronowicz (par exemple, un analogue de la mesure de Haar n'existe *pas* automatiquement) mais on achève également un analogue de la dualité de Pontryagin et tous les exemples de groupes quantiques antérieurement construits y trouvent un cadre théorique précis.

ii) *Espaces homogènes et groupes de transformations*. Rappelons brièvement la philosophie générale du programme d'Erlangen [108], initiée par F. Klein dans la seconde moitié du XIXème siècle. L'idée fondatrice est celle de construire la géométrie à partir de la notion de groupe et d'invariant. Ainsi, un groupe est vu comme un groupe de *transformations d'un espace* et ses éléments représentent les symétries qui préservent la géométrie de l'espace. Si nous demandons de plus que l'action du groupe sur l'espace soit *transitive*, il n'y aura pas de points distingués et nous obtenons une notion d'*isotropie* car les points de l'espace forment une seule orbite selon le groupe. Nous parlons ainsi d'*espace homogène*.

Un exemple éclairant est le suivant. Étant donné $n \in \mathbb{N}$ considérons la sphère de dimension $n-1$ dans $\mathbb{R}^n$ et notons-la $\mathbb{S}^{n-1}$. Considérons aussi le groupe orthogonal de $\mathbb{R}^n$, $SO(n,\mathbb{R})$. Nous savons que ce groupe préserve les distances, les angles et aussi les rotations de $\mathbb{R}^n$. Ainsi, l'action naturelle de ce groupe sur la sphère $\mathbb{S}^{n-1}$ préserve sa géométrie et donc $\mathbb{S}^{n-1}$ doit être vu comme un espace homogène selon la philosophie de F. Klein. En effet, un exercice élémentaire est de prouver que l'action de $SO(n,\mathbb{R})$ sur $\mathbb{S}^{n-1}$ est transitive. De même, nous obtenons une description explicite de la sphère en tant qu'espace quotient

$$\mathbb{S}^{n-1} \cong SO(n,\mathbb{R})/SO(n-1,\mathbb{R})$$

Lorsque nous considérons la sphère $\mathbb{S}^{2n-1}$ dans $\mathbb{C}^n$, sa description comme espace homogène est donnée par le groupe spécial *unitaire*

$$\mathbb{S}^{2n-1} \cong SU(n)/SU(n-1)$$

Étant donné la similitude des groupes quantiques avec les groupes classiques, nous voulons aussi mener dans le cadre quantique une géométrie basée sur les espaces homogènes. Nous parlerons ainsi d'*espace homogène quantique*.

D'une part, l'espace géométrique sous-jacent sur lequel le groupe quantique va agir devra être un "espace quantique", c'est à dire, une "C^*-algèbre" d'après la dualité de Gelfand. La notion d'action d'un groupe quantique *compact* sur une C^*-algèbre a été introduite par P. Podleś dans [156] et utilisée ultérieurement par plusieurs auteurs comme F. Boca dans [25], M. Marciniak dans [126], S. Wang dans [217], S. Vaes dans [194] ou C. Pinzari-J. E. Roberts dans [153]. Une classe importante d'actions de groupes quantiques compacts est celle des actions *ergodiques* avec lesquelles on développe une théorie spectrale proche de celle pour les groupes compacts classiques. Dans le cadre de cette thèse les actions ergodiques sont spécialement intéressantes en relation avec le phénomène de torsion (d'après R. Meyer et R. Nest [133], [131]) et avec la notion d'*équivalence monoïdale* (d'après J. Bichon, A. De Rijdt et S. Vaes [23]). En outre, les actions ergodiques s'avèrent en étroite relation avec les espaces homogènes quantiques grâce aux travaux de K. De Commer et M. Yamashita [50], [51].

D'autre part, la théorie des espaces homogènes dans le cas classique entraîne naturellement travailler avec des sous-groupes du groupe de transformations en question. Ceci représente

un obstacle lorsque nous voulons imiter cette théorie dans le cadre quantique, de sorte que différentes stratégies ont été proposées pour achever cette généralisation [52], [102]. En effet, dans [155] P. Podleś définit une sphère quantique $\mathbb{S}_q^2$ (avec $q \in (0,1)$) qui mérite la nomenclature d'espace homogène de $SU_q(2)$, mais ces espaces ne peuvent pas être définis en termes des sous-groupes quantiques de $SU_q(2)$.

Remarquons que la théorie des espaces homogènes a un but purement géométrique. En particulier, les espaces homogènes à l'issu des groupes de Lie donnent lieu à une approche de théorie de groupes de la géométrie différentielle. Dans ce sens, nous voudrions aborder la géométrie non commutative d'A. Connes suivant l'approche des espaces homogènes quantiques. Plusieurs travaux se sont faits à ce sujet. Mentionnons les travaux de D. Goswani [72] et les travaux de J. Bhowmick et D. Goswani [74], [73], [75], [20], [170] où est étudié l'analogue quantique du groupe d'isométries riemanniennes d'une variété différentielle usuelle (notamment, on caractérise $SO_q(3)$ comme le groupe quantique d'isométries d'une variété riemannienne non commutative). Bien que la géométrie différentielle non commutative développée par A. Connes concerne les variétés réelles, il y a eu quelques tentatives pour construire une *géométrie complexe non commutative*. Premièrement, la structure complexe sur les sphères de P. Podleś a été étudiée [105] suivant des travaux précédents de S. Majid [125] et de S. Schwartz-A. Polishchuk [158]. Deuxièmement, on a étudié la structure complexe sur le plan projectif complexe [107], [106] suivant des travaux précédents de F. D'Andrea et L. Dąbrowski [46]. Une nouvelle approche à la géométrie complexe non commutative en termes d'espaces homogènes quantiques est développée depuis très récemment par R. Buachalla [30] où un cadre non commutative pour la géométrie de Kähler est établit.

Dans le même ordre de questions, nous pouvons nous demander quel devrait être le groupe d'automorphismes d'un espace quantique. Étant donné qu'un espace quantique est une C^*-algèbre A, la définition de $Aut(A)$ issu de la géométrie ne peut pas être celle des transformations ou permutations de ses points (si X_n denote l'espace fini de n-points, alors par définition nous avons $Aut(X_n) = S_n$, le groupe de permutations de n-éléments). Nous devons donc suivre une nouvelle approche. Pour cela nous remarquons que si X est un espace quelconque, son groupe d'automorphismes $Aut(X)$ vérifie la propriété universelle suivante : si G est un groupe qui agit sur X, alors il existe un unique homomorphisme de groupes $G \longrightarrow Aut(X)$ compatible avec les actions respectives sur X. Cette observation a permis à S. Wang de donner une définition abstraite de *groupe quantique d'automorphismes* dans [217] qui devient une description explicite pour le cas des C^*-algèbres de dimension finie.

iii) *Combinatoire et géométrie*. La théorie de S. L. Woronowicz sur les groupes quantiques compacts s'est avérée très productive grâce à sa simplicité et sa richesse. Ainsi, plusieurs théories classiques sur les groupes ont été transportées dans le cas quantique suivant l'approche de Woronowicz. Cela a donné lieu à une large variété de thématiques qui sont maintenant étudiées avec les groupes quantiques.

La *théorie géométrique des groupes*, qui étudie le rapport entre les propriétés des groupes et la géométrie des espaces où ces groupes agissent, a vu son jour à l'issu de l'étude des groupes libres et des présentations de groupes. Ceci était basé sur des approches purement combinatoires selon W. von Dyck [213]. Durant la première moitié du XXème siècle, plusieurs mathématiciens comme M. Dehn, J. Nielsen, J. H. Whitehead ou E. van Kampen ont commencé à introduire des approches géométriques pour l'étude des groupes. En 1977, J.-P. Serre développe une théorie élégante et fructueuse sur les groupes agissant sur des arbres en généralisant des propriétés

connues sur les produits libres et les extensions HNN. Elle s'appelle *théorie de Bass-Serre* [169] et elle a été étudiée en détaille dans le cadre des groupes quantiques discrets par R. Vergnioux [206], [207], par P. Fima [63] et par P. Fima-A. Freslon [64] en obtenant notamment une généralisation dans le cadre quantique de la *"K-moyennabilité des groupes discrets agissant sur des arbres avec des stabilisateurs moyennables"* (résultat dû à P. Julg et A. Valette [90]).

Un point de vue différent pour aborder la géométrie d'un groupe serait une étude analytique du groupe en question. Cela donne lieu à une connexion directe avec la théorie des C^*-algèbres et donc un passage naturel vers les groupes quantiques. Dans ce sens on parle de *propriétés d'approximation de groupes (quantiques)*.

Bien que les véritables origines de la théorie des propriétés d'approximation doit se placer dans les pionniers travaux d'A. Grothendieck sur les produits tensoriels topologiques d'espaces vectoriels topologiques [76], [77] ; on pourra placer les origines des propriétés d'approximation des groupes dans les années 30 avec la définition de *moyennabilité* de J. von Neumann [214] dans le contexte de la célèbre *paradoxe de Banach-Tarski* [8]. Ensuite, diverses propriétés sont apparues avec différentes applications dans différents domaines. La notion de *nucléarité* pour les C^*-algèbres [116], [39] est la propriété d'approximation pour les C^*-algèbres la plus proche de la théorie originale d'A. Grothendieck. Dire qu'une C^*-algèbre A est nucléaire équivaut à dire que tous les produits tensoriels de C^*-algèbres par A donnent la même C^*-algèbre. Mentionnons d'autres propriétés d'approximation importantes. La *propriété de Haagerup* [81], considérée comme un affaiblissement de la moyennabilité, s'avère spécialement intéressante en vertu des profondes conséquences analytiques et topologiques [82]. La *propriété* (T) *de Kazhdan* [104] doit être vue comme un opposé direct à la propriété de Haagerup et elle pourrait éventuellement aider à la construction d'obstructions pour des théorèmes d'isomorphismes. La *moyennabilité faible* [79], [80] a donné lieu quant à elle à des résultats de classification d'algèbres de von Neumann et aussi elle a permis le développement des techniques de rigidité-déformation de S. Popa [145], [146], [195].

Le lien entre les différentes propriétés d'approximation a été beaucoup étudié et à l'heure actuelle il existent encore des questions ouvertes à ce sujet. Notamment le lien entre la propriété de Haagerup et la moyennabilité faible. Dans le contexte de classification de C^*-algèbres et du calcul des groupes de K-théorie, les propriétés de moyennabilité et de nucléarité ont été définies aussi au niveau de la K-théorie. En effet, J. Cuntz introduit dans [44] la notion de *K-moyennabilité* permettant de donner des raisons conceptuelles aux calculs de la K-théorie des C^*-algèbres du groupe libre et G. Skandalis introduit dans [172] la notion de *K-nucléarité*, qui a permis le développement de l'étude de la KK-théorie des produits libres par E. Germain [70], [71].

En ce qui concerne les groupes quantiques, les propriétés d'approximation ont été également abordées de façon fructueuse pendant les dernières années. Tandis que les notions de moyennabilité et de K-moyennabilité trouvent une généralisation flagrante pour le cas quantique dans le contexte de la théorie de S. L. Woronowicz, les propriétés de Haagerup et la propriété (T) de Kazhdan nécessitent un traitement plus minutieux. P. Fima a introduit la propriété (T) de Kazhdan pour les groupes quantiques discrets dans [62] et M. Daws P. Fima, A. Skalski et S. White ont introduit la propriété de Haagerup pour les groupes quantiques localement compacts dans [47]. Mentionnons quelques travaux remarquables sur ce sujet : [26], [27], [32], [119], [189]. Pour un aperçu général de cette théorie nous mentionnons l'exposé très détaillé [28].

Les probabilités libres offrent une perspective combinatoire et intéressante sur les groupes quantiques. Cette théorie a été initiée par D. Voiculescu dans les années 80-90 [209] et largement développée par D. Voiculescu R. Speicher. Elle a pour but d'étudier les variables aléatoires dans un cadre non commutatif suivant la notion d'*indépendance libre*. Le livre [179] servira au lecteur intéressé pour une introduction complète à ce sujet.

Le lien entre la probabilité libre et les groupes quantiques se fait par le biais de la notion de *groupe quantique "easy"* introduite par T. Banica et R. Speicher [178] et étudiée ultérieurement par plusieurs auteurs [176], [177], [68]. En effet, les groupes quantiques "easy" possèdent par définition une structure combinatoire intrinsèque (les entrelaceurs de ses représentations irréductibles sont indexés par des partitions d'ensembles finis), ce qui permet d'étudier la notion d'indépendance libre selon les actions du groupe quantique de permutations S_N^+ .

L'approche combinatoire des groupes quantiques a permis notamment $i)$ une description complète de la théorie des représentations de S_N^+ et les groupes quantiques libres avec des conséquences concernant la moyennabilité [9], [10], [12], [11] [13], [14] $ii)$ la construction d'un produit en couronne par un groupe quantique d'automorphismes [66] et $iii)$ l'étude de sa propriété de Haagerup [186]. Les ingrédients principaux pour ces résultats sont les partitions non-croisées, les diagrammes de Temperley-Lieb [150] et les algèbres planaires [88]. Cela donne tout de mème une forte connexion avec la théorie des sous-facteurs de type II_1.

D'une part, nous observons que les groupes quantiques peuvent être vus comme des analogues non commutatifs des groupes, de sorte que beaucoup de propriétés et résultats classiques peuvent être transportés dans ce cadre plus général. D'autre part, nous observons également que la perte de commutativité représente parfois un véritable phénomène d'obstruction et nous sommes obligés de créer des nouvelles stratégies afin d'obtenir une complète généralisation des théories classiques dans un cadre non commutatif.

Il est important de dire que la théorie des groupes quantiques ne représente pas uniquement une jolie généralisation au cadre non commutatif de résultats classiques connus, mais ils permettent d'avoir aussi une nouvelle perspective des problèmes classiques qui pourra éventuellement aboutir à des solutions. Cela entraîne notamment un nouveau paradigme, ce qui est le cas de la problématique étudiée dans la présente thèse.

La *conjecture de Baum-Connes* a été formulée en 1982 par P. Baum et A. Connes [18]. Nous ne connaissons pas encore un contre-exemple pour la conjecture originale mais il est connu que celle avec coefficients est fausse [83]. Pour cette raison nous allons nous référer à la conjecture de Baum-Connes avec coefficients comme la *propriété de Baum-Connes*. Le but de la conjecture est de comprendre le rapport entre deux groupes de K-théorie de nature différente, ce qui pourra établir une forte connexion entre la géométrie et la topologie dans un contexte généralisé de la théorie de l'indice. Plus précisément, si G est un groupe localement compact et A est une G-C^*-algèbre, alors la propriété de Baum-Connes pour G avec coefficients dans A affirme que le morphisme d'assemblage

$$\mu_A^G : K_*^{top}(G;A) \longrightarrow K_*(G \underset{r}{\ltimes} A)$$

est un isomorphisme, où $K_*^{top}(G;A) := R\Big(KK_*^G(\underline{E}G, A)\Big) := \lim\limits_{\substack{Y \subset \underline{E}G \\ G-\text{compact}}} \Big(KK_*^G(C_0(Y), A)\Big)$ est la K-homologie équivariante de G à support compact avec coefficients dans A et $K_*(G \underset{r}{\ltimes} A)$ est la

K-théorie du produit croisé réduit $G \underset{r}{\ltimes} A$. Cette propriété a été prouvée pour une large classe de groupes. Mentionnons les remarquables travaux de N. Higson et G. Kasparov [82] sur les groupes ayant la propriété de Haagerup et les travaux de V. Lafforgue [115] sur les groupes hyperboliques.

La K-homologie équivariante à support compact $K_*^{top}(G;A)$ est, bien entendu, l'object géométrique obtenu à partir de l'espace classifiant des actions propres de G. Donc il est *a priori* plus facile à calculer que le groupe $K_*(G \underset{r}{\ltimes} A)$, qui est quant à lui de nature analytique et donc moins souple dans sa structure. Néanmoins, le groupe $K_*^{top}(G;A)$ crée parfois des problèmes non-triviaux. Pour cette raison, R. Meyer et R. Nest proposent en 2006 une nouvelle formulation de la propriété de Baum-Connes dans un cadre catégoriel approprié [132]. Plus précisément, si maintenant $\mathscr{KK}^G$ désigne la catégorie de Kasparov G-équivariante et $F(A) := K_*(G \underset{r}{\ltimes} A)$ est le foncteur homologique sur $\mathscr{KK}^G$ qui définit le membre de droite du morphisme d'assemblage, alors R. Meyer et R. Nest montrent dans [132] que le morphisme d'assemblage μ_A^G est équivalent à la transformation naturelle

$$\eta_A^G : \mathbb{L}F(A) \longrightarrow F(A),$$

où $\mathbb{L}F$ est la localisation du foncteur F par rapport à une paire convenable de sous-catégories complémentaires $(\mathscr{L}, \mathscr{N})$. Plus précisément, $\mathscr{L}$ est la sous-catégorie de $\mathscr{KK}^G$ des *G-C^*-algèbres compactement induites* et $\mathscr{N}$ est la sous-catégorie de $\mathscr{KK}^G$ des *G-C^*-algèbres compactement contractiles*.

Cette reformulation permet notamment d'éviter toute construction géométrique. Nous pouvons donc remplacer G par un groupe *quantique* localement compact $\mathbb{G}$. Le problème pour aboutir à une formulation complète dans le cas quantique est la *torsion* d'un tel groupe quantique. En effet, si Γ est un groupe discret, sa torsion est complètement déterminée par ses sous-groupes finis tandis que si $\widehat{\mathbb{G}}$ est un groupe quantique discret, la notion de torsion n'est pas du tout un problème trivial et elle a été introduite pour la première fois par R. Meyer et R. Nest dans [133], [131] et récemment re-interprétée par Y. Arano et K. De Commer en termes d'anneaux de fusion dans [3]. Néanmoins, le phénomène de torsion dans le cas quantique est loin d'être complètement compris. Ainsi, l'actuelle formulation de la propriété de Baum-Connes concerne uniquement les *groupes quantiques discrets sans torsion*. Dans ce sens nous pouvons parler de *propriété de Baum-Connes quantique*.

La propriété de Baum-Connes s'avère spécialement intéressante pour un groupe discret Γ et elle a été beaucoup étudiée depuis sa formulation. En outre, grâce aux travaux de G. Kasparov [96], de G. Kasparov et G. Skandalis [99], [100] et de J.-L. Tu [191], [192], [193] nous disposons d'une méthode "constructive" pour aborder la conjecture, nommée *méthode Dirac-dual Dirac*. Elle a été utilisée dans la plupart des preuves connues actuellement. La stratégie est basée sur un théorème de J.-L. Tu [191] qui affirme que *"si Γ est un groupe discret et A est une Γ-C^*-algèbre propre, alors la propriété de Baum-Connes pour Γ avec coefficients dans A est satisfaite"*. Ainsi, la méthode Dirac-dual Dirac consiste à construire

i) une Γ-C^*-algèbre propre A,

ii) un élément $\alpha \in KK^\Gamma(A, \mathbb{C})$ (nommé *élément Dirac*),

iii) un élément $\beta \in KK^\Gamma(\mathbb{C}, A)$ (nomé *élément dual-Dirac*),

tels que $\beta \otimes_A \alpha = \mathbb{1}_\mathbb{C}$ dans $KK^\Gamma(\mathbb{C}, \mathbb{C})$. Cette situation garantit la vérification de la propriété de Baum-Connes pour Γ *avec coefficients*. Plus généralement, nous pouvons nous contenter de

construire un élément $\gamma := \beta \otimes_A \alpha \in KK^\Gamma(\mathbb{C}, \mathbb{C})$ vérifiant une certaine condition topologique concernant les Γ-espaces propres (nommé alors *élément* γ). Dans cette situation, un résultat de J.-L. Tu [191] affirme que le morphisme d'assemblage μ_B^Γ est *injectif*, pour toute Γ-C^*-algèbre B. Des considérations analogues sont vraies pour un groupe localement compact G quelconque.

La situation idéale lorsque $\gamma = \mathbb{1}_\mathbb{C}$ dans $KK^G(\mathbb{C}, \mathbb{C})$ se traduit dans le contexte catégoriel de Meyer-Nest par dire que $\mathscr{L} = \mathscr{K}\mathscr{K}^G$ et dans ce cas nous disons que G *vérifie la propriété de Baum-Connes forte*. Notons que la définition de la sous-catégorie $\mathscr{L}$ dépend directement des sous-groupes compacts de G, c'est à dire, de la torsion de G lorsque G est discret. Comme conséquence, la vérification ou non de la propriété de Baum-Connes *forte* pour un groupe quantique discret $\widehat{\mathbb{G}}$ dépend directement de la torsion d'un tel groupe quantique.

Lorsque G vérifie la propriété de Baum-Connes, des nombreuses propriétés géométriques liées à G en découlent [136], notamment la *conjecture de Novikov*. Étant donné que la géométrie liée à un groupe quantique $\mathbb{G}$ doit être vue comme une notion "virtuelle" d'après la dualité de Gelfand, ce type de conséquences n'auront pas un analogue dans le contexte quantique. Nous devrons plutôt nous concentrer sur les applications concernant la structure des C^*-algèbres. Dans ce sens, rappelons la *conjecture de Kadison-Kaplansky*: *"si Γ est un groupe discret sans torsion, alors $C_r^*(\Gamma)$ ne possède pas d'idempotents non triviaux"*.

Une application plus directe de la propriété de Baum-Connes serait le calcul explicite de la K-théorie de la C^*-algèbre réduite d'un groupe discret Γ. En effet, puisque le groupe $K_*^{top}(\Gamma; \mathbb{C})$ est construit de façon géométrique, un expert en topologie pourra appliquer des techniques de suites spectrales afin de le calculer explicitement. Ainsi, l'éventuel isomorphisme de l'application d'assemblage donnerait le calcul correspondant pour la K-théorie de $C_r^*(\Gamma)$. Observons toutefois que la reformulation de R. Meyer et R. Nest de la propriété de Baum-Connes change essentiellement cette perspective : le groupe de nature topologique $K_*^{top}(\Gamma; \mathbb{C})$ sera maintenant un groupe de K-théorie dont la construction n'est pas tellement parlante comme dans le cas de l'utilisation des espaces classifiants. De toutes manières, le contexte catégoriel développé par R. Meyer et R. Nest [134], [131] fournit également des outils, en s'appuyant sur des techniques d'algèbre homologique, pour faire aboutir un tel calcul.

Dans ce contexte nous pouvons rappeler un problème classique en algèbre d'opérateurs et qui reste ouvert à l'heure actuelle : *étant donnés $n, m \in \mathbb{N}$ et si $\mathbb{F}_n$ et $\mathbb{F}_m$ désignent les groupes libres à n et m générateurs, respectivement ; est-ce que $\mathcal{L}(\mathbb{F}_n) \cong \mathcal{L}(\mathbb{F}_m)$ si et seulement si $n = m$?*, où $\mathcal{L}(\Gamma)$ denote l'algèbre de von Neumann associée à un groupe discret Γ. Si au lieu de considérer les algèbres de von Neumann nous considérons les C^*-algèbres réduites, la question a été répondue affirmativement grâce au calcul direct des groupes de K-théorie. En effet, M. Pimsner et D. Voiculescu ont montré dans [152] que $K_0(C_r^*(\mathbb{F}_n)) = \mathbb{Z}$ et que $K_1(C_r^*(\mathbb{F}_n)) = \mathbb{Z}_n$. Il faut remarquer que J. Cuntz avait déjà calculé dans [43] la K-théorie pour les C^*-algèbres *maximales* d'un groupe libre en obtenant le même résultat que M. Pimsner et D. Voiculescu. Cela l'avait motivé à introduire la notion de *K-moyennabilité* dans [44], lui permettant de donner une raison conceptuelle à la distinction des C^*-algèbres réduites et maximales du groupe libre au niveau de la K-théorie.

Ces deux remarquables problématiques en algèbre d'opérateurs : existence d'idempotents non triviaux et calcul des groupes de K-théorie, ont été abordées avec beaucoup de succès dans le contexte des groupes quantiques. Dans le "bestiaire" des groupes quantiques nous pouvons signaler les groupes quantiques compacts $SU_q(2)$ (avec $q \in (0, 1)$), $O^+(n)$, $U^+(n)$ et S_N^+ comme étant les exemples fondamentaux de la théorie et donc ils fournissent la première cible lorsque nous voulons résoudre des problèmes précis.

C. Voigt a commencé en 2010 une série d'articles (avec diverses collaborations avec R. Nest et R. Vergnioux) où il a obtenu des résultats favorables au développement des problématiques précédentes [140], [208], [210], [211], [212].

D'une part, l'étude du phénomène de torsion pour un groupe quantique montre que $\widehat{SU_q(2)}$, $\widehat{O^+(n)}$ et $\widehat{U^+(n)}$ sont sans torsion tandis que $\widehat{S_N^+}$ non (et sa torsion est due uniquement à l'action ergodique canonique de S_N^+ sur $\mathbb{C}^N$). Ils possèdent également la propriété quantique de Baum-Connes *forte*. Or, en rapport avec la conjecture de Kadison-Kaplansky, nous obtenons le résultat suivant : l'analogue de cette conjecture est vrai pour $O^+(n)$ mais non pour $SU_q(2)$ [210]. Autrement dit, $C_r^*(O^+(n))$ ne contient pas d'idempotents non triviaux tandis que $C(SU_q(2))$ en contient beaucoup.

D'autre part, une fois que la propriété de Baum-Connes forte est établie pour les groupes quantiques duaux $\widehat{SU_q(2)}$, $\widehat{O^+(n)}$, $\widehat{U^+(n)}$ et $\widehat{S_N^+}$; la propriété de K-moyenabbilité en découle automatiquement et les techniques d'algèbre homologique développées par R. Meyer et R. Nest ont permis à C. Voigt d'obtenir des calculs précis sur la K-théorie des C^*-algèbres de ces groupes quantiques (et aussi d'un groupe quantique libre et du groupe quantique d'automorphismes d'une C^*-algèbre de dimension finie) :

	$\mathbf{K_0}$	$\mathbf{K_1}$
$\mathbf{SU_q(2)}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbf{O^+(n)}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbf{U^+(n)}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$
$\mathbf{U_1^+ * \ldots * U_k^+ * O_1^+ * \ldots * O_l^+}$	$\mathbb{Z}$	$\mathbb{Z}^{2k}\oplus\mathbb{Z}^{l}$
$\mathbf{S_N^+}$	$\mathbb{Z}^{N^2-2N+2}$	$\mathbb{Z}$
$\mathbf{Qut}(\mathcal{M}_\mathbf{n}(\mathbb{C}))$	$\mathbb{Z}\oplus\mathbb{Z}_n$	$\mathbb{Z}$
$\mathbf{Qut}\Big(\mathcal{M}_{\mathbf{n_1}}(\mathbb{C})\oplus\ldots\oplus\mathcal{M}_{\mathbf{n_r}}(\mathbb{C})\Big)$	$\mathbb{Z}^{(r-1)^2+1}\oplus\mathbb{Z}^{2r-1}_{gcd(n_1,\ldots,n_r)}$	$\mathbb{Z}$

En particulier, le calcul de la K-théorie pour S_N^+ entraîne le résultat de classification suivant : $C_r^*(S_N^+)\cong C_r^*(S_{N'}^+)$ si et seulement si $N=N'$ [212].

Une manière effective de produire de nouveaux groupes consiste à faire des opérations entre deux ou plus groupes donnés. Ainsi nous pouvons signaler les constructions remarquables suivantes : produit direct, produit semi-direct, biproduit croisé, produit libre et produit en couronne. Dans un premier temps, il paraît raisonnable d'analyser quelle est la classe de groupes dans laquelle la propriété de Baum-Connes reste vraie. Il s'agit d'une question élémentaire, mais certaines de ces constructions posent des difficultés non triviales et méritent ainsi une analyse soigneuse.

Dans ce contexte, une série d'articles de J. Chabert, S. Echterhoff et H. Oyono-Oyono ont établi la stabilité de la propriété de Baum-Connes par quelques constructions de groupes. Plus précisément, l'article [34] de J. Chabert établit que la propriété de Baum-Connes est stable pour un produit semi-direct de groupes localement compacts sous certaines hypothèses. Toutefois, un an plus tard et en collaboration avec S. Echterhoff, ils obtiennent dans l'article [35] le même résultat que précédemment mais en remarquant qu'une des hypothèses initiales était superflue. Dans ce même article, J. Chabert et S. Echterhoff ont étudié également la stabilité de la propriété de Baum-Connes pour les sous-groupes fermés d'un groupe localement compact et pour les produits directs de groupes localement compacts. En ce qui concerne le cas des groupes discrets, H. Oyono-Oyono les a étudié en détail dans [141] [142], [144] et [143]. Dans le dernier, il démontre que la propriété de Baum-Connes est stable par rapport aux extensions de groupes discrets sous certaines conditions et

des conséquences remarquables s'en tirent. Notamment, la propriété de Baum-Connes est stable par les extensions centrales de groupes discrets, par les produits semi-directs (sous les hypothèses analogues à celles de Chabert-Echterhoff) et par les produits directs de groupes discrets. Une application directe de ces résultats est la suivante : *"tous les groupes des tresses pures vérifient la propriété de Baum-Connes"*.

Un premier objectif de la présente thèse a été la généralisation de ces propriétés de stabilité pour les groupes quantiques. En effet, les constructions mentionnées ci-dessus peuvent être réalisées pour les groupes quantiques compacts. Plus précisément, les travaux de S. Wang [216], [215] donnent une version quantique du produit direct, semi-direct et libre de groupes. L'article de J. Bichon [22] donne une version quantique du produit en couronne par un groupe de permutations (qui a été généralisée pour un groupe quantique d'automorphismes quelconque dans [66] par P. Fima et L. Pittau). L'article de S. Vaes et L. I. Vainerman [196] donne une version très générale de la notion d'*extension* pour les groupes quantiques localement compacts en obtenant une notion de biproduit croisé dans ce contexte. L'article de P. Fima, K. Mukherjee et I. Patri [65] donne quant à lui une version beaucoup plus précise de cette construction appelée *biproduit croisé compact*. Donc on se demande si la propriété de Baum-Connes quantique est stable par ces constructions de groupes quantiques.

D'une part, R. Vergnioux et C. Voigt ont montré dans [208] que la propriété de Baum-Connes forte est stable par un produit libre de groupes quantiques discrets. Pour cela, R. Vergnioux et C. Voigt suivent les idées de G. Kasparov et G. Skandalis dans [99] afin d'appliquer la méthode Dirac-dual Dirac dans un contexte quantique de la théorie de Bass-Serre. En particulier, $\widehat{O^+(n)}$ et $\widehat{U^+(n)}$ vérifient la propriété quantique de Baum-Connes *forte*.

D'autre part, pour un produit direct quantique, un produit semi-direct quantique et un biproduit croisé compact, nous présentons ici une preuve suivant la philosophie des cas classiques de J. Chabert, S. Echterhoff et H. Oyono-Oyono expliqués ci-dessus tout en utilisant la machinerie de Meyer-Nest. Il est opportun de signaler que l'hypothèse *sans torsion* force un biproduit croisé compact à devenir un produit semi-direct quantique. Ainsi, la propriété de Baum-Connes correspondante ne fournit pas d'information remarquable dans ce cas, le cas *avec torsion* présentant des problèmes théoriques majeures. L'étude de la propriété de Baum-Connes pour un produit direct quantique donne lieu à une connexion avec la formule de Künneth de manière analogue à ce qui a été démontré dans [37] par J. Chabert, S. Echterhoff and H. Oyono-Oyono pour les groupes localement compact classiques.

Lorsque nous analysons les articles originaux des cas classiques, nous observons que les difficultés se trouvent lors du traitement du groupe de K-homologie équivariante en relation avec l'opération réalisée avec les groupes. Cela entraîne un traitement minutieux et technique du membre de gauche du morphisme d'assemblage. Grâce à l'approche catégoriel de Meyer-Nest, ces technicités disparaissent et nous sommes capables d'obtenir une preuve plus conceptuelle. Si nous voulons expliquer de manière simple la stratégie générale que nous avons suivi, nous pouvons dire qu'il s'agit de *transporter l'opération en question entre les groupes (quantiques) vers une décomposition convenable de la catégorie de Kasparov*.

À la connaissance de l'auteur, il est une question ouverte de savoir si la propriété *usuelle* de Baum-Connes quantique est préservée par passage aux sous-groupes quantiques. Mais la condition de *divisible* garantit l'invariance de la propriété de Baum-Connes *forte* [208]. Nous montrons que le même résultat d'invariance est vrai pour la propriété *usuelle* de Baum-Connes. En ce qui concerne la stabilité de la propriété de Baum-Connes pour une extension quelconque de groupes quantiques au sens de Vaes-Vainerman, cela reste en dehors du champ de la présente thèse dû aux difficultés techniques de cette construction par rapport à la formulation de la propriété de Baum-Connes

quantique en toute généralité.

Concernant un produit en couronne d'un groupe quantique compact $\mathbb{G}$ dont le dual vérifie la propriété de Baum-Connes forte par S_N^+, $\mathbb{G} \wr_* S_N^+$, la propriété de stabilité découle de la description complète de la torsion d'un tel produit en couronne. En particulier, un produit en couronne libre n'est *jamais* sans torsion. Pour cela, nous réalisons d'abord une complète classification des actions de torsion pour un produit libre quantique, ce qui permet de donner une formulation adéquate de la propriété de Baum-Connes *forte* pour un produit en couronne libre.

Cette étude détaillée d'un produit en couronne libre, nous a permis d'achever une application majeure, à savoir un calcul explicite de K-théorie. Plus précisément, nous menons à bien le calcul de la K-théorie, pour trois choix pertinents du groupes quantique $\mathbb{G}$, pour le groupe quantique compact de Lemeux-Tarrago qui est monoïdallement équivalent à un produit en couronne libre [120]. Il est judicieux de signaler qu'aussi bien la propriété d'invariance pour un produit en couronne libre que les calculs de K-théorie intègrent un travail de collaboration avec A. Freslon [127].

Afin d'achever les objectifs précédemment exposés nous avons essayé de faire le mémoire aussi auto-contenu et détaillé que possible selon l'organisation suivante.

Le premier chapitre *Background* consiste à établir tous les outils nécessaires pour aborder les problèmes de la propriété de Baum-Connes, de la torsion et du calcul de la K-théorie. Notamment, nous donnons un exposé très détaillé sur la machinerie de Meyer-Nest afin de présenter aussi nettement que possible la reformulation de la propriété de Baum-Connes et son analogue dans le cas quantique. Nous détaillons également la théorie générale des groupes quantiques compacts d'après S. L. Woronowicz et la construction explicite des produits croisés dans ce cadre.

Le deuxième chapitre *Construction of Compact Quantum Groups* présente les exemples les plus importants de groupes quantiques compacts en regroupant tous les résultats connus et d'intérêt pour la présente thèse durant les 30 dernières années.

Le troisième chapitre *Stability properties for the BCc* doit être vu comme le "noyau" de la thèse où nous développons la stabilité de la propriété de Baum-Connes sous certaines constructions de groupes quantiques. Notamment, nous traitons les cas d'intérêt suivants : un groupe quantique compact, le dual d'un groupe de Lie compact, connexe et avec groupe fondamental sans torsion, un produit direct quantique, un produit semi-direct quantique, un biproduit croisé compact, un produit libre quantique et un produit en couronne libre. Dans chacune de ces situations, nous analysons également le phénomène de torsion et la propriété de K-moyennabilité.

Le quatrième chapitre *An application: the K-theory for the Lemeux-Tarrago's* $\widehat{\mathbb{H}}_q < \mathbb{G} * \widehat{SU_q}(2)$ peut être vu comme l'apogée de la recherche réalisée en cette thèse. Nous illustrons une application majeure des propriétés de stabilité obtenues dans le chapitre 3, à savoir nous menons à bien le calcul explicite de la K-théorie de $C(\mathbb{H}_q)$ pour trois choix pertinents de $\mathbb{G}$: a) lorsque $\mathbb{G}$ est un groupe orthogonal libre, b) lorsque $\mathbb{G}$ est un groupe quantique libre et c) lorsque $\mathbb{G}$ est le groupe classique libre à n générateurs.

Le cinquième et dernier chapitre *Conclusion: open questions and possible lines of attack* est consacré à nous questionner sur les résultats obtenus et sur les résultats à obtenir idéalement. Notamment, cela tourne autour de quatre axes : i) stabilité de la propriété de Baum-Connes (resp. *forte*) pour les constructions traités dans le chapitre 3, ii) stratégie des *tores maximaux* pour achever la propriété de Baum-Connes *forte*, iii) stratégie pour les calculs de la K-théorie dans des nouveaux examples issu d'un produit semi-direct quantique ou un produit en couronne libre et iv)

problématique de la formulation de la propriété de Baum-Connes pour un groupe quantique discret arbitraire (non nécessairement sans torsion).

En dehors de ces chapitres, nous avons inclut dans ce mémoire deux annexes qui servent d'appui à tous les résultats présentés au long du document. L'annexe A est consacré à des faits généraux sur différents objets de base tels comme les C^*-algèbres, les algèbres de von Neumann, les modules hilbertiens ou les algèbres de multiplicateurs. L'annexe B est consacré à des faits généraux sur les catégories et les catégories tensorielles.

Chapter 1

Background

In the first chapter we establish all the foundations for the good understanding and development of the thesis.

First of all, in Section 1.1 we point out all the conventions and notations used in the document and they should be kept in mind. Section 1.2 is devoted to the detailed exposition of the general categorical framework for the Meyer-Nest's Baum-Connes conjecture reformulation, which is complemented by Section 1.7 where we apply this machinery for (torsion-free) discrete quantum groups and establish thus the quantum counterpart of the Baum-Connes conjecture.

To this end, there are three aspects that must be exposed as well. Firstly, Section 1.3 gives the general picture of compact quantum groups in the sense of S. L. Woronowicz (representation and spectral theory) including the study of induced actions from discrete quantum subgroups. Secondly, in Section 1.5 we give a detailed proof of the universal property of a (reduced) crossed product by a discrete quantum group and investigate further properties of interest for the study of the Baum-Connes conjecture's stability properties. Thirdly, in Section 1.6 we present the notion of torsion for a discrete quantum group.

1.1 Conventions and notations

Conventions

- All locally compact groups are supposed to be *second countable*.
- All C^*-algebras are supposed to be *separable* and all ideals of C^*-algebras are supposed to be *two-sided closed ideals*. All Hilbert modules are supposed to be *countably generated*.
- We use the symbol $\circ$ to denote the composition of maps. If the context is clear, then we omit such a symbol.
- We use the symbol $\sqcup$ to denote the disjoint unions.
- If $\mathcal{S}$ is any set, then we denote by $\#\mathcal{S}$ the cardinal of the set $\mathcal{S}$ (possibly ∞). The symbol $\varnothing$ stands for the empty set.
- If $\mathscr{G}$ is a set of generators and $\mathscr{R}$ is a set of relations on the generators, the corresponding universal C^*-algebra (if it exists) is denoted by $C^*\langle \mathscr{G} \mid \mathscr{R} \rangle$.

  If A is a C^*-algebra and $\mathcal{S}$ is a subset of elements in A, then we write $\langle \mathcal{S} \rangle := C^*\langle \mathcal{S} \cup \mathcal{S}^* \rangle$ for the corresponding C^*-subalgebra of A generated by $\mathcal{S}$, that is, the intersection of all C^*-subalgebras of A containing $\mathcal{S}$. In this case, the elements of $\mathcal{S}$ are called *generators* of $\langle \mathcal{S} \rangle$.
- If E is a $\mathbb{C}$-vector space and $\mathcal{S}$ is a subset of vectors of E, then we write $span\{S\}$ for the corresponding $\mathbb{C}$-vector subspace generated by $\mathcal{S}$.
- If $(E, ||\cdot||)$ is a normed $\mathbb{C}$-vector space and $\mathcal{S}$ is a subset of vectors in E, then we write $\overline{span}\{\mathcal{S}\} := [span\{\mathcal{S}\}]$ for the corresponding normed $\mathbb{C}$-vector subspace generated by $\mathcal{S}$.

  In particular, if $F \subset E$ is a vector subspace, we write $[F] := \overline{F}^{||\cdot||}$ for the $||\cdot||$-closure of F in E.

  We write E^* for the corresponding *dual space*.
- The commutativity of functor's diagrams are considered with respect to the notion of *equivalence of functors*.
- If $\mathscr{C}$ is a category and $\mathcal{S}$ is a subset of objects in $\mathscr{C}$, then we write $\langle \mathcal{S} \rangle$ for the corresponding full subcategory of $\mathscr{C}$ generated by $\mathcal{S}$, that is, the intersection of all full subcategories of $\mathscr{C}$ containing $\mathcal{S}$.
- All additive categories are supposed to have *countable direct sums*. If F is an *additive* functor on such an additive category, it is, by definition, compatible with *finite* direct sums. Whenever we require F to be compatible with *infinite (countable)* direct sums, it will be explicitly indicated.
- We use the symbol $\mathscr{A}$ to denote abelian categories. The category of abelian groups is denoted by $\mathscr{A}b$ and the (locally small) category of sets is denoted by Set.

- If H is a Hilbert space and $\mathcal{S} \subset \mathcal{B}(H)$ is any subset of operators on H, then we denote by $\mathcal{S}'$ the commutant of $\mathcal{S}$, that is,

$$\mathcal{S}' := \{T \in \mathcal{B}(H) \mid T \circ X = X \circ T, \text{ for all } X \in \mathcal{S}\}$$

  We denote by $\mathcal{S}'' := \left(\mathcal{S}'\right)'$ the bicommutant of $\mathcal{S}$.

- If A is a C^*-algebra, we denote by A^+ the convex cone of its positive elements.

- If M is a von Neumann algebra, we denote by M_* the pre-dual of M.

- If A is a C^*-algebra, H is a Hilbert A-module and $N \subset H$ is a Hilbert A-submodule, then given $\xi \in H$ we denote by $[\xi] \in H/N$ the image of ξ by the quotient map. This notation shall be applied in other contexts with quotients.

- If H_1, H_2 are Hilbert spaces, we denote by $\mathcal{B}(H_1, H_2)$ the C^*-algebra of linear operators between H_1 and H_2 and by $\mathcal{K}(H_1, H_2)$ the C^*-algebra of compact operators between H_1 and H_2. If $H_1 = H = H_2$, we write $\mathcal{B}(H)$ for the linear operators on H and $\mathcal{K}(H)$ for the compact operators on H.

- If A is a C^*-algebra and H_1, H_2 are Hilbert A-modules, we denote by $\mathcal{L}_A(H_1, H_2)$ the C^*-algebra of adjointable operators between H_1 and H_2 and by $\mathcal{K}_A(H_1, H_2)$ the C^*-algebra of the compact adjointable operators between H_1 and H_2. If $H_1 = H = H_2$, we write $\mathcal{L}_A(H)$ for the adjointable operators on H and $\mathcal{K}_A(H)$ for the compact adjointable operators on H.

- Hilbert A-modules are considered to be *right A-modules*, so that the corresponding inner products are considered to be conjugate-linear on the left and linear on the right.

- If H is a finite dimensional Hilbert space and $\{\xi_1, \ldots, \xi_{dim(H)}\}$ is an orthonormal basis in H, the associated *matrix units* in $\mathcal{B}(H)$ are denoted by $\{m_{ij}\}_{i,j=1,\ldots,dim(H)}$ and we have the following relations for all $i, j, k, r, s = 1, \ldots, dim(H)$,

$$m_{ij} \cdot \xi_k = \xi_i \delta_{jk},\ \xi_k \cdot m_{ij} = \xi_j \delta_{ki} \text{ and } m_{ir} \cdot m_{sj} = \delta_{r,s} m_{ij}$$

  The *coordinate linear forms* on $\mathcal{B}(H)$ with respect to the basis $\{\xi_1, \ldots, \xi_{dim(H)}\}$ are denoted by $\omega_{\xi_i,\xi_j} := \omega_{i,j}$ and defined by

$$\omega_{\xi_i,\xi_j}(T) := \langle \xi_i, T(\xi_j) \rangle, \text{ for all } i, j = 1, \ldots, dim(H) \text{ and all } T \in \mathcal{B}(H)$$

  Remark that $\omega_{i,j}^* = \omega_{j,i}$, for all $i, j = 1, \ldots, dim(H)$. Notice that the analogous definition is valid for any vectors $\xi, \eta \in H$: $\omega_{\xi,\eta}(T) := \langle \xi, T(\eta) \rangle$, for all $T \in \mathcal{B}(H)$.

- Let H be a complex vector space. We denote by $\overline{H}$ its complex conjugate, whose underlying additive group is the same as the one of H and whose scalar multiplication is given by complex conjugation of scalars. In order to distinguish the vectors of H from those of $\overline{H}$, given an element ξ of their common underlying additive group, we write $\overline{\xi}$ to specify that it is a vector in $\overline{H}$. We denote by H^* its dual vector space formed by all linear functionals on H. If H is a Hilbert space, then we still denote by H^* its *topological* dual space formed by all *continuous* linear functionals on H and we refer to it simply as dual space of H.

Let H be a Hilbert space. The inner product of $\overline{H}$ is defined by $\langle \overline{\eta}, \overline{\xi} \rangle := \langle \xi, \eta \rangle$, for all $\xi, \eta \in H$. Every linear operator $T \in \mathcal{B}(H)$ gives rise to a *linear* operator $\overline{T} \in \mathcal{B}(\overline{H})$ defined by the same action of T, that is, $\overline{T}(\overline{\xi}) := \overline{T(\xi)}$, for all $\overline{\xi} \in \overline{H}$. We denote by $j : \mathcal{B}(H) \longrightarrow \mathcal{B}(H^*)$ the map that sends an operator to its dual. Namely, $j(T)(\omega) := \omega \circ T$, for all $T \in \mathcal{B}(H)$ and all $\omega \in H^*$. Sometimes, the dual operator of $T \in \mathcal{B}(H)$ is denoted by T^t.

We recall that, by virtue of the celebrated Riesz-Fréchet representation theorem, $\overline{H}$ is identified to H^* by means of the *linear* isometry $\rho : \overline{H} \xrightarrow{\sim} H^*$ defined by $\rho(\overline{\xi}) := \langle \cdot, \xi \rangle$, for all $\overline{\xi} \in \overline{H}$. By means of this identification, the scalar product on H^* is given by $\langle \rho(\overline{\xi}), \rho(\overline{\eta}) \rangle := \langle \overline{\xi}, \overline{\eta} \rangle$, for every $\xi, \eta \in H$. Moreover, the map $j : \mathcal{B}(H) \longrightarrow \mathcal{B}(\overline{H})$ is given by

$$j(T)(\overline{\xi}) = \overline{T^*(\xi)},$$

for all $T \in \mathcal{B}(H)$ and all $\overline{\xi} \in \overline{H}$, where $T^* \in \mathcal{B}(H)$ denotes the adjoint operator of T. For this, we have just to remark that $\rho \circ T^* = T^t \circ \rho$, which is a straightforward computation. Observe that j is an anti-multiplicative linear $*$-homomorphism.

By abuse of notation we still denote by $j : \mathcal{B}(H) \otimes C \longrightarrow \mathcal{B}(\overline{H}) \otimes C$ the map defined by $j \otimes *$, where C is any C^*-algebra and $*$ is the conjugation map on C.

Assume that H is finite dimensional with orthonormal basis $\{\xi_1, \ldots, \xi_{dim(H)}\}$. The corresponding dual basis in $H^* \cong \overline{H}$ is denoted by $\{\xi_1^*, \ldots, \xi_{dim(H)}^*\} := \{\omega_1, \ldots, \omega_{dim(H)}\}$.

Remark that $\omega_{\xi_i^*, \xi_j^*}(j(T)) = \omega^*_{\xi_i, \xi_j}(T^*)$, for all $i, j = 1, \ldots, dim(H)$ and all $T \in \mathcal{B}(H)$, where we use the identification $\overline{\xi}_i \overset{\rho}{\cong} \xi_i^*$, for all $i = 1, \ldots, dim(H)$.

- The *algebraic* tensor product between two structures is denoted by $\odot$.

  The *minimal* tensor product between two C^*-algebras is denoted by $\otimes$.

  The *maximal* tensor product between C^*-algebras is denoted by $\underset{\max}{\otimes}$.

  The *exterior* tensor product between two Hilbert modules is denoted by $\otimes$.

  The *interior* tensor product between two Hilbert modules with respect to a $*$-homomorphism ϕ is denoted by $\underset{\phi}{\otimes}$.

  In any of the previous cases, the *elementary tensors* in the corresponding tensor product are denoted simply by $\otimes$ and the context will distinguish the specific situation in which we are working. The elementary tensors of a *interior* tensor product will be usually denoted by $\underset{\phi}{\otimes}$ for more clarity of the exposition.

- If A and B are two C^*-algebras, $\Sigma : A \otimes B \longrightarrow B \otimes A$ denotes the flip map. We use the same symbol Σ for the flip between tensor product of Hilbert modules.

  The symbol Σ is used as well for the suspension functor in the framework of triangulated categories. The context will distinguish the specific situation in which we are working.

- Given a family of normed vector spaces $\{(E_i, ||\cdot||)\}_{i \in I}$, an element $x := (x_i)_{i \in I} \in \prod_{i \in I} E_i$ is said to vanish at ∞ if for every $\epsilon > 0$ there exists a finite subset $J \subset I$ such that $||x_i|| < \epsilon$, for all $i \in I \backslash J$. In this case we write $x \xrightarrow{\infty} 0$ and we define

$$\bigoplus_{i \in I}^{c_0} E_i := \{x \in \prod_{i \in I} E_i \mid x \xrightarrow{\infty} 0\}$$

If $\{(E_i, ||\cdot||)\}_{i\in I}$ is a family of involutive Banach algebras, $\bigoplus_{i\in I}^{c_0} E_i$ is an involutive Banach algebra with the supremum norm.

- Given a family of normed vector spaces $\{(E_i, ||\cdot||)\}_{i\in I}$, an element $x := (x_i)_{i\in I} \in \prod_{i\in I} E_i$ is said to be bounded if $(||x_i||)_{i\in I} \in l^\infty(I)$. We define

$$\bigoplus_{i\in I}^{l^\infty} E_i := \{x \in \prod_{i\in I} E_i \mid x \text{ is bounded}\}$$

If $\{(E_i, ||\cdot||)\}_{i\in I}$ is a family of involutive Banach algebras, $\bigoplus_{i\in I}^{l^\infty} E_i$ is an involutive Banach algebra with the supremum norm.

Leg numbering notation

- Let A be a unital C^*-algebra and consider the tensor product $A \otimes A$. The leg numbering notation indicates the three obvious ways in which we can embed $A \otimes A$ in $A \otimes A \otimes A$. More precisely, given $x \in A \otimes A$ we write

$$x_{12} := x \otimes 1_A,\ x_{13} := (id_A \otimes \Sigma)(x_{12}),\ x_{23} := 1_A \otimes x$$

Likewise, if $T : A \otimes A \longrightarrow A \otimes A$ is any $*$-homomorphism, we can consider the corresponding legs of T as $*$-homomorphisms from $A \otimes A \otimes A$ to itself in accordance with the obvious embeddings mentioned above. More precisely, we write

$$T_{12} := T \otimes id_A,\ T_{13} := (id_A \otimes \Sigma) \circ T_{12} \circ (id_A \otimes \Sigma),\ T_{23} := id_A \otimes T$$

1.1.1 Remarks. 1. Sometimes, the leg numbering notation x_{ij} or T_{ij} can be found in this document as $\big[x\big]_{ij}$ or $\big[T\big]_{ij}$. The reason for this is only "esthetic" in order to stress the use of the corresponding legs and facilitate the reading of computations.

2. If A is not unital, the same notations stand using the multiplier algebra of A, $M(A)$, which is a unital C^*-algebra.

3. More generally, given $n, m \in \mathbb{N}$ with $n \leqslant m$ consider a faithful map of sets $\iota : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$. This defines an embedding $A^{\otimes n} \hookrightarrow A^{\otimes m}$. Given $x \in A^{\otimes n}$, its image in $A^{\otimes m}$ following the embedding ι is written in leg numbering notation as $x_{\iota(1),\ldots,\iota(n)}$. Given a $*$-homomorphism $T : A^{\otimes n} \longrightarrow A^{\otimes n}$, its leg following the embedding ι is written as $T_{\iota(1),\ldots,\iota(n)}$ meaning that T only acts only on the places $\iota(1), \ldots, \iota(n)$ in $A^{\otimes m}$.

4. Notice that similar notations are adopted when we consider tensor product of *different* C^*-algebras.

- Let H be a Hilbert space and A a unital C^*-algebra. The leg numbering notation can be applied also to the tensor product $H \otimes A$ following the obvious embeddings when we add a copy of A by the right. More precisely, given $x \in H \otimes A$ we define

$$x_{12} := x \otimes 1_A,\ x_{13} := (id_H \otimes \Sigma)(x_{12}),\ x_{23} := (\Sigma \otimes id_A)(x_{13})$$

Likewise, if $T : H \otimes A \longrightarrow H \otimes A$ is any homomorphism, we can consider the corresponding legs of T as homomorphisms from $H \otimes A \otimes A$ (or $A \otimes H \otimes A$) to itself in accordance with the obvious embeddings mentioned above. More precisely, we write

$$T_{12} := T \otimes id_A,\ T_{13} := (id_H \otimes \Sigma) \circ T_{12} \circ (id_H \otimes \Sigma),\ T_{23} := (id_A \otimes T) \circ (\Sigma \otimes id_A)$$

Sweedler notation

Let A be a unital C^*-algebra and consider the tensor product $A \odot A$. The Sweedler notation simplifies the expression of any element in this tensor product. Namely, we know that every element $x \in A \odot A$ can be written (in a *not* unique fashion) under the form $x = \sum_{i=1}^{n} a_{1i} \otimes a_{2i}$, for some $n \in \mathbb{N}$ and some $a_{1i}, a_{2i} \in A$ for all $i = 1, \ldots, n$. In this way, Sweedler notation suggests the following writing

$$x := a_{(1)} \otimes a_{(2)},$$

where the subscripts (1) and (2) point out the order of the factors in the tensor product. Notice that it is important to keep in mind this order. In particular, the Sweedler notation yields the following writing for the flip of x,

$$\Sigma(x) := a_{(2)} \otimes a_{(1)}$$

1.1.2 Remarks.

1. If $A \odot A$ is dense in $A \otimes A$, then by abuse of notation we still use the Sweedler notation for elementary tensors in the analytical tensor product. Namely, notice that by the density condition we can perform computations on the algebraic level using Sweedler notations and pass then to the analytical level.

2. If A is not unital, then we may work with its multiplier algebra $M(A)$, which is a unital C^*-algebra. However, every element in $x \in M(A \odot A)$ can *not* be written as a finite sum $x = \sum_{i=1}^{n} a_{1i} \otimes a_{2i}$, for some $n \in \mathbb{N}$ and some $a_{1i}, a_{2i} \in M(A)$ for all $i = 1, \ldots, n$.

   Nevertheless, Sweedler notation can be still applied with a very precise meaning and we must be careful in this case. See Notation 2.1.7 in [188] or [203] for more details.

3. Notice that similar notations are adopted when we consider tensor products of *different* C^*-algebras.

Multiplicative unitaries

The celebrated theory of *multiplicative unitaries* (developed by S. Baaj and G. Skandalis [7]) is not really used in this dissertation. Nevertheless, some notations and notions concerning multiplicative unitaries are mentioned in the document. We collect here the elementary language that we need for the convenience of the reader. We refer to the original article [7] or to Chapter 7 and Chapter 9 of [188] for a comprehensive and detailed exposition of the subject (we refer as well to [230] and [175]).

Let H be a Hilbert space and $V \in \mathcal{B}(H \otimes H)$ be a multiplicative unitary on H, that is, a unitary operator such that $V_{12} \circ V_{13} \circ V_{23} = V_{23} \circ V_{12}$. If V is a multiplicative unitary on H, then $V^{op} := \widehat{V} := \Sigma \circ V^* \circ \Sigma$ is again a multiplicative unitary on H. We have the following objects:

$$\begin{array}{rccl} \widehat{\Delta}_V : & \mathcal{B}(H) & \longrightarrow & \mathcal{B}(H \otimes H) \\ & T & \longmapsto & \widehat{\Delta}_V(T) := V^* \circ (id \otimes T) \circ V \end{array}$$

$$\begin{array}{rccl} \Delta_V : & \mathcal{B}(H) & \longrightarrow & \mathcal{B}(H \otimes H) \\ & T & \longmapsto & \Delta_V(T) := V \circ (T \otimes id) \circ V^* \end{array}$$

$$\widehat{A}_0(V) := \{\widehat{a}_\omega := (id \otimes \omega)(V) \mid \omega \in \mathcal{B}(H)_*\} \subset \mathcal{B}(H)$$
$$A_0(V) := \{a_\omega := (\omega \otimes id)(V) \mid \omega \in \mathcal{B}(H)_*\} \subset \mathcal{B}(H)$$

$$\widehat{A}(V) := \|\cdot\| - \text{closure of } \widehat{A}_0(V) \text{ in } \mathcal{B}(H)$$
$$A(V) := \|\cdot\| - \text{closure of } A_0(V) \text{ in } \mathcal{B}(H)$$

The pairs $(\widehat{A}(V), \widehat{\Delta}_V)$ and $(A(V), \Delta_V)$ are called *left leg of* V and *right leg of* V, respectively.

1.2 Triangulated categories

Triangulated categories are one of the main achievements in abstract homotopy theory during the 60's thanks to the work of A. Grothendieck and J. L. Verdier. Roughly speaking, a triangulated category is an additive category with a *translation functor* and a distinguished class of triangles. The main motivation for this notion is the axiomatization and understanding of the *derived category*.

If $\mathscr{C}$ is any additive category, let us consider the corresponding homotopy category $\mathscr{H}(\mathscr{C})$ (see Theorem B.2.11 for a definition). If $\mathscr{C}$ is abelian, it is not guarantee that $\mathscr{H}(\mathscr{C})$ is again an abelian category (take for instance $\mathscr{C}$ as the category of abelian groups). However, it can be shown that $\mathscr{H}(\mathscr{C})$ is a *triangulated category*, so that distinguished triangles should be regarded as the replacement of short exact sequences (indeed, it is not very hard to show that in an abelian triangulated category, every short exact sequence splits). In a simple way, the main problem consists in the construction of homology theories in order to describe obstructions for a given property. Hence, the derived category of $\mathscr{C}$ is the localization of $\mathscr{H}(\mathscr{C})$ with respect to quasi-isomorphisms and, in this way, it gives a general and abstract description of homology theories and its triangulated structure gives the appropriate understanding of the manipulation of long exact sequences.

The goal of this section is not to enter in the details of this abstract homological picture, but to introduce the general framework and the main useful results for the purpose of the present dissertation. Good references for more details are [138], [85] or [205].

An other example of triangulated category that is interesting for us is the *Kasparov category* (see Section 1.2.3), which allows the application of these abstract categorical theories to non commutative geometry. The most important innovation is the homological algebra developed in such a triangulated category. This homological algebra has been developed by R. Meyer and R. Nest in [134] and [131] inspired by previous work of different authors as A. Beligiannis [19], J. D. Christensen [40] and S. Eilenberg and J.C. Moore [57]. In the triangulated context, the corresponding homological algebra is always relative to some fixed *ideal* $\mathcal{J}$, which will be supposed

to be the *homomorphism-kernel* of some triangulated functor. We can then translate the classical notions of homological algebra relatively to such ideal $\mathcal{J}$, but sometimes we must be careful in the definitions in order to obtain a notion that only depends on the ideal and not in the functor itself. In Section 1.2.2 we give a first approach to this homological algebra in order to justify properly the reformulation of the Baum-Connes property in this categorical framework. In Section 1.2.4 we carry out a deeper analysis of this homological algebra in order to treat the derived category picture in the triangulated framework and obtain thus some tools for the existence of useful exact sequences.

1.2.1 Elementary facts

1.2.1.1 Definition. Let $\mathcal{T}$ be an additive category equipped with an additive auto-equivalence Σ (that is, a stable additive category). A triangle in $\mathcal{T}$ is a sequence of objects and homomorphisms in $\mathcal{T}$ of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$$

1.2.1.2 Definition. Let $\mathcal{T}$ be an additive category equipped with an additive auto-equivalence Σ. A homomorphism between two triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma(X')$ in $\mathcal{T}$ is the data of three homomorphisms $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$, $h : Z \longrightarrow Z'$ such that the following diagram is commutative

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma(X) \arrow[d, "\Sigma(f)"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma(X')
\end{tikzcd}$$

The triangles are said to be isomorphic if f, g, h are isomorphisms.

1.2.1.3 Definition. A triangulated category is the data $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ where $\mathcal{T}$ is an additive category, Σ is an additive auto-equivalence of $\mathcal{T}$ called *suspension functor* and Δ_Σ is a class of triangles in $\mathcal{T}$ called *distinguished triangles* such that

i) every triangle isomorphic to a distinguished one is a distinguished triangle and for every object $X \in Obj(\mathcal{T})$ the triangle

$$X \xrightarrow{id} X \longrightarrow 0 \longrightarrow \Sigma(X)$$

is a distinguished triangle,

ii) for every homomorphism $f : X \longrightarrow Y$ in $\mathcal{T}$, there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma(X)$$

where the object C_f is called *cone of* f and the corresponding distinguished triangle is called *cone triangle of* f,

iii) (*rotation axiom*) a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ is a distinguished one if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X) \xrightarrow{-\Sigma(u)} \Sigma(Y)$ is a distinguished one,

iv) if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma(X')$ are two distinguished triangles in $\mathcal{T}$, then every commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed] & \Sigma(X) \arrow[d, "\Sigma(f)"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma(X')
\end{tikzcd}$$

meaning that $g \circ u = u' \circ f$ can be completed into a homomorphism of triangles.

v) (*octahedron axiom*) if $X \xrightarrow{u} Y \longrightarrow Z' \longrightarrow \Sigma(X)$, $Y \xrightarrow{v} Z \longrightarrow X' \longrightarrow \Sigma(Y)$, $X \xrightarrow{v \circ u} Z \longrightarrow Y' \longrightarrow \Sigma(X)$ are distinguished triangles in $\mathcal{T}$, then there exists a distinguished triangle $Z' \longrightarrow Y' \longrightarrow X' \longrightarrow \Sigma(Z')$ such that the following diagram is commutative

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal, "id_X"'] & Y \arrow[r] \arrow[d, "v"] & Z' \arrow[r] \arrow[d, dashed] & \Sigma(X) \arrow[d, equal, "id_{\Sigma(X)}"] \\
X \arrow[r, "v \circ u"'] \arrow[d, "u"'] & Z \arrow[r] \arrow[d, equal, "id_Z"] & Y' \arrow[r] \arrow[d, dashed] & \Sigma(X) \arrow[d, "\Sigma(u)"] \\
Y \arrow[r, "v"'] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, equal, "id_{X'}"] & \Sigma(Y) \arrow[d] \\
Z' \arrow[r, dashed] & Y' \arrow[r, dashed] & X' \arrow[r, dashed] & \Sigma(Z')
\end{tikzcd}$$

1.2.1.4 Remarks. 1. Given a homomorphism $f : X \longrightarrow Y$ in $\mathcal{T}$, it is shown later on (see Remarks 1.2.1.18) that the cone triangle of f is unique up to isomorphism, which explains the notation C_f for the object of the axiom (*ii*) of the preceding definition.

2. Notice that if $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ is a triangulated category, we can consider the *opposite* category $\mathcal{T}^{op}$ which is again a triangulated one with suspension functor Σ^{-1}. Moreover, a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ in $\mathcal{T}$ becomes the triangle $\Sigma^{-1}(X) \xrightarrow{w^{op}} Z \xrightarrow{v^{op}} Y \xrightarrow{u^{op}} X$ in $\mathcal{T}^{op}$.

1.2.1.5 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category. A triangulated subcategory of $\mathcal{T}$ is an additive full subcategory $\mathcal{S}$ of $\mathcal{T}$ such that

i) every object of $\mathcal{T}$ isomorphic to an object in $\mathcal{S}$ is an object of $\mathcal{S}$,

ii) $\Sigma(\mathcal{S}) \subset \mathcal{S}$,

iii) if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ is a distinguished triangle in $\mathcal{T}$ with $X, Y \in Obj(\mathcal{S})$, then $Z \in Obj(\mathcal{S})$.

1.2.1.6 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category. A thick subcategory of $\mathcal{T}$ is a triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ such that if $X, Y \in Obj(\mathcal{T})$ are such that $X \oplus Y \in Obj(\mathcal{S})$, then $X, Y \in Obj(\mathcal{S})$.

1.2.1.7 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category. A localizing subcategory of $\mathcal{T}$ is a triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ such that every countable direct sum of objects of $\mathcal{S}$ is an object of $\mathcal{S}$.

1.2.1.8 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category. If $\mathcal{S} \subset Obj(\mathcal{T})$ is any class of objects in $\mathcal{T}$, we denote by $\langle \mathcal{S} \rangle$ the smallest triangulated subcategory of $\mathcal{T}$ such that

i) the objects of $\mathcal{S}$ are in $\langle \mathcal{S} \rangle$,

ii) every countable direct sum of objects of $\mathcal{S}$ is an object of $\langle \mathcal{S} \rangle$,

iii) the subcategory $\langle \mathcal{S} \rangle$ is thick.

In that case, we say that $\langle \mathcal{S} \rangle$ is the subcategory of $\mathcal{T}$ generated by $\mathcal{S}$.

1.2.1.9 Remarks. 1. Fix a class of objects $\mathcal{S}$ in a triangulated category $\mathcal{T}$. Then $\langle \mathcal{S} \rangle$ is well-defined. Namely, it is the *intersection of all triangulated subcategories of $\mathcal{T}$ satisfying the properties (i), (ii) and (iii) of the preceding definition.* Indeed, let $\mathcal{G} := \bigcap_{i \in I} \mathcal{S}_i$ such an intersection. Notice that $Obj(\mathcal{G}) = \bigcap_{i \in I} Obj(\mathcal{S}_i)$ by definition.

- $\mathcal{G}$ is clearly a triangulated subcategory of $\mathcal{T}$.
- $\mathcal{G}$ satisfies the properties (i), (ii) and (iii) of the preceding definition. Namely,
  i) by assumption we have that $\mathcal{S} \subset Obj(\mathcal{S}_i)$, for all $i \in I$. Hence, $S \subset Obj(\mathcal{G})$,
  ii) if D is a countable direct sum of objects of $\mathcal{S}$, then by assumption we have that $D \in Obj(\mathcal{S}_i)$, for all $i \in I$. Hence, $D \in Obj(\mathcal{G})$,
  iii) by assumption we have that $\mathcal{S}_i$ is thick for all $i \in I$. This means that for all $X, Y \in Obj(\mathcal{T})$ such that $X \oplus Y \in Obj(\mathcal{S}_i)$, we have $X, Y \in Obj(\mathcal{S}_i)$. This is true for every $i \in I$. Hence, whenever $X, Y \in Obj(\mathcal{T})$ are such that $X \oplus Y \in Obj(\mathcal{G})$, we have $X, Y \in Obj(\mathcal{G})$. In other words, $\mathcal{G}$ is a thick subcategory

2. By virtue of the rotation axiom of a triangulated category, it is clear that the axiom (iii) of a triangulated subcategory can be formulated with any pair of objects of a given distinguished triangle. More precisely, let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ be a distinguished triangle in $\mathcal{T}$ and let $\mathcal{S}$ be an additive full subcategory of $\mathcal{T}$, then, thanks to the rotation axiom, the following assertions are equivalent

   a) $X, Y \in Obj(\mathcal{S}) \Rightarrow Z \in Obj(\mathcal{S})$
   b) $X, Z \in Obj(\mathcal{S}) \Rightarrow Y \in Obj(\mathcal{S})$
   c) $Y, Z \in Obj(\mathcal{S}) \Rightarrow X \in Obj(\mathcal{S})$

3. We can show that any localizing subcategory is automatically a thick subcategory (see Remark 3.2.7 in [138]). Notice that [138] works in a more general framework. Namely, a localization subcategory is defined with respect to a given cardinal. We restrict our attention to the countable cardinal $\aleph_0$ as indicated in Section 1.1. Recall that all our additive categories are supposed to have *countable direct sums.*

4. With these definitions and the preceding remark, we observe that $\langle \mathcal{S} \rangle$ is the *localizing subcategory generated by* $\mathcal{S}$ meaning that $\langle \mathcal{S} \rangle$ is the smallest triangulated category containing the objects of $\mathcal{S}$ and stable with respect to countable direct sums.

1.2.1.10 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ and $(\mathcal{T}', \Sigma', \Delta_{\Sigma'})$ two triangulated categories and $F : \mathcal{T} \longrightarrow \mathcal{T}'$ an additive functor. We say that F is a triangulated functor if

i) F is stable meaning that $F \circ \Sigma \cong \Sigma' \circ F$,

ii) F transforms distinguished triangles into distinguished triangles.

1.2.1.11 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category, $\mathscr{A}$ any abelian category and $F : \mathcal{T} \longrightarrow \mathscr{A}$ a covariant (resp. contravariant) additive functor. We say that F is a homological (resp. co-homological) functor if for every distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ in $\mathcal{T}$, the sequence $F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z)$ is exact in $\mathscr{A}$.

1.2.1.12 Note. When triangulated structures are involved, triangulated or stable additive categories are needed and when exact sequences are involved, abelian categories are needed. Moreover, we shall work with *stable* abelian categories, so with *stable* homological functors. All these assumptions are clear by the context on each statement of the theory.

1.2.1.13 Remark. Given a triangulated functor $F : \mathcal{T} \longrightarrow \mathcal{T}'$ (or a stable homological functor $F : \mathcal{T} \longrightarrow \mathscr{A}$) we have a natural additive full subcategory of $\mathcal{T}$ defined by

$$\ker_{Obj}(F) := \{X \in Obj(\mathcal{T}) \mid F(X) \cong 0\}$$

In fact, $\ker_{Obj}(F)$ is a *thick subcategory* of $\mathcal{T}$. Namely,

- $\ker_{Obj}(F)$ is a triangulated subcategory of $\mathcal{T}$.

  i) If $X \in Obj(\mathcal{T})$ is isomorphic to an object $K \in \ker_{Obj}(F)$, then we have that $F(X) \cong F(K) \cong 0$, whence $X \in \ker_{Obj}(F)$.

  ii) Consider $K \in \ker_{Obj}(F)$. Since F is a triangulated (resp. stable homological) functor, then F is stable and we have

  $$F(\Sigma(K)) \cong \Sigma'(F(K)) \cong \Sigma'(0) \cong 0,$$

  whence $\Sigma(K) \in \ker_{Obj}(F)$.

  iii) Let $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma(X)$ be a distinguished triangle in $\mathcal{T}$ with $X, Y \in \ker_{Obj}(F)$. Since F is a triangulated (resp. stable homological) functor, then we have the following distinguished triangle in $\mathcal{T}'$ (resp. short exact sequence in $\mathscr{A}$)

  $$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow \Sigma'(F(X))$$

  $$(\text{resp. } F(X) \longrightarrow F(Y) \longrightarrow F(Z))$$

  By assumption we have that $F(X) \cong 0 \cong F(Y)$, which implies that the above distinguished triangle (resp. short exact sequence) is isomorphic to the distinguished triangle $0 \longrightarrow 0 \longrightarrow F(Z) \longrightarrow 0$ (resp. to the short exact sequence $0 \longrightarrow 0 \longrightarrow F(Z)$). In particular, $F(Z) \cong 0$ whence $Z \in \ker_{Obj}(F)$.

- $\ker_{Obj}(F)$ is thick. Consider two objects $X, Y \in Obj(\mathcal{T})$ such that $X \oplus Y \in \ker_{Obj}(F)$. Since F is an additive functor, then we have that $0 \cong F(X \oplus Y) \cong F(X) \oplus F(Y)$. This implies that $F(X) \cong 0 \cong F(Y)$ by definition of direct sum object.

Observe that $\ker_{Obj}(F)$ is a *localizing* subcategory whenever F is compatible with countable direct sums.

1.2.1.14 Remark. By virtue of the rotation axiom, if $F : \mathcal{T} \longrightarrow \mathscr{A}$ is a homological (resp. co-homological) functor, then for every distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ there exists a long exact sequence in $\mathscr{A}$

$$\ldots \to F(\Sigma^{i-1}(Z)) \to F(\Sigma^i(X)) \to F(\Sigma^i(Y)) \to F(\Sigma^i(Z)) \to F(\Sigma^{i+1}(X)) \to \ldots$$

$$(\text{resp. } \ldots \leftarrow F(\Sigma^{i-1}(Z)) \leftarrow F(\Sigma^i(X)) \leftarrow F(\Sigma^i(Y)) \leftarrow F(\Sigma^i(Z)) \leftarrow F(\Sigma^{i+1}(X)) \leftarrow \ldots)$$

1.2.1.15 Lemma. *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category. If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ is any distinguished triangle in $\mathcal{T}$, then we always have*

$$v \circ u = 0 = w \circ v$$

Proof. Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ be any distinguished triangle in $\mathcal{T}$. Firstly we observe that, by virtue of the rotation axiom, it is enough to prove that $v \circ u = 0$. Now, applying again the rotation axiom, we know that $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X) \xrightarrow{-\Sigma(u)} \Sigma(Y)$ is a distinguished triangle in $\mathcal{T}$ as well.

Apply the axiom (i) of a triangulated category to the object Z, so that $Z \xrightarrow{id} Z \longrightarrow 0 \longrightarrow \Sigma(Z)$ is a distinguished triangle in $\mathcal{T}$. We have then the following commutative diagram,

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "v"] & Z \arrow[r, "w"] \arrow[d, equal, "id"] & \Sigma(X) \arrow[r, "-\Sigma(u)"] \arrow[d, dashed] & \Sigma(Y) \arrow[d, "\Sigma(v)"] \\
Z \arrow[r, "id"'] & Z \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \Sigma(Z)
\end{tikzcd}$$

which can be completed into a homomorphism of triangles by virtue of the axiom (iv) of a triangulated category. In particular, we have $0 = \Sigma(v) \circ (-\Sigma(u)) = -\Sigma(v \circ u)$ so that $v \circ u = 0$ because Σ is an auto-equivalence in $\mathcal{T}$. ■

1.2.1.16 Proposition (Long exact sequence for homomorphisms)**.** *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category. Given an object $T \in Obj(\mathcal{T})$, the homomorphism functor $Hom_{\mathcal{T}}(T, \cdot\,)$ is homological and the homomorphism functor $Hom_{\mathcal{T}}(\,\cdot\,, T)$ is co-homological.*

Consequently, given any distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ in $\mathcal{T}$, there exist long exact sequences of abelian groups

$$\begin{aligned}
\ldots \to Hom_{\mathcal{T}}(T, \Sigma^{i-1}(Z)) &\to Hom_{\mathcal{T}}(T, \Sigma^i(X)) \to Hom_{\mathcal{T}}(T, \Sigma^i(Y)) \\
&\to Hom_{\mathcal{T}}(T, \Sigma^i(Z)) \to Hom_{\mathcal{T}}(T, \Sigma^{i+1}(X)) \to \ldots
\end{aligned}$$

$$\begin{aligned}
\ldots \leftarrow Hom_{\mathcal{T}}(\Sigma^{i-1}(Z), T) &\leftarrow Hom_{\mathcal{T}}(\Sigma^i(X), T) \leftarrow Hom_{\mathcal{T}}(\Sigma^i(Y), T) \\
&\leftarrow Hom_{\mathcal{T}}(\Sigma^i(Z), T) \leftarrow Hom_{\mathcal{T}}(\Sigma^{i+1}(X), T) \leftarrow \ldots
\end{aligned}$$

Proof. Let's prove the result for the functor $Hom_{\mathcal{T}}(T,\ \cdot\ )$ (the argument for $Hom_{\mathcal{T}}(\ \cdot\ ,T)$ is completely analogous). Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ be any distinguished triangle in $\mathcal{T}$. We have to show that

$$Hom_{\mathcal{T}}(T,X) \xrightarrow{u_*} Hom_{\mathcal{T}}(T,Y) \xrightarrow{v_*} Hom_{\mathcal{T}}(T,Z)$$

is an exact sequence of abelian groups.

Thanks to the previous lemma we know that $v \circ u = 0$ and so $(u \circ v)_* = v_* \circ u_* = 0$. In other words, $Im(u_*) \subset \ker(v_*)$. In order to show that $\ker(v_*) \subset Im(u_*)$, take any $f \in \ker(v_*)$.

Given the distinguished triangle above, the rotation axiom assures that $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X) \xrightarrow{-\Sigma(u)} \Sigma(Y)$ is a distinguished triangle in $\mathcal{T}$ as well. Given an object $T \in Obj(\mathcal{T})$, consider the corresponding distinguished triangle given by the axiom (i) of a triangulated category: $T \xrightarrow{id} T \longrightarrow 0 \longrightarrow \Sigma(T)$. Apply again the rotation axiom so that $T \xrightarrow{0} 0 \longrightarrow \Sigma(T) \xrightarrow{-id} \Sigma(T)$ is a distinguished triangle in $\mathcal{T}$ as well.

Since $f \in \ker(v_*)$ the following diagram is commutative,

$$\begin{tikzcd}
T \arrow[r, "0"] \arrow[d, "f"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \Sigma(T) \arrow[r, "-id"] \arrow[d, dashed, "h"] & \Sigma(T) \arrow[d, "\Sigma(f)"] \\
Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & \Sigma(X) \arrow[r, "\Sigma(u)"'] & \Sigma(Y)
\end{tikzcd}$$

and hence it can be completed via h into a homomorphism of triangles by virtue of the axiom (iv) of a triangulated category. In particular, we have $\Sigma(f) \circ (-id) = -\Sigma(u) \circ h$, that is, $\Sigma(f) = \Sigma(u) \circ h$ and so $f = u \circ \Sigma^{-1}(h)$. In other words, $f = u_*(\Sigma^{-1}(h)) \in Im(u_*)$. ■

1.2.1.17 Lemma (Five's lemma for triangulated categories)**.** *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and let*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma(X) \arrow[d, "\Sigma(f)"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma(X')
\end{tikzcd}$$

be a homomorphism of distinguished triangles in $\mathcal{T}$.

If f and g are isomorphisms, so it is h and consequently the triangles are isomorphic.

Proof. Fix the object $Z' \in Obj(\mathcal{T})$ and consider the homological functor $Hom_{\mathcal{T}}(Z',\ \cdot\ )$. By virtue of the preceding proposition, we can consider the following commutative diagram with exact lines,

$$\begin{tikzcd}
Hom_{\mathcal{T}}(Z',X) \arrow[r] \arrow[d, "f_*"] & Hom_{\mathcal{T}}(Z',Y) \arrow[r] \arrow[d, "g_*"] & Hom_{\mathcal{T}}(Z',Z) \arrow[r] \arrow[d, "h_*"] & Hom_{\mathcal{T}}(Z',\Sigma(X)) \arrow[r] \arrow[d, "\Sigma(f_*)"] & Hom_{\mathcal{T}}(Z',\Sigma(Y)) \arrow[d, "\Sigma(g_*)"] \\
Hom_{\mathcal{T}}(Z',X') \arrow[r] & Hom_{\mathcal{T}}(Z',Y') \arrow[r] & Hom_{\mathcal{T}}(Z',Z') \arrow[r] & Hom_{\mathcal{T}}(Z',\Sigma(X')) \arrow[r] & Hom_{\mathcal{T}}(Z',\Sigma(Y'))
\end{tikzcd}$$

Since f and g are isomorphisms by assumption, then f_*, g_*, $\Sigma(f_*)$ and $\Sigma(g_*)$ are isomorphisms as well. Hence the five's lemma for abelian categories guarantees that h_* is an isomorphism as well.

In particular, given $id_{Z'} \in Hom_{\mathcal{T}}(Z', Z')$ there exists a unique homomorphism $q \in Hom_{\mathcal{T}}(Z', Z)$ such that $id_{Z'} = h_*(q) = h \circ q$. In other words, q is a right inverse for h. Now, the same argument as above using the homological functor $Hom_{\mathcal{T}}(\,\cdot\,, Z')$ yields a left inverse for h and so we conclude that h is an isomorphism. ■

1.2.1.18 Remarks. 1. The five's lemma for triangulated categories can be stated for a more general class of triangles *not necessary distinguished*. Namely, it is enough to consider triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ such that

a) $v \circ v = 0 = w \circ v$

b) the homological functors $Hom_{\mathcal{T}}(T,\,\cdot\,)$ and $Hom_{\mathcal{T}}(\,\cdot\,, T)$ yield long exact sequences

$$\begin{aligned} \ldots \to Hom_{\mathcal{T}}(T,X) &\xrightarrow{u_*} Hom_{\mathcal{T}}(T,Y) \xrightarrow{v_*} Hom_{\mathcal{T}}(T,Z) \\ &\xrightarrow{w_*} Hom_{\mathcal{T}}(T,\Sigma(X)) \xrightarrow{\Sigma(u)_*} Hom_{\mathcal{T}}(T,\Sigma(Y)) \to \ldots \end{aligned}$$

$$\begin{aligned} \ldots \leftarrow Hom_{\mathcal{T}}(X,T) &\xleftarrow{u^*} Hom_{\mathcal{T}}(Y,T) \xleftarrow{v^*} Hom_{\mathcal{T}}(Z,T) \\ &\xleftarrow{w^*} Hom_{\mathcal{T}}(\Sigma(X),T) \xleftarrow{\Sigma(u)^*} Hom_{\mathcal{T}}(\Sigma(Y),T) \leftarrow \ldots \end{aligned}$$

for every object $T \in Obj(\mathcal{T})$.

For more details we refer to Chapter 1 of [138] (for instance, see Caution 1.1.16 and Proposition 1.1.20).

2. By virtue of the five's lemma, it is clear that the cone of a given homomorphism is unique up to isomorphism. Namely, given a homomorphism $f : X \longrightarrow Y$ in $\mathcal{T}$, consider two different cone triangle for f, say

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma(X) \text{ and } X \xrightarrow{f} Y \longrightarrow C'_f \longrightarrow \Sigma(X)$$

We can consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal, "id_X"] & Y \arrow[r] \arrow[d, equal, "id_Y"] & C_f \arrow[r] \arrow[d, dashed] & \Sigma(X) \arrow[d, equal, "id_{\Sigma(X)}"] \\ X \arrow[r, "f"'] & Y \arrow[r] & C'_f \arrow[r] & \Sigma(X) \end{tikzcd}$$

which can be completed into a homomorphism of triangles. Since id_X and id_Y are isomorphisms, the five's lemma yield that $C_f \cong C'_f$ as we wanted to show. In fact, the whole cone triangle associated to a homomorphism is unique up to isomorphism.

1.2.1.19 Proposition (Direct sum of distinguished triangles)**.** *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category. If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma(X')$ are two distinguished triangles in $\mathcal{T}$, then*

$$X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y' \xrightarrow{v \oplus v'} Z \oplus Z' \xrightarrow{w \oplus w'} \Sigma(X \oplus X')$$

is a distinguished triangle in $\mathcal{T}$.

As a result, for every objects $X, Y \in Obj(\mathcal{T})$ the triangle $Y \longrightarrow X \oplus Y \longrightarrow X \longrightarrow \Sigma(Y)$ is a distinguished one and so every triangulated subcategory is stable under finite direct sums.

Proof. Consider the cone triangle associated to the homomorphism $X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y'$ given by the axiom (ii) of a triangulated category,

$$X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y' \longrightarrow C_{u \oplus u'} \longrightarrow \Sigma(X \oplus X')$$

Consider the following commutative diagrams whose lines are distinguished triangles,

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "{(id_X, 0)}"] & Y \arrow[r, "v"] \arrow[d, "{(id_Y, 0)}"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & \Sigma(X) \arrow[d, "{\Sigma(id_X, 0)}"] \\
X \oplus X' \arrow[r, "u \oplus u'"'] & Y \oplus Y' \arrow[r] & C_{u \oplus u'} \arrow[r] & \Sigma(X \oplus X')
\end{tikzcd}$$

$$\begin{tikzcd}
X' \arrow[r, "u'"] \arrow[d, "{(0, id_{X'})}"] & Y' \arrow[r, "v'"] \arrow[d, "{(0, id_{Y'})}"] & Z' \arrow[r, "w'"] \arrow[d, dashed, "h'"] & \Sigma(X) \arrow[d, "{\Sigma(0, id_{X'})}"] \\
X \oplus X' \arrow[r, "u \oplus u'"'] & Y \oplus Y' \arrow[r] & C_{u \oplus u'} \arrow[r] & \Sigma(X \oplus X')
\end{tikzcd}$$

which can be completed into a homomorphism of triangles via h and h', respectively thanks to the axiom (iv) of a triangulated category. Consider thus the following commutative diagram,

$$\begin{tikzcd}
X \oplus X' \arrow[r, "u \oplus u'"] \arrow[d, equal, "{(id_X, id_{X'})}"] & Y \oplus Y' \arrow[r, "v \oplus v'"] \arrow[d, equal, "{(id_Y, id_{Y'})}"] & Z \oplus Z' \arrow[r, "w \oplus w'"] \arrow[d, "{(h, h')}"] & \Sigma(X \oplus X') \arrow[d, equal, "{\Sigma(id_X, id_{X'})}"] \\
X \oplus X' \arrow[r, "u \oplus u'"'] & Y \oplus Y' \arrow[r] & C_{u \oplus u'} \arrow[r] & \Sigma(X \oplus X')
\end{tikzcd}$$

which is actually an isomorphism of triangles by virtue of the five's lemma. Indeed, notice that the top triangle (that is, the direct sum triangle) satisfies the properties of Remark 1.2.1.18 (since the homological functor $Hom_{\mathcal{T}}$ preserves direct sums), so that the five's lemma still holds. In other words, the direct sum triangle $X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y' \xrightarrow{v \oplus v'} Z \oplus Z' \xrightarrow{w \oplus w'} \Sigma(X \oplus X')$ of the statement is isomorphic to the distinguished triangle $X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y' \longrightarrow C_{u \oplus u'} \longrightarrow \Sigma(X \oplus X')$ and so the axiom (i) of a triangulated category yields the conclusion.

Finally, remark that the triangle $Y \longrightarrow X \oplus Y \longrightarrow X \longrightarrow \Sigma(Y)$ is clearly the direct sum of the distinguished triangles $0 \longrightarrow X \longrightarrow X \longrightarrow 0$ and $Y \longrightarrow Y \longrightarrow 0 \longrightarrow \Sigma(Y)$. Hence it is a distinguished one by the first part of the proof. ■

1.2.1.20 Proposition (Split triangles)**.** *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category. If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ is a distinguished triangle with $w = 0$, then the triangle splits meaning that u admits a section, say s, and v admits a retraction, say r.*

As a consequence, we have two isomorphisms of distinguished triangles

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal, "id_X"'] & Y \arrow[r, "v"] \arrow[d, "\wr\ (s, v)"] & Z \arrow[r, "w"] \arrow[d, equal, "id_Z"] & \Sigma(X) \arrow[d, equal, "id_{\Sigma(X)}"] \\
X \arrow[r, "{(id_X, 0)}"'] & X \oplus Z \arrow[r, "{(0, id_Z)}"'] & Z \arrow[r, "w"'] & \Sigma(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "{(id_X, 0)}"] \arrow[d, equal, "id_X"'] & X \oplus Z \arrow[r, "{(0, id_Z)}"] \arrow[d, "\wr\ (u + r)"] & Z \arrow[r, "w"] \arrow[d, equal, "id_Z"] & \Sigma(X) \arrow[d, equal, "id_{\Sigma(X)}"] \\
X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & \Sigma(X)
\end{tikzcd}$$

In particular, we have $Y \cong X \oplus Z$ via (s, v) and $X \oplus Z \cong Y$ via $u + r$. These isomorphisms coincide if and only if $s \circ r = 0$.

Besides, $X \overset{u}{\cong} Y$ if and only if $Z \cong 0$. Consequently, any homomorphism $f : X \longrightarrow Y$ in $\mathcal{T}$ is an isomorphism if and only if $C_f \cong 0$.

Proof. Let us prove that u admits a section, that is, a homomorphism $s : Y \longrightarrow X$ such that $s \circ u = id_X$. Consider the distinguished triangle associated to the object X given by the axiom (i) of a triangulated category : $X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow \Sigma(X)$. Since $w = 0$ by assumption, we can consider the following commutative diagram,

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal, "id_X"] & Y \arrow[r, "v"] \arrow[d, dashed, "s"] & Z \arrow[r, "w"] \arrow[d, "0"] & \Sigma(X) \arrow[d, equal, "\Sigma(id_X)"] \\
X \arrow[r, "id_X"'] & X \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \Sigma(X)
\end{tikzcd}$$

which can be completed into a homomorphism of triangles via s by virtue of the axiom (iv) of a triangulated category. In particular, we have $s \circ u = id_X$ as required.

In order to show that v admits a retraction, that is, a homomorphism $r : Z \longrightarrow Y$ such that $v \circ r = id_Z$ we do the same argument as above using the object Z, the rotation axiom and the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"] & Z \arrow[r, "id_Z"] \arrow[d, dashed, "r"] & Z \arrow[r, "0"] \arrow[d, equal, "id_Z"] & \Sigma(0) \arrow[d, "\Sigma(0)"] \\
X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & \Sigma(X)
\end{tikzcd}$$

Next, consider the following distinguished triangles

$$X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow \Sigma(X) \text{ and } 0 \longrightarrow Z \xrightarrow{id_Z} Z \longrightarrow \Sigma(0)$$

By virtue of Proposition 1.2.1.19 the direct sum triangle

$$X \xrightarrow{(id_X, 0)} X \oplus Z \xrightarrow{(0, id_Z)} Z \xrightarrow{0} \Sigma(X)$$

is a distinguished triangle in $\mathcal{T}$. Hence, the diagrams of the statement are homomorphisms of distinguished triangles. Observe that by virtue of the five's lemma we can extend them into isomorphisms of distinguished triangles.

Notice that the composition

$$X \oplus Z \xrightarrow{(u+r)} Y \xrightarrow{(s,v)} X \oplus Z$$

can be expressed by construction as

$$(s, v) \circ (u + r) = (s \circ (u + r), v \circ (u + r)) = (id_X + s \circ r, 0 + id_Z),$$

where we use that $v \circ u = 0$ thanks to Lemma 1.2.1.15. Therefore, the isomorphisms $Y \cong X \oplus Z$ and $X \oplus Z \cong Y$ obtained above coincide if and only if $s \circ r = 0$.

Assume now that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ is any distinguished triangle in $\mathcal{T}$. If $Z \cong 0$, then it is clear that $w = 0$ and then the preceding result assures that $Y \cong X \oplus 0 \cong X$. Conversely, suppose that $X \overset{u}{\cong} Y$ is an isomorphism. Consider the following commutative diagram,

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal, "id_X"] & Y \arrow[r, "v"] \arrow[d, "u^{-1}"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & \Sigma(X) \arrow[d, equal, "\Sigma(id_X)"] \\
X \arrow[r, equal, "id_X"'] & X \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \Sigma(X)
\end{tikzcd}$$

which can be completed, via h, into an isomorphism of triangles by virtue of the axiom (iv) of a triangulated category and the five's lemma for triangulated categories. In other words, $Z \cong 0$.

Finally, the consequence of the statement is clear applying the axiom (ii) of a triangulated category. ■

In this framework, we can actually formulated a "categorical Baum-Connes property" for a given triangulated category. Let us show here the general process that we'll adapt suitably later in order to get the classical Baum-Connes property in the context of the Kasparov's theory (which will be regarded as a triangulated category of course).

1.2.1.21 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{S}$ a thick subcategory of $\mathcal{T}$.

A $\mathcal{S}$-quasi-isomorphism in $\mathcal{T}$ is any homomorphism f in $\mathcal{T}$ whose cone object C_f is an object in $\mathcal{S}$.

The localization of $\mathcal{T}$ with respect to the class of $\mathcal{S}$-quasi-isomorphisms is denoted by $\mathcal{T}/\mathcal{S}$.

1.2.1.22 Remark. The general theory of localization of categories attempt to imitate the classical theory of localization of rings. Its existence in the context of triangulated categories is a celebrated theorem of J. L. Verdier and we refer to Theorem B.1.20 for the precise statement.

Here it is important to do the following observation. Let $F : \mathcal{T} \longrightarrow \mathcal{T}'$ be a triangulated functor between two triangulated categories. Assume that $\mathscr{N} \subset \mathcal{T}$ is a thick subcategory such that $\mathscr{N} \subset \ker_{Obj}(F)$. Take a $\mathscr{N}$-quasi-isomorphism in $\mathcal{T}$, say $f : X \longrightarrow Y$ with $X, Y \in Obj(\mathcal{T})$. Consider the corresponding cone triangle (given by the axiom (ii) of a triangulated category),

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma(X)$$

Since f is a $\mathscr{N}$-quasi-isomorphism, then $C_f \in Obj(\mathscr{N}) \subset \ker_{Obj}(F)$, so that $F(C_f) \cong 0$. Since F is a triangulated functor, the following is again a distinguished triangle

$$F(X) \xrightarrow{F(f)} F(Y) \longrightarrow F(C_f) \longrightarrow \Sigma(F(X)),$$

where we have to remark that the cone object is unique up to isomorphism, so that it must be $F(C_f) \cong C_{F(f)}$. Hence $C_{F(f)} \cong 0$. By Proposition 1.2.1.20 we have that $F(f)$ is an isomorphism in $\mathcal{T}'$.

In other words, *$\mathscr{N}$-quasi-isomorphisms are transformed into isomorphisms* through F. This means in particular that any homomorphism in $\mathscr{N} \subset \mathcal{T}$ is transformed into an isomorphism in $\mathcal{T}/\mathscr{N}$ through the Verdier localization functor $Q : \mathcal{T} \longrightarrow \mathcal{T}/\mathscr{N}$, which explains the nomenclature of Definition 1.2.1.21.

1.2.1.23 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{S}$ a thick subcategory of $\mathcal{T}$.

- The left orthogonal complement of $\mathcal{S}$ is the thick subcategory of $\mathcal{T}$ defined by

$$\mathcal{S}^{\vdash} := \{X \in Obj(\mathcal{T}) \mid Hom_{\mathcal{T}}(X, S) = (0)\ \forall S \in Obj(\mathcal{S})\}$$

- The right orthogonal complement of $\mathcal{S}$ is the thick subcategory of $\mathcal{T}$ defined by

$$\mathcal{S}^{\dashv} := \{X \in Obj(\mathcal{T}) \mid Hom_{\mathcal{T}}(S, X) = (0)\ \forall S \in Obj(\mathcal{S})\}$$

1.2.1.24 Remark. If $\mathcal{S}$ is a localizing subcategory of $\mathcal{T}$, so it is $\mathcal{S}^{\vdash}$ and $\mathcal{S}^{\dashv}$ because the homomorphism functors are always compatible with countable direct sums.

1.2.1.25 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathscr{L}$, $\mathscr{N}$ two thick subcategories of $\mathcal{T}$. We say that $(\mathscr{L}, \mathscr{N})$ is a complementary pair of thick subcategories if

i) $\mathscr{L} \subset \mathscr{N}^{\vdash}$,

ii) for evey object $X \in Obj(\mathcal{T})$ there exists a distinguished triangle of the form

$$L \xrightarrow{u} X \xrightarrow{v} N \xrightarrow{w} \Sigma(L)$$

for some $L \in Obj(\mathscr{L})$ and $N \in Obj(\mathscr{N})$. Such a distinguished triangle will be called $(\mathscr{L}, \mathscr{N})$-*triangle associated to* X in the sequel.

The following result contains the main elementary properties of complementary pair of thick subcategories. The proof uses routine arguments in combination with the preceding results. For more details we refer to Proposition 2.9 in [132].

1.2.1.26 Lemma (Fundamental lemma about complementary pair of thick subcategories)**.** *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $(\mathscr{L}, \mathscr{N})$ a complementary pair of thick subcategories. The following hold*

i) $\mathscr{L} = \mathscr{N}^{\vdash}$ and $\mathscr{N} = \mathscr{L}^{\dashv}$.

ii) Given an object $X \in Obj(\mathcal{T})$, the associated $(\mathscr{L}, \mathscr{N})$-triangle $L \xrightarrow{u} X \xrightarrow{v} N \xrightarrow{w} \Sigma(L)$ is unique up to canonical isomorphism and depends functorially on X. In particular, its entries define functors

$$L : \mathcal{T} \longrightarrow \mathscr{L} \text{ and } N : \mathcal{T} \longrightarrow \mathscr{N},$$

which are unique up to natural isomorphism. Precisely, if $L' : \mathcal{T} \longrightarrow \mathscr{L}$ and $N' : \mathcal{T} \longrightarrow \mathscr{N}$ are two other functors such that for every object $X \in Obj(\mathcal{T})$ there exists a unique up to a canonical isomorphism $(\mathscr{L}, \mathscr{N})$-triangle $L'(X) \longrightarrow X \longrightarrow N'(X) \longrightarrow \Sigma(L'(X))$ which depends functorially on X, then L and L' are naturally isomorphic and N and N' are naturally isomorphic.

iii) The functors $L, N : \mathcal{T} \longrightarrow \mathcal{T}$ are triangulated.

iv) The functors $L, N : \mathcal{T} \longrightarrow \mathcal{T}$ descend to triangulated functors

$$L : \mathcal{T}/\mathscr{N} \longrightarrow \mathscr{L} \text{ and } N : \mathcal{T}/\mathscr{L} \longrightarrow \mathscr{N}$$

which are the inverses of the natural equivalences $\mathscr{L} \hookrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathscr{N}$ and $\mathscr{N} \hookrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathscr{L}$.

1.2.1.27 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathscr{C}$ any category. Fix a complementary pair of thick subcategories $(\mathscr{L}, \mathscr{N})$ in $\mathcal{T}$ and a functor $F : \mathcal{T} \longrightarrow \mathscr{C}$.

- The localization of F with respect to $(\mathscr{L}, \mathscr{N})$ is the functor
$$\mathbb{L}F := F \circ L : \mathcal{T} \longrightarrow \mathscr{C}$$
- The obstruction of F with respect to $(\mathscr{L}, \mathscr{N})$ is the functor
$$\mathbb{O}F := F \circ N : \mathcal{T} \longrightarrow \mathscr{C}$$

1.2.1.28 Remarks.

1. If F is a triangulated or homological functor, so it is $\mathbb{L}F$ and $\mathbb{O}F$.
2. Since the functors L and N descend to triangulated functors on the quotient categories, the same is true for the localization and obstruction of F:
$$\mathbb{L}F : \mathcal{T}/\mathscr{N} \longrightarrow \mathscr{C} \text{ and } \mathbb{O}F : \mathcal{T}/\mathscr{L} \longrightarrow \mathscr{C}$$
Moreover, by construction we have
$$\mathbb{L}F_{|\mathscr{N}} = 0 \text{ and } \mathbb{O}F_{|\mathscr{L}} = 0$$
To see this we have just to remark that the functor L (resp. N) factorizes via the the localization functor Q of $\mathcal{T}/\mathscr{N}$ (resp. $\mathcal{T}/\mathscr{L}$). But $\mathscr{N} = \ker_{Obj}(Q)$ (resp. $\mathscr{L} = \ker_{Obj}(Q)$), so that
$$F(L(\mathscr{N})) = F(P \circ Q(\mathscr{N})) = P(0) = 0$$
$$(\text{resp. } F(N(\mathscr{L})) = F(P \circ Q(\mathscr{L})) = P(0) = 0),$$
where we follow the same notations as in the preceding lemma.
3. There exists a natural transformation
$$\eta : \mathbb{L}F \longrightarrow F$$
Namely, given any object $X \in Obj(\mathcal{T})$, consider the corresponding $(\mathscr{L}, \mathscr{N})$-triangle $L(X) \xrightarrow{u} X \xrightarrow{v} N(X) \xrightarrow{w} \Sigma(L(X))$. So, it suffices to take $\eta_X := F(u)$. Indeed, if $f : X \longrightarrow Y$ is any homomorphism in $\mathcal{T}$, then by functoriality of the $(\mathscr{L}, \mathscr{N})$-triangles, we have the following homomorphism of distinguished triangles
$$\begin{tikzcd}
L(X) \arrow[r, "u"] \arrow[d, "L(f)"'] & X \arrow[r, "v"] \arrow[d, "f"] & N(X) \arrow[r, "w"] \arrow[d, "N(f)"] & \Sigma(L(X)) \arrow[d, "\Sigma(L_f)"] \\
L(Y) \arrow[r, "u'"'] & Y \arrow[r, "v'"'] & N(Y) \arrow[r, "w'"'] & \Sigma(L(Y))
\end{tikzcd}$$
where $L(Y) \xrightarrow{u'} Y \xrightarrow{v'} N(Y) \xrightarrow{w'} \Sigma(L(Y))$ is the $(\mathscr{L}, \mathscr{N})$-triangle associated to the object Y. In particular, we have $f \circ u = u' \circ L(f)$ so $F(f) \circ F(u) = F(u') \circ F(L(f))$. In other words, $F(f) \circ \eta_X = \eta_Y \circ \mathbb{L}F(f)$.
Likewise, we have a natural transformation
$$\nu : F \longrightarrow \mathbb{O}F$$
taking $\nu_X := F(v)$, for all object $X \in Obj(\mathcal{T})$.

1.2.1.29 Theorem (Universal property of localization)**.** *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathscr{C}$ a triangulated (resp. stable abelian) category. Fix a complementary pair of thick subcategories $(\mathscr{L}, \mathscr{N})$ in $\mathcal{T}$ and a triangulated (resp. stable homological) functor $F : \mathcal{T} \longrightarrow \mathscr{C}$.*

If $G : \mathcal{T} \longrightarrow \mathscr{C}$ is any triangulated (resp. stable homological) functor such that $G_{|\mathscr{N}} = 0$ and G is equipped with a natural transformation $G \xrightarrow{\eta'} F$, then there exists a unique natural factorization

$$\begin{tikzcd} G \arrow[r, "\eta'"] \arrow[dr, dashed] & F \\ & \mathbb{L}F \arrow[u, "\eta"'] \end{tikzcd}$$

Consequently, the natural transformation $\eta : \mathbb{L}F \longrightarrow F$ is invertible if and only if $F_{|\mathscr{N}} = 0$.

Proof. Given any object $X \in Obj(\mathcal{T})$, consider the corresponding $(\mathscr{L}, \mathscr{N})$-triangle $L(X) \xrightarrow{u} X \xrightarrow{v} N(X) \xrightarrow{w} \Sigma(L(X))$ and apply the functor G of the statement. We obtain then a distinguished triangle (respectively, a long exact sequence)

$$G(L(X)) \xrightarrow{G(u)} G(X) \longrightarrow G(N(X)) \longrightarrow \Sigma(G(L(X)))$$

$$(\text{resp. } G(L(X)) \xrightarrow{G(u)} G(X) \longrightarrow G(N(X)))$$

Since $G_{|\mathscr{N}} = 0$, we have that $G(N(X)) \cong 0$ which implies that $G(L(X)) \overset{G(u)}{\cong} G(X)$ (see Proposition 1.2.1.20).

Next, let's apply the natural transformation $G \xrightarrow{\eta'} F$ of the statement to the object $L(X)$, so that we have $\eta'_{L(X)} : G(L(X)) \longrightarrow \mathbb{L}F(X)$ and we define a natural transformation $G \longrightarrow \mathbb{L}F$ by the composition

$$G(X) \overset{G(u)}{\cong} G(L(X)) \xrightarrow{\eta'_{L(X)}} \mathbb{L}F(X)$$

By construction, we obtain a commutative diagram

$$\begin{tikzcd} G(X) \arrow[r, phantom, "\overset{G(u)}{\cong}"] & G(L(X)) \arrow[r, "\eta'_X"] \arrow[dr, "\eta'_{L(X)}"'] & F(X) \\ & & \mathbb{L}F(X) \arrow[u, "\eta_X"'] \end{tikzcd}$$

as required.

Finally, suppose that $\eta : \mathbb{L}F \longrightarrow F$ is invertible, then there exists a natural transformation $m : F \longrightarrow \mathbb{L}F$ such that $\eta \circ m \cong id$. In other words, $id : F \longrightarrow F$ satisfies the above universal property and so it must be $F_{|\mathscr{N}} = 0$. Conversly, if $F_{|\mathscr{N}} = 0$, then $F(L(X)) \cong F(X)$ for every object $X \in Obj(\mathcal{T})$ via η_X because F is a triangulated (resp. stable homological) functor. In other words, the natural tranformation $\eta : \mathbb{L}F \longrightarrow F$ is invertible. ■

1.2.1.30 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathscr{C}$ a triangulated (resp. abelian) category. Fix a complementary pair of thick subcategories $(\mathscr{L}, \mathscr{N})$ in $\mathcal{T}$ and a triangulated (resp. homological) functor $F : \mathcal{T} \longrightarrow \mathscr{C}$.

The categorical Baum-Connes assembly map for $\mathcal{T}$ with respect to $(\mathscr{L}, \mathscr{N}, F)$ is the natural transformation

$$\eta : \mathbb{L}F \longrightarrow F$$

Thus the categorical Baum-Connes property for $\mathcal{T}$ with respect to $(\mathscr{L}, \mathscr{N}, F)$ consists in investigate wheather η is invertible. Of course, it is very improbable that such a conjecture will be true in full generality. But we can realized this conjecture in a more concrete framework, where there may be reasons that become interesting the study of this problem. Namely, in the context of the Kasparov's theory. In order to establish this relationship, we need the following result.

1.2.1.31 Lemma. *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category, $\mathcal{P}$ any class of objects in $\mathcal{T}$ and $\mathscr{N}$ a localizing subcategory in $\mathcal{T}$ such that $(\langle \mathcal{P} \rangle, \mathscr{N})$ is a complementary pair of localizing subcategories in $\mathcal{T}$.*

Let $F, G : \mathcal{T} \longrightarrow \mathscr{A}$ be two homological functors compatible with countable direct sums and $\Phi : G \longrightarrow F$ a natural transformation.

If $G_{|\mathscr{N}} = 0$ and $\Phi_X : G(X) \longrightarrow F(X)$ is an isomorphism for every object $X \in \mathcal{P}$, then Φ descent to a unique natural equivalence

$$G \cong \mathbb{L}F$$

Proof. Since $G_{|\mathscr{N}} = 0$, universal property of localization yields that there exists a unique natural transformation $G \longrightarrow \mathbb{L}F$ such that

$$\begin{tikzcd} G \arrow[r, "\Phi"] \arrow[dr] & F \\ & \mathbb{L}F \arrow[u, "\eta"'] \end{tikzcd}$$

is a commutative diagram.

Since $(\langle \mathcal{P} \rangle, \mathscr{N})$ is a complementary pair of localizing subcategories in $\mathcal{T}$, we can apply the fundamental lemma about complementary pairs (recall Lemma 1.2.1.26). Hence there exists a triangulated functor $L : \mathcal{T}/\mathscr{N} \longrightarrow \langle \mathcal{P} \rangle$ such that $L(X) \cong X$, for all $X \in \mathcal{P}$. In particular, we have $\mathbb{L}F(X) \cong F(X)$, for all $X \in \mathcal{P}$.

Now, by assumption, we have that $\Phi_X : G(X) \longrightarrow F(X)$ is an isomorphism for all $X \in \mathcal{P}$. Hence we can write $G(X) \cong F(X) \cong \mathbb{L}F(X)$, for all $X \in \mathcal{P}$. In other words, the natural transformation $G \longrightarrow \mathbb{L}F$ given by universal property is a natural *equivalence* on $\mathcal{P}$. Since G and F are supposed to be compatible with countable direct sums, the same conclusion is true on $\langle \mathcal{P} \rangle$. To conclude, recall from the proof of the preceding theorem that the natural transformation $G \longrightarrow \mathbb{L}F$ is defined by the composition $G(X) \xrightarrow{G(u)^{-1}} G(L(X)) \xrightarrow{\Phi_{L(X)}} \mathbb{L}F(X)$, for all $X \in Obj(\mathcal{T})$. Since Φ is a natural equivalence on $\langle \mathcal{P} \rangle$, we obtain that $G \cong \mathbb{L}F$ as claimed.

■

1.2.2 Meyer-Nest's homological algebra

First of all, observe that the main tool for the categorification of the Baum-Connes property explained in the preceding section is the *choice* of a complementary pair of thick subcategories.

This pair is then used to localize a fixed functor. Hence, it is advisable to provide methods by which we can construct such complementary pairs in a given triangulated category. The Meyer-Nest's idea is to develop an adapted homological algebra in a triangulated category (very close to the one that we develop for any abelian category) in such a way that we can obtain complementary pairs from projective objects.

1.2.2.1 Definition. Let $\mathcal{T}$ any additive category. An ideal $\mathcal{J}$ in $\mathcal{T}$ is a family of homomorphism subgroups $\{\mathcal{J}(X,Y)\}_{X,Y\in Obj(\mathcal{T})} \subset Hom_{\mathcal{T}}(X,Y)$ such that

$$Hom_{\mathcal{T}}(Z,W)\circ\mathcal{J}(Y,Z)\circ Hom_{\mathcal{T}}(X,Y)\subset\mathcal{J}(X,W),$$

for all $X,Y,Z,W\in Obj(\mathcal{T})$.

We say that $\mathcal{J}$ is additive if $\mathcal{J}$ is compatible with countable direct sums meaning that $\mathcal{J}(\bigoplus_i X_i,Y)\cong\prod_i\mathcal{J}(X_i,Y)$ through the canonical map $Hom_{\mathcal{T}}(\bigoplus_i X_i,Y)\longrightarrow\prod_i Hom_{\mathcal{T}}(X_i,Y)$. for all $X_i,Y\in Obj(\mathcal{T})$.

1.2.2.2 Remarks.

1. Given any additive (resp. compatible with countable direct sums) functor $F:\mathcal{T}\longrightarrow\mathcal{T}'$ we have an obvious ideal (resp. additive ideal) in $\mathcal{T}$ defined for all $X,Y\in Obj(\mathcal{T})$ by
$$\ker_{Hom}(F)(X,Y):=\{f\in Hom_{\mathcal{T}}(X,Y)\mid F(f)\cong 0\}$$
2. If $\{\mathcal{J}_l\}_{l\in I}$ is a family of (additive) ideals in $\mathcal{T}$, then it is straightforward to see that their intersection $\mathcal{J}:=\bigcap_{l\in I}\mathcal{J}_l$ is a (additive) ideal in $\mathcal{T}$. More precisely, for every $X,Y\in Obj(\mathcal{T})$ we define
$$\mathcal{J}(X,Y):=\bigcap_{l\in I}\mathcal{J}_l(X,Y)\subset Hom_{\mathcal{T}}(X,Y)$$

1.2.2.3 Definition. Let $(\mathcal{T},\Sigma,\Delta_\Sigma)$ be a triangulated category. An homological ideal in $\mathcal{T}$ is an ideal $\mathcal{J}$ in $\mathcal{T}$ for which there exists an stable homological functor $F:\mathcal{T}\longrightarrow\mathscr{A}$ such that $\mathcal{J}=\ker_{Hom}(F)$.

1.2.2.4 Remark. It is a non trivial fact, but we can show by virtue of the Freyd's theorem (see Remark 2.21 in [134] for the details) that an homological ideal can be realized as $\ker_{Hom}(F)$ with F *triangulated*. This is important for this dissertation because when we apply this general theory to the Kasparov category, the corresponding ideal that we choose comes from a triangulated functor (namely, the restriction functor).

For simplicity in the exposition, we carry out all the following arguments concerning homological ideals with respect a *stable homological functor*. Nevertheless, the same results hold when we work with *triangulated functors* instead. Indeed, it is enough to consider distinguished triangles instead of exact sequences and apply the *split triangles* strategy using Proposition 1.2.1.20.

1.2.2.5 Remark. If $\{\mathcal{J}_l\}_{l\in I}$ is a family of homological ideals in $\mathcal{T}$ where $\mathcal{J}_l:=\ker_{Hom}(F_l)$ with stable homological (resp. triangulated) functors $F_l:\mathcal{T}\longrightarrow\mathscr{A}_l$ for every $l\in I$, then their intersection $\mathcal{J}:=\bigcap_{l\in I}\mathcal{J}_l$ is a homological ideal. More precisely, we define the following stable homological (resp. triangulated) functor

$$F:=\left(F_l\right)_{l\in I}:\mathcal{T}\longrightarrow\prod_{l\in I}\mathscr{A}_l,$$

so that $\mathcal{J}=\ker_{Hom}(F)$ by construction.

1.2.2.6 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J}$ a homological ideal in $\mathcal{T}$. Let $f : X \longrightarrow Y$ be a homomorphism in $\mathcal{T}$ and write $X \xrightarrow{f} Y \xrightarrow{v} C_f \xrightarrow{w} \Sigma(X)$ for the cone triangle associated to f. We say that

- f is a $\mathcal{J}$-epimorphism if $v \in \mathcal{J}(Y, C_f)$,
- f is a $\mathcal{J}$-monomorphism if $w \in \mathcal{J}(C_f, \Sigma(X))$,
- f is a $\mathcal{J}$-isomorphism if f is both a $\mathcal{J}$-epimorphism and a $\mathcal{J}$-monomorphism.

1.2.2.7 Proposition. *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J} := \ker_{Hom}(F)$ a homological ideal in $\mathcal{T}$ with $F : \mathcal{T} \longrightarrow \mathscr{A}$. If $f : X \longrightarrow Y$ is a homomorphism in $\mathcal{T}$ and $X \xrightarrow{f} Y \xrightarrow{v} C_f \xrightarrow{w} \Sigma(X)$ is its cone triangle, then we have that*

i) f is a $\mathcal{J}$-epimorphism if and only if $F(f)$ is an epimorphism,

ii) f is a $\mathcal{J}$-monomorphism if and only if $F(f)$ is a monomorphism,

iii) f is a $\mathcal{J}$-isomorphism if and only if $F(f)$ is an isomorphism.

Proof. Given the distinguished triangle $X \xrightarrow{f} Y \xrightarrow{v} C_f \xrightarrow{w} \Sigma(X)$, we construct a long exact sequence on $\mathscr{A}$ by applying the functor F (because it is a stable homological one)

$$\begin{aligned} \ldots \to F(\Sigma^{-1}(C_f)) \overset{F(\Sigma^{-1}(w))}{\to} F(X) \overset{F(f)}{\to} F(Y) \overset{F(v)}{\to} F(C_f) \\ \overset{F(w)}{\to} F(\Sigma(X)) \overset{F(\Sigma(f))}{\to} F(\Sigma(Y)) \to \ldots \end{aligned}$$

In this situation, we have

i) f is a $\mathcal{J}$-epimorphism $\Leftrightarrow v \in \mathcal{J}(Y, C_f) \Leftrightarrow F(v) = 0 \Leftrightarrow F(f)$ is an epimorphism.

ii) f is a $\mathcal{J}$-monomorphism $\Leftrightarrow w \in \mathcal{J}(C_f, \Sigma(X)) \Leftrightarrow \Sigma^{-1}(w) \in \mathcal{J}(\Sigma^{-1}(C_f), X) \Leftrightarrow F(\Sigma^{-1}(w)) = 0 \Leftrightarrow$ $F(f)$ is a monomorphism.

iii) the statement *"f is a $\mathcal{J}$-isomorphism if and only if $F(f)$ is an isomorphism"* is obvious thanks to the previous ones.

■

1.2.2.8 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J}$ an homological ideal in $\mathcal{T}$. Let $(X_n, d_n)_{n\in\mathbb{Z}}$ a chain complex in $\mathcal{T}$ and for each $n \in \mathbb{Z}$ consider the cone triangle associated to the differential d_n, say $X_n \xrightarrow{d_n} X_{n+1} \xrightarrow{v_n} C_n \xrightarrow{w_n} \Sigma(X_n)$.

We say that (X_n, d_n) is $\mathcal{J}$-exact at degree n if the composition $C_n \xrightarrow{w_n} \Sigma(X_n) \xrightarrow{\Sigma(v_{n+1})} \Sigma(C_{n+1})$ is a homomorphism in $\mathcal{J}(C_n, \Sigma(C_{n+1}))$.

We say that (X_n, d_n) is $\mathcal{J}$-exact if (X_n, d_n) is $\mathcal{J}$-exact at degree n, for all $n \in \mathbb{Z}$.

1.2.2.9 Proposition. *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J} := \ker_{Hom}(F)$ a homological ideal in $\mathcal{T}$ with $F : \mathcal{T} \longrightarrow \mathscr{A}$.*

A chain complexe $(X_n, d_n)_{n\in\mathbb{Z}}$ in $\mathcal{T}$ is $\mathcal{J}$-exact at degree n if and only if $F(X_{n+1}) \xrightarrow{F(d_{n+1})} F(X_n) \xrightarrow{F(d_n)} F(X_{n-1})$ is a short exact sequence in $\mathscr{A}$.

Proof. First of all, consider the short sequence $F(X_{n+1}) \overset{F(d_{n+1})}{\longrightarrow} F(X_n) \overset{F(d_n)}{\longrightarrow} F(X_{n-1})$. Since $(X_n, d_n)_{n\in\mathbb{Z}}$ is a chain complexe, we have $d_n \circ d_{n+1} = 0$, so that $F(d_n) \circ F(d_{n+1}) = 0$, that is, $Im(F(d_{n+1})) \subset \ker(F(d_n))$, for all $n \in \mathbb{Z}$. It remains to show that the converse inclusion holds if and only if $(X_n, d_n)_{n\in\mathbb{Z}}$ is $\mathcal{J}$-exact.

Next, consider the cone triangle associated to the differential d_n, say $X_n \xrightarrow{d_n} X_{n+1} \xrightarrow{v_n} C_n \xrightarrow{w_n} \Sigma(X_n)$. Since F is a stable homological functor, we consider the following long exact sequence in $\mathscr{A}$

$$\begin{aligned} \ldots \to F(\Sigma^{-1}(C_n)) \overset{F(\Sigma^{-1}(w_n))}{\to} F(X_n) \overset{F(d_n)}{\to} F(X_{n+1}) \overset{F(v_n)}{\to} F(C_n) \\ \overset{F(w_n)}{\to} F(\Sigma(X_n)) \overset{F(\Sigma(d_n))}{\to} F(\Sigma(X_{n+1})) \to \ldots \end{aligned}$$

which implies in particular that

$$Im(\Sigma^{-1}(F(w_n))) = \ker(F(d_n)) \text{ and } Im(F(d_{n+1})) = \ker(F(v_{n+1})) \tag{1.2.1}$$

By definition, $(X_n, d_n)_{n\in\mathbb{Z}}$ is a $\mathcal{J}$-exact chain complexe in $\mathcal{T}$ at degree n if and only if the composition

$$C_n \xrightarrow{w_n} \Sigma(X_n) \xrightarrow{\Sigma(v_{n+1})} \Sigma(C_{n+1})$$

is a homomorphism in $\mathcal{J}(C_n, \Sigma(C_{n+1}))$, which is equivalent to say that the composition

$$\Sigma^{-1}(C_n) \xrightarrow{\Sigma^{-1}(w_n)} X_n \xrightarrow{v_{n+1}} C_{n+1}$$

is a homomorphism in $\mathcal{J}(\Sigma^{-1}(C_n), C_{n+1})$, which is equivalent to say that the composition

$$\Sigma^{-1}(F(C_n)) \xrightarrow{\Sigma^{-1}(F(w_n))} F(X_n) \xrightarrow{F(v_{n+1})} F(C_{n+1})$$

is the zero homomorphism in $\mathscr{A}$, that is,

$$Im(\Sigma^{-1}(F(w_n))) \subset \ker(F(v_{n+1}))$$

In other words, by virtue of the identities (1.2.1) above, we have that $(X_n, d_n)_{n\in\mathbb{Z}}$ is a $\mathcal{J}$-exact chain complexe in $\mathcal{T}$ at degree n if and only if $\ker(F(d_n)) \subset Im(F(d_{n+1}))$, which finishes the proof. ■

1.2.2.10 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J}$ an homological ideal in $\mathcal{T}$. A homological functor $G : \mathcal{T} \longrightarrow \mathscr{A}$ is said to be $\mathcal{J}$-exact if $\mathcal{J} \subset \ker_{Hom}(G)$.

1.2.2.11 Proposition. *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J}$ an homological ideal in $\mathcal{T}$. If $G : \mathcal{T} \longrightarrow \mathscr{A}$ is a stable homological functor, then the following statements are equivalent*

i) G is $\mathcal{J}$-exact.

ii) G transforms $\mathcal{J}$-epimorphisms into epimorphisms.

iii) G transforms $\mathcal{J}$-monomorphisms into monomorphisms.

iv) G transforms $\mathcal{J}$-exact chain complexes into long exact sequences.

Proof. Let $f : X \longrightarrow Y$ be a homomorphism in $\mathcal{T}$ and consider its cone triangle $X \xrightarrow{f} Y \xrightarrow{v} C_f \xrightarrow{w} \Sigma(X)$. Next, consider the long exact sequence

$$\ldots \longrightarrow G(\Sigma^{-1}(C_f)) \overset{G(\Sigma^{-1}(w))}{\rightarrow} G(X) \overset{G(f)}{\rightarrow} G(Y) \overset{G(v)}{\rightarrow} G(C_f) \overset{G(w)}{\rightarrow} G(\Sigma(X)) \longrightarrow \ldots$$

In this situation, it is clear by definition that *G is $\mathcal{J}$-exact if and only if $G(f)$ is an epimorphism (resp. monomorphism) whenever f is a $\mathcal{J}$-epimorphism (resp. $\mathcal{J}$-monomorphism).*

Next, let (X_n, d_n) be a $\mathcal{J}$-exact chain complex in $\mathcal{T}$, which means that the composition

$$C_n \xrightarrow{w_n} \Sigma(X_n) \xrightarrow{\Sigma(v_{n+1})} \Sigma(C_{n+1})$$

is a homomorphism in $\mathcal{J}(C_n, \Sigma(C_{n+1}))$, for all $n \in \mathbb{Z}$ where C_n is the cone of the differential $d_n : X_n \longrightarrow X_{n+1}$.

If G is a $\mathcal{J}$-exact functor, then $\mathcal{J} \subset \ker_{Hom}(G)$ and so we have $G(\Sigma(v_{n+1}) \circ w_n) = 0$. In this situation, the same argument of Proposition 1.2.2.9 yields that $(G(X_n), G(d_n))$ is a long exact sequence. This shows $(i) \Rightarrow (iv)$.

Conversely, suppose that G transforms $\mathcal{J}$-exact chain complexes into long exact sequences and let $f : X \longrightarrow Y$ any homomorphism in $\mathcal{J}(X, Y)$ with $X, Y \in Obj(\mathcal{T})$. We have to show that $G(f) = 0$. Indeed, we can consider the following $\mathcal{J}$-exact chain complex,

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots 0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots 0 \longrightarrow 0 \longrightarrow \ldots$$

By assumption, we have a long exact sequence,

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots 0 \longrightarrow G(X) \xrightarrow{G(f)} G(Y) \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots 0 \longrightarrow 0 \longrightarrow \ldots,$$

which implies clearly that $G(f) = 0$ as required. This shows $(iv) \Rightarrow (i)$

■

1.2.2.12 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J}$ a homological ideal in $\mathcal{T}$. A $\mathcal{J}$-projective object in $\mathcal{T}$ is an object $P \in Obj(\mathcal{T})$ such that the homomorphism functor $Hom_{\mathcal{T}}(P, \cdot\,) : \mathcal{T} \longrightarrow \mathscr{A}b$ is $\mathcal{J}$-exact.

The class of $\mathcal{J}$-projective objects in $\mathcal{T}$ is denoted by $\mathfrak{p}_{\mathcal{J}}$.

1.2.2.13 Remarks.

1. Observe that the suspension functor and the direct sum functor are triangulated, so that the class of $\mathcal{J}$-projective objects is closed under suspension, retractions and direct sums. This implies that all the objects in the minimal localizing subcategory $\langle \mathfrak{p}_{\mathcal{J}} \rangle$ generated by $\mathcal{J}$-projective objects are $\mathcal{J}$-projective objects as well.

2. If $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ is a triangulated category and $\mathcal{J}$ is a homological ideal in $\mathcal{T}$, then given an object $P \in Obj(\mathcal{T})$ we have

$$P \in \mathfrak{p}_{\mathcal{J}} \Leftrightarrow \mathcal{J}(P, Y) = (0), \text{ for all object } Y \in Obj(\mathcal{T})$$

Indeed, assume that P is a $\mathcal{J}$-projective object and take $f \in \mathcal{J}(P, Y)$ any homomorphism in the ideal $\mathcal{J}$ where Y is any object in $\mathcal{T}$. We have just to write

$$f = f \circ id_P = f_*(id_P),$$

so that $f = 0$, since $\mathcal{J}(P,Y) \subset \ker_{Hom}(Hom_{\mathcal{T}}(P, \ \cdot \ ))(P,Y)$ by assumption.

Conversely, assume that $\mathcal{J}(P,Y) = (0)$, for all object $Y \in Obj(\mathcal{T})$. Consider any object $Z \in Obj(\mathcal{T})$ and take any $f \in \mathcal{J}(Y,Z)$. By definition, given any $h \in Hom_{\mathcal{T}}(P,Y)$ we have

$$f_*(h) = f \circ h = id_Z \circ f \circ h,$$

which implies that $f_*(h) \in \mathcal{J}(P,Z)$ by definition of ideal of an additive category. So, by hypothesis, we obtain that $f_*(h) = 0$. In other words, $f \in \ker_{Hom}(Hom_{\mathcal{T}}(P, \ \cdot \ ))$. Since $f \in \mathcal{J}(Y,Z)$ and Y, Z are arbitrary objects in $\mathcal{T}$, we deduce that P is $\mathcal{J}$-projective.

1.2.2.14 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J}$ an homological ideal in $\mathcal{T}$. Consider an object $X \in Obj(\mathcal{T})$.

- A simple $\mathcal{J}$-projective resolution for X is a $\mathcal{J}$-epimorphism $\pi : P \longrightarrow X$ where $P \in \mathfrak{p}_{\mathcal{J}}$.
- A $\mathcal{J}$-projective resolution for X is a $\mathcal{J}$-exact chain complexe $(P_n, d_n)_{n \in \mathbb{Z}^+}$ in $\mathcal{T}$ with $P_n \in \mathfrak{p}_{\mathcal{J}}$ for all $n \in \mathbb{Z}^+$ and with $d_0 : P_0 \longrightarrow X$. In this case we write $P_\bullet \longrightarrow X$ for such a resolution.

We say that $\mathcal{T}$ has enough $\mathcal{J}$-projectives if every object $X \in Obj(\mathcal{T})$ admits a simple $\mathcal{J}$-projective resolution.

In this theoretic context, we can state the main theorem about the construction of complementary pairs from $\mathcal{J}$-projective objects in a given triangulated category.

1.2.2.15 Theorem (Complementary pairs from $\mathcal{J}$-projective objects)**.** *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J} = \ker_{Hom}(F)$ an additive homological ideal in $\mathcal{T}$.*

If $\mathcal{T}$ has enough $\mathcal{J}$-projective objects, then $(\langle \mathfrak{p}_{\mathcal{J}} \rangle, \ker_{Obj}(F))$ is a complementary pair of localizing subcategories in $\mathcal{T}$.

1.2.2.16 Remark. Notice that the converse of the above theorem is also true. Namely, suppose that $(\langle \mathfrak{p}_{\mathcal{J}} \rangle, \ker_{Obj}(F))$ is a complementary pair of localizing subcategories in $\mathcal{T}$. In this case, for every object $X \in Obj(\mathcal{T})$ there exists a (unique up to isomorphism) distinguished triangle of the form

$$L \xrightarrow{u} X \xrightarrow{v} N \xrightarrow{w} \Sigma(L),$$

with $L \in \langle \mathfrak{p}_{\mathcal{J}} \rangle$ and $N \in \ker_{Obj}(F)$. Thanks to Remarks 1.2.2.13 we know that L is also a $\mathcal{J}$-projective object. Next, we claim that the homomorphism $u : L \longrightarrow X$ is a $\mathcal{J}$-epimorphism, which yields the existence of a simple $\mathcal{J}$-projective resolution for X.

Indeed, consider the following long exact sequence

$$\ldots \longrightarrow F(\Sigma^{-1}(N)) \xrightarrow{F(\Sigma^{-1}(w))} F(L) \xrightarrow{F(u)} F(X) \xrightarrow{F(v)} F(N) \longrightarrow \ldots$$

Since $N \in \ker_{Obj}(F)$, we have $F(N) \cong 0$ and the above exact sequence implies that $F(v) = 0$ and so $F(u)$ is an epimorphism, which means that u is a $\mathcal{J}$-epimorphism.

Following the notations of the preceding section, we may put $\mathscr{L} := \langle \mathfrak{p}_{\mathcal{J}} \rangle$ and $\mathscr{N} := \ker_{Obj}(F)$. First of all, notice that $\mathscr{L} = \langle \mathfrak{p}_{\mathcal{J}} \rangle$ is a localizing subcategory of $\mathcal{T}$ by definition and $\mathscr{N} = \ker_{Obj}(F)$ is a localizing subcategory as well because F is supposed to be compatible with countable direct

sums (since $\mathcal{J}$ is, by assumption, an *additive* homological ideal), consequently $\ker_{Obj}(F)$ is a localizing subcategory and Remark 1.2.1.24 assures that its orthogonal complement is localizing as well. Remember that in our framework, a localizing subcategory is automatically thick (see Remark 1.2.1.9).

In order to show that this is a complementary pair of localizing subcategories in $\mathcal{T}$, we have to check the two axioms of Definition 1.2.1.25. The first one is easy to prove.

1.2.2.17 Lemma. *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J} = \ker_{Hom}(F)$ an additive homological ideal in $\mathcal{T}$. Then we have*

$$\langle \mathfrak{p}_{\mathcal{J}} \rangle \subset (\ker_{Obj}(F))^{\vdash}$$

Proof. Take any $\mathcal{J}$-projective object, say $P \in \mathfrak{p}_{\mathcal{J}}$. By definition, we have $\ker_{Hom}(F) \subset \ker_{Hom}(Hom_{\mathcal{T}}(P, \cdot))$. Take any object $N \in \ker_{Obj}(F)$, which means that $F(N) \cong 0$ and so $F(id_N) = 0$, that is, $id_N \in \ker_{Hom}(F)(N, N)$. Hence we have also that $Hom_{\mathcal{T}}(P, id_N) = (id_N)_* = 0$. Now, if $g \in Hom_{\mathcal{T}}(P, N)$ is any homomorphism between P and N, we can write

$$g = id_N \circ g = (id_N)_*(g) = Hom_{\mathcal{T}}(P, id_N)(g),$$

whence $g = 0$. Since this is true for every homomorphism $g \in Hom_{\mathcal{T}}(P, N)$, any object $N \in \ker_{Obj}(F)$ and any object $P \in \mathfrak{p}_{\mathcal{J}}$, we deduce that $\mathfrak{p}_{\mathcal{J}} \subset (\ker_{Obj}(F))^{\vdash}$, by definition of left orthogonal complement.

Finally, remark that $(\ker_{Obj}(F))^{\vdash}$ is a localizing subcategory. In other words, $(\ker_{Obj}(F))^{\vdash}$ is stable under countable direct sums and so we conclude that $\langle \mathfrak{p}_{\mathcal{J}} \rangle \subset (\ker_{Obj}(F))^{\vdash}$ by definition of *localizing subcategory generated by.* ■

Concerning the second axiom of the definition of a complementary pair of localizing subcategories, that is, the existence of a $(\langle \mathfrak{p}_{\mathcal{J}} \rangle, \ker_{Obj}(F))$-triangle for every object in $\mathcal{T}$, the proof is not at all trivial and it requires a more delicate study. We refer to Theorem 3.21 in [131] for a proof and more details.

In this situation, there arises a natural question: how can we construct $\mathcal{J}$-projective objects in a given triangulated category? We observe that the result of the preceding theorem depends strongly on the choice of the functor F to construct the corresponding complementary pair. In this way, we may think that such $\mathcal{J}$-projective objects could be constructed directly using the functor F itself. More precisely, we can show that whenever F admits a *"gentle"* adjoint functor F^+, the localizing subcategory of $\mathcal{J}$-projective objects can be realized as the localizing subcategory of objects of the form $F^+(Obj(\mathcal{P}))$, for some specific subcategory $\mathcal{P}$.

Let us describe with more details this process. Compare the following definition with the classical one (see Definition B.1.13).

1.2.2.18 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category, $\mathscr{C}$ an additive category and $F : \mathcal{T} \longrightarrow \mathscr{C}$ a functor. Let $\mathcal{P} \subset \mathscr{C}$ be a full subcategory.

A partially defined left adjoint for F is a functor $F^{\vdash} : \mathcal{P} \longrightarrow \mathcal{T}$ such that for every objects $X \in Obj(\mathcal{T})$ and $Y \in Obj(\mathcal{P})$ we have an isomorphism

$$Hom_{\mathcal{T}}(F^{\vdash}(Y), X) \cong Hom_{\mathscr{C}}(Y, F(X))$$

such that $\psi : Hom_{\mathcal{T}}(F^{\vdash}(\cdot), \ \cdot \ ) \longrightarrow Hom_{\mathscr{C}}(\ \cdot \ , F(\cdot))$ is a natural isomorphism in each variable.

1.2.2.19 Remark. Let us remark that for every object $Y \in Obj(\mathcal{P})$, the object $F^{\vdash}(Y) \in Obj(\mathcal{T})$ is a $\mathcal{J}$-projective one for $\mathcal{J} := \ker_{Hom}(F)$, whenever F is a stable homological functor.

Indeed, given an object $Y \in Obj(\mathcal{P})$, it is enough to show, thanks to the definition of adjoint functor, that $\mathcal{J} \subset \ker_{Hom}(Hom_{\mathscr{C}}(Y, F(\cdot)))$, which is straightforward because if $f : X_1 \longrightarrow X_2$ is a homomorphism in $\mathcal{T}$ with $X_1, X_2 \in Obj(\mathcal{T})$, then the corresponding homomorphism $Hom_{\mathscr{C}}(Y, F(X_1)) \longrightarrow Hom_{\mathscr{C}}(Y, F(X_2))$ is given by the left composition by $F(f)$, so that if $f \in \mathcal{J}(X_1, X_2)$, it is clear that $F(f)_*$ is the zero map.

In conclusion, $F^{\vdash}(Obj(\mathcal{P})) \subset \mathfrak{p}_{\mathcal{J}}$ with $\mathcal{J} := \ker_{Hom}(F)$.

The following theorem can be found in Proposition 3.37 of [134], but we include here a proof of a simpler statement for the convenience of the exposition.

1.2.2.20 Theorem. *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J} = \ker_{Hom}(F)$ an additive homological ideal in $\mathcal{T}$ with $F : \mathcal{T} \longrightarrow \mathscr{A}$.*

Let $\mathcal{P} \subset \mathscr{A}$ be a full subcategory such that

i) for any object $X \in Obj(\mathcal{T})$ there exists an epimorphism $P \longrightarrow F(X)$ for some $P \in Obj(\mathcal{P})$,

ii) F admits a partially defined left adjoint on $\mathcal{P}$, say $F^{\vdash} : \mathcal{P} \longrightarrow \mathcal{T}$.

In this situation, $\mathcal{T}$ has enough $\mathcal{J}$-projective objects and consequently $\Big(\langle F^{\vdash}(Obj(\mathcal{P}))\rangle, \ker_{Obj}(F)\Big)$ is a complementary pair of localizing subcategories.

Proof. First of all, thanks to Remark 1.2.2.19 above we know that $F^{\vdash}(Obj(\mathcal{P})) \subset \mathfrak{p}_{\mathcal{J}}$.

Next, given any object $X \in Obj(\mathcal{T})$ we have to construct a $\mathcal{J}$-epimorphism $\pi : P \longrightarrow X$ for some $P \in \mathfrak{p}_{\mathcal{J}}$. By assumption, given $X \in Obj(\mathcal{T})$ there exists an epimorphism $\pi' : P' \longrightarrow F(X)$ for some $P' \in Obj(\mathcal{P})$, that is, we have an epimorphism $\pi' \in Hom_{\mathscr{C}}(P', F(X))$. Put $P := F^{\vdash}(P') \in \mathfrak{p}_{\mathcal{J}}$.

Since $F^{\vdash}$ is a (partially defined) left adjoint of F on $\mathcal{P}$, the adjointness relation yields a homomorphism $\pi \in Hom_{\mathcal{T}}(P, X)$. We must check that π is a $\mathcal{J}$-epimorphism, which is equivalent to show that $F(\pi)$ is an epimorphism in $\mathscr{A}$.

Consider the identity homomorphism $id_{F^{\vdash}(P')} \in Hom_{\mathcal{T}}(F^{\vdash}(P'), F^{\vdash}(P'))$ which yields, by adjointness, a map $\alpha \in Hom_{\mathscr{C}}(P', F(F^{\vdash}(P'))) = Hom_{\mathscr{C}}(P', F(P))$. Since the adjointness relation is a *natural* isomorphism, we have the following commutative diagram

$$\begin{tikzcd}
Hom_{\mathcal{T}}(F^{\vdash}(P'), F^{\vdash}(P')) \arrow[r, "\psi"] \arrow[d, "\pi_*"'] & Hom_{\mathscr{C}}(P', F(P)) \arrow[d, "F(\pi)_*"] \\
Hom_{\mathcal{T}}(F^{\vdash}(P'), X) \arrow[r, "\psi"'] & Hom_{\mathscr{C}}(P', F(X))
\end{tikzcd}$$

As a consequence, we have that

$$F(\pi)_*(\psi(id_P)) = \psi(\pi_*(id_P)) \Leftrightarrow F(\pi) \circ \alpha = \pi',$$

where π' is an epimorphism and the formula $F(\pi) \circ \alpha = \pi'$ implies that $F(\pi)$ is an epimorphism as well.

At this stage, Theorem 1.2.2.15 guarantees that $(\langle \mathfrak{p}_{\mathcal{J}} \rangle, \ker_{Obj}(F))$ is a complementary pair of localizing subcategories in $\mathcal{T}$. It remains to show that $\langle \mathfrak{p}_{\mathcal{J}} \rangle = \langle F^{\vdash}(Obj(\mathcal{P}))\rangle$.

Let $X \in \mathfrak{p}_{\mathcal{J}}$ be any *$\mathcal{J}$-projective* object in $\mathcal{T}$. Thanks to the preceding argument, there exists a $\mathcal{J}$-epimorphism $\pi : P \longrightarrow X$ with $P \in F^{\vdash}(Obj(\mathcal{P}))$. Let's embed π into a distinguished triangle, say $P \xrightarrow{\pi} X \longrightarrow C_\pi \longrightarrow \Sigma(P)$ which can be supposed of the form $C \longrightarrow P \xrightarrow{\pi} X \xrightarrow{w} \Sigma(C)$ for some object $C \in Obj(\mathcal{T})$ by virtue of the rotation axiom. In this case, observe that we have $w \in \mathcal{J}(X, \Sigma(C)) = (0)$ because X is $\mathcal{J}$-projective (recall Remark 1.2.2.13) and π is a $\mathcal{J}$-epimorphism (recall Definition 1.2.2.6). Hence Proposition 1.2.1.20 assures that the triangle splits and we have $P \cong C \oplus X \in F^{\vdash}(Obj(\mathcal{P}))$. In other words, X is a *retract* of P and $\langle F^{\vdash}(Obj(\mathcal{P}))\rangle$ is by definition the smallest localizing subcategory containing the objects $F^{\vdash}(Obj(\mathcal{P}))$ so it is thick, that is, stable by retracts. In conclusion, $X \in \langle F^{\vdash}(Obj(\mathcal{P}))\rangle$, which ends the proof. ■

It is important to observe that the above proof is quite constructive once the adjoint functor is known. Indeed, in that case the projective objects are exactly *direct summands of* $F^{\vdash}(Obj(\mathcal{P}))$. For the case of the Kasparov category with respect to a *discrete* group G we'll have that the corresponding projective objects are exactly *direct summands of induced C^*-algebras by finite subgroups* (see Theorem 1.2.3.11 below).

1.2.3 Reformulation of the Baum-Connes conjecture

The Kasparov's category

Here we introduce the equivariant Kasparov's category. It consists in observing the triangulated aspect of the Kasparov's theory using, of course, elementary facts of the KK-theory.

1.2.3.1 Definition. Let G be a (second countable) locally compact group. The G-equivariant Kasparov's category, denoted by $\mathscr{K}\mathscr{K}^G$, is the additive category defined by

- The objects of $\mathscr{K}\mathscr{K}^G$ are the separable G-C^*-algebras.
- The homomorphisms between two (separable) G-C^*-algebras are given by the corresponding G-equivariant Kasparov triples. More precisely,

$$Hom_{\mathscr{K}\mathscr{K}^G}(A, B) := KK^G(A, B),$$

  for all (separable) G-C^*-algebras A, B. The composition of two homomorphisms in $\mathscr{K}\mathscr{K}^G$ being the Kasparov product of the corresponding Kasparov triples.

1.2.3.2 Remarks. 1. Since all our G-C^*-algebras are supposed to be separable, then the corresponding Kasparov groups are abelian and the properties of the Kasparov product give to $\mathscr{K}\mathscr{K}^G$ the additive character. More precisely, any G-C^*-algebra KK^G-equivalent to the zero algebra is the zero element in $\mathscr{K}\mathscr{K}^G$. The well known classical Kasparov theory assures that the Kasparov product is bilinear with respect to the sum of Kasparov triples. Finally, if A_1 and A_2 are two G-C^*-algebras, the direct sum $A_1 \oplus A_2$ is a G-C^*-algebra in the obvious way and we have two natural inclusions, $\iota_1 : A_1 \hookrightarrow A_1 \oplus A_2$, $\iota_2 : A_2 \hookrightarrow A_1 \oplus A_2$ and two natural projections $p_1 : A_1 \oplus A_2 \twoheadrightarrow A_1$, $p_2 : A_1 \oplus A_2 \twoheadrightarrow A_2$. In this way we have that $KK^G(A_1 \oplus A_2, B) \cong KK^G(A_1, B) \times KK^G(A_2, B)$, for all G-C^*-algebra B. Notice as well that the Kasparov category admits *countable direct sums* thanks to the separability condition.

2. Observe that in $\mathscr{K}\mathscr{K}^G$ the word *homomorphism (respectively, isomorphism)* will mean a true homomorphism (respectively, isomorphism) between C^*-algebras or any Kasparov triple between C^*-algebras (respectively, any KK-equivalence between C^*-algebras).

Next, we wish to give a triangulated structure to $\mathscr{K}\mathscr{K}^G$. Firstly, our suspension functor will be just the suspension of C^*-algebras. Namely,

- if (A, α) is any G-C^*-algebra, then we define

$$\Sigma(A) := \{h \in C([0,1], A) \mid h(0) = 0 = h(1)\} \cong C_0(\mathbb{R}) \otimes A$$

- if (A, α), (B, β) are two G-C^*-algebras and $f := (H, \pi, F) \in KK^G(A, B)$ any G-equivariant Kasparov triple for (A, B), then we define

$$\Sigma(f) := (C_0(\mathbb{R}) \otimes H, id \otimes \pi, id \otimes F) \in KK^G(\Sigma(A), \Sigma(B)),$$

  where $C_0(\mathbb{R})$ is equipped with the trivial action of G.

1.2.3.3 Remark. Notice that whenever (A, α) is a G-C^*-algebra, the suspension $\Sigma(A)$ is again a G-C^*-algebra. Namely, we define the action $\widetilde{\alpha} : G \longrightarrow Aut(\Sigma(A))$ by

$$\widetilde{\alpha}_g(h)(t) := \alpha_g(h(t)),$$

for all $g \in G$, all $h \in C([0,1], A)$ and all $t \in [0,1]$.

Notice as well that, by virtue of Bott periodicity, Σ establish an (additive) auto-equivalence. Indeed, by Bott periodicity we have that $K_0^G(A) \cong K_1^G(\Sigma(A))$, for all G-C^*-algebra A and the general Kasparov theory yields the following identifications

$$K_0^G(A) \cong KK^G(\mathbb{C}, A) \text{ and } K_1^G(\Sigma(A)) \cong KK^G(\mathbb{C}, \Sigma^2(A))$$

Therefore, we obtain that A is KK^G-equivalent to $\Sigma^2(A)$, for all G-C^*-algebra A. See [24] for more details.

Next, we have to define the class of distinguished triangles in $\mathscr{K}\mathscr{K}^G$. Given a $*$-homomorphism between two G-C^*-algebras, say $\varphi : A \longrightarrow B$, recall the definition of its cone

$$C_\varphi := \{(a, h) \in A \times C_0((0,1], B) \mid \varphi(a) = h(1)\},$$

which is a G-C^*-algebra in an obvious way.

In this situation, we have a natural exact sequence of G-C^*-algebras,

$$\Sigma(B) \hookrightarrow C_\varphi \twoheadrightarrow A$$

1.2.3.4 Definition. Let $(\mathscr{K}\mathscr{K}^G, \Sigma)$ be the G-equivariant Kasparov Category. A standard triangle or mapping cone triangle in $\mathscr{K}\mathscr{K}^G$ is a triangle in $\mathscr{K}\mathscr{K}^G$ of the form

$$\Sigma(B) \longrightarrow C_\varphi \longrightarrow A \xrightarrow{\varphi} B,$$

where φ is a $*$-homomorphism between G-C^*-algebras.

The class of all triangles in $\mathscr{K}\mathscr{K}^G$ isomorphic to a mapping cone triangle is denoted by Δ_Σ.

1.2.3.5 Definition. Let $(\mathscr{K}\mathscr{K}^G, \Sigma)$ be the G-equivariant Kasparov Category. An extension triangle in $\mathscr{K}\mathscr{K}^G$ is a triangle in $\mathscr{K}\mathscr{K}^G$ of the form

$$\Sigma(A) \longrightarrow B \xrightarrow{\iota} E \xrightarrow{p} A,$$

where $B \overset{\iota}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} A$ is G-equivariant semi-split extension of G-C^*-algebras and the homomorphism $\Sigma(A) \longrightarrow B$ in $\mathscr{K}\mathscr{K}^G$ is given by the class of the extension $B \overset{\iota}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} A$ in $Ext^{-1}(A, B) \cong KK^G(\Sigma(A), B)$ (recall Remark A.1.8).

The class of all triangles in $\mathscr{K}\mathscr{K}^G$ isomorphic to an extension triangle is denoted by Ξ_Σ.

1.2.3.6 Remark. Notice that in the Kasparov category, triangles are considered to be diagrams in $\mathscr{K}\mathscr{K}^G$ of the form $\Sigma(B) \longrightarrow C \longrightarrow A \longrightarrow B$, which are the *opposite* with respect to the notations followed in the preceding sections. However, having in mind that the suspension of C^*-algebras is such that $\Sigma = \Sigma^{-1}$ in $\mathscr{K}\mathscr{K}^G$, Remarks 1.2.1.4 yields the coherence of our notations.

1.2.3.7 Lemma. *Let $(\mathscr{K}\mathscr{K}^G, \Sigma)$ the G-equivariant Kasparov category. Then we have*

$$\Delta_\Sigma = \Xi_\Sigma$$

Proof. Firstly, we see that $\Delta_\Sigma \subset \Xi_\Sigma$. Let $\varphi : A \longrightarrow B$ be G-equivariant $*$-homomorphism and consider the corresponding mapping cone triangle $\Sigma(B) \longrightarrow C_\varphi \longrightarrow A \xrightarrow{\varphi} B$. Recall now the definition of the cylinder of φ,

$$Z_\varphi := \{(a, h) \in A \times C([0, 1], B) \mid \varphi(a) = h(1)\}$$

We claim that Z_φ and A are G-equivariant homotopic equivalent, which means that there exist two G-equivariant $*$-homomorphisms $f : A \longrightarrow Z_\varphi$ and $g : Z_\varphi \longrightarrow A$ such that $f \circ g$ is G-equivariant homotopic equivalent to id_{Z_φ} and $g \circ f$ is G-equivariant homotopic equivalent to id_A. Namely, put

$$\begin{array}{rccl} f: & A & \longrightarrow & Z_\varphi \\ & a & \longmapsto & f(a) := (a, \varphi(a)) \end{array} \qquad \begin{array}{rccl} g: & Z_\varphi & \longrightarrow & A \\ & (a, h) & \longmapsto & g(a, h) := a \end{array}$$

It is clear that $g \circ f = id_A$ and the collection of $*$-homomorphisms

$$\begin{array}{rccl} \varphi_t: & Z_\varphi & \longrightarrow & Z_\varphi \\ & (a, h) & \longmapsto & \varphi_t(a, h) := \big(a, h\big(\cdot\ t + (1 - t)\big)\big), \end{array}$$

for all $t \in [0, 1]$ is a continuous G-equivariant homotopy between $f \circ g$ and id_{Z_φ}. Precisely, $\varphi_0 = f \circ g$, $\varphi_1 = id_{Z_\varphi}$ and the map $t \mapsto \varphi_t(a, h)$ is continous for every $(a, h) \in Z_\varphi$.

Next, consider the obvious G-equivariant semi-split extension $C_\varphi \hookrightarrow Z_\varphi \overset{\widetilde{p}}{\twoheadrightarrow} B$, where $\widetilde{p}(a, h) := h(0)$. This yields an extension triangle $\Sigma(B) \longrightarrow C_\varphi \longrightarrow Z_\varphi \xrightarrow{\widetilde{p}} B$.

Moreover, the homotopy between Z_φ and A described before yields an isomorphism of distinguished triangles in $\mathscr{K}\mathscr{K}^G$

$$\begin{tikzcd} \Sigma(B) \arrow[r] \arrow[d] & C_\varphi \arrow[r] \arrow[d, equal] & A \arrow[r, "\varphi"] \arrow[d, "f"] & B \arrow[d, equal] \\ \Sigma(B) \arrow[r] & C_\varphi \arrow[r] & Z_\varphi \arrow[r, "\widetilde{p}"'] & B \end{tikzcd}$$

Hence the mapping cone triangle is isomorphic to the extension triangle given by the cylinder.

Conversely, let's see $\Xi_\Sigma \subset \Delta_\Sigma$. Consider a G-equivariant semi-split extension $B \overset{\iota}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} A$ and the corresponding extension triangle $\Sigma(A) \longrightarrow B \xrightarrow{\iota} E \xrightarrow{p} A$. Consider as well the G-equivariant semi-split extension of the cylinder of p, $C_p \hookrightarrow Z_p \overset{\widetilde{p}}{\twoheadrightarrow} A$, and the corresponding extension triangle, $\Sigma(A) \longrightarrow C_p \longrightarrow Z_p \xrightarrow{\widetilde{p}} A$.

Next, consider the G-equivariant homotopy between Z_p and E as before, which is given by means of

$$\begin{array}{rccl} f: & E & \longrightarrow & Z_p \\ & e & \longmapsto & f(e) := (e, p(e)) \end{array} \qquad \begin{array}{rccl} g: & Z_\varphi & \longrightarrow & E \\ & (e,h) & \longmapsto & g(e,h) := e \end{array}$$

Observe that the map f restricts to a map

$$\begin{array}{rccl} f_|: & B & \longrightarrow & C_p \\ & b & \longmapsto & f_|(b) := (\iota(b), 0) \end{array}$$

Hence, the homotopy between Z_p and E described before yields an isomorphism of distinguished triangles in $\mathscr{K}\mathscr{K}^G$

$$\begin{tikzcd} \Sigma(A) \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "f_|"] & E \arrow[r, "p"] \arrow[d, "f"] & A \arrow[d, equal] \\ \Sigma(A) \arrow[r] & C_p \arrow[r] & Z_p \arrow[r, "\widetilde{p}"'] & A \end{tikzcd}$$

In particular, $B \cong C_p$ in $\mathscr{K}\mathscr{K}^G$. Observe that $\widetilde{p}$ is G-equivariant homotopic equivalent to $p \circ g$ because $\widetilde{p} \circ f = p$ and $f \circ g$ is G-equivariant homotopic equivalent to id_{Z_p}. In this way, C_p is G-equivariant homotopic equivalent to $C_{\widetilde{p}}$, which allows to consider the bottom distinguished triangle of the above diagram as a mapping cone triangle and the proof is complete. ■

The preceding lemma allows to show that the G-equivariant Kasparov category is triangulated using either Δ_Σ or Ξ_Σ for defining the corresponding distinguished triangles. We refer to Appendix A in [132] for the details.

1.2.3.8 Theorem. *Let G be a (second countable) locally compact group. The G-equivariant Kasparov's category $\mathscr{K}\mathscr{K}^G$ equipped with the C^*-algebra suspension functor Σ and the class of distinguished triangles $\Delta_\Sigma (= \Xi_\Sigma)$ given by the mapping cone triangles (or the extension triangles) is a triangulated category.*

1.2.3.9 Remark. Since $(\mathscr{K}\mathscr{K}^G, \Sigma, \Delta_\Sigma)$ is a triangulated category *all* the elementary facts that we have established in Section 1.2.1 can be applied to $\mathscr{K}\mathscr{K}^G$. In particular, it is important to observe that for any (separable) G-C^*-algebra $T \in Obj(\mathscr{K}\mathscr{K}^G)$, the homomorphism functors $KK^G(T, \cdot\,)$ and $KK^G(\,\cdot\,, T)$ are homological, so that for every distinguished triangle in $\mathscr{K}\mathscr{K}^G$ we can construct long exact sequences in KK-theory (see Remark 1.2.1.14 and Proposition 1.2.1.16).

Choice of the complementary pair

In order to construct a suitable complementary pair of localizing subcategories in $\mathscr{K}\mathscr{K}^G$, we are going to apply the Meyer-Nest's homological algebra developed previously. It is important to say that the treatment with projective objects in the Kasparov's category needs of course deep results of

locally compact groups and Kasparov's theory that are out of the scope of this survey chapter and so we may refer to the original papers [132] and [131] for all the details. Despite of this difficulty, the case in which G is a discrete group can be easily established and we'll explain the argument. Moreover, the discrete case is specially interesting for the present thesis.

Given a (second countable) locally compact group G and a compact subgroup $H < G$, we have two natural functors that we can consider. Namely,

- Restriction functor:
$$Res_H^G : \mathscr{K}\mathscr{K}^G \longrightarrow \mathscr{K}\mathscr{K}^H,$$
which is just a forgetful functor.

- Induction functor:
$$Ind_H^G : \mathscr{K}\mathscr{K}^H \longrightarrow \mathscr{K}\mathscr{K}^G,$$
defined on the objects level as follows: if (B, β) is any (separable) H-C^*-algebra, then
$$Ind_H^G(B,\beta) := \big\{ f \in C_b(G,B) \mid f(gh) = h^{-1}f(g)\ \forall g \in G, h \in H \\ \text{and } (gH \mapsto ||f(g)||) \in C_0(G/H) \big\},$$
equipped with the action of G by left translations is a (separable) G-C^*-algebra. It is well known that this association is functorial with respect to the KK-theory, so that if $g \in KK^H(B, B')$ is a Kasparov triple between two H-C^*-algebras (B, β) and (B', β'), then there exists a Kasparov triple $Ind_H^G(g) \in KK^G(Ind_H^G(B,\beta), Ind_H^G(B',\beta'))$ and so the this yields the existence of the induction functor Ind_H^G above (see Section 3 in [98] or Theorem 20.5.4 in [24] for more details about this construction).

Let us write $\mathcal{F}$ for the family of all compact subgroups of G. Then we define the following subcategories of $\mathscr{K}\mathscr{K}^G$

- Compactly Contractible objects:
$$\mathcal{CC} := \{A \in Obj(\mathscr{K}\mathscr{K}^G) \mid Res_H^G(A) \cong 0\ \forall H \in \mathcal{F}\} = \bigcap_{H \in \mathcal{F}} \ker_{Obj}(Res_H^G)$$

- Compactly Induced objects:
$$\langle \mathcal{CI} \rangle := \langle \{A \in Obj(\mathscr{K}\mathscr{K}^G) \mid A \cong Ind_H^G(B) \text{ with } H \in \mathcal{F},\ B \in Obj(\mathscr{K}\mathscr{K}^H)\} \rangle$$

1.2.3.10 Remarks. 1. Notice that the restriction functor Res_H^G is just a forgetful one, so it is clear that it is compatible with countable direct sums. And it is straightforward to see that Res_H^G is a triangulated functor. Namely, it is clear applying the definitions that $\Sigma(Res_H^G(A)) = Res_H^G(\Sigma(A))$, for all $A \in Obj(\mathscr{K}\mathscr{K}^G)$ and that $Res_H^G(C_f) = C_{Res_H^G(f)}$, for all $*$-homomorphism $f : A \longrightarrow B$ between G-C^*-algebras. In other words, Res_H^G is a stable functor that transforms mapping cone triangles into mapping cone triangles, that is, Res_H^G is a triangulated functor. Hence, thanks to Remark 1.2.1.13, we have that $\ker_{Obj}(Res_H^G)$ is a localizing subcategory of $\mathscr{K}\mathscr{K}^G$, for every compact subgroup $H < G$; as a result, $\mathcal{CC} = \bigcap_{H \in \mathcal{F}} \ker_{Obj}(Res_H^G)$ is also a localizing subcategory of $\mathscr{K}\mathscr{K}^G$ (see Remark 1.2.1.9).

2. The induction functor is more complicated than the restriction one. For instance, Ind_H^G is not at all compatible with (finite or countable) direct sums. In this way, $\mathcal{CI}$ is just a *class of objects* in $\mathscr{K}\mathscr{K}^G$. Hence we are forced to take the corresponding localizing subcategory generated by compactly induced objects.

3. Following the notations of the preceding sections, we may put $\mathscr{L} := \langle \mathcal{CI} \rangle$ and $\mathscr{N} := \mathcal{CC}$. In order to apply the Meyer-Nest machinery, we want to show that $(\langle \mathcal{CI} \rangle, \mathcal{CC})$ is a complementary pair of localizing subcategories in $\mathscr{K}\mathscr{K}^G$. To this end, the strategy consists in constructing projective objects in $\mathscr{K}\mathscr{K}^G$ with respect an additive homological ideal. Namely, this ideal is simply $\mathcal{J} := \bigcap_{H \in \mathcal{F}} \ker_{Hom}(Res_H^G)$ (recall Theorem 1.2.2.15).

   Notice that Res_H^G is a triangulated functor compatible with countable direct sums, so $\ker_{Hom}(Res_H^G)$ is an additive homological ideal in $\mathscr{K}\mathscr{K}^G$ by virtue of Remark 1.2.2.2 and Remark 1.2.2.4. As a result, $\mathcal{J} := \bigcap_{H \in \mathcal{F}} \ker_{Hom}(Res_H^G)$ is also an additive homological ideal (see Remark 1.2.2.2 and Remark 1.2.2.5).

1.2.3.11 Theorem. *Let G be a (second countable) locally compact group and $\mathcal{F}$ the family of all compact subgroups of G. If $\mathcal{J} := \bigcap_{H \in \mathcal{F}} \ker_{Hom}(Res_H^G)$, then*

i) $\langle \mathcal{CI} \rangle = \langle \mathfrak{p}_{\mathcal{J}} \rangle$,

ii) $\mathscr{K}\mathscr{K}^G$ has enough $\mathcal{J}$-projective objects.

Therefore, $(\langle \mathcal{CI} \rangle, \mathcal{CC})$ is a complementary pair of localizing subcategories in $\mathscr{K}\mathscr{K}^G$.

Proof. We distinguish two cases.

A) The general case in which G is any (second countable) locally compact group requires, on the one hand, structural results about compact subgroups in locally compact groups (for instance, *existence of a maximal compact subgroup in a almost connected group*) and, on the other hand, deep results in KK-theory due to Kasparov (for instance, the relationship between the functors Res and $Res \circ Ind$ in the context of maximal compact subgroups. See Theorem 5.8 in [98] and Lemma 3.3 in [132]). With these preliminaries (which can be found in Section 3 of [132]), some work yields that

   - $\mathcal{CC} = (\langle \mathcal{CI} \rangle)^{\dashv}$, so $\langle \mathcal{CI} \rangle \subset \mathcal{CC}^{\vdash}$
   - and $\mathcal{CI} \subset \mathfrak{p}_{\mathcal{J}}$

   (see Proposition 4.4 in [132] for a proof). In order to achieve the conclusion of the statement, we want to apply Theorem 1.2.2.15. However, when G is a locally compact group, the restriction functor does not always have an adjoint functor. Nevertheless, we can define an adjoint functor for the restriction on *enough compact subgroups*, so that Theorem 1.2.2.15 can be applied. For a proof following these ideas, we refer to Theorem 7.3 in [131]. For a more analytical and direct approach to this proof, we refer to Proposition 4.6 and Theorem 4.7 in [132].

B) The case in which G is a discrete group is much easier to establish. In this case, the family $\mathcal{F}$ of all compact subgroups of G becomes the family of all *finite* subgroups of G. Since G is

discrete, $\mathcal{F}$ is formed, in particular, by *closed* subgroups of G, so that the classical Frobenius reciprocity in KK-theory (due to Wassermann in 1983, [220]) can be applied and we have that for every $H \in \mathcal{F}$

$$KK^G(Ind_H^G(A), B) \cong KK^H(A, Res_H^G(B)),^1$$

for all H-C^*-algebra A and all G-C^*-algebra B. In other words, the functors Res_H^G and Ind_H^G are adjoints in $\mathscr{K}\mathscr{K}^G$, for all finite subgroup $H \in \mathcal{F}$. More precisely, Ind_H^G is a left adjoint functor for Res_H^G, for all finite subgroup $H \in \mathcal{F}$.

Let's consider the following triangulated functor,

$$F := (Res_H^G)_{H\in\mathcal{F}} : \mathscr{K}\mathscr{K}^G \longrightarrow \prod_{H\in\mathcal{F}} \mathscr{K}\mathscr{K}^H,$$

which is defined in an obvious way. With these notations, it is clear that

$$\mathcal{CC} = \ker_{Obj}(F) \text{ and } \mathcal{J} = \ker_{Hom}(F)$$

Let's define the functor

$$F^{\vdash} : \prod_{H\in\mathcal{F}} \mathscr{K}\mathscr{K}^H \longrightarrow \mathscr{K}\mathscr{K}^G$$

- on objects by

$$F^{\vdash}\big((A_H)_{H\in\mathcal{F}}\big) := \bigoplus_{H\in\mathcal{F}} Ind_H^G(A_H),$$

  for all $(A_H)_{H\in\mathcal{F}} \in Obj(\prod_{H\in\mathcal{F}} \mathscr{K}\mathscr{K}^H)$.

  Remark that since G is discrete, $\mathcal{F}$ is a countable set. Hence the above direct sum is countable and $F^{\vdash}$ is well-defined.

- and on homomorphisms by functoriality of the induction functor.

We claim that $F^{\vdash}$ is a left adjoint functor for F. Indeed, we have just to apply the fact that Ind_H^G is a left adjoint functor for Res_H^G, for all finite subgroup $H \in \mathcal{F}$ by virtue of the Frobenius reciprocity and the structure of a product category,

$$\begin{aligned} KK^G(F^{\vdash}\big((A_H)_{H\in\mathcal{F}}\big), B) &\cong \prod_{H\in\mathcal{F}} KK^G(\bigoplus_{H\in\mathcal{H}} Ind_H^G(A_H), B) \\ &\cong \prod_{H\in\mathcal{F}} KK^G(Ind_H^G(A_H), B) \\ &\cong \prod_{H\in\mathcal{F}} KK^H(A_H, Res_H^G(B)) = \Big(\prod_{H\in\mathcal{F}} KK^H\Big)(A_H, F(B)) \end{aligned}$$

Therefore, Theorem 1.2.2.20 assures that $\mathscr{K}\mathscr{K}^G$ has enough $\mathcal{J}$-projective objects and consequently, by Theorem 1.2.2.15, $(\langle \mathfrak{p}_{\mathcal{J}} \rangle, \mathcal{CC})$ is a complementary pair in $\mathscr{K}\mathscr{K}^G$ with $\langle \mathfrak{p}_{\mathcal{J}} \rangle = \langle F^{\vdash}\big(Obj(\prod_{H\in\mathcal{F}} \mathscr{K}\mathscr{K}^H)\big)\rangle$.

[1] In the discrete case we can prove directly this adjointness property (using Theorem B.1.15) instead of using the stronger result of Wassermann (see for instance Section 3.2 in [132] for more details). Namely, the argument is analogous to the one given for the discrete quantum groups, see Lemma 1.7.2.4.

To conclude, observe that $\langle \mathcal{CI} \rangle$ is the minimal localizing subcategory containing the compactly induced objects by definition, so it contains objects of the form $F^{\vdash}\big(Obj(\prod_{H\in\mathcal{F}} \mathscr{K}\mathscr{K}^H)\big)$, so we have $\langle F^{\vdash}\big(Obj(\prod_{H\in\mathcal{F}} \mathscr{K}\mathscr{K}^H)\big)\rangle \subset \langle \mathcal{CI} \rangle$ and, by minimality, this inclusion must be an equality. This yields the conclusion of the statement.

■

1.2.3.12 Remark. Since $(\langle \mathcal{CI} \rangle, \mathcal{CC})$ is a complementary pair of localizing subcategories in $\mathscr{K}\mathscr{K}^G$, the fundamental lemma about complementary pairs (see Lemma 1.2.1.26) can be applied, so that in particular we have two triangulated functors

$$L : \mathscr{K}\mathscr{K}^G \longrightarrow \langle \mathcal{CI} \rangle \text{ and } N : \mathscr{K}\mathscr{K}^G \longrightarrow \mathcal{CC}$$

such that for any G-C^*-algebra $A \in Obj(\mathscr{K}\mathscr{K}^G)$ there exists a (unique up to isomorphism) distinguished triangle of the form

$$\Sigma(N(A)) \longrightarrow L(A) \xrightarrow{D} A \longrightarrow N(A),$$

where D is called *Dirac homomorphism*.

This nomenclature is used according to the classical Dirac-dual Dirac method. More precisely, the homomorphism D of the above distinguished triangle must be regarded as an element $D \in KK^G(L(A), A)$. Let us denote by $D_{\mathbb{C}} \in KK^G(L(\mathbb{C}), \mathbb{C})$ the Dirac homomorphism corresponding to the trivial G-C^*-algebra $\mathbb{C}$. We call $D_{\mathbb{C}}$ *Dirac element*. Assume that there exists an element $\eta_{\mathbb{C}} \in KK^G(\mathbb{C}, L(\mathbb{C}))$ such that $D_{\mathbb{C}} \underset{\mathbb{C}}{\otimes} \eta_{\mathbb{C}} = \mathbb{1}_{L(\mathbb{C})} \in KK^G(L(\mathbb{C}), L(\mathbb{C}))$. We call $\eta_{\mathbb{C}}$ *dual Dirac element*. In this situation, the element $\gamma_{\mathbb{C}} := \eta_{\mathbb{C}} \underset{L(\mathbb{C})}{\otimes} D_{\mathbb{C}} \in KK^G(\mathbb{C}, \mathbb{C})$, called γ-*element*, is an idempotent. We can show that

$$\gamma = \mathbb{1}_{\mathbb{C}} \iff \langle \mathcal{CI} \rangle = \mathscr{K}\mathscr{K}^G$$

In other words, the categorical framework of Meyer-Nest provides a categorical formulation of the classical Dirac-dual Dirac method. The main difference is that in this case the Dirac element is fixed, so that we only have to construct the dual Dirac element in order to apply the method.

We refer to Section 8 of [132] for all the details about the Dirac-dual Dirac method in the categorical framework.

Reformulation

Finally, we have just to choose an appropriated triangulated or homological functor on $\mathscr{K}\mathscr{K}^G$ whose localization with respect to our complementary pair $(\langle \mathcal{CI} \rangle, \mathcal{CC})$ yields the classical Baum-Connes assembly map. To this end, we consider the functor defining the right-hand side of the classical Baum-Connes assembly map, that is,

$$\begin{array}{rccl} F_* : & \mathscr{K}\mathscr{K}^G & \longrightarrow & \mathscr{A}b^{\mathbb{Z}/2} \\ & (A, \alpha) & \longmapsto & F_*(A, \alpha) := K_*(G \underset{\alpha,r}{\ltimes} A), \text{ with } * = 0, 1 \end{array}$$

where $\mathscr{A}b^{\mathbb{Z}/2}$ denotes the abelian category of $\mathbb{Z}/2$-graded groups of $\mathscr{A}b$, which is equipped with the obvious suspension functor $\Sigma_{\mathscr{A}b^{\mathbb{Z}/2}}$ given by shifting the grading. Observe that F_* is a stable thanks to Bott periodicity. Indeed,

$$\begin{aligned} F_*(\Sigma(A,\alpha)) &= K_*(G \underset{r}{\ltimes} \Sigma(A)) = K_*(G \underset{r}{\ltimes} C_0(\mathbb{R}) \otimes A) \cong K_*(C_0(\mathbb{R}) \otimes G \underset{\alpha,r}{\ltimes} A) \\ &= K_*(\Sigma(G \underset{\alpha,r}{\ltimes} A)) \cong K_{*+1}(G \underset{\alpha,r}{\ltimes} A) = \Sigma_{\mathscr{A}b^{\mathbb{Z}/2}}(K_*(G \underset{\alpha,r}{\ltimes} A)) \\ &= \Sigma_{\mathscr{A}b^{\mathbb{Z}/2}}(F_*(A,\alpha)), \end{aligned}$$

for all $(A,\alpha) \in Obj(\mathscr{K}\mathscr{K}^G)$. Moreover, F_* is a homological functor compatible with countable direct sums. Indeed, for all $(A,\alpha) \in Obj(\mathscr{K}\mathscr{K}^G)$ we have

$$F_0(A,\alpha) = K_0(G \underset{\alpha,r}{\ltimes} A) \cong KK(\mathbb{C}, G \underset{\alpha,r}{\ltimes} A) = KK(\mathbb{C},\cdot) \circ G \underset{r}{\ltimes} \cdot\ (A,\alpha)$$

$$F_1(A,\alpha) = K_1(G \underset{\alpha,r}{\ltimes} A) \cong KK(\mathbb{C}, \Sigma(G \underset{\alpha,r}{\ltimes} A)) = KK(\mathbb{C},\cdot) \circ G \underset{r}{\ltimes} \cdot\ \Sigma(A,\alpha),$$

where the descent functor $G \underset{r}{\ltimes} \cdot : \mathscr{K}\mathscr{K}^G \longrightarrow \mathscr{K}\mathscr{K}$ is triangulated and compatible with countable direct sums and the homomorphism functor $KK(\mathbb{C},\cdot) : \mathscr{K}\mathscr{K} \longrightarrow \mathscr{A}b$ is homological and compatible with countable direct sums. Hence the compositions above are homological functors compatible with countable direct sums.

By abuse of notation we write simply $F_* := F$. In this situation, Definition 1.2.1.30 can be applied to our context.

1.2.3.13 Definition. Let G be a (second countable) locally compact group and fix the homological functor $F : \mathscr{K}\mathscr{K}^G \longrightarrow \mathscr{A}b^{\mathbb{Z}/2}$ defined by $F(A,\alpha) := K_*(G \underset{\alpha,r}{\ltimes} A)$, for all $(A,\alpha) \in Obj(\mathscr{K}\mathscr{K}^G)$.

The categorical Baum-Connes assembly map for G is the categorical Baum-Connes assembly map for $\mathscr{K}\mathscr{K}^G$ with respect to $(\langle \mathcal{CI} \rangle, \mathcal{CC}, F)$, that is, the natural tranformation

$$\eta^G : \mathbb{L}F \longrightarrow F$$

1.2.3.14 Definition. Let G be a (second countable) locally compact group.

- We say that G satisfies the (categorical) Baum-Connes property (with coefficients) if η^G is a natural equivalence.
- We say that G satisfies the *strong* (categorical) Baum-Connes property if $\langle \mathcal{CI} \rangle = \mathscr{K}\mathscr{K}^G$.

1.2.3.15 Remark. It is clear that the strong (categorical) Baum-Connes property implies the (categorical) Baum-Connes property by virtue of the uniqueness of the $(\langle \mathcal{CI} \rangle, \mathcal{CC})$-triangles. Notice, by the way, that G always satisfies the (categorical) Baum-Connes property with coefficients in objects of $\langle \mathcal{CI} \rangle$.

1.2.3.16 Note. It is worth mentioning that what has been called classically *strong* Baum-Connes conjecture is some weaker condition with respect to the above definition. Namely, following the *Dirac-dual Dirac method* explained in the introduction of this dissertation, when there exists a γ-element $\gamma \in KK^G(\mathbb{C},\mathbb{C})$, it defines a projection in $End(K_*(G \underset{r}{\ltimes} B))$, for all G-C^*-algebra B by means of the descent homomorphism. Moreover, the image of μ_B^G is precisely the image of this

projection in $K_*(G \underset{r}{\ltimes} B)$ (see [97], [114], [78] for more details). Hence, classically we say that G satisfies the *strong* Baum-Connes conjecture with coefficients if the image of this projection is invertible in $KK(G \underset{r}{\ltimes} B, G \underset{r}{\ltimes} B)$, which is weaker than the above definition as explained in Remark 1.2.3.12.

One of the main achievement of this categorical formulation of the Baum-Connes property is not only a new approach to the conjecture, but we can establish different criterion to decide whether the conjecture is true or not regarding only the family of compact subgroups of G, which may be useful in practice. Namely, we have the following

1.2.3.17 Theorem (Baum-Connes conjecture Reformulation)**.** *Let G be a (second countable) locally compact group and $\mathcal{F}$ the family of all compact subgroups of G. The following assertions are equivalent*

i) G satisfies the Baum-Connes property, that is, μ_A^G is an isomorphism, for all (separable) G-C^*-algebra A.

ii) G satisfies the categorical Baum-Connes property, that is, η^G is a natural equivalence.

iii) $K_*(G \underset{\alpha,r}{\ltimes} A) = (0)$, for all $(A, \alpha) \in Obj(\mathcal{CC})$.

iv) $K_*(G \underset{\alpha,r}{\ltimes} A) = (0)$, for all G-C^*-algebra (A, α) that is H-contractible for all $H \in \mathcal{F}$.

v) If (A, α) is a G-C^*-algebra such that $K_*(H \underset{\alpha,r}{\ltimes} A) = (0)$ for all $H \in \mathcal{F}$, then $K_*(G \underset{\alpha,r}{\ltimes} A) = (0)$.

vi) If $f : A \longrightarrow B$ is a $*$-homomorphism between G-C^*-algebras that is a H-homotopy equivalence for all $H \in \mathcal{F}$, then f induces an isomorphism $K_*(G \underset{r}{\ltimes} A) \cong K_*(G \underset{r}{\ltimes} B)$.

vii) If $\psi \in KK^G(A, B)$ is any Kasparov triple between two G-C^*-algebras that induces an isomorphism $K_*(H \underset{r}{\ltimes} A) \cong K_*(H \underset{r}{\ltimes} B)$ for all $H \in \mathcal{F}$, then ψ induces an isomorphism $K_*(G \underset{r}{\ltimes} A) \cong K_*(G \underset{r}{\ltimes} B)$.

viii) If $\psi \in KK^G(A, B)$ is any Kasparov triple between two G-C^*-algebras that is invertible in $KK^H(A, B)$ for all $H \in \mathcal{F}$, then ψ induces an isomorphism $K_*(G \underset{r}{\ltimes} A) \cong K_*(G \underset{r}{\ltimes} B)$.

Proof. - $(i) \Leftrightarrow (ii)$. Consider the two following homological functors compatible with countable direct sums,

$$H : \mathscr{K}\mathscr{K}^G \longrightarrow \mathscr{A}b^{\mathbb{Z}/2} \text{ and } F : \mathscr{K}\mathscr{K}^G \longrightarrow \mathscr{A}b^{\mathbb{Z}/2},$$

defined on objects by

$$H(A) := R\Big(KK_*^G(\underline{E}G, A)\Big) \text{ and } F(A) := K_*(G \underset{r}{\ltimes} A),$$

for all G-C^*-algebra A; and on homomorphisms by functoriality.

In this way, the classical assembly map of the Baum-Connes property, $\mu_A^G : R\Big(KK_*^G(\underline{E}G, A)\Big) \longrightarrow K_*(G \underset{r}{\ltimes} A)$ can be regarded as a natural transformation between H and F, say $\mu^G : H \longrightarrow F$.

Next, we have to use some well-known (but *non-trivial*) results about the assembly map. On the one hand, G satisfies the Baum-Connes property with coefficients in compactly induced C^*-algebras, that is, $\mu^G_{|\mathcal{CI}}$ is an isomorphism (see [36] and [35] for more details). On the other hand, Theorem 7.1 of [132] assures that we have an isomorphism of categories $\mathscr{K}\mathscr{K}^G/\mathcal{CC} \cong R\Big(KK^G_*(\underline{E}G, \ \cdot \ )\Big)$, which implies that $H_{|\mathcal{CC}} = 0$ by definition of the Verdier quotient.

In this situation, Lemma 1.2.1.31 can be applied, so that μ^G induces a natural equivalence $H \cong \mathbb{L}F$, which yields the equivalence between the classical assembly map μ^G and the categorical one η^G.

- $(ii) \Leftrightarrow (iii)$. This is true by virtue of the universal property of localization (see Theorem 1.2.1.29).

- $(iii) \Leftrightarrow (iv)$. In [130] R. Meyer performs a description of *equivariant* Kasparov theory in terms of generalized homomorphisms following the pioneer work of J. Cuntz [45]. Given a G-C^*-algebra A, R. Meyer defines a universal C^*-algebra q_sA so that, if B is an other G-C^*-algebra, the Kasparov group $KK^G(A,B)$ is identified with the set of homotopy classes of G-equivariant homomorphisms from $\mathcal{K}(L^2(G)\otimes l^2(\mathbb{N}))\otimes q_sA$ to $\mathcal{K}(L^2(G)\otimes l^2(\mathbb{N}))\otimes B$, denoted by $[\mathcal{K}(L^2(G)\otimes l^2(\mathbb{N}))\otimes q_sA, \mathcal{K}(L^2(G)\otimes l^2(\mathbb{N}))\otimes B]$ (see Theorem 5.5 in [130]).

  If $A \in \mathcal{CC}$, then $Res^G_H(A) \cong 0$ in $\mathscr{K}\mathscr{K}^H$ for all $H \in \mathcal{F}$. This means that there exists an invertible Kasparov triple $\mathcal{E}_H \in KK^H(Res^G_H(A), 0)$, for all $H \in \mathcal{F}$. Since $KK^H(Res^G_H(A), 0) \cong [\mathcal{K}(L^2(H)\otimes l^2(\mathbb{N}))\otimes q_sRes^G_H(A), 0]$, then the invertible Kasparov triple $\mathcal{E}_H$ is transformed into a H-equivariant homotopy equivalence between $\mathcal{K}(L^2(H)\otimes l^2(\mathbb{N}))\otimes q_sRes^G_H(A)$ and 0, which means that $\mathcal{K}(L^2(G)\otimes l^2(\mathbb{N}))\otimes q_sA$ is a G-C^*-algebra which is H-contractible for all $H \in \mathcal{F}$.

  In other words, if $A \in \mathcal{CC}$, then A is KK^G-equivalent to a G-C^*-algebra which is H-contractible for all compact subgroup $H \in \mathcal{F}$. This yields obviously the equivalence $(iii) \Leftrightarrow (iv)$.

- $(iv) \Leftrightarrow (v)$. Let's see the following implications,

  - $(v) \Rightarrow (iii)$. Let (A, α) be a G-C^*-algebra in $\mathcal{CC}$. By definition, we have that $Res^G_H(A) \cong 0$, for all $H \in \mathcal{F}$. As a consequence, $H \underset{\alpha_|, r}{\ltimes} Res^G_H(A) \cong 0$ and so $K_*(H \underset{\alpha_|, r}{\ltimes} Res^G_H(A)) \cong (0)$. Since this is true for all compact subgroup $H \in \mathcal{F}$, the hypothesis of (v) implies that $K_*(G \underset{\alpha, r}{\ltimes} A) \cong (0)$ and we get (iii).

  - $(ii) \Rightarrow (v)$. If G is an arbitrary topological group, then there always exists an open almost connected subgroup $U < G$. Namely, if $G_0 < G$ denotes the connected component of the neutral element, it is well-known that G/G_0 is always a totally disconnected group (see Theorem 7.3 in [56] for a proof). This means that G/G_0 contains a compact open subgroup, say $\overline{U}$. If $q : G \longrightarrow G/G_0$ denotes the canonical quotient map, then $U := q^{-1}(\overline{U})$ is an almost connected subgroup of G since $U/G_0 = q(U) = \overline{U}$ is compact. If G is a locally compact group, a compact subgroup $K < G$ is said to be *large* if it is a maximal compact subgroup of some open almost connected subgroup of G. In Lemma 3.1 of [132] it is shown that any compact subgroup of G is contained in a large one. Hence, in the definitions of the localizing subcategories $\langle \mathcal{CI} \rangle$ and $\mathcal{CC}$ it is enough to consider *large compact subgroups*.

Large compact subgroups are also *smooth* following the terminology of Section 3 in [132], which allows to apply Theorem 9.3 of [132] in order to show the implication $(ii) \Rightarrow (v)$. More precisely, assume that η^G is a natural equivalence and take a G-C^*-algebra (A, α) such that $K_*(H \underset{\alpha_|, r}{\ltimes} Res^G_H(A)) = (0)$ for all large compact subgroup H. Theorem 9.3 of [132] implies that $\mathbb{L}F(A) = K_*(G \underset{\alpha, r}{\ltimes} L(A)) = (0)$, which implies by our assumption that $F(A) = K_*(G \underset{\alpha, r}{\ltimes} A) = (0)$.

In other words, we have that $(iv) \Leftrightarrow (iii) \Leftrightarrow (ii) \Leftrightarrow (v)$.

- $(vii) \Rightarrow (v)$. Let's see the following implications,
  - $(vii) \Rightarrow (viii)$. Let $\psi \in KK^G(A, B)$ be any Kasparov triple between two G-C^*-algebras that is invertible in $KK^H(A, B)$ for all $H \in \mathcal{F}$. In this case, A and B have the same H-equivariant K-theory, for all $H \in \mathcal{F}$. That is, $K^H_*(A) \cong K^H_*(B)$ or, thanks to Green-Julg theorem, we have also that $K_*(H \underset{r}{\ltimes} A) \cong K_*(H \underset{r}{\ltimes} B)$, for all $H \in \mathcal{F}$. The hypothesis of (vii) implies that $K_*(G \underset{r}{\ltimes} A) \cong K_*(G \underset{r}{\ltimes} B)$ and we get $(viii)$.
  - $(viii) \Rightarrow (vi)$. Let $f : A \longrightarrow B$ be a $*$-homomorphism between G-C^*-algebras that is a H-homotopy equivalence for all $H \in \mathcal{F}$. Then the corresponding Kasparov triple $[f] \in KK^G(A, B)$ induces an invertible element in $KK^H(A, B)$, for all $H \in \mathcal{F}$ (since KK-theory is a homotopy invariant). The hypothesis of $(viii)$ implies that $K_*(G \underset{r}{\ltimes} A) \cong K_*(G \underset{r}{\ltimes} B)$ and we get (vi).
  - $(vi) \Rightarrow (iv)$. Let A be any H-contractible G-C^*-algebra for all $H \in \mathcal{F}$. This means that the zero homomorphism $0 \longrightarrow A$ is a H-homotopy equivalence, for all $H \in \mathcal{F}$. The hypothesis of (vi) implies that $(0) \cong K_*(G \underset{r}{\ltimes} 0) \cong K_*(G \underset{r}{\ltimes} A)$ and we get (iv).

  In other words, we have that $(vii) \Rightarrow (viii) \Rightarrow (vi) \Rightarrow (iv) \Leftrightarrow (v)$.

- $(v) \Rightarrow (vii)$. Let $\psi \in KK^G(A, B)$ be any Kasparov triple between two G-C^*-algebras that induces an isomorphism $K_*(H \underset{r}{\ltimes} A) \cong K_*(H \underset{r}{\ltimes} B)$ for all $H \in \mathcal{F}$.

  Given the homomorphism $\psi : A \longrightarrow B$ in the Kasparov category $\mathscr{K}\mathscr{K}^G$, apply the axiom (ii) of a triangulated category and consider the cone triangle associated to ψ,

$$\Sigma(B) \longrightarrow C_\psi \longrightarrow A \xrightarrow{\psi} B$$

  Now, given a compact subgroup $H \in \mathcal{F}$, consider the homological functor of homomorphisms with respect to $\mathbb{C}$ in $\mathscr{K}\mathscr{K}^G$ given by $K_*(H \underset{r}{\ltimes} \cdot)$. Consider the following long exact sequence

$$\begin{aligned} \ldots \to K_*(H \underset{r}{\ltimes} \Sigma(B)) \to K_*(H \underset{r}{\ltimes} C_\psi) \to K_*(H \underset{r}{\ltimes} A) \to K_*(H \underset{r}{\ltimes} B) \\ \to K_*(H \underset{r}{\ltimes} \Sigma^{-1}(C_\psi)) \to \ldots \end{aligned}$$

  Since $K_*(H \underset{r}{\ltimes} A) \cong K_*(H \underset{r}{\ltimes} B)$ for all $H \in \mathcal{F}$, we deduce that $K_*(H \underset{r}{\ltimes} C_\psi) \cong (0) \cong K_*(H \underset{r}{\ltimes} \Sigma^{-1}(C_\psi))$, for all $H \in \mathcal{F}$. The hypothesis of (v) implies that $K_*(G \underset{r}{\ltimes} C_\psi) \cong (0) \cong K_*(G \underset{r}{\ltimes} \Sigma^{-1}(C_\psi))$ as well.

Now, consider the homological functor of homomorphisms with respect to $\mathbb{C}$ in $\mathscr{K}\mathscr{K}^G$ given by $K_*(G \underset{r}{\ltimes} \cdot)$. Consider the following long exact sequence

$$\ldots \to K_*(G \underset{r}{\ltimes} \Sigma(B)) \to K_*(G \underset{r}{\ltimes} C_\psi) \to K_*(G \underset{r}{\ltimes} A) \to K_*(G \underset{r}{\ltimes} B) \\ \to K_*(G \underset{r}{\ltimes} \Sigma^{-1}(C_\psi)) \to \ldots$$

Since $K_*(G \underset{r}{\ltimes} C_\psi) \cong (0) \cong K_*(G \underset{r}{\ltimes} \Sigma^{-1}(C_\psi))$, we deduce that $K_*(G \underset{r}{\ltimes} A) \cong K_*(G \underset{r}{\ltimes} B)$ and we get (vii).

■

1.2.4 Meyer-Nest's homological algebra *revisited*

In this section we want to give a closer inspection of the Meyer-Nest's homological algebra introduced in Section 1.2.2. More precisely, the goal here is to define *derived functors* in a similar way as we do for abelian categories. Hence, spectral sequences can be established and for some concrete situations we obtain useful results for K-group computations as soon as we restrict ourselves to the Kasparov category, which will be explained with more details in Chapter 5. For more details about spectral sequences and related subjects we refer to [134] and [131].

1.2.4.1 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J}$ a homological ideal in $\mathcal{T}$. A distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ in $\mathcal{T}$ is called $\mathcal{J}$-exact if $w \in \mathcal{J}(Z, \Sigma(X))$.

1.2.4.2 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J}$ a homological ideal in $\mathcal{T}$. An object $X \in Obj(\mathcal{T})$ is called $\mathcal{J}$-contractible if $id_X \in \mathcal{J}(X, X)$.

1.2.4.3 Proposition. *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J} := \ker_{Hom}(F)$ a homological ideal in $\mathcal{T}$ with $F : \mathcal{T} \longrightarrow \mathscr{A}$.*

i) A distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ in $\mathcal{T}$ is $\mathcal{J}$-exact if and only if $0 \longrightarrow F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \longrightarrow 0$ is a short exact sequence in $\mathscr{A}$.

ii) An object $X \in Obj(\mathcal{T})$ is $\mathcal{J}$-contractible if and only if $0 \longrightarrow X$ is a $\mathcal{J}$-isomorphism. In other words, $\mathcal{J}$-contractible objects are exactly the objects in $\ker_{Obj}(F)$.

iii) A homomorphism $f : X \longrightarrow Y$ in $\mathcal{T}$ is a $\mathcal{J}$-isomorphism if and only if C_f is a $\mathcal{J}$-contractible object.

Proof. i) Given the distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$, consider the corresponding long exact sequence in $\mathscr{A}$,

$$\ldots \to F(\Sigma^{-1}(Z)) \xrightarrow{F(\Sigma^{-1}(w))} F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{F(w)} F(\Sigma(X)) \to \ldots$$

Hence, the distinguished triangle above is $\mathcal{J}$-exact if and only if $w \in \mathcal{J}(Z, \Sigma(X))$, which means that $F(w) = 0$ and also that $F(\Sigma(w)) = 0$ and these conditions say that the above long exact sequence yields the following short exact sequence, $0 \longrightarrow F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \longrightarrow 0$.

ii) By virtue of Proposition 1.2.2.7, $0 \longrightarrow X$ is a $\mathcal{J}$-isomorphism if and only if $F(X) \cong 0$, that is, $X \in \ker_{Obj}(F)$.

Suppose that X is a $\mathcal{J}$-contractible object, then $id_X \in \mathcal{J}(X, X)$, that is, $F(id_X) = id_{F(X)} = 0$, which implies that $F(X) \cong 0$. Conversely, suppose that $X \in \ker_{Obj}(F)$, then $F(X) \cong 0$ and so $id_{F(X)} = F(id_X) = 0$, that is, $id_X \in \mathcal{J}(X, X)$

iii) By virtue of Proposition 1.2.2.7, $f : X \longrightarrow Y$ in $\mathcal{T}$ is a $\mathcal{J}$-isomorphism if and only if $F(f) : F(X) \longrightarrow F(Y)$ is an isomorphism, which is equivalent to say that $F(C_f) \cong 0$ using the short exact sequence $F(X) \xrightarrow{F(f)} F(Y) \longrightarrow F(C_f)$ given by the image under F of the cone triangle associated to f. The preceding property yields the conclusion.

■

1.2.4.4 Lemma. *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J} := \ker_{Hom}(F)$ a homological ideal in $\mathcal{T}$ with $F : \mathcal{T} \longrightarrow \mathscr{A}$.*

A 3-chain complex

$$\ldots 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \ldots$$

is $\mathcal{J}$-exact if and only if there exists a $\mathcal{J}$-exact distinguished triangle $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma(X')$ and a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "\alpha"] & Y' \arrow[r, "v'"] \arrow[d, "\beta"] & Z' \arrow[d, "\gamma"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

where the homomorphisms α, β and γ are $\mathcal{J}$-isomorphisms. Actually, α and β are identity homomorphisms.

Proof. Assume that there exists a $\mathcal{J}$-exact distinguished triangle $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma(X')$ and a commutative diagram as in the statement. Since α, β and γ are $\mathcal{J}$-isomorphisms, then $F(\alpha), F(\beta)$ and $F(\gamma)$ are isomorphisms (see Proposition 1.2.2.7). Since F is a stable homological functor, we have a short exact sequence

$$0 \longrightarrow F(X') \xrightarrow{F(u')} F(Y') \xrightarrow{F(v')} F(Z') \longrightarrow 0$$

Hence the commutativity of the diagram of the statement yields a short exact sequence

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0,$$

which is equivalent to the $\mathcal{J}$-exactness of the chain complex

$$\ldots 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \ldots$$

by virtue of Proposition 1.2.2.9.

Conversely, assume that $\ldots 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \ldots$ is a $\mathcal{J}$-exact complex in $\mathcal{T}$. By Proposition 1.2.2.9 this is equivalent to say that

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$$

is a short exact sequence. In particular, $F(f)$ is a monomorphism, which is equivalent to say that the homomorphism $f : X \longrightarrow Y$ is a $\mathcal{J}$-monomorphism.

Next, consider the cone triangle associated to f, say

$$X \xrightarrow{f} Y \xrightarrow{v} C_f \xrightarrow{w} \Sigma(X)$$

Since f is $\mathcal{J}$-monomorphism, we have that $w \in \mathcal{J}(C_f, \Sigma(X))$, which means by definition that the cone triangle associated to f is $\mathcal{J}$-exact.

Take $\alpha := id_X$ and $\beta := id_Y$. Consider the homological functor $Hom_{\mathcal{T}}(\ \cdot\ , Z)$ given by the right composition, which yields the following short exact sequence

$$Hom_{\mathcal{T}}(X, Z) \xleftarrow{f^*} Hom_{\mathcal{T}}(Y, Z) \xleftarrow{v^*} Hom_{\mathcal{T}}(C_f, Z)$$

Remark that $g \circ f = 0$ because of the chain complex condition. In other words, we have that $f^*(g) = 0$, that is, $g \in \ker(f^*) = Im(v^*)$. Hence, there exists a homomorphism $\gamma \in Hom_{\mathcal{T}}(C_f, Z)$ such that $v^*(\gamma) = \gamma \circ v = g$.

In this situation, we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "v"] \arrow[d, "\beta"] & C_f \arrow[d, "\gamma"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Observe that its image under F is also a commutative diagram with $F(\alpha)$ and $F(\beta)$ isomorphisms, so that $F(\gamma)$ is as well an isomorphism (by virtue of Five's lemma), that is, γ is a $\mathcal{J}$-isomorphism. ■

The following theorem is the triangulated counterpart of the existence of projective resolutions as it is known for abelian categories. Its proof is very close to the classical one and we refer to Proposition 3.26 of [134] for more details.

1.2.4.5 Theorem. *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J}$ a homological ideal in $\mathcal{T}$.*

i) If $\mathcal{T}$ has enough $\mathcal{J}$-projective objects, then every object of $\mathcal{T}$ has a unique up to homotopy $\mathcal{J}$-projective resolution.

ii) Given two objects $X, X' \in Obj(\mathcal{T})$, let $P_\bullet \longrightarrow X$ and $P'_\bullet \longrightarrow X'$ be two $\mathcal{J}$-projective resolutions. Every homomorphism $f : X \longrightarrow X'$ induces a unique up to homotopy homomorphism of chain complexes $\widetilde{f} : P_\bullet \longrightarrow P'_\bullet$.

As a consequence we have a functor

$$\mathscr{P} : \mathcal{T} \longrightarrow \mathscr{H}(\mathcal{T}),$$

where $\mathscr{H}(\mathcal{T})$ denotes the homotopy category of $\mathcal{T}$.

iii) Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ be a $\mathcal{J}$-exact distinguished triangle in $\mathcal{T}$. There exists a canonical homomorphism $\eta : \mathscr{P}(Z) \longrightarrow \Sigma(\mathscr{P}(X))$ such that $\mathscr{P}(X) \xrightarrow{\mathscr{P}(u)} \mathscr{P}(Y) \xrightarrow{\mathscr{P}(v)} \mathscr{P}(Z) \xrightarrow{\mathscr{P}(w)} \mathscr{P}(X)[1]$ is a distinguished triangle on $\mathscr{H}(\mathcal{T})$.

Given a triangulated category $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ and a homological ideal $\mathcal{J}$ in $\mathcal{T}$, assume that $\mathcal{T}$ has enough $\mathcal{J}$-projective objects. Hence, given any object $X \in Obj(\mathcal{T})$ consider the corresponding $\mathcal{J}$-projective resolution for X, say $(P_\bullet, \delta_\bullet)$ with $\delta_0 : P_0 \longrightarrow X$ a $\mathcal{J}$-epimorphism.

Next, let $\mathscr{A}$ be an abelian category and consider a (covariant) additive functor $F : \mathcal{T} \longrightarrow \mathscr{A}$. The chain complex $(P_\bullet, \delta_\bullet)$ in $\mathcal{T}$ is then transformed into a chain complexe $(F(P_\bullet), F(\delta_\bullet))$ in $\mathscr{A}$,

$$\ldots \xrightarrow{F(\delta_3)} F(P_2) \xrightarrow{F(\delta_2)} F(P_1) \xrightarrow{F(\delta_1)} F(P_0) \xrightarrow{F(\delta_0)} F(X) \tag{1.2.2}$$

Consider now the following chain complex,

$$\ldots \xrightarrow{F(\delta_3)} F(P_2) \xrightarrow{F(\delta_2)} F(P_1) \xrightarrow{F(\delta_1)} F(P_0) \longrightarrow 0 \tag{1.2.3}$$

In this situation, we can consider thus the corresponding homology functor and we put

$$\mathbb{L}_n F(X) := H_n\big((1.2.3)\big) = \ker(F(\delta_n))/Im(F(\delta_{n+1})),$$

for every $n \in \mathbb{Z}^+$, where

$$\mathbb{L}_0 F(X) = \ker(F(P_0) \longrightarrow 0)/Im(F(\delta_1)) = F(P_0)/Im(F(\delta_1)) = coker(F(\delta_1))$$

Since the association to each object the corresponding $\mathcal{J}$-projective resolution is functorial (thanks to Theorem 1.2.4.5), the above construction yields a functor

$$\mathbb{L}_n F : \mathcal{T} \longrightarrow \mathscr{A},$$

for every $n \in \mathbb{Z}^+$. It is called *nth left derived functor.*

1.2.4.6 Remark. Under the same assumptions as above, suppose in addition that F is a homological functor. If

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$$

is $\mathcal{J}$-exact triangle in $\mathcal{T}$, then Theorem 1.2.4.5 gives a distinguished triangle

$$\mathscr{P}(X) \xrightarrow{\mathscr{P}(u)} \mathscr{P}(Y) \xrightarrow{\mathscr{P}(v)} \mathscr{P}(Z) \xrightarrow{\mathscr{P}(w)} \Sigma(\mathscr{P}(X))$$

in the homotopy category of $\mathcal{T}$ and the application of the functor F yields a short exact sequence of chain complexes. Now, it is well known that the nth homology functor is homological so that we obtain a long exact sequence in homology, that is, a long exact sequence between the derived functors

$$\begin{aligned}\ldots \longrightarrow \mathbb{L}_n F(X) \longrightarrow \mathbb{L}_n F(Y) \longrightarrow \mathbb{L}_n F(Z) \longrightarrow \mathbb{L}_{n-1} F(X)\\ \longrightarrow \ldots \longrightarrow \mathbb{L}_1 F(Z) \longrightarrow \mathbb{L}_0 F(X) \longrightarrow \mathbb{L}_0 F(Y) \longrightarrow \mathbb{L}_0 F(Z) \longrightarrow 0\end{aligned}$$

1.2.4.7 Proposition. *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J}$ a homological ideal in $\mathcal{T}$ such that $\mathcal{T}$ has enough $\mathcal{J}$-projective objects. If $F : \mathcal{T} \longrightarrow \mathscr{A}$ is a stable homological functor, then the following assertions are equivalent*

i) F is $\mathcal{J}$-exact.

ii) $\mathbb{L}_0 F(X) \cong F(X)$ and $\mathbb{L}_n F(X) = 0$, for all $n > 0$ and all $X \in Obj(\mathcal{T})$.

iii) $\mathbb{L}_0F(X) \cong F(X)$, *for all* $X \in Obj(\mathcal{T})$.

Proof. By virtue of Proposition 1.2.2.11, F is $\mathcal{J}$-exact if and only if it transforms $\mathcal{J}$-exact chain complexes into long exact sequences, which can be applied to $\mathcal{J}$-projective resolutions. Hence the complex (1.2.2) is exact with $F(\delta_0)$ an epimorphism so that

$$\mathbb{L}_nF(X) = 0 \text{ for all } n > 0 \text{ and}$$

$$\mathbb{L}_0F(X) = F(P_0)/Im(F(\delta_1)) = F(P_0)/\ker(F(\delta_0)) = F(X)$$

This shows $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Conversely, suppose that $\mathbb{L}_0F(X) \cong F(X)$, for all $X \in Obj(\mathcal{T})$. Observe that if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ is a $\mathcal{J}$-exact triangle in $\mathcal{T}$, then v is a $\mathcal{J}$-epimorphism (use the characterization of Proposition 1.2.4.3). By the previous remark we can construct the following long exact sequence

$$\ldots \longrightarrow \mathbb{L}_1F(Z) \longrightarrow \mathbb{L}_0F(X) \longrightarrow \mathbb{L}_0F(Y) \longrightarrow \mathbb{L}_0F(Z) \longrightarrow 0,$$

which is actually

$$\ldots \longrightarrow \mathbb{L}_1F(Z) \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow 0,$$

thanks to our assumption. Hence, $F(v)$ is an epimorphism. Since the objects Y and Z are arbitrary we deduce that the functor F transforms $\mathcal{J}$-epimorphisms into epimorphisms under the assumption of (iii) and this characterizes the $\mathcal{J}$-exact (see Proposition 1.2.2.11) functors obtaining thus (i). ■

1.2.4.8 Remark. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J} := \ker_{Hom}(F')$ an additive homological ideal in $\mathcal{T}$ with $F' : \mathcal{T} \longrightarrow \mathscr{A}'$ such that $\mathcal{T}$ has enough $\mathcal{J}$-projective objects. In this situation, Theorem 1.2.2.15 guarantees that $(\langle \mathfrak{p}_{\mathcal{J}} \rangle, \ker_{Obj}(F'))$ is a complementary pair of localizing subcategories in $\mathcal{T}$. Notice that objects in $\ker_{Obj}(F')$ are exactly the $\mathcal{J}$-contractible objects (see Proposition 1.2.4.3).

Hence, given a complementary pair of localizing subcategories we can define the localization of any given functor as in Definition 1.2.1.27. Precisely, if $F : \mathcal{T} \longrightarrow \mathscr{A}$ is a stable homological functor, then we define its localization with respect to $(\langle \mathfrak{p}_{\mathcal{J}} \rangle, \ker_{Obj}(F'))$ as the functor

$$\mathbb{L}F := F \circ L : \mathcal{T} \longrightarrow \mathscr{A},$$

where $L : \mathcal{T} \longrightarrow \langle \mathfrak{p}_{\mathcal{J}} \rangle$ is the canonical triangulated functor associated to the complementary pair.

Recall as well that such localization enjoys the following universal property (see Theorem 1.2.1.29): there exists a natural transformation $\eta : \mathbb{L}F \longrightarrow F$ such that $\mathbb{L}F_{|\ker_{Obj(F')}} = 0$.

We claim that the 0th left derived functor $\mathbb{L}_0F$ coincides with the localization functor $\mathbb{L}F$ with respect to the projective complementary pair of localizing subcategories.

To see this, it is enough to check the universal property for $\mathbb{L}_0F$. Indeed,

- there exists a natural transformation $\eta' : \mathbb{L}_0F \longrightarrow F$ given by the augmentation map. More precisely, given any object $X \in Obj(\mathcal{T})$, consider the corresponding $\mathcal{J}$-projective resolution, say $(P_\bullet, \delta_\bullet)$ with augmentation map $\delta_0 : P_0 \longrightarrow X$, which is a $\mathcal{J}$-epimorphism. Since (1.2.2) is a chain complex, we have that $Im(F(\delta_1)) \subset \ker(F(\delta_0))$ so that we can write

  $$\mathbb{L}_0F(X) = F(P_0)/Im(F(\delta_1)) \longrightarrow F(P_0)/\ker(F(\delta_0)) \longrightarrow F(X)$$

  And this association is clearly natural.

- the 0th left derived functor is $\mathcal{J}$-exact, that is, $\mathcal{J} \subset \ker_{Hom}(\mathbb{L}_0 F)$. More precisely, let $f : X \longrightarrow Y$ be any homomorphism in $\mathcal{T}$ and consider $\mathcal{J}$-projective resolutions for each object, say $(P_\bullet, \delta_\bullet)$ for X and $(Q_\bullet, \rho_\bullet)$ for Y. If $f \in \mathcal{J}(X, Y)$, then the zero homomorphism $0 : P_0 \longrightarrow Q_0$ defines a homomorphism of chain complexes lifting f, that is, we have a commutative diagram

$$\begin{tikzcd} \dots \arrow[r] & P_0 \arrow[r, "\delta_0"] \arrow[d, dashed, "0"'] & X \arrow[d, "f"] \\ \dots \arrow[r] & Q_0 \arrow[r, "\rho_0"'] & Y \end{tikzcd}$$

Indeed, since $f \in \mathcal{J}(X, Y)$ we have that $f \circ \delta_0 \in \mathcal{J}(P_0, Y)$ by definition of ideal and since P_0 is a $\mathcal{J}$-projective object, then it must be $f \circ \delta_0 = 0$ (recall Remark 1.2.2.13). Consequently, $\mathbb{L}_0 F(X) \xrightarrow{\mathbb{L}_0 F(f)} \mathbb{L}_0 F(Y)$ is the zero homomorphism. In particular, $\mathbb{L}_0 F(id_X) = 0$ for every $\mathcal{J}$-contractible object X. In other words, $\mathbb{L}_0 F_{|\ker_{Obj}(F')} = 0$.

1.2.4.9 Theorem. *Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category and $\mathcal{J}$ a homological ideal in $\mathcal{T}$. Let $F : \mathcal{T} \longrightarrow \mathscr{A}$ be a stable homological functor.*

If $X \in Obj(\mathcal{T})$ is an object such that

i) X has a $\mathcal{J}$-projective resolution of length 1,

ii) $Hom_{\mathcal{T}}(X, Y) = (0)$, for all $\mathcal{J}$-contractible object Y

then there exists a natural short exact sequence

$$0 \longrightarrow \mathbb{L}_0 F(X) \longrightarrow F(X) \longrightarrow \mathbb{L}_1 F(\Sigma(X)) \longrightarrow 0$$

Proof. Let $0 \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} X \longrightarrow 0$ be a $\mathcal{J}$-projective resolution for X of length 1, which can be viewed as a $\mathcal{J}$-exact 3-chain complex. By virtue of Lemma 1.2.4.4, there exists a $\mathcal{J}$-exact distinguished triangle $P_1 \xrightarrow{\delta_1} P_0 \longrightarrow \widetilde{X} \longrightarrow \Sigma(P_1)$ and a commutative diagram

$$\begin{tikzcd} P_1 \arrow[r, "\delta_1"] \arrow[d, equal, "\alpha"] & P_0 \arrow[r] \arrow[d, equal, "\beta"] & \widetilde{X} \arrow[d, "\gamma"] \\ P_1 \arrow[r, "\delta_1"'] & P_0 \arrow[r, "\delta_0"'] & X \end{tikzcd}$$

where γ is a $\mathcal{J}$-isomorphism. Consider the cone triangle associated to $\gamma : \widetilde{X} \longrightarrow X$, say $\widetilde{X} \xrightarrow{\gamma} X \xrightarrow{v} C_\gamma \longrightarrow \Sigma(\widetilde{X})$ and apply the rotation axiom so that $\Sigma^{-1}(C_\gamma) \longrightarrow \widetilde{X} \xrightarrow{\gamma} X \xrightarrow{v} C_\gamma$ is again a distinguished triangle. Since γ is a $\mathcal{J}$-isomorphism, then C_γ is a $\mathcal{J}$-contractible object (see Proposition 1.2.4.3). Hence, thanks to the assumptions on the object X, we have that $Hom_{\mathcal{T}}(X, C_\gamma) = (0)$ which implies that $v = 0$. In other words, the above triangle splits and we have $X \cong \widetilde{X} \oplus C_\gamma$ (see Proposition 1.2.1.20).

Next, remark that this decomposition yields that $Hom_{\mathcal{T}}(C_\gamma, C_\gamma) \subset Hom_{\mathcal{T}}(X, C_\gamma) = (0)$. In particular, $id_{C_\gamma} = 0$ so $C_\gamma \cong 0$. In other words, γ is an isomorphism and we have $X \cong \widetilde{X}$.

In this way, we obtain a distinguished triangle given by

$$P_1 \xrightarrow{\delta_1} P_0 \longrightarrow X \longrightarrow \Sigma(P_1)$$

Since F is a stable homological functor, we obtain as well a long exact sequence

$$\ldots \to F(\Sigma^{-1}(X)) \to F(P_1) \overset{F(\delta_1)}{\to} F(P_0) \to F(X) \to F(\Sigma(P_1)) \overset{F(\Sigma(\delta_1))}{\to} F(\Sigma(P_0)) \to \ldots$$

which yields the following natural short exact sequence

$$coker(F(\delta_1)) \hookrightarrow F(X) \twoheadrightarrow \ker(F(\Sigma(\delta_1)))$$

And by definition of the (left) derived functors we have,

$$\mathbb{L}_0 F(X) \hookrightarrow F(X) \twoheadrightarrow \mathbb{L}_1 F(\Sigma(X)),$$

where we should remark that our resolution is of length 1, so that

$$\mathbb{L}_1 F(\Sigma(X)) = \ker(F(\Sigma(\delta_1)))/Im(0 \longrightarrow \Sigma(P_1)) \cong \ker(F(\Sigma(\delta_1)))$$

■

1.2.4.10 Remarks. 1. In the proof of the preceding theorem, the conclusion is achieved thanks to the fact that γ is actually an *isomorphism* and not only a $\mathcal{J}$-equivalence. In this sense, the above theorem is still valid for an object $X \in Obj(\mathcal{T})$ whenever

   i) X has a $\mathcal{J}$-projective resolution of length 1,

   ii) every $\mathcal{J}$-isomorphism for X is an isomorphism.

2. It is obvious that the statement of the preceding theorem is more general in the sense that we obtain a natural short exact sequence

$$\mathbb{L}_0 F(\Sigma^i(X)) \hookrightarrow F(\Sigma^i(X)) \twoheadrightarrow \mathbb{L}_1 F(\Sigma^{i+1}(X)),$$

   for all $i \in \mathbb{Z}$, which is straightforward by the above proof.

1.3 Compact Quantum Groups

Let us introduce the theory of compact quantum groups in the sense of S. L. Woronowicz by giving two fundamental examples that should be regarded as the reference ones.

On the one hand, let G be a compact group and consider its (unital) C^*-algebra of continuous functions $C(G)$. We have a canonical $*$-isomorphism $C(G) \otimes C(G) \cong C(G \times G)$ and we can define thus a unital $*$-homomorphism $\Delta : C(G) \longrightarrow C(G \times G)$ by

$$\Delta(f)(x, y) := f(xy),$$

for all $f \in C(G)$ and all $x, y \in G$. This map satisfies two important properties,

i) $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$, thanks to the associativity of the internal law of G.

ii) $(C(G) \otimes 1)\Delta(C(G))$ and $(1 \otimes C(G))\Delta(G)$ are linearly dense in $C(G \times G)$. Indeed, it is enough to remark that the space $(C(G) \otimes 1)\Delta(C(G))$ is spanned by functions of the form $(x, y) \mapsto f_1(x)f_2(xy)$. These functions separate points in G, so that the Stone-Weierstrass theorem yields the conclusion.

In this way, the data $\mathbb{G} := (C(G), \Delta)$ with the above properties is a *compact quantum group* in the sense of Woronowicz.

Besides, assume that A is a unital *commutative* C^*-algebra equipped with a unital $*$-homomorphism $\Delta : A \longrightarrow A \otimes A$ satisfying the analogue of the two properties (i) and (ii) above. In this case, we can show by applying the Gelfand duality that there exists a compact group G such that the corresponding compact quantum group $\mathbb{G}$ identifies to the data (A, Δ). In this sense, we obtain a bijective correspondence between compact classical groups and commutative compact quantum groups.

$$\left\{\text{Compact Classical Groups}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{Commutative}\\ \text{Compact Quantum Groups}\end{array}\right\}$$

On the other hand, let Γ be a discrete group and consider its reduced C^*-algebra $C^*_r(\Gamma)$, which is by definition the closed linear span of the operators λ_γ with $\gamma \in \Gamma$, where λ is the left regular representation of Γ on $l^2(\Gamma)$. In this way, we can define a unital $*$-homomorphism $\Delta_r : C^*_r(\Gamma) \longrightarrow C^*_r(\Gamma) \otimes C^*_r(\Gamma)$ such that

$$\Delta_r(\lambda_\gamma) := \lambda_\gamma \otimes \lambda_\gamma,$$

for all $\gamma \in \Gamma$. This map satisfies obviously the analogue of the two properties (i) and (ii) above.

In this way, the data $\mathbb{\Gamma}_r := (C^*_r(\Gamma), \Delta_r)$ is a *compact quantum group* in the sense of Woronowicz.

An important observation is that the same construction as above can be made by taking the *universal* C^*-algebra of Γ and we can form the compact quantum group $\mathbb{\Gamma}_m := (C^*_m(\Gamma), \Delta_m)$. By construction, there exists a canonical surjective $*$-homomorphism $\tau : C^*_m(\Gamma) \twoheadrightarrow C^*_r(\Gamma)$.

Notice by the way that the map Δ is invariant under the flip map $\Sigma : C^*_r(\Gamma) \otimes C^*_r(\Gamma) \longrightarrow C^*_r(\Gamma) \otimes C^*_r(\Gamma)$, that is, $\Delta = \Sigma \circ \Delta =: \Delta^{op}$. We say then that Δ *is co-commutative*. Besides, assume that A is a unital C^*-algebra equipped with a unital *co-commutative* $*$-homomorphism $\Delta : A \longrightarrow A \otimes A$ satisfying the analogue of the two properties (i) and (ii) above. In this case, we can show that there exists a discrete group Γ such that the data (A, Δ) sits between the $\mathbb{\Gamma}_m$ and $\mathbb{\Gamma}_r$ as compact quantum groups. Specifically, there exists surjective homomorphisms of compact quantum groups $C^*_m(\Gamma) \overset{\tau_m}{\twoheadrightarrow} A \overset{\tau_r}{\twoheadrightarrow} C^*_r(\Gamma)$ such that $\tau = \tau_r \circ \tau_m$.

Remark that if Γ was *amenable*, there wouldn't be any difference between its reduced and maximal C^*-algebras. This is the situation for the *compact* case. In this sense, we obtain a bijective correspondence between discrete amenable classical groups and co-commutative compact quantum groups.

$$\left\{\text{Discrete Amenable Classical Groups}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{Co-commutative}\\ \text{Compact Quantum Groups}\end{array}\right\}$$

One of the main achievement of the Woronowicz's theory with respect to earlier attempts is a large repertory of examples that are neither commutative nor co-commutative. For this, Chapter

2 give a complete overview of the main "genuine" examples of compact quantum groups. In the present section we give, in contrast, the theoretical context for compact quantum groups in the sense of S. L. Woronowicz. Thus we develop, as in detail as possible, the representation and the duality theory for compact quantum groups.

Finally, let us recall that in the classical case the Pontryagin dual of a commutative compact group is a commutative discrete group and *viceversa*. Sometimes the notations and nomenclature in the literature are misleading. For this reason we wish to clarify this point with the following diagrams.

<u>Classical Abelian Groups</u>

$$\begin{tikzcd}[column sep=small]
 & \Gamma,\ \text{D.G.} \arrow[dl, squiggly] \arrow[dr, squiggly] & \\
\mathbb{\Gamma} := (C_r^*(\Gamma), \Delta_r) & & \widehat{\mathbb{\Gamma}} := (c_0(\Gamma), \Delta) \\
\text{C.Q.G.},\ \widehat{\Gamma} & & \text{D.Q.G.},\ \Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & G,\ \text{C.G.} \arrow[dl, squiggly] \arrow[dr, squiggly] & \\
\mathbb{G} := (C(G), \Delta) & & \widehat{\mathbb{G}} := (C_r^*(G), \Delta_r) \\
\text{C.Q.G.},\ G & & \text{D.Q.G.},\ \widehat{G}
\end{tikzcd}$$

Assume that G is an abelian compact group. In this case, $\widehat{G} =: \Gamma$ is an abelian discrete group by virtue of the Pontryagin duality (remark that, with these notations, $G \cong \widehat{\Gamma}$). Besides, we have the following isomorphisms of C^*-algebras

$$c_0(\Gamma) \cong C_r^*(G) \text{ and } C(G) \cong C_r^*(\Gamma),$$

so that the above notations are consistent in the following sense

$$\mathbb{\Gamma} \cong \mathbb{G} \longleftrightarrow \widehat{\Gamma} \cong G \text{ and } \widehat{\mathbb{\Gamma}} \cong \widehat{\mathbb{G}} \longleftrightarrow \Gamma \cong \widehat{G}$$

If we want to have a more precise picture in our mind, we'll think about the well known Pontryagin duality between $\mathbb{Z}$ and S^1.

Therefore, we extend these notations for any (locally) compact group and also for any (locally) compact quantum group.

<u>Classical Groups</u>

$$\begin{tikzcd}[column sep=small]
 & G,\ \text{L.C.G.} \arrow[dl, squiggly] \arrow[dr, squiggly] & \\
\mathbb{G} := (C_0(G), \Delta) & & \widehat{\mathbb{G}} := (C_r^*(G), \Delta_r) \\
\text{L.C.Q.G.},\ G & & \text{L.C.Q.G.},\ \widehat{G}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & G,\ \text{C.G.} \arrow[dl, squiggly] \arrow[dr, squiggly] & \\
\mathbb{G} := (C(G), \Delta) & & \widehat{\mathbb{G}} := (C_r^*(G), \Delta_r) \\
\text{C.Q.G.},\ G & & \text{D.Q.G.},\ \widehat{G}
\end{tikzcd}$$

<u>Quantum Groups</u>

$$\begin{tikzcd}[column sep=small]
 & \mathbb{G},\ \text{L.C.Q.G.} \arrow[dl, squiggly] \arrow[dr, squiggly] & \\
\mathbb{G} := (C_0(\mathbb{G}), \Delta) & & \widehat{\mathbb{G}} := (C_r^*(\mathbb{G}), \widehat{\Delta}) \\
\text{L.C.Q.G.} & & \text{L.C.Q.G.}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathbb{G},\ \text{C.Q.G.} \arrow[dl, squiggly] \arrow[dr, squiggly] & \\
\mathbb{G} := (C(\mathbb{G}), \Delta) & & \widehat{\mathbb{G}} := (c_0(\widehat{\mathbb{G}}), \widehat{\Delta}) \\
\text{C.Q.G.} & & \text{D.Q.G.}
\end{tikzcd}$$

1.3.1 Woronowicz's theory

1.3.1.1 Definition. A compact quantum group $\mathbb{G}$ is the data $(C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital C^*-algebra and $\Delta : C(\mathbb{G}) \longrightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$ is a unital $*$-homomorphism such that

i) Δ is co-associative meaning that the diagram

$$\begin{tikzcd} C(\mathbb{G}) \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C(\mathbb{G}) \otimes C(\mathbb{G}) \arrow[d, "id\otimes\Delta"] \\ C(\mathbb{G}) \otimes C(\mathbb{G}) \arrow[r, "\Delta\otimes id"] & C(\mathbb{G}) \otimes C(\mathbb{G}) \otimes C(\mathbb{G}) \end{tikzcd}$$

is commutative.

ii) Δ satisfies the cancellation property meaning that

$$[(C(\mathbb{G}) \otimes 1)\Delta(C(\mathbb{G}))] = C(\mathbb{G}) \otimes C(\mathbb{G}) = [(1 \otimes C(\mathbb{G}))\Delta(C(\mathbb{G}))]$$

1.3.1.2 Note. The object $\mathbb{G}$ defined above has different names in the literature. Namely, such $\mathbb{G}$ may be called *compact topological quantum group*, *C^*-Woronowicz algebra*, *C^*-algebraic compact quantum group* or *unital bisimplifiable C^*-bialgebra*.

Other common language that is used in this dissertation is the following: the data $(C(\mathbb{G}), \Delta)$ of the preceding definition is also called a *unital Hopf C^*-algebra*. A Hopf C^*-algebra is the data $\mathbb{S} := (S, \Delta)$, where S is a C^*-algebra and $\Delta : S \longrightarrow \widetilde{M}(S \otimes S)$ is a $*$-homomorphism satisfying the analogue properties of the above definition. It is important to remark that in the context of general Hopf C^*-algebras we don't use the usual multiplier algebra $M(\cdot)$, but the refinement $\widetilde{M}(\cdot)$ (see Definition A.4.4 and [6], [206] for more details).

For any Hopf C^*-algebra $\mathbb{S} = (S, \Delta)$ we write $\mathbb{S}^{cop} := (S, \Delta^{cop})$ for the corresponding *co-opposite Hopf C^*-algebra*, where $\Delta^{cop} := \Sigma \circ \Delta$.

1.3.1.3 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ and $\mathbb{G}' = (C(\mathbb{G}'), \Delta')$ be two compact quantum groups. A Hopf $*$-homomorphism from $\mathbb{G}$ to $\mathbb{G}'$, denoted by $f : \mathbb{G} \longrightarrow \mathbb{G}'$, is a unital $*$-homomorphism $f : C(\mathbb{G}') \longrightarrow C(\mathbb{G})$ such that

$$\Delta \circ f = (f \otimes f)\Delta'$$

In particular, if $\mathbb{G} = \mathbb{G}'$, such a f is called Hopf $*$-automorphism.

1.3.1.4 Note. It turns out that this early definition must be suitably modified in order to take into account the analytical subtleties for giving an appropriated notion of *morphism of compact quantum groups*. We will be more precise later on.

The most remarkable result in compact quantum group theory is the next theorem whose proof can be found in Theorem 1.2.1 of [139], Theorem 4.4 of [124], Theorem 5.1.6 of [188] or in the original paper of Woronowicz, Section 2 of [231].

1.3.1.5 Theorem. *For any compact quantum group $\mathbb{G} = (C(\mathbb{G}), \Delta)$, there exists a unique state $h_{\mathbb{G}} \in C(\mathbb{G})^*$ such that*

$$(h_{\mathbb{G}} \otimes id)\Delta(a) = h_{\mathbb{G}}(a)1 = (id \otimes h_{\mathbb{G}})\Delta(a),$$

for all $a \in C(\mathbb{G})$. The state $h_{\mathbb{G}}$ is called the Haar state of $\mathbb{G}$.

1.3.1.6 Remarks. 1. If $\tau : C(\mathbb{G}) \longrightarrow C(\mathbb{G})$ is a Hopf $*$-automorphism and $h_{\mathbb{G}}$ is the Haar state of $\mathbb{G}$, then we have $h_{\mathbb{G}} \circ \tau = h_{\mathbb{G}}$. Indeed, we have just to apply the uniqueness of the Haar state under the condition $(h_{\mathbb{G}} \otimes id)\Delta(a) = h_{\mathbb{G}}(a)1 = (id \otimes h_{\mathbb{G}})\Delta(a)$, for all $a \in C(\mathbb{G})$. So, for all $a \in C(\mathbb{G})$ we have

$$h_{\mathbb{G}}(\tau(a))1 = (h_{\mathbb{G}} \otimes id)\Delta(\tau(a)) = (h_{\mathbb{G}} \otimes id)(\tau \otimes \tau)\Delta(a) = (h_{\mathbb{G}} \circ \tau \otimes \tau)\Delta(a)$$

Since τ is a *unital* $*$-automorphism of $C(\mathbb{G})$, the above equation is equivalent to the following one,

$$h_{\mathbb{G}}(\tau(a))1 = h_{\mathbb{G}}(\tau(a))\tau^{-1}(1) = (id \otimes \tau^{-1})(h_{\mathbb{G}} \circ \tau \otimes \tau)\Delta(a) = (h_{\mathbb{G}} \circ \tau \otimes id)\Delta(a)$$

In the same way we show that $h_{\mathbb{G}}(\tau(a))1 = (id \otimes h_{\mathbb{G}} \circ \tau)\Delta(a)$, for all $a \in C(\mathbb{G})$.

2. Given a compact quantum group $\mathbb{G} = (C(\mathbb{G}), \Delta)$ with Haar state $h_{\mathbb{G}}$, we can perform the associated GNS construction, which is denoted by $(L^2(\mathbb{G}), \lambda, \Omega)$. Here we adopt the standard convention for the inner product on $L^2(\mathbb{G})$. Namely, for all $a, b \in C(\mathbb{G})$ we put $\langle \lambda(a), \lambda(b) \rangle := h_{\mathbb{G}}(a^*b)$.

1.3.1.7 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group. A (unitary) representation of $\mathbb{G}$ on a Hilbert space H is an invertible (resp. unitary) element $w \in M(\mathcal{K}(H) \otimes C(\mathbb{G}))$ sucht that

$$(id \otimes \Delta)(w) = w_{12}w_{13}$$

The collection of all *unitary finite dimensional* representations of $\mathbb{G}$ is denoted by $Rep(\mathbb{G})$.

1.3.1.8 Remarks. 1. Let C be any C^*-algebra. If $\mathbb{S} := (S, \Delta)$ is any Hopf C^*-algebra (not necessarily unital), we can give the notion of *(resp. unitary) representation of* $\mathbb{S}$ *on an Hilbert* C*-module* H (or *(resp. unitary) co-representation of* S *on* H). This means an invertible (resp. unitary) element $V \in \mathcal{L}_C(H \otimes S) = M(\mathcal{K}(H) \otimes S)$ such that $(id \otimes \Delta)(V) = V_{12}V_{13}$.

2. Assume that H is a *finite dimensional* Hilbert space. In this case we have $\mathcal{B}(H) = \mathcal{K}(H)$ and $M(\mathcal{K}(H) \otimes C(\mathbb{G})) \cong \mathcal{B}(H) \otimes C(\mathbb{G})$.

   Therefore, if $w \in \mathcal{B}(H) \otimes C(\mathbb{G})$ is a *finite dimensional* representation of $\mathbb{G}$ on H, consider $\{\xi_1, \ldots, \xi_n\}$ an orthonormal basis for H and $\{m_{i,j}\}_{i,j=1,\ldots,n}$ the corresponding matrix units in $\mathcal{B}(H)$.

   We define the *matrix coefficients* of w as the elements

$$w_{i,j} := (\omega_{\xi_i,\xi_j} \otimes id_{C(\mathbb{G})})(w) \in C(\mathbb{G}),$$

   for all $i, j = 1 \ldots, n$ where ω_{ξ_i,ξ_j} are the coordinate linear forms on $\mathcal{B}(H)$ defined by $\omega_{\xi_i,\xi_j}(T) := \langle \xi_i, T(\xi_j) \rangle$, for all $T \in \mathcal{B}(H)$.

   In this situation, the representation w can be written in coordinates as

$$w = \sum_{i,j=1}^{n} m_{ij} \otimes w_{ij}$$

and the condition of the definition above becomes

$$\Delta(w_{i,j}) = \sum_{k=1}^{n} w_{ik} \otimes w_{kj},$$

for all $i, j = 1, \ldots, n$.

In particular, if w is unitary, the following relations hold

$$\sum_{k=1}^{n} w_{ki}^* \; w_{kj} = \delta_{ij} 1_{C(\mathbb{G})} = \sum_{k=1}^{n} w_{ik} \; w_{jk}^*,$$

for all $i, j = 1, \ldots, n$

Hence, given a (unitary) finite dimensional representation w of $\mathbb{G}$ on H, we have constructed an invertible (resp. unitary) matrix $(w_{ij})_{i,j} \in \mathcal{M}_n(C(\mathbb{G}))$ satisfying the formula above. Such a matrix is called *representation matrix of* $\mathbb{G}$ *on* H.

3. Let $w \in \mathcal{B}(H) \otimes C(\mathbb{G})$ be a finite dimensional representation of $\mathbb{G}$ on H. With the same notations as above, we define the *character of* w *in* $\mathbb{G}$ to be the element

$$\chi_{\mathbb{G}}(w) := \sum_{i=1}^{n} w_{ii} \in C(\mathbb{G})$$

1.3.1.9 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and w, v two (unitary) representations of $\mathbb{G}$ on Hilbert spaces H_w and H_v, respectively. The direct sum representation of $\mathbb{G}$, denoted by $w \oplus v$, is the (unitary) representation of $\mathbb{G}$ on $H_w \oplus H_v$ defined as

$$w \oplus v := w + v \in M(\mathcal{K}(H_w \oplus H_v) \otimes C(\mathbb{G}))$$

1.3.1.10 Remark. Assume that both H_w and H_v are finite dimensional Hilbert spaces. Let us compute the matrix coefficients of the direct sum representation.

Let $\{\xi_1, \ldots, \xi_{dim(H_w)}\}$ and $\{\eta_1, \ldots, \eta_{dim(H_v)}\}$ be orthonormal basis for H_w and H_v, respectively and let $\{m_{i,j}\}_{i,j=1,\ldots,dim(H_w)}$ and $\{n_{l,k}\}_{l,k=1,\ldots,dim(H_v)}$ be the corresponding matrix units in $\mathcal{B}(H_w)$ and $\mathcal{B}(H_v)$, respectively. We know that $\{\xi_1, \ldots, \xi_{dim(H_w)}, \eta_1, \ldots, \eta_{dim(H_v)}\}$ is an orthonormal basis for $H_w \oplus H_v$. Denote by I_w and by I_v the set of indices corresponding to the vector basis of H_w and H_v, respectively.

A straightforward computation shows that

$$\omega_{\zeta_r, \zeta_s} = \begin{cases} \omega_{\xi_r, \xi_s}, \text{ if } r, s \in I_w \\ \omega_{\eta_r, \eta_s}, \text{ if } r, s \in I_v \\ 0, \text{otherwise} \end{cases},$$

for all $r, s \in I_w \sqcup I_v$. So that we obtain that the matrix coefficients of the direct sum representation are exactly

$$(w \oplus v)_{r,s} = \begin{cases} w_{r,s}, \text{ if } r, s \in I_w \\ v_{r,s}, \text{ if } r, s \in I_v \\ 0, \text{otherwise} \end{cases},$$

for all $r, s \in I_w \sqcup I_v$. It is clear that

$$\chi_{\mathbb{G}}(w \oplus v) = \chi_{\mathbb{G}}(w) + \chi_{\mathbb{G}}(v)$$

1.3.1.11 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and w, v two (unitary) representations of $\mathbb{G}$ on Hilbert spaces H_w and H_v, respectively. The tensor product representation of $\mathbb{G}$, denoted by $w \bigoplus v$, is the (unitary) representation of $\mathbb{G}$ on $H_w \otimes H_v$ defined as

$$w \bigoplus v := w_{13} v_{23} \in M(\mathcal{K}(H_w \otimes H_v) \otimes C(\mathbb{G}))$$

1.3.1.12 Remark. Assume that both H_w and H_v are finite dimensional Hilbert spaces. Let us compute the matrix coefficients of the tensor product representation.

Let $\{\xi_1, \ldots, \xi_{dim(H_w)}\}$ and $\{\eta_1, \ldots, \eta_{dim(H_v)}\}$ be orthonormal basis for H_w and H_v, respectively and let $\{m_{i,j}\}_{i,j=1,\ldots,dim(H_w)}$ and $\{n_{k,l}\}_{k,l=1,\ldots,dim(H_v)}$ be the corresponding matrix units in $\mathcal{B}(H_w)$ and $\mathcal{B}(H_v)$, respectively. We know that $\{\zeta_r := \xi_i \otimes \eta_k\}_{\substack{i=1,\ldots,dim(H_w)\\ k=1,\ldots,dim(H_v)}}$ is an orthonormal basis for $H_w \otimes H_v$. Denote by I_w and by I_v the set of indices corresponding to the vector basis of H_w and H_v, respectively.

A straightforward computation shows that

$$\omega_{\zeta_r, \zeta_s} = \omega_{\xi_i, \xi_j} \otimes \omega_{\eta_k, \eta_l},$$

for all $r = (i, k), s = (j, l) \in I_w \times I_v$. So that we obtain that the matrix coefficients of the tensor product representation are exactly

$$(w \bigoplus v)_{r,s} = w_{i,j}\ v_{k,l},$$

for all $r = (i, k), s = (j, l) \in I_w \times I_v$. It is clear that

$$\chi_{\mathbb{G}}(w \bigoplus v) = \chi_{\mathbb{G}}(w)\chi_{\mathbb{G}}(v)$$

1.3.1.13 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and w, v two (unitary) representations of $\mathbb{G}$ on Hilbert spaces H_w and H_v, respectively. An intertwiner between w and v is a linear operator $\Phi : H_w \longrightarrow H_v$ such that

$$(\Phi \otimes 1_{C(\mathbb{G})})w = v(\Phi \otimes 1_{C(\mathbb{G})})$$

The space of all intertwiners between w and v is denoted by $Mor(w, v)$. In particular, if $w = v$, we write $End(w) := Mor(w, w)$.

1.3.1.14 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group.

i) Two (unitary) representations of $\mathbb{G}$ w, v are called (unitary) equivalents if $Mor(w, v)$ contains an invertible (resp. unitary) operator.

ii) A (unitary) representation w of $\mathbb{G}$ is called irreducible if $End(w) = \mathbb{C}$.

The set of all unitary equivalence classes of irreducible unitary finite dimensional representations of $\mathbb{G}$ is denoted by $Irr(\mathbb{G})$.

1.3.1.15 Note. If $x \in Irr(\mathbb{G})$ is such a class, then we use the symbol $w^x \in \mathcal{B}(H_x) \otimes C(\mathbb{G})$ to denote an irreducible unitary finite dimensional representation of $\mathbb{G}$ representing x. Once such a representative is fixed, we write

- H_x for the corresponding finite dimensional representation Hilbert space,
- $n_x := dim(x) := dim(H_x)$,
- $Mor(x, y)$ for the space of all intertwiners, up to unitary equivalence, between representatives w^x and w^y, where $y \in Irr(\mathbb{G})$,
- $\chi_{\mathbb{G}}(x) := \chi_{\mathbb{G}}(w^x)$.

For any compact quantum group $\mathbb{G}$ there always exists a distinguished irreducible representation. Namely, the trivial representation $\epsilon := 1_{\mathbb{C}} \otimes 1_{C(\mathbb{G})} \in \mathbb{C} \otimes C(\mathbb{G})$. By abuse of notation, we still denote by ϵ the corresponding class in $Irr(\mathbb{G})$.

1.3.1.16 Remark. If w, v are two unitary representations of $\mathbb{G}$, then it is easy to show that $\Phi^* \in Mor(v, w)$ whenever $\Phi \in Mor(w, v)$. Indeed,

$$\begin{aligned}\Phi \in Mor(w,v) &\Leftrightarrow (\Phi \otimes 1_{C(\mathbb{G})})w = v(\Phi \otimes 1_{C(\mathbb{G})}) \Leftrightarrow w^*(\Phi^* \otimes 1_{C(\mathbb{G})}) = (\Phi^* \otimes 1_{C(\mathbb{G})})v^* \\ &\Leftrightarrow ww^*(\Phi^* \otimes 1_{C(\mathbb{G})})v = w(\Phi^* \otimes 1_{C(\mathbb{G})})v^*v \\ &\Leftrightarrow (\Phi^* \otimes 1_{C(\mathbb{G})})v = w(\Phi^* \otimes 1_{C(\mathbb{G})}) \Leftrightarrow \Phi^* \in Mor(v,w)\end{aligned}$$

In particular, $End(w)$ is a C^*-algebra for all unitary representation w of $\mathbb{G}$. Notice that the set of *compact* intertwiners of w is a C^*-subalgebra of $End(w)$.

1.3.1.17 Proposition (Quantum Schur's lemma)**.** *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group. If $x, y \in Irr(\mathbb{G})$, then*

- *either $x = y$ and $Mor(x, y)$ is a one dimensional vector space*
- *or $Mor(x, y) = (0)$.*

Proof. Given the classes $x, y \in Irr(\mathbb{G})$, take two representatives w^x and w^y, respectively. Suppose that $\Phi : H_x \longrightarrow H_y$ is a non-zero intertwiner between w^x and w^y. We are going to see that w^x and w^y are unitary equivalent and that $Mor(w^x, w^y)$ is a one dimensional vector space.

Indeed, since w^x and w^y are unitaries, then we have $\Phi^* \in Mor(w^y, w^x)$. So we have $\Phi^* \circ \Phi \in End(w^x)$ and $\Phi \circ \Phi^* \in End(w^y)$. But w^x and w^y are irreducible by assumption, then $End(w^x) = \mathbb{C} = End(w^y)$. In other words, there exist some scalars $\alpha, \beta \in \mathbb{C}\backslash\{0\}$ such that $\Phi^* \circ \Phi = \alpha\ id_{H_x}$ and $\Phi \circ \Phi^* = \beta\ id_{H_y}$. This means that the operator Φ is unitary up to a constant. Finally, if $\Psi \in Mor(w^x, w^y)$ is another non-zero intertwiner between w^x and w^y, then we have $\Phi^* \circ \Psi \in End(w^x) = \mathbb{C}$, hence $\Psi \in \Phi \cdot \mathbb{C}$ and so $Mor(w^x, w^y)$ is a one dimensional vector space generated by the intertwiner realizing the equivalence between w^x and w^y. ■

1.3.1.18 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and w a (unitary) finite dimensional representation of $\mathbb{G}$ on a Hilbert space H. The contragredient or adjoint representation of w, denoted by $\overline{w}$, is the finite dimensional representation of $\mathbb{G}$ on $\overline{H}$ defined as

$$\overline{w} := (j \otimes id_{C(\mathbb{G})})(w^{-1}) \in \mathcal{B}(\overline{H}) \otimes C(\mathbb{G}),$$

where $j : \mathcal{B}(H) \longrightarrow \mathcal{B}(\overline{H})$ is the anti-multiplicative linear $*$-homomorphism that sends an operator to its dual.

1.3.1.19 Remark. Let $\{\xi_1, \ldots, \xi_{dim(H)}\}$ be an orthonormal basis of H and $\{\omega_1, \ldots, \omega_{dim(\overline{H})}\}$ be its dual basis of the dual space $\overline{H}$. If $\{w_{i,j}\}_{i,j=1,\ldots,dim(H)}$ are the matrix coefficients of w with respect to the basis $\{\xi_1, \ldots, \xi_{dim(H)}\}$, then it is straightforward to see that the matrix coefficients of $\overline{w}$ with respect to the dual basis $\{\omega_1, \ldots, \omega_{dim(\overline{H})}\}$ are exactly

$$\overline{w}_{i,j} = w^*_{i,j},$$

for all $i, j = 1, \ldots, dim(\overline{H})$ whenever w is *unitary*. It is clear that

$$\chi_{\mathbb{G}}(\overline{w}) = \chi_{\mathbb{G}}(w)^*$$

Since Δ is a $*$-homomorphism it is clear that $\overline{w}$ is still an element of $\mathcal{B}(\overline{H}) \otimes C(\mathbb{G})$ satisfying the condition $(id \otimes \Delta)(\overline{w}) = \overline{w}_{12}\overline{w}_{13}$. For a general finite dimensional representation w (not necessarily unitary), this condition still holds because j is a conjugate $*$-homomorphism, so that

$$(id \otimes \Delta)(\overline{w}) = (j \otimes id_{C(\mathbb{G})})w_{13}^{-1}w_{12}^{-1} = \overline{w}_{12}\overline{w}_{13}$$

However, the invertibility of such an element $\overline{w}$ needs some work to be established (see Proposition 1.3.11 in [139] for a proof). And it is important to remark that it is not guarantee *a priori* that such an element $\overline{w}$ is still a *unitary* one even if w itself is unitary. Actually, $\overline{w}$ is well-defined as a unitary representation *only* in the set of irreducible classes meaning that if $x \in Irr(\mathbb{G})$, then we can prove that $\overline{w^x}$ defined above is unitary equivalent to a unitary irreducible representation defining then a class $\overline{x} \in Irr(\mathbb{G})$. We refer to Definition 1.4.5 in [139] or to Proposition 6.10 in [124] for the details.

The representation theory of a compact quantum group is very closed to the one of a classical compact group. For instance, we can prove that every finite dimensional representation of a compact quantum group decomposes as a direct sum of irreducible representations, then $Irr(\mathbb{G})$ provides all relevant information about representations of $\mathbb{G}$. Let us establish the main results of the representation theory of a compact quantum group.

1.3.1.20 Lemma. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and let $w \in M(\mathcal{K}(H_w) \otimes C(\mathbb{G}))$, $v \in M(\mathcal{K}(H_v) \otimes C(\mathbb{G}))$ be two representations of $\mathbb{G}$ on Hilbert spaces H_w and H_v, respectively.*

If $\Phi : H_w \longrightarrow H_v$ is any linear (resp. compact) operator, then the linear (resp. compact) operator $S := (id \otimes h_{\mathbb{G}})(w^*(\Phi \otimes 1_{C(\mathbb{G})})v)$ is such that

$$S \otimes 1_{C(\mathbb{G})} = w^*(S \otimes 1_{C(\mathbb{G})})v$$

In particular, if w is a unitary *representation, then $S \in Mor(w, v)$.*

Proof. Since the co-multiplication Δ is a unital $*$-homomorphism, we can write

$$\begin{aligned}(id \otimes \Delta)(w^*(\Phi \otimes 1_{C(\mathbb{G})})v) &= (id \otimes \Delta)(w)^*(id \otimes \Delta)(\Phi \otimes 1_{C(\mathbb{G})})(id \otimes \Delta)(v) \\ &= w_{13}^* w_{12}^* (\Phi \otimes 1_{C(\mathbb{G})} \otimes 1_{C(\mathbb{G})})v_{12}v_{13}\end{aligned}$$

If now we apply to this equation the operator $(id \otimes h_{\mathbb{G}} \otimes id)$, we get

$$\begin{aligned}(id \otimes h_{\mathbb{G}} \otimes id)\big((id \otimes \Delta)(w^*(\Phi \otimes 1_{C(\mathbb{G})})v)\big) &= (id \otimes (h_{\mathbb{G}} \otimes id) \circ \Delta)(w^*(\Phi \otimes 1_{C(\mathbb{G})})v) \\ &= (id \otimes h_{\mathbb{G}} \cdot 1_{C(\mathbb{G})}))(w^*(\Phi \otimes 1_{C(\mathbb{G})})v) = S \otimes 1_{C(\mathbb{G})}\end{aligned}$$

$$(id \otimes h_{\mathbb{G}} \otimes id)\big(w_{13}^* w_{12}^* (\Phi \otimes 1_{C(\mathbb{G})} \otimes 1_{C(\mathbb{G})}) v_{12} v_{13}\big) \cong w^*(id \otimes h_{\mathbb{G}})(w^*(\Phi \otimes 1_{C(\mathbb{G})})v \otimes 1_{C(\mathbb{G})})v$$
$$= w^*(S \otimes 1_{C(\mathbb{G})})v,$$

which ends the proof. ■

1.3.1.21 Proposition (Quantum Maschke's theorem)**.** *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group.*

i) Every finite dimensional representation of $\mathbb{G}$ is equivalent to a unitary finite dimensional representation.

ii) Every finite dimensional representation of $\mathbb{G}$ is equivalent to a direct sum of finite dimensional unitary irreducible representations.

Proof. i) Let $w \in \mathcal{B}(H) \otimes C(\mathbb{G})$ be a finite dimensional representation of $\mathbb{G}$ on a finite dimensional Hilbert space H. Consider the identity operator id_H and apply Lemma 1.3.1.20 above, so that the operator $S := (id \otimes h_{\mathbb{G}})(w^* w)$ is such that $S \otimes 1_{C(\mathbb{G})} = w^*(S \otimes 1_{C(\mathbb{G})})w$.

Now, since $w \in \mathcal{B}(H) \otimes C(\mathbb{G})$ is an invertible element, the same is true for $w^* w$ and then we can write $w^* w > \epsilon$ for some $\epsilon > 0$. Since $(id \otimes h_{\mathbb{G}})$ is positive, we can also write $S > \epsilon$. In this case, we can consider the square root $S^{1/2}$.

We claim that the element

$$v := (S^{1/2} \otimes 1_{C(\mathbb{G})})w(S^{-1/2} \otimes 1_{C(\mathbb{G})}) \in \mathcal{B}(H) \otimes C(\mathbb{G})$$

is a unitary representation of $\mathbb{G}$ on H. Indeed,

$$(id \otimes \Delta)(v) = (S^{1/2} \otimes 1_{C(\mathbb{G})} \otimes 1_{C(\mathbb{G})})w_{12}w_{13}(S^{-1/2} \otimes 1_{C(\mathbb{G})} \otimes 1_{C(\mathbb{G})}) = v_{12}v_{13}$$

$$\begin{aligned} v^* v &= (S^{-1/2} \otimes 1_{C(\mathbb{G})})w^*(S^{1/2} \otimes 1_{C(\mathbb{G})})(S^{1/2} \otimes 1_{C(\mathbb{G})})w(S^{-1/2} \otimes 1_{C(\mathbb{G})}) \\ &= (S^{-1/2} \otimes 1_{C(\mathbb{G})})w^*(S \otimes 1_{C(\mathbb{G})})w(S^{-1/2} \otimes 1_{C(\mathbb{G})}) \\ &= (S^{-1/2} \otimes 1_{C(\mathbb{G})})(S \otimes 1_{C(\mathbb{G})})(S^{-1/2} \otimes 1_{C(\mathbb{G})}) = id \otimes 1_{C(\mathbb{G})}, \end{aligned}$$

By definition of v, it is clear that $S^{1/2} \in Mor(w, v)$. Since $S^{1/2}$ is an invertible operator, we conclude that w is equivalent to the unitary representation v.

ii) If $w \in \mathcal{B}(H) \otimes C(\mathbb{G})$ is a finite dimensional representation of $\mathbb{G}$ on the finite dimensional Hilbert space H, we can assume, without loose of generality, that w is unitary thanks to the statement (i) proven above.

In this case, we know that $End(w)$ is a C^*-algebra formed of compact operators. Hence, let $p_1, \ldots, p_r \in End(w)$ be a finite number of minimal mutually orthogonal projections such that $id_H = p_1 + \ldots + p_r$. In this situation, we have an orthogonal decomposition for H, say $H = H_1 \oplus \ldots \oplus H_r$ where $H_l := p_l(H)$, for all $l = 1, \ldots, r$.

For every $l = 1, \ldots, r$ we define the element

$$w_l := w(p_l \otimes 1_{C(\mathbb{G})}) \in \mathcal{B}(H_l) \otimes C(\mathbb{G})$$

It is straightforward to see that every w_l is still a unitary representation of $\mathbb{G}$ (because w_l is just the restriction of w to the subspace H_l) and by construction we have that $w = \bigoplus_{l=1}^{r} w_l$.

To conclude, we have to show that each unitary representation w_l is irreducible. Indeed, since $End(w_l)$ is a finite dimensional C^*-algebra for all $l = 1, \ldots, r$, it suffices to see that given any projection $P \in End(w_l)$, then either $P = 0$ or $P = p_l$, for all $l = 1, \ldots, r$. By construction we have that $Im(P) \subset Im(p_l)$, so $p \leqslant p_l$. Moreover, $P \in End(w)$ because

$$(P \otimes id)w \overset{(*)}{=} (P \otimes 1)(p_l \otimes 1)w = (P \otimes 1)w_l = w_l(P \otimes 1) = w(p_l \otimes 1)(P \otimes 1) \overset{(*)}{=} w(P \otimes 1),$$

where in $(*)$ we use the fact that $P \circ p_l = P$. Hence, minimality of p_l implies that either $P = 0$ or $P = p_l$, for all $l = 1, \ldots, r$.

■

1.3.1.22 Remarks. 1. Notice that the above proposition (in particular the decomposition into a direct sum of finite dimensional unitary irreducible representations) is still true for *infinite dimensional* representations of $\mathbb{G}$ (see for instance Theorem 1.5.4 in [139] or Theorem 5.3.3 in [188]). This is why we can restrict our attention to the study of *finite dimensional unitary irreducible* representations.

2. Let w a (unitary) representation of $\mathbb{G}$ on a Hilbert space H. On the one hand, a subspace $K \subset H$ is said to be *w-invariant* if $(p_K \otimes 1)w(p_K \otimes 1) = w(p_K \otimes 1)$, where p_K denotes the orthogonal projection of H onto K.

   It is possible to characterize this definition in the following way. $K \subset H$ is w-invariant if and only if $\Big((id \otimes \omega)w\Big)(k) = k\omega(1)$, for all $k \in K$ and all linear form $\omega \in C(\mathbb{G})^*$. We refer to Proposition 5.2.8 in [188] or to page 25 in [139] for a proof.

   On the other hand, the definition of irreducible representation given in Definition 1.3.1.14 can be characterized in the following way. A (unitary) representation w is irreducible if and only if there are no proper w-invariant subspaces besides 0 and H. We refer to page 25 in [139] or to Proposition 5.2.7 in [188] for a proof.

3. Finally, it is worth pointing out that calling Proposition 1.3.1.17 *"Quantum Schur's lemma"* is a bit misleading. Indeed, classically the Schur's lemma states that if a unitary representation if irreducible, then its commutant is trivial. However, this is exactly our original definition of irreducible representation (Definition 1.3.1.14). If we want to be rigorous, we may show Proposition 1.3.1.17 starting from the definition of irreducible representation given by the above characterization in terms of invariant subspaces. This can be done as in the classical case and we refer to Proposition 5.3.4 in [188] or to Lemma 6.6 in [124] for a proof.

1.3.1.23 Proposition. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group. If $w \in M(\mathcal{K}(H) \otimes C(\mathbb{G}))$ is a unitary representation of $\mathbb{G}$ on a Hilbert space H, then*

$$P := (id \otimes h_{\mathbb{G}})w$$

is the orthogonal projection onto the subspace of all w-invariant vectors.

Proof. First of all, thanks to the preceding proposition and the preceding remarks we can assume that w is irreducible.

Since w is a unitary representation of $\mathbb{G}$, we have $(id \otimes \Delta)w = w_{12}w_{13}$. If we apply $(id \otimes id \otimes h_{\mathbb{G}})$ to both sides of this equation, we get that $P \otimes id = w(P \otimes id)$ because

$$(id \otimes id \otimes h_{\mathbb{G}})(id \otimes \Delta)w = (id \otimes h_{\mathbb{G}})w \otimes id = P \otimes id$$

$$(id \otimes id \otimes h_{\mathbb{G}})w_{12}w_{13} = (id \otimes id)w(id \otimes h_{\mathbb{G}})w \otimes id = w(P \otimes id)$$

This relation shows that the range of P is formed by w-invariant vectors because given any vector $\xi \in H$ and any linear form $\omega \in C(\mathbb{G})^*$ we write

$$\Big((id \otimes \omega)w\Big)P(\xi) = (id \otimes \omega)w(P(\xi) \otimes 1) = (id \otimes \omega)(P(\xi) \otimes 1) = P(\xi)\omega(1)$$

Next, if w is a non-trivial irreducible representation, then H does not contain any proper w-invariant vector, so that $P = 0$. If w is trivial, then $w = id \otimes 1$ and $P = id$. In any case, P is the orthogonal projection onto the subspace of all w-invariant vectors and the proof is complete. ■

Next we can establish the analogue of the Peter-Weyl theory in the framework of compact quantum groups. In particular, the proof of orthogonality relations can be found in Theorem 1.4.3 of [139], Proposition 5.3.8 of [188] or Section 6 of [231].

1.3.1.24 Theorem (Quantum Schur's orthogonality relations)**.** *Let $\mathbb{G}$ be a compact quantum group with Haar state $h_{\mathbb{G}}$. Fix an irreducible representation $x \in Irr(\mathbb{G})$.*

i) We have that

$$Mor(\epsilon, w^x \oplus w^{\overline{x}}) \neq 0 \neq Mor(\epsilon, w^{\overline{x}} \oplus w^x)$$

and consequently there exist non-trivial invariant vectors in $H_x \otimes H_{\overline{x}}$ and $H_{\overline{x}} \otimes H_x$, say E_x and $E_{\overline{x}}$, respectively.

The vector E_x, canonically associated to an invertible anti-linear map $J_x : H_x \longrightarrow H_{\overline{x}}$, and the vector $E_{\overline{x}}$, canonically associated to an invertible anti-linear map $J_{\overline{x}} : H_{\overline{x}} \longrightarrow H_x$; are defined up to non-zero multiplicative factors. We choose these factors in such a way that

$$\langle E_x, E_x \rangle = \langle E_{\overline{x}}, E_{\overline{x}} \rangle \text{ and } J_{\overline{x}} = J_x^{-1}$$

In that case, the operator $Q_x := J_x^* J_x$ is uniquely determined, $Tr(Q_x) = Tr(Q_x^{-1})$ and $Q_{\overline{x}} = Q_x^{-1}$. We define the quantum dimension of x *as the number*

$$dim_q(x) := Tr(Q_x)$$

ii) If $(w^x_{i,j})_{i,j=1,\ldots,n_x}$ are the matrix coefficients of w^x with respect to an orthonormal basis of H_x, then we have

$$h_{\mathbb{G}}(w^x_{k,l}(w^x)^*_{i,j}) = \frac{\delta_{k,i}(Q_x)_{j,l}}{dim_q(x)} \text{ and } h_{\mathbb{G}}((w^x)^*_{i,j}w^x_{k,l}) = \frac{\delta_{j,l}(Q_x^{-1})_{k,i}}{dim_q(x)}$$

iii) Let $y \in Irr(\mathbb{G})$ be an other irreducible representation of $\mathbb{G}$ with matrix coefficients $(w^y_{k,l})_{i,j=1,\ldots,n_y}$. If y is not *equivalent to x, then*

$$h_{\mathbb{G}}(w^y_{k,l}(w^x)^*_{i,j}) = 0 = h_{\mathbb{G}}((w^x)^*_{i,j}w^y_{k,l})$$

1.3.1.25 Remarks. 1. The preceding orthogonality relations yield clearly that for every irreducible representations $x, y \in Irr(\mathbb{G})$ we have

$$h_{\mathbb{G}}\Big(\chi_{\mathbb{G}}(x)^*\chi_{\mathbb{G}}(y)\Big) = h_{\mathbb{G}}\Big(\chi_{\mathbb{G}}(x)\chi_{\mathbb{G}}(y)^*\Big) = \begin{cases} 1, \text{ if } x = y \\ 0, \text{ otherwise} \end{cases}$$

2. It is worth pointing out some elementary facts about the invariant vectors E_x, $E_{\overline{x}}$ of the preceding theorem. We refer to Section 6 of [231] for more details.

   Let X, Y be two Hilbert spaces. It is well-known (see for instance [1]) that the tensor product $X \otimes Y$ can be identified with space of all anti-linear maps from X into Y. Precisely, for any element $E \in X \otimes Y$ there exists a unique anti-linear map $J_E : X \longrightarrow Y$ such that

   $$\langle E, \xi \otimes \eta \rangle = \langle J_E(\xi), \eta \rangle = \langle J_E^*(\eta), \xi \rangle,$$

   for all $\xi \in X$, $\eta \in Y$ where the second equality defines the conjugation for J_E.

   If both X and Y are finite dimensional with $dim(X) = rank(E) = dim(Y)$, then J_E is invertible and a straightforward computation by taking orthonormal basis on X and Y shows that $||E||^2 = Tr(J_E^* J_E)$.

   If $x \in Irr(\mathbb{G})$, then the preceding theorem guarantees that there exists a non-trivial $\overline{x} \oplus x$-invariant vector $E_{\overline{x}} \in H_{\overline{x}} \otimes H_x$ and a non-trivial $x \oplus \overline{x}$-invariant vector $E_x \in H_x \otimes H_{\overline{x}}$ which are unique up to a non-zero multiplicative factor (which is chosen as in the statement of the preceding theorem). Let us introduce the following operator on $H_{\overline{x}} \otimes H_x$

   $$\begin{array}{rccl} T_{E_{\overline{x}}}: & H_{\overline{x}} \otimes H_x & \longrightarrow & H_{\overline{x}} \otimes H_x \\ & \zeta & \longmapsto & T_{E_{\overline{x}}}(\zeta) := \langle E_{\overline{x}}, \zeta \rangle E_{\overline{x}} \end{array}$$

   In this way, Proposition 1.3.1.23 applied to this context yields that

   $$(id \otimes h_{\mathbb{G}})(\overline{x} \oplus y) = \frac{\delta_{x,y}}{||E_x||^2} T_{E_{\overline{x}}} = \frac{\delta_{x,y}}{dim_q(x)} T_{E_{\overline{x}}},$$

   for every $x, y \in Irr(\mathbb{G})$. Next, using the $x \oplus \overline{x}$-equivariance of E_x a straightforward computation by taking orthonormal basis on H_x and $H_{\overline{x}}$ shows that

   $$\Big((\omega_{J_x^*(\eta), \xi} \otimes id) w^x \Big)^* = (\omega_{\eta, J_x(\xi)} \otimes id) w^{\overline{x}},$$

   for all $\xi \in H_x$, $\eta \in H_{\overline{x}}$.

As in the classical theory, we can also define an analogue of the *left regular representation*. The proof of the following theorem can be found in Theorem 1.5.2 and Theorem 1.5.3 of [139], Section 5 of [124] or Section 4 of [231].

1.3.1.26 Theorem (Left regular representation)**.** *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group.*

i) There exists a unique unitary operator $W \in \mathcal{B}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ such that

$$W^*(\xi \otimes \lambda(a)\Omega) = (\lambda \otimes \lambda) \circ \Delta(a)(\xi \otimes \Omega),$$

for all $\xi \in L^2(\mathbb{G})$ and all $a \in C(\mathbb{G})$.

ii) W is a unitary representation of $\mathbb{G}$ on $L^2(\mathbb{G})$ meaning that $W \in M(\mathcal{K}(L^2(\mathbb{G}) \otimes \lambda(C(\mathbb{G})))$ is a unitary element such that $(id \otimes \Delta)(W) = W_{12} W_{13}$. The element W is called left regular representation *of $\mathbb{G}$.*

iii) The space span$\{(id \otimes \omega)(W) \mid \omega \in \mathcal{K}(L^2(\mathbb{G}))^*\}$ is dense in $\lambda(C(\mathbb{G}))$ and the co-multiplication Δ is given by

$$(\lambda \otimes \lambda)\Delta(a) = W^*(1 \otimes \lambda(a))W,$$

for all $a \in C(\mathbb{G})$.

iv) W is a multiplicative unitary on $L^2(\mathbb{G})$ in the sense of Baaj-Skandalis, that is, $W \in \mathcal{B}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ is a unitary operator satisfying the pentagonal equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$. It is called fundamental unitary of $\mathbb{G}$ *and denoted by $W_{\mathbb{G}}$.*

1.3.1.27 Remark. In a similar way, we can define the *right regular representation of* $\mathbb{G}$. More precisely, we can imitate the preceding theorem in order to show that there exists a unique unitary operator $V \in \mathcal{B}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ such that

$$V(\lambda(a)\Omega \otimes \xi) = (\lambda \otimes \lambda) \circ \Delta(a)(\Omega \otimes \xi),$$

for all $\xi \in L^2(\mathbb{G})$ and all $a \in C(\mathbb{G})$. Such a unitary is denoted by $V_{\mathbb{G}}$ and it satisfies the analogue properties of those of $W_{\mathbb{G}}$. In particular, the co-multiplication of $\mathbb{G}$ is characterized by the formula

$$(\lambda \otimes \lambda)\Delta(a) = V_{\mathbb{G}}(\lambda(a) \otimes 1)V_{\mathbb{G}}^*,$$

for all $a \in C(\mathbb{G})$.

The following result says that inside every compact quantum group there exists an *algebraic* compact quantum group meaning that there always exists a (unital) $*$-Hopf algebra with an invariant state with respect to the co-multiplication (see Chapter 3 in [188] for more details). Namely,

1.3.1.28 Proposition. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group. Denote by $Pol(\mathbb{G}) := \mathbb{C}[\mathbb{G}]$ the linear span of matrix coefficients of all finite dimensional representations of $\mathbb{G}$. We have that*

i) $Pol(\mathbb{G})$ is a dense unital $*$-subalgebra of $C(\mathbb{G})$,

ii) the set $\{w^x_{i,j}\}_{\substack{x \in Irr(\mathbb{G}) \\ i,j=1,\dots,n_x}}$ provides a basis of $Pol(\mathbb{G})$,

iii) $(Pol(\mathbb{G}), \Delta, \varepsilon, S)$ is a $*$-Hopf algebra where

- *ε, the co-unit, is given by $\varepsilon(w^x_{i,j}) := \delta_{i,j}$, for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, n_x$,*
- *S, the antipode, is given by $S(w^x_{i,j}) := (w^x_{i,j})^*$, for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, n_x$,*

iv) the Haar state $h_{\mathbb{G}}$ is faithful on $Pol(\mathbb{G})$,

v) if $\tau : C(\mathbb{G}) \longrightarrow C(\mathbb{G})$ is a Hopf $*$-automorphism of $\mathbb{G}$, then $\tau(Pol(\mathbb{G})) = Pol(\mathbb{G})$. Moreover, $\{\tau(w^x_{i,j})\}_{i,j=1,\dots,n_x}$ is an irreducible representation of $\mathbb{G}$ whenever $x \in Irr(\mathbb{G})$.

Proof. i) First of all, we observe that $Pol(\mathbb{G})$ is a unital $*$-subalgebra of $C(\mathbb{G})$ by virtue of remarks 1.3.1.12 and 1.3.1.19. The density of $Pol(\mathbb{G})$ in $C(\mathbb{G})$ is proven as follows.

Consider the left regular representation W of $\mathbb{G}$ on the Hilbert space $L^2(\mathbb{G})$ and decompose it in a direct sum of finite dimensional unitary irreducible representations (recall Remark 1.3.1.22), so that

$$L^2(\mathbb{G}) = \bigoplus_{\alpha \in I} H_\alpha \text{ and } W = \bigoplus_{\alpha \in I} w_\alpha$$

For each $\alpha \in I$, put $n_\alpha := dim(H_\alpha)$ and fix an orthonormal basis $\{\xi_1^\alpha, \dots, \xi_{n_\alpha}^\alpha\}$ in H_α. Given $\alpha, \beta \in I$, we define the following linear form on $\mathcal{B}(L^2(\mathbb{G}))$

$$\omega_{\xi_i^\alpha, \xi_j^\beta}(T) := \langle \xi_i^\alpha, T(\xi_j^\beta) \rangle,$$

for all $i = 1, \dots, n_\alpha$, all $j = 1, \dots, n_\beta$ and for all $T \in \mathcal{B}(H)$. So by the decomposition of W we have

$$(id \otimes \omega_{\xi_i^\alpha, \xi_j^\beta})(W) = \begin{cases} 0, \text{ if } \alpha \neq \beta \\ (id \otimes \omega_{\xi_i^\alpha, \xi_j^\alpha})(w_\alpha) = (w_\alpha)_{i,j}, \text{ if } \alpha = \beta \end{cases}$$

Remark that $span\{\omega_{\xi_i^\alpha, \xi_j^\beta} \mid \alpha, \beta \in I,\ i = 1, \dots, n_\alpha,\ j = 1, \dots, n_\beta\}$ defines a dense subspace in $\mathcal{K}(L^2(\mathbb{G}))^*$. Moreover, Theorem 1.3.1.26 guarantees that $span\{(id \otimes \omega)(W) \mid \omega \in \mathcal{K}(L^2(\mathbb{G}))^*\}$ is dense in $C(\mathbb{G})$, so that the formula above concludes the claim.

ii) Since every finite dimensional representation decomposes in a direct sum of unitary irreducible ones, then the set $\mathscr{B} := \{w_{i,j}^x\}_{\substack{x \in Irr(\mathbb{G}) \\ i,j=1,\dots,n_x}}$ provides a system of generators for $Pol(\mathbb{G})$. Let's see these elements are linearly independent.

Since the operator Q_x is an invertible positive self-adjoint one for each $x \in Irr(\mathbb{G})$, then we choose orthonormal basis of the corresponding H_x that diagonalizes Q_x. We take the matrix coefficients in $\mathscr{B}$ with respect to theses basis.

Let $\sum\limits_{\substack{x \in Irr(\mathbb{G}) \\ i,j=1,\dots,n_x}} \lambda_{i,j}^x\ w_{i,j}^x$ be a finite linear combination of elements of $\mathscr{B}$ and assume that $\sum\limits_{\substack{x \in Irr(\mathbb{G}) \\ i,j=1,\dots,n_x}} \lambda_{i,j}^x\ w_{i,j}^x = 0$. Given an other class $y \in Irr(\mathbb{G})$ and index $k, l = 1, \dots, n_y$, we apply the orthogonality relations and write

$$\begin{aligned} \sum_{\substack{x \in Irr(\mathbb{G}) \\ i,j=1,\dots,n_x}} \lambda_{i,j}^x\ w_{i,j}^x = 0 \Rightarrow & \sum_{\substack{x \in Irr(\mathbb{G}) \\ i,j=1,\dots,n_x}} \lambda_{i,j}^x\ (w^y)_{k,l}^*\ w_{i,j}^x = 0 \\ \Rightarrow & \sum_{\substack{x \in Irr(\mathbb{G}) \\ i,j=1,\dots,n_x}} \lambda_{i,j}^x\ h_{\mathbb{G}}((w^y)_{k,l}^*\ w_{i,j}^x) = \sum_{\substack{x \in Irr(\mathbb{G}) \\ i,j=1,\dots,n_x}} \lambda_{i,j}^x\ \delta_{y,x} \frac{\delta_{l,j}(Q_x^{-1})_{i,k}}{dim_q(x)} \\ = & \frac{1}{dim_q(y)} \sum_i \lambda_{i,l}^x\ (Q_y^{-1})_{i,k} = 0, \end{aligned}$$

since this is true for every $y \in Irr(\mathbb{G})$ and we have chosen diagonalizing basis for every operator Q_x for all $x \in Irr(\mathbb{G})$, we conclude that $\lambda_{i,l}^x = 0$ for all $i, l = 1, \dots, n_x$ and all $x \in Irr(\mathbb{G})$, which ends the proof.

iii) First of all, by virtue of Remark 1.3.1.8 we have that $\Delta(Pol(\mathbb{G})) \subset Pol(\mathbb{G}) \odot Pol(\mathbb{G})$. In order to show that the homomorphism ε and S defined in the statement satisfy the co-unit and antipode axioms, respectively; we are going to check the corresponding axioms on the basis of $Pol(\mathbb{G})$ and conclude by linearity. Namely, for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, n_x$ we have

$$(id \otimes \varepsilon)\Delta(w_{i,j}^x) = (id \otimes \varepsilon) \sum_{k=1}^{n_x} w_{ik}^x \otimes w_{kj} = \sum_{k=1}^{n_x} w_{ik}^x \varepsilon(w_{kj}) = w_{i,j}^x$$

$$m \circ (id \otimes S)\Delta(w^x_{i,j}) = m \circ (id \otimes S)\sum_{k=1}^{n_x} w^x_{ik} \otimes w_{kj} = \sum_{k=1}^{n_x} w^x_{ik}\ S(w_{kj}) = \delta_{i,j} = \varepsilon(w^x_{i,j})$$

iv) Given $a \in Pol(\mathbb{G})$ we have to show that $h_{\mathbb{G}}(a^*a) > 0$ whenever $a \neq 0$. By orthogonality relations and statement (ii) above we have showed that the set $\{w^x_{i,j}\}_{\substack{x\in Irr(\mathbb{G})\\ i,j=1,\dots,n_x}}$ provides an orthogonal basis of $Pol(\mathbb{G})$ with respect to the sesquilinear form $\langle a, b\rangle := h_{\mathbb{G}}(a^*b)$, for all $a, b \in Pol(\mathbb{G})$. Therefore, we can suppose that $a = w^x_{i,j}$ for some $x \in Irr(\mathbb{G})$ and some $i, j = 1, \dots, n_x$. In this case we have just to apply orthogonality relations: $h_{\mathbb{G}}((w^x)^*_{ij} w^x_{ij}) = \frac{(Q_x^{-1})_{i,i}}{dim_q(x)} > 0$.

v) Given the Hopf $*$-automorphism τ, consider the set $\{\tau(w^x_{i,j})\}_{\substack{x\in Irr(\mathbb{G})\\ i,j=1,\dots,n_x}}$. We have the following formula

$$\Delta(\tau(w^x_{i,j})) = (\tau \otimes \tau)\Delta(w^x_{i,j}) = (\tau \otimes \tau)\sum_{k=1}^{n} w_{ik} \otimes w_{kj} = \sum_{k=1}^{n} \tau(w_{ik}) \otimes \tau(w_{kj}),$$

for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, n_x$. This means that $(id \otimes \tau)(w^x)$ is again a unitary finite dimensional representation of $\mathbb{G}$ on H_x, for all $x \in Irr(\mathbb{G})$. Hence $\tau(Pol(\mathbb{G})) \subset Pol(\mathbb{G})$. It is straightforward to see that τ^{-1} is still a quantum automorphism of $\mathbb{G}$, so that we have $\tau(Pol(\mathbb{G})) = Pol(\mathbb{G})$. Moreover, $\{\tau(w^x_{i,j})\}_{i,j=1,\dots,n_x}$ is an irreducible representation of $\mathbb{G}$ whenever $x \in Irr(\mathbb{G})$ because the Haar state $h_{\mathbb{G}}$ is preserved by τ (recall Remarks 1.3.1.6) and we have

$$h_{\mathbb{G}}\Big(\chi_{\mathbb{G}}(\tau(w^x))^*\chi_{\mathbb{G}}(\tau(w^x))\Big) = h_{\mathbb{G}}\Big(\tau(\chi_{\mathbb{G}}(x)^*\chi_{\mathbb{G}}(x))\Big) = h_{\mathbb{G}}\Big(\chi_{\mathbb{G}}(x)^*\chi_{\mathbb{G}}(x)\Big)$$

■

Thus we observe that the matrix coefficients of finite dimensional representations of a compact quantum group $\mathbb{G}$ play an important role in the theory. In fact, we are going to show that these matrices characterize completely such a compact quantum group.

1.3.1.29 Proposition. *Let A be a unital C^*-algebra equipped with a unital $*$-homomorphism $\Delta : A \longrightarrow A \otimes A$. The following assertions are equivalent*

i) $\mathbb{G} := (A, \Delta)$ *is a compact quantum group.*

ii) There exist an index set I and a collection of invertible matrices $w^\alpha \in \mathcal{M}_{n_\alpha}(A)$, for all $\alpha \in I$ with $n_\alpha \in \mathbb{N}$ such that

- *the $*$-subalgebra of A generated by the matrix coefficients $\{w^\alpha_{i,j}\}_{\substack{\alpha\in I\\ i,j=1,\dots,n_\alpha}}$ is dense in A,*
- *the conjugate matrices $\overline{w^\alpha} := \Big((w^\alpha_{i,j})^*\Big)_{i,j}$ are invertible for all $\alpha \in I$,*
- *for all $\alpha \in I$ we have $\Delta(w^\alpha_{i,j}) = \sum_{k=1}^{n} w^\alpha_{ik} \otimes w^\alpha_{kj}$ for all $i, j = 1, \dots, n_\alpha$.*

iii) There exist an index set I and a collection of invertible matrices $w^\alpha \in \mathcal{M}_{n_\alpha}(A)$, for all $\alpha \in I$ with $n_\alpha \in \mathbb{N}$ such that

- *the subalgebra of A generated by the matrix coefficients $\{w^\alpha_{i,j}\}_{\substack{\alpha\in I\\ i,j=1,\dots,n_\alpha}}$ is dense in A,*

- *for all $\alpha\in I$ we have $\Delta(w^\alpha_{i,j}) = \sum_{k=1}^{n} w^\alpha_{ik}\otimes w^\alpha_{kj}$ for all $i,j=1,\dots,n_\alpha$.*

Proof. The implication $(i)\Rightarrow(ii)$ follows from Proposition 1.3.1.28 because if $\mathbb{G}=(A,\Delta)$ is a compact quantum group, then $Pol(\mathbb{G})$ is the dense unital $*$-algebra in A generated by the matrix coefficients of all finite dimensional representations. Actually, a basis for $Pol(\mathbb{G})$ is given by the coefficients of unitary irreducible finite dimensional representations. Recall by Remark 1.3.1.19, that the conjugate representation matrix (of a unitary one) is also an invertible one.

The implication $(ii)\Rightarrow(iii)$ is clear: we take simply the collection $\{w^\alpha\}_{\alpha\in I}\cup\{\overline{w^\alpha}\}_{\alpha\in I}$.

Let's see the implication $(iii)\Rightarrow(i)$. Given $\alpha\in I$, consider the coefficients $(w^\alpha_{ij})_{i,j=1,\dots,n_\alpha}$. Thanks to our assumption we have

$$(id\otimes\Delta)\Delta(w^\alpha_{ij}) = \sum_{k,l} w^\alpha_{ik}\otimes w^\alpha_{kl}\otimes w^\alpha_{lj} = (\Delta\otimes id)\Delta(w^\alpha_{ij}),$$

for all $i,j=1,\dots,n_\alpha$. So Δ is co-associative.

Now let $(v^\alpha_{ij})_{i,j}$ be the inverse matrix of $(w^\alpha_{ij})_{i,j}$ and write

$$1_A\otimes w^\alpha_{ij} = \sum_l \delta_{jl} 1_A\otimes w^\alpha_{li} = \sum_{l,k} v^\alpha_{jk}\ w^\alpha_{kl}\otimes w^\alpha_{li} = \sum_k (v^\alpha_{jk}\otimes 1_A)\Delta(w^\alpha_{ki}) \in (A\otimes 1_A)\Delta(A) \tag{1.3.1}$$

Remark as well that given $a,a'\in A$ such that

$$1_A\otimes a = \sum_k (b_k\otimes 1_A)\Delta(c_k) \text{ and } 1_A\otimes a' = \sum_{k'} (b'_{k'}\otimes 1_A)\Delta(c'_{k'}),$$

for some $b_k, b'_{k'}, c_k, c'_{k'}\in A$, then an easy computation yields that

$$1_A\otimes aa' = \sum_{k,k'} (b'_{k'}b_k\otimes 1_A)\Delta(c_k c'_{k'})$$

In other words, the space $\left\{a\in A \mid 1_A\otimes a = \sum_k (b_k\otimes 1_A)\Delta(c_k) \text{ for some } b_k,c_k\in A\right\}$ is a subalgebra of A.

By assumption we know that the subalgebra of A generated by the matrix coefficients $\{w^\alpha_{i,j}\}_{\substack{\alpha\in I\\ i,j=1,\dots,n_\alpha}}$ is dense in A, so the above space is dense in A thanks to the computations (1.3.1). Therefore, $(A\otimes 1_A)\Delta(A)$ is dense in $A\otimes A$ (similarly for $(1_A\otimes A)\Delta(A)$). We conclude that (A,Δ) is a compact quantum group. ■

For our purpose it will be useful to have explicit formulas for the GNS construction of the Haar sate. In this sense, we have the next result which can be found in [231]. We include here a proof for the convenience of the exposition.

1.3.1.30 Proposition. *Let $\mathbb{G}=(C(\mathbb{G}),\Delta)$ be a compact quantum group and $(L^2(\mathbb{G}),\lambda,\Omega)$ the GNS construction associated to the Haar state $h_{\mathbb{G}}$. We have a canonical identification*

$$L^2(\mathbb{G}) \cong \bigoplus_{x\in Irr(\mathbb{G})} (H_x\otimes H_{\overline{x}})$$

and the formula

$$\lambda\big((\omega_{\xi,\xi'}\otimes id_{C(\mathbb{G})})(w^x)\big)(\Omega)=\frac{1}{\sqrt{dim_q(x)}}\,\xi\otimes J_x(\xi'),$$

holds for all $x\in Irr(\mathbb{G})$ *and all* $\xi,\xi'\in H_x$.

Proof. Given $x\in Irr(\mathbb{G})$, we define the following linear map

$$\begin{array}{rccl}\psi_x: & H_x\otimes H_{\overline{x}} & \longrightarrow & Pol(\mathbb{G})\\ & \xi\otimes\eta & \longmapsto & \psi_x(\xi\otimes\eta):=(\omega_{J_x^*(\eta),\xi}\otimes id)(w^x)\end{array}$$

Remark that given an element $w^x_{\xi,\xi'}:=(\omega_{\xi,\xi'}\otimes id_{C(\mathbb{G})})(w^x)\in Pol(\mathbb{G})$ with $\xi,\xi'\in H_x$, then the element $\xi'\otimes J_x(\xi)\in H_x\otimes H_{\overline{x}}$ is such that $\psi_x(\xi'\otimes J_x(\xi))=w^x_{\xi,\xi'}$ by definition/construction.

Remember that $Pol(\mathbb{G})$ is generated by the elements $\{w^x_{\xi,\xi'}\}_{\substack{x\in Irr(\mathbb{G})\\ \xi,\xi'\in H_x}}$, so that any element of $Pol(\mathbb{G})$ can be write as a sum of the form $\sum\limits_{x\in Irr(\mathbb{G})}\psi_x(\zeta_x)$ with $\zeta_x\in H_x\otimes H_{\overline{x}}$, where only a finite number of terms in the sum are non zero.

Given $x\in Irr(\mathbb{G})$ define the following linear map

$$\begin{array}{rccl}\Psi_x: & H_x\otimes H_{\overline{x}} & \longrightarrow & L^2(\mathbb{G})\\ & \zeta_x & \longmapsto & \Psi_x(\zeta_x):=\lambda(\psi_x(\zeta_x))\Omega\end{array}$$

Since $Pol(\mathbb{G})$ is dense in $C(\mathbb{G})$ and Ω is a cyclic vector for λ, then the above discussion yields the decomposition

$$L^2(\mathbb{G})=\bigoplus_{x\in Irr(\mathbb{G})}\Psi_x(H_x\otimes H_{\overline{x}})$$

Remark that this is actually an orthogonal decomposition. Indeed, by definition of the GNS construction, for every $x,y\in Irr(\mathbb{G})$ and every $\zeta_x\in H_x\otimes H_{\overline{x}}$, $\zeta_y\in H_y\otimes H_{\overline{y}}$ we can write

$$\begin{aligned}\langle\Psi_y(\zeta_y),\Psi_x(\zeta_x)\rangle&=\langle\lambda(\psi_y(\zeta_y))\Omega,\lambda(\psi_x(\zeta_x))\Omega\rangle\\&=\langle\Omega,\lambda(\psi_y(\zeta_y))^*\lambda(\psi_x(\zeta_x))\Omega\rangle\\&=\langle\Omega,\lambda\big(\psi_y(\zeta_y)^*\psi_x(\zeta_x)\big)\Omega\rangle\\&=h_{\mathbb{G}}\big(\psi_y(\zeta_y)^*\psi_x(\zeta_x)\big)=\frac{\delta_{y,x}}{dim_q(x)}\langle\zeta_y,\zeta_x\rangle\end{aligned}\tag{1.3.2}$$

In other words, the operator Ψ_x is a multiple of an isometry and the spaces $\{\Psi_x(H_x\otimes H_{\overline{x}})\}_{x\in Irr(\mathbb{G})}$ are pairwise orthogonal. In the preceding computation we must clarify the last equality. Let $\alpha,\beta\in Irr(\mathbb{G})$ and fix orthonormal basis $\{\xi_1^{\overline{\alpha}},\dots,\xi_{n_{\overline{\alpha}}}^{\overline{\alpha}}\}$ and $\{\eta_1^{\beta},\dots,\eta_{n_\beta}^{\beta}\}$ of $H_{\overline{\alpha}}$ and H_β, respectively. Accordingly, we consider the orthonormal basis $\{\zeta_r:=\xi_i^{\overline{\alpha}}\otimes\eta_k^{\beta}\}_{\substack{i=1,\dots,n_{\overline{\alpha}}\\ k=1,\dots,n_\beta}}$ on $H_{\overline{\alpha}}\otimes H_\beta$. Given vectors $\xi,\xi'\in H_{\overline{\alpha}}$ and $\eta,\eta'\in H_\beta$ consider their corresponding coordinate expressions, say

$$\xi=\sum_{i=1}^{n_{\overline{\alpha}}}\lambda_i\xi_i^{\overline{\alpha}},\ \xi'=\sum_{j=1}^{n_{\overline{\alpha}}}\lambda'_j\xi_j^{\overline{\alpha}},\ \eta=\sum_{k=1}^{n_\beta}\mu_k\eta_k^{\beta}\text{ and }\eta'=\sum_{l=1}^{n_\beta}\mu'_l\eta_l^{\beta},$$

so that it is clear that

$$\omega_{\xi,\xi'}=\sum_{i,j}\overline{\lambda}_i\lambda'_j\omega_{i,j}\text{ and }\omega_{\eta,\eta'}=\sum_{k,l}\overline{\mu}_k\mu'_l\omega_{k,l}$$

Hence we write the following,

$$\begin{aligned}
h_{\mathbb{G}}\big((\omega_{\xi,\xi'}\otimes id)w^{\overline{\alpha}}(\omega_{\eta,\eta'}\otimes id)w^{\beta}\big) &= \sum_{i,j,k,l}\overline{\lambda}_i\lambda'_j\overline{\mu}_k\mu'_l h_{\mathbb{G}}\big((\omega_{i,j}\otimes id)w^{\overline{\alpha}}(\omega_{k,l}\otimes id)w^{\beta}\big)\\
&= \sum_{r=(i,k),s=(j,l)}\overline{\lambda}_i\lambda'_j\overline{\mu}_k\mu'_l(\omega_{r,s}\otimes id)(id\otimes h_{\mathbb{G}})w^{\overline{\alpha}}\oplus w^{\beta}\\
&= \sum_{r=(i,k),s=(j,l)}\overline{\lambda}_i\lambda'_j\overline{\mu}_k\mu'_l(\omega_{r,s}\otimes id)(id\otimes h_{\mathbb{G}})w^{\overline{\alpha}}_{13}w^{\beta}_{23}\\
&\overset{(1)}{=} \sum_{r=(i,k),s=(j,l)}\overline{\lambda}_i\lambda'_j\overline{\mu}_k\mu'_l\ \omega_{r,s}\ \frac{\delta_{\alpha,\beta}}{dim_q(\alpha)}T_{E_{\overline{\alpha}}}\\
&= \frac{\delta_{\alpha,\beta}}{dim_q(\alpha)}\sum_{r=(i,k),s=(j,l)}\overline{\lambda}_i\lambda'_j\overline{\mu}_k\mu'_l\ \langle\zeta_r, T_{E_{\overline{\alpha}}}(\zeta_s)\rangle\\
&= \frac{\delta_{\alpha,\beta}}{dim_q(\alpha)}\sum_{i,j,k,l)}\overline{\lambda}_i\lambda'_j\overline{\mu}_k\mu'_l\ \langle\xi^{\overline{\alpha}}_i\otimes\eta^{\beta}_k, T_{E_{\overline{\alpha}}}(\xi^{\overline{\alpha}}_j\otimes\eta^{\beta}_l)\rangle\\
&= \frac{\delta_{\alpha,\beta}}{dim_q(\alpha)}\sum_{i,j,k,l)}\overline{\lambda}_i\lambda'_j\overline{\mu}_k\mu'_l\ \langle\xi^{\overline{\alpha}}_i\otimes\eta^{\beta}_k, \langle E_{\overline{\alpha}}, \xi^{\overline{\alpha}}_j\otimes\eta^{\beta}_l\rangle E_{\overline{\alpha}}\rangle\\
&= \frac{\delta_{\alpha,\beta}}{dim_q(\alpha)}\sum_{i,j,k,l)}\overline{\lambda}_i\lambda'_j\overline{\mu}_k\mu'_l\ \langle\xi^{\overline{\alpha}}_i\otimes\eta^{\beta}_k, E_{\overline{\alpha}}\rangle\langle E_{\overline{\alpha}}, \xi^{\overline{\alpha}}_j\otimes\eta^{\beta}_l\rangle\\
&= \frac{\delta_{\alpha,\beta}}{dim_q(\alpha)}\sum_{i,j,k,l)}\overline{\lambda}_i\lambda'_j\overline{\mu}_k\mu'_l\ \langle\eta^{\beta}_k, J_{\overline{\alpha}}(\xi^{\overline{\alpha}}_i)\rangle\langle J_{\overline{\alpha}}(\xi^{\overline{\alpha}}_j), \eta^{\beta}_l\rangle\\
&= \frac{\delta_{\alpha,\beta}}{dim_q(\alpha)}\langle\eta, J_{\overline{\alpha}}(\xi)\rangle\langle J_{\overline{\alpha}}(\xi'), \eta'\rangle,
\end{aligned}$$

where in (1) we use Remark 1.3.1.25. Using again Remark 1.3.1.25 we can apply the preceding computations to the elements $\psi_x(\xi\otimes\eta)^*$ and $\psi_y(\xi'\otimes\eta')$ for any $\xi\otimes\eta\in H_x\otimes H_{\overline{x}}$ and $\xi'\otimes\eta'\in H_y\otimes H_{\overline{y}}$ and we obtain

$$h_{\mathbb{G}}\big(\psi_x(\xi\otimes\eta)^*\psi_y(\xi'\otimes\eta')\big) = \frac{\delta_{x,y}}{dim_q(x)}\langle\xi\otimes\eta, \xi'\otimes\eta'\rangle,$$

which justifies the last equality in formula (1.3.2).

In order to get the decomposition of the statement, it is enough to consider the following map

$$\begin{array}{rccl}
\Psi: & \bigoplus_{x\in Irr(\mathbb{G})}(H_x\otimes H_{\overline{x}}) & \longrightarrow & L^2(\mathbb{G})\\
& \big(\zeta_x\big)_{x\in Irr(\mathbb{G})} & \longmapsto & \Psi\big(\zeta_x\big)_{x\in Irr(\mathbb{G})} := \Big(\sqrt{dim_q(x)}\ \Psi_x(\zeta_x)\Big)_{x\in Irr(\mathbb{G})},
\end{array}$$

which defines an isometric isomorphism of Hilbert spaces. Through this identification, the formula of the statement is easy to establish. Indeed,

$$\lambda\big((\omega_{\xi,\xi'}\otimes id)(w^x)\big)(\Omega) = \lambda(\psi_x(\xi\otimes J_x(\xi')))\Omega \cong \frac{1}{\sqrt{dim_q(x)}}\ \xi\otimes J_x(\xi'),$$

for all $\xi,\xi'\in H_x$. ■

1.3.1.31 Remarks. 1. Notice that the operator Q_x is an invertible positive self-adjoint operator. In that case, we can choose an orthonormal basis $\{\xi_1^x, \dots, \xi_{n_x}^x\}$ of H_x that diagonalises Q_x. Let $\{\omega_1^x, \dots, \omega_{n_x}^x\}$ be its dual basis in the dual space $\overline{H}_x$.

For every $j = 1, \dots, n_x$, let $\lambda_j^x \in \mathbb{R}^+$ be the eigenvalue of Q_x associated to the vector ξ_j^x meaning that

$$Q_x(\xi_j^x) = \lambda_j^x \ \xi_j^x$$

Recall from Theorem 1.3.1.24 that $Q_x = J_x^* J_x$ and $J_{\overline{x}} = J_x^{-1}$, so that the singular values of J_x and $J_{\overline{x}}$ are $\{\sqrt{\lambda_j^{\overline{x}}}\}_{j=1,\dots,n_x}$ and $\{\frac{1}{\sqrt{\lambda_j^{\overline{x}}}}\}_{j=1,\dots,n_x}$, respectively which means that

$$J_x(\xi_j^x) = \sqrt{\lambda_j^x} \ \xi_j^{\overline{x}} \text{ and } J_{\overline{x}}(\xi_j^{\overline{x}}) = \frac{1}{\sqrt{\lambda_j^{\overline{x}}}} \ \xi_j^x,$$

where $\{\xi_1^{\overline{x}}, \dots, \xi_{n_{\overline{x}}}^{\overline{x}}\}$ is an orthonormal basis of $H_{\overline{x}}$.

The well-known Riesz-Fréchet identification $\overline{H}_x \cong H_x^*$ yields that any continuous linear functional $\omega \in H_x^*$ is of the form $\langle \cdot, \xi \rangle$, for some $\overline{\xi} \in \overline{H}_x$ (recall Section 1.1). Hence the dual Hilbert space $\overline{H}_x$ is identified to $H_{\overline{x}}$ by means of the map

$$\begin{array}{ccc} \overline{H}_x & \longrightarrow & H_{\overline{x}} \\ \omega = \langle \cdot, \xi \rangle & \longmapsto & J_x(\xi), \end{array}$$

which is an isomorphism because J_x is a bijective anti-linear map. In coordinates, this identification yields

$$\omega_j^x \mapsto \sqrt{\lambda_j^x} \ \xi_j^{\overline{x}},$$

for all $j = 1, \dots, n_x$. By duality, we have as well $\overline{H}_{\overline{x}} \cong H_x$ via $J_{\overline{x}}$ so that if $\{\omega_1^{\overline{x}}, \dots, \omega_{n_{\overline{x}}}^{\overline{x}}\}$ denotes the dual basis of $\{\xi_1^{\overline{x}}, \dots, \xi_{n_{\overline{x}}}^{\overline{x}}\}$ in $\overline{H}_{\overline{x}}$, then the above identification yields

$$\omega_j^{\overline{x}} \mapsto \frac{1}{\sqrt{\lambda_j^{\overline{x}}}} \ \xi_j^x,$$

for all $j = 1, \dots, n_x$.

Since $\{w_{i,j}^x\}_{\substack{x \in Irr(\mathbb{G}) \\ i,j=1,\dots,n_x}}$ is a basis of $Pol(\mathbb{G})$, it is interesting to have a formula for $\lambda(w_{i,j}^x)(\Omega)$. Fix a diagonalisation basis of Q_x as above, then the formula obtained in Proposition 1.3.1.30 yields to the following one for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, n_x$

$$\lambda(w_{i,j}^x)(\Omega) = \frac{1}{\sqrt{dim_q(x)}} \ \xi_i^x \otimes J_x(\xi_j^x) = \frac{\sqrt{\lambda_j^{\overline{x}}}}{\sqrt{dim_q(x)}} \ \xi_i^x \otimes \xi_j^{\overline{x}}$$

2. The non-trivial invariant vectors $E_x \in H_x \otimes H_{\overline{x}}$, $E_{\overline{x}} \in H_{\overline{x}} \otimes H_x$ of Theorem 1.3.1.24 can be regarded as intertwiner operators, $\Phi_x \in Mor(\epsilon, x \oplus \overline{x})$ and $\Phi_{\overline{x}} \in Mor(\epsilon, \overline{x} \oplus x)$ defined by $\Phi_x(1) = E_x$ and $\Phi_{\overline{x}}(1) = E_{\overline{x}}$ and they are called *canonical intertwiners.*

   From now on we we don't distinguish the non-invariant vectors from the corresponding intertwiner operators and we choose the notation Φ_x and $\Phi_{\overline{x}}$ for them. In this way, a straightforward computation yields the following coordinate expressions for these vectors

$$\Phi_x = \sum_{k=1}^{n_x} \sqrt{\lambda_k^x} \ \xi_k^x \otimes \xi_k^{\overline{x}} \text{ and } \Phi_{\overline{x}} = \sum_{k=1}^{n_x} \frac{1}{\sqrt{\lambda_k^x}} \ \xi_k^{\overline{x}} \otimes \xi_k^x,$$

where we use the same notations as above. A family of basis $\{\xi_1^x, \ldots, \xi_{n_x}^x\}$, $\{\xi_1^{\overline{x}}, \ldots, \xi_{n_x}^{\overline{x}}\}$ satisfying the formulae above is called *canonical orthonormal basis.*

3. Notice that the cyclic vector $\Omega = \Lambda(1_{C(\mathbb{G})})$ of the GNS construction of $h_{\mathbb{G}}$ can be identified, via the canonical isomorphism of the preceding proposition, to the canonical unit vector in $H_\epsilon \otimes H_{\overline{\epsilon}} = \mathbb{C}$.

Given a compact quantum group $\mathbb{G}$ we can construct two new compact quantum groups: the reduced and the universal/maximal picture of $\mathbb{G}$. Both pictures have the same representation theory as $\mathbb{G}$. In this sense, both pictures represent the same object under the representation theory point of view. More precisely, we have the following theorem.

1.3.1.32 Theorem. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group.*

i) The C^*-enveloping algebra of $Pol(\mathbb{G})$ is a compact quantum group denoted by $\mathbb{G}_m = (C_m(\mathbb{G}), \Delta_m)$ and called maximal picture of $\mathbb{G}$*, where Δ_m is the extension of $\Delta_| : Pol(\mathbb{G}) \longrightarrow C_m(\mathbb{G}) \otimes C_m(\mathbb{G})$ to $C_m(\mathbb{G})$. Moreover, we have that $Pol(\mathbb{G}_m) \cong Pol(\mathbb{G})$. The co-unit $\varepsilon : Pol(\mathbb{G}) \longrightarrow \mathbb{C}$ extends to a character on $C_m(\mathbb{G})$ which is still denoted by ε and satisfies the relation*

$$(\varepsilon \otimes id) \circ \Delta_m = id = (id \otimes \varepsilon) \circ \Delta_m$$

If $\mathbb{H} = (C(\mathbb{H}), \Theta)$ is another compact quantum group and $f : Pol(\mathbb{H}) \longrightarrow C(\mathbb{G})$ is a $*$-homomorphism intertwining the co-multiplications, then it extends to a Hopf $*$-homomorphism $f_m : \mathbb{G} \longrightarrow \mathbb{H}_m$.

ii) The C^*-algebra $\lambda(C(\mathbb{G})) \subset \mathcal{B}(L^2(\mathbb{G}))$ is a compact quantum group denoted by $\mathbb{G}_r = (C_r(\mathbb{G}), \Delta_r)$ and called reduced picture of $\mathbb{G}$*, where Δ_r is the extension of $\Delta_| : Pol(\mathbb{G}) \longrightarrow C_r(\mathbb{G}) \otimes C_r(\mathbb{G})$ to $C_r(\mathbb{G})$. Moreover, we have that $Pol(\mathbb{G}_r) \cong Pol(\mathbb{G})$. The Haar state h_r on $C_r(\mathbb{G})$ is faithful.*

Besides, since any Hopf $*$-automorphism $\tau : C(\mathbb{G}) \longrightarrow C(\mathbb{G})$ preserves the Haar state, then its restriction $\tau_| : Pol(\mathbb{G}) \longrightarrow Pol(\mathbb{G})$ extends to a Hopf $*$-automorphism $\tau_r : C_r(\mathbb{G}) \longrightarrow C_r(\mathbb{G})$.

iii) If ι_m, ι, ι_r denote the canonical embedding of $Pol(\mathbb{G})$ in $C_m(\mathbb{G})$, $C(\mathbb{G})$ and $C_r(\mathbb{G})$, respectively; then we have the following commutative diagram

$$\begin{tikzcd}
 & Pol(\mathbb{G}) \arrow[dl, hook, "\iota_m"'] \arrow[d, hook, "\iota"] \arrow[dr, hook, "\iota_r"] & \\
C_m(\mathbb{G}) \arrow[r, two heads, "\tau_m"'] & C(\mathbb{G}) \arrow[r, two heads, "\lambda"'] & C_r(\mathbb{G})
\end{tikzcd}$$

where the horizontal maps are surjective homomorphisms of compact quantum groups.

Proof. i) First of all, observe that $Pol(\mathbb{G})$ is spanned by the matrix coefficients of *unitary* matrices over $Pol(\mathbb{G})$. These are actually unitary representations of $Pol(\mathbb{G})$, so that for all $x \in Irr(\mathbb{G})$, we have $||w_{i,j}^x|| \leqslant 1$, for all $i, j = 1, \ldots, n_x$ where the norm is taken with respect to the sesquilinear form $\langle a, b \rangle := h_{\mathbb{G}}(a^*b)$, for all $a, b \in Pol(\mathbb{G})$. Thus it is licit to consider the enveloping algebra of $Pol(\mathbb{G})$, say $C_m(\mathbb{G})$. Next, the co-multiplication $\Delta_| : Pol(\mathbb{G}) \longrightarrow C_m(\mathbb{G}) \otimes C_m(\mathbb{G})$ extends to a unital $*$-homomorphism $\Delta_m : C_m(\mathbb{G}) \longrightarrow C_m(\mathbb{G}) \otimes C_m(\mathbb{G})$ by universality. By construction

it is clear that $\mathbb{G}_m := (C_m(\mathbb{G}), \Delta_m)$ is a compact quantum group such that $Pol(\mathbb{G}_m) \cong Pol(\mathbb{G})$. Likewise, the co-unit map $\varepsilon : Pol(\mathbb{G}) \longrightarrow \mathbb{C}$ extends to a character on $C_m(\mathbb{G})$ by universality. The relation satisfied by the character ε is true thanks to the co-unit axiom and the density of $Pol(\mathbb{G})$ in $C_m(\mathbb{G})$.

ii) If $(L^2(\mathbb{G}), \lambda, \Omega)$ denotes the GNS construction for the Haar state $h_{\mathbb{G}}$, then we put $C_r(\mathbb{G}) := \lambda(C(\mathbb{G})) \subset \mathcal{B}(L^2(\mathbb{G}))$. For all $a \in Pol(\mathbb{G})\backslash\{0\}$ we have

$$0 < h(a^*a) = ||\lambda(a)\Omega||^2 \leqslant ||\lambda(a)||^2$$

because $h_{\mathbb{G}|_{Pol(\mathbb{G})}}$ is faithful and so we obtain that λ is faithful on $Pol(\mathbb{G})$. This means that we have a natural embedding $\iota_r : Pol(\mathbb{G}) \hookrightarrow C_r(\mathbb{G})$ and that $C_r(\mathbb{G})$ is the operator completion of $Pol(\mathbb{G})$ in $\mathcal{B}(L^2(\mathbb{G}))$.

Now, we have to show that the co-multiplication $\Delta_| : Pol(\mathbb{G}) \longrightarrow C_r(\mathbb{G}) \otimes C_r(\mathbb{G})$ extends to a co-multiplication on $C_r(\mathbb{G})$. For this, consider the left regular representation $W \in M(\mathcal{K}(L^2(\mathbb{G})) \otimes C(\mathbb{G}))$ and put $W_r := (id \otimes \lambda)(W) \in \mathcal{B}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$. Then by the theorem 1.3.1.26 we have

$$\Delta_r(a) = W_r^*(1 \otimes a)W_r,$$

for all $a \in C_r(\mathbb{G})$ where $\Delta_r := (id \otimes \lambda)\Delta$.

By construction, $\mathbb{G}_r := (C_r(\mathbb{G}), \Delta_r)$ is a compact quantum group such that $Pol(\mathbb{G}_r) \cong Pol(\mathbb{G})$ with Haar state defined by $h_r(x) = \langle \Omega, x \rangle$, for all $x \in C_r(\mathbb{G})$. We can show that h_r is faithful, see Corollary 1.7.5 of [139] or Theorem 5.4.5 of [188] for a proof.

iii) It is straightforward.

■

1.3.1.33 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group.

- We say that $\mathbb{G}$ is a *maximal* compact quantum group if τ_m is an isomorphism.
- We say that $\mathbb{G}$ is a *reduced* compact quantum group if τ_r is an isomorphism.

Recall Note 1.3.1.4. The main problem with the early Definition 1.3.1.3 is that it is not well-behaved, in general, with the duality between compact and discrete quantum groups (see Theorem 1.3.1.36 below). A homomorphism between locally compact groups always induces a homomorphism between the corresponding *maximal* C^*-algebras. But this is not the case for the reduced ones. For instance, let G be a *classical* locally compact group and consider the constant map from G to the trivial group $\{e\}$, $G \longrightarrow \{e\}$. This induces a map between the corresponding *reduced* C^*-algebras, $C_r^*(G) \longrightarrow C_r^*(\{e\}) \cong \mathbb{C}$, if and only if G is amenable. Recall from the introduction to Section 1.3 that $C_r^*(G)$ must be regarded as the generalized Pontryagin dual of G. We refer to [48] and [135] for more details.

For this reason, Definition 1.3.1.3 must be refined as follows in order to give an appropriated notion of morphism of compact quantum groups.

1.3.1.34 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ and $\mathbb{G}' = (C(\mathbb{G}'), \Delta')$ be two compact quantum groups. A quantum homomorphism from $\mathbb{G}$ to $\mathbb{G}'$, denoted by $f : \mathbb{G} \longrightarrow \mathbb{G}'$, is a unital $*$-homomorphism $f_m : C_m(\mathbb{G}') \longrightarrow C_m(\mathbb{G})$ such that

$$\Delta_m \circ f_m = (f_m \otimes f_m)\Delta'_m$$

In particular, if $\mathbb{G} = \mathbb{G}'$, such a f is called quantum automorphism. The group of all quantum automorphism of $\mathbb{G}$ is denoted by $Aut(\mathbb{G})$.

1.3.1.35 Remark. Notice that, by Theorem 1.3.1.32, a unital $*$-homomorphism $f_m : C_m(\mathbb{G}') \longrightarrow C_m(\mathbb{G})$ such that $\Delta_m \circ f_m = (f_m \otimes f_m)\Delta'_m$ is equivalent to a unital $*$-homomorphism $f : Pol(\mathbb{G}') \longrightarrow Pol(\mathbb{G})$ such that $\Delta_| \circ f = (f \otimes f)\Delta'_|$.

Finally, given a compact quantum group $\mathbb{G}$ we can define its quantum Pontryagin dual $\widehat{\mathbb{G}}$. It is important to notice that such a dual is *not longer* a compact quantum group in the sense of Woronowicz, but a *locally compact quantum group* in the sense of Kustermans-Vaes (we refer to Section 1.3.2 for the main definitions and results). This object can be described in several different ways.

First of all, recall that the main reason to develop the theory of quantum groups was to give a general framework in which the Pontryagin duality of locally compact abelian groups holds in full generality. Hence, the theory developed by J. Kustermans and S. Vaes in [113] for *locally compact quantum groups* seems to be the right context to understand the duality between compact and discrete quantum groups. However, the Kustermans-Vaes's theory is quite technical and too much powerful for our objectives. A different approach to locally compact quantum groups is the theory developed by S. Baaj and G. Skandalis in [7] using *multiplicative unitaries*. In this case, any ("nice" enough) multiplicative unitary gives rise to two quantum groups in duality. Recall that any locally compact quantum group has a fundamental unitary and so we can define its dual using the corresponding right/left leg. In particular, if $\mathbb{G}$ is a compact quantum group, the left regular representation $W_{\mathbb{G}}$ of $\mathbb{G}$ on $L^2(\mathbb{G})$ (Theorem 1.3.1.26) is its fundamental unitary and we could define $\widehat{\mathbb{G}}$ to be left leg of W and apply the Baaj-Skandalis theory to conclude the corresponding duality.

Nevertheless, the context of compact quantum groups of Woronowicz is quite explicit thanks to its rich representation theory as we have just presented. Therefore, such a Pontryagin dual can be built in a more precise way avoiding these technical and general theories. Namely, we have the following result.

1.3.1.36 Theorem-Definition. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and define the following C^*-algebra*

$$c_0(\widehat{\mathbb{G}}) := \bigoplus_{x \in Irr(\mathbb{G})}^{c_0} \mathcal{B}(H_x)$$

We have that

i) $M(c_0(\widehat{\mathbb{G}})) = \bigoplus_{x \in Irr(\mathbb{G})}^{l^\infty} \mathcal{B}(H_x)$ *and we denote it by* $l^\infty(\widehat{\mathbb{G}})$,

ii) the element $\mathscr{V} := \bigoplus_{x \in Irr(\mathbb{G})} w^x$ *is a unitary element in* $M(c_0(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}))$ *satisfying the formula*

$$(id \otimes \Delta)\mathscr{V} = \mathscr{V}_{12}\mathscr{V}_{13},$$

which means that $\mathscr{V}$ *is a (infinite dimensional) representation of* $\mathbb{G}$,

iii) there exists a $*$-homomorphism $\widehat{\Delta} : c_0(\widehat{\mathbb{G}}) \longrightarrow M(c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{G}}))$ *such that given* $x \in Irr(\mathbb{G})$ *and* $T \in \mathcal{B}(H_x)$ *we have*

$$\widehat{\Delta}(T) \circ \Phi = \Phi \circ T,$$

for every $y, z \in Irr(\mathbb{G})$ *and every* $\Phi \in Mor(x, y \oplus z)$,

iv) the pair $(c_0(\widehat{\mathbb{G}}), \widehat{\Delta})$ *is a reduced* C^**-locally compact quantum group whose left and right Haar weights are given respectively by*

$$\widehat{h}_L(a) = \sum_{x \in Irr(\mathbb{G})} dim_q(x)\ Tr\big(Q_x\big(ap_x\big)\big)$$

$$\widehat{h}_R(a) = \sum_{x \in Irr(\mathbb{G})} dim_q(x)\ Tr\big(Q_x^{-1}\big(ap_x\big)\big),$$

for all $a \in c_0(\widehat{\mathbb{G}})$ *where* p_x *with* $x \in Irr(\mathbb{G})$ *denotes the minimal central projection of* $c_0(\widehat{\mathbb{G}})$ *on* $\mathcal{B}(H_x)$*,*

v) the co-multiplication $\widehat{\Delta}$ *extends to a normal map* $\widehat{\Delta} : l^\infty(\widehat{\mathbb{G}}) \longrightarrow l^\infty(\widehat{\mathbb{G}}) \otimes l^\infty(\widehat{\mathbb{G}})$*, so that* $(l^\infty(\widehat{\mathbb{G}}), \widehat{\Delta}, \widehat{h}_L, \widehat{h}_R)$ *is a vN-locally compact quantum group,*

The locally compact quantum group $\widehat{\mathbb{G}} = (c_0(\widehat{\mathbb{G}}), \widehat{\Delta}, \widehat{h}_L, \widehat{h}_R)$ *is called the* (quantum Pontryagin) dual of $\mathbb{G}$*. In general, a* discrete quantum group *is the dual of a compact quantum group.*

1.3.1.37 Remark. By virtue of the preceding theorem, a discrete quantum group $\widehat{\mathbb{G}}$ is completely defined by the corresponding compact quantum group $\mathbb{G}$ (and *vice versa*). More precisely, given a compact quantum group $\mathbb{G}$, the corresponding $*$-Hopf algebra $Pol(\mathbb{G})$ yields the definition of $\widehat{\mathbb{G}}$. Conversely, given a discrete quantum group $\widehat{\mathbb{G}}$, we can recover, by definition, the $*$-Hopf algebra $Pol(\mathbb{G})$. Hence, under the representation theory point of view, we recover the object $\mathbb{G}$ itself. But notice that the C^*-norm on $Pol(\mathbb{G})$ is not unique as it has been pointed out in Theorem 1.3.1.32.

1.3.1.38 Remarks. For a precise treatment of discrete quantum groups with respect to the above picture, we refer to [231], [232], [124] and Section 1 in [197]. Let us give some general remarks.

1. If $(L^2(\mathbb{G}), \lambda, \Omega)$ is the GNS construction for the Haar state $h_{\mathbb{G}}$, then the GNS space representation of the left Haar weight $\widehat{h}_L$ can be identified to $L^2(\mathbb{G})$. Actually, we can give a precise description of the corresponding GNS construction, denoted by $(L^2(\mathbb{G}), \widehat{\Lambda}_L, \widehat{\lambda})$ (see Section 1 in [197] for more details). For the present dissertation we only need the description of the corresponding GNS-representation $\widehat{\lambda} : c_0(\widehat{\mathbb{G}}) \longrightarrow \mathcal{B}(L^2(\mathbb{G}))$, which satisfies the following formula
$$\widehat{\lambda}(a)\zeta_x = (ap_x \otimes id_{H_{\overline{x}}})\zeta_x,$$
for all $x \in Irr(\mathbb{G})$, all $a \in c_0(\widehat{\mathbb{G}})$ and all $\zeta_x \in H_x \otimes H_{\overline{x}}$ (recall Proposition 1.3.1.30). In particular, we have
$$\widehat{\lambda}(p_\epsilon)\lambda(w^x_{i,j})\Omega = h_{\mathbb{G}}(w^x_{i,j})\Omega,$$
for all $a \in c_0(\widehat{G})$, $x \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$, where ϵ denotes the trivial representation.

2. As in the compact case, we can show that there exists a unique unitary operator $W_{\widehat{\mathbb{G}}} \in \mathcal{B}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ such that
$$W^*_{\widehat{\mathbb{G}}}(\widehat{\Lambda}_L(a) \otimes \widehat{\Lambda}_L(b)) = (\widehat{\Lambda}_L \otimes \widehat{\Lambda}_L)\widehat{\Delta}(b)(a \otimes 1),$$
for all $a, b \in \mathscr{N}_{\widehat{h}_L}$. Moreover,

- $W_{\widehat{\mathbb{G}}} \in \mathcal{B}(c_0(\widehat{\mathbb{G}}) \otimes \mathcal{K}(L^2(\mathbb{G})))$ is a unitary representation of $\widehat{\mathbb{G}}$ on $L^2(\mathbb{G})$ called *left regular representation of* $\widehat{\mathbb{G}}$.
- $W_{\widehat{\mathbb{G}}}$ is a multiplicative unitary on $L^2(\mathbb{G})$ in the sense of Baaj-Skandalis called *fundamental unitary of* $\widehat{\mathbb{G}}$.
- $W_{\widehat{\mathbb{G}}} = (id \otimes \lambda)(\mathscr{V})$.

And following the Baaj-Skandalis picture (see Section 1.3.2) we have

$$c_0(\widehat{\mathbb{G}}) \cong A(W_{\mathbb{G}}) = \overline{(\mathcal{B}(L^2(\mathbb{G}))_* \otimes id)(W_{\mathbb{G}})}^{||\cdot||} \text{ and } l^\infty(\widehat{\mathbb{G}}) \cong A_0(W_{\mathbb{G}})''$$

$$\widehat{\Delta}(a) = \widehat{W}_{\mathbb{G}}^*(id \otimes a)\widehat{W}_{\mathbb{G}}, \text{ for all } a \in c_0(\widehat{\mathbb{G}}),$$

where $\widehat{W}_{\mathbb{G}} := W_{\widehat{\mathbb{G}}} = \Sigma W_{\mathbb{G}}^* \Sigma$.

3. Finally, it is important to say that a discrete quantum group is equipped naturally with a co-unit $\widehat{\varepsilon}$, that is, a linear map $\varepsilon_{\widehat{\mathbb{G}}} := \widehat{\varepsilon} : c_0(\widehat{\mathbb{G}}) \longrightarrow \mathbb{C}$ such that

$$(\widehat{\varepsilon} \otimes id)\widehat{\Delta} = id = (id \otimes \widehat{\varepsilon})\widehat{\Delta}$$

Namely, in the picture of the above theorem we have $\widehat{\varepsilon}(x) := p_\epsilon \; x \; p_\epsilon$, for all $x \in c_0(\widehat{\mathbb{G}})$.

1.3.1.39 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group.

- We say that $\mathbb{G}$ is *co-amenable* (or that $\widehat{\mathbb{G}}$ is *amenable*) if $\tau := \tau_r \circ \tau_m$ is an isomorphism.
- We say that $\mathbb{G}$ is *co-K-amenable* (or that $\widehat{\mathbb{G}}$ is *K-amenable*) if $\tau := \tau_r \circ \tau_m$ is a K-equivalence.

1.3.1.40 Remark. The well-known characterization of amenability for classical groups (see [29] for more details) can be translated for compact quantum groups (see Theorem 2.7.10 of [139] for a proof). Namely, if $\mathbb{G}$ is a compact quantum group, then the following assertions are equivalent.

i) $\mathbb{G}$ is co-amenable.

ii) the co-unit map $\varepsilon : Pol(\mathbb{G}) \longrightarrow \mathbb{C}$ extends to a character on $C_r(\mathbb{G})$.

iii) the Haar state of $\mathbb{G}_m$ is faithful.

1.3.1.41 Remark. The well-known characterization of K-amenability of J. Cuntz for classical groups (see [44] for more details) can be translated for compact quantum groups (see Theorem 5.14 of [206] for a proof). Namely, if $\mathbb{G}$ is a compact quantum group, then the following assertions are equivalent.

i) $\mathbb{G}$ is co-K-amenable.

ii) The canonical map $\widetilde{\tau} : \widehat{\mathbb{G}} \underset{m}{\ltimes} A \longrightarrow \widehat{\mathbb{G}} \underset{r}{\ltimes} A$ is a K-equivalence, for every $\widehat{\mathbb{G}}$-C^*-algebra A.

iii) There exists an element $\alpha \in KK(C_r(\mathbb{G}), \mathbb{C})$ such that $[\tau] \underset{C_r(\mathbb{G})}{\otimes} \alpha = [\varepsilon]$, where $\varepsilon : Pol(\mathbb{G}) \longrightarrow \mathbb{C}$ denotes the co-unit of $\mathbb{G}$ whose extension to $C_m(\mathbb{G})$ is still denoted by ε.

1.3.1.42 Remark. Observe that if G is a compact (classical) group, it is automatically amenable and then we have automatically $C^*_m(G) \cong C^*_r(G)$ denoted simply by $C(G)$. In the quantum case, the analogous property is encoded under the notion of *co-amenability* of $\mathbb{G}$ whenever $\mathbb{G}$ is a (locally) compact quantum group.

Thus, a *compact* quantum group $\mathbb{G}$ is always *amenable* in the sense that it admits a left invariant mean (namely, the Haar state of $\mathbb{G}$). But this doesn't imply, in general, the co-amenability, that is, that the canonical surjection $\tau : C_m(\mathbb{G}) \twoheadrightarrow C_r(\mathbb{G})$ is an $*$-isomorphism. Actually, if $\mathbb{G}$ is any *locally compact* quantum group, the connexion between amenability and co-amenability is as follows

$$\widehat{\mathbb{G}} \text{ co-amenable } \Longrightarrow \mathbb{G} \text{ amenable}$$

$$\mathbb{G} \text{ amenable } \overset{?}{\Longrightarrow} \widehat{\mathbb{G}} \text{ co-amenable}$$

The second implication has been proven by R. Tomatsu in [189] for *discrete* quantum groups.

1.3.2 Locally compact case

The goal of this section is just to present the main definitions and results of the locally compact groups theory in order to have an overlook of the general theory with respect to the Woronowicz's, and so the compact's, one. Therefore, we don't give any proof in that regard and, for more information and details, we refer to the original paper [113] of J. Kustermans and S. Vaes; also Chapter 8 of [188] provides a very useful reference to get introduced into this subject. In order to understand the following presentation, it is advisable to keep in mind the elementary notions about multiplicative unitaries of Baaj-Skandalis recalled in Section 1.1 and the elementary notions about von Neumann algebras recalled in Section A.2.

1.3.2.1 Definition. A vN-locally compact quantum group $\mathbb{G}$ is the data (M, Δ, ϕ, ψ) where M is a von Neumann algebra, $\Delta : M \longrightarrow M \otimes M$ is a normal unital $*$-homomorphism called *co-multiplication* and ϕ, ψ are normal semi-finite faithful weights on M called *left Haar weight* and *right Haar weight*, respectively; such that

i) Δ is co-associative meaning that the diagram

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\Delta"] & M \otimes M \arrow[d, "id\otimes\Delta"] \\ M \otimes M \arrow[r, "\Delta\otimes id"] & M \otimes M \otimes M \end{tikzcd}$$

is commutative.

ii) ϕ is left invariant with respect to Δ meaning that

$$\phi\big((\omega \otimes id)\Delta(a)\big) = \omega(1_M)\phi(a),$$

for all $a \in M^+$ and all $\omega \in M^+_*$.

iii) ψ is right invariant with respect to Δ meaning that

$$\psi\big((id \otimes \omega)\Delta(a)\big) = \omega(1_M)\psi(a),$$

for all $a \in M^+$ and all $\omega \in M^+_*$.

1.3.2.2 Definition. A reduced C^*-locally compact quantum group $\mathbb{G}$ is the data (A, Δ, ϕ, ψ) where (A, Δ) is Hopf C^*-algebra and ϕ, ψ are faithful KMS-weights on A called *left Haar weight* and *right Haar weight*, respectively; such that

i) the sets
$$\{(\omega \otimes id)\Delta(a) \mid \omega \in A_+^*, a \in A\} \text{ and } \{(id \otimes \omega)\Delta(a) \mid \omega \in A_+^*, a \in A\}$$
are linearly dense in A.

ii) ϕ is left invariant with respect to Δ meaning that
$$\phi\big((\omega \otimes id)\Delta(a)\big) = \omega(1)\phi(a),$$
for all $a \in A_+$ and all $\omega \in A_+^*$.

iii) ψ is right invariant with respect to Δ meaning that
$$\psi\big((id \otimes \omega)\Delta(a)\big) = \omega(1)\psi(a),$$
for all $a \in A_+$ and all $\omega \in A_+^*$.

1.3.2.3 Theorem. *Let $\mathbb{G} = (A, \Delta, \phi, \psi)$ be a reduced C^*-locally compact quantum group. If $(L^2(\mathbb{G}), \Lambda_\phi, \pi_\phi)$ is the GNS construction of the left Haar weight ϕ, then*

i) there exists a multiplicative unitary $W \in \mathcal{B}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ on $L^2(\mathbb{G})$ such that for all $x, y \in \mathscr{N}_\phi$
$$\Delta(y)(x \otimes 1) \in \mathscr{N}_{\phi\otimes\phi} \text{ and } W^*(\Lambda_\phi(x) \otimes \Lambda_\phi(y)) = (\Lambda_\phi \otimes \Lambda_\phi)\big(\Delta(y)(x \otimes 1)\big)$$
The operator $W =: W_{\mathbb{G}}$ is called fundamental unitary of $\mathbb{G}$*;*

ii) the fundamental unitary of $\mathbb{G}$ is well-behaved. As a consequence, the left and right legs of W are Hopf C^*-algebras. Moreover, the explicit description of the left leg is given by
$$\widehat{A}(W_{\mathbb{G}}) \cong \overline{\pi_\phi(A)}^{||\cdot||} \text{ and } \widehat{\Delta}_{W_{\mathbb{G}}} \circ \pi_\phi = (\pi_\phi \otimes \pi_\phi) \circ \Delta$$

1.3.2.4 Note. If W is a multiplicative unitary on a Hilbert space H, then its left and right legs, $(\widehat{A}(W), \widehat{\Delta}_W)$ and $(A(W), \Delta_W)$, are always subalgebras of $\mathcal{B}(H)$ but they are not necessarily C^*-algebras. We say that W is well-behaved if its left and right legs are Hopf C^*-algebras and $W \in M(\widehat{A}(W) \otimes A(W))$.

It is advisable to have general criteria that assure the well-behaved condition for a given multiplicative unitary. There are several of such criteria. For instance, *regularity* introduced by S. Baaj and G. Skandalis in [7] or *manageability* introduced by S. L. Woronowicz in [230].

We refer to Chapter 7 of [188] for more details about these notions.

1.3.2.5 Definition. Let $\mathbb{G} = (A, \Delta, \phi, \psi)$ be a reduced C^*-locally compact quantum group.

- The *algebra of continuous functions vanishing at ∞ of* $\mathbb{G}$ is by definition the left leg of the fundamental unitary of $\mathbb{G}$,
$$C_0(\mathbb{G}) := \widehat{A}(W_{\mathbb{G}}) \subset \mathcal{B}(L^2(\mathbb{G}))$$

- The *reduced algebra of* $\mathbb{G}$ is by definition the right leg of the fundamental unitary of $\mathbb{G}$,

$$C_r^*(\mathbb{G}) := A(W_{\mathbb{G}}) \subset \mathcal{B}(L^2(\mathbb{G}))$$

1.3.2.6 Remark. Since the fundamental unitary of $\mathbb{G}$ is well-behaved, its legs are Hopf C^*-algebras, that is, $C_0(\mathbb{G})$, $C_r^*(\mathbb{G})$ are Hopf C^*-algebras and we have $W \in M(C_0(\mathbb{G}) \otimes C_r^*(\mathbb{G}))$. The corresponding co-multiplications are given precisely by

$$\Delta_{\mathbb{G}}(x) := \widehat{\Delta}_W(x) = W^*(id \otimes x)W, \text{ for all } x \in C_0(\mathbb{G})$$

$$\widehat{\Delta}_{\mathbb{G}}(x) := \Delta_W(x) = W^{op^*}(id \otimes x)W^{op}, \text{ for all } x \in C_r^*(\mathbb{G}),$$

where we recall that $W^{op} := \widehat{W} := \Sigma W^* \Sigma$.

1.3.2.7 Theorem (Quantum Pontryagin duality)**.** *Let $\mathbb{G} = (A, \Delta, \phi, \psi)$ be a reduced C^*-locally compact quantum group.*

i) The algebra of continuous functions vanishing at ∞ of $\mathbb{G}$ is a reduced C^*-locally compact quantum group such that $\mathbb{G} = (C_0(\mathbb{G}), \Delta_{\mathbb{G}})$. The fundamental unitary of $\mathbb{G}$ is given by $W_{\mathbb{G}}$.

ii) The reduced algebra of $\mathbb{G}$ is a reduced C^*-locally compact quantum group called quantum dual of $\mathbb{G}$ *and we put $\widehat{\mathbb{G}} := (C_r^*(\mathbb{G}), \widehat{\Delta}_{\mathbb{G}})$. The fundamental unitary of $\widehat{\mathbb{G}}$ is given by $W_{\widehat{\mathbb{G}}} = \Sigma \circ W_{\mathbb{G}}^* \circ \Sigma$, which is denoted by $\widehat{W}_{\mathbb{G}}$.*

iii) There exists a canonical isomorphism of reduced C^*-locally compact quantum groups $\widehat{\widehat{\mathbb{G}}} \cong \mathbb{G}$ called quantum Pontryagin duality*.*

1.3.2.8 Note. By construction, both $C_0(\mathbb{G})$ and $C_r^*(\mathbb{G})$ describe the *reduced* pictures of $\mathbb{G}$ and $\widehat{\mathbb{G}}$, respectively (because these C^*-algebras are constructed by means of the GNS constructions of the Haar weights, that is, both $\mathbb{G}$ and $\widehat{\mathbb{G}}$ are represented on the GNS-Hilbert space representation of their Haar weights). In accordance with Theorem 1.3.1.32, it is possible to show that both $\mathbb{G}$ and $\widehat{\mathbb{G}}$ admit *universal pictures*, denoted by $\mathbb{G}_m := (C_0^m(\mathbb{G}), \Delta_m)$ and $\widehat{\mathbb{G}}_m := (C_m^*(\mathbb{G}), \widehat{\Delta}_m)$, respectively. Precisely, $C_0^m(\mathbb{G})$ and $C_m^*(\mathbb{G})$ are defined as enveloping C^*-algebras of appropriated $*$-subalgebras of $C_r^*(\mathbb{G})^* = A(W_{\mathbb{G}})^*$ and $C_0(\mathbb{G})^* = \widehat{A}(W_{\mathbb{G}})^*$, respectively. As in the compact case, there exist canonical surjective $*$-homomorphisms $\tau_m : C_0^m(\mathbb{G}) \longrightarrow C_0(\mathbb{G})$ and $\widehat{\tau}_m : C_m^*(\mathbb{G}) \longrightarrow C_r^*(\mathbb{G})$, which intertwine the corresponding co-multiplications. Moreover, it is possible to show that there exist unitary elements $\mathcal{W}_{\mathbb{G}} \in M(C_0^m(\mathbb{G}) \otimes C_r^*(\mathbb{G}))$ and $\widehat{\mathcal{W}}_{\mathbb{G}} \in M(C_0(\mathbb{G}) \otimes C_m^*(\mathbb{G}))$ such that

$$(id \otimes \widehat{\Delta}_{\mathbb{G}})(\mathcal{W}_{\mathbb{G}}) = \big(\mathcal{W}_{\mathbb{G}}\big)_{12}\big(\mathcal{W}_{\mathbb{G}}\big)_{13} \text{ and } (\Delta_{\mathbb{G}} \otimes id)(\widehat{\mathcal{W}}_{\mathbb{G}}) = \big(\widehat{\mathcal{W}}_{\mathbb{G}}\big)_{13}\big(\mathcal{W}_{\mathbb{G}}\big)_{23}$$

These results are technically involved and we refer to [111] for the full details. In particular, if $\mathbb{G}$ is a *compact* quantum group, its discrete dual $\widehat{\mathbb{G}}$ defined in Theorem 1.3.1.36, which is a reduced picture of $\widehat{\mathbb{G}}$, admits a universal picture $\widehat{\mathbb{G}}_m := (c_0^m(\widehat{\mathbb{G}}), \widehat{\Delta})$. But, by Remark 1.3.1.42, $\widehat{\mathbb{G}}$ is always co-amenable so that $c_0^m(\widehat{\mathbb{G}})$ is always canonically isomorphic to $c_0(\widehat{\mathbb{G}})$. Moreover, there exists a unitary element $\mathcal{W}_{\mathbb{G}} \in M(C_m(\mathbb{G}) \otimes c_0(\widehat{\mathbb{G}}))$ such that

$$(id \otimes \widehat{\Delta}_{\mathbb{G}})(\mathcal{W}_{\mathbb{G}}) = \big(\mathcal{W}_{\mathbb{G}}\big)_{12}\big(\mathcal{W}_{\mathbb{G}}\big)_{13}$$

In accordance with [111], the unitary $\mathcal{W}_{\mathbb{G}}$ allows to produce unitary representations of $\widehat{\mathbb{G}}$. Hence it must be regarded as a *universal representation of* $\mathbb{G}$. The connection with the fundamental unitary of $\mathbb{G}$ introduced in Theorem 1.3.1.26 is given by the formula $(\tau_m \otimes id)\mathcal{W}_{\mathbb{G}} = W_{\mathbb{G}}$.

1.3.2.9 Remark. In the formalism of multiplicative unitaries, the quantum Pontryagin duality can be easily established. Namely, it is enough to check that the multiplicative unitary associated to $\mathbb{G}$ is the same to the one associated to $\widehat{\widehat{\mathbb{G}}}$, which is straightforward,

$$W_{\widehat{\widehat{\mathbb{G}}}} = \Sigma \circ W^*_{\widehat{\mathbb{G}}} \circ \Sigma = \Sigma \circ (\Sigma \circ (W^*_{\mathbb{G}})^* \circ \Sigma) \circ \Sigma = W_{\mathbb{G}}$$

1.3.2.10 Remark. It is possible to pass from the picture of Definition 1.3.2.1 to the picture of Definition 1.3.2.2 by taking the bicommutant of the leg defining the locally compact quantum group. We can prove that there exists a bijective correspondence (up to isomorphism) between all vN-locally compact quantum groups and all reduced C^*-locally compact quantum groups.

This correspondence is realized as follows. Given a C^*-locally compact quantum group $\mathbb{G} = (C_0(\mathbb{G}), \Delta_{\mathbb{G}})$ and its dual $\widehat{\mathbb{G}} := (C^*_r(\mathbb{G}), \widehat{\Delta}_{\mathbb{G}})$, we put

$$L^\infty(\mathbb{G}) := \overline{\widehat{A}_0(W_{\mathbb{G}})}^{\sigma-w} \supset C_0(\mathbb{G})$$

$$L^\infty(\widehat{\mathbb{G}}) := \mathcal{L}(\mathbb{G}) := \overline{A_0(W_{\mathbb{G}})}^{\sigma-w} \supset C^*_r(\mathbb{G})$$

Hence, $(L^\infty(\mathbb{G}), \Delta_{\mathbb{G}})$ and $(\mathcal{L}(\mathbb{G}), \widehat{\Delta}_{\mathbb{G}})$ are vN-locally compact quantum groups. See Section 8.3.4 in [188] for more details. In particular we have that

$$L^\infty(\mathbb{G}) = \overline{Pol(\mathbb{G})}^{\sigma-w} = C_r(\mathbb{G})''$$

$$Pol(\widehat{\mathbb{G}}) := A_0(W_{\mathbb{G}}) \Rightarrow l^\infty(\widehat{\mathbb{G}}) := \mathcal{L}(\mathbb{G}) = \overline{Pol(\widehat{\mathbb{G}})}^{\sigma-w} = c_0(\widehat{\mathbb{G}})''$$

whenever $\mathbb{G}$ is a compact quantum group (see Theorem 7.2.14 in [188]).

1.4 Actions of Quantum Groups

Let us introduce the theory of actions of quantum groups by giving one example coming from the classical case, which should be regarded as the reference one.

Let G be a locally compact group. An action of G on a C^*-algebra A is a continuous group homomorphism

$$\begin{array}{rccc} \alpha : & G & \longrightarrow & Aut(A) \\ & g & \longmapsto & \alpha_g \end{array}$$

which means that $\alpha_{gg'} = \alpha_g \circ \alpha_{g'}$ for all $g, g' \in G$ and that the application $G \longrightarrow A$ given by $g \mapsto \alpha_g(a)$ is continuous for all $a \in A$. In this situation, the data (A, α) is called G-C^*-algebra. Recall from Section 1.3 that the data $\mathbb{G} := (C_0(G), \Delta)$ defines a locally compact quantum group.

The notion of G-C^*-algebra can be dualize in the following way. Given the locally compact group G and the C^*-algebra A, we denote by $ev_g : C_0(G) \longrightarrow \mathbb{C}$ the evaluation map on $g \in G$ and by $C_b(G, A)$ the C^*-algebra of bounded continuous functions on G with values on A. This can be identified with a C^*-subalgebra of $M(A \otimes C_0(G))$. More precisely, it is well-known that $A \otimes C_0(G) \cong C_0(G, A)$, so that $M(C_0(G, A)) \cong C_b(G, M(A))$ by virtue of a classical result of Akerman-Pedersen-Tomiyama (see Corollary 3.4 of [2]). Hence, it is not difficult to show that we have $C_b(G, A) \cong \widetilde{M}(A \otimes C_0(G)) \subset M(A \otimes C_0(G))$ (recall Definition A.4.4).

Let us define the following non-degenerate $*$-homomorphism

$$\begin{array}{rccl} \phi_\alpha : & A & \longrightarrow & \widetilde{M}(A \otimes C_0(G)) \\ & a & \longmapsto & \phi_\alpha(a);\ \phi_\alpha(a)(g) := \alpha_g(a), \text{ for all } g \in G \end{array}$$

This map satisfies several important properties (see Theorem 9.2.4 in [188] for a proof),

i) ϕ_α is injectif,

ii) $(\phi_\alpha \otimes id_{C_0(G)}) \circ \phi_\alpha = (id_A \otimes \Delta) \circ \phi_\alpha$,

iii) $[\phi_\alpha(A)(1 \otimes C_0(G))] = A \otimes C_0(G)$.

In this way, the data (A, ϕ_α) with the above properties is a $\mathbb{G}$-C^**-algebra* and the homomorphism ϕ_α is called either a *(right) action of* $\mathbb{G}$ *on* A or a *(left) co-action of* $C_0(G)$ *on* A.

Conversely, if A is a C^*-algebra equipped with a non-degenerate $*$-homomorphism $\phi : A \longrightarrow \widetilde{M}(A \otimes C_0(G))$ satisfying the analogue of the properties (i), (ii) and (iii) above, then we can show (see Theorem 9.2.4 in [188]) that the map

$$\begin{array}{rccl} (\alpha_\phi)_g : & A & \longrightarrow & M(A) \\ & a & \longmapsto & (\alpha_\phi)_g(a) := (id_A \otimes ev_g)(\phi(a)) \end{array}$$

defines an automorphism of A for all $g \in G$ such that the map $G \longrightarrow Aut(A)$, $g \mapsto (\alpha_\phi)_g$, defines an action of G on A. For a fact, in the preceding setting we only need to consider either the property (i) or the property (iii) (see Theorem 9.2.4 in [188] for more details).

In other words, we obtain a bijective correspondence between G-C^*-algebras and $\mathbb{G}$-C^*-algebras, whenever G is a locally compact group and $\mathbb{G} = (C_0(G), \Delta)$ is the corresponding locally compact quantum group.

$$\left\{ \begin{array}{c} \text{(left) } G\text{-}C^*\text{-algebras} \\ G \overset{\alpha}{\curvearrowright} A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{(right) } \mathbb{G}\text{-}C^*\text{-algebras} \\ \mathbb{G} \overset{\phi}{\curvearrowright} A \end{array} \right\}$$

In the present section we formalize the preceding notion of $\widehat{\mathbb{G}}$-C^*-algebra when $\mathbb{G}$ is a compact quantum group. In this case, we'll observe that the standard definition can be translated into a more explicite picture using the representation theory approach for $\widehat{\mathbb{G}}$ presented in Theorem 1.3.1.36. We give also an overview of the well-known spectral theory for actions of compact quantum groups. Finally we discuss the notion of *discrete* quantum subgroup and the corresponding theory of induced actions.

1.4.1 Actions of Discrete Quantum Groups

For more details about actions of quantum groups we refer to the original article [157], the book [188] or the surveys [49], [174].

1.4.1.1 Definition. Let $\mathbb{G}$ be a compact quantum group and A a C^*-algebra. We say that A is a (resp. injective) left $\widehat{\mathbb{G}}$-C^*-algebra if there exists a non-degenerate $*$-homomorphism $\alpha : A \longrightarrow \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A)$ such that

i) (resp. α is injective),

ii) α intertwines the co-multiplication meaning that the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\alpha"] & \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A) \arrow[d, "id_{c_0(\widehat{\mathbb{G}})} \otimes \alpha"] \\ \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A) \arrow[r, "\widehat{\Delta} \otimes id_A"'] & M(c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{G}}) \otimes A) \end{tikzcd}$$

is commutative,

iii) α satisfies the cancellation property meaning that

$$[\alpha(A)(c_0(\widehat{\mathbb{G}}) \otimes 1)] = c_0(\widehat{\mathbb{G}}) \otimes A$$

Such a homomorphism is called a *(resp. injective) left action of* $\widehat{\mathbb{G}}$ *on* A or a *right co-action of* $c_0(\widehat{\mathbb{G}})$ *on* A.

1.4.1.2 Note. A *right* action of $\widehat{\mathbb{G}}$ on A (or a *left* co-action of $c_0(\widehat{\mathbb{G}})$ on A) is a non-degenerate $*$-homomorphism $\alpha : A \longrightarrow \widetilde{M}(A \otimes c_0(\widehat{\mathbb{G}}))$ satisfying the analogue properties of the preceding definition. We must point out that the subalgebra $\widetilde{M}$ used for right actions of $\widehat{\mathbb{G}}$ is lightly different from the one used for left actions $\widehat{\mathbb{G}}$. Indeed, here we have to exchange S and A of Definition A.4.4.

In the present thesis, an action of a discrete quantum group $\widehat{\mathbb{G}}$ is supposed to be an *injective left* one unless the contrary is explicitly indicated. Hence, we refer to such actions simply as *action of* $\widehat{\mathbb{G}}$.

1.4.1.3 Remarks. 1. If $\alpha : A \longrightarrow \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A)$ is an *injective* action of $\widehat{\mathbb{G}}$ on A, then the following formula holds

$$(\widehat{\varepsilon} \otimes id_A) \circ \alpha = id_A,$$

where $\widehat{\varepsilon}$ is co-unit of $\widehat{\mathbb{G}}$. Indeed,

$$\begin{aligned} &(\widehat{\Delta} \otimes id_A)\alpha = (id_{c_0(\widehat{\mathbb{G}})} \otimes \alpha)\alpha \\ &\Rightarrow (\widehat{\varepsilon} \otimes id_{c_0(\widehat{\mathbb{G}})} \otimes id_A)(\widehat{\Delta} \otimes id_A)\alpha = (\widehat{\varepsilon} \otimes id_{c_0(\widehat{\mathbb{G}})} \otimes id_A)(id_{c_0(\widehat{\mathbb{G}})} \otimes \alpha)\alpha \\ &\Rightarrow (id_{c_0(\widehat{\mathbb{G}})} \otimes id_A)\alpha = (\widehat{\varepsilon} \otimes \alpha)\alpha \\ &\Rightarrow \alpha = (\widehat{\varepsilon} \otimes \alpha)\alpha \Rightarrow id_A = (\widehat{\varepsilon} \otimes id_A)\alpha, \end{aligned}$$

where the last implication is true thanks to the injectivity of α.

2. Let $\mathbb{G}$ and $\mathbb{H}$ be two compact quantum groups. If (A, α) is a $\widehat{\mathbb{G}}$-C^*-algebra and (B, β) is a $\widehat{\mathbb{H}}$-C^*-algebra, then the tensor product $A \otimes B$ is equipped with an action of $\widehat{\mathbb{G} \times \mathbb{H}}$ (see Section 2.2 for more details) given by the following composition,

$$\begin{aligned} \delta : A \otimes B \xrightarrow{\alpha \otimes \beta} &\widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A) \otimes \widetilde{M}(c_0(\widehat{\mathbb{H}}) \otimes B) \subset \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A \otimes c_0(\widehat{\mathbb{H}}) \otimes B) \\ &\overset{\Sigma_{23}}{\cong} \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{H}}) \otimes A \otimes B) = \widetilde{M}(c_0(\widehat{\mathbb{F}}) \otimes A \otimes B) \end{aligned}$$

By abuse of notation we denote this composition simply by $\delta := \alpha \otimes \beta$.

1.4.1.4 Definition. Let $\mathbb{G}$ be a compact quantum group and (A,α), (B,β) two $\widehat{\mathbb{G}}$-C^*-algebras. A non-degenerate $*$-homomorphism $\varphi : A \longrightarrow M(B)$ is called $\widehat{\mathbb{G}}$-equivariant if the following diagram commutes,

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & M(B) \arrow[d, "\beta"] \\ \widetilde{M}(c_0(\widehat{\mathbb{G}})\otimes A) \arrow[r, "id_{c_0(\widehat{\mathbb{G}})}\otimes\varphi"'] & \widetilde{M}(c_0(\widehat{\mathbb{G}})\otimes B) \end{tikzcd}$$

1.4.1.5 Proposition. *Let $\mathbb{G}$ be a compact quantum group and A a C^*-algebra. If $\alpha : A \longrightarrow \widetilde{M}(c_0(\widehat{\mathbb{G}})\otimes A)$ is a non-degenerate $*$-homomorphism, then*

- *for every irreducible representation $x \in Irr(\mathbb{G})$ we define the following $*$-homomorphism*

$$\begin{array}{rccl} \alpha^x : & A & \longrightarrow & \mathcal{B}(H_x)\otimes A \\ & a & \longmapsto & \alpha^x(a) := \alpha(a)(p_x\otimes id_A), \end{array}$$

  where p_x denotes the minimal central projection of $c_0(\widehat{\mathbb{G}})$ on $\mathcal{B}(H_x)$.

- *and for every unitary finite dimensional representation $u \in Rep(\mathbb{G})$ we define the following $*$-homomorphism*

$$\begin{array}{rccl} \alpha^u : & A & \longrightarrow & \mathcal{B}(H_u)\otimes A \\ & a & \longmapsto & \alpha^u(a) := \sum\limits_{\substack{k \\ x\subset u}} (\Phi_k\otimes id_A)\alpha^x(a)(\Phi_k^*\otimes id_A), \end{array}$$

  where we consider the decomposition of u into direct sum of irreducible representations, that is, for every $x \subset u$ denote by $p_x^u \in \mathcal{B}(H_u)$ the corresponding orthogonal finite dimensional projection such that $\sum\limits_{x\subset u} p_x^u = id_{H_u}$ and for every $k = 1,\ldots,dim\Big(Mor(x,u)\Big)$ consider a family of intertwiners $\Phi_k \in Mor(x,u)$ such that $\Phi_k^*\Phi_k = id_{H_x}$ and $\sum\limits_k \Phi_k\Phi_k^* = p_x^u$.

The following assertions are equivalent.

i) *(A,α) is a $\widehat{\mathbb{G}}$-C^*-algebra with (resp. injective) action $\alpha : A \longrightarrow \widetilde{M}(c_0(\widehat{\mathbb{G}})\otimes A)$.*

ii) *The family of $*$-homomorphisms $\alpha^x : A \longrightarrow \mathcal{B}(H_x)\otimes A$ for every $x \in Irr(\mathbb{G})$ is such that*

  a) *(resp. $\alpha^\epsilon = id_A$),*

  b) *$(\Phi\otimes id_A)\alpha^x(a) = (id_{H_z}\otimes\alpha^y)(\alpha^z(a))(\Phi\otimes id_A)$, for all $a \in A$, all $x,y,z \in Irr(\mathbb{G})$ and all $\Phi \in Mor(x, y\oplus z)$,*

  c) *$[\alpha^x(A)(H_x\otimes A)] = H_x\otimes A$, for all $x \in Irr(\mathbb{G})$.*

iii) *The family of $*$-homomorphisms $\alpha^u : A \longrightarrow \mathcal{B}(H_u)\otimes A$ for every $u \in Rep(\mathbb{G})$ is such that*

  a) *(resp. $\alpha^\epsilon = id_A$),*

  b) *$\alpha^{u\oplus v} = (id_{H_u}\otimes\alpha^v)\circ\alpha^u$, for all $u,v \in Rep(\mathbb{G})$,*

  c) *$(\Phi\otimes id_A)\alpha^u(a) = \alpha^v(a)(\Phi\otimes id_A)$, for all $a \in A$, all $u,v \in Rep(\mathbb{G})$ and all $\Phi \in Mor(u,v)$,*

d) $[\alpha^u(A)(H_u \otimes A)] = H_u \otimes A$, *for all* $u \in Rep(\mathbb{G})$.

Proof. - $(i) \Leftrightarrow (ii)$. Assume that (A, α) is a $\widehat{\mathbb{G}}$-C^*-algebra with (resp. injective) action $\alpha : A \longrightarrow \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A)$. Notice that the $*$-homomorphism α^x of the statement is well defined because for all $a \in A$ we have $\alpha(a) \in \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A)$, so that $\alpha(a)(p_x \otimes id_A) \in \mathcal{B}(H_x) \otimes A$, for all $x \in Irr(\mathbb{G})$.

a) (resp. $\alpha^\epsilon = id_A$ whenever α is injective. Namely, in this case the formula obtained in Remarks 1.4.1.3 allows to write

$$\alpha^\epsilon(a) = \alpha(a)(p_\epsilon \otimes id_A) = (p_\epsilon \otimes id_A)\alpha(a)(p_\epsilon \otimes id_A) = (\widehat{\varepsilon} \otimes id_A)\alpha(a) = a,$$

for all $a \in A$).

b) For all $x, y, z \in Irr(\mathbb{G})$ and all $\Phi \in Mor(x, y \oplus z)$ we recall from Theorem 1.3.1.36 that $\widehat{\Delta}(T) \circ \Phi = \Phi \circ T$, for all $T \in \mathcal{B}(H_x)$. Hence, for all $a \in A$, we write the following,

$$\begin{aligned}(\Phi \otimes id_A)\alpha^x(a) &= (\Phi \otimes id_A)\alpha(a)(p_x \otimes id_A) = (\widehat{\Delta} \otimes id_A)\alpha(a)(p_y \otimes p_z \otimes id_A)(\Phi \otimes id_A)\\ &= (id_{c_0(\widehat{\mathbb{G}})} \otimes \alpha)\alpha(a)(p_y \otimes p_z \otimes id_A)(\Phi \otimes id_A)\\ &= (id_{H_z} \otimes \alpha)(p_y \otimes id_{H_z} \otimes id_A)\alpha(a)(p_z \otimes id_A)(\Phi \otimes id_A)\\ &= (id_{H_z} \otimes \alpha^y)(\alpha^z(a))(\Phi \otimes id_A)\end{aligned}$$

c) Observe that $\mathcal{B}(H_x) \otimes A \subset M(\mathcal{B}(H_x) \otimes A) \cong \mathcal{L}_A(H_x \otimes A)$, for all $x \in Irr(\mathbb{G})$. By the cancellation property of α we have,

$$\begin{aligned}[\alpha^x(A)(H_x \otimes A)] &= [\alpha(A)(p_x \otimes id_A)(c_0(\widehat{\mathbb{G}})) \otimes 1] = [\alpha(A)(c_0(\widehat{\mathbb{G}})) \otimes 1](p_x \otimes id_A)\\ &= c_0(\widehat{\mathbb{G}}) \otimes A(p_x \otimes id_A) = H_x \otimes A,\end{aligned}$$

for all $x \in Irr(\mathbb{G})$.

Conversely, assume that $\left(\alpha^x\right)_{x \in Irr(\mathbb{G})}$ is a family of $*$-homomorphisms as in the statement. Then we define the following non-degenerate $*$-homomorphism

$$\begin{array}{rccl}\alpha : & A & \longrightarrow & \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A)\\ & a & \longmapsto & \alpha(a) := \bigoplus\limits_{x \in Irr(\mathbb{G})}^{c_0} \alpha^x(a),\end{array}$$

where we notice that $\alpha(a) \in \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A)$.

a) (resp. α is injective whenever every $\alpha^\epsilon = id_A$. Namely, by definition of α we can write

$$(\widehat{\varepsilon} \otimes id_A)\alpha(a) = (\widehat{\varepsilon} \otimes id_A)\Big(\bigoplus_{x \in Irr(\mathbb{G})}^{c_0} \alpha^x(a)\Big) = \alpha^\epsilon(a) = a,$$

for all $a \in A$. This clearly implies the injectivity of α).

b) Given $y, z \in Irr(\mathbb{G})$, denote by $p_x^{y \oplus z} \in \mathcal{B}(H_y \otimes H_z)$ the orthogonal finite dimensional projection associated to an irreducible $x \subset y \oplus z$ such that $\sum\limits_{x \subset y \oplus z} p_x^{y \oplus z} = id_{H_y \otimes H_z}$. For

every $k = 1, \dots, dim\Big(Mor(x, y \bigoplus z)\Big)$ consider a family of intertwiners $\Phi_k \in Mor(x, y \bigoplus z)$ such that $\Phi_k^* \Phi_k = id_{H_x}$ and $\sum_k \Phi_k \Phi_k^* = p_x^{y \bigoplus z}$. From our assumption we can write the following for every $k = 1, \dots, dim\Big(Mor(x, y \bigoplus z)\Big)$ and every $a \in A$

$$\begin{aligned}
(\Phi_k \otimes id_A)\alpha^x(a) &= (id_{H_z} \otimes \alpha^y)(\alpha^z(a))(\Phi_k \otimes id_A) \\
&\Rightarrow (\Phi_k \otimes id_A)\alpha^x(a)(\Phi_k^* \otimes id_A) = (id_{H_z} \otimes \alpha^y)(\alpha^z(a))(\Phi_k \Phi_k^* \otimes id_A) \\
&\Rightarrow \sum_k (\Phi_k \otimes id_A)\alpha^x(a)(\Phi_k^* \otimes id_A) = (id_{H_z} \otimes \alpha^y)(\alpha^z(a))(p_x^{y \bigoplus z} \otimes id_A) \\
&\Rightarrow \sum_{\substack{k \\ x \subset y \bigoplus z}} (\Phi_k \otimes id_A)\alpha^x(a)(\Phi_k^* \otimes id_A) = (id_{H_z} \otimes \alpha^y)(\alpha^z(a)) \\
&\Leftrightarrow (\widehat{\Delta} \otimes id_A) \sum_{\substack{k \\ x \subset y \bigoplus z}} \alpha^x(a)(\Phi_k \Phi_k^* \otimes id_A) = (id_{H_z} \otimes \alpha^y)(\alpha^z(a)) \\
&\Leftrightarrow (\widehat{\Delta} \otimes id_A) \sum_{x \subset y \bigoplus z} \alpha^x(a)(p_x^{y \bigoplus z} \otimes id_A) = (id_{H_z} \otimes \alpha^y)(\alpha^z(a))
\end{aligned}$$

Since this is true for any $y, z \in Irr(\mathbb{G})$, we deduce that $(\widehat{\Delta} \otimes id_A) \circ \alpha = (id_{c_0(\widehat{\mathbb{G}})} \otimes \alpha) \circ \alpha$

c)

$$\begin{aligned}
[\alpha(A)(c_0(\widehat{\mathbb{G}}) \otimes 1)] &= [\bigoplus_{x \in Irr(\mathbb{G})}^{c_0} \alpha^x(A)(\mathcal{B}(H_x) \otimes A)] = \bigoplus_{x \in Irr(\mathbb{G})}^{c_0} [\alpha^x(A)(\mathcal{B}(H_x) \otimes A)] \\
&= \bigoplus_{x \in Irr(\mathbb{G})}^{c_0} \mathcal{B}(H_x) \otimes A = c_0(\widehat{\mathbb{G}}) \otimes A
\end{aligned}$$

- $(ii) \Leftrightarrow (iii)$. Assume that $\big(\alpha^x\big)_{x \in Irr(\mathbb{G})}$ is a family of $*$-homomorphisms as in the statement. Given a unitary finite dimensional representation $u \in Rep(\mathbb{G})$, consider its decomposition into direct sum of irreducible representations, so that for every $x \subset u$ denote by $p_x^u \in \mathcal{B}(H_u)$ the corresponding orthogonal finite dimensional projection such that $\sum_{x \subset u} p_x^u = id_{H_u}$. For every $k = 1, \dots, dim\Big(Mor(x, u)\Big)$ consider a family of intertwiners $\Phi_k \in Mor(x, u)$ such that $\Phi_k^* \Phi_k = id_{H_x}$ and $\sum_k \Phi_k \Phi_k^* = p_x^u$. Consider the family of $*$-homomorphism $\big(\alpha^u\big)_{u \in Rep(\mathbb{G})}$ of the statement.

  a) (resp. $\alpha^\epsilon = id_A$).

  b) Given $u, v \in Rep(\mathbb{G})$, remark that $x \subset u \bigoplus v$ is irreducible if and only if $x \subset y \bigoplus z$ with $y \subset u$ and $z \subset v$ irreducibles.
  Given $x \subset u \bigoplus v$, denote by $p_x^{u \bigoplus v} \in \mathcal{B}(H_u \otimes H_v)$ the corresponding orthogonal finite dimensional projection and consider a family of intertwiners $\Phi_k \in Mor(x, u \bigoplus v)$ such that $\Phi_k^* \Phi_k = id_{H_x}$ and $\sum_k \Phi_k \Phi_k^* = p_x^{u \bigoplus v}$, for all $k = 1, \dots, dim\Big(Mor(x, u \bigoplus v)\Big)$. Given $y \subset u$, denote by $p_y^u \in \mathcal{B}(H_u)$ the corresponding orthogonal finite dimensional projection and consider a family of intertwiners $\Psi_i \in Mor(y, u)$ such that $\Psi_i^* \Psi_i = id_{H_y}$ and

$\sum_i \Psi_i \Psi_i^* = p_y^u$, for all $i = 1, \ldots, dim\Big(Mor(y,u)\Big)$. Likewise, given $z \subset v$, denote by $p_z^v \in \mathcal{B}(H_v)$ the corresponding orthogonal finite dimensional projection and consider a family of intertwiners $\Psi'_j \in Mor(z,v)$ such that $\Psi_j'^* \Psi'_j = id_{H_z}$ and $\sum_j \Psi'_j \Psi_j'^* = p_z^v$, for all $j = 1, \ldots, dim\Big(Mor(z,v)\Big)$. Remark that the composition $H_x \xrightarrow{\Phi_k} H_u \otimes H_v \cong H_v \otimes H_u \xrightarrow{\Psi_j'^* \otimes \Psi_i^*} H_z \otimes H_y$ is clearly an operator in $Mor(x, z \bigoplus y)$ and by construction we have that $\sum_{\substack{k \\ x \subset y \bigoplus z}} (\Psi_j'^* \otimes \Psi_i^*)\Phi_k \Phi_k^* (\Psi'_j \otimes \Psi_i) = id_{H_z} \otimes id_{H_y}$. Then for all $a \in A$ we write

$$
\begin{aligned}
\alpha^{u \bigoplus v}(a) &= \sum_{\substack{k \\ x \subset u \bigoplus v}} (\Phi_k \otimes id_A)\alpha^x(a)(\Phi_k^* \otimes id_A) \\
&= \sum_{\substack{i \\ y \subset u}} \sum_{\substack{j \\ z \subset v}} \sum_{\substack{k \\ x \subset y \bigoplus z}} (\Psi'_j \Psi_j'^* \otimes \Psi_i \Psi_i^* \otimes id_A)(\Phi_k \otimes id_A)\alpha^x(a)(\Phi_k^* \otimes id_A) \\
&(\Psi'_j \Psi_j'^* \otimes \Psi_i \Psi_i^* \otimes id_A) \\
&= \sum_{\substack{i \\ y \subset u}} \sum_{\substack{j \\ z \subset v}} \sum_{\substack{k \\ x \subset y \bigoplus z}} (\Psi'_j \otimes \Psi_i \otimes id_A)((\Psi_j'^* \otimes \Psi_i^*)\Phi_k \otimes id_A)\alpha^x(a) \\
&(\Phi_k^*(\Psi'_j \otimes \Psi_i) \otimes id_A)(\Psi_j'^* \otimes \Psi_i^* \otimes id_A) \\
&\overset{(1)}{=} \sum_{\substack{i \\ y \subset u}} \sum_{\substack{j \\ z \subset v}} \sum_{\substack{k \\ x \subset y \bigoplus z}} (\Psi'_j \otimes \Psi_i \otimes id_A)(id_{H_z} \otimes \alpha^y)(\alpha^z(a))((\Psi_j'^* \otimes \Psi_i^*)\Phi_k \otimes id_A) \\
&(\Phi_k^*(\Psi'_j \otimes \Psi_i) \otimes id_A)(\Psi_j'^* \otimes \Psi_i^* \otimes id_A) \\
&= \sum_{\substack{i \\ y \subset u}} \sum_{\substack{j \\ z \subset v}} \sum_{\substack{k \\ x \subset y \bigoplus z}} (\Psi'_j \otimes \Psi_i \otimes id_A)(id_{H_z} \otimes \alpha^y)(\alpha^z(a)) \\
&((\Psi_j'^* \otimes \Psi_i^*)\Phi_k \Phi_k^*(\Psi'_j \otimes \Psi_i) \otimes id_A)(\Psi_j'^* \otimes \Psi_i^* \otimes id_A) \\
&= \sum_{\substack{i \\ y \subset u}} \sum_{\substack{j \\ z \subset v}} (\Psi'_j \otimes \Psi_i \otimes id_A)(id_{H_z} \otimes \alpha^y)(\alpha^z(a))(\Psi_j'^* \otimes \Psi_i^* \otimes id_A) \\
&= \sum_{\substack{i \\ y \subset u}} \sum_{\substack{j \\ z \subset v}} \Big(id_{H_u} \otimes (\Psi'_j \otimes id_A)\alpha^y(\Psi_j'^* \otimes id_A)\Big)\Big((\Psi_i \otimes id_A)\alpha^z(a)(\Psi_i^* \otimes id_A)\Big) \\
&= \Big(\sum_{\substack{j \\ z \subset v}} id_{H_u} \otimes (\Psi'_j \otimes id_A)\alpha^z(\Psi_j'^* \otimes id_A)\Big)\Big(\sum_{\substack{i \\ y \subset u}} (\Psi_i \otimes id_A)\alpha^y(a)(\Psi_i^* \otimes id_A)\Big) \\
&= (id_{H_u} \otimes \alpha^v)(\alpha^u(a)),
\end{aligned}
$$

where in (1) we apply the properties of the family $(\alpha^x)_{x \in Irr(\mathbb{G})}$ with $(\Psi_j'^* \otimes \Psi_i^*)\Phi_k \in Mor(x, z \bigoplus y)$.

c) Let $u, v \in Rep(\mathbb{G})$ be unitary finite dimensional representations of $\mathbb{G}$. Given $y \subset u$, denote by $p_y^u \in \mathcal{B}(H_u)$ the corresponding orthogonal finite dimensional projection and consider a family of intertwiners $\Psi_i \in Mor(y,u)$ such that $\Psi_i^* \Psi_i = id_{H_y}$ and

$\sum_i \Psi_i \Psi_i^* = p_y^u$, for all $i = 1, \ldots, dim\Big(Mor(y,u)\Big)$. Likewise, given $z \subset v$, denote by $p_z^v \in \mathcal{B}(H_v)$ the corresponding orthogonal finite dimensional projection and consider a family of intertwiners $\Psi'_j \in Mor(z,v)$ such that $\Psi_j'^* \Psi'_j = id_{H_z}$ and $\sum_j \Psi'_j \Psi_j'^* = p_z^v$, for all $j = 1, \ldots, dim\Big(Mor(z,v)\Big)$.

Given $\Phi \in Mor(u,v)$, then the composition $H_y \xrightarrow{\Psi_i} H_u \xrightarrow{\Phi} H_v \xrightarrow{\Psi_j'^*} H_z$ is clearly an operator in $Mor(y,z)$, which is denoted by $\Phi_{i,j}$. Then for all $a \in A$ we write

$$\begin{aligned}
(\Phi \otimes id_A)\alpha^u(a) &= (\Phi \otimes id_A) \sum_{\substack{i \\ y \subset u}} (\Psi_i \otimes id_A)\alpha^y(a)(\Psi_i^* \otimes id_A) \\
&= \sum_{\substack{i \\ y \subset u}} (id_{H_v} \otimes id_A)(\Phi\Psi_i \otimes id_A)\alpha^y(a)(\Psi_i^* \otimes id_A) \\
&= \sum_{\substack{i \\ y \subset u}} \sum_{\substack{j \\ z \subset v}} (\Psi'_j \Psi_j'^* \otimes id_A)(\Phi\Psi_i \otimes id_A)\alpha^y(a)(\Psi_i^* \otimes id_A) \\
&= \sum_{\substack{i \\ y \subset u}} \sum_{\substack{j \\ z \subset v}} (\Psi'_j \otimes id_A)(\Phi_{i,j} \otimes id_A)\alpha^y(a)(\Psi_i^* \otimes id_A) \\
&\overset{(1)}{=} \sum_{\substack{i \\ y \subset u}} \sum_{\substack{j \\ z \subset v}} (\Psi'_j \otimes id_A)(id_{H_z} \otimes \alpha^\epsilon)\alpha^z(a)(\Phi_{i,j} \otimes id_A)(\Psi_i^* \otimes id_A) \\
&= \sum_{\substack{i \\ y \subset u}} \sum_{\substack{j \\ z \subset v}} (\Psi'_j \otimes id_A)\alpha^z(a)(\Psi_j'^* \Phi\Psi_i\Psi_i^* \otimes id_A) \\
&= \sum_{\substack{j \\ z \subset v}} (\Psi'_j \otimes id_A)\alpha^z(a)(\Psi_j'^* \otimes id_A)(\Phi \otimes id_A) = \alpha^v(a)(\Phi \otimes id_A),
\end{aligned}$$

where in (1) we apply the properties of the family $(\alpha^x)_{x \in Irr(\mathbb{G})}$ with $\Phi_{i,j} \in Mor(y,z) = Mor(y, \epsilon \bigoplus z)$.

d) By the hypothesis of the family $(\alpha^x)_{x \in Irr(\mathbb{G})}$ we have,

$$\begin{aligned}
[\alpha^u(A)(H_u \otimes A)] &= \Big[\sum_{\substack{k \\ x \subset u}} (\Phi_k \otimes id_A)\alpha^x(A)(\Phi_k^* \otimes id_A)(H_u \otimes A)\Big] \\
&= \Big[\sum_{\substack{k \\ x \subset u}} (\Phi_k \otimes id_A)\alpha^x(A)(\Phi_k^* \otimes id_A)(\Phi_k\Phi_k^*) \otimes A\Big] \\
&= \sum_{\substack{k \\ x \subset u}} (\Phi_k \otimes id_A)\Big[\alpha^x(A)(H_x \otimes A)\Big](\Phi_k^* \otimes id_A) \\
&= \sum_{\substack{k \\ x \subset u}} (\Phi_k \otimes id_A)(H_x \otimes A)(\Phi_k^* \otimes id_A) = H_u \otimes A,
\end{aligned}$$

for all $u \in Rep(\mathbb{G})$.

Conversely, assume that $\left(\alpha^u\right)_{u\in Rep(\mathbb{G})}$ is a family of $*$-homomorphisms as in the statement. In particular, the family $\left(\alpha^x\right)_{x\in Irr(\mathbb{G})}$ satisfies the properties (a), (b), (c) and (d) of (iii). Let us check the properties (a), (b) and (c) of (ii).

a) (resp. $\alpha^\epsilon = id_A$).

b) Given $x, y, z \in Irr(\mathbb{G})$ and $\Phi \in Mor(x, y \bigoplus z)$ we write the following using the properties (b) and (c) of (iii)

$$(\Phi \otimes id_A)\alpha^x(a) = \alpha^{y\bigoplus z}(a)(\Phi \otimes id_A) = (id_{H_z} \otimes \alpha^y)(\alpha^z(a))(\Phi \otimes id_A),$$

for all $a \in A$.

c) $[\alpha^x(A)(H_x \otimes A)] = H_x \otimes A$, for all $x \in Irr(\mathbb{G})$ because this is true for any unitary finite dimensional representation.

■

1.4.1.6 Remark. The preceding characterization yields straightforwardly the following reformulation of Definition 1.4.1.4. Let $\mathbb{G}$ be a compact quantum group and (A, α), (B, β) two $\widehat{\mathbb{G}}$-C^*-algebras. A non-degenerate $*$-homomorphism $\varphi : A \longrightarrow M(B)$ is called $\widehat{\mathbb{G}}$-equivariant if one of the following equivalent conditions holds.

i) The following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & M(B) \arrow[d, "\beta"] \\ \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A) \arrow[r, "id_{c_0(\widehat{\mathbb{G}})}\otimes\varphi"'] & \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes B) \end{tikzcd}$$

ii) The following diagram commutes for every $x \in Irr(\mathbb{G})$

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha^x"] & M(B) \arrow[d, "\beta^x"] \\ \mathcal{B}(H_x) \otimes A \arrow[r, "id_{\mathcal{B}(H_x)}\otimes\varphi"'] & \mathcal{B}(H_x) \otimes B \end{tikzcd}$$

iii) The following diagram commutes for every $u \in Rep(\mathbb{G})$

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha^u"] & M(B) \arrow[d, "\beta^u"] \\ \mathcal{B}(H_u) \otimes A \arrow[r, "id_{\mathcal{B}(H_u)}\otimes\varphi"'] & \mathcal{B}(H_u) \otimes B \end{tikzcd}$$

1.4.1.7 Definition. Let $\mathbb{G}$ be a compact quantum group. A graded $\widehat{\mathbb{G}}$-C^*-algebra is a $\widehat{\mathbb{G}}$-C^*-algebra $(A, (\alpha^u)_{u\in Rep(\mathbb{G})})$ such that

i) A is a graded C^*-algebra meaning that there exist two closed self-adjoint linear subspaces of A, $A^{(0)}$ and $A^{(1)}$, such that $A = A^{(0)} \oplus A^{(1)}$ and $A^{(i)}A^{(j)} \subset A^{(i+j)\ mod.2}$, for all $i, j = 0, 1$.

ii) for every $u \in Rep(\mathbb{G})$ we have $\alpha^u(A^{(i)}) \subset \mathcal{B}(H_u) \otimes A^{(i)}$, for all $i = 0, 1$.

The elements in $A^{(i)}$ are called *homogeneous of degree* i, for all $i = 0, 1$. We say that A is trivially graded if $A^{(1)} = 0$.

1.4.1.8 Remark. This definition is equivalent to the standard one given Remarks A.3.21. More precisely, if $A = A^{(0)} \oplus A^{(1)}$ is a graded C^*-algebra with graduation automorphism $\theta \in Aut(A)$ and $(A, (\alpha^u)_{u\in Rep(\mathbb{G})}) = (A, \alpha)$ is a $\widehat{\mathbb{G}}$-C^*-algebra, then the following assertions are equivalent.

i) For every $u \in Rep(\mathbb{G})$ we have $\alpha^u(A^{(i)}) \subset \mathcal{B}(H_u) \otimes A^{(i)}$, for all $i = 0, 1$.

ii) For every $x \in Irr(\mathbb{G})$ we have $\alpha^x(A^{(i)}) \subset \mathcal{B}(H_x) \otimes A^{(i)}$, for all $i = 0, 1$.

iii) For every $u \in Rep(\mathbb{G})$ we have $(id_{\mathcal{B}(H_u)} \otimes \theta)\alpha^u(a) = \alpha^u(\theta(a))$, for all $a \in A$.

iv) For every $x \in Irr(\mathbb{G})$ we have $(id_{\mathcal{B}(H_x)} \otimes \theta)\alpha^x(a) = \alpha^x(\theta(a))$, for all $a \in A$.

v) $\alpha(A^{(i)}) \subset \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A^{(i)})$, for all $i = 0, 1$.

vi) $(id_{c_0(\widehat{\mathbb{G}})} \otimes \theta)\alpha(a) = \alpha(\theta(a))$, for all $a \in A$.

Namely,

- $(i) \Leftrightarrow (ii)$. The implication $(i) \Rightarrow (ii)$ is obvious. For the converse, recall from Proposition 1.4.1.5 that given $u \in Rep(\mathbb{G})$, we have

$$\alpha^u(a) = \sum_{\substack{k \\ x\subset u}} (\Phi_k \otimes id_A)\alpha^x(a)(\Phi_k^* \otimes id_A),$$

  where $\Phi_k \in Mor(x, u)$ for all $k = 1, \dots dim\Big(Mor(x, u)\Big)$ with $x \subset u$ are the family of intertwiners associated to the decomposition of of u into direct sum of irreducible representations.

- $(iii) \Leftrightarrow (iv)$. The same argument yields this equivalence.

- $(i) \Leftrightarrow (iii)$. Since $A = A^{(0)} \oplus A^{(1)}$ is a graded C^*-algebra with graduation automorphism $\theta \in Aut(A)$, we know that

$$A^{(0)} = \{a \in A \mid \theta(a) = a\} \text{ and } A^{(1)} = \{a \in A \mid \theta(a) = -a\}$$

  Assume that (i) holds. Given $a \in A$, write $a = x + y$ with $x \in A^{(0)}$ and $y \in A^{(1)}$. By assumption we have that $\alpha^u(x) \in \mathcal{B}(H_u) \otimes A^{(0)}$ and $\alpha^u(y) \in \mathcal{B}(H_u) \otimes A^{(1)}$. Hence,

$$\begin{aligned}(id_{\mathcal{B}(H_u)} \otimes \theta)\alpha^u(a) &= (id_{\mathcal{B}(H_u)} \otimes \theta)\alpha^u(x) + (id_{\mathcal{B}(H_u)} \otimes \theta)\alpha^u(y) \\ &= \alpha^u(x) - \alpha^u(y) = \alpha^u(x - y) = \alpha^u(\theta(a)),\end{aligned}$$

which yields (iii). Conversely, assume that (iii) holds. Given $a \in A^{(0)}$ (resp. $a \in A^{(1)}$), we have $\theta(a) = a$ (resp. $\theta(a) = -a$). Hence the relation $(id_{\mathcal{B}(H_u)} \otimes \theta)\alpha^u(a) = \alpha^u(\theta(a))$ from the assumption becomes $(id_{\mathcal{B}(H_u)} \otimes \theta)\alpha^u(a) = \alpha^u(a)$ (resp. $(id_{\mathcal{B}(H_u)} \otimes \theta)\alpha^u(a) = -\alpha^u(a)$), which means that $\alpha^u(a) \in \mathcal{B}(H_u) \otimes A^{(0)}$ (resp. $\alpha^u(a) \in \mathcal{B}(H_u) \otimes A^{(1)}$).

- $(ii) \Leftrightarrow (v)$. Recall from Proposition 1.4.1.5 that

$$\alpha(a) = \bigoplus_{x \in Irr(\mathbb{G})}^{c_0} \alpha^x(a) \text{ and } \alpha^x(a) = \alpha(a)(p_x \otimes id_A),$$

for all $a \in A$ and $x \in Irr(\mathbb{G})$. Hence the equivalence $(ii) \Leftrightarrow (v)$ is obvious.

- $(v) \Leftrightarrow (vi)$. This has been observed in Remarks A.3.21.

By virtue of the preceding characterization of action of a discrete quantum group on a C^*-algebra, we shall give explicit expressions in coordinates.

Let us introduce some useful notations for the sequel. Let α be an *injective* action of $\widehat{\mathbb{G}}$ on A, which is equivalent to give a family of $*$-homomorphisms $(\alpha^u)_{u \in Rep(\mathbb{G})}$, $\alpha^u : A \longrightarrow \mathcal{B}(H_u) \otimes A$ as in Proposition 1.4.1.5. If $\{\xi_1^u, \ldots, \xi_{n_u}^u\}$ is an orthonormal basis of H_u, then we define the *matrix coefficients of α with respect to $u \in Rep(\mathbb{G})$* as follows

$$\alpha_{i,j}^u(a) := (\omega_{\xi_i^u, \xi_j^u} \otimes id_A)(\alpha^u(a)) \in A,$$

for all $a \in A$ and all $i, j = 1 \ldots, n_u$. We observe the following elementary facts.

- If $\{m_{i,j}^u\}_{i,j=1,\ldots,n_u}$ are the matrix units in $\mathcal{B}(H_u)$ associated to the basis $\{\xi_1^u, \ldots, \xi_{n_u}^u\}$, then we have the following expression in coordinates

$$\alpha^u(a) = \sum_{i,j=1}^{n_u} m_{i,j}^u \otimes \alpha_{i,j}^u(a),$$

for all $a \in A$.

- The maps $\alpha_{i,j}^u = (\omega_{\xi_i^u, \xi_j^u} \otimes id_A) \circ \alpha^u : A \longrightarrow A$ are *completely bounded*, for all $i, j = 1, \ldots, n_u$ and *completely positive* whenever $i = j$.

- Since α^u is a $*$-homomorphism for all $u \in Rep(\mathbb{G})$, then for all $a \in A$ and all $i, j = 1, \ldots, n_u$ we have $\alpha_{i,j}^u(a)^* = \alpha_{j,i}^u(a^*)$. Namely,

$$\alpha_{i,j}^u(a)^* = \big((\omega_{\xi_i^u, \xi_j^u} \otimes id_A)\alpha^u(a)\big)^* = (\omega_{\xi_i^u, \xi_j^u}^* \otimes id_A)\alpha^u(a^*) = (\omega_{\xi_j^u, \xi_i^u} \otimes id_A)\alpha^u(a^*) = \alpha_{j,i}^u(a^*)$$

- Since α^u is a $*$-homomorphism for all $u \in Rep(\mathbb{G})$, then for all $a, b \in A$ and all $i, j = 1, \ldots, n_u$ we have $\alpha_{i,j}^u(ab) = \sum_{k=1}^{n_u} \alpha_{i,k}^u(a)\alpha_{k,j}^u(b)$. Namely

$$\begin{aligned}
\alpha^u(ab) = \alpha^u(a)\alpha^u(b) &= \Big(\sum_{i,r=1}^{n_u} m_{i,r}^u \otimes \alpha_{i,r}^u(a)\Big)\Big(\sum_{s,j=1}^{n_u} m_{s,j}^u \otimes \alpha_{s,j}^u(b)\Big) \\
&= \sum_{i,r,s,j} m_{i,r}^u m_{s,j}^u \otimes \alpha_{i,r}^u(a)\alpha_{s,j}^u(b) \\
&= \sum_{i,r,s,j} \delta_{r,s} m_{i,j}^u \otimes \alpha_{i,r}^u(a)\alpha_{s,j}^u(b) = \sum_{i,j=1}^{n_u} \sum_{k=1}^{n_u} m_{i,j}^u \otimes \alpha_{i,k}^u(a)\alpha_{k,j}^u(b)
\end{aligned}$$

- If $(A, (\alpha^u)_{u\in Rep(\mathbb{G})})$ is a *graded* $\widehat{\mathbb{G}}$-C^*-algebra with graduation automorphism $\theta \in Aut(A)$, then the formula $(id_{\mathcal{B}(H_x)} \otimes \theta)\alpha^x = \alpha^x \circ \theta$, with $x \in Irr(\mathbb{G})$ can be written in coordinates with respect to a canonical orthonormal basis. More precisely we have the following formula,

$$\theta(\alpha^x_{i,j}(a)) = \alpha^x_{i,j}(\theta(a)),$$

for all $x \in Irr(\mathbb{G})$, $a \in A$, $i, j = 1, \ldots, n_x$.

- Given an irreducible representation $x \in Irr(\mathbb{G})$, then the conditions (b) and (c) of the family $(\alpha^u)_{u\in Rep(\mathbb{G})}$ can be written explicitly in coordinates for the canonical intertwiner $\Phi_x \in Mor(\epsilon, x\oplus\overline{x})$. Indeed, if $\{\xi^x_1, \ldots, \xi^x_{n_x}\}$ is an orthonormal basis of H_x that diagonalizes the operator Q_x and $\{\omega^x_1, \ldots, \omega^x_{n_x}\}$ is its dual basis in $\overline{H}_x$, then, using the coordinate expression of Φ_x from Remarks 1.3.1.31, a straightforward computation yields that the formula

$$(\Phi_x \otimes id_A)\alpha^\epsilon(a) = (id_{H_{\overline{x}}} \otimes \alpha^x)(\alpha^{\overline{x}}(a))(\Phi_x \otimes id_A),$$

for all $a \in A$ can be written as

$$\sum_{k=1}^{n_x} \frac{\sqrt{\lambda^x_k}}{\sqrt{\lambda^x_i}}\ \alpha^x_{i,k} \circ \alpha^{\overline{x}}_{j,k} = \delta_{i,j}\ id_A$$

Analogously, the corresponding formulas for $\Phi_{\overline{x}} \in Mor(\epsilon, \overline{x} \oplus x)$, $\Phi^*_x \in Mor(x \oplus \overline{x}, \epsilon)$, $\Phi^*_{\overline{x}} \in Mor(\overline{x} \oplus x, \epsilon)$ can be written respectively as follows

$$\sum_{k=1}^{n_x} \frac{\sqrt{\lambda^x_i}}{\sqrt{\lambda^x_k}}\ \alpha^{\overline{x}}_{i,k} \circ \alpha^x_{j,k} = \delta_{i,j}\ id_A;\ \sum_{k=1}^{n_x} \frac{\sqrt{\lambda^x_k}}{\sqrt{\lambda^x_i}}\ \alpha^x_{k,i} \circ \alpha^{\overline{x}}_{k,j} = \delta_{i,j}\ id_A \text{ and } \sum_{k=1}^{n_x} \frac{\sqrt{\lambda^x_i}}{\sqrt{\lambda^x_k}}\ \alpha^{\overline{x}}_{k,i} \circ \alpha^x_{k,j} = \delta_{i,j}\ id_A$$

1.4.1.9 Remark. If $V \in M(C \otimes c_0(\widehat{\mathbb{G}})) \cong \mathcal{L}_C(C \otimes c_0(\widehat{\mathbb{G}}))$ is a unitary representation of $\widehat{\mathbb{G}}$ on C (recall Remarks 1.3.1.8), where C is any C^*-algebra (or even more generally, when C is any Hilbert module), then we use the analogous notations. Namely, we put

$$V^x_{i,j} := (id_C \otimes \omega_{\xi^x_i, \xi^x_j})V(id_C \otimes p_x) \in M(C),$$

for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$. More generally, if $u \in Rep(\mathbb{G})$, then we put

$$V^u_{r,s} := \sum_{\substack{k \\ z\subset u}} (id_C \otimes \omega_{\xi^u_r, \xi^u_s})(id_C \otimes \Phi_k)V(id_C \otimes \Phi^*_k) \in M(C),$$

for all $r, s = 1, \ldots, n_u$ where we follow the usual notations: $p^u_z \in \mathcal{B}(H_u)$ is the corresponding orthogonal finite dimensional projection such that $\sum_{z\subset u} p^u_z = id_{H_u}$ and for every $k = 1, \ldots, dim\Big(Mor(z, u)\Big)$ consider a family of intertwiners $\Phi_k \in Mor(z, u)$ such that $\Phi^*_k\Phi_k = id_{H_z}$ and $\sum_k \Phi_k\Phi^*_k = p^u_z$.

1.4.1.10 Proposition. *Let $\widehat{\mathbb{G}} = (c_0(\widehat{\mathbb{G}}), \widehat{\Delta})$ be a discrete quantum group and C any C^*-algebra.*

i) If $V \in M(C \otimes c_0(\widehat{\mathbb{G}}))$ is a unitary representation of $\widehat{\mathbb{G}}$ on C, then there exists a non-degenerate $*$-homomorphism $\phi_V : C_m(\mathbb{G}) \longrightarrow M(C)$ such that

$$\phi_V(w^x_{i,j}) = V^x_{i,j},$$

for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$.

ii) If $\phi : C_m(\mathbb{G}) \longrightarrow M(C)$ is a non-degenerate $*$-homomorphism, then there exists a unitary representation of $\widehat{\mathbb{G}}$ on C, $V_\phi \in M(C \otimes c_0(\widehat{\mathbb{G}}))$ such that

$$(V_\phi)^x_{i,j}(c) = \phi(w^x_{i,j})(c),$$

for all $c \in C$, $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$.

Proof. i) Let $V \in M(C \otimes c_0(\widehat{\mathbb{G}}))$ be a unitary operator such that $(id_C \otimes \widehat{\Delta})V = V_{12}V_{13}$. On the algebraic level, $Pol(\mathbb{G})$, we define the linear map ϕ_V on the basis as follows

$$\phi_V(w^x_{i,j}) := V^x_{i,j} \in M(C),$$

for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$ (recall, by Proposition 1.3.1.28, that the set of matrix coefficients of classes of irreducible finite dimensional representations forms a basis of $Pol(\mathbb{G})$).

Let us show that ϕ_V is a non-degenerate $*$-homomorphism. Take orthonormal basis $\{\xi^x_1, \ldots, \xi^x_{n_x}\}$ and $\{\xi^y_1, \ldots, \xi^y_{n_y}\}$ of H_x and H_y, respectively and denote by $\{\omega^x_1, \ldots, \omega^x_{n_x}\}$, $\{\omega^y_1, \ldots, \omega^y_{n_y}\}$ the respective dual basis in $\overline{H}_x$ and $\overline{H}_y$, respectively. By Remark 1.3.1.12 and Remark 1.3.1.19 we know that if $x, y \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$ and $k, l = 1, \ldots, n_y$, then we have

$$w^x_{i,j}\ w^y_{k,l} = w^{x \bigoplus y}_{r,s} \text{ and } (w^x_{i,j})^* = w^{\overline{x}}_{i,j},$$

where $r := (i, k)$ and $s := (j, l)$ are the indices corresponding to the usual basis of the tensor product $H_x \otimes H_y$, that is, $\{\zeta^{x \bigoplus y}_r := \xi^x_i \otimes \xi^y_k\}_{\substack{i=1,\ldots,n_x \\ k=1,\ldots,n_y}}$ and $\{\zeta^{x \bigoplus y}_s := \xi^x_j \otimes \xi^y_l\}_{\substack{j=1,\ldots,n_x \\ l=1,\ldots,n_y}}$.

Moreover, let us decompose $x \bigoplus y$ into a direct sum of irreducible representations: if $z \subset x \bigoplus y$, denote by $p^{x \bigoplus y}_z \in \mathcal{B}(H_{x \bigoplus y})$ the corresponding orthogonal finite dimensional projection such that $\sum\limits_{z \subset u} p^{x \bigoplus y}_z = id_{H_{x \bigoplus y}}$ and for every $k = 1, \ldots, dim\Big(Mor(z, x \bigoplus y)\Big)$ consider a family of intertwiners $\Phi_k \in Mor(z, x \bigoplus y)$ such that $\Phi^*_k \Phi_k = id_{H_z}$ and $\sum\limits_k \Phi_k \Phi^*_k = p^{x \bigoplus y}_z$. Hence, we write

$$\begin{aligned}
\phi_V(w^x_{i,j})\phi_V(w^y_{k,l}) &= V^x_{i,j}\ V^y_{k,l} = (id_C \otimes \omega_{\xi^x_i, \xi^x_j})V(id_C \otimes p_x)(id_C \otimes \omega_{\xi^x_k, \xi^x_l})V(id_C \otimes p_y) \\
&= (id_C \otimes \omega_{\xi^x_i, \xi^x_j} \otimes \omega_{\xi^x_k, \xi^x_l})V_{12}V_{13}(id_C \otimes p_x \otimes p_y) \\
&= (id_C \otimes \omega_{\xi^x_i, \xi^x_j} \otimes \omega_{\xi^x_k, \xi^x_l})(id_C \otimes \widehat{\Delta})(V)(id_C \otimes p_x \otimes p_y) \\
&= \sum_{\substack{k \\ z \subset x \bigoplus y}} (id_C \otimes \omega_{\xi^x_i, \xi^x_j} \otimes \omega_{\xi^x_k, \xi^x_l})(id_C \otimes \widehat{\Delta})(V)(id_C \otimes \Phi_k \Phi^*_k) \\
&= \sum_{\substack{k \\ z \subset x \bigoplus y}} (id_C \otimes \omega_{\xi^x_i, \xi^x_j} \otimes \omega_{\xi^x_k, \xi^x_l})(id_C \otimes \Phi_k)V(id_C \otimes \Phi^*_k) \\
&= \sum_{\substack{k \\ z \subset x \bigoplus y}} (id_C \otimes \omega_{\zeta^{x \bigoplus y}_r, \zeta^{x \bigoplus y}_s})(id_C \otimes \Phi_k)V(id_C \otimes \Phi^*_k) \\
&= V^{x \bigoplus y}_{r,s} = \phi_V(w^{x \bigoplus y}_{r,s}) = \phi_V(w^x_{i,j}\ w^y_{k,l})
\end{aligned}$$

$$\begin{aligned}\phi_V\big((w^x_{i,j})^*\big) &= \phi_V(w^{\overline{x}}_{i,j}) = V^{\overline{x}}_{i,j} = (id\otimes\omega_{\omega^x_i,\omega^x_j})j(V)(id\otimes p_{\overline{x}})\\ &= (id\otimes\omega_{(\xi^x_i)*,(\xi^x_j)*})j(V)(id\otimes p_{\overline{x}})\\ &= (id\otimes\omega^*_{\xi^x_i,\xi^x_j})V^*(id\otimes p_{\overline{x}})\\ &= \Big((id\otimes\omega_{\xi^x_i,\xi^x_j})V(id\otimes p_x)\Big)^* = \big(V^x_{i,j}\big)^* = \phi_V(w^x_{i,j})^*,\end{aligned}$$

which shows that ϕ_V is a $*$-homomorphism.

Finally, since V is a unitary operator, we know that $V(C\otimes c_0(\widehat{\mathbb{G}}))$ is linearly dense in $C\otimes c_0(\widehat{\mathbb{G}})$, so that $(id\otimes\omega_{\xi^x_i,\xi^x_j})V(C\otimes p_x) = \phi_V(w^x_{i,j})(C)$ is linearly dense in C for all $x\in Irr(\mathbb{G})$ and all $i,j=1,\dots,n_x$. In other words, the map $\phi_V : Pol(\mathbb{G})\longrightarrow M(C)$ is non-degenerate and we extend it into a non-degenerate $*$-homomorphism $\phi_V : C_m(\mathbb{G})\longrightarrow M(C)$.

ii) Let $\phi : C_m(\mathbb{G})\longrightarrow M(C)$ be a non-degenerate $*$-homomorphism. If $\mathcal{W}_{\mathbb{G}}\in M(C_m(\mathbb{G})\otimes c_0(\widehat{\mathbb{G}}))$ is the universal representation of $\mathbb{G}$ (recall Note 1.3.2.8), then we put

$$V_\phi := (\phi\otimes id)\mathcal{W}_{\mathbb{G}}$$

Since ϕ is a non-degenerate $*$-homomorphism and $\mathcal{W}_{\mathbb{G}}$ is a unitary operator, V_ϕ is a unitary operator as well. The relation $(id_C\otimes\widehat{\Delta})V_\phi = (V_\phi)_{12}(V_\phi)_{13}$ holds thanks to the analogous relation for $\mathcal{W}_{\mathbb{G}}$. Namely,

$$\begin{aligned}&(id_C\otimes\widehat{\Delta})\mathcal{W}_{\mathbb{G}} = (\mathcal{W}_{\mathbb{G}})_{12}(\mathcal{W}_{\mathbb{G}})_{13}\\ &\qquad\Rightarrow (\phi\otimes id\otimes id)(id_C\otimes\widehat{\Delta})\mathcal{W}_{\mathbb{G}} = (\phi\otimes id\otimes id)(\mathcal{W}_{\mathbb{G}})_{12}(\mathcal{W}_{\mathbb{G}})_{13}\\ &\qquad\Leftrightarrow (id_C\otimes\widehat{\Delta})V_\phi = (V_\phi)_{12}(V_\phi)_{13}\end{aligned}$$

Routine computations show the relation of the statement $(V_\phi)^x_{i,j}(c) = \phi(w^x_{i,j})(c)$, for all $c\in C$, $x\in Irr(\mathbb{G})$ and all $i,j=1,\dots,n_x$. The proof is therefore complete.

■

1.4.2 Spectral theory for Compact Quantum Groups

Classical spectral theory for compact groups consists in the study of classification of ergodic actions on von Neumann algebras through the irreducible decomposition of the group representations. It was deeply studied by R. Høegh-Krohn, M. B. Landstad and E. Størmer in [84] and by A. Wassermann in [223], [221], [222]. The quantum counterpart was initiated by F. Boca in [25] and by M. B. Landstad in [118] and further studied by R. Tomatsu in [190] where he gives the complete classification of ergodic actions of $SU_q(2)$ and by J. Bichon, A. De Rijdt, S. Vaes in [23] where they introduce the notion of *monodial equivalence* in order to produce new ergodic actions.

Hence, spectral theory shall be a useful strategy for the study of torsion-freeness in the context of quantum groups. We refer to the original articles mentioned before or to the survey [49] for more details.

1.4.2.1 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and A a (resp. unital) C^*-algebra. We say that A is a (resp. injective) right $\mathbb{G}$-C^*-algebra if there exists a non-degenerate $*$-homomorphism $\delta : A \longrightarrow M(A \otimes C(\mathbb{G}))$ such that

i) (resp. δ is injective),

ii) δ intertwines the co-multiplication meaning that the diagram

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\delta"'] & M(A \otimes C(\mathbb{G})) \arrow[d, "\delta \otimes id_{C(\mathbb{G})}"] \\ M(A \otimes C(\mathbb{G})) \arrow[r, "id_A \otimes \Delta"'] & M(A \otimes C(\mathbb{G}) \otimes C(\mathbb{G})) \end{tikzcd}$$

is commutative,

iii) δ satisfies the cancellation property meaning that

$$[\delta(A)(1 \otimes C(\mathbb{G}))] = A \otimes C(\mathbb{G})$$

Such a homomorphism is called a *(resp. injective) right action of* $\mathbb{G}$ *on* A or a *left co-action of* $C(\mathbb{G})$ *on* A.

1.4.2.2 Note. A *left* action of $\mathbb{G}$ on A (or a *right* co-action of $C(\mathbb{G})$ on A) is a non-degenerate $*$-homomorphism $\delta : A \longrightarrow M(C(\mathbb{G}) \otimes A)$ satisfying the analogue properties of the preceding definition.

In the present thesis, an action of a compact quantum group $\mathbb{G}$ is supposed to be a *right* one unless the contrary is explicitly indicated. Hence, we refer to such actions simply as *action of* $\mathbb{G}$.

1.4.2.3 Remarks. 1. If $\mathbb{G} = (C(\mathbb{G}), \Delta)$ is a compact quantum group, then every C^*-algebra A is equipped with an action of $\mathbb{G}$, namely the *trivial action* defined as

$$\begin{array}{rccl} trv: & A & \longrightarrow & M(A \otimes C(\mathbb{G})) \\ & a & \longmapsto & trv(a) := a \otimes 1_{C(\mathbb{G})} \end{array}$$

2. If $\mathbb{G} = (C(\mathbb{G}), \Delta)$ is a compact quantum group, then $C(\mathbb{G})$ is naturally a $\mathbb{G}$-C^*-algebra via the co-multiplication. More precisely, the co-multiplication

$$\Delta : C(\mathbb{G}) \longrightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$$

is by definition an action of $\mathbb{G}$ on A. It is called *regular action of* $\mathbb{G}$.

3. If $\mathbb{G} = (C(\mathbb{G}), \Delta)$ is a compact quantum group and $u \in \mathcal{B}(H) \otimes C(\mathbb{G})$ is a unitary representation of $\mathbb{G}$ on a finite dimensional Hilbert space H, then $\mathcal{B}(H)$ is naturally a $\mathbb{G}$-C^*-algebra via the *adjoint action with respect to* u. More precisely,

$$\begin{array}{rccl} Ad_u: & \mathcal{B}(H) & \longrightarrow & \mathcal{B}(H) \otimes C(\mathbb{G}) \\ & T & \longmapsto & Ad_u(T) := u(T \otimes 1_{C(\mathbb{G})})u^* \end{array}$$

is an action of $\mathbb{G}$ on $\mathcal{B}(H)$.

1.4.2.4 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and (A, δ), (B, δ') two $\mathbb{G}$-C^*-algebras. A non-degenerate $*$-homomorphism $\varphi : A \longrightarrow M(B)$ is called $\mathbb{G}$-equivariant if the following diagram commutes,

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\delta"] & M(B) \arrow[d, "\delta'"] \\ M(A \otimes C(\mathbb{G})) \arrow[r, "\varphi \otimes id_{C(\mathbb{G})}"'] & M(B \otimes C(\mathbb{G})) \end{tikzcd}$$

1.4.2.5 Remark. On the one hand, the injectivity condition of a compact quantum group action is a delicate axiom that some authors include in the definition and others don't (we refer to [174] for a detailed exposition of this subject). Nevertheless, when we work in the context of spectral theory, we can prove (see Proposition 3.22 in [49]) that if δ is not injective, we can restrict δ to an *injective* action of $\mathbb{G}$ on $A/\ker(\delta)$ so that (A, δ) and $(A/\ker(\delta), \delta_|)$ have the same algebraic core.

On the other hand, according to Theorem 1.3.1.32 there are two pictures that represent the same compact quantum group $\mathbb{G}$, the universal one $\mathbb{G}_m$ and the reduced one $\mathbb{G}_r$. In this way, it is important to specify if an action of $\mathbb{G}$ is an action of $\mathbb{G}_m$ or an action of $\mathbb{G}_r$. Let us explain this problem more precisely.

Let A be a C^*-algebra and $\delta_0 : A \longrightarrow A \odot Pol(\mathbb{G}) \subset A \odot C(\mathbb{G})$ a co-action of $Pol(\mathbb{G})$, which means that δ_0 is a $*$-homomorphism such that

i) δ_0 intertwines the co-multiplication meaning that $(\delta_0 \otimes id_{Pol(\mathbb{G})})\delta_0 = (id_A \otimes \Delta)\delta_0$,

ii) $(id_A \otimes \varepsilon) \circ \delta_0 = id_A$.

In this situation, it is clear by universality that δ_0 extends to a $*$-homomorphism $\delta : A \longrightarrow A \otimes C_m(\mathbb{G})$ which intertwines the co-multiplication of $\mathbb{G}$ as in Definition 1.4.2.1. Let us show that the above condition (ii) implies the cancellation property for δ. It is clear that $\delta_0(A)(1_A \odot Pol(\mathbb{G})) \subset A \odot Pol(\mathbb{G})$. Now, given $a \in A$ we write

$$\begin{aligned} & m(id_A \otimes S)(\delta_0 \otimes id_{Pol(\mathbb{G})})\delta_0(a) = m(id_A \otimes S)(id_A \otimes \Delta)\delta_0(a) = a \otimes 1_{Pol(\mathbb{G})} \\ & \quad \Leftrightarrow \delta_0(a_{(1)})(1_A \otimes S(a_{(2)})) = a \otimes 1_{Pol(\mathbb{G})}, \end{aligned}$$

where $m : Pol(\mathbb{G}) \otimes Pol(\mathbb{G}) \longrightarrow Pol(\mathbb{G})$ denotes the multiplication map and $S : Pol(\mathbb{G}) \longrightarrow Pol(\mathbb{G})$ denotes the antipode of the $*$-Hopf algebra $Pol(\mathbb{G})$ (see Proposition 1.3.1.28). Notice that we have used as well the Sweedler notation as explained in Section 1.1. This shows that $\delta_0(A)(1_A \odot Pol(\mathbb{G})) = A \odot Pol(\mathbb{G})$ and the cancellation property for δ follows.

In other words, δ_0 extends to an action of $\mathbb{G}_m$ on A. Observe that we can not guarantee in general that δ_0 extends to an action of $\mathbb{G}_r$. However, the spectral theory for compact quantum groups allows to solve this issue. Namely, given a $\mathbb{G}$-C^*-algebra (A, δ) as in Definition 1.4.2.1, the action δ restricts to a co-action $\delta_{Pol(\mathbb{G})}$ of $Pol(\mathbb{G})$ on the algebraic core of A, $\mathcal{A}_{\mathbb{G}}$ (see Proposition 1.4.2.12). This co-action extends to an (injective) action of $\mathbb{G}_m$ on $C^*(\mathcal{A}_{\mathbb{G}})$ and also to an (injective) action of $\mathbb{G}_r$ on some C^*-algebra $\mathcal{A}_{\mathbb{G},r}$. In both cases, the algebraic core and the fixed points space are preserved. We refer to Theorem 4.2 and Theorem 4.6 in [49] for a proof of these results.

1.4.2.6 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and (A, δ) a $\mathbb{G}$-C^*-algebra. The δ-fixed points space of A is the C^*-algebra defined as

$$A^\delta := \{a \in A \mid \delta(a) = a \otimes 1_{C(\mathbb{G})}\}$$

If A is unital, we say that

i) δ is an ergodic action if $A^\delta = \mathbb{C}1_A$.

ii) δ is a torsion action if δ is ergodic and A is finite dimensional.

1.4.2.7 Remark. If $\mathbb{G} = (C(\mathbb{G}), \Delta)$ is a compact quantum group and $u \in \mathcal{B}(H) \otimes C(\mathbb{G})$ is a unitary representation of $\mathbb{G}$ on a finite dimensional Hilbert space H, then by definition of adjoint action it is clear that

$$\mathcal{B}(H)^{Ad_u} = End(u)$$

1.4.2.8 Note. For simplicity of the exposition, all $\mathbb{G}$-C^*-algebras considered in this section are supposed to be *unital* for the development of the spectral theory for compact quantum groups.

1.4.2.9 Proposition. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and (A, δ) a $\mathbb{G}$-C^*-algebra. There always exists a non-degenerate δ-invariant conditional expectation*

$$\begin{array}{rccl} \mathbb{E}_\delta : & A & \longrightarrow & A^\delta \\ & a & \longmapsto & \mathbb{E}_\delta(a) := (id_A \otimes h_{\mathbb{G}})\delta(a) \end{array}$$

Moreover, $\mathbb{E}_\delta$ is a state on A whenever δ is ergodic and it is denoted by h_A.

Proof. First of all, let's check that $\mathbb{E}_\delta$ is well-defined. Namely, given $a \in A$ we write

$$\begin{aligned} \delta(\mathbb{E}_\delta(a)) &= \delta\big((id_A \otimes h_{\mathbb{G}})\delta(a)\big) = (id_A \otimes id_{C(\mathbb{G})} \otimes h_{\mathbb{G}})(\delta \otimes id_{C(\mathbb{G})})\delta(a) \\ &= (id_A \otimes id_{C(\mathbb{G})} \otimes h_{\mathbb{G}})(id_A \otimes \Delta)\delta(a) = \Big(id_A \otimes \big((id_{C(\mathbb{G})} \otimes h_{\mathbb{G}}) \circ \Delta\big)\Big)\delta(a) \\ &= (id_A \otimes h_{\mathbb{G}} 1_{C(\mathbb{G})})\delta(a) = \mathbb{E}_\delta(a) \otimes 1_{C(\mathbb{G})} \end{aligned}$$

Next, since δ is a $*$-homomorphism and $h_{\mathbb{G}}$ is a state, it is clear that $\mathbb{E}_\delta$ is a completely contractive positive map. It is straightforward to see that $\mathbb{E}_\delta$ is a projection and a A^δ-bimodule map. Namely, given $a, a' \in A^\delta$ and $c \in A$ we write

$$\mathbb{E}_\delta(a) = (id_A \otimes h_{\mathbb{G}})\delta(a) = (id_A \otimes h_{\mathbb{G}})(a \otimes 1_{C(\mathbb{G})}) = a$$

$$\mathbb{E}_\delta(aca') = \mathbb{E}_\delta(a)\mathbb{E}_\delta(c)\mathbb{E}_\delta(a') = a\mathbb{E}_\delta(c)a'$$

Remark that $\mathbb{E}_\delta$ is also a non-degenerate map because if $\{e_n\}_{n\in\mathbb{N}}$ is an approximate unit for A, then for every $b \in A$ we have

$$\lim_{n\to\infty} \mathbb{E}_\delta(e_n)b = \lim_{n\to\infty} (id_A \otimes h_{\mathbb{G}})(\delta(e_n)(b \otimes 1_{C(\mathbb{G})})) = b$$

Let us check the δ-invariance of $\mathbb{E}_\delta$. Given $a \in A$ we have to show that $(\mathbb{E}_\delta \otimes id_{C(\mathbb{G})})\delta(a) = \mathbb{E}_\delta(a) \otimes 1_{C(\mathbb{G})}$. Namely,

$$\begin{aligned} (\mathbb{E}_\delta \otimes id_{C(\mathbb{G})})\delta(a) &= ((id_A \otimes h_{\mathbb{G}})\delta \otimes id_{C(\mathbb{G})})\delta(a) = (id_A \otimes h_{\mathbb{G}} \otimes id_{C(\mathbb{G})})(\delta \otimes id_{C(\mathbb{G})})\delta(a) \\ &= (id_A \otimes h_{\mathbb{G}} \otimes id_{C(\mathbb{G})})(id_A \otimes \Delta)\delta(a) = (id_A \otimes (h_{\mathbb{G}} \otimes id_{C(\mathbb{G})})\Delta)\delta(a) \\ &= (id_A \otimes h_{\mathbb{G}} 1_{C(\mathbb{G})})\delta(a) = (id_A \otimes h_{\mathbb{G}})\delta(a) \otimes 1_{C(\mathbb{G})} = \mathbb{E}_\delta(a) \otimes 1_{C(\mathbb{G})} \end{aligned}$$

Finally, suppose that δ is an ergodic action. In this case we have $A^\delta = \mathbb{C}1_A$ and the conditional expectation $\mathbb{E}_\delta$ becomes a state because $\mathbb{E}_\delta(1_A) = 1$ and it is completely positive (so positive). ■

1.4.2.10 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and (A, δ) a $\mathbb{G}$-C^*-algebra. Given an irreducible representation $x \in Irr(\mathbb{G})$, the x-isotypical component of δ or the x-spectral space of δ is the following vector space

$$Mor(x, \delta) := \{T : H_x \longrightarrow A \mid T \text{ is linear such that } \delta(T(\xi)) = (T \otimes id_{C(\mathbb{G})})w^x(\xi \otimes 1_{C(\mathbb{G})})\}$$

By abuse of language, we call x-spectral space of δ also the following vector subspace of A

$$\mathcal{A}_x := span\{T(\xi) \mid \xi \in H_x,\ T \in Mor(x, \delta)\}$$

The Podleś subalgebra of A or algebraic core of A is the following vector subspace of A

$$\mathcal{A}_{\mathbb{G}} := span\{\mathcal{A}_x \mid x \in Irr(\mathbb{G})\}$$

1.4.2.11 Remarks. 1. By definition it is clear that the ϵ-spectral space of δ is exactly the fixed points space of A, that is, $Mor(\epsilon, \delta) = A^\delta$.

2. Given an irreducible representation $x \in Irr(\mathbb{G})$, there exists a natural identification

$$K_x \cong Mor(x, \delta),$$

where $K_x := \{X \in \overline{H}_x \otimes A \mid (id \otimes \delta)(X) = \big[X\big]_{12}\big[w^x\big]_{13}\}$. This identification is done by associating to any $X \in K_x$ the following linear map

$$\begin{array}{rccl} T_X : & H_x & \longrightarrow & A \\ & \xi & \longmapsto & T_X(\xi) := X(\xi \otimes 1_A) \end{array}$$

3. Given an irreducible representation $x \in Irr(\mathbb{G})$, the x-spectral space of δ, $\mathcal{A}_x$, is a *closed* subspace of A. Indeed, let $\{\xi_1^x, \ldots, \xi_{n_x}^x\}$ be an orthonormal basis for H_x that diagonalizes the canonical operator Q_x with positive eigenvalues $\{\lambda_i^x\}_{i=1,\ldots,n_x}$ and define the following element of $C(\mathbb{G})$ (actually, an element of $Pol(\mathbb{G})$)

$$\chi_x := dim_q(x) \sum_{i=1}^{n_x} w^x_{\xi_i^x, (Q_x)^{-1}(\xi_i^x)} \in C(\mathbb{G}),$$

where $w^x_{\xi_i^x, (Q_x)^{-1}(\xi_i^x)} := (\omega_{\xi_i^x, (Q_x)^{-1}(\xi_i^x)} \otimes id_{C(\mathbb{G})})(w^x)$, for all $i = 1, \ldots, n_x$ (recall Remark 1.3.1.8 for these notations). Notice that $(Q_x)^{-1}(\xi_i^x) = \frac{1}{\lambda_i^x}\xi_i^x$ (recall Remarks 1.3.1.31), so that $w^x_{\xi_i^x, (Q_x)^{-1}(\xi_i^x)} = \frac{1}{\lambda_i^x} w^x_{\xi_i^x, \xi_i^x}$, for all $i = 1, \ldots, n_x$. Define now the following linear map

$$\begin{array}{rccl} E_x : & A & \longrightarrow & A \\ & a & \longmapsto & E_x(a) := (id_A \otimes h_{\mathbb{G}})(\delta(a)(1_A \otimes \chi_x^*)) \end{array}$$

We claim that

$$\mathcal{A}_x = \{a \in A \mid E_x(a) = a\},$$

which yields the closedness of $\mathcal{A}_x$ in A. Namely, given any linear map $T \in Mor(x,\delta)$ and any vector $\xi \in H_x$ we have

$$\begin{aligned}
E_x(T(\xi)) &= (id_A \otimes h_{\mathbb{G}})(\delta(T(\xi))(1_A \otimes \chi_x^*)) \\
&= (id_A \otimes h_{\mathbb{G}})\big((T \otimes id_{C(\mathbb{G})})w^x(\xi \otimes id_{C(\mathbb{G})})(1_A \otimes \chi_x^*)\big) \\
&= (T \otimes h_{\mathbb{G}})w^x\Big(\xi \otimes dim_q(x)\sum_{i=1}^{n_x}(w^x_{\xi_i^x,(Q_x)^{-1}(\xi_i^x)})^*\Big) \\
&= (T \otimes h_{\mathbb{G}})\Big(\sum_{l,k} m_{l,k} \otimes w^x_{l,k}\Big)\Big(\xi \otimes dim_q(x)\sum_{i=1}^{n_x}\frac{1}{\lambda_i^x}(w^x_{\xi_i^x,\xi_i^x})^*\Big) \\
&= \sum_{l,k} T(m_{l,k}(\xi)) \otimes dim_q(x)\sum_{l,k,i}\frac{1}{\lambda_i^x}h_{\mathbb{G}}(w^x_{l,k}(w^x_{\xi_i^x,\xi_i^x})^*) \\
&= \sum_{l,k} T(m_{l,k}(\xi)) \otimes dim_q(x)\sum_{l,k,i}\frac{1}{\lambda_i^x}\frac{\delta_{l,i}(Q_x)_{i,k}}{dim_q(x)} \\
&= \sum_{l,k} T(m_{l,k}(\xi)) \otimes \sum_{l,k}\frac{1}{\lambda_l^x}(Q_x)_{l,k} \\
&= \sum_{k} T(m_{k,k}(\xi)) \otimes \sum_{k}\frac{1}{\lambda_k^x}(Q_x)_{k,k} = T(\xi) \otimes \frac{1}{dim_q(x)}dim_q(x) = T(\xi),
\end{aligned}$$

which shows that $\mathcal{A}_x \subset \{a \in A \mid E_x(a) = a\}$ by linearity. Conversely, given any $a \in A$ such that $E_x(a) = a$ and any $\eta \in H_x$ define the following linear map

$$\begin{array}{rccl}
T: & H_x & \longrightarrow & A \\
 & \xi & \longmapsto & T(\xi) := (id_A \otimes h_{\mathbb{G}})(\delta(a)(1_A \otimes (w^x_{\xi,\eta})^*)),
\end{array}$$

so that we have

$$\begin{aligned}
\delta(T(\xi)) &= \delta\Big((id_A \otimes h_{\mathbb{G}})(\delta(a)(1_A \otimes (w^x_{\xi,\eta})^*))\Big) \\
&= (id_A \otimes id_{C(\mathbb{G})} \otimes h_{\mathbb{G}})(\delta \otimes id_{C(\mathbb{G})})\delta(a)(1_A \otimes 1_A \otimes (w^x_{\xi,\eta})^*) \\
&= (id_A \otimes id_{C(\mathbb{G})} \otimes h_{\mathbb{G}})(id_A \otimes \Delta)\delta(a)(1_A \otimes 1_A \otimes (w^x_{\xi,\eta})^*) \\
&= \Big(id_A \otimes \big((id_{C(\mathbb{G})} \otimes h_{\mathbb{G}}) \circ \Delta\big)\Big)\delta(a)(1_A \otimes (w^x_{\xi,\eta})^*) \\
&= \sum_{i=1}^{n_x}(id_A \otimes id_{C(\mathbb{G})} \otimes h_{\mathbb{G}})\big[\delta(a)\big]_{13}(1_A \otimes w^x_{\xi_i,\xi} \otimes (w^x_{\xi_i,\eta})^*) \\
&= \sum_{i=1}^{n_x} T(\xi_i) \otimes w^x_{\xi_i,\xi} = (T \otimes id_{C(\mathbb{G})})w^x(\xi \otimes 1_{C(\mathbb{G})}),
\end{aligned}$$

where $\{\xi_1^x, \ldots, \xi_{n_x}^x\}$ is an orthonormal basis of H_x. This shows that $T \in Mor(x,\delta)$. Since χ_x is a linear combination of $w^x_{\xi,\eta}$'s, then we deduce that $E_x(A) \subset \mathcal{A}_x$.

1.4.2.12 Proposition. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and (A, δ) a $\mathbb{G}$-C^*-algebra. The algebraic core of A, $\mathcal{A}_{\mathbb{G}}$, is a dense $*$-subalgebra of A which is unital whenever A is unital.*

Moreover, we have that

i) the action $\delta : A \longrightarrow A \otimes C(\mathbb{G})$ *restricts to an action of* $(Pol(\mathbb{G}), \Delta_|)$ *on* $\mathcal{A}_\mathbb{G}$*, say*

$$\delta_{Pol(\mathbb{G})} : \mathcal{A}_\mathbb{G} \longrightarrow \mathcal{A}_\mathbb{G} \odot Pol(\mathbb{G}),$$

which is ergodic whenever δ *is ergodic.*

ii) the canonical conditional expectation $\mathbb{E}_\delta$ *is faithful on* $\mathcal{A}_\mathbb{G}$*. Consequently, the algebraic core of* A *admits the following* spectral decomposition

$$\mathcal{A}_\mathbb{G} = \bigoplus_{x \in Irr(\mathbb{G})} \mathcal{A}_x$$

Proof. First of all, if A is unital, then $\mathcal{A}_\mathbb{G}$ contains the unit of A because given the trivial representation $\epsilon \in Irr(\mathbb{G})$ (which is represented by the element $1 \otimes 1_{C(\mathbb{G})} \in \mathbb{C} \otimes C(\mathbb{G})$) the linear map

$$\begin{array}{rccc} T_\epsilon : & \mathbb{C} & \longrightarrow & A \\ & z & \longmapsto & T_\epsilon(z) := z \cdot 1_A \end{array}$$

is clearly a linear map in $Mor(\epsilon, \delta)$. Indeed,

$$\delta(T_\epsilon(z)) = \delta(z \cdot 1_A) = z(1_A \otimes 1_{C(\mathbb{G})}) = (T_\epsilon \otimes id_{C(\mathbb{G})})(1 \otimes 1_{C(\mathbb{G})})(z \otimes id_{C(\mathbb{G})})$$

Hence, $1_A = T_\epsilon(1) \in \mathcal{A}_\mathbb{G}$. Next, given two irreducible representations $x, y \in Irr(\mathbb{G})$, consider the elements $a := T(\xi), b := S(\eta) \in \mathcal{A}_\mathbb{G}$ with $T \in Mor(x, \delta)$, $S \in Mor(y, \delta)$ and $\xi \in H_x$, $\eta \in H_y$. Note that

$$ab := m \circ (T \otimes S)(\xi \otimes \eta),$$

where m denotes the multiplication homomorphism $m : \mathcal{A}_\mathbb{G} \odot \mathcal{A}_\mathbb{G} \longrightarrow \mathcal{A}_\mathbb{G}$. Observe that $m \circ (T \otimes S) \in Mor(x \oplus y, \delta)$ since for every $\xi \in H_x$ and $\eta \in H_y$ we have

$$\begin{aligned} \delta\big(m \circ (T \otimes S)(\xi \otimes \eta)\big) &= \delta(T(\xi))\delta(S(\eta)\big) = \\ &= (T \otimes id_{C(\mathbb{G})})w^x(\xi \otimes id_{C(\mathbb{G})})(S \otimes id_{C(\mathbb{G})})w^y(\eta \otimes id_{C(\mathbb{G})}) \\ &= (\big[T\big]_1\big[S\big]_2 \otimes id_{C(\mathbb{G})})(\big[w^x\big]_{13}\big[w^y\big]_{23})(\xi \otimes \eta \otimes id_{C(\mathbb{G})}) \\ &= (m \circ (T \otimes S) \otimes id_{C(\mathbb{G})})(w^x \oplus w^y)(\xi \otimes \eta \otimes id_{C(\mathbb{G})}) \end{aligned}$$

This shows that $\mathcal{A}_\mathbb{G}$ is a subalgebra of A. Next, given $x \in Irr(\mathbb{G})$ and $a := T(\xi)$ with $T \in Mor(x, \delta)$ and $\xi \in H_x$, we define $a^* := T^+(\xi)^*$, where $T^+ : \xi^* \mapsto (T(\xi))^*$, for all $\xi \in H_x$. Observe that

$$\begin{aligned} \delta(T^+(\xi^*)) = \delta\big((T(\xi))^*\big) &= \Big((T \otimes id_{C(\mathbb{G})})w^x(\xi \otimes id_{C(\mathbb{G})})\Big)^* \\ &= (T^+ \otimes id_{C(\mathbb{G})})w^{\overline{x}}(\xi^* \otimes id_{C(\mathbb{G})}), \end{aligned}$$

so that $\eta^* \mapsto T(\eta)^*$ is in $Mor(\overline{x}, \delta)$. This shows that $\mathcal{A}_\mathbb{G}$ is a $*$-subalgebra. Finally, $\mathcal{A}_\mathbb{G}$ is dense in A. Indeed, by cancellation property of δ and since $Pol(\mathbb{G})$ is dense in $C(\mathbb{G})$ we have

$$A = A \otimes \mathbb{C} \subset A \otimes C(\mathbb{G}) = [\delta(A)(1 \otimes C(\mathbb{G}))] = [\delta(A)(1 \otimes Pol(\mathbb{G}))]$$

This implies that $A \subset \big[(id_A \otimes h_\mathbb{G})\delta(A)(1 \otimes Pol(\mathbb{G}))\big]$. By Remark 1.4.2.11 the space $(id_A \otimes h_\mathbb{G})\delta(A)(1 \otimes Pol(\mathbb{G}))$ is contained in $\mathcal{A}_\mathbb{G} \subset A$, whence $\mathcal{A}_\mathbb{G}$ is dense in A.

Moreover,

i) let us show that the action δ restricts to an action $\delta_{Pol(\mathbb{G})} : \mathcal{A}_{\mathbb{G}} \longrightarrow \mathcal{A}_{\mathbb{G}} \odot Pol(\mathbb{G})$, which is ergodic whenever δ is ergodic. Namely, given an irreducible representation $x \in Irr(\mathbb{G})$ consider the element $a := T(\xi) \in \mathcal{A}_{\mathbb{G}}$ with $T \in Mor(x, \delta)$ and $\xi \in H_x$ and write

$$\delta(a) = \delta(T(\xi)) = (T \otimes id_{C(\mathbb{G})})w^x(\xi \otimes id_{C(\mathbb{G})}) \in \mathcal{A}_x \odot Pol(\mathbb{G}) \subset \mathcal{A}_{\mathbb{G}} \odot Pol(\mathbb{G}),$$

so that $\delta_{Pol(\mathbb{G})}$ is well-defined and it is straightforward that δ intertwines the co-multiplication of $Pol(\mathbb{G})$. Let us check the co-unit condition, that is, the formula

$$(id_{\mathcal{A}_{\mathbb{G}}} \otimes \varepsilon) \circ \delta = id_{\mathcal{A}_{\mathbb{G}}},$$

where ε is co-unit of $Pol(\mathbb{G})$. Given an irreducible representation $x \in Irr(\mathbb{G})$ consider the element $a := T(\xi) \in \mathcal{A}_{\mathbb{G}}$ with $T \in Mor(x, \delta)$ and $\xi \in H_x$ and write

$$\begin{aligned}(id_{\mathcal{A}_{\mathbb{G}}} \otimes \varepsilon)\delta(T(\xi)) &= (id_{\mathcal{A}_{\mathbb{G}}} \otimes \varepsilon)(T \otimes id_{C(\mathbb{G})})w^x(\xi \otimes id_{C(\mathbb{G})}) \\ &= T\Big((id_{H_x} \otimes \varepsilon)w^x(\xi \otimes id_{C(\mathbb{G})})\Big) = T(\xi),\end{aligned}$$

where the last equality can be shown by taken an orthonormal basis of H_x and by applying the definition of ε from Proposition 1.3.1.28.

ii) let us show that the canonical conditional expectation $\mathbb{E}_\delta$ is faithful on $\mathcal{A}_{\mathbb{G}}$. Namely, let $a \in \mathcal{A}_{\mathbb{G}}$ be an element such that $\mathbb{E}_\delta(a^*a) = 0$. Consider any positive linear form $\phi : A \longrightarrow \mathbb{C}$ and write

$$0 = \phi(\mathbb{E}_\delta(a^*a)) = \phi\big((id_A \otimes h_{\mathbb{G}})\delta(a^*a)\big) = h_{\mathbb{G}}\big((\phi \otimes id_{C(\mathbb{G})})\delta(a^*a)\big),$$

where we observe that $(\phi \otimes id_{C(\mathbb{G})})\delta(a^*a) \in Pol(\mathbb{G})$ because δ restricts to an action of $(Pol(\mathbb{G}), \Delta_|)$ on $\mathcal{A}_{\mathbb{G}}$. Besides, it is a positive element (because ϕ is positive) and $h_{\mathbb{G}}$ is a state which is faithful on $Pol(\mathbb{G})$, so it must be $(\phi \otimes id_{C(\mathbb{G})})\delta(a^*a) = 0$. This is true for every positive linear form ϕ and $\delta(a^*a) = \delta(a)^*\delta(a)$ is a positive element in $A \otimes C(\mathbb{G})$. Hence it must be $\delta(a^*a) = 0$. Apply the co-unit of $Pol(\mathbb{G})$: $0 = (id_A \otimes \varepsilon)\delta(a^*a) = a^*a$, so it must be $a = 0$.

Finally, remark that we have shown by the way that $\delta(\mathcal{A}_x) \subset \mathcal{A}_x \odot Pol(\mathbb{G})$. Hence, given two non-equivalent irreducible representations, say $x, y \in Irr(\mathbb{G})$ we have that $\mathbb{E}_\delta(a^*b) = 0$, for all $a \in \mathcal{A}_x$ and all $b \in \mathcal{A}_y$ thanks to the orthogonality relations, which yields the decomposition of the statement.

■

One of the main achievement in spectral theory of quantum groups is the following theorem due to F. Boca [25] and improved later by in J. Bichon, A. De Rijdt and S. Vaes in [23] and by R. Tomatsu in [190]. See for instance Theorem 6.11 in [49] for a proof.

1.4.2.13 Theorem. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and (A, δ) a $\mathbb{G}$-C^*-algebra. If δ is an ergodic action and $x \in Irr(\mathbb{G})$ is an irreducible representation of $\mathbb{G}$, then the x-spectral space of δ, $\mathcal{A}_x$, is finite dimensional and we have*

$$dim(\mathcal{A}_x) \leqslant dim_q(x)$$

Next, it is advisable to establish some notations and auxiliary results, which will be applied later on in this dissertation. It is important to recall notations and formulas of Remarks 1.3.1.31.

1.4.2.14 Definition. Let $\mathbb{G}$ be a compact quantum group and (A, δ) a unital $\mathbb{G}$-C^*-algebra. Consider irreducible representations of $\mathbb{G}$, say $x, y, z \in Irr(\mathbb{G})$ and an intertwiner $\Phi \in Mor(z, x \oplus y)$. Given $X \in K_x$ and $Y \in K_y$ we define their spectral product with respect to Φ as the element

$$X \underset{\Phi}{\otimes} Y := \Big(\big[X\big]_{13} \big[Y\big]_{23} \Big) (\Phi \otimes 1_A) \in K_z,$$

where $\big[X\big]_{13}$ and $\big[Y\big]_{23}$ are the corresponding legs of X and Y in $\overline{H}_x \otimes \overline{H}_y \otimes A$.

1.4.2.15 Remarks. 1. Let us check that $X \underset{\Phi}{\otimes} Y$ is indeed an element of the z-spectral space K_z

$$\begin{aligned}
(id \otimes \delta)(X \underset{\Phi}{\otimes} Y) &= (id \otimes \delta)(\big[X\big]_{13} \big[Y\big]_{23})(\Phi \otimes 1_A) \\
&= (id \otimes \delta)(\big[X\big]_{13})(id \otimes \delta)(\big[Y\big]_{23})(\Phi \otimes 1_A) \\
&= \big[\big[X\big]_{13}\big]_{13} \big[w^x\big]_{14} \big[\big[Y\big]_{23}\big]_{23} \big[w^y\big]_{24} (\Phi \otimes 1_A) \\
&= \big[\big[X\big]_{13}\big]_{13} \big[\big[Y\big]_{23}\big]_{23} \big[w^x\big]_{14} \big[w^y\big]_{24} (\Phi \otimes 1_A) \\
&= \big[\big[X\big]_{13}\big]_{13} \big[\big[Y\big]_{23}\big]_{23} \big[w^{x \oplus y}\big]_{124} (\Phi \otimes 1_A) \\
&= \big[\big[X\big]_{13}\big]_{13} \big[\big[Y\big]_{23}\big]_{23} (\Phi \otimes 1_{C(\mathbb{G})}) \big[w^z\big]_{13} \\
&= \big[X \underset{\Phi}{\otimes} Y\big]_{12} \big[w^z\big]_{13}
\end{aligned}$$

2. Given $X \in K_x$ and $Y \in K_y$, the analogous computation as above yields the following formula

$$(id \otimes \delta)\Big(\big[X\big]_{13} \big[Y\big]_{23} \Big) = \Big[\big[X\big]_{13} \big[Y\big]_{23} \Big]_{12} \big[w^{x \oplus y}\big]_{13},$$

where the legs of the right hand side of the identity are considered in $\overline{H}_{x \oplus y} \otimes A \otimes C(\mathbb{G})$. Hence, the decomposition of $x \oplus y$ in direct sum of irreducible representations, say $\{z_k\}_{k=1,\ldots,r}$ for some $r \in \mathbb{N}$ yields that

$$\big[X\big]_{13} \big[Y\big]_{23} \in \bigoplus_{k=1}^{r} K_{z_k}$$

Namely, by virtue of the decomposition $x \oplus y = \bigoplus_{k=1}^{r} z_k$ we have the identification $H_{x \oplus y} \cong H_{z_1} \oplus \ldots \oplus H_{z_r}$. If $\{p_k\}_{k=1,\ldots,r} \subset \mathcal{B}(H_{x \oplus y})$ denotes the set of mutually orthogonal finite-dimensional projections with sum $id_{H_{x \oplus y}}$, then by construction we have

$$w^{z_k} = w^{x \oplus y}(p_k \otimes 1_{C(\mathbb{G})}),\ \forall k = 1, \ldots, r$$

$$\big[X\big]_{13} \big[Y\big]_{23} = \sum_{k=1}^{r} (p_k \otimes 1_A)\Big(\big[X\big]_{13} \big[Y\big]_{23} \Big) = \sum_{k=1}^{r} (p_k \otimes 1_A)\Big(\big[X\big]_{13} \big[Y\big]_{23} \Big)(p_k \otimes id_A),$$

where we remark that

$$(id \otimes \delta)\Big((p_k \otimes 1_A)\Big(\big[X\big]_{13} \big[Y\big]_{23} \Big)(p_k \otimes 1_A) \Big) = \Big[(p_k \otimes 1_A)\Big(\big[X\big]_{13} \big[Y\big]_{23} \Big) \Big]_{12} \big[w^{z_k}\big]_{13},$$

for all $k = 1, \ldots, r$ thanks to the formula above.

1.4.2.16 Definition. Let $\mathbb{G}$ be a compact quantum group and (A, δ) a unital $\mathbb{G}$-C^*-algebra. Given an irreducible representation $x \in Irr(\mathbb{G})$ and an element $X \in K_x$, we define its spectral conjugate as the element

$$X^{\#} := (* \circ J_x \otimes id_A)(X^*) \in K_{\overline{x}}$$

1.4.2.17 Remarks. 1. Let us write in coordinates the spectral conjugate $X^{\#}$ (recall Remarks 1.3.1.31). Namely, given an irreducible representation of $\mathbb{G}$, say $x \in Irr(\mathbb{G})$, fix an orthonormal basis $\{\xi_1^x, \ldots, \xi_{n_x}^x\}$ of H_x that diagonalizes the canonical operator $Q_x = J_x^* J_x$ and let $\{\omega_1^x, \ldots, \omega_{n_x}^x\}$ be its dual basis in the dual space $\overline{H}_x$.

If the coordinate expression of an element $X \in K_x \subset \overline{H}_x \otimes A$ is given by $X = \sum_{i=1}^{n_x} \omega_i^x \otimes a_i$, for some $a_i \in A$, for all $i = 1, \ldots, n_x$; then we have

$$X^{\#} = \sum_{i=1}^{n_x} J_x(\xi_i^x)^* \otimes a_i^* = \sum_{i=1}^{n_x} (\sqrt{\lambda_i^x}\ \xi_i^{\overline{x}})^* \otimes a_i^*,$$

where $\{\xi_1^{\overline{x}}, \ldots, \xi_{n_x}^{\overline{x}}\}$ is an orthonormal basis of $H_{\overline{x}}$ as in Remarks 1.3.1.31.

2. We must check that $X^{\#}$ is indeed an element of the $\overline{x}$-spectral space $K_{\overline{x}}$.

$$\begin{aligned}
(id \otimes \delta)(X^{\#}) &= (id \otimes \delta)\Big(\sum_{i=1}^{n_x} J_x(\xi_i^x)^* \otimes a_i^*\Big) = \sum_{i=1}^{n_x} (\sqrt{\lambda_i^x}\ \xi_i^{\overline{x}})^* \otimes \delta(a_i^*) \\
&= \sum_{i=1}^{n_x} \Big(\frac{\sqrt{\lambda_i^x}\ (\omega_i^x)^*}{\sqrt{\lambda_i^x}}\Big) \otimes \delta(a_i)^* = \Big(\sum_{i=1}^{n_x} \omega_i^x \otimes \delta(a_i)\Big)^* \\
&= \Big((id \otimes \delta)(X)\Big)^* = \Big(\big[X\big]_{12}\big[w^x\big]_{13}\Big)^* = \Big(\big[X\big]_{12}\Big)^* \Big(\big[w^x\big]_{13}\Big)^* \\
&= \Big(\sum_{i=1}^{n_x} \omega_i^x \otimes a_i \otimes 1_{C(\mathbb{G})}\Big)^* \ \big[w^{\overline{x}}\big]_{13} \\
&= \sum_{i=1}^{n_x} \Big(\frac{\sqrt{\lambda_i^x}\ \omega_i^x}{\sqrt{\lambda_i^x}}\Big)^* \otimes a_i^* \otimes 1_{C(\mathbb{G})}\ \big[w^{\overline{x}}\big]_{13} \\
&= \sum_{i=1}^{n_x} (\sqrt{\lambda_i^x}\ \xi_i^{\overline{x}})^* \otimes a_i^* \otimes 1_{C(\mathbb{G})}\ \big[w^{\overline{x}}\big]_{13} \\
&= \sum_{i=1}^{n_x} J_x(\xi_i^x)^* \otimes a_i^* \otimes 1_{C(\mathbb{G})}\ \big[w^{\overline{x}}\big]_{13} = \big[X^{\#}\big]_{12}\big[w^{\overline{x}}\big]_{13}
\end{aligned}$$

3. Moreover, the association $X \longmapsto X^{\#}$ is anti-linear thanks to the anti-linearity of J_x and the vector space structure of the dual space $\overline{H}_x$. Indeed, given $\lambda \in \mathbb{C}$ we write

$$
\begin{aligned}
(\lambda\ X)^{\#} &= \Big(\lambda \sum_{i=1}^{n_x} \omega_i^x \otimes a_i\Big)^{\#} = \Big(\sum_{i=1}^{n_x} \lambda\ \omega_i^x \otimes a_i\Big)^{\#} \\
&\overset{(1)}{=} \sum_{i=1}^{n_x} J_x(\overline{\lambda}\ \xi_i^x)^* \otimes a_i^* = \sum_{i=1}^{n_x} \overline{\lambda}\ J_x(\xi_i^x)^* \otimes a_i^* \\
&= \overline{\lambda} \sum_{i=1}^{n_x} J_x(\xi_i^x)^* \otimes a_i^* = \overline{\lambda}\ X^{\#},
\end{aligned}
$$

where in (1) we use the fact that the transition between H_x and H_x^* is anti-linear (recall Section 1.1). Notice that the same formula is true if we introduce the scalar $\lambda \in \mathbb{C}$ in the second term of the tensor product.

4. Observe that we can consider the canonical intertwiners $\Phi_x \in Mor(\epsilon, x \oplus \overline{x})$ and $\Phi_{\overline{x}} \in Mor(\epsilon, \overline{x} \oplus x)$. In this case we have by construction that

$$
X \underset{\Phi_x}{\otimes} X^{\#} \in K_\epsilon \text{ and } X^{\#} \underset{\Phi_{\overline{x}}}{\otimes} X \in K_\epsilon
$$

Recall that, by definition, we have $K_\epsilon = A^\delta$. Hence, if δ is an ergodic action of $\mathbb{G}$ on A, we have $K_\epsilon = \mathbb{C}1_A$ and $X \underset{\Phi_x}{\otimes} X^{\#}$, $X^{\#} \underset{\Phi_{\overline{x}}}{\otimes} X$ are scalar multiples of 1_A. More precisely, we have the following coordinate expressions

$$
\begin{aligned}
X \underset{\Phi_x}{\otimes} X^{\#} &= \sum_{i,j} \omega_i^x \otimes J_x(\xi_j^x)^* \circ \Phi_x \otimes a_i a_j^* = \sum_{i,j} \omega_i^x \otimes (\sqrt{\lambda_j^x}\ \xi_j^{\overline{x}})^* \circ \Phi_x \otimes a_i a_j^* \\
&= \sum_{i,j} \omega_i^x \otimes \sqrt{\lambda_j^x}\ \omega_j^{\overline{x}} \circ \Phi_x \otimes a_i a_j^* \\
&= \sum_{i,j} \omega_i^x \otimes \sqrt{\lambda_j^x}\ \omega_j^{\overline{x}} \circ \Big(\sum_{k=1}^{n_x} \sqrt{\lambda_k^x}\ \xi_k^x \otimes \xi_k^{\overline{x}}\Big) \otimes a_i a_j^* \\
&= \sum_{i,j} \sqrt{\lambda_k^x}\ \delta_{i,k} \otimes \sqrt{\lambda_j^x}\ \delta_{j,k} \otimes a_i a_j^* = \sum_{i=1}^{n_x} \lambda_i^x\ a_i a_i^*
\end{aligned}
$$

$$
\begin{aligned}
X^{\#} \underset{\Phi_{\overline{x}}}{\otimes} X &= \sum_{i,j} J_x(\xi_j^x)^* \otimes \omega_i^x \circ \Phi_{\overline{x}} \otimes a_j^* a_i = \sum_{i,j} (\sqrt{\lambda_j^x}\ \xi_j^{\overline{x}})^* \otimes \omega_i^x \circ \Phi_{\overline{x}} \otimes a_j^* a_i \\
&= \sum_{i,j} \sqrt{\lambda_j^x}\ \omega_j^{\overline{x}} \otimes \omega_i^x \circ \Phi_{\overline{x}} \otimes a_j^* a_i \\
&= \sum_{i,j} \sqrt{\lambda_j^x}\ \omega_j^{\overline{x}} \otimes \omega_i^x \circ \Big(\sum_{k=1}^{n_x} \frac{1}{\sqrt{\lambda_k^x}}\ \xi_k^{\overline{x}} \otimes \xi_k^x\Big) \otimes a_j^* a_i \\
&= \sum_{i,j} \frac{\sqrt{\lambda_j^x}}{\sqrt{\lambda_k^x}} \delta_{j,k} \otimes \delta_{i,k} \otimes a_i a_j^* = \sum_{i=1}^{n_x} a_i^* a_i
\end{aligned}
$$

1.4.2.18 Lemma. *Let $\mathbb{G}$ be a compact quantum group and (A,δ) a ergodic $\mathbb{G}$-C^*-algebra. Given irreducible representations $x,y \in Irr(\mathbb{G})$ and non-zero elements $X \in K_x$, $Y \in K_y$; then there exist an irreducible representation $z \in Irr(\mathbb{G})$ and an intertwiner $\Phi \in Mor(z, x \bigoplus y)$ such that $X \underset{\Phi}{\otimes} Y \neq 0$.*

Proof. Let's fix orthonormal basis $\{\xi_1^x, \dots, \xi_{n_x}^x\}$ of H_x and $\{\xi_1^y, \dots, \xi_{n_y}^y\}$ of H_y that diagonalizes the canonical operators $Q_x = J_x^* J_x$ and $Q_y = J_y^* J_y$, respectively; with eigenvalues $\{\lambda_i^x\}_{i=1,\dots,n_x}$ and $\{\mu_j^y\}_{j=1,\dots,n_y}$, respectively. Denote by $\{\omega_1^x, \dots, \omega_{n_x}^x\}$ and $\{\omega_1^y, \dots, \omega_{n_y}^y\}$ the corresponding dual basis of $\overline{H}_x$ and $\overline{H}_y$, respectively. The elements X and Y are written with respect to these basis under the form

$$X = \sum_{i=1}^{n_x} \omega_i^x \otimes a_i \text{ and } Y = \sum_{j=1}^{n_y} \omega_j^y \otimes b_j,$$

for some $a_i, b_j \in A$ for each $i = 1, \dots, n_x$ and $j = 1, \dots, n_y$ so that their spectral product with respect to Φ is given by

$$X \underset{\Phi}{\otimes} Y = \sum_{i,j} \omega_i^x \otimes \omega_j^y \circ \Phi \otimes a_i b_j$$

Suppose that for all irreducible representation $z \in Irr(\mathbb{G})$ and all intertwiner $\Phi \in Mor(z, x \bigoplus y)$ we have $X \underset{\Phi}{\otimes} Y = 0$, that is,

$$X \underset{\Phi}{\otimes} Y = \sum_{i,j} \omega_i^x \otimes \omega_j^y \circ \Phi \otimes a_i b_j = 0$$

Multiplying by $\Phi^* \otimes 1_A$ by the right, the equality still holds

$$\sum_{i,j} \omega_i^x \otimes \omega_j^y \circ \Phi\Phi^* \otimes a_i b_j = 0 \tag{1.4.1}$$

This is true for *every* irreducible representation $z \in Irr(\mathbb{G})$ and *every* intertwiner $\Phi \in Mor(z, x \bigoplus y)$. Given the unitary representation $x \bigoplus y$, let's consider its decomposition in direct sum of irreducible representations (recall Proposition 1.3.1.21). Given $z \subset x \bigoplus y$ an irreducible representation of this decomposition, denote by $p_z^{x \bigoplus y} \in \mathcal{B}(H_x \otimes H_y)$ the corresponding orthogonal finite-dimensional projection such that

$$\sum_{z \subset x \bigoplus y} p_z^{x \bigoplus y} = id_{H_x \otimes H_y}$$

For every $k = 1, \dots, dim\big(Mor(z, x \bigoplus y)\big)$ consider a family of intertwiners $\Phi_k \in Mor(z, x \bigoplus y)$ such that

$$\Phi_k^* \Phi_k = id_{H_z} \text{ and } \sum_{k=1}^{dim\big(Mor(z, x \bigoplus y)\big)} \Phi_k \Phi_k^* = p_z^{x \bigoplus y}$$

Hence we write down the identity (1.4.1) above for these intertwiners Φ_k for each $k = 1, \dots, dim\big(Mor(z, x \bigoplus y)\big)$ and sum over k. We get

$$\sum_{i,j} \omega_i^x \otimes \omega_j^y \circ p_z^{x \bigoplus y} \otimes a_i b_j = 0$$

Next, we can sum over $z \subset x \oplus y$ (since this is true for *every* irreducible representation of $\mathbb{G}$). We get

$$\sum_{i,j} \omega_i^x \otimes \omega_j^y \otimes a_i b_j = 0, \tag{1.4.2}$$

which implies that $a_i b_j = 0$, for all $i = 1, \ldots, n_x$ and all $j = 1, \ldots, n_y$.

Since δ is an ergodic action by assumption we have that

$$X^{\#} \underset{\Phi_{\overline{x}}}{\otimes} X = \sum_{i=1}^{n_x} a_i^* a_i = \lambda\ 1_A \text{ and } Y \underset{\Phi_y}{\otimes} Y^{\#} = \sum_{j=1}^{n_y} \mu_j^y\ b_j b_j^* = \mu\ 1_A,$$

for some $\lambda, \mu \in \mathbb{C}$. Since X and Y are supposed to be non-zero, there exist at least one $i = 1, \ldots, n_x$ and one $j = 1, \ldots, n_y$ such that $a_i \neq 0$ and $b_j \neq 0$. Consequently, $X^{\#} \underset{\Phi_{\overline{x}}}{\otimes} X \neq 0$ and $Y \underset{\Phi_y}{\otimes} Y^{\#} \neq 0$. In other words, we have $\lambda \neq 0$ and $\mu \neq 0$. Using equation (1.4.2) above, we get

$$0 \neq \lambda\ \mu = \Big(\sum_{i=1}^{n_x} a_i^* a_i\Big)\Big(\sum_{j=1}^{n_y} \mu_j^y\ b_j b_j^*\Big) = \sum_{i,j} \mu_j^y\ a_i^* a_i b_j b_j^* = 0,$$

a contradiction. ■

1.4.2.19 Remark. It is important to observe the following. The argument used in the proof of the preceding lemma shows that the irreducible representation $z \in Irr(\mathbb{G})$ and the intertwiner $\Phi \in Mor(z, x \oplus y)$ such that $X \underset{\Phi}{\otimes} Y \neq 0$ are related to $x \oplus y$ in the following way. Given $z \subset x \oplus y$, denote by $p_z^{x \oplus y} \in \mathcal{B}(H_x \otimes H_y)$ the corresponding orthogonal finite dimensional projection and consider a family of intertwiners $\Phi_k \in Mor(z, x \oplus y)$ such that $\Phi_k^* \Phi_k = id_{H_z}$ and $\sum_k \Phi_k \Phi_k^* = p_z^{x \oplus y}$, for all $k = 1, \ldots, dim\Big(Mor(z, x \oplus y)\Big)$. The preceding lemma guarantees that for every $z \subset x \oplus y$ there exists an intertwiner $\Phi_k \in Mor(z, x \oplus y)$ such that $X \underset{\Phi_k}{\otimes} Y \neq 0$.

1.4.3 Induced actions from Discrete Quantum Subgroups

Induction functor concerning quantum groups is a delicate notion whose more general definition for *locally compact quantum groups* can be found in Theorem 7.2 of [194]. However, the treatment of the compact/discrete case yields a more concrete description (see Section 3 of [208] for a detailed exposition). Here we include an overview of this approach for the convenience of the exposition.

First of all, we need a notion of *quantum subgroup*. For the compact case, this notion was introduced by P. Podleś in [157] and further studied by S. Wang in [215], [219]. The general locally compact case is more involved and there is no unanimous definition. S. Vaes introduced in [194] a notion of *closed quantum subgroup* in the context of von Neumann algebras in order to develop all the induction and homogeneous spaces machinery and in [48] S. L. Woronowicz proposed a different approach in the context of C^*-algebras and closer to the initial idea of P. Podleś. Fortunately, both definitions coincide in the case of discrete quantum groups, which is specially interesting for this dissertation and admits, by the way, a more concrete and algebraic picture as we can see in [207].

1.4.3.1 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta_{\mathbb{G}})$ and $\mathbb{H} = (C(\mathbb{H}), \Delta_{\mathbb{H}})$ be two compact quantum groups. We say that $\mathbb{H}$ is a compact quantum subgroup of $\mathbb{G}$, denoted by $\mathbb{H} < \mathbb{G}$, if there exists a surjective

$*$-homomorphism $\rho : C_m(\mathbb{G}) \twoheadrightarrow C_m(\mathbb{H})$ such that

$$\Delta_{\mathbb{H}} \circ \rho = (\rho \otimes \rho)\Delta_{\mathbb{G}}$$

In this case we define the left and right coset spaces as

$$C(\mathbb{G}/\mathbb{H}) := \{a \in C(\mathbb{G}) \mid (id \otimes \rho)\Delta_{\mathbb{G}}(a) = a \otimes 1_{C(\mathbb{G})}\}$$

$$C(\mathbb{H}\backslash\mathbb{G}) := \{a \in C(\mathbb{G}) \mid (\rho \otimes id)\Delta_{\mathbb{G}}(a) = 1_{C(\mathbb{G})} \otimes a\}$$

respectively. We say that $\mathbb{H}$ is a *normal* (compact) quantum group of $\mathbb{G}$ if $C(\mathbb{G}/\mathbb{H}) = C(\mathbb{H}\backslash\mathbb{G})$.

The trivial quantum subgroup of $\mathbb{G}$, denoted by $\mathbb{E}$, is given by

$$\mathbb{E} := (C(\{\bullet\}), \Delta_{\mathbb{E}}) \cong \mathbb{C}$$

with the co-unit map $\varepsilon_{\mathbb{G}} : C_m(\mathbb{G}) \twoheadrightarrow \mathbb{C}$ as the corresponding surjective $*$-homomorphism.

1.4.3.2 Definition. Let $\widehat{\mathbb{G}} = (c_0(\widehat{\mathbb{G}}), \widehat{\Delta}_{\mathbb{G}})$ and $\widehat{\mathbb{H}} = (c_0(\widehat{\mathbb{H}}), \widehat{\Delta}_{\mathbb{H}})$ be two discrete quantum groups. We say that $\widehat{\mathbb{H}}$ is a discrete quantum subgroup of $\widehat{\mathbb{G}}$, denoted by $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$, if there exists an injective homomorphism of $*$-Hopf algebras $\iota : Pol(\mathbb{H}) \hookrightarrow Pol(\mathbb{G})$.

Some important structural properties of discrete quantum subgroups are gathered in the following theorem.

1.4.3.3 Theorem. *Let $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ be discrete quantum groups such that $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$.*

i) (R. Vergnioux, [207]) There exists a faithful conditional expectation

$$E_{\mathbb{H}} : C_r(\mathbb{G}) \longrightarrow C_r(\mathbb{H})$$

which sends $Pol(\mathbb{G})$ to $Pol(\mathbb{H})$ such that

- $(id \otimes E_{\mathbb{H}}) \circ \Delta_{\mathbb{G}} = \Delta_{\mathbb{H}} \circ E_{\mathbb{H}} = (E_{\mathbb{H}} \otimes id) \circ \Delta_{\mathbb{G}}$
- $h_{\mathbb{H}} \circ E_{\mathbb{H}} = h_{\mathbb{G}}$, *so that* $(id \otimes E_{\mathbb{H}})(w^x) = \begin{cases} w^x, \text{ if } x \in Irr(\mathbb{H}) \\ 0, \text{ otherwise} \end{cases}$, *for all* $x \in Irr(\mathbb{G})$.

ii) (P. Fima, [62]) Define the central projection $p_{\mathbb{H}} := \sum_{z \in Irr(\mathbb{H})} p_z$. Then we have that

- $\widehat{\Delta}_{\mathbb{G}}(p_{\mathbb{H}})(p_{\mathbb{H}} \otimes 1) = p_{\mathbb{H}} \otimes p_{\mathbb{H}}$,
- $M\big(c_0(\widehat{\mathbb{H}})\big) = p_{\mathbb{H}}\big(M\big(c_0(\widehat{\mathbb{G}})\big)\big)$,
- $\widehat{\Delta}_{\mathbb{H}}(a) = \widehat{\Delta}_{\mathbb{G}}(a)(p_{\mathbb{H}} \otimes p_{\mathbb{H}})$, *for all* $a \in c_0(\widehat{\mathbb{G}})$,
- *if $\widehat{h}_{L,\mathbb{G}}$ denotes the left Haar weight for $\widehat{\mathbb{G}}$, then the map $a \mapsto \widehat{h}_{L,\mathbb{G}}\big(p_{\mathbb{H}}(a)\big)$ for all $a \in c_0(\widehat{\mathbb{G}})$, defines a left Haar weight for $\widehat{\mathbb{H}}$ and it is denoted by $\widehat{h}_{L,\mathbb{H}}$.*

The definition of discrete quantum subgroup admits several equivalent formulations which are useful in practice. We refer to [48] and [194] for more details about this notion for general locally compact quantum groups.

1.4.3.4 Proposition. *Let $\widehat{\mathbb{H}}$ and $\widehat{\mathbb{G}}$ be two discrete quantum groups. The following assertions are equivalent.*

i) $\widehat{\mathbb{H}}$ *is a discrete quantum subgroup of* $\widehat{\mathbb{G}}$, $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$.

ii) $\mathscr{R}ep(\mathbb{H})$ *is a full subcategory of* $\mathscr{R}ep(\mathbb{G})$ *containing the trivial representation and stable by direct sums, tensor product and adjoint operations.*

iii) There exists an injective $*$*-homomorphism* $\iota^r_{\mathbb{H}} : C_r(\mathbb{H}) \hookrightarrow C_r(\mathbb{G})$ *that intertwines the co-multiplications.*

iv) There exists an injective $*$*-homomorphism* $\iota^m_{\mathbb{H}} : C_m(\mathbb{H}) \hookrightarrow C_m(\mathbb{G})$ *that intertwines the co-multiplications.*

v) There exists a non-degenerate $*$*-homomorphism* $\widehat{\rho} : c_0(\widehat{\mathbb{G}}) \longrightarrow M(c_0(\widehat{\mathbb{H}}))$ *such that* $\widehat{\rho}(c_0(\widehat{\mathbb{G}})) = c_0(\widehat{\mathbb{H}})$ *and that intertwines the co-multiplications.*

vi) There exists a normal injective $*$*-homomorphism* $\widetilde{\iota} : L^\infty(\mathbb{H}) \hookrightarrow L^\infty(\mathbb{G})$ *that intertwines the co-multiplications.*

Proof. - $(i) \Leftrightarrow (ii)$. This is true by virtue of Tannaka-Krein-Woronowicz duality (see Theorem B.3.16). In addition, the Tannaka-Krein reconstruction theorem of Woronowicz describe explicitly the discrete quantum subgroup $\widehat{\mathbb{H}}$ in terms of the corresponding full subcategory $\mathscr{R}ep(\mathbb{H})$. Namely, $c_0(\widehat{\mathbb{H}})$ is obtained as the sum of matrix blocks of $c_0(\widehat{\mathbb{G}})$ corresponding to the irreducible representations in $Irr(\mathbb{H})$ (see Remark B.3.17).

- $(i) \Rightarrow (iii)$. Since we have an injective homomorphism of $*$-Hopf algebras $\iota : Pol(\mathbb{H}) \hookrightarrow Pol(\mathbb{G})$, then we obtain by completion a subalgebra $A := \overline{\iota(Pol(\mathbb{H}))}$ of $C_r(\mathbb{G})$ which is stable under the co-multiplication of $\mathbb{G}$ (that is, $\Delta_{\mathbb{G}}(A) \subset A \otimes A$). Hence $(A, \Delta_{\mathbb{G}})$ is a compact quantum group. Recall that the Haar state of $\mathbb{G}$ is faithful on $C_r(\mathbb{G})$, so its restriction is still faithful on A. We deduce that $A = C_r(\mathbb{H})$. Notice that the inclusion of C^*-algebras $C_r(\mathbb{H}) \hookrightarrow C_r(\mathbb{G})$ is compatible with the co-multiplications by construction.

- $(i) \Leftarrow (iii)$. If A is a C^*-subalgebra of $C_r(\mathbb{G})$ stable under the co-multiplication of $\mathbb{G}$ such that $\mathbb{H} := (A, \Delta)$ is a compact quantum group, then $A = C_r(\mathbb{H})$ because its Haar state is faithful. Hence any representation of $\mathbb{H}$ induces a representation of $\mathbb{G}$ and so $\widehat{\mathbb{H}}$ is a discrete quantum subgroup of $\widehat{\mathbb{G}}$.

- $(i) \Rightarrow (iv)$. Since we have an injective homomorphism of $*$-Hopf algebras $\iota : Pol(\mathbb{H}) \hookrightarrow Pol(\mathbb{G})$, then we identify $Pol(\mathbb{H})$ to a $*$-Hopf subalgebra of $Pol(\mathbb{G})$, namely $Pol(\mathbb{H}) \cong \iota(Pol(\mathbb{H})) \subset Pol(\mathbb{G})$.

  By universal property, the homomorphism of $*$-Hopf algebras $\iota : Pol(\mathbb{H}) \hookrightarrow Pol(\mathbb{G}) \subset C_m(\mathbb{G})$ extends to a homomorphism of C^*-algebras, say $\iota^m_{\mathbb{H}} : C_m(\mathbb{H}) \longrightarrow C_m(\mathbb{G})$, which intertwines the co-multiplications. We have to show that $\iota^m_{\mathbb{H}}$ is still injective. For this, we shall show that $C_m(\mathbb{H}) = C^*(\iota(Pol(\mathbb{H})))$, which is straightforward by universal property.

- $(i) \Leftarrow (iv)$. This is straightforward because the inclusion $C_m(\mathbb{H}) \hookrightarrow C_m(\mathbb{G})$ intertwining the co-multiplications yields, by definition of the maximal compact quantum groups, an obvious inclusion of $*$-Hopf algebras $Pol(\mathbb{H}) \hookrightarrow Pol(\mathbb{G})$.

- $(ii) \Rightarrow (v)$. As explained in the equivalence $(i) \Leftrightarrow (ii)$ above, $c_0(\widehat{\mathbb{H}})$ is obtained as the sum of matrix blocks of $c_0(\widehat{\mathbb{G}})$ corresponding to the irreducible representations in $Irr(\mathbb{H})$ by the Tannaka-Krein reconstruction theorem of Woronowicz. More precisely, we have

$$c_0(\widehat{\mathbb{G}}) = \bigoplus_{x \in Irr(\mathbb{G})}^{c_0} \mathcal{B}(H_x) \text{ and } c_0(\widehat{\mathbb{H}}) = \bigoplus_{x \in Irr(\mathbb{H})}^{c_0} \mathcal{B}(H_x) \subset c_0(\widehat{\mathbb{G}})$$

  Hence, we consider the canonical surjection (so a non-degenerate $*$-homomorphism) $\widehat{\rho} : c_0(\widehat{\mathbb{G}}) \twoheadrightarrow c_0(\widehat{\mathbb{H}}) \subset M(c_0(\widehat{\mathbb{H}}))$ given by the central projection $p_{\mathbb{H}} = \sum_{z \in Irr(\mathbb{H})} p_z$. In addition, $\widehat{\rho}$ intertwines the co-multiplications by construction.

- $(v) \Rightarrow (vi)$. See Theorem 3.3, Theorem 3.6 and Theorem 6.2 in [48].

- $(vi) \Rightarrow (i)$. Recall that $L^\infty(\mathbb{H}) := C_r(\mathbb{H})''$ and $L^\infty(\mathbb{G}) := C_r(\mathbb{G})''$ (see Remark 1.3.2.10), where

$$C_r(\mathbb{H}) = \overline{Pol(\mathbb{H})}^{||\cdot||_{\mathcal{B}(L^2(\mathbb{H}))}} \subset \overline{Pol(\mathbb{H})}^{\sigma-w} = C_r(\mathbb{H})''$$

$$C_r(\mathbb{G}) = \overline{Pol(\mathbb{G})}^{||\cdot||_{\mathcal{B}(L^2(\mathbb{G}))}} \subset \overline{Pol(\mathbb{G})}^{\sigma-w} = C_r(\mathbb{G})''$$

  Since $\widetilde{\iota} : L^\infty(\mathbb{H}) \hookrightarrow L^\infty(\mathbb{G})$ is a normal injective $*$-homomorphism, then by continuity it restricts to an injective $*$-homomorphism $\iota : Pol(\mathbb{H}) \hookrightarrow Pol(\mathbb{G})$. Since $\widetilde{\iota}$ intertwines the co-multiplications by assumption, then ι is a homomorphism of $*$-Hopf algebras.

■

1.4.3.5 Remarks. 1. Let $\widehat{\mathbb{G}}$, $\widehat{\mathbb{H}}$ be discrete quantum groups. If $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$ is a discrete quantum subgroup, then we have the canonical surjection $\widehat{\rho} : c_0(\widehat{\mathbb{G}}) \to M(c_0(\widehat{\mathbb{H}}))$ by the preceding proposition. Assume that A is any $\widehat{\mathbb{G}}$-C^*-algebra with (left) action $\alpha : A \longrightarrow \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A)$. Then we can *restrict* the action α in order to have an action $\alpha_{\widehat{\mathbb{H}}}$ of $\widehat{\mathbb{H}}$ on A. This means precisely that the composition with the canonical surjection $\widehat{\rho}$ yields an action of $\widehat{\mathbb{H}}$ on A,

$$\alpha_{\widehat{\mathbb{H}}} : A \xrightarrow{\alpha} \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A) \xrightarrow{\widehat{\rho} \otimes id_A} \widetilde{M}(c_0(\widehat{\mathbb{H}}) \otimes A)$$

  Routine computations show that the pair $(A, \alpha_{\widehat{\mathbb{H}}})$ satisfies Definition 1.4.1.1.

2. Let $\widehat{\mathbb{G}}$, $\widehat{\mathbb{H}}$ be discrete quantum groups. If $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$ is a discrete quantum subgroup, then we have $C_m(\mathbb{H}) \subset C_m(\mathbb{G})$ by the preceding proposition. Assume that A is any $\mathbb{H}$-C^*-algebra with (right) action $\delta : A \longrightarrow M(A \otimes C_m(\mathbb{H}))$. Since $C_m(\mathbb{H}) \subset C_m(\mathbb{G})$, we can obviously extend δ into a action of $\mathbb{G}$ on A, say $\widetilde{\delta} : A \longrightarrow M(A \otimes C_m(\mathbb{H})) \subset M(A \otimes C_m(\mathbb{G}))$. Observe that $\widetilde{\delta}$ is ergodic whenever δ is ergodic.

   For this reason, we write $Ind_{\mathbb{H}}^{\mathbb{G}}(\delta) := Ind_{\mathbb{H}}^{\mathbb{G}}(A, \delta)$ for the same C^*-algebra A equipped now with the action $\widetilde{\delta}$. We must be careful to not mislead $Ind_{\mathbb{H}}^{\mathbb{G}}(\cdot)$ with $Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(\cdot)$, which will be defined in Theorem 1.4.3.8 below. The difference will be clear by the context.

Let us summarize the induction process in the quantum setting following [208]. For the full details of the next results, we refer to Section 3 in [208]. It is worth mentioning that the construction given in [208] uses the algebraic approach for quantum groups in the sense of A. Van Daele. Namely, if $\mathbb{G}$ is a compact quantum group, the algebraic level of the right leg of $W_{\mathbb{G}}$, $Pol(\widehat{\mathbb{G}}) := A_0(W_{\mathbb{G}})$, must be regarded as a dense multiplier Hopf $*$-algebra inside $c_0(\widehat{\mathbb{G}})$. For more details about these notions we refer to the original articles [201], [202] and to Chapter 2 of [188].

1.4.3.6 Definition. Let $\widehat{\mathbb{H}} = (c_0(\widehat{\mathbb{H}}), \widehat{\Delta}_{\mathbb{H}})$ be a discrete quantum subgroup of a discrete quantum group $\widehat{\mathbb{G}} = (c_0(\widehat{\mathbb{G}}), \widehat{\Delta}_{\mathbb{G}})$. The algebra of all functions on the homogeneous space $\widehat{\mathbb{G}}/\widehat{\mathbb{H}}$ is defined as

$$\mathcal{F}(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}) := \{a \in M\big(Pol(\widehat{\mathbb{G}})\big) \mid (id \odot \widehat{\rho})\widehat{\Delta}_{\mathbb{G}}(a) = a \odot 1\},$$

where $\widehat{\rho} : c_0(\widehat{\mathbb{G}}) \twoheadrightarrow M(c_0(\widehat{\mathbb{H}}))$ is the canonical projection and $M\big(Pol(\widehat{\mathbb{G}})\big)$ denotes the algebraic multipliers of $Pol(\widehat{\mathbb{G}})$ in the sense of A. Van Daele [202]. Moreover we define

$$Pol(\mathbb{G}/\mathbb{H}) := Pol(\mathbb{G}) \underset{Pol(\mathbb{H})}{\odot} \mathbb{C}$$

1.4.3.7 Remark. Notice that both $Pol(\mathbb{G})$ and $\mathbb{C}$ are $Pol(\mathbb{H})$-modules. Indeed, $Pol(\mathbb{H})$ acts on $Pol(\mathbb{G})$ via the canonical injection $\iota : Pol(\mathbb{H}) \hookrightarrow Pol(\mathbb{G})$ and $Pol(\mathbb{H})$ acts on $\mathbb{C}$ via the co-unit $\varepsilon_{\mathbb{H}} : Pol(\mathbb{H}) \longrightarrow \mathbb{C}$.

In order to state the definition of induced actions by discrete quantum subgroups, we need some nomenclature. Consider a discrete quantum subgroup $\widehat{\mathbb{H}}$ of a discrete quantum group $\widehat{\mathbb{G}}$ (recall Proposition 1.4.3.4 above)

- For each $x \in Irr(\mathbb{G})$ we define

  $$Pol(\mathbb{G})_x := span\{w^x_{ij} \mid i, j = 1, \ldots, n_x\} \subset Pol(\mathbb{G})$$

  Hence, by definition of tensor product of representations, we have

  $$Pol(\mathbb{G})_x \cdot Pol(\mathbb{G})_y = \bigoplus_{z \subset x \oplus y} Pol(\mathbb{G})_z,$$

  for all $x, y \in Irr(\mathbb{G})$.

- We define the following equivalence relation on $Irr(\mathbb{G})$: given $x, y \in Irr(\mathbb{G})$

  $$x \underset{L}{\sim} y \Leftrightarrow y \subset x \oplus z \Leftrightarrow z \subset \overline{x} \oplus y, \text{ for some } z \in Irr(\mathbb{H})$$

  In this case, we write $Irr(\mathbb{G})/\underset{L}{\sim} := Irr(\mathbb{G})/Irr(\mathbb{H})$ for the corresponding quotient space. Likewise, given $x, y \in Irr(\mathbb{G})$ we define

  $$x \underset{R}{\sim} y \Leftrightarrow y \subset z \oplus x \Leftrightarrow z \subset y \oplus \overline{x}, \text{ for some } z \in Irr(\mathbb{H})$$

  In this case, we write $\underset{R}{\sim} \backslash Irr(\mathbb{G}) := Irr(\mathbb{H}) \backslash Irr(\mathbb{G})$ for the corresponding quotient space.

- For every class $[x] \in Irr(\mathbb{G})/\underset{L}{\sim}$ we define

$$Pol(\mathbb{G})_{[x]} := \bigoplus_{x' \in [x]} Pol(\mathbb{G})_{x'}$$

In particular, we have $Pol(\mathbb{G})_{[\epsilon]} = Pol(\mathbb{H})$.

- With these notations we have

$$Pol(\mathbb{G}) = \bigoplus_{[x] \in Irr(\mathbb{G})/\underset{L}{\sim}} Pol(\mathbb{G})_{[x]} \text{ and } M(Pol(\widehat{\mathbb{G}})) = \prod_{[x] \in Irr(\mathbb{G})/\underset{L}{\sim}} \mathcal{B}(H_x)$$

- Given $x \in Irr(\mathbb{G})$, denote by $p_x \in M(Pol(\widehat{\mathbb{G}}))$ the central projection on $\mathcal{B}(H_x)$, which is associated to the identity in $\mathcal{B}(H_x)$. Next, given $[x] \in Irr(\mathbb{G})/\underset{L}{\sim}$, we define

$$p_{[x]} := \sum_{x' \in [x]} p_{x'} \in M(Pol(\widehat{\mathbb{G}}))$$

Hence, by construction we have that $\sum\limits_{[x] \in Irr(\mathbb{G})/\underset{L}{\sim}} p_{[x]} = id$. Moreover, we can show that $p_{[x]} \in \mathcal{F}(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$.

- We define the following spaces

$$\mathcal{F}_c(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}) := \bigoplus_{[x] \in Irr(\mathbb{G})/\underset{L}{\sim}} p_{[x]}\Big(\mathcal{F}(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})\Big) \subset M(Pol(\widehat{\mathbb{G}})) \text{ and } c_0(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}) := \overline{\mathcal{F}_c(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})}^{||\cdot||_{\mathcal{B}(L^2(\mathbb{G}))}}$$

- If B is a $*$-algebra (resp. C^*-algebra), we define the following $*$-subalgebra of $M(Pol(\widehat{\mathbb{G}}) \odot B)$ - which denotes the algebraic multipliers of $Pol(\widehat{\mathbb{G}}) \odot B$ in the sense of A. Van Daele [202] - (resp. $M(c_0(\widehat{\mathbb{G}}) \otimes B)$)

$$\mathcal{F}(\widehat{\mathbb{G}}, B) := \prod_{x \in Irr(\mathbb{G})} \mathcal{B}(H_x) \odot B \ \Big(\text{ resp. } c_b(\widehat{\mathbb{G}}, B) := \bigoplus_{x \in Irr(\mathbb{G})}^{l^\infty} \mathcal{B}(H_x) \otimes B \subset \mathcal{F}(\widehat{\mathbb{G}}, B)\Big)$$

If B is equipped with a (injective) left co-action of $Pol(\widehat{\mathbb{H}})$, say $\beta : B \longrightarrow M(Pol(\widehat{\mathbb{H}}) \odot B)$, then we define

$$\mathcal{F}(\widehat{\mathbb{G}}, B)^{\widehat{\mathbb{H}}} := \{f \in \mathcal{F}(\widehat{\mathbb{G}}, B) \mid (\overline{\Delta} \odot id)(f) = (id \odot \beta)(f)\},$$

where $\overline{\Delta} = (id \odot \widehat{\rho})\widehat{\Delta}_{\mathbb{G}}$ is the action of $\widehat{\mathbb{H}}$ on $\widehat{\mathbb{G}}$ by right translations. Remark that $p_{[x]} \odot 1$ is central in $M(\mathcal{F}(\widehat{\mathbb{G}}, B)^{\widehat{\mathbb{H}}})$, for every $[x] \in Irr(\mathbb{G})/\underset{L}{\sim}$. In particular, if $B := \mathbb{C}$ equipped with the trivial co-action of $Pol(\widehat{\mathbb{H}})$, then it is clear that $\mathcal{F}(\widehat{\mathbb{G}}, \mathbb{C})^{\widehat{\mathbb{H}}} = \mathcal{F}(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$.

1.4.3.8 Theorem-Definition. *Let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup of a discrete quantum group $\widehat{\mathbb{G}}$. If (B,β) is a $\widehat{\mathbb{H}}$-algebra, the algebraic induced algebra of (B,β) is the following $\widehat{\mathbb{G}}$-algebra,*

$$alg.Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B,\beta) := \bigoplus_{[x]\in Irr(\mathbb{G})/_{\sim_L}} p_{[x]}\Big(\mathcal{F}(\widehat{\mathbb{G}},B)^{\widehat{\mathbb{H}}}\Big) \subset \mathcal{F}(\widehat{\mathbb{G}},B)^{\widehat{\mathbb{H}}}$$

with action $alg.Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(\beta) := \widehat{\Delta}_{\mathbb{G}} \odot id_B$.

Besides, if (B,β) is a $\widehat{\mathbb{H}}$-C^*-algebra, we have that $alg.Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B,\beta) \subset c_b(\widehat{\mathbb{G}},B)$. We define the induced algebra of (B,β) as the closure of the algebraic induction inside $c_b(\widehat{\mathbb{G}},B)$, that is,

$$Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B,\beta) := \overline{alg.Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B,\beta)} \subset M(c_0(\widehat{\mathbb{G}})\otimes B)$$

In other words, we have that

$$Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B,\beta) = \bigoplus_{x\in Irr(\mathbb{G})}^{c_0} p_{[x]}\Big(alg.Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B,\beta)\Big),$$

which is a $\widehat{\mathbb{G}}$-C^*-algebra with action $Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(\beta) := \widehat{\Delta}_{\mathbb{G}} \otimes id_B$.

In particular, if $B=\mathbb{C}$ with the trivial action, then $Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(\mathbb{C}) = c_0(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}) := \overline{alg.Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(\mathbb{C})}$.

1.4.3.9 Remark. As shown in Proposition 3.5 of [208], the preceding notion of induced $\widehat{\mathbb{H}}$-C^*-algebras coincides with the analogous notion given by S. Vaes in Theorem 7.2 in [194] for the general case of locally compact quantum groups and closed quantum subgroups. More precisely, using the same notations as in Section 1.3.2, $Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)$ is the unique C^*-subalgebra of

$$\widetilde{B} := \{X \in M\big(\mathcal{K}(L^2(\mathbb{G}))\otimes B\big) \mid X \in (l^\infty(\widehat{\mathbb{G}})'\otimes 1)' \text{ and } (\overline{\Delta}\otimes id_B)(X) = (id\otimes\beta)(X)\}$$

such that

i) $\widehat{\Delta}\otimes id_B : Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B) \longrightarrow M(c_0(\widehat{\mathbb{G}})\otimes Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B))$ is an action of $\widehat{\mathbb{G}}$ on $Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)$.

ii) $\widehat{\Delta}\otimes id_B : \widetilde{B} \longrightarrow M(c_0(\widehat{\mathbb{G}})\otimes Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B))$ is a well-defined $*$-homomorphism which is strictly continuous on the unit ball of $\widetilde{B}$.

iii) $Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B) \subset \widetilde{B}$ is non-degenerate, meaning that $\big[Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)(L^2(\mathbb{G})\otimes B)\big] = L^2(\mathbb{G})\otimes B$.

Using this explicit description of induced C^*-algebras, we see that any $\widehat{\mathbb{H}}$-C^*-algebra B is a retract of $Res_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}\Big(Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)\Big)$. In other words, there exist $\widehat{\mathbb{H}}$-equivariant homomorphisms $B \longrightarrow Res_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}\Big(Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)\Big)$ and $Res_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}\Big(Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)\Big) \longrightarrow B$, whose composition yields the identity on B.

Let us give more details of this retraction. Given $b\in B$ we claim that the element $\beta(b) \in \widetilde{M}(c_0(\widehat{\mathbb{H}})\otimes B)$ is an element in $\widetilde{B}$. On the one hand, since $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$ then we have by Theorem 1.4.3.3 that $M\big(c_0(\widehat{\mathbb{H}})\big) = \widehat{\rho}\big(M\big(c_0(\widehat{\mathbb{G}})\big)\big) = \widehat{\rho}(l^\infty(\widehat{\mathbb{G}}))$ so that it is clear that $[\beta(b), y\otimes 1] = 0$ for every $y\in l^\infty(\widehat{\mathbb{G}})'$, that is, $\beta(b)\in(l^\infty(\widehat{\mathbb{G}})'\otimes 1)'$. On the other hand, using the formulae of Theorem 1.4.3.3 we write

$$(id\otimes\beta)(\beta(b)) = (\widehat{\Delta}_{\mathbb{H}}\otimes id_B)\beta(b) = \big(\widehat{\Delta}_{\mathbb{G}}(\widehat{\rho}\otimes\widehat{\rho})\otimes id_B\big)\beta(b) = \big((id\otimes\widehat{\rho})\widehat{\Delta}_{\mathbb{G}}\otimes id_B\big)\beta(b) = (\overline{\Delta}\otimes id_B)(\beta(b)),$$

which yields the claim and allows to define the following $\widehat{\mathbb{H}}$-equivariant homomorphism

$$\begin{array}{ccc} B & \longrightarrow & Res_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}\Big(Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)\Big) \\ b & \longmapsto & \beta(b) \end{array}$$

Next, we define the following homomorphism

$$\begin{array}{ccc} Res_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}\Big(Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)\Big) & \longrightarrow & B \\ x & \longmapsto & (\widehat{\varepsilon}\otimes id_B)(x), \end{array}$$

where $\widehat{\varepsilon}_{\mathbb{H}} : c_0(\widehat{\mathbb{H}}) \longrightarrow \mathbb{C}$ is the co-unit of $\widehat{\mathbb{H}}$ (recall Remark 1.3.1.38). Notice that, according to Note 1.4.1.2, all actions of discrete quantum groups are supposed to be *injective*. Hence, as we have showed in Remarks 1.4.1.3, the formula $(\widehat{\varepsilon}_{\mathbb{H}}\otimes id_B)\circ\beta = id_B$ holds, which implies that the preceding map is $\widehat{\mathbb{H}}$-equivariant and that B is a retract of $Res_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}\Big(Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)\Big)$.

1.5 Crossed Products by Discrete (Quantum) Groups

In order to study the Baum-Connes property for a quantum group, we have to deal with *reduced* crossed products. This is why we have to define such a crossed product by a *quantum* group. We can give a very general construction in the context of multiplicative unitaries in the sense of Baaj-Skandalis (see for instance [7], [206], [188]). But using the rich representation theory of a compact quantum group as we have presented in Section 1.3, we can give an explicit construction of a crossed product by a *discrete* quantum group which is, by the way, very close to the classical one. Actually we're going to establish a *universal* property for the *reduced* crossed product by a *discrete quantum group* that will be very useful for our purpose.

1.5.1 Classical crossed products

For more information about classical crossed products, we refer to Section 2 and Section 7.2 of [147] or Section 4.1 of [29]. We include here a proof of the *universal property of the reduced crossed product by a discrete group* for the convenience of the exposition. Afterwards, the *quantum* case will imitate this construction. To this end it is advisable to recall the definition of strict completely positive maps and the corresponding KSGNS construction (see Theorem A.3.11 and Remark A.3.12).

1.5.1.1 Theorem-Definition. *Let Γ be a discrete group and (A,α) a Γ-C^*-algebra.*

There exists a C^*-algebra P with a non-degenerate faithful $*$-homomorphism $\pi : A \longrightarrow P$, a group homomorphism $u : \Gamma \longrightarrow \mathcal{U}(M(P))$ and a non-degenerate completely positive KSGNS-faithful map $E : P \longrightarrow M(A)$ such that

i) $u_\gamma\pi(a)u_\gamma^* = \pi(\alpha_\gamma(a))$, *for all* $\gamma\in\Gamma$ *and all* $a\in A$,

ii) $P = C^*\langle\pi(a)u_\gamma : a\in A$ *and* $\gamma\in\Gamma\rangle$,

iii) $E(\pi(a)u_\gamma) = a\delta_{\gamma,e}$ *for all* $\gamma\in\Gamma$ *and all* $a\in A$,

In addition, P is unique up to a canonical isomorphism meaning that for any C^*-algebra Q with a triple (ρ, v, E') where $\rho : A \longrightarrow Q$ is a non-degenerate faithful $*$-homomorphism, $v : \Gamma \longrightarrow \mathcal{U}(M(Q))$ is a group homomorphism and $E' : Q \longrightarrow M(A)$ is a strict completely positive KSGNS-faithful map satisfying the analogous properties $(i), (ii)$ and (iii) above, there exists a (necessarily unique) $*$-isomorphism $\psi : P \longrightarrow Q$ such that

$$\psi(\pi(a)u_\gamma) = \rho(a)v_\gamma,$$

for all $\gamma \in \Gamma$ and all $a \in A$. Moreover, E' is a non-degenerate map and we have $E = E' \circ \psi$.

The C^*-algebra P constructed in this way is called reduced crossed product of A by Γ *and is denoted by $\Gamma \underset{\alpha,r}{\ltimes} A$.*

Proof. We construct P as an explicit C^*-algebra of operators. Given the group Γ, we consider the left regular representation of Γ, $\lambda : \Gamma \hookrightarrow \mathcal{B}(l^2(\Gamma))$ (which is a unitary faithful representation). Consider the Hilbert A-module $l^2(\Gamma) \otimes A$.

Define the homomorphisms π and u of the statement. For π we consider the representation of A on $l^2(\Gamma) \otimes A$ "twisting" by the action α. More precisely, we put

$$\begin{array}{rccc} \pi : & A & \longrightarrow & \mathcal{L}_A(l^2(\Gamma) \otimes A) \\ & a & \longmapsto & \pi(a) \end{array}$$

such that

$$\pi(a)(\delta_\gamma \otimes b) := \delta_\gamma \otimes \alpha_{\gamma^{-1}}(a)b,$$

for all $b \in A$ and all $\gamma \in \Gamma$. Remark that this formula extends to the whole tensor product $l^2(\Gamma) \otimes A$ because Γ acts by *automophisms* on A. Moreover, $\pi(a)$ is well-defined as an adjointable operator on $l^2(\Gamma) \otimes A$ for all $a \in A$ whose adjoint is given by $\pi(a)^* := \pi(a^*)$, for all $a \in A$. Indeed, given $\gamma \in \Gamma$ and $b, b' \in A$ we have

$$\begin{aligned} \langle \pi(a)(\delta_\gamma \otimes b), \delta_\gamma \otimes b' \rangle &= \langle \delta_\gamma \otimes \alpha_{\gamma^{-1}}(a)b, \delta_\gamma \otimes b' \rangle = \langle \delta_\gamma, \delta_\gamma \rangle \big(\alpha_{\gamma^{-1}}(a)b\big)^* b' \\ &= \langle \delta_\gamma, \delta_\gamma \rangle b^* \alpha_{\gamma^{-1}}(a^*)b' = \langle \delta_\gamma \otimes b, \delta_\gamma \otimes \alpha_{\gamma^{-1}}(a^*)b' \rangle \\ &= \langle \delta_\gamma \otimes b, \pi(a^*)(\delta_\gamma \otimes b') \rangle, \end{aligned}$$

for all $a \in A$. Since Γ is acting on A by automorphisms, π is a *faithful* $*$-homomorphism and it is non-degenerate by construction.

For u we consider the representation of Γ on $l^2(\Gamma) \otimes A$ induced by λ. Precisely,

$$\begin{array}{rccc} u := \lambda \otimes id_A : & \Gamma & \longrightarrow & \mathcal{L}_A(l^2(\Gamma) \otimes A) \\ & \gamma & \longmapsto & u_\gamma := \lambda_\gamma \otimes id_A \end{array}$$

such that

$$u_\gamma(\delta_{\gamma'} \otimes b) = \lambda_\gamma \otimes id_A(\delta_{\gamma'} \otimes b) := \lambda_\gamma(\delta_{\gamma'}) \otimes b = \delta_{\gamma\gamma'} \otimes b,$$

for all $b \in A$ and all $\gamma \in \Gamma$. By construction, u is a unitary representation of Γ.

In this situation, we easily check the formula $u_\gamma \pi(a) u_\gamma^* = \pi(\alpha_\gamma(a))$, for all $\gamma \in \Gamma$ and all a A. Indeed,

$$\begin{aligned} u_\gamma \pi(a) u_\gamma^* \Big(\delta_{\gamma'} \otimes b\Big) &= u_\gamma \pi(a) \Big(\delta_{\gamma^{-1}\gamma'} \otimes b\Big) = u_\gamma \Big(\delta_{\gamma^{-1}\gamma'} \otimes \alpha_{\gamma'^{-1}\gamma}(a)b\Big) \\ &= \delta_{\gamma'} \otimes \alpha_{\gamma'^{-1}\gamma}(a)b = \delta_{\gamma'} \otimes \alpha_{\gamma'^{-1}}(\alpha_\gamma(a))b \\ &= \pi(\alpha_\gamma(a)) \Big(\delta_{\gamma'} \otimes b\Big), \end{aligned}$$

which is true for all $b \in A$ and all $\gamma' \in \Gamma$.

Thus we define

$$P := C^*\langle \pi(a)u_\gamma : a \in A \text{ et } \gamma \in \Gamma\rangle \subset \mathcal{L}_A(l^2(\Gamma) \otimes A)$$

To conclude the construction of P as in the statement, we have to define a non-degenerate completely positive KSGNS-faithful map $E : P \longrightarrow M(A)$ satisfying the formula $E(\pi(a)u_\gamma) = a\delta_{\gamma,e}$ for all $\gamma \in \Gamma$ and all $a \in A$. With this aim in mind, we define directly the associated KSGNS construction (recall Remark A.3.12). Namely, let us define the following A-linear map

$$\begin{array}{rccl} \Upsilon : & A & \longrightarrow & l^2(\Gamma) \otimes A \\ & a & \longmapsto & \Upsilon(a) := \delta_e \otimes a \end{array}$$

Define the A-linear operator $\Upsilon^* : \lambda(\mathbb{C}[\Gamma])\delta_e \odot A \longrightarrow A$ by the formula

$$\Upsilon^*(\lambda(\delta_\gamma)\delta_e \otimes a) = \chi_e(\gamma)a,$$

for all $\gamma \in \Gamma$ and all $a \in A$, where χ_e is the positive type function defining the GNS construction for the left regular representation λ, that is, the characteristic function of $\{e\}$. Remark that Υ^* is bounded,

$$\begin{aligned} ||\Upsilon^*(\lambda(\delta_\gamma)\delta_e \otimes a)||^2 &= ||\langle \Upsilon^*(\lambda(\delta_\gamma)\delta_e \otimes a), \Upsilon^*(\lambda(\delta_\gamma)\delta_e \otimes a)\rangle|| \\ &= ||\langle \chi_e(\gamma)a, \chi_e(\gamma)a\rangle|| = |\chi_e(\gamma)|^2 \; ||a^*a|| \\ &\leqslant \chi_e(\gamma^{-1}\gamma)||a^*a|| = ||\chi_e(\gamma^{-1}\gamma)a^*a|| \\ &= ||\langle \delta_e, \lambda(\delta_\gamma^*\delta_\gamma)\delta_e\rangle a^*a|| \\ &= ||\langle \delta_e \otimes a, \lambda(\delta_\gamma^*\delta_\gamma)\delta_e \otimes a\rangle|| \\ &= ||\langle \delta_e \otimes a, \lambda(\delta_\gamma)^*\lambda(\delta_\gamma)\delta_e \otimes a\rangle|| \\ &= ||\langle \lambda(\delta_\gamma)\delta_e \otimes a, \lambda(\delta_\gamma)\delta_e \otimes a\rangle|| \\ &= ||\lambda(\delta_\gamma)\delta_e \otimes a||^2 \end{aligned}$$

Therefore we can extend the above formula to the whole $l^2(\Gamma) \otimes A$ obtaining a bounded operator $\Upsilon^* : l^2(\Gamma) \otimes A \longrightarrow A$. Moreover, Υ and Υ^* defined in this way are adjoint,

$$\begin{aligned} \langle \Upsilon(a), \lambda(\delta_g)\delta_e \otimes b\rangle &= \langle \delta_e \otimes a, \lambda(\delta_g)\delta_e \otimes b\rangle = \chi_e(\gamma)a^*b \\ &= \langle a, \chi_e(\gamma)b\rangle = \langle a, \Upsilon^*(\lambda(\delta_\gamma)\delta_e \otimes b)\rangle, \end{aligned}$$

for all $\gamma \in \Gamma$ and all $a, b \in A$.

In other words, Υ is an adjointable operator between A and $l^2(\Gamma) \otimes A$ whose adjoint is Υ^* satisfying the formulas above. Hence, we define the following completely positive map

$$\begin{array}{rccl} E : & P & \longrightarrow & M(A) \\ & X & \longmapsto & E(X) := \Upsilon^* \circ X \circ \Upsilon \end{array}$$

We assure that the triple $(l^2(\Gamma) \otimes A, id, \Upsilon)$ is the KSGNS construction for E. We only have to prove that $l^2(\Gamma) \otimes A = \overline{span}\{P\Upsilon(A)\}$. By construction, it is enough to show that $\delta_\gamma \otimes a \in \overline{P\Upsilon(A)}$, for all $a \in A$ and all $\gamma \in \Gamma$. Indeed,

$$\delta_\gamma \otimes a = u_\gamma(\delta_e \otimes a) = u_\gamma\Upsilon(a) = \lim_{r\to\infty}\big(\pi(e_r)u_\gamma\Upsilon(a)\big) \in \overline{P\Upsilon(A)},$$

where $\{e_r\}_{r\in\mathbb{N}}$ is an approximate unit for A, so that $\{\pi(e_r)\}_{r\in\mathbb{N}}$ is an approximate unit for $\pi(A)$.

Finally, it is straightforward that the formula $E(\pi(a)u_\gamma) = a\delta_{\gamma,e}$ holds for all $\gamma \in \Gamma$ and all $a \in A$. Indeed,

$$\begin{aligned} E(\pi(a)u_\gamma)(b) &= \Upsilon^*\big(\pi(a)u_\gamma\Upsilon(b)\big) = \Upsilon^*\big(\pi(a)u_\gamma(\delta_e\otimes b)\big) \\ &= \Upsilon^*\big(\pi(a)(\delta_\gamma\otimes b)\big) = \Upsilon^*\big(\delta_\gamma\otimes\alpha_{\gamma^{-1}}(a)b\big) \\ &= \chi_e(\gamma)\alpha_{\gamma^{-1}}(a)b = \delta_{\gamma,e}\alpha_{\gamma^{-1}}(a)b = a\delta_{\gamma,e}\ b, \end{aligned}$$

where we have just used the definition of our KSGNS construction. Since this is true for all $b \in B$, we conclude the required formula.

Observe that, by KSGNS construction, E is just a *strict* completely positive map (recall Section A.3 for the details). But, thanks to the property $E(\pi(a)) = a$, for all $a \in A$ that we have just proved, it is clear that E is actually a *non-degenerate* completely positive map as assured in the statement.

Now, let us establish the uniqueness of such a construction. Suppose that Q is another C^*-algebra with a triple (ρ, v, E') where $\rho : A \longrightarrow Q$ is a non-degenerate faithful $*$-homomorphism, $v : \Gamma \longrightarrow \mathcal{U}(M(Q))$ is a group homomorphism and $E' : Q \longrightarrow M(A)$ is a strict completely positive KSGNS-faithful map satisfying the analogous properties $(i), (ii)$ and (iii) of the triple (π, u, E) associated to P. We have to show that there exists a (necessarily unique) $*$-isomorphism $\psi : P \longrightarrow Q$ such that $\psi(\pi(a)u_\gamma) = \rho(a)v_\gamma$ for all $\gamma \in \Gamma$ and all $a \in A$.

Given the non-degenerate completely positive KSGNS-faithful maps $E : P \longrightarrow M(A)$ and $E' : Q \longrightarrow M(A)$, consider their KSGNS constructions; say $(l^2(\Gamma)\otimes A, id, \Upsilon)$ and (K, σ, Υ'), respectively. This means in particular that $l^2(\Gamma)\otimes A = \overline{span}\{P\Upsilon(A)\}$, $\sigma : Q \longrightarrow \mathcal{L}_A(K)$ is a non-degenerate faithful $*$-homomorphism such that $K = \overline{span}\{\sigma(Q)\Upsilon'(A)\}$ and that $E'(Y) = (\Upsilon')^* \circ \sigma(Y) \circ \Upsilon'$, for all $Y \in Q$.

Firstly, we define a unitary operator $\mathscr{U} : l^2(\Gamma)\otimes A \longrightarrow K$ such that $\mathscr{U}\big(X\Upsilon(b)\big) = \sigma(Y)\Upsilon'(b)$, for all *generators* $X \in P$, $Y \in Q$ and all $b \in B$. Notice that this formula defines a bounded operator with dense range by construction.

Namely, take $b, b' \in A$ and $X := \pi(a)u_\gamma, X' := \pi(a')u_{\gamma'} \in P$, $Y := \rho(a)v_\gamma, Y' := \rho(a')v_{\gamma'} \in Q$

with $a, a' \in A$ and $\gamma, \gamma' \in \Gamma$ and write the following

$$\begin{aligned}
&\langle \mathscr{U}\Big(X\Upsilon(b)\Big), \mathscr{U}\Big(X'\Upsilon(b')\Big)\rangle \\
&\qquad = \langle Y\Upsilon'(b), Y'\Upsilon'(b')\rangle = \langle \rho(a)v_\gamma\Upsilon'(b), \rho(a')v_{\gamma'}\Upsilon'(b')\rangle \\
&\qquad = \langle \Upsilon'(b), (v_\gamma)^*\rho(a^*)\rho(a')v_{\gamma'}\Upsilon'(b')\rangle \\
&\qquad = \langle \Upsilon'(b), (v_\gamma)^*\rho(a^*a')v_{\gamma'}\Upsilon'(b')\rangle \\
&\qquad = \langle b, (\Upsilon')^*\big((v_\gamma)^*\rho(a^*a')v_{\gamma'}\Upsilon'(b')\big)\rangle \\
&\qquad = \langle b, E'\big((v_\gamma)^*\rho(a^*a')v_{\gamma'}\big)(b')\rangle \\
&\qquad = \langle b, E'\Big(\rho(\alpha_{\gamma^{-1}}(a^*a'))(v_\gamma)^*v_{\gamma'}\Big)(b')\rangle \\
&\qquad = \langle b, E'\Big(\rho(\alpha_{\gamma^{-1}}(a^*a'))v_{\gamma^{-1}\gamma'}\Big)(b')\rangle \\
&\qquad = \langle b, \alpha_{\gamma^{-1}}(a^*a')\delta_{\gamma^{-1}\gamma',e}\ b'\rangle \\
&\qquad = \langle b, E\Big(\pi(\alpha_{\gamma^{-1}}(a^*a'))u_{\gamma^{-1}\gamma'}\Big)(b')\rangle \\
&\qquad = \langle b, E\Big(\pi(\alpha_{\gamma^{-1}}(a^*a'))(u_\gamma)^*u_{\gamma'}\Big)(b')\rangle \\
&\qquad = \langle b, E\big((u_\gamma)^*\pi(a^*a')u_{\gamma'}\big)(b')\rangle \\
&\qquad = \langle b, \Upsilon^*\big((u_\gamma)^*\pi(a^*a')u_{\gamma'}\Upsilon(b')\big)\rangle \\
&\qquad = \langle \Upsilon(b), (u_\gamma)^*\pi(a^*a')u_{\gamma'}\Upsilon(b')\rangle \\
&\qquad = \langle \Upsilon(b), (u_\gamma)^*\pi(a^*)\pi(a')u_{\gamma'}\Upsilon(b')\rangle \\
&\qquad = \langle \pi(a)u_\gamma\Upsilon(b), \pi(a')u_{\gamma'}\Upsilon(b')\rangle = \langle X\Upsilon(b), X'\Upsilon(b')\rangle
\end{aligned}$$

Since the range of $\mathscr{U}$ is dense in K by construction, the preceding computation shows that $\mathscr{U}$ defines actually a unitary operator. Doing the identification $Q \cong \sigma(Q)$, we define the following $*$-isomorphism

$$\begin{array}{rccl}
\psi: & P & \longrightarrow & Q \\
 & X & \longmapsto & \psi(X) := \mathscr{U} \circ X \circ \mathscr{U}^*
\end{array}$$

By construction it is clear that the formula $\psi(\pi(a)u_\gamma) = \rho(a)v_\gamma$ holds for all $\gamma \in \Gamma$ and all $a \in A$. Namely, given $\eta := Y\Upsilon'(b) \in K$ with $Y = \rho(a')v_{\gamma'} \in Q$ with $a', b \in A$ and $\gamma' \in \Gamma$, one has

$$\begin{aligned}
\psi(\pi(a)u_\gamma)(\eta) &= \mathscr{U}\Big(\pi(a)u_\gamma\mathscr{U}^*(\eta)\Big) = \mathscr{U}\Big(\pi(a)u_\gamma\mathscr{U}^*(Y\Upsilon'(b))\Big) \\
&= \mathscr{U}\Big(\pi(a)u_\gamma\ \pi(a')u_{\gamma'}\Upsilon(b)\Big) = \rho(a)v_\gamma\ \rho(a')v_{\gamma'}\Upsilon'(b)) \\
&= \rho(a)v_\gamma\ Y\Upsilon'(b) = \rho(a)v_\gamma(\eta)
\end{aligned}$$

Moreover, by assumption we have $E'(\rho(a)v_\gamma) = a\delta_{\gamma,e}$ for all $\gamma \in \Gamma$ and all $a \in A$ so $E'(\rho(a)) = a$, for all $a \in A$. It is clear then that E' is in fact a non-degenerate map.

Finally, the relation $E = E' \circ \psi$ holds: the isomorphism ψ satisfies the identity $\psi(\pi(a)u_\gamma) = \rho(a)v_\gamma$, for all $\gamma \in \Gamma$ and all $a \in A$; so one has,

$$E(\pi(a)u_\gamma) = a\delta_{\gamma,e} = E'(\rho(a)v_\gamma) = E' \circ \psi(\pi(a)u_\gamma),$$

for all $\gamma \in \Gamma$ and all $a \in A$. In other words, the relation holds on generators of P, so it holds on P. ■

It is important to observe that we can construct a C^*-algebra P_m satisfying the analogous statement of the preceding theorem without the claims concerning the map E. In this case case, we give the following definition.

1.5.1.2 Definition. Let Γ be a discrete group and (A, α) a Γ-C^*-algebra. The maximal crossed product of A by Γ, denoted by $\Gamma \underset{\alpha,m}{\ltimes} A$, is a C^*-algebra equipped with a non-degenerate faithful $*$-homomorphism $\pi : A \longrightarrow \Gamma \underset{\alpha,m}{\ltimes} A$ and a group homomorphism $u : \Gamma \longrightarrow \mathcal{U}(M(\Gamma \underset{\alpha,m}{\ltimes} A))$ such that

i) $u_\gamma \pi(a) u_\gamma^* = \pi(\alpha_\gamma(a))$, for all $\gamma \in \Gamma$ and all $a \in A$,

ii) $\Gamma \underset{\alpha,m}{\ltimes} A = C^*\langle \pi(a)u_\gamma : a \in A \text{ and } \gamma \in \Gamma\rangle$,

1.5.1.3 Remark. As for the reduced crossed product constructions, $\Gamma \underset{\alpha,m}{\ltimes} A$ is unique up to a canonical isomorphism meaning that for any C^*-algebra Q with a non-degenerate faithful $*$-homomorphism $\rho : A \longrightarrow Q$ and a group homomorphism $v : \Gamma \longrightarrow \mathcal{U}(M(Q))$ satisfying the analogous properties (i) and (ii) above, there exists a (necessarily unique) $*$-isomorphism $\psi : \Gamma \underset{\alpha,m}{\ltimes} A \longrightarrow Q$ such that

$$\psi(\pi(a)u_\gamma) = \rho(a)v_\gamma,$$

for all $\gamma \in \Gamma$ and all $a \in A$.

1.5.1.4 Remark. It is advisable to collect some well known facts about the crossed products by discrete groups.

- If $(A, \alpha) = (\mathbb{C}, trv)$ is the C^*-algebra of complex numbers equipped with the trivial action trv of Γ, then we have that $\Gamma \underset{trv,r}{\ltimes} \mathbb{C} = C_r^*(\Gamma)$ and $\Gamma \underset{trv,m}{\ltimes} \mathbb{C} = C_m^*(\Gamma)$, by universal property. More generally, if (A, trv) is a C^*-algebra equipped with the trivial action trv of Γ, then we have that $\Gamma \underset{trv,r}{\ltimes} A = A \otimes C_r^*(\Gamma)$ and $\Gamma \underset{trv,m}{\ltimes} A = A \underset{max}{\otimes} C_m^*(\Gamma)$, by universal property.

- Constructions of the reduced and maximal crossed products are functorial. More precisely, if (A, α) and (B, β) are two Γ-C^*-algebras and $\varphi : A \longrightarrow M(B)$ is a Γ-equivariant $*$-homomorphism, then there exist unique $*$-homomorphisms

$$\widetilde{\varphi} := id \underset{r}{\ltimes} \varphi : \Gamma \underset{\alpha,r}{\ltimes} A \longrightarrow M(\Gamma \underset{\beta,r}{\ltimes} B)$$

$$\widetilde{\varphi} := id \underset{m}{\ltimes} \varphi : \Gamma \underset{\alpha,m}{\ltimes} A \longrightarrow M(\Gamma \underset{\beta,m}{\ltimes} B)$$

such that

$$\widetilde{\varphi}(\pi(a)u_\gamma) = \rho(\varphi(a))v_\gamma,$$

for all $a \in A$ and all $\gamma \in \Gamma$.

1.5.1.5 Remark. If G is a *locally compact group* and (A, α) is a G-C^*-algebra, then we can also construct the corresponding reduced and maximal crossed products

$$G \underset{\alpha,r}{\ltimes} A \text{ and } G \underset{\alpha,m}{\ltimes} A$$

The construction is technically more involved and we refer to [147] for more details. We give here a definition for the convenience of the exposition. We denote by μ the left Haar measure on G and by Δ the corresponding modular function.

- Let $C_c(G, A)$ be the $\mathbb{C}$-vector space of continuous functions $G \longrightarrow A$ with compact support. We equip $C_c(G, A)$ with the following convolution product and involution

$$f \star g(x) := \int_G f(y)\alpha_y\big(g(y^{-1}x)\big)d\mu(y) \ , \ \forall x \in G, \ \forall f, g \in C_c(G, A)$$

$$f^*(x) := \Delta(x^{-1})\alpha_x\big(f(x^{-1})^*\big) \ , \ \forall x \in G, \ \forall f \in C_c(G, A)$$

  It is well known that given any covariant representation (π, U) of A on a Hilbert space H (that is, $\pi : A \longrightarrow \mathcal{B}(H)$ is a $*$-homomorphism and $U : G \longrightarrow \mathcal{U}(H)$ is a group homomorphism such that $\pi(\alpha_g(a)) = U_g \circ \pi(a) \circ U_g^*$, for all $g \in G$ and all $a \in A$), then there exists a unique representation of $C_c(G, A)$ on H, which is denoted by $U \ltimes \pi$ and it is faithful whenever π is faithful.

  In this situation we define

$$G \underset{\alpha,m}{\ltimes} A := \overline{C_c(G, A)}^{||\cdot||_{max}},$$

  where

$$||f||_{max} := \sup\{||U \ltimes \pi(f)|| \mid (\pi, U) \text{ is a covariant representation of } A\},$$

  for all $f \in C_c(G, A)$.

- Let $\lambda : G \longrightarrow \mathcal{B}(L^2(G))$ be the left regular representation of G. If $\pi_0 : A \hookrightarrow \mathcal{B}(H)$ is any faithful representation of A on a Hilbert space H, then we consider the obvious representation $\lambda := \lambda \otimes id_H : G \longrightarrow \mathcal{B}(L^2(G) \otimes H)$ and we define the representation $\pi : A \longrightarrow \mathcal{B}(L^2(G) \otimes H)$ by the formula

$$\pi(a)(f \otimes \xi)(g) := \Big(f \otimes \pi_0(\alpha_{g^{-1}}(a))(\xi)\Big)(g) \ , \ f \in L^2(G), \forall \xi \in H, g \in G$$

  A straightforward computation yields that the pair (π, λ) defines a *covariant* representation of A on $L^2(G) \otimes H$ and the corresponding representation of $C_c(G, A)$ on $L^2(G) \otimes H$, denoted by $\lambda \ltimes \pi$, is faithful because π_0 is faithful.

  In this situation we define

$$G \underset{\alpha,r}{\ltimes} A := \overline{\lambda \ltimes \pi\big(C_c(G, A)\big)}^{||\cdot||},$$

  where $||\cdot||$ denotes the operator norm in $\mathcal{B}(L^2(G) \otimes H)$. We can prove that this definition is independent of the choice of π_0.

The analogous results stated in Remark 1.5.1.4 still hold for the crossed products $G \underset{\alpha,r}{\ltimes} A$ and $G \underset{\alpha,m}{\ltimes} A$.

1.5.2 Quantum crossed products

The next result concerning the *universal property of the reduced crossed product by a discrete quantum group* is well known by specialists. In order to give a proof it is advisable to recall the definition of strict completely positive maps, the corresponding KSGNS construction (see Theorem A.3.11 and Remark A.3.12) and the characterization of a discrete quantum group action given in Proposition 1.4.1.5.

1.5.2.1 Theorem-Definition. *Let $\mathbb{G}$ be a compact quantum group and (A,α) a $\widehat{\mathbb{G}}$-C^*-algebra.*

There exists a C^*-algebra P with a non-degenerate $*$-homomorphism $\pi : A \longrightarrow P$, a unitary representation $U \in M(c_0(\widehat{\mathbb{G}}) \otimes P)$ and a non-degenerate completely positive KSGNS-faithful map $E : P \longrightarrow M(A)$ such that

i) $\pi(a)U^u_{i,j} = \sum_{k=1}^{n_u} U^u_{i,k}\pi(\alpha^u_{k,j}(a))$, *for all $u \in Rep(\mathbb{G})$, all $a \in A$ and all $i,j = 1,\dots,n_u$,*

ii) $P = C^*\langle \pi(a)U^u_{i,j} : a \in A, u \in Rep(\mathbb{G}), i,j = 1,\dots,n_u\rangle$,

iii) $E(\pi(a)U^u_{i,j}) = \delta_{u,\epsilon}a$ *for all $u \in Irr(\mathbb{G})$ and all $a \in A$,*

In addition, P is unique up to a canonical isomorphism meaning that for any C^*-algebra Q with a triple (ρ, V, E') where $\rho : A \longrightarrow Q$ is a non-degenerate $*$-homomorphism, $V \in M(c_0(\widehat{\mathbb{G}}) \otimes Q)$ is a unitary representation and $E' : Q \longrightarrow M(A)$ is a strict completely positive KSGNS-faithful map satisfying the analogous properties $(i), (ii)$ and (iii) above, there exists a (necessarily unique) $*$-isomorphism $\psi : P \longrightarrow Q$ such that

$$\psi(\pi(a)U^u_{i,j}) = \rho(a)V^u_{i,j},$$

for all $u \in Rep(\mathbb{G})$, all a A and all $i,j = 1,\dots,n_u$. Moreover, E' is a non-degenerate map and we have $E = E' \circ \psi$.

The C^*-algebra P constructed in this way is called reduced crossed product of A by $\widehat{\mathbb{G}}$ *and is denoted by $\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A$.*

Proof. First of all, notice that the statement is proven once it is proven for any $x \in Irr(\mathbb{G})$.

If $(L^2(\mathbb{G}), \widehat{\lambda}, \Omega)$ denotes the GNS construction associated to the left Haar weight $\widehat{h}_L$ of $\widehat{\mathbb{G}}$, then $\widehat{\lambda} \otimes id_A : c_0(\widehat{\mathbb{G}}) \otimes A \longrightarrow \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$ is a non-degenerate $*$-homomorphism and we define

$$\begin{array}{rccl} \pi : & A & \longrightarrow & \mathcal{L}_A(L^2(\mathbb{G}) \otimes A) \\ & a & \longmapsto & \pi(a) := (\widehat{\lambda} \otimes id_A) \circ \alpha(a), \end{array}$$

for all $a \in A$.

If $(L^2(\mathbb{G}), \lambda, \Omega)$ denotes the GNS construction associated to the Haar state $h_{\mathbb{G}}$, then we define

$$U := (id_{c_0(\widehat{\mathbb{G}})} \otimes \lambda)(\mathscr{V}) \otimes id_A \in M(c_0(\widehat{\mathbb{G}}) \otimes \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)),$$

where $\mathscr{V}$ is a unitary representation of $\widehat{\mathbb{G}}$ defined by $\mathscr{V} := \bigoplus^{c_0}_{x \in Irr(\mathbb{G})} w^x \in M(c_0(\widehat{\mathbb{G}}) \otimes C_r(\mathbb{G}))$ with $w^x \in \mathcal{B}(H_x) \otimes C(\mathbb{G})$, for all $x \in Irr(\mathbb{G})$ (see Theorem 1.3.1.36).

Fix an orthonormal basis $\{\xi^x_1, \dots, \xi^x_{n_x}\}$ on H_x, for every $x \in Irr(\mathbb{G})$. Straightforward computations yield the following expressions

- $U^x = (id_{\mathcal{B}(H_x)} \otimes \lambda)(w^x) \otimes id_A \in \mathcal{B}(H_x) \otimes \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$. Namely,

$$\begin{aligned} U^x &= U(p_x \otimes id_{\mathcal{L}_A(L^2(\mathbb{G})\otimes A)}) = \Big((id_{c_0(\hat{\mathbb{G}})} \otimes \lambda)(\mathscr{V}) \otimes id_A\Big)(p_x \otimes id_{\mathcal{L}_A(L^2(\mathbb{G})\otimes A)}) \\ &= (id_{\mathcal{B}(H_x)} \otimes \lambda)(w^x) \otimes id_A \end{aligned}$$

- $U^x_{i,j} = \lambda(w^x_{i,j}) \otimes id_A \in \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$, for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$. Namely,

$$\begin{aligned} U^x_{i,j} &= (\omega_{\xi^x_i,\xi^x_j} \otimes id_{\mathcal{L}_A(L^2(\mathbb{G})\otimes A)})(U^x) \\ &= (\omega_{\xi^x_i,\xi^x_j} \otimes id_{\mathcal{L}_A(L^2(\mathbb{G})\otimes A)})\Big((id_{\mathcal{B}(H_x)} \otimes \lambda)w^x \otimes id_A\Big) \\ &= (\omega_{\xi^x_i,\xi^x_j} \otimes id_{\mathcal{L}_A(L^2(\mathbb{G})\otimes A)})\Big((id_{\mathcal{B}(H_x)} \otimes \lambda)\big(\sum_{k,l=1}^{n_x} m^x_{k,l} \otimes w^x_{k,l}\big) \otimes id_A\Big) \\ &= (\omega_{\xi^x_i,\xi^x_j} \otimes id_{\mathcal{L}_A(L^2(\mathbb{G})\otimes A)})\Big(\sum_{k,l=1}^{n_x} m^x_{k,l} \otimes \lambda(w^x_{k,l}) \otimes id_A\Big) \\ &= \sum_{k,l=1}^{n_x} \omega_{\xi^x_i,\xi^x_j}(m^x_{k,l})\lambda(w^x_{k,l}) \otimes id_A \\ &= \sum_{\substack{k=1\\ l=j}}^{n_x} \delta_{k,i}\lambda(w^x_{k,j}) \otimes id_A = \lambda(w^x_{i,j}) \otimes id_A \end{aligned}$$

In this situation, we can check the formula $\pi(a)U^x_{i,j} = \sum_{k=1}^{n_x} U^x_{i,k}\pi(\alpha^x_{k,j}(a))$, for all $x \in Irr(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \ldots, n_x$. Indeed,

$$\begin{aligned} \sum_{k=1}^{n_x} U^x_{i,k}\pi(\alpha^x_{k,j}(a)) &= \sum_{k=1}^{n_x} (\omega_{\xi^x_i,\xi^x_k} \otimes id)\Big(((id \otimes \lambda)(\mathscr{V}) \otimes id_A)(p_x \otimes id)\Big)\big((\widehat{\lambda} \otimes id_A) \circ \alpha(\alpha^x_{k,j}(a))\big) \\ &= \sum_{k=1}^{n_x} (\omega_{\xi^x_i,\xi^x_k} \otimes id)\Big(((\widehat{\lambda} \otimes \lambda)(\mathscr{V}) \otimes id_A)(p_x \otimes id)\Big)\big(\alpha(\alpha^x_{k,j}(a))\big) \\ &= \sum_{k=1}^{n_x} (\omega_{\xi^x_i,\xi^x_k} \otimes id)\Big(((\widehat{\lambda} \otimes \lambda)(\mathscr{V}) \otimes id_A)(p_x \otimes id)\Big)\big(\alpha\big((\omega_{\xi^x_k,\xi^x_j} \otimes id)\alpha(a)(p_x \otimes id)\big)\big) \\ &= \sum_{k=1}^{n_x} (\omega_{\xi^x_i,\xi^x_k} \otimes id)(\omega_{\xi^x_k,\xi^x_j} \otimes id)\Big((p_x \otimes id)((\widehat{\lambda} \otimes \lambda)(\mathscr{V}) \otimes id_A)\Big)\big((id \otimes \alpha)\alpha(a)(p_x \otimes id)\big)\big) \\ &\overset{(1)}{=} \sum_{k=1}^{n_x} (\omega_{\xi^x_i,\xi^x_k} \otimes id)(\omega_{\xi^x_k,\xi^x_j} \otimes id)\Big((p_x \otimes id)((\widehat{\lambda} \otimes \lambda)(\mathscr{V}) \otimes id_A)\Big)\big((\widehat{\Delta} \otimes id)\alpha(a)(p_x \otimes id)\big)\big) \\ &\overset{(2)}{=} \sum_{k=1}^{n_x} (\omega_{\xi^x_i,\xi^x_k} \otimes id)(\omega_{\xi^x_k,\xi^x_j} \otimes id)\Big((p_x \otimes id)(1 \otimes \alpha(a))(\widehat{\lambda} \otimes \lambda)(\mathscr{V}) \otimes id_A)\Big)(p_x \otimes id) \\ &= (\omega_{\xi^x_i,\xi^x_j} \otimes id \otimes id)\Big((1 \otimes \alpha(a))(\widehat{\lambda} \otimes \lambda)(\mathscr{V}) \otimes id_A)\Big)(p_x \otimes id) \\ &= (\widehat{\lambda} \otimes id)\alpha(a)(\omega_{\xi^x_i,\xi^x_j} \otimes id \otimes id)\Big((id \otimes \lambda)(\mathscr{V}) \otimes id_A)\Big)(p_x \otimes id) = \pi(a)U^x_{ij}, \end{aligned}$$

where the equality (1) holds because α is a left action of $\widehat{\mathbb{G}}$ on A and the equality (2) holds because of the definition of the co-multiplication $\widehat{\Delta}$ of $\widehat{\mathbb{G}}$ in terms of its fundamental unitary (observe that $\widehat{W}_{\mathbb{G}} = (id \otimes \lambda)(\mathscr{V})$ as stated in Theorem 1.3.1.36).

Thus we define

$$P := C^*\langle \pi(a)U^x_{i,j} : a \in A, x \in Irr(\mathbb{G}), i,j = 1,\dots,n_x\rangle \subset \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$$

To conclude the construction of P as in the statement, we have to define a non-degenerate completely positive KSGNS-faithful map $E : P \longrightarrow M(A) = \mathcal{L}_A(A)$ satisfying the formula $E(\pi(a)U^x_{i,j}) = a\delta_{x,\epsilon}$ for all $x \in Irr(\mathbb{G})$, all $a \in A$ and all $i,j = 1,\dots,n_x$. We are going to define directly the associated KSGNS construction (recall Remark A.3.12). Namely, let us define the following A-linear map

$$\begin{array}{rccl} \Upsilon : & A & \longrightarrow & L^2(\mathbb{G}) \otimes A \\ & a & \longmapsto & \Upsilon(a) := \Omega \otimes a \end{array}$$

Define the A-linear operator $\Upsilon^* : \lambda(Pol(\mathbb{G}))\Omega \odot A \longrightarrow A$ by the formula

$$\Upsilon^*(\lambda(w^x_{i,j})\Omega \otimes a) = h_{\mathbb{G}}(w^x_{i,j})a,$$

for all $x \in Irr(\mathbb{G})$, all $i,j = 1,\dots,n_x$ and all $a \in A$. Remark that Υ^* is bounded,

$$\begin{aligned} ||\Upsilon^*(\lambda(w^x_{i,j})\Omega \otimes a)||^2 &= ||\langle \Upsilon^*(\lambda(w^x_{i,j})\Omega \otimes a), \Upsilon^*(\lambda(w^x_{i,j})\Omega \otimes a)\rangle|| \\ &= ||\langle h_{\mathbb{G}}(w^x_{i,j})a, h_{\mathbb{G}}(w^x_{i,j})a\rangle|| = ||\overline{h_{\mathbb{G}}(w^x_{i,j})}h_{\mathbb{G}}(w^x_{i,j})a^*a|| \\ &= ||\ |h_{\mathbb{G}}(w^x_{i,j})|^2\ a^*a|| = |h_{\mathbb{G}}(w^x_{i,j})|^2||a^*a|| \\ &\overset{(1)}{\leqslant} h_{\mathbb{G}}((w^x_{i,j})^*w^x_{i,j})||a^*a|| = ||h_{\mathbb{G}}((w^x_{i,j})^*w^x_{i,j})\ a^*a|| \\ &= ||\langle \Omega, \lambda((w^x_{i,j})^*w^x_{i,j})\Omega\rangle a^*a|| \\ &= ||\langle \Omega, \lambda(w^x_{i,j})^*\lambda(w^x_{i,j})\Omega\rangle a^*a|| \\ &= ||\langle \lambda(w^x_{i,j})\Omega, \lambda(w^x_{i,j})\Omega\rangle a^*a|| \\ &= ||\langle \lambda(w^x_{i,j})\Omega \otimes a, \lambda(w^x_{i,j})\Omega \otimes a\rangle|| \\ &= ||\lambda(w^x_{i,j})\Omega \otimes a||^2, \end{aligned}$$

where in (1) we apply the Schwarz inequality to the Haar state. Therefore we can extend the above formula to the whole $L^2(\mathbb{G}) \otimes A$ obtaining a bounded operator $\Upsilon^* : L^2(\mathbb{G}) \otimes A \longrightarrow A$. Moreover, Υ and Υ^* defined in this way are adjoint,

$$\begin{aligned} \langle \Upsilon(a), \lambda(w^x_{i,j})\Omega \otimes b\rangle &= \langle \Omega \otimes a, \lambda(w^x_{i,j})\Omega \otimes b\rangle = h(w^x_{i,j})a^*b \\ &= \langle a, h(w^x_{i,j})b\rangle = \langle a, \Upsilon^*(\lambda(w^x_{i,j})\Omega \otimes b)\rangle, \end{aligned}$$

for all $x \in Irr(\mathbb{G})$, all $i,j = 1,\dots,n_x$ and all $a,b \in A$.

In other words, Υ is an adjointable operator between A and $L^2(\mathbb{G}) \otimes A$ whose adjoint is Υ^* satisfying the formulas above. Hence, we define the following completely positive map

$$\begin{array}{rccl} E : & P & \longrightarrow & M(A) \\ & X & \longmapsto & E(X) := \Upsilon^* \circ X \circ \Upsilon \end{array}$$

We claim that the triple $(L^2(\mathbb{G})\otimes A, id, \Upsilon)$ is the KSGNS construction for E. We only have to prove that $L^2(\mathbb{G})\otimes A = \overline{span}\{P\Upsilon(A)\}$. By construction, it is enough to show that $\lambda(w^x_{i,j})\Omega\otimes a \in \overline{P\Upsilon(A)}$ for all $a\in A$, all $x\in Irr(\mathbb{G})$ and all $i,j=1,\dots,n_x$. Namely,

$$\begin{aligned}\lambda(w^x_{i,j})\Omega\otimes a &= (\lambda(w^x_{i,j})\otimes id_A)(\Omega\otimes a) = U^x_{i,j}(\Omega\otimes a)\\ &= U^x_{i,j}\Upsilon(a) = \lim_{r\to\infty}\big(\pi(e_r)U^x_{i,j}\Upsilon(a)\big)\in\overline{P\Upsilon(A)},\end{aligned}$$

where $\{e_r\}_{r\in\mathbb{N}}$ is an approximate unit for A, so that $\{\pi(e_r)\}_{r\in\mathbb{N}}$ is an approximate unit for $\pi(A)$.

Finally, a direct computation shows that the formula $E(\pi(a)U^x_{i,j}) = a\delta_{x,\epsilon}$ holds for all $x\in Irr(\mathbb{G})$, all $a\in A$ and all $i,j=1,\dots,n_x$. Namely, fix an orthonormal basis $\{\xi^x_1,\dots,\xi^x_{n_x}\}$ of H_x diagonalizing the canonical operator Q_x with eigenvalues $\{\lambda^x_j\}_{j=1,\dots,n_x}$, so that the formula $\lambda(w^x_{i,j})\Omega = \frac{\sqrt{\lambda^x_j}}{\sqrt{dim_q(x)}}\xi^x_i\otimes\omega^{\overline{x}}_j$ holds for all $i,j=1,\dots,n_x$ where $\{\omega^{\overline{x}}_1,\dots,\omega^{\overline{x}}_{n_x}\}$ is the dual basis of $\{\xi^x_1,\dots,\xi^x_{n_x}\}$ in the dual space $H_{\overline{x}}$ (recall Remarks 1.3.1.31). We write

$$\begin{aligned}E(\pi(a)U^x_{i,j})(b) &= \Upsilon^*\big(\pi(a)U^x_{i,j}(\Upsilon(b))\big) = \Upsilon^*\big(\pi(a)U^x_{i,j}(\Omega\otimes b)\big)\\
&= \Upsilon^*\big(\pi(a)(\lambda(w^x_{i,j})\otimes id_A)(\Omega\otimes b)\big) = \Upsilon^*\big(\pi(a)\big(\lambda(w^x_{i,j})\Omega\otimes b\big)\big)\\
&= \Upsilon^*\Big((\widehat{\lambda}\otimes id_A)\circ\alpha(a)\big(\lambda(w^x_{i,j})\Omega\otimes b\big)\Big)\\
&= \Upsilon^*\Big((p_x\otimes id_A)\Big[(\widehat{\lambda}\otimes id_A)\circ\alpha(a)\Big](p_x\otimes id_A)\big(\lambda(w^x_{i,j})\Omega\otimes b\big)\Big)\\
&= \Upsilon^*\Big((p_x\otimes id_A)\Big[(\widehat{\lambda}\otimes id_A)\circ\alpha^x(a)\Big]\big(\lambda(w^x_{i,j})\Omega\otimes b\big)\Big)\\
&= \Upsilon^*\Big((p_x\otimes id_A)\Big[(\widehat{\lambda}\otimes id_A)\circ\sum_{i,j=1}^{n_x} m^x_{i,j}\otimes\alpha^x_{i,j}(a)\Big]\Big(\Big(\frac{\sqrt{\lambda^x_j}}{\sqrt{dim_q(x)}}\xi^x_i\otimes\omega^x_j\Big)\otimes b\Big)\Big)\\
&= \Upsilon^*\Big(\sum_{i,j=1}^{n_x}\big(m^x_{i,j}\otimes id_{H_{\overline{x}}}\otimes\alpha^x_{i,j}(a)\big)\Big(\Big(\frac{\sqrt{\lambda^x_j}}{\sqrt{dim_q(x)}}\xi^x_i\otimes\omega^x_j\Big)\otimes b\Big)\Big)\\
&= \Upsilon^*\Big(\frac{\sqrt{\lambda^x_j}}{\sqrt{dim_q(x)}}\sum_{i,j=1}^{n_x}\delta_{j,i}\xi^x_i\otimes\omega^x_j\otimes\alpha^x_{i,j}(a)b\Big)\\
&= \Upsilon^*\Big(\frac{\sqrt{\lambda^x_i}}{\sqrt{dim_q(x)}}\xi^x_i\otimes\omega^x_i\otimes\alpha^x_{i,i}(a)b\Big) = \Upsilon^*\Big(\lambda(w^x_{i,i})\Omega\otimes\alpha^x_{i,i}(a)b\Big)\\
&= h_{\mathbb{G}}(w^x_{i,i})\alpha^x_{i,i}(a)b = \alpha^x_{i,i}(a)b\delta_{x,\epsilon} = ab\delta_{x,\epsilon},\end{aligned}$$

where we use the orthogonality relations (and the definition of the KSGNS construction). Notice that, as we warned in Note 1.4.1.2, an action of a discrete quantum group is supposed to be *injective* so that we have also $\alpha^\epsilon = id_A$ as shown in Proposition 1.4.1.5. Since it is true for all $b\in A$, we conclude the required formula.

Observe that by KSGNS construction, E is just a *strict* completely positive map (recall Section A.3 for the details). But, thanks to the property $E(\pi(a)) = a$, for all $a\in A$ that we have just proved, it is clear that E is actually a *non-degenerate* completely positive map as assured in the statement.

Next, let us establish the uniqueness of such a construction. Suppose Q is another C^*-algebra with a triple (ρ, V, E') where $\rho : A\longrightarrow Q$ is a non degenerate $*$-homomorphism, $V\in M(c_0(\widehat{\mathbb{G}})\otimes Q)$

is a unitary representation and $E' : Q \longrightarrow M(A)$ is a strict completely positive KSGNS-faithful map satisfying the analogous properties $(i), (ii)$ and (iii) of the triple (π, U, E) associated to P. We have to show that there exists a (necessarily unique) $*$-isomorphism $\psi : P \longrightarrow Q$ such that $\psi(\pi(a)U^x_{i,j}) = \rho(a)V^x_{i,j}$, for all $x \in Irr(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \ldots, n_x$.

Given the strict completely positive KSGNS-faithful maps $E : P \longrightarrow M(A)$ and $E' : Q \longrightarrow M(A)$, consider their KSGNS constructions; say $(L^2(\mathbb{G}) \otimes A, id, \Upsilon)$ and (K, σ, Υ'), respectively. This means in particular that $L^2(\mathbb{G}) \otimes A = \overline{span}\{P\Upsilon(A)\}$, $\sigma : Q \longrightarrow \mathcal{L}_A(K)$ is a non-degenerate faithful $*$-homomorphism such that $K = \overline{span}\{\sigma(Q)\Upsilon'(A)\}$ and that $E'(Y) = (\Upsilon')^* \circ \sigma(Y) \circ \Upsilon'$, for all all $Y \in Q$.

Define a unitary operator $\mathscr{U} : L^2(\mathbb{G}) \otimes A \longrightarrow K$. If such an operator exists, it must verify the formula $\mathscr{U}\big(X\Upsilon(b)\big) = \sigma(Y)\Upsilon'(b)$, for all $X = \pi(a)U^x_{i,j} \in P$, $Y = \rho(a)V^x_{i,j} \in Q$ and all $b \in A$.

Actually, a straightforward computation shows that the formula above defines an isometry. Indeed, doing the identification $Q \cong \sigma(Q)$ (by virtue of the faithfulness of the KSGNS construction), let's take $X = \pi(a)U^x_{i,j}, X' = \pi(a')U^{x'}_{i,j} \in P$, $Y = \rho(a)V^x_{i,j}, Y' = \rho(a')V^{x'}_{i,j} \in Q$, $b, b' \in A$ and write

$$
\begin{aligned}
\langle \mathscr{U}\Big(X\Upsilon(b)\Big), \mathscr{U}\Big(X'\Upsilon(b')\Big)\rangle &= \langle Y\Upsilon'(b), Y'\Upsilon'(b')\rangle \\
&= \langle \rho(a)V^x_{i,j}\Upsilon'(b), \rho(a')V^{x'}_{i,j}\Upsilon'(b')\rangle = \langle \Upsilon'(b), (V^x_{i,j})^*\rho(a^*)\rho(a')V^{x'}_{i,j}\Upsilon'(b')\rangle \\
&= \langle b, (\Upsilon')^*\big((V^x_{i,j})^*\rho(a^*a')V^{x'}_{i,j}\Upsilon'(b')\big)\rangle = \langle b, E'\big((V^x_{i,j})^*\rho(a^*a')V^{x'}_{i,j}\big)(b')\rangle \\
&= \langle b, E'\big(V^{\overline{x}}_{i,j}\rho(a^*a')V^{x'}_{i,j}\big)(b')\rangle = \langle b, E'\Big(\sum_{k=1}^{n_x}\rho(\alpha^x_{j,k}(a^*a'))V^{\overline{x}}_{i,k}V^{x'}_{i,j}\Big)(b')\rangle \\
&\overset{(1)}{=} \langle b, E'\Big(\Big(\sum_{k=1}^{n_x}\rho(\alpha^{x\oplus\epsilon}_{j,k}(a^*a'))V^{\overline{x}\oplus x'}_{r,t}\Big)(b')\rangle = \langle b, \sum_t \alpha^{x\oplus\epsilon}_{j,k}(a^*a')\delta_{\overline{x}\oplus x',\epsilon}\ b'\rangle \\
&= \langle b, E\Big(\sum_{k=1}^{n_x}\pi(\alpha^{x\oplus\epsilon}_{j,k}(a^*a'))U^{\overline{x}\oplus x'}_{r,t}\Big)(b')\rangle \\
&= \langle b, E\Big(\sum_{k=1}^{n_x}\pi(\alpha^x_{j,k}(a^*a'))U^{\overline{x}}_{i,k}\ U^{x'}_{i,j}\Big)(b')\rangle = \langle b, E\big(U^{\overline{x}}_{i,j}\pi(a^*a')U^{x'}_{i,j}\big)(b')\rangle \\
&= \langle b, E\big((U^x_{i,j})^*\pi(a^*a')U^{x'}_{i,j}\big)(b')\rangle = \langle b, \Upsilon^*\big((U^x_{i,j})^*\pi(a^*a')U^{x'}_{i,j}\Upsilon(b')\big)\rangle \\
&= \langle \Upsilon(b), (U^x_{i,j})^*\pi(a^*a')U^{x'}_{i,j}\Upsilon(b')\rangle = \langle \Upsilon(b), (U^x_{i,j})^*\pi(a^*)\pi(a')U^{x'}_{i,j}\Upsilon(b')\rangle \\
&= \langle \pi(a)U^x_{i,j}\Upsilon(b), \pi(a')U^{x'}_{i,j}\Upsilon(b')\rangle = \langle X\Upsilon(b), X'\Upsilon(b')\rangle,
\end{aligned}
$$

where it should be noticed that in (1) we use the index notation $r := (i, i)$, $t := (k, j)$ in order to write down properly the coefficients for the tensor product $\overline{x} \oplus x'$.

Doing again the identification $Q \cong \sigma(Q)$, we define

$$
\begin{array}{rccl}
\psi : & P & \longrightarrow & Q \\
 & X & \longmapsto & \psi(X) := \mathscr{U} \circ X \circ \mathscr{U}^*
\end{array}
$$

It is clear that ψ is a $*$-isomorphism and the formula $\psi(\pi(a)U^x_{i,j}) = \rho(a)V^x_{i,j}$ for all $x \in Irr(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \ldots, dim(x)$ is easily checked. Namely, given $\eta := Y\Upsilon'(b) \in K$ with

$Y = \rho(a')V^{x'}_{i,j} \in Q$ with $a', b \in A$, $x' \in Irr(\mathbb{G})$ and $i, j = 1, \ldots, n_x$, one has

$$\begin{aligned}\psi(\pi(a)U^x_{i,j})(\eta) &= \mathscr{U}\left(\pi(a)U^x_{i,j}\mathscr{U}^*(\eta)\right) = \mathscr{U}\left(\pi(a)U^x_{i,j}\mathscr{U}^*(Y\Upsilon'(b))\right)\\ &= \mathscr{U}\left(\pi(a)U^x_{i,j}\ \pi(a')U^{x'}_{i,j}\Upsilon(b)\right) = \rho(a)V^x_{i,j}\ \rho(a')V^{x'}_{i,j}\Upsilon'(b))\\ &= \rho(a)V^x_{i,j}\ Y\Upsilon'(b) = \rho(a)V^x_{i,j}(\eta)\end{aligned}$$

Moreover, by assumption we have $E'(\rho(a)V^x_{i,j}) = a\delta_{\gamma,e}$ for all $x \in Irr(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \ldots, n_x$ and so $E'(\rho(a)) = a$ for all $a \in A$; then it is clear that E' is in fact a *non-degenerate* map. The relation $E = E' \circ \psi$ holds: the isomorphism ψ satisfies the identity $\psi(\pi(a)U^x_{i,j}) = \rho(a)V^x_{i,j}$, for all $x \in Irr(\mathbb{G})$, all a A and all $i, j = 1, \ldots, n_x$; so one has,

$$E(\pi(a)U^x_{i,j}) = a\delta_{\gamma,e} = E'(\rho(a)V^x_{i,j}) = E' \circ \psi(\pi(a)U^x_{i,j}),$$

for all $x \in Irr(\mathbb{G})$, all a A and all $i, j = 1, \ldots, n_x$. ■

It is important to observe that we can construct a C^*-algebra P_m satisfying the analogous statement of the preceding theorem without the claims concerning the map E. In this case case, we give the following definition.

1.5.2.2 Definition. Let $\mathbb{G}$ be a compact quantum group and (A, α) a $\widehat{\mathbb{G}}$-C^*-algebra. The maximal crossed product of A by $\widehat{\mathbb{G}}$, denoted by $\widehat{\mathbb{G}} \underset{\alpha,m}{\ltimes} A$, is a C^*-algebra equipped with a non-degenerate $*$-homomorphism $\pi : A \longrightarrow \widehat{\mathbb{G}} \underset{\alpha,m}{\ltimes} A$ and a unitary representation $U \in M(c_0(\widehat{\mathbb{G}}) \otimes \widehat{\mathbb{G}} \underset{\alpha,m}{\ltimes} A)$ such that

i) $\pi(a)U^u_{i,j} = \sum_{k=1}^{n_u} U^u_{i,k}\pi(\alpha^u_{k,j}(a))$, for all $u \in Rep(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \ldots, n_u$,

ii) $P = C^*\langle \pi(a)U^u_{i,j} : a \in A, x \in Rep(\mathbb{G}), i, j = 1, \ldots, n_u\rangle$,

1.5.2.3 Remark. As in for the reduced crossed product construction, $\widehat{\mathbb{G}} \underset{\alpha,m}{\ltimes} A$ is unique up to a canonical isomorphism meaning that for any C^*-algebra Q with a non-degenerate $*$-homomorphism $\rho : A \longrightarrow Q$ and a unitary representation $V \in M(c_0(\widehat{\mathbb{G}}) \otimes Q)$ satisfying the analogous properties (i) and (ii) above, there exists a (necessarily unique) $*$-isomorphism $\psi : \widehat{\mathbb{G}} \underset{\alpha,m}{\ltimes} A \longrightarrow Q$ such that

$$\psi(\pi(a)U^u_{i,j}) = \rho(a)V^u_{i,j},$$

for all $u \in Rep(\mathbb{G})$, all a A and all $i, j = 1, \ldots, n_u$.

1.5.2.4 Remark. It is worth mentioning that the proof of the preceding theorem has been completely written down using coordinate expressions. This allows, on the one hand, to imitate directly the proof of the construction of a reduced crossed product by a classical discrete group (Theorem 1.5.1.1). On the other hand, this approach has been chosen in order to obtain explicit formulae for the subsequent results (Section 1.5.3, Section 2.3 and Section 2.4).

However, we can give a more common and conceptual statement whose proof follows the analogous computation of the preceding one without taking coordinate expressions. More precisely, under the same assumptions of Theorem 1.5.2.1, the conditions (i), (ii) and (iii) are equivalent to the following ones

i') $(U^u)^*(id_{\mathcal{B}(H_u)} \otimes \pi(a))U^u = (id_{\mathcal{B}(H_u)} \otimes \pi)\alpha^u(a)$, for all $a \in A$ and all $u \in Rep(\mathbb{G})$.

ii') $P = C^*\langle \pi(a)(\omega \otimes id)(U^u) : \omega \in \mathcal{B}(H_u)_*\rangle$,

iii') $(id_{\mathcal{B}(H_u)} \otimes E)\Big((id_{\mathcal{B}(H_u)}) \otimes \pi(a))U^u\Big) = p^u_\epsilon \otimes a$, for all $u \in Irr(\mathbb{G})$ and all $a \in A$, where $p^u_\epsilon \in \mathcal{B}(H_u)$ denotes the orthogonal projection onto the subspace of u-invariant vectors.

Besides, the map $E : \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \longrightarrow M(A)$ of the statement, where we recall that $\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \subset \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$, can be defined simply by restricting $\Omega \otimes id_A : \mathcal{L}_A(L^2(\mathbb{G}) \otimes A) \longrightarrow M(A)$ given by $\Omega \otimes id_A(X) := \langle \Omega \otimes id_A, X(\Omega \otimes id_A)$, for all $X \in \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$.

1.5.2.5 Remark. Let us briefly explain the functoriality of the reduced crossed product construction in the quantum group setting. Remark that the maximal crossed product construction is also functorial. For more details about this we refer to Section 4.3 of [206].

Let $\mathbb{G}$ be a compact quantum group, (A, α), (B, β) two $\widehat{\mathbb{G}}$-C^*-algebras and $\varphi : A \longrightarrow B$ a $\widehat{\mathbb{G}}$-equivariant $*$-homomorphism. In this situation, there exists a $*$-homomorphism $\mathcal{Z}(\varphi) := id \underset{r}{\ltimes} \varphi : \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \longrightarrow \widehat{\mathbb{G}} \underset{\beta,r}{\ltimes} B$ such that

$$\mathcal{Z}(\varphi)(\pi_\alpha(a)(U^\alpha)^x_{i,j}) = \pi_\beta(\varphi(a)(U^\beta)^x_{i,j}),$$

for all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$ where $(\pi_\alpha, U^\alpha, E_\alpha)$ and $(\pi_\beta, U^\beta, E_\beta)$ are the canonical triples associated to the reduced crossed products $\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A$ and $\widehat{\mathbb{G}} \underset{\beta,r}{\ltimes} B$, respectively (given by Theorem 1.5.2.1).

The $*$-homomorphism $\mathcal{Z}(\varphi)$ above is nothing but the restriction of

$$\begin{array}{ccc} \mathcal{L}_A(L^2(\mathbb{G}) \otimes A) & \longrightarrow & \mathcal{L}_B(L^2(\mathbb{G}) \otimes B) \\ T & \longmapsto & \mathscr{U}_\varphi(T \underset{\varphi}{\otimes} id_B)\mathscr{U}_\varphi^{-1} \end{array}$$

where $\mathscr{U}_\varphi : L^2(\mathbb{G}) \otimes A \underset{\varphi}{\otimes} B \xrightarrow{\sim} L^2(\mathbb{G}) \otimes B$ is the canonical isometry of Hilbert modules such that $\mathscr{U}_\varphi(\xi \otimes a \underset{\varphi}{\otimes} b) = \xi \otimes \varphi(a)b$, for all $\xi \in L^2(\mathbb{G})$, all $a \in A$ and all $b \in B$.

Finally, observe that $\mathcal{Z}(\varphi) = id \underset{r}{\ltimes} \varphi$ is, by construction, compatible with the elements of the canonical triples in the following sense

$$\mathcal{Z}(\varphi)(\pi_\alpha(a)) = \pi_\beta(\varphi(a)),\ \mathcal{Z}(\varphi)((U^\alpha)^x_{i,j}) = (U^\beta)^x_{i,j},\ E_\beta \circ \mathcal{Z}(\varphi) = E_\alpha \circ \varphi,$$

for all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$.

1.5.3 Further properties

In order to reach stability properties of the Baum-Connes property, we need "good" stability properties of the reduced crossed product with respect to some operations between C^*-algebras and likewise "good" stability properties of the reduced crossed product with respect to the cone of $*$-homomorphisms. Hence, this section is devoted to the study of several of such properties that will be used later on.

1.5.3.1 Proposition. *Let $\mathbb{G}$, $\mathbb{H}$ be two compact quantum groups and let $\mathbb{F} := \mathbb{G} \times \mathbb{H}$ be the corresponding quantum direct product of $\mathbb{G}$ and $\mathbb{H}$ (as in Theorem 2.2.1).*

If (A, α) is a $\widehat{\mathbb{G}}$-C^*-algebra and (B, β) is a $\widehat{\mathbb{H}}$-C^*-algebra, then there exists a canonical $*$-isomorphism

$$\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} C \cong \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \otimes \widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B,$$

where $C := A \otimes B$ is the $\widehat{\mathbb{F}}$-C^*-algebra with action $\delta := \alpha \otimes \beta$.

Proof. The isomorphism of the statement is simply induced by the canonical map

$$\mathcal{L}_A(L^2(\mathbb{G}) \otimes A) \otimes \mathcal{L}_B(L^2(\mathbb{H}) \otimes B) \longrightarrow \mathcal{L}_{A\otimes B}\big(L^2(\mathbb{G}) \otimes L^2(\mathbb{H}) \otimes A \otimes B\big) = \mathcal{L}_{A\otimes B}(L^2(\mathbb{F}) \otimes C)$$

Let us check studiously the universal property of $\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} C$ for the C^*-algebra $\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \otimes \widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B$ following Theorem 1.5.2.1. In other words, we have to construct a triple (ρ, V, E) associated to the C^*-algebra $\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \otimes \widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B$ satisfying the analogue properties of the triple $(\pi_\delta, U, E_\delta)$ associated to the reduced crossed product $\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} C$.

If $(\pi_\alpha, U', E_\alpha)$ is the triple associated to $\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A$ and $(\pi_\beta, U'', E_\beta)$ is the triple associated to $\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B$, then we define

- $\rho : C \longrightarrow \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \otimes \widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B$ as the tensor product $\rho := \pi_\alpha \otimes \pi_\beta$ (recall Theorem A.1.11),
- $V \in M\big(c_0(\widehat{\mathbb{F}}) \otimes (\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \otimes \widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B)\big)$ through the corresponding non-degenerate $*$-homomorphims (recall Proposition 1.4.1.10). For this, recall that $C_m(\mathbb{F}) = C_m(\mathbb{G}) \underset{\max}{\otimes} C_m(\mathbb{H})$ (see Theorem 2.2.1), so that we can apply the universal property of the maximal tensor product (see Theorem A.1.12) and we put

$$\phi_V := \phi_{U'} \times \phi_{U''} : C_m(\mathbb{F}) \to M(\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A) \otimes M(\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B) \subset M(\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \otimes \widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B),$$

- $E : \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \otimes \widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B \longrightarrow M(C)$ as the tensor product $E := E_\alpha \otimes E_\beta$ (recall Theorem A.1.11).

Remark that, by construction, we have clearly

$$V^x_{i,j} = (U')^y_{i',j'} \otimes (U'')^z_{i'',j''} \text{ and } \delta^x_{k,j}(c) = \alpha^y_{k',j'}(c) \otimes \beta^z_{k'',j''}(a),$$

for all $x \in Irr(\mathbb{F})$ such that $x = \big(y\big)_{13}\big(z\big)_{24}$ with $y \in Irr(\mathbb{G})$ and $z \in Irr(\mathbb{H})$ (recall Theorem 2.2.1), all $c \in C = A \otimes B$ and for all $i, j, k = 1, \ldots, n_x$, $i', j', k' = 1, \ldots, n_y$, $i'', j'', k'' = 1, \ldots, n_z$.

To conclude, we have to check the following.

i) $\rho(c)V^x_{i,j} = \sum_{k=1}^{n_x} V^x_{i,k}\rho\big(\delta^x_{k,j}(c)\big)$, for all $x \in Irr(\mathbb{F})$, all $a \in C$ and all $i,j = 1,\dots,n_x$. Indeed, given $c = a\otimes b \in C = A\otimes B$ and $x = \big(x_1\big)_{13}\big(x_2\big)_{24} \in Irr(\mathbb{F}) = \big(Irr(\mathbb{G})\big)_{13}\big(Irr(\mathbb{H})\big)_{24}$ we write

$$\begin{aligned}
\rho(c)V^x_{i,j} &= (\pi_\alpha(a)\otimes\pi_\beta(b))\big((U')^y_{i',j'}\otimes(U'')^z_{i'',j''}\big)\\
&= \pi_\alpha(a)(U')^y_{i',j'}\otimes\pi_\beta(b)(U'')^z_{i'',j''}\\
&= \sum_{k'=1}^{n_y}(U')^y_{i',k'}\pi_\alpha\big((\alpha)^y_{k',j'}(a)\big)\otimes\sum_{k''=1}^{n_z}(U'')^z_{i'',k''}\pi_\beta\big((\beta)^z_{k'',j''}(b)\big)\\
&= \sum_{k',k''}\Big((U')^y_{i',k'}\otimes(U'')^z_{i'',k''}\Big)\Big(\pi_\alpha\big((\alpha)^y_{k',j'}(a)\big)\otimes\pi_\beta\big((\beta)^z_{k'',j''}(b)\big)\Big)\\
&= \sum_{k=1}^{n_x}V^x_{i,k}\Big(\pi_\alpha\otimes\pi_\alpha\big((\alpha)^y_{k',j'}(a)\otimes(\beta)^z_{k'',j''}(b)\big)\Big)\\
&= \sum_{k=1}^{n_x}V^x_{i,k}\rho\big(\delta^x_{k,j}(c)\big)
\end{aligned}$$

ii) $E := E_\alpha\otimes E_\beta$ is always a KSGNS-faithful map. To this end, we give the explicit KSGNS-construction of E. If $(L^2(\mathbb{G})\otimes A, id, \Upsilon')$ is the KSGNS-construction for E_α and $(L^2(\mathbb{H})\otimes B, id, \Upsilon'')$ is the KSGNS-construction for E_β (recall Theorem 1.5.2.1), then we consider the triple $(L^2(\mathbb{G})\otimes L^2(\mathbb{H})\otimes C, id\otimes id, \Upsilon := \Upsilon'\otimes\Upsilon'')$. It is straightforward to check that $E = \Upsilon^*\circ id\otimes id\circ\Upsilon$, so that this triple is the KSGNS-construction for E. Hence, the KSGNS-faithfulness for E follows.

■

The following result is well known for classical groups.

1.5.3.2 Proposition. *Let $\mathbb{G}$ be a compact quantum group. If (A,α) is a $\widehat{\mathbb{G}}$-C^*-algebra and B is any C^*-algebra, then there exists a canonical $*$-isomorphism*

$$\widehat{\mathbb{G}} \underset{id\otimes\alpha,r}{\ltimes} (B\otimes A) \cong B\otimes\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A,$$

where $id\otimes\alpha : B\otimes A \longrightarrow \widetilde{M}(c_0(\widehat{\mathbb{G}})\otimes B\otimes A)$ denotes, by abuse of notation, the action given by the composition $(\Sigma_{12}\otimes id_A)(id_B\otimes\alpha)$.

Proof. The isomorphism of the statement is simply induced by the canonical map

$$B\otimes\mathcal{L}_A(L^2(\mathbb{G})\otimes A) \longrightarrow \mathcal{L}_{B\otimes A}\big(L^2(\mathbb{G})\otimes B\otimes A\big)$$

Let us check studiously the universal property of $\widehat{\mathbb{G}} \underset{id\otimes\alpha,r}{\ltimes} (B\otimes A)$ for the C^*-algebra $B\otimes\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A$ following Theorem 1.5.2.1. In other words, we are going to construct a triple $(\overline{\rho},\overline{V},\overline{E})$ associated to the C^*-algebra $B\otimes\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A$ satisfying the analogue properties of the triple $(\widetilde{\pi},\widetilde{U},\widetilde{E})$ associated to the reduced crossed product $\widehat{\mathbb{G}} \underset{id\otimes\alpha,r}{\ltimes} (B\otimes A)$. Namely, let's put

- $\overline{\rho} : B \otimes A \longrightarrow B \otimes \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A$ as the tensor product $id_B \otimes \pi$.

- $\overline{V} \in M(c_0(\widehat{\mathbb{G}}) \otimes B \otimes \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A)$ as the unitary U_{13}.

- $\overline{E} : B \otimes \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \longrightarrow M(B \otimes A)$ as the tensor product $\iota_B \otimes E$.

where (π, U, E) is the canonical triple associated to the reduced crossed product $\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A$ and $\iota_B : B \longrightarrow M(B)$ is the canonical injection.

Remark that, by construction, we have clearly $\overline{V}^x_{i,j} = id_B \otimes U^x_{i,j}$, for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$. To conclude, we have to check the following.

i) $\overline{\rho}(b \otimes a)\overline{V}^x_{i,j} = \sum\limits_{k=1}^{n_x} \overline{V}^x_{i,k}\overline{\rho}\big((id \otimes \alpha)^x_{k,j}(b \otimes a)\big)$, for all $x \in Irr(\mathbb{F})$, all $b \otimes a \in B \otimes A$ and all $i, j = 1, \ldots, n_x$. Indeed,

$$\begin{aligned}
\overline{\rho}(b \otimes a)\overline{V}^x_{i,j} &= (b \otimes \pi(a))(id_B \otimes U^x_{i,j}) = b \otimes \pi(a)U^x_{i,j} \\
&= b \otimes \sum_{k=1}^{n_x} U^x_{i,k}\pi(\alpha^x_{k,j}(a)) = \sum_{k=1}^{n_x} b \otimes U^x_{i,k}\pi(\alpha^x_{k,j}(a)) \\
&= \sum_{k=1}^{n_x} (id_B \otimes U^x_{i,k})(b \otimes \pi(\alpha^x_{k,j}(a))) \\
&= \sum_{k=1}^{n_x} (id_B \otimes U^x_{i,k})\overline{\rho}(b \otimes \alpha^x_{k,j}(a)) \\
&= \sum_{k=1}^{n_x} \overline{V}^x_{i,k}\overline{\rho}\big((id \otimes \alpha)^x_{k,j}(b \otimes a)\big)
\end{aligned}$$

ii) $\overline{E} = \iota_B \otimes E$ is a KSGNS-faithful map. Namely, if $(L^2(\mathbb{G}) \otimes A, id, \Upsilon)$ denotes the KSGNS-construction for E (recall Theorem 1.5.2.1), then it is straightforward to check that the triple $(B \otimes L^2(\mathbb{G}) \otimes A, \iota_B \otimes id, \overline{\Upsilon} := id_B \otimes \Upsilon)$ is the KSGNS-construction for $\overline{E}$. Hence, the KSGNS-faithfulness for $\overline{E}$ follows.

■

1.5.3.3 Proposition. *Let $\mathbb{G}$ and $\mathbb{H}$ be two compact quantum groups. If A_0 is a $\widehat{\mathbb{G}}$-C^*-algebra and $\phi : B \longrightarrow B'$ is a $\widehat{\mathbb{H}}$-equivariant $*$-homomorphism, then there exists a canonical $*$-isomorphism*

$$A_0 \otimes C_\phi \cong C_{id \otimes \phi},$$

where C_ϕ denotes the cone of the $*$-homomorphism ϕ and $C_{id\otimes\phi}$ the cone of the induced $*$-homomorphism $id_{A_0} \otimes \phi : A_0 \otimes B \longrightarrow A_0 \otimes B'$.

Proof. Recall the definitions of the cones associated to the $*$-homomorphisms ϕ and $id \otimes \phi$ of the statement.

$$C_\phi := \{(b, g) \in B \times C_0\big((0, 1], B'\big) \mid \phi(b) = g(1)\}$$

$$C_{id\otimes\phi} := \{(x, h) \in A_0 \otimes B \times C_0\big((0, 1], A_0 \otimes B'\big) \mid (id \otimes \phi)(x) = h(1)\}$$

Hence the identification of the statement is simply induced by the canonical identification $A_0 \otimes C_0\big((0,1], B'\big) \cong C_0\big((0,1], A_0 \otimes B'\big)$. Namely, this identification is such that

$$a \otimes f \mapsto \Big(t \mapsto a \otimes f(t)\Big),$$

for all $a \in A_0$, all $f \in C_0\big((0,1], B'\big)$ and all $t \in (0,1]$, which preserves the definition of the cones. More precisely, if $a \in A_0$ and $(b,g) \in C_\phi$, then the element $\big((a \otimes b), h\big)$ where $h(t) := a \otimes g(t)$, for all $t \in (0,1]$, lies in $C_{id\otimes\phi}$ because $h(1) = a \otimes g(1) = a \otimes \phi(b) = (id \otimes \phi)(a \otimes b)$. ■

1.5.3.4 Proposition. *Let $\mathbb{G}$ be a compact quantum group and (A,α), (B,β) two $\widehat{\mathbb{G}}$-C^*-algebras. If $\varphi : A \longrightarrow B$ is any $\widehat{\mathbb{G}}$-equivariant $*$-homomorphism, then the corresponding cone C_φ is a $\widehat{\mathbb{G}}$-C^*-algebra with action*

$$\begin{array}{rccl} \delta : & C_\varphi & \longrightarrow & M(c_0(\widehat{\mathbb{G}}) \otimes C_\varphi) \\ & (a,h) & \longmapsto & \delta(a,h) := (\alpha(a), \beta \circ h) \end{array}$$

Given an irreducible representation $x \in Irr(\mathbb{G})$, the matrix coefficients of δ^x with respect to an orthonormal basis of H_x are given by

$$\delta^x_{i,j}(a,h) := \big(\alpha^x_{i,j}(a), \beta^x_{i,j} \circ h\big) \in C_\varphi,$$

for all $(a,h) \in C_\varphi$ and all $i,j = 1,\dots,n_x$.

Proof. First, recall the definition of the cone associated to φ

$$C_\varphi := \{(a,h) \in A \times C_0\big((0,1], B\big) \mid \varphi(a) = h(1)\}$$

In order to define the action δ as in the statement, use the canonical identification $\big(c_0(\widehat{\mathbb{G}}) \otimes A\big) \times \big(c_0(\widehat{\mathbb{G}}) \otimes C_0\big((0,1], B\big)\big) \cong c_0(\widehat{\mathbb{G}}) \otimes \big(A \times C_0\big((0,1], B\big)\big)$ and recall that the multiplier algebra is compatible with direct products (see Proposition A.4.3). In this situation, we have

$$\begin{array}{rccl} \delta : & C_\varphi & \longrightarrow & M\Big(\big(c_0(\widehat{\mathbb{G}}) \otimes A\big) \times \big(c_0(\widehat{\mathbb{G}}) \otimes C_0\big((0,1], B\big)\big)\Big) \\ & & & \cong M\Big(c_0(\widehat{\mathbb{G}}) \otimes \big(A \times C_0\big((0,1], B\big)\big)\Big) \\ & (a,h) & \longmapsto & \delta(a,h) := (\alpha(a), \beta \circ h) \end{array}$$

Observe that δ takes its values in $M(c_0(\widehat{\mathbb{G}}) \otimes C_\varphi)$ thanks to the $\widehat{\mathbb{G}}$-equivariance of φ:

$$(id \otimes \varphi)(\alpha(a)) = \beta(\varphi(a)) = \beta(h(1)),$$

for all $(a,h) \in C_\varphi$. It is straightforward to check that δ defines a left action of $\widehat{\mathbb{G}}$ on C_φ.

Fix an irreducible representation $x \in Irr(\mathbb{G})$ and an orthonormal basis $\{\xi^x_1, \dots, \xi^x_{n_x}\}$ of H_x. Let's compute the corresponding matrix coefficients of δ^x.

$$\begin{aligned} \delta^x_{i,j}(a,h) &= (\omega_{\xi^x_i,\xi^x_j} \otimes id_{C_\varphi})(\delta(a,h)(p_x \otimes id_{C_\varphi})) \\ &= ((\omega_{\xi^x_i,\xi^x_j}, \omega_{\xi^x_i,\xi^x_j}) \otimes id_{C_\varphi})((\alpha(a), \beta \circ h)((p_x, p_x) \otimes id_{C_\varphi})) \\ &= \big(\alpha^x_{i,j}(a), \beta^x_{i,j} \circ h\big), \end{aligned}$$

for all $(a,h) \in C_\varphi$. ■

The following result is well-known for classical groups.

1.5.3.5 Proposition. *Let $\mathbb{G}$ be a compact quantum group and (A,α), (B,β) two $\widehat{\mathbb{G}}$-C^*-algebras. If $\varphi : A \longrightarrow B$ is any $\widehat{\mathbb{G}}$-equivariant $*$-homomorphism, then there exists a canonical $*$-isomorphism*

$$\widehat{\mathbb{G}} \underset{r}{\ltimes} C_\varphi \cong C_{id\ltimes\varphi},$$

where C_φ denotes the cone of the $*$-homomorphism φ and $C_{id\ltimes\varphi}$ the cone of the induced $*$-homomorphism $id \ltimes \varphi : \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \longrightarrow \widehat{\mathbb{G}} \underset{\beta,r}{\ltimes} B$.

Proof. First, recall the definitions of our cones

$$C_\varphi := \{(a,h) \in A \times C_0\big((0,1],B\big) \mid \varphi(a) = h(1)\}$$

$$C_{id\ltimes\varphi} := \{(X,\widetilde{h}) \in \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \times C_0\big((0,1],\widehat{\mathbb{G}} \underset{\beta,r}{\ltimes} B\big) \mid id \ltimes \varphi(X) = \widetilde{h}(1)\}$$

and observe that if (A,α), (B,β) are $\widehat{\mathbb{G}}$-C^*-algebras, then (C_φ,δ) is again a $\widehat{\mathbb{G}}$-C^*-algebra by virtue of the preceding proposition, where the action δ is such that

$$\delta^x_{i,j}(a,h) := \big(\alpha^x_{i,j}(a), \beta^x_{i,j} \circ h\big) \in C_\varphi,$$

for all $(a,h) \in C_\varphi$, all $x \in Irr(\mathbb{G})$ and all $i,j = 1,\ldots,n_x$.

The isomorphism of the statement is simply the restriction of the canonical isomorphism

$$\widehat{\mathbb{G}} \underset{\delta,r}{\ltimes} \big(A \times C_0\big((0,1]\big) \otimes B\big) \cong \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \times C_0\big((0,1]\big) \otimes \widehat{\mathbb{G}} \underset{\beta,r}{\ltimes} B$$

Let us show studiously that the C^*-algebra $C_{id\ltimes\varphi}$ satisfies the universal property of the reduced crossed product $\widehat{\mathbb{G}} \underset{r}{\ltimes} C_\varphi$. To do so, we have to define a triple $(\overline{\rho},\overline{V},\overline{E})$ where $\overline{\rho} : C_\varphi \longrightarrow C_{id\ltimes\varphi}$ is a non-degenerate $*$-homomorphism, $\overline{V} \in M(c_0(\widehat{\mathbb{G}}) \otimes C_{id\ltimes\varphi})$ is a unitary representation and $\overline{E} : C_{id\ltimes\varphi} \longrightarrow M(C_\varphi)$ is a strict completely positive KSGNS-faithful map satisfying the corresponding properties $(i),(ii)$ and (iii) of Theorem 1.5.2.1.

Given the reduced crossed products $\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A$ and $\widehat{\mathbb{G}} \underset{\beta,r}{\ltimes} B$, consider the corresponding canonical associated triples, say $(\pi_\alpha, U^\alpha, E_\alpha)$ and $(\pi_\beta, U^\beta, E_\beta)$, respectively. This means precisely that

- $\pi_\alpha : A \longrightarrow \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A$ is a non-degenerate $*$-homomorphism, $U^\alpha \in M(c_0(\widehat{\mathbb{G}}) \otimes \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A)$ is a unitary representation and $E_\alpha : \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \longrightarrow M(A)$ is a non-degenerate completely positive KSGNS-faithful map satisfying the corresponding properties $(i),(ii)$ and (iii) of Theorem 1.5.2.1.
- $\pi_\beta : B \longrightarrow \widehat{\mathbb{G}} \underset{\beta,r}{\ltimes} B$ is a non-degenerate $*$-homomorphism, $U^\beta \in M(c_0(\widehat{\mathbb{G}}) \otimes \widehat{\mathbb{G}} \underset{\beta,r}{\ltimes} B)$ is a unitary representation and $E_\beta : \widehat{\mathbb{G}} \underset{\beta,r}{\ltimes} B \longrightarrow M(B)$ is a non-degenerate completely positive KSGNS-faithful map satisfying the corresponding properties $(i),(ii)$ and (iii) of Theorem 1.5.2.1.

With all these objects we can easily define the required triple $(\overline{\rho}, \overline{V}, \overline{E})$. Namely,

- for the non-degenerate $*$-homomorphism $\overline{\rho}$ let's put

$$\begin{array}{rccl} \overline{\rho}: & C_\varphi & \longrightarrow & C_{id\ltimes\varphi} \\ & (a,h) & \longmapsto & \overline{\rho}(a,h) := (\pi_\alpha(a), \pi_\beta \circ h), \end{array}$$

  which is a non-degenerate $*$-homomorphism and it is well defined because

$$id \ltimes \varphi(\pi_\alpha(a)) = \pi_\beta(\varphi(a)) = \pi_\beta(h(1)) = \pi_\beta \circ h(1),$$

  for all $(a,h) \in C_\varphi$.

- for the unitary representation $\overline{V} \in M(c_0(\widehat{\mathbb{G}}) \otimes C_{id\ltimes\varphi})$ we define, in an equivalent way, a non-degenerate $*$-homomorphism $\phi_{\overline{V}} : C_m(\mathbb{G}) \longrightarrow M(C_{id\ltimes\varphi})$ such that $\overline{V} = (id_{c_0(\widehat{\mathbb{G}})} \otimes \phi_{\overline{V}})(W_{\mathbb{G}})$. Let's put

$$\begin{array}{rccl} \phi_{\overline{V}}: & C_m(\mathbb{G}) & \longrightarrow & M(C_{id\ltimes\varphi}) = \mathcal{L}_{C_{id\ltimes\varphi}}(C_{id\ltimes\varphi}) \\ & c & \longmapsto & \phi_{\overline{V}}(c) := \big(\phi_{U^\alpha}(c)\cdot\ , \phi_{U^\beta}(c)\cdot\ \big), \end{array}$$

  which is a non-degenerate $*$-homomorphism and it is well defined because

$$id \ltimes \varphi(\phi_{U^\alpha}(c)X) = id \ltimes \varphi(\phi_{U^\alpha}(c)) id \ltimes \varphi(X) = \phi_{U^\beta}(c)\widetilde{h}(1),$$

  for all $(X, \widetilde{h}) \in C_{id\ltimes\varphi}$.

  Observe that, by construction, it is clear that for all $x \in Irr(\mathbb{G})$ and all $i,j = 1,\ldots,n_x$ we have

$$\overline{V}^x_{i,j} = \big((U^\alpha)^x_{i,j}\cdot\ , (U^\beta)^x_{i,j}\cdot\ \big) \in M(C_{id\ltimes\varphi})$$

- for the strict completely positive KSGNS-faithful map $\overline{E}$ let's put

$$\begin{array}{rccl} \overline{E}: & C_{id\ltimes\varphi} & \longrightarrow & M(C_\varphi) = \mathcal{L}_{C_\varphi}(C_\varphi) \\ & (X, \widetilde{h}) & \longmapsto & \overline{E}(X, \widetilde{h}) := \big(E_\alpha(X)\cdot\ , E_\beta \circ \widetilde{h}\cdot\ \big) \end{array}$$

  what is a non-degenerate completely positive map (since so are E_α and E_β) and it is well defined because

$$\varphi(E_\alpha(X)a) = \varphi(E_\alpha(X))\varphi(a) = E_\beta(id \ltimes \varphi(X))h(1) = E_\beta(\widetilde{h}(1))h(1),$$

  for all $(a,h) \in C_\varphi$.

By construction we have

$$C_{id\ltimes\varphi} = C^*\langle \overline{\rho}(a,h)\overline{V}^x_{i,j} : (a,h) \in C_\varphi, x \in Irr(\mathbb{G}), i,j = 1\ldots,n_x\rangle$$

To conclude we have to check the following,

i) $\overline{\rho}(a,h)\overline{V}^x_{i,j} = \sum_{k=1}^{n_x} \overline{V}^x_{i,k}\overline{\rho}(\delta^x_{k,j}(a,h))$, for all $(a,h) \in C_\varphi$, all $x \in Irr(\mathbb{G})$ and all $i,j = 1,\ldots,n_x$. Indeed, for all $t \in (0,1]$ we write

$$\begin{aligned}
\overline{\rho}(a,h(t))\overline{V}^x_{i,j} &= (\pi_\alpha(a),\pi_\beta(h(t)))\big((U^\alpha)^x_{i,j}\cdot\,,(U^\beta)^x_{i,j}\,\cdot\,\big)\\
&= \big(\pi_\alpha(a)(U^\alpha)^x_{i,j}\cdot\,,\pi_\beta(h(t))(U^\beta)^x_{i,j}\,\cdot\,\big)\\
&= \Big(\sum_{k=1}^{n_x}(U^\alpha)^x_{i,k}\pi_\alpha(\alpha^x_{k,j}(a))\cdot\,,\sum_{k=1}^{n_x}(U^\beta)^x_{i,k}\pi_\beta(\beta^x_{k,j}(h(t)))\,\cdot\,\Big)\\
&= \sum_{k=1}^{n_x}\Big(\big((U^\alpha)^x_{i,k}\cdot\,,(U^\beta)^x_{i,k}\,\cdot\,\big)\big(\pi_\alpha(\alpha^x_{k,j}(a)),\pi_\beta(\beta^x_{k,j}(h(t)))\big)\Big)\\
&= \sum_{k=1}^{n_x}\overline{V}^x_{i,k}\overline{\rho}(\alpha^x_{k,j}(a),\beta^x_{k,j}(h(t)))\\
&= \sum_{k=1}^{n_x}\overline{V}^x_{i,k}\overline{\rho}(\delta^x_{k,j}(a,h(t)))
\end{aligned}$$

ii) $\overline{E}$ is a KSGNS-faithful map such that $\overline{E}(\overline{\rho}(a,h)\overline{V}^x_{i,j}) = (a,h)\delta_{x,\epsilon}$, for all $(a,h) \in C_\varphi$, all $x \in Irr(\mathbb{G})$ and all $i,j = 1,\ldots,n_x$. For all $t \in (0,1]$ we have

$$\begin{aligned}
\overline{E}(\overline{\rho}(a,h(t))\overline{V}^x_{i,j}) &= \overline{E}\Big((\pi_\alpha(a),\pi_\beta(h(t)))\big((U^\alpha)^x_{i,j}\cdot\,,(U^\beta)^x_{i,j}\,\cdot\,\big)\Big)\\
&= \overline{E}\Big(\pi_\alpha(a)(U^\alpha)^x_{i,j}\cdot\,,\pi_\beta(h(t))(U^\beta)^x_{i,j}\,\cdot\,\Big)\\
&= \Big(E_\alpha(\pi_\alpha(a)(U^\alpha)^x_{i,j}\cdot\,)\cdot\,,E_\beta(\pi_\beta(h(t))(U^\beta)^x_{i,j}\cdot\,)\,\cdot\,\Big)\\
&= \Big(a\delta_{x,\epsilon}\cdot\,,h(t)\delta_{x,\epsilon}\,\cdot\,\Big) = (a,h(t))\delta_{x,\epsilon}.
\end{aligned}$$

Concerning the KSGNS-faithfulness, we are going to exhibit directly the KSGNS-construction for our $\overline{E} : C_{id\ltimes\varphi} \longrightarrow M(C_\varphi) = \mathcal{L}_{C_\varphi}(C_\varphi)$. Define

$$\mathscr{H} := \{(\xi,\eta) \in L^2(\mathbb{G})\otimes A \times C_0\big((0,1],L^2(\mathbb{G})\otimes B\big) \mid \eta(1) = \mathscr{U}_\varphi(\xi \underset{\varphi}{\otimes} id_B)\},$$

where $\mathscr{U}_\varphi$ is the canonical isometry between $L^2(\mathbb{G})\otimes A \underset{\varphi}{\otimes} B$ and $L^2(\mathbb{G})\otimes B$ defined in Remark 1.5.2.5

Next, put the following adjointable operator between C_φ and $\mathscr{H}$

$$\begin{array}{rccl}
\Upsilon : & C_\varphi & \longrightarrow & \mathscr{H}\\
& (a,h) & \longmapsto & \Upsilon(a,h) := \big(\Upsilon_\alpha(a),\Upsilon_\beta\circ h\big),
\end{array}$$

which is well defined because of the definition of $\mathscr{U}_\varphi$:

$$\Upsilon_\beta\circ h(1) = \Upsilon_\beta(\varphi(a)) = \Omega\otimes\varphi(a) = \mathscr{U}_\varphi(\Omega\otimes a \underset{\varphi}{\otimes} id_B) = \mathscr{U}_\varphi(\Upsilon_\alpha(a) \underset{\varphi}{\otimes} id_B)$$

and whose adjoint is simply given by

$$\Upsilon^* := (\Upsilon^*_\alpha,\Upsilon^*_\beta\circ\eta),$$

for all $\eta \in C_0\big((0,1], L^2(\mathbb{G}) \otimes B\big)$.

Finally, we define the following representation of $C_{id \ltimes \varphi}$ on $\mathscr{H}$

$$\begin{array}{rccl} \sigma: & C_{id\ltimes\varphi} & \longrightarrow & \mathcal{L}_{C_\varphi}(\mathscr{H}) \\ & (X, \widetilde{h}) & \longmapsto & \sigma(X,\widetilde{h}) := \big(X\cdot\,, \widetilde{h}\cdot\ \big), \end{array}$$

which is well defined because of the definition of $\mathscr{U}_\varphi$. Indeed, for every $(\xi, \eta) \in \mathscr{H}$ we have

$$\begin{aligned} \widetilde{h}\cdot\eta(1) &= \widetilde{h}(1)\eta(1) = id \ltimes \varphi(X)\mathscr{U}_\varphi(\xi \underset{\varphi}{\otimes} id_B) \\ &= \mathscr{U}_\varphi(X \underset{\varphi}{\otimes} id_B)\mathscr{U}_\varphi(\xi \underset{\varphi}{\otimes} id_B) = \mathscr{U}_\varphi(X\xi \underset{\varphi}{\otimes} id_B) \end{aligned}$$

Note that σ is faithful thanks to the faithfulness of the KSGNS constructions of E_α and E_β.

Consider the following Hilbert C_φ-submodule of $\mathscr{H}$

$$\mathscr{G} := \overline{span}\{\sigma(C_{id\ltimes\varphi})\Upsilon(C_\varphi)\}$$

Hence, to conclude, observe that for all $(X, \widetilde{h}) \in C_{id\ltimes\varphi}$, all $(a, h) \in C_\varphi$ and all $t \in (0,1]$ we have

$$\begin{aligned} \Upsilon^* \circ \sigma(X,\widetilde{h}) \circ \Upsilon(a, h(t)) &= \Upsilon^*\big(\sigma(X,\widetilde{h})(\Upsilon_\alpha(a), \Upsilon_\beta(h(t)))\big) \\ &= \Upsilon^*\big(X(\Upsilon_\alpha(a)), \widetilde{h}(t)\Upsilon_\beta(h(t))\big) \\ &= \big(\Upsilon^*_\alpha(X(\Upsilon_\alpha(a))), \Upsilon^*_\beta(\widetilde{h}(t)\Upsilon_\beta(h(t)))\big) \\ &= \big(E_\alpha(X)(a), E_\beta(\widetilde{h}(t))h(t)\big) = \overline{E}(X,\widetilde{h})(a, h(t)) \end{aligned}$$

■

Recall Section 1.4.3. An important result in this context is the *Green's Imprimitivity theorem*. In the classical case, this result states the following: let G be a locally compact group and $H < G$ a closed subgroup. If (B, β) is any H-C^*-algebra, then we have the natural Morita equivalences

$$G \underset{m}{\ltimes} Ind_H^G(B,\beta) \underset{M}{\sim} H \underset{m}{\ltimes} B \text{ and } G \underset{r}{\ltimes} Ind_H^G(B,\beta) \underset{M}{\sim} H \underset{r}{\ltimes} B$$

As a consequence, the well-known Green-Julg theorem yields

$$K_*(G \underset{r}{\ltimes} Ind_H^G(B,\beta)) \cong K_*(H \underset{r}{\ltimes} B) \cong K_*^H(B),$$

whenever H is compact.

We refer to [147] and [161] for a detailed exposition about the theory of induced C^*-algebras and imprimitivity theorems for classical locally compact groups. The quantum counterpart has been established in Theorem 7.3 in [194] by S. Vaes for a general locally compact quantum group and a closed quantum subgroup. The corresponding statement in the discrete quantum case is the following.

1.5.3.6 Theorem (Quantum Green's imprimitivity theorem)**.** *Let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup of a discrete quantum group $\widehat{\mathbb{G}}$. If (B,β) is a $\widehat{\mathbb{H}}$-C^*-algebra, then there exists a natural equivariant Morita equivalence*

$$\widehat{\mathbb{G}} \underset{r}{\ltimes} Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B,\beta) \underset{M}{\sim} \widehat{\mathbb{H}} \underset{r}{\ltimes} B,$$

where equivariance is understood with respect to the dual actions on the crossed products.

1.6 Torsion phenomena in the quantum setting

In the categorical formulation of the Baum-Connes conjecture given by R. Meyer and R. Nest for a locally compact group G we use the family $\mathcal{F}$ of compact subgroups of G (see Section 1.2.3). If G is discrete, then the family $\mathcal{F}$ is formed by the *finite* subgroups of G, which is exactly the torsion of G. Hence, the torsion of a discrete group allows to define an obvious complementary pair of subcategories (see Theorem 1.2.3.11), which yields the definition of the categorical assembly map.

In this way, we may investigate the torsion phenomena for discrete quantum groups in order to construct the analogous complementary pair of subcategories and so the corresponding quantum assembly map. It turns out that torsion for a discrete quantum group $\widehat{\mathbb{G}}$ can appeared under different exotic fashions. Hence, "quantum torsion" is more complicated than "classical torsion" and we don't have yet a complete understanding of this phenomena in order to handle it in the general categorical framework of Meyer-Nest. The first notion of torsion for a discrete quantum group was introduced by R. Meyer and R. Nest [133], [131] and recently re-interpreted by Y. Arano and K. De Commer in terms of fusion rings [3].

As we shall explain in Section 1.7.2, the current formulation of the Baum-Connes conjecture for quantum groups deals only with *torsion-free discrete quantum groups*. Moreover, in Chapter 3 we investigate the torsion phenomena for some constructions of quantum groups (quantum direct product, quantum semi-direct product, compact bicrossed product, free product and free wreath product) in order to tackle the corresponding stability properties of the Baum-Connes property. In some cases, the torsion phenomena of a discrete quantum group can be successfully controlled in order to give a suitable Baum-Connes property formulation. For instance, this is the case for the *quantum automorphism group* [212] and the *free wreath product* Section 3.7.

1.6.1 Torsion *à la Meyer-Nest*

For more details about the following statements we refer to Section 4 in [212].

1.6.1.1 Definition. Let $\mathbb{G}$ be a compact quantum group. The discrete quantum group $\widehat{\mathbb{G}}$ is called *torsion-free* (or *torsion-free in the sense of Meyer-Nest*) if every finite dimensional $\mathbb{G}$-C^*-algebra is $\mathbb{G}$-equivariantly isomorphic to a direct sum of $\mathbb{G}$-C^*-algebras which are $\mathbb{G}$-equivariantly Morita equivalent to the trivial $\mathbb{G}$-C^*-algebra $\mathbb{C}$.

1.6.1.2 Remark. Let $\mathbb{G}$ be a compact quantum group and let u be a unitary representation of $\mathbb{G}$ on a finite dimensional Hilbert space H. As we have indicated in Remark 1.4.2.3, $\mathcal{B}(H) = \mathcal{K}(H)$ is an obvious $\mathbb{G}$-C^*-algebra with the adjoint action induced by u, which is denoted by Ad_u.

By definition of Morita equivalence (see Definition A.3.15), it is straightforward to show that a discrete quantum group $\widehat{\mathbb{G}}$ is torsion-free if and only if for any finite dimensional $\mathbb{G}$-C^*-algebra (A, δ) there exist $l \in \mathbb{N}$ and finite dimensional unitary representations $u_i \in \mathcal{B}(H_i) \otimes C(\mathbb{G})$, for all $i = 1, \ldots, l$ such that

$$(A, \delta) \cong (\mathcal{B}(H_1), Ad(u_1)) \oplus \ldots \oplus (\mathcal{B}(H_l), Ad(u_l))$$

In other words, $A \cong \mathcal{K}(H_1) \oplus \ldots \oplus \mathcal{K}(H_l)$ as $\mathbb{G}$-C^*-algebras.

The following is an elementary but useful result.

1.6.1.3 Lemma. *Let $\mathbb{G}$ be a compact quantum group. Let A be a simple C^*-algebra and $p \in A$ a non zero projection. The following properties hold.*

i) A is Morita equivalent to pAp by means of Ap. More precisely, the $*$-homomorphism

$$\begin{array}{rccl} \pi : & A & \longrightarrow & \mathcal{K}_{pAp}(Ap) \\ & a & \longmapsto & \pi(a),\ \pi(a)(bp) := abp, \text{ for all } b \in A \end{array}$$

is an isomorphism.

ii) If δ is an action of $\mathbb{G}$ on A and $p \in A^\delta$, then A is $\mathbb{G}$-equivariantly Morita equivalent to pAp by means of Ap. More precisely, the isomorphism π of (i) is $\mathbb{G}$-equivariant.

Moreover, if A is unital and $p \in A^\delta$ is minimal, then (pAp, δ) is an ergodic action.

Proof. i) First of all, routine computations show that Ap is a Hilbert pAp-module with inner product defined by

$$\langle ap, bp \rangle := pa^*bp,$$

for all $a, b \in A$. Next consider the two-sided closed ideal $\overline{ApA}$ of A. Since A is simple and p is non zero by assumption, it must be $A = \overline{ApA}$.

The map π defined in the statement is clearly a homomorphism. Given $a \in A$, let us describe the adjoint operator of $\pi(a)$. We claim that $\pi(a)^* = \pi(a^*)$. Indeed, for all $a, b, b' \in A$ we write

$$\langle \pi(a)(bp), b'p \rangle = \langle abp, b'p \rangle = pb^*a^*b'p$$

$$\langle bp, \pi(a^*)(b'p) \rangle = \langle bp, a^*b'p \rangle = pb^*a^*b'p,$$

which yields the claim. Accordingly, π is a $*$-homomorphism.

Since A is simple by assumption, π must be injective. Let us show that π is surjective. Given $a, b \in A$, we claim that the finite rank operator $\Theta_{ap,bp} \in \mathcal{K}_{pAp}(Ap)$ is in the image of π. Indeed, for all $c \in A$ we write

$$\Theta_{ap,bp}(cp) = ap\langle bp, cp \rangle = ap(pb^*cp) = (apb^*)cp = \pi(apb^*)(cp),$$

which yields the claim. To conclude, notice that $A = \overline{ApA}$ which means that every element in A can be approximated by elements of the form apb with $a, b \in A$. Since every compact operator in $\mathcal{K}_{pAp}(Ap)$ can be approximated by finite rank operators $\Theta_{ap,bp}$ with $a, b \in A$, the preceding computation yields that $\pi(A) = \mathcal{K}_{pAp}(Ap)$ and the proof is complete.

ii) (A, δ) is a $\mathbb{G}$-C^*-algebra. Since $p \in A^\delta$ by assumption, then both pAp and Ap are $\mathbb{G}$-C^*-algebras by restricting δ. In order to show that π is $\mathbb{G}$-equivariant, we have to show that the diagram

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "\delta"] & \mathcal{K}_{pAp}(Ap) \arrow[d, "\widetilde{\delta}"] \\ M(A \otimes C(\mathbb{G})) \arrow[r, "\pi \otimes id_{C(\mathbb{G})}"'] & M(\mathcal{K}_{pAp}(Ap) \otimes C(\mathbb{G})) \end{tikzcd}$$

is commutative where $\widetilde{\delta}$ is the action induced by δ and determined by $\widetilde{\delta}(\Theta_{ap,bp}) = \Theta_{\delta(ap),\delta(bp)}$, for all $a, b \in A$ (see Remarks 1.7.1.7). Indeed, computations of (i) and δ-invariance of p allow to write

$$apb^* \overset{\pi}{\mapsto} \pi(apb^*) = \Theta_{ap,bp} \overset{\widetilde{\delta}}{\mapsto} \Theta_{\delta(ap),\delta(bp)} = \Theta_{\delta(a)(p\otimes 1),\delta(b)(p\otimes 1)}$$

$$apb^* \overset{\delta}{\mapsto} \delta(apb^*) = \delta(a)(p \otimes 1)\delta(b)^* \overset{\pi\otimes id}{\mapsto} \Theta_{\delta(a)(p\otimes 1),\delta(b)(p\otimes 1)},$$

which yields the claim.

Moreover, assume that A is unital and that $p \in A^\delta$ is minimal. In this case, it is clear that $p \in (pAp)^\delta$, so $(pAp)^\delta = \mathbb{C}1_A$ by minimality.

■

Let us show the following useful characterization for torsion-freeness of a discrete quantum group.

1.6.1.4 Theorem-Definition. *Let $\mathbb{G}$ be a compact quantum group. The following assertions are equivalent.*

i) $\widehat{\mathbb{G}}$ is torsion-free.

ii) Let (A, δ) be a finite dimensional $\mathbb{G}$-C^*-algebra.

- *If δ is ergodic, then A is simple. In other words, there are no non-simple ergodic finite dimensional $\mathbb{G}$-C^*-algebras. In this case, we say that $\widehat{\mathbb{G}}$ is* permutation torsion-free.
- *If A is simple, then there exists a finite dimensional unitary representation (u, H) of $\mathbb{G}$ such that $A \cong \mathcal{K}(H)$ as $\mathbb{G}$-C^*-algebras. In this case, we say that $\widehat{\mathbb{G}}$ is* projective torsion-free.

iii) Every finite dimensional ergodic action of $\mathbb{G}$ is $\mathbb{G}$-equivariantly Morita equivalent to the trivial $\mathbb{G}$-C^*-algebra $\mathbb{C}$.

Proof. - $(i) \Rightarrow (ii)$. Assume that $\widehat{\mathbb{G}}$ is torsion-free, which means that given a finite dimensional $\mathbb{G}$-C^*-algebra (A, δ) there exist $l \in \mathbb{N}$ and finite dimensional unitary representations $u_i \in \mathcal{B}(H_i) \otimes C(\mathbb{G})$, for all $i = 1, \ldots, l$ such that $A \cong \mathcal{K}(H_1) \oplus \ldots \oplus \mathcal{K}(H_l)$ as $\mathbb{G}$-C^*-algebras.

If A is simple, the it is clear that $\widehat{\mathbb{G}}$ is projective torsion-free.

Assume that A is non-simple and δ is ergodic. Since A is non-simple, it must be $l > 1$. But in this case, the direct sum $(\mathcal{K}(H_1), Ad_{u_1}) \oplus \ldots \oplus (\mathcal{K}(H_l), Ad_{u_l})$ is not an ergodic $\mathbb{G}$-C^*-algebra, which contradicts the ergodicity of δ. Hence a non-simple ergodic $\mathbb{G}$-C^*-algebra can not exist so that $\widehat{\mathbb{G}}$ is permutation torsion-free.

- $(ii) \Rightarrow (i)$. Conversely, assume that $\widehat{\mathbb{G}}$ is both permutation and projective torsion-free. Let (A, δ) be a finite dimensional $\mathbb{G}$-C^*-algebra. If A is simple, then the result follows from the projective torsion-freeness assumption. If δ is ergodic, then the result follows from the torsion and projective torsion-freeness assumption. Suppose then that A is not simple and δ is not ergodic.

  Let us write A under the form $A = \mathcal{M}_{k_1}(\mathbb{C}) \oplus \ldots \oplus \mathcal{M}_{k_n}(\mathbb{C})$, for some $k_1, \ldots, k_n, n \in \mathbb{N}$. Observe that the subalgebra of δ-fixed points of A, A^δ, is clearly δ-invariant. Let $P_1, \ldots, P_r$ be mutually orthogonal minimal projections in A^δ such that $P_1 + \ldots + P_r = id$. For each $j = 1, \ldots, r$ consider the decomposition of P_j following the blocks of A and write $P_j = (p_1^j, \ldots, p_n^j)$.

  Since $P_j \in A^\delta$ is δ-invariant, then $P_j A P_j$ is a $\mathbb{G}$-C^*-algebra. Since P_j is minimal in A^δ, then it is ergodic. Since $\widehat{\mathbb{G}}$ is both permutation and projective torsion-free by assumption, we have that there exists a finite dimensional unitary representation (u_j, H_j) of $\mathbb{G}$ such that $P_j A P_j \cong \mathcal{K}(H_j)$ as $\mathbb{G}$-C^*-algebras. This means in particular that there exists a single $i = 1, \ldots, n$ such that $p_i^j \neq 0$. Consequently, P_j is supported on single matrix block for all $j = 1, \ldots, r$. In other words, since $P_1 + \ldots + P_r = id$, we can identify $P_1, \ldots, P_r$ with the minimal central projections of A on each matrix block, $p_1, \ldots, p_n$ (so $r = n$). In particular, each minimal central projection of A is δ-invariant, which allows to assume that A is a single matrix block, that is, A is simple. In this case, since $\widehat{\mathbb{G}}$ is supposed to be projective torsion-free, the result follows.

- $(ii) \Rightarrow (iii)$. This is straightforward. Namely, assume that $\widehat{\mathbb{G}}$ is both permutation and projective torsion-free. If (A, δ) is a finite dimensional ergodic action of $\mathbb{G}$, then A is simple by the permutation torsion-freeness assumption. By projective torsion-freeness assumption, we know in this case that there exists a unitary representation (u, H) of $\mathbb{G}$ such that $A \cong \mathcal{K}(H)$ as $\mathbb{G}$-C^*-algebras, which yields the claim.

- $(iii) \Rightarrow (ii)$. Let (A, δ) be any finite dimensional $\mathbb{G}$-C^*-algebra.

  If δ is ergodic, then by assumption A is $\mathbb{G}$-equivariantly Morita equivalent to the trivial $\mathbb{G}$-C^*-algebra $\mathbb{C}$, which implies in particular that A is simple.

  If A is simple, we are going to show that A is $\mathbb{G}$-equivariantly Morita equivalent to a finite dimensional ergodic action of $\mathbb{G}$. Namely, consider the subalgebra of δ-fixed points of A, A^δ, which is clearly δ-invariant. Let $P_1, \ldots, P_r$ be non zero mutually orthogonal minimal projections in A^δ. For every $j = 1, \ldots, r$ consider the two-sided closed ideal $\overline{AP_jA}$ of A. Since A is simple and P_j is non zero, it must be $A = \overline{AP_jA}$, which is true for every $j = 1, \ldots, r$ so that it is enough to consider a single non zero minimal projection $p \in A^\delta$. In this situation, Lemma 1.6.1.3 assures that $A \underset{\mathbb{G}-M}{\sim} pAp$, where (pAp, δ) is a finite dimensional ergodic action. Hence, our assumption yields that A is $\mathbb{G}$-equivariantly Morita equivalent to the trivial $\mathbb{G}$-C^*-algebra $\mathbb{C}$, which completes the proof.

■

The preceding characterization allows to give the following reformulation of Definition 1.6.1.1 by using Remark 1.6.1.2 and the terminology of Definition 1.4.2.6.

1.6.1.5 Definition. Let $\mathbb{G}$ be a compact quantum group. The discrete quantum group $\widehat{\mathbb{G}}$ is called *torsion-free* if any torsion action of $\mathbb{G}$ is $\mathbb{G}$-equivariantly Morita equivalent to the trivial $\mathbb{G}$-C^*-algebra $\mathbb{C}$.

It leads as well to the following definition.

1.6.1.6 Definition. Let $\mathbb{G}$ be a compact quantum group.

- A non-simple ergodic finite dimensional $\mathbb{G}$-C^*-algebra is called torsion action of permutation type. If such a $\mathbb{G}$-C^*-algebra exists, we say that $\widehat{\mathbb{G}}$ has *torsion of permutation type*.
- A simple ergodic finite dimensional $\mathbb{G}$-C^*-algebra which is *not* $\mathbb{G}$-equivariantly Morita equivalent to the trivial $\mathbb{G}$-C^*-algebra $\mathbb{C}$ is called torsion action of projective type. If such a $\mathbb{G}$-C^*-algebra exists, we say that $\widehat{\mathbb{G}}$ has *torsion of projective type*.

1.6.1.7 Remark. If $\widehat{\mathbb{G}}$ is a discrete quantum group that has a non-trivial finite discrete quantum subgroup, then $\widehat{\mathbb{G}}$ is *not* torsion-free. Indeed, in this case $Pol(\mathbb{G})$ contains a non-trivial finite dimensional Hopf $*$-algebra, say $Pol(\Lambda)$. So its co-multiplication

$$\Delta_\Lambda : Pol(\Lambda) \longrightarrow Pol(\Lambda) \otimes Pol(\Lambda) \subset Pol(\Lambda) \otimes Pol(\mathbb{G})$$

defines an ergodic action of $\mathbb{G}$ on $Pol(\Lambda)$. If it was isomorphic to an adjoint action associated to a representation (u, H), then the co-unit over $Pol(\Lambda)$ would give a character on $\mathcal{B}(H)$, which implies that H is one-dimensional and so $Pol(\Lambda) = \mathbb{C}$, a contradiction.

In other words, *regular actions of finite discrete quantum subgroups of $\widehat{\mathbb{G}}$ yield torsion of permutation type for $\widehat{\mathbb{G}}$.*

Of course, if Γ is a classical discrete group, then the usual torsion-freeness for Γ is equivalent to the torsion-freeness for $\widehat{\mathbb{\Gamma}}$ in the sense of Meyer-Nest. We have precisely the following result (see Proposition 4.2 of [212] for a proof).

1.6.1.8 Theorem. *Let Γ be a classical discrete group. For every torsion action of $\widehat{\Gamma}$, (A, δ), there exists a finite subgroup $\Lambda < \Gamma$ and a normalized 2-cocycle $\omega \in Z^2(\Lambda, S^1)$ such that $A \underset{\widehat{\Gamma}-M}{\sim} C^*_\omega(\Lambda)$, where $C^*_\omega(\Lambda)$ is the twisted group C^*-algebra of Λ.*

As a consequence, Γ is torsion-free in the classical sense if and only if $\widehat{\mathbb{\Gamma}}$ is torsion-free in the sense of Meyer-Nest.

If Γ is a classical discrete group, then the torsion of Γ is given exactly by the finite subgroups of Γ. The above theorem guarantees that the torsion of Γ corresponds exactly to the quantum torsion of $\widehat{\mathbb{\Gamma}}$. In other words, if Γ is a classical discrete group, then all torsion of the discrete *quantum* group $\widehat{\mathbb{\Gamma}}$ is *only of permutation type* arising from finite dimensional discrete quantum subgroups. More exotic torsion phenomena can appear whenever we work with *genuine* discrete quantum groups. For instance we can regard duals of *classical* compact groups.

Let G be a classical compact group. Then $\widehat{G}$ is a discrete *quantum* group. In this situation, the quantum torsion phenomena of $\widehat{G}$ can be related to the topology of G. We refer to [181] and [56] for the general theory of topological groups (and particularly, for locally compact groups). More precisely, we have the following result.

1.6.1.9 Theorem. *Let G be a classical compact group.*

i) G is connected if and only if $\widehat{G}$ is permutation torsion-free.

ii) G has no non-trivial normalized 2-cocycles if and only if $\widehat{G}$ is projective torsion-free.

Proof. i) Assume that G is connected. Let δ be an ergodic action of G on a finite dimensional C^*-algebra A, which will be written under the form $A = \mathcal{M}_{k_1}(\mathbb{C}) \oplus \ldots \oplus \mathcal{M}_{k_n}(\mathbb{C})$, for some $k_1, \ldots, k_n, n \in \mathbb{N}$. We have to show that A is simple.

The stabilizer group of any block $\mathcal{M}_{k_j}(\mathbb{C})$, which is always a closed subgroup of G, is an open subgroup of G. For the latter, observe that the ergodic action of G on A yields a transitive action of G on the center $Z(A) = \mathbb{C}^n \cong \mathbb{C}p_1 \oplus \ldots \oplus \mathbb{C}p_n$, where $p_1, \ldots, p_n$ are the minimal central projections of A on each matrix block. In other words, we have a *transitive* action of G on the discret set of n-points $\{1, \ldots, n\}$. Hence $\{1, \ldots, n\} = \mathcal{O}_{p_j} \cong G/Stab_G(p_j)$ and $Stab_G(p_j)$ must be open.

To conclude we observe that the action of G on A preserves the individual matrix blocks. Indeed, if there was some $i = 1, \ldots, n$ such that $\mathcal{M}_{k_i}(\mathbb{C})$ is *not* preserved by δ, then the stabilizer group of the block $\mathcal{M}_{k_i}(\mathbb{C})$ is a closed and open subgroup of G, which is neither empty (because the identity element e is always a stabilizer) nor G itself (because δ does not preserve the block $\mathcal{M}_{k_i}(\mathbb{C})$). Since G is connected by assumption, this conclusion is impossible.

Since δ preserves the individual matrix blocks and it is ergodic, it must be $n = 1$. Hence, A is simple.

For the converse, let us assume that G is not connected. We will show that $\widehat{G}$ *has* torsion of permutation type.

Let G_0 be the connected component of the identity element e, which is always a closed normal subgroup of G. Consequently, G/G_0 is a compact group because G is supposed to be compact. In addition, it is well-known that the quotient space G/G_0 is always a totally disconnected topological space (for instance, see Theorem 7.3 in [56] for a proof). In other words, G/G_0 is a pro-finite group (for instance, see Theorem 19.9 in [181]).

Remark that G/G_0 is *non-trivial* because G is not connected by assumption. Thus, let N be a normal open subgroup of G/G_0 and consider the corresponding quotient group, $F := (G/G_0)/N$. Notice that, since G/G_0 is compact, F is a finite group. Hence, the C^*-algebra of its continuous functions, $C(F)$, is a commutative non-simple (because F is not trivial) finite dimensional C^*-algebra, which defines a finite discrete quantum subgroup of $\widehat{G}$ by means of the canonical quotient homomorphisms $G \twoheadrightarrow G/G_0 \twoheadrightarrow F$.

ii) Assume that G has a non-trivial normalized 2-cocycle $\omega \in H^2(G, S^1)$. Denote by λ the usual left regular representation of G on $L^2(G)$, which is given by

$$\begin{array}{rccl} \lambda: & G & \longrightarrow & \mathcal{B}(L^2(G)) \\ & g & \longmapsto & \lambda_g,\ \lambda_g(f)(x) := f(g^{-1}x), \text{ for all } f \in L^2(G) \text{ and } x \in G. \end{array}$$

Define the following continuous map

$$\begin{array}{rccl} \lambda^\omega: & G & \longrightarrow & \mathcal{B}(L^2(G)) \\ & g & \longmapsto & \lambda^\omega_g,\ \lambda^\omega_g(f)(x) := \omega(x^{-1}, g)\lambda_g(f)(x), \text{ for all } f \in L^2(G) \text{ and } x \in G. \end{array}$$

A straightforward computation shows that $(\lambda_g^\omega)^*(f)(x) = \overline{\omega(x^{-1}g^{-1},g)}\lambda_{g^{-1}}(f)(x)$, for all $f \in L^2(G)$ and $g, x \in G$. Namely, for all $h \in L^2(G)$ we write

$$\begin{aligned}\langle \lambda_g^\omega(f), h\rangle &= \int_G \lambda_g^\omega(f)(x)\overline{h(x)}\ d\mu(x) = \int_G \omega(x^{-1},g)f(g^{-1}x)\overline{h(x)}\ d\mu(x)\\ &= \int_G \omega(y^{-1}g^{-1},g)f(y)\overline{h(gy)}\ d\mu(y) = \int_G f(y)\overline{\overline{\omega(y^{-1}g^{-1},g)}h(gy)}\ d\mu(y)\\ &= \langle f, \overline{\omega((\cdot)^{-1}g^{-1},g)}h(g\cdot)\rangle = \langle f, \overline{\omega((\cdot)^{-1}g^{-1},g)}\lambda_{g^{-1}}(h)\rangle,\end{aligned}$$

where we have put the change $y := g^{-1}x$ and μ denotes the *left* Haar measure on G. This formula shows that, for every $g \in G$, λ_g^ω is a unitary operator on $L^2(G)$. Namely, for all $f \in L^2(G)$ and all $x \in G$ we write

$$\begin{aligned}(\lambda_g^\omega)^*\Big(\lambda_g^\omega(f)(x)\Big) &= (\lambda_g^\omega)^*\Big(\omega(x^{-1},g)\lambda_g(f)(x)\Big) = (\lambda_g^\omega)^*\Big(\omega(x^{-1},g)f(g^{-1}x)\Big)\\ &= \overline{\omega(x^{-1}g^{-1},g)}\omega(x^{-1}g^{-1},g)f(g^{-1}gx) = f(x)\end{aligned}$$

Hence $(\lambda_g^\omega)^* \circ \lambda_g^\omega = id_{L^2(G)}$ and a similar computation shows that $\lambda_g^\omega \circ (\lambda_g^\omega)^* = id_{L^2(G)}$, for all $g \in G$. Moreover, the map λ^ω defines a *unitary projective representations of G on $L^2(G)$ with associated multiplier ω*, that is, we have $\lambda^\omega : G \longrightarrow \mathcal{U}(L^2(G))/\mathbb{C}^*id$ and the formula $\lambda_{g_1g_2}^\omega = \overline{\omega(g_1,g_2)}\lambda_{g_1}^\omega \circ \lambda_{g_2}^\omega$ holds for all $g_1, g_2 \in G$. Namely, for all $f \in L^2(G)$ and all $x \in G$ we write

$$\lambda_{g_1g_2}^\omega(f)(x) = \omega(x^{-1},g_1g_2)\lambda_{g_1g_2}(f)(x) = \omega(x^{-1},g_1g_2)f(g_2^{-1}g_1^{-1}x)$$

$$\begin{aligned}\lambda_{g_1}^\omega\Big(\lambda_{g_2}^\omega(f)(x)\Big) &= \lambda_{g_1}^\omega\Big(\omega(x^{-1},g_2)\lambda_{g_2}(f)(x)\Big) = \lambda_{g_1}^\omega\Big(\omega(x^{-1},g_2)f(g_2^{-1}x)\Big)\\ &= \omega(x^{-1},g_1)\omega(x^{-1}g_1,g_2)f(g_2^{-1}g_1^{-1}x)\end{aligned}$$

and we recall the 2-cocycle equation $\omega(g_1,g_2)\omega(g_1g_2,g_3) = \omega(g_1,g_2g_3)\omega(g_2,g_3)$, for all $g_1, g_2, g_3 \in G$, which applied to the preceding computation yields the formula for λ^ω. Accordingly, the map

$$\begin{array}{rccl} Ad_{\lambda^\omega}: & G & \longrightarrow & Aut\big(\mathcal{B}(L^2(G))\big)\\ & g & \longmapsto & = \big(Ad_{\lambda^\omega}\big)_g,\ \big(Ad_{\lambda^\omega}\big)_g(T) = \lambda_g^\omega \circ T \circ (\lambda_g^\omega)^*, \text{ for all } T \in \mathcal{B}(L^2(G))\end{array}$$

defines an action of G on $\mathcal{B}(L^2(G))$.

The theory of projective representations of compact groups is completely analogous to classical theory of *usual* representations of compact groups. In particular, the projective representation λ^ω defined above (called *left regular projective representation of G with respect to ω*) decomposes as a direct sum of irreducible unitary projective representations obtaining an analogue of the celebrated Peter-Weyl theorem (we refer to [38] for more details). In other words, there exists a non-zero finite dimensional Hilbert subspace $K \subset L^2(G)$ such that $\lambda_g^\omega(K) = K$, for every $g \in G$. In this case, we define an action of G on $\mathcal{B}(K)$ by restricting the adjoint action of λ^ω defined above

$$\begin{array}{rccl} \alpha^\omega: & G & \longrightarrow & Aut\big(\mathcal{B}(K)\big)\\ & g & \longmapsto & \alpha_g^\omega := \big(Ad_{\lambda^\omega}\big)_{g\ |K},\ \alpha_g^\omega(T) = \lambda_g^\omega \circ T \circ (\lambda_g^\omega)^*, \text{ for all } T \in \mathcal{B}(K)\end{array}$$

Hence, the pair $(\mathcal{B}(K), \alpha^\omega)$ is a simple ergodic finite dimensional G-C^*-algebra. We claim that this action is a torsion action of projective type, that is, it is *not* G-equivariantly Morita equivalent to the trivial G-C^*-algebra $\mathbb{C}$. Indeed, if this was not the case, which means that $\widehat{G}$ would be projective torsion-free, then there would exist a finite dimensional unitary representation (u, H) of G such that $\mathcal{B}(K) \cong \mathcal{B}(H)$ as G-C^*-algebras. More precisely, there exists a $*$-isomorphism $\psi : \mathcal{B}(K) \longrightarrow \mathcal{B}(H)$ such that $\psi \circ \alpha_g^\omega = Ad_{u_g} \circ \psi$, for all $g \in G$. Observe that the commutator $[u_g^* \circ \psi(\lambda_g^\omega), \psi(T)]$ is trivial for every $g \in G$ and every $T \in \mathcal{B}(K)$. This implies that $u_g^* \circ \psi(\lambda_g^\omega) \in S^1 id$, for all $g \in G$ (notice that the operator $u_g^* \circ \psi(\lambda_g^\omega)$ is unitary for all $g \in G$).

Accordingly, we define the following continuous function

$$\begin{array}{rccl} f: & G & \longrightarrow & S^1 \\ & g & \longmapsto & f(g), \text{ such that } f(g)id = u_g^* \circ \psi(\lambda_g^\omega) \Leftrightarrow f(g)u_g = \psi(\lambda_g^\omega) \end{array}$$

We claim that this function trivializes the 2-cocycle ω, which is a contradiction by our assumption. Namely, for every $g_1, g_2 \in G$ we write

$$\psi(\lambda_{g_1 g_2}^\omega) = \psi\big(\omega(g_1, g_2)\lambda_{g_1}^\omega \lambda_{g_2}^\omega\big) = \omega(g_1, g_2)\psi(\lambda_{g_1}^\omega \lambda_{g_2}^\omega) = \omega(g_1, g_2)f(g_1)f(g_2)u_{g_1}u_{g_2}$$

$$\psi(\lambda_{g_1 g_2}^\omega) = f(g_1 g_2)u_{g_1 g_2} = f(g_1 g_2)u_{g_1}u_{g_2}$$

so that $\omega(g_1, g_2)f(g_1)f(g_2) = f(g_1 g_2) \Leftrightarrow \omega(g_1, g_2) = \frac{f(g_1 g_2)}{f(g_1)f(g_2)}$, for all $g_1, g_2 \in G$, which yields the claim. Therefore $\widehat{G}$ must not be projective torsion-free and the non-trivial 2-cocycle ω gives rise to a non-trivial torsion action of projective type α^ω.

Conversely, assume that there exists a simple ergodic finite dimensional G-C^*-algebra A which is *not* G-equivariantly Morita equivalent to the trivial one, say (A, δ) and write $A = \mathcal{B}(H)$ with H a finite dimensional Hilbert space. We are going to construct explicitly a non-trivial 2-cocycle of G.

Let $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal basis of H and let $\{\xi_1^*, \ldots, \xi_n^*\}$ its dual basis in H^*. By uniqueness of the trace Tr on $\mathcal{B}(H)$, we have $Tr \circ \delta_g = Tr$, for all $g \in G$. Consequently, $\delta_g(\xi_i \xi_i^*)$ is a rank one projection, for all $i = 1, \ldots, n$. For every $g \in G$ and every $i = 1, \ldots, n$, let η_i^g be a norm one vector in $Im(\delta_g(\xi_i \xi_i^*))$, so that $\delta_g(\xi_i \xi_i^*) = \eta_i^g (\eta_i^g)^*$, for all $g \in G$ and all $i = 1, \ldots, n$. Since

$$\delta_g(\xi_i \xi_i^*)\delta_g(\xi_j \xi_j^*) = \delta_{i,j} \cdot \eta_i^g (\eta_j^g)^*,$$

for all $g \in G$ and all $i, j = 1, \ldots, n$, then $\{\eta_1^g, \ldots, \eta_n^g\}$ is an orthonormal basis of H, for all $g \in G$.

For every $g \in G$, we denote by u_g the unitary in $\mathcal{B}(H)$ such that

$$u_g(\xi_i) = \eta_i^g,$$

for all $i = 1, \ldots, n$. A straightforward computation shows that

$$\delta_g(T) = u_g \circ T \circ u_g^*,$$

for all $g \in G$ and all $T \in \mathcal{B}(H)$. In particular, we can write the following

$$u_{gh} \circ T \circ u_{gh}^* = \delta_{gh}(T) = \delta_g\big(\delta_h(T)\big) = u_g \circ u_h \circ T \circ u_h^* \circ u_g^*,$$

for all $g, h \in G$ and all $T \in \mathcal{B}(H)$. This formula shows clearly that $u_h^* \circ u_g^* \circ u_{gh} \in Z(\mathcal{B}(H)) = \mathbb{C}id_H$, so that we write $u_h^* \circ u_g^* \circ u_{gh} = \omega(g,h)id_H$ with $\omega(g,h) \in \mathbb{C}^*$. Since this is true for every $g, h \in G$, we define

$$\begin{array}{rccc} \omega: & G \times G & \longrightarrow & S^1 \\ & (g,h) & \longmapsto & \omega(g,h), \end{array}$$

by the previous construction (up to normalization). A straightforward computation shows that ω is a 2-cocycle of G. Moreover it must be non-trivial (non-cohomologous to the trivial 2-cocycle). Otherwise, there would exist a continuous fonction $f : G \longrightarrow S^1$ such that $\omega(g,h) = f(gh)^{-1}f(g)f(h)$, for all $g, h \in G$. The group homomorphism

$$\begin{array}{rccc} v: & G & \longrightarrow & \mathcal{B}(H) \\ & g & \longmapsto & v_g := f(g)u_g \end{array}$$

defines a unitary representation of G on H and by construction we have $\delta_g(T) = v_g \circ T \circ v_g^*$, for all $T \in \mathcal{B}(H)$. This would imply that $A = \mathcal{B}(H)$ is G-equivariantly Morita equivalent to $\mathbb{C}$, which is impossible because of our assumption. Hence ω is a non-trivial normalized 2-cocycle of G and the proof is complete.

■

1.6.1.10 Note. The following bijective correspondence is well-known

$$\left\{ \begin{array}{c} \textit{projective representations of } G \\ \pi : G \longrightarrow \mathcal{B}(H)/\mathbb{C}^* id \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{normalized 2-cocycles} \\ \omega \in H^2(G, S^1) \end{array} \right\},$$

where a projective representation of G on a Hilbert space H is a *continuous* map $\pi : G \longrightarrow \mathcal{B}(H)$ such that $\pi_{g_1 g_2} = \omega(g_1, g_2)\pi_{g_1}\pi_{g_2}$, for all $g_1, g_2 \in G$. For a detailed exposition about the elementary theory of projective representations we refer to [38].

A typical example that illustrates the preceding theorem is the following. Consider the classical rotation group $G := SO(3)$, which is connected. Then $\widehat{SO(3)}$ is *not* torsion-free in the sense of Meyer-Nest. Namely, $\widehat{SO(3)}$ has torsion of projective type because of the well-known projective representation of $SO(3)$ on $\mathbb{C}^2$ given by its universal covering group $SU(2)$. Let us be more precise. It is a classical result that $SO(3) \cong SU(2)/\mathbb{Z}_2$, where $\mathbb{Z}_2 \cong Z(SU(2))$ is the center of the special unitary group $SU(2)$. Consider the following action of $SU(2)$ on $\mathcal{M}_2(\mathbb{C})$

$$\begin{array}{rccc} \delta: & SU(2) \times \mathcal{M}_2(\mathbb{C}) & \longrightarrow & \mathcal{M}_2(\mathbb{C}) \\ & (N, M) & \longmapsto & \delta_N(M) := NMN^*, \end{array}$$

which is obviously trivial on $\mathbb{Z}_2 \cong Z(SU(2))$. Hence, δ descends to an action of $SO(3)$ on $\mathcal{M}_2(\mathbb{C})$, so that $(\mathcal{M}_2(\mathbb{C}), \delta)$ defines a simple ergodic finite dimensional $SO(3)$-C^*-algebra, which is *not* equivariantly Morita equivalent to to $\mathbb{C}$. Indeed, if $\mathcal{M}_2(\mathbb{C})$ was *equivariantly* Morita equivalent to $\mathbb{C}$, then it would exist a Hilbert space H such that $\mathcal{M}_2(\mathbb{C}) \cong \mathcal{K}(H)$. Because of dimension reasons, it must be $dim(H) = 2$. Thus, we assume without loss of generality, that $H = \mathbb{C}^2$.

Observe that $SO(3)$ acts on $\mathbb{C}^2$ by means of its projective representation. More precisely, consider the usual representation of $SU(2)$ on $\mathbb{C}^2$ given by matrix multiplication, $\pi : SU(2) \longrightarrow \mathcal{B}(\mathbb{C}^2)$. It

is well-known that $SU(2)$ is the universal covering of $SO(3)$, so that the corresponding universal covering map $\widetilde{p}$ yields the projective representation announced before.

$$\begin{tikzcd} SU(2) \arrow[r, "\pi"] \arrow[d, two heads, "\widetilde{p}"'] & \mathcal{B}(\mathbb{C}^2) \\ SO(3) \arrow[ur, dashed, "\widetilde{\pi}"'] & \end{tikzcd}$$

This prevents $\mathcal{M}_2(\mathbb{C})$ to be $SO(3)$-equivariantly Morita equivalent to $\mathbb{C}$.

Furthermore, this example can be translated in the quantum setting. On the one hand, it is possible to give a quantum version of the rotation group as explained in Remarks 2.1.4 by deforming with a parameter $q \in [-1,1]\backslash\{0\}$ the C^*-algebra of its continuous functions. We denote such a deformation by $SO_q(3)$. On the other hand, P. Soltan gives in [173] an explicit construction of torsion actions of projective type of $SO_q(3)$ on $\mathcal{M}_2(\mathbb{C})$, which is the analogue to the one described above in the classical case. In addition, it is known that every quantum automorphism group $Qut(A,\omega)$ is monoidally equivalent to $SO_q(3)$, for some $q \in (0,1]$ (recall Theorem B.3.19). The projective torsion of $\widehat{SO_q(3)}$ is turned into the permutation torsion of $\widehat{Qut(A,\omega)}$ given by its defining action (see Lemma 4.4 in [212] for more details)

1.6.2 Torsion *à l'Arano-De Commer*

The re-interpretation of torsion for quantum groups by Y. Arano and K. De Commer follows a categorical and combinatorial approach through the notion of fusions ring. Indeed, associated to any discrete quantum group we have an obvious fusion ring arising from its irreducible representations.

It is advisable to keep in mind notations and definitions from Section B.3. In particular, given a compact quantum group $\mathbb{G}$, we denote by $\mathscr{R}ep(\mathbb{G})$ the corresponding rigid C^*-tensor category, which is called *representation category of* $\mathbb{G}$. Given a subset $\mathcal{S} \subset Irr(\mathbb{G})$, we denote by $\mathscr{C} := \langle \mathcal{S} \rangle$ the smallest full subcategory of $\mathscr{R}ep(\mathbb{G})$ containing $\mathcal{S}$. If, in addition, $\mathscr{C}$ contains the trivial representation and it is closed under taking tensor product and contragredient representations, by Tannaka-Krein-Woronowicz duality (see Theorem B.3.16 and Remark B.3.17), there is an associated C^*-subalgebra $C(\mathbb{H})$ such that restricting the coproduct to $C(\mathbb{H})$ endows it with the structure of compact quantum group $\mathbb{H}$. Moreover, $\mathscr{R}ep(\mathbb{H})$ naturally identifies with $\mathscr{C}$ and we say that $\widehat{\mathbb{H}}$ is the quantum subgroup of $\widehat{\mathbb{G}}$ generated by $\mathcal{S}$.

Let us recall the main definitions an results about fusion rings in order to summarize the work [3] by Y. Arano and K. De Commer. We refer to [3] and [61] for more details of the subject. By the convenience of the exposition, the next presentation have been adapted with respect to the special case of fusion rings coming from discrete quantum groups (which is the relevant one for the present dissertation), so that these definitions are equivalent with the standard ones.

Let $(I, \mathbb{1})$ be a pointed set with distinguished element $\mathbb{1}$, called *unit of* I. We equip I with an involution

$$\begin{array}{ccc} I & \longrightarrow & I \\ \alpha & \longmapsto & \overline{\alpha}, \end{array}$$

such that $\overline{\mathbb{1}} = \mathbb{1}$ and we say that $(I, \mathbb{1})$ is an *involutive pointed set*. Denote by $\mathbb{Z}_I$ the *free $\mathbb{Z}$-module with basis I*, that is to say, every element in $\mathbb{Z}_I$ is a unique finite $\mathbb{Z}$-linear combination of elements of I. The addition operation in $\mathbb{Z}_I$ is denoted by $\oplus$. A *ring structure* $\otimes$ on $\mathbb{Z}_I$ is given by constants $\lambda^i_{\alpha,\alpha'} \in \mathbb{N} \cup \{0\}$ for all $\alpha, \alpha', i \in I$, called *fusion rules*, such that

$$\alpha \otimes \alpha' = \sum_{i \in I} \lambda^i_{\alpha,\alpha'} \cdot i,$$

where all but finitely many terms vanish. This rule extends obviously to any element of $\mathbb{Z}_I$ and it can be regarded as an action of $\mathbb{Z}_I$ on itself; we call it *regular action of* $\mathbb{Z}_I$. We write $i \subset \alpha \otimes \alpha'$ whenever $\lambda^i_{\alpha,\alpha'} \neq 0$.

1.6.2.1 Definition. Let $(I, \mathbb{1})$ be an involutive pointed set. Let $(\mathbb{Z}_I, \oplus)$ be the corresponding free $\mathbb{Z}$-module equipped with a ring structure $\otimes$. We say that $\mathbb{Z}_I$ is a I-based ring if

i) $\overline{\alpha \otimes \alpha'} = \overline{\alpha'} \otimes \overline{\alpha}$, for all $\alpha, \alpha' \in I$,

ii) $\mathbb{1} \subset \overline{\alpha} \otimes \alpha'$ if and only if $\alpha = \alpha'$, for all $\alpha, \alpha' \in I$.

- A dimension function on a I-based ring $(\mathbb{Z}_I, \oplus, \otimes)$ is a unital ring homomorphism $d : \mathbb{Z}_I \longrightarrow \mathbb{R}$ such that

  i) $d(\alpha) > 0$, for all $\alpha \in I$,

  ii) $d(\overline{\alpha}) = d(\alpha)$, for all $\alpha \in I$.

- A fusion ring is a I-based ring endowed with a dimension function, $(\mathbb{Z}_I, \oplus, \otimes, d)$.

The main examples and constructions of fusion rings which are interesting for the present dissertation are the following.

a) Let Γ be a discrete group. Define $(I, \mathbb{1}) := (\Gamma, e)$ as the pointed set with involution given by the inverse in the group. We define the *fusion ring of* Γ as the Γ-based ring $\mathbb{Z}_\Gamma$ with ring structure and dimension function given by

$$g \otimes h := gh \text{ and } d(\gamma) := 1,$$

for all $g, h, \gamma \in \Gamma$. This ring is denoted by $Fus(\Gamma)$ or $R(\Gamma)$ and sometimes we refer to it as the *representation ring of* Γ.

b) Let $\widehat{\mathbb{G}}$ be a discrete quantum group. Define $(I, \mathbb{1}) := (Irr(\mathbb{G}), \epsilon)$ as the pointed set with involution given by the adjoint representation. We define the *fusion ring of* $\widehat{\mathbb{G}}$ as the $Irr(\mathbb{G})$-based ring $\mathbb{Z}_{Irr(\mathbb{G})}$ with fusion rules and dimension function given by

$$N^z_{x,y} := dim\Big(Mor((z, x \oplus y))\Big) \text{ and } d(x) := dim(x) \ ,$$

for all $x, y, z \in Irr(\mathbb{G})$. In other words, the ring structure is given simply by the tensor product of representations and so by the corresponding fusion rules. This ring is denoted by $Fus(\widehat{\mathbb{G}})$ or $R(\mathbb{G})$ and sometimes we refer to it as the *representation ring of* $\mathbb{G}$.

c) Let $R_1 := (\mathbb{Z}_{I_1}, \oplus, \otimes, d_1)$ and $R_2 := (\mathbb{Z}_{I_2}, \oplus, \otimes, d_2)$ be two fusion rings. We define the tensor product of $\mathbb{Z}_{I_1}$ and $\mathbb{Z}_{I_2}$ as the free $\mathbb{Z}$-module $\mathbb{Z}_{I_1} \underset{\mathbb{Z}}{\odot} \mathbb{Z}_{I_2}$, which is a fusion ring denoted by $R_1 \otimes R_2$ with

- basis $I_1 \underset{\mathbb{Z}}{\odot} I_2$,
- unit $\mathbb{1}_1 \odot \mathbb{1}_2$,
- involution $\overline{x \odot y} = \overline{x} \odot \overline{y}$, for all $x \in \mathbb{Z}_{I_1}$ and all $y \in \mathbb{Z}_{I_2}$,
- and dimension function $d(\alpha \odot \beta) = d_1(\alpha) d_2(\beta)$, for all $\alpha \in I_1$ and all $\beta \in I_2$.

d) Let $R_1 := (\mathbb{Z}_{I_1}, \oplus, \otimes, d_1)$ and $R_2 := (\mathbb{Z}_{I_2}, \oplus, \otimes, d_2)$ be two fusion rings. We define the free product of R_1 and R_2, denoted by $R_1 * R_2$, as the free $\mathbb{Z}$-module with basis $I := I_1 * I_2$, which is a fusion ring with

- basis $I := I_1 * I_2$, which is the set of alternating words (possibly empty) in $I_1 \backslash \{\mathbb{1}_1\}$ and $I_2 \backslash \{\mathbb{1}_2\}$. I is an involutive pointed set with the empty word as the distinguished element and involution given by inverting order and acting as involution of I_i on each letter,
- unit the *empty word*,
- ring structure $\otimes$ such that
  a) if $\omega, \omega' \in I = I_1 * I_2$ are words such that ω ends in I_1 and ω' starts in I_2 (or *vice-versa*), then
  $$\omega \otimes \omega' := \omega\omega'$$
  b) if $\omega = \zeta\alpha, \omega' = \alpha'\zeta' \in I = I_1 * I_2$ are words such that $\alpha, \alpha' \in I_1$ (or in I_2), then
  $$\omega \otimes \omega' = \bigoplus_{\substack{t \subset \alpha \otimes \alpha' \\ t \neq \mathbb{1}_1}} \lambda^t_{\alpha,\alpha'} \zeta t \zeta' \oplus \lambda^{\mathbb{1}_1}_{\alpha,\alpha'} \zeta \otimes \zeta'$$
- and dimension function d uniquely determined by $d_{|_{I_i}} = d_i$, for all $i = 1, 2$.

e) Let $R := (\mathbb{Z}_I, \oplus, \otimes, d)$ be a fusion ring. If $L \subset I$ is subset such that $(L, \mathbb{1})$ is an involutive pointed set such that $\lambda^i_{\alpha,\alpha'} = 0$, for all $\alpha, \alpha' \in L$ and all $i \in I \backslash L$, then we obtain by restriction of $\otimes$ and d a fusion ring $S := (\mathbb{Z}_L, \oplus, \otimes_|, d_|)$, which is called *fusion subring of* R and we write $S \subset R$.

For instance, given any basis element $\alpha \in I$ we can consider the *fusion ring generated by* α, which is the smallest fusion subring of R containing α.

Next, let J be any set and consider the corresponding free $\mathbb{Z}$-module with basis J, $\mathbb{Z}_J$. Assume that $(\mathbb{Z}_I, \oplus)$ is equipped with a ring structure $\otimes$ following the same notations as above. Then a $\mathbb{Z}_I$*-module structure* on $\mathbb{Z}_J$, denoted by the same symbol $\otimes$ by abuse of notation, is given by constants $\lambda^j_{\alpha,\beta} \in \mathbb{N} \cup \{0\}$ for all $\alpha \in I$, $\beta, j \in J$ such that

$$\alpha \otimes \beta = \sum_{j \in J} \lambda^j_{\alpha,\beta} \cdot j,$$

where all but finitely many terms vanish. This rule extends obviously to any element of $\mathbb{Z}_I$ and $\mathbb{Z}_J$ and it can be regarded as an action of $\mathbb{Z}_I$ on $\mathbb{Z}_j$; we say that $\mathbb{Z}_J$ is a $\mathbb{Z}_I$-module. We write $j \subset \alpha \otimes \beta$ whenever $\lambda^j_{\alpha,\beta} \neq 0$.

1.6.2.2 Definition. Let $(I, \mathbb{1})$ be an involutive pointed set and let J be any set. Let $(\mathbb{Z}_I, \oplus)$ be the free $\mathbb{Z}$-module with basis I equipped with a ring structure $\otimes$. Let $(\mathbb{Z}_J, \oplus)$ be the free $\mathbb{Z}$-module with basis J equipped with a $\mathbb{Z}_I$-module structure $\otimes$. We say that $\mathbb{Z}_J$ is a J-based module if the following condition holds,

$$\beta \subset \alpha \otimes j \text{ if and only if } j \subset \overline{\alpha} \otimes \beta,$$

for all $\alpha \in I$, $\beta, j \in J$.

- A J-based module $(\mathbb{Z}_J, \oplus, \otimes)$ is said to be co-finite if for all $\beta, j \in J$, the set $\{\alpha \in I \mid j \subset \alpha \otimes \beta\}$ is finite.
- A J-based module $(\mathbb{Z}_J, \oplus, \otimes)$ is said to be connected if for all $\beta, j \in J$, there exists $\alpha \in I$ such that $j \subset \alpha \otimes \beta$.
- A J-based module is said to be a torsion module, if it is co-finite and connected.
- If $(\mathbb{Z}_I, \oplus, \otimes, d_I)$ is a fusion ring, a compatible dimension function on a J-based module $(\mathbb{Z}_J, \oplus, \otimes)$ is a linear map $d_J : \mathbb{Z}_J \longrightarrow \mathbb{R}$ such that
  i) $d_J(\beta) > 0$, for all $\beta \in J$,
  ii) $d_J(\alpha \otimes \beta) = d_I(\alpha) d_J(\beta)$, for all $\alpha \in I$ and all $\beta \in J$.
- A fusion module is a J-based module over $\mathbb{Z}_I$ endowed with a compatible dimension function, $(\mathbb{Z}_J, \oplus, \otimes, d_J)$.

1.6.2.3 Remarks. 1. If $M := (\mathbb{Z}_J, \oplus, \otimes)$ is a co-finite based module over a fusion ring $R := (\mathbb{Z}_I, \oplus, \otimes, d_I)$ as in the above definition, then we can define a $\mathbb{Z}_I$-valued bilinear form on M by

$$\langle \beta, \beta' \rangle := \sum_{i \in I} \lambda_{\overline{i}, \beta}^{\beta'} \cdot i,$$

for all $\beta, \beta' \in J$. This rule extends obviously to any pair of elements of M. Notice the following,

- For any $\alpha \in I$, $\beta, \beta' \in J$ we have $\langle \alpha \otimes \beta, \beta' \rangle = \alpha \otimes \langle \beta, \beta' \rangle$.
- For any $j_0 \in J$, the map $d_J : M \longrightarrow \mathbb{R}$ defined by $d_J(\beta) := d_I(\langle \beta, j_0 \rangle)$, for all $\beta \in J$ is a compatible dimension function on M.

2. Let $R := (\mathbb{Z}_I, \oplus, \otimes, d)$ be a fusion ring. The regular action of $\mathbb{Z}_I$ turns R into a I-based module over R and its dimension function is a compatible dimension function on the corresponding module. It follows from Definition 1.6.2.1 that R is itself a co-finite and connected fusion module. The corresponding $\mathbb{Z}_I$-valued bilinear form on R is given simply by

$$\langle \alpha, \alpha' \rangle = \alpha \otimes \overline{\alpha'},$$

for all $\alpha, \alpha' \in I$. In this way, we say that R is equipped with the *standard fusion module structure*.

3. Let R be a fusion ring and let $S \subset R$ be a fusion subring of R. By restriction, it is clear that R can be viewed as a fusion S-module. If N is a fusion S-module, the tensor product $M := R \underset{S}{\otimes} N$ is naturally a fusion R-module. We denote this module by

$$Ind_S^R(N) := R \underset{S}{\otimes} N$$

and we call it *induced R-module from N*.

Notice that this construction imitates the classical algebraic construction of induced representation from a subgroup. Moreover, if $\widehat{\mathbb{H}}$ is a discrete quantum subgroup of a discrete quantum group $\widehat{\mathbb{G}}$, then we have $C(\mathbb{H}) \subset C(\mathbb{G})$ and by Tannaka-Krein-Woronowicz duality, $\mathscr{R}ep(\mathbb{H})$ is a full subcategory of $\mathscr{R}ep(\mathbb{G})$. Hence, $R(\mathbb{H})$ is a fusion subring of $R(\mathbb{G})$. In this case, we use the notation $Ind_{R(\mathbb{H})}^{R(\mathbb{G})}(\cdot)$

1.6.2.4 Definition. Let R be a fusion ring. A standard fusion module is any fusion R-module which is isomorphic to R with its standard fusion module structure.

1.6.2.5 Remark. Let R be a fusion ring and $S \subset R$ a fusion subring. Assume that N is a standard fusion S-module. Then $N \cong S$ as in the definition above. It is clear that the induction of N is again a standard fusion R-module because

$$Ind_S^R(N) = R \underset{S}{\otimes} N \cong R \underset{S}{\otimes} S \cong R$$

1.6.2.6 Definition. A fusion ring R is said to be torsion-free if any torsion R-module is standard.

In particular, a discrete quantum group $\widehat{\mathbb{G}}$ is called strong torsion-free (or *torsion-free in the sense of Arano-De Commer*) if $Fus(\widehat{\mathbb{G}})$ is torsion-free.

1.6.2.7 Remark. Let us explain the motivation for the preceding definition. Let $\mathbb{G}$ be a compact quantum group and let (A, δ) be a torsion action of $\mathbb{G}$. Then the *category of $\mathbb{G}$-equivariant Hilbert A-modules*, say $\mathscr{H}$, (see Section 1.7.1 for more details) can be regarded as a *module C^*-category over $\mathscr{R}ep(\mathbb{G})$*. As a consequence, its Grothendieck group is a based module over $Fus(\widehat{\mathbb{G}})$ and it is not hard to see that

- the finite-dimensionality of A implies that this module is co-finite
- and the ergodicity of δ implies that this module is connected.

In other words, any torsion action of $\mathbb{G}$ gives rise to a torsion module in $\mathscr{H}$ over $Fus(\widehat{\mathbb{G}})$. Moreover, if (A, δ) is the trivial torsion action, then we have $\mathscr{H} = \mathscr{R}ep(\mathbb{G})$ by construction and the corresponding module is simply $Fus(\widehat{\mathbb{G}})$.

Notice that if $\widehat{\mathbb{H}}$ is a discrete quantum subgroup of a discrete quantum group $\widehat{\mathbb{G}}$, then the induction of actions $Ind_{\mathbb{H}}^{\mathbb{G}}(\cdot)$ explained in Remarks 1.4.3.5 corresponds to the induction of modules explained in Remarks 1.6.2.3 through the above categorical interplay.

Of course, if Γ is a classical discrete group, then the usual torsion-freeness for Γ is equivalent to the strong torsion-freeness for $\widehat{\Gamma}$ in the sense above. Moreover, we can show that the torsion-freeness in the sense of Arano-De Commer implies the torsion-freeness in the sense of Meyer-Nest (see

Theorem 2.8 in [3] for a proof). However, the converse may not hold because it is not clear that any torsion $Fus(\widehat{\mathbb{G}})$-module can be traced back to a torsion action of $\mathbb{G}$.

The previous remark brings us to give a more general notion of (strong) torsion-freeness in the framework of C^*-tensor categories, which has been done in [3] by Y. Arano and K. De Commer. Furthermore, this approach of torsion-freeness turns out to be very helpful in order to exhibit the torsion-freeness in the sense of Meyer-Nest for a given discrete quantum group. Indeed, this approach allows to tackle the problem through fusion modules (for instance, Section 3.6.1 illustrates this process).

Let us give some elementary definitions and constructions related to latter idea. We refer to [3], [50] and [121] for more precisions and details.

Let $\mathscr{C}$ be a rigid C^*-tensor category. We can associate to $\mathscr{C}$ a fusion ring, denoted by $Fus(\mathscr{C})$, with basis given by the irreducible objects of $\mathscr{C}$, fusion rules analogous to the fusion rules of a discrete quantum group and dimension function defined in [121].

Let $\mathscr{M}$ be a $\mathscr{C}$-module C^*-category. Observe that, associated to $\mathscr{M}$, we have a J-based $Fus(\mathscr{C})$-module, denoted by $Fus(\mathscr{M})$, where J is the set of equivalence classes of irreducible objects in $\mathscr{M}$.

In this situation, we say that

- $\mathscr{M}$ is *co-finite* if $Fus(\mathscr{M})$ is co-finite as J-based $Fus(\mathscr{C})$-module,
- $\mathscr{M}$ is *connected* if $Fus(\mathscr{M})$ is connected as J-based $Fus(\mathscr{C})$-module,
- $\mathscr{M}$ is a *torsion $\mathscr{C}$-module C^*-category* if $Fus(\mathscr{M})$ is a torsion $Fus(\mathscr{C})$-module.

1.6.2.8 Remark. We can show (see Lemma 3.10 in [3] for a proof) that if $Fus(\mathscr{M}) \cong Fus(\mathscr{C})$ as based modules, then $\mathscr{M} \cong \mathscr{C}$ as $\mathscr{C}$-module C^*-categories.

The following definition must be regarded as an abstraction of the notion of torsion-freeness in the sense of Meyer-Nest. Indeed, it is compatible with Definition 1.6.1.1 as shown in Proposition 3.4 of [3].

1.6.2.9 Definition. A rigid C^*-tensor category $\mathscr{C}$ is said to be torsion-free if every non-trivial torsion $\mathscr{C}$-module C^*-category is equivalent to $\mathscr{C}$ as $\mathscr{C}$-module C^*-categories.

1.6.2.10 Note. It is important to warn that the preceding definition is rather a characterization (Lemma 3.11 in [3]) of the original definition of torsion-freeness for a rigid C^*-tensor category (Definition 3.7 in [3]). Since the original definition requires more categorical preliminaries, we prefer to give directly the preceding one, which is enough for our purpose.

Among all the results obtained by Y. Arano and K. De Commer in [3], the following two (see Theorem 1.26 and Proposition 1.28, respectively in [3] for a proof) are particularly useful for the present dissertation.

1.6.2.11 Theorem. *Let $R_1 := (\mathbb{Z}_{I_1}, \oplus, \otimes, d_1)$ and $R_2 := (\mathbb{Z}_{I_2}, \oplus, \otimes, d_2)$ be torsion-free fusion rings. If $R_1 \otimes R_2$ is* not *torsion-free, then R_1 and R_2 have non-trivial isomorphic finite fusion subrings.*

1.6.2.12 Proposition. *Let $R := (\mathbb{Z}_I, \oplus, \otimes, d)$ and $S := (\mathbb{Z}_L, \oplus, \otimes_|, d_|)$ be fusion rings such that S is a fusion subring of R. Assume that R is torsion-free.*

If S is divisible *in R (which means that $R \cong \bigoplus S$ as based S-modules), then S is torsion-free.*

1.7 KK-theory in the quantum setting

Given a (second countable) locally compact group G, the corresponding G-equivariant Kasparov theory has been (and will be) presupposed for this dissertation and standard references for the necessary material on this subject are [86], [224], [24] or [164] (we can refer as well to the original articles of J. Cuntz, G. G. Kasparov and G. Skandalis, see for example [171], [45], [95], [97], [98]).

Given a locally compact *quantum* group $\mathbb{G}$ we can construct a *quantum* $\mathbb{G}$-equivariant Kasparov theory which imitates all the classical constructions and definitions. In this section we are going to present this quantum picture of KK-theory for the convenience of the exposition, so that it shall recall as well the classical well known Kasparov theory.

In the early work [6], S. Baaj and G. Skandalis define an equivariant KK-theory with respect to any *Hopf C^*-algebra*, extend the Kasparov product into this framework and give a particular version of the Baaj-Skandalis duality for locally compact groups and its duals. In addition, it is possible to give a more general perspective of this quantum KK-theory working with a *weak Kac system* (in the sense of R. Vergnioux) instead of working directly with a Hopf C^*-algebra. For a well detailed exposition of this we refer to Chapter 3 and Chapter 5 in [206].

1.7.1 Quantum Kasparov's theory and Baaj-Skandalis Duality

In order to simplify notations and to have a general perspective of the quantum Kasparov's theory, we work with any Hopf C^*-algebra $\mathbb{S} := (S, \Delta)$. However, in the context of the present dissertation such a Hopf C^*-algebra is supposed to be either $\widehat{\mathbb{G}} = (c_0(\widehat{\mathbb{G}}), \widehat{\Delta})$ or $\mathbb{G} = (C(\mathbb{G}), \Delta)$, where $\mathbb{G}$ is a compact quantum group. In order to understand the following presentation, it is advisable to keep in mind elementary notions about Hilbert modules recalled in Section A.3 and the corresponding notion of multipliers (see Definition A.4.5 and Definition A.4.6).

1.7.1.1 Definition. Let $\mathbb{S} = (S, \Delta)$ be a Hopf C^*-algebra and (A, δ) a $\mathbb{S}$-C^*-algebra. A left $\mathbb{S}$-equivariant Hilbert A-module is a Hilbert A-module H equipped with a linear map $\delta_H : H \longrightarrow \widetilde{M}(S \otimes H)$ such that

i) $\delta_H(\xi \cdot a) = \delta_H(\xi) \circ \delta(a)$, for all $\xi \in H$ and all $a \in A$,

ii) $\delta(\langle \xi, \eta \rangle) = \langle \delta_H(\xi), \delta_H(\eta) \rangle$, for all $\xi, \eta \in H$,

iii) $(id_S \otimes \delta_H) \circ \delta_H = (\Delta \otimes id_H) \circ \delta_H$,

iv) $[\delta_H(H)(S \otimes A)] = S \otimes H$.

Such a map is called a *left action of* $\mathbb{S}$ *on* H or a *right co-action of* S *on* H.

1.7.1.2 Remark. If (H, δ_H) is a $\mathbb{S}$-equivariant Hilbert A-module, then the action δ_H is an isometry hence *injective* whenever δ_A is injective.

1.7.1.3 Note. A *right* action of $\mathbb{S}$ on H (or a *left* co-action of S on H) is a linear map $\delta_H : H \longrightarrow \widetilde{M}(H \otimes S)$ satisfying the analogue properties of the preceding definition.

In the present thesis, an action of a Hopf C^*-algebra is supposed to be a *left* one unless the contrary is explicitly indicated. Hence, we refer to such actions simply as *action of* $\mathbb{S}$.

1.7.1.4 Note. There are some different terminologies in the literature that must be clarified here. Let $\mathbb{S} = (S, \Delta)$ be a Hopf C^*-algebra and consider $\mathbb{C}$ as a trivial $\mathbb{S}$-C^*-algebra. In this case we consider a Hilbert space H. A linear map $\delta_H : H \longrightarrow M(S \otimes H)$ satisfying (ii), (iii) and (iv) of the definition above (observe that (i) is automatically satisfied in this case) is called sometimes *unitary representation of* $\mathbb{S}$ *on* H. Namely, it is easy to show (see Proposition 5.2.2 in [188] for a proof) that such a linear map is equivalent to a unitary operator $w \in M(\mathcal{K}(H) \otimes S)$ such that $(id \otimes \Delta)(w) = w_{12}w_{13}$. This correspondence is realized by means of the following relation

$$\delta_H(\xi) = w(id_S \otimes \xi),$$

for all $\xi \in H$. For an arbitrary Hilbert A-module, the analogous correspondence is established in Proposition 1.7.1.6 below.

Hence, if we do these constructions for a compact quantum group $\mathbb{G}$, we obtain that a unitary representation of $\mathbb{G}$ on a Hilbert space H is equivalent to a linear map $\delta_H : H \longrightarrow M(C(\mathbb{G}) \otimes H)$ satisfying (ii), (iii) and (iv) of the definition above. In other words, a unitary representation of $\mathbb{G}$ on a Hilbert space H is exactly the unitary admissible of Proposition 1.7.1.6 below associated to an action δ_H of $\mathbb{G}$ on H.

Remark as well that, in general, if $\delta_H : H \longrightarrow \widetilde{M}(S \otimes H)$ is an action of $\mathbb{S}$ on a Hilbert A-module H where A is equipped with the *trivial* action of $\mathbb{S}$, then the admissible unitary associated to (H, δ_H) (see Proposition 1.7.1.6 below) is a unitary operator $V_H \in M(\mathcal{K}(H) \otimes S)$ such that $(id \otimes \Delta)(V_H) = (V_H)_{12}(V_H)_{13}$. Such an operator is called *unitary representation of* $\mathbb{S}$ *on* H (recall Remarks 1.3.1.8).

Remark finally that we can take in particular the C^*-algebra A itself as a Hilbert A-module. In this case, the above definition restricts to the usual definition of an action of $\mathbb{G}$ or $\widehat{\mathbb{G}}$ on A (notice that conditions (i) and (ii) are automatically fulfilled because such an action is a $*$-homomorphism; recall Definition 1.4.1.1 and Definition 1.4.2.1).

1.7.1.5 Remark. Let (H, δ_H) be a $\mathbb{S}$-quivariant Hilbert A-module. In order to handle properly the admissible operator associated to H (which we introduce in the next proposition) it is advisable to notice the following canonical identifications of Hilbert A-modules.

- $(S \otimes H) \underset{id_S \otimes \delta}{\otimes} (S \otimes S \otimes A) \cong S \otimes (H \underset{\delta}{\otimes} (S \otimes A))$
- $\big(H \underset{\delta}{\otimes} (S \otimes A)\big) \underset{id_S \otimes \delta}{\otimes} \big(S \otimes S \otimes A\big) \cong H \underset{\delta^2}{\otimes} (S \otimes S \otimes A)$
- $\big(H \underset{\delta}{\otimes} (S \otimes A)\big) \underset{\Delta \otimes id_A}{\otimes} \big(S \otimes S \otimes A\big) \cong H \underset{\delta^2}{\otimes} (S \otimes S \otimes A)$
- $S \otimes S \otimes H \cong \big(S \underset{\Delta}{\otimes} (S \otimes S)\big) \otimes H \cong \big(S \otimes H\big) \underset{\Delta \otimes id_A}{\otimes} \big(S \otimes S \otimes A\big)$

The proof of these identifications is straightforward.

1.7.1.6 Proposition. *Let $\mathbb{S} = (S, \Delta)$ be a Hopf C^*-algebra and (A, δ) a $\mathbb{S}$-C^*-algebra. If H is any Hilbert A-module and $\delta_H : H \longrightarrow \widetilde{M}(S \otimes H)$ is a linear map, the following assertions are equivalent*

i) $\delta_H : H \longrightarrow \widetilde{M}(S \otimes H)$ is an action of $\mathbb{S}$ on H.

ii) There exists a unique unitary operator (called admissible operator for H*)*
$V_H \in \mathcal{L}_{S \otimes A}(H \underset{\delta}{\otimes} (S \otimes A), S \otimes H)$ such that

- $\delta_H(\xi) = V_H \circ T_\xi$, *for all* $\xi \in H$ *where* $T_\xi \in \mathcal{L}_{S\otimes A}(S\otimes A, H \underset{\delta}{\otimes} (S\otimes A))$ *is defined by* $T_\xi(x) := \xi \underset{\delta}{\otimes} x$, *for all* $x \in S\otimes A$;
- $(id_S \otimes V_H) \circ (V_H \underset{id_S\otimes\delta}{\otimes} id_{S\otimes S\otimes A}) = V_H \underset{\Delta\otimes id_A}{\otimes} id_{S\otimes S\otimes A}$.

Proof. Suppose that $\delta_H : H \longrightarrow \widetilde{M}(S\otimes H)$ is an action of $\mathbb{S}$ on H. Define the following linear operator

$$\begin{array}{rccl} W: & H\odot(S\otimes A) & \longrightarrow & S\otimes H \\ & \xi\otimes x & \longmapsto & W(\xi\otimes x) := \delta_H(\xi)(x), \end{array}$$

which can be extended to the internal tensor product (recall Theorem A.3.5) thanks to axioms (i) and (ii) of Definition 1.7.1.1. Let $V_H : H \underset{\delta}{\otimes} (S\otimes A) \longrightarrow S\otimes H$ be such extension. Thanks to axiom (iv) of Definition 1.7.1.1, V_H is surjective. Define thus $W^*(\delta_H(\xi)(x)) := \xi\otimes x$, for all $\xi \in H$ and all $x \in S\otimes A$. The extension of W^* to $S\otimes H$ is the adjoint of V_H. We deduce that $V_H \in \mathcal{L}_{S\otimes A}(H\underset{\delta}{\otimes}(S\otimes A), S\otimes H)$ is an adjointable operator and $V_H^* = V_H^{-1}$. Notice that the equation $\delta_H(\xi) = V_H \circ T_\xi$ is satisfied for all $\xi \in H$ by construction.

Conversely, suppose that there exists a unitary operator $V_H \in \mathcal{L}_{S\otimes A}(H \underset{\delta}{\otimes} (S\otimes A), S\otimes H)$ such that $\delta_H(\xi) = V_H \circ T_\xi$, for all $\xi \in H$. In this case, it is straightforward to check the axioms (i), (ii) and (iv) of Definition 1.7.1.1 for the linear map $\delta_H : H \longrightarrow \widetilde{M}(S\otimes H)$.

Finally, it remains to show that axiom (iii) of Definition 1.7.1.1 is equivalent to the condition $(id_S \otimes V_H) \circ (V_H \underset{id_S\otimes\delta}{\otimes} id_{S\otimes S\otimes A}) = V_H \underset{\Delta\otimes id_A}{\otimes} id_{S\otimes S\otimes A}$. Some computations show that for all $\xi \in H$ we have

$$(id_S \otimes \delta_H) \circ \delta_H(\xi) = (id_S \otimes V_H) \circ (V_H \underset{id_S\otimes\delta}{\otimes} id_{S\otimes S\otimes A}) \circ T'_\xi$$

$$(\Delta \otimes id_H) \circ \delta_H(\xi) = (V_H \underset{\Delta\otimes id_A}{\otimes} id_{S\otimes S\otimes A}) \circ T'_\xi,$$

where we define $T'_\xi \in \mathcal{L}_{S\otimes S\otimes A}(S\otimes S\otimes A, H \underset{\delta^2}{\otimes} (S\otimes S\otimes A))$ by $T'_\xi(x) := \xi \underset{\delta^2}{\otimes} x$, for all $x \in S\otimes S\otimes A$. We have used as well the natural identifications of Remark 1.7.1.5. These relations yield clearly the equivalence required.

■

1.7.1.7 Remarks. 1. If (H, δ_H) is a $\mathbb{S}$-equivariant Hilbert A-module, the equation $(id_S \otimes V_H) \circ (V_H \underset{id_S\otimes\delta}{\otimes} id_{S\otimes S\otimes A}) = V_H \underset{\Delta\otimes id_A}{\otimes} id_{S\otimes S\otimes A}$ of the admissible operator means the commutativity of the following diagram by virtue of the natural identifications of Remark 1.7.1.5.

$$\begin{tikzcd} H \underset{\delta^2}{\otimes} (S\otimes S\otimes A) \arrow[rr, "V_H \underset{\Delta\otimes id_A}{\otimes} id_{S\otimes S\otimes A}"] \arrow[d, "V_H \underset{id_S\otimes\delta}{\otimes} id_{S\otimes S\otimes A}"'] & & S\otimes S\otimes H \\ (S\otimes H) \underset{id_S\otimes\delta}{\otimes} (S\otimes S\otimes A) \arrow[urr, "id_S \otimes V_H"'] & & \end{tikzcd}$$

2. If (H, δ_H) is a $\mathbb{S}$-equivariant Hilbert A-module, then $\mathcal{K}_A(H)$ is a $\mathbb{S}$-C^*-algebra with the adjoint action with respect to the admissible operator V_H associated to (H, δ_H)

$$\begin{array}{rccl} \widetilde{\delta}_H := Ad_{V_H} : & \mathcal{K}_A(H) & \longrightarrow & \widetilde{M}(S \otimes \mathcal{K}_A(H)) \\ & T & \longmapsto & Ad_{V_H}(T) := V_H \circ (T \underset{\delta}{\otimes} id_{S\otimes A}) \circ V_H^*, \end{array}$$

Observe that this action can be described directly by the formula

$$\widetilde{\delta}_H(\Theta_{\xi,\eta}) := \delta_H(\xi)\delta_H(\eta)^* = \Theta_{\delta_H(\xi),\delta_H(\eta)} \in \mathcal{K}_A\big(\widetilde{M}(S \otimes H)\big) \subset \widetilde{M}(S \otimes \mathcal{K}_A(H)),$$

for all $\xi, \eta \in H$.

1.7.1.8 Definition. Let $\mathbb{S} = (S, \Delta)$ be a Hopf C^*-algebra, (A, δ_A) and (B, δ_B) (resp. graded)$\mathbb{S}$-C^*-algebras. If (H, δ_H) is a (resp. graded) $\mathbb{S}$-equivariant Hilbert B-module, a (resp. graded) $\mathbb{S}$-equivariant representation of A on H is a (resp. graded) $*$-homomorphism $\pi : A \longrightarrow \mathcal{L}_B(H)$ such that

$$\delta_H \circ \pi(a)(\xi) = (id_S \otimes \pi)(\delta_A(a)) \circ \delta_H(\xi),$$

for all $a \in A$ and all $\xi \in H$.

In this case we say that (H, δ_H, π) is a *(resp. graded)* $\mathbb{S}$*-equivariant* (A, B)*-bimodule.*

1.7.1.9 Note. Let A and B be C^*-algebras. If H is a Hilbert B-module equipped with a representation of A on H, $\pi : A \longrightarrow \mathcal{L}_B(H)$, the pair (H, π) is called an (A, B)-bimodule and sometimes the following notation is useful

$${}_AH_B,$$

where the right action of B on H is given by the corresponding Hilbert B-module structure and the left action of A on H is given by the representation π.

1.7.1.10 Remarks. 1. Let (H, δ_H, π) be $\mathbb{S}$-equivariant (A, B)-bimodule. If V_H denotes the admissible operator associated to the action δ_H, then the $\mathbb{S}$-equivariance condition for π of the above definition is expressed in a equivalent way by the following equation

$$(id_S \otimes \pi)(\delta_A(a)) = V_H(\pi(a) \underset{\delta}{\otimes} id_{S\otimes A})V_H^*,$$

for all $a \in A$.

2. Let $\mathbb{S} = (S, \Delta)$ be a Hopf C^*-algebra, (A, δ_A) and (B, δ_B) $\mathbb{S}$-C^*-algebras. Let (H, δ_H) be a $\mathbb{S}$-equivariant Hilbert A-module and (K, δ_K) $\mathbb{S}$-equivariant Hilbert B-module. If $\pi : A \longrightarrow \mathcal{L}_B(K)$ is a $\mathbb{S}$-equivariant representation, then routine arguments show that the interior tensor product $H \underset{\pi}{\otimes} K$ is a $\mathbb{S}$-equivariant Hilbert B-module with action $\delta_\pi : H \underset{\pi}{\otimes} K \longrightarrow M(S \otimes H \underset{\pi}{\otimes} K)$ such that

$$\delta_\pi(\xi \underset{\pi}{\otimes} \eta) = (\delta_H(\xi) \otimes id_B) \circ \delta_K(\eta),$$

for all $\xi \in H$ and all $\eta \in K$.

1.7.1.11 Definition. Let $\mathbb{S} = (S, \Delta)$ be a Hopf C^*-algebra, (A, δ_A) and (B, δ_B) graded $\mathbb{S}$-C^*-algebras. A $\mathbb{S}$-equivariant Kasparov (A, B)-bimodule or a $\mathbb{S}$-equivariant Kasparov triple for (A, B) is the data $((H, \delta_H), \pi, F)$ where (H, δ_H, π) is a graded $\mathbb{S}$-equivariant (A, B)-bimodule and $F \in \mathcal{L}_B(H)$ is a degree one operator such that

i) (H, π, F) is a usual Kasparov (A, B)-bimodule

ii) for all $x \in S \otimes A$ we have

$$(V_H(F \underset{\delta_B}{\otimes} id_{S\otimes B})V_H^* - id_S \underset{\mathbb{C}}{\otimes} F)(id_S \otimes \pi)(x) \in \mathcal{K}_{S\otimes B}(S \otimes B),$$

where V_H is the admissible operator associated to (H, δ_H).

The set of all $\mathbb{S}$-equivariant Kasparov (A, B)-bimodules is denoted by $\mathbb{E}^{\mathbb{S}}(A, B)$.

If (H, π, F) is a usual *degenerate* Kasparov (A, B)-bimodule and $(V_H(F \underset{\delta_B}{\otimes} id_{S\otimes B})V_H^* - id_S \underset{\mathbb{C}}{\otimes} F)(id_S \otimes \pi)(x) = 0$ for all $x \in S \otimes A$, then we say (H, π, F) is a degenerate $\mathbb{S}$-equivariant Kasparov (A, B)-bimodule. The set of all degenerate $\mathbb{S}$-equivariant Kasparov (A, B)-bimodules is denoted by $\mathbb{D}^{\mathbb{S}}(A, B)$.

1.7.1.12 Definition. Let $\mathbb{S} = (S, \Delta)$ be a Hopf C^*-algebra, (A, δ_A) and (B, δ_B) graded $\mathbb{S}$-C^*-algebras. Two $\mathbb{S}$-equivariant Kasparov (A, B)-bimodules (H, π, F) and (H', π', F') are said to be unitary equivalent if there exists an isomorphism of graded Hilbert B-modules $U : H \longrightarrow H'$ such that

i) U intertwines the representations, $\pi'(a) \circ U = U \circ \pi(a)$, for all $a \in A$,

ii) U intertwines the operators, $F' \circ U = U \circ F$,

iii) U is compatible with the admisible operators, $V_{H'}(U \underset{\delta_B}{\otimes} id)V_H^* = id_S \underset{\mathbb{C}}{\otimes} U$.

In this case we write $(H, \pi, F) \cong (H', \pi', F')$.

1.7.1.13 Definition. Let $\mathbb{S} = (S, \Delta)$ be a Hopf C^*-algebra, (A, δ_A) and (B, δ_B) graded $\mathbb{S}$-C^*-algebras. Two $\mathbb{S}$-equivariant Kasparov (A, B)-bimodules (H, π, F) and (H', π', F') are said to be homotopic if there exists $\mathbb{S}$-equivariant Kasparov $(A, C([0, 1]) \otimes B)$-bimodule $\mathcal{E}$ such that

i) $\mathcal{E}_{ev_0} \cong (H, \pi, F)$,

ii) $\mathcal{E}_{ev_1} \cong (H', \pi', F')$.

In this case we write $(H, \pi, F) \sim_h (H', \pi', F')$.

1.7.1.14 Remarks. 1. Observe that if (B, δ_B) is a $\mathbb{S}$-C^*-algebra, then $C([0, 1]) \otimes B$ is also a $\mathbb{S}$-C^*-algebra with action $\delta' : C([0, 1]) \otimes B \longrightarrow M(S \otimes C([0, 1]) \otimes B)$ defined by

$$\delta'(f)(t) := \delta_B(f(t)),$$

for all $f \in C([0, 1]) \otimes B \cong C([0, 1], B)$ and all $t \in [0, 1]$.

2. Given $t \in [0, 1]$, $ev_t : C([0, 1]) \otimes B \cong C([0, 1], B) \longrightarrow B$ denotes the evaluation map. Hence, given a $\mathbb{S}$-equivariant Kasparov $(A, C([0, 1]) \otimes B)$-bimodule $\mathcal{E} := (E, \rho, G)$, we can construct the *pushout of $\mathcal{E}$ with respect to ev_t*. More precisely, it is a $\mathbb{S}$-equivariant Kasparov (A, B)-bimodule given by $\mathcal{E}_{ev_t} := (E_{ev_t}, \rho_{ev_t}, G_{ev_t})$, where E_{ev_t} is the usual pushout Hilbert module (see Theorem A.3.5 and Theorem A.3.22 for more precisions).

Using the analogous arguments as in the classical case, we obtain thus the corresponding *quantum Kasparov groups.*

1.7.1.15 Theorem-Definition. *Let $\mathbb{S} = (S, \Delta)$ be a Hopf C^*-algebra, (A, δ_A) and (B, δ_B) graded $\mathbb{S}$-C^*-algebras. The quotient set of $\mathbb{S}$-equivariant Kasparov triples for (A, B) by the homotopy relation is an abelian group with the direct sum of $\mathbb{S}$-equivariant Kasparov triples and with zero element represented by the degenerated $\mathbb{S}$-equivariant Kasparov triples.*

We write

$$KK^{\mathbb{S}}(A, B) := \big(\mathbb{E}^{\mathbb{S}}(A, B)/\sim_h, \oplus\big)$$

for such a group and we call it $\mathbb{S}$-equivariant Kasparov group for (A, B).

The whole formal theory concerning the Kasparov groups can be generalized in the setting of Hopf C^*-algebras and so for any locally compact quantum group. We collect in the next theorem the most relevant results concerning functoriality and the Kasparov product (for a detailed proof of these facts we refer to [206], [6], [86] and [24]).

1.7.1.16 Theorem. *Let $\mathbb{S} = (S, \Delta)$ be a Hopf C^*-algebra.*

i) If $\varphi : A \longrightarrow B$ is a $\mathbb{S}$-equivariant $*$-homomorphism, then it defines a $\mathbb{S}$-equivariant Kasparov triple. Namely,

$$[\varphi] := [(B, \varphi, 0)]$$

In particular, we write $\mathbb{1}_A := [id_A] \in KK^{\mathbb{S}}(A, A)$.

ii) if A_1, A_2, B are $\mathbb{S}$-C^*-algebras and $f : A_1 \longrightarrow A_2$ is a $\mathbb{S}$-equivariant $*$-homomorphism, then there exists a homomorphism of abelian groups

$$\begin{array}{rccl} f^* : & KK^{\mathbb{S}}(A_2, B) & \longrightarrow & KK^{\mathbb{S}}(A_1, B) \\ & [(H, \pi, F)] & \longmapsto & f^*\big([(H, \pi, F)]\big) := [(H, \pi \circ f, F)] \end{array}$$

iii) if A, B_1, B_2 are $\mathbb{S}$-C^*-algebras and $g : B_1 \longrightarrow B_2$ is a $\mathbb{S}$-equivariant $*$-homomorphism, then there exists a homomorphism of abelian groups

$$\begin{array}{rccl} g_* : & KK^{\mathbb{S}}(A, B_1) & \longrightarrow & KK^{\mathbb{S}}(A, B_2) \\ & [(H, \pi, F)] & \longmapsto & g_*\big([(H, \pi, F)]\big) := [(H \underset{g}{\otimes} B_2, \pi \underset{g}{\otimes} id, F \underset{g}{\otimes} id)] \end{array}$$

iv) if A, B, C are $\mathbb{S}$-C^*-algebras, then there exists a bilinear map (called Kasparov product*)*

$$\begin{array}{rccl} \underset{C}{\otimes} : & KK^{\mathbb{S}}(A, C) \times KK^{\mathbb{S}}(C, B) & \longrightarrow & KK^{\mathbb{S}}(A, B) \\ & (\mathcal{E}, \mathcal{E}') & \longmapsto & \mathcal{E} \underset{C}{\otimes} \mathcal{E}' \end{array}$$

such that

- *$\underset{C}{\otimes}$ is associative meaning that if C' is another $\mathbb{S}$-C^*-algebra, then*

$$(x \underset{C}{\otimes} y) \underset{C'}{\otimes} z = x \underset{C}{\otimes} (y \underset{C'}{\otimes} z),$$

for all $x \in KK^{\mathbb{S}}(A, C)$, $y \in KK^{\mathbb{S}}(C, C')$ and $z \in KK^{\mathbb{S}}(C', B)$.

- $\mathbb{1}_A$ *is the neutral element by the left for the operation* $\underset{A}{\otimes}$,
- $[f] \underset{C}{\otimes} y = f^*(y)$, *for all* $*$*-homomorphism* $f : A \to C$ *and all* $y \in KK^{\mathbb{S}}(C, B)$,
- $x \underset{C}{\otimes} [g] = g_*(x)$, *for all* $x \in KK^{\mathbb{S}}(A, C)$ *and all* $*$*-homomorphism* $g : C \to B$.

v) If A, A', B, B', C *are* C^**-algebras, then the exterior tensor product of Kasparov triples induces two group homomorphisms*

$$\begin{array}{rccc} \tau_R^C : & KK(A,B) & \longrightarrow & KK(A \otimes C, B \otimes C) \\ & \mathcal{E} & \longmapsto & \tau_R^C(\mathcal{E}) := \mathcal{E} \otimes C \end{array}$$

$$\begin{array}{rccc} \tau_L^C : & KK(A,B) & \longrightarrow & KK(C \otimes A, C \otimes B) \\ & \mathcal{E} & \longmapsto & \tau_L^C(\mathcal{E}) := C \otimes \mathcal{E} \end{array}$$

such that

- $\tau_R^C([\varphi]) = [\varphi \otimes id_C]$, *for all* $*$*-homomorphism* $\varphi : A \longrightarrow B$,
- $\tau_L^C([\varphi]) = [id_C \otimes \varphi]$, *for all* $*$*-homomorphism* $\varphi : A \longrightarrow B$.

Consequently, there exists a bilinear map

$$\begin{array}{rccc} \tau^C : & KK(A, A' \otimes C) \times KK(C \otimes B, B') & \longrightarrow & KK(A \otimes B, A' \otimes B') \\ & (\mathcal{E}, \mathcal{G}) & \longmapsto & \tau^C(\mathcal{E}, \mathcal{G}) := \tau_R^B(\mathcal{E}) \underset{A' \otimes C \otimes B}{\otimes} \tau_L^{A'}(\mathcal{G}), \end{array}$$

which is contravariantly functorial in A *and* B *and covariantly functorial in* A' *and* B'.

If $C := \mathbb{C}$, *the bilinear map* $\tau^{\mathbb{C}}$ *defines what we call a tensor product of Kasparov triples and we write* $\tau^{\mathbb{C}}(\mathcal{E}, \mathcal{G}) =: \mathcal{E} \otimes \mathcal{G}$, *for all* $(\mathcal{E}, \mathcal{G}) \in KK(A, A') \times KK(B, B')$.

1.7.1.17 Remark. In particular, if $\mathbb{G}$ is any *locally compact quantum group* (for instance a *compact* or a *discrete* quantum group), it is licit to consider the corresponding Kasparov category $\mathscr{KK}^{\mathbb{G}}$ defined as in the classical case (recall Section 1.2.3). Hence, in the same way as before, $\mathscr{KK}^{\mathbb{G}}$ is actually a *triangulated category.* The problem appears when we want to apply the general categorical machinery to this triangulated category due to the more complicated structure of such a *quantum* group. In this sense we need some technical restrictions to formulate a *quantum* Baum-Connes property.

1.7.1.18 Definition. Let $\mathbb{S} = (S, \Delta)$ be a Hopf C^*-algebra, (A, δ_A) and (B, δ_B) graded $\mathbb{S}$-C^*-algebras. We say that A and B are $\mathbb{S}$-equivariantly KK-equivalent if there exist elements $\alpha \in KK^{\mathbb{S}}(A, B)$ and $\beta \in KK^{\mathbb{S}}(B, A)$ such that

$$\alpha \underset{B}{\otimes} \beta = \mathbb{1}_A \text{ and } \beta \underset{A}{\otimes} \alpha = \mathbb{1}_B$$

Finally, in the context of this *quantum* Kasparov theory we can obtain a generalization of the classical Takesaki-Takai duality called *Baaj-Skandalis duality* (see Theorem 9.5.11 in [188] for a more general statement in the context of multiplicative unitaries in the sense of Baaj-Skandalis). This duality is useful to pass from the Kasparov category of a given *compact* quantum group $\mathbb{G}$ to the Kasparov category corresponding to the *discrete* quantum group $\widehat{\mathbb{G}}$. For this, we have firstly to establish a quantum version of the *descent homomorphism* (see Section 5 in [206] or [6] for the full details).

1.7.1.19 Theorem-Definition. *Let $\mathbb{G}$ be a compact quantum group, (A,δ_A) and (B,δ_B) graded $\widehat{\mathbb{G}}$-C^*-algebras. Given a $\widehat{\mathbb{G}}$-equivariant Kasparov triple (H,π,F), then the data*

$$(\widehat{\mathbb{G}} \underset{r}{\ltimes} H, id \underset{r}{\ltimes} \pi, F \underset{\iota_B}{\otimes} id),$$

where $\widehat{\mathbb{G}} \underset{r}{\ltimes} H := H \underset{\iota_B}{\otimes} (\widehat{\mathbb{G}} \underset{r}{\ltimes} B)$, $id \underset{r}{\ltimes} \pi : \widehat{\mathbb{G}} \underset{r}{\ltimes} A \longrightarrow \mathcal{L}_{\widehat{\mathbb{G}} \underset{r}{\ltimes} B}(\widehat{\mathbb{G}} \underset{r}{\ltimes} H)$ and $F \underset{\iota_B}{\otimes} id \in \mathcal{L}_{\widehat{\mathbb{G}} \underset{r}{\ltimes} B}(\widehat{\mathbb{G}} \underset{r}{\ltimes} H)$ are defined by functoriality; is a Kasparov triple in $\mathbb{E}(\widehat{\mathbb{G}} \underset{r}{\ltimes} A, \widehat{\mathbb{G}} \underset{r}{\ltimes} B)$ called descent triple.

Moreover, there exists a group homomorphism

$$j_{\widehat{\mathbb{G}}} : KK^{\widehat{\mathbb{G}}}(A,B) \longrightarrow KK(\widehat{\mathbb{G}} \underset{r}{\ltimes} A, \widehat{\mathbb{G}} \underset{r}{\ltimes} B),$$

which is compatible with the Kasparov product, that is, if (C,δ_C) is another $\widehat{\mathbb{G}}$-C^*-algebra and $\alpha \in KK^{\widehat{\mathbb{G}}}(A,C)$, $\beta \in KK^{\widehat{\mathbb{G}}}(C,B)$, then we have

$$j_{\widehat{\mathbb{G}}}(\alpha \underset{C}{\otimes} \beta) = j_{\widehat{\mathbb{G}}}(\alpha) \underset{\widehat{\mathbb{G}} \underset{r}{\ltimes} C}{\otimes} j_{\widehat{\mathbb{G}}}(\beta) \text{ and } j_{\widehat{\mathbb{G}}}(\mathbb{1}_A) = \mathbb{1}_{\widehat{\mathbb{G}} \underset{r}{\ltimes} A}$$

The homomorphism $j_{\widehat{\mathbb{G}}}$ is called descent homomorphism with respect to $\widehat{\mathbb{G}}$. *In an analogous way we define the* descent homomorphism with respect to $\mathbb{G}$, *denoted by $j_{\mathbb{G}}$.*

1.7.1.20 Theorem (Baaj-Skandalis duality)**.** *Let $\mathbb{G}$ be a compact quantum group. If (A,α) is any $\mathbb{G}$-C^*-algebra, then the reduced crossed product $\mathbb{G} \underset{\alpha,r}{\ltimes} A$ is naturally a $\widehat{\mathbb{G}}^{cop}$-$C^*$-algebra with action $\widehat{\alpha} := (\widehat{\Delta}^{cop} \otimes id) \circ \alpha$ so that we have a $\mathbb{G}$-equivariant isomorphism*

$$\widehat{\mathbb{G}} \underset{\widehat{\alpha},r}{\ltimes} (\mathbb{G} \underset{\alpha,r}{\ltimes} A) \cong A \otimes \mathcal{K}(L^2(\mathbb{G}))$$

that identifies the bi-dual action $\widehat{\widehat{\alpha}}$ with the action

$$\widetilde{\alpha} := Ad_{id_A \otimes W_{\mathbb{G}}} \circ \left[\alpha\right]_{13} : A \otimes \mathcal{K}(L^2(\mathbb{G})) \longrightarrow M(C(\mathbb{G}) \otimes A \otimes \mathcal{K}(L^2(\mathbb{G})))$$

As a consequence, for every $\mathbb{G}$-C^*-algebras (A,α) and (B,β) we have a canonical isomorphism of abelian groups

$$J_{\mathbb{G}} : KK^{\mathbb{G}}(A,B) \xrightarrow{\sim} KK^{\widehat{\mathbb{G}}^{cop}}(\mathbb{G} \underset{\alpha,r}{\ltimes} A, \mathbb{G} \underset{\beta,r}{\ltimes} B)$$

which is compatible with the Kasparov product, that is, if (C,γ) is another $\mathbb{G}$-C^*-algebra and $\alpha \in KK^{\mathbb{G}}(A,C)$, $\beta \in KK^{\mathbb{G}}(C,B)$, then we have

$$J_{\mathbb{G}}(\alpha \underset{C}{\otimes} \beta) = J_{\mathbb{G}}(\alpha) \underset{\mathbb{G} \underset{r}{\ltimes} C}{\otimes} J_{\mathbb{G}}(\beta) \text{ and } J_{\mathbb{G}}(\mathbb{1}_A) = \mathbb{1}_{\mathbb{G} \underset{r}{\ltimes} A}$$

In an analogous way we define the isomorphism of abelian groups $J_{\widehat{\mathbb{G}}}$, which is the inverse of $J_{\mathbb{G}}$.

Moreover, we have that $j_{\mathbb{G}} = \mathcal{O}_{\widehat{\mathbb{G}}} \circ J_{\mathbb{G}}$ and $j_{\widehat{\mathbb{G}}} = \mathcal{O}_{\mathbb{G}} \circ J_{\widehat{\mathbb{G}}}$, where $\mathcal{O}_{\widehat{\mathbb{G}}}$ and $\mathcal{O}_{\mathbb{G}}$ are the obvious forgetful functors.

In particular, we have the following.

- *If (A,trv) is a trivial $\mathbb{G}$-C^*-algebra, then $\mathbb{G} \underset{trv,r}{\ltimes} A \cong c_0(\widehat{\mathbb{G}}) \otimes A$.*

- *If $(\mathbb{C},trv)$ is considered as a trivial $\mathbb{G}$-C^*-algebra, then $\mathbb{G} \underset{trv,r}{\ltimes} \mathbb{C} \cong c_0(\widehat{\mathbb{G}})$.*

1.7.1.21 Remark. Baaj-Skandalis duality yields in particular a canonical $*$-isomorphism

$$\widehat{\mathbb{G}} \underset{\widehat{\Delta},r}{\ltimes} c_0(\widehat{\mathbb{G}}) \cong \widehat{\mathbb{G}} \underset{\widehat{trv},r}{\ltimes} \big(\mathbb{G} \underset{trv,r}{\ltimes} \mathbb{C}\big) \cong \mathcal{K}(L^2(\mathbb{G}))$$

It is advisable to give an explicit description of this identification. For this, denote by $(\pi_{\widehat{\Delta}}, U_{\widehat{\Delta}}, E_{\widehat{\Delta}})$ the canonical triple associated to the reduced crossed product $\widehat{\mathbb{G}} \underset{\widehat{\Delta},r}{\ltimes} c_0(\widehat{\mathbb{G}})$ following Theorem 1.5.2.1, which is given precisely by $(\widehat{\lambda}, \widehat{W}_{\mathbb{G}} \otimes id_{c_0(\widehat{\mathbb{G}})}, \Omega \otimes id_{c_0(\widehat{\mathbb{G}})})$.

We are going to show that the C^*-algebra $\mathcal{K}(L^2(\mathbb{G}))$ satisfies the universal property of $\widehat{\mathbb{G}} \underset{\widehat{\Delta},r}{\ltimes} c_0(\widehat{\mathbb{G}})$.

On the one hand, consider the canonical left regular representations $\widehat{\lambda} : c_0(\widehat{\mathbb{G}}) \longrightarrow \mathcal{B}(L^2(\mathbb{G}))$ and $\lambda : C(\mathbb{G}) \longrightarrow \mathcal{B}(L^2(\mathbb{G}))$ defined by the GNS construction of $\widehat{h}_L$ and $h_{\mathbb{G}}$, respectively. By definition of the fundamental unitary $\widehat{W}_{\mathbb{G}} \in M(c_0(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}))$ (recall Theorem 1.3.1.36 and Remark 1.3.1.38), we have that

$$\widehat{W}_{\mathbb{G}}^*(id_{c_0(\widehat{\mathbb{G}})} \otimes \widehat{\lambda}(a))\widehat{W}_{\mathbb{G}} = (id_{c_0(\widehat{\mathbb{G}})} \otimes \widehat{\lambda})\widehat{\Delta}(a) \text{ and } \big(\widehat{W}_{\mathbb{G}}^x\big)_{i,j} = \lambda(w_{i,j}^x),$$

for all $a \in c_0(\widehat{\mathbb{G}})$, $x \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$.

On the other hand, consider the following C^*-algebra

$$\mathscr{C} := C^*\langle \widehat{\lambda}(a)\lambda(w_{i,j}^x) \mid a \in c_0(\widehat{\mathbb{G}}), x \in Irr(\mathbb{G}), i, j = 1, \ldots, n_x\rangle \subset \mathcal{B}(L^2(\mathbb{G}))$$

We claim that $\mathscr{C} = \mathcal{K}(L^2(\mathbb{G}))$. Notice that the operators $\widehat{\lambda}(a)$ with $a \in c_0(\widehat{\mathbb{G}})$ are compact on $L^2(\mathbb{G})$. Indeed, it is enough to observe that $Im(\widehat{\lambda}(ap_x)) \subset H_x \otimes H_{\overline{x}}$, for all $x \in Irr(\mathbb{G})$. Consequently, the operators $\widehat{\lambda}(a)\lambda(w_{i,j}^x)$ with $a \in c_0(\widehat{\mathbb{G}})$, $x \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$ are compact on $L^2(\mathbb{G})$. Conversely, to show that $\mathcal{K}(L^2(\mathbb{G})) \subset \mathscr{C}$ it is enough to show that $\xi\xi^* \in \mathscr{C}$, for all $\xi \in L^2(\mathbb{G})$. Take $\xi := \lambda(w_{i,j}^x)\Omega$ and write

$$\begin{aligned}\xi\xi^*(\lambda(w_{k,l}^y)\Omega) &= \xi\langle \lambda(w_{k,l}^y)\Omega, \lambda(w_{i,j}^x)\Omega\rangle = \xi h_{\mathbb{G}}\big((w_{i,j}^x)^* w_{k,l}^y\big) \\ &= \lambda(w_{i,j}^x)\widehat{\lambda}(p_\epsilon)\lambda((w_{i,j}^x)^* w_{k,l}^y))\Omega = \lambda(w_{i,j}^x)\widehat{\lambda}(p_\epsilon)\lambda((w_{i,j}^x)^*)(\lambda(w_{k,l}^y)\Omega),\end{aligned}$$

for all $y \in Irr(\mathbb{G})$, $k, l = 1, \ldots, n_y$. To conclude, observe that the set $\{\xi \in L^2(\mathbb{G}) \mid \xi\xi^* \in \mathscr{C}\}$ is closed, which is a routine computation.

In order to define a non-degenerate completely positive KSGNS-faithful map $\widehat{E} : \mathcal{K}(L^2(\mathbb{G}) \longrightarrow M(c_0(\widehat{\mathbb{G}}))$ such that $\widehat{E}(\widehat{\lambda}(a)\lambda(w_{i,j}^x)) = \delta_{x,\epsilon}a$, for all $a \in c_0(\widehat{\mathbb{G}})$, $x \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$, we are going to define directly the associated KSGNS construction (recall Remark A.3.12). Define the following $c_0(\widehat{\mathbb{G}})$-linear map

$$\begin{array}{rccc} \Upsilon : & c_0(\widehat{\mathbb{G}}) & \longrightarrow & L^2(\mathbb{G}) \otimes c_0(\widehat{\mathbb{G}}) \\ & a & \longmapsto & \Upsilon(x) := \Omega \otimes a \end{array}$$

Define the $c_0(\widehat{\mathbb{G}})$-linear operator $\Upsilon^* : \lambda(Pol(\mathbb{G}))\Omega \odot c_0(\widehat{\mathbb{G}}) \longrightarrow c_0(\widehat{\mathbb{G}})$ by the formula

$$\Upsilon^*(\lambda(w_{i,j}^x)\Omega \otimes a) = h_{\mathbb{G}}(w_{i,j}^x)a,$$

for all $x \in Irr(\mathbb{G})$, all $i, j = 1, \ldots, n_x$ and all $a \in c_0(\widehat{\mathbb{G}})$. Applying Schwarz inequality to the Haar state, a routine computation shows that Υ^* is bounded. Therefore we can extend the above formula

to the whole $L^2(\mathbb{G})\otimes c_0(\widehat{\mathbb{G}})$ obtaining a bounded operator $\Upsilon^* : L^2(\mathbb{G})\otimes c_0(\widehat{\mathbb{G}}) \longrightarrow c_0(\widehat{\mathbb{G}})$. Moreover, it is clear that Υ and Υ^* defined in this way are adjoint.

Next, we define the following faithful representation of $\mathcal{K}(L^2(\mathbb{G}))$ on $L^2(\mathbb{G})\otimes c_0(\widehat{\mathbb{G}})$,

$$\begin{array}{rccl} \vartheta: & \mathcal{K}(L^2(\mathbb{G})) & \longrightarrow & \mathcal{L}_{c_0(\widehat{\mathbb{G}})}(L^2(\mathbb{G})\otimes c_0(\widehat{\mathbb{G}})) \\ & T & \longmapsto & \vartheta(T) := T\otimes id_{c_0(\widehat{G})} \end{array}$$

and the following completely positive map

$$\begin{array}{rccl} \widehat{E}: & \mathcal{K}(L^2(\mathbb{G})) & \longrightarrow & M(c_0(\widehat{\mathbb{G}})) \\ & T & \longmapsto & \widehat{E}(T) := \Upsilon^*\circ\vartheta(T)\circ\Upsilon \end{array}$$

We claim that the data $(L^2(\mathbb{G})\otimes c_0(\widehat{\mathbb{G}}), \vartheta, \Upsilon)$ is the KSGNS construction for $\widehat{E}$. We only have to prove that $L^2(\mathbb{G})\otimes c_0(\widehat{\mathbb{G}}) = \overline{span}\{\vartheta(\mathcal{K}(L^2(\mathbb{G})))\Upsilon(c_0(\widehat{G}))\}$. By construction, it is enough to show that $\lambda(w^x_{i,j})\Omega\otimes a \in \overline{\vartheta(\mathcal{K}(L^2(\mathbb{G})))\Upsilon(c_0(\widehat{G}))}$ for all $x\in Irr(\mathbb{G})$, all $i,j=1,\dots,n_x$ and $a\in c_0(\widehat{\mathbb{G}})$. Namely,

$$\begin{aligned} \lambda(w^x_{i,j})\Omega\otimes a &= (\lambda(w^x_{i,j})\otimes id_{c_0(\widehat{G})})(\Omega\otimes a) = (\lambda(w^x_{i,j})\otimes id_{c_0(\widehat{G})})\Upsilon(a) \\ &= \lim_{r\to\infty}\big(\widehat{\lambda}(e_r)\lambda(w^x_{i,j})\otimes id_{c_0(\widehat{G})})\Upsilon(a)\big) \in \overline{\vartheta(\mathcal{K}(L^2(\mathbb{G})))\Upsilon(c_0(\widehat{G}))}, \end{aligned}$$

where $\{e_r\}_{r\in\mathbb{N}}$ is an approximate unit for $c_0(\widehat{G})$, so that $\{\widehat{\lambda}(e_r)\}_{r\in\mathbb{N}}$ is an approximate unit for $\widehat{\lambda}(c_0(\widehat{G}))$. To conclude, let us check the formula $\widehat{E}(\widehat{\lambda}(a)\lambda(w^x_{i,j})) = \delta_{x,\epsilon}a$, for all $a\in c_0(\widehat{\mathbb{G}})$, $x\in Irr(\mathbb{G})$, $i,j=1,\dots,n_x$. Namely,

$$\begin{aligned} \widehat{E}(\widehat{\lambda}(a)\lambda(w^x_{i,j}))(b) &= \Upsilon^*\big(\vartheta(\widehat{\lambda}(a)\lambda(w^x_{i,j}))\big(\Upsilon(b)\big)\big) = \Upsilon^*\big(\vartheta(\widehat{\lambda}(a)\lambda(w^x_{i,j}))\big(\Upsilon(b)\big)\big) \\ &= \Upsilon^*\big(\vartheta(\widehat{\lambda}(a)\lambda(w^x_{i,j}))\big(\Omega\otimes b\big)\big) = \Upsilon^*\big(\widehat{\lambda}(a)\lambda(w^x_{i,j})\otimes id_{c_0(\widehat{G})}\big(\Omega\otimes b\big)\big) \\ &= \Upsilon^*\big(\widehat{\lambda}(a)\lambda(w^x_{i,j})\Omega\otimes b\big) = \Upsilon^*\Big(\widehat{\lambda}(a)\frac{\sqrt{\lambda^{\overline{x}}_j}}{\sqrt{dim_q(x)}}\ \xi^x_i\otimes\xi^{\overline{x}}_j\otimes b\Big) \\ &= \Upsilon^*\Big(\frac{\sqrt{\lambda^{\overline{x}}_j}}{\sqrt{dim_q(x)}}\ (ap_x\otimes id_{H_{\overline{x}}})\xi^x_i\otimes\xi^{\overline{x}}_j\otimes b\Big) \\ &= (ap_x\otimes id_{H_{\overline{x}}})\Upsilon^*\Big(\lambda(w^x_{i,j})\Omega\otimes b\Big) = (ap_x\otimes id_{H_{\overline{x}}})h_{\mathbb{G}}(w^x_{i,j})b = ab\delta_{x,\epsilon}, \end{aligned}$$

which is true for every $b\in c_0(\widehat{\mathbb{G}})$ and shows the formula.

In conclusion, the triple $(\widehat{\lambda}, \widehat{W}_{\mathbb{G}}, \widehat{E})$ associated to $\mathcal{K}(L^2(\mathbb{G}))$ in the sense of Theorem 1.5.2.1 together with the preceding computations yield, by universal property of $\widehat{\mathbb{G}} \underset{\widehat{\Delta},r}{\ltimes} c_0(\widehat{\mathbb{G}})$, that there exists a canonical $*$-isomorphism

$$\psi : \widehat{\mathbb{G}} \underset{\widehat{\Delta},r}{\ltimes} c_0(\widehat{\mathbb{G}}) \longrightarrow \mathcal{K}(L^2(\mathbb{G}))$$

such that

$$\psi(\pi_{\widehat{\Delta}}(a)\big(U^x_{\widehat{\Delta}}\big)_{i,j}) = \widehat{\lambda}(a)\lambda(w^x_{i,j}),$$

for all $a\in c_0(\widehat{\mathbb{G}})$, $x\in Irr(\mathbb{G})$, $i,j=1,\dots,n_x$.

1.7.1.22 Theorem (Quantum Green-Julg theorem)**.** *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group.*

i) Let (A, δ_A) and (B, δ_B) be graded $\mathbb{G}$-C^*-algebras. If δ_A is the trivial action, then there exists a group isomorphism

$$KK^{\mathbb{G}}(A, B) \cong KK(A, \mathbb{G} \underset{r}{\ltimes} B)$$

ii) Let (A, δ_A) and (B, δ_B) graded $\widehat{\mathbb{G}}$-C^*-algebras. If δ_B is the trivial action, then there exists a group isomorphism

$$KK^{\widehat{\mathbb{G}}}(A, B) \cong KK(\widehat{\mathbb{G}} \underset{m}{\ltimes} A, B)$$

1.7.1.23 Remark. The following ring isomorphism is well-known

$$R(\mathbb{G}) \cong KK^{\mathbb{G}}(\mathbb{C}, \mathbb{C}),$$

where $R(\mathbb{G})$ is the representation ring of $\mathbb{G}$ (introduced in Section 1.6.2) equipped with the tensor product of irreducible representations as ring product and $KK^{\mathbb{G}}(\mathbb{C}, \mathbb{C})$ is the $\mathbb{G}$-equivariant Kasparov group for $(\mathbb{C}, \mathbb{C})$ equipped with the Kasparov product as ring product.

Observe that given any $\widehat{\mathbb{G}}$-C^*-algebra B we can write

$$K_0(B) \cong K_0(B \otimes \mathcal{K}(L^2(\mathbb{G}))) = KK(\mathbb{C}, B \otimes \mathcal{K}(L^2(\mathbb{G}))) \cong KK(\mathbb{C}, \mathbb{G} \underset{r}{\ltimes} (\widehat{\mathbb{G}} \underset{r}{\ltimes} B)) \cong KK^{\mathbb{G}}(\mathbb{C}, \widehat{\mathbb{G}} \underset{r}{\ltimes} B)$$

by combining Baaj-Skandalis duality and quantum Green-Julg theorem. Consequently, the Kasparov product induces an $R(\mathbb{G})$-module structure on $K_0(B)$ for any $\widehat{\mathbb{G}}$-C^*-algebra B,

$$R(\mathbb{G}) \times K_0(B) \longrightarrow K_0(B)$$

and if C is any $\widehat{\mathbb{G}}$-C^*-algebra, then every element $\mathcal{E} \in KK^{\widehat{\mathbb{G}}}(B, C)$ induces an $R(\mathbb{G})$-module homomorphism $\mathcal{E}_* : K_0(B) \longrightarrow K_0(C)$ by right Kasparov product with $\mathcal{E}$.

It is advisable to point out two particular cases.

a) If $B := c_0(\widehat{\mathbb{G}})$, then the above identification yields $K_0(c_0(\widehat{\mathbb{G}})) \cong R(\mathbb{G})$ and the preceding $R(\mathbb{G})$-module structure on $K_0(c_0(\widehat{\mathbb{G}}))$ is simply the action of $R(\mathbb{G})$ on itself by multiplication of irreducible representations.

b) If $B := \mathbb{C}$, then we have $K_0(\mathbb{C}) \cong \mathbb{Z}$ and the preceding $R(\mathbb{G})$-module structure on $K_0(\mathbb{C})$ is induced by the dimension function on irreducible representations.

   More precisely, notice that, by definition, $R(\mathbb{G})$ has an additive basis indexed by $Irr(\mathbb{G})$. In this way, we identify $R(\mathbb{G})$ with the free abelian group $\mathbb{Z}[Irr(\mathbb{G})]$ and we define the dimension function $\mathbb{Z}[Irr(\mathbb{G})] \longrightarrow \mathbb{Z}$ to be such that $x \mapsto n_x$, for every $x \in Irr(\mathbb{G})$.

1.7.2 Quantum Baum-Connes conjecture

In this section we want to formulate a version of the Baum-Connes property for a *discrete quantum group* $\widehat{\mathbb{G}}$. In order to do so, we are going to imitate the reformulation of R. Meyer and R. Nest for classical groups as we have explained in Section 1.2.3.

Recall that for this we use the family $\mathcal{F}$ of compact subgroups of the given locally compact group G in order to define the complementary pair of subcategories $(\mathscr{L}, \mathscr{N}) := (\langle \mathcal{CI} \rangle, \mathcal{CC})$. Then we define the corresponding *categorical* assembly map. The first issue that we find here is that we can not translate the notion of *compact subgroup* in the quantum setting. For this reason, we have to restrict our attention to the *discrete* case for which the compact subgroups become the finite ones and so the family $\mathcal{F}$ describes actually the torsion of the considered discrete group. In this way, the quantum formulation deals with the family $\mathcal{F}$ of torsion C^*-algebras for $\widehat{\mathbb{G}}$. Observe that, *a priori*, the discrete quantum subgroups do not give all possible torsion; other exotic phenomena can occur (recall Section 1.6.1).

Since torsion phenomena in the quantum setting is more complicated than in the classical case, a *quantum* Baum-Connes property concerns currently *only torsion-free discrete quantum groups.*

As a result, if $\mathbb{G}$ is a compact quantum group such that $\widehat{\mathbb{G}}$ is a torsion-free discrete quantum group, the family $\mathcal{F}$ is formed in this case just by the trivial quantum subgroup $\mathbb{E}$ (which is, by the way, the only finite quantum subgroup of $\widehat{\mathbb{G}}$, recall Remark 1.6.1.7). That is, $\mathcal{F} = \{\mathbb{E}\}$.

Hence the subcategories $\mathscr{L}$ and $\mathscr{N}$ in the quantum picture are described precisely as follows:

- <u>(Quantum) Compactly Contractible objects</u>:

$$\mathscr{N} := \mathscr{N}_{\widehat{\mathbb{G}}} := \mathcal{CC}_{Qut.} := \{A \in Obj(\mathscr{KK}^{\widehat{\mathbb{G}}}) \mid A \cong 0 \text{ in } \mathscr{KK} \text{ as a trivial } \widehat{\mathbb{G}}\text{-}C^*\text{-algebra}\}$$

- <u>(Quantum) Compactly Induced objects</u>:

$$\mathscr{L} := \mathscr{L}_{\widehat{\mathbb{G}}} := \langle \mathcal{CI}_{Qut.} \rangle := \langle \{A \in Obj(\mathscr{KK}^{\widehat{\mathbb{G}}}) \mid A \cong c_0(\widehat{\mathbb{G}}) \otimes B \text{ for some } B \in Obj(\mathscr{KK})\} \rangle$$

1.7.2.1 Note. Sometimes, the study of the Baum-Connes property of a discrete quantum group requires to work in a more restricted category. Namely, the Kasparov category associated to the *Drinfeld double*, $\mathscr{KK}^{\mathcal{D}(\widehat{\mathbb{G}})}$, where there exists a notion of tensor product (see Remark A.3.24). In this way, we may define the following subcategories

$$\mathscr{N}_{\widehat{\mathbb{G}}-YD} := \{A \in Obj(\mathscr{KK}^{\mathcal{D}(\widehat{\mathbb{G}})}) \mid A \cong 0 \text{ in } \mathscr{KK} \text{ as a trivial } \widehat{\mathbb{G}}\text{-}C^*\text{-algebra}\}$$

$$\mathscr{L}_{\widehat{\mathbb{G}}-YD} := \langle \{A \in Obj(\mathscr{KK}^{\mathcal{D}(\widehat{\mathbb{G}})}) \mid A \cong c_0(\widehat{\mathbb{G}}) \otimes B \text{ for some } B \in Obj(\mathscr{KK})\} \rangle$$

We observe that the $\widehat{\mathbb{G}}$-C^*-algebras of the form $c_0(\widehat{\mathbb{G}}) \otimes B$ with $B \in Obj(\mathscr{KK})$ are automatically $\widehat{\mathbb{G}}$-YD-C^*-algebras by taking the dual co-multiplication. Therefore, when $\widehat{\mathbb{G}}$ is torsion-free, the canonical forgetful functor $\mathscr{KK}^{\mathcal{D}(\widehat{\mathbb{G}})} \longrightarrow \mathscr{KK}^{\widehat{\mathbb{G}}}$ sends $\mathscr{L}_{\widehat{\mathbb{G}}-YD}$ to $\mathscr{L}_{\widehat{\mathbb{G}}}$.

1.7.2.2 Remarks. 1. Since in our framework we have $\mathcal{F} = \{\mathbb{E}\}$, there exists only *one* restriction functor $Res_{\mathbb{E}}^{\widehat{\mathbb{G}}}$ and *one* induction functor $Ind_{\mathbb{E}}^{\widehat{\mathbb{G}}}$.

Restriction functor is always a forgetful functor and, in our case, we forget *completely* any action of $\widehat{\mathbb{G}}$ on the C^*-algebra considered, this is why we may write $Res_{\mathbb{E}}^{\widehat{\mathbb{G}}}(A) = A$, for any $A \in Obj(\mathscr{KK}^{\widehat{\mathbb{G}}})$.

Induction functor concerning quantum groups is a more delicate notion as we have noticed in Section 1.4, but in our framework the only quantum *subgroup* that we consider is the trivial one, so that, as in the classical case, the induction from such a subgroup yields a $\widehat{\mathbb{G}}$-C^*-algebra of the form $Ind_{\mathbb{E}}^{\widehat{\mathbb{G}}}(B) \cong c_0(\widehat{\mathbb{G}}) \otimes B$, for some C^*-algebra B.

2. In order to apply the Meyer-Nest machinery, we want to show that $(\langle \mathcal{CI}_{Qut.}\rangle, \mathcal{CC}_{Qut.})$ is a complementary pair of localizing subcategories in $\mathscr{KK}^{\widehat{\mathbb{G}}}$. To this end, the strategy consists in constructing projective objects in $\mathscr{KK}^{\widehat{\mathbb{G}}}$ with respect to an additive homological ideal. In our situation, this ideal is simply $\mathcal{J} := \ker_{Hom}(Res_{\mathbb{E}}^{\widehat{\mathbb{G}}})$ (recall Theorem 1.2.2.20 and Theorem 1.2.3.11).

Notice that these constructions (that arise naturally from the classical case developed in Section 1.2.3) are located in the Kasparov category with respect to $\widehat{\mathbb{G}}$, $\mathscr{KK}^{\widehat{\mathbb{G}}}$. But sometimes it is interesting and useful to work in the Kasparov category associated to the *compact* quantum group, that is, in $\mathscr{KK}^{\mathbb{G}}$. We can easily do the translation from one category to the other using Baaj-Skandalis duality. More precisely, we have the next result.

1.7.2.3 Proposition. *Let $\mathbb{G}$ be a compact quantum group such that $\widehat{\mathbb{G}}$ is a torsion-free discrete quantum group. The Baaj-Skandalis duals of the pair $(\langle \mathcal{CI}_{Qut.}\rangle, \mathcal{CC}_{Qut.})$ in $\mathscr{KK}^{\widehat{\mathbb{G}}}$ is the pair $(\langle \widehat{\mathcal{CI}}_{Qut.}\rangle, \widehat{\mathcal{CC}}_{Qut.})$ in $\mathscr{KK}^{\mathbb{G}}$, where*

$$\widehat{\mathscr{L}} := \mathscr{L}_{\mathbb{G}} = \langle \widehat{\mathcal{CI}}_{Qut.}\rangle = \langle \{A \in Obj(\mathscr{KK}^{\mathbb{G}}) \mid A \text{ with trivial action of } \mathbb{G}\}\rangle$$

$$\widehat{\mathscr{N}} := \mathscr{N}_{\mathbb{G}} = \widehat{\mathcal{CC}}_{Qut.} = \{A \in Obj(\mathscr{KK}^{\mathbb{G}}) \mid \mathbb{G} \underset{r}{\ltimes} A \cong 0 \text{ in } \mathscr{KK} \text{ as a trivial } \widehat{\mathbb{G}}^{cop}\text{-}C^*\text{-algebra}\}$$

Proof. Recall that Baaj-Skandalis duality (Theorem 1.7.1.20) states a canonical equivalence of categories

$$\mathscr{KK}^{\mathbb{G}} \cong \mathscr{KK}^{\widehat{\mathbb{G}}^{cop}}$$

via the crossed product functor. More precisely, any $\mathbb{G}$-C^*-algebra (A, α) is equivalent to the $\widehat{\mathbb{G}}^{cop}$-$C^*$-algebra given by $\mathbb{G} \underset{\alpha,r}{\ltimes} A$ with action $\widehat{\alpha} := (\widehat{\Delta}^{cop} \otimes id) \circ \alpha$.

Under this duality it is clear the identification $\mathcal{CC}_{Qut.} \cong \widehat{\mathcal{CC}}_{Qut.}$.

For the identification $\langle \mathcal{CI}_{Qut.}\rangle \cong \langle \widehat{\mathcal{CI}}_{Qut.}\rangle$ notice that given any C^*-algebra B, the action of $\widehat{\mathbb{G}}^{cop}$ on $Ind_{\mathbb{E}}^{\widehat{\mathbb{G}}}(B) = c_0(\widehat{\mathbb{G}}) \otimes B$ is given simply by $\widehat{\Delta}^{cop} \otimes id_B$. If we apply Baaj-Skandalis duality, we obtain the $\mathbb{G}$-C^*-algebra

$$\begin{aligned}\widehat{\mathbb{G}} \underset{\widehat{\Delta}^{cop}\otimes id_B, r}{\ltimes} (c_0(\widehat{\mathbb{G}}) \otimes B) &\cong (\widehat{\mathbb{G}} \underset{\widehat{\Delta}^{cop}, r}{\ltimes} c_0(\widehat{\mathbb{G}})) \otimes B \\ &= \big(\widehat{\mathbb{G}} \underset{\widehat{\Delta}^{cop}, r}{\ltimes} (\mathbb{G} \underset{trv, r}{\ltimes} \mathbb{C})\big) \otimes B \cong \mathbb{C} \otimes \mathcal{K}(L^2(\mathbb{G})) \otimes B \cong B\end{aligned}$$

with action $(\Delta \otimes id) \circ (\widehat{\Delta}^{cop} \otimes id_B)$. By virtue of Baaj-Skandalis duality (see Theorem 1.7.1.20) we know that this action is identified to the action $Ad_{W_{\mathbb{G}}}(\widehat{\Delta}^{cop} \otimes id_B)$ of $\mathbb{G}$ on $\mathbb{C} \otimes \mathcal{K}(L^2(\mathbb{G})) \otimes B$. Next, recall that $\widehat{\Delta}^{cop}(\cdot) = W_{\mathbb{G}}(id \otimes \cdot)W_{\mathbb{G}}^*$. Hence, $\widehat{\mathbb{G}} \underset{\widehat{\Delta}^{cop}\otimes id_B, r}{\ltimes} (c_0(\widehat{\mathbb{G}}) \otimes B) \cong B$ is a $\mathbb{G}$-C^*-algebra with trivial action.

$$\begin{aligned}Ad_{W_{\mathbb{G}}}(\widehat{\Delta}^{cop} \otimes id_B) &= W_{\mathbb{G}}^*\big(W_{\mathbb{G}}(id \otimes \cdot)W_{\mathbb{G}}^* \otimes id_B\big)W_{\mathbb{G}} \\ &= \big(W_{\mathbb{G}}^* W_{\mathbb{G}}(id \otimes \cdot)W_{\mathbb{G}}^* W_{\mathbb{G}}\big) \otimes id_B = id_B \otimes 1\end{aligned}$$

In other words, $\mathcal{CI}_{Qut.} \cong \widehat{\mathcal{CI}}_{Qut.}$ under Baaj-Skandalis duality. ■

It remains to show that the pair $(\langle \mathcal{CI}_{Qut.}\rangle, \mathcal{CC}_{Qut.})$ is actually a *complementary pair of localizing subcategories in* $\mathscr{KK}^{\widehat{\mathbb{G}}}$. To do this, the strategy consists in imitating the classical discrete case for which we use the *adjoint approach* for constructing projective objects (recall Theorem 1.2.2.20 and Theorem 1.2.3.11). In this way, the adjointness property between the induction and the restriction functors is the main ingredient that we need. This can be showed using the explicit description of induced actions stated in Section 1.4 and the characterization of Theorem B.1.15 for adjoint functors. A detailed proof can be found in Proposition 6.2 of [208].

1.7.2.4 Lemma. *Let $\mathbb{G}$ be a compact quantum group and $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$ any discrete quantum subgroup. The restriction functor $Res_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}$ and the induction functor $Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}$ are adjoint in the following sense,*

$$KK^{\widehat{\mathbb{G}}}(Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B), A) \cong KK^{\widehat{\mathbb{H}}}(B, Res_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(A)),$$

for all $\widehat{\mathbb{G}}$-C^*-algebra A and all $\widehat{\mathbb{H}}$-C^*-algebra B.

In particular, this is true for the trivial quantum subgroup $\mathbb{E}$.

1.7.2.5 Remark. First of all, observe that the preceding formula can be written under the form

$$KK^{\widehat{\mathbb{G}}}(A, Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)) \cong KK^{\widehat{\mathbb{H}}}(Res_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(A), B),$$

for all $\widehat{\mathbb{G}}$-C^*-algebra A and all $\widehat{\mathbb{H}}$-C^*-algebra B.

Moreover, this adjointness property can be established in a more general framework. Namely, if $\mathbb{G}$ is any *locally compact* quantum group and $\mathbb{H} < \mathbb{G}$ is a *co-compact* quantum subgroup (which means that the C^*-algebra $c_0(\mathbb{G}/\mathbb{H})$ of the corresponding homogeneous space is *unital*), then we can show that $Ind_{\mathbb{H}}^{\mathbb{G}}$ is a left adjoint of $Res_{\mathbb{H}}^{\mathbb{G}}$ in the sense above whenever $\mathbb{G}$ is *co-amenable*. The proof is essentially the same as the one for the above statement (under some regularity condition) and we refer to Proposition 4.7 in [140] for a proof.

1.7.2.6 Theorem. *Let $\mathbb{G}$ be a compact quantum group such that $\widehat{\mathbb{G}}$ is a torsion-free discrete quantum group. If $\mathcal{J} = \ker_{Hom}(Res_{\mathbb{E}}^{\widehat{\mathbb{G}}})$, then*

i) $\langle \mathcal{CI}_{Qut.}\rangle = \langle \mathfrak{p}_{\mathcal{J}}\rangle$,

ii) $\mathscr{KK}^{\widehat{\mathbb{G}}}$ has enough $\mathcal{J}$-projective objects.

Therefore, $(\langle \mathcal{CI}_{Qut.}\rangle, \mathcal{CC}_{Qut.})$ is a complementary pair of localizing subcategories in $\mathscr{KK}^{\widehat{\mathbb{G}}}$.

Proof. By virtue of the preceding lemma, we have that

$$KK^{\widehat{\mathbb{G}}}(Ind_{\mathbb{E}}^{\widehat{\mathbb{G}}}(A), B) \cong KK(A, Res_{\mathbb{E}}^{\widehat{\mathbb{G}}}(B)),$$

for all C^*-algebra A and all $\widehat{\mathbb{G}}$-C^*-algebra B. In other words, the functors $Res_{\mathbb{E}}^{\widehat{\mathbb{G}}}$ and $Ind_{\mathbb{E}}^{\widehat{\mathbb{G}}}$ are adjoints in $\mathscr{KK}^{\widehat{\mathbb{G}}}$. More precisely, $Ind_{\mathbb{E}}^{\widehat{\mathbb{G}}}$ is a left adjoint functor for $Res_{\mathbb{E}}^{\widehat{\mathbb{G}}}$.

In this way, we have by definition that

$$\mathcal{CC}_{Qut.} = \ker_{Obj}(Res_{\mathbb{E}}^{\widehat{\mathbb{G}}}) \text{ and } \mathcal{J} = \ker_{Hom}(Res_{\mathbb{E}}^{\widehat{\mathbb{G}}})$$

Therefore, Theorem 1.2.2.20 assures that $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ has enough $\mathcal{J}$-projective objects and consequently, by Theorem 1.2.2.15, $(\langle \mathfrak{p}_{\mathcal{J}}\rangle, \mathcal{CC}_{Qut.})$ is a complementary pair in $\mathscr{K}\mathscr{K}^{G}$ with $\langle \mathfrak{p}_{\mathcal{J}}\rangle = \langle (Res^{\widehat{\mathbb{G}}}_{\mathbb{E}})^{\vdash}\big(Obj(\mathscr{K}\mathscr{K})\big)\rangle = \langle Ind^{\widehat{\mathbb{G}}}_{\mathbb{E}}\big(Obj(\mathscr{K}\mathscr{K})\big)\rangle$.

To conclude, observe that $\langle \mathcal{CI}_{Qut.}\rangle$ is the minimal localizing subcategory containing the compactly induced objects by definition, so it contains objects of the form $Ind^{\widehat{\mathbb{G}}}_{\mathbb{E}}\big(Obj(\mathscr{K}\mathscr{K})\big)$, so we have $\langle Ind^{\widehat{\mathbb{G}}}_{\mathbb{E}}\big(Obj(\mathscr{K}\mathscr{K})\big)\rangle \subset \langle \mathcal{CI}_{Qut.}\rangle$ and, by minimality, this inclusion must be an equality. This yields the conclusion of the statement. ■

1.7.2.7 Remark. Since $(\langle \mathcal{CI}_{Qut.}\rangle, \mathcal{CC}_{Qut})$ is a complementary pair of localizing subcategories in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$, the fundamental lemma about complementary pairs (recall Lemma 1.2.1.26) can be applied, so that in particular we have two triangulated functors

$$L : \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \longrightarrow \langle \mathcal{CI}_{Qut.}\rangle \text{ and } N : \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \longrightarrow \mathcal{CC}_{Qut.}$$

such that for any $\widehat{\mathbb{G}}$-C^*-algebra $A \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}})$ there exists a (unique up to isomorphism) distinguished triangle of the form

$$\Sigma(N(A)) \longrightarrow L(A) \xrightarrow{D} A \longrightarrow N(A),$$

where D is called *quantum Dirac homomorphism*.

The analogous remark about the Dirac-dual Dirac method as in Remark 1.2.3.12 can be applied for the quantum case. Namely, the Dirac homomorphism corresponding to the trivial $\widehat{\mathbb{G}}$-C^*-algebra $\mathbb{C}$ is denoted by $D_{\mathbb{C}} \in KK^{\widehat{\mathbb{G}}}(L(\mathbb{C}), \mathbb{C})$. We call $D_{\mathbb{C}}$ *Dirac element*. Assume that there exists an element $\eta_{\mathbb{C}} \in KK^{\widehat{\mathbb{G}}}(\mathbb{C}, L(\mathbb{C}))$ such that $D_{\mathbb{C}} \underset{\mathbb{C}}{\otimes} \eta_{\mathbb{C}} = \mathbb{1}_{L(\mathbb{C})} \in KK^{\widehat{\mathbb{G}}}(L(\mathbb{C}), L(\mathbb{C}))$. We call $\eta_{\mathbb{C}}$ *dual Dirac element*. In this situation, the element $\gamma_{\mathbb{C}} := \eta_{\mathbb{C}} \underset{L(\mathbb{C})}{\otimes} D_{\mathbb{C}} \in KK^{\widehat{\mathbb{G}}}(\mathbb{C}, \mathbb{C})$, called γ-*element*, is an idempotent. We can show that

$$\gamma = \mathbb{1}_{\mathbb{C}} \iff \langle \mathcal{CI}_{Qut.}\rangle = \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$$

In this situation, we are able to establish the quantum version of the classical Baum-Connes property. Namely,

1.7.2.8 Definition. Let $\mathbb{G}$ be a compact quantum group such that $\widehat{\mathbb{G}}$ is a torsion-free discrete quantum group and fix the homological functor $F : \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \longrightarrow \mathscr{A}b^{\mathbb{Z}/2}$ defined by $F(A, \alpha) := K_*(\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A)$, for all $(A, \alpha) \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}})$.

The quantum Baum-Connes assembly map for $\widehat{\mathbb{G}}$ is the categorical Baum-Connes assembly map for $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ with respect to $(\langle \mathcal{CI}_{Qut.}\rangle, \mathcal{CC}_{Qut.}, F)$, that is, the natural transformation

$$\eta^{\widehat{\mathbb{G}}} : \mathbb{L}F \longrightarrow F$$

1.7.2.9 Definition. Let $\mathbb{G}$ be a compact quantum group such that $\widehat{\mathbb{G}}$ is a torsion-free discrete quantum group.

- We say that $\widehat{\mathbb{G}}$ satisfies the (quantum) Baum-Connes property (with coefficients) if $\eta^{\widehat{\mathbb{G}}}$ is a natural equivalence.
- We say that $\widehat{\mathbb{G}}$ satisfies the *strong* (quantum) Baum-Connes property if $\langle \mathcal{CI}_{Qut.}\rangle = \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$.

1.7.2.10 Remark. It is clear that the strong (quantum) Baum-Connes property implies the (quantum) Baum-Connes property by virtue of the uniqueness of the $(\langle \mathcal{CI}_{Qut.}\rangle, \mathcal{CC}_{Qut.})$-triangles. Notice, by the way, that $\widehat{\mathbb{G}}$ always satisfies the (categorical) Baum-Connes property with coefficients in objects of $\langle \mathcal{CI}_{Qut.}\rangle$.

Finally, using the universal property of the localization (Theorem 1.2.1.29) we obtain that $\widehat{\mathbb{G}}$ satisfies the (quantum) Baum-Connes property with coefficients if and only if $K_*(\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A) = (0)$, for all compactly contractible object $(A, \alpha) \in Obj(\mathcal{CC}_{Qut.})$.

1.7.2.11 Remark. From the strong Baum-Connes property and the torsion-freeness assumption, we get the K-amenability property automatically. Indeed, suppose that $\widehat{\mathbb{G}}$ is any torsion-free discrete quantum group satisfying the quantum *strong* Baum-Connes property $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} = \mathscr{L}_{\widehat{\mathbb{G}}}$. Given a compactly induced $\widehat{\mathbb{G}}$-C^*-algebra in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ of the form $A := c_0(\widehat{\mathbb{G}}) \otimes C$, we write

$$\widehat{\mathbb{G}} \underset{m}{\ltimes} A = \widehat{\mathbb{G}} \underset{m}{\ltimes} (c_0(\widehat{\mathbb{G}}) \otimes C) \overset{(*)}{\cong} \mathbb{K} \otimes C \overset{(*)}{\cong} \widehat{\mathbb{G}} \underset{r}{\ltimes} (c_0(\widehat{\mathbb{G}}) \otimes C) = \widehat{\mathbb{G}} \underset{r}{\ltimes} A,$$

where in $(*)$ we use Proposition 1.5.3.2 and apply Baaj-Skandalis duality to obtain the corresponding Morita equivalence. Since the crossed product functor is compatible with countable direct sums, we can replace A for *any* $\widehat{\mathbb{G}}$-C^*-algebra in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} = \mathscr{L}_{\widehat{\mathbb{G}}}$. Hence the canonical map $\widehat{\mathbb{G}} \underset{m}{\ltimes} A \longrightarrow \widehat{\mathbb{G}} \underset{r}{\ltimes} A$ is a K-equivalence for all $\widehat{\mathbb{G}}$-C^*-algebra A. In other words, $\widehat{\mathbb{G}}$ is K-amenable (recall Remark 1.3.1.41).

It is important to say that the *torsion-freeness* assumption is needed for the formal framework of the theory. That is, this is the case for which the classical constructions can be imitated without restrictions. Of course, a better understanding of the torsion phenomena in the quantum setting should allow to develop a satisfactory theory in which a quantum Baum-Connes property for any discrete quantum group can be stated.

Some of the typical examples of compact quantum groups presented in the next chapter have discrete duals that fail to be torsion-free. However, since the main goal is the K-group computation for C^*-algebras defining quantum groups, the strategy to achieve this (as we'll explain more precisely in Chapter 4) consists in proving the *strong* Baum-Connes property, so that we can use suitably Baaj-Skandalis duality and monoidal equivalences in order to show $\langle \mathcal{CI}_{Qut.}\rangle = \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ for which we don't need, *a priori*, either the existence of the assembly map or the formal framework developed previously, but only an appropriate definition of $\langle \mathcal{CI}_{Qut.}\rangle$. For instance, this is the case for the *quantum automorphism group* [212] and the *free wreath product* Section 3.7.

CHAPTER 2

Construction of Compact Quantum Groups

The second chapter of this thesis is devoted to illustrate the Woronowicz's theory with the main examples and constructions of compact quantum groups.

In Section 2.1 we give the definition and the main related results of concrete compact quantum groups as the free orthogonal quantum group $O^+(n)$ (including the q-deformation of $SU(2)$), the free unitary quantum group $U^+(n)$ and the quantum automorphism group $Qut(A, \omega)$ (including the quantum permutation group S_N^+).

The next sections describe the main construction processes in order to obtain new compact quantum groups imitating the classical setting. Namely, in Section 2.2 we explain the construction and the main properties of a direct product of two compact quantum groups following the work of S. Wang [216]. In Section 2.3 we explain the construction and the main properties of a the semi-direct product of a compact quantum group by a discrete group following the work of S. Wang [216]. In Section 2.4 we explain the construction and the main properties of a compact bicrossed product in the sense of P. Fima-K. Mukherjee-I. Patri [65] (which is a very concrete picture of the more general object defined in [196] by S. Vaes and L. Vainerman). In Section 2.5 we explain the construction and the main properties of a free product of two compact quantum groups following the work of S. Wang [215]. In Section 2.6 we explain the construction and the main properties of a free wreath product of a compact quantum group by S_N^+ following the work of J. Bichon **??**.

As we have already noticed in the introduction of the dissertation, the fundamental examples $SU_q(2)$, $O^+(n)$, $U^+(n)$ and S_N^+ above have been the main source of satisfactory results for the quantum Baum-Connes property in accordance with the work of C. Voigt and his collaborators. Likewise, the classical constructions of new groups as direct products, semi-direct products or free products are a natural class of groups in which the Baum-Connes property is stable in accordance with the work of J. Chabert-S. Echterhoff [35] and H. Oyono-Oyono [143].

In this sense, the present chapter should be regarded as the natural prelude for the core of the thesis contained in Chapter 3.

2.1 Typical examples

In this section we recall the definitions of the main examples of compact quantum groups that are interesting with respect to the framework of the present dissertation. In order to have a general perspective of each of these compact quantum groups, we collect (without proofs but with the corresponding references) the main results known until now due to different authors throughout the last thirty years.

A general method for defining compact quantum groups is by giving a *fundamental representation* of the compact quantum group itself. In this way we obtain what we call a *compact matrix quantum group*.

2.1.1 Definition. A compact matrix quantum group $\mathbb{G}$ is the data $(C(\mathbb{G}), \Delta, u)$ where $C(\mathbb{G})$ is a unital C^*-algebra, $\Delta : C(\mathbb{G}) \longrightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$ is a unital $*$-homomorphism and $u = (u_{ij})_{i,j=1,\ldots,n} \in \mathcal{M}_n(C(\mathbb{G}))$ is a matrix called *fundamental representation of* $\mathbb{G}$ such that

i) $C(\mathbb{G})$ is generated, as $*$-algebra, by the matrix coefficients of u,

ii) for all $i, j = 1, \ldots, n$ we have $\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$,

iii) the matrices u and $\overline{u} := (u_{ij}^*)_{i,j=1,\ldots,n}$ are invertible.

2.1.2 Remark. The analogue argument as the one given in Proposition 1.3.1.29 yields that every compact matrix quantum group is a compact quantum group in the usual sense. The difference between these two approaches is that in the above one we *choose a concrete* representation for the quantum group and so we regard it directly as a "matrix quantum group". In this way, the choice of a *unitary* representation allows to define the corresponding quantum group by using generators and relations and taking thus the corresponding universal (unital) C^*-algebra with the natural co-multiplication satisfying the definition above. Observe that this strategy could fail because *not every $*$-algebra admits a C^*-enveloping algebra.* However, in the examples that we are interested in, the corresponding generators form a fundamental representation that is *unitary*, which assures that the corresponding C^*-enveloping algebra is well-defined.

Free orthogonal quantum group

The following definition is due to A. van Daele and S. Wang [204].

2.1.3 Definition. Fix a natural number $n \in \mathbb{N}$ and an invertible matrix $Q \in GL_n(\mathbb{C})$ such that $Q\overline{Q} \in \mathbb{R}Id$. The free orthogonal quantum group is the compact matrix quantum group $O^+(Q) := (A_o(Q), \Delta_o, v)$ whose fundamental representation $v = (v_{ij})_{i,j=1,\ldots,n}$ satisfies the following relations

i) v is a unitary matrix,

ii) $v = Q\overline{v}Q^{-1}$.

If $Q = Id \in GL_n(\mathbb{C})$, we write $O^+(Id) := O^+(n)$.

2.1.4 Remarks. 1. We claim that any compact matrix quantum group $\mathbb{G} = (C(\mathbb{G}), \Delta, w)$ such that w is irreducible and $w \cong \overline{w}$ is a compact quantum *subgroup* of some free orthogonal quantum group $O^+(Q)$. Indeed, given the fundamental representation w of $\mathbb{G}$, which is equivalent to its contragredient representation $\overline{w}$ by assumption, we can find an invertible matrix $Q \in GL_n(\mathbb{C})$ such that $w = Q\overline{w}Q^{-1}$ is unitary. Observe that we can write

$$w = Q\overline{w}Q^{-1} \Rightarrow w = Q\overline{(Q\overline{w}Q^{-1})}Q^{-1} = (Q\overline{Q})w(Q\overline{Q})^{-1}$$

If $Q\overline{Q} \notin \mathbb{C}id$, then the Quantum Schur's lemma (recall Proposition 1.3.1.17) would say that w is a *reducible* representation, which is impossible by our assumption. Hence, it must be $Q\overline{Q} \in \mathbb{C}id$ and an easy linear algebra computation yields that this condition implies that $Q\overline{Q} \in \mathbb{R}id$.

Hence, by universality we can define a (*surjective*) $*$-homomorphism $f : A_o(Q) \twoheadrightarrow C(\mathbb{G})$ such that $f(v_{ij}) = w_{ij}$, for all $i, j = 1, \ldots, n$. This map is clearly compatible with the co-multiplications of each compact quantum group and so $\mathbb{G}$ is a compact quantum subgroup of $O^+(Q)$.

2. The q-deformation of $SU(2)$, whose definition is due to S.L. Woronowicz [227], is a particular example of free orthogonal quantum group. Let us be more precise.

Fix a real number $q \in [-1, 1]\backslash\{0\}$ (called *deformation parameter*). The quantum $SU(2)$ group is the compact matrix quantum group $SU_q(2) := (C(SU_q(2)), \Delta, u)$ whose fundamental representation $u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$ is unitary meaning precisely that the following relations hold

$$\alpha^*\alpha + \gamma^*\gamma = 1;\ \alpha\alpha^* + q^2\gamma^*\gamma = 1;\ \alpha\gamma = q\gamma\alpha;\ \alpha\gamma^* = q\gamma^*\alpha,$$

where α, γ are two letters called *generators of* $SU_q(2)$. The co-multiplication Δ of $SU_q(2)$, the co-unit ε and the antipode S corresponding to the $*$-Hopf algebra $Pol(SU_q(2))$ are such that

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma \text{ and } \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma,$$

$$\varepsilon(\alpha) = 1 \text{ and } \varepsilon(\gamma) = 0,$$

$$S(\alpha) = \alpha^*,\ S(\alpha^*) = \alpha,\ S(\gamma) = -q\gamma \text{ and } S(\gamma^*) = -q^{-1}\gamma$$

It is important to make the following observation. Consider the unit circle

$$S^1 := \{z \in \mathbb{C} \mid |z| = 1\},$$

which is an abelian compact group. Thus its continuous functions $C(S^1)$ is naturally a compact quantum group with the co-multiplication given by the multiplication in the group. Let us write $\mathbb{T} := (C(S^1), \Delta_{\mathbb{T}})$.

If z denotes the function on S^1 given by $z \longmapsto z$, for all $z \in S^1$; then we put $\rho_{\mathbb{T}}(\alpha) := z$ and $\rho_{\mathbb{T}}(\gamma) := 0$. It is clear by the definitions that the relation $(\rho_{\mathbb{T}} \otimes \rho_{\mathbb{T}}) \circ \Delta = \Delta_{\mathbb{T}} \circ \rho_{\mathbb{T}}$ holds on generators of $SU_q(2)$ and so by universality we can extend $\rho_{\mathbb{T}}$ (in a unique fashion) into a homomorphism of compact quantum groups,

$$\rho_{\mathbb{T}} : SU_q(2) \longrightarrow \mathbb{T}$$

Again by universality there exists a unique $*$-homomorphism

$$\theta_\zeta : SU_q(2) \longrightarrow \mathbb{C},$$

such that $\theta_\zeta(\alpha) = \zeta$ and $\theta_\zeta(\gamma) = 0$, for each $\zeta \in S^1$.

Suppose that $w \in \mathcal{B}(H) \otimes C(SU_q(2))$ is a finite dimensional representation of $SU_q(2)$ on a Hilbert space H. Since $\rho_{\mathbb{T}}$ is a quantum homomorphism, then $(id \otimes \rho_{\mathbb{T}})(w)$ is a finite dimensional representation of $\mathbb{T}$ on H. Notice that the compact *quantum* group $\mathbb{T}$ comes from the *classical* compact group S^1, so that such representation corresponds to the following classical representation of S^1,

$$\begin{array}{rccl} \pi_w : & S^1 & \longrightarrow & \mathcal{B}(H) \\ & \zeta & \longmapsto & \pi_w(\zeta) := (id \otimes \theta_\zeta)(w) \end{array}$$

Decompose π_w as a sum of irreducibles, say $H \cong \bigoplus_{k\in\mathbb{Z}} H_k$, where we recall that the space $H_k := \{\xi \in H \mid \pi_w(\zeta)(\xi) = \xi\zeta^k \text{ for all } \zeta \in S^1\}$ is called *kth weight space of* w and the parameter k is called *a weight of* w whenever $H_k \neq 0$. The *weight function of* w is defined by

$$\begin{array}{rccl} W_w : & \mathbb{Z} & \longrightarrow & \mathbb{N} \\ & k & \longmapsto & W_w(k) := dim(H_k) \end{array}$$

3. From $SU_q(2)$ we can obtain another relevant compact quantum group: the q-deformation of $SO(3)$, whose definition is due to P. Podleś [155], [157].

   Given a real number $q \in [-1, 1]\backslash\{0\}$, the quantum $SO(3)$ group is the compact matrix quantum group $SO_q(3) := (C(SO_q(3)), \Delta, u)$ whose fundamental representation is the following unitary

$$u = \begin{pmatrix} (\alpha^*)^2 & -(q^2+1)\alpha^*\gamma^* & -q\gamma^2 \\ \gamma^*\alpha^* & 1-(q^2+1)\gamma^*\gamma & \alpha\gamma \\ -q(\gamma^*)^2 & -(q^2+1)\gamma^*\alpha & \alpha^2 \end{pmatrix},$$

   where α, γ are two letters called *generators of* $SO_q(3)$. It is possible to describe $C(SO_q(3))$ as the universal C^*-algebra generated by five elements and satisfying a list of twenty relations. We refer to [155] and [157] for more precisions. There are several properties of $SO_q(3)$ that must be mentioned.

   - We have $C(SO_q(3)) = C\Big(SU_q(2)/\mathbb{Z}_2\Big)$. Moreover, $\widehat{SO_q(3)}$ is a discrete quantum subgroup of $\widehat{SU_q(2)}$, meaning that $C(SO_q(3)) \subset C(SU_q(2))$ (recall Definition 1.4.3.2). More precisely, $C(SO_q(3))$ is generated by the coefficients of $u_{SU_q(2)}^{\oplus 2}$, where $u_{SU_q(2)}$ denotes the fundamental representation of $SU_q(2)$.
   - For $q = 1$ we recover the classical rotation group $SO(3)$. Moreover, $SO_q(3) \cong SO_{-q}(3)$, for all $q \in [-1, 1]\backslash\{0\}$. In particular, the Podleś $SO_{-1}(3)$ is nothing but the classical rotation group.

The main results about the free orthogonal quantum group are stated in the following theorem.

2.1.5 Theorem. *Fix a natural number $n \in \mathbb{N}$ and an invertible matrix $Q \in GL_n(\mathbb{C})$ such that $Q\overline{Q} \in \mathbb{R}id$.*

i) *(S. Wang, [218]) Let $Q' \in GL_n(\mathbb{C})$ such that $Q'\overline{Q'} \in \mathbb{R}id$. The compact quantum groups $O^+(Q)$ and $O^+(Q')$ are isomorphic if and only if there exists a unitary matrix $U \in U_n(\mathbb{C})$ such that $Q' = U^tQU$.*

ii) *(S.L. Woronowicz, [227] or [188]) For each $r \in \mathbb{N}$ there exists a unique (up to equivalence) irreducible representation matrix $w_r \in \mathcal{M}_{r+1}(C(SU_q(2)))$ with weight function given by*

$$W_r(k) := \begin{cases} 1, \text{ if } k \in \{-r, 2-r, \ldots, r-2, r\} \\ 0, \text{ otherwise} \end{cases}$$

for all $k \in \mathbb{Z}$.

Every finite dimensional representation of $SU_q(2)$ is completely determined, up to equivalence, by its weight function. Moreover the following fusion rules hold

$$w_1 = u;\ w_r \oplus w_s = w_{|r-s|} \oplus w_{|r-s|+2} \oplus \ldots \oplus w_{r+s-2} \oplus w_{r+s},$$

for all $r, s \in \mathbb{N}$.

iii) *(T. Banica, [9]) There exists a familiy of pairwise inequivalent irreducible representation $\{x_r\}_{r\in\mathbb{N}}$ of $O^+(Q)$ that are exactly the irreducible representations of $SU_q(2)$ such that the following fusion rules hold*

$$x_0 = 1;\ x_1 = v;\ x_r \oplus x_s = x_{|r-s|} \oplus x_{|r-s|+2} \oplus \ldots \oplus x_{r+s-2} \oplus x_{r+s},$$

for all $r, s \in \mathbb{N}$. Every irreducible representation of $O^+(Q)$ is equivalent to x_r, for some $r \in \mathbb{N}$.

iv) *(T. Banica, [9] or [139]) $O^+(Q)$ is co-amenable for $n = 2$ and* not *co-amenable if $n > 2$.*

In particular, $SU_q(2)$ is co-amenable (see also E. Bédos, J. G. Murphy and L. Tuset, [31] or [139]).

v) *(C. Voigt, [210]) $\widehat{O^+(Q)}$ is torsion-free and satisfies the* strong *(quantum) Baum-Connes property. As a result, $\widehat{O^+(Q)}$ is K-amenable.*

In particular, $\widehat{SU_q(2)}$ is torsion-free and satisfies the strong *(quantum) Baum-Connes property (see also [211]).*

vi)
- *(C. Voigt, [210]) The K-theory of $O^+(Q)$ is given by*

$$K_0(A_0(Q)) = \mathbb{Z} \text{ and } K_1(A_0(Q)) = \mathbb{Z}$$

- *(T. Masuda, Y. Nakagami and J. Watanabe, [128]; C. Voigt, [210]) The K-theory of $SU_q(2)$ is given by*

$$K_0(C(SU_q(2))) = \mathbb{Z} \text{ and } K_1(C(SU_q(2))) = \mathbb{Z}$$

Free unitary quantum group

The following definition is due to A. van Daele and S. Wang [204].

2.1.6 Definition. Fix a natural number $n \in \mathbb{N}$ and an invertible matrix $Q \in GL_n(\mathbb{C})$. The free unitary quantum group is the compact matrix quantum group $U^+(Q) := (A_u(Q), \Delta_u, u)$ whose fundamental representation $u = (u_{ij})_{i,j=1,\dots,n}$ satisfies the following relations

i) u is a unitary matrix,

ii) $Q\overline{u}Q^{-1}$ is a unitary matrix.

If $Q = Id \in GL_n(\mathbb{C})$, we write $U^+(Id) := U^+(n)$.

2.1.7 Remark. We claim that any compact matrix quantum group $\mathbb{G} = (C(\mathbb{G}), \Delta, w)$ is a compact quantum *subgroup* of some free unitary quantum group $U^+(Q)$. Indeed, given the fundamental representation w of $\mathbb{G}$, consider its contragredient representation $\overline{w}$, which is always unitary equivalent to a unitary representation (recall Remark 1.3.1.19 or Remark 1.3.1.21). In other words, there always exists an invertible matrix $Q \in GL_n(\mathbb{C})$ such that $Q\overline{w}Q^{-1}$ is unitary. Hence, by universality we can define a (*surjective*) $*$-homomorphism $f : A_u(Q) \twoheadrightarrow C(\mathbb{G})$ such that $f(u_{ij}) = w_{ij}$, for all $i, j = 1, \dots, n$. This map is clearly compatible with the co-multiplications of each compact quantum group and so $\mathbb{G}$ is a compact quantum subgroup of $U^+(Q)$.

The main results about the free unitary quantum group are stated in the following theorem.

2.1.8 Theorem. *Fix a natural number $n \in \mathbb{N}$ and an invertible matrix $Q \in GL_n(\mathbb{C})$.*

i) (S. Wang, [218]) Let $Q' \in GL_n(\mathbb{C})$ and assume that Q and Q' are positive *matrices. Let $(\lambda_1, \dots, \lambda_n)$ and $(\lambda'_1, \dots, \lambda'_n)$ the (positive) eigen values of Q and Q', respectively. The compact quantum groups $U^+(Q)$ and $U^+(Q')$ are isomorphic if and only if $(\lambda_1, \dots, \lambda_n) = (\lambda'_1, \dots, \lambda'_n)$ or $(\lambda_n^{-1}, \dots, \lambda_1^{-1}) = (\lambda'_1, \dots, \lambda'_n)$.*

ii) (T. Banica, [10]) There exists a family of pairwise inequivalent irreducible representations $\{x_\gamma\}_{\gamma \in \mathbb{N} * \mathbb{N}}$ of $U^+(Q)$ such that the following fusion rules hold

$$x_e = 1;\ x_\alpha = u;\ x_\beta = \overline{u};\ x_r \oplus x_s = \sum_{\substack{a,b,c \in \mathbb{N} * \mathbb{N} \\ r = ac,\, s = \overline{c}b}} x_{ab},$$

for all $r, s \in \mathbb{N}$ where $\alpha, \beta \in \mathbb{N} * \mathbb{N}$ are the canonical generators. Every irreducible representation of $U^+(Q)$ is equivalent to x_γ, for some $\gamma \in \mathbb{N} * \mathbb{N}$.

iii) (T. Banica, [10]) $U^+(Q)$ is not *co-amenable.*

iv) (R. Vergnioux and C. Voigt, [208]) $\widehat{U^+(Q)}$ satisfies the strong (quantum) Baum-Connes property. As a result, $\widehat{U^+(Q)}$ is K-amenable.

v) (R. Vergnioux and C. Voigt, [208]) The K-theory of $U^+(Q)$ is given by

$$K_0(A_u(Q)) = \mathbb{Z} \text{ and } K_1(A_u(Q)) = \mathbb{Z} \oplus \mathbb{Z}$$

Quantum automorphisms group

The quantum permutation group of N-points and the quantum automorphisms group of $\mathcal{M}_n(\mathbb{C})$ fit in a more general context. Namely, they are particular cases of a *quantum automorphism group of a quantum or noncommutative space*. The following definition is due to S. Wang [217].

2.1.9 Definition. Fix a C^*-algebra A and a continuous linear form $\omega \in A^*$. The quantum automorphism group of (A, ω) is a compact quantum group $Qut(A, \omega)$ acting on A and preserving ω such that if $\mathbb{G}$ is another compact quantum group acting on A and preserving ω, then there exists a unique quantum group homomorphism $\mathbb{G} \longrightarrow Qut(A, \omega)$ preserving ω.

If the linear form ω is clear by the context, we write simply $Qut(A, \omega) := Qut(A)$.

If such a quantum automorphism group exists, it is automatically *unique*. The main problem is thus to establish the existence of such a quantum group. We might fail if we want the existence in full generality, but under some assumptions on the pair (A, ω) we achieve interesting results. Namely, whenever A is finite dimensional and ω is a δ-form, we can perform a satisfactory theory. Let us be more precise.

Given a C^*-algebra A, fix a faithful state ω on A, that is, a continuous linear form $\omega : A \longrightarrow \mathbb{C}$ such that

i) $\omega(aa^*) \geqslant 0$, for all $a \in A$ and $\omega(1) = 1$,

ii) $\omega(aa^*) \neq 0$, for all *non-zero* $a \in A$,

iii) (sometimes such state is *tracial*, meaning that $\omega(ab) = \omega(ba)$, for all $a, b \in A$).

In this case, we can perform the corresponding GNS construction, which allows in particular to equip A (and hence $A \otimes A$) with a Hilbert space structure whose inner product is such that $\langle x, y \rangle = \omega(x^* y)$, for all $x, y \in A$. The multiplication of A can be regarded as a linear homomorphism $m : A \otimes A \longrightarrow A$.

Fix now a positive number $\delta > 0$. We say that ω is a *δ-form* if $mm^* = \delta^2 id_A$.

If A is in addition a finite dimensional C^*-algebra, it is well known that it can be expressed as a direct sum of matrices say $A = \mathcal{M}_{n_1}(\mathbb{C}) \oplus \ldots \oplus \mathcal{M}_{n_r}(\mathbb{C})$. Hence, the corresponding quantum automorphism group can be explicitly described in terms of generators and relations via a fundamental representation u (see and compare Theorem 5.1 in [217] and Theorem 1.1 in [13]).

2.1.10 Remarks. 1. If $\{e_1, \ldots, e_N\}$ denotes the canonical orthonormal basis of $\mathbb{C}^N$, for some $N \in \mathbb{N}$, then there exists a natural action of $Qut(\mathbb{C}^N)$ on $\mathbb{C}^N$ given by

$$\begin{array}{rccl} \alpha : & \mathbb{C}^N & \longrightarrow & C(Qut(\mathbb{C}^N)) \otimes \mathbb{C}^N \\ & e_j & \longmapsto & \alpha(e_j) := \sum\limits_{i=1}^{N} u_{ij} \otimes e_i, \text{ for all } j = 1, \ldots, N, \end{array}$$

where $u := (u_{ij})_{i,j=1,\ldots,N}$ is the fundamental representation of $Qut(\mathbb{C}^N)$.

The *quantum permutation group of N-points* is defined as the quantum automorphism group of $\mathbb{C}^N$ with the uniform probability measure on N-points as a trace, which is a δ-form for $\delta := \sqrt{n}$ (see Theorem 3.1 in [217] for more details). We write

$$S_N^+ := Qut(\mathbb{C}^N)$$

It is important to make the following observation. Consider the classical permutation group S_N, which is a finite discrete group. Thus its continuous functions $C(S_N)$ is naturally a compact quantum group. By abuse of notation, we write $S_N := (C(S_N), \Delta_{S_N})$.

For every $i, j = 1, \ldots, N$ consider the following subset of S_N,

$$S_{ij} := \{\sigma \in S_N \mid \sigma(j) = i\}$$

and denote by $\chi_{S_{ij}} =: \chi_{ij}$ its characteristic function, which can be regarded as a continuous function on S_N. Put $\rho_{S_N}(u_{ij}) := \chi_{ij}$, for all $i, j = 1, \ldots, N$. It is clear by the definitions that the relation $(\rho_{S_N} \otimes \rho_{S_N}) \circ \Delta = \Delta_{S_N} \circ \rho_{S_N}$ holds on generators of S_N^+ and so by universality we can extend ρ_{S_N} (in a unique fashion) into a homomorphism of compact quantum groups,

$$\rho_{S_N} : S_N^+ \longrightarrow S_N$$

We can show that for $N = 1, 2, 3$ we have $S_N^+ \cong S_N$ as compact quantum groups (because in this case the magic unitaries defining S_N^+ must mutually commute). For $N \geqslant 4$, $C(S_N^+)$ is not commutative and infinite dimensional, which prevent ρ_{S_N} to be a isomorphism. In this sense we say that *quantum permutations* exist only for $N \geqslant 4$. For a proof of these facts we refer to [217] or [16].

2. If $\{m_{ij}\}_{i,j=1,\ldots,n}$ denotes the canonical orthonormal basis of matrix units of $\mathcal{M}_n(\mathbb{C})$, for some $n \in \mathbb{N}$, there exists a natural action of $Qut(\mathcal{M}_n(\mathbb{C}))$ on $\mathcal{M}_n(\mathbb{C})$ given by

$$\begin{array}{rccl} \alpha : & \mathcal{M}_n(\mathbb{C}) & \longrightarrow & C(Qut(\mathcal{M}_n(\mathbb{C}))) \otimes \mathcal{M}_n(\mathbb{C}) \\ & m_{ij} & \longmapsto & \alpha(m_{ij}) := \sum\limits_{k,l=1}^{n} u_{ij}^{kl} \otimes m_{kl}, \text{ for all } i, j = 1, \ldots, n, \end{array}$$

where $u = (u_{ij}^{lk})_{i,j,l,k=1,\ldots,n}$ is the fundamental representation of $Qut(\mathcal{M}_n(\mathbb{C}))$.

Actually, we can show (see Theorem 4.1 in [217] for a proof) that $Qut(\mathcal{M}_n(\mathbb{C}))$ is the quantum automorphism group of $\mathcal{M}_n(\mathbb{C})$ with the canonical (normalized) trace as a linear form $\omega := tr.$, which is a δ-form (for a well-chosen $\delta > 0$, see Proposition 2.1 in [13] for the details).

3. By virtue of the above remark about $S_N^+ = Qut(\mathbb{C}^N)$, all quantum automorphism groups of C^*-algebras of dimension $1, 2$ or 3 are *classical* automorphism groups. For dimension $N = 4$, we have two possibilities.

   a) either we consider the C^*-algebra $\mathbb{C}^4$ whose quantum automorphism group is given by $S_4^+ \cong SO_{-1}(3)$ thanks to the result [15] due to T. Banica and J. Bichon. Here the quantum group $SO_{-1}(3)$ is a 2-cocycle twist of the classical group $SO(3)$, so it is different from the Podleś q-deformation introduced in Remarks 2.1.4

   b) or we consider the C^*-algebra $\mathcal{M}_2(\mathbb{C})$ whose quantum automorphism group is given by $Qut(\mathcal{M}_2(\mathbb{C})) \cong SO_q(3)$, for a unique $q \in (0, 1]$ by virtue of a P. M. Soltan's result [173].

   Hence, for the treatment of the quantum automorphism group we can restrict our attention to the $\geqslant$ 4th-dimensional case.

The main results about the quantum permutation group are stated in the following theorem.

2.1.11 Theorem. *Fix a finite dimensional C^*-algebra A with $dim(A) = n \geqslant 4$ and a δ-form $\omega \in A^*$.*

i) (T. Banica, [13]) For each $r \in \mathbb{N}$ there exists a unique (up to equivalence) irreducible representation w_r of $Qut(A, \omega)$ such that

$$w_0 = \epsilon,\ w_0 \oplus w_1 = u \text{ and } w_r = \overline{w}_r, \text{ for all } r \in \mathbb{N}$$

Moreover, the following (recursive) fusion rules hold

$$w_1 \ominus w_r = w_{r-1} \oplus w_r \oplus w_{r+1},$$

for all $r \in \mathbb{N}$.

ii) (C. Voigt, [212]) $\widehat{Qut(A, \omega)}$ is not *torsion-free. Moreover, the trivial action on $\mathbb{C}$ and the canonical action of $Qut(A, \omega)$ on A are the unique, up to Morita equivalence, torsion actions.*

In particular, we have that

- *$\widehat{S_N^+}$ is* not *torsion-free and its unique, up to Morita equivalence, torsion actions are the trivial action on $\mathbb{C}$ and the canonical action on $\mathbb{C}^N$,*
- *(see also [211]) $\widehat{Qut(\mathcal{M}_n(\mathbb{C}))}$ is* not *torsion-free and its unique, up to Morita equivalence, torsion actions are the trivial action on $\mathbb{C}$ and the canonical action on $\mathcal{M}_n(\mathbb{C})$.*

iii) (T. Banica, [13]) S_N^+ is co-amenable if and only if $N \leqslant 4$.

iv) (C. Voigt, [212]) $\widehat{Qut(A, \omega)}$ satisfies the strong *(quantum) Baum-Connes property and it is K-amenable.*

In particular, we have that

- *$\widehat{S_N^+}$ satisfies the* strong *(quantum) Baum-Connes property and it is K-amenable,*
- *(see also [211]) $\widehat{Qut(\mathcal{M}_n(\mathbb{C}))}$ satisfies the* strong *(quantum) Baum-Connes property and it is K-amenable.*

v) (C. Voigt, [212]) If we write $A = \mathcal{M}_{n_1}(\mathbb{C}) \oplus \ldots \oplus \mathcal{M}_{n_r}(\mathbb{C})$, the K-theory of $Qut(A, \omega)$ is given by

$$K_0(C(Qut(A, \omega))) = \mathbb{Z}^{(r-1)^2+1} \oplus \mathbb{Z}_d^{2r-1} \text{ and } K_1(C(Qut(A, \omega))) = \mathbb{Z},$$

where $d := gcd(n_1, \ldots, n_r)$.

In particular, we have that

- *the K-theory of S_N^+ is given by*

$$K_0(C(S_N^+)) = \mathbb{Z}^{N^2-2N+2} \text{ and } K_1(C(S_N^+)) = \mathbb{Z}$$

  As a consequence, $C(S_N^+) \cong C(S_{N'}^+)$ if and only if $N = N'$.
- *(see also [211]) the K-theory of $Qut(\mathcal{M}_n(\mathbb{C}))$ is given by*

$$K_0\big(C(Qut(\mathcal{M}_n(\mathbb{C})))\big) = \mathbb{Z} \oplus \mathbb{Z}_n \text{ and } K_1\big(C(Qut(\mathcal{M}_n(\mathbb{C})))\big) = \mathbb{Z}$$

2.2 Quantum direct product

We introduce the direct product of two compact quantum groups and we analyze some structure properties of this object which are useful for our purpose. We may recall the leg and Sweedler notations from Section 1.1.

The following theorem is due to S. Wang [216].

2.2.1 Theorem-Definition. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta_{\mathbb{G}})$ and $\mathbb{H} = (C(\mathbb{H}), \Delta_{\mathbb{H}})$ be two compact quantum groups.*

There exists a unique unital $*$-homomorphism

$$\Theta : C_m(\mathbb{G}) \underset{max}{\otimes} C_m(\mathbb{H}) \longrightarrow C_m(\mathbb{G}) \underset{max}{\otimes} C_m(\mathbb{H}) \otimes C_m(\mathbb{G}) \underset{max}{\otimes} C_m(\mathbb{H})$$

such that

$$\Theta(a \otimes b) = \Delta_{\mathbb{G}}(a)_{(1)} \otimes \Delta_{\mathbb{H}}(b)_{(1)} \otimes \Delta_{\mathbb{G}}(a)_{(2)} \otimes \Delta_{\mathbb{H}}(b)_{(2)},$$

for all $a \in C_m(\mathbb{G})$ and all $b \in C_m(\mathbb{H})$.

Besides, we have that

i) $\mathbb{F} := (C_m(\mathbb{G}) \underset{max}{\otimes} C_m(\mathbb{H}), \Theta)$ *is a compact quantum group,*

ii) the Haar state on $\mathbb{F}$ is given by $h_{\mathbb{F}} = h_{\mathbb{G}} \underset{max}{\otimes} h_{\mathbb{H}}$, where $h_{\mathbb{G}}$ and $h_{\mathbb{H}}$ are the Haar states on $\mathbb{G}$ and $\mathbb{H}$, respectively,

iii) the maximal picture of $\mathbb{F}$ is given by $C_m(\mathbb{F}) = C_m(\mathbb{G}) \underset{max}{\otimes} C_m(\mathbb{H})$,

iv) the reduced picture of $\mathbb{F}$ is given by $C_r(\mathbb{F}) = C_r(\mathbb{G}) \otimes C_r(\mathbb{H})$,

v) the irreducible representations of $\mathbb{F}$ are described as follows: for every irreducible representation $y \in Irr(\mathbb{F})$, take a representative $w^y \in \mathcal{B}(H_y) \otimes C(\mathbb{F})$. There exist unique irreducible representations $x \in Irr(\mathbb{G})$ and $z \in Irr(\mathbb{H})$ such that if $w^x \in \mathcal{B}(H_x) \otimes C(\mathbb{G})$ and $w^z \in \mathcal{B}(H_z) \otimes C(\mathbb{H})$ are respective representatives of x and z, then we have

$$w^y \cong \left[w^x\right]_{13}\left[w^z\right]_{24} \in \mathcal{B}(H_x \otimes H_z) \otimes C(\mathbb{F}),$$

where $\left[w^x\right]_{13}$ and $\left[w^z\right]_{24}$ are the corresponding legs of w^x and w^z, respectively inside $\mathcal{B}(H_x) \otimes \mathcal{B}(H_z) \otimes C_m(\mathbb{G}) \underset{\max}{\otimes} C_m(\mathbb{H})$. In this case we write $w^y := w^{(x,z)}$.

In other words, the irreducible representations described above provide a complete set of mutually inequivalent irreducible representations of $\mathbb{F}$. For this reason we write $Irr(\mathbb{F}) = \left[Irr(\mathbb{G})\right]_{13} \left[Irr(\mathbb{H})\right]_{24}$.

The compact quantum group $\mathbb{F}$ constructed in this way is called quantum direct product of $\mathbb{G}$ and $\mathbb{H}$ *and is denoted by $\mathbb{G} \times \mathbb{H}$.*

2.2.2 Remark. The representation theory of a quantum direct product $\mathbb{F}$ described in the theorem above allows to give some explicit expressions which are useful for subsequent computations.

- Let $x, x' \in Irr(\mathbb{G})$ and $z, z' \in Irr(\mathbb{H})$ irreducible representations of $\mathbb{G}$ and $\mathbb{H}$ and consider the corresponding irreducible representations of $\mathbb{F}$, say $y := (x, z), y' := (x', z') \in Irr(\mathbb{F})$. Thanks to the theorem above we know that
$$w^y = \big[w^x\big]_{13}\big[w^z\big]_{24} \text{ and } w^{y'} = \big[w^{x'}\big]_{13}\big[w^{z'}\big]_{24},$$
where the legs are considered inside $\mathcal{B}(H_x)\otimes\mathcal{B}(H_z)\otimes C_m(\mathbb{G}) \underset{\max}{\otimes} C_m(\mathbb{H})$ and $\mathcal{B}(H_{x'})\otimes\mathcal{B}(H_{z'})\otimes C_m(\mathbb{G}) \underset{\max}{\otimes} C_m(\mathbb{H})$, respectively.
The flip map $H_z \otimes H_{x'} \longrightarrow H_{x'} \otimes H_z$ yields the following obvious identification
$$w^{y\oplus y'} := w^y \oplus w^{y'} = \big[w^x \oplus w^{x'}\big]_{13}\big[w^z \oplus w^{z'}\big]_{24}$$
- The C^*-algebra of the quantum discrete dual of $\mathbb{F}$ can be written as
$$c_0(\widehat{\mathbb{F}}) \cong c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{H}})$$
Indeed,
$$\begin{aligned} c_0(\widehat{\mathbb{F}}) &= \bigoplus_{y\in Irr(\mathbb{F})}^{c_0} \mathcal{B}(H_y) = \bigoplus_{(x,z)\in Irr(\mathbb{G})\times Irr(\mathbb{H})}^{c_0} \mathcal{B}(H_x \otimes H_z) \\ &\cong \Big(\bigoplus_{x\in Irr(\mathbb{G})}^{c_0} \mathcal{B}(H_x)\Big) \otimes \Big(\bigoplus_{z\in Irr(\mathbb{H})}^{c_0} \mathcal{B}(H_z)\Big) = c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{H}}) \end{aligned}$$
- The fundamental multiplicative unitary of $\mathbb{F}$ can be written as
$$W_{\mathbb{F}} = \big[W_{\mathbb{G}}\big]_{13}\big[W_{\mathbb{H}}\big]_{24}$$
Indeed, by definition we have
$$\begin{aligned} W_{\mathbb{G}} &= \bigoplus_{x\in Irr(\mathbb{G})} w^x \in M(c_0(\widehat{\mathbb{G}}) \otimes C_r(\mathbb{G})) \\ &\Longrightarrow \big[W_{\mathbb{G}}\big]_{13} = \bigoplus_{x\in Irr(\mathbb{G})} \big[w^x\big]_{13} \in M(c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{H}}) \otimes C(\mathbb{F})) \\ W_{\mathbb{H}} &= \bigoplus_{z\in Irr(\mathbb{H})} w^z \in M(c_0(\widehat{\mathbb{H}}) \otimes C_r(\mathbb{H})) \\ &\Longrightarrow \big[W_{\mathbb{G}}\big]_{24} = \bigoplus_{z\in Irr(\mathbb{H})} \big[w^z\big]_{24} \in M(c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{H}}) \otimes C(\mathbb{F})), \end{aligned}$$
where we understand the legs (13) and (24) in the space $c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{H}}) \otimes C_m(\mathbb{G}) \underset{\max}{\otimes} C_m(\mathbb{H})$. In this way, we have
$$\begin{aligned} W_{\mathbb{F}} &= \bigoplus_{y\in Irr(\mathbb{F})} w^y = \bigoplus_{(x,z)\in Irr(\mathbb{G})\times Irr(\mathbb{H})} \big[w^x\big]_{13}\big[w^z\big]_{24} \\ &= \Big(\bigoplus_{x\in Irr(\mathbb{G})} \big[w^x\big]_{13}\Big)\Big(\bigoplus_{z\in Irr(\mathbb{H})} \big[w^z\big]_{24}\Big) = \big[W_{\mathbb{G}}\big]_{13}\big[W_{\mathbb{H}}\big]_{24} \end{aligned}$$

From now on, $\mathbb{G} = (C(\mathbb{G}), \Delta_{\mathbb{G}})$ and $\mathbb{H} = (C(\mathbb{H}), \Delta_{\mathbb{H}})$ denote compact quantum groups and $\mathbb{F} := \mathbb{G} \times \mathbb{H}$ denotes the corresponding quantum direct product as in the theorem above.

2.2.3 Proposition. *The canonical injections*

$$\iota^r_{\mathbb{G}} : C_r(\mathbb{G}) \hookrightarrow C_r(\mathbb{F}) \text{ and } \iota^r_{\mathbb{H}} : C_r(\mathbb{H}) \hookrightarrow C_r(\mathbb{F})$$

are such that

$$(\iota^r_{\mathbb{G}} \otimes \iota^r_{\mathbb{G}}) \circ \Delta_{\mathbb{G}} = \Theta \circ \iota^r_{\mathbb{G}} \text{ and } (\iota^r_{\mathbb{H}} \otimes \iota^r_{\mathbb{H}}) \circ \Delta_{\mathbb{H}} = \Theta \circ \iota^r_{\mathbb{H}}$$

In other words, $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are discrete quantum subgroups of $\widehat{\mathbb{F}}$.

Proof. We have just to remark that $\Theta = (id \otimes \Sigma \otimes id) \circ \Delta_{\mathbb{G}} \otimes \Delta_{\mathbb{H}}$ by definition, where $\Sigma : C(\mathbb{G}) \otimes C(\mathbb{H}) \longrightarrow C(\mathbb{H}) \otimes C(\mathbb{G})$ denotes the flip map.

Hence, the canonical inclusions $\iota^r_{\mathbb{G}}$ and $\iota^r_{\mathbb{H}}$ intertwine the corresponding co-multiplications by construction.

Consequently, $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are discrete quantum subgroups of $\widehat{\mathbb{F}}$ as explained in Proposition 1.4.3.4. ■

2.2.4 Remark. Furthermore, the representation theory of $\mathbb{F}$ yields that $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are divisible in $\widehat{\mathbb{F}}$ (recall the equivalence relation defined in terms of discrete quantum subgroups in Section 1.4.3 and see Definition 3.2.1 and Lemma 3.2.2 for more details).

Namely, take an irreducible representation $y := (x, z) \in Irr(\mathbb{F})$ with $x \in Irr(\mathbb{G})$ and $z \in Irr(\mathbb{H})$. Then $x = (x, \epsilon_{\mathbb{H}}) \in [y]$ in $Irr(\mathbb{H}) \backslash Irr(\mathbb{F})$ because $(x, z) \ominus (\overline{x}, \epsilon_{\mathbb{H}}) = (\epsilon_{\mathbb{G}}, z) = z \in Irr(\mathbb{H})$. Likewise, we have that $z = (\epsilon_{\mathbb{G}}, z) \in [y]$ in $Irr(\mathbb{G}) \backslash Irr(\mathbb{F})$ because $(x, z) \ominus (\epsilon_{\mathbb{G}}, \overline{z}) = (x, \epsilon_{\mathbb{H}}) = x \in Irr(\mathbb{G})$.

Consequently, $\widehat{\mathbb{G}}$ is divisible in $\widehat{\mathbb{F}}$ because for all $s \in Irr(\mathbb{G})$ we have that $s \ominus (\epsilon, z) = (s, \epsilon_{\mathbb{H}}) \ominus (\epsilon_{\mathbb{G}}, z) = (s, z) \in Irr(\mathbb{F})$. Likewise, $\widehat{\mathbb{H}}$ is divisible in $\widehat{\mathbb{F}}$ because for all $s \in Irr(H)$ we have that $(x, \epsilon_{\mathbb{H}}) \ominus s = (x, \epsilon_{\mathbb{H}}) \ominus (\epsilon_{\mathbb{G}}, s) = (x, s) \in Irr(\mathbb{F})$.

2.3 Quantum semi-direct product

We introduce the semi-direct product of a compact quantum group by a discrete group and analyze some structure properties of this compact quantum group which are useful for our purpose. We may recall the crossed product constructions from Section 1.5.1 and Section 1.5.

In order to do so, the discrete group acts on the compact quantum group by *quantum* automorphisms. In this way, it is licit to form a natural crossed product that defines the C^*-algebra of the resulting compact quantum group. This construction is due to S. Wang [216]. For a proof of Theorem 2.3.2 below we refer to [216] and [65].

2.3.1 Definition. Let $\mathbb{G}$ be a compact quantum group and Γ be a discrete group. We say that Γ acts on $\mathbb{G}$ by quantum automorphisms if there exists a group homomorphism $\alpha : \Gamma \longrightarrow Aut(\mathbb{G})$.

In that case we say that α is a (quantum) action of Γ on $\mathbb{G}$.

2.3.2 Theorem-Definition. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and Γ be a discrete group acting on $\mathbb{G}$ by quantum automorphisms with action α.*

There exists a unique unital $*$-homomorphism

$$\Theta : \Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G}) \longrightarrow \Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G}) \otimes \Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G})$$

such that

$$\Theta(\pi(a)) = (\pi \otimes \pi)\big(\Delta_{\mathbb{G}}(a)\big) \text{ and } \Theta(u_\gamma) = u_\gamma \otimes u_\gamma,$$

for all $a \in C_m(\mathbb{G})$ and all $\gamma \in \Gamma$, where $\pi : C_m(\mathbb{G}) \longrightarrow \Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G})$ is the unital faithful $*$-homomorphism and $u : \Gamma \longrightarrow \mathcal{U}(M(\Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G})))$ the group homomorphism defining the crossed product.

Besides, we have that

i) $\mathbb{F} := (\Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G}), \Theta)$ *is a compact quantum group,*

ii) the Haar state on $\mathbb{F}$ is given by $h_{\mathbb{F}} := h_{\mathbb{G}} \circ E \circ \kappa$, where $h_{\mathbb{G}}$ is the Haar state of $\mathbb{G}$, $\kappa : \Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G}) \twoheadrightarrow \Gamma \underset{\alpha,r}{\ltimes} C_r(\mathbb{G})$ is the canonical surjection and $E : \Gamma \underset{\alpha,r}{\ltimes} C_r(\mathbb{G}) \to C_r(\mathbb{G})$ is the canonical conditional expectation,

iii) the maximal picture of $\mathbb{F}$ is given by $C_m(\mathbb{F}) = \Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G})$,

iv) the reduced picture of $\mathbb{F}$ is given by $C_r(\mathbb{F}) = \Gamma \underset{\alpha,r}{\ltimes} C_r(\mathbb{G})$,

v) the irreducible representations of $\mathbb{F}$ are described as follows: for every irreducible representation $y \in Irr(\mathbb{F})$, take a representative $w^y \in \mathcal{B}(H_y) \otimes C(\mathbb{F})$. There exist unique $\gamma \in \Gamma$ and $x \in Irr(\mathbb{G})$ such that if $w^\gamma \in \mathbb{C} \otimes C_r^*(\Gamma)$ and $w^x \in \mathcal{B}(H_x) \otimes C(\mathbb{G})$ are respective representatives of γ and x, then we have

$$w^y \cong v^\gamma \bigoplus v^x \in \mathcal{B}(\mathbb{C} \otimes H_x) \otimes C(\mathbb{F}),$$

where $v^\gamma := (id \otimes u)(w^\gamma) \in \mathbb{C} \otimes C(\mathbb{F})$ and $v^x := (id \otimes \pi)(w^x) \in \mathcal{B}(H_x) \otimes C(\mathbb{F})$. In this case we write $w^y := w^{(\gamma,x)}$.

In other words, the tensor product of irreducible representations of Γ by irreducible representations of $\mathbb{G}$ provide a complete set of mutually inequivalent irreducible representations of $\mathbb{F}$. For this reason we write $Irr(\mathbb{F}) = \Gamma \bigoplus Irr(\mathbb{G})$.

The compact quantum group $\mathbb{F}$ constructed in this way is called quantum semi-direct product of $\mathbb{G}$ by Γ *and is denoted by $\Gamma \underset{\alpha}{\ltimes} \mathbb{G}$.*

2.3.3 Remark. The representation theory of a quantum semi-direct product $\mathbb{F}$ described in the theorem above allows to give some explicit expressions which are useful for subsequent computations. For instance, it is advisable to give an explicit description of $\widehat{\mathbb{F}}$ in terms of Theorem 1.3.1.36.

- First of all, since α is an action of Γ on $\mathbb{G}$ by quantum automorphisms, then for every $\gamma \in \Gamma$, we have that $(id \otimes \alpha_\gamma)(w^x)$ is an irreducible unitary finite dimensional representation of $\mathbb{G}$ on H_x whenever $x \in Irr(\mathbb{G})$ (recall Proposition 1.3.1.28). Hence there exists a unique

class $\alpha_\gamma(x) \in Irr(\mathbb{G})$ such that $(id \otimes \alpha_\gamma)(w^x) \cong w^{\alpha_\gamma(x)}$. Since $dim(\alpha_\gamma(x)) = dim(x)$ we can assume that $w^{\alpha_\gamma(x)} \in \mathcal{B}(H_x) \otimes C(\mathbb{G})$, for all $\gamma \in \Gamma$ (if this is not the case, we might change the representative of $\alpha_\gamma(x)$ by an appropriate one in the orbit of x).

Hence, there exists a unique, up to a multiplicative factor in S^1, unitary operator $V_{\gamma,x} \in \mathcal{U}(H_x)$ such that

$$(id \otimes \alpha_\gamma)(w^x) = (V_{\gamma,x} \otimes id) w^{\alpha_\gamma(x)} (V^*_{\gamma,x} \otimes id)$$

Notice that it is clear that $\alpha_e(x) = x$, for all $x \in Irr(\mathbb{G})$ and that $\alpha_\gamma(\epsilon) = \epsilon$, for all $\gamma \in \Gamma$. Therefore, we can choose the multiplicative factor defining $V_{\gamma,x}$ such that $V_{e,x} = id_{H_x}$, for all $x \in Irr(\mathbb{G})$ and $V_{\gamma,\epsilon} = 1_{\mathbb{C}}$, for all $\gamma \in \Gamma$. We keep this choice for the sequel.

- Let $\gamma, \gamma' \in \Gamma$ and $x, x' \in Irr(\mathbb{G})$ be irreducible representations of Γ and $\mathbb{G}$ and consider the corresponding irreducible representations of $\mathbb{F}$, say $y := (\gamma, x), y' := (\gamma', x') \in Irr(\mathbb{F})$. Thanks to the theorem above we know that

$$w^y = v^\gamma \oplus v^x \text{ and } w^{y'} = v^{\gamma'} \oplus v^{x'},$$

where $v^\gamma := (id \otimes u)(w^\gamma), v^{\gamma'} := (id \otimes u)(w^{\gamma'}) \in \mathbb{C} \otimes C(\mathbb{F})$ and $v^x := (id \otimes \pi)(w^x), v^{x'} := (id \otimes \pi)(w^{x'}) \in \mathcal{B}(H_x) \otimes C(\mathbb{F})$.

A straightforward computation yields the following

$$w^{y \oplus y'} := w^y \oplus w^{y'} = v^{\gamma\gamma'} \oplus \big((V_{\gamma'^{-1}} \otimes id) v^{\alpha_{\gamma'^{-1}}(x)} (V^*_{\gamma'^{-1}} \otimes id) \oplus v^{x'}\big),$$

where $v^{\alpha_{\gamma'^{-1}}(x)} := (id \otimes \pi \circ \alpha_{\gamma'^{-1}})(w^x) \in \mathcal{B}(H_x) \otimes C(\mathbb{F})$. Indeed, let $\{\xi^x_1, \ldots, \xi^x_{n_x}\}$ be an orthonormal basis for H_x and $\{m_{i,j}\}_{i,j=1,\ldots,n_x}$ the corresponding matrix units in $\mathcal{B}(H_x)$. Likewise, let $\{\xi^{x'}_1, \ldots, \xi^{x'}_{n_{x'}}\}$ be an orthonormal basis for $H_{x'}$ and $\{m'_{k,l}\}_{k,l=1,\ldots,n_{x'}}$ the corresponding matrix units in $\mathcal{B}(H_{x'})$. Using the relation between the homomorphisms $\pi : C_m(\mathbb{G}) \longrightarrow \Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G})$ and $u : \Gamma \longrightarrow \mathcal{U}(M(\Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G})))$ defining the crossed product $C(\mathbb{F}) = \Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G})$ we can write the following

$$\begin{aligned}
w^y \oplus w^{y'} &= \big[v^\gamma \oplus v^x\big]_{13} \big[v^{\gamma'} \oplus v^{x'}\big]_{23} \\
&= \Big[\sum_{i,j=1}^{n_x} 1_{\mathbb{C}} \otimes m_{i,j} \otimes u_\gamma \pi(w^x_{ij})\Big]_{13} \Big[\sum_{k,l=1}^{n_{x'}} 1_{\mathbb{C}} \otimes m'_{k,l} \otimes u_{\gamma'} \pi(w^{x'}_{kl})\Big]_{23} \\
&= \sum_{i,j,k,l} 1_{\mathbb{C}} \otimes m_{i,j} \otimes 1_{\mathbb{C}} \otimes m'_{k,l} \otimes u_\gamma \pi(w^x_{ij}) u_{\gamma'} \pi(w^{x'}_{kl}) \\
&= \sum_{i,j,k,l} 1_{\mathbb{C}} \otimes m_{i,j} \otimes 1_{\mathbb{C}} \otimes m'_{k,l} \otimes u_\gamma u_{\gamma'} \pi\big(\alpha_{\gamma'^{-1}}(w^x_{ij})\big) \pi(w^{x'}_{kl}) \\
&= \sum_{i,j,k,l} 1_{\mathbb{C}} \otimes 1_{\mathbb{C}} \otimes m_{i,j} \otimes m'_{k,l} \otimes u_\gamma u_{\gamma'} \pi\big(\alpha_{\gamma'^{-1}}(w^x_{ij})\big) \pi(w^{x'}_{kl}) \\
&= \big[1_{\mathbb{C}} \otimes 1_{\mathbb{C}} \otimes u_\gamma u_{\gamma'}\big]_{13} \Big[\sum_{i,j,k,l} m_{ij} \otimes m'_{kl} \otimes \pi\big(\alpha_{\gamma'^{-1}}(w^x_{ij})\big) \pi(w^{x'}_{kl})\Big]_{23} \\
&= \big[v^\gamma \oplus v^{\gamma'}\big]_{13} \big[(V_{\gamma'^{-1}} \otimes id) v^{\alpha_{\gamma'^{-1}}(x)} (V^*_{\gamma'^{-1}} \otimes id) \oplus v^{x'}\big]_{23} \\
&= \big(v^\gamma \oplus v^{\gamma'}\big) \oplus \big((V_{\gamma'^{-1}} \otimes id) v^{\alpha_{\gamma'^{-1}}(x)} (V^*_{\gamma'^{-1}} \otimes id) \oplus v^{x'}\big) \\
&= v^{\gamma\gamma'} \oplus \big((V_{\gamma'^{-1}} \otimes id) v^{\alpha_{\gamma'^{-1}}(x)} (V^*_{\gamma'^{-1}} \otimes id) \oplus v^{x'}\big),
\end{aligned}$$

which is the formula announced above. Consequently, the decomposition of $y \oplus y'$ into direct sum of irreducible representations depends only on the corresponding decomposition of $\alpha_{\gamma'^{-1}}(x) \oplus x'$. More precisely, if $\{x_k\}_{k=1,\dots,r}$ is such a decomposition for $\alpha_{\gamma'^{-1}}(x) \oplus x'$, then the formula above implies that the corresponding decomposition for $y \oplus y'$ is given by $\{(\gamma\gamma', x_k)\}_{k=1,\dots,r}$.

2.3.4 Lemma. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and Γ be a discrete group acting on $\mathbb{G}$ by quantum automorphisms with action α. Let $\mathbb{F} := \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$ be the corresponding quantum semi-direct product.*

i) For all $\gamma, g, h \in \Gamma$ and all $x, y, z \in Irr(\mathbb{G})$, we have

$$h_{\mathbb{F}}\Big(\chi_{\mathbb{F}}(\gamma, x)^* \chi_{\mathbb{F}}\big((g, y) \oplus (h, z)\big)\Big) = \begin{cases} h_{\mathbb{G}}\Big(\chi_{\mathbb{G}}(x)^* \chi_{\mathbb{G}}\big(\alpha_{h^{-1}}(y) \oplus z\big)\Big), \text{ if } \gamma = gh \\ 0, \text{ otherwise} \end{cases}$$

ii) For all $\gamma, g, h \in \Gamma$ and all $x, y, z \in Irr(\mathbb{G})$, we have

$$Mor\Big((\gamma, x), (g, y) \oplus (h, z)\Big) \cong \begin{cases} Mor\Big(x, \alpha_{h^{-1}}(y) \oplus z\Big), \text{ if } \gamma = gh \\ 0, \text{ otherwise} \end{cases}$$

iii) The dual discrete quantum group $\widehat{\mathbb{F}} = (c_0(\widehat{\mathbb{F}}), \widehat{\Theta})$ is given precisely by

$$c_0(\widehat{\mathbb{F}}) \cong c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})$$

and $\widehat{\Theta} : c_0(\widehat{\mathbb{F}}) \longrightarrow M(c_0(\widehat{\mathbb{F}}) \otimes c_0(\widehat{\mathbb{F}}))$ such that

$$\widehat{\Theta}(\delta_\gamma \otimes a)\big(p_{(g,y)} \otimes p_{(h,z)}\big) = \delta_{\gamma,gh}\ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big((V_{h^{-1},y} \otimes p_z)\widehat{\Delta}(a)(p_y \otimes p_z)(V^*_{h^{-1},y} \otimes p_z)\Big)_{24},$$

for all $\gamma, g, h \in \Gamma$, all $a \in c_0(\widehat{\mathbb{G}})$ and all $y, z \in Irr(\mathbb{G})$.

Proof. i) The elementary properties of the character together with the definition of the Haar state of $\mathbb{F}$ given by Theorem 2.3.2 allow to write the desired formula,

$$\begin{aligned} h_{\mathbb{F}}\Big(\chi_{\mathbb{F}}(\gamma, x)^* \chi_{\mathbb{F}}\big((g, y) \oplus (h, z)\big)\Big) &= h_{\mathbb{F}}\Big(\chi_{\mathbb{F}}(v^\gamma \oplus v^x)^* \chi_{\mathbb{F}}\big(v^{gh} \oplus \big(v^{\alpha_{h^{-1}}(y)} \oplus v^z\big)\big)\Big) \\ &= h_{\mathbb{F}}\Big(u_{\gamma^{-1}} \chi_{\mathbb{F}}(v^x)^* u_{gh} \chi_{\mathbb{F}}\big(v^{\alpha_{h^{-1}}(y)} \oplus v^z\big)\Big) \\ &= h_{\mathbb{F}}\Big(u_{\gamma^{-1}gh} \chi_{\mathbb{F}}(v^x)^* \chi_{\mathbb{F}}\big(v^{\alpha_{h^{-1}}(y)} \oplus v^z\big)\Big) \\ &= \delta_{\gamma^{-1}gh,e}\ h_{\mathbb{G}}\Big(\chi_{\mathbb{G}}(w^x)^* \chi_{\mathbb{G}}\big(w^{\alpha_{h^{-1}}(y)} \oplus w^z\big)\Big) \\ &= \delta_{\gamma^{-1}gh,e}\ h_{\mathbb{G}}\Big(\chi_{\mathbb{G}}(x)^* \chi_{\mathbb{G}}\big(\alpha_{h^{-1}}(y) \oplus z\big)\Big), \end{aligned}$$

for all $\gamma, g, h \in \Gamma$ and all $x, y, z \in Irr(\mathbb{G})$ where $e \in \Gamma$ denotes the identity element of Γ.

ii) On the one hand, given $\gamma, g, h \in \Gamma$ and all $x, y, z \in Irr(\mathbb{G})$, the computation of $Mor\Big((\gamma, x), (g, y) \oplus (h, z)\Big)$ reduces to case when $\gamma = gh$ thanks to the formula of (i). On the other hand, by Remark 2.3.3 we know that, for all $\gamma \in \Gamma$ and all $x \in Irr(\mathbb{G})$, there exists a unique (up to a multiplicative factor) unitary operator $V_{\gamma,x} \in \mathcal{U}(H_x)$ such that $(id \otimes \alpha_\gamma)(w^x) = (V_{\gamma,x} \otimes id_{C(\mathbb{G})}) w^{\alpha_\gamma(x)} (V^*_{\gamma,x} \otimes id_{C(\mathbb{G})})$.
Hence for all $\gamma, g, h \in \Gamma$ and all $x, y, z \in Irr(\mathbb{G})$ such that $\gamma = gh$, we define

$$\begin{array}{rccl} \psi: & Mor\Big(x, \alpha_{h^{-1}}(y) \oplus z\Big) & \longrightarrow & Mor\Big((\gamma, x), (g, y) \oplus (h, z)\Big) \\ & \Phi & \longmapsto & \psi(\Phi) := (V_{h^{-1},y} \otimes id_{H_z}) \circ \Phi, \end{array}$$

which is a linear isomorphism with inverse $\psi^{-1}(\widetilde{\Phi}) = (V^*_{h^{-1},y} \otimes id_{H_z}) \circ \widetilde{\Phi}$, for all $\widetilde{\Phi} \in Mor\Big((\gamma, x), (g, y) \oplus (h, z)\Big)$. Let us check that ψ is well-defined. Namely, we have to show that $(\psi(\Phi) \otimes id_{C(\mathbb{F})}) w^{(\gamma,x)} = w^{(g,y)} \oplus w^{(h,z)} (\psi(\Phi) \otimes id_{C(\mathbb{F})})$,

$$\begin{aligned} & w^{(g,y)} \oplus w^{(h,z)} (\psi(\Phi) \otimes id_{C(\mathbb{F})}) \\ &= v^{gh} \oplus \Big((V_{h^{-1},y} \otimes id_{C(\mathbb{F})}) v^{\alpha_{h^{-1}}(y)} (V^*_{h^{-1},y} \otimes id_{C(\mathbb{F})}) \oplus v^z \Big) (\psi(\Phi) \otimes id_{C(\mathbb{F})}) \\ &= v^{gh} \oplus \Big((V_{h^{-1},y} \otimes id_{H_z} \otimes id_{C(\mathbb{F})}) v^{\alpha_{h^{-1}}(y)} \oplus v^z (V^*_{h^{-1},y} \otimes id_{H_z} \otimes id_{C(\mathbb{F})}) \Big) (\psi(\Phi) \otimes id_{C(\mathbb{F})}) \\ &= v^{\gamma} \oplus \Big((V_{h^{-1},y} \otimes id_{H_z} \otimes id_{C(\mathbb{F})}) v^{\alpha_{h^{-1}}(y)} \oplus v^z (\Phi \otimes id_{C(\mathbb{F})}) \Big) \\ &= v^{\gamma} \oplus \Big((V_{h^{-1},y} \otimes id_{H_z} \otimes id_{C(\mathbb{F})}) (\Phi \otimes id_{C(\mathbb{F})}) v^x \Big) \\ &= \Big((V_{h^{-1},y} \otimes id_{H_z} \otimes id_{C(\mathbb{F})}) (\Phi \otimes id_{C(\mathbb{F})}) v^{\gamma} \oplus v^x \Big) = (\psi(\Phi) \otimes id_{C(\mathbb{F})}) w^{(\gamma,x)}, \end{aligned}$$

where we have used the computations of Remark 2.3.3.

iii) The description of the C^*-algebra of the quantum discrete dual of $\mathbb{F}$ is easy to establish. Indeed,

$$\begin{aligned} c_0(\widehat{\mathbb{F}}) &= \bigoplus_{y \in Irr(\mathbb{F})}^{c_0} \mathcal{B}(H_y) = \bigoplus_{(\gamma,x) \in \Gamma \times Irr(\mathbb{G})}^{c_0} \mathcal{B}(\mathbb{C} \otimes H_x) \\ &\cong \Big(\bigoplus_{\gamma \in \Gamma}^{c_0} \mathbb{C} \Big) \otimes \Big(\bigoplus_{x \in Irr(\mathbb{G})}^{c_0} \mathcal{B}(H_x) \Big) = c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}}) \end{aligned}$$

Let us compute the dual co-multiplication $\widehat{\Theta}$ using the identification $c_0(\widehat{\mathbb{F}}) \cong c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})$ above. Following Theorem 1.3.1.36, $\widehat{\Theta}$ is completely determined by the relation

$$\widehat{\Theta}(S) \circ \widetilde{\Phi} = \widetilde{\Phi} \circ S,$$

for all $S \in \mathcal{B}(H_{(\gamma,x)}) \cong \mathcal{B}(H_x)$ with $(\gamma, x) \in Irr(\mathbb{F})$, $\widetilde{\Phi} \in Mor\Big((\gamma, x), (g, y) \oplus (h, z)\Big)$ and all $(g, y), (h, z) \in Irr(\mathbb{F})$. Thanks to the isomorphism ψ given in (ii) we write

$$\begin{aligned} \widehat{\Theta}(S) \circ \widetilde{\Phi} = \widetilde{\Phi} \circ S &= \delta_{\gamma,gh} \ (V_{h^{-1},y} \otimes id_{H_z}) \psi^{-1}(\widetilde{\Phi}) \circ S \\ &= \delta_{\gamma,gh} \ (V_{h^{-1},y} \otimes id_{H_z}) \widehat{\Delta}(S)(p_y \otimes p_z) \circ \psi^{-1}(\widetilde{\Phi}) \\ &= \delta_{\gamma,gh} \ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z) \Big((V_{h^{-1},y} \otimes id_{H_z}) \widehat{\Delta}(S)(p_y \otimes p_z)(V^*_{h^{-1},y} \otimes id_{H_z}) \Big)_{24} \circ \widetilde{\Phi} \end{aligned}$$

Next, for every $(\gamma, x) \subset (g,y) \bigoplus (h,z)$ denote by $p^{(g,y)\bigoplus(h,z)}_{(\gamma,x)} \in \mathcal{B}(H_{(g,y)\bigoplus(h,z)})$ the corresponding orthogonal finite dimensional projection on the direct sum of subrepresentations of $(g,y)\bigoplus(h,z)$ which are isomorphic to (γ, x). This projection is such that $\sum\limits_{(\gamma,x)\subset(g,y)\bigoplus(h,z)} p^{(g,y)\bigoplus(h,z)}_{(\gamma,x)} = id_{H_{(g,y)\bigoplus(h,z)}}$ and for every $k = 1, \ldots, dim\Big(Mor((\gamma,x),(g,y)\bigoplus(h,z))\Big)$ consider a family of intertwiners $\widetilde{\Phi}_k \in Mor((\gamma,x),(g,y)\bigoplus(h,z))$ such that $\widetilde{\Phi}^*_k\widetilde{\Phi}_k = id_{H_{(\gamma,x)}}$ and $\sum\limits_k \widetilde{\Phi}_k\widetilde{\Phi}^*_k = p^{(g,y)\bigoplus(h,z)}_{(\gamma,x)}$. Hence, for all $a \in c_0(\widehat{\mathbb{F}})$ we write the following,

$$\begin{aligned}
\widehat{\Theta}(a) \circ p^{(g,y)\bigoplus(h,z)}_{(\gamma,x)} &= \sum_k \widehat{\Theta}(a) \circ \widetilde{\Phi}_k\widetilde{\Phi}^*_k \\
&= \sum_k \delta_{\gamma,gh}\ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big((V_{h^{-1},y} \otimes id_{H_z})\widehat{\Delta}(a_\gamma)(p_y \otimes p_z)(V^*_{h^{-1},y} \otimes id_{H_z})\Big)_{24} \circ \widetilde{\Phi}_k\widetilde{\Phi}^*_k \\
&= \delta_{\gamma,gh}\ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big(V_{h^{-1},y} \otimes id_{H_z})\widehat{\Delta}(a_\gamma)(p_y \otimes p_z)(V^*_{h^{-1},y} \otimes id_{H_z})\Big)_{24} \circ p^{(g,y)\bigoplus(h,z)}_{(\gamma,x)},
\end{aligned}$$

where $a_\gamma := (\delta_\gamma \otimes id_{c_0(\widehat{\mathbb{G}})})a \in c_0(\widehat{\mathbb{G}})$, where δ_γ denotes the minimal central projection of $c_0(\Gamma)$ on $\mathcal{B}(H_\gamma)$. Next, for all $\gamma \in \Gamma$, $a \in c_0(\widehat{\mathbb{G}})$ and all $(g,y),(h,z) \in Irr(\mathbb{F})$ we write the following

$$\begin{aligned}
\widehat{\Theta}(\delta_\gamma \otimes a)\big(p_{(g,y)} \otimes p_{(h,z)}\big) &= \sum_{(\gamma,x)\subset(g,y)\bigoplus(h,z)} \widehat{\Theta}(\delta_\gamma \otimes a) \circ p^{(g,y)\bigoplus(h,z)}_{(\gamma,x)} \\
= \sum_{(\gamma,x)\subset(g,y)\bigoplus(h,z)} &\delta_{\gamma,gh}\ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big((V_{h^{-1},y} \otimes id_{H_z})\widehat{\Delta}(a)(p_y \otimes p_z)(V^*_{h^{-1},y} \otimes id_{H_z})\Big)_{24} \circ p^{(g,y)\bigoplus(h,z)}_{(\gamma,x)} \\
= \delta_{\gamma,gh}\ &(\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big((V_{h^{-1},y} \otimes id_{H_z})\widehat{\Delta}(a)(p_y \otimes p_z)(V^*_{h^{-1},y} \otimes id_{H_z})\Big)_{24}
\end{aligned}$$

which is the formula of the statement.

■

From now on, $\mathbb{G} = (C(\mathbb{G}), \Delta)$ denotes a compact quantum group, Γ denotes a discrete group acting on $\mathbb{G}$ by quantum automorphisms with action α and $\mathbb{F} := \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$ denotes the corresponding quantum semi-direct product as in the theorem above. Remark as well that Γ can be regarded as a *discrete quantum group* and we write $\Gamma = (c_0(\Gamma), \widehat{\Delta}_\Gamma)$.

2.3.5 Proposition. *There exist non-degenerate $*$-homomorphisms*

$$\rho_{\widehat{\mathbb{G}}} : c_0(\widehat{\mathbb{F}}) \longrightarrow M(c_0(\widehat{\mathbb{G}})) \text{ and } \rho_\Gamma : c_0(\widehat{\mathbb{F}}) \longrightarrow M(c_0(\Gamma))$$

such that $\rho_{\widehat{\mathbb{G}}}(c_0(\widehat{\mathbb{F}})) = c_0(\widehat{\mathbb{G}})$, $\rho_\Gamma(c_0(\widehat{\mathbb{F}})) = c_0(\Gamma)$ and

$$\widehat{\Delta} \circ \rho_{\widehat{\mathbb{G}}} = (\rho_{\widehat{\mathbb{G}}} \otimes \rho_{\widehat{\mathbb{G}}})\widehat{\Theta} \text{ and } \widehat{\Delta}_\Gamma \circ \rho_\Gamma = (\rho_\Gamma \otimes \rho_\Gamma)\widehat{\Theta}$$

In other words, $\widehat{\mathbb{G}}$ and Γ are discrete quantum subgroups of $\widehat{\mathbb{F}}$. As a result, if (A,δ) is any $\widehat{\mathbb{F}}$-C^*-algebra, then $(A,\delta_{\widehat{\mathbb{G}}})$ is a $\widehat{\mathbb{G}}$-C^*-algebra and (A,δ_Γ) is a Γ-C^*-algebra with actions

$$\delta_{\widehat{\mathbb{G}}} := (\rho_{\widehat{\mathbb{G}}} \otimes id_A) \circ \delta \text{ and } \delta_\Gamma := (\rho_\Gamma \otimes id_A) \circ \delta$$

Proof. In order to define the $*$-homomorphisms $\rho_{\widehat{\mathbb{G}}}$ and ρ_Γ of the statement, consider the co-unit homomorphisms of Γ and $\widehat{\mathbb{G}}$. Namely,

$$\begin{array}{rccc} \varepsilon_\Gamma : & c_0(\Gamma) & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & \varepsilon_\Gamma(f) := f(e) \end{array} \qquad \begin{array}{rccc} \varepsilon_{\widehat{\mathbb{G}}} : & c_0(\widehat{\mathbb{G}}) & \longrightarrow & \mathbb{C} \\ & a & \longmapsto & \varepsilon_{\widehat{\mathbb{G}}}(a) := p_\epsilon \; a \; p_\epsilon, \end{array}$$

where $e \in \Gamma$ is the identity element of Γ, $\epsilon \in Irr(\mathbb{G})$ is the trivial representation of $\mathbb{G}$ and p_ϵ the corresponding central projection from $c_0(\widehat{\mathbb{G}})$ into $H_\epsilon = \mathbb{C}$ (recall as well Remark 1.3.1.38).

Taking into account the decomposition $c_0(\widehat{\mathbb{F}}) = c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})$, we put

$$\rho_{\widehat{\mathbb{G}}} := \varepsilon_\Gamma \otimes id_{c_0(\widehat{\mathbb{G}})} \text{ and } \rho_\Gamma := id_{c_0(\Gamma)} \otimes \varepsilon_{\widehat{\mathbb{G}}}$$

It is clear that the above $\rho_{\widehat{\mathbb{G}}}$ and ρ_Γ are surjective $*$-homomorphisms on $c_0(\widehat{\mathbb{G}})$ and $c_0(\Gamma)$ because they are projections on the corresponding component of $c_0(\widehat{\mathbb{F}}) = c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})$. Therefore, $\rho_{\widehat{\mathbb{G}}} : c_0(\widehat{\mathbb{F}}) \longrightarrow M(c_0(\widehat{\mathbb{G}}))$ and $\rho_\Gamma : c_0(\widehat{\mathbb{F}}) \longrightarrow M(c_0(\Gamma))$ are non-degenerate $*$-homomorphisms.

Moreover, using the explicit description of $\widehat{\Theta}$ given by Lemma 2.3.4, it is straightforward to check that $\rho_{\widehat{\mathbb{G}}}$ and ρ_Γ intertwine the corresponding co-multiplications. Namely, for all $\gamma \in \Gamma$, all $a \in c_0(\widehat{\mathbb{G}})$ and all $(g,y),(h,z) \in Irr(\mathbb{F})$ we write

$$\begin{aligned}
&(\rho_{\widehat{\mathbb{G}}} \otimes \rho_{\widehat{\mathbb{G}}})\widehat{\Theta}(\delta_\gamma \otimes a)\big(p_{(g,y)} \otimes p_{(h,z)}\big) \\
&\quad = (\rho_{\widehat{\mathbb{G}}} \otimes \rho_{\widehat{\mathbb{G}}})\Big(\delta_{\gamma,gh}\ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big((V_{h^{-1},y} \otimes id_{H_z})\widehat{\Delta}(a)(p_y \otimes p_z)(V^*_{h^{-1},y} \otimes id_{H_z})\Big)_{24}\Big) \\
&\quad = (\varepsilon_\Gamma \otimes id \otimes \varepsilon_\Gamma \otimes id)\Big(\delta_{\gamma,gh}\ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big((V_{h^{-1},y} \otimes id_{H_z})\widehat{\Delta}(a)(p_y \otimes p_z)(V^*_{h^{-1},y} \otimes id_{H_z})\Big)_{24}\Big) \\
&\quad = (V_{e,y} \otimes p_z)\widehat{\Delta}(a)(p_y \otimes p_z)(V^*_{e,y} \otimes p_z) = \widehat{\Delta}(a)(p_y \otimes p_z) \\
&\quad = \widehat{\Delta}(\varepsilon_\Gamma \otimes id)(\delta_\gamma \otimes a)(p_y \otimes p_z) = \widehat{\Delta}\big(\rho_{\widehat{\mathbb{G}}}(\delta_\gamma \otimes a)(p_{(g,y)} \otimes p_{(h,z)})\big),
\end{aligned}$$

and

$$\begin{aligned}
&(\rho_\Gamma \otimes \rho_\Gamma)\widehat{\Theta}(\delta_\gamma \otimes a)\big(p_{(g,y)} \otimes p_{(h,z)}\big) \\
&\quad = (\rho_\Gamma \otimes \rho_\Gamma)\Big(\delta_{\gamma,gh}\ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big((V_{h^{-1},y} \otimes id_{H_z})\widehat{\Delta}(a)(p_y \otimes p_z)(V^*_{h^{-1},y} \otimes id_{H_z})\Big)_{24}\Big) \\
&\quad = (id \otimes \varepsilon_{\widehat{\mathbb{G}}} \otimes id \otimes \varepsilon_{\widehat{\mathbb{G}}})\Big(\delta_{\gamma,gh}\ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big((V_{h^{-1},y} \otimes id_{H_z})\widehat{\Delta}(a)(p_y \otimes p_z)(V^*_{h^{-1},y} \otimes id_{H_z})\Big)_{24}\Big) \\
&\quad = \delta_{\gamma,gh}\ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big((V_{h^{-1},\epsilon} \otimes p_\epsilon)\widehat{\Delta}(a)(p_\epsilon \otimes p_\epsilon)(V^*_{h^{-1},\epsilon} \otimes p_\epsilon)\Big)_{24} \\
&\quad = \delta_{\gamma,gh}\ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big(\widehat{\Delta}(a)(p_\epsilon \otimes p_\epsilon)\Big)_{24} = \delta_{\gamma,gh}\ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big((\varepsilon_{\widehat{\mathbb{G}}} \otimes \varepsilon_{\widehat{\mathbb{G}}})\widehat{\Delta}(a)\Big)_{24} \\
&\quad = \delta_{\gamma,gh}\ (\delta_g \otimes p_y \otimes \delta_h \otimes p_z)\Big((id \otimes \varepsilon_{\widehat{\mathbb{G}}})(a)\Big)_4 = \widehat{\Delta}_\Gamma(id \otimes \varepsilon_{\widehat{\mathbb{G}}})(\delta_\gamma \otimes a)\big(p_{(g,y)} \otimes p_{(h,z)}\big) \\
&\quad = \widehat{\Delta}_\Gamma\big(\rho_\Gamma(\delta_\gamma \otimes a)(p_{(g,y)} \otimes p_{(h,z)}\big)
\end{aligned}$$

■

2.3.6 Remark. Since Γ is a classical group, a Γ-C^*-algebra is equivalent to a C^*-algebra equipped with a co-action of $c_0(\Gamma)$ as explained in the introduction of Section 1.4. This correspondence

explains the abuse of language used in the preceding proposition. Indeed, the non-degenerate $*$-homomorphism $\delta_\Gamma := (\rho_\Gamma \otimes id_A) \circ \delta : A \longrightarrow M(c_0(\Gamma) \otimes A)$ defined above is a *co-action* of $(c_0(\Gamma), \widehat{\Delta}_\Gamma)$. Thanks to the characterization of Proposition 1.4.1.5, the latter is equivalent to give a family of $*$-homomorphisms $\delta_\Gamma^\gamma : A \longrightarrow A$, for all $\gamma \in \Gamma$ satisfying $\delta_\Gamma^e = id_A$ and $\delta_\Gamma^{\gamma\gamma'} = \delta_\Gamma^{\gamma'} \circ \delta_\Gamma^\gamma$ for all $\gamma, \gamma' \in \Gamma$, among other properties. Hence, the map

$$\begin{array}{rcl} \Gamma & \longrightarrow & Aut(A) \\ \gamma & \longmapsto & \big(\delta_\Gamma\big)_\gamma,\ \big(\delta_\Gamma\big)_\gamma(a) := \delta_\Gamma^{\gamma^{-1}}(a) \end{array}$$

defines an action of Γ on A. By abuse of notation, we denote this action by δ_Γ and the difference between the action and the co-action will be clear by the context.

2.3.7 Corollary. *The following properties hold*

i) The C^*-algebra $c_0(\widehat{\mathbb{F}}) = c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})$ is a $\widehat{\mathbb{G}}$-C^*-algebra with action $\widehat{\Theta}_{\widehat{\mathbb{G}}} := (\rho_{\widehat{\mathbb{G}}} \otimes id_{c_0(\widehat{\mathbb{F}})}) \circ \widehat{\Theta}$ such that

$$\widehat{\Theta}_{\widehat{\mathbb{G}}}(\delta_\gamma \otimes a)\big(p_y \otimes \delta_h \otimes p_z\big) = \delta_{\gamma,h}\ (p_y \otimes \delta_h \otimes p_z)\Big((V_{h^{-1},y} \otimes p_z)\widehat{\Delta}(a)(p_y \otimes p_z)(V^*_{h^{-1},y} \otimes p_z)\Big)_{13},$$

for all $\gamma, h \in \Gamma$, all $a \in c_0(\widehat{\mathbb{G}})$ and all $y, z \in Irr(\mathbb{G})$.

ii) The C^*-algebra $c_0(\widehat{\mathbb{F}}) = c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})$ is a Γ-C^*-algebra with action $\widehat{\Theta}_\Gamma := (\rho_\Gamma \otimes id_{c_0(\widehat{\mathbb{F}})}) \circ \widehat{\Theta}$ such that

$$\widehat{\Theta}_\Gamma(\delta_\gamma \otimes a)\big(\delta_g \otimes \delta_h \otimes p_z\big) = \delta_{\gamma,gh}\ (\delta_g \otimes \delta_h \otimes p_z)(id \otimes id \otimes a),$$

for all $\gamma, g, h \in \Gamma$, all $a \in c_0(\widehat{\mathbb{G}})$ and all $z \in Irr(\mathbb{G})$.

iii) If $\eta : c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}}) \longrightarrow \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}}))$ denotes the action of $\widehat{\mathbb{G}}$ on $c_0(\widehat{\mathbb{F}}) = c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})$ given by the composition $(\Sigma_{12} \otimes id_{c_0(\widehat{\mathbb{G}})}) \circ (id_{c_0(\Gamma)} \otimes \widehat{\Delta})$, then

$$\widehat{\Theta}_{\widehat{\mathbb{G}}}(\cdot) = (\mathscr{U} \otimes id_{c_0(\widehat{\mathbb{G}})})\eta(\cdot)(\mathscr{U}^* \otimes id_{c_0(\widehat{\mathbb{G}})}),$$

where $\mathscr{U} \in \mathcal{U}\big(M(c_0(\widehat{\mathbb{G}}) \otimes c_0(\Gamma))\big)$ is the unitary such that $\mathscr{U}(p_x \otimes \delta_\gamma) = V_{\gamma^{-1},x} \otimes \delta_\gamma$, for every $x \in Irr(\mathbb{G})$ and every $\gamma \in \Gamma$.

Proof. For the formulas in (i) and (ii) we have just to apply Lemma 2.3.4 and Proposition 2.3.5.

Let us show that the action $\widehat{\Theta}_{\widehat{\mathbb{G}}}$ is the conjugation of the action η by the unitary $\mathscr{U} \in \mathcal{U}\big(M(c_0(\widehat{\mathbb{G}}) \otimes c_0(\Gamma))\big)$ defined in the statement. Namely, for all $\gamma, h \in \Gamma$, $a \in c_0(\widehat{\mathbb{G}})$ and all $y, z \in Irr(\mathbb{G})$ we write

$$\begin{aligned} &(\mathscr{U} \otimes id_{c_0(\widehat{\mathbb{G}})})\eta(\delta_\gamma \otimes a)\big(p_y \otimes \delta_h \otimes p_z\big)(\mathscr{U}^* \otimes id_{c_0(\widehat{\mathbb{G}})}) \\ &\qquad = (\mathscr{U} \otimes p_z))\big(p_y \otimes \delta_\gamma \otimes p_z)\big)\big(\widehat{\Delta}(a)\big)_{13}\big(p_y \otimes \delta_h \otimes p_z\big)(\mathscr{U}^* \otimes p_z) \\ &\qquad = \delta_{\gamma,h}\ (p_y \otimes \delta_h \otimes p_z)\Big((V_{h^{-1},y} \otimes p_z)\widehat{\Delta}(a)(p_y \otimes p_z)(V^*_{h^{-1},y} \otimes p_z))\Big)_{13} \\ &\qquad = \widehat{\Theta}_{\widehat{\mathbb{G}}}(\delta_\gamma \otimes a)\big(p_{(g,y)} \otimes p_{(h,z)}\big), \end{aligned}$$

which shows that $\widehat{\Theta}_{\widehat{\mathbb{G}}}(\cdot) = (\mathscr{U} \otimes id_{c_0(\widehat{\mathbb{G}})})\eta(\cdot)(\mathscr{U}^* \otimes id_{c_0(\widehat{\mathbb{G}})})$.

■

2.3.8 Remark. In accordance with Remark 2.3.6, let us give the expression of $\widehat{\Theta}_\Gamma$ as a true action of Γ on $c_0(\Gamma)\otimes c_0(\widehat{\mathbb{G}})$ and not as a co-action of $(c_0(\Gamma),\widehat{\Delta}_\Gamma)$ as done in the previous corollary. By applying the characterization of Proposition 1.4.1.5 and the formula obtained in the previous corollary, for every $\gamma, r\in\Gamma$ and $a\in c_0(\widehat{\mathbb{G}})$ we write

$$\begin{aligned}\widehat{\Theta}_\Gamma^\gamma(\delta_r\otimes a) &= \widehat{\Theta}_\Gamma(\delta_r\otimes a)(\delta_\gamma\otimes id_{c_0(\widehat{\mathbb{F}})})\\ &= \sum_{\substack{s,t\\ r=st}}(\delta_s\otimes\delta_t\otimes id_{c_0(\widehat{\mathbb{G}})})(id_{c_0(\Gamma)}\otimes id_{c_0(\Gamma)}\otimes a)(\delta_\gamma\otimes id_{c_0(\widehat{\mathbb{F}})}) = \delta_{\gamma^{-1}r}\otimes a\end{aligned}$$

Hence, the corresponding action of Γ on $c_0(\Gamma)\otimes c_0(\widehat{\mathbb{G}})$, still denoted by $\widehat{\Theta}_\Gamma$, is given by

$$\left(\widehat{\Theta}_\Gamma\right)_\gamma(\delta_r\otimes a) = \widehat{\Theta}_\Gamma^{\gamma^{-1}}(\delta_r\otimes a) = \delta_{\gamma r}\otimes a,$$

for all $\gamma, r\in\Gamma$ and $a\in c_0(\widehat{\mathbb{G}})$.

2.3.9 Remark. By applying Proposition 1.4.3.4 we know that Γ and $\widehat{\mathbb{G}}$ are discrete quantum subgroups of $\widehat{\mathbb{F}}$ if and only if there exist injections

$$\iota_\Gamma^r : C_r^*(\Gamma)\hookrightarrow C_r(\mathbb{F}) \text{ and } \iota_{\mathbb{G}}^r : C_r(\mathbb{G})\hookrightarrow C_r(\mathbb{F})$$

$$\left(\text{or } \iota_\Gamma^m : C_m^*(\Gamma)\hookrightarrow C_m(\mathbb{F}) \text{ and } \iota_{\mathbb{G}}^m : C_m(\mathbb{G})\hookrightarrow C_m(\mathbb{F})\right)$$

that intertwine the corresponding co-multiplications. We can give an explicit description of these injections for a quantum semi-direct product.

The co-unit map $\varepsilon_{\mathbb{G}} : Pol(\mathbb{G})\longrightarrow\mathbb{C}$ extends to a (α-invariant) character on $C_m(\mathbb{G})$, which we always denote by $\varepsilon_{\mathbb{G}} : C_m(\mathbb{G})\longrightarrow\mathbb{C}$.

Recall that $C_m(\mathbb{F}) = \Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G}) = C^*\langle\pi(a)u_\gamma : a\in C_m(\mathbb{G}), \gamma\in\Gamma\rangle$. So, with the help of the α-invariant character above, we can identify $C_m^*(\Gamma)$ with the subalgebra of $C_m(\mathbb{F})$ generated by $\{u_\gamma : \gamma\in\Gamma\}$ by universal property (see Remark 3.6 in [65] for more details).

Likewise, recall that $C_r(\mathbb{F}) = \Gamma \underset{\alpha,r}{\ltimes} C_r(\mathbb{G}) = C^*\langle\pi(a)u_\gamma : a\in C_r(\mathbb{G}), \gamma\in\Gamma\rangle$ is equipped with a GNS-faithful conditional expectation $E : \Gamma \underset{\alpha,r}{\ltimes} C_r(\mathbb{G})\longrightarrow C_r(\mathbb{G})$, which restricted to the subalgebra generated by $\{u_\gamma : \gamma\in\Gamma\}$ is just $E(u_\gamma) = \delta_{\gamma,e}\in\mathbb{C}$. Remember as well that $u_\gamma = \lambda_\gamma\otimes id_{C_r(\mathbb{G})}\cong\left[\lambda_\gamma\right]_1$ in $\Gamma \underset{\alpha,r}{\ltimes} C_r(\mathbb{G})\subset\mathcal{L}_{C_r(\mathbb{G})}(l^2(\Gamma)\otimes C_r(\mathbb{G}))$; so that this subalgebra is identified canonically to $C_r^*(\Gamma) = \Gamma \underset{trv,r}{\ltimes} \mathbb{C}$ by universal property (here trv denotes the trivial action).

In conclusion, we consider the following canonical injections

$$\iota_\Gamma^r : C_r^*(\Gamma)\hookrightarrow C_r(\mathbb{F}) \text{ and } \iota_\Gamma^m : C_m^*(\Gamma)\hookrightarrow C_m(\mathbb{F})$$

By definition of the co-multiplication Θ of the quantum semi-direct product $\mathbb{F}$, it is clear that the canonical injections ι_Γ^r and ι_Γ^m intertwine the corresponding co-multiplications. Observe that, by construction, we have the following commutative diagrams

$$\begin{tikzcd} C_m(\mathbb{F}) \arrow[r, two heads, "\tau_{\mathbb{F}}"] & C_r(\mathbb{F}) \\ C_m^*(\Gamma) \arrow[u, hook, "\iota_\Gamma^m"] \arrow[r, two heads, "\tau_\Gamma"'] & C_r^*(\Gamma) \arrow[u, hook, "\iota_\Gamma^r"'] \end{tikzcd} \qquad \begin{tikzcd} C_m(\mathbb{F}) \arrow[r, "\varepsilon_{\mathbb{F}}"] & \mathbb{C} \\ C_m^*(\Gamma) \arrow[u, hook, "\iota_\Gamma^m"] \arrow[ur, "\varepsilon_\Gamma"'] & \end{tikzcd}$$

where $\tau_{\mathbb{F}} : C_m(\mathbb{F}) \twoheadrightarrow C_r(\mathbb{F})$, $\tau_\Gamma : C^*_m(\Gamma) \twoheadrightarrow C^*_r(\Gamma)$ are the canonical surjections and $\varepsilon_{\mathbb{F}} : Pol(\mathbb{F}) \longrightarrow \mathbb{C}$, $\varepsilon_\Gamma : \Gamma \longrightarrow \mathbb{C}$ are the co-unit of $\mathbb{F}$ and Γ, respectively whose extension to $C_m(\mathbb{F})$ and $C^*_m(\Gamma)$ are still denoted by $\varepsilon_{\mathbb{F}}$ and ε_Γ, respectively.

Notice by the way that we have also canonical injections

$$\iota^r_{\mathbb{G}} : C_r(\mathbb{G}) \hookrightarrow C_r(\mathbb{F}) \text{ and } \iota^m_{\mathbb{G}} : C_m(\mathbb{G}) \hookrightarrow C_m(\mathbb{F})$$

Again, it is clear, by the definition of the co-multiplication Θ, that the canonical injections $\iota^r_{\mathbb{G}}$ and $\iota^m_{\mathbb{G}}$ intertwine the corresponding co-multiplications. Again, we have commutative diagrams

$$\begin{tikzcd} C_m(\mathbb{F}) \arrow[r, two heads, "\tau_{\mathbb{F}}"] & C_r(\mathbb{F}) \\ C_m(\mathbb{G}) \arrow[u, hook, "\iota^m_{\mathbb{G}}"] \arrow[r, two heads, "\tau_{\mathbb{G}}"'] & C_r(\mathbb{G}) \arrow[u, hook, "\iota^r_{\mathbb{G}}"'] \end{tikzcd} \qquad \begin{tikzcd} C_m(\mathbb{F}) \arrow[r, "\varepsilon_{\mathbb{F}}"] & \mathbb{C} \\ C_m(\mathbb{G}) \arrow[u, hook, "\iota^m_{\mathbb{G}}"] \arrow[ur, "\varepsilon_{\mathbb{G}}"'] & \end{tikzcd}$$

where $\tau_{\mathbb{G}} : C_m(\mathbb{G}) \twoheadrightarrow C_r(\mathbb{G})$ is the canonical surjection and $\varepsilon_{\mathbb{G}} : Pol(\mathbb{G}) \longrightarrow \mathbb{C}$ is the co-unit of $\mathbb{G}$ whose extension to $C_m(\mathbb{G})$ is still denoted by $\varepsilon_{\mathbb{G}}$.

2.3.10 Remark. Furthermore, the representation theory of $\mathbb{F}$ yields that $\widehat{\mathbb{G}}$ and Γ are divisible in $\widehat{\mathbb{F}}$ (recall the equivalence relation defined in terms of discrete quantum subgroups in Section 1.4.3 and see Definition 3.2.1 and Lemma 3.2.2 for more details).

Namely, take an irreducible representation $y := (\gamma, x) \in Irr(\mathbb{F})$ with $\gamma \in \Gamma$ and $x \in Irr(\mathbb{G})$. Then $\gamma = (\gamma, \epsilon_{\mathbb{G}}) \in [y]$ in $Irr(\mathbb{F})/Irr(\mathbb{G})$ because $(\gamma^{-1}, \epsilon_{\mathbb{G}}) \oplus (\gamma, x) = (e, x) = x \in Irr(\mathbb{G})$. Likewise, we have that $x = (e, x) \in [y]$ in $\Gamma \backslash Irr(\mathbb{F})$ because $(\gamma, x) \oplus (e, \overline{x}) = (\gamma, \epsilon_{\mathbb{G}}) = \gamma \in \Gamma$.

Consequently, $\widehat{\mathbb{G}}$ is divisible in $\widehat{\mathbb{F}}$ because for all $s \in Irr(\mathbb{G})$ we have that $(\gamma, \epsilon_{\mathbb{G}}) \oplus s = (\gamma, \epsilon_{\mathbb{G}}) \oplus (e, s) = (\gamma, s) \in Irr(\mathbb{F})$. Likewise, Γ is divisible in $\widehat{\mathbb{F}}$ because for all $s \in \Gamma$ we have that $s \oplus (e, x) = (s, \epsilon_{\mathbb{G}}) \oplus (e, x) = (s, x) \in Irr(\mathbb{F})$.

In order to carry out our study it is advisable to set some notations. If (A, δ) is a $\widehat{\mathbb{F}}$-C^*-algebra, we know that $(A, \delta_{\widehat{\mathbb{G}}})$ is a $\widehat{\mathbb{G}}$-C^*-algebra and that (A, δ_Γ) is a Γ-C^*-algebra by virtue of the preceding proposition. Hence we can form the corresponding reduced crossed products,

$$\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A \subset \mathcal{L}_A(L^2(\mathbb{F}) \otimes A),\ \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A \subset \mathcal{L}_A(L^2(\mathbb{G}) \otimes A),\ \Gamma \underset{\delta_\Gamma,r}{\ltimes} A \subset \mathcal{L}_A(l^2(\Gamma) \otimes A)$$

Let us establish adapted notations for these crossed products following the general constructions from Theorem 1.5.1.1 and Theorem 1.5.2.1. These notations will be used in the sequel.

- There exist a non-degenerate $*$-homomorphism $\pi_\delta : A \longrightarrow \widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A$, a unitary representation $V \in M(c_0(\widehat{\mathbb{F}}) \otimes \widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A)$ and a non-degenerate completely positive KSGNS-faithful map $E_\delta : \widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A \longrightarrow M(A)$ such that

$$\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A = C^*\langle \pi_\delta(a) V^y_{i,j} : a \in A, y \in Irr(\mathbb{F}), i, j = 1, \ldots, n_y \rangle$$

Recall from Proposition 1.4.1.10 that the unitary representation $V \in M(c_0(\widehat{\mathbb{F}}) \otimes \widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A)$ is canonically associated to a non-degenerate $*$-homomorphism $\phi_V : C_m(\mathbb{F}) \longrightarrow M(\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A)$ such that

$$\phi_V(w_{ij}^y) = V_{ij}^y,$$

for all $y \in Irr(\mathbb{F})$ and all $i, j = 1, \ldots, n_y$.

- There exist a non-degenerate $*$-homomorphism $\pi_{\delta_{\widehat{\mathbb{G}}}} : A \longrightarrow \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$, a unitary representation $U \in M(c_0(\widehat{\mathbb{G}}) \otimes \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A)$ and a non-degenerate completely positive KSGNS-faithful map $E_{\delta_{\widehat{\mathbb{G}}}} : \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A \longrightarrow M(A)$ such that

$$\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A = C^*\langle \pi_{\delta_{\widehat{\mathbb{G}}}}(a) U_{i,j}^x : a \in A, x \in Irr(\mathbb{G}), i, j = 1, \ldots, n_x \rangle$$

  Recall from Proposition 1.4.1.10 that the unitary representation $U \in M(c_0(\widehat{\mathbb{G}}) \otimes \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A)$ is canonically associated to a non-degenerate $*$-homomorphism $\phi_U : C_m(\mathbb{G}) \longrightarrow M(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A)$ such that

$$\phi_U(w_{ij}^x) = U_{ij}^x,$$

  for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$.

- There exist a non-degenerate faithful $*$-homomorphism $\sigma : A \longrightarrow \Gamma \underset{\delta_\Gamma,r}{\ltimes} A$, a group homomorphism $\nu : \Gamma \longrightarrow \mathcal{U}(M(\Gamma \underset{\delta_\Gamma,r}{\ltimes} A))$ defined by $\nu_\gamma = \lambda_\gamma \otimes id_A$, for all $\gamma \in \Gamma$ and a non-degenerate completely positive KSGNS-faithful map $E : \Gamma \underset{\delta_\Gamma,r}{\ltimes} A \longrightarrow M(A)$ such that

$$\Gamma \underset{\delta_\Gamma,r}{\ltimes} A = C^*\langle \sigma(a)\nu_\gamma : a \in A, \gamma \in \Gamma \rangle$$

2.3.11 Lemma. *Put $\mathscr{C} := C^*\langle \pi_\delta(a) V_{i,j}^{(e,x)} : a \in A, x \in Irr(\mathbb{G}), i, j = 1, \ldots, n_x \rangle$. If (A, δ) is a $\widehat{\mathbb{F}}$-C^*-algebra, there exists a canonical $*$-isomorphism*

$$\psi : \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A \xrightarrow{\sim} \mathscr{C},$$

such that

$$\psi(\pi_{\delta_{\widehat{\mathbb{G}}}}(a)) = \pi_\delta(a) \text{ and } \psi(U_{i,j}^x) = V_{i,j}^{(e,x)},$$

for all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$.

Proof. In order to prove the (canonical) $*$-isomorphism $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A \cong \mathscr{C}$ of the statement we shall show that the C^*-algebra $\mathscr{C}$ satisfies the universal property of $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$.

Notice that $\mathscr{C}$ is actually a C^*-subalgebra of $\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A$ by construction. Hence we can restrict the elements defining $\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A$ to check such universal property. In particular, we define

$$V^e := (id_{c_0(\widehat{\mathbb{F}})} \otimes \lambda_{\mathbb{F}})\Big(\bigoplus_{x\in Irr(\mathbb{G})}^{c_0} w^{(e,x)}\Big) \otimes id_A \in M(c_0(\widehat{\mathbb{F}}) \otimes \mathscr{C}),$$

where we must remark that, by construction, we have $\big(V^e\big)^x_{i,j} = V^{(e,x)}_{i,j}$, for all $x \in Irr(\mathbb{G})$ and all $i,j = 1,\ldots,n_x$.

By definition of $\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A$, the formulas

$$\pi_\delta(a)V^y_{i,j} = \sum_k V^y_{i,k}\pi_\delta(\delta^y_{k,j}(a)) \text{ and } E_\delta(\pi_\delta(a)V^y_{i,j}) = a\ \delta_{y,\epsilon_{\mathbb{F}}}$$

hold for all $a \in A$, all $y \in Irr(\mathbb{F})$ and all $i,j = 1,\ldots,n_y$. In particular, it holds for $y := e \oplus x$ with $x \in Irr(\mathbb{G})$.

It remains to justify that the restriction $E_{\delta|}$ is always KSGNS-faithful. We know that $(L^2(\mathbb{F}) \otimes A, id, \Upsilon_{\mathbb{F}})$ is the KSGNS construction for E_δ. Consider the Hilbert A-module defined as $\mathscr{H} := \overline{span}\{\mathscr{C}\Upsilon_{\mathbb{F}}(A)\} \subset L^2(\mathbb{F}) \otimes A$. Observe that $\mathscr{C} \xrightarrow{id_|} \mathcal{L}_A(\mathscr{H})$, so $(\mathscr{H}, id_|, \Upsilon_{\mathbb{F}})$ is the KSGNS construction for the restriction $E_{\delta|}$, whence the faithfulness required.

This discussion shows that $\mathscr{C}$ is a C^*-algebra equipped with a triple $(\pi_{\delta|}, V^e, E_{\delta|})$ satisfying the analogue properties of the triple $(\pi_{\delta_{\widehat{\mathbb{G}}}}, U, E_{\delta_{\widehat{\mathbb{G}}}})$ associated to $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$. The universal property property of $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$ yields the canonical $*$-isomorphism of the statement. ■

2.3.12 Proposition. *If (A,δ) is a $\widehat{\mathbb{F}}$-C^*-algebra, then $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$ is a Γ-C^*-algebra with action $\partial : \Gamma \to Aut(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A)$ such that*

$$\partial_\gamma(\pi_{\delta_{\widehat{\mathbb{G}}}}(a)) = \pi_\delta\big((\delta_\Gamma)_\gamma(a)\big) \text{ and } \partial_\gamma(U^x_{i,j}) = \phi_U\big(\alpha_\gamma(w^x_{i,j})\big),$$

for all $\gamma \in \Gamma$, all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i,j = 1,\ldots,n_x$.

Proof. First of all, using the lemma above, we use systematically the canonical identifications $\pi_{\delta_{\widehat{\mathbb{G}}}}(a) \cong \pi_\delta(a)$ and $U^x_{i,j} \cong V^{(e,x)}_{i,j}$, for all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i,j = 1,\ldots,n_x$.

Fix an element $\gamma \in \Gamma$. In order to define an automorphism $\partial_\gamma : \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A \longrightarrow \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$, we shall define a non-degenerate $*$-homomorphism $\widetilde{\pi}^\gamma : A \longrightarrow \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$ and a unitary representation $\widetilde{U}^\gamma \in M(c_0(\widehat{\mathbb{G}}) \otimes \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A)$.

Let's put

$$\widetilde{\pi}^\gamma(a) := \pi_\delta\big(\delta^\gamma_\Gamma(a)\big), \text{ for all } a \in A$$

$$\widetilde{U}^\gamma := (id_{c_0(\widehat{\mathbb{G}})} \otimes \lambda_{\mathbb{G}})\Big(\bigoplus_{x\in Irr(\mathbb{G})}^{c_0} (id_{\mathcal{B}(H_x)} \otimes \alpha_\gamma)w^x\Big) \otimes id_A$$

Observe that α is a quantum action so that $\alpha_\gamma(Pol(\mathbb{G})) = Pol(\mathbb{G})$, for all $\gamma \in \Gamma$ (recall Proposition 1.3.1.28). Hence it is licit to consider the above definition of $\widetilde{U}^\gamma$.

Before going on with the proof, we need some useful remarks:

1. Doing the canonical identification $C(\mathbb{G}) \cong \lambda_{\mathbb{G}}(C(\mathbb{G}))$ in $\mathcal{B}(L^2(\mathbb{G}))$ we have that

$$\big(\widetilde{U}^\gamma\big)^x_{i,j} = \alpha_\gamma(w^x_{i,j}) \otimes id_A = \phi_U\big(\alpha_\gamma(w^x_{i,j})\big),$$

   for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$. Likewise, since $\pi : C_m(\mathbb{G}) \longrightarrow \Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G})$ is a unital *faithful* $*$-homomorphism we can do the canonical identification $C_m(\mathbb{G}) \cong \pi(C_m(\mathbb{G})$. As a consequence, the relation between π and u coming from the definition of the crossed product can be written on $Pol(\mathbb{G})$ as $u_\gamma w^x_{i,j} u^*_\gamma = \alpha_\gamma(w^x_{i,j})$, for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$.

2. By construction we have that $Im(\pi_\delta) \subset \mathcal{L}_A(L^2(\mathbb{F}) \otimes A)$ and $Im(\sigma) \subset \mathcal{L}_A(l^2(\Gamma) \otimes A)$. By Remark 2.3.3 we recall that $L^2(\mathbb{F}) = l^2(\Gamma) \otimes L^2(\mathbb{G})$. In this way, we can consider the leg (13) of σ inside $L^2(\mathbb{F}) \otimes A$ and then we can compare $\pi_\delta(a)$ and $\big[\sigma(a)\big]_{13}$ for every $a \in A$.

   Doing the canonical identification $c_0(\widehat{\mathbb{G}}) \cong \widehat{\lambda}_{\mathbb{G}}(c_0(\widehat{\mathbb{G}}))$ in $\mathcal{B}(L^2(\mathbb{G}))$, a straightforward computation yields naturally to $\pi_\delta(a) = \big[\sigma(a)\big]_{13}$, for all $a \in A$.

3. By construction we have that $Im(u) \subset \mathcal{L}_A(l^2(\Gamma) \otimes C_m(\mathbb{G}))$ and $Im(\nu) \subset \mathcal{L}_A(l^2(\Gamma) \otimes A)$. Precisely, we have $u_\gamma = \lambda_\gamma \otimes id_{C_m(\mathbb{G})}$ and $\nu_\gamma = \lambda_\gamma \otimes id_A$, for all $\gamma \in \Gamma$. Thus, inside the space $l^2(\Gamma) \otimes C_m(\mathbb{G}) \otimes A$, it is clear that $\big[u_\gamma\big]_{12} = \big[\nu_\gamma\big]_{13}$, for all $\gamma \in \Gamma$.

In order to apply the universal property of $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$ and conclude the existence of the automorphism ∂_γ of the statement, we have to check the formula

$$\widetilde{\pi}^\gamma(a)\big(\widetilde{U}^\gamma\big)^x_{i,j} = \sum_{k=1}^{n_x} \big(\widetilde{U}^\gamma\big)^x_{i,k} \widetilde{\pi}^\gamma(\delta^{(e,x)}_{k,j}(a)),$$

for all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$. Namely,

$$\begin{aligned}
\widetilde{\pi}^{\gamma}(a)\big(\widetilde{U}^{\gamma}\big)^{x}_{i,j} &= \pi_{\delta}(\delta^{\gamma}_{\Gamma}(a))\alpha_{\gamma}(w^{x}_{i,j}) \otimes id_A = \big[\sigma(\delta^{\gamma}_{\Gamma}(a))\big]_{13}\big[\alpha_{\gamma}(w^{x}_{i,j}) \otimes id_A\big]_{23} \\
&= \big[\nu_{\gamma}\sigma(a)\nu_{\gamma^{-1}}\big]_{13}\big[u_{\gamma}w^{x}_{i,j}u_{\gamma^{-1}} \otimes id_A\big]_{23} \\
&= \big[\nu_{\gamma}\big]_{13}\big[\sigma(a)\big]_{13}\big[w^{x}_{i,j} \otimes id_A\big]_{23}\big[u_{\gamma^{-1}}\big]_{12} \\
&= \big[\nu_{\gamma}\big]_{13}\pi_{\delta}(a)w^{x}_{i,j} \otimes id_A\big[u_{\gamma^{-1}}\big]_{12} \\
&= \big[\nu_{\gamma}\big]_{13}\Big(\sum_{k=1}^{n_x} w^{x}_{i,k} \otimes id_A\pi_{\delta}(\delta^{(e,x)}_{k,j}(a))\Big)\big[u_{\gamma^{-1}}\big]_{12} \\
&= \big[\nu_{\gamma}\big]_{13}\Big(\sum_{k=1}^{n_x} \big[w^{x}_{i,k} \otimes id_A\big]_{23}\big[\sigma(\delta^{(e,x)}_{k,j}(a))\big]_{13}\Big)\big[u_{\gamma^{-1}}\big]_{12} \\
&= \sum_{k=1}^{n_x} \big[\nu_{\gamma}\big]_{13}\big[w^{x}_{i,k} \otimes id_A\big]_{23}\big[\nu_{\gamma^{-1}}\sigma\Big(\delta^{\gamma}_{\Gamma}\big(\delta^{(e,x)}_{k,j}(a)\big)\Big)\nu_{\gamma}\big]_{13}\big[u_{\gamma^{-1}}\big]_{12} \\
&= \sum_{k=1}^{n_x} \big[\nu_{\gamma}\big]_{13}\big[w^{x}_{i,k} \otimes id_A\big]_{23}\big[\nu_{\gamma^{-1}}\big]_{13}\big[\sigma\Big(\delta^{\gamma}_{\Gamma}\big(\delta^{(e,x)}_{k,j}(a)\big)\Big)\big]_{12} \\
&= \sum_{k=1}^{n_x} u_{\gamma}w^{x}_{i,k}u_{\gamma^{-1}} \otimes id_A\pi_{\delta}\Big(\delta^{\gamma}_{\Gamma}\big(\delta^{(e,x)}_{k,j}(a)\big)\Big) \\
&= \sum_{k=1}^{n_x} \alpha_{\gamma}(w^{x}_{i,k}) \otimes id_A\pi_{\delta}\Big(\delta^{\gamma}_{\Gamma}\big(\delta^{(e,x)}_{k,j}(a)\big)\Big) \\
&= \sum_{k=1}^{n_x} \big(\widetilde{U}^{\gamma}\big)^{x}_{i,j}\widetilde{\pi}^{\gamma}(\delta^{(e,x)}_{k,j}(a))
\end{aligned}$$

To conclude the relations of the statement we have just to apply Remark 2.3.6. ■

2.3.13 Remark. The action ∂ of Γ on $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$ constructed in the preceding proposition can be defined in a more direct manner. Namely, given $\gamma \in \Gamma$ we define the automorphism $\partial_{\gamma} : \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A \longrightarrow \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$ by

$$\partial_{\gamma} := Ad_{\phi_V(u_{\gamma})},$$

where $Ad_{(\cdot)}$ denotes the adjoint map. This defines clearly an invertible map for each $\gamma \in \Gamma$ (because ϕ_V is a $*$-homomorphism and elements u_{γ} are unitaries). It remains to show that the space $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$ is preserved.

On the one hand, we have

$$\partial_{\gamma}\big(\phi_V(w^{x}_{i,j})\big) = \phi_V(u_{\gamma})\phi_V(w^{x}_{i,j})\phi^{*}_V(u_{\gamma}) = \phi_V\big(u_{\gamma}w^{x}_{i,j}u^{*}_{\gamma}\big) = \phi_V\big(\alpha_{\gamma}(w^{x}_{i,j})\big) \in \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A,$$

for all $\gamma \in \Gamma$, $x \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$.

On the other hand, the relations of the reduced crossed product $\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A$ following Theorem 1.5.2.1 are precisely

$$\pi_{\delta}(a)\phi_V(u_{\gamma}v^{x}_{i,j}) = \sum_{k=1}^{n_x} \phi_V(u_{\gamma}v^{x}_{i,k})\pi_{\delta}(\delta^{y}_{k,j}(a)),$$

for all $y := (\gamma, x) \in Irr(\mathbb{F})$, $a \in A$ and all $i, j = 1, \ldots, n_x$. In particular, if we take $x := \epsilon$ this formula becomes

$$\pi_{\delta_{\widehat{\mathbb{G}}}}(a)\phi_V(u_\gamma) = \phi_V(u_\gamma)\pi_{\delta_{\widehat{\mathbb{G}}}}\big(\delta_\Gamma^\gamma(a)\big)$$
$$\Leftrightarrow \big(\partial_\gamma(\pi_{\delta_{\widehat{\mathbb{G}}}}(a))\big)^* = \phi_V^*(u_\gamma)\pi_{\delta_{\widehat{\mathbb{G}}}}(a)\phi_V(u_\gamma) = \pi_{\delta_{\widehat{\mathbb{G}}}}\big(\delta_\Gamma^\gamma(a)\big) \in \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A,$$

for all $\gamma \in \Gamma$, $a \in A$. In particular, we have $\phi_V(u_\gamma)\pi_{\delta_{\widehat{\mathbb{G}}}}(a)\phi_V^*(u_\gamma) = \pi_{\delta_{\widehat{\mathbb{G}}}}\big(\delta_\Gamma^{\gamma^{-1}}(a)\big)$. Hence, Remark 2.3.6 yields that the action ∂ is such that

$$\partial_\gamma\big(\pi_{\delta_{\widehat{\mathbb{G}}}}(a)U_{i,j}^x\big) = \pi_\delta\big(\big(\delta_\Gamma\big)_\gamma(a)\big)\phi_U\big(\alpha_\gamma(w_{i,j}^x)\big),$$

for all $\gamma \in \Gamma$, all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \ldots, n_x$

In order to finish our study it is advisable to set some notations. If (A, δ) is a $\widehat{\mathbb{F}}$-C^*-algebra, the above proposition assures that $(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A, \partial)$ is a Γ-C^*-algebra. Recall as well that we have a canonical $*$-isomorphism $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A \cong \mathscr{C}$ thanks to Lemma 2.3.11. Hence we can form the corresponding reduced crossed product,

$$\Gamma \underset{\partial,r}{\ltimes} \mathscr{C} \subset \mathcal{L}_{\mathscr{C}}(l^2(\Gamma) \otimes \mathscr{C})$$

We shall use the following adapted notations for this crossed product following the general construction from Theorem 1.5.1.1: there exist a non-degenerate faithful $*$-homomorphism $\varrho : \mathscr{C} \longrightarrow \Gamma \underset{\partial,r}{\ltimes} \mathscr{C} \subset \mathcal{L}_A(l^2(\Gamma) \otimes \mathscr{C})$, a group homomorphism $\vartheta : \Gamma \longrightarrow \mathcal{U}(M(\Gamma \underset{\partial,r}{\ltimes} \mathscr{C}))$ defined by $\vartheta_\gamma := \lambda_\gamma \otimes id_{\mathscr{C}}$, for all $\gamma \in \Gamma$ and a non-degenerate completely positive KSGNS-faithful map $\mathscr{E} : \Gamma \underset{\partial,r}{\ltimes} \mathscr{C} \longrightarrow M(\mathscr{C})$ such that

$$\Gamma \underset{\partial,r}{\ltimes} \mathscr{C} = C^*\langle \varrho(c)\vartheta_\gamma : c \in \mathscr{C}, \gamma \in \Gamma \rangle$$

2.3.14 Theorem (Associativity for a quantum semi-direct product)**.** *Let $\mathbb{F} = \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$ be the quantum semi-direct product of a compact quantum group $\mathbb{G}$ by a discrete group Γ acting by quantum automorphisms on $\mathbb{G}$ with action α.*

If (A, δ) is a $\widehat{\mathbb{F}}$-C^*-algebra, then there exists a canonical $*$-isomorphism

$$\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A \cong \Gamma \underset{\partial,r}{\ltimes} \Big(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A\Big)$$

Proof. In order to prove the canonical $*$-isomorphism $\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A \cong \Gamma \underset{\partial,r}{\ltimes} \mathscr{C}$ of the statement, we shall apply the universal property of the reduced crossed product $\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A$. In other words, we are going to construct a triple $(\overline{\rho}, \overline{V}, \overline{E})$ associated to the reduced crossed product $\Gamma \underset{\partial,r}{\ltimes} \mathscr{C}$ satisfying the analogues properties of the triple $(\pi_\delta, V, E_\delta)$ associated to $\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A$. Namely, let's put

- $\overline{\rho} : A \longrightarrow \Gamma \underset{\partial,r}{\ltimes} \mathscr{C}$ as the composition $A \xrightarrow{\pi_\delta} \mathscr{C} \xrightarrow{\varrho} \Gamma \underset{\partial,r}{\ltimes} \mathscr{C}$,

- $\overline{V} \in M(c_0(\widehat{\mathbb{F}}) \otimes \Gamma \underset{\partial,r}{\ltimes} \mathscr{C})$ as the unitary V itself $(*)$,

- $\overline{E} : \Gamma \underset{\partial,r}{\ltimes} \mathscr{C} \longrightarrow A$ as the composition $\Gamma \underset{\partial,r}{\ltimes} \mathscr{C} \xrightarrow{\mathscr{E}} M(\mathscr{C}) \xrightarrow{E_{\delta|}} M(A)$.

Notice that it is licit to define $\overline{V}$ as above in $(*)$ because, by the well-known correspondence from Proposition 1.4.1.10, giving a unitary $\overline{V} \in M(c_0(\widehat{\mathbb{F}}) \otimes \Gamma \underset{\partial,r}{\ltimes} \mathscr{C})$ is equivalent to give a non-degenerate $*$-homomorphism $\phi_{\overline{V}} : C_m(\mathbb{F}) \longrightarrow M(\Gamma \underset{\partial,r}{\ltimes} \mathscr{C})$ such that $\overline{V} = (id_{c_0(\widehat{\mathbb{F}})} \otimes \phi_{\overline{V}})(W_{\mathbb{F}})$. Hence we take simply

$$\phi_{\overline{V}} := id \ltimes \phi_U : \Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G}) \longrightarrow M(\Gamma \underset{\partial,r}{\ltimes} \mathscr{C}),$$

where we use the canonical identification of Lemma 2.3.11 and that $C_m(\mathbb{F}) = \Gamma \underset{\alpha,m}{\ltimes} C_m(\mathbb{G})$ by construction. In this situation, we easily check that $\phi_{\overline{V}}$ induces actually the unitary V itself. Namely, we have

$$\begin{aligned} V &= (id_{c_0(\widehat{\mathbb{F}})} \otimes \lambda_{\mathbb{F}})\Big(\bigoplus_{y \in Irr(\mathbb{F})}^{c_0} w^y\Big) \otimes id_A \\ &= (id_{c_0(\Gamma)} \otimes id_{c_0(\widehat{\mathbb{G}})} \otimes \lambda \otimes \lambda_{\mathbb{G}})\Big(\bigoplus_{(\gamma,x) \in \Gamma \times Irr(\mathbb{G})}^{c_0} w^{(\gamma,x)}\Big) \otimes id_A, \end{aligned}$$

which can be seen as an element in $M(c_0(\widehat{\mathbb{F}}) \otimes \mathcal{L}_{\mathscr{C}}(l^2(\Gamma) \otimes \mathscr{C}))$.

Notice as well that $\overline{\rho}$ is a non-degenerate $*$-homomorphism because it is a composition of non-degenerate $*$-homomorphisms and $\overline{E}$ is a non-degenerate completely positive map because it is a composition of non-degenerate completely positive maps.

To conclude we have to check the following.

i) $\overline{\rho}(a)\overline{V}^y_{i,j} = \sum_{k=1}^{n_y} \overline{V}^y_{i,k}\overline{\rho}(\delta^y_{k,j}(a))$, for all $y \in Irr(\mathbb{F})$, all $a \in A$ and all $i,j = 1,\ldots,n_y$. Indeed, it suffices to remember that ϱ is a non-degenerate *faithful* $*$-homomorphism and to apply the corresponding formula on $\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A$.

$$\begin{aligned} \overline{\rho}(a)\overline{V}^y_{i,j} &= \varrho(\pi_\delta(a))V^y_{i,j} \cong \pi_\delta(a)V^y_{i,j} = \sum_{k=1}^{n_y} V^y_{i,k}\pi_\delta(\delta^y_{k,j}(a)) \\ &\cong \sum_{k=1}^{n_y} V^y_{i,k}\varrho\big(\pi_\delta(\delta^y_{k,j}(a))\big) = \sum_{k=1}^{n_y} \overline{V}^y_{i,k}\overline{\rho}(\delta^y_{k,j}(a)) \end{aligned}$$

ii) $\overline{E} = E_{\delta|} \circ \mathscr{E}$ is always a KSGNS-faithful map. It suffices to observe that, by all previous constructions, we can be reduced to the more simple situation in which $C \subset B \subset A$ are C^*-algebras with conditional expectations $A \xrightarrow{E_A} B \xrightarrow{E_B} C$ in such a way that both E_A and E_B are KSGNS-faithful. In this situation, it is straightforward to see that the composition $E := E_B \circ E_A : A \longrightarrow C$ is always a KSGNS-faithful conditional expectation recall Remark A.3.13. Namely,

Since $\varrho : \mathscr{C} \longrightarrow \Gamma \underset{\partial,r}{\ltimes} \mathscr{C}$ is faithful, we have that $\mathscr{C} \cong \varrho(\mathscr{C}) \subset \Gamma \underset{\partial,r}{\ltimes} \mathscr{C}$. Moreover, we have $\mathscr{E}(\varrho(c)) = c$, for all $c \in \mathscr{C}$ by definition of the reduced crossed product.
In other words, $\mathscr{E} : \Gamma \underset{\partial,r}{\ltimes} \mathscr{C} \longrightarrow M(\mathscr{C})$ is a (non-degenerate) completely positive map and also a projection from $\mathscr{C} \cong \varrho(\mathscr{C})$ to $\Gamma \underset{\partial,r}{\ltimes} \mathscr{C}$; that is, a conditional expectation.

Thanks again to the definition of the reduced crossed product, we have that $E_\delta(\pi_\delta(a)) = a$, for all $a \in A$. In this way, $E_\delta : \widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A \longrightarrow M(\pi_\delta(A))$ is also a conditional expectation and so it is $E_{\delta|} : \mathscr{C} \longrightarrow M(\pi_\delta(A))$.

Therefore we have the sequence of C^*-algebras $\pi_\delta(A) \subset \mathscr{C} \subset \Gamma \underset{\partial,r}{\ltimes} \mathscr{C}$ with conditional expectations $\Gamma \underset{\partial,r}{\ltimes} \mathscr{C} \xrightarrow{\mathscr{E}} M(\mathscr{C}) \xrightarrow{E_{\delta|}} M(\pi_\delta(A))$. Since $\mathscr{E}$ and $E_{\delta|}$ are KSGNS-faithful by assumption, we obtain the KSGNS-faithfulness of $\overline{E}$ as required.

■

The preceding results are true for any $\widehat{\mathbb{F}}$-C^*-algebra. We can apply them to the case of the $\widehat{\mathbb{F}}$-C^*-algebra $(c_0(\widehat{\mathbb{F}}), \widehat{\Theta})$, which is particularly interesting for our purpose. Recall from Lemma 2.3.4 that we have the identification $c_0(\widehat{\mathbb{F}}) = c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})$ and that the dual co-multiplication $\widehat{\Theta}$ has been explicitly described in terms of this identification (recall Lemma 2.3.4). Next, we want to describe explicitly the action ∂ of Γ on $\widehat{\mathbb{G}} \underset{\widehat{\Theta}_{\widehat{\mathbb{G}}}}{\ltimes} \big(c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})\big)$ given by Proposition 2.3.12.

Let us set some notations. We denote by $(\pi_{\widehat{\Theta}}, U_{\widehat{\Theta}}, E_{\widehat{\Theta}})$, $(\pi_{\widehat{\Delta}}, U_{\widehat{\Delta}}, E_{\widehat{\Delta}})$, $(\pi_\eta, U_\eta, E_\eta)$ the canonical triples (following Theorem 1.5.2.1) associated to the reduced crossed products $\widehat{\mathbb{G}} \underset{\widehat{\Theta}_{\widehat{\mathbb{G}}}}{\ltimes} \big(c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})\big)$, $\widehat{\mathbb{G}} \underset{\widehat{\Delta}}{\ltimes} c_0(\widehat{\mathbb{G}})$, $\widehat{\mathbb{G}} \underset{\eta}{\ltimes} \big(c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})\big)$, respectively. We denote by $\mathscr{U} \in \mathcal{U}\big(M(c_0(\widehat{\mathbb{G}}) \otimes c_0(\Gamma))\big)$ the unitary introduced in Corollary 2.3.7 such that $\mathscr{U}(p_x \otimes \delta_\gamma) = V_{\gamma^{-1},x} \otimes \delta_\gamma$, for every $x \in Irr(\mathbb{G})$ and every $\gamma \in \Gamma$.

2.3.15 Lemma. *The following properties hold.*

i) There exists a canonical $*$-isomorphism

$$\widehat{\mathbb{G}} \underset{\widehat{\Theta}_{\widehat{\mathbb{G}}}}{\ltimes} \big(c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})\big) \cong c_0(\Gamma) \otimes \widehat{\mathbb{G}} \underset{\widehat{\Delta}}{\ltimes} c_0(\widehat{\mathbb{G}})$$

which is Γ-equivariant, where $c_0(\Gamma) \otimes \widehat{\mathbb{G}} \underset{\widehat{\Delta}}{\ltimes} c_0(\widehat{\mathbb{G}})$ is equipped with the action μ of Γ such that

$$\mu_\gamma\big((\delta_r \otimes \pi_{\widehat{\Delta}}(a))W^x_{i,j}\big) = (\delta_{\gamma r} \otimes \pi_{\widehat{\Delta}}(a))\phi_W\big(\alpha_\gamma(w^x_{i,j})\big),$$

for all $a \in c_0(\widehat{\mathbb{G}})$, $\gamma \in \Gamma$, $x \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$, where

$$W := \big(U_{\widehat{\Delta}}\big)_{13}\big(id_{c_0(\widehat{\mathbb{G}})} \otimes (id_{c_0(\Gamma)} \otimes \pi_{\widehat{\Delta}})\big)(\mathscr{U}^* \otimes id_{c_0(\widehat{\mathbb{G}})}) \in M\big(c_0(\widehat{\mathbb{G}}) \otimes c_0(\Gamma) \otimes \widehat{\mathbb{G}} \underset{\widehat{\Delta}}{\ltimes} c_0(\widehat{\mathbb{G}})\big)$$

ii) There exists a canonical Γ-equivariant $*$-isomorphism

$$c_0(\Gamma)\otimes\widehat{\mathbb{G}}\underset{\widehat{\Delta}}{\ltimes}c_0(\widehat{\mathbb{G}})\cong c_0(\Gamma)\otimes\mathcal{K}(L^2(\mathbb{G})),$$

where $c_0(\Gamma)\otimes\widehat{\mathbb{G}}\underset{\widehat{\Delta}}{\ltimes}c_0(\widehat{\mathbb{G}})$ is equipped with the action μ defined in (i), $c_0(\Gamma)$ is equipped with the action induced by co-multiplication $\widehat{\Delta}_\Gamma$ and $\mathcal{K}(L^2(\mathbb{G}))$ is equipped with the adjoint action of Γ with respect to the unitary representation of Γ on $L^2(\mathbb{G})$ induced by α.

In particular, there exists a canonical Γ-equivariant Morita equivalence

$$c_0(\Gamma)\otimes\widehat{\mathbb{G}}\underset{\widehat{\Delta}}{\ltimes}c_0(\widehat{\mathbb{G}})\underset{\Gamma-M}{\sim}c_0(\Gamma)$$

Proof. i) Recall that $\eta=(\Sigma_{12}\otimes id_{c_0(\widehat{\mathbb{G}})})\circ(id_{c_0(\Gamma)}\otimes\widehat{\Delta})$. By Proposition 1.5.3.2 we know that there exists a canonical $*$-isomorphism $\widehat{\mathbb{G}}\underset{\eta}{\ltimes}\big(c_0(\Gamma)\otimes c_0(\widehat{\mathbb{G}})\big)\cong c_0(\Gamma)\otimes\widehat{\mathbb{G}}\underset{\widehat{\Delta}}{\ltimes}c_0(\widehat{\mathbb{G}})$, where the latter is equipped with the triple $\big(id_{c_0(\Gamma)}\otimes\pi_{\widehat{\Delta}},\big(U_{\widehat{\Delta}}\big)_{13},id_{c_0(\Gamma)}\otimes E_{\widehat{\Delta}}\big)$. Next, if we replace $\big(U_{\widehat{\Delta}}\big)_{13}$ by the unitary W of the statement, the corresponding triple is again associated to $c_0(\Gamma)\otimes\widehat{\mathbb{G}}\underset{\widehat{\Delta}}{\ltimes}c_0(\widehat{\mathbb{G}})$ in the sense of Theorem 1.5.2.1.

Observe that both triple give rise to isomorphic underlying C^*-algebras by means of the unitary $\mathscr{U}$. We claim that the C^*-algebra described in terms of the triple $\big(id_{c_0(\Gamma)}\otimes\pi_{\widehat{\Delta}},W,id_{c_0(\Gamma)}\otimes E_{\widehat{\Delta}}\big)$ is identified to the reduced crossed product $\widehat{\mathbb{G}}\underset{\widehat{\Theta}_{\widehat{\mathbb{G}}}}{\ltimes}\big(c_0(\Gamma)\otimes c_0(\widehat{\mathbb{G}})\big)$. Let us check its universal property.

On the one hand, for all $a\in c_0(\Gamma)\otimes c_0(\widehat{\mathbb{G}})$ we write

$$\begin{aligned}
&W^*\big(id_{c_0(\widehat{\mathbb{G}})}\otimes(id_{c_0(\Gamma)}\otimes\pi_{\widehat{\Delta}})(a)\big)W\\
&\quad=\big(id_{c_0(\widehat{\mathbb{G}})}\otimes(id_{c_0(\Gamma)}\otimes\pi_{\widehat{\Delta}})\big)(\mathscr{U}\otimes id_{c_0(\widehat{\mathbb{G}})})\\
&\quad\big(U^*_{\widehat{\Delta}}\big)_{13}\big(id_{c_0(\widehat{\mathbb{G}})}\otimes(id_{c_0(\Gamma)}\otimes\pi_{\widehat{\Delta}})(a)\big)\big(U_{\widehat{\Delta}}\big)_{13}\\
&\quad\big(id_{c_0(\widehat{\mathbb{G}})}\otimes(id_{c_0(\Gamma)}\otimes\pi_{\widehat{\Delta}})\big)(\mathscr{U}^*\otimes id_{c_0(\widehat{\mathbb{G}})})\\
&\quad\overset{(1)}{=}\big(id_{c_0(\widehat{\mathbb{G}})}\otimes(id_{c_0(\Gamma)}\otimes\pi_{\widehat{\Delta}})\big)(\mathscr{U}\otimes id_{c_0(\widehat{\mathbb{G}})})\\
&\quad\big(id_{c_0(\widehat{\mathbb{G}})}\otimes(id_{c_0(\Gamma)}\otimes\pi_{\widehat{\Delta}}\big)(\Sigma_{12}\otimes id_{c_0(\widehat{\mathbb{G}})})(id_{c_0(\Gamma)}\otimes\widehat{\Delta})(a)\\
&\quad\big(id_{c_0(\widehat{\mathbb{G}})}\otimes(id_{c_0(\Gamma)}\otimes\pi_{\widehat{\Delta}})\big)(\mathscr{U}^*\otimes id_{c_0(\widehat{\mathbb{G}})})\\
&\quad=\big(id_{c_0(\widehat{\mathbb{G}})}\otimes(id_{c_0(\Gamma)}\otimes\pi_{\widehat{\Delta}})\big)\Big((\mathscr{U}\otimes id_{c_0(\widehat{\mathbb{G}})})\eta(a)(\mathscr{U}^*\otimes id_{c_0(\widehat{\mathbb{G}})})\Big)\\
&\quad\overset{(2)}{=}\big(id_{c_0(\widehat{\mathbb{G}})}\otimes(id_{c_0(\Gamma)}\otimes\pi_{\widehat{\Delta}})\big)\widehat{\Theta}_{\widehat{\mathbb{G}}}(a),
\end{aligned}$$

where in (1) we have used Proposition 1.5.3.2 and in (2) we have used that $\widehat{\Theta}_{\widehat{\mathbb{G}}}$ is conjugate of η by $\mathscr{U}$ thanks to Corollary 2.3.7. On the other hand, a routine computation yields the following expression for all $x\in Irr(\mathbb{G})$, $i,j=1,\dots,n_x$

$$W^x_{i,j}=\sum_{\gamma\in\Gamma}\delta_\gamma\otimes\big(U^x_{\widehat{\Delta}}\big)_{i,j}\big(V^*_{\gamma,x}\big)_{i,j}\in M\big(c_0(\Gamma)\otimes\widehat{\mathbb{G}}\underset{\widehat{\Delta}}{\ltimes}c_0(\widehat{\mathbb{G}})\big),$$

so that for all $r \in \Gamma$, $a \in c_0(\widehat{\mathbb{G}})$, $x \in Irr(\mathbb{G})$, $i,j = 1,\dots,n_x$ we have

$$\begin{aligned}(id_{c_0(\Gamma)} \otimes E_{\widehat{\Delta}})\Big((id_{c_0(\Gamma)} \otimes \pi_{\widehat{\Delta}})(\delta_r \otimes a)W^x_{i,j}\Big) \\ = (id_{c_0(\Gamma)} \otimes E_{\widehat{\Delta}})\Big(\delta_r \otimes \pi_{\widehat{\Delta}}(a)\big(U^x_{\widehat{\Delta}}\big)_{i,j}\big(V^*_{\gamma,x}\big)_{i,j}\Big) \\ = \delta_r \otimes a\ \delta_{x,\epsilon}\big(V^*_{\gamma,x}\big)_{i,j} = \delta_r \otimes a\ \delta_{x,\epsilon} = E_{\widehat{\Theta}}(\pi_{\widehat{\Theta}}(\delta_r \otimes a)\big(U^x_{\widehat{\Theta}}\big)_{i,j})\end{aligned}$$

Hence, by universal property of $\widehat{\mathbb{G}} \underset{\widehat{\Theta}_{\widehat{\mathbb{G}}}}{\ltimes} \big(c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})\big)$ there exists a canonical $*$-isomorphism

$$\psi : \widehat{\mathbb{G}} \underset{\widehat{\Theta}_{\widehat{\mathbb{G}}}}{\ltimes} \big(c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})\big) \xrightarrow{\sim} c_0(\Gamma) \otimes \widehat{\mathbb{G}} \underset{\widehat{\Delta}}{\ltimes} c_0(\widehat{\mathbb{G}})$$

such that

$$\psi\big(\pi_{\widehat{\Theta}}(\delta_\gamma \otimes a)\big(U^x_{\widehat{\Theta}}\big)_{i,j}\big) = (\delta_\gamma \otimes \pi_{\widehat{\Delta}}(a))W^x_{i,j},$$

for all $\gamma \in \Gamma$, $a \in c_0(\widehat{\mathbb{G}})$, $x \in Irr(\mathbb{G})$, $i,j = 1,\dots,n_x$. Moreover, for all $\gamma, r \in \Gamma$, $a \in c_0(\widehat{\mathbb{G}})$, $x \in Irr(\mathbb{G})$, $i,j = 1,\dots,n_x$ we write

$$\pi_{\widehat{\Theta}}(\delta_r \otimes a)\big(U^x_{\widehat{\Theta}}\big)_{i,j} \overset{\partial_\gamma}{\mapsto} \pi_{\widehat{\Theta}}\big((\widehat{\Theta}_\Gamma)_\gamma(\delta_r \otimes a)\big)\phi_{U_{\widehat{\Theta}}}(\alpha_\gamma(w^x_{i,j})) \overset{\psi}{\mapsto} (\delta_{\gamma r} \otimes \pi_{\widehat{\Delta}}(a))\phi_W(\alpha_\gamma(w^x_{i,j}))$$

$$\pi_{\widehat{\Theta}}(\delta_r \otimes a)\big(U^x_{\widehat{\Theta}}\big)_{i,j} \overset{\psi}{\mapsto} (\delta_r \otimes \pi_{\widehat{\Delta}}(a))W^x_{i,j} \overset{\mu_\gamma}{\mapsto} (\delta_{\gamma r} \otimes \pi_{\widehat{\Delta}}(a))\phi_W(\alpha_\gamma(w^x_{i,j})),$$

which yields that μ is a well-defined action of Γ on $c_0(\Gamma) \otimes \widehat{\mathbb{G}} \underset{\widehat{\Delta}}{\ltimes} c_0(\widehat{\mathbb{G}})$ and thus the Γ-equivariance of the statement.

ii) We are going to establish a canonical $*$-isomorphism $\psi : c_0(\Gamma) \otimes \widehat{\mathbb{G}} \underset{\widehat{\Delta}}{\ltimes} c_0(\widehat{\mathbb{G}}) \xrightarrow{\sim} c_0(\Gamma) \otimes \mathcal{K}(L^2(\mathbb{G}))$. For this, remark firstly that the canonical triple $(\pi_{\widehat{\Delta}}, U_{\widehat{\Delta}}, E_{\widehat{\Delta}})$ associated to $\widehat{\mathbb{G}} \underset{\widehat{\Delta}}{\ltimes} c_0(\widehat{\mathbb{G}})$ following Theorem 1.5.2.1 is exactly $(\widehat{\lambda}, \widehat{W}_{\mathbb{G}} \otimes id_{c_0(\widehat{\mathbb{G}})}, \Omega \otimes id_{c_0(\widehat{\mathbb{G}})})$.

Moreover, it is well-known that $\widehat{\mathbb{G}} \underset{\widehat{\Delta}}{\ltimes} c_0(\widehat{\mathbb{G}}) \cong \mathcal{K}(L^2(\mathbb{G}))$ and for the latter we have the triple $(\widehat{\lambda}, \widehat{W}_{\mathbb{G}}, \widehat{E})$ following Remark 1.7.1.21. In particular, we have

$$\mathcal{K}(L^2(\mathbb{G})) = C^*\langle \widehat{\lambda}(a)\lambda(w^x_{i,j}) \mid a \in c_0(\widehat{\mathbb{G}}), x \in Irr(\mathbb{G}), i.j = 1,\dots,n_x\rangle$$

Since Γ acts on $\mathbb{G}$ by quantum automorphisms with action α, then $L^2(\mathbb{G})$ is equipped with the action of Γ such that $\gamma \cdot \lambda(w^{x'}_{k,l})\Omega = \lambda\big(\alpha_\gamma(w^{x'}_{k,l})\big)\Omega$, for all $\gamma \in \Gamma$, $x' \in Irr(\mathbb{G})$, $k,l = 1,\dots,n_{x'}$. Therefore, the corresponding action on $\mathcal{K}(L^2(\mathbb{G}))$ (recall Remark A.3.0.3) is such that $\gamma \cdot \widehat{\lambda}(a)\lambda(w^x_{i,j}) = \widehat{\lambda}(a)\lambda\big(\alpha_\gamma(w^x_{i,j})\big)$, for all $a \in c_0(\widehat{\mathbb{G}})$, $x \in Irr(\mathbb{G})$, $i,j = 1,\dots,n_x$, which is a straightforward computation.

Since $c_0(\Gamma) \otimes \widehat{\mathbb{G}} \underset{\widehat{\Delta}}{\ltimes} c_0(\widehat{\mathbb{G}})$ has been described in (ii) with the help of the unitary $\mathscr{U}$, then we consider now the triple $(id_{c_0(\Gamma)} \otimes \widehat{\lambda}, \widetilde{W}, id_{c_0(\Gamma)} \otimes \widehat{E})$ associated to $c_0(\Gamma) \otimes \mathcal{K}(L^2(\mathbb{G}))$ where

$$\widetilde{W} := \big(\widehat{W}_{\mathbb{G}}\big)_{13}\big(id_{c_0(\widehat{\mathbb{G}})} \otimes (id_{c_0(\Gamma)} \otimes \widehat{\lambda})\big)(\mathscr{U}^* \otimes id_{c_0(\widehat{\mathbb{G}})}) \in M\big(c_0(\widehat{\mathbb{G}}) \otimes c_0(\Gamma) \otimes \mathcal{K}(L^2(\mathbb{G})\big)$$

Observe that both triple $(id_{c_0(\Gamma)} \otimes \widehat{\lambda}, \big(\widehat{W}_{\mathbb{G}}\big)_{13}, id_{c_0(\Gamma)} \otimes \widehat{E})$ and $(id_{c_0(\Gamma)} \otimes \widehat{\lambda}, \widetilde{W}, id_{c_0(\Gamma)} \otimes \widehat{E})$ give rise to isomorphic underlying C^*-algebras by means of the unitary $\mathscr{U}$. We claim that the C^*-algebra $c_0(\Gamma) \otimes \mathcal{K}(L^2(\mathbb{G}))$ described in terms of the triple $\big(id_{c_0(\Gamma)} \otimes \pi_{\widehat{\Delta}}, \widetilde{W}, id_{c_0(\Gamma)} \otimes E_{\widehat{\Delta}}\big)$ is identified to the reduced crossed product $\widehat{\mathbb{G}} \underset{\widehat{\Theta}_{\widehat{\mathbb{G}}}}{\ltimes} \big(c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})\big)$. Let us check its universal property.

On the one hand, for all $a \in c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})$ we write

$$\begin{aligned}
&\widetilde{W}^*\big(id_{c_0(\widehat{\mathbb{G}})} \otimes (id_{c_0(\Gamma)} \otimes \widehat{\lambda})(a)\big)\widetilde{W} \\
&\quad = \big(id_{c_0(\widehat{\mathbb{G}})} \otimes (id_{c_0(\Gamma)} \otimes \widehat{\lambda})\big)(\mathscr{U} \otimes id_{c_0(\widehat{\mathbb{G}})}) \\
&\quad \big(\widehat{W}^*_{\mathbb{G}}\big)_{13}\big(id_{c_0(\widehat{\mathbb{G}})} \otimes (id_{c_0(\Gamma)} \otimes \pi_{\widehat{\Delta}})(a)\big)\big(\widehat{W}_{\mathbb{G}}\big)_{13} \\
&\quad \big(id_{c_0(\widehat{\mathbb{G}})} \otimes (id_{c_0(\Gamma)} \otimes \widehat{\lambda})\big)(\mathscr{U}^* \otimes id_{c_0(\widehat{\mathbb{G}})}) \\
&\quad \overset{(1)}{=} \big(id_{c_0(\widehat{\mathbb{G}})} \otimes (id_{c_0(\Gamma)} \otimes \widehat{\lambda})\big)(\mathscr{U} \otimes id_{c_0(\widehat{\mathbb{G}})}) \\
&\quad \big(id_{c_0(\widehat{\mathbb{G}})} \otimes (id_{c_0(\Gamma)} \otimes \widehat{\lambda})(\Sigma_{12} \otimes id_{c_0(\widehat{\mathbb{G}})})(id_{c_0(\Gamma)} \otimes \widehat{\Delta})(a) \\
&\quad \big(id_{c_0(\widehat{\mathbb{G}})} \otimes (id_{c_0(\Gamma)} \otimes \widehat{\lambda})\big)(\mathscr{U}^* \otimes id_{c_0(\widehat{\mathbb{G}})}) \\
&\quad = \big(id_{c_0(\widehat{\mathbb{G}})} \otimes (id_{c_0(\Gamma)} \otimes \widehat{\lambda})\big)\Big((\mathscr{U} \otimes id_{c_0(\widehat{\mathbb{G}})})\eta(a)(\mathscr{U}^* \otimes id_{c_0(\widehat{\mathbb{G}})})\Big) \\
&\quad \overset{(2)}{=} \big(id_{c_0(\widehat{\mathbb{G}})} \otimes (id_{c_0(\Gamma)} \otimes \widehat{\lambda})\big)\widehat{\Theta}_{\widehat{\mathbb{G}}}(a),
\end{aligned}$$

where in (1) we have used the definition of $\widehat{W}_{\mathbb{G}}$ (recall Theorem 1.3.1.36) and in (2) we have used that $\widehat{\Theta}_{\widehat{\mathbb{G}}}$ is conjugate of η by $\mathscr{U}$ thanks to Corollary 2.3.7. On the other hand, a routine computation yields the following expression for all $x \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$

$$\widetilde{W}^x_{i,j} = \sum_{\gamma \in \Gamma} \delta_\gamma \otimes \big(\widehat{W}^x_{\mathbb{G}}\big)_{i,j}\big(V^*_{\gamma,x}\big)_{i,j} \in M\big(c_0(\Gamma) \otimes \mathcal{K}(L^2(\mathbb{G}))\big),$$

so that for all $r \in \Gamma$, $a \in c_0(\widehat{\mathbb{G}})$, $x \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$ we have

$$\begin{aligned}
&(id_{c_0(\Gamma)} \otimes \widehat{E})\Big((id_{c_0(\Gamma)} \otimes \widehat{\lambda})(\delta_r \otimes a)\widetilde{W}^x_{i,j}\Big) \\
&\qquad = (id_{c_0(\Gamma)} \otimes \widehat{E})\Big(\delta_r \otimes \widehat{\lambda}(a)\big(\widehat{W}^x_{\mathbb{G}}\big)_{i,j}\big(V^*_{\gamma,x}\big)_{i,j}\Big) \\
&\qquad = \delta_r \otimes a\ \delta_{x,\epsilon}\big(V^*_{\gamma,x}\big)_{i,j} = \delta_r \otimes a\ \delta_{x,\epsilon} = E_{\widehat{\Theta}}(\pi_{\widehat{\Theta}}(\delta_r \otimes a)\big(U^x_{\widehat{\Theta}}\big)_{i,j})
\end{aligned}$$

Hence, by property (i) and universal property of $\widehat{\mathbb{G}} \underset{\widehat{\Theta}_{\widehat{\mathbb{G}}}}{\ltimes} \big(c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})\big)$ there exists a canonical $*$-isomorphism

$$\psi : c_0(\Gamma) \otimes \widehat{\mathbb{G}} \underset{\widehat{\Delta}}{\ltimes} c_0(\widehat{\mathbb{G}}) \xrightarrow{\sim} c_0(\Gamma) \otimes \mathcal{K}(L^2(\mathbb{G}))$$

such that

$$\psi\big((\delta_\gamma \otimes \pi_{\widehat{\Delta}}(a))W^x_{i,j}\big) = (\delta_\gamma \otimes \widehat{\lambda}(a))\widetilde{W}^x_{i,j},$$

for all $\gamma \in \Gamma$, $a \in c_0(\widehat{\mathbb{G}})$, $x \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$.

Finally, let us study the Γ-equivariance condition. By definition, the action of Γ on $c_0(\Gamma) \otimes \mathcal{K}(L^2(\mathbb{G}))$ is such that for all $\gamma, r \in \Gamma$, $a \in c_0(\widehat{\mathbb{G}})$, $i, j = 1, \ldots, n_x$

$$\gamma \cdot \big(\delta_r \otimes \widehat{\lambda}(a)\lambda(w^x_{i,j})\big) = \delta_{\gamma r} \otimes \widehat{\lambda}(a)\lambda\big(\alpha_\gamma(w^x_{i,j})\big),$$

which allows to show the Γ-equivariance of the above $*$-isomorphism because for all $\gamma, r \in \Gamma$, $a \in c_0(\widehat{\mathbb{G}})$, $x \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$ we write

$$(\delta_r \otimes \pi_{\widehat{\Delta}}(a))W^x_{i,j} \overset{\mu_\gamma}{\mapsto} (\delta_{\gamma r} \otimes \pi_{\widehat{\Delta}}(a))\phi_W\big(\alpha_\gamma(w^x_{i,j})\big) \overset{\psi}{\mapsto} (\delta_{\gamma r} \otimes \widehat{\lambda}(a))\phi_{\widetilde{W}}\big(\alpha_\gamma(w^x_{i,j})\big)$$

$$(\delta_r \otimes \pi_{\widehat{\Delta}}(a))W^x_{i,j} \overset{\psi}{\mapsto} (\delta_r \otimes \widehat{\lambda}(a))\widetilde{W}^x_{i,j} \overset{\gamma\cdot}{\mapsto} (\delta_{\gamma r} \otimes \widehat{\lambda}(a))\phi_{\widetilde{W}}\big(\alpha_\gamma(w^x_{i,j})\big)$$

■

2.4 Compact bicrossed product

We introduce the compact bicrossed product of a matched pair of a discrete group and a compact group and we analyze some structure properties of this compact quantum group which are useful for our purpose.

It is important to say that the bicrossed product construction have had different approaches throughout the history and that we are interested in a very concrete case. More precisely, in the fundamental work [94], G. I. Kac introduced the notion of matched pair of *finite groups* in order to study the classification of extensions of finite groups. In the context of multiplictive unitaries, S. Baaj and G. Skandalis give in [7] a generalization of the Kac's work defining the notion of matched pair of *locally compact groups*. Finally, the work of S. Vaes and L. Vainerman [196] give a very general framework for the bicrossed product defining the notion of matched pair of *locally compact quantum groups*. This allows in particular to develop a very technical theory by which we can give a satisfactory notion of *extension of locally compact quantum groups*.

If we restrict our attention to a matched pair of a discrete group and a compact group (we say *compact matched pair*), the resulting object is a compact quantum group and we can investigate in a much more clear fashion the properties of its representation theory and approximation properties as we can see in the work [65] due to P. Fima, K. Mukherjee and I. Patri. Actually, we refer to [65] for more details about *compact matched pairs* and specifically for a proof of Theorem 2.4.1 below defining the compact bicrossed product.

Let Γ be a discrete group and G a compact group such that the pair (Γ, G) is a matched pair (see [7] or [65] for a precise definition). Then we have that

- there exists a continuous left action of Γ on the topological space G, $\alpha : \Gamma \times G \longrightarrow G$,
- there exists a continuous right action of G on the topological space Γ, $\beta : G \times \Gamma \longrightarrow \Gamma$
- and both actions α and β are related in the following way: for every $\gamma \in \Gamma$ and every $g \in G$, we have

$$\gamma g = \alpha_\gamma(g)\beta_g(\gamma)$$

In particular, if $e \in \Gamma$ denotes the identity element of Γ, then $\beta_g(e) = e$, for all $g \in G$. Hence $\#[e] = 1$, where $[e] \in \Gamma/G$ is the corresponding class in the orbit space. Observe that $\alpha_e = id_G$ and $\beta_e = id_\Gamma$, where e denotes either the identity element in Γ or in G, respectively.

In this situation, we write $(\Gamma, G, \alpha, \beta)$ for a compact matched pair with associated actions α and β according to the statements above. For every class $[\gamma] \in \Gamma/G$ in the orbit space, we define the following *clopen* subsets of G (see [65] for more details)

$$A_{r,s} := \{g \in G : \beta_g(r) = s\},$$

for every $r, s \in [\gamma]$. Consider as well its characteristic function, say $\mathbb{1}_{A_{r,s}} =: \mathbb{1}_{r,s}$, for all $r, s \in [\gamma]$. We can show that $\left(\mathbb{1}_{r,s}\right)_{r,s\in[\gamma]} \in \mathcal{M}_{\#[\gamma]}(\mathbb{C}) \otimes C(G)$ is a magic unitary and a unitary representation of G (again, see [65] for more details).

We shall use all these notations for the sequel.

2.4.1 Theorem-Definition. *Let $(\Gamma, G, \alpha, \beta)$ be a compact matched pair.*

There exists a unique unital $*$-homomorphism

$$\Theta : \Gamma \underset{\alpha,m}{\ltimes} C(G) \longrightarrow \Gamma \underset{\alpha,m}{\ltimes} C(G) \otimes \Gamma \underset{\alpha,m}{\ltimes} C(G)$$

such that

$$\Theta(\pi(a)) = (\pi \otimes \pi)\big(\Delta_G(a)\big) \text{ and } \Theta(u_\gamma) = \sum_{r\in[\gamma]} u_\gamma \alpha(\mathbb{1}_{\gamma,r}) \otimes u_r,$$

for all $a \in C_m(G)$ and all $\gamma \in \Gamma$, where $\pi : C(G) \longrightarrow \Gamma \underset{\alpha,m}{\ltimes} C(G)$ is the unital faithful $*$-homomorphism and $u : \Gamma \longrightarrow \mathcal{U}(M(\Gamma \underset{\alpha,m}{\ltimes} C(G)))$ the group homomorphism defining the crossed product.

Besides, we have that

i) $\mathbb{F} := (\Gamma \underset{\alpha,m}{\ltimes} C(G), \Theta)$ is a compact quantum group,

ii) the Haar state on $\mathbb{F}$ is given by $h_{\mathbb{F}} := h_G \circ E \circ \kappa$, where h_G is the Haar integral of G, $\kappa : \Gamma \underset{\alpha,m}{\ltimes} C(G) \twoheadrightarrow \Gamma \underset{\alpha,r}{\ltimes} C(G)$ is the canonical surjection and $E : \Gamma \underset{\alpha,r}{\ltimes} C(G) \to C(G)$ is the canonical conditional expectation,

iii) the maximal picture of $\mathbb{F}$ is given by $C_m(\mathbb{F}) = \Gamma \underset{\alpha,m}{\ltimes} C(G)$,

iv) the reduced picture of $\mathbb{F}$ is given by $C_r(\mathbb{F}) = \Gamma \underset{\alpha,r}{\ltimes} C(G)$,

The compact quantum group $\mathbb{F}$ constructed in this way is called compact bicrossed product of the matched pair (Γ, G) *and is denoted by $\Gamma_\alpha \bowtie_\beta G$.*

2.4.2 Note. It is possible to give an explicit description of the representation theory of $\Gamma_\alpha \bowtie_\beta G$ in the same spirit of Theorem 2.3.2 for the quantum semi-direct product construction. Nevertheless, the presence of the action β (which is trivial in the quantum semi-direct product case) makes the classification of the irreducible representations of $\Gamma_\alpha \bowtie_\beta G$ more complicated.

Using this classification it is possible to carry out the analogous study of Section 2.3 in order to achieve the *associativity property for a compact bicrossed product*. Since the classification of irreducible representations of $\Gamma_\alpha \bowtie_\beta G$ given in [65] is not correct and its rectification is a work in progress, we do not develop here the associativity property for a compact bicrossed product.

2.4.3 Remark. Since G is a classical compact group, we have an obvious injective $*$-homomorphism $\iota_G : C(G) \hookrightarrow \Gamma \underset{\alpha,m}{\ltimes} C(G)$. Moreover, the definition of the co-multiplication Θ of $\mathbb{F}$ yields clearly that ι_G intertwines the corresponding co-multiplication. This shows, thanks to Proposition 1.4.3.4, that $\widehat{G}$ is a discrete quantum subgroup of $\widehat{\mathbb{F}}$.

As in Remark 2.3.9 we define canonical injections $\iota_\Gamma^m : C_m^*(\Gamma) \hookrightarrow C_m(\mathbb{F})$ and $\iota_\Gamma^r : C_r^*(\Gamma) \hookrightarrow C_r(\mathbb{F})$. However, the definition of the co-multiplication Θ of the compact bicrossed product $\mathbb{F}$ prevent ι_Γ^r and ι_Γ^m from intertwining the corresponding co-multiplications. Hence, Γ can not be a quantum subgroup of $\mathbb{F}$ (compare with the quantum semi-direct product case, see Proposition 2.3.5 and Remark 2.3.9). However, these injections are clearly compatible with the canonical surjections and co-units as in Remark 2.3.9.

2.5 Quantum free product

We introduce the free product of two compact quantum groups and we analyze some structure properties of this object which are useful for our purpose. We may recall the maximal free product construction for C^*-algebras from Theorem A.1.10.

The following theorem is due to S. Wang [215].

2.5.1 Theorem-Definition. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta_\mathbb{G})$ and $\mathbb{H} = (C(\mathbb{H}), \Delta_\mathbb{H})$ be two compact quantum groups.*

There exists a unique unital $*$-homomorphism

$$\Theta : C_m(\mathbb{G}) * C_m(\mathbb{H}) \longrightarrow C_m(\mathbb{G}) * C_m(\mathbb{H}) \otimes C_m(\mathbb{G}) * C_m(\mathbb{H})$$

such that

$$\Theta(\nu_\mathbb{G}(a)) = (\nu_\mathbb{G} \otimes \nu_\mathbb{G})\big(\Delta_\mathbb{G}(a)\big) \text{ and } \Theta(\nu_\mathbb{H}(b)) = (\nu_\mathbb{H} \otimes \nu_\mathbb{H})\big(\Delta_\mathbb{H}(b)\big),$$

for all $a \in C_m(\mathbb{G})$ and all $b \in C_m(\mathbb{H})$, where $\nu_\mathbb{G} : C(\mathbb{G}) \longrightarrow C_m(\mathbb{G}) * C_m(\mathbb{H})$ and $\nu_\mathbb{H} : C(\mathbb{H}) \longrightarrow C_m(\mathbb{G}) * C_m(\mathbb{H})$ denote the canonical inclusions.

Besides, we have that

i) $\mathbb{F} := (C_m(\mathbb{G}) * C_m(\mathbb{H}), \Theta)$ is a compact quantum group,

ii) the Haar state on $\mathbb{F}$ is given by $h_\mathbb{F} = h_\mathbb{G} * h_\mathbb{H}$, where $h_\mathbb{G}$ and $h_\mathbb{H}$ are the Haar states on $\mathbb{G}$ and $\mathbb{H}$, respectively,

iii) *the maximal picture of* $\mathbb{F}$ *is given by* $C_m(\mathbb{F}) = C_m(\mathbb{G}) * C_m(\mathbb{H})$,

iv) *the irreducible representations of* $\mathbb{F}$ *are described as follows: for every irreducible representation* $y \in Irr(\mathbb{F})$, *take a representative* $w^y \in \mathcal{B}(H_y) \otimes C(\mathbb{F})$. *There exist a natural number* $n \in \mathbb{N}$ *and irreducible representations* $\zeta_1, \dots, \zeta_n$ *either in* $Irr(\mathbb{G})$ *or in* $Irr(\mathbb{H})$ *such that if* $w^{\zeta_i} \in \mathcal{B}(H_i) \otimes C(\mathbb{G}$ *or* $\mathbb{H})$ *are respective representatives of* ζ_i, *for all* $i = 1, \dots, n$, *then we have*

$$w^y \cong w^{\zeta_{l_1}} \oplus w^{\zeta_{l_2}} \oplus \dots \oplus w^{\zeta_{l_n}} \in \mathcal{B}(H_{l_1} \otimes \dots H_{l_n}) \otimes C(\mathbb{F})$$

in such a way that $l_i \in \{1, \dots, n\}$ *with* $l_i \neq l_j$ *for all* $i \neq j$ *and if* $\zeta_{l_i} \in Irr(\mathbb{G})$ *then* $\zeta_{l_{i+1}} \in Irr(\mathbb{H})$ *(and* vice-versa*), for all* $i, j = 1, \dots, n-1$.

In other words, the words with letters in $Irr(\mathbb{G})$ *and* $Irr(\mathbb{H})$ *(according the above formulae) provide a complete set of mutually inequivalent irreducible representations of* $\mathbb{F}$. *For this reason we write* $Irr(\mathbb{F}) = Irr(\mathbb{G}) * Irr(\mathbb{H})$.

In addition, the fusion rules are described in the following way.

a) *If* $y, y' \in Irr(\mathbb{F}) = Irr(\mathbb{G}) * Irr(\mathbb{H})$ *are words such that* y *ends in* $Irr(\mathbb{G})$ *and* y' *starts in* $Irr(\mathbb{H})$ *(or* vice-versa*), then*

$$y \oplus y' = yy'$$

is an irreducible representation of $\mathbb{F}$.

b) *If* $y = \zeta x, y' = x'\zeta' \in Irr(\mathbb{F}) = Irr(\mathbb{G}) * Irr(\mathbb{H})$ *are words such that* $x, x' \in Irr(\mathbb{G})$ *(or in* $Irr(\mathbb{H})$*), then*

$$y \oplus y' = \bigoplus_{t \subset x \oplus x'} \zeta t \zeta' \oplus \delta_{\overline{x}, x'}(\zeta \oplus \zeta'),$$

where the sum runs over all non-trivial irreducible representations $t \in Irr(\mathbb{G})$ *(or in* $Irr(\mathbb{H})$*) contained in* $x \oplus x'$ *with multiplicity.*

The compact quantum group $\mathbb{F}$ *constructed in this way is called* quantum free product of $\mathbb{G}$ and $\mathbb{H}$ *and is denoted by* $\mathbb{G} * \mathbb{H}$.

2.5.2 Remark. By definition of a quantum free product of $\mathbb{G}$ and $\mathbb{H}$, it is clear that both $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are discrete quantum subgroups of $\widehat{\mathbb{G} * \mathbb{H}}$ with canonical inclusions given by $\nu_{\mathbb{G}}$ and $\nu_{\mathbb{H}}$, respectively (recall Proposition 1.4.3.4).

Observe that both $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are divisible in $\widehat{\mathbb{F}}$. Let us show that, for example, $\widehat{\mathbb{G}}$ is divisible in $\widehat{\mathbb{F}}$ (the proof for $\widehat{\mathbb{H}}$ is analogous). Take any irreducible representation of $\mathbb{F}$, which is given by an alternating word in $Irr(\mathbb{G})$ and $Irr(\mathbb{H})$, say

$$y := x_{i_1} z_{i_2} \dots x_{i_{n-1}} z_{i_n}$$

By definition, a representative of y in $Irr(\mathbb{G}) \backslash Irr(\mathbb{F})$ (resp. in $Irr(\mathbb{F}) / Irr(\mathbb{G})$) is an irreducible representation $y' \in Irr(\mathbb{F})$ such that $y \oplus \overline{y'}$ (resp. $\overline{y'} \oplus y$) contains an irreducible representation of $\mathbb{G}$ (inside $\mathbb{F}$). The latter is possible if and only if the tensor product $y \oplus \overline{y'}$ (resp. $\overline{y'} \oplus y$) reduces to a single letter in $Irr(\mathbb{G})$.

Assume that y starts in $Irr(\mathbb{G})$, then it is enough to put $y' := z_{i_2} \dots x_{i_{n-1}} z_{i_n}$. The fusion rules of a quantum free product yield that $y \oplus \overline{y'} = x_{i_1} \in Irr(\mathbb{G})$.

In this situation, given any $s \in Irr(\mathbb{G})$ we have to prove that $s \oplus y' \in Irr(\mathbb{F})$. Since y starts in $Irr(\mathbb{G})$, then y' starts in $Irr(\mathbb{H})$, so that the fusion rules of a quantum free product yield that $s \oplus y' = sy' \in Irr(\mathbb{F})$.

Assume that y starts in $Irr(\mathbb{H})$ and ends in $Irr(\mathbb{G})$, then it is enough to put $y' := x_{i_1} z_{i_2} \dots x_{i_{n-1}}$. The fusion rules of a quantum free product yield that $\overline{y'} \oplus y = z_{i_n} \in Irr(\mathbb{G})$. Since y ends in $Irr(\mathbb{G})$, then y' ends in $Irr(\mathbb{H})$ and the fusion rules of a quantum free product yield that $y' \oplus s = y's \in Irr(\mathbb{F})$, for every $s \in Irr(\mathbb{G})$.

Assume that y starts and ends in $Irr(\mathbb{H})$, then we can not choose any representative y' of y either in $Irr(\mathbb{G}) \backslash Irr(\mathbb{F})$ or in $Irr(\mathbb{F})/Irr(\mathbb{G})$ such that either $y \oplus \overline{y'}$ or $\overline{y'} \oplus y$ reduces to a single letter in $Irr(\mathbb{G})$. In other words, the class of $[y]$ is formed only by y itself (notice that $y \otimes \overline{y} = \epsilon_{\mathbb{H}} \cong \epsilon_{\mathbb{G}}$ in $Irr(\mathbb{F})$). In this case, it is obvious that $s \oplus y' = sy' \in Irr(\mathbb{F})$, for every $s \in Irr(\mathbb{G})$.

2.5.3 Definition. A free quantum group is a compact quantum group that is a free product of free unitary and free orthogonal quantum groups

$$U^+(P_1) * \dots * U^+(P_k) * O^+(Q_1) * \dots * O^+(Q_l),$$

for some $k, l \in \mathbb{N}$ where $P_i \in GL_{m_i}(\mathbb{C})$ with $m_i \geqslant 2$ for all $i = 1, \dots, k$ and $Q_j \in GL_{n_j}(\mathbb{C})$ with $n_j \geqslant 2$ satisfies $Q_j \overline{Q}_j = \pm id$ for all $j = 1, \dots, l$.

2.5.4 Remark. Observe that if in the preceding definition we take $l = 0$, $m_i = 1$ and $P_i = id \in GL_1(\mathbb{C})$, for all $i = 1, \dots, k$, then the corresponding free quantum group reduces to a *classical* free group on k generators.

In this sense, the preceding definition is a generalization of the classical case. Namely, the classical free group on n generators can be written as a free product of n copies of $\mathbb{Z}$, $\mathbb{F}_n = \mathbb{Z} * \dots * \mathbb{Z}$. In the quantum setting, we need to allow different building blocks for constructing free quantum groups.

The quantum free product construction and particularly the case of a free quantum group have been successfully studied by R. Vergnioux and C. Voigt [208] in relation with the Baum-Connes property and the K-theory computations. In Section 3.6 we will give an overview of the Vergnioux-Voigt's work concerning the Baum-Connes property. Let us state here the corresponding result concerning the K-theory computations. Notice that the following result is a generalization of the well-known Pimsner-Voiculescu exact sequence [151], [152] for free quantum groups.

2.5.5 Theorem (R. Vergnioux and C. Voigt, [208])**.** *Let $\mathbb{G} := U^+(P_1) * \dots * U^+(P_k) * O^+(Q_1) * \dots * O^+(Q_l)$ be a free quantum group. Then $\widehat{\mathbb{G}}$ is K-amenable and for every $\widehat{\mathbb{G}}$-C^*-algebra A there exists a six-term exact sequence*

$$\begin{tikzcd}
\bigoplus_{r=1}^{2k+l} K_0(A) \arrow[r] & K_0(A) \arrow[r] & K_0(\widehat{\mathbb{G}} \ltimes A) \arrow[d] \\
K_1(\widehat{\mathbb{G}} \ltimes A) \arrow[u] & K_1(A) \arrow[l] & \bigoplus_{r=1}^{2k+l} K_1(A) \arrow[l]
\end{tikzcd}$$

As a consequence, we have that the K-theory of $\mathbb{G} = U^+(P_1) * \dots * U^+(P_k) * O^+(Q_1) * \dots * O^+(Q_l)$ is given by

$$K_0(C(\mathbb{G})) = \mathbb{Z} \text{ and } K_1(C(\mathbb{G})) = \mathbb{Z}^{2k} \oplus \mathbb{Z}^l$$

2.6 Free wreath product

We introduce the wreath product of a compact quantum group by S_N^+ and we analyze some structure properties of this object that are useful for our purpose.

It is important to recall that in the classical case, a wreath product can be regarded as a generalization of a semi-direct product that provides new and interesting examples of groups in order to study automorphism groups. With this philosophy in mind, we want to give a *quantum* version of such an object. The first definition and construction of a *free wreath product* was given by J. Bichon in [22] where he defines the wreath product of a compact quantum group $\mathbb{G}$ by a quantum permutation group S_N^+, denoted by $\mathbb{G} \wr_* S_N^+$. Recently, the work of P. Fima and L. Pittau in [66] gives a generalization of this construction replacing S_N^+ by any quantum automorphism group $Qut(A, \omega)$, defining thus the compact quantum group $\mathbb{G} \wr_* Qut(A, \omega)$.

In the context of the present thesis, the original definition of J. Bichon is enough for our purpose. The main construction of the following theorem can be found in [22] and we refer to the work of F. Lemeux and P. Tarrago in [120] for the corresponding representation theory and more details.

2.6.1 Theorem-Definition. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and fix a natural number $N \in \mathbb{N}$. Given the free product $\mathbb{G}^{*N}$, denote by $\nu_k : C(\mathbb{G}) \longrightarrow C(\mathbb{G}^{*N})$ the canonical faithful $*$-homomorphism corresponding to the k-th component, for each $k = 1, \dots, N$. Denote by $u = (u_{ij})_{i,j=1,\dots,N}$ the fundamental representation of S_N^+.*

There exists a unique $*$-homomorphism

$$\Theta : C(\mathbb{G}^{*N}) * C(S_N^+)/I \longrightarrow C(\mathbb{G}^{*N}) * C(S_N^+)/I \otimes C(\mathbb{G}^{*N}) * C(S_N^+)/I,$$

where I is the closed two-sided ideal in $C(\mathbb{G}^{*N}) * C(S_N^+)$ generated by $\{\nu_k(a)u_{ki} - u_{ki}\nu_k(a) \mid a \in C(\mathbb{G})\}_{i,k=1,\dots,N}$; such that

$$\Theta(\nu_i(a)) = \sum_{k=1}^{N} \nu_i(a_{(1)})u_{ik} \otimes \nu_k(a_{(2)}) \text{ and } \Theta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj},$$

for all $a \in C(\mathbb{G})$ and all $i = 1, \dots N$ and for all $i, j = 1, \dots N$.

Besides, we have that

i) $\mathbb{F} := (C(\mathbb{G}^{*n}) * C(S_N^+)/I, \Theta)$ is a compact quantum group,

ii) If $\mathbb{G}$ is a maximal compact quantum group, then $\mathbb{F}$ too,

iii) the irreducible representations of $\mathbb{F}$ are described as follows: for every irreducible representation $x \in Irr(\mathbb{G})$, take a representative $w^x \in \mathcal{B}(H_x) \otimes C(\mathbb{G})$, denote by $(w^x)^{(k)}$ a representative of x in the k-th component of $\mathbb{G}^{*N}$ and put

$$r(x) := \left(u_{ij}\nu_k((w_{ij}^x)^{(k)})\right)_{k,i,j=1,\dots,N}$$

Irreducible representations of $\mathbb{F}$ are subrepresentations of the tensor products

$$r(x_1) \oplus \dots \oplus r(x_n),$$

for all $n \in \mathbb{N}$. In other words, the irreducible representation of $\mathbb{F}$ are labelled by words over $Irr(\mathbb{G})$. For this reason we write $Irr(\mathbb{F}) = \langle Irr(\mathbb{G})\rangle$.

In addition, the fusion rules are described in the following way. If $x, y \in \langle Irr(\mathbb{G})\rangle$ and $\omega(x), \omega(y) \in Irr(\mathbb{F})$ denotes the corresponding irreducible representations of $\mathbb{F}$, then

$$\omega(x) \oplus \omega(y) = \Big(\sum_{\substack{x=ut\\ y=\bar{t}v}} \omega(u,v)\Big) \oplus \Big(\sum_{\substack{x=ut,\ u\neq\varnothing\\ y=\bar{t}v,\ v\neq\varnothing}} \omega(u \oplus v)\Big),$$

where we must point out the notation.

- *$\langle Irr(\mathbb{G})\rangle$ is equipped with an involution defined by*

$$\overline{(x_1, \ldots, x_n)} := (\overline{x}_n, \ldots, \overline{x}_1),$$

  for all $x_1, \ldots, x_n \in Irr(\mathbb{G})$.
- *If $u, v \in \langle Irr(\mathbb{G})\rangle$ are two words, we denote by (u, v) their concatenation.*
- *Given two words $u := (x_1, \ldots, x_n), v := (y_1, \ldots, y_m) \in \langle Irr(\mathbb{G})\rangle$, we denote by $u \oplus v$ the following operation*

$$u \oplus v := \bigoplus_{z \subset x_n \oplus y_1} (x_1, \ldots, x_{n-1}, z, y_2, \ldots, y_m)$$

The compact quantum group $\mathbb{F}$ constructed in this way is called free wreath product of $\mathbb{G}$ by S_N^+ *and is denoted by $\mathbb{G} \wr_* S_N^+$.*

We will see in Section 3.7.1 that a free wreath product $\mathbb{G} \wr_* S_N^+$ is *never* torsion-free. Consequently, it is not clear, from a theoretical point of view, how to formulate the Baum-Connes property for it. As we have already pointed out in the end of Section 1.7.2, one possible strategy is to consider a suitable monoidal equivalence between $\mathbb{G} \wr_* S_N^+$ and some compact quantum group $\mathbb{H}$, for which it is easier to establish or to formulate the Baum-Connes property. In this sense, the following result is crucial in order to develop the study of the torsion phenomena and the Baum-Connes property for a free wreath product as we have done in Section 3.7.

2.6.2 Theorem (F. Lemeux and P. Tarrago, [120])**.** *Let $\mathbb{G}$ be a compact quantum group. If u denotes the fundamental representation of $SU_q(2)$, let $\mathbb{H}_q$ be the compact quantum group generated by uxu for all $x \in Irr(\mathbb{G})$. The following properties hold.*

i) $\widehat{\mathbb{H}}_q$ is a discrete quantum subgroup of $\mathbb{G} * \widehat{SU_q}(2)$.

ii) If $N \geqslant 4$, there exists $0 < |q| < 1$ with $q + q^{-1} = \sqrt{N}$ such that $\mathbb{H}_q$ is monoidally equivalent to $\mathbb{G} \wr_* S_N^+$.

2.6.3 Remark. In order to carry out a study of the torsion phenomena and the Baum-Connes property for a free wreath product $\mathbb{G} \wr_* S_N^+$, it is advisable to give some explicite expressions concerning the representation theory of $\mathbb{G} \wr_* S_N^+$ and $\mathbb{H}_q$. We refer to Section 5 of [120] for more details.

First of all, the precise description of the Lemeux-Tarrago's compact quantum group $\mathbb{H}_q := (C(\mathbb{H}_q), \Delta_q)$ of the preceding theorem is the following.

- $C(\mathbb{H}_q)$ is generated, as $*$-algebra, by the matrix coefficients of representations of the form uxu, for all $x \in Irr(\mathbb{G})$. In other words, $\mathbb{H}_q$ is defined by Tannaka-Krein-Woronowicz duality where $\mathscr{R}ep(\mathbb{H}_q)$ is identified with the smallest full subcategory of $\mathscr{R}ep(\mathbb{G} * SU_q(2))$ containing $\{uxu \mid x \in Irr(\mathbb{G})\} \subset Irr(\mathbb{G} * SU_q(2))$.
- The co-multiplication $\Delta_q : C(\mathbb{H}_q) \longrightarrow C(\mathbb{H}_q) \otimes C(\mathbb{H}_q)$ is such that

$$\Delta_q(u_{ij}x_{rs}u_{kl}) = \Delta_{SU_q(2)}(u_{ij})\Delta_{\mathbb{G}}(x_{rs})\Delta_{SU_q(2)}(u_{kl}),$$

  for all $i, j, k, l = 1, 2$, all $r, s = 1, \ldots, n_x$ and all $x \in Irr(\mathbb{G})$.

This means that the irreducible representations of $\mathbb{H}_q$ are subrepresentations of the tensor products

$$(ux_1u) \oplus \ldots \oplus (ux_nu),$$

with $x_1, \ldots, x_n \in Irr(\mathbb{G})$ and $n \in \mathbb{N}$; which decomposes as a direct sum of irreducible representations of the form

$$u^{l_1}z_1u^{l_2} \ldots z_{l_{n-1}}u^{l_n} \in Irr(\mathbb{G}) * Irr(SU_q(2)),$$

where l_1 and l_n are *odd integers*, for all $i = 2, \ldots, n-1$ l_i are *even integers* and for all $i = 1, \ldots, l_{n-1}$ $z_i \subset x_{i_1} \oplus \ldots \oplus x_{i_t}$ is irreducible for some $i_1, \ldots, i_t \in \{1, \ldots, n\}$.

In this way, we have a bijection between $Irr(\mathbb{H}_q)$ and the free monoid $\langle Irr(\mathbb{G})\rangle$ formed by the words over $Irr(\mathbb{G})$, so that any irreducible representation of $\mathbb{H}_q$ is *labelled* by a word in $\langle Irr(\mathbb{G})\rangle$. This description of $Irr(\mathbb{H}_q)$, together with the fusion rules of a free product of compact quantum groups (see Theorem 2.5.1), allows to give an explicit description of the inclusion $\Lambda : Fus(\widehat{\mathbb{H}}_q) \hookrightarrow Fus(\mathbb{G} * \widehat{SU_q}(2))$ given by $\widehat{\mathbb{H}}_q < \mathbb{G} * \widehat{SU_q}(2)$ of the preceding theorem.

- If $x_1 = \ldots = x_n = \epsilon_{\mathbb{G}}$, then $\Lambda(\epsilon_{\mathbb{G}}^n) = u^{2n}$, for all $n \in \mathbb{N}$.
- If $x_i \neq \epsilon_{\mathbb{G}}$, for all $i = 1, \ldots, n$, then $\Lambda(x_1 \ldots x_n) = ux_1u^2x_2u^2 \ldots u^2x_nu$, for all $n \in \mathbb{N}$.
- In general, if ω is any word over $Irr(\mathbb{G})$, we can write it under the form $\omega = \epsilon_{\mathbb{G}}^{n_1}\omega_1\epsilon_{\mathbb{G}}^{n_2}\omega_2 \ldots \omega_k\epsilon_{\mathbb{G}}^{n_{k+1}}$, where each ω_i does not contain $\epsilon_{\mathbb{G}}$. Denote by $\widetilde{\Lambda}(\omega_i)$ the same expression as $\Lambda(\omega_i)$ from the preceding case except that the first and last u are removed. Then, we have $\Lambda(\omega) = u^{2n_1+1}\widetilde{\Lambda}(\omega_1)u^{2n_2+1}\widetilde{\Lambda}(\omega_2) \ldots u^{2n_k+1}\widetilde{\Lambda}(\omega_k)u^{2n_{k+1}+1}$, for all $k \in \mathbb{N}$.

Finally, using the above description of $Irr(\mathbb{H}_q)$, the fusion rules of $\mathbb{H}_q$ are described as follows: if $x_1, \ldots, x_n, y_1, \ldots, y_m \in Irr(\mathbb{G})$, then

$$(x_1 \ldots x_n) \oplus (y_1 \ldots y_m) = \Big(\sum_{z \subset x_n \oplus y_1} x_1 \ldots x_{n-1}\ z\ y_2 \ldots y_m\Big) \oplus \delta_{x_n, \overline{y}_1}\ (x_1 \ldots x_{n-1}) \oplus (y_2 \ldots y_m)$$

2.6.4 Remark. Notice that the discrete quantum group $\widehat{\mathbb{H}}_q$ might *not be divisible* in $\mathbb{G} * \widehat{SU_q}(2)$ in general. Namely, as we will see in Section 3.7.1, the dual of the free wreath product $\mathbb{G} \wr_* S_N^+$ is *never* torsion-free, even when $\widehat{\mathbb{G}}$ is torsion-free. Hence, $\widehat{\mathbb{H}}_q$ is *never* torsion-free, by monoidal equivalence. But, if $\widehat{\mathbb{G}}$ is torsion-free, then it is well-known that $\mathbb{G} * \widehat{SU_q}(2)$ is torsion-free (by combining Theorem 2.1.5 and Theorem 3.6.1.1). If $\widehat{\mathbb{H}}_q$ was divisible in $\mathbb{G} * \widehat{SU_q}(2)$, then $\widehat{\mathbb{H}}_q$ would be torsion-free provided that torsion-freeness is preserved by divisible discrete quantum groups as conjectured in the end of Section 3.2.1.

This shortcoming makes the proof of the *strong* Baum-Connes property for $\widehat{\mathbb{G} \wr_* S_N^+}$ of Section 3.7.2 a non trivial result. Namely, if $\widehat{\mathbb{G}}$ is torsion-free and satisfies the *strong* Baum-Connes property, then it is well-known that $\mathbb{G} * \widehat{SU_q}(2)$ is torsion-free and satisfies the *strong* Baum-Connes property as well (by combining Theorem 2.1.5 and Theorem 3.6.2.3). If $\widehat{\mathbb{H}}_q$ was divisible in $\mathbb{G} * \widehat{SU_q}(2)$, then $\widehat{\mathbb{H}}_q$ would satisfy automatically the *strong* Baum-Connes property by Theorem 3.2.2.1. Thus, $\widehat{\mathbb{G} \wr_* S_N^+}$ too by monoidal equivalence (recall Theorem B.3.19).

CHAPTER 3

Stability properties for the Quantum Baum-Connes property

This chapter should be regarded as the core of the thesis. Here we develop the main contributions of the present dissertation.

In Section 3.1 we care about the *compact* case. Namely, we study the Baum-Connes property for any compact quantum group and for the duals of compact Lie groups which are connected and have torsion-free fundamental group following the work of R. Meyer and R. Nest [133].

In Section 3.2 we recall that the *divisible* condition of a discrete quantum subgroup is a sufficient one to conclude that the strong Baum-Connes property passes to quantum subgroups following the work of R. Vergnioux and C. Voigt [208] and also to conclude that the strong torsion-freeness passes to quantum subgroups following the work of Y. Arano and K. De Commer [3]. Moreover, we establish the analogue stability result for the *usual* Baum-Connes property. The K-amenability property for a discrete quantum subgroup is analyzed.

In Section 3.3 we establish the quantum counterpart of the stability for the Baum-Connes property for a direct product in connexion with the *Künneth formula* as studied in [37] by J. Chabert, S. Echterhoff and H. Oyono-Oyono for classical locally compact groups. In addition, the K-amenability property for such a construction is analyzed.

In Section 3.4 we establish the quantum counterpart of the stability for the Baum-Connes property for a semi-direct product obtained by H. Oyono-Oyono [143] in the classical discrete case. In Section 3.5 we observe that, under the torsion-freeness assumption, the analogue study of the Baum-Connes property for a compact bicrossed product in the sense of P. Fima-K. Mukherjee-I. Patri [65] reduces to the quantum semi-direct product studied earlier. In addition, the K-amenability property for both constructions is analyzed.

In Section 3.6 we recall that the Baum-Connes property is stable under the free product construction for quantum groups following the work of R. Vergnioux and C. Voigt [208]. Our contribution here is the complete classification of the torsion actions for a quantum free product,

which is part of a collaboration work with A. Freslon [127].

In Section 3.7 we show that the Baum-Connes property is stable under the free wreath product construction using the well-known monoidal equivalence given by F. Lemeux and P. Tarrago in [120] and being inspired by the pioneering work of C. Voigt in [212]. In addition the K-amenability property for such a construction is analyzed. The whole content of Section 3.7 is a collaboration work with A. Freslon [127].

3.1 The Baum-Connes property for *compact* quantum groups

The compact case is one of the first situations that we have to analyze in relation with the Baum-Connes property. Indeed, from a topological point of view, compact groups are particularly interesting and rich concerning the representation theory and the approximation properties.

In this sense, it is well-known that classical compact groups always satisfy the Baum-Connes property. We can see this in different ways. On the one hand, we can prove it directly by recalling that the corresponding classifying space for proper actions is a *singleton*. On the other hand, we can apply the celebrated and strong result of Higson-Kasparov [82] to conclude that, actually, compact groups satisfy the *strong* Baum-Connes property.

Let us analyze the Baum-Connes property for compact *quantum* groups.

3.1.1 Theorem. *Let $\mathbb{G}$ be a compact quantum group such that $\widehat{\mathbb{G}}$ is a torsion-free discrete quantum group. Consider the homological functor $\widetilde{F} : \mathscr{K}\mathscr{K}^{\mathbb{G}} \longrightarrow \mathscr{A}b^{\mathbb{Z}/2}$ defined by $\widetilde{F}(A,\delta) := K_*(\mathbb{G} \underset{\delta,r}{\ltimes} A)$, for all $(A,\delta) \in Obj(\mathscr{K}\mathscr{K}^{\mathbb{G}})$.*

Then $\mathbb{G}$ satisfies the Baum-Connes property with respect to $(\mathscr{L}_{\mathbb{G}}, \mathscr{N}_{\mathbb{G}}, \widetilde{F})$.

Proof. Recall from Section 1.7.2 that, under the torsion-freeness assumption, the complementary pair of subcategories $(\mathscr{L}_{\mathbb{G}}, \mathscr{N}_{\mathbb{G}})$ is given by Baaj-Skandalis duality by

$$\mathscr{L}_{\mathbb{G}} = \langle \{A \in Obj(\mathscr{K}\mathscr{K}^{\mathbb{G}}) \mid A \text{ with trivial action of } \mathbb{G}\} \rangle$$

$$\mathscr{N}_{\mathbb{G}} = \{A \in Obj(\mathscr{K}\mathscr{K}^{\mathbb{G}}) \mid \mathbb{G} \underset{r}{\ltimes} A \cong 0 \text{ in } \mathscr{K}\mathscr{K} \text{ as a trivial } \widehat{\mathbb{G}}^{cop}\text{-}C^*\text{-algebra}\}$$

Denote by $(\widetilde{L}, \widetilde{N})$ the canonical triangulated functors associated to this complementary pair. Consider the homological functor $\widetilde{F}$ of the statement and the categorical assembly map with respect to $(\mathscr{L}_{\mathbb{G}}, \mathscr{N}_{\mathbb{G}}, \widetilde{F})$ (recall Definition 1.2.1.30)

$$\widetilde{\eta} : \mathbb{L}\widetilde{F} \longrightarrow \widetilde{F},$$

where by definition we have $\mathbb{L}\widetilde{F} = \widetilde{F} \circ \widetilde{L}$.

We have to prove that $\widetilde{\eta}$ is a natural equivalence. By universal property of localization (recall Theorem 1.2.1.29 and Theorem 1.2.3.17) this is equivalent to say that $\widetilde{F}_{|\mathscr{N}_{\mathbb{G}}} = 0$, which is obvious by definition of $\mathscr{N}_{\mathbb{G}}$.

■

Once the Baum-Connes property for the compact case is clarified, we have to care about discrete quantum groups, which arise as duals of compact (quantum) groups. As we have already explained at different stages of the dissertation, the torsion phenomena is one of the main obstacles for this study.

An important source of examples of topological groups is *Lie groups*. Moreover, they are in the origins of the quantum groups theory from the physical point of view. In this sense, it is reasonable to look at *duals of compact Lie groups*, which should be regarded as a particular case of discrete

quantum groups. This is the object of study of the article [133] by R. Meyer and R. Nest. More precisely, they give a concrete application of the abstract formulation of the Baum-Connes property for duals of *compact connected Lie groups*. In particular, they are able to manage properly the quantum torsion phenomena in this situation.

Let us present here the *torsion-free* version of the main theorem of [133] in order to exhibit the strategy followed by Meyer-Nest in such a situation. Firstly, a combination of some classical well-known results [129], [165], [180] yields the following lemma (see Proposition 2.1 of [133] for a proof).

3.1.2 Lemma. *Let G be a compact connected Lie group with torsion-free fundamental group. If $T < G$ denotes the maximal torus of G, then there exists $N \in \mathbb{N}$ such that*

$$\mathbb{C}^N \cong C(G/T) \text{ in } \mathscr{KK}^G$$

3.1.3 Theorem. *Let G be a compact connected Lie group with torsion-free fundamental group. Then $\widehat{G}$ satisfies the strong Baum-Connes property. As a consequence, $\widehat{G}$ satisfies the Baum-Connes property.*

Proof. Given the discrete quantum group $\widehat{G}$, consider the corresponding $\widehat{G}$-equivariant Kasparov category $\mathscr{KK}^{\widehat{G}}$ with the usual complementary pair of localizing subcategories $(\mathscr{L}, \mathscr{N})$ as explained in Section 1.7.2. By definition, $\widehat{G}$ satisfies the strong Baum-Connes property if and only if $\mathscr{L} = \mathscr{KK}^{\widehat{G}}$. Since $\mathscr{L}$ and $\mathscr{N}$ are complementary, this condition is equivalent to the condition $\mathscr{N} = (0)$.

In other words, we have to see the following: given $B \in Obj(\mathscr{KK}^{\widehat{G}})$ such that $B \cong 0$ in $\mathscr{KK}$, then $B \cong 0$ in $\mathscr{KK}^{\widehat{G}}$.

Next, by Baaj-Skandalis duality this statement is equivalent to the following one: given $A \in Obj(\mathscr{KK}^G)$ such that $G \underset{r}{\ltimes} A \cong 0$ in $\mathscr{KK}$, then $A \cong 0$ in $\mathscr{KK}^G$.

Therefore, consider any G-C^*-algebra $A \in Obj(\mathscr{KK}^G)$ and denote by $T < G$ the maximal torus of the compact Lie group G. Apply the preceding lemma to write the following chain of equivalences.

$$\begin{aligned} G \underset{r}{\ltimes} A \cong 0 \text{ in } \mathscr{KK} &\Leftrightarrow G \underset{r}{\ltimes} (A \otimes \mathbb{C}^N) \cong 0 \text{ in } \mathscr{KK} \\ &\Leftrightarrow G \underset{r}{\ltimes} (A \otimes C(G/T)) \cong 0 \text{ in } \mathscr{KK} \\ &\Leftrightarrow T \underset{r}{\ltimes} Res_T^G(A) \cong 0 \text{ in } \mathscr{KK}, \end{aligned}$$

where the last one is true by virtue of the obvious Morita equivalence $A \otimes C(G/T) \underset{M}{\sim} Ind_T^G(Res_T^G(A))$, so that the Green's Imprimitivity theorem guarantees that $T \underset{r}{\ltimes} Res_T^G(A) \underset{M}{\sim} G \underset{r}{\ltimes} (A \otimes C(G/T))$.

The preceding lemma yields also the following chain of equivalences.

$$\begin{aligned} A \cong 0 \text{ in } \mathscr{KK}^G &\Leftrightarrow A \otimes \mathbb{C}^N \cong 0 \text{ in } \mathscr{KK}^G \\ &\Leftrightarrow A \otimes C(G/T) \cong 0 \text{ in } \mathscr{KK}^G \\ &\Leftrightarrow Ind_T^G(Res_T^G(A)) \cong 0 \text{ in } \mathscr{KK}^G, \end{aligned}$$

where the last one is true by virtue of the same Morita equivalence as above.

These computations assure that $\widehat{G}$ satisfies the strong Baum-Connes property whenever $\widehat{T}$ satisfies the strong Baum-Connes property, which is obviously true. Indeed, the maximal torus T is isomorphic to a n-dimensional torus $\mathbb{T}^N$ for some $n \in \mathbb{N}$ and the classical Pontryagin duality yields

$$\widehat{\mathbb{T}^n} \cong \mathbb{Z}^n \text{ and } \mathbb{T}^n \cong \widehat{\mathbb{Z}^n},$$

where $\widehat{\mathbb{T}^n} \cong \mathbb{Z}^n$ satisfies the *strong* Baum-Connes property by virtue of the Higson-Kasparov theorem [82] since it is amenable. ■

3.2 The Baum-Connes property for a quantum subgroup

In this section we recall that the *divisible* condition of a discrete quantum subgroup is a sufficient one to conclude that the *strong* Baum-Connes property passes to quantum subgroups following the work of R. Vergnioux and C. Voigt [208]. We will see that the same is true for the *usual* Baum-Connes property.

The divisibility condition for a quantum discrete subgroup may be regarded as a condition of existence of a section for the canonical quotient map. In this way, every classical discrete subgroup $H < G$ is divisible. We refer to Section 4 of [208] for the full details and we state here the main results. Notations and definitions of Section 1.4.3 concerning induced actions and discrete quantum subgroups may be kept in mind for what follows.

3.2.1 Definition. Let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup of a discrete quantum group $\widehat{\mathbb{G}}$. We say that $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$ is divisible if there exists a $\widehat{\mathbb{H}}$-equivariant $*$-isomorphism

$$c_0(\widehat{\mathbb{G}}) \cong c_0(\widehat{\mathbb{H}}) \otimes c_0(\widehat{\mathbb{H}} \backslash \widehat{\mathbb{G}})$$

with respect to the restricted action on the left hand side and the action given by co-multiplication on $c_0(\widehat{\mathbb{H}})$ on the right hand side.

This notion can be characterized in terms of the representation theory of the quantum groups involved. More precisely, we have the following result.

3.2.2 Lemma. *Let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup of a discrete quantum group $\widehat{\mathbb{G}}$. The following assertions are equivalent.*

i) $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$ *is divisible.*

ii) For each equivalence class $\alpha \in Irr(\mathbb{H}) \backslash Irr(\mathbb{G})$ there exists a representation $l_\alpha \in \alpha$ such that $s \oplus l_\alpha$ is irreducible, for all $s \in Irr(\mathbb{H})$.

iii) For each equivalence class $\alpha \in Irr(\mathbb{G}) / Irr(\mathbb{H})$ there exists a representation $r_\alpha \in \alpha$ such that $r_\alpha \oplus s$ is irreducible, for all $s \in Irr(\mathbb{H})$.

3.2.1 Torsion property

As we have already explained, the torsion phenomena for discrete quantum groups is closely related with the *strong* Baum-Connes property (see Section 1.7.2 and Chapter 5 for more precisions). The *divisibility* condition is sufficient to state the *strong* torsion-freeness (torsion-freeness in the sense of Arano-De Commer) of a discrete quantum subgroup from the *strong* torsion-freeness of the discrete quantum group. More precisely, we have the following result due to Y. Arano and K. De Commer (see Proposition 1.28 in [3]).

3.2.1.1 Theorem (Y. Arano and K. De Commer, [3])**.** *Let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup of a discrete quantum group $\widehat{\mathbb{G}}$. If $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$ is divisible and $\widehat{\mathbb{G}}$ is strong torsion-free, then $\widehat{\mathbb{H}}$ is strong torsion-free.*

It is important to observe that usual torsion-freeness (that is, torsion-freeness in the sense of Meyer-Nest) is *not* preserved, in general, by discrete quantum subgroups. For instance, we have explained in Section 2.1 that $\widehat{SO_q(3)} < \widehat{SU_q(2)}$. Moreover, while $\widehat{SU_q(2)}$ is torsion-free by [210], $\widehat{SO_q(3)}$ is *not* torsion-free by [173] (recall the discussion in the end of Section 1.6.1).

In relation with the results obtained in [127], we can consider an other example more complicated. Let $\mathbb{G}$ be a compact quantum group such that $\widehat{\mathbb{G}}$ is torsion-free. Then the dual of the free product $\mathbb{G} * SU_q(2)$ is torsion-free (because $\widehat{SU_q(2)}$ is torsion-free for all $q \in (-1,1)\backslash\{0\}$ as it is shown in [210] and torsion-freeness is preserved by free product as it is shown in [3]). Consider the Lemeux-Tarrago's discrete quantum subgroup $\widehat{\mathbb{H}}_q < \mathbb{G} * \widehat{SU_q}(2)$ which is such that $\mathbb{H}_q$ is monoidally equivalent to the free wreath product $\mathbb{G} \wr_* S_N^+$ (see [120] or Section 2.6 for more details). It is explained in [127] (and in Section 3.7) that the dual of $\mathbb{G} \wr_* S_N^+$ is *never* torsion-free. Hence $\widehat{\mathbb{H}}_q$ neither (because torsion-freeness is preserved under monoidally equivalence as it is shown in [210] or [163]).

It is reasonable to expect that torsion-freeness (in the sense of Meyer-Nest) is preserved under *divisible* discrete quantum subgroups. Inspired by the study carried out in Section 3.6.1, we expect to be able to apply the techniques from [3] for proving the following stability result: given a compact quantum group $\mathbb{G}$, $\widehat{\mathbb{G}}$ is torsion-free if and only if every divisible discrete quantum subgroup $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$ is torsion-free.

3.2.2 The Baum-Connes property

In order to legitimate the Baum-Connes property formulation for a discrete quantum group $\widehat{\mathbb{G}}$, we need $\widehat{\mathbb{G}}$ to be torsion-free. The same is true for a discrete quantum subgroup $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$. Hence, we must keep the preceding section in mind. For simplicity we assume that every discrete quantum group considered in what follows is torsion-free.

Next, let us set up the adapted notations of the categorical framework of Meyer-Nest to this situation. Consider the equivariant Kasparov categories associated to $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$, say

$$\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \text{ and } \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}},$$

respectively. The canonical suspension functors for each of the above Kasparov categories are denoted simply by Σ.

Notice that from now on, the word *homomorphism (respectively, isomorphism)* will mean *homomorphism (respectively, isomorphism) in the corresponding Kasparov category*; it can be either a true homomorphism (respectively, isomorphism) between C^*-algebras or any Kasparov triple between C^*-algebras (respectively, any KK-equivalence between C^*-algebras).

Consider the canonical complementary pair of localizing subcategories in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ and $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}}$, say

$$(\mathscr{L}_{\widehat{\mathbb{G}}}, \mathscr{N}_{\widehat{\mathbb{G}}}) \text{ and } (\mathscr{L}_{\widehat{\mathbb{H}}}, \mathscr{N}_{\widehat{\mathbb{H}}}),$$

respectively. In this way, the canonical triangulated functors associated to these complementary pairs are denoted by (L, N), (L', N'), respectively.

Consider the homological functors defining the *quantum* Baum-Connes assembly maps for $\widehat{\mathbb{F}}$, $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$. Namely,

$$\begin{array}{rccl} F: & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} & \longrightarrow & \mathscr{A}b^{\mathbb{Z}/2} \\ & (A,\alpha) & \longmapsto & F(A) := K_*(\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A) \end{array} \qquad \begin{array}{rccl} F': & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}} & \longrightarrow & \mathscr{A}b^{\mathbb{Z}/2} \\ & (B,\beta) & \longmapsto & F'(B) := K_*(\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B) \end{array}$$

Therefore, the quantum assembly maps for $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are given by the following natural transformations

$$\eta^{\widehat{\mathbb{G}}} : \mathbb{L}F \longrightarrow F \text{ and } \eta^{\widehat{\mathbb{H}}} : \mathbb{L}F' \longrightarrow F',$$

where by definition we have

$$\mathbb{L}F = F \circ L \text{ and } \mathbb{L}F' = F' \circ L'$$

By the torsion-freeness assumption, the subcategory $\mathscr{L}_{\widehat{\mathbb{G}}}$ (resp. $\mathscr{L}_{\widehat{\mathbb{H}}}$) is easily described as the *localizing subcategory of $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ (resp. $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}}$) generated by the objects of the form $c_0(\widehat{\mathbb{G}}) \otimes C$ (resp. $c_0(\widehat{\mathbb{H}}) \otimes C$) with C any C^*-algebra in the Kasparov category $\mathscr{K}\mathscr{K}$.*

In relation with the *strong* Baum-Connes property, the divisibility condition of a discrete quantum subgroup guarantees that the restriction functor preserves the localizing subcategories of compactly induced C^*-algebras. More precisely, if $\widehat{\mathbb{H}}$ is a *divisible* discrete quantum subgroup of a discrete quantum group $\widehat{\mathbb{G}}$, then by definition we have a $\widehat{\mathbb{H}}$-equivariant $*$-isomorphism as in Definition 3.2.1

$$c_0(\widehat{\mathbb{G}}) \cong c_0(\widehat{\mathbb{H}}) \otimes c_0(\widehat{\mathbb{H}} \backslash \widehat{\mathbb{G}})$$

Both $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are supposed to be torsion-free. Hence the description of the subcategories $\mathscr{L}_{\widehat{\mathbb{G}}}$ and $\mathscr{L}_{\widehat{\mathbb{H}}}$ yields clearly that

$$Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(\mathscr{L}_{\widehat{\mathbb{G}}}) \subset \mathscr{L}_{\widehat{\mathbb{H}}},$$

whenever $\widehat{\mathbb{H}}$ is divisible in $\widehat{\mathbb{G}}$.

Assume that $\widehat{\mathbb{G}}$ satisfies the *strong* Baum-Connes property. Let $B \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}})$ be any $\widehat{\mathbb{H}}$-C^*-algebra. Since $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} = \mathscr{L}_{\widehat{\mathbb{G}}}$ by assumption, then $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(B) \in \mathscr{L}_{\widehat{\mathbb{G}}}$. Since $\widehat{\mathbb{H}}$ is divisible in $\widehat{\mathbb{G}}$, the preceding discussion yields that $Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\Big(Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(B)\Big) \in \mathscr{L}_{\widehat{\mathbb{H}}}$. Next, the explicit description of induced C^*-algebras with respect to discrete quantum subgroups (see Section 1.4.3), yields that B is a retract of $Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\Big(Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(B)\Big)$. Hence $B \in \mathscr{L}_{\widehat{\mathbb{H}}}$ as well because $\mathscr{L}_{\widehat{\mathbb{H}}}$ is, by definition, closed under retracts. In other words, we have proved the following result.

3.2.2.1 Theorem (R. Vergnioux and C. Voigt, [208])**.** *Let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup of a discrete quantum group $\widehat{\mathbb{G}}$. Assume that $\widehat{\mathbb{G}}$ is torsion-free.*

If $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$ is divisible and $\widehat{\mathbb{G}}$ satisfies the strong Baum-Connes property, then $\widehat{\mathbb{H}}$ satisfies the strong Baum-Connes property.

To the best knowledge of the author it is open to know if the *strong* Baum-Connes property is preserved by quantum subgroups in general. As well as the usual Baum-Connes property. However, the divisibility condition is still sufficient to state the usual Baum-Connes property for a discrete quantum subgroup from the usual Baum-Connes property of the discrete quantum group.

3.2.2.2 Lemma. *Let $\mathbb{G}$, $\mathbb{H}$ be two compact quantum groups. If $\widehat{\mathbb{G}}$ is torsion-free and $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$ is a divisible torsion-free discrete quantum subgroup, then the following properties hold.*

i) $Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(\mathscr{L}_{\widehat{\mathbb{G}}}) \subset \mathscr{L}_{\widehat{\mathbb{H}}}$ and $Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(\mathscr{N}_{\widehat{\mathbb{G}}}) \subset \mathscr{N}_{\widehat{\mathbb{H}}}$. Hence, $Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}} \circ L = L' \circ Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}$ and $Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}} \circ N = N' \circ Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}$.

ii) $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(\mathscr{L}_{\widehat{\mathbb{H}}}) \subset \mathscr{L}_{\widehat{\mathbb{G}}}$ and $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(\mathscr{N}_{\widehat{\mathbb{H}}}) \subset \mathscr{N}_{\widehat{\mathbb{G}}}$. Hence, $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}} \circ L' = L \circ Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}$ and $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}} \circ N' = N \circ Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}$.

Consequently, $Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}$ transforms the assembly map for $\widehat{\mathbb{G}}$ into the assembly map for $\widehat{\mathbb{H}}$ and $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}$ transforms the assembly map for $\widehat{\mathbb{H}}$ into the assembly map for $\widehat{\mathbb{G}}$.

Proof. i) Since $\widehat{\mathbb{H}}$ is divisible in $\widehat{\mathbb{G}}$, then $c_0(\widehat{\mathbb{G}}) = c_0(\widehat{\mathbb{H}}) \otimes c_0(\widehat{\mathbb{H}} \backslash \widehat{\mathbb{G}})$ as $\widehat{\mathbb{H}}$-C^*-algebras. Hence, it is clear that $Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(\mathscr{L}_{\widehat{\mathbb{G}}}) \subset \mathscr{L}_{\widehat{\mathbb{H}}}$. Take $N \in \mathscr{N}_{\widehat{\mathbb{G}}}$, then we have that $Res^{\widehat{\mathbb{G}}}_{\mathbb{E}}(N) = 0$. Restriction by stages yields that $0 = Res^{\widehat{\mathbb{G}}}_{\mathbb{E}}(N) = Res^{\widehat{\mathbb{H}}}_{\mathbb{E}}\big(Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(N)\big)$, which means that $Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(N) \in \mathscr{N}_{\widehat{\mathbb{H}}}$.

Given any $A \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}})$, its corresponding $(\mathscr{L}_{\widehat{\mathbb{G}}}, \mathscr{N}_{\widehat{\mathbb{G}}})$-triangle is transformed into a distinguished triangle by restriction because $Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}$ is a triangulated functor. We have just seen that restriction functor preserves the subcategories $\mathscr{L}$ and $\mathscr{N}$. Hence the distinguished triangle given by restriction is actually a $(\mathscr{L}_{\widehat{\mathbb{H}}}, \mathscr{N}_{\widehat{\mathbb{H}}})$-triangle for $Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(A)$. By uniqueness of these distinguished triangles we get the relations.

ii) Take a generator $Ind^{\widehat{\mathbb{H}}}_{\mathbb{E}}(C) \in \mathscr{L}_{\widehat{\mathbb{H}}}$ with $C \in Obj(\mathscr{K}\mathscr{K})$. Induction by stages (see Proposition 2.7 in [140] for a proof) yields that $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(Ind^{\widehat{\mathbb{H}}}_{\mathbb{E}}(C)\big) = Ind^{\widehat{\mathbb{G}}}_{\mathbb{E}}(C)$, which is again a generator in $\mathscr{L}_{\widehat{\mathbb{G}}}$. Hence, we also have $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(\mathscr{L}_{\widehat{\mathbb{H}}}) \subset \mathscr{L}_{\widehat{\mathbb{G}}}$.

Take $N' \in \mathscr{N}_{\widehat{\mathbb{H}}}$. Recall that, since $\mathscr{L}_{\widehat{\mathbb{H}}}$ and $\mathscr{N}_{\widehat{\mathbb{H}}}$ are complementary, then we have $\mathscr{N}_{\widehat{\mathbb{H}}} = \mathscr{L}_{\widehat{\mathbb{H}}}^{\dashv}$. Accordingly, $KK^{\widehat{\mathbb{H}}}(L', N') = (0)$, for all $L' \in \mathscr{L}_{\widehat{\mathbb{H}}}$. By virtue of property (i) above, we can take $L' := Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(L)$ for any $L \in \mathscr{L}_{\widehat{\mathbb{G}}}$. Hence, the adjointness property between restriction and induction functor of Lemma 1.7.2.4, yields that

$$KK^{\widehat{\mathbb{G}}}(L, Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(N')) = KK^{\widehat{\mathbb{H}}}(Res^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(L), N') = (0),$$

for all $L \in \mathscr{L}_{\widehat{\mathbb{G}}}$, which means that $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(N') \in \mathscr{L}_{\widehat{\mathbb{G}}}^{\dashv} = \mathscr{N}_{\widehat{\mathbb{G}}}$.

Given any $B \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}})$, its corresponding $(\mathscr{L}_{\widehat{\mathbb{H}}}, \mathscr{N}_{\widehat{\mathbb{H}}})$-triangle is transformed into a distinguished triangle by induction because $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}$ is a triangulated functor. We have just seen that induction functor preserves the subcategories $\mathscr{L}$ and $\mathscr{N}$. Hence the distinguished triangle given by induction is actually a $(\mathscr{L}_{\widehat{\mathbb{G}}}, \mathscr{N}_{\widehat{\mathbb{G}}})$-triangle for $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(B)$. By uniqueness of these distinguished triangles we get the relations.

■

3.2.2.3 Proposition. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group. Assume that $\widehat{\mathbb{G}}$ is torsion-free.*

$\widehat{\mathbb{G}}$ satisfies the quantum Baum-Connes property if and only if every divisible torsion-free discrete quantum subgroup $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$ satisfies the quantum Baum-Connes property.

Proof. Assume that $\widehat{\mathbb{G}}$ satisfies the quantum Baum-Connes property and consider a divisible discrete quantum subgroup $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$.

By assumption, $\widehat{\mathbb{G}}$ satisfies the quantum Baum-Connes property with coefficients. In particular, we have a natural isomorphism $\eta^{\widehat{\mathbb{G}}}_{Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(B)} : K_*\big(\widehat{\mathbb{G}} \ltimes L(Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(B))\big) \longrightarrow K_*\big(\widehat{\mathbb{G}} \ltimes Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(B)\big)$, for all $B \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}})$.

Thanks to the preceding lemma $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}} \circ L' = L \circ Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}$, so that we have a natural isomorphism $\eta^{\widehat{\mathbb{G}}}_{Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(B)} : K_*\big(\widehat{\mathbb{G}} \ltimes Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(L'(B)\big) \longrightarrow K_*\big(\widehat{\mathbb{G}} \ltimes Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(B)\big)$, for all $B \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}})$.

By virtue of the quantum Green's Imprimitivity theorem (see Theorem 1.5.3.6 or Theorem 7.3 in [194] for a proof) we have a natural Morita equivalence $\widehat{\mathbb{G}} \ltimes Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(B) \underset{M}{\sim} \widehat{\mathbb{H}} \ltimes B$ for all $B \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}})$, which yields an isomorphism between $\widehat{\mathbb{G}} \ltimes Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(B)$ and $\widehat{\mathbb{H}} \ltimes B$ in $\mathscr{K}\mathscr{K}$.

Moreover, the induction functor transforms the assembly map for $\widehat{\mathbb{H}}$ into the assembly map for $\widehat{\mathbb{G}}$ by the preceding lemma. More precisely, given $B \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}})$ if

$$\Sigma(N'(B)) \longrightarrow L'(B) \xrightarrow{u'} B \longrightarrow N'(B) \tag{3.2.1}$$

is the $(\mathscr{L}_{\widehat{\mathbb{H}}}, \mathscr{N}_{\widehat{\mathbb{H}}})$-triangle for B, then

$$\Sigma(Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(N'(B)\big)) \longrightarrow Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(L'(B)\big) \xrightarrow{Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(u')} Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(B\big) \longrightarrow Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(N'(B)\big) \tag{3.2.2}$$

is the $(\mathscr{L}_{\widehat{\mathbb{G}}}, \mathscr{N}_{\widehat{\mathbb{G}}})$-triangle for $Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(B\big)$.

Apply the triangulated functors $\widehat{\mathbb{H}} \underset{r}{\ltimes} \cdot$ and $\widehat{\mathbb{G}} \underset{r}{\ltimes} \cdot$ to the triangles (3.2.1) and (3.2.2), respectively so that we get the following distinguished triangles in $\mathscr{K}\mathscr{K}$

$$\Sigma(\widehat{\mathbb{H}} \underset{r}{\ltimes} N'(B)) \longrightarrow \widehat{\mathbb{H}} \underset{r}{\ltimes} L'(B) \xrightarrow{\widehat{\mathbb{H}} \ltimes u'} \widehat{\mathbb{H}} \underset{r}{\ltimes} B \longrightarrow \widehat{\mathbb{H}} \underset{r}{\ltimes} N'(B)$$

$$\Sigma\big(\widehat{\mathbb{G}} \underset{r}{\ltimes} Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(N'(B)\big)\big) \longrightarrow \widehat{\mathbb{G}} \underset{r}{\ltimes} Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(L'(B)\big) \xrightarrow{\widehat{\mathbb{G}} \ltimes Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(u')} \widehat{\mathbb{G}} \underset{r}{\ltimes} Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(B\big) \longrightarrow \widehat{\mathbb{G}} \underset{r}{\ltimes} Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(N'(B)\big)$$

Since the isomorphism between $\widehat{\mathbb{G}} \ltimes Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(B)$ and $\widehat{\mathbb{H}} \ltimes B$ in $\mathscr{K}\mathscr{K}$ is natural by the quantum Green's Imprimitivity theorem, then we get an isomorphism of distinguished triangles

$$\begin{tikzcd}
\Sigma(\widehat{\mathbb{H}} \underset{r}{\ltimes} N'(B)) \arrow[r] \arrow[d, "\wr"] & \widehat{\mathbb{H}} \underset{r}{\ltimes} L'(B) \arrow[r, "\widehat{\mathbb{H}} \ltimes u'"] \arrow[d, "\wr"] & \widehat{\mathbb{H}} \underset{r}{\ltimes} B \arrow[r] \arrow[d, "\wr"] & \widehat{\mathbb{H}} \underset{r}{\ltimes} N'(B) \arrow[d, "\wr"] \\
\Sigma\big(\widehat{\mathbb{G}} \underset{r}{\ltimes} Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(N'(B)\big)\big) \arrow[r] & \widehat{\mathbb{G}} \underset{r}{\ltimes} Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(L'(B)\big) \arrow[r, "\widehat{\mathbb{G}} \ltimes Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}(u')"'] & \widehat{\mathbb{G}} \underset{r}{\ltimes} Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(B\big) \arrow[r] & \widehat{\mathbb{G}} \underset{r}{\ltimes} Ind^{\widehat{\mathbb{G}}}_{\widehat{\mathbb{H}}}\big(N'(B)\big)
\end{tikzcd}$$

which allows to consider the following commutative diagram,

$$\begin{tikzcd}
\mathbb{L}F'(B) \arrow[r, "\eta_B^{\widehat{\mathbb{H}}}"] \arrow[d, "\wr"] & F'(B) \arrow[d, "\wr"] \\
\mathbb{L}F\big(Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(L'(B))\big) \arrow[r, "\eta^{\widehat{\mathbb{G}}}_{Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)}"'] & F\big(Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)\big)
\end{tikzcd}$$

Since $\eta^{\widehat{\mathbb{G}}}_{Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)}$ is an isomorphism for all $B \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}})$, we conclude that the same is true for $\eta_B^{\widehat{\mathbb{H}}}$, that is, $\widehat{\mathbb{H}}$ satisfies the quantum Baum-Connes property with coefficients.

The converse is obvious and the proof is complete. ■

3.2.2.4 Remark. For classical groups it is well-known that the Baum-Connes property is preserved by *closed* subgroups. It was showed by J. Chabert and S. Echterhoff in [35]. To this end they showed that the induction homomorphism $K_*^{top}(H;B) \longrightarrow K_*^{top}(G;Ind_H^G(B))$ is *always bijective* (see Theorem 2.2 in [35] for a proof). In our case, this result is encoded in the identification $K_*(\widehat{\mathbb{H}} \ltimes L'(B)) \cong K_*(\widehat{\mathbb{G}} \ltimes L(Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(B)))$, obtained by the property $Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}} \circ L' = L \circ Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}$ plus the quantum Green's Imprimitivity theorem.

3.2.2.5 Remark. Assume that $\widehat{\mathbb{G}}$ is any discrete quantum group not necessarily torsion-free. We have a reasonable definition for the localizing subcategory $\mathscr{L}_{\widehat{\mathbb{G}}}$ of compactly induced C^*-algebras as it is explained in Section 4.1.2. This definition takes into account the torsion phenomena of $\widehat{\mathbb{G}}$. Hence, the abstract conditions $Res_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(\mathscr{L}_{\widehat{\mathbb{G}}}) \subset \mathscr{L}_{\widehat{\mathbb{H}}}$ and $Ind_{\widehat{\mathbb{H}}}^{\widehat{\mathbb{G}}}(\mathscr{L}_{\widehat{\mathbb{H}}}) \subset \mathscr{L}_{\widehat{\mathbb{G}}}$, where $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$ is some discrete quantum subgroup, must be checked also for all torsion actions and not only for the trivial torsion action of the torsion-free case, as it has been done in Lemma 3.2.2.2. An example of this process can be found in Theorem 3.7.2.6.

3.2.3 K-amenability property

Here we care about a property of own interest, namely the K-amenability of a quantum subgroup. More precisely we have the following result.

3.2.3.1 Theorem. *Let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup of a discrete quantum group $\widehat{\mathbb{G}}$. If $\mathbb{G}$ is co-K-amenable, then $\mathbb{H}$ is co-K-amenable.*

Proof. Since $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$, then we have canonical injections between their reduced C^*-algebras and also between the maximal ones (recall Proposition 1.4.3.4). Let's put

$$\iota_{\mathbb{H}}^r : C_r(\mathbb{H}) \hookrightarrow C_r(\mathbb{G}) \text{ and } \iota_{\mathbb{H}}^m : C_m(\mathbb{H}) \hookrightarrow C_m(\mathbb{G})$$

for such injections, which intertwine the corresponding co-multiplications.

Next, let $\tau_{\mathbb{G}} : C_m(\mathbb{G}) \twoheadrightarrow C_r(\mathbb{G})$, $\tau_{\mathbb{H}} : C_m(\mathbb{H}) \twoheadrightarrow C_r(\mathbb{H})$ denote the canonical surjections and $\varepsilon_{\mathbb{G}} : Pol(\mathbb{G}) \longrightarrow \mathbb{C}$, $\varepsilon_{\mathbb{H}} : Pol(\mathbb{H}) \longrightarrow \mathbb{C}$ the co-units of $\mathbb{G}$ and $\mathbb{H}$, respectively whose extension to

$C_m(\mathbb{G})$ and $C_m(\mathbb{H})$ are still denoted by $\varepsilon_{\mathbb{G}}$ and $\varepsilon_{\mathbb{H}}$, respectively. Hence, by construction we have the following commutative diagrams

$$\begin{tikzcd} C_m(\mathbb{G}) \arrow[r, two heads, "\tau_{\mathbb{G}}"] & C_r(\mathbb{G}) \\ C_m(\mathbb{H}) \arrow[u, hook, "\iota^m_{\mathbb{H}}"] \arrow[r, two heads, "\tau_{\mathbb{H}}"'] & C_r(\mathbb{H}) \arrow[u, hook, "\iota^r_{\mathbb{H}}"'] \end{tikzcd} \qquad \begin{tikzcd} C_m(\mathbb{G}) \arrow[r, "\varepsilon_{\mathbb{G}}"] & \mathbb{C} \\ C_m(\mathbb{H}) \arrow[u, hook, "\iota^m_{\mathbb{H}}"] \arrow[ur, "\varepsilon_{\mathbb{H}}"'] & \end{tikzcd} \tag{3.2.3}$$

Since $\mathbb{G}$ is supposed to be co-K-amenable, then Remark 1.3.1.41 assures that there exists an element $\alpha_{\mathbb{G}} \in KK(C_r(\mathbb{G}), \mathbb{C})$ such that $[\tau_{\mathbb{G}}] \underset{C_r(\mathbb{G})}{\otimes} \alpha_{\mathbb{G}} = [\varepsilon_{\mathbb{G}}]$. In order to show that $\mathbb{H}$ is also co-K-amenable, let us define

$$\alpha_{\mathbb{H}} := [\iota^r_{\mathbb{H}}] \underset{C_r(\mathbb{G})}{\otimes} \alpha_{\mathbb{G}} \in KK(C_r(\mathbb{H}), \mathbb{C})$$

Hence, diagrams (3.2.3) above yields that $[\tau_{\mathbb{H}}] \underset{C_r(\mathbb{H})}{\otimes} \alpha_{\mathbb{H}} = [\varepsilon_{\mathbb{H}}]$. Indeed,

$$\begin{aligned} [\tau_{\mathbb{H}}] \underset{C_r(\mathbb{H})}{\otimes} \alpha_{\mathbb{H}} &= [\tau_{\mathbb{H}}] \underset{C_r(\mathbb{H})}{\otimes} [\iota^r_{\mathbb{H}}] \underset{C_r(\mathbb{G})}{\otimes} \alpha_{\mathbb{G}} = [\iota^r_{\mathbb{H}} \circ \tau_{\mathbb{H}}] \underset{C_r(\mathbb{G})}{\otimes} \alpha_{\mathbb{G}} \\ &= [\tau_{\mathbb{G}} \circ \iota^m_{\mathbb{H}}] \underset{C_r(\mathbb{G})}{\otimes} \alpha_{\mathbb{G}} = [\iota^m_{\mathbb{H}}] \underset{C_m(\mathbb{G})}{\otimes} [\tau_{\mathbb{G}}] \underset{C_r(\mathbb{G})}{\otimes} \alpha_{\mathbb{G}} \\ &= [\iota^m_{\mathbb{H}}] \underset{C_m(\mathbb{G})}{\otimes} [\varepsilon_{\mathbb{G}}] = [\varepsilon_{\mathbb{G}} \circ \iota^m_{\mathbb{H}}] = [\varepsilon_{\mathbb{H}}] \end{aligned}$$

Therefore, Remark 1.3.1.41 yields the co-K-amenability for $\mathbb{H}$.

■

3.3 The Baum-Connes property for a quantum direct product

In this section we are going to study the *strong* Baum-Connes property for the dual of a direct product of compact quantum groups in terms of the involved compact quantum groups. Recall that H. Oyono-Oyono has already successfully studied the stability of the Baum-Connes property for classical discrete groups in [143]. The locally compact case has also been studied by J. Chabert and S. Echterhoff in [35].

It is important to remark that our study must be restricted to the *strong* version of the Baum-Connes property. Indeed, we will see that the usual Baum-Connes property for a quantum direct product is closely related to the *Künneth formula* (as pointed already out in [37] by J. Chabert, S. Echterhoff and H. Oyono-Oyono for classical locally compact groups) so that more hypothesis are needed in order to establish the corresponding stability property.

In order to do so, we carry the same notations of Section 2.2 on. Let $\mathbb{F} := \mathbb{G} \times \mathbb{H}$ be a quantum direct product, where $\mathbb{G}$ and $\mathbb{H}$ are two compact quantum groups.

3.3.1 Torsion property

We have already explained that all our *discrete quantum groups are supposed to be torsion-free* in order to study the corresponding quantum Baum-Connes property. For this reason it is advisable to study in more detail the torsion phenomena of the dual of a quantum direct product $\mathbb{F} = \mathbb{G} \times \mathbb{H}$ in terms of the involved quantum groups.

First of all, the description of the irreducible representations of $\mathbb{F} = \mathbb{G} \times \mathbb{H}$ allows to give a suitable decomposition of its fusion ring so that we can study the *strong* torsion-freeness of $\widehat{\mathbb{F}}$ in terms of the *strong* torsion-freeness of $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$. Namely, we have

3.3.1.1 Proposition. *Let $\mathbb{F} = \mathbb{G} \times \mathbb{H}$ be a quantum direct product. The fusion ring of $\widehat{\mathbb{F}}$ decomposes as*

$$Fus(\widehat{\mathbb{F}}) = Fus(\widehat{\mathbb{G}}) \otimes Fus(\widehat{\mathbb{H}})$$

Consequently, if $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are strong torsion-free, then $\widehat{\mathbb{F}}$ is strong torsion-free.

Proof. Recall Section 2.2. The representation theory for $\mathbb{F}$ yields the decomposition $Irr(\mathbb{F}) = \big[Irr(\mathbb{G})\big]_{13}\big[Irr(\mathbb{H})\big]_{24}$, so that $Irr(\mathbb{F})$ can be regarded as the tensor product of $Irr(\mathbb{G})$ and $Irr(\mathbb{H})$ as *based rings* (recall Section 1.6.2). Indeed, given $y \in Irr(\mathbb{F})$, take $x \in Irr(\mathbb{G})$ and $z \in Irr(\mathbb{H})$ such that $y = \big[x\big]_{13}\big[z\big]_{24}$. If $w^{\overline{y}}$, $w^{\overline{x}}$ and $w^{\overline{z}}$ are representatives of $\overline{y}$, $\overline{x}$ and $\overline{z}$, respectively; then we have

- $\epsilon_{\mathbb{F}} = \epsilon_{\mathbb{G}} \bigoplus \epsilon_{\mathbb{H}}$
- $w^{\overline{y}} = \overline{\big[w^x\big]_{13}\big[w^z\big]_{24}} = \big[w^{\overline{x}}\big]_{13}\big[w^{\overline{z}}\big]_{24}$
- $d(w^y = \big[w^x\big]_{13}\big[w^z\big]_{24}) = d(w^x)d(w^z)$

In conclusion, the decomposition $Fus(\widehat{\mathbb{F}}) = Fus(\widehat{\mathbb{G}}) \otimes Fus(\widehat{\mathbb{H}})$ holds.

Finally, assume that $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are strong torsion-free. Then they are in particular torsion-free. By Remark 1.6.1.7, $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ can *not* contain finite discrete quantum subgroups. Hence, $Fus(\widehat{\mathbb{G}})$ and $Fus(\widehat{\mathbb{H}})$ can *not* contain finite fusion subrings. Theorem 1.6.2.11 assures thus that $Fus(\widehat{\mathbb{G}}) \otimes Fus(\widehat{\mathbb{H}}) = Fus(\widehat{\mathbb{F}})$ is torsion-free. In other words, $\widehat{\mathbb{F}}$ is strong torsion-free as we wanted to show. ■

Although this result implies in particular the torsion-freeness of $\widehat{\mathbb{F}}$ in the sense of Meyer-Nest (whenever $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are *strong* torsion-free), it is interesting to obtain the torsion-freeness in the sense of Meyer-Nest *directly* from the torsion-freeness in the sense of Meyer-Nest of $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$.

3.3.1.2 Theorem (Y. Arano and K. De Commer, [3])**.** *Let $\mathbb{F} := \mathbb{G} \times \mathbb{H}$ be the quantum direct product. If $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are torsion-free, then $\widehat{\mathbb{F}}$ is torsion-free.*

3.3.1.3 Note. The converse of the preceding statement would be true whenever the torsion-freeness is preserved under divisible discrete quantum subgroups as conjectured in the end of Section 3.2.1.

3.3.2 The Baum-Connes property

In order to legitimate the Baum-Connes property formulation for the dual of a quantum direct product $\mathbb{F} = \mathbb{G} \times \mathbb{H}$, we need $\widehat{\mathbb{F}}$ to be torsion-free. And, in order to legitimate the Baum-Connes property formulation for $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$, we need these discrete quantum groups to be torsion-free. Hence, we must keep the preceding section in mind and, for simplicity, we assume that $\widehat{\mathbb{F}}$, $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are all torsion-free.

Next, let us set up the adapted notations of the categorical framework of Meyer-Nest to this situation. Consider the equivariant Kasparov categories associated to $\widehat{\mathbb{F}}$, $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$, say

$$\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}},\ \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \text{ and } \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}},$$

respectively. The canonical suspension functors for each of the above Kasparov categories are denoted simply by Σ.

Notice that from now on, the word *homomorphism (respectively, isomorphism)* will mean *homomorphism (respectively, isomorphism) in the corresponding Kasparov category*; it can be either a true homomorphism (respectively, isomorphism) between C^*-algebras or any Kasparov triple between C^*-algebras (respectively, any KK-equivalence between C^*-algebras).

Consider the canonical complementary pair of localizing subcategories in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$, $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ and $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}}$, say

$$(\mathscr{L}_{\widehat{\mathbb{F}}}\mathscr{N}_{\widehat{\mathbb{F}}}),\ (\mathscr{L}_{\widehat{\mathbb{G}}}\mathscr{N}_{\widehat{\mathbb{G}}}) \text{ and } (\mathscr{L}_{\widehat{\mathbb{H}}}\mathscr{N}_{\widehat{\mathbb{H}}}),$$

respectively. In this way, the canonical triangulated functors associated to these complementary pairs are denoted by (L, N), (L', N') and (L'', N''), respectively.

Consider the homological functors defining the *quantum* Baum-Connes assembly maps for $\widehat{\mathbb{F}}$, $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$. Namely,

$$\begin{array}{rccl} F: & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}} & \longrightarrow & \mathscr{A}b^{\mathbb{Z}/2} \\ & (C,\delta) & \longmapsto & F(C) := K_*(\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} C) \end{array}$$

$$\begin{array}{rccl} F': & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} & \longrightarrow & \mathscr{A}b^{\mathbb{Z}/2} \\ & (A,\alpha) & \longmapsto & F'(A) := K_*(\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A) \end{array} \qquad \begin{array}{rccl} F'': & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}} & \longrightarrow & \mathscr{A}b^{\mathbb{Z}/2} \\ & (B,\beta) & \longmapsto & F''(B) := K_*(\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B) \end{array}$$

Therefore, the quantum assembly maps for $\widehat{\mathbb{F}}$, $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are given by the following natural transformations

$$\eta^{\widehat{\mathbb{F}}} : \mathbb{L}F \longrightarrow F,\ \eta^{\widehat{\mathbb{G}}} : \mathbb{L}F' \longrightarrow F' \text{ and } \eta^{\widehat{\mathbb{H}}} : \mathbb{L}F'' \longrightarrow F'',$$

where by definition we have

$$\mathbb{L}F = F \circ L,\ \mathbb{L}F' = F' \circ L' \text{ and } \mathbb{L}F'' = F'' \circ L''$$

By the torsion-freeness assumption, the subcategory $\mathscr{L}_{\widehat{\mathbb{F}}}$ (resp. $\mathscr{L}_{\widehat{\mathbb{G}}}$, $\mathscr{L}_{\widehat{\mathbb{H}}}$) is easily described as the *localizing subcategory of $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$ (resp. $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$, $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}}$) generated by the objects of the form $c_0(\widehat{\mathbb{F}}) \otimes C$ (resp. $c_0(\widehat{\mathbb{G}}) \otimes C$, $c_0(\widehat{\mathbb{H}}) \otimes C$) with C any C^*-algebra in the Kasparov category $\mathscr{K}\mathscr{K}$.*

Now, we are ready to start the study of the strong Baum-Connes property for a quantum direct product. To this end, we consider the following functor

$$\begin{array}{rccl} \mathcal{Z}: & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \times \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}} & \longrightarrow & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}} \\ & (A,\alpha) \times (B,\beta) & \longmapsto & \mathcal{Z}(A,B) := (C := A \otimes B, \delta := \alpha \otimes \beta), \end{array}$$

which is defined on homomorphisms by functoriality thanks to Theorem 1.7.1.16. Let us be more precise in the definition of $\mathcal{Z}$ as a *functor* between $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \times \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}}$ and $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$. For this we have to keep in mind all the notations and results obtained in Section 2.2.

Consider an object in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \times \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}}$, say $(A,\alpha) \times (B,\beta) \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}) \times Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}})$. The tensor product $A \otimes B$ is a C^*-algebra equipped with the following action of $\widehat{\mathbb{F}}$,

$$\begin{aligned} A \otimes B &\xrightarrow{\alpha\otimes\beta} \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A) \otimes \widetilde{M}(c_0(\widehat{\mathbb{H}}) \otimes B) \subset \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes A \otimes c_0(\widehat{\mathbb{H}}) \otimes B) \\ &\overset{\Sigma}{\cong} \widetilde{M}(c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{H}}) \otimes A \otimes B) = \widetilde{M}(c_0(\widehat{\mathbb{F}}) \otimes A \otimes B) \end{aligned}$$

By abuse of notation we denote this composition simply by $\alpha \otimes \beta$. Hence, $\mathcal{Z}$ is well-defined on objects. Let us explain its definition on homomorphisms.

Let $(A',\alpha') \times (B',\beta') \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}) \times Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}})$ an other object in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \times \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}}$.

Let $\mathcal{X} := ((H,\delta_H),\pi,F) \in KK^{\widehat{\mathbb{G}}}(A,A')$ and $\mathcal{Y} := ((H',\delta_{H'}),\pi',F') \in KK^{\widehat{\mathbb{H}}}(B,B')$ be two Kasparov triples so that $\mathcal{X} \times \mathcal{Y} \in KK^{\widehat{\mathbb{G}}}(A,A') \times KK^{\widehat{\mathbb{H}}}(B,B')$ is a homomorphism between (A,B) and (A',B') in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \times \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}}$.

Hence the corresponding homomorphism between $\mathcal{Z}(A,B)$ and $\mathcal{Z}(A',B')$ in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$ is defined in the following way.

$$\mathcal{Z}(\mathcal{X},\mathcal{Y}) := \tau_R^B(\mathcal{X}) \underset{A'\otimes B}{\otimes} \tau_L^{A'}(\mathcal{Y}) =: \mathcal{X} \otimes \mathcal{Y}$$

Observe that by Theorem 1.7.1.16 we know that the Kasparov triple $\mathcal{X} \otimes \mathcal{Y}$ defined above is an element in $KK(A \otimes B, A' \otimes B')$. Next, it is easy to establish an action of $\widehat{\mathbb{F}}$ on $\mathcal{X} \otimes \mathcal{Y}$. Indeed,

- since $(A,\alpha),(A',\alpha')$ are $\widehat{\mathbb{G}}$-C^*-algebras and $(B,\beta),(B',\beta')$ are $\widehat{\mathbb{H}}$-C^*-algebras, then $(A \otimes B, \alpha\otimes\beta)$, $(A' \otimes B, \alpha' \otimes \beta)$, $(A' \otimes B', \alpha' \otimes \beta')$ are $\widehat{\mathbb{F}}$-C^*-algebras.
- Since $\mathcal{X}$ is $\widehat{\mathbb{G}}$-equivariant Kasparov triple and (B,β) is a $\widehat{\mathbb{H}}$-C^*-algebra, then $\tau_R^B(\mathcal{X}) = \mathcal{X} \otimes B \in KK^{\widehat{\mathbb{F}}}(A \otimes B, A' \otimes B)$ is a $\widehat{\mathbb{F}}$-equivariant Kasparov triple (recall Remark A.3.23).
- Since $\mathcal{Y}$ is $\widehat{\mathbb{H}}$-equivariant Kasparov triple and (A',α') is a $\widehat{\mathbb{G}}$-C^*-algebra, then $\tau_L^{A'}(\mathcal{Y}) = A' \otimes \mathcal{Y} \in KK^{\widehat{\mathbb{F}}}(A' \otimes B, A' \otimes B')$ is a $\widehat{\mathbb{F}}$-equivariant Kasparov triple (recall Remark A.3.23).

Consequently, since $\mathcal{X} \otimes \mathcal{Y}$ defined above is a Kasparov product of two $\widehat{\mathbb{F}}$-equivariant Kasparov triples, then it is as well a $\widehat{\mathbb{F}}$-equivariant one, so that $\mathcal{X} \otimes \mathcal{Y} \in KK^{\widehat{\mathbb{F}}}(A \otimes B, A' \otimes B')$. Hence, $\mathcal{Z}$ is well-defined on homomorphisms.

3.3.2.1 Lemma. *The functor*

$$\begin{array}{rccl} \mathcal{Z}: & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \times \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}} & \longrightarrow & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}} \\ & (A,\alpha) \times (B,\beta) & \longmapsto & \mathcal{Z}(A,B) := (C := A \otimes B, \delta := \alpha \otimes \beta) \end{array}$$

is such that $\mathcal{Z}(\mathscr{L}_{\widehat{\mathbb{G}}} \times \mathscr{L}_{\widehat{\mathbb{H}}}) \subset \mathscr{L}_{\widehat{\mathbb{F}}}$ and $\mathcal{Z}(\mathscr{N}_{\widehat{\mathbb{G}}} \times \mathscr{N}_{\widehat{\mathbb{H}}}) \subset \mathscr{N}_{\widehat{\mathbb{F}}}$.

If $(A_0,\alpha_0) \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}})$ is a given $\widehat{\mathbb{G}}$-C^*-algebra, the functor

$$\begin{array}{rccl} {}_{A_0}\mathcal{Z}: & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}} & \longrightarrow & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}} \\ & (B,\beta) & \longmapsto & {}_{A_0}\mathcal{Z}(B) := \mathcal{Z}(A_0,B) \end{array}$$

is triangulated such that ${}_{A_0}\mathcal{Z}(\mathcal{N}_{\widehat{\mathbb{H}}}) \subset \mathcal{N}_{\widehat{\mathbb{F}}}$.

If $(B_0, \beta_0) \in Obj(\mathcal{KK}^{\widehat{\mathbb{H}}})$ *is a given* $\widehat{\mathbb{H}}$*-C*-algebra, the functor*

$$\begin{array}{rccc} \mathcal{Z}_{B_0}: & \mathcal{KK}^{\widehat{\mathbb{G}}} & \longrightarrow & \mathcal{KK}^{\widehat{\mathbb{F}}} \\ & (A, \alpha) & \longmapsto & \mathcal{Z}_{B_0}(A) := \mathcal{Z}(A, B_0) \end{array}$$

is triangulated such that $\mathcal{Z}_{B_0}(\mathcal{N}_{\widehat{\mathbb{G}}}) \subset \mathcal{N}_{\widehat{\mathbb{F}}}$.

Proof. Firstly, let us show that $\mathcal{Z}(\mathcal{L}_{\widehat{\mathbb{G}}} \times \mathcal{L}_{\widehat{\mathbb{H}}}) \subset \mathcal{L}_{\widehat{\mathbb{F}}}$. Namely, since all our discrete quantum groups are supposed to be torsion-free, then we know that $\mathcal{L}_{\widehat{\mathbb{G}}}$, $\mathcal{L}_{\widehat{\mathbb{H}}}$ and $\mathcal{L}_{\widehat{\mathbb{F}}}$ are the localizing subcategories generated by the objects of the form $c_0(\widehat{\mathbb{G}}) \otimes C_1$, $c_0(\widehat{\mathbb{H}}) \otimes C_2$ and $c_0(\widehat{\mathbb{F}}) \otimes C_3$ in $\mathcal{KK}^{\widehat{\mathbb{G}}}$, $\mathcal{KK}^{\widehat{\mathbb{H}}}$ and $\mathcal{KK}^{\widehat{\mathbb{F}}}$, respectively where $C_1, C_2, C_3 \in Obj(\mathcal{KK})$. Recall as well that $c_0(\widehat{\mathbb{F}}) = c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{H}})$ by virtue of the representation theory of $\mathbb{F} = \mathbb{G} \times \mathbb{H}$ (see Section 2.2). Hence we write

$$\begin{aligned} \mathcal{Z}\big(c_0(\widehat{\mathbb{G}}) \otimes C_1, c_0(\widehat{\mathbb{H}}) \otimes C_2\big) &= c_0(\widehat{\mathbb{G}}) \otimes C_1 \otimes c_0(\widehat{\mathbb{H}}) \otimes C_2 \\ &\cong c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{H}}) \otimes C_1 \otimes C_2 = c_0(\widehat{\mathbb{F}}) \otimes C_3 \in Obj(\mathcal{L}_{\widehat{\mathbb{F}}}), \end{aligned}$$

where $C_3 := C_1 \otimes C_2 \in Obj(\mathcal{KK})$. This shows that $\mathcal{Z}$ sends generators of $\mathcal{L}_{\widehat{\mathbb{G}}} \times \mathcal{L}_{\widehat{\mathbb{H}}}$ to generators of $\mathcal{L}_{\widehat{\mathbb{F}}}$, so $\mathcal{Z}$ sends generators of $\mathcal{L}_{\widehat{\mathbb{G}}} \times \mathcal{L}_{\widehat{\mathbb{H}}}$ to $\mathcal{L}_{\widehat{\mathbb{F}}}$. Next, since $\mathcal{Z}$ is compatible with countable direct sums, it is clear that the subcategory $\mathcal{Z}(\mathcal{L}_{\widehat{\mathbb{G}}} \times \mathcal{L}_{\widehat{\mathbb{H}}})$ is contained in the localizing subcategory generated by the objects of the form $\mathcal{Z}$("generator of $\mathcal{L}_{\widehat{\mathbb{G}}} \times \mathcal{L}_{\widehat{\mathbb{H}}}$"). The latter is the smallest triangulated subcategory containing the objects of the form $\mathcal{Z}$("generator of $\mathcal{L}_{\widehat{\mathbb{G}}} \times \mathcal{L}_{\widehat{\mathbb{H}}}$") and stable with respect to countable direct sums. $\mathcal{L}_{\widehat{\mathbb{F}}}$ is a triangulated subcategory containing the objects of the form $\mathcal{Z}$("generator of $\mathcal{L}_{\widehat{\mathbb{G}}} \times \mathcal{L}_{\widehat{\mathbb{H}}}$") by the discussion above and stable with respect to countable direct sums by definition. Hence, by minimality, we have that the localizing subcategory generated by the objects of the form $\mathcal{Z}$("generator of $\mathcal{L}_{\widehat{\mathbb{G}}} \times \mathcal{L}_{\widehat{\mathbb{H}}}$") is contained in $\mathcal{L}_{\widehat{\mathbb{F}}}$, which yields the claim.

Secondly, let us show that $\mathcal{Z}(\mathcal{N}_{\widehat{\mathbb{G}}} \times \mathcal{N}_{\widehat{\mathbb{H}}}) \subset \mathcal{N}_{\widehat{\mathbb{F}}}$. For this we have to notice that the restriction functor is obviously compatible with the tensor functor $\mathcal{Z}$. Given $A \in Obj(\mathcal{N}_{\widehat{\mathbb{G}}})$ and $B \in Obj(\mathcal{N}_{\widehat{\mathbb{H}}})$, we can write

$$Res_{\mathbb{E}}^{\widehat{\mathbb{F}}}\big(\mathcal{Z}(A, B)\big) = Res_{\mathbb{E}}^{\widehat{\mathbb{F}}}(A \otimes B) = Res_{\mathbb{E}}^{\widehat{\mathbb{G}}}(A) \otimes Res_{\mathbb{E}}^{\widehat{\mathbb{H}}}(B) \cong 0,$$

so that $\mathcal{Z}(A, B) \in Obj(\mathcal{N}_{\widehat{\mathbb{F}}})$.

Next, fix a $\widehat{\mathbb{G}}$-C^*-algebra $(A_0, \alpha_0) \in Obj(\mathcal{KK}^{\widehat{\mathbb{G}}})$ and consider the functor ${}_{A_0}\mathcal{Z}$ of the statement (which is well defined on homomorphisms in an analogous way as $\mathcal{Z}$ by using Theorem 1.7.1.16). In order to show that ${}_{A_0}\mathcal{Z}$ is triangulated, we are going to show that ${}_{A_0}\mathcal{Z}$ is compatible with the suspension functors of the corresponding Kasparov categories and that ${}_{A_0}\mathcal{Z}$ preserves mapping cone triangles.

For the first claim, given $(B, \beta) \in Obj(\mathcal{KK}^{\widehat{\mathbb{H}}})$ we have

$$\begin{aligned} {}_{A_0}\mathcal{Z}(\Sigma(B)) &= {}_{A_0}\mathcal{Z}\big(C_0(\mathbb{R}) \otimes B\big) = A_0 \otimes C_0(\mathbb{R}) \otimes B \\ &\cong C_0(\mathbb{R}) \otimes A_0 \otimes B \overset{(1)}{\cong} \Sigma(A_0 \otimes B) = \Sigma({}_{A_0}\mathcal{Z}(B)), \end{aligned}$$

where the identification (1) is simply induced by the canonical identification $A_0 \otimes C_0\big([0,1], B\big) \cong C_0\big([0,1], A_0 \otimes B\big)$. Let us show that ${}_{A_0}\mathcal{Z}$ preserves mapping cone triangles. Consider a mapping cone triangle in $\mathcal{KK}^{\widehat{\mathbb{H}}}$, say

$$\Sigma(B') \longrightarrow C_\phi \longrightarrow B \xrightarrow{\phi} B',$$

where $\phi : B \longrightarrow B'$ is a $\widehat{\mathbb{H}}$-equivariant $*$-homomorphism. Apply the functor ${}_{A_0}\mathcal{Z}$ so that we obtain the following diagram

$$\Sigma(A_0 \otimes B') \longrightarrow A_0 \otimes C_\phi \longrightarrow A_0 \otimes B \xrightarrow{id\otimes\phi} A_0 \otimes B',$$

where $A_0 \otimes C_\phi \cong C_{id\otimes\phi}$ by virtue of Proposition 1.5.3.3. Hence the above diagram is again a mapping cone triangle in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$. Moreover, if now $B \in Obj(\mathscr{N}_{\widehat{\mathbb{H}}})$, then $Res^{\widehat{\mathbb{F}}}_{\mathbb{E}}\big({}_{A_0}\mathcal{Z}(B)\big) = Res^{\widehat{\mathbb{G}}}_{\mathbb{E}}(A_0) \otimes Res^{\widehat{\mathbb{H}}}_{\mathbb{E}}(B) \cong 0$, which implies that ${}_{A_0}\mathcal{Z}(B) \in \mathscr{N}_{\widehat{\mathbb{H}}}$. The same argument can be applied to the functor $\mathcal{Z}_{B_0}$ of the statement for a given $\widehat{\mathbb{H}}$-C^*-algebra $(B_0, \beta_0) \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}})$ and the proof is complete. ■

3.3.2.2 Lemma. *Let $\mathbb{F} = \mathbb{G} \times \mathbb{H}$ be a quantum direct product of compact quantum groups such that $\widehat{\mathbb{F}}$, $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are a torsion-free discrete quantum groups.*

i) For all $\widehat{\mathbb{G}}$-C^*-algebra (A, α) and all $\widehat{\mathbb{H}}$-C^*-algebra (B, β) there exists a Kasparov triple

$$\psi \in KK^{\widehat{\mathbb{F}}}\big(L'(A) \otimes L''(B), L(A \otimes B)\big)$$

such that the following diagram is commutative

$$\begin{tikzcd}
\widehat{\mathbb{G}} \ltimes_r L'(A) \otimes \widehat{\mathbb{H}} \ltimes_r L''(B) \arrow[r, "\Psi"] \arrow[d, "\widehat{\mathbb{G}} \ltimes_r u' \otimes \widehat{\mathbb{H}} \ltimes_r u''"'] & \widehat{\mathbb{F}} \ltimes_r L(A \otimes B) \arrow[d, "\widehat{\mathbb{F}} \ltimes_r u"] \\
\widehat{\mathbb{G}} \ltimes_r A \otimes \widehat{\mathbb{H}} \ltimes_r B \arrow[r, "\cong"] & \widehat{\mathbb{F}} \ltimes_r (A \otimes B)
\end{tikzcd} \tag{3.3.1}$$

where $\Psi := \widehat{\mathbb{F}} \ltimes_r \psi$ and u', u'', u are the Dirac homomorphisms for A, B, $A \otimes B$, respectively.

ii) For all $\widehat{\mathbb{G}}$-C^*-algebra $(A_0, \alpha_0) \in \mathscr{L}_{\widehat{\mathbb{G}}}$ and all $\widehat{\mathbb{H}}$-C^*-algebra (B, β) there exists an invertible *Kasparov triple*

$${}_{A_0}\psi \in KK^{\widehat{\mathbb{F}}}\big(A_0 \otimes L''(B), L(A_0 \otimes B)\big)$$

such that the following diagram is commutative

$$\begin{tikzcd}
\widehat{\mathbb{G}} \ltimes_r A_0 \otimes \widehat{\mathbb{H}} \ltimes_r L''(B) \arrow[r, "{}_{A_0}\Psi", "\sim"'] \arrow[d, "\widehat{\mathbb{F}} \ltimes_r {}_{A_0}\mathcal{Z}(u'')"'] & \widehat{\mathbb{F}} \ltimes_r L(A_0 \otimes B) \arrow[d, "\widehat{\mathbb{F}} \ltimes_r u"] \\
\widehat{\mathbb{G}} \ltimes_r A_0 \otimes \widehat{\mathbb{H}} \ltimes_r B \arrow[r, "\cong"] & \widehat{\mathbb{F}} \ltimes_r (A_0 \otimes B)
\end{tikzcd} \tag{3.3.2}$$

where ${}_{A_0}\Psi := \widehat{\mathbb{F}} \ltimes_r {}_{A_0}\psi$ and u'', u are the Dirac homomorphism for B, $A_0 \otimes B$, respectively.

Proof. First of all, we recall that for all $\widehat{\mathbb{G}}$-C^*-algebra (A,α) and all $\widehat{\mathbb{H}}$-C^*-algebra (B,β) we have a canonical $*$-isomorphism $\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} (A\otimes B) \cong \widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \otimes \widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B$ by Proposition 1.5.3.1.

i) Given a $\widehat{\mathbb{G}}$-C^*-algebra (A,α), consider the corresponding $(\mathscr{L}_{\widehat{\mathbb{G}}},\mathscr{N}_{\widehat{\mathbb{G}}})$-triangle, say $\Sigma(N'(A)) \longrightarrow L'(A) \xrightarrow{u'} A \longrightarrow N'(A)$. Given a $\widehat{\mathbb{H}}$-C^*-algebra (B,β), consider the corresponding $(\mathscr{L}_{\widehat{\mathbb{H}}},\mathscr{N}_{\widehat{\mathbb{H}}})$-triangle, say $\Sigma(N''(B)) \longrightarrow L''(B) \xrightarrow{u''} B \longrightarrow N''(B)$. Consider the $(\mathscr{L}_{\widehat{\mathbb{F}}},\mathscr{N}_{\widehat{\mathbb{F}}})$-triangle of the $\widehat{\mathbb{F}}$-C^*-algebra $\mathcal{Z}(A\otimes B) = A\otimes B$, say

$$\Sigma(N(A\otimes B)) \longrightarrow L(A\otimes B) \xrightarrow{u} A\otimes B \longrightarrow N(A\otimes B)$$

Let's fix the object $\mathcal{Z}(L'(A),L''(B)) = L'(A)\otimes L''(B) =: T \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}})$ and take the long exact sequence associated to the above triangle with respect to the object T. Namely,

$$\begin{aligned}\ldots \to KK^{\widehat{\mathbb{F}}}(T,\Sigma(N(A\otimes B))) \to KK^{\widehat{\mathbb{F}}}(T,L(A\otimes B)) \overset{(u)_*}{\to}\\ \to KK^{\widehat{\mathbb{F}}}(T,A\otimes B) \to KK^{\widehat{\mathbb{F}}}(T,N(A\otimes B)) \to \ldots\end{aligned}$$

Since $(L'(A),L''(B)) \in \mathscr{L}_{\widehat{\mathbb{G}}} \times \mathscr{L}_{\widehat{\mathbb{H}}}$, then $T \in \mathscr{L}_{\widehat{\mathbb{F}}}$ by Lemma 3.3.2.1. But, by definition of complementary pair, we have $\mathscr{L}_{\widehat{\mathbb{F}}} \subset \mathscr{N}_{\widehat{\mathbb{F}}}^{\vdash}$. In particular, we obtain $KK^{\widehat{\mathbb{F}}}(T,\Sigma(N(A\otimes B))) = (0) = KK^{\widehat{\mathbb{F}}}(T,N(A\otimes B))$. Hence the above long exact sequence yields the isomorphism $KK^{\widehat{\mathbb{F}}}(T,L(A\otimes B)) \overset{(u)_*}{\cong} KK^{\widehat{\mathbb{F}}}(T,A\otimes B)$. Take $\psi := (u)_*^{-1}(\mathcal{Z}(u',u''))$. Consequently, we have $u\circ\psi = \mathcal{Z}(u',u'') = u'\otimes u''$, by definition.

If we put $\Psi := \widehat{\mathbb{F}} \underset{r}{\ltimes} \psi : \widehat{\mathbb{F}} \underset{r}{\ltimes} (L'(A)\otimes L''(B)) \longrightarrow \widehat{\mathbb{F}} \underset{r}{\ltimes} L(A\otimes B)$, then the functoriality of constructions and the definition of the element ψ yields straightforwardly the diagram (3.3.1) of the statement.

ii) Given a $\widehat{\mathbb{G}}$-C^*-algebra $(A_0,\alpha_0) \in \mathscr{L}_{\widehat{\mathbb{G}}}$ and a $\widehat{\mathbb{H}}$-C^*-algebra (B,β), consider the corresponding $(\mathscr{L},\mathscr{N})$-triangles as above.

The same argument as in (i) by replacing $L'(A)$ by A_0 yields the existence of a Kasparov triple ${}_{A_0}\psi \in KK^{\widehat{\mathbb{F}}}\big(A_0\otimes L''(B), L(A_0\otimes B)\big)$ such that diagram (3.3.2) of the statement is commutative. Namely, put ${}_{A_0}\psi := (u)_*^{-1}({}_{A_0}\mathcal{Z}(u''))$.

Let us show that the Kasparov triple ${}_{A_0}\psi$ is invertible. If we apply the triangulated (by Lemma 3.3.2.1) functor ${}_{A_0}\mathcal{Z}$ to the $(\mathscr{L}_{\widehat{\mathbb{H}}},\mathscr{N}_{\widehat{\mathbb{H}}})$-triangle of B, we get the following distinguished triangle in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$

$$\Sigma(A_0\otimes N''(B)) \longrightarrow A_0\otimes L''(B) \overset{{}_{A_0}\mathcal{Z}(u'')}{\longrightarrow} A_0\otimes B \longrightarrow A_0\otimes N''(B),$$

where $A_0\otimes L''(B) = \mathcal{Z}(A_0,L''(B)) \in \mathscr{L}_{\widehat{\mathbb{F}}}$ because $A_0 \in \mathscr{L}_{\widehat{\mathbb{G}}}$, $L''(B) \in \mathscr{L}_{\widehat{\mathbb{H}}}$ and we apply Lemma 3.3.2.1 and $A_0\otimes N''(B) \in \mathscr{N}_{\widehat{\mathbb{F}}}$ because ${}_{A_0}\mathcal{Z}(\mathscr{N}_{\widehat{\mathbb{H}}}) \subset \mathscr{N}_{\widehat{\mathbb{F}}}$ by Lemma 3.3.2.1. In other words, the above is a $(\mathscr{L}_{\widehat{\mathbb{F}}},\mathscr{N}_{\widehat{\mathbb{F}}})$-triangle for $A_0\otimes B$. Hence, by uniqueness of this kind of distinguished

triangles, we have the following isomorphism of distinguished triangles in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$,

$$\begin{tikzcd}
\Sigma(A_0 \otimes L''(B)) \arrow[r] \arrow[d, "\wr"] & A_0 \otimes L''(B) \arrow[r, "{}_{A_0}\mathcal{Z}(u'')"] \arrow[d, "\wr\ {}_{A_0}\psi"] & A_0 \otimes B \arrow[r] \arrow[d, equal, "id"] & A_0 \otimes N''(B)) \arrow[d, "\wr"] \\
\Sigma'(N(A_0 \otimes B) \arrow[r] & L(A_0 \otimes B) \arrow[r, "u"'] & A_0 \otimes B \arrow[r] & N(A_0 \otimes B)
\end{tikzcd}$$

which yields in particular the invertibility of ${}_{A_0}\psi$ as claimed.

■

3.3.2.3 Theorem. *Let $\mathbb{F} = \mathbb{G} \times \mathbb{H}$ be a quantum direct product of compact quantum groups such that $\widehat{\mathbb{F}}$, $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are a torsion-free discrete quantum groups.*

i) If $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ satisfy the strong quantum Baum-Connes property, then $\widehat{\mathbb{F}}$ satisfies the quantum Baum-Connes property with coefficients in $A \otimes B$, for every $A \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}})$ and $B \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}})$.

ii) If $\widehat{\mathbb{F}}$ satisfies the strong *quantum Baum-Connes property, then $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ satisfy the* strong *quantum Baum-Connes property.*

iii) If $\widehat{\mathbb{F}}$ satisfies the quantum Baum-Connes property, then $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ satisfy the quantum Baum-Connes property with coefficients.

Proof. i) Given $A \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}})$ and $B \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}})$ consider the commutative diagram (3.3.1) of the preceding lemma,

$$\begin{tikzcd}
\widehat{\mathbb{G}} \underset{r}{\ltimes} L'(A) \otimes \widehat{\mathbb{H}} \underset{r}{\ltimes} L''(B) \arrow[r, "\Psi"] \arrow[d, "\widehat{\mathbb{G}} \underset{r}{\ltimes} u' \otimes \widehat{\mathbb{H}} \underset{r}{\ltimes} u''"'] & \widehat{\mathbb{F}} \underset{r}{\ltimes} L(A \otimes B) \arrow[d, "\widehat{\mathbb{F}} \underset{r}{\ltimes} u"] \\
\widehat{\mathbb{G}} \underset{r}{\ltimes} A \otimes \widehat{\mathbb{H}} \underset{r}{\ltimes} B \arrow[r, "\cong"] & \widehat{\mathbb{F}} \underset{r}{\ltimes} (A \otimes B)
\end{tikzcd} \tag{3.3.3}$$

where $\Psi = \widehat{\mathbb{F}} \underset{r}{\ltimes} \psi$ with $\psi = (u)_*^{-1}(\mathcal{Z}(u', u''))$.

Since $\widehat{\mathbb{G}}$ satisfies the *strong* quantum Baum-Connes property by assumption, then any Dirac homomorphism for A is an isomorphism, that is, $L'(A) \overset{u'}{\cong} A \in \mathscr{L}_{\widehat{\mathbb{G}}}$. In other words, $u' \in KK^{\widehat{\mathbb{G}}}(L'(A), A)$ is an invertible Kasparov triple. Accordingly, $\tau_R^{L''(B)}(u') \in KK^{\widehat{\mathbb{F}}}(L'(A) \otimes L''(B), A \otimes L''(B))$ is also an invertible Kasparov triple.

Recall that by definition we have

$$\mathcal{Z}(u', u'') = u' \otimes u'' = \tau_R^{L''(B)}(u') \underset{A_0 \otimes L''(B)}{\otimes} \tau_L^A(u'') \in KK^{\widehat{\mathbb{F}}}(L'(A) \otimes L''(B), A \otimes B)$$

$$_A\mathcal{Z}(u'') = \tau_L^A(u'') \in KK^{\widehat{\mathbb{F}}}(A \otimes L''(B), A \otimes B)$$

This two elements can be identified via the Kasparov multiplication $\tau_R^{L''(B)}(u') \underset{A\otimes L''(B)}{\otimes} (\,\cdot\,)$. In other words, the element ψ can be identified to the element $_A\psi$ via this Kasparov multiplication. The latter is invertible by (ii) of Lemma 3.3.2.2, which yields that Ψ in diagram (3.3.3) is invertible as well.

Next, since $\widehat{\mathbb{H}}$ satisfies the *strong* quantum Baum-Connes property by assumption, then any Dirac homomorphism for B is an isomorphism, that is, $L''(B) \overset{u''}{\cong} B \in \mathscr{L}_{\widehat{\mathbb{H}}}$. Hence, the element $\widehat{\mathbb{G}} \underset{r}{\ltimes} u' \otimes \widehat{\mathbb{H}} \underset{r}{\ltimes} u''$ of diagram (3.3.3) is invertible. The commutativity of (3.3.3) yields that $\widehat{\mathbb{F}} \underset{r}{\ltimes} u$ is an isomorphism in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$, which implies that the assembly map $\eta^{\widehat{\mathbb{F}}}_{A\otimes B}$ is invertible, that is, $\widehat{\mathbb{F}}$ satisfies the quantum Baum-Connes property with coefficients in $A \otimes B$.

ii) We have just to recall that $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are divisible torsion-free discrete quantum subgroups of $\widehat{\mathbb{F}}$ as explained in Remark 2.2.4. Therefore, Theorem 3.2.2.1 yields the assertion.

iii) In this case we apply Proposition 3.2.2.3.

■

3.3.2.4 Remark. It is worth mentioning the following. The element ψ constructed in (i) of Lemma 3.3.2.2 is such that

$$\begin{tikzcd}
L'(A) \otimes L''(B) \arrow[r, "\psi"] \arrow[d, "u' \otimes u''"'] & L(A \otimes B) \arrow[d, "u"] \\
A \otimes B \arrow[r, equal] & A \otimes B
\end{tikzcd}$$

The argument followed in (i) of the preceding theorem yields actually that both ψ and $u' \otimes u''$ are isomorphisms, which implies that u is also an isomorphism by the commutativity of the above diagram. In other words, we have proved that the $\widehat{\mathbb{F}}$-C^*-algebras of the form $A \otimes B$, where A is a $\widehat{\mathbb{G}}$-C^*-algebra and B is a $\widehat{\mathbb{H}}$-C^*-algebra, are actually in the subcategory $\mathscr{L}_{\widehat{\mathbb{F}}}$, which yields of course the conclusion given in (i) of the preceding theorem.

Taking crossed products in the preceding arguments has been done just for convenience of the presentation in order to make appear more clearly the corresponding assembly maps.

The above theorem yields immediately the connexion of the *usual* quantum Baum-Connes property for $\widehat{\mathbb{F}} = \widehat{\mathbb{G} \times \mathbb{H}}$ with the *Künneth formula* as announced in the introduction. Let A be a C^*-algebra, we say that A satisfies the Künneth formula if for every C^*-algebra B with free abelian K-group $K_*(B)$, the canonical homomorphism $K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B)$ is an isomorphism. Observe that this homomorphism is natural in A and B and it can be described in terms of the Kasparov product. We refer to Section 23 of [24] for more details.

3.3.2.5 Corollary. *Let $\mathbb{F} = \mathbb{G} \times \mathbb{H}$ be a quantum direct product of compact quantum groups such that $\widehat{\mathbb{F}}$, $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are torsion-free discrete quantum groups.*

For all $\widehat{\mathbb{G}}$-C^*-algebra (A, α) and all $\widehat{\mathbb{H}}$-C^*-algebra (B, β) the following diagram is commutative

$$\begin{tikzcd}
K_*\big(\widehat{\mathbb{G}} \underset{r}{\ltimes} L'(A) \otimes \widehat{\mathbb{H}} \underset{r}{\ltimes} L''(B)\big) \arrow[r, "K_*(\Psi)"] \arrow[d, "K_*\big(\widehat{\mathbb{G}} \underset{r}{\ltimes} u' \otimes \widehat{\mathbb{H}} \underset{r}{\ltimes} u''\big)"'] & \mathbb{L}F(A \otimes B) \arrow[d, "\eta^{\widehat{\mathbb{F}}}_{A\otimes B}"] \\
K_*\big(\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \otimes \widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B\big) \arrow[r, "\cong"] & F(A \otimes B)
\end{tikzcd}$$

Denote by $\mathcal{N}$ the class of C^*-algebras satisfying the Künneth formula.

i) If either $\widehat{\mathbb{G}}$ satisfies the strong Baum-Connes property, $\widehat{\mathbb{H}}$ satisfies the Baum-Connes property with coefficients in B or $\widehat{\mathbb{H}}$ satisfies the strong Baum-Connes property, $\widehat{\mathbb{G}}$ satisfies the Baum-Connes property with coefficients in A; either $\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A$ or $\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B$ belong to the class $\mathcal{N}$ and either $\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} L'(A)$ (and $K_*(\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} L''(B))$ is free abelian) or $\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} L''(B)$ (and $K_*(\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} L'(A))$ is free abelian) belong to the class $\mathcal{N}$, then $\widehat{\mathbb{F}}$ satisfies the Baum-Connes property with coefficients in $A \otimes B$.

ii) If $\widehat{\mathbb{G}}$ satisfies the strong Baum-Connes property, $\widehat{\mathbb{H}}$ satisfies the Baum-Connes property with coefficients in $\mathbb{C}$, either $C_r(\mathbb{G})$ or $C_r(\mathbb{H})$ belong to the class $\mathcal{N}$ and either $\widehat{\mathbb{G}} \underset{r}{\ltimes} L'(\mathbb{C})$ (and $K_*(\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} L''(\mathbb{C}))$ is free abelian) or $\widehat{\mathbb{H}} \underset{r}{\ltimes} L''(\mathbb{C})$ (and $K_*(\widehat{\mathbb{G}} \underset{\beta,r}{\ltimes} L'(\mathbb{C}))$ is free abelian) belong to the class $\mathcal{N}$, then $\widehat{\mathbb{F}}$ satisfies the Baum-Connes property with coefficients in $\mathbb{C}$.

Proof. The commutative diagram of the statement is obtained by simply applying the functor $K_*(\cdot)$ to diagram (3.3.1) from Lemma 3.3.2.2.

i) Let A be a $\widehat{\mathbb{G}}$-C^*-algebra and B a $\widehat{H}$-C^*-algebra. Assume that

- $\widehat{\mathbb{G}}$ satisfies the *strong* Baum-Connes property,
- $\widehat{\mathbb{H}}$ satisfies the Baum-Connes property with coefficients in B,
- $\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A \in \mathcal{N}$,
- $\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} L'(A) \in \mathcal{N}$ and $K_*(\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} L''(B))$ is free abelian.

The last condition guarantees that $K_*(\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B)$ is free abelian too because the Dirac homomorphism for B, $L''(B) \xrightarrow{u''} B$, induces a group homomorphism $K_*(\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} L''(B)) \longrightarrow K_*(\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B)$ by functoriality. Hence, by Künneth formula we have natural isomorphisms

$$K_*\big(\widehat{\mathbb{G}} \underset{r}{\ltimes} L'(A) \otimes \widehat{\mathbb{H}} \underset{r}{\ltimes} L''(B)\big) \cong K_*\big(\widehat{\mathbb{G}} \underset{r}{\ltimes} L'(A)\big) \otimes K_*\big(\widehat{\mathbb{H}} \underset{r}{\ltimes} L''(B)\big)$$

and

$$K_*\big(\widehat{\mathbb{G}} \underset{r}{\ltimes} A \otimes \widehat{\mathbb{H}} \underset{r}{\ltimes} B\big) \cong K_*\big(\widehat{\mathbb{G}} \underset{r}{\ltimes} A\big) \otimes K_*\big(\widehat{\mathbb{H}} \underset{r}{\ltimes} B\big)$$

which allows to write the commutative diagram of the statement as follows

$$\begin{tikzcd}
K_*\big(\widehat{\mathbb{G}} \underset{r}{\ltimes} L'(A)\big) \otimes K_*\big(\widehat{\mathbb{H}} \underset{r}{\ltimes} L''(B)\big) \arrow[r, "K_*(\Psi)"] \arrow[d, "\eta_A^{\widehat{\mathbb{G}}} \otimes \eta_B^{\widehat{\mathbb{H}}}"'] & \mathbb{L}F(A \otimes B) \arrow[d, "\eta_{A\otimes B}^{\widehat{\mathbb{F}}}"] \\
K_*\big(\widehat{\mathbb{G}} \underset{\alpha,r}{\ltimes} A\big) \otimes K_*\big(\widehat{\mathbb{H}} \underset{\beta,r}{\ltimes} B\big) \arrow[r, "\cong"] & F(A \otimes B)
\end{tikzcd}$$

Since $\widehat{\mathbb{G}}$ satisfies the *strong* Baum-Connes property, it satisfies the Baum-Connes property with coefficients in A. $\widehat{\mathbb{H}}$ satisfies the Baum-Connes property with coefficients in B by assumption. Hence $\eta_A^{\widehat{\mathbb{G}}}$ and $\eta_B^{\widehat{\mathbb{H}}}$ are isomorphisms. Since $\widehat{\mathbb{G}}$ satisfies the *strong* Baum-Connes property, the same argument as in Theorem 3.3.2.3 shows that Ψ is invertible, so $K_*(\Psi)$ of the above diagram is an isomorphism. We conclude that $\eta_{A\otimes B}^{\widehat{\mathbb{F}}}$ is an isomorphism by commutativity of the above diagram, which yields the claim.

ii) This is a particular case of (i).

■

3.3.1 Remark. It is important to say that the Künneth formula is a non trivial problem studied by several authors (see for instance [37], [168], [24] for more details) in connexion with the Baum-Connes property and the K-theory of tensor product of C^*-algebras.

Let us recall the following theorem of J. Chabert, S. Echterhoff and H. Oyono-Oyono contained in [37]: *"Let G, H be two locally compact groups. If G and H satisfy the Baum-Connes property with coefficients in $\mathbb{C}$ and either $C_r^*(G)$ or $C_r^*(H)$ belong to the class $\mathcal{N}$, then $G \times H$ satisfies the Baum-Connes property with coefficients in $\mathbb{C}$".*

The above corollary is thus a generalization of the Chabert-Echterhoff-Oyono-Oyono's result for (torsion-free) discrete quantum groups. However, we would like to remove the *strong Baum-Connes property* assumption from the statement.

If we wanted to give a more optimal result concerning the Baum-Connes property for a quantum direct product, we would have to carry out a detailed study about the Künneth formula in the *equivariant quantum setting*, which is out of the scope of the present dissertation. In particular, we would like to find sufficient conditions to a crossed product to belong to the class $\mathcal{N}$.

3.3.3 K-amenability property

Here we care about a property of own interest, namely the K-amenability of a quantum direct product. More precisely we have the following result.

3.3.3.1 Theorem. *Let $\mathbb{F} = \mathbb{G} \times \mathbb{H}$ be a quantum direct product of compact quantum groups. Then $\mathbb{F}$ is co-K-amenable if and only if $\mathbb{G}$ and $\mathbb{H}$ are co-K-amenable.*

Proof. Assume that $\mathbb{F}$ is co-K-amenable. This means that there exists an element $\alpha_{\mathbb{F}} \in KK(C_r(\mathbb{F}), \mathbb{C})$ such that

$$[\tau_{\mathbb{F}}] \underset{C_r(\mathbb{F})}{\otimes} \alpha_{\mathbb{F}} = [\varepsilon_{\mathbb{F}}] \in KK(C_m(\mathbb{F}), \mathbb{C}),$$

where $\tau_{\mathbb{F}} : C_m(\mathbb{F}) \twoheadrightarrow C_r(\mathbb{F})$ is the canonical surjection and $\varepsilon_{\mathbb{F}} : Pol(\mathbb{F}) \longrightarrow \mathbb{C}$ is the co-unit of $\mathbb{F}$ whose extension to $C_m(\mathbb{F})$ is still denoted by $\varepsilon_{\mathbb{F}}$ (recall Remark 1.3.1.41).

By virtue of Proposition 2.2.3 and we know that $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are discrete quantum subgroups of $\widehat{\mathbb{F}}$ via the canonical injections

$$\iota^r_{\mathbb{G}} : C_r(\mathbb{G}) \hookrightarrow C_r(\mathbb{F}) \text{ and } \iota^r_{\mathbb{H}} : C_r(\mathbb{H}) \hookrightarrow C_r(\mathbb{F})$$

Hence, by virtue of Theorem 3.2.3.1 we conclude that $\mathbb{G}$ and $\mathbb{H}$ are co-K-amenable with elements

$$\alpha_{\mathbb{G}} := [\iota^r_{\mathbb{G}}] \underset{C_r(\mathbb{F})}{\otimes} \alpha_{\mathbb{F}} \in KK(C_r(\mathbb{G}), \mathbb{C})$$

$$\alpha_{\mathbb{H}} := [\iota^r_{\mathbb{H}}] \underset{C_r(\mathbb{F})}{\otimes} \alpha_{\mathbb{F}} \in KK(C_r(\mathbb{H}), \mathbb{C})$$

Conversely, assume that both $\mathbb{G}$ and $\mathbb{H}$ are co-K-amenable. This means that there exist elements $\alpha_{\mathbb{G}} \in KK(C_r(\mathbb{G}), \mathbb{C})$ and $\alpha_{\mathbb{H}} \in KK(C_r(\mathbb{H}), \mathbb{C})$ such that

$$[\tau_{\mathbb{G}}] \underset{C_r(\mathbb{G})}{\otimes} \alpha_{\mathbb{G}} = [\varepsilon_{\mathbb{G}}] \text{ and } [\tau_{\mathbb{H}}] \underset{C_r(\mathbb{H})}{\otimes} \alpha_{\mathbb{H}} = [\varepsilon_{\mathbb{H}}]$$

where $\tau_{\mathbb{G}} : C_m(\mathbb{G}) \twoheadrightarrow C_r(\mathbb{G})$, $\tau_{\mathbb{H}} : C_m(\mathbb{H}) \twoheadrightarrow C_r(\mathbb{H})$ are the canonical surjections and $\varepsilon_{\mathbb{G}} : Pol(\mathbb{G}) \longrightarrow \mathbb{C}$, $\varepsilon_{\mathbb{H}} : Pol(\mathbb{H}) \longrightarrow \mathbb{C}$ are the co-units of $\mathbb{G}$ and $\mathbb{H}$, respectively whose extensions to $C_m(\mathbb{G})$ and $C_m(\mathbb{H})$ are still denoted by $\varepsilon_{\mathbb{G}}$ and $\varepsilon_{\mathbb{H}}$, respectively (recall Remark 1.3.1.41).

By using the canonical injections $\iota^r_{\mathbb{G}} : C_r(\mathbb{G}) \hookrightarrow C_r(\mathbb{F})$ and $\iota^r_{\mathbb{H}} : C_r(\mathbb{H}) \hookrightarrow C_r(\mathbb{F})$, we observe that $\tau_{\mathbb{F}} = \tau_{\mathbb{G}} \times \tau_{\mathbb{H}}$ by universal property of the maximal tensor product. We have as well that $\varepsilon_{\mathbb{F}} = \varepsilon_{\mathbb{G}} \times \varepsilon_{\mathbb{H}}$. If $\pi : C_m(\mathbb{G}) \underset{\max}{\otimes} C_m(\mathbb{H}) \twoheadrightarrow C_m(\mathbb{G}) \otimes C_m(\mathbb{H})$ denotes the canonical surjection, then by universal property of the maximal tensor product we have the following commutative diagrams

$$\begin{tikzcd} C_m(\mathbb{F}) \arrow[r, two heads, "\tau_{\mathbb{F}}"] \arrow[d, two heads, "\pi"'] & C_r(\mathbb{F}) \\ C_m(\mathbb{G}) \otimes C_m(\mathbb{H}) \arrow[ur, "\tau_{\mathbb{G}} \otimes \tau_{\mathbb{H}}"'] & \end{tikzcd} \qquad \begin{tikzcd} C_m(\mathbb{F}) \arrow[r, "\varepsilon_{\mathbb{F}}"] \arrow[d, two heads, "\pi"'] & \mathbb{C} \\ C_m(\mathbb{G}) \otimes C_m(\mathbb{H}) \arrow[ur, "\varepsilon_{\mathbb{G}} \otimes \varepsilon_{\mathbb{H}}"'] & \end{tikzcd}$$

where $\tau_{\mathbb{G}} \otimes \tau_{\mathbb{H}} : C_m(\mathbb{G}) \otimes C_m(\mathbb{H}) \twoheadrightarrow C_r(\mathbb{G}) \otimes C_r(\mathbb{H})$ is the tensor product of the canonical surjections $\tau_{\mathbb{G}}$ and $\tau_{\mathbb{H}}$ and $\varepsilon_{\mathbb{G}} \otimes \varepsilon_{\mathbb{H}} : C_m(\mathbb{G}) \otimes C_m(\mathbb{H}) \longrightarrow \mathbb{C}$ is the tensor product of the co-units $\varepsilon_{\mathbb{G}}$ and $\varepsilon_{\mathbb{H}}$ (recall Theorem A.1.11).

In this way, the canonical surjection $\tau_{\mathbb{F}} : C_m(\mathbb{F}) \twoheadrightarrow C_r(\mathbb{F})$ and the co-unit $\varepsilon_{\mathbb{F}} : C_m(\mathbb{F}) \longrightarrow \mathbb{C}$ can be written, as Kasparov bimodules, under the following form

$$[\tau_{\mathbb{F}}] = [\pi] \underset{C_m(\mathbb{G}) \otimes C_m(\mathbb{H})}{\otimes} [\tau_{\mathbb{G}} \otimes \tau_{\mathbb{H}}] = \pi^*\big([\tau_{\mathbb{G}} \otimes \tau_{\mathbb{H}}]\big) \in KK(C_m(\mathbb{F}), C_r(\mathbb{F})),$$

$$[\varepsilon_{\mathbb{F}}] = [\pi] \underset{C_m(\mathbb{G}) \otimes C_m(\mathbb{H})}{\otimes} [\varepsilon_{\mathbb{G}} \otimes \varepsilon_{\mathbb{H}}] = \pi^*\big([\varepsilon_{\mathbb{G}} \otimes \varepsilon_{\mathbb{H}}]\big) \in KK(C_m(\mathbb{F}), \mathbb{C}),$$

By using Theorem 1.7.1.16 we define the following element

$$\alpha_{\mathbb{F}} := \alpha_{\mathbb{G}} \otimes \alpha_{\mathbb{H}} := \tau_R^{C_r(\mathbb{H})}(\alpha_{\mathbb{G}}) \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}}) \in KK(C_r(\mathbb{F}), \mathbb{C})$$

and we claim that

$$[\tau_{\mathbb{F}}] \underset{C_r(\mathbb{F})}{\otimes} \alpha_{\mathbb{F}} = [\varepsilon_{\mathbb{F}}] \in KK(C_m(\mathbb{F}), \mathbb{C}),$$

which yields the result. Indeed, by applying the elementary properties of the Kasparov product of Theorem 1.7.1.16 we write the following

$$\begin{aligned}
[\tau_{\mathbb{F}}] \underset{C_r(\mathbb{F})}{\otimes} \alpha_{\mathbb{F}} &= \Big(\pi^*\big([\tau_{\mathbb{G}} \otimes \tau_{\mathbb{H}}]\big)\Big) \underset{C_r(\mathbb{F})}{\otimes} \Big(\tau_R^{C_r(\mathbb{H})}(\alpha_{\mathbb{G}}) \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\Big) \\
&= \pi^*\Big(\big([\tau_{\mathbb{G}} \otimes \tau_{\mathbb{H}}]\big) \underset{C_r(\mathbb{F})}{\otimes} \big(\tau_R^{C_r(\mathbb{H})}(\alpha_{\mathbb{G}}) \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\big)\Big) \\
&= \pi^*\Big(\big[(\tau_{\mathbb{G}} \otimes id_{C_r(\mathbb{H})})(id_{C_m(\mathbb{G})} \otimes \tau_{\mathbb{H}})\big] \underset{C_r(\mathbb{F})}{\otimes} \big(\tau_R^{C_r(\mathbb{H})}(\alpha_{\mathbb{G}}) \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\big)\Big) \\
&= \pi^*\Big(\big([id_{C_m(\mathbb{G})} \otimes \tau_{\mathbb{H}}] \underset{C_m(\mathbb{G})\otimes C_r(\mathbb{H})}{\otimes} [\tau_{\mathbb{G}} \otimes id_{C_r(\mathbb{H})}]\big) \underset{C_r(\mathbb{F})}{\otimes} \big(\tau_R^{C_r(\mathbb{H})}(\alpha_{\mathbb{G}}) \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\big)\Big) \\
&= \pi^*\Big([id_{C_m(\mathbb{G})} \otimes \tau_{\mathbb{H}}] \underset{C_m(\mathbb{G})\otimes C_r(\mathbb{H})}{\otimes} \big([\tau_{\mathbb{G}} \otimes id_{C_r(\mathbb{H})}] \underset{C_r(\mathbb{F})}{\otimes} \tau_R^{C_r(\mathbb{H})}(\alpha_{\mathbb{G}})\big) \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\Big) \\
&= \pi^*\Big([id_{C_m(\mathbb{G})} \otimes \tau_{\mathbb{H}}] \underset{C_m(\mathbb{G})\otimes C_r(\mathbb{H})}{\otimes} \big(\tau_{\mathbb{G}} \otimes id_{C_r(\mathbb{H})}\big)^* \big(\tau_R^{C_r(\mathbb{H})}(\alpha_{\mathbb{G}})\big) \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\Big) \\
&= \pi^*\Big([id_{C_m(\mathbb{G})} \otimes \tau_{\mathbb{H}}] \underset{C_m(\mathbb{G})\otimes C_r(\mathbb{H})}{\otimes} \tau_R^{C_r(\mathbb{H})}\big([\tau_{\mathbb{G}}] \underset{C_r(\mathbb{G})}{\otimes} \alpha_{\mathbb{G}}\big) \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\Big) \\
&= \pi^*\Big([id_{C_m(\mathbb{G})} \otimes \tau_{\mathbb{H}}] \underset{C_m(\mathbb{G})\otimes C_r(\mathbb{H})}{\otimes} \tau_R^{C_r(\mathbb{H})}\big([\varepsilon_{\mathbb{G}}]\big) \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\Big) \\
&= \pi^*\Big([id_{C_m(\mathbb{G})} \otimes \tau_{\mathbb{H}}] \underset{C_m(\mathbb{G})\otimes C_r(\mathbb{H})}{\otimes} [\varepsilon_{\mathbb{G}} \otimes id_{C_r(\mathbb{H})}] \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\Big) \\
&= \pi^*\Big(\big[(\varepsilon_{\mathbb{G}} \otimes id_{C_r(\mathbb{H})})(id_{C_m(\mathbb{G})} \otimes \tau_{\mathbb{H}})\big] \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\Big) = \pi^*\Big([\varepsilon_{\mathbb{G}} \otimes \tau_{\mathbb{H}}] \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\Big) \\
&= \pi^*\Big(\big[(id_{\mathbb{C}} \otimes \tau_{\mathbb{H}})(\varepsilon_{\mathbb{G}} \otimes id_{C_m(\mathbb{H})})\big] \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\Big) \\
&= \pi^*\Big(\big([\varepsilon_{\mathbb{G}} \otimes id_{C_m(\mathbb{H})}] \underset{C_m(\mathbb{H})}{\otimes} [id_{\mathbb{C}} \otimes \tau_{\mathbb{H}}]\big) \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\Big) \\
&= \pi^*\Big([\varepsilon_{\mathbb{G}} \otimes id_{C_m(\mathbb{H})}] \underset{C_m(\mathbb{H})}{\otimes} \big([id_{\mathbb{C}} \otimes \tau_{\mathbb{H}}] \underset{C_r(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\big)\Big) \\
&= \pi^*\Big([\varepsilon_{\mathbb{G}} \otimes id_{C_m(\mathbb{H})}] \underset{C_m(\mathbb{H})}{\otimes} \big(id_{\mathbb{C}} \otimes \tau_{\mathbb{H}}\big)^* \big(\tau_L^{\mathbb{C}}(\alpha_{\mathbb{H}})\big)\Big) \\
&= \pi^*\Big([\varepsilon_{\mathbb{G}} \otimes id_{C_m(\mathbb{H})}] \underset{C_m(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}\big([\tau_{\mathbb{H}}] \underset{C_r(\mathbb{H})}{\otimes} \alpha_{\mathbb{H}}\big)\Big) = \pi^*\Big([\varepsilon_{\mathbb{G}} \otimes id_{C_m(\mathbb{H})}] \underset{C_m(\mathbb{H})}{\otimes} \tau_L^{\mathbb{C}}\big([\varepsilon_{\mathbb{H}}]\big)\Big) \\
&= \pi^*\Big([\varepsilon_{\mathbb{G}} \otimes id_{C_m(\mathbb{H})}] \underset{C_m(\mathbb{H})}{\otimes} [id_{\mathbb{C}} \otimes \varepsilon_{\mathbb{H}}]\Big) = \pi^*\Big(\big[(id_{\mathbb{C}} \otimes \varepsilon_{\mathbb{H}})(\varepsilon_{\mathbb{G}} \otimes id_{C_m(\mathbb{H})})\big]\Big) \\
&= \pi^*\Big([\varepsilon_{\mathbb{G}} \otimes \varepsilon_{\mathbb{H}}]\Big) = [\varepsilon_{\mathbb{F}}]
\end{aligned}$$

■

3.4 The Baum-Connes property for a quantum semi-direct product

In this section we are going to study the Baum-Connes property for the dual of a quantum semi-direct product of a compact quantum group by a discrete group in terms of the involved (quantum) groups. Recall that J. Chabert has already studied the Baum-Connes property for this construction in the context of *classical* locally compact groups in [34]. Likewise, J. Chabert and S. Echterhoff studied again the same property for a semi-direct product of classical locally compact groups in [35] removing a superfluous hypothesis from the earlier work [34] (we will be more precise about this later on). It is important to say that H. Oyono-Oyono has also studied some permanence properties of the Baum-Connes property for *classical discrete* groups. Namely, he obtained in [143] that the Baum-Connes property is stable under the semi-direct product construction with the analogous hypothesis to the ones of the locally compact case of Chabert-Echterhoff.

Thus, we give here a generalization of this result for the dual of a quantum semi-direct product of a compact quantum group by a discrete group as defined in Section 2.3. The strategy to reach this result consists in doing the translation of the arguments of the articles [34], [35] into the categorical framework of Meyer-Nest. We observe in particular that this new perspective avoids all technical problems appeared in the classical situation with respect to the treatment of the left-hand side of the assembly map, which yields a more conceptual proof.

In order to do so, we carry the same notations of Section 2.3 on. Let $\mathbb{F} = \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$ be a quantum semi-direct product, where Γ is a classical discrete group and $\mathbb{G}$ is a compact quantum group.

3.4.1 Torsion property

We have already explained that all our *discrete quantum groups are supposed to be torsion-free* in order to study the corresponding quantum Baum-Connes property. For this reason it is advisable to study in more detail the torsion phenomena of the dual of a quantum semi-direct product $\mathbb{F} = \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$ in terms of the involved (quantum) groups.

Let us study the torsion in the sense of Meyer-Nest. For the following result it is advisable to keep in mind the spectral theory for compact quantum groups explained in Section 1.4.2.

3.4.1.1 Theorem. *Let $\mathbb{F} = \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$ be the quantum semi-direct product of $\mathbb{G}$ by Γ. If Γ and $\widehat{\mathbb{G}}$ are torsion-free, then $\widehat{\mathbb{F}}$ is torsion-free.*

Proof. Let A be a unital finite dimensional C^*-algebra equipped with a right torsion action of $\mathbb{F}$, say $\delta : A \longrightarrow A \otimes C(\mathbb{F})$.

Let's define

$$\Lambda := \{\gamma \in \Gamma \mid \exists\ x \in Irr(\mathbb{G}) \text{ such that } K_{(\gamma,x)} \neq 0\},$$

where $K_{(\gamma,x)}$ denotes the spectral subspace associated to the representation $(\gamma, x) =: y \in Irr(\mathbb{F})$. We claim that Λ is a finite subgroup of Γ. Indeed, Λ is a subgroup of Γ because given $g, h \in \Lambda$, let $X_g \in K_{(g,x_g)}$ and $X_h \in K_{(h,x_h)}$ be some non-zero elements for the corresponding irreducible representations $x_g, x_h \in Irr(\mathbb{G})$. Put $y_g := (g, x_g), y_h := (h, x_h) \in Irr(\mathbb{F})$. By virtue of Lemma

1.4.2.18, there exist an irreducible representation $z := (\gamma, x) \in Irr(\mathbb{F})$ and an intertwiner $\Phi \in Mor(z, y_g \oplus y_h)$ such that $X_g \underset{\Phi}{\otimes} X_h \neq 0$.

Besides, the proof of Lemma 1.4.2.18 shows that z is an irreducible representation of the decomposition of $y_g \oplus y_h$ in direct sum of irreducible representations. Thanks to the fusion rules of a quantum semi-direct product we have that $w^{y_g} \oplus w^{y_h} = v^{gh} \oplus (v^{\alpha_{h^{-1}}(x_g)} \oplus v^{x_h})$. Next, consider the decomposition in direct sum of irreducible representations of the tensor product $\alpha_{h^{-1}}(x_g) \oplus x_h$, say $\{x_k\}_{k=1,\dots,r}$ for some $r \in \mathbb{N}$. Hence we write $w^{y_g \oplus y_h} = \bigoplus_{k=1}^{r} v^{gh} \oplus x_k$. As a result, the irreducible representation $z = (\gamma, x) \in Irr(\mathbb{F})$ found above must be of the form (gh, x_k) for some $k = 1, \dots, r$. Recall that $X_g \underset{\Phi}{\otimes} X_h \in K_z$ by definition. This shows that $gh = \gamma \in \Lambda$ as required. Moreover, Λ is finite because A is finite dimensional.

Thanks to the torsion-freeness of Γ, Λ is just the trivial subgroup $\{e\}$. Hence, for every $y \in Irr(\mathbb{F})$, $K_y \neq 0$ implies $y = (e, x)$ for some $x \in Irr(\mathbb{G})$. Consequently, the spectral decomposition for $A = \mathcal{A}_{\mathbb{F}}$ becomes $A = \bigoplus_{x \in Irr(\mathbb{G})} \mathcal{A}_{(e,x)} = \mathcal{A}_{\mathbb{G}}$ and the action δ takes its values on $A \otimes \pi(C_m(\mathbb{G}))$ so that δ is actually an action of $\mathbb{G}$ on A. Since $\widehat{\mathbb{G}}$ is torsion-free by assumption, we achieve the conclusion. ■

3.4.1.2 Note. The converse of the preceding statement would be true whenever the torsion-freeness is preserved under divisible discrete quantum subgroups as conjectured in the end of Section 3.2.1.

3.4.2 The Baum-Connes property

In order to legitimate the Baum-Connes property formulation for the dual of a quantum semi-direct product $\mathbb{F} = \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$, we need $\widehat{\mathbb{F}}$ to be torsion-free. And, in order to legitimate the Baum-Connes property formulation for Γ and $\widehat{\mathbb{G}}$, we need these discrete (quantum) groups to be torsion-free. Hence, we must keep the preceding section in mind and, for simplicity, we assume that $\widehat{\mathbb{F}}$, Γ and $\widehat{\mathbb{G}}$ are all torsion-free.

Next, let us set up the adapted notations to this situation of the categorical framework of Meyer-Nest. Consider the equivariant Kasparov categories associated to $\widehat{\mathbb{F}}$ and Γ, say

$$\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}} \text{ and } \mathscr{K}\mathscr{K}^{\Gamma},$$

respectively. The canonical suspension functors for each of the above Kasparov categories are denoted simply by Σ.

Notice that from now on, the word *homomorphism (respectively, isomorphism)* will mean *homomorphism (respectively, isomorphism) in the corresponding Kasparov category*; it can be either a true homomorphism (respectively, isomorphism) between C^*-algebras or any Kasparov triple between C^*-algebras (respectively, any KK-equivalence between C^*-algebras).

Consider the canonical complementary pair of localizing subcategories in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$ and $\mathscr{K}\mathscr{K}^{\Gamma}$, say

$$(\mathscr{L}_{\widehat{\mathbb{F}}}, \mathscr{N}_{\widehat{\mathbb{F}}}) \text{ and } (\mathscr{L}_{\Gamma}, \mathscr{N}_{\Gamma}),$$

respectively. In this way, the canonical triangulated functors associated to these complementary pairs are denoted by (L, N) and (L', N'), respectively.

Consider the homological functors defining the (*quantum*) Baum-Connes assembly maps for $\widehat{\mathbb{F}}$ and Γ. Namely,

$$\begin{array}{rccl} F: & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}} & \longrightarrow & \mathscr{A}b^{\mathbb{Z}/2} \\ & (A,\delta) & \longmapsto & F(A) := K_*(\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A) \end{array} \qquad \begin{array}{rccl} F': & \mathscr{K}\mathscr{K}^{\Gamma} & \longrightarrow & \mathscr{A}b^{\mathbb{Z}/2} \\ & (B,\beta) & \longmapsto & F'(B) := K_*(\Gamma \underset{\beta,r}{\ltimes} B) \end{array}$$

Therefore, the (quantum) assembly maps for $\widehat{\mathbb{F}}$ and for Γ are given by the following natural transformations

$$\eta^{\widehat{\mathbb{F}}} : \mathbb{L}F \longrightarrow F \text{ and } \eta^{\Gamma} : \mathbb{L}F' \longrightarrow F',$$

where, by definition, we have

$$\mathbb{L}F = F \circ L \text{ and } \mathbb{L}F' = F' \circ L'$$

By the torsion-freeness assumption, the subcategory $\mathscr{L}_{\widehat{\mathbb{F}}}$ (resp. $\mathscr{L}_{\Gamma}$) is easily described as the *localizing subcategory of $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$ (resp. $\mathscr{K}\mathscr{K}^{\Gamma}$) generated by the objects of the form $c_0(\widehat{\mathbb{F}}) \otimes C$ (resp. $c_0(\Gamma) \otimes C$) with C any C^*-algebra in the Kasparov category $\mathscr{K}\mathscr{K}$.*

Now, we are now ready to start the study of the Baum-Connes property for a quantum semi-direct product. To this end, consider the following functor

$$\begin{array}{rccl} \mathcal{Z}: & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}} & \longrightarrow & \mathscr{K}\mathscr{K}^{\Gamma} \\ & (A,\delta) & \longmapsto & \mathcal{Z}(A) := (\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A, \partial), \end{array}$$

which is defined on homomorphisms by functoriality. Let us be more precise in the definition of $\mathcal{Z}$ as a *functor* between $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$ and $\mathscr{K}\mathscr{K}^{\Gamma}$. For this we have to keep in mind all notations and results obtained in Section 2.3.

Given any $\widehat{\mathbb{F}}$-C^*-algebra $(A,\delta) \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}})$, we regard it either as an object in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ or as an object in $\mathscr{K}\mathscr{K}^{\Gamma}$ by restricting the action δ as explained in Proposition 2.3.5. More precisely, we can consider the following

$$(A, \delta_{\widehat{\mathbb{G}}}) \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}) \text{ and } (A, \delta_{\Gamma}) \in Obj(\mathscr{K}\mathscr{K}^{\Gamma})$$

In this way, it is licit to consider the crossed product $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$, which is now a Γ-C^*-algebra with action ∂ by virtue of Proposition 2.3.12.

Hence $\mathcal{Z}$ is well-defined on objects. Let us explain its definition on homomorphisms. Let $(B,\nu) \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}})$ be an other $\widehat{\mathbb{F}}$-C^*-algebra and consider the corresponding objects in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ and in $\mathscr{K}\mathscr{K}^{\Gamma}$,

$$(B, \nu_{\widehat{\mathbb{G}}}) \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}),\ (B, \nu_{\Gamma}) \in Obj(\mathscr{K}\mathscr{K}^{\Gamma}) \text{ and } (\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B, \partial') \in Obj(\mathscr{K}\mathscr{K}^{\Gamma})$$

Let $\mathcal{X} := ((H,\delta_H),\pi,F) \in KK^{\widehat{\mathbb{F}}}(A,B)$ be a homomorphism between A and B in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$. Restricting again the corresponding action, we can regard this Kasparov triple either as an element in $KK^{\widehat{\mathbb{G}}}(A,B)$ or as an element in $KK^{\Gamma}(A,B)$. We denote both elements simply by $\mathcal{X}$ and we write

$$\mathcal{X} := ((H, \delta_{H,\widehat{\mathbb{G}}}), \pi, F) \in KK^{\widehat{\mathbb{G}}}(A,B) \text{ and } \mathcal{X} := ((H, \delta_{H,\Gamma}), \pi, F) \in KK^{\Gamma}(A,B)$$

In this way, the descent homomorphism with the functoriality of the crossed product yield the existence of a Kasparov triple $\widehat{\mathbb{G}} \underset{r}{\ltimes} \mathcal{X} \in KK(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A, \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B)$, which is given precisely by (recall Theorem 1.7.1.19)

$$\widehat{\mathbb{G}} \underset{r}{\ltimes} \mathcal{X} := \big(H \underset{\pi_\nu}{\otimes} \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B, id \underset{r}{\ltimes} \pi, F \underset{\pi_\nu}{\otimes} id\big)$$

Next, it is easy to establish an action of Γ on $\widehat{\mathbb{G}} \underset{r}{\ltimes} \mathcal{X}$ in order to get a Γ-*equivariant* Kasparov triple between $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$ and $\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B$. Indeed, since $\mathcal{X}$ is also a Γ-equivariant Kasparov triple with action $\delta_{H,\Gamma}$ and $\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B$ is also a Γ-C^*-algebra with action ∂', we can equip $H \underset{\pi_\nu}{\otimes} \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B$ with the corresponding diagonal action, say

$$\begin{array}{rccl} \sigma: & \Gamma & \longrightarrow & Aut(H \underset{\pi_\nu}{\otimes} \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B) \\ & \gamma & \longmapsto & \sigma_\gamma := \big(\delta_{H,\Gamma}\big)_\gamma \underset{\pi_\nu}{\otimes} \partial'_\gamma, \end{array}$$

where $Aut(H \underset{\pi_\nu}{\otimes} \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B)$ denotes the group of *invertible* bounded linear transformations of $H \underset{\pi_\nu}{\otimes} \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B$ (*not necessarily module homomorphisms*). It turns out that this action makes $\widehat{\mathbb{G}} \underset{r}{\ltimes} \mathcal{X}$ a Γ-equivariant Kasparov triple, so that we have actually that $\widehat{\mathbb{G}} \underset{r}{\ltimes} \mathcal{X} \in KK^\Gamma(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A, \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B)$. Hence $\mathcal{Z}$ is well-defined on homomorphisms. Let us precise the computations to show the Γ-equivariance of $\widehat{\mathbb{G}} \underset{r}{\ltimes} \mathcal{X}$. Since we have already that $\widehat{\mathbb{G}} \underset{r}{\ltimes} \mathcal{X} \in KK(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A, \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B)$ by Theorem 1.7.1.19, we only have to check the Γ-equivariance properties of this Kasparov triple (see for instance [24] for more details about the equivariant Kasparov KK-theory). Namely,

- First of all, observe that the linear map

$$\xi \underset{\pi_\nu}{\odot} x \mapsto \big(\delta_{H,\Gamma}\big)_\gamma(\xi) \underset{\pi_\nu}{\otimes} \partial'_\gamma(x),\ \xi \in H,\ x \in \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B$$

  is bounded for all $\gamma \in \Gamma$ because ∂'_γ is bounded on $\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B$ for all $\gamma \in \Gamma$ and we have

$$\begin{aligned} \big\langle \big(\delta_{H,\Gamma}\big)_\gamma(\xi) \underset{\pi_\nu}{\otimes} \partial'_\gamma(x), \big(\delta_{H,\Gamma}\big)_\gamma(\xi) \underset{\pi_\nu}{\otimes} \partial'_\gamma(x)\big\rangle &= \big\langle \partial'_\gamma(x), \pi_\nu\Big(\big\langle \big(\delta_{H,\Gamma}\big)_\gamma(\xi), \big(\delta_{H,\Gamma}\big)_\gamma(\xi)\big\rangle\Big)\partial'_\gamma(x)\big\rangle \\ &\overset{(1)}{=} \big\langle \partial'_\gamma(x), \pi_\nu\Big(\big(\nu_\Gamma\big)_\gamma\langle \xi,\xi\rangle\Big)\partial'_\gamma(x)\big\rangle \overset{(2)}{=} \big\langle \partial'_\gamma(x), \partial'_\gamma\Big(\pi_\nu\big(\langle \xi,\xi\rangle\big)\Big)\partial'_\gamma(x)\big\rangle \\ &= \big\langle \partial'_\gamma(x), \partial'_\gamma\Big(\pi_\nu\big(\langle \xi,\xi\rangle\big)x\Big)\big\rangle \\ &\overset{(3)}{=} \partial'_\gamma\Big(\big\langle x, \pi_\nu\big(\langle \xi,\xi\rangle\big)x\big\rangle\Big) = \partial'_\gamma\Big(\big\langle \xi \underset{\pi_\nu}{\odot} x, \xi \underset{\pi_\nu}{\odot} x\big\rangle\Big), \end{aligned}$$

  where in (1) we use the fact that $\mathcal{X} = ((H, \delta_{H,\Gamma}), \pi, F) \in KK^\Gamma(A, B)$, in (2) we use the definition of the action ∂' of Γ on $\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B$ and in (3) we regard $\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B$ as a Γ-equivariant Hilbert $\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B$-module with its action ∂'. Hence it extends to a bounded linear map on

$H \underset{\pi_\nu}{\otimes} \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B$, whence the definition of σ. Moreover, this formula yields clearly that $\sigma_\gamma^{-1} = \sigma_{\gamma^{-1}}$, for all $\gamma \in \Gamma$.

- $H \underset{\pi_\nu}{\otimes} \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B$ is a Γ-equivariant Hilbert $\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B$-module. Namely, for all $\xi, \eta \in H$ and all $x, y \in \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B$ we have

$$\begin{aligned}
\sigma_\gamma\big((\xi \underset{\pi_\nu}{\otimes} x) \cdot y)\big) &= \sigma_\gamma(\xi \underset{\pi_\nu}{\otimes} xy) = \big(\delta_{H,\Gamma}\big)_\gamma(\xi) \underset{\pi_\nu}{\otimes} \partial'_\gamma(xy) \\
&= \big(\delta_{H,\Gamma}\big)_\gamma(\xi) \underset{\pi_\nu}{\otimes} \partial'_\gamma(x)\partial'_\gamma(y) = \Big(\big(\delta_{H,\Gamma}\big)_\gamma(\xi) \underset{\pi_\nu}{\otimes} \partial'_\gamma(x)\Big) \cdot \partial'_\gamma(y) \\
&= \sigma_\gamma\big((\xi \underset{\pi_\nu}{\otimes} x) \cdot \partial'_\gamma(y),
\end{aligned}$$

and

$$\begin{aligned}
\langle \sigma_\gamma(\xi \underset{\pi_\nu}{\otimes} x), \sigma_\gamma(\eta \underset{\pi_\nu}{\otimes} y)\rangle &= \langle \big(\delta_{H,\Gamma}\big)_\gamma(\xi) \underset{\pi_\nu}{\otimes} \partial'_\gamma(x), \big(\delta_{H,\Gamma}\big)_\gamma(\eta) \underset{\pi_\nu}{\otimes} \partial'_\gamma(y)\rangle \\
&= \langle \partial'_\gamma(x), \pi_\nu\Big(\langle \big(\delta_{H,\Gamma}\big)_\gamma(\xi), \big(\delta_{H,\Gamma}\big)_\gamma(\eta)\rangle\Big)\partial'_\gamma(y)\rangle \\
&\overset{(1)}{=} \langle \partial'_\gamma(x), \pi_\nu\Big(\big(\nu_\Gamma\big)_\gamma\langle \xi, \eta\rangle\Big)\partial'_\gamma(y)\rangle \\
&\overset{(2)}{=} \langle \partial'_\gamma(x), \partial'_\gamma\Big(\pi_\nu\big(\langle \xi, \eta\rangle\big)\Big)\partial'_\gamma(y)\rangle = \langle \partial'_\gamma(x), \partial'_\gamma\Big(\pi_\nu\big(\langle \xi, \eta\rangle\big)y\Big)\rangle \\
&\overset{(3)}{=} \partial'_\gamma\Big(\langle x, \pi_\nu\big(\langle \xi, \eta\rangle\big)y\rangle\Big) = \partial'_\gamma\Big(\langle \xi \underset{\pi_\nu}{\otimes} x, \eta \underset{\pi_\nu}{\otimes} y\rangle\Big),
\end{aligned}$$

where in (1) we use the fact that $\mathcal{X} = ((H, \delta_{H,\Gamma}), \pi, F) \in KK^\Gamma(A, B)$, in (2) we use the definition of the action ∂' of Γ on $\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B$ and in (3) we regard $\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B$ as a Γ-equivariant Hilbert $\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B$-module with its action ∂'.

- The representation $id \underset{r}{\ltimes} \pi : \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}},r}}{\ltimes} A \longrightarrow \mathcal{L}_{\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B}\big(H \underset{\pi_\nu}{\otimes} \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B\big)$ is Γ-equivariant. Indeed, for all $\gamma \in \Gamma$, $a \in A$, $x \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$ we write

$$\begin{aligned}
id \underset{r}{\ltimes} \pi\big(\partial'_\gamma(\pi_\delta(a)U^x_{ij})\big) &= id \underset{r}{\ltimes} \pi\Big(\pi_\delta\big((\delta_\Gamma)_\gamma(a)\big)\phi_U\big(\alpha_\gamma(w^x_{ij})\big)\Big) \\
&= \pi_{\mathcal{K}}\big(\pi\big((\delta_\Gamma)_\gamma(a)\big)\big)\phi_W\big(\alpha_\gamma(w^x_{ij})\big) \\
&\overset{(1)}{=} \big(\delta_{H,\Gamma}\big)_\gamma \pi_{\mathcal{K}}\big(\pi(a)\big)\big(\delta_{H,\Gamma}\big)_{\gamma^{-1}}\partial'_\gamma\big(W^x_{ij}\big) \\
&= \big(\delta_{H,\Gamma}\big)_\gamma \pi_{\mathcal{K}}\big(\pi(a)\big)\big(\delta_{H,\Gamma}\big)_{\gamma^{-1}}\partial'_\gamma\big(W^x_{ij} \cdot \partial'_{\gamma^{-1}}\big) \\
&= \big(\delta_{H,\Gamma}\big)_\gamma \underset{\pi_\nu}{\otimes} \partial'_\gamma \circ \pi_{\mathcal{K}}\big(\pi(a)\big)W^x_{ij} \circ \big(\delta_{H,\Gamma}\big)_{\gamma^{-1}} \underset{\pi_\nu}{\otimes} \partial'_{\gamma^{-1}} \\
&= \sigma_\gamma \circ \pi_{\mathcal{K}}\big(\pi(a)\big)W^x_{ij} \circ \sigma_{\gamma^{-1}} \\
&= \sigma_\gamma \circ id \underset{r}{\ltimes} \pi(\pi_\delta(a)U^x_{ij}) \circ \sigma_{\gamma^{-1}},
\end{aligned}$$

where in (1) we use the fact that $\mathcal{X} = ((H, \delta_{H,\Gamma}), \pi, F) \in KK^{\Gamma}(A, B)$ and the definition of the action ∂' of Γ on $\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B$. Moreover it is important to remark that for the preceding computations we have taken into account the identification

$$\mathcal{L}_{\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B}\big(H \underset{\pi_\nu}{\otimes} \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B\big) = M\big(\mathcal{K}_{\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B}\big(H \underset{\pi_\nu}{\otimes} \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B\big)\big) \cong M\big(\widehat{\mathbb{G}} \underset{Ad_{V_{H,\widehat{\mathbb{G}}}},r}{\ltimes} \mathcal{K}_B(H)\big),$$

given by Lemma 5.2 in [206], which yields the definition of $id \underset{r}{\ltimes} \pi$ by functoriality of the reduced crossed product (recall Remark 1.5.2.5). In this way, $\pi_{\mathcal{K}} : \mathcal{K}_B(H) \longrightarrow \widehat{\mathbb{G}} \underset{Ad_{V_{H,\widehat{\mathbb{G}}}},r}{\ltimes} \mathcal{K}_B(H)$ denotes the non-degenerate $*$-homomorphism and $W \in M\big(c_0(\widehat{\mathbb{G}}) \otimes \widehat{\mathbb{G}} \underset{Ad_{V_{H,\widehat{\mathbb{G}}}},r}{\ltimes} \mathcal{K}_B(H)\big)$ the unitary representation associated to the reduced crossed product construction $\widehat{\mathbb{G}} \underset{Ad_{V_{H,\widehat{\mathbb{G}}}},r}{\ltimes} \mathcal{K}_B(H)$ following Theorem 1.5.2.1.

- We have $\Big(\sigma_\gamma \circ F \underset{\pi_\nu}{\otimes} id \circ \sigma_{\gamma^{-1}} - F \underset{\pi_\nu}{\otimes} id\Big) id \underset{r}{\ltimes} \pi(x) \in \mathcal{K}_{\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B}\big(H \underset{\pi_\nu}{\otimes} \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B\big)$, for all $\gamma \in \Gamma$, $x \in \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}},r}}{\ltimes} A$. Indeed, for all $\gamma \in \Gamma$, $a \in A$, $x \in Irr(\mathbb{G})$, $i, j = 1, \ldots, n_x$ we write

$$\begin{aligned}
&\Big(\sigma_\gamma \circ F \underset{\pi_\nu}{\otimes} id \circ \sigma_{\gamma^{-1}} - F \underset{\pi_\nu}{\otimes} id\Big) id \underset{r}{\ltimes} \pi\big(\pi_\delta(a) U^x_{ij}\big)\big) \\
&\qquad = \Big(\sigma_\gamma \circ F \underset{\pi_\nu}{\otimes} id \circ \sigma_{\gamma^{-1}} - F \otimes id\Big)\big(\pi_{\mathcal{K}}\big(\pi(a)\big) W^x_{ij}\big) \\
&\qquad = \Big(\big(\delta_{H,\Gamma}\big)_\gamma \underset{\pi_\nu}{\otimes} \partial'_\gamma \circ F \underset{\pi_\nu}{\otimes} id \circ \big(\delta_{H,\Gamma}\big)_{\gamma^{-1}} \underset{\pi_\nu}{\otimes} \partial'_{\gamma^{-1}} - F \underset{\pi_\nu}{\otimes} id\Big)\big(\pi_{\mathcal{K}}\big(\pi(a)\big) W^x_{ij}\big) \\
&\qquad = \big(\delta_{H,\Gamma}\big)_\gamma F \big(\delta_{H,\Gamma}\big)_{\gamma^{-1}} (\pi_{\mathcal{K}}(\pi(a))) \partial'_\gamma\big(\partial'_{\gamma^{-1}}(W^x_{ij})\big) - F(\pi_{\mathcal{K}}\big(\pi(a)\big) W^x_{ij} \\
&\qquad = \Big(\big(\delta_{H,\Gamma}\big)_\gamma F \big(\delta_{H,\Gamma}\big)_{\gamma^{-1}} (\pi_{\mathcal{K}}(\pi(a))) - F(\pi_{\mathcal{K}}\big(\pi(a)\big)\Big) W^x_{ij} \\
&\qquad = \Big(\Big(\big(\delta_{H,\Gamma}\big)_\gamma \circ F \circ \big(\delta_{H,\Gamma}\big)_{\gamma^{-1}} - F\Big) \pi_{\mathcal{K}}\big(\pi(a)\big)\Big) W^x_{ij},
\end{aligned}$$

which is compact because $\mathcal{X} = ((H, \delta_{H,\Gamma}), \pi, F) \in KK^{\Gamma}(A, B)$ so that $\Big(\big(\delta_{H,\Gamma}\big)_\gamma \circ F \circ \big(\delta_{H,\Gamma}\big)_{\gamma^{-1}} - F\Big)\pi(a)$ is compact for all $\gamma \in \Gamma$, $a \in A$. Again we have used the identification $\mathcal{L}_{\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B}\big(H \underset{\pi_\nu}{\otimes} \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B\big) \cong M\big(\widehat{\mathbb{G}} \underset{Ad_{V_{H,\widehat{\mathbb{G}}}},r}{\ltimes} \mathcal{K}_B(H)\big)$ and the same notations as above.

3.4.2.1 Remark. Notice that the functor above is well defined at the level of equivariant Kasparov groups. Indeed, let $\mathcal{X} := ((H, \delta_H), \pi, F), \mathcal{X}' := ((H', \delta_{H'}), \pi', F') \in \mathbb{E}^{\widehat{\mathbb{F}}}(A, B)$ two $\widehat{\mathbb{F}}$-equivariant Kasparov triple which are homotopic by means of $\mathcal{E} := ((\mathcal{E}, \delta_{\mathcal{E}}), \rho, L) \in \mathbb{E}^{\widehat{\mathbb{F}}}(A, C([0,1]) \otimes B)$. Remark that $\mathcal{X}$ and $\mathcal{X}'$ will be also homotopic with respect to the restriction actions to $\widehat{\mathbb{G}}$ and Γ. By the well-known descent homomorphism, $\mathcal{Z}(\mathcal{E}) \in \mathbb{E}\big(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}},r}}{\ltimes} A, C([0,1]) \otimes \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B\big)$ yields a homotopy between $\mathcal{Z}(\mathcal{X})$ and $\mathcal{Z}(\mathcal{X}')$. If we equipped $\mathcal{Z}(\mathcal{X})$, $\mathcal{Z}(\mathcal{X}')$ and $\mathcal{Z}(\mathcal{E})$ with the diagonal actions $\tau := (\delta_{H,\Gamma}) \otimes \partial'$, $\tau' := (\delta_{H',\Gamma}) \otimes \partial'$ and $\widetilde{\tau} := (\delta_{\mathcal{E},\Gamma}) \otimes \partial'$, then a straightforward computation yields that $\mathcal{Z}(\mathcal{X})$ and $\mathcal{Z}(\mathcal{X}')$ are *equivariantly* homotopic by means of $\mathcal{Z}(\mathcal{E}) \in \mathbb{E}^{\Gamma}\big(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}},r}}{\ltimes} A, C([0,1]) \otimes \widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}},r}}{\ltimes} B\big)$.

3.4.2.2 Lemma. *The functor*

$$\begin{array}{rccc} \mathcal{Z}: & \mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}} & \longrightarrow & \mathscr{K}\mathscr{K}^{\Gamma} \\ & (A,\delta) & \longmapsto & \mathcal{Z}(A) := (\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A, \partial) \end{array}$$

is triangulated such that $\mathcal{Z}(\mathscr{L}_{\widehat{\mathbb{F}}}) \subset \mathscr{L}_{\Gamma}$.

Proof. First of all, using Proposition 1.5.3.2 it is straightforward to see the stability of $\mathcal{Z}$ with respect to the canonical suspension functors of the corresponding Kasparov categories. Indeed,

$$\mathcal{Z}(\Sigma(A)) = \widehat{\mathbb{G}} \underset{r}{\ltimes} \Sigma(A) \cong \widehat{\mathbb{G}} \underset{r}{\ltimes} \big(C_0(\mathbb{R}) \otimes A\big) \cong C_0(\mathbb{R}) \otimes \widehat{\mathbb{G}} \underset{r}{\ltimes} A = \Sigma(\mathcal{Z}(A)),$$

for all $\widehat{\mathbb{F}}$-C^*-algebra (A,δ).

Next, consider any mapping cone triangle in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$, say

$$\Sigma(B) \longrightarrow C_{\varphi} \longrightarrow A \xrightarrow{\varphi} B,$$

where $\varphi : A \longrightarrow B$ is a $\widehat{\mathbb{F}}$-equivariant $*$-homomorphism. Apply the functor $\mathcal{Z}$ of the statement. We obtain the following diagram

$$\Sigma(\mathcal{Z}(B)) \longrightarrow \widehat{\mathbb{G}} \underset{r}{\ltimes} C_{\varphi} \longrightarrow \widehat{\mathbb{G}} \underset{r}{\ltimes} A \xrightarrow{\mathcal{Z}(\varphi)} \widehat{\mathbb{G}} \underset{r}{\ltimes} B,$$

where $\mathcal{Z}(\varphi)$ is nothing but $id \ltimes \varphi$. By virtue of Proposition 1.5.3.5 we have a canonical $*$-isomorphism $\widehat{\mathbb{G}} \underset{r}{\ltimes} C_{\varphi} \cong C_{\mathcal{Z}(\varphi)}$, so that the above diagram is again a mapping cone triangle in $\mathscr{K}\mathscr{K}^{\Gamma}$. In other words, the functor $\mathcal{Z}$ transforms mapping cone triangles into mapping cone triangles and thus it is triangulated.

Let us show that $\mathcal{Z}(\mathscr{L}_{\widehat{\mathbb{F}}}) \subset \mathscr{L}_{\Gamma}$. Namely, since all our discrete quantum groups are supposed to be torsion-free, then we know that $\mathscr{L}_{\widehat{\mathbb{F}}}$ is the localizing subcategory of $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$ generated by the objects of the form $c_0(\widehat{\mathbb{F}}) \otimes C$ with C any C^*-algebra in the Kasparov category $\mathscr{K}\mathscr{K}$. Likewise, $\mathscr{L}_{\Gamma}$ is by definition the localizing subcategory of $\mathscr{K}\mathscr{K}^{\Gamma}$ generated by the objects of the form $Ind_{\{e\}}^{\Gamma}(B)$ with B any C^*-algebra in the Kasparov category $\mathscr{K}\mathscr{K}$. Recall as well that $c_0(\widehat{\mathbb{F}}) \cong c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})$ by virtue of the representation theory of $\mathbb{F} = \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$ (see Section 2.3). Hence we write

$$\begin{aligned} \mathcal{Z}(c_0(\widehat{\mathbb{F}}) \otimes C) &= \widehat{\mathbb{G}} \underset{\widehat{\Theta}_{\widehat{\mathbb{G}}},r}{\ltimes} \big(c_0(\widehat{\mathbb{F}}) \otimes C\big) \cong \widehat{\mathbb{G}} \underset{\widehat{\Theta}_{\widehat{\mathbb{G}}},r}{\ltimes} \big(c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}}) \otimes C\big) \\ &\overset{(1)}{\cong} \widehat{\mathbb{G}} \underset{\widehat{\Theta}_{\widehat{\mathbb{G}}},r}{\ltimes} \big(c_0(\Gamma) \otimes c_0(\widehat{\mathbb{G}})\big) \otimes C \overset{(2)}{\cong} c_0(\Gamma) \otimes C \end{aligned}$$

where in (1) we use Proposition 1.5.3.2 and in (2) we use the Γ-equivariant Morita equivalence given by Lemma 2.3.15. In other words, $\mathcal{Z}(c_0(\widehat{\mathbb{F}}) \otimes C)$ is a Γ-C^*-algebra in $\mathscr{K}\mathscr{K}^{\Gamma}$ induced by the trivial subgroup $\{e\} < \Gamma$, which yields the claim. ■

3.4.1 Remark. Consider the following functors:

$$\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}} \xrightarrow{j_{\widehat{\mathbb{F}}}} \mathscr{K}\mathscr{K} \text{ and } \mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}} \xrightarrow{\mathcal{Z}} \mathscr{K}\mathscr{K}^{\Gamma} \xrightarrow{j_{\Gamma}} \mathscr{K}\mathscr{K},$$

where $j_{\widehat{\mathbb{F}}}$ is the descent functor with respect to $\widehat{\mathbb{F}}$ and j_Γ is the descent functor with respect to Γ.

By virtue of Theorem 2.3.14, we know that for all $\widehat{\mathbb{F}}$-C^*-algebra $(A,\delta) \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}})$ there exists an isomorphism $\eta_A : \widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A \xrightarrow{\sim} \Gamma \underset{\partial,r}{\ltimes} \left(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A\right)$ in $\mathscr{K}\mathscr{K}$.

Actually, we obtain a natural equivalence between the functors above. More precisely, given two $\widehat{\mathbb{F}}$-C^*-algebras $(A,\delta),(B,\nu) \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}})$ and a Kasparov triple $\mathcal{X} \in KK^{\widehat{\mathbb{F}}}(A,B)$, the following diagram in $\mathscr{K}\mathscr{K}$ is commutative

$$\begin{tikzcd}
\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A \arrow[r, "\widehat{\mathbb{F}} \underset{r}{\ltimes} \mathcal{X}"] \arrow[d, "\eta_A"'] & \widehat{\mathbb{F}} \underset{\nu,r}{\ltimes} B \arrow[d, "\eta_B"] \\
\Gamma \underset{\partial,r}{\ltimes} \left(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A\right) \arrow[r, "\Gamma \underset{r}{\ltimes} \left(\widehat{\mathbb{G}} \underset{r}{\ltimes} \mathcal{X}\right)"'] & \Gamma \underset{\partial',r}{\ltimes} \left(\widehat{\mathbb{G}} \underset{\nu_{\widehat{\mathbb{G}}},r}{\ltimes} B\right)
\end{tikzcd}$$

which is a routine computation. Hence, we have $F \cong F' \circ \mathcal{Z}$.

Observe that the argument used in Lemma 3.3.2.2 can be performed in a more general framework. Indeed, the following result must be regarded as an abstraction of Lemma 3.3.2.2.

3.4.2.3 Lemma. *Let $(\mathcal{T},\Sigma)$, $(\mathcal{T}',\Sigma')$ be two triangulated categories. Let $(\mathscr{L}_\mathcal{T},\mathscr{N}_\mathcal{T})$ and $(\mathscr{L}_{\mathcal{T}'},\mathscr{N}_{\mathcal{T}'})$ be two complementary pairs of localizing subcategories in $\mathcal{T}$ and $\mathcal{T}'$, respectively. Denote by (L,N) and by (L',N') respectively, the canonical triangulated functors associated the these complementary pairs. Let $F : \mathcal{T} \longrightarrow \mathcal{A}b$ and $F' : \mathcal{T}' \longrightarrow \mathcal{A}b$ be two homological functors.*

If $\mathcal{Z} : \mathcal{T} \longrightarrow \mathcal{T}'$ is a triangulated functor such that $F \cong F' \circ \mathcal{Z}$ and $\mathcal{Z}(\mathscr{L}_\mathcal{T}) \subset \mathscr{L}_{\mathcal{T}'}$, then for all object $X \in Obj(\mathcal{T})$ there exists a homomorphism

$$\psi \in Hom_{\mathcal{T}'}(\mathcal{Z}(L(X)), L'(\mathcal{Z}(X)))$$

such that the following diagram is commutative

$$\begin{tikzcd}
\mathbb{L}F(X) \arrow[r, "\Psi"] \arrow[d, "\eta_X"'] & \mathbb{L}F'(\mathcal{Z}(X)) \arrow[d, "\eta'_{\mathcal{Z}(X)}"] \\
F(X) \arrow[r, "\cong"] & F'(\mathcal{Z}(X))
\end{tikzcd}$$

where $\Psi = F'(\psi)$. If moreover $\mathcal{Z}(\mathscr{N}_\mathcal{T}) \subset \mathscr{N}_{\mathcal{T}'}$, then ψ is an isomorphism.

Proof. Given an object $X \in Obj(\mathcal{T})$, consider the corresponding distinguished triangle with respect to the complementary pair $(\mathscr{L}_\mathcal{T},\mathscr{N}_\mathcal{T})$, say $\Sigma(N(X)) \longrightarrow L(X) \xrightarrow{u} X \longrightarrow N(X)$. Consider the distinguished $(\mathscr{L}_{\mathcal{T}'},\mathscr{N}_{\mathcal{T}'})$-triangle associated to the object $\mathcal{Z}(X) \in Obj(\mathcal{T}')$ say

$$\Sigma'(N'(\mathcal{Z}(X))) \longrightarrow L'(\mathcal{Z}(X)) \xrightarrow{u'} \mathcal{Z}(X) \longrightarrow N'(\mathcal{Z}(X)) \tag{3.4.1}$$

Let's fix the object $\mathcal{Z}(L(X)) =: T \in Obj(\mathcal{T}')$ and take the long exact sequence associated to the above triangle with respect to the object T. Namely,

$$\begin{aligned}\ldots \to Hom_{\mathcal{T}'}(T, \Sigma'(N'(\mathcal{Z}(X)))) \to Hom_{\mathcal{T}'}(T, L'(\mathcal{Z}(X))) \overset{(u')_*}{\to} \\ \to Hom_{\mathcal{T}'}(T, \mathcal{Z}(X)) \to Hom_{\mathcal{T}'}(T, N'(\mathcal{Z}(X))) \to \ldots\end{aligned}$$

Since $L(X) \in \mathscr{L}_{\mathcal{T}}$ and we have $\mathcal{Z}(\mathscr{L}_{\mathcal{T}}) \subset \mathscr{L}_{\mathcal{T}'}$ by assumption, then $T \in \mathscr{L}_{\mathcal{T}'}$. But, by definition of complementary pair, we have $\mathscr{L}_{\mathcal{T}'} \subset \mathscr{N}_{\mathcal{T}'}^{\dashv}$. In particular, we obtain $Hom_{\mathcal{T}'}(T, \Sigma'(N'(\mathcal{Z}(X)))) = (0) = Hom_{\mathcal{T}'}(T, N'(\mathcal{Z}(X)))$. Hence the above long exact sequence yields the isomorphism $Hom_{\mathcal{T}'}(T, L'(\mathcal{Z}(X))) \overset{(u')_*}{\cong} Hom_{\mathcal{T}'}(T, \mathcal{Z}(X))$. Hence, just take $\psi := (u')_*^{-1}(\mathcal{Z}(u))$.

Next, put $\Psi := F'(\psi) : F'\big(\mathcal{Z}(L(X))\big) \longrightarrow F'\big(L'(\mathcal{Z}(X))\big)$. The functoriality of constructions and the definition of the element ψ above yields straightforwardly the diagram of the statement.

If moreover we have $\mathcal{Z}(\mathscr{N}_{\mathcal{T}}) \subset \mathscr{N}_{\mathcal{T}'}$, then the functor $\mathcal{Z}$ transforms a $(\mathscr{L}_{\mathcal{T}}, \mathscr{N}_{\mathcal{T}})$-triangle for X into a $(\mathscr{L}_{\mathcal{T}'}, \mathscr{N}_{\mathcal{T}'})$-triangle for $\mathcal{Z}(X)$. Since the distinguished triangles associated to a complementary pair are unique up to a isomorphism, we have an isomorphism of distinguished triangles between (3.4.1) and the image of $\Sigma(N(X)) \longrightarrow L(X) \xrightarrow{u} X \longrightarrow N(X)$ by $\mathcal{Z}$,

$$\begin{tikzcd}
\Sigma'(\mathcal{Z}(N(X))) \arrow[r] \arrow[d, "\wr"] & \mathcal{Z}(L(X)) \arrow[r, "\mathcal{Z}(u)"] \arrow[d, "\wr\ \psi"] & \mathcal{Z}(X) \arrow[r] \arrow[d, equal, "id"] & \mathcal{Z}(N(X)) \arrow[d, "\wr"] \\
\Sigma'(N'(\mathcal{Z}(X))) \arrow[r] & L'(\mathcal{Z}(X)) \arrow[r, "u'"'] & \mathcal{Z}(X) \arrow[r] & N'(\mathcal{Z}(X))
\end{tikzcd}$$

■

Let us apply the preceding lemma to our particular situation with the triangulated functor $\mathcal{Z}$ introduced in Lemma 3.4.2.2 and the homological functors F, F' introduced in the introduction of this section. Then we obtain that for every $\widehat{\mathbb{F}}$-C^*-algebra (A, δ) there exists an element $\psi \in KK^{\Gamma}(\widehat{\mathbb{G}} \underset{r}{\ltimes} L(A), L'(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}}, r}{\ltimes} A))$ such that the following diagram is commutative

$$\begin{tikzcd}
\mathbb{L}F(A) \arrow[r, "\Psi"] \arrow[d, "\eta_A^{\widehat{\mathbb{F}}}"'] & \mathbb{L}F'(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}}, r}{\ltimes} A) \arrow[d, "\eta^{\Gamma}_{\widehat{\mathbb{G}} \underset{r}{\ltimes} A}"] \\
F(A) \arrow[r, "\cong"] & F'(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}}, r}{\ltimes} A)
\end{tikzcd} \tag{3.4.2}$$

where $\Psi := F'(\psi)$, $\eta_A^{\widehat{\mathbb{F}}}$ is the assembly map for $\widehat{\mathbb{F}}$ with coefficients in A and $\eta^{\Gamma}_{\widehat{\mathbb{G}} \ltimes_r A}$ is the assembly map for Γ with coefficients in $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}}, r}{\ltimes} A$. Precisely, if $\Sigma(N(A)) \longrightarrow L(A) \xrightarrow{u} A \longrightarrow N(A)$ is a $(\mathscr{L}_{\widehat{\mathbb{F}}}, \mathscr{N}_{\widehat{\mathbb{F}}})$-triangle associated to A and $\Sigma(N'(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}}, r}{\ltimes} A)) \longrightarrow L'(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}}, r}{\ltimes} A) \xrightarrow{u'} \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}}, r}{\ltimes} A \longrightarrow N'(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}}, r}{\ltimes} A)$

is a $(\mathscr{L}_\Gamma, \mathscr{N}_\Gamma)$-triangle associated to $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$, then $\psi := (u')_*^{-1}(\mathcal{Z}(u))$. Moreover, the definition of the element ψ yields the following commutative diagram

$$\begin{tikzcd}
 & \widehat{\mathbb{G}} \underset{r}{\ltimes} L(A) \arrow[r, "\psi"] \arrow[d, "\mathcal{Z}(u) = \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} u"'] & L'(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A) \arrow[d, "u'"] \\
 & \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A \arrow[r, equal] & \widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A
\end{tikzcd} \tag{3.4.3}$$

We can now conclude our study with the following theorem, generalizing the result [34] of J. Chabert as we have discussed in the introduction.

3.4.2.4 Theorem. *Let $\mathbb{F} = \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$ be a quantum semi-direct product. Assume that $\widehat{\mathbb{F}}$, Γ and $\widehat{\mathbb{G}}$ are torsion-free discrete quantum groups. Let (A, δ) be a $\widehat{\mathbb{F}}$-C^*-algebra.*

$\widehat{\mathbb{F}}$ satisfies the quantum Baum-Connes property with coefficients in A if and only if Γ satisfies the Baum-Connes property with coefficients in $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$ and $\widehat{\mathbb{G}}$ satisfies the quantum Baum-Connes property with coefficients in A.

Proof. Assume that $\widehat{\mathbb{F}}$ satisfies the quantum Baum-Connes property. Since Γ and $\widehat{\mathbb{G}}$ are divisible torsion-free discrete quantum subgroups of $\widehat{\mathbb{F}}$ thanks to Remark 2.3.10, then by Proposition 3.2.2.3 they satisfy the Baum-Connes property.

Conversely, assume that $\widehat{\mathbb{G}}$ satisfies the quantum Baum-Connes property with coefficients in A and Γ satisfies the Baum-Connes property with coefficients in $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$. By the torsion-freeness assumption, the *only* finite subgroup of Γ is the trivial one, $\{e\} < \Gamma$. It is obvious that the trivial group $\{e\}$ satisfies the Baum-Connes property. We can do the same preceding constructions with $\mathbb{F}_{\{e\}} := \{e\} \underset{\alpha_|}{\ltimes} \mathbb{G} = \mathbb{G}$. In particular, we have the following commutative diagram,

$$\begin{tikzcd}
\mathbb{L}F_{\{e\}}(A) \arrow[r, "\Psi_{\{e\}}"] \arrow[d, "\eta_A^{\widehat{\mathbb{F}}_{\{e\}}}"'] & \mathbb{L}F'_{\{e\}}(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A) \arrow[d, "\eta^{\{e\}}_{\widehat{\mathbb{G}} \underset{r}{\ltimes} A}"] \\
F_{\{e\}}(A) \arrow[r, equal] & F'_{\{e\}}(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A)
\end{tikzcd}$$

where $F_{\{e\}}$ et $F'_{\{e\}}$ are the analogous functors to F and F' defined with respect to $\widehat{\mathbb{F}}_{\{e\}} = \widehat{\mathbb{G}}$ and $\{e\}$, respectively. Likewise, we have $\Psi_{\{e\}} := F'_{\{e\}}(\psi)$ where $\psi \in KK^\Gamma(\widehat{\mathbb{G}} \underset{r}{\ltimes} L(A), L'(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A))$ is the element constructed from Lemma 3.4.2.3. In this situation we have $\eta_A^{\widehat{\mathbb{F}}_{\{e\}}} = \eta_A^{\widehat{\mathbb{G}}}$. Since $\widehat{\mathbb{G}}$ satisfies

the quantum Baum-Connes property by assumption, then $\eta_A^{\widehat{\mathbb{F}}_{\{e\}}} = \eta_A^{\widehat{\mathbb{G}}}$ is a natural isomorphism. Hence, $\Psi_{\{e\}}$ is a natural isomorphism. This means in particular that $\widehat{\mathbb{G}} \underset{r}{\ltimes} L(A)$ and $L'(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A)$ are K-equivalent via the element ψ of Lemma 3.4.2.3. Therefore the same element induces a K-equivalence between $\Gamma \underset{r}{\ltimes} L'\Big(\widehat{\mathbb{G}} \underset{r}{\ltimes} L(A)\Big)$ and $\Gamma \underset{r}{\ltimes} L'(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A)$ by virtue of Theorem 9.3 in [132].

Observe that $L(A) \in \mathscr{L}_{\widehat{\mathbb{F}}}$, so $\widehat{\mathbb{G}} \underset{r}{\ltimes} L(A) = \mathcal{Z}(L(A)) \in \mathscr{L}_{\Gamma}$ thanks to Lemma 3.4.2.2. Hence we have $L'\Big(\widehat{\mathbb{G}} \underset{r}{\ltimes} L(A)\Big) \cong \widehat{\mathbb{G}} \underset{r}{\ltimes} L(A)$ in $\mathscr{K}\mathscr{K}^{\Gamma}$. In other words, $\Gamma \underset{r}{\ltimes} \Big(\widehat{\mathbb{G}} \underset{r}{\ltimes} L(A)\Big)$ is K-equivalent to $\Gamma \underset{r}{\ltimes} L'(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A)$ via the element ψ. That is, $\Psi = F'(\psi)$ is an isomorphism.

To conclude, we use the commutative diagramme (3.4.2). Namely, since Γ satisfies the Baum-Connes property with coefficients in $\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A$ by assumption, then $K_*(\Gamma \underset{r}{\ltimes} L'(\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A)) \cong K_*(\Gamma \underset{\partial,r}{\ltimes} (\widehat{\mathbb{G}} \underset{\delta_{\widehat{\mathbb{G}}},r}{\ltimes} A))$ via $\eta^{\Gamma}_{\widehat{\mathbb{G}} \underset{r}{\ltimes} A}$. By using the associativity for quantum semi-direct products from Theorem 2.3.14 we get $K_*(\widehat{\mathbb{F}} \underset{r}{\ltimes} L(A)) \xrightarrow{\sim} K_*(\widehat{\mathbb{F}} \underset{\delta,r}{\ltimes} A)$ via $\eta^{\Gamma}_{\widehat{\mathbb{G}} \underset{r}{\ltimes} A} \circ \Psi$. So $\mathbb{L}F(A) \cong F(A)$ through $\eta_A^{\widehat{\mathbb{F}}}$ thanks to the commutativity of diagram (3.4.2). That is, $\widehat{\mathbb{F}}$ satisfies the quantum Baum-Connes property with coefficients in A. ■

3.4.2.5 Remark. The argument of the preceding theorem can be applied when Γ has more finite subgroups than the trivial one. Indeed, we could do the argument with the quantum semi-direct products given by $\mathbb{F}_{\Lambda} := \Lambda \underset{\alpha_|}{\ltimes} \mathbb{G}$ for every finite subgroup $\Lambda < \Gamma$. In that case, the claim "$\Psi = F'(\psi)$ *is an isomorphism*", which is used in order to conclude using the commutative diagram (3.4.2), can be achieved by applying Theorem 9.3 in [132]. The problem with this case is that the finite subgroups of Γ provide torsion of $\widehat{\mathbb{F}}$. Hence the theoretical framework for the quantum Baum-Connes property fails. It is reasonable to expect that the same stability property holds for any quantum semi-direct product (not necessarily torsion-free) once the Baum-Connes property can be formulated properly without the torsion-freeness assumption.

Let us analyze the stability of the *strong* version of the Baum-Connes property.

3.4.2.6 Theorem. *Let $\mathbb{F} = \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$ be a quantum semi-direct product such that $\widehat{\mathbb{F}}$, Γ and $\widehat{\mathbb{G}}$ is a torsion-free discrete quantum groups.*

If $\widehat{\mathbb{F}}$ satisfies the strong *quantum Baum-Connes property, then Γ satisfies the* strong *Baum-Connes property and $\widehat{\mathbb{G}}$ satisfies the* strong *quantum Baum-Connes property.*

Proof. Assume that $\widehat{\mathbb{F}}$ satisfies the *strong* quantum Baum-Connes property. Since Γ and $\widehat{\mathbb{G}}$ are *divisible* torsion-free discrete quantum subgroups of $\widehat{\mathbb{F}}$ thanks to Remark 2.3.10, then they satisfy the *strong* Baum-Connes property by virtue of Theorem 3.2.2.1.

■

3.4.2 Remark. As we have already mentioned, the original result of J. Chabert in [34] was not optimal. The precise statement of [34] concerning the Baum-Connes property for a semi-direct product is the following

"Let $F = \Gamma \ltimes G$ be a semi-direct product of two locally compact groups equipped with a γ-element. Assume that

i) Γ has a compact-open subgroup,

ii) for any compact subgroup $\Lambda < \Gamma$ the group $F_\Lambda := \Lambda \ltimes G$ satisfies the Baum-Connes property with coefficients.

Then if the Baum-Connes property holds for Γ, it holds for G too."

On the one hand, the hypothesis (i) about the existence of a compact-open subgroup of Γ is automatically fulfilled in the discrete case (taking the *singleton* of the identity element $\{e\}$). On the other hand, the hypothesis about the existence of a γ-element for $F = \Gamma \ltimes G$ is just a way to use the J.-L. Tu's result [191], which assures the injectivity of the corresponding assembly map under the γ-element assumption. In this sense, it is a superfluous hypothesis as shown later in [35].

The initial strategy of J. Chabert was defining a *partial* descent homomorphism following the semi-direct product construction (see Section 2 in [34]). In this way, our commutative diagram (3.4.2) is the analogous one to the diagram obtained in Proposition 3.10 of [34].

Notice that in the above statement, J. Chabert only obtains one implication for the Baum-Connes property. In Theorem 3.4.2.4 we obtain an *"if and only if"* result. This is possible thanks to the isomorphism of Ψ in diagram (3.4.2), which follows from Theorem 9.3 of [132] as explained in the proof above. In Corollary 3.4 of [35], J. Chabert and S. Echterhoff also obtain both implications for the Baum-Connes property. In fact, H. Oyono-Oyono obtained an *"if and only if"* result for *discrete* groups in [143], which partly inspired the work [35].

3.4.3 K-amenability property

Here we care about a property of own interest, namely the K-amenability of a quantum semi-direct product. More precisely we have the following result.theo.

3.4.3.1 Theorem. *Let $\mathbb{F} = \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$ be a quantum semi-direct product. Then $\mathbb{F}$ is co-K-amenable if and only if Γ is K-amenable and $\mathbb{G}$ is co-K-amenable.*

Proof. Assume that $\mathbb{F}$ is co-K-amenable. This means that there exists an element $\alpha_{\mathbb{F}} \in KK(C_r(\mathbb{F}), \mathbb{C})$ such that

$$[\tau_{\mathbb{F}}] \underset{C_r(\mathbb{F})}{\otimes} \alpha_{\mathbb{F}} = [\varepsilon_{\mathbb{F}}] \in KK(C_m(\mathbb{F}), \mathbb{C}),$$

where $\tau_{\mathbb{F}} : C_m(\mathbb{F}) \twoheadrightarrow C_r(\mathbb{F})$ is the canonical surjection and $\varepsilon_{\mathbb{F}} : Pol(\mathbb{F}) \longrightarrow \mathbb{C}$ is the co-unit of $\mathbb{F}$ whose extension to $C_m(\mathbb{F})$ is still denoted by $\varepsilon_{\mathbb{F}}$.

By virtue of Remark 2.3.9 we know that Γ and $\widehat{\mathbb{G}}$ are discrete quantum subgroups of $\widehat{\mathbb{F}}$ via the canonical injections

$$\iota_\Gamma^r : C_r^*(\Gamma) \hookrightarrow C_r(\mathbb{F}) \text{ and } \iota_{\mathbb{G}}^r : C_r(\mathbb{G}) \hookrightarrow C_r(\mathbb{F})$$

Hence, by virtue of Theorem 3.2.3.1 we conclude that Γ and $\widehat{\mathbb{G}}$ are K-amenable with elements

$$\alpha_\Gamma := [\iota_\Gamma^r] \underset{C_r(\mathbb{F})}{\otimes} \alpha_{\mathbb{F}} \in KK(C_r^*(\Gamma), \mathbb{C})$$

$$\alpha_{\mathbb{G}} := [\iota_{\mathbb{G}}^r] \underset{C_r(\mathbb{F})}{\otimes} \alpha_{\mathbb{F}} \in KK(C_r(\mathbb{G}), \mathbb{C})$$

Conversely, assume that Γ is K-amenable and that $\mathbb{G}$ is co-K-amenable. By virtue of the K-amenability characterization of J. Cuntz (see Theorem 2.1 in [44]), the surjection $\Gamma \underset{m}{\ltimes} A \twoheadrightarrow \Gamma \underset{r}{\ltimes} A$ induces a KK-equivalence for every Γ-C^*-algebra A. In particular,

$$\Gamma \underset{m}{\ltimes} C_m(\mathbb{G}) \twoheadrightarrow \Gamma \underset{r}{\ltimes} C_m(\mathbb{G})$$

induces a KK-equivalence. Since $\mathbb{G}$ is co-K-amenable, then the canonical surjection

$$\tau_{\mathbb{G}} : C_m(\mathbb{G}) \twoheadrightarrow C_r(\mathbb{G}),$$

which is Γ-equivariant, induces a Γ-equivariant KK-equivalence. If j_Γ denotes the descent homomorphism with respect to Γ, which is compatible with the Kasparov product, then it is clear that $[id \ltimes \tau_{\mathbb{G}}] = j_\Gamma([\tau_{\mathbb{G}}]) \in KK(\Gamma \underset{r}{\ltimes} C_m(\mathbb{G}), \Gamma \underset{r}{\ltimes} C_r(\mathbb{G}))$ is an invertible element. In other words, the composition

$$\Gamma \underset{m}{\ltimes} C_m(\mathbb{G}) \twoheadrightarrow \Gamma \underset{r}{\ltimes} C_m(\mathbb{G}) \overset{id \ltimes \tau_{\mathbb{G}}}{\rightarrow} \Gamma \underset{r}{\ltimes} C_r(\mathbb{G})),$$

which is precisely $\tau_{\mathbb{F}}$, induces a KK-equivalence. Hence $\mathbb{F}$ is co-K-amenable. ■

3.5 The Baum-Connes property for a compact bicrossed product

It is worth mentioning that a compact bicrossed product in the sense of [65] is a special case of a more general construction called *bicrossed product*. This object has been studied in detail in [196] by S. Vaes and L. I. Vainerman giving a great general notion of *extension of locally compact quantum groups*. In this way, the quantum semi-direct product and the compact bicrossed product may be regarded as part of a same class of objects. The Baum-Connes property for extensions of locally compact groups has been successfully studied by J. Chabert and S. Echterhoff in [35] and also by H. Oyono-Oyono in [143] for discrete groups. For this reason it is licit to think about a permanence property of the *quantum Baum-Connes property for extensions of quantum groups*. However, such a property remains out of the scope of the present dissertation because of the torsion phenomena problems and the technical difficulties of the construction of [196].

We carry the same notations of Section 2.4 on. Let $\mathbb{F} = \Gamma_\alpha \bowtie_\beta G$ be a compact bicrossed product, where $(\Gamma, G, \alpha, \beta)$ is a compact matched pair.

In order to legitimate the Baum-Connes property formulation for the dual of a compact bicrossed product $\mathbb{F} = \Gamma_\alpha \bowtie_\beta G$, we need $\widehat{\mathbb{F}}$ to be torsion-free. In this way, we do the following crucial observation.

3.5.1 Proposition. *Let $\mathbb{F} = \Gamma_\alpha \bowtie_\beta G$ be a compact bicrossed product of the matched pair (Γ, G). If $\widehat{\mathbb{F}}$ is torsion-free, then the action β is trivial. Consequently, $\mathbb{F} = \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$ is a quantum semi-direct product with $\mathbb{G} := (C(G), \Delta)$.*

Proof. Let G_0 be the connected component of the identity element e, which is always a closed normal subgroup of G. Consequently, G/G_0 is a finite group because G is supposed to be compact. Its dual is therefore a finite discrete quantum subgroup of $\widehat{G}$. The latter is a discrete quantum subgroup of $\widehat{\mathbb{F}}$ as explained in Remark 2.4.3. Since $\widehat{\mathbb{F}}$ is torsion-free by assumption, then G/G_0 must be the trivial group (recall Remark 1.6.1.7). Hence G must be connected, which forces β to be the trivial action. ■

As we have pointed out several times, the torsion-freeness assumption is a theoretical obstruction for the (current) quantum Baum-Connes property formulation for discrete quantum groups. The preceding proposition shows that this hypothesis forces the compact bicrossed product to be a torsion-free quantum semi-direct product. Therefore, the analogous results from Section 3.4 still hold for the dual of a torsion-free compact bicrossed product (particularly, Theorem 3.4.2.4 and Theorem 3.4.2.6 still hold).

In this sense, the torsion case is the interesting one. The general analogous strategy used in Section 3.4 may be applied for a compact bicrossed product provided that an *associativity property* can be established (recall Note 2.4.2), which has been one of the main ingredients to reach Theorem 3.4.2.4. It is reasonable to expect that the same stability property holds for any compact bicrossed product (not necessarily torsion-free) once the Baum-Connes property can be formulated without the torsion-freeness assumption, which is the the main obstacle to achieve such a result.[1]

3.5.1 K-amenability property

Here we care about a property of own interest, namely the K-amenability of a compact bicrossed product. Observe that this property is completely independent of the torsion-freeness assumption. The analogous result to Theorem 3.4.3.1 for a quantum semi-direct product can be established. Recall by Remark 2.4.3 that Γ is not a quantum subgroup of $\widehat{\mathbb{F}}$ anymore. Moreover notice that G is a classical compact group, so it is automatically amenable (so, K-amenable). The corresponding statement for the compact bicrossed product is the following (compare with Theorem 3.4.3.1 and see Remark 1.3.1.42).

3.5.1.1 Theorem. *Let $\mathbb{F} = \Gamma_\alpha \bowtie_\beta G$ be a compact bicrossed product. Then $\mathbb{F}$ is co-K-amenable if and only if Γ is K-amenable.*

Proof. Given the compact bicrossed product $\mathbb{F}$, consider the canonical surjection $\tau_{\mathbb{F}} : C_m(\mathbb{F}) \twoheadrightarrow C_r(\mathbb{F})$, where we recall that $C_m(\mathbb{F}) = \Gamma \underset{\alpha,m}{\ltimes} C(G)$ and $C_r(\mathbb{F}) = \Gamma \underset{\alpha,r}{\ltimes} C(G)$. Likewise, given the discrete group Γ, consider the canonical surjection $\tau_\Gamma : C_m^*(\Gamma) \twoheadrightarrow C_r^*(\Gamma)$.

Assume that $\mathbb{F}$ is co-K-amenable. This means that the canonical surjection $\tau_{\mathbb{F}} : C_m(\mathbb{F}) \twoheadrightarrow C_r(\mathbb{F})$ induces a KK-equivalence, that is, the induced element $[\tau_{\mathbb{F}}] \in KK(C_m(\mathbb{F}), C_r(\mathbb{F}))$ is invertible. Let us denote by $\mathcal{X} \in KK(C_r(\mathbb{F}), C_m(\mathbb{F}))$ its inverse, so that we have

$$[\tau_{\mathbb{F}}] \underset{C_r(\mathbb{F})}{\otimes} \mathcal{X} = 1_{C_m(\mathbb{F})} \text{ and } \mathcal{X} \underset{C_m(\mathbb{F})}{\otimes} [\tau_{\mathbb{F}}] = 1_{C_r(\mathbb{F})}$$

Since G is a classical compact group, we have the canonical (α-invariant) character on $C(G)$ defined precisely by

$$\begin{array}{rccl} \varepsilon_G : & C(G) & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & \varepsilon(f) := f(e) \end{array}$$

[1] This is a work in progress in collaboration with P. Fima.

Consider thus the following (unital) $*$-homomorphisms

$$\varepsilon_\Gamma^m := \Gamma \underset{\alpha,m}{\ltimes} \varepsilon_G : C_m(\mathbb{F}) \longrightarrow C_m^*(\Gamma)$$

$$\varepsilon_\Gamma^r := \Gamma \underset{\alpha,r}{\ltimes} \varepsilon_G : C_r(\mathbb{F}) \longrightarrow C_r^*(\Gamma)$$

In this situation, we have the following commutative diagram

$$\begin{tikzcd} C_m(\mathbb{F}) \arrow[r, "\varepsilon_\Gamma^m"] \arrow[d, two heads, "\tau_{\mathbb{F}}"'] & C_m^*(\Gamma) \arrow[d, two heads, "\tau_\Gamma"] \\ C_r(\mathbb{F}) \arrow[r, "\varepsilon_\Gamma^r"'] & C_r^*(\Gamma) \end{tikzcd} \tag{3.5.1}$$

Recall that $C_m(\mathbb{F}) = \Gamma \underset{\alpha,m}{\ltimes} C(G) = C^*\langle \pi(f)u_\gamma : f \in C(G), \gamma \in \Gamma\rangle$. So, with the help of the α-invariant character above, we can identify $C_m^*(\Gamma)$ with the subalgebra of $C_m(\mathbb{F})$ generated by $\{u_\gamma : \gamma \in \Gamma\}$ by universal property (see Remark 3.6 in [65] for more details). Hence, we consider the canonical injection $\iota_m : C_m^*(\Gamma) \hookrightarrow C_m(\mathbb{F})$, which is such that

$$\varepsilon_\Gamma^m \circ \iota_m = id_{C_m^*(\Gamma)} \tag{3.5.2}$$

Likewise, recall that $C_r(\mathbb{F}) = \Gamma \underset{\alpha,r}{\ltimes} C(G) = C^*\langle \pi(f)u_\gamma : f \in C(G), \gamma \in \Gamma\rangle$ is equipped with a conditional expectation $E : \Gamma \underset{\alpha,r}{\ltimes} C(G) \longrightarrow C(G)$ which restricted to the subalgebra generated by $\{u_\gamma \ : \ \gamma \in \Gamma\}$ is just $E(u_\gamma) = \delta_{\gamma,e} \in \mathbb{C}$. Remember as well that $u_\gamma = \lambda_\gamma \otimes id_{C(G)}$ in $\Gamma \underset{\alpha,r}{\ltimes} C(G) \subset \mathcal{L}_{C(G)}(l^2(\Gamma) \otimes C(G))$; so that this subalgebra will be identified canonically to $C_r^*(\Gamma) = \Gamma \underset{trv,r}{\ltimes} \mathbb{C}$ by universal property. Hence, we consider the canonical injection $\iota_r : C_r^*(\Gamma) \hookrightarrow C_r(\mathbb{F})$, which is such that

$$\varepsilon_\Gamma^r \circ \iota_r = id_{C_r^*(\Gamma)} \tag{3.5.3}$$

Finally, remark that the following equation holds by construction

$$\tau_{\mathbb{F}} \circ \iota_m = \iota_r \circ \tau_\Gamma \tag{3.5.4}$$

Given the induced element $[\tau_\Gamma] \in KK(C_m^*(\Gamma), C_r^*(\Gamma))$, we claim that the element

$$\mathcal{Y} := [\iota_r] \underset{C_r(\mathbb{F})}{\otimes} \mathcal{X} \underset{C_m(\mathbb{F})}{\otimes} [\varepsilon_\Gamma^m] \in KK(C_r^*(\Gamma), C_m^*(\Gamma))$$

is its inverse and hence Γ is K-amenable. Indeed, using the equations (3.5.1), (3.5.2), (3.5.3), (3.5.4) above and the definition of $\mathcal{X}$ we can write the following

$$\begin{aligned} \mathcal{Y} \underset{C_m^*(\Gamma)}{\otimes} [\tau_\Gamma] &= [\iota_r] \underset{C_r(\mathbb{F})}{\otimes} \mathcal{X} \underset{C_m(\mathbb{F})}{\otimes} [\varepsilon_\Gamma^m] \underset{C_m^*(\Gamma)}{\otimes} [\tau_\Gamma] = [\iota_r] \underset{C_r(\mathbb{F})}{\otimes} \mathcal{X} \underset{C_m(\mathbb{F})}{\otimes} [\tau_\Gamma \circ \varepsilon_\Gamma^m] \\ &= [\iota_r] \underset{C_r(\mathbb{F})}{\otimes} \mathcal{X} \underset{C_m(\mathbb{F})}{\otimes} [\varepsilon_\Gamma^r \circ \tau] = [\iota_r] \underset{C_r(\mathbb{F})}{\otimes} \mathcal{X} \underset{C_m(\mathbb{F})}{\otimes} [\tau] \underset{C_r(\mathbb{F})}{\otimes} [\varepsilon_\Gamma^r] \\ &= [\iota_r] \underset{C_r(\mathbb{F})}{\otimes} [\varepsilon_\Gamma^r] = [\varepsilon_\Gamma^r \circ \iota_r] = 1_{C_r^*(\Gamma)} \end{aligned}$$

$$\begin{aligned}[\tau_\Gamma] \underset{C_r^*(\Gamma)}{\otimes} \mathcal{Y} &= [\tau_\Gamma] \underset{C_r^*(\Gamma)}{\otimes} [\iota_r] \underset{C_r(\mathbb{F})}{\otimes} \mathcal{X} \underset{C_m(\mathbb{F})}{\otimes} [\varepsilon_\Gamma^m] = [\iota_r \circ \tau_\Gamma] \underset{C_r(\mathbb{F})}{\otimes} \mathcal{X} \underset{C_m(\mathbb{F})}{\otimes} [\varepsilon_\Gamma^m] \\ &= [\tau \circ \iota_m] \underset{C_r(\mathbb{F})}{\otimes} \mathcal{X} \underset{C_m(\mathbb{F})}{\otimes} [\varepsilon_\Gamma^m] = [\iota_m] \underset{C_m(\mathbb{F})}{\otimes} [\tau] \underset{C_r(\mathbb{F})}{\otimes} \mathcal{X} \underset{C_m(\mathbb{F})}{\otimes} [\varepsilon_\Gamma^m] \\ &= [\iota_m] \underset{C_m(\mathbb{F})}{\otimes} [\varepsilon_\Gamma^m] = [\varepsilon_\Gamma^m \circ \iota_m] = 1_{C_m^*(\Gamma)}\end{aligned}$$

Conversely, assume that Γ is K-amenable, then by virtue of the K-amenability characterization of J. Cuntz (see Theorem 2.1 in [44]), the surjection $\tau_{\mathbb{F}} : \Gamma \underset{\alpha,m}{\ltimes} C(G) \twoheadrightarrow \Gamma \underset{\alpha,r}{\ltimes} C(G)$ induces a KK-equivalence, whence the K-amenability of $\mathbb{F}$. ■

3.5.1.2 Remark. It is important to notice that the preceding proof can be simplified by using the same argument as in Theorem 3.4.3.1. Indeed, even if Γ is not longer a quantum discrete subgroup of the compact bicrossed product $\widehat{\mathbb{F}}$, we have observed in Remarks 2.4.3 that there exist still canonical injections $\iota_\Gamma^m : C_m^*(\Gamma) \hookrightarrow C_m(\mathbb{F})$ and $\iota_\Gamma^r : C_r^*(\Gamma) \hookrightarrow C_r(\mathbb{F})$ compatible with the corresponding canonical surjections and co-units as in Remark 2.3.9. This allows to define the element $\alpha_\Gamma := [\iota_\Gamma^r] \underset{C_r(\mathbb{F})}{\otimes} \alpha_{\mathbb{F}} \in KK(C_r^*(\Gamma), \mathbb{C})$ provided that $\mathbb{F}$ is co-K-amenable with element $\alpha_{\mathbb{F}} \in KK(C_r(\mathbb{F}), \mathbb{C})$ such that $[\tau_{\mathbb{F}}] \underset{C_r(\mathbb{F})}{\otimes} \alpha_{\mathbb{F}} = [\varepsilon_{\mathbb{F}}] \in KK(C_m(\mathbb{F}), \mathbb{C})$. In this way, the analogous computation as in Theorem 3.2.3.1 yields that $[\tau_\Gamma] \underset{C_r^*(\Gamma)}{\otimes} \alpha_\Gamma = [\varepsilon_\Gamma]$, which yields the K-amenability for Γ by Remark 1.3.1.41.

3.6 The Baum-Connes property for a quantum free product

In this section we recall that the *strong* Baum-Connes property is stable under the free product construction for quantum groups following the work of R. Vergnioux and C. Voigt [208]. It is important to say that the strategy of R. Vergnioux and C. Voigt in [208] to achieve such a stability property follows earlier work of G. Kasparov and G. Skandalis [99] and R. Vergnioux [206], [207]. More precisely, they apply the Dirac-dual Dirac method in a quantum version of the Bass-Serre theory. In this sense, the Vergnioux-Voigt's approach is very constructive and gives lot of information in the process of proving the (strong) Baum-Connes property for a free product of (torsion-free) discrete quantum groups.

In order to do so, we carry the same notations of Section 2.5 on. Let $\mathbb{F} := \mathbb{G} * \mathbb{H}$ be a quantum free product, where $\mathbb{G}$ and $\mathbb{H}$ are compact quantum groups.

3.6.1 Torsion property

We have already explained that all our *discrete quantum groups are supposed to be torsion-free* in order to study the corresponding quantum Baum-Connes property. For this reason it is advisable to study in more detail the torsion phenomena of the dual of a quantum free product $\mathbb{F} = \mathbb{G} * \mathbb{H}$ in terms of the involved quantum groups.

It is important to say that, in the original article [208], R. Vergnioux and C. Voigt expected that all free quantum groups must be torsion-free, but they did not give any proof. Nevertheless, the case of the free *orthogonal* quantum group was already proven in [210] by C. Voigt thanks to the invariance of the torsion-freeness under monoidal equivalence (Theorem B.3.19) and the fact that $\widehat{SU_q(2)}$ is torsion-free, which is also shown in [210].

The general case has been recently studied by Y. Arano and K. De Commer in [3] and they have obtained a positive answer to this problem using the approach of *fusion rings* and C^*-tensor categories (recall Section 1.6.2 for definitions and more details).

3.6.1.1 Theorem (Y. Arano and K. De Commer, [3]). *i) Let $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ be two discrete quantum groups and let $\widehat{\mathbb{F}} := \widehat{\mathbb{G} * \mathbb{H}}$ be the dual of the corresponding quantum free product.*

If $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are (resp. strong) torsion-free, then $\widehat{\mathbb{F}}$ is (resp. strong) torsion-free.

ii) Let $n > 1$ and $Q \in GL_n(\mathbb{C})$. Then $\widehat{U^+(Q)}$ is strong torsion-free. As a consequence, it is torsion-free.

Using the combinatorial methods introduced in [3] by Y. Arano and K. De Commer (recall Section 1.6.2 for definitions and more details), we can give a more precise picture of the torsion phenomena for a quantum free product. Namely, we can classify the torsion actions of a quantum free product, which will be very useful in Section 3.7.1 where we will investigate the torsion phenomena for a free wreath product. The following results have been obtained as a collaboration with A. Freslon [127].

First of all, given a quantum free product $\mathbb{F} = \mathbb{G} * \mathbb{H}$, it is clear that both $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are discrete quantum subgroups of $\widehat{\mathbb{F}}$. In particular, we have $C(\mathbb{G}) \subset C(\mathbb{F})$ and $C(\mathbb{H}) \subset C(\mathbb{F})$, so that any action of $\mathbb{G}$ (resp. $\mathbb{H}$) can be extended to an action of $\mathbb{F}$ in the sense of Remarks 1.4.3.5. Thus, it is reasonable to expect that torsion actions of the quantum free product all are induced from either torsion actions of $\mathbb{G}$ or torsion actions of $\mathbb{H}$. This is the goal of this section.

The definition of induced module (recall Remarks 1.6.2.3) and the construction of the free product of fusion rings yield immediately the following result.

3.6.1.2 Lemma. *Let R_1 be a I_1-based ring and R_2 be a I_2-based ring. If N is a J-based R_1-module, then the induced module $Ind_{R_1}^{R_1 * R_2}(N)$ is a $\widetilde{J}$-based $R_1 * R_2$-module with basis*

$$\widetilde{J} := \{\omega j \mid \omega \in W(I_1, I_2) \cup \varnothing \text{ and } j \in J\} \cong W(I_1, I_2) \underset{R_1}{\odot} J,$$

where $W(I_1, I_2)$ denotes the set of alternating words in I_1 and I_2 ending in I_2 and the obvious action of $R_1 * R_2$.

By definition of induced module, it is clear that the induction of standard modules yields standard modules. We shall show that the induction of non-standard torsion modules yields again a non-standard one.

3.6.1.3 Definition. Let R be a I-based ring and M a J-based R-module. Given a basis element $\beta \in J$ we define the stabilizer of β by

$$Stab(\beta) := \{\alpha \in I \mid \beta \subset \alpha \otimes \beta\}$$

The preceding definition yields the following immediate observations.

i) If M is co-finite, then $Stab(\beta)$ is finite for all $\beta \in J$.

ii) If M is standard, that is $M \cong R$, then there exists exactly one element with trivial stabilizer, which is the one corresponding to the unit of I. Namely, by definition of based ring we have that
$$Stab(\mathbb{1}) = \{\alpha \in I \mid \mathbb{1} \subset \alpha \otimes \mathbb{1}\} = \{\mathbb{1}\}$$
Moreover, if there was some $\alpha' \in I$ such that $Stab(\alpha) = \{x\}$ is a *singleton*, then for all $\alpha \in I$, $\alpha' \subset \alpha \otimes \alpha'$ implies $\alpha = x$. In particular, $\mathbb{1}$ is such that $\alpha' \subset \mathbb{1} \otimes \alpha' = \alpha'$. So it must be $\alpha = \mathbb{1} = x$.

3.6.1.4 Lemma. *Let R be a fusion ring with basis I and M a fusion R-module with basis J. M is standard if and only there exists a basis element $j_0 \in J$ with trivial stabilizer.*

Proof. If M is a standard fusion R-module, then we have explained above that $Stab(\mathbb{1}) = \{\mathbb{1}\}$ and that $\mathbb{1}$ is the only basis element with trivial stabilizer.

Conversely, assume that M is a fusion R-module with basis J such that there exists a basis element $j_0 \in J$ with trivial stabilizer. Recall from Remarks 1.6.2.3 that we have a bilinear form on M such that
$$\langle \beta, \beta' \rangle := \sum_{i \in I} \lambda^{\beta'}_{\bar{i},\beta} \cdot i,$$
for all $\beta, \beta' \in J$. Since $Stab(j_0) = \{\mathbb{1}\}$ by assumption, which means that $\lambda^{j_0}_{\alpha,j_0} \neq 0 \Leftrightarrow \alpha = \mathbb{1}$, then by definition we have
$$\langle j_0, j_0 \rangle = \lambda^{j_0}_{\mathbb{1},j_0} \cdot \mathbb{1} = \mathbb{1}$$
Given any $\alpha \in I \backslash \{\mathbb{1}\}$, set $\alpha \otimes j_0 = \sum\limits_{k=1}^{n} \lambda_k j_k$. Hence we write the following
$$\sum_{k=1}^{n} \lambda_k \langle j_k, j_0 \rangle = \langle \alpha \otimes j_0, j_0 \rangle = \alpha \otimes \langle j_0, j_0 \rangle = \alpha \otimes \mathbb{1} = \alpha$$
Since $\langle j_k, j_0 \rangle$ always contains α and the coefficients λ_k are non-negative integer, it must be $k = 1$. In other words, we have $\alpha \otimes j_0 = j_1$, which is a basis element. Hence, in order to define an isomorphism of fusion R-modules $M \longrightarrow R$, it is enough to send the basis element $j_0 \in J$ to the unit $\mathbb{1} \in I$. ■

3.6.1.5 Lemma. *Let R_1 be a fusion ring with basis I_1 and R_2 be a fusion ring with basis I_2. If N is a torsion R_1-module with basis J, then the induced module $M := Ind_{R_1}^{R_1 * R_2}(N)$ splits as a R_1-module into a direct sum of N and standard modules.*

In particular, $Ind_{R_1}^{R_1 * R_2}(N) \cong Ind_{R_1}^{R_1 * R_2}(N')$ as $R_1 * R_2$-modules if and only if $N \cong N'$ as R_1-modules.

Proof. Let P be an arbitrary R_1-submodule of M and let ω be an alternating word in I_1 and I_2 of minimal length such that there exists a basis element $j \in J$ satisfying $\omega j \in P$, which exists by Lemma 3.6.1.2.

Assume that $\omega \neq \varnothing$ and write $\omega = \alpha_k \dots \alpha_1$. Notice that we can not have $\alpha_k \in I_1$ since otherwise $\alpha_{k-1} \dots \alpha_1 \subset \overline{\alpha}_k \otimes \omega j \in P$ because P is a R_1-module, which contradicts the minimality

of ω. Thus, $\alpha_k \in I_2$ and the definition of the ring structure on $R_1 * R_2$ together with Lemma 3.6.1.2 imply that P is standard with respect to R_1. Indeed, it is enough to remark that $Stab(\omega j) = \{\mathbb{1}_1\}$, where we recall that ωj is a basis element in $\widetilde{J}$. If $\alpha \in Stab(\omega j) \cap I_1$, then $\omega j \subset \alpha \otimes \omega j = \alpha\omega j$, where $\alpha\omega$ is now a new word and so $\alpha\omega j$ is a new basis element in $\widetilde{J}$. Hence, it must be $\omega = \alpha\omega$, that is, $\alpha = \mathbb{1}_1$.

If $\omega = \varnothing$, then P contains a basis element of N, which implies $P = N$.

In particular, if $Ind_{R_1}^{R_1 * R_2}(N) \longrightarrow Ind_{R_1}^{R_1 * R_2}(N')$ is an isomorphism of $R_1 * R_2$-modules, it can be seen as an isomorphism of R_1-modules, which preserves standard modules. Hence it must send N isomorphically to N' and the proof is completed. ■

3.6.1.6 Proposition. *Let R_1 be a fusion ring with basis I_1 and R_2 be a fusion ring with basis I_2. If M is a torsion $R_1 * R_2$-module with basis J, then M is induced from a module over one of the factors.*

Proof. If M is standard, then we can write either $M \cong Ind_{R_1}^{R_1 * R_2}(R_1)$ or $M \cong Ind_{R_2}^{R_1 * R_2}(R_2)$ and the proof is completed.

Let $e \in J$ be a basis element and denote by N_1^e (resp. N_2^e) the R_1-submodule (resp. R_2-submodule) generated by the action of R_1 (resp. R_2) on e. The proof of Theorem 1.25 in [3] shows that if both N_1^e and N_2^e are standard for all $e \in J$, then M is itself standard.

Let us assume that M is *not standard* and assume that $e \in J$ is such that N_1^e is *not standard* (the case where N_2^e is not standard is similiar). We are going to prove that M is isomorphic to $Ind_{R_1}^{R_1 * R_2}(N_1^e)$. Recall from Lemma 3.6.1.2 that the latter has the basis $\widetilde{J_1^e}$, where J_1^e denotes the basis of N_1^e.

The natural candidate for the isomorphism is the following: given any word $\omega \in W(I_1, I_2)$ and any basis element $j \in J_1^e$, it should send the basis element $\omega j \in \widetilde{J_1^e}$ to $\omega \otimes j$. For this, we must show that such an element $\omega \otimes j$ is still a basis element of M. Let us prove this by induction on the length of $\omega := \alpha_k \dots \alpha_1$.

- For $k = 1$, take a (non-trivial) word of length 1 in $W(I_1, I_2)$, say $\alpha_2 \in I_2 \backslash \{\mathbb{1}_2\}$. Assume that $\alpha_2 \otimes j$ is *not a basis element*. This means that we can take α_2 such that $j \subset \alpha_2 \otimes j$. Let α_1 be a non-trivial element in $Stab(j) \cap I_1$, which exists by Lemma 3.6.1.4 because N_1^e is not standard. Then, for any integer $l \in \mathbb{N}$, $(\alpha_1\alpha_2)^{\otimes l} \in I_1 * I_2$ is a non-trivial stabilizer of j,

$$j \subset \alpha_1 \otimes j \subset \alpha_1\alpha_2 \otimes j \subset \alpha_1\alpha_2\alpha_1 \otimes j \subset \dots$$

  Consequently, we obtain infinitely many stabilizers, which contradicts the co-finiteness of M. Thus, $Stab(j) \cap I_1 = \{\mathbb{1}_1\}$, which implies that the R_1-module generated by j must be standard and the corresponding isomorphism sends j to $\mathbb{1}_1$ (by Lemma 3.6.1.4). In particular, $\alpha_2 \otimes j \cong \alpha_2$ is a basis element for all $\alpha_2 \in I_2 \backslash \{\mathbb{1}_2\}$.

- Assume now that for some $k \in \mathbb{N}$, $\omega \otimes j$ is a basis element with $\omega = \alpha_k \dots \alpha_1 \in W(I_1, I_2)$ and $j \in J_1^e$.

- Without loss of generality, assume that $\alpha_k \in I_2 \backslash \{\mathbb{1}_2\}$. Put $j' := \alpha_k \dots \alpha_1 \otimes j$, which is a basis element by the induction hypothesis. By the same argument as before, $Stab(j') \cap I_1 = \{\mathbb{1}_1\}$, so that $N_1^{j'}$ is standard for R_1. In particular, for any $\alpha_{k+1} \in I_1 \backslash \{\mathbb{1}_1\}$, $\alpha_{k+1} \otimes j'$ is a basis element and the claim is proved.

Now, let $\varphi : Ind_{R_1}^{R_1 * R_2}(N_1^e) \longrightarrow M$ be the homomorphism sending ωj to $\omega \otimes j$, for any word $\omega \in W(I_1, I_2)$ and any $j \in J_1^e$.

Since M is connected by assumption, the homomorphism φ is clearly surjective. Let us show that φ is injective. Let $\omega, \omega' \in W(I_1, I_2)$ with $\omega \neq \omega'$ and let $j, j' \in J_1^e$ be basis elements with $j \neq j'$ such that $\varphi(\omega j) = \varphi(\omega' j')$, that is, $\omega \otimes j = \omega' \otimes j'$. In particular, we have that $j' \subset (\overline{\omega'} \otimes \omega) \otimes j$. By definition of the ring structure on $R_1 * R_2$, we observe that $\overline{\omega'} \otimes \omega$ is a direct sum of non-empty words starting and ending in $I_2 \backslash \{\mathbb{1}_2\}$. In particular, there exists $\omega'' \in W(I_1, I_2)$ such that $j' \subset \omega'' \otimes j$. But such an element $\omega'' \otimes j$ is a basis element, so it must be $j' = \omega'' \otimes j$. Now, if $\alpha, \alpha' \in I_1 \backslash \{\mathbb{1}_1\}$ are stabilizers of j and j', respectively, then we get for any $l \in \mathbb{N}$ a stabilizer of j given by

$$\left(\overline{\omega''} \alpha' \omega'' \alpha\right)^{\otimes l} = \left(\overline{\omega''} \otimes \alpha' \otimes \omega'' \otimes \alpha\right)^{\otimes l},$$

which contradicts the co-finiteness of M. Thus, it must be $\omega = \omega'$ and $j = j'$. The proof is then complete. ■

The preceding results have been obtained at the level of fusion modules for which we have used the general combinatoric description of the free product of fusion rings. In order to state the corresponding result for a quantum free product at the level of torsion actions we have to recast the preceding proof in the setting of module C^*-categories as we have illustrated in Remark 1.6.2.7. This is, by the way, the strategy used in [3] for showing that a free product of torsion-free discrete quantum groups is torsion-free (see Theorem 3.16 in [3] for more details).

3.6.1.7 Theorem. *Let $\mathbb{G}$ and $\mathbb{H}$ be compact quantum groups. There is a one-to-one correspondence, up to equivariant Morita equivalence, between torsion actions of $\mathbb{G} * \mathbb{H}$ and torsion actions induced from $\mathbb{G}$ or $\mathbb{H}$.*

Proof. Let (A, δ) be a torsion action of $\mathbb{G} * \mathbb{H}$ and consider the corresponding module C^*-category over $\mathscr{R}ep(\mathbb{G} * \mathbb{H})$, say $\mathscr{H}$, whose objects are $\mathbb{G} * \mathbb{H}$-equivariant Hilbert A-modules.

Given any irreducible object $X \in Obj(\mathscr{H})$, we denote by $\mathscr{H}_{\mathbb{G}}^X$ (resp. $\mathscr{H}_{\mathbb{H}}^X$) the module C^*-category generated by X and the action of $\mathscr{R}ep(\mathbb{G})$ (resp. $\mathscr{R}ep(\mathbb{H})$).

By the proof of Proposition 3.6.1.6, there is an object X such that either $\mathscr{H}_{\mathbb{G}}^X$ or $\mathscr{H}_{\mathbb{H}}^X$ contains a non-standard fusion module whenever (A, δ) is not trivial. Without loss of generality, let us say that this is case for $\mathscr{H}_{\mathbb{G}}^X$. By Lemma 3.10 in [3], the module C^*-category is equivalent to the one of the trivial action if its associated fusion module is standard. The same reasoning as for fusion modules yields therefore an equivalence between $\mathscr{H}$ and the module C^*-category induced from $\mathscr{H}_{\mathbb{G}}^X$.

To conclude, notice that by the general result of [50], there exists a torsion action (A', δ') of $\mathbb{G}$ such that the associated module C^*-category is $\mathscr{H}_{\mathbb{G}}^X$. Thus, $\mathscr{H}$ is equivalent to the module C^*-category associated to the induced action $Ind_{\mathbb{G}}^{\mathbb{G} * \mathbb{H}}(\delta')$ and again by [50] the actions are equivariantly Morita equivalent.

Consider now two induced torsion actions which are equivariantly Morita equivalent. If they are induced from different factors, then the fusion module of the one induced from $\mathbb{H}$ is a direct sum of standard modules when restricted to $\mathbb{G}$. Thus, the fusion module associated to the one induced by $\mathbb{G}$ is isomorphic to a direct sum of standard module with respect to $R(\mathbb{G})$. Since all its submodules are also standard with respect to $R(\mathbb{H})$, we conclude by Theorem 1.25 in [3] that the actions are trivial.

This leaves us with the case of two induced torsion actions from the same factor which are equivariantly Morita equivalent. Without loss of generality, let us say that (A, δ) and (A', δ') are two torsion actions of $\mathbb{G}$ such that $Ind_{\mathbb{G}}^{\mathbb{G}*\mathbb{H}}(\delta)$ and $Ind_{\mathbb{G}}^{\mathbb{G}*\mathbb{H}}(\delta')$ are equivariantly Morita equivalent. The same reasoning as in Lemma 3.6.1.5 shows that in both associated module C^*-categories, the module C^*-subategory coming from the original action is the only one to be non-trivial over $\mathscr{R}ep(\mathbb{G})$. The equivalence of categories must therefore restrict to an equivalence between these subcategories and we conclude by [50]. ■

3.6.2 The Baum-Connes property

In order to legitimate the Baum-Connes property formulation for the dual of a quantum free product $\mathbb{F} = \mathbb{G} * \mathbb{H}$, we need $\widehat{\mathbb{F}}$ to be torsion-free. And, in order to legitimate the Baum-Connes property formulation for $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$, we need these discrete quantum groups to be torsion-free. Hence, we must keep the preceding section in mind and we assume that $\widehat{\mathbb{F}}$, $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are all torsion-free.

The aim of this section is to summarize the strategy carried out by R. Vergnioux and C. Voigt in [208]. Here we collect the main results and definitions and we refer to Section 5 and Section 6 of [208] for the full details.

Given a locally compact quantum group $\mathbb{G}$ we denote by $\mathcal{D}(\mathbb{G})$ the *Drinfeld quantum double of* $\mathbb{G}$ (see Remark A.3.24 and [140] for more details). Given two discrete quantum groups $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$, let us explain the notations concerning the quantum Bass-Serre theory for the quantum free product $\widehat{\mathbb{F}} = \widehat{\mathbb{G} * \mathbb{H}}$.

The tree associated to $\widehat{\mathbb{F}}$ is defined by

$$l^2(X) := l^2(X^{(0)}) \oplus l^2(X^{(1)}),$$

where

$$l^2(X^{(0)}) := l^2(\widehat{\mathbb{F}}/\widehat{\mathbb{G}}) \oplus l^2(\widehat{\mathbb{F}}/\widehat{\mathbb{H}}) \text{ and } l^2(X^{(1)}) := l^2(\widehat{\mathbb{F}}),$$

where we remark that X may represent the *quantum* tree associated to $\widehat{\mathbb{G}}$, so that in the quantum context it must be regarded as a *virtual object*.

Consider the one-dimensional affine space $E := \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1\}$ and write $\mathbb{C}l_1$ for the one-dimensional Clifford algebra.

Finally, observe that the compact operators on $l^2(X)$, $\mathcal{K}(l^2(X))$, can be viewed as a *graded* C^*-algebra following the decomposition of the definition of $l^2(X)$. Recall that we use the symbol $\widehat{\oplus}$ for the *graded tensor product*.

3.6.2.1 Theorem (R. Vergnioux and C. Voigt, [208])**.** *Let $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ be two discrete quantum groups and let $\widehat{\mathbb{F}} = \widehat{\mathbb{G} * \mathbb{H}}$ be the dual of the corresponding quantum free product.*

i) There exists a $\mathcal{D}(\widehat{\mathbb{F}})$-$C^*$-algebra $\mathcal{A}_X \subset \big(C_0(E) \otimes \mathbb{C}l_1\big)\widehat{\oplus}\mathcal{K}(l^2(X))$, which is $KK^{\mathcal{D}(\widehat{\mathbb{F}})}$-equivalent to $\mathcal{P} := C_0(\mathbb{R}) \otimes \mathcal{B}_X$, for some ungraded $\mathcal{D}(\widehat{\mathbb{G}})$-$C^*$-algebra $\mathcal{B}_X \subset C_0(E) \otimes \mathcal{K}(l^2(X))$.

We define the Dirac element for $\widehat{\mathbb{F}}$ *as the element*

$$D \in KK^{\mathcal{D}(\widehat{\mathbb{F}})}(\mathcal{A}_X, \mathbb{C})$$

defined as the composition of the canonical inclusion $\mathcal{A}_X \hookrightarrow \big(C_0(E) \otimes \mathbb{C}l_1\big)\widehat{\oplus}\mathcal{K}(l^2(X))$ with the Bott periodicity isomorphism and the equivariant Morita equivalence $\mathcal{K}(l^2(X)) \underset{M}{\sim} \mathbb{C}$.

ii) There exists un element $\gamma \in KK^{\mathcal{D}(\widehat{\mathbb{F}})}(\mathbb{C}, \mathbb{C})$ *such that* $\gamma = \mathbb{1}_{\mathbb{C}}$.

iii) There exists a dual-Dirac element for D*, that is, an element* $\eta \in KK^{\mathcal{D}(\widehat{\mathbb{F}})}(\mathbb{C}, \mathcal{A}_X)$ *such that*

$$\eta \underset{\mathcal{A}_X}{\otimes} D = \mathbb{1}_{\mathbb{C}}$$

3.6.2.2 Remark. Notice that the Dirac element for $\widehat{\mathbb{F}}$ is defined with respect to the C^*-algebra $\mathcal{A}_X$. But this one is $KK^{\mathcal{D}(\widehat{\mathbb{F}})}$-equivalent to $\mathcal{P}$, which is *ungraded*. By abuse of notation we still write

$$D \in KK^{\mathcal{D}(\widehat{\mathbb{F}})}(\mathcal{P}, \mathbb{C})$$

for the Dirac element. Moreover, we can show that this element is still invertible (see Lemma 6.4 and Lemma 6.5 in [208] for more details).

A combination of the preceding results and the equivariant Poincaré duality for quantum groups developed by R. Meyer and C. Voigt in [140] yields the stability of the Baum-Connes property for quantum free products, which generalizes the classical case studied in [144], [193].

3.6.2.3 Theorem (R. Vergnioux and C. Voigt, [208])**.** *Let* $\widehat{\mathbb{G}}$ *and* $\widehat{\mathbb{H}}$ *be two discrete quantum groups such that* $\widehat{\mathbb{G}}$ *and* $\widehat{\mathbb{H}}$ *are torsion-free. Let* $\widehat{\mathbb{F}} = \widehat{\mathbb{G} * \mathbb{H}}$ *be the dual of the corresponding quantum free product.*

If $\widehat{\mathbb{G}}$ *and* $\widehat{\mathbb{H}}$ *satisfies the strong Baum-Connes property, then* $\widehat{\mathbb{F}}$ *satisfies the strong Baum-Connes property.*

Next, recall the following facts:

- The strong Baum-Connes property is invariant under monoidal equivalences (Theorem B.3.19).
- Divisible discrete quantum subgroups of a discrete quantum group satisfying the strong Baum-Connes property satisfy the strong Baum-Connes property (Theorem 3.2.2.1).
- For all $n > 1$, all $Q \in GL_n(\mathbb{C})$ with $Q\overline{Q} = \pm id$ and all $q \in [-1, 1]\backslash\{0\}$, $SU_q(2)$ is monoidally equivalent to $O^+(Q)$ (Theorem B.3.19).
- $\widehat{SU_q(2)}$ satisfies the strong Baum-Connes property, for all $q \in [-1, 1]\backslash\{0\}$ (Theorem 2.1.5).
- $\mathbb{Z}$ satisfies the strong Baum-Connes property [82].
- For all $n > 1$, all $Q \in GL_n(\mathbb{C})$ with $Q\overline{Q} = \pm id$, $\widehat{U^+(Q)} \subset \widehat{O^+(Q)} * \mathbb{Z}$ is divisible (Proposition 4.3 in [208]).

With all these properties in mind, it is not very hard to establish that duals of free unitary quantum groups satisfy the strong Baum-Connes property (see Theorem 6.8 in [208] for more details).

3.6.2.4 Theorem (R. Vergnioux and C. Voigt, [208])**.** *Let* $n > 1$ *and* $Q \in GL_n(\mathbb{C})$*. Then* $\widehat{U^+(Q)}$ *satisfy the strong Baum-Connes property.*

3.6.2.5 Remark. As we have pointed out in Remark 2.5.2, both $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are *divisible* discrete quantum subgroups of $\widehat{\mathbb{G} * \mathbb{H}}$. Consequently, if both $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are torsion-free and $\widehat{\mathbb{G} * \mathbb{H}}$ satisfies the (resp. *strong*) Baum-Connes property, then both $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ satisfy the (resp. *strong*) Baum-Connes property by applying (resp. Theorem 3.2.2.1) Proposition 3.2.2.3.

3.6.3 K-amenability property

Here we care about a property of own interest, namely the K-amenability of a quantum free product.

On the one hand, we recall Remark 1.7.2.11 where we have noticed that the torsion-freeness assumption and the *strong* Baum-Connes property for a discrete quantum group implies automatically the K-amenability of the discrete quantum group.

On the other hand, as explained in the introduction of this dissertation, several work of R. Vergnioux [206], [207], P. Fima [63] and P. Fima-A. Freslon [64] have been made for studying the K-amenability property for quantum groups acting on *quantum* trees using a quantum version of the Bass-Serre theory generalizing the classical case [90] studied by P. Julg and A. Valette.

Therefore, from Remark 1.7.2.11 and the work mentioned above, we obtain the following result.

3.6.3.1 Theorem. *i) Let $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ be two discrete quantum groups and let $\widehat{\mathbb{F}} = \widehat{\mathbb{G} * \mathbb{H}}$ be the dual of the corresponding quantum free product.*

a) $\mathbb{G}$ and $\mathbb{H}$ are co-amenable if and only if $\mathbb{F}$ is co-K-amenable.

b) If $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are torsion-free and satisfy the strong Baum-Connes property, then $\mathbb{F}$ is co-K-amenable.

ii) Let $n > 1$ and $Q \in GL_n(\mathbb{C})$. Then $\widehat{U^+(Q)}$ is K-amenable.

iii) Let $n > 1$ and $Q \in GL_n(\mathbb{C})$ with $Q\overline{Q} \in \mathbb{R}id$. Then $\widehat{O^+(Q)}$ is K-amenable.

3.7 The Baum-Connes property for a free wreath product

First of all, the whole content of this section together with Section 4.2 is a collaboration work with A. Freslon [127].

In this section we are going to study the Baum-Connes property for the dual of a free wreath product, $\mathbb{G} \wr_* S_N^+$, of a compact quantum group $\mathbb{G}$ by S_N^+ (with $N \geqslant 4$) in terms of $\mathbb{G}$ and S_N^+.

The Baum-Connes property for a classical wreath product has been studied under different perspectives during the last years. More precisely, Y. Cornulier, Y. Stalder and A. Valette have studied the stability of the Haagerup property for the wreath product construction in [42]. They have showed that if G and H are two countable groups with the Haagerup property, then their wreath product $G \wr H$ has the Haagerup property too. Consequently, the celebrated result of Higson-Kasparov [82] yields in this case that $G \wr H$ satisfies the *strong* Baum-Connes property.

This result allows to include an extensive list of groups satisfying the *strong* Baum-Connes property. For instance, this is true for the lamplighter group $\mathbb{Z}/\mathbb{Z}_2 \wr \mathbb{Z}$ or, more generally, for a wreath product $G \wr H$, where G is any finite group and $H := \mathbb{F}_n$ is the free group on n generators. However, the *abstract* conclusion about the Baum-Connes property for this kind of groups does not bring any information about the explicit computation of the K-theory and the K-homology groups of the corresponding C^*-algebra. Recently, R. Flores, S. Pooya and A. Valette [67] and S. Pooya

[160] have provided an explicit proof of the Baum-Connes property by computing both sides of the assembly map in this two cases, respectively.

The quantum case presents a first issue : the presence of non-trivial torsion for the dual of a free wreath product $\mathbb{G} \wr_* S_N^+$ (even when $\widehat{\mathbb{G}}$ is torsion-free). In Section 3.7.1 we classify all torsion actions of $\mathbb{G} \wr_* S_N^+$.

As we have already mentioned, the torsion-phenomena prevents a proper theoretical formulation of the *quantum* Baum-Connes property, which is the second main issue. Nevertheless, as it is explained in full detail in Section 4.1, there exists a reasonable choice for the analogue of the *localizing subcategory of compactly induced C^*-algebras* in the quantum setting, taking into account all torsion actions of the compact quantum group. This choice turns out to be the appropriated one in order to analyze the *strong* Baum-Connes property for a free wreath product, as it is shown in Section 3.7.2. A major application of these results is the explicit computation of the K-theory of the C^*-algebras associated to several free wreath products, for which we refer to Section 4.2.

We carry the same notations of Section 2.6 on. Let $\mathbb{F} := \mathbb{G} \wr_* S_N^+$ be a free wreath product, where $\mathbb{G}$ is a compact quantum group and $N \geqslant 4$ is a natural number. We denote by $\mathbb{H}_q$ the Lemeux-Tarrago's compact quantum group which is monoidally equivalent to $\mathbb{G} \wr_* S_N^+$ (recall Theorem 2.6.2). We denote by $\Lambda : R(\mathbb{H}_q) \hookrightarrow R(\mathbb{G} * SU_q(2))$ the corresponding inclusion described in Remark 2.6.3.

3.7.1 Torsion property

In contrast to the compact quantum group constructions studied in preceding sections, the free wreath product construction is *never torsion-free* because $\widehat{S_N^+}$ is *never torsion-free*, so that every torsion action of $\mathbb{G} \wr_* S_N^+$ is induced by the defining action of S_N^+ on $\mathbb{C}^N$, which is a torsion one. This section is dedicated to show in detail these affirmations. For this we should recall notations and results from Section 1.6.2 and Section 3.6.1.

3.7.1.1 Lemma. *Let $\mathbb{G}$ be a compact quantum group. If N is a torsion $R(\mathbb{G})$-module with basis J and $M := Ind_{R(\mathbb{G})}^{R(\mathbb{G})*R(SU_q(2))}(N)$, then M contains a unique non-standard torsion $R(\mathbb{H}_q)$-module.*

Proof. Let $j \in J$ be a basis element of N and let $\omega \in W\Big(Irr(\mathbb{G}), Irr(SU_q(2))\Big)$ be a word ending in $Irr(SU_q(2))$. Recall from Lemma 3.6.1.2 that $\omega j \in \widetilde{J}$ is a basis element of M. Denote by $N(\omega j)$ the $R(\widehat{\mathbb{H}}_q)$-submodule generated by ωj. Let ω' be a word of minimal length such that $\omega' j \in N(\omega j)$.

If $\omega' = \varnothing$, then for any non-trivial word $\omega'' \in W\Big(Irr(\mathbb{G}), Irr(SU_q(2))\Big)$, we have $\Lambda(\omega'') \otimes j = \Lambda(\omega'')j$, which is a basis element, so that $N(\omega j) = N(j)$ is standard.

If ω' starts in $Irr(\mathbb{G})$, then for any non-trivial word $\omega'' \in W\Big(Irr(\mathbb{G}), Irr(SU_q(2))\Big)$, we have $\Lambda(\omega'') \otimes \omega' j = \Lambda(\omega'')\omega' j$, which is a basis element, so that $N(\omega j) = N(\omega' j)$ is standard.

Therefore, let us assume that $\omega' = u^{n_0} x_1 \dots x_k u^{n_k}$, for some $k \in \mathbb{N}$. If $n_0 > 1$, we see that $u^{n_0-2} x_1 \dots x_k u^{n_k} j \subset \overline{u^2} \otimes \omega' j \in N(\omega j)$ because $u^2 \in Irr(\mathbb{H}_q)$ by definition. This contradicts the minimality of ω'. Hence, we can assume that $n_0 = 1$.

If $k > 0$, then see that $u^{n_1} x_2 \dots x_k u^{n_k} \in N(\omega j)$ by tensoring ω' by $u\overline{x}_1 u$. This contradicts the minimality of ω'. Hence, it must be $\omega' = u$.

Since N is by assumption a connected $R(\mathbb{G})$-module, for any basis element $j' \in J$, there exists $x \in Irr(\mathbb{G})$ such that $j' \subset x \otimes j$. Therefore, $uj' \subset x \otimes uj \subset \Lambda(x) \otimes uj = uxu \otimes uj \in N(uj)$ since

$uxu \in Irr(\mathbb{H}_q)$ by definition. In other words, $N(uj') = N(uj)$ and we obtain in this way a *torsion* module, which is unique by construction and we denote it by N_u. Moreover, it is *non-standard* because u^2 is clearly a non-trivial stabilizer of uj, which is the generator of N_u. ■

3.7.1.2 Theorem. *Let $\mathbb{G}$ be a compact quantum group and $N \geqslant 4$ a natural number. The equivariant Morita equivalence classes of non-trivial torsion actions of $\mathbb{G} \wr_* S_N^+$ are in one-to-one correspondence with* all *the equivariant Morita equivalence classes of torsion actions of $\mathbb{G}$.*

In particular, a free wreath product $\mathbb{G} \wr_* S_N^+$ is never torsion-free *and $(\mathbb{C}^N, \alpha_N)$ is the only, up to equivariant Morita equivalence, non-trivial torsion action of $\mathbb{G} \wr_* S_N^+$ whenever $\widehat{\mathbb{G}}$ is torsion-free.*

Proof. First of all, thanks to the monoidal equivalence between $\mathbb{G} \wr_* S_N^+$ and $\mathbb{H}_q$, it is enough to study the torsion actions of $\mathbb{H}_q$.

Let (A, δ) be a non-trivial torsion action of $\mathbb{H}_q$. Then it is clear that $Ind_{\mathbb{H}_q}^{\mathbb{G}*SU_q(2)}(\delta)$ is again a torsion action of $\mathbb{G} * SU_q(2)$. By Theorem 3.6.1.7, $Ind_{\mathbb{H}_q}^{\mathbb{G}*SU_q(2)}(\delta)$ is equivariantly Morita equivalent to a torsion action induced from $\mathbb{G}$ (since $\widehat{SU_q(2)}$ is torsion-free). By Lemma 3.7.1.1, the restriction of such an action to $\mathbb{H}_q$ has exactly one non-trivial summand. Thus, this summand is equivariantly Morita equivalent to (A, δ).

Moreover, if two torsion actions of $\mathbb{H}_q$ are equivariantly Morita equivalent, then the same holds for their induction to $\mathbb{G} * SU_q(2)$, so that by Theorem 3.6.1.7 the original actions of $\mathbb{G}$ are also equivariantly Morita equivalent.

In particular, $\mathbb{G} \wr_* S_N^+$ is *never torsion-free* because the trivial action of $\mathbb{G}$ (which is a torsion action) gives rise to a non-trivial torsion action of the free wreath product. Let us describe explicitly this action.

Consider the quantum subgroup of $\widehat{\mathbb{H}}_q$ generated by u^2, which is isomorphic to $\widehat{SO_q(3)}$. As we have mentioned in Remarks 2.1.10, we have an isomorphism $Qut(\mathcal{M}_2(\mathbb{C})) \cong SO_q(3)$. Hence, thanks to Theorem 2.1.11, the only non-trivial torsion action of $SO_q(3)$ is the canonical action of $Qut(\mathcal{M}_2(\mathbb{C}))$ on $\mathcal{M}_2(\mathbb{C})$, say α_q (see as well [173]).

By definition, $\widehat{SO_q(3)} < \widehat{SU_q(2)}$ and since $\widehat{SU_q(2)}$ is torsion-free, $Ind_{SO_q(3)}^{SU_q(2)}(\alpha_q)$ must be the trivial action (up to equivariantly Morita equivalence). By what we have proven before, $Ind_{SO_q(3)}^{\mathbb{H}_q(2)}(\alpha_q)$ must be a non-trivial torsion action whose induction to $\mathbb{G} * SU_q(2)$ is trivial. Under the monoidal equivalence between $\mathbb{H}_q$ and $\mathbb{G} \wr_* S_N^+$, $(\mathcal{M}_2, \alpha_q)$ becomes the canonical action of S_N^+ on $\mathbb{C}^N$. Inducing this action to $\mathbb{G} \wr_* S_N^+$ yields a non-trivial torsion action $(\mathbb{C}^N, \alpha_N)$, which is precisely the one obtained from the trivial action of $\mathbb{G}$. If $\mathbb{G}$ is torsion-free, this is the only source of torsion in the free wreath product. ■

3.7.1.3 Remark. The proof of the preceding theorem shows precisely which is the *only* torsion action of a free wreath product $\mathbb{G} \wr_* S_N^+$, where $\widehat{\mathbb{G}}$ is torsion-free. Namely, it is the defining action of S_N^+ on $\mathbb{C}^{\mathbb{N}}$ induced to $\mathbb{G} \wr_* S_N^+$. If α_q denotes the torsion action of $SO_q(3)$ on $\mathcal{M}_2(\mathbb{C})$, then the monoidal equivalence between $\mathbb{H}_q$ and $\mathbb{G} \wr_* S_N^+$ yields a correspondence between these torsion actions,

$$Ind_{S_N^+}^{\mathbb{G}\wr_* S_N^+}(\alpha_N) \longleftrightarrow Ind_{SO_q(3)}^{\mathbb{H}_q(2)}(\alpha_q)$$

3.7.2 The Baum-Connes property

By virtue of the preceding section, the dual of a free wreath product $\mathbb{F} = \mathbb{G} \wr_* S_N^+$ is *never torsion-free*, even if $\widehat{\mathbb{G}}$ is torsion-free. Hence, the theoretical framework developed in Section 1.7.2 for a quantum Baum-Connes property formulation can not be applied. Let $\mathscr{K}\mathscr{K}^{\mathbb{G}\wr_* S_N^+}$ be the equivariant Kasparov category associated to the compact quantum group $\mathbb{G} \wr_* S_N^+$. Being inspired by the pioneering work of C. Voigt in [212], we can re-define the localizing subcategory of quantum compactly induced C^*-algebras $\mathscr{L}_{\mathbb{F}} := \widehat{\mathscr{L}_{\widehat{\mathbb{F}}}}$ by taking into account the torsion phenomena of the discrete dual of the free wreath product,

$$\mathscr{L}_{\mathbb{F}} := \langle \{T \otimes B \text{ with } T \in \mathbb{T}or(\widehat{\mathbb{F}}) \text{ and } B \in Obj(\mathscr{K}\mathscr{K})\}\rangle \subset \mathscr{K}\mathscr{K}^{\mathbb{G}\wr_* S_N^+}$$

$$\mathscr{L}_{\widehat{\mathbb{F}}} := \langle \{\mathbb{F} \underset{r}{\ltimes} T \otimes B \text{ with } T \in \mathbb{T}or(\widehat{\mathbb{F}}) \text{ and } B \in Obj(\mathscr{K}\mathscr{K})\}\rangle \subset \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}\wr_* S_N^+}}$$

3.7.2.1 Definition. Let $\mathbb{G}$ be a compact quantum group and $N \geqslant 4$ a natural number. We say that $\widehat{\mathbb{F}} := \widehat{\mathbb{G} \wr_* S_N^+}$ satisfies the *strong* Baum-Connes property if

$$\mathscr{L}_{\widehat{\mathbb{F}}} = \mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$$

Recall that, by virtue of Theorem 2.6.2, there exists a discrete quantum subgroup $\widehat{\mathbb{H}}_q < \mathbb{G} * \widehat{SU_q}(2)$ such that $\mathbb{H}_q$ is monoidally equivalent to $\mathbb{G} \wr_* S_N^+$. Denote by $\mathscr{K}\mathscr{K}^{\mathbb{G}\wr_* S_N^+}$ the corresponding equivariant Kasparov category. Therefore, $\widehat{\mathbb{H}}_q$ is neither torsion-free because all torsion actions of monoidally equivalent compact quantum groups are in bijection as it is shown in [163]. In this way, we can re-define the localizing subcategory of quantum compactly induced C^*-algebras $\mathscr{L}_{\mathbb{H}_q} := \widehat{\mathscr{L}_{\widehat{\mathbb{H}}_q}}$ by taking into account the torsion phenomena of the discrete dual of the Lemeux-Tarrago's $\mathbb{H}_q$,

$$\mathscr{L}_{\mathbb{H}_q} := \langle \{S \otimes B \text{ with } S \in \mathbb{T}or(\widehat{\mathbb{H}}_q) \text{ and } B \in Obj(\mathscr{K}\mathscr{K})\}\rangle \subset \mathscr{K}\mathscr{K}^{\mathbb{H}_q}$$

$$\mathscr{L}_{\widehat{\mathbb{H}}_q} := \langle \{\mathbb{H}_q \underset{r}{\ltimes} S \otimes B \text{ with } S \in \mathbb{T}or(\widehat{\mathbb{H}}_q) \text{ and } B \in Obj(\mathscr{K}\mathscr{K})\}\rangle \subset \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}_q}$$

3.7.2.2 Definition. Let $\mathbb{G}$ be a compact quantum group and $N \geqslant 4$ a natural number. Let $\mathbb{H}_q$ the Lemeux-Tarrago's compact quantum group monoidally equivalent to $\mathbb{G} \wr_* S_N^+$. We say that $\mathbb{H}_q$ satisfies the *strong* Baum-Connes property if

$$\mathscr{L}_{\widehat{\mathbb{H}}_q} = \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}_q}$$

By virtue of the monoidal equivalence between $\mathbb{H}_q$ and $\mathbb{G} \wr_* S_N^+$, we know that the corresponding equivariant Kasparov categories, $\mathscr{K}\mathscr{K}^{\mathbb{H}_q}$ and $\mathscr{K}\mathscr{K}^{\mathbb{G}\wr_* S_N^+}$, are equivalent (see [210] for more details). Moreover, the choice of our localizing subcategories of quantum compactly induced C^*-algebras together with the bijective correspondence between torsion actions from [163] and Baaj-Skandalis duality yield the following

3.7.2.3 Lemma. $\widehat{\mathbb{G} \wr_* S_N^+}$ satisfies the strong Baum-Connes property if and only if $\widehat{\mathbb{H}}_q$ satisfies the strong Baum-Connes property.

3.7.2.4 Remark. In other words, we re-define the *strong* Baum-Connes property by requiring the abstract condition $\mathscr{L}_{\widehat{\mathbb{F}}} = \mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$, which is consistent with the torsion-free case as it is pointed out in Remark 4.1.

However, the formal statement for the *usual* Baum-Connes property in the framework of Meyer-Nest requires the definition of an other localizing subcategory $\mathscr{N}_{\widehat{\mathbb{F}}}$ such that the pair $(\mathscr{L}_{\widehat{\mathbb{F}}}, \mathscr{N}_{\widehat{\mathbb{F}}})$ is complementary. The definition of such a $\mathscr{N}_{\widehat{\mathbb{F}}}$ must be the the right orthogonal complement of $\mathscr{L}_{\widehat{\mathbb{F}}}$ (recall Definition 1.2.1.23). But it is not clear, *a priori*, that this pair is complementary in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$. These difficulties have been out of the scope of the present dissertation, but we give a more precise overview of this problem in Chapter 5.

Since $\widehat{\mathbb{H}}_q$ is a discrete quantum subgroup of the dual of quantum free product and for the later the strong Baum-Connes property has already been studied in Section 3.6, we will work with $\widehat{\mathbb{H}}_q$ and not with $\widehat{\mathbb{G} \wr_* S_N^+}$ itself.

3.7.2.5 Remark. Before carrying on, it is advisable to describe explicitly the generator objects of the localizing subcategory $\mathscr{L}_{\widehat{\mathbb{H}}_q}$.

Assume that $\widehat{\mathbb{G}}$ is torsion-free. Then the only, up to equivariant Morita equivalence, non-trivial torsion action of $\mathbb{G} \wr_* S_N^+$ is, by virtue of Theorem 3.7.1.2, the defining action of S_N^+ on $\mathbb{C}^N$ induced to $\mathbb{G} \wr_* S_N^+$. As we have pointed out in Remark 3.7.1.3, this torsion action is in correspondence with the torsion action of $\mathbb{H}_q$ given by $(\mathcal{M}_2(\mathbb{C}), \alpha_q)$. Let us describe the Baaj-Skandalis dual of this torsion action, that is, the crossed product $\mathbb{H}_q \underset{r}{\ltimes} \mathcal{M}_2(\mathbb{C})$.

Let us define the subset $J_u := \mathcal{S} \subset Irr(\mathbb{G} * SU_q(2))$ as the set of irreducible representations of $\mathbb{G} * SU_q(2)$ generated by the action of $Irr(\mathbb{H}_q)$ on u, that is,

$$J_u := \mathcal{S} := \{y \in Irr(\mathbb{G} * SU_q(2)) \mid y \subset h \oplus u \text{ with } h \in Irr(\mathbb{H}_q)\}$$

Then we put

$$A_u := A_q := \bigoplus_{y \in \mathcal{S}}^{c_0} \mathcal{B}(H_y) \subset c_0\big(\mathbb{G} * \widehat{SU_q}(2)\big)$$

and observe that, by virtue of Theorem 3.7.1.2, we have the following decomposition

$$c_0\big(\mathbb{G} * \widehat{SU_q}(2)\big) = A_q \oplus \Big(\bigoplus_{\mathbb{N}} c_0(\widehat{\mathbb{H}}_q)\Big)$$

as $\widehat{\mathbb{H}}_q$-C^*-algebras. Again by Theorem 3.7.1.2 we observe that $\widehat{\mathbb{H}}_q \underset{r}{\ltimes} A_q$ must be equivariantly Morita equivalent to $(\mathcal{M}_2(\mathbb{C}), \alpha_q)$ (because $\mathbb{G} * \widehat{SU_q}(2)$ is torsion-free, so that $\mathbb{G} * SU_q(2)$ only admits the *trivial* torsion action, which restricted to an action of $\mathbb{H}_q$ gives the corresponding torsion action of $\mathbb{H}_q$). In particular, A_q is equivariantly Morita equivalent to $\mathbb{H}_q \underset{r}{\ltimes} \mathcal{M}_2(\mathbb{C})$.

In other words, we have showed that

$$\mathscr{L}_{\widehat{\mathbb{H}}_q} := \langle \{A_q \otimes B \text{ with } B \in Obj(\mathscr{K}\mathscr{K})\} \cup \{c_0(\widehat{\mathbb{H}}_q) \otimes B \text{ with } B \in Obj(\mathscr{K}\mathscr{K})\}\rangle$$

again by virtue of Theorem 3.7.1.2.

3.7.2.6 Theorem. *Let $\mathbb{G}$ be a compact quantum group and $N \geqslant 4$ a natural number. If $\widehat{\mathbb{G}}$ is torsion-free and satisfies the* strong *Baum-Connes property, then $\mathbb{G} \wr_* \widehat{S_N^+}$ satisfies the* strong *Baum-Connes property.*

Proof. By virtue of Lemma 3.7.2.3 it is enough to prove that $\widehat{\mathbb{H}}_q$ satisfies the *strong* Baum-Connes property.

Recall that $\widehat{\mathbb{H}}_q < \mathbb{G} * \widehat{SU_q}(2)$ is a discrete quantum subgroup of the dual of a quantum free product. $\widehat{\mathbb{G}}$ is torsion-free and satisfies the strong Baum-Connes property by assumption and $\widehat{SU_q(2)}$ is torsion-free and satisfies the strong Baum-Connes property by [210], [211] (recall as well Theorem 2.1.5). Hence, $\mathbb{G} * \widehat{SU_q}(2)$ is again torsion-free by Theorem 3.6.1.1 and satisfies the strong Baum-Connes property by Theorem 3.6.2.3. This means that $\mathscr{L}_{\mathbb{G}*\widehat{SU_q}(2)} = \mathscr{K}\mathscr{K}^{\mathbb{G}*\widehat{SU_q}(2)}$.

Let $B \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}_q})$ any $\widehat{\mathbb{H}}_q$-C^*-algebra. We have $Ind_{\widehat{\mathbb{H}}_q}^{\mathbb{G}*\widehat{SU_q}(2)}(B) \in \mathscr{L}_{\mathbb{G}*\widehat{SU_q}(2)}$. Thanks to Remark 3.7.2.5 we know that $c_0(\mathbb{G} * \widehat{SU_q}(2)) = A_q \oplus \Big(\bigoplus_{\mathbb{N}} c_0(\widehat{\mathbb{H}}_q)\Big)$ as $\widehat{\mathbb{H}}_q$-C^*-algebras. Hence it is clear that the restriction functor preserves the localizing subcategories of compactly induced C^*-algebras, that is, $Res_{\widehat{\mathbb{H}}_q}^{\mathbb{G}*\widehat{SU_q}(2)}\Big(\mathscr{L}_{\mathbb{G}*\widehat{SU_q}(2)}\Big) \subset \mathscr{L}_{\widehat{\mathbb{H}}_q}$. In particular, $Res_{\widehat{\mathbb{H}}_q}^{\mathbb{G}*\widehat{SU_q}(2)}\Big(Ind_{\widehat{\mathbb{H}}_q}^{\mathbb{G}*\widehat{SU_q}(2)}(B)\Big) \in \mathscr{L}_{\widehat{\mathbb{H}}_q}$.

Finally, using the explicit description of induced C^*-algebras given in Section 1.4.3 we know that B is a retract of $Res_{\widehat{\mathbb{H}}_q}^{\mathbb{G}*\widehat{SU_q}(2)}\Big(Ind_{\widehat{\mathbb{H}}_q}^{\mathbb{G}*\widehat{SU_q}(2)}(B)\Big)$. Since $\mathscr{L}_{\widehat{\mathbb{H}}_q}$ is closed under retracts by definition of localizing subcategory, then we obtain that $B \in \mathscr{L}_{\widehat{\mathbb{H}}_q}$, which completes the proof. ■

3.7.3 K-amenability property

Here we care about a property of own interest, namely the K-amenability of a free wreath product. In contrast to the compact quantum group constructions studied in preceding sections, here the K-amenability property will be particularly useful for the K-theory computations in Section 4.2.

In Remark 1.7.2.11 we have noticed that the K-amenability property is automatically fulfilled for every *torsion-free* discrete quantum group $\widehat{\mathbb{G}}$ satisfying the *strong* Baum-Connes property. Recall that the argument used in Remark 1.7.2.11 consists in proving that the action of $\widehat{\mathbb{G}}$ on the Baaj-Skandalis dual of the trivial torsion action is always amenable.

Following the discussion of the preceding section, an obvious generalization of this fact can be carried out in the torsion case.

3.7.3.1 Definition. Let $\widehat{\mathbb{G}}$ be any discrete quantum group. A $\widehat{\mathbb{G}}$-C^*-agebra P is said to be proper almost homogenous if it is equivariantly Morita equivalent to $\mathbb{G} \underset{r}{\ltimes} T$, for some torsion action $T \in \mathbb{T}or(\widehat{\mathbb{G}})$.

3.7.3.2 Note. The terminology *proper almost homogeneous* has been firstly introduced in [212] by C. Voigt and inspired by the work of R. Meyer and R. Nest in [133], where they formulate and establish the *strong* Baum-Connes property for duals of classical compact connected groups.

3.7.3.3 Lemma (C. Voigt, [212])**.** *Let $\widehat{\mathbb{G}}$ be any discrete quantum group. If P is a proper almost homogenous $\widehat{\mathbb{G}}$-C^*-algebra, then $\widehat{\mathbb{G}}$ acts amenably on P.*

3.7.3.4 Remark. In addition, if $\widehat{\mathbb{G}'}$ is another discrete quantum subgroup such that $\mathbb{G}$ and $\mathbb{G}'$ are monoidally equivalent, then P is proper almost homogenous with respect to $\widehat{\mathbb{G}}$ if and only if it is with respect to $\widehat{\mathbb{G}'}$.

3.7.3.5 Theorem. *Let $\widehat{\mathbb{G}}$ be any discrete quantum group. Denote by $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ the corresponding equivariant Kasparov category and put*

$$\mathscr{L}_{\mathbb{G}} := \langle \{T \otimes B \text{ with } T \in \mathbb{T}or(\widehat{\mathbb{G}}) \text{ and } B \in Obj(\mathscr{K}\mathscr{K})\}\rangle \subset \mathscr{K}\mathscr{K}^{\mathbb{G}}$$

$$\mathscr{L}_{\widehat{\mathbb{G}}} := \langle \{\mathbb{G} \underset{r}{\ltimes} T \otimes B \text{ with } T \in \mathbb{T}or(\widehat{\mathbb{G}}) \text{ and } B \in Obj(\mathscr{K}\mathscr{K})\}\rangle \subset \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$$

If $\widehat{\mathbb{G}}$ satisfies the strong *Baum-Connes property, which means that $\mathscr{L}_{\widehat{\mathbb{G}}} = \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$, then $\widehat{\mathbb{G}}$ is automatically K-amenable.*

Proof. Indeed, it enough to observe that every generator of $\mathscr{L}_{\widehat{\mathbb{G}}}$ is a proper almost homogeneous $\widehat{\mathbb{G}}$-C^*-algebra and so $\widehat{\mathbb{G}}$ acts amenably on generators of $\mathscr{L}_{\widehat{\mathbb{G}}} = \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ by Lemma 3.7.3.3. This concludes the proof because the crossed product functor is compatible with countable direct sums. ■

3.7.3.6 Corollary. *Let $\mathbb{G}$ be a compact quantum group and $N \geqslant 4$ a natural number. If $\widehat{\mathbb{G}}$ is torsion-free and satisfies the* strong *Baum-Connes property, then $\mathbb{G} \wr_* \widehat{S_N^+}$ is K-amenable.*

CHAPTER 4

An application: the K-theory for the *Lemeux-Tarrago's* $\widehat{\mathbb{H}}_q < \mathbb{G} * \widehat{SU_q}(2)$

The fourth chapter is dedicated to illustrate the general and abstract theory about the *quantum* Baum-Connes formulation, as it has been presented in the present thesis, with a major application: explicit K-theory computations. More concretely, we achieve the computation of the K-theory groups for some free wreath products by using the results obtained in Section 3.7.

In Section 4.1 we explain the different general strategies for computing the K-theory of C^*-algebras defining compact quantum groups by using the categorical framework of Meyer-Nest for the quantum Baum-Connes property. In other words, we describe in detail the general method used by C. Voigt and his collaborators to this end.

In Section 4.2 we perform, inspired by the pioneering work of C. Voigt and his collaborators, an explicit K-theory computation for some free wreath product. More precisely, we compute the K-theory groups of the C^*-algebra $C(\mathbb{H}_q)$, where $\widehat{\mathbb{H}}_q < \mathbb{G} * \widehat{SU_q}(2)$ is the Lemeux-Tarrago's discrete quantum subgroup which is such that $\mathbb{H}_q$ is monoidally equivalent to $\mathbb{G} \wr_* S_N^+$. This is done in three different situations: a) when $\mathbb{G}$ is a free orthogonal quantum group, b) when $\mathbb{G}$ is a free quantum group and c) when $\mathbb{G} := \mathbb{F}_n$ is the classical free group on n generators. The whole content of Section 4.2 is a collaboration work with A. Freslon [127].

4.1 Strategies for K-theory computations

The work of C. Voigt and his collaborators [140], [208], [210], [211], [212] have been very fruitful with respect to the K-theory computations of C^*-algebras associated to compact quantum groups. Let us explain the general strategy used by C. Voigt to this end.

The categorical framework adopted by R. Meyer and R. Nest in their approach for the Baum-Connes property is also a powerful tool for the K-group computations. The notion of *spectral sequence* coming from algebraic topology and algebraic geometry turns out to be a very useful method to compute homology groups. In this way, the general idea in the Meyer-Nest's work is to develop an adapted homological algebra for triangulated categories with which we can establish appropriated spectral sequences. Once we restrict ourselves to the Kasparov category, these spectral sequences reveal a strong connexion with the KK-groups. Nevertheless, it is important to say that using spectral sequences in full generality in this context is still a very hard problem to solve for doing explicit computations. But in some concrete situations, spectral sequences restrict to much more handle sequences, which allows such explicit computations.

In order to understand what follows, Section 1.2 and specially Section 1.2.4 contain the main material that we need.

First of all, we have already explained that, for the moment, a quantum version of the Baum-Connes property must be formulated for *discrete* quantum groups. Likewise, all along in this dissertation we have stressed that the *torsion phenomena* in the quantum setting is one of the main obstacles in order to formulate properly a quantum version of the Baum-Connes property. Recall that, doing the comparison with the classical discrete case, the main problem comes from the definition of the localizing subcategory $\mathscr{L}$. Indeed, we must be careful in the manipulation of the induction functor.

Although we have already noticed in the end of Section 1.7.2 that some typical examples of compact quantum groups have discrete duals that fail to be torsion-free, the corresponding K-group computations may avoid this shortcoming.

In this way, if $\mathbb{G}$ is a compact quantum group, we shall distinguish two main complementary situations for the K-group computations of $C(\mathbb{G})$: $\widehat{\mathbb{G}}$ *is torsion-free* and $\widehat{\mathbb{G}}$ *is* not *torsion-free*.

4.1.1 Torsion-free discrete quantum group case

The "algorithm" for the K-theory computations is the following.

i) We show that $\widehat{\mathbb{G}}$ satisfies the *strong* Baum-Connes property. This means that $\mathscr{L}_{\widehat{\mathbb{G}}} = \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$. For this we may use different techniques. For instance, Poincaré duality and Dirac-dual Dirac method is used for the quantum free product case or the maximal torus argument is used for $\widehat{SU_q(2)}$ (and also for duals of connected compact Lie groups).

ii) Since $\widehat{\mathbb{G}}$ is torsion-free and it satisfies the *strong* Baum-Connes property, we know that $\mathbb{G}$ is automatically co-K-amenable as explained in Remark 1.7.2.11. This means in particular that $C_m(\mathbb{G})$ is isomorphic to $C_r(\mathbb{G})$ at the level of the K-theory, so that we do not make any difference between these C^*-algebras in the corresponding Kasparov category and we write simply $C(\mathbb{G})$.

By virtue of the K-amenability characterization of J. Cuntz, which is still true in the quantum setting (see Remark 1.3.1.41), we know that there doesn't exist any difference between reduced and maximal crossed product at the level of the K-theory, so that we write simply $\widehat{\mathbb{G}} \ltimes \ \cdot$ for both the reduced and the maximal crossed product functor.

iii) We consider the homological ideal $\mathcal{J} := \ker_{Hom}(Res_{\mathbb{E}}^{\widehat{\mathbb{G}}})$ and we construct a $\mathcal{J}$-projective resolution of *length* 1 for $\mathbb{C}$ in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$, say

$$0 \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} \mathbb{C} \longrightarrow 0$$

4.1.1.1 Remark. On the one hand, since $\widehat{\mathbb{G}}$ is supposed to be *torsion-free*, $\mathscr{L}_{\widehat{\mathbb{G}}}$ is described as the localizing subcategory of $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ generated by the objects of the form $c_0(\widehat{\mathbb{G}}) \otimes C$ with C any C^*-algebra in the Kasparov category $\mathscr{K}\mathscr{K}$. In particular, $c_0(\widehat{\mathbb{G}}) \otimes \mathbb{C} \cong c_0(\widehat{\mathbb{G}}) \in \mathscr{L}_{\widehat{\mathbb{G}}}$.

On the other hand, by virtue of Theorem 1.7.2.6, we know that $\mathscr{L}_{\widehat{\mathbb{G}}}$ is actually the localizing subcategory of $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ generated by *$\mathcal{J}$-projective objects.*

In this way, the most obvious candidates for the $\mathcal{J}$-projective resolution above are $P_0 := c_0(\widehat{\mathbb{G}})$ and $P_1 := c_0(\widehat{\mathbb{G}})$. Otherwise, we may take suitably modifications of these P_0 and P_1 depending on the concrete situation.

Notice by the way that, given an irreducible representation $x \in Irr(\mathbb{G})$, there exists an obvious map from $c_0(\widehat{\mathbb{G}})$ to itself in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ induced by the co-multiplication $\widehat{\Delta}$. More precisely, we have the following composition

$$T_x : c_0(\widehat{\mathbb{G}}) \xrightarrow{\widehat{\Delta}} M(c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{G}})) \xrightarrow{id \otimes p_x} M(c_0(\widehat{\mathbb{G}}) \otimes \mathcal{B}(H_x)) \cong c_0(\widehat{\mathbb{G}}),$$

where the last identification is the evident Morita equivalence. Moreover, it is straightforward to see that the corresponding induced map at the level of the K-theory, $(T_x)_* : K_0(c_0(\widehat{\mathbb{G}})) \longrightarrow K_0(c_0(\widehat{\mathbb{G}}))$, identifies with the right multiplication by $\overline{x}$ under the usual identification $K_0(c_0(\widehat{\mathbb{G}})) \cong R(\mathbb{G})$. So we put $r_{\overline{x}} := (T_x)_*$, for all $x \in Irr(\mathbb{G})$.

Likewise, the left regular representation of $\widehat{\mathbb{G}}$, $\widehat{\lambda} : c_0(\widehat{\mathbb{G}}) \longrightarrow \mathcal{K}(L^2(\mathbb{G}))$, yields a map in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ between $c_0(\widehat{\mathbb{G}})$ and $\mathbb{C}$ (notice that $\mathcal{K}(L^2(\mathbb{G})) \cong \mathbb{C}$ in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$). Moreover, we recall that the $R(\mathbb{G})$-module structure on $\mathbb{Z}$ is induced by the dimension function. In other words, it is straightforward to see that the corresponding induced map at the level of the K-theory, $\widehat{\lambda}_* : K_0(c_0(\widehat{\mathbb{G}})) \longrightarrow K_0(\mathbb{C})$ is given by the dimension function ε, which is such that $\varepsilon(x) = dim(x)$ for all $x \in Irr(\mathbb{G})$ under the usual identification $K_0(c_0(\widehat{\mathbb{G}})) \cong R(\mathbb{G})$ and $K_0(\mathbb{C}) \cong \mathbb{Z}$. So we put $\varepsilon := \widehat{\lambda}_*$, for all $x \in Irr(\mathbb{G})$.

These identifications are in accordance with Remark 1.7.1.23.

iv) Observe that, by virtue of the adjointness between induction and restriction functors (recall Lemma 1.7.2.4), we have that $KK^{\widehat{\mathbb{G}}}(A, B) = (0)$, for all $A \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}})$ and for all $B \in \mathscr{N}_{\widehat{\mathbb{G}}}$. Indeed, since $\widehat{\mathbb{G}}$ satisfy the strong Baum-Connes property by step (i), it is enough to check this for $\widehat{\mathbb{G}}$-C^*-algebras $A \in \mathscr{L}_{\widehat{\mathbb{G}}}$. Hence, we write

$$KK^{\widehat{\mathbb{G}}}(Ind_{\mathbb{E}}^{\widehat{\mathbb{G}}}(B), A) = KK(B, Res_{\mathbb{E}}^{\widehat{\mathbb{G}}}(A)) = (0),$$

for all $B \in \mathscr{N}_{\widehat{\mathbb{G}}}$.

Observe that, thanks to the *strong* Baum-Connes property, this is also true for $A := \mathbb{C} \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}})$.

v) Step (iii) and the observation of step (iv) allow to apply Theorem 1.2.4.9. Hence, we know that there exists a short exact sequence

$$\mathbb{L}_0 F(\Sigma^i(\mathbb{C})) \hookrightarrow F(\Sigma^i(\mathbb{C})) \twoheadrightarrow \mathbb{L}_1 F(\Sigma^{i+1}(\mathbb{C})),$$

where $F : \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \longrightarrow \mathscr{A}b^{\mathbb{Z}/2}$ is the homological functor given by $F(A) := K_*(\widehat{\mathbb{G}} \underset{r}{\ltimes} A)$, for all $A \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}})$.

In the Kasparov category $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ and thanks to Bott periodicity, this sequence is precisely the following six-terms exact sequence,

$$\begin{tikzcd}
K_0(\widehat{\mathbb{G}} \ltimes P_1) \arrow[r, "K_0(\widehat{\mathbb{G}} \ltimes \delta_1)"] & K_0(\widehat{\mathbb{G}} \ltimes P_0) \arrow[r, "K_0(\widehat{\mathbb{G}} \ltimes \delta_0)"] & K_0(C(\mathbb{G})) \arrow[d] \\
K_1(C(\mathbb{G})) \arrow[u] & K_1(\widehat{\mathbb{G}} \ltimes P_0) \arrow[l, "K_1(\widehat{\mathbb{G}} \ltimes \delta_0)"] & K_1(\widehat{\mathbb{G}} \ltimes P_1) \arrow[l, "K_1(\widehat{\mathbb{G}} \ltimes \delta_1)"]
\end{tikzcd}$$

vi) This may yield the computation of $K_0(C(\mathbb{G}))$ and $K_1(C(\mathbb{G}))$ since P_0 and P_1 shall be chosen in such a way that we know the K-groups of the crossed products $\widehat{\mathbb{G}} \ltimes P_0$ and $\widehat{\mathbb{G}} \ltimes P_1$. Indeed, recall Remark 4.1.1.1 above.

vii) It is important to observe the following. Sometimes, it seems that working in the *"compact category"* $\mathscr{K}\mathscr{K}^{\mathbb{G}}$ (doing Baaj-Skandalis duality) is more convenient. In this case, we may modify step (iii) above by constructing a $\mathcal{I}$-projective resolution of length 1 for $C(\mathbb{G})$ for an appropriated homological ideal $\mathcal{I}$ in $\mathscr{K}\mathscr{K}^{\mathbb{G}}$. Therefore, we obtain an analogous 6-terms exact sequence as above by applying directly Theorem 1.2.4.9 with the *K-theory functor*.

4.1.2 Torsion discrete quantum group case

The "algorithm" for the K-theory computations is the following.

i) We carry out a complete classification of the torsion for $\widehat{\mathbb{G}}$. Let us denote by $\mathbb{T}or(\widehat{\mathbb{G}})$ the set of all torsion actions of $\mathbb{G}$. Recall that a torsion action of $\mathbb{G}$ is the pair (A, δ), where A is a unital C^*-algebra and δ is an ergodic action of $\mathbb{G}$ on A.

ii) By analogy with the classical locally compact case, we may re-define the corresponding localizing subcategory $\langle \mathcal{CI}_{Qut.} \rangle$ by adding the $\widehat{\mathbb{G}}$-C^*-algebras arising from the torsion phenomena studied in step (i). More precisely, we may put

$$\mathscr{L}_{\mathbb{G}} := \widehat{\mathscr{L}_{\widehat{\mathbb{G}}}} := \langle \widehat{\mathcal{CI}}_{Qut.} \rangle := \langle \{T \otimes B \text{ with } T \in \mathbb{T}or(\widehat{\mathbb{G}}) \text{ and } B \in Obj(\mathscr{K}\mathscr{K})\} \rangle,$$

whose Baaj-Skandalis dual is denoted by $\mathscr{L}_{\widehat{\mathbb{G}}} := \langle \mathcal{CI}_{Qut.} \rangle$.

4.1 Remark. Observe that this definition is consistent with the definitions and constructions of Section 1.7.2 for the torsion-free case. Indeed, assume that $\widehat{\mathbb{G}}$ is a torsion-free discrete quantum group. In this case, the unique torsion action of $\mathbb{G}$ is the trivial one. Hence $\mathbb{T}or(\widehat{\mathbb{G}}) = \{\mathbb{C}\}$ and the above definition yields

$$\begin{aligned}\langle \widehat{\mathcal{CI}}_{Qut.} \rangle &= \langle \{\mathbb{C} \otimes B \text{ with the trivial action of } \mathbb{G} \text{ on } \mathbb{C} \text{ and } B \in Obj(\mathscr{K}\mathscr{K})\}\rangle \\ &\cong \langle \{B \in Obj(\mathscr{K}\mathscr{K}^{\mathbb{G}}) \mid B \text{ with trivial action of } \mathbb{G}\}\rangle,\end{aligned}$$

which corresponds exactly to $\langle \mathcal{CI}_{Qut.} \rangle = \langle \{c_0(\widehat{\mathbb{G}}) \otimes B \text{ with } B \in Obj(\mathscr{K}\mathscr{K})\}\rangle$ by Baaj-Skandalis duality (recall Theorem 1.7.1.20 and Proposition 1.7.2.3).

iii) We show that $\widehat{\mathbb{G}}$ satisfies the *strong* Baum-Connes property. This means that $\mathscr{L}_{\widehat{\mathbb{G}}} = \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$. Notice that from a theoretical point of view, we do not have any right to speak about *"quantum Baum-Connes property"* for the torsion case. Nevertheless, we can re-define the *strong* Baum-Connes property for this case by requiring the abstract condition $\mathscr{L}_{\widehat{\mathbb{G}}} = \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$, where $\mathscr{L}_{\widehat{\mathbb{G}}}$ depends now on the torsion phenomena of $\widehat{\mathbb{G}}$ as established in step (ii).

For proving this, we shall have two possibilities

a) either we prove it directly using adapted techniques coming from the torsion-free case

b) or we prove it using *monoidal equivalences*. Indeed, we have already explained in Theorem B.3.19 that torsion actions of monoidallly equivalent discrete quantum groups are in bijective correspondence. In this way, if $\widehat{\mathbb{F}}$ is a discrete quantum group that is monoidally equivalent to $\widehat{\mathbb{G}}$, then we have

$$\mathbb{T}or(\widehat{\mathbb{G}}) \cong \mathbb{T}or(\widehat{\mathbb{F}}) \text{ and } \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \cong \mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$$

If $\widehat{\mathbb{F}}$ is "well-related" with an other discrete quantum quantum group for which we know the strong Baum-Connes property, the monoidal equivalence may yield the property for $\widehat{\mathbb{G}}$ itself. For instance, this is the situation for the *quantum automorphism group of a finite dimensional C^*-algebra* [212] or for the *free wreath product* as it has been shown in Section 3.7.

iv) Since $\mathscr{L}_{\widehat{\mathbb{G}}}$ takes into account the torsion phenomena of $\widehat{\mathbb{G}}$, Remark 1.7.2.11 can *not* be applied in a straight fashion, so that in the torsion case the K-amenability is not guaranteed *a priori*. However, the notion of *proper almost homogeneous $\widehat{\mathbb{G}}$-C^*-algebra* (see Definition 4.5 and Lemma 4.6 in [212] for more details) allows to generalize Remark 1.7.2.11 in the torsion case as it has been shown in Theorem 3.7.3.5.

In other words, our definition of the subcategory $\mathscr{L}_{\widehat{\mathbb{G}}}$ and thus the corresponding definition of *strong* Baum-Connes property guarantees the K-amenability property whenever $\widehat{\mathbb{G}}$ satisfies the *strong* Baum-Connes property.

Therefore, we write simply $C(\mathbb{G})$ for both the reduced and the maximal C^*-algebra associated to $\mathbb{G}$. Likewise, we write simply $\widehat{\mathbb{G}} \ltimes \cdot$ for both the reduced and the maximal crossed product functor.

v) We consider a homological ideal $\mathcal{J}$ and we construct a $\mathcal{J}$-projective resolution of *length* 1 for $\mathbb{C}$ in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$, say

$$0 \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} \mathbb{C} \longrightarrow 0$$

4.1.2.1 Remarks. 1. Notice that, because of the torsion of $\widehat{\mathbb{G}}$, we can not do an obvious choice of the homological ideal $\mathcal{J}$. Its definition depends thus on the concrete situation in which we are working.

2. Notice that, because of the torsion of $\widehat{\mathbb{G}}$, we can not do an obvious choice of the $\mathcal{J}$-projective objects P_0 and P_1. However, following Remark 4.1.1.1 and the definition of the subcategory $\mathscr{L}_{\widehat{\mathbb{G}}}$ in step (ii), the reasonable choice should be a suitable combination of $c_0(\widehat{\mathbb{G}})$ and objects $T \in \mathbb{T}or(\widehat{\mathbb{G}})$. For instance this is the situation for the *quantum automorphism group of a finite dimensional C^*-algebra* [212].

   Nevertheless, remark that we must check that such combinations defining P_0 and P_1 are actually $\mathcal{J}$-projective objects (in terms of the choice of the homological ideal $\mathcal{J}$). In other words, we have to stress that in the torsion case we don't have *a priori* an analogue of Theorem 1.7.2.6, so that we don't have *a priori* that $\mathscr{L}_{\widehat{\mathbb{G}}}$ is generated by *$\mathcal{J}$-projective objects*.

vi) We prove that $KK^{\widehat{\mathbb{G}}}(A, B) = (0)$, for all $A \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}})$ and for all $\mathcal{J}$-contractible object B in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$.

   Notice that this step is automatically fulfilled in the torsion-free case thanks to the adjointness between induction and restriction functors and the obvious choice of the homological ideal $\mathcal{J}$.

vii) Step (v) and step (vi) allow to apply Theorem 1.2.4.9. Hence, we know that there exists a short exact sequence

$$\mathbb{L}_0 F(\Sigma^i(\mathbb{C})) \hookrightarrow F(\Sigma^i(\mathbb{C})) \twoheadrightarrow \mathbb{L}_1 F(\Sigma^{i+1}(\mathbb{C})),$$

   where $F : \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \longrightarrow \mathscr{A}b^{\mathbb{Z}/2}$ is the homological functor given by $F(A) := K_*(\widehat{\mathbb{G}} \underset{r}{\ltimes} A)$, for all $A \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}})$.

   In the Kasparov category $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$ and thanks to Bott periodicity, this sequence is precisely the following 6-terms exact sequence,

$$\begin{tikzcd}
K_0(\widehat{\mathbb{G}} \ltimes P_1) \arrow[r, "K_0(\widehat{\mathbb{G}} \ltimes \delta_1)"] & K_0(\widehat{\mathbb{G}} \ltimes P_0) \arrow[r, "K_0(\widehat{\mathbb{G}} \ltimes \delta_0)"] & K_0(C(\mathbb{G})) \arrow[d] \\
K_1(C(\mathbb{G})) \arrow[u] & K_1(\widehat{\mathbb{G}} \ltimes P_0) \arrow[l, "K_1(\widehat{\mathbb{G}} \ltimes \delta_0)"] & K_1(\widehat{\mathbb{G}} \ltimes P_1) \arrow[l, "K_1(\widehat{\mathbb{G}} \ltimes \delta_1)"]
\end{tikzcd}$$

viii) This may yield the computation of $K_0(C(\mathbb{G}))$ and $K_1(C(\mathbb{G}))$ since P_0 and P_1 shall be chosen in such a way that we know the K-groups of the crossed products $\widehat{\mathbb{G}} \ltimes P_0$ and $\widehat{\mathbb{G}} \ltimes P_1$. Indeed, recall Remark 4.1.2.1 above and observe that now we have to know in addition the K-theory of $\mathbb{T}or(\widehat{\mathbb{G}})$.

ix) It is important to observe the following. Sometimes, it seems that working in the *"compact category"* $\mathscr{K}\mathscr{K}^{\mathbb{G}}$ (doing Baaj-Skandalis duality) is more convenient. In this case, we may

modify step (v) above by constructing a $\mathcal{I}$-projective resolution of length 1 for $C(\mathbb{G})$ for an appropriated homological ideal $\mathcal{I}$ in $\mathscr{K}\mathscr{K}^{\mathbb{G}}$. Therefore, we obtain an analogous 6-terms exact sequence as above by applying directly Theorem 1.2.4.9 with the K-*theory functor*.

From the preceding panorama we may conclude the following. The *torsion-free* case is well-understood up to natural choices of the $\mathcal{J}$-projective resolution for $\mathbb{C}$. However, the *torsion* case presents deep difficulties since we have to adapt the general theoretical framework in order to give a meaning to the *"quantum Baum-Connes property"*. In addition, the torsion phenomena yields new C^*-algebras for which neither there exists a natural choice for the $\mathcal{J}$-projective resolution for $\mathbb{C}$ nor the corresponding K-groups are known *a priori*.

For this reason, even if we solve the theoretical issue for a general formulation of a *quantum Baum-Connes property*, the study of the torsion phenomena remains a central stage that we can not avoid for the K-theory computations.

The general torsion-free-strategy has been successfully applied for free quantum groups [208] and for $SU_q(2)$ [210] obtaining in particular the following K-groups

$$K_0(C(U^+(n))) = \mathbb{Z} \text{ and } K_1(C(U^+(n))) = \mathbb{Z} \oplus \mathbb{Z}$$

$$K_0(C(O^+(n))) = \mathbb{Z} \text{ and } K_1(C(O^+(n))) = \mathbb{Z}$$

Likewise, the general torsion-strategy has been successfully applied in a ingenious way for a quantum automorphism group of a finite dimensional C^*-algebra [212] obtaining in particular the following K-groups

$$K_0(C(S_N^+)) = \mathbb{Z}^{N^2-2N+2} \text{ and } K_1(C(S_N^+)) = \mathbb{Z}$$

4.2 The *Lemeux-Tarrago's* $\widehat{\mathbb{H}}_q < \mathbb{G} * \widehat{SU_q}(2)$

First of all, the whole content of this section is a collaboration work with A. Freslon [127]. We will follow the same notations from Section 2.6 and Section 3.7.

The main motivation for this work has been the computation of the K-theory of the C^*-algebra defining a free wreath product $\mathbb{G} \wr_* S_N^+$, where $\mathbb{G}$ is a compact quantum group and $N \geqslant 4$ is a natural number. However, the research carried out in [127] has not shed light on how to apply the methods described in the preceding section in order to perform such a computation. To the best knowledge of the author, the cohomological dimension of a free wreath product (see [21] for more details) is not known yet, so that it is reasonable to expect that the K-theory of $C^*(\mathbb{G} \wr_* S_N^+)$ can be computed by means of a spectral sequence (that is to say, *it is not enough to construct projective resolutions of length* 1), which differs from the examples known up to the present.

Despite this setback, we can work with the Lemeux-Tarrago's compact quantum subgroup $\mathbb{H}_q$, which is monidally equivalent to the free wreath product $\mathbb{G} \wr_* S_N^+$. The reason for this is that $\widehat{\mathbb{H}}_q$ is a discrete quantum subgroup of $\mathbb{G} * \widehat{SU_q}(2)$, which is the dual of a free product of compact quantum

groups. Hence, our strategy consists in combining the work [208] by R. Vergnioux and C. Voigt with the work [211] by C. Voigt.

Let us be more precise. Let $\mathbb{G}$ be a compact quantum group such that $\widehat{\mathbb{G}}$ is torsion-free and satisfies the *strong* Baum-Connes conjecture. Put $\mathbb{F} := \mathbb{G} * SU_q(2)$. We shall begin by particularizing the steps described in Section 4.1.1 and Section 4.1.2 using the results from Section 3.7.

i) The torsion for the dual of the free wreath product $\mathbb{G} \wr_* S_N^+$ is completely classified. Hence, the torsion for $\widehat{\mathbb{H}}_q$ too by monoidal equivalence. In particular, the only, up to equivariant Morita equivalence, torsion action of $\mathbb{H}_q$ is given by $(\mathcal{M}_2(\mathbb{C}), \alpha_q)$ (see Remark 3.7.1.3).

ii) The dual of the free wreath product $\mathbb{G} \wr_* S_N^+$ satisfies the *strong* Baum-Connes property. Hence, $\widehat{\mathbb{H}}_q$ too by monoidal equivalence (see Theorem 3.7.2.6). In particular, $\mathscr{L}_{\widehat{\mathbb{H}}_q} = \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}_q}$, where

$$\mathscr{L}_{\widehat{\mathbb{H}}_q} := \langle \{\mathbb{H}_q \underset{r}{\ltimes} \mathcal{M}_2(\mathbb{C}) \otimes B \text{ with } B \in Obj(\mathscr{K}\mathscr{K})\} \cup \{c_0(\widehat{\mathbb{H}}_q) \otimes B \text{ with } B \in Obj(\mathscr{K}\mathscr{K})\} \rangle$$

iii) The dual of the free wreath product $\mathbb{G} \wr_* S_N^+$ is K-amenable. Hence, $\widehat{\mathbb{H}}_q$ too by monoidal equivalence (see Corollary 3.7.3.6). For this reason we write simply $C(\mathbb{H}_q)$ for both the reduced and the maximal C^*-algebra associated to $\mathbb{H}_q$. Likewise, we write simply $\mathbb{H}_q \ltimes \ \cdot$ for both the reduced and the maximal crossed product functor.

iv) Since $\widehat{\mathbb{G}}$ is torsion-free by assumption and $\widehat{SU_q(2)}$ is torsion-free (recall Theorem 2.1.5), then $\widehat{\mathbb{F}} = \mathbb{G} * \widehat{SU_q}(2)$ is still torsion-free by Theorem 3.6.1.1. In this situation, let us consider the usual homological ideal $\mathcal{J} := \ker_{Hom}\left(Res_{\mathbb{E}}^{\mathbb{G} * \widehat{SU_q}(2)}\right)$. Assume that we have a $\mathcal{J}$-projective resolution of length 1 for $\mathbb{C}$ in $\mathscr{K}\mathscr{K}^{\mathbb{G} * \widehat{SU_q}(2)}$, say

$$0 \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} \mathbb{C} \longrightarrow 0$$

Moreover, $\widehat{\mathbb{G}}$ satisfies the *strong* Baum-Connes property by assumption and $\widehat{SU_q(2)}$ too (recall Theorem 2.1.5). Hence $\widehat{\mathbb{F}} = \mathbb{G} * \widehat{SU_q}(2)$ satisfies the *strong* Baum-Connes property by Theorem 3.6.2.3. So, by virtue of Theorem 1.2.4.9 there exists a distinguished triangle in $\mathscr{K}\mathscr{K}^{\mathbb{G} * \widehat{SU_q}(2)}$, which can be written, up to $\mathcal{J}$-isomorphism, under the form

$$P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} \mathbb{C} \longrightarrow \Sigma(P_1)$$

v) Given the discrete quantum group $\widehat{\mathbb{H}}_q < \mathbb{G} * \widehat{SU_q}(2) = \widehat{\mathbb{F}}$, let us restrict the preceding distinguished triangle to a distinguished triangle in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}_q}$,

$$Res_{\widehat{\mathbb{H}}_q}^{\widehat{\mathbb{F}}}(P_1) \xrightarrow{Res(\delta_1)} Res_{\widehat{\mathbb{H}}_q}^{\widehat{\mathbb{F}}}(P_0) \xrightarrow{Res(\delta_0)} \mathbb{C} \longrightarrow \Sigma\Big(Res_{\widehat{\mathbb{H}}_q}^{\widehat{\mathbb{F}}}(P_1)\Big) \tag{4.2.1}$$

vi) Finally, applying the homological functor $F : \mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}_q} \longrightarrow \mathscr{A}b^{\mathbb{Z}/2}$ given by $F(B) := K_*(\widehat{\mathbb{H}}_q \ltimes B)$,

for all $B \in Obj(\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}_q})$ leads to the following six-term exact sequence

$$\begin{tikzcd}
K_0\Big(\widehat{\mathbb{H}}_q \ltimes Res^{\widehat{\mathbb{F}}}_{\widehat{\mathbb{H}}_q}(P_1)\Big) \arrow[r] & K_0\Big(\widehat{\mathbb{H}}_q \ltimes Res^{\widehat{\mathbb{F}}}_{\widehat{\mathbb{H}}_q}(P_0)\Big) \arrow[r] & K_0(C(\mathbb{H}_q)) \arrow[d] \\
K_1(C(\mathbb{H}_q)) \arrow[u] & K_1\Big(\widehat{\mathbb{H}}_q \ltimes Res^{\widehat{\mathbb{F}}}_{\widehat{\mathbb{H}}_q}(P_0)\Big) \arrow[l] & K_1\Big(\widehat{\mathbb{H}}_q \ltimes Res^{\widehat{\mathbb{F}}}_{\widehat{\mathbb{H}}_q}(P_1)\Big) \arrow[l]
\end{tikzcd} \tag{4.2.2}$$

To sum up, our strategy consists in constructing a $\mathcal{J}$-projective resolution of length 1 for $\mathbb{C}$ in $\mathscr{K}\mathscr{K}^{\mathbb{G}*\widehat{SU_q}(2)}$ and then restricting the corresponding distinguished triangle in $\mathscr{K}\mathscr{K}^{\mathbb{G}*\widehat{SU_q}(2)}$ to a distinguished triangle in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}_q}$, which leads a six-term exact sequence by applying the functor $K_*(\widehat{\mathbb{H}}_q \ltimes \ \cdot)$, which allows to compute the K-theory of $C(\mathbb{H}_q)$. This process is plausible because $\mathbb{G} * \widehat{SU_q}(2)$ is torsion-free and satisfies the strong Baum-Connes property. Moreover, for some choices of $\mathbb{G}$ we can give an explicit $\mathcal{J}$-projective resolution by applying the result [208] by R. Vergnioux and C. Voigt. Namely,

a) if $\mathbb{G} := O^+(n)$ is a free orthogonal quantum group with $n \geqslant 2$, then Theorem 7.1 in [208] suggests

$$P_1 := c_0\Big(\widehat{O^+(n) * SU_q(2)}\Big) \oplus c_0\Big(\widehat{O^+(n) * SU_q(2)}\Big) \text{ and } P_0 := c_0\Big(\widehat{O^+(n) * SU_q(2)}\Big)$$

$$\delta_1 := (T_u - dim(u)id) \oplus (T_v - dim(v)id) \text{ and } \delta_0 := \widehat{\lambda},$$

where u denotes de fundamental representation of $SU_q(2)$, v the fundamental representation of $O^+(n)$ and $\widehat{\lambda}$ the left regular representation. Recall that the homomorphisms T_u and T_v were defined in Remark 4.1.1.1.

b) if $\mathbb{G} := U^+(P_1) * \ldots * U^+(P_k) * O^+(Q_1) * \ldots * O^+(Q_l)$ is a free product of free unitary and free orthogonal quantum groups, where $P_i \in GL_{m_i}(\mathbb{C})$ with $m_i \geqslant 2$ for all $i = 1, \ldots, k$ and $Q_j \in GL_{n_j}(\mathbb{C})$ with $n_j \geqslant 2$ satisfies $Q_j\overline{Q}_j = \pm id$ for all $j = 1, \ldots, l$; then Theorem 7.1 in [208] suggests

$$P_1 := \bigoplus_{r=1}^{2k+l+1} c_0\Big(\mathbb{G} * \widehat{SU_q}(2)\Big) \text{ and } P_0 := c_0\Big(\mathbb{G} * \widehat{SU_q}(2)\Big)$$

$$\delta_1 := \bigoplus_{i=1}^{k}\Big((T_{u_i} - dim(u_i)id) \oplus (T_{\overline{u_i}} - dim(u_i)id)\Big) \oplus \bigoplus_{j=1}^{l}\Big((T_{v_j} - dim(v_j)id)\Big) \oplus (T_u - dim(u)id)$$

$$\text{and } \delta_0 := \widehat{\lambda},$$

where u denotes de fundamental representation of $SU_q(2)$, u_i the fundamental representation of $U^+(P_i)$ for every $i = 1, \ldots, k$, v_j the fundamental representation of $O^+(Q_j)$ for every $j = 1, \ldots, l$ and $\widehat{\lambda}$ the left regular representation.

c) if $\mathbb{G} := \mathbb{F}_n$ is the classical free group on n generators, then an appropriated modification of the resolution given in [208] suggests to take

$$P_1 := \bigoplus_{r=1}^{n+1} c_0\Big(\mathbb{F}_n * \widehat{SU_q}(2)\Big) \text{ and } P_0 := c_0\Big(\mathbb{F}_n * \widehat{SU_q}(2)\Big)$$

$$\delta_1 := \bigoplus_{i=1}^{n}(T_{a_i} - id) \ \oplus \ (T_u - 2\ id) \text{ and } \delta_0 := \widehat{\lambda},$$

where u denotes the fundamental representation of $SU_q(2)$, $a_1, \ldots, a_n$ the canonical generators of $\mathbb{F}_n$ and $\widehat{\lambda}$ the left regular representation.

It is important to remark that in this case we must check that the corresponding diagram $P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} \mathbb{C}$ defines actually a $\mathcal{J}$-projective resolution of length 1 for $\mathbb{C}$ in $\mathscr{K}\mathscr{K}^{\mathbb{F}_n * \widehat{SU_q}(2)}$ because we can not apply directly the result [208]. We will be more precise later on.

In the first stage of this process we work with $\mathbb{G} * \widehat{SU_q}(2)$, which is torsion-free and then we only have to restrict the corresponding resolution to $\widehat{\mathbb{H}}_q$. It seems that the torsion phenomena of $\widehat{\mathbb{H}}_q$ does not play any role in this computation. But this is not true at all. Namely, the process of *restriction* of the step (v) above implies the computation of the distinguished triangle (4.2.1), for which the torsion phenomena of $\widehat{\mathbb{H}}_q$ is central. This computation is our next goal.

4.2.1 Note. This *restriction strategy* has already been used by C. Voigt for computing the K-theory of the quantum automorphism group of matrices. See Theorem 5.2 in [211] for more details.

4.2.1 Preliminary computations

Given the quantum free product $\mathbb{F} := \mathbb{G} * SU_q(2)$, recall from Section 2.5 that $c_0(\mathbb{G} * \widehat{SU_q}(2)) = \bigoplus_{y \in Irr(\mathbb{G} * SU_q(2))}^{c_0} \mathcal{B}(H_y)$, where $Irr(\mathbb{G} * SU_q(2)) = Irr(\mathbb{G}) * Irr(SU_q(2))$. Denote by u the fundamental representation of $SU_q(2)$, which can be viewed as an irreducible representation of the quantum free product, $u = \epsilon_{\mathbb{G}} * u$. Denote by p_u the minimal central projection of $c_0(\mathbb{G} * \widehat{SU_q}(2))$ on $\mathcal{B}(H_u)$. As in Remark 4.1.1.1 we consider the following composition

$$T_u : c_0(\widehat{\mathbb{F}}) \xrightarrow{\widehat{\Theta}} M(c_0(\widehat{\mathbb{F}}) \otimes c_0(\widehat{\mathbb{F}})) \xrightarrow{id \otimes p_u} M(c_0(\widehat{\mathbb{F}}) \otimes \mathcal{B}(H_u)) \cong c_0(\widehat{\mathbb{F}}),$$

where the last identification is the canonical Morita equivalence and Θ denotes the co-multiplication of the quantum free product $\mathbb{F} = \mathbb{G} * SU_q(2)$. This composition is a homomorphism in $\mathscr{K}\mathscr{K}^{\mathbb{G} * \widehat{SU_q}(2)}$, so that it is viewed as an equivariant Kasparov triple $T_u \in KK^{\widehat{\mathbb{F}}}(c_0(\widehat{\mathbb{F}}), c_0(\widehat{\mathbb{F}}))$. We can restrict this element to a homomorphism in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}_q}$ because $\widehat{\mathbb{H}}_q < \widehat{\mathbb{F}}$. By abuse of notation, the element $Res^{\widehat{\mathbb{F}}}_{\widehat{\mathbb{H}}_q}(T_u) \in KK^{\widehat{\mathbb{H}}_q}(c_0(\widehat{\mathbb{F}}), c_0(\widehat{\mathbb{F}}))$ is still denoted by T_u.

More precisely, the equivariant Kasparov triple T_u is defined by the Hilbert $c_0(\widehat{\mathbb{F}})$-module $H := c_0(\widehat{\mathbb{F}}) \otimes H_u$ with right action of $\widehat{\mathbb{F}}$ given simply by multiplication of the first tensor, that is, the action $\delta_H : H \longrightarrow \widetilde{M}(H \otimes c_0(\widehat{\mathbb{F}}))$ is such that $\delta_H(a \otimes \eta)(b) = a \cdot b \otimes \eta$, for all $a, b \in c_0(\widehat{\mathbb{F}})$ and all $\eta \in H_u$. Moreover, the $*$-homomorphism $(id \otimes p_u) \circ \widehat{\Theta} : c_0(\widehat{\mathbb{F}}) \longrightarrow M(c_0(\widehat{\mathbb{F}}) \otimes \mathcal{B}(H_u)) = \mathcal{L}_{c_0(\widehat{\mathbb{F}})}(H)$

defines an equivariant representation of $c_0(\widehat{\mathbb{F}})$ on H. Recall Section 1.7.1 for the general definitions of equivariant Kasparov triples in the quantum setting.

Consider the descent homomorphism with respect to $\widehat{\mathbb{H}}_q$,

$$j_{\widehat{\mathbb{H}}_q} : KK^{\widehat{\mathbb{H}}_q}(c_0(\widehat{\mathbb{F}}), c_0(\widehat{\mathbb{F}})) \longrightarrow KK(\widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{F}}), \widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{F}}))$$

We want to understand the image of T_u under $j_{\widehat{\mathbb{H}}_q}$. For this, we must describe $c_0(\widehat{\mathbb{F}})$ as a $\widehat{\mathbb{H}}_q$-C^*-algebra. In other words, we are going to compute $Res^{\widehat{\mathbb{F}}}_{\widehat{\mathbb{H}}_q}\Big(c_0(\widehat{\mathbb{F}})\Big)$. Let $\omega \in Irr(\mathbb{G}) * Irr(SU_q(2)) = Irr(\mathbb{F})$ be a word with letters in $Irr(\mathbb{G}) \sqcup Irr(SU_q(2))$ which is

- either empty
- or *starts* in $Irr(\mathbb{G})$.

For instance, the fundamental representation u is a plausible word ω in the sense of the preceding definition. Given any word ω as above, we define the following subset of $Irr(\mathbb{F})$,

$$J_\omega := \{y \in Irr(\mathbb{F}) \mid y \subset h \oplus \omega \text{ with } h \in Irr(\mathbb{H}_q)\}$$

Notice that $J_\varnothing = Irr(\mathbb{H}_q)$ by definition. Given any word ω as above, we define the following C^*-algebra,

$$A_\omega := \bigoplus_{y \in J_\omega}^{c_0} \mathcal{B}(H_y)$$

Let us give two important observations.

- Given a word ω as above, we write

$$\begin{aligned} c_0(\widehat{\mathbb{H}}_q) \otimes \mathcal{B}(H_\omega) &= \Big(\bigoplus_{h \in Irr(\mathbb{H}_q)}^{c_0} \mathcal{B}(H_h)\Big) \otimes \mathcal{B}(H_\omega) = \bigoplus_{h \in Irr(\mathbb{H}_q)}^{c_0} \mathcal{B}(H_h \otimes H_\omega) \\ &\cong \bigoplus_{y \subset h \oplus \omega}^{c_0} \mathcal{B}(H_y) = \bigoplus_{y \in J_\omega}^{c_0} \mathcal{B}(H_y) = A_\omega, \end{aligned}$$

  which implies that A_ω is equivariantly Morita equivalent to $c_0(\widehat{\mathbb{H}}_q)$. Consequently, Baaj-Skandalis duality yields the following isomorphism in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{H}}_q}$

$$\widehat{\mathbb{H}}_q \ltimes A_\omega \cong \mathbb{C} \otimes \mathcal{K}(L^2(\mathbb{H}_q)),$$

  for all word ω as above.

- Since $\widehat{\mathbb{H}}_q$ is a discrete quantum subgroup of $\widehat{\mathbb{F}} = \mathbb{G} * \widehat{SU_q}(2)$, then we know from Proposition 1.4.3.4 that

$$c_0(\widehat{\mathbb{F}}) = \Big(\bigoplus_{h \in Irr(\mathbb{H}_q)}^{c_0} \mathcal{B}(H_h)\Big) \oplus \Big(\bigoplus_{y \in Irr(\mathbb{F}) \setminus Irr(\mathbb{H}_q)}^{c_0} \mathcal{B}(H_y)\Big)$$

But we have $J_\varnothing = Irr(\mathbb{H}_q)$, so that $Irr(\mathbb{F})\backslash Irr(\mathbb{H}_q) = \bigoplus_{\omega\neq\varnothing} J_\omega$ because $\mathbb{H}_q$ is generated, by definition, by the words of the form uxu with $x \in Irr(\mathbb{G})$ (recall Theorem 2.6.2). Hence we write

$$\begin{aligned} c_0(\widehat{\mathbb{F}}) &= \Big(\bigoplus_{y\in J_\varnothing}^{c_0} \mathcal{B}(H_y)\Big) \oplus \Big(\bigoplus_{\omega\neq\varnothing}\Big(\bigoplus_{y\in J_\omega}^{c_0} \mathcal{B}(H_y)\Big)\Big) = A_\varnothing \oplus \Big(\bigoplus_{\substack{\omega\neq\varnothing\\ \omega\neq u}} A_\omega\Big) \oplus A_u \\ &= A_u \oplus \Big(\bigoplus_{\omega\neq u} A_\omega\Big) \end{aligned} \tag{4.2.3}$$

In other words, we have obtained that $Res^{\widehat{\mathbb{F}}}_{\widehat{\mathbb{H}}_q}\Big(c_0(\widehat{\mathbb{F}})\Big) = A_u \oplus \Big(\bigoplus_{\omega\neq u} A_\omega\Big)$. By abuse of notation we still denote by $c_0(\widehat{\mathbb{F}})$ this restriction. Consequently, Baaj-Skandalis duality yields the following isomorphism in $\mathscr{KK}^{\widehat{\mathbb{H}}_q}$

$$\widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{F}}) \cong \mathbb{C}\otimes\mathcal{K}(L^2(\mathbb{H}_q)) \oplus \Big(\bigoplus_{\omega\neq u} \mathbb{C}\otimes\mathcal{K}(L^2(\mathbb{H}_q))\Big),$$

which, at the level of K-theory, yields

$$K_0(\widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{F}})) = \mathbb{Z}_u \oplus \Big(\bigoplus_{\omega\neq u} \mathbb{Z}_\omega\Big) \text{ and } K_1(\widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{F}})) = (0),$$

where $\mathbb{Z}_u$ denotes the copy of $\mathbb{Z}$ corresponding to A_u and $\mathbb{Z}_\omega$ denotes the copy of $\mathbb{Z}$ corresponding to A_ω with $\omega \neq u$. Let us denote by e_u and by e_ω the unit of $\mathbb{Z}_u$ and $\mathbb{Z}_\omega$, respectively.

The description of $c_0(\widehat{\mathbb{F}})$ as a $\widehat{\mathbb{H}}_q$-C^*-algebra given in (4.2.3) yields that the element $T_u \in KK^{\widehat{\mathbb{H}}_q}(c_0(\widehat{\mathbb{F}}), c_0(\widehat{\mathbb{F}}))$ *splits as a direct sum*, so that we can study each summand separately. With the same notations as above, we can compute the element $j_{\widehat{\mathbb{H}}_q}(T_u)$ *at the level of K-theory*, which will be denoted in the same way by abuse of notation.

4.2.1.1 Proposition. *The element $\partial_u := j_{\widehat{\mathbb{H}}_q}(T_u)$ acts on the basis as follows*

i) if $\omega \neq \varnothing$ and ω ends in $Irr(\mathbb{G})$, then $\partial_u(e_{\omega u^k}) = e_{\omega u^{k+1}} + e_{\omega u^{k-1}}$,

ii) if $\omega \neq \varnothing$ and ω ends in $Irr(\mathbb{G})$, then $\partial_u(e_\omega) = e_{\omega u}$,

iii) if $\omega = u$, then $\partial_u(e_u) = 2e_\varnothing$,

iv) if $\omega = \varnothing$, then $\partial_u(e_\varnothing) = 2e_u$.

Proof. First of all, remark that given any $a \in \mathcal{B}(H_y)$ with $y \in Irr(\mathbb{F})$, then by construction we have $T_u(a) = (id\otimes p_u)\widehat{\Theta}(a) \in \bigoplus_{y\subset z\oplus u}^{c_0} \mathcal{B}(H_z)\otimes\mathcal{B}(H_u)$. Therefore, we have four possibilities.

i) If $y \in J_{\omega u^k}$ with $\omega \neq \varnothing$ ending in $Irr(\mathbb{G})$, then either $z \in J_{\omega u^{k+1}}$ or $z \in J_{\omega u^{k-1}}$. Indeed, $y \in J_{\omega u^k}$ means that $y \subset h \oplus \omega u^k$, for some $h \in Irr(\mathbb{H}_q)$. Since $y \subset z \oplus u$, then it must be either $z \in J_{\omega u^{k+1}}$ or $z \in J_{\omega u^{k-1}}$ because applying the fusion rules of $SU_q(2)$ we have

$$\text{either } z \subset h \oplus \omega u^{k+1} \Rightarrow y \subset h \oplus \omega u^{k+1} \oplus u = (h \oplus \omega u^k) \oplus (h \oplus \omega u^{k+2}), \text{ for some } h \in Irr(\mathbb{H}_q)$$

$$\text{or } z \subset h \oplus \omega u^{k-1} \Rightarrow y \subset h \oplus \omega u^{k-1} \oplus u = (h \oplus \omega u^{k-2}) \oplus (h \oplus \omega u^k), \text{ for some } h \in Irr(\mathbb{H}_q)$$

Therefore, $\partial_u(e_{\omega u^k}) \in \mathbb{Z}e_{\omega u^{k+1}} \oplus \mathbb{Z}e_{\omega u^{k-1}}$.

ii) If $y \in J_\omega$ with $\omega \neq \varnothing$ ending in $Irr(\mathbb{G})$, then $z \in J_{\omega u}$. Indeed, $y \in J_\omega$ means that $y \subset h \oplus \omega$, for some $h \in Irr(\mathbb{H}_q)$. Since $y \subset z \oplus u$, then it must be $z \in J_{\omega u}$ because applying the fusion rules of $SU_q(2)$ we have

$$z \subset h \oplus \omega u \Rightarrow y \subset h \oplus \omega u \oplus u = (h \oplus \omega) \oplus (h \oplus u^2), \text{ for some } h \in Irr(\mathbb{H}_q)$$

Therefore, $\partial_u(e_\omega) \in \mathbb{Z}e_{\omega u}$.

iii) If $y \in J_u$, then $z \in J_\varnothing$. Indeed, $y \in J_u$ means that $y \subset h \oplus u$, for some $h \in Irr(\mathbb{H}_q)$. Since $y \subset z \oplus u$, then it is clear that it must be $z \in J_\varnothing = Irr(\mathbb{H}_q)$.

Therefore, $\partial_u(e_u) \in \mathbb{Z}e_\varnothing$.

iv) If $y \in J_\varnothing$, then $z \in J_u$. Indeed, $y \in J_\varnothing = Irr(\mathbb{H}_q)$ means that $y = h \in Irr(\mathbb{H}_q)$. Since $y \subset z \oplus u$, then it must be $z \in J_u$ because applying the fusion rules of $SU_q(2)$ we have

$$z \subset h \oplus u \Rightarrow y \subset h \oplus u \oplus u = h \oplus (h \oplus u^2), \text{ with } h := y \in Irr(\mathbb{H}_q)$$

Therefore, $\partial_u(e_\varnothing) \in \mathbb{Z}e_u$.

Let us analyze the first case, so that consider a *non-empty* word ω ending in $Irr(\mathbb{G})$ and consider an irreducible representation $y \in J_{\omega u^k}$. By the above discussion, we know that $T_u(a) \in \bigoplus_{y \subset z \oplus u}^{c_0} \mathcal{B}(H_z) \otimes \mathcal{B}(H_u)$ with either $z \in J_{\omega u^{k+1}}$ or $z \in J_{\omega u^{k-1}}$, for all $a \in \mathcal{B}(H_y)$. Hence, in order to study $\partial_u = j_{\widehat{\mathbb{H}}_q}(T_u)$, we have to understand the image by the descent homomorphism of the Kasparov triple associated to T_u corresponding to the term associated to either ωu^{k+1} or to ωu^{k-1} in the direct sum decomposition of $c_0(\widehat{\mathbb{F}})$ (recall the formula (4.2.3) and that each term A_ω is isomorphic to $c_0(\widehat{\mathbb{H}}_q) \otimes \mathcal{B}(H_\omega)$). In other words, we have to understand either the module

$$\widehat{\mathbb{H}}_q \ltimes \Big(c_0(\widehat{\mathbb{H}}_q) \otimes \mathcal{B}(H_{\omega u^{k+1}}) \otimes H_u\Big) \in KK\Big(\widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{H}}_q) \otimes \mathcal{B}(H_{\omega u^k}), \widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{H}}_q) \otimes \mathcal{B}(H_{\omega u^{k+1}})\Big),$$

equipped with the right action given simply by multiplication on the first two tensors and the representation of $\widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{H}}_q) \otimes \mathcal{B}(H_{\omega u^k})$ given by multiplication for the crossed product and $(p_{\omega u^{k+1}} \otimes p_u) \circ \widehat{\Theta}$ for $\mathcal{B}(H_{\omega u^k})$; or the module

$$\widehat{\mathbb{H}}_q \ltimes \Big(c_0(\widehat{\mathbb{H}}_q) \otimes \mathcal{B}(H_{\omega u^{k-1}}) \otimes H_u\Big) \in KK\Big(\widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{H}}_q) \otimes \mathcal{B}(H_{\omega u^k}), \widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{H}}_q) \otimes \mathcal{B}(H_{\omega u^{k-1}})\Big),$$

equipped with the right action given simply by multiplication on the first two tensors and the representation of $\widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{H}}_q) \otimes \mathcal{B}(H_{\omega u^k})$ given by multiplication for the crossed product and $(p_{\omega u^{k-1}} \otimes p_u) \circ \widehat{\Theta}$ for $\mathcal{B}(H_{\omega u^k})$.

Let us concentrate on the first one and write it in the following way as *bimodule*,

$$_{\widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{H}}_q)\otimes\mathcal{B}(H_{\omega u^k})}\Big(\widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{H}}_q) \otimes \mathcal{B}(H_{\omega u^{k+1}}) \otimes H_u\Big)_{\widehat{\mathbb{H}}_q \ltimes c_0(\widehat{\mathbb{H}}_q)\otimes\mathcal{B}(H_{\omega u^{k+1}})}$$

Notice that the crossed product appearing here is isomorphic to the compact operators by Takesaki-Takai duality, so that we can remove it.

$$_{\mathcal{B}(H_{\omega u^k})}\Big(\mathcal{B}(H_{\omega u^{k+1}}) \otimes H_u\Big)_{\mathcal{B}(H_{\omega u^{k+1}})}$$

Next, remark that $H_{\omega u^{k+1}}$ is a finite dimensional Hilbert space. Hence, $\mathcal{B}(H_{\omega u^{k+1}})$ is Morita equivalent to $\mathbb{C}$ by means of $H_{\omega u^{k+1}}$, so that we write the following

$$\begin{aligned}
_{\mathcal{B}(H_{\omega u^k})}\Big(\mathcal{B}(H_{\omega u^{k+1}}) \otimes H_u\Big)_{\mathcal{B}(H_{\omega u^{k+1}})} &\cong_{\mathcal{B}(H_{\omega u^k})} \Big(H_{\omega u^{k+1}} \underset{\mathcal{B}(H_{\omega u^{k+1}})}{\otimes} \mathcal{B}(H_{\omega u^{k+1}}) \otimes H_u\Big)_{\mathbb{C}} \\
&\cong_{\mathcal{B}(H_{\omega u^k})} \Big(H_{\omega u^{k+1}} \otimes H_u\Big)_{\mathbb{C}},
\end{aligned}$$

where the representation of $\mathcal{B}(H_{\omega u^k})$ is given simply by the embedding of $\mathcal{B}(H_{\omega u^k})$ as a corner in $\mathcal{B}(H_{\omega u^{k+1}} \otimes H_u)$. Next, let us apply Morita equivalence on the left action in analogous way. We have

$$\begin{aligned}
_{\mathcal{B}(H_{\omega u^k})}\Big(H_{\omega u^{k+1}} \otimes H_u\Big)_{\mathbb{C}} &\cong_{\mathbb{C}} \Big(H_{\omega u^k} \underset{\mathcal{B}(H_{\omega u^k})}{\otimes} \big(H_{\omega u^{k+1}} \otimes H_u\big)\Big)_{\mathbb{C}} \\
&\cong_{\mathbb{C}} \Big(H_{\omega u^k} \underset{\mathcal{B}(H_{\omega u^k})}{\otimes} \big(H_{\omega u^k} \oplus H_{\omega u^{k+2}}\big)\Big)_{\mathbb{C}} \\
&=_{\mathbb{C}} \Big(H_{\omega u^k} \underset{\mathcal{B}(H_{\omega u^k})}{\otimes} H_{\omega u^k}\Big)_{\mathbb{C}},
\end{aligned}$$

where for the last equality we remark that $\mathcal{B}(H_{\omega u^k})$ acts by zero on $H_{\omega u^{k+2}}$ (recall that $\mathcal{B}(H_{\omega u^k})$ is embedded as a corner in $\mathcal{B}(H_{\omega u^{k+1}} \otimes H_u) = \mathcal{B}(H_{\omega u^k} \oplus H_{\omega u^{k+2}})$).

In other words, we have obtained that the module corresponding to the term ωu^{k+1} identifies with $1 \in \mathbb{Z} = K_0(\mathbb{C}) = KK(\mathbb{C}, \mathbb{C})$ in the copy $\mathbb{Z}_{\omega u^{k+1}}$, that is, with $e_{\omega u^{k+1}}$. The same computation works for the module corresponding to the term ωu^{k-1}. Hence, by virtue of the above discussion, we have $\partial_u(e_{\omega u^k}) = e_{\omega u^{k+1}} + e_{\omega u^{k-1}}$, as claimed in the statement.

The same computation works for the case (ii) of the statement. For the cases (iii) and (iv) of the statement, it is clear that e_u and $e_\varnothing$ are exchanged. Moreover, the action of ∂_u on $\mathbb{Z}e_\varnothing \oplus \mathbb{Z}e_u$ does not depend on the compact quantum group $\mathbb{G}$, so that it is enough to do the computation when $\mathbb{G}$ is trivial. In that case, we have $\mathbb{H}_q = SO_q(3)$ and the result was proven in [211].

■

4.2.1.2 Lemma. *For a word $\omega \neq \varnothing$ and $\omega \neq u$ which ends in $Irr(\mathbb{G})$ we put*

$$E_\omega := span\{e_{\omega u^k} \mid k \in \mathbb{N}\} \text{ and } E_u := \mathbb{Z}e_\varnothing \oplus \mathbb{Z}e_u$$

Put $d_u := \partial_u - 2\ id$. The following properties hold.

i) The image of E_u by d_u is $2\mathbb{Z}(e_\varnothing - e_u)$.

ii) The image of E_ω by d_u is the free module spanned by the vectors $\xi_k := e_{\omega u^{k+1}} - a_k e_\omega$ for all $k \in \mathbb{N}$, where $(a_n)_{n\in\mathbb{N}}$ is the sequence defined recursively by $a_0 := 2$, $a_1 := 3$ and $a_{k+1} := 2a_k - a_{k-1}$, for all $k \geqslant 1$.

Proof. i) Indeed, using the formulas obtained in Proposition 4.2.1.1 we write

$$d_u(e_\varnothing) = (\partial_u - 2\ id)(e_\varnothing) = \partial_u(e_\varnothing) - 2e_\varnothing = 2e_u - 2e_\varnothing \in 2\mathbb{Z}(e_\varnothing - e_u)$$

$$d_u(e_u) = (\partial_u - 2\ id)(e_u) = \partial_u(e_u) - 2e_u = 2e_\varnothing - 2e_u \in 2\mathbb{Z}(e_\varnothing - e_u)$$

Hence $d_u(E_u) = 2\mathbb{Z}(e_\varnothing - e_u)$ as claimed.

ii) For this we proceed by induction showing that the image of the span of $\{d_u(e_{\omega u^l})\}_{0\leqslant l\leqslant k}$ equals the span of $\{\xi_l\}_{0\leqslant l\leqslant k}$.

- For $k = 0$ and $k = 1$ we have respectively the following

$$d_u(e_\omega) = (\partial_u - 2\ id)(e_\omega) = e_{\omega u} - 2e_\omega = \xi_0$$

$$d_u(e_{\omega u}) = (\partial_u - 2\ id)(e_{\omega u}) = e_{\omega u^2} + e_\omega - 2e_{\omega u} = e_{\omega u^2} - 3e_\omega + 4e_\omega - 2e_{\omega u} = \xi_1 - 2\xi_0$$

- Assume that for every $0 \leqslant l \leqslant k$ the span of $\{d_u(e_{\omega u^l})\}$ equals the span of $\{\xi_l\}$.
- Then we write

$$\begin{aligned} d_u(e_{\omega u^{k+1}}) &= (\partial_u - 2\ id)(e_{\omega u^{k+1}}) = e_{\omega u^{k+2}} + e_{\omega u^k} - 2e_{\omega u^{k+1}} \\ &= e_{\omega u^{k+2}} + (\xi_{k-1} + a_{k-1}e_\omega) - 2(\xi_k + a_k e_\omega) \\ &= e_{\omega u^{k+2}} + (a_{k-1} - 2a_k)e_\omega + \xi_{k-1} - 2\xi_k \\ &= e_{\omega u^{k+2}} - a_{k+1}e_\omega + \xi_{k-1} - 2\xi_k \\ &= \xi_{k+1} + \xi_{k-1} - 2\xi_k \in span\{\xi_l\}_{0\leqslant l\leqslant k+1} \end{aligned}$$

Since the family $\{\xi_k\}_{k\in\mathbb{N}}$ is free, the result follows.

■

After these preliminary computations, we can carry out the study of the K-theory of $C(\mathbb{H}_q)$ for the different choices of $\mathbb{G}$ as explained in the beginning of this section.

4.2.2 $\mathbb{G} := O^+(n)$ is a free orthogonal quantum group

If $\mathbb{G} := O^+(n)$ is a free orthogonal quantum group with $n \geqslant 2$ and we put $\mathbb{F} := O^+(n) * SU_q(2)$, then Theorem 7.1 in [208] suggests

$$P_1 := c_0(\widehat{\mathbb{F}}) \oplus c_0(\widehat{\mathbb{F}}) \text{ and } P_0 := c_0(\widehat{\mathbb{F}})$$

$$\delta_1 := (T_u - dim(u)id) \oplus (T_v - dim(v)id) \text{ and } \delta_0 := \widehat{\lambda},$$

where u denotes de fundamental representation of $SU_q(2)$, v the fundamental representation of $O^+(n)$ and $\widehat{\lambda}$ the left regular representation.

Hence, the preliminary computations above yield that diagram (4.2.2) becomes now the following six-term exact sequence

$$\begin{tikzcd}
\left(\mathbb{Z}_u \oplus \left(\bigoplus_{\omega\neq u}\mathbb{Z}_\omega\right)\right)^{\oplus 2} \arrow[r, "d_u \oplus d_v"] & \mathbb{Z}_u \oplus \left(\bigoplus_{\omega\neq u}\mathbb{Z}_\omega\right) \arrow[r] & K_0(C(\mathbb{H}_q)) \arrow[d] \\
K_1(C(\mathbb{H}_q)) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where $d_v := \partial_v - n\ id$ and $\partial_v := j_{\widehat{\mathbb{H}}_q}(T_v)$. Therefore, we have to compute

$$K_0(C(\mathbb{H}_q)) = coker(d_u \oplus d_v) \text{ and } K_1(C(\mathbb{H}_q)) = \ker(d_u \oplus d_v)$$

In order to so, we need an analogue of Proposition 4.2.1.1 for ∂_v. Let us denote by f_u and by f_ω the unit of $\mathbb{Z}_u$ and $\mathbb{Z}_\omega$, respectively of the *second copy* of $\mathbb{Z}_u \oplus \left(\bigoplus_{\omega\neq u}\mathbb{Z}_\omega\right)$ coming from P_1.

4.2.2.1 Proposition. *The element $\partial_v := j_{\widehat{\mathbb{H}}_q}(T_v)$ acts on the basis as follows*

i) for all word ω, $\partial_v(f_{\omega v^k}) = f_{\omega v^{k+1}} + f_{\omega v^{k-1}}$,

ii) if $\omega = \varnothing$ or ω ends in $Irr(SU_q(2))$, then $\partial_v(f_\omega) = f_{\omega v}$,

iii) if $\omega = u$, then $\partial_v(f_u) = nf_u$,

Proof. The first two cases follow from the same computations as in Proposition 4.2.1.1. Let us show the third one.

As in Proposition 4.2.1.1, remark that given any $a \in \mathcal{B}(H_y)$ with $y \in Irr(\mathbb{F})$, then by construction we have $T_v(a) = (id \otimes p_v)\widehat{\Theta}(a) \in \bigoplus_{y\subset z\oplus v}^{c_0} \mathcal{B}(H_z) \otimes \mathcal{B}(H_v)$. In particular, the only representation z such that $u \subset z \oplus v$ is uv. Moreover, $uv \in J_u$ because $uv \subset uvu \oplus u = uv \oplus uvu^2$, where we take $h := uvu \in Irr(\mathbb{H}_q)$. Hence, it is clear that $\partial_v(f_u) \in \mathbb{Z}f_u$.

In order to study $\partial_v = j_{\widehat{\mathbb{H}}_q}(T_v)$, we have to understand the image by the descent homomorphism of the Kasparov triple associated to T_v corresponding to the term associated to u (recall the formula (4.2.3)). In other words, we have to understand the following module

$$\widehat{\mathbb{H}}_q \ltimes A_u \otimes H_v \in KK(\widehat{\mathbb{H}}_q \ltimes A_u, \widehat{\mathbb{H}}_q \ltimes A_u),$$

equipped with the right action given simply by multiplication on the first tensor and the representation of $\widehat{\mathbb{H}}_q \ltimes A_u$ given by $\widehat{\mathbb{H}}_q \ltimes (id \otimes p_v)\widehat{\Theta}$. Let us write it in the following way as *bimodule*,

$${}_{\widehat{\mathbb{H}}_q \ltimes A_u}\left(\widehat{\mathbb{H}}_q \ltimes A_u \otimes H_v\right)_{\widehat{\mathbb{H}}_q \ltimes A_u}$$

By virtue of Remark 3.7.2.5, this yields the same KK-theory element as

$${}_{\mathcal{M}_2(\mathbb{C})}\left(\mathcal{M}_2(\mathbb{C}) \otimes H_v\right)_{\mathcal{M}_2(\mathbb{C})},$$

where the right action is simply multiplication on the first tensor and the representation of $\mathcal{M}_2(\mathbb{C})$ is faithful.

Next, remark that $\mathcal{M}_2(\mathbb{C})$ is Morita equivalent to $\mathbb{C}$ by means of $\mathbb{C}^2$, so that we write the following

$$\begin{aligned}
{}_{\mathcal{M}_2(\mathbb{C})}\Big(\mathcal{M}_2(\mathbb{C}) \otimes H_v\Big)_{\mathcal{M}_2(\mathbb{C})} &\cong_{\mathcal{M}_2(\mathbb{C})} \Big(\mathbb{C}^2 \underset{\mathcal{M}_2(\mathbb{C})}{\otimes} \mathcal{M}_2(\mathbb{C}) \otimes H_v\Big)_{\mathbb{C}} \\
&\cong_{\mathcal{M}_2(\mathbb{C})} \Big(\mathbb{C}^2 \otimes H_v\Big)_{\mathbb{C}} \\
&\cong_{\mathbb{C}} \Big(\mathbb{C}^2 \underset{\mathcal{M}_2(\mathbb{C})}{\otimes} \big(\mathbb{C}^2 \otimes H_v\big)\Big)_{\mathbb{C}}
\end{aligned}$$

By faithfulness of the representation of $\mathcal{M}_2(\mathbb{C})$, the latter Hilbert space has dimension n. In other words, the module corresponding to the term u identifies with $n \in \mathbb{Z} = K_0(\mathbb{C}) = KK(\mathbb{C}, \mathbb{C})$ in the copy $\mathbb{Z}_u$, that is, with nf_u and the proof is complete. ■

4.2.2.2 Lemma. *The following properties hold.*

i) The kernel of $d_u \oplus d_v$ is $\mathbb{Z}(e_\varnothing + e_u) \oplus \mathbb{Z} f_u$.

ii) The image of d_v is spanned by the vectors $\eta_k := e_{\omega v^{k+1}} - b_k e_\omega$ for all $k \in \mathbb{N}$, where ω is a word ending in $Irr(SU_q(2))$ (possibly empty) and $(b_n)_{n\in\mathbb{N}}$ is the sequence defined recursively by $b_0 := n$, $b_1 := n^2 - 1$ and $b_{k+1} := nb_k - b_{k-1}$, for all $k \geqslant 1$.

Proof. i) We are going to prove that the vectors $(e_\varnothing + e_u) \oplus f_u$ provide a $\mathbb{Z}$-basis for $\ker(d_u \oplus d_v)$. Consider an element

$$x := \lambda_u e_u + \mu_u f_u + \sum_{\omega \neq u} \big(\lambda_\omega e_\omega + \mu_\omega f_\omega\big) \in \mathbb{Z}_u \oplus \mathbb{Z}_u \oplus \Big(\bigoplus_{\omega\neq u} \mathbb{Z}_\omega\Big) \oplus \Big(\bigoplus_{\omega\neq u} \mathbb{Z}_\omega\Big)$$

and assume that $(d_u \oplus d_v)(x) = 0$.

Denote by L the *maximal length* of the words appearing in this sum with non-zero coefficient.

If $L > 1$, then a word ω of length L must be either of the form $\omega' u^k$ or $\omega' v^k$. Assume that ω is of the form $\omega' u^k$ and let k_0 be maximal so that $\omega' u^{k_0}$ occurs in the element x. So either $\lambda_\omega \neq 0$ or $\mu_\omega \neq 0$. If $\lambda_\omega \neq 0$, then

$$\lambda_\omega e_{\omega' u^{k_0+1}} - (d_u \oplus d_v)(x)$$

must be a linear combination of basis vectors non including $e_{\omega' u^{k_0+1}}$, which is impossible because $(d_u \oplus d_v)(x) = 0$. If $\mu_\omega \neq 0$, then

$$\mu_\omega f_{\omega' u^{k_0+1}} - (d_u \oplus d_v)(x)$$

must be a linear combination of basis vectors non including $f_{\omega' u^{k_0+1}}$, which is impossible because $(d_u \oplus d_v)(x) = 0$.

The same argument works when ω is of the form $\omega' v^k$.

If $L = 1$ and $\omega = v^k$, then we get the same contradiction. In conclusion, the sum only contains terms associated to $\omega = u$ or $\omega = \varnothing$. So we write

$$x = \lambda_u e_u + \mu_u f_u + \lambda_\varnothing e_\varnothing + \mu_\varnothing f_\varnothing$$

and we compute its image by $d_u \oplus d_v$,

$$\begin{aligned}(d_u \oplus d_v)(x) &= d_u(\lambda_u e_u + \lambda_\varnothing e_\varnothing) + d_v(\mu_u f_u + \mu_\varnothing f_\varnothing)\\ &= \lambda_u(2e_\varnothing - 2e_u) + \lambda_\varnothing(2e_u - 2e_\varnothing) + \mu_u(n\ f_u - n\ f_u) + \mu_\varnothing(f_v - n\ f_\varnothing)\\ &= (2\lambda_u - 2\lambda_\varnothing)e_\varnothing + (2\lambda_\varnothing - 2\lambda_u)e_u + \mu_\varnothing(f_v - n\ f_\varnothing)\end{aligned}$$

Since $(d_u \oplus d_v)(x) = 0$ by assumption, the above computation implies $\lambda_u = \lambda_\varnothing$ and $\mu_\varnothing = 0$. In other words, $(e_u + e_\varnothing) \oplus f_u$ is a $\mathbb{Z}$-basis for $\ker(d_u \oplus d_v)$, which ends the proof.

ii) The analogous argument as the one of Lemma 4.2.1.2 yields the claim of the statement.

■

4.2.2.3 Theorem. *Let $n \geqslant 2$ and $N \geqslant 4$ be natural numbers. Let $O^+(n) \wr_* S_N^+$ be the free wreath product of the free orthogonal quantum group $O^+(n)$ by S_N^+. If $\mathbb{H}_q$ denotes the corresponding Lemeux-Tarrago's compact quantum group which is monoidally equivalent to $O^+(n) \wr_* S_N^+$, then*

$$K_0(C(\mathbb{H}_q)) = \mathbb{Z} \oplus \mathbb{Z}_2 \text{ and } K_1(C(\mathbb{H}_q)) = \mathbb{Z}^2$$

Proof. Thanks to Lemma 4.2.2.2, we have

$$K_1(C(\mathbb{H}_q)) = \ker(d_u \oplus d_v) \cong \mathbb{Z} \oplus \mathbb{Z}$$

Let us compute $K_0(C(\mathbb{H}_q)) = coker(d_u \oplus d_v)$. Denote by π the canonical quotient map $\mathbb{Z}_u \oplus \Big(\bigoplus_{\omega \neq u} \mathbb{Z}_\omega\Big) \twoheadrightarrow \mathbb{Z}_u \oplus \Big(\bigoplus_{\omega \neq u} \mathbb{Z}_\omega\Big)/Im(d_u \oplus d_v) = coker(d_u \oplus d_v)$.

If we consider a word of the form ωu^k of length *at least* 2 (so $\omega \neq \varnothing$), then we have that

$$\pi(e_{\omega u^k}) = a_{k-1}\pi(e_\omega)$$

Namely, since $\omega \neq \varnothing$, then $e_{\omega u^k} \in E_\omega$ and Lemma 4.2.1.2 assures that $d_u(E_\omega)$ is spanned by the vectors $\xi_k = e_{\omega u^{k+1}} - a_k e_\omega$, for all $k \in \mathbb{N}$. This relation yields in particular $e_{\omega u^k} - \xi_{k-1} = a_{k-1}e_\omega$. Since $\xi_{k-1} \in Im(d_u) \subset Im(d_u \oplus d_v)$ by construction, then we obtain the formula above.

If we consider a word of the form ωv^k for any ω (possibly empty), then we have that

$$\pi(e_{\omega v^k}) = b_{k-1}\pi(e_\omega)$$

Namely, Lemma 4.2.2.2 assures that $Im(d_v)$ is spanned by the vectors $\eta_k = e_{\omega v^{k+1}} - b_k e_\omega$, for all $k \in \mathbb{N}$. This relation yields in particular $e_{\omega v^k} - \eta_{k-1} = b_{k-1}e_\omega$. Since $\eta_{k-1} \in Im(d_v) \subset Im(d_u \oplus d_v)$ by construction, then we obtain the formula above.

In conclusion, $coker(d_u \oplus d_v)$ is spanned by the vectors $\pi(e_\omega)$ with ω a word of length *at least* 1. But notice that $\pi(e_{v^k}) = b_{k-1}\pi(e_\varnothing)$. Hence we have to consider only the vectors $\pi(e_u)$ and $\pi(e_\varnothing)$ and $coker(d_u \oplus d_v)$ has rank at most two.

Observe that $e_u \notin Im(d_v)$ by construction. Moreover, recall from Lemma 4.2.1.2 that $d_u(E_u) = 2\mathbb{Z}(e_\varnothing - e_u) \subset Im(d_u) \subset Im(d_u \oplus d_v)$. Hence, $2\pi(e_\varnothing) = 2\pi(e_u)$.

In other words, the vector $\pi(e_u)$ is free in $coker(d_u \oplus d_v)$ and it generates a copy of $\mathbb{Z}$ inside $coker(d_u \oplus d_v)$ and the vector $\pi(e_\varnothing)$ satisfies the relation $2\pi(e_\varnothing) = 2\pi(e_u)$ inside $coker(d_u \oplus d_v)$, so that it generates a copy of $\mathbb{Z}_2$ inside $coker(d_u \oplus d_v)$. In conclusion,

$$K_0(C(\mathbb{H}_q)) = coker(d_u \oplus d_v) \cong \mathbb{Z} \oplus \mathbb{Z}_2,$$

generated by $\pi(e_\varnothing + e_u)$ and $\pi(e_\varnothing - e_u)$. ■

4.2.3 $\mathbb{G} := U_1^+ * \ldots * U_k^+ * O_1^+ * \ldots * O_l^+$ is a free quantum group

If $\mathbb{G} := U^+(P_1) * \ldots * U^+(P_k) * O^+(Q_1) * \ldots * O^+(Q_l)$ is a free product of free unitary and free orthogonal quantum groups, where $P_i \in GL_{m_i}(\mathbb{C})$ with $m_i \geqslant 2$ for all $i = 1, \ldots, k$ and $Q_j \in GL_{n_j}(\mathbb{C})$ with $n_j \geqslant 2$ satisfies $Q_j\overline{Q}_j = \pm id$ for all $j = 1, \ldots, l$ and we put $\mathbb{F} := \mathbb{G} * SU_q(2)$, then Theorem 7.1 in [208] suggests

$$P_1 := \bigoplus_{r=1}^{2k+l+1} c_0(\widehat{\mathbb{F}}) \text{ and } P_0 := c_0(\widehat{\mathbb{F}})$$

$$\delta_1 := \bigoplus_{i=1}^{k}\Big((T_{u_i} - dim(u_i)id) \ \oplus \ (T_{\overline{u_i}} - dim(u_i)id)\Big) \oplus \bigoplus_{j=1}^{l}\Big((T_{v_j} - dim(v_j)id)\Big) \ \oplus \ (T_u - dim(u)id)$$

$$\text{and } \delta_0 := \widehat{\lambda},$$

where u denotes de fundamental representation of $SU_q(2)$, u_i the fundamental representation of $U^+(P_i)$ for every $i = 1, \ldots, k$, v_j the fundamental representation of $O^+(Q_j)$ for every $j = 1, \ldots, l$ and $\widehat{\lambda}$ the left regular representation.

Hence, the preliminary computations above yield that diagram (4.2.2) becomes now the following six-term exact sequence

$$\begin{tikzcd}
\Big(\mathbb{Z}_u \oplus \Big(\bigoplus_{\omega \neq u} \mathbb{Z}_\omega\Big)\Big)^{\oplus 2k+l+1} \arrow[r, "d"] & \mathbb{Z}_u \oplus \Big(\bigoplus_{\omega \neq u} \mathbb{Z}_\omega\Big) \arrow[r] & K_0(C(\mathbb{H}_q)) \arrow[d] \\
K_1(C(\mathbb{H}_q)) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where $d := \bigoplus_{i}^{2k}\big(d_{u_i} \oplus d_{\overline{u_i}}\big) \oplus \bigoplus_{j}^{l}\big(d_{v_j}\big) \oplus d_u$ with

$$d_{u_i} := \partial_{u_i} - m_i \ id \text{ and } \partial_{u_i} := j_{\widehat{\mathbb{H}}_q}(T_{u_i}), \text{ for all } i = 1, \ldots, k$$

$$d_{v_j} := \partial_{v_j} - n_j \ id \text{ and } \partial_{v_j} := j_{\widehat{\mathbb{H}}_q}(T_{v_j}), \text{ for all } j = 1, \ldots, l$$

Therefore, we have to compute

$$K_0(C(\mathbb{H}_q)) = coker(d) \text{ and } K_1(C(\mathbb{H}_q)) = \ker(d)$$

and the same computations as before yield the following result.

4.2.3.1 Theorem. *Let $m_i \geqslant 2$, $n_j \geqslant 2$ and $N \geqslant 4$ be natural numbers, for all $i = 1, \ldots, k$ and $j = 1, \ldots, l$. Let $\mathbb{G} := U^+(P_1) * \ldots * U^+(P_k) * O^+(Q_1) * \ldots * O^+(Q_l)$ be a free product of free unitary and free orthogonal quantum groups as above. Let $\mathbb{G} \wr_* S_N^+$ be the free wreath product of the quantum free product $\mathbb{G}$ by S_N^+. If $\mathbb{H}_q$ denotes the corresponding Lemeux-Tarrago's compact quantum group which is monoidally equivalent to $\mathbb{G} \wr_* S_N^+$, then*

$$K_0(C(\mathbb{H}_q)) = \mathbb{Z} \oplus \mathbb{Z}_2 \text{ and } K_1(C(\mathbb{H}_q)) = \mathbb{Z}^{2k+l+1}$$

4.2.4 $\mathbb{G} := \mathbb{F}_n$ is the classical free group on n generators

If $\mathbb{G} := \mathbb{F}_n$ is the classical free group on n generators and we put $\mathbb{F} := \mathbb{F}_n * SU_q(2)$, then an appropriated modification of the resolution given in [208] suggests to take

$$P_1 := \bigoplus_{r=1}^{n+1} c_0(\widehat{\mathbb{F}}) \text{ and } P_0 := c_0(\widehat{\mathbb{F}})$$

$$\delta := \delta_1 := \bigoplus_{i=1}^{n} (T_{a_i} - id) \ \oplus \ (T_u - 2\ id) \text{ and } \delta_0 := \widehat{\lambda},$$

where u denotes the fundamental representation of $SU_q(2)$, $a_1, \ldots, a_n$ the canonical generators of $\mathbb{F}_n$ and $\widehat{\lambda}$ the left regular representation.

First of all, let us check that the complex $0 \longrightarrow P_1 \xrightarrow{\delta} P_0 \xrightarrow{\lambda} \mathbb{C} \longrightarrow 0$ defines actually a $\mathcal{J}$-projective resolution of length 1 for $\mathbb{C}$ in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{F}}}$, where $\mathcal{J} := \ker_{Hom}\left(Res_{\mathbb{E}}^{\widehat{\mathbb{F}}}\right)$ as usual. For this, it is enough to check that the application of the functor $KK^{\widehat{\mathbb{F}}}(c_0(\widehat{\mathbb{F}}),\ \cdot\ )$ yields an exact sequence of abelian groups. More precisely, we obtain the following diagram

$$0 \longrightarrow KK^{\widehat{\mathbb{F}}}\left(c_0(\widehat{\mathbb{F}}), \bigoplus_{r=1}^{n+1} c_0(\widehat{\mathbb{F}})\right) \longrightarrow KK^{\widehat{\mathbb{F}}}(c_0(\widehat{\mathbb{F}}), c_0(\widehat{\mathbb{F}})) \longrightarrow KK^{\widehat{\mathbb{F}}}(c_0(\widehat{\mathbb{F}}), \mathbb{C}) \longrightarrow 0$$

that is to say,

$$0 \longrightarrow \bigoplus_{r=1}^{n+1} R(\mathbb{F}) \xrightarrow{\delta'} R(\mathbb{F}) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where $\delta' := \bigoplus_{i=1}^{n} (r_{a_i^{-1}} - id) \ \oplus \ (r_u - 2\ id)$ ("r" denotes the right multiplication operator) and ε is the map induced by the dimension function (recall Remark 4.1.1.1 and notice that $u \cong \overline{u}$ because $SU_q(2)$ is a free orthogonal quantum group, see Remarks 2.1.4).

4.2.4.1 Lemma. *With the same notations as above, the diagram*

$$0 \longrightarrow \bigoplus_{r=1}^{n+1} R(\mathbb{F}) \xrightarrow{\delta'} R(\mathbb{F}) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is an exact sequence of abelian groups.

Proof. First of all, notice that ε is such that $\varepsilon(a_i) = 1$, for all $i = 1, \ldots, n$ and $\varepsilon(u) = 2$. So the surjectivity of ε is clear. Moreover, by construction it is also clear that $Im(\delta') \subset \ker \varepsilon$.

Let us show that δ' is injective. Denote by e_ω^k the basis element corresponding to the representation ω in the k-th copy of $R(\mathbb{F})$. Denote by x a finite linear combination of such elements in $\ker(\delta')$. Moreover, consider a word ω of *maximal length* appearing in x with non-zero coefficient.

If it appears in the last component, then either ω ends in $\mathbb{F}_n$ and ωu appears in $\delta'(\omega)$ or ω ends with some u^k and then, taking k maximal, ωu (which ends with u^{k+1}) appears in $\delta'(\omega)$. In both cases, the same argument as in Lemma 4.2.2.2 yields a contradiction.

Therefore, we may assume that ω appears in of the first n components and, without loss of generality, let us assume that ω appears in the first component. If ω ends with some u^k, then the

same argument as above yields a contradiction. Thus, we have $\omega = \omega'\gamma$, for some $\gamma \in \mathbb{F}_n$ and we may assume that γ is of maximal length. The same argument as above implies that γ must end with a_1. Then $\delta'(\omega\gamma) = \omega\gamma a_1^{-1} - \omega\gamma = 0$, so that $\omega\gamma$ appears in $\delta'(x)$, which means that x must contain terms whose images by δ' simplify with $\omega\gamma$. But by definition of free groups, such a term must be of the form $\omega(\gamma a_i)$, which contradicts the maximality of γ.

In conclusion, δ' is injective. It remains to show that $\ker(\varepsilon) \subset Im(\delta')$. For this we observe that the vectors $x_\omega := \omega - dim(\omega)$ with ω any word, form a basis of $\ker(\varepsilon)$. Denote by π the canonical quotient map $\ker(\varepsilon) \twoheadrightarrow \ker(\varepsilon)/Im(\delta')$. Let us consider a non-empty word.

If it is of the form ωu^k, then the fusion rules of $SU_q(2)$ yields the equality

$$\omega u^k = (r_u - 2\ id)(\omega u^{k-1}) - \omega u^{k-2} + 2\omega u^{k-1},$$

(with the convention $\omega u^{-1} = 0$) so that

$$\begin{aligned} x_{\omega u^k} &= (r_u - 2\ id)(\omega u^{k-1}) - \omega u^{k-2} + 2\omega u^{k-1} - dim(\omega u^k) \\ &= (r_u - 2\ id)(\omega u^{k-1}) - \big(x_{\omega u^{k-2}} + dim(\omega u^{k-2})\big) + \big(2x_{\omega u^{k-1}} + 2dim(\omega u^{k-1})\big) - dim(\omega u^k) \\ &= (r_u - 2\ id)(\omega u^{k-1}) - x_{\omega u^{k-2}} + 2x_{\omega u^{k-1}}, \end{aligned}$$

so that $\pi(x_{\omega u^k}) \in \mathbb{Z}\pi(\omega u^{k-2}) \oplus \mathbb{Z}\pi(\omega u^{k-1})$ (since $(r_u - 2\ id)(\omega u^{k-1}) \in Im(\delta')$). Applying this inductively, we see that $\pi(\omega u^k) \in \mathbb{Z}\pi(\omega)$.

If the word is of the form ωa_l^k, then using the equality

$$\omega a_l^k - \omega a_l^{k+1} = (r_{a_l^{-1}} - id)(\omega a_l^{k+1})$$

we see that we can increase or decrease k depending on its sign until we get $\pi(\omega a_l^k) \in \mathbb{Z}\pi(\omega)$.

We have shown that all the basis elements have the same image by π. Since $x_u = u - 2 = (r_u - 2\ id)(\epsilon)$ is the image of the trivial representation, its image by π is zero and the proof is complete. ■

In this situation, we can apply the same strategy as before. Hence, the preliminary computations above yield that diagram (4.2.2) becomes now the following six-term exact sequence

$$\begin{tikzcd} \Big(\mathbb{Z}_u \oplus \Big(\bigoplus_{\omega \neq u} \mathbb{Z}_\omega\Big)\Big)^{\oplus n+1} \arrow[r, "d"] & \mathbb{Z}_u \oplus \Big(\bigoplus_{\omega \neq u} \mathbb{Z}_\omega\Big) \arrow[r] & K_0(C(\mathbb{H}_q)) \arrow[d] \\ K_1(C(\mathbb{H}_q)) \arrow[u] & 0 \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

where $d := \bigoplus_i^n (d_{a_i}) \oplus d_u$ with

$$d_{a_i} := \partial_{a_i} - id \text{ and } \partial_{a_i} := j_{\widehat{\mathbb{H}}_q}(T_{a_i}), \text{ for all } i = 1, \ldots, n$$

Therefore, we have to compute

$$K_0(C(\mathbb{H}_q)) = coker(d) \text{ and } K_1(C(\mathbb{H}_q)) = \ker(d)$$

and the same computations as before yield the following result.

4.2.4.2 Theorem. *Let $n \geqslant 1$ and $N \geqslant 4$ be natural numbers. Let $\mathbb{F}_n \wr_* S_N^+$ be the free wreath product of the classical free group on n generators $\mathbb{F}_n$ by S_N^+. If $\mathbb{H}_q$ denotes the corresponding Lemeux-Tarrago's compact quantum group which is monoidally equivalent to $\mathbb{F}_n \wr_* S_N^+$, then*

$$K_0(C(\mathbb{H}_q)) = \mathbb{Z} \oplus \mathbb{Z}_2 \text{ and } K_1(C(\mathbb{H}_q)) = \mathbb{Z}^{n+1}$$

CHAPTER 5

Conclusion: open questions and possible lines of attack

The fifth and last chapter of the thesis is devoted to question ourselves about the results that we have obtained. This chapter must be regarded as a *compendium* of the main questions, problems and goals that the author has encountered during the whole research period of the present dissertation. Most of the following subjects are part of future research projects of the author.

In Section 5.1 we care about the stability of the (resp. *strong*) Baum-Connes property for constructions of Chapter 3. In Section 5.2 we care about the maximal torus strategy for proving the *strong* Baum-Connes property. In Section 5.3 we care about the K-theory computations of C^*-algebras associated to concrete examples of compact quantum groups guided by the results of Chapter 3 and Chapter 4. In Section 5.4 we care about the formulation of the Baum-Connes property for arbitrary discrete quantum groups (torsion-free or not).

5.1 Stability of the Baum-Connes property

Notice that one of the main application of the *quantum* Baum-Connes property is the computation of the K-theory of C^*-algebras arising from compact quantum groups. To this end, the *strong* Baum-Connes property is the first requirement that we need. For this reason it is interesting to analyze the strong version of the Baum-Connes property for the constructions of Chapter 3. From a theoretical point of view, the *usual* Baum-Connes property is the original problem to solve and it would be also interesting to know if we can establish directly this weaker version.

- *Compact groups*. If G is a *classical* compact group, then it is automatically amenable and so it satisfies the *strong* Baum-Connes property by applying the celebrated result of Higson-Kasparov [82]. Accordingly, G satisfies the usual Baum-Connes property. In the quantum setting, Theorem 3.1.1 gives a model of the Kasparov category $\mathscr{KK}^{\mathbb{G}}$ in terms of the complementary pair associated to $\widehat{\mathbb{G}}$ under torsion-freeness assumption.

  As a matter of fact, the amenability property is one of the main differences between compact classical groups and compact quantum groups as we have explained in Remark 1.3.1.42. In this sense, the first question would be the following: *if $\mathbb{G}$ is a co-K-amenable compact quantum group such that $\widehat{\mathbb{G}}$ is torsion-free, then does $\mathbb{G}$ satisfy automatically the strong Baum-Connes property ?*

  Recall from Remark 1.3.1.42 that if $\widehat{\mathbb{G}}$ is amenable, then $\mathbb{G}$ is automatically co-amenable (and so co-K-amenable). We can thus wonder about the relation between the Baum-Connes property for a compact quantum group and the one for its discrete dual. Observe the following obvious implications in the classical commutative case:

  $$\text{If } G \text{ is compact abelian} \Rightarrow \widehat{G} \text{ is discrete abelian} \Rightarrow \widehat{G} \text{ satisfies the strong BC property}$$

  $$\text{If } \Gamma \text{ is discrete abelian} \Rightarrow \widehat{\Gamma} \text{ is compact abelian} \Rightarrow \widehat{\Gamma} \text{ satisfies the strong BC property}$$

  As it has been shown in [133] (see as well Theorem 3.1.3), if G is a classical *connected* compact group (which satisfies the strong Baum-Connes property), then the discrete quantum group $\widehat{G}$ satisfies the strong Baum-Connes property.

  From these results it is reasonable to consider the following questions.

  i) *Given a classical compact group G (which satisfies the strong Baum-Connes property), does its discrete dual $\widehat{G}$ satisfy the strong Baum-Connes property?*

  ii) *Given a compact co-amenable (or even co-K-amenable) quantum group $\mathbb{G}$ such that $\widehat{\mathbb{G}}$ is torsion-free, does its discrete dual $\widehat{\mathbb{G}}$ satisfy the strong Baum-Connes property? Can we remove the torsion-freeness assumption?*

  iii) Inspired by Remark 1.3.1.42, the preceding question can be formulated as follows: *if $\widehat{\mathbb{G}}$ is an amenable torsion-free discrete quantum group, does it satisfy the strong Baum-Connes property? Can we remove the torsion-freeness assumption?*

  iv) *Can we prove a quantum version of the Higson-Kasparov theorem? Namely, can we prove that a torsion-free discrete quantum group satisfying the Haagerup property satisfies directly the* strong *Baum-Connes property? Can we remove the torsion-freeness assumption?*

- *Quantum subgroups*. On the one hand, we know from Theorem 3.2.2.1 and Proposition 3.2.2.3 that the (resp. strong) Baum-Connes property is hereditary for *divisible* discrete quantum subgroups. But, to the best knowledge of the author, it is still open to know if the (resp. strong) Baum-Connes property is preserved by arbitrary discrete quantum subgroups. Moreover, up to the present, it has not been shown a discrete quantum group that does *not* satisfy the (resp. strong) Baum-Connes property.

  On the other hand, we know from [3] that *strong* torsion-freeness is preserved under *divisible* discrete quantum subgroups. We have conjectured in the end of Section 3.2.1 that the same stability result must hold for the torsion-freeness in the sense of Meyer-Nest. Observe that, in general, discrete quantum groups involved in constructions of other discrete quantum groups are discrete quantum *subgroups* of the resulting object. In this way, if the divisibility condition assured the stability of torsion-freeness in the sense of Meyer-Nest, then we could simplify the statements of Chapter 3 concerning the stability of the Baum-Connes property by removing the assumptions of torsion-freeness of *all* discrete quantum groups involved.

- *Quantum direct product*. On the one hand, we have studied the *strong* Baum-property for a quantum direct product in terms of the quantum groups involved obtaining the stability result of Theorem 3.3.2.3.

  On the other hand, we wanted to study directly the *usual* Baum-Connes property for a quantum direct product in terms of the quantum groups involved. However, we have observed that such a stability result is closely related to the *Künneth formula* (see Corollary 3.3.2.5 and Remark 3.3.1) as in the classical case [37]. Nevertheless, the quantum counterpart contained in this dissertation is not a totally optimal conclusion.

  In this way, we open a first line of work beyond the present dissertation. Namely, it would be interesting to establish a *quantum setting for the Künneth formula* in order to find sufficient conditions to a crossed product (by a discrete quantum group) to belong to the class $\mathcal{N}$. More precisely, we are inspired by the work of J. Chabert, S. Echterhoff and H. Oyono-Oyono in [37]. They develop an abstract categorical framework for the Künneth formula in terms of what they call *"Künneth functors"*. If G is a locally compact group, then this approach allows to give an *equivariant* version of the class $\mathcal{N}$, say $\mathcal{N}_G$. One of the main theorems in [37] is the following: *"Let A be a G-C^*-algebra. If $K \ltimes A \in \mathcal{N}$, for all compact subgroup $K < G$, then $A \in \mathcal{N}_G$"*.

  It is reasonable to extend these definitions and constructions for discrete quantum groups. *Can we replace G by a discrete quantum group $\widehat{\mathbb{G}}$ (perhaps under the torsion-freeness assumption) ?*

- *Quantum semi-direct product and compact bicrossed product*. We have studied the *usual* Baum-Connes property for a quantum semi-direct product in terms of the quantum groups involved. In this way, Theorem 3.4.2.4 is the quantum counterpart of the result [34] by J. Chabert. Under the torsion-freeness assumption, a compact bicrossed product becomes a quantum semi-direct product as explained in Section 3.5. Hence, the corresponding Baum-Connes property does not give any relevant information.

  The *strong* version of the Baum-Connes property has been studied in Theorem 3.4.2.6. Notice that it is not clear *a priori* (even with torsion-freeness assumption) that we can recover the *strong* Baum-Connes property for $\widehat{\mathbb{F}}$ from the strong Baum-Connes property for Γ and $\widehat{\mathbb{G}}$.

In this sense, the stability of the *strong* Baum-Connes property for a quantum semi-direct product has been half-achieved. However, the classification of torsion actions of $\widehat{\mathbb{F}}$ must be the key point in order to complete this stability result and we can tackle this classification problem using a similar strategy as the one used for the free product construction in Section 3.6.1.

In other words, the torsion phenomena turns out to be crucial for the study of the strong Baum-Connes property, which leads us to the problem of a Baum-Connes property formulation without the torsion-freeness assumption. In such a situation, the compact bicrossed product plays a much more interesting role as we have pointed out in the end of Section 3.5.

- *Quantum free product*. As we have explained in Section 3.6, R. Vergnioux and C. Voigt have proven in [208] that the *strong* Baum-Connes property is stable under the quantum free product construction.

  To this end, they adapt geometrical strategies from classical well-known results in order to apply directly the Dirac-dual Dirac method. However, once the discrete dual of such a quantum free product is torsion-free (which has been investigated in detail in Section 3.6.1), it is legitimate to speak about the *usual* Baum-Connes property. *Can we establish directly the* usual *Baum-Connes property for a quantum free product in terms of the* usual *Baum-Connes property of the quantum groups involved?*

- *Free wreath product*. In Theorem 3.7.2.6 we have showed that the dual of a free wreath product $\mathbb{G} \wr_* S_N^+$ with $N \geqslant 4$ satisfies the *strong* Baum-Connes property whenever $\widehat{\mathbb{G}}$ is torsion-free and satisfies the *strong* Baum-Connes property.

  To this end, we have had to give a suitable definition of *strong* Baum-Connes property taking into account the torsion phenomena of $\widehat{\mathbb{G} \wr_* S_N^+}$. Once we can give a proper formulation for the *usual* Baum-Connes property without the torsion-freeness assumption, we could consider the question, *can we establish the* usual *Baum-Connes property for a free wreath product from the* usual *Baum-Connes property of the quantum groups involved?*

- *Quantum group extension*. In the classical case, both a direct product and a semi-direct product can be viewed as group extensions. Moreover, the notion of extension for locally compact quantum groups has been established in [196] by S. Vaes and L. I. Vainerman in terms of the bicrossed product construction.

  We know thanks to the work [143] of H. Oyono-Oyono that the Baum-Connes conjecture is stable under central extensions of discrete groups. *Can we establish the stability of the Baum-Connes porperty for a general quantum group extension?*

5.2 Maximal torus strategy

If G is a compact connected Lie group with torsion-free fundamental group, then $\widehat{G}$ satisfies automatically the strong Baum-Connes property as explained in Theorem 3.1.3. Notice that the

torsion-freeness assumption of the fundamental group is *not* necessary since R. Meyer and R. Nest are able to manage the quantum torsion phenomena in this case [133].

It is important to have in mind the proof of Theorem 3.1.3. Namely, given the compact Lie group G, we consider the corresponding maximal torus $T < G$. Hence, using Lemma 3.1.2 it turns out that the *strong* Baum-Connes property for $\widehat{T}$ is enough to guarantee the *strong* Baum-Connes property for $\widehat{G}$. And the *strong* Baum-Connes property for $\widehat{T} \cong \mathbb{Z}^n$ with $n \in \mathbb{N}$ is immediate.

This strategy for duals of compact Lie groups have been imitated by C. Voigt in order to prove the *strong* Baum-Connes property for $\widehat{SU_q(2)}$ [210], [211]. These examples bring several observations out.

a) Given a compact quantum group $\mathbb{G}$, we shall find a maximal torus $T < \mathbb{G}$.

b) Given the maximal torus T, we shall carry KK-theory computations out in order to understand the object $C(\mathbb{G}/T)$ in $\mathscr{KK}^{\mathbb{G}}$.

c) If $\mathbb{G} := \mathbb{G}_q$ is a q-deformation of a compact connected Lie group, then there always exists a maximal torus $T < \mathbb{G}_q$ but the KK-theory computations for $C(\mathbb{G}_q/T)$ remain the main obstacle to imitate the above arguments. For $\mathbb{G}_q := SU_q(2)$ we have to deal with the KK-theory of the Podleś sphere. We can find a very detailed treatment of this in Section 4 of [210]. It should be expected that these arguments can be turned out to be general enough to be applied to any q-deformation of a compact connected Lie group.

Observe that for q-deformations, the corresponding maximal torus corresponds to a maximal torus in the usual sense. However, we can not expect the same for a general compact quantum group. In this way, we have to face several issues.

i) *What is the precise notion of* maximal torus *in the quantum setting?* For this, we refer to [17] where we can find a detailed survey about the subject. Roughly speaking, given any compact quantum group $\mathbb{G}$ with $\widehat{\mathbb{G}} < \widehat{U^+(n)}$, we can construct a family of duals subgroups $\{\widehat{\Gamma}_Q < \widehat{\mathbb{G}}\}_{Q \in U(n)}$ that plays the role of *maximal tori* of the classical case. For instance, we have the following remarkable examples

$$\mathbb{G} := U^+(n) \rightsquigarrow \Gamma_Q = \mathbb{F}_n$$

$$\mathbb{G} := O^+(n) \rightsquigarrow \Gamma_Q = \mathbb{F}_n / \langle \left(QQ^t\right)_{ij} \neq 0 \Rightarrow g_i g_j = 1 \rangle,$$

for all $Q \in U(n)$, where $\mathbb{F}_n = \langle g_1, \ldots, g_n \rangle$ is the free group on n generators.

ii) We shall deal with KK-theory computations for $C(\mathbb{G}/\Gamma_Q)$, for every $Q \in U(n)$.

iii) We obtain a family of groups parametrized by unitaries playing the role of maximal tori *but* we can not construct a *unique formal maximal torus*.

iv) Once we have obtained this family, we have to adapt the arguments of Theorem 3.1.3 to this more general setting. In particular, we shall prove that the corresponding Γ_Q satisfy the *strong* Baum-Connes property to achieve the conclusion for $\widehat{\mathbb{G}}$.

For instance, observe that the free group $\mathbb{F}_n$ satisfies clearly the *strong* Baum-Connes property and we have stated in (i) above that the family of maximal tori for the free unitary quantum group is given simply by the free group. In this sense, we can think about a different method from the

the one used by R. Vergnioux and C. Voigt in [208] as explained in Section 3.6 in order to prove the *strong* Baum-Connes property for $\widehat{U^+}(n)$.

Observe finally that the key point of the argument of [133] is that the compact connected Lie group G contains some subgroup $T < G$ that satisfies the strong Baum-Connes property and such that the object $C(G/T)$ is well understood in $\mathscr{K}\mathscr{K}^G$ in terms of the torsion phenomena of $\widehat{G}$. Let $\mathbb{G}$ be a compact quantum group. We contemplate proving the strong Baum-Connes property for $\widehat{\mathbb{G}}$ using the maximal torus strategy. The above observation leads us to suggest not searching for a maximal torus, but searching for an appropriated discrete quantum subgroup $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$. Namely, $\widehat{\mathbb{H}}$ should be such that it satisfies the strong Baum-Connes property and the object $C(\mathbb{G}/\mathbb{H})$ is well understood in $\mathscr{K}\mathscr{K}^{\mathbb{G}}$ in terms of the torsion phenomena of $\widehat{\mathbb{G}}$.

5.3 K-theory computations

One of the main application of the quantum Baum-Connes property is the computation of the K-theory of C^*-algebras arising from compact quantum groups. In this way, the stability results obtained in Chapter 3 lead us to consider concrete examples of compact quantum groups for which we contemplate computing the K-theory for the corresponding defining C^*-algebras.

The present dissertation has already illustrated a concrete application. Namely, Chapter 4 has been devoted to compute the K-theory of $C(\mathbb{H}_q)$, where $\mathbb{H}_q$ is Lemeux-Tarrago's compact quantum group which is monoidal equivalent to the free wreath product $\mathbb{G} \wr_* S_N^+$ with $N \geqslant 4$. We recall that such a computation has been carried out for three different choices of $\mathbb{G}$: a) when $\mathbb{G}$ is a free orthogonal quantum group, b) when $\mathbb{G}$ is a free quantum group and c) when $\mathbb{G}$ is the classical free group on n generators.

Other concrete and interesting examples that we can consider are the following.

i) If $\mathbb{F} := \Gamma \underset{\alpha}{\ltimes} \mathbb{G}$ denotes the quantum semi-direct product of a compact quantum group $\mathbb{G}$ by a discrete group Γ, we shall find *concrete* examples of this construction arising from known (quantum) groups. Let us be more precise following [216].

   a) <u>*Examples coming from classical groups*</u>. Let G be a compact group and consider a discrete subgroup of G, say $H < G$. There always exists an action of H on G by inner automorphisms, say α. In this way we can construct the corresponding quantum semi-direct product,

   $$\mathbb{F} := H \underset{\alpha}{\ltimes} G,$$

   which is, in general, a genuine compact quantum group. If both G and H are finite groups, this construction yields examples of finite quantum groups. But in this case the corresponding $\widehat{\mathbb{F}}$ is *not torsion-free*. For instance, the finite subgroups of H yield torsion for $\widehat{\mathbb{F}}$ by Theorem 3.4.1.1. Hence, we should do a preliminary analysis of the torsion phenomena in order to perform the corresponding K-theory computation.

   b) <u>*General construction*</u>. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group. We describe the general construction with the following steps.

- Define the following set

$$X(C(\mathbb{G})) := \{\phi : C(\mathbb{G}) \longrightarrow \mathbb{C},\ *\text{-homomorphism} \neq 0\},$$

which is a compact group whenever it is not empty. Suppose that for a given $\mathbb{G}$, $X(C(\mathbb{G})) \neq \varnothing$.
- For every $\phi \in X(C(\mathbb{G}))$ we define the following automorphisms of $C(\mathbb{G})$,

$$\lambda_\phi := (\phi^{-1} \otimes id) \circ \Delta \text{ and } \rho_\phi := (id \otimes \phi) \circ \Delta,$$

where ϕ^{-1} denotes the inverse of ϕ in the group $X(C(\mathbb{G}))$.
- It is straightforward to show that λ_ϕ and ρ_ϕ commute, so that we define the following quantum automorphism of $\mathbb{G}$

$$\alpha_\phi := \lambda_\phi \circ \rho_\phi,$$

for every $\phi \in X(C(\mathbb{G}))$. In other words, we have just defined an action $\alpha : X(C(\mathbb{G})) \longrightarrow Aut(\mathbb{G})$ of $X(C(\mathbb{G}))$ on $\mathbb{G}$ by quantum automorphisms.
- In this situation we can construct the corresponding quantum semi-direct product,

$$\mathbb{F} := X(C(\mathbb{G})) \underset{\alpha}{\ltimes} \mathbb{G}$$

Observe that, in this general construction, the condition $X(C(\mathbb{G})) \neq \varnothing$ is the key point in order to obtain the compact quantum group $\mathbb{F} := X(C(\mathbb{G})) \underset{\alpha}{\ltimes} \mathbb{G}$. The typical compact quantum groups as $SU_q(2)$, $O^+(n)$ or $U^+(n)$ (recall Section 2.1) satisfy this condition. Besides, it is possible to describe explicitly the corresponding compact group $X(C(\mathbb{G}))$ (and even the corresponding action α) for these typical examples (see Section 4 in [216] for more details)

$$\mathbb{G} := SU_{-1}(2) \rightsquigarrow X\big(C(SU_{-1}(2))\big) = \left\{ \begin{pmatrix} x & -\overline{y} \\ y & \overline{x} \end{pmatrix} \in SU(2) \mid xy = 0 \right\}$$

$$\mathbb{G} := O^+(n) \rightsquigarrow X\big(C(O^+(n))\big) = O(n)$$

$$\mathbb{G} := U^+(n) \rightsquigarrow X\big(C(U^+(n))\big) = U(n)$$

In particular, we contemplate computing the following K-theory groups,

$$K_0\Big(C\big(O(n) \underset{\alpha}{\ltimes} O^+(n)\big)\Big) = ? \text{ and } K_1\Big(C\big(O(n) \underset{\alpha}{\ltimes} O^+(n)\big)\Big) = ?$$

$$K_0\Big(C\big(U(n) \underset{\alpha}{\ltimes} U^+(n)\big)\Big) = ? \text{ and } K_1\Big(C\big(U(n) \underset{\alpha}{\ltimes} U^+(n)\big)\Big) = ?$$

ii) As we have already explained in Section 4.2, the main motivation of the collaboration work with A. Freslon [127] has been the computation of the K-theory of the C^*-algebra defining a free wreath product $\mathbb{G} \wr_* S_N^+$ with $N \geqslant 4$. But, to the best knowledge of the author, there is not an obvious way to apply the methodologies from the work of R. Vergnioux and C. Voigt to the free wreath product case. However, it is an open question to know the cohomological dimension of $\mathbb{G} \wr_* S_N^+$ [21]. This, together with the research carried out in [127], yields the

expectation of computing the corresponding K-theory by means of a spectral sequence, which differs from the examples known up to the present.

To this end, the first step should be to adapt some classical homological notions into the framework of triangulated categories in the spirit of Section 1.2.2 and Section 1.2.4 such as the cohomological dimension. Next, inspired by Section 5 in [134] and guided by the analysis of the torsion phenomena of $\widehat{\mathbb{G}\wr_* S_N^+}$ performed in Section 3.7.1, we should carry out a study of the functor Ext for $\widehat{\mathbb{G}\wr_* S_N^+}$. This, together with the general theory of the *Universal Coefficient Theorem*, should suggest an advisable spectral sequence.

5.4 Formulation of the Baum-Connes property for arbitrary quantum groups

First of all, we focus on a less ambitious problem than the one announced in the title. Namely, we focus on the the Baum-Connes property formulation for arbitrary *discrete* quantum groups.

We have already pointed out several times throughout the present dissertation that the torsion-freeness assumption of a discrete quantum group $\widehat{\mathbb{G}}$ is crucial in order to give a satisfactory formulation of the Baum-Connes property in the quantum setting, as it has been done in Section 1.7.2.

The main issue to such a formulation is that it is not clear

a) either what should be the homological ideal $\mathscr{J}$ in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$, which would define the localizing subcategory $\mathscr{N}$ of *(quantum) compactly contractible objects*

b) or what should be the localizing subcategory $\mathscr{L}$ of *(quantum) compactly induced objects*.

In Section 1.6.1 we have illustrated, with duals of classical compact groups, that the torsion phenomena in the quantum setting is not restricted to the study of discrete quantum subgroups, but other more exotic phenomena can occur. For this reason, it is *not enough* to take

a) either $\mathscr{J}$ as the homological ideal by means of the finite discrete quantum subgroups of $\widehat{\mathbb{G}}$ by analogy with the classical case of discrete groups (recall Section 1.2.3)

b) or $\mathscr{L}$ by means of the finite discrete quantum subgroups of $\widehat{\mathbb{G}}$ by analogy with the classical case of discrete groups (recall Section 1.2.3)

If the *only* torsion phenomena for $\widehat{\mathbb{G}}$ comes from finite discrete quantum subgroups, the Meyer-Nest's strategy can be applied *verbatim* as in the torsion-free case of Section 1.7.2 because the induction and restriction functors with respect to discrete quantum subgroups are always adjoint by Lemma 1.7.2.4, which allows the construction of $\mathscr{J}$-projective objects. Otherwise, when $\widehat{\mathbb{G}}$ admits other types of torsion, the Meyer-Nest's strategy must be suitably modified.

Essentially, there are two main ways to tackle this problem. On the one hand, we should find directly an advisable candidate for both $\mathscr{L}$ and $\mathscr{N}$ and prove that they form a complementary pair of localizing subcategories in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$. Notice that it is enough to know *one* of the localizing subcategories of the pair because the other one is completely determined by taking the orthogonal

complement by virtue of Lemma 1.2.1.26. On the other hand, we should find either a stable abelian category or a triangulated category $\mathscr{C}$ and respectively, either a stable homological functor or a triangulated functor $F : \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \longrightarrow \mathscr{C}$. Moreover, this functor must admit a *partially defined adjoint functor* in the sense of Definition 1.2.2.18. In this situation, the choice of the complementary pair $(\mathscr{L}, \mathscr{N})$ is given by Theorem 1.2.2.20.

In any case, the whole torsion phenomena for $\widehat{\mathbb{G}}$ must be taken into account for achieving a satisfactory Baum-Connes property formulation. In this way, inspired by the pioneering work by R. Meyer and R. Nest [133] and by C. Voigt [212], we have announced in Section 4.1.2 that there exists an appropriated candidate for the localizing subcategory $\mathscr{L}_{\widehat{\mathbb{G}}}$ of *compactly induced objects* when $\widehat{\mathbb{G}}$ has torsion. Indeed, this choice of $\mathscr{L}_{\widehat{\mathbb{G}}}$ allows to give an abstract categorical sense to the *strong* Baum-Connes property for: *i*) the dual of a classical compact connected group [133], *ii*) the quantum automorphism group [212] and *iii*) the free wreath product Section 3.7.

In connexion with the above discussion, we suggest the following questions.

a) With the purpose of defining the homological ideal $\mathscr{J}$ and then of constructing $\mathscr{J}$-projective objects in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$, *could it be possible to choose a stable abelian or triangulated category $\mathscr{C}_\delta$ and to construct a, respectively, stable homological or triangulated functor*

$$F_\delta : \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \longrightarrow \mathscr{C},$$

for every torsion action (A, δ) of $\mathbb{G}$ such that the functor

$$F := \Big(F_\delta\Big)_{\delta \in \mathbb{T}or(\widehat{\mathbb{G}})} : \mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}} \longrightarrow \prod_{\delta \in \mathbb{T}or(\widehat{\mathbb{G}})} \mathscr{C}_\delta$$

admits a partially defined adjoint functor?

b) *Could it be possible to show directly that the localizing subcategory $\mathscr{L}_{\widehat{\mathbb{G}}}$ suggested in Section 4.1.2 together with its orthogonal complement $\mathscr{L}_{\widehat{\mathbb{G}}}^{\dashv}$ form a complementary pair of localizing subcategories in $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}}$?*

If G is a classical compact connected group, the subcategory $\mathscr{L}_{\widehat{G}}$ is explicitly described in [133]. *Can we do a similar description without the connectedness assumption?*

Appendices

APPENDIX A

Generalities

The goal of this appendix is just to establish some nomenclature of well-known theories as C^*-algebras, von Neumann algebras, Hilbert modules and multiplier algebras. We don't give any proof and we refer to the corresponding references for the details.

As we have explained in the introduction, the main object of interest in this dissertation is C^*-algebras. They are the underlying structure for compact quantum groups (in the sense of Woronowicz) and (quantum) KK-theory. Nevertheless, the general notion of locally compact quantum group in the sense of Kustermans-Vaes, that is currently accepted to be the most proper one, fit in the context of von Neumann algebras and it needs a very technical theory background. We don't need in this dissertation to go into these details, but in order to give to the notion of compact quantum group (in the sense of Woronowicz) an understandable perspective inside the general theory (Section 1.3.2), it is advisable to establish some language of von Neumann algebras. Finally, Hilbert modules are the elementary objects for the study of KK-theory so it is important to establish the main definitions and constructions that we use in the dissertation.

A.1 Elements of C^*-algebras

Standard references for the necessary material on this subject are [53], [137], [148] and [167] (see also [29] and [154]).

Let A be a $*$-algebra. A C^*-norm (resp. C^*-semi-norm) on A is a norm (resp. semi-norm) p on A such that $p(ab) \leqslant p(a)p(b)$, $p(a^*) = p(a)$ and $p(a^*a) = p(a)^2$, for all $a, b \in A$.

- If C is any C^*-algebra and $\varphi : A \longrightarrow C$ is a $*$-homomorphism, then it is clear that the

application

$$\begin{array}{rccl} p: & A & \longrightarrow & \mathbb{R}^+ \\ & a & \longmapsto & p(a) := ||\varphi(a)|| \end{array}$$

is a C^*-semi-norm on A. It is a C^*-norm whenever φ is injective.

- If $\mathcal{P}$ is a family of C^*-semi-norms on A, then the subset

$$A_{\mathcal{P}} := \{a \in A \mid \sup_{p \in \mathcal{P}}\{p(a)\} < \infty\}$$

is a $*$-subalgebra of A and the appllicatoin

$$\begin{array}{rccl} \widetilde{p}: & A_{\mathcal{P}} & \longrightarrow & \mathbb{R}^+ \\ & a & \longmapsto & \widetilde{p}(a) := \sup_{p \in \mathcal{P}}\{p(a)\} \end{array}$$

is a C^*-semi-norm on $A_{\mathcal{P}}$.

A.1.1 Definition. Let A be a $*$-algebra. The enveloping C^*-algebra of A, denoted by $C^*(A)$, is the separated completion of A by the C^*-semi-norm

$$\begin{array}{rccl} ||\cdot||_{max}: & A & \longrightarrow & \mathbb{R}^+ \\ & a & \longmapsto & ||a||_{max} := \sup\{||\pi(a)|| \mid \ \pi \text{ is a representation of } A\} \end{array}$$

whenever this supremum is finite.

A.1.2 Remarks. 1. It is well-known that every $*$-Banach algebra admits a C^*-enveloping algebra.

2. If A is a $*$-algebra and $C^*(A)$ exists, then the enveloping C^*-algebra satisfies the following universal property. For every $*$-homomorphism $\varphi : A \longrightarrow C$, where C is a C^*-algebra, there exists a unique $*$-homomorphism $\widetilde{\varphi} : C^*(A) \longrightarrow C$ such that $\widetilde{\varphi} \circ j = \varphi$, where $j : A \longrightarrow C^*(A)$ denotes the canonical map.

A.1.3 Definition. Let A be a C^*-algebra. An approximate unit for A is a increasing sequence $\{e_n\}_{n \in \mathbb{N}}$ of positive elements in the closed unit ball of A such that

$$\lim_{n \longrightarrow \infty} ae_n = a = \lim_{n \longrightarrow \infty} e_n a,$$

for all $a \in A$.

A.1.4 Remark. It is well known that every C^*-algebra admits an approximate unit as a generalized sequence. If A is in addition *separable*, such approximate unit can be realized as a *sequence*. Recall that all C^*-algebras of this dissertation are supposed to be separable.

In this case, A is called sometimes *σ-unital* and it can be shown (see [148] for a proof) that the existence of a countable approximate unit is equivalent to the existence of a strictly positive element, that is, an element $h \in A$ such that $\phi(h) > 0$, for all state $\phi \in A^*$ (a positive element $h \in A$ is strictly positive if and only if hA is dense in A).

A.1.5 Definition. Let A be a C^*-algebra and $I \subset A$ an ideal. We say that I is essential if the following condition holds,

$$\forall a \in A,\ aI = (0) \Rightarrow a = 0$$

A.1.6 Remark. If I is in addition a *two-sided closed ideal*, then I is itself a C^*-subalgebra of A as well as the quotient A/I. Essential ideals are important for the notion of *multiplier algebra*.

A.1.7 Definition. Let A, B and E three C^*-algebras. We say that E is an extension of A by B if there is a short exact sequence of C^*-algebras,

$$0 \longrightarrow B \xrightarrow{\iota} E \xrightarrow{p} A \longrightarrow 0,$$

meaning more precisely that ι is an injective $*$-homomorphism, p is a surjective $*$-homomorphism and $Im(\iota) = \ker(p)$.

We say that B is a *split extension* if there exists a $*$-homomorphism $s : A \longrightarrow E$ (called *section of* p) such that $p \circ s = id_A$. If such a section is just a completely positive linear contraction, we say that B is a *semi-split extension*.

The set of all extensions of A by B is denoted by $\mathcal{E}xt(A, B)$.

A.1.8 Remark. It is important to recall that, by means of the *Busby invariant*, $\mathcal{E}xt(A, B)$ is an additive commutative semi-group and, by defining an appropriate notion of *equivalent extensions*, the corresponding quotient $Ext(A, B) := \mathcal{E}xt(A, B)/\sim$ is an additive commutative semi-group with neutral element.

Denote by $Ext^{-1}(A, B)$ the group of invertible classes in $Ext(A, B)$. It can be shown that, whenever A is separable, a class of extension in $Ext(A, B)$ is invertible if and only if the extension is *semi-split*. Moreover, it is a well-known result that $Ext^{-1}(A, B) \cong KK(\Sigma(A), B)$.

The analogous constructions and results hold when we require *equivariance* conditions with respect to a group action. We refer to [24], [86] and [187] for more information about the theory of extensions of C^*-algebras and for the proof the results stated in this remark.

A.1.9 Theorem. *Let A be a C^*-algebra. The following properties hold.*

i) The relation defined on projections of A by

$$p \geqslant q \Leftrightarrow pq = q = qp$$

is a partial order.

ii) Given two projections $p, q \in A$ we have that $p \geqslant q$ if and only if $pAp \supseteq qAq$. In particular, $pAp = qAq$ if and only if $p = q$.

iii) Given a projection $p \in A$, if pAp is finite dimensional and $dim(pAp) > 1$, then there exists a non zero projection $q \in A$ such that $p \geqslant q$.

iv) If A is unital and finite dimensional, then it has a minimal non zero projection with respect to the order $\geqslant$. If p denotes such a projection, then we have $pAp = \mathbb{C}p$.

v) A is finite dimensional if and only if $A \cong \mathcal{M}_{k_1}(\mathbb{C}) \oplus \ldots \oplus \mathcal{M}_{k_n}(\mathbb{C})$, for some $k_1, \ldots, k_n, n \in \mathbb{N}$.

vi) A is simple if and only if $A \cong \mathcal{M}_k(\mathbb{C})$, for some $k \in \mathbb{N}$.

Some standard constructions for C^*-algebras are the following.

A.1.10 Theorem-Definition. *Let A and B two C^*-algebras. There exists a unique (up to a canonical $*$-isomorphism) C^*-algebra P with $*$-homomorphisms $j_A : A \longrightarrow P$ and $j_B : B \longrightarrow P$ such that*

i) $P = C^*\langle j_A(A), j_B(B)\rangle$

ii) For any C^*-algebra Q with $*$-homomorphisms $\phi_A : A \longrightarrow Q$ and $\phi_B : B \longrightarrow Q$, there exists a (necessarily unique) $*$-homomorphism $\psi : P \longrightarrow Q$ such that

$$\psi \circ j_A = \phi_A \text{ and } \psi \circ j_B = \phi_B$$

The C^*-algebra P constructed in this way is called maximal free product of A and B *and is denoted by $A * B$. The $*$-homomorphisms j_A and j_B are called* canonical inclusions.

A.1.11 Theorem-Definition. *Let A and B two C^*-algebras. If $\pi : A \longrightarrow \mathcal{B}(H)$ and $\rho : B \longrightarrow \mathcal{B}(K)$ are two faithful representations of A and B on Hilbert spaces H and K, respectively; then there exists a unique $*$-homomorphism $\lambda : A \odot B \longrightarrow \mathcal{B}(H \otimes K)$ such that*

$$\lambda(a \otimes b) = \pi(a) \otimes \rho(b),$$

for all $a \in A$ and all $b \in B$.

The application

$$\begin{array}{rccl} ||\cdot||_{min} : & A \odot B & \longrightarrow & \mathbb{R}^+ \\ & x & \longmapsto & ||x||_{min} := ||\lambda(x)|| \end{array}$$

defines a C^*-norm on $A \odot B$, called minimal norm. *Moreover, $||\cdot||_{min}$ does not depend on the choice of the faithful representations π and ρ.*

The completion of $A \odot B$ with respect to the minimal norm is denoted by $A \otimes B$ and it is called minimal tensor product of A and B.

Moreover, if C, D are two C^*-algebras and $\varphi : A \longrightarrow C$, $\psi : B \longrightarrow D$ are two $*$-homomorphisms, then there exists a unique $*$-homomorphism

$$\varphi \otimes \psi : A \otimes B \longrightarrow C \otimes D$$

A.1.12 Theorem-Definition. *Let A and B two C^*-algebras. Let $||\cdot||_\beta$ be any C^*-norm on $A \odot B$ and let $A \otimes_\beta B$ the corresponding completion. If $\gamma : A \otimes_\beta B \longrightarrow \mathcal{B}(H)$ is a non-degenerate representation of $A \otimes_\beta B$ on a Hilbert space H, then there exist unique non-degenerate representations $\pi : A \longrightarrow \mathcal{B}(H)$ and $\rho : B \longrightarrow \mathcal{B}(H)$ such that*

$$\gamma(a \otimes b) = \pi(a)\rho(b),$$

for all $a \in A$ and all $b \in B$. Moreover, π and ρ commute in $\mathcal{B}(H)$.

The application

$$\begin{array}{rccl} ||\cdot||_{max} : & A \odot B & \longrightarrow & \mathbb{R}^+ \\ & x & \longmapsto & ||x||_{max} := \sup\{||\gamma(x)|| \mid \gamma \text{ is a non-degenerate rep. of } A \odot B\} \end{array}$$

defines a C^*-norm on $A \odot B$, called maximal norm.

The completion of $A \odot B$ with respect to the maximal norm is denoted by $A \underset{\max}{\otimes} B$ and it is called maximal tensor product of A and B.

Moreover, if C is a C^*-algebra and $\varphi : A \longrightarrow C$, $\psi : B \longrightarrow C$ are two $*$-homomorphisms with commuting ranges, then there exists a unique $*$-homomorphism

$$\varphi \times \psi : A \underset{\max}{\otimes} B \longrightarrow C$$

such that $\varphi \times \psi(a \otimes b) = \varphi(a)\psi(b)$, for all $a \in A$ and all $b \in B$.

It is advisable to establish some nomenclature about states and weights on a C^*-algebras.

A.1.13 Definition. Let A be a C^*-algebra. A linear form $\phi : A \longrightarrow \mathbb{C}$ is called

- positive if $\phi(aa^*) \geqslant 0$, for all $a \in A$,
- normalized if $||\phi|| = 1$,
- faithful if $\phi(aa^*) \neq 0$ for all non-zero $a \in A$,
- tracial if $\phi(ab) = \phi(ba)$, for all $a, b \in A$.
- state if ϕ is positive and normalized.

A.1.14 Remark. Let A be any $*$-algebra. Consider a non-zero positive linear form $\phi : A \longrightarrow \mathbb{C}$. For any $a, b \in A$ the following formula holds

$$\phi(a^*b) = \overline{\phi(b^*a)}$$

Indeed, given $a, b \in A$ we write

$$\begin{aligned} 0 \leqslant \phi\big((a+b)^*(a+b)\big) &= \phi\big(a^*a + a^*b + b^*a + b^*b\big) \\ &= \phi(a^*a) + \phi(a^*b) + \phi(b^*a) + \phi(b^*b) \end{aligned}$$

Since $\phi(a^*a) \geqslant 0$ and $\phi(b^*b) \geqslant 0$, it must be $0 = Im\big(\phi(a^*b) + \phi(b^*a)\big) = Im(\phi(a^*b)) + Im(\phi(b^*a))$. Since this is true for all $b \in B$, it is also true for ib. Hence we have that $0 = Im(i\phi(a^*b)) + Im(\bar{i}\phi(b^*a)) = Re(\phi(a^*b)) - Re(\phi(b^*a))$. In other words, we have that $Im(\phi(a^*b)) = -Im(\phi(b^*a))$ and $Re(\phi(a^*b)) = Re(\phi(b^*a))$, whence the formula.

In particular, any positive linear form $\phi : A \longrightarrow \mathbb{C}$ is involutive whenever A is unital.

A.1.15 Proposition. *Let A be a C^*-algebra.*

i) If $\phi : A \longrightarrow \mathbb{C}$ is a positive linear form, then ϕ satisfies the Schwarz inequality meaning that

$$|\phi(b^*a)|^2 \leqslant \phi(a^*a)\phi(b^*b),$$

for all $a, b \in A$. Moreover, $||\phi|| = \phi(1_A)$ whenever A is unital.

ii) If $\phi : A \longrightarrow \mathbb{C}$ is a positive linear form, then it is bounded.

iii) If $\phi : A \longrightarrow \mathbb{C}$ is a bounded linear form, then ϕ is positive if and only if $||\phi|| = \lim_{n\to\infty} \phi(e_n)$, for every approximate unit $\{e_n\}_{n\in\mathbb{N}}$ of A. In particular, a bounded linear form on a unital C^*-algebra A is positive if and only if $\phi(1) = ||\phi||$.

iv) If $B \subset A$ is a C^*-subalgebra of A and $\phi : B \longrightarrow \mathbb{C}$ is a positive linear form, then there exists a positive linear functional $\widetilde{\phi} : A \longrightarrow \mathbb{C}$ extending ϕ and such that $||\widetilde{\phi}|| = ||\phi||$.

The GNS construction is the main result that we have to recall,

A.1.16 Theorem (GNS construction for states on C^*-algebras)**.** *Let A be a C^*-algebra. Given a (faithful) state $\phi : A \longrightarrow \mathbb{C}$ there exists a (unique up to a unitary transformation) triple $(H_\phi, \Lambda_\phi, \pi_\phi)$ where H_ϕ is a Hilbert space, $\Lambda_\phi : A \longrightarrow H_\phi$ is a linear map with dense image and $\pi_\phi : A \longrightarrow \mathcal{B}(H_\phi)$ is a (faithful) representation such that*

i) $\phi(a^*b) = \langle \Lambda_\phi(a), \Lambda_\phi(b)\rangle$, for all $a, b \in A$,

ii) $\pi_\phi(a)(\Lambda_\phi(b)) = \Lambda_\phi(ab)$, for all $a, b \in A$.

Moreover, there exists a unique norm 1 vector $\Omega \in H_\phi$ (called cyclic vector*) such that*

i) $\Lambda_\phi(a) = \pi_\phi(a)(\Omega)$, for all $a \in A$,

ii) $\phi(a) = \langle \Omega, \pi_\phi(a)(\Omega)\rangle$, for all $a \in A$.

A.1.17 Definition. Let A be a C^*-algebra. A function $\phi : A^+ \longrightarrow [0, +\infty]$ is called a weight if

i) $\phi(a + b) = \phi(a) + \phi(b)$, for all $a, b \in A^+$,

ii) $\phi(ra) = r\phi(a)$, for all $a \in A^+$ and all $r \in [0, +\infty)$.

In that case, we write

- $\mathcal{M}_\phi^+ := \{a \in A^+ \mid \phi(a) < +\infty\}$ for the *positive ϕ-integrable elements*,
- $\mathcal{N}_\phi := \{a \in A^+ \mid \phi(a^*a) < +\infty\}$ for the *ϕ-square-integrable elements*,
- $\mathcal{M}_\phi :=$ *span* $\mathcal{M}_\phi^+ =$ *span* $\mathcal{N}_\phi^*\mathcal{N}_\phi$ for the *ϕ-integrable elements.*

A.1.18 Definition. Let A be a C^*-algebra. A weight $\phi : A^+ \longrightarrow [0, +\infty]$ is called

- faithful if $\phi(a) \neq 0$ for all non-zero $a \in A^+$,
- lower semi-continuous if the subset $\{a \in A^+ \mid \phi(a) \leqslant \lambda\} \subset A$ is closed for all $\lambda \in \mathbb{R}^+$,
- densely defined if $\mathcal{M}_\phi^+$ is dense in A^+ (or, equivalently, if $\mathcal{N}_\phi$ or $\mathcal{M}_\phi$ is dense in A).

The C^*-algebraic picture of locally compact quantum groups needs the notion of KMS-weights. We refer to [113], [148] or [184] for more details about them. We give here the definition for the convenience of the exposition.

A.1.19 Definition. Let A be a C^*-algebra. A KMS-weight on A is a non-zero densely defined and lower semi-continuous weight $\phi : A^+ \longrightarrow [0, +\infty]$ for which there exists a norm continuous one-parameter group σ on A (called *modular automorphism group for* ϕ) such that

i) $\phi \circ \sigma_t = \phi$, for all $t \in \mathbb{R}$,

ii) $\phi(x^*x) = \phi(\sigma_{i/2}(x)\sigma_{i/2}(x)^*)$, for all $x \in Dom(\sigma_{i/2})$.

The same as for states on C^*-algebras, we can establish a GNS construction for weights.

A.1.20 Theorem (GNS construction for weights on C^*-algebras)**.** *Let A be a C^*-algebra. Given a (faithful) weight $\phi : A^+ \longrightarrow [0, +\infty]$ there exists a (unique up to a unitary transformation) triple $(H_\phi, \Lambda_\phi, \pi_\phi)$ where H_ϕ is a Hilbert space, $\Lambda_\phi : \mathcal{N}_\phi \longrightarrow H_\phi$ is a linear map with dense image and $\pi_\phi : A \longrightarrow \mathcal{B}(H_\phi)$ is a (faithful) representation such that*

i) $\phi(a^*b) = \langle \Lambda_\phi(a), \Lambda_\phi(b) \rangle$, for all $a, b \in \mathcal{N}_\phi$,

ii) $\pi_\phi(c)(\Lambda_\phi(b)) = \Lambda_\phi(cb)$, for all $c \in A$ and all $b \in \mathcal{N}_\phi$.

Next, some nomenclature about complete positivity is required for the construction of quantum crossed product by a discrete quantum group (namely, the notion of *conditional expectation*).

A.1.21 Definition. Let A and B two C^*-algebras. A linear map $\varphi : A \longrightarrow B$ is called

- completely positive (*c.p.* for short) if for any $n \in \mathbb{N}$ the obvious linear map $\varphi_n : \mathcal{M}_n(A) \longrightarrow \mathcal{M}_n(B)$ is positive. If A, B and φ are in addition unital, then it is called unital completely positive (*u.c.p.* for short).
- completely bounded (*c.b.* for short) if

$$||\varphi||_{cb} := \sup_{n \in \mathbb{N}} \{||\varphi_n||\} < \infty$$

- contractive completely positive (*c.c.p.* for short) if φ is c.p. and $||\varphi||_{cb} \leqslant 1$.

A.1.22 Remark. Any $*$-homomorphism and any positive linear form is c.p.

The classical GNS construction can be generalized for c.p. maps with the celebrated Stinespring's theorem (see Theorem 1.5.3 in [29] for a proof).

A.1.23 Theorem (Stinespring)**.** *Let A be a unital C^*-algebra and H a Hilbert space. If $\varphi : A \longrightarrow \mathcal{B}(H)$ is a c.p. map, then there exists a triple $(H_\varphi, \pi_\varphi, V_\varphi)$ where H_φ is a Hilbert space, $\pi_\varphi : A \longrightarrow \mathcal{B}(H_\varphi)$ is a representation and $V_\varphi : H \longrightarrow H_\varphi$ is a bounded operator such that*

$$\varphi(a) = V_\varphi^* \circ \pi_\varphi(a) \circ V_\varphi,$$

for all $a \in A$. In particular, $||\varphi|| = ||V_\varphi^* \circ V_\varphi|| = ||\varphi(1)||$.

Moreover, if $\pi_\varphi(A)V_\varphi(H)$ is a dense subspace in H_φ, then the triple $(H_\varphi, \pi_\varphi, V_\varphi)$ is unique up to a unitary transformation. Such a triple is called Stinespring dilation of φ.

A.1.24 Definition. Let A be a C^*-algebra and $B \subset A$ a C^*-subalgebra. A conditional expectation from A to B is a linear map $E : A \longrightarrow B$ such that

i) E is a projection, that is, $E(b) = b$ for all $b \in B$,

ii) E is a B-bimodule map, that is, $E(bab') = bE(a)b'$ for all $b, b' \in B$ and all $a \in A$,

iii) E is c.c.p.

We say that E is faithful if the following condition holds

$$\forall a \in A^+,\ E(a) = 0 \ \Rightarrow a = 0$$

A.2 Elements of von Neumann algebras

Standard references for the necessary material on this subject are [54], [167], [183], [184] or Chapter 4 in [137] (see also [89]).

Let H be a Hilbert space. The space $\mathcal{B}(H)$ of linear operator on H can be equipped with the following convex topology. For every $\xi, \eta \in H \otimes l^2(\mathbb{N})$ we define the following semi-norm on $\mathcal{B}(H)$,

$$\widetilde{\phi}_{\xi,\eta}(T) := |\langle \eta, (T \otimes id)\xi \rangle|,$$

for all $T \in \mathcal{B}(H)$. The topology on $\mathcal{B}(H)$ defined by the family of semi-norms $\{\widetilde{\phi}_{\xi,\eta}\}_{\xi,\eta \in H \otimes l^2(\mathbb{N})}$ is called *σ-weak topology* or *ultraweak topology*. If $E \subset \mathcal{B}(H)$ is a subspace, its σ-weak closure is denoted by $\overline{E}^{\sigma-w}$.

A.2.1 Definition. Let H and K be two Hilbert spaces. If $M \subset \mathcal{B}(H)$ and $N \subset \mathcal{B}(K)$ are two von Neumann algebras, a linear map $\varphi : M \longrightarrow N$ is called normal if it is continuous with respect to the σ-weak topologies on M and N, respectively.

A.2.2 Definition. Let H be a Hilbert space. If $M \subset \mathcal{B}(H)$ is a von Neumann algebra, the pre-dual of M, denoted by M_*, is the following closed vector subspace of the dual space of M

$$M_* := \{\omega \in M^* \mid \omega \text{ is } \sigma\text{-weakly continuous}\}$$

A.2.3 Definition. Let H be a Hilbert space. If $M \subset \mathcal{B}(H)$ is a von Neumann algebra, a function $\phi : M^+ \longrightarrow [0, +\infty]$ is called a weight if

i) $\phi(a + b) = \phi(a) + \phi(b)$, for all $a, b \in M^+$,

ii) $\phi(ra) = r\phi(a)$, for all $a \in M^+$ and all $r \in [0, +\infty)$.

In that case, we write

- $\mathcal{M}_\phi^+ := \{a \in M^+ \mid \phi(a) < +\infty\}$ for the *positive ϕ-integrable elements*,

- $\mathcal{N}_\phi := \{a \in M^+ \mid \phi(a^*a) < +\infty\}$ for the *ϕ-square-integrable elements*,
- $\mathcal{M}_\phi :=$ *span* $\mathcal{M}_\phi^+ =$ *span* $\mathcal{N}_\phi^*\mathcal{N}_\phi$ for the *ϕ-integrable elements*.

A.2.4 Definition. Let H be a Hilbert space and $M \subset \mathcal{B}(H)$ a von Neumann algebra. A weight $\phi : M^+ \longrightarrow [0, +\infty]$ is called

- faithful if $\phi(a) \neq 0$ for all non-zero $a \in M^+$,
- normal if the subset $\{a \in M^+ \mid \phi(a) < \lambda\} \subset M$ is ultraweakly closed for all $\lambda \in [0, +\infty)$,
- semi-finite if $\mathcal{M}_\phi^+$ is dense in M^+ (or, equivalently, if $\mathcal{N}_\phi$ or $\mathcal{M}_\phi$ is dense in M) with respect to the ultraweakly topology.

The same as for states on C^*-algebras, we can establish a GNS construction for weights on von Neumann algebras. The celebrated Tomita-Takesaki theory [182] provides a detailed description of such a GNS construction which is, by the way, one of the main ingredients for the proper development of the locally compact quantum groups theory.

A.2.5 Theorem (GNS construction for weights on von Neumann algebras)**.** *Let H be a Hilbert space and $M \subset \mathcal{B}(H)$ a von Neumann algebra. Given a (faithful) weight $\phi : M^+ \longrightarrow [0, +\infty]$ there exists a (unique up to a unitary transformation) triple $(H_\phi, \Lambda_\phi, \pi_\phi)$ where H_ϕ is a Hilbert space, $\Lambda_\phi : \mathcal{N}_\phi \longrightarrow H_\phi$ is a linear map with dense image and $\pi_\phi : M \longrightarrow \mathcal{B}(H_\phi)$ is a (faithful) representation such that*

i) $\phi(a^*b) = \langle \Lambda_\phi(a), \Lambda_\phi(b) \rangle$, *for all* $a, b \in \mathcal{N}_\phi$,

ii) $\pi_\phi(c)(\Lambda_\phi(b)) = \Lambda_\phi(cb)$, *for all* $c \in M$ *and all* $b \in \mathcal{N}_\phi$.

Moreover, if ϕ is normal and semi-finite, then Λ_ϕ is closed with respect to the ultraweakly topology on M and the weak topology on H_ϕ.

A.3 Elements of Hilbert modules

Standard references for the necessary material on this subject are [117], [224], [86], [24] (see also [101] and [161]).

A.3.1 Remarks. 1. Every C^*-algebra A is a Hilbert A-module with inner product given by $\langle a, b \rangle := a^*b$, for all $a, b \in A$.

2. If A is a C^*-algebra, $\{e_n\}_{n\in\mathbb{N}}$ is an approximate unit for A and H is a Hilbert A-module, then it is easy to see that
$$\lim_{n\to\infty} \xi e_n = \xi = \lim_{n\to\infty} e_n \xi,$$
for all $\xi \in H$. As a consequence, HA is dense in H.

3. Let A be a C^*-algebra and H a Hilbert A-module. Put $\langle H, H\rangle := span\{\langle \xi, \eta\rangle \mid \xi, \eta \in H\}$. We can show that $\overline{\langle H, H\rangle}$ is a two-sided closed ideal in A such that $H\langle H, H\rangle$ is dense in H. If moreover $\langle H, H\rangle$ is dense in A, then we say that H is a *full* Hilbert A-module.

A.3.2 Definition. Let A be a C^*-algebra and H a Hilbert A-module. We say that H is countably generated if there exists a countable set $\{\xi_n\}_{n\in\mathbb{N}}$ in H such that $span\{\xi_n a \mid a \in A\}$ is dense in H.

In this case, the set $\{\xi_n\}_{n\in\mathbb{N}}$ is called *set of generators for* H.

A.3.3 Definition. Let A be a C^*-algebra and H, K Hilbert A-modules.

- An adjointable operator between H and K is a linear map $T : H \longrightarrow K$ for which there exists a (necessary unique) linear map $T^* : K \longrightarrow H$ such that

  $$\langle T(\xi), \eta\rangle = \langle \xi, T^*(\eta)\rangle,$$

  for all $\xi \in H$ and all $\eta \in K$.

  In this case T^* is called *adjoint operator of* T and the C^*-algebra of all adjointable operators between H and K is denoted by $\mathcal{L}_A(H, K)$.

- A finite rank operator between K and H is an adjointable operator between K and H of the form

  $$\begin{array}{rccl} \theta_{\xi,\eta} : & K & \longrightarrow & H \\ & \zeta & \longmapsto & \theta_{\xi,\eta}(\zeta) := \xi\langle \eta, \zeta\rangle, \end{array}$$

  for some given vectors $\xi \in H$ and $\eta \in K$.

  We define the set of *compact operators between* K *and* H as the following two-sided closed essential ideal of $\mathcal{L}_A(K, H)$

  $$\mathcal{K}_A(K, H) := \overline{span}\{\theta_{\xi,\eta} \mid \xi \in H, \eta \in K\}$$

A.3.4 Remark. Sometimes it is useful to work with other topologies than the norm one on $\mathcal{L}_A(H, K)$. Namely, the *strict topology* which is the topology given by the following seminorms

$$T \mapsto ||T(\xi)|| \text{ and } T \mapsto ||T^*(\eta)||,$$

with $\xi \in H$ and $\eta \in K$ (see Chapter 1 in [117] for more details).

A.3.5 Theorem (Standard Hilbert module constructions)**.** *i) (Exterior tensor product) Let A, B be two C^*-algebras. If H is a Hilbert A-module and K a Hilbert B-module, the exterior tensor product of H and K, denoted by $H \otimes K$, is the Hilbert $A \otimes B$-module defined as the completion of $H \odot K$ with respect to the $A \otimes B$-valued inner product given by*

$$\langle \xi \otimes \eta, \xi' \otimes \eta'\rangle := \langle \xi, \eta\rangle \otimes \langle \xi', \eta'\rangle,$$

for all $\xi, \xi' \in H$ and all $\eta, \eta' \in K$.

Moreover, there exists a injective $*$-homomorphism

$$j : \mathcal{L}_A(H) \otimes \mathcal{L}_B(K) \longrightarrow \mathcal{L}_{A\otimes B}(H \otimes K)$$

such that $j(T \otimes S)(\xi \otimes \eta) = T(\xi) \otimes S(\eta)$, for all $T \in \mathcal{L}_A(H), S \in \mathcal{L}_B(K)$ and all $\xi \in H, \eta \in K$. Moreover, $j\big(\mathcal{K}_A(H) \otimes \mathcal{K}_B(K)\big) = \mathcal{K}_{A\otimes B}(H \otimes K)$.

ii) (Interior tensor product) Let A, B be two C^*-algebras. If H is a Hilbert A-module, K a Hilbert B-module and $\phi : A \longrightarrow \mathcal{L}_B(K)$ is a $*$-homomorphism, the interior tensor product of H and K with respect to ϕ, denoted by $H \underset{\phi}{\otimes} K$, is the Hilbert B-module defined as the completion of $H \odot K$ with respect to the B-valued inner product given by

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle := \langle \eta, \phi(\langle \xi, \xi' \rangle)\eta' \rangle,$$

for all $\xi, \xi' \in H$ and all $\eta, \eta' \in K$.

Moreover, there exists a $*$-homomorphism

$$j : \mathcal{L}_A(H) \longrightarrow \mathcal{L}_B(H \underset{\phi}{\otimes} K)$$

such that $j(T) = T \underset{\phi}{\otimes} id$, for all $T \in \mathcal{L}_A(H)$. If ϕ is injective, then so is j.

The $*$-homomorphism j transforms compact operators into compact operators whenever $\phi : A \longrightarrow \mathcal{K}_B(K)$ is a $*$-homomorphism. In this case, if ϕ is injective (resp. surjective), then so is j.

iii) (Pushout) Let A, B be two C^*-algebras. If H is a Hilbert A-module and $\varphi : A \longrightarrow B$ is a surjective $*$-homomorphism, the pushout of H by φ, denoted by H_φ, is the Hilbert B-module defined as the completion of H/N_φ, where $N_\varphi := \{\xi \in H \mid \varphi(\langle \xi, \xi \rangle) = 0\}$, with respect to the B-valued inner product given by

$$\langle [\xi], [\eta] \rangle := \varphi(\langle \xi, \eta \rangle),$$

for all $\xi, \eta \in H$.

Moreover, there exists a $*$-homomorphism

$$\begin{array}{rccl} j: & \mathcal{L}_A(H) & \longrightarrow & \mathcal{L}_B(H_\varphi) \\ & T & \longmapsto & j(T) := T_f,\ T_f([\xi]) := [T(\xi)],\ \textit{for all } \xi \in H \end{array}$$

The $*$-homomorphism j transforms compact operators into compact operators.

A.3.6 Theorem (Kasparov's stability theorem)**.** *Let A be a C^*-algebra and H a Hilbert A-module. If H is countably generated, then*

$$H \oplus H_A \cong H_A,$$

where $H_A := \bigoplus_{i \in I} A$.

As a consequence, H is countably generated if and only if $\mathcal{K}_A(H)$ is σ-unital.

A.3.7 Definition. Let A, B be two C^*-algebras and H a Hilbert B-module. A $*$-homomorphism $\varphi : A \longrightarrow \mathcal{L}_B(H)$ is called non-degenerate if $\varphi(A)H$ is dense in H.

A.3.8 Remark. We can give several equivalent conditions for the notion of non-degenerate $*$-homomorphism (see Proposition 2.5 in [117] for a proof). Namely, $\varphi : A \longrightarrow \mathcal{L}_B(H)$ is a non-degenerate $*$-homomorphism if and only if there exists an approximate unit $\{e_n\}_{n \in \mathbb{N}}$ of A such that $\lim_{n \to \infty} \varphi(e_n) = id_H$ with respect to the *strict topology*.

A.3.9 Definition. Let A, B be two C^*-algebras and H a Hilbert B-module. A completely positive map $\varphi : A \longrightarrow \mathcal{L}_B(H)$ is called strict if there exists an approximate unit $\{e_n\}_{n\in\mathbb{N}}$ of A such that $\{\varphi(e_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}_B(H)$ with respect to the strict topology.

A.3.10 Remark. Of course, non-degeneracy implies strictness and strictness is always fulfilled whenever A is unital.

A far-reaching generalization of the classical GNS construction and Stinespring dilation was given by G. G. Kasparov in [101] for strict completely positive maps in the framework of Hilbert modules (see Theorem 5.6 in [117] for a proof).

A.3.11 Theorem (KSGNS construction)**.** *Let A, B be two C^*-algebras and H a Hilbert B-module. If $\varphi : A \longrightarrow \mathcal{L}_B(H)$ is a strict completely positive map, then there exists a (unique up to a unitary transformation) triple $(H_\varphi, V_\varphi, \pi_\varphi)$ where H_φ is a Hilbert B-module, $V_\varphi \in \mathcal{L}_B(H, H_\varphi)$ is an adjointable operator and $\pi_\varphi : A \longrightarrow \mathcal{L}_B(H_\varphi)$ is a representation such that*

i) $\varphi(a) = V_\varphi^* \circ \pi_\varphi(a) \circ V_\varphi$, for all $a \in A$,

ii) $\pi_\varphi(A) \circ V_\varphi(H)$ is dense in H_φ.

We say that φ is KSGNS-faithful if the corresponding representation π_φ is faithful.

A.3.12 Remark. Let A, B be two C^*-algebras and H, E Hilbert B-modules. If $\pi : A \longrightarrow \mathcal{L}_B(E)$ is a non-degenerate representation and $V \in \mathcal{L}_B(H, E)$ is an adjointable operator, then the map $\varphi : A \longrightarrow \mathcal{L}_B(H)$ defined by

$$\varphi(a) := V^* \circ \pi(a) \circ V, \text{ for all } a \in A$$

is a strict completely positive map in the sense of Definition A.3.9.

This observation with the theorem above assure that *a strict completely positive map is fully determined by its KSGNS-construction.* This fact is used in the construction of reduced crossed products by a discrete (quantum) group (see Theorem 1.5.1.1 and Theorem 1.5.2.1).

A.3.13 Remark. We can apply this construction to (strict) conditional expectations, which has been used for the explicit construction of the reduced crossed product by a discrete (quantum) group.

Let $C \subset B \subset A$ be C^*-algebras with conditional expectations $A \xrightarrow{E_A} B \xrightarrow{E_B} C$. Assume that both E_A and E_B are KSGNS-faithful. We claim that the composition $E := E_B \circ E_A : A \longrightarrow C$ is a KSGNS-faithful conditional expectation from A to C.

Namely, given $a \in A$ suppose that $E((ax)^*ax) = 0$, for all $x \in A$. In particular, this is true for $x := yb$ for some $y \in A$ and $b \in B \subset A$ and we have $E((ayb)^*ayb) = 0$, for all $b \in B$ and all $y \in A$. In other words,

$$E_B\big(E_A((ayb)^*ayb)\big) = E_B\big(E_A((b^*y^*a^*ayb)\big) = E_B\big(b^*E_A(y^*a^*ay)b\big) = 0,$$

for all $b \in B$ and all $y \in A$. Since $E_A((ay)^*ay) \geqslant 0$ (because E is c.p.) and E_B is KSGNS-faithful, it must be $E_A(y^*a^*ay) = 0$ for all $y \in A$. Since E_A is also KSGNS-faithful, we conclude that $a = 0$ and hence $E = E_B \circ E_A$ is KSGNS-faithful as claimed.

A.3.14 Definition. Let A, B two C^*-algebras. We say that A and B are stably isomorphic if $\mathcal{K} \otimes A \cong \mathcal{K} \otimes B$.

A.3.15 Definition. Let A, B two C^*-algebras. We say that A and B are Morita equivalent if there exists a full Hilbert A-module H such that $B \cong \mathcal{K}_A(H)$.

In this case we write $A \underset{M}{\sim} B$.

A.3.16 Remark. We can show that *Morita equivalence* is an equivalence relation between C^*-algebras. In addition, if our C^*-algebras are supposed to be σ-unital, then the notions of *stably isomorphism* and *Morita equivalence* are the same. We refer to Chapter 7 of [117] for more details.

We can give a more algebraic approach of Morita equivalence by means of the notion of *imprimitivity bimodules*. In this way, we can show that A and B are Morita equivalent if and only if there exists and (A, B)-bimodule ${}_AX_B$ (called *imprimitivity bimodule*) such that the categories of Hilbert A-modules and Hilbert B-modules are equivalent through the functor of tensor product by ${}_AX_B$. We refer to [161] and [162] for more details.

In order to present the Kasparov KK-theory of Section 1.7, it is advisable to recall the notions of *graded Hilbert modules*. For more details we refer to [24].

A.3.17 Definition. Let G be a locally compact group. A G-C^*-algebra is a C^*-algebra A with a group homomorphism $\alpha : G \longrightarrow Aut(A)$, $g \longmapsto \alpha_g$, such that for every $a \in A$ the map

$$\begin{array}{ccc} G & \longrightarrow & A \\ g & \longmapsto & \alpha_g(a) \end{array}$$

is norm-continuous.

A.3.18 Definition. Let G be a locally compact group and A a G-C^*-algebra. A G-equivariant Hilbert A-module is a Hilbert A-module H with a continuous action

$$\begin{array}{ccc} G \times H & \longrightarrow & H \\ (g, \xi) & \longmapsto & g \cdot \xi, \end{array}$$

such that

i) $g \cdot (\xi + \eta) = g \cdot \xi + g \cdot \eta$, for all $g \in G$ and all $\xi, \eta \in H$,

ii) $g \cdot (\xi \cdot a) = (g \cdot \xi) g \cdot a$, for all $g \in G$, all $\xi \in H$ and all $a \in A$,

iii) $\langle g \cdot \xi, g \cdot \eta \rangle = g \cdot \langle \xi, \eta \rangle$, for all $g \in G$ and all $\xi, \eta \in H$.

A.3.0.3 Remark. If H is a G-equivariant Hilbert A-module as in the above definition, then $\mathcal{L}_A(H)$ is a G-C^*-algebra with the following action

$$g \cdot T := L_g \circ T \circ L_{g^{-1}},$$

for all $g \in G$ and all $T \in \mathcal{L}_A(H)$, where $L_g \in \mathcal{L}_A(H)$ denotes the operator defined by $L_g(\xi) := g \cdot \xi$, for all $\xi \in H$. Observe that this action restricts to an action of G on $\mathcal{K}_A(H)$.

If A, B are two G-C^*-algebras, we say that A and B are *G-equivariantly Morita equivalent* if there exists a full G-equivariant Hilbert A-module H such that $B \cong \mathcal{K}_A(H)$ as G-C^*-algebras. In this case we write $A \underset{G-M}{\sim} B$.

A.3.19 Definition. Let G be a locally compact group and A a G-C^*-algebra. A graded G-C^*-algebra is a G-C^*-algebra A such that

i) A is a graded C^*-algebra meaning that there exist two closed self-adjoint linear subspaces of A, $A^{(0)}$ and $A^{(1)}$, such that $A = A^{(0)} \oplus A^{(1)}$ and $A^{(i)}A^{(j)} \subset A^{(i+j)\ mod.2}$, for all $i, j = 0, 1$.

ii) $g \cdot A^{(i)} \in A^{(i)}$, for all $g \in G$ and all $i = 0, 1$.

The elements in $A^{(i)}$ are called *homogeneous of degree* i, for all $i = 0, 1$. We say that A is trivially graded if $A^{(1)} = 0$.

A.3.20 Definition. Let G be a locally compact group and A a graded G-C^*-algebra. A graded G-equivariant Hilbert A-module is a G-equivariant Hilbert A-module H such that

i) H is a graded Hilbert A-module meaning that there exist two linear subspaces, $H^{(0)}$ and $H^{(1)}$, such that $H = H^{(0)} \oplus H^{(1)}$, $H^{(i)}A^{(j)} \subset H^{(i+j)\ mod.2}$, and $\langle H^{(i)}, H^{(j)} \rangle \subset A^{(i+j)\ mod.2}$, for all $i, j = 0, 1$.

ii) $g \cdot H^{(i)} \in H^{(i)}$, for all $g \in G$ and all $i = 0, 1$.

The elements in $H^{(i)}$ are called *homogeneous of degree* i, for all $i = 0, 1$. We say that H is trivially graded if $H^{(1)} = 0$.

A.3.21 Remarks. 1. If $\varphi : A \longrightarrow B$ is a G-equivariant homomorphism between G-graded G-C^*-algebras, then we say that φ is *graded* if $\varphi(A^{(i)}) \subset B^{(i)}$, for all $i = 0, 1$. If $T : H \longrightarrow E$ is a G-equivariant operator between G-graded G-equivariant Hilbert A-modules, then we say that T is *graded* if $T(H^{(i)}) \subset E^{(i)}$, for all $i = 0, 1$.

2. Let A be any G-C^*-algebra. It is straightforward to see that giving a (resp. G-equivariant) graduation on A in the sense of Definition A.3.19 is equivalent to give an (resp. G-equivariant) automorphism $\theta \in Aut(A)$ such that $\theta^2 = id_A$. Moreover we have that

$$A^{(0)} = \{a \in A \mid \theta(a) = a\} \text{ and } A^{(1)} = \{a \in A \mid \theta(a) = -a\}$$

3. Let H be any G-equivariant Hilbert A-module. It is straightforward to see that giving a (resp. G-equivariant) graduation on H in the sense of Definition A.3.20 is equivalent to give a (resp. G-equivariant) operator $R \in \mathcal{L}_A(H)$ such that

   - $R^2 = id_H$,
   - $R(\xi a) = R(\xi)\theta(a)$, for all $\xi \in H$ and all $a \in A$,
   - $\langle R(\xi), R(\eta) \rangle = \theta(\langle \xi, \eta \rangle)$, for all $\xi, \eta \in H$.

   Moreover we have that

$$H^{(0)} = \{\xi \in H \mid R(\xi) = \xi\} \text{ and } H^{(1)} = \{\xi \in H \mid R(\xi) = -\xi\}$$

4. Let H be a G-graded G-equivariant Hilbert A-module. The G-C^*-algebra of its adjointable operators $\mathcal{L}_A(H)$ is equipped with a G-equivariant graduation induced by the automorphism

$$\begin{array}{rccl} \theta_H : & \mathcal{L}_A(H) & \longrightarrow & \mathcal{L}_A(H) \\ & T & \longmapsto & \theta_H(T) := R \circ T \circ R^{-1} \end{array}$$

   Notice that θ_H preserves compact operators. Hence, $\mathcal{K}_A(H)$ is also a G-graded G-C^*-algebra.

5. It is possible to obtain the graded version of the standard Hilbert module constructions of Theorem A.3.5 in the obvious way. See [24] for more details.

6. If we replace G by any Hopf C^*-algebra $\mathbb{S} := (S, \Delta)$, then the preceding definitions are still valid. We have just to adapt in the obvious way the equivariant conditions (ii) of Definition A.3.19 and (iii) of Definition A.3.20. More precisely,

   - A *graded* $\mathbb{S}$-C^*-algebra is a $\mathbb{S}$-C^*-algebra (A, δ) such that A is a graded C^*-algebra with graduation automorphism $\theta \in Aut(A)$ and one of the following equivalent conditions holds,
     i) $\delta(A^{(i)}) \subset \widetilde{M}(S \otimes A^{(i)})$, for all $i = 0, 1$.
     ii) $(id_S \otimes \theta)\delta(a) = \delta(\theta(a))$, for all $a \in A$.
   - Let (A, δ) be a graded $\mathbb{S}$-C^*-algebra. A *graded* $\mathbb{S}$-equiariant Hilbert A-module is a $\mathbb{S}$-equivariant Hilbert A-module (H, δ_H) such that H is a graded Hilbert A-module with graduation operator $R \in \mathcal{L}_A(H)$ and one of the following equivalent conditions holds,
     i) $\delta_H(H^{(i)}) \subset \widetilde{M}(S \otimes H^{(i)})$, for all $i = 0, 1$.
     ii) $(id_S \otimes R)\delta_H(\xi) = \delta_H(R(\xi))$, for all $\xi \in H$.
   - Let (A, δ) be a graded $\mathbb{S}$-C^*-algebra and (H, δ_H) a graded $\mathbb{S}$-equivariant Hilbert A-module. If $V_H \in \mathcal{L}_{S\otimes A}(H \underset{\delta}{\otimes} (S \otimes A), S \otimes H)$ denotes the admissible operator associated to δ_H, then $\mathcal{K}_A(H)$ is a $\mathbb{S}$-C^*-algebra with action Ad_{V_H} (as stated in Remarks 1.7.1.7). Moreover, the condition of grade-preserving of the action δ_H is equivalent to say that the admissible operator V_H is of degree 0.
   - If A, B are two $\mathbb{S}$-C^*-algebras, we say that A and B are $\mathbb{S}$-*equivariantly Morita equivalent* if there exists a full $\mathbb{S}$-equivariant Hilbert A-module H such that $B \cong \mathcal{K}_A(H)$ as $\mathbb{S}$-C^*-algebras. In this case we write $A \underset{\mathbb{S}-M}{\sim} B$.

In the context of the Kasparov theory, the constructions of Hilbert modules stated in Theorem A.3.5 can be performed for Kasparov triples. It is advisable to give the precise statement.

A.3.22 Theorem (Standard Kasparov triple constructions). *i) (Exterior tensor product) Let A, B, C be graded C^*-algebras. If $\mathcal{E} := (H, \pi, F)$ is a Kasparov (A, B)-bimodule, the exterior tensor product of $\mathcal{E}$ and C, denoted by $\mathcal{E} \otimes C$, is the Kasparov $(A \otimes C, B \otimes C)$-bimodule defined as the triple*

$$\mathcal{E} \otimes C := (H \otimes C, j(\pi) =: \pi \otimes id, j(F) =: F \otimes id),$$

where $j : \mathcal{L}_B(H) \otimes C \longrightarrow \mathcal{L}_{B\otimes C}(H \otimes C)$ is the canonical injective $*$-homomorphism such that $j(T \otimes C)(\xi \otimes c) = T(\xi) \otimes Cc$, for all $T \in \mathcal{L}_B(H)$ and all $\xi \in H, c \in C$.

In an analogous fashion, we define the Kasparov $(C \otimes A, C \otimes B)$-bimodule $C \otimes \mathcal{E}$.

ii) (Interior tensor product) Let A, B, C be graded C^*-algebras. If $\mathcal{E} := (H, \pi, F)$ is a Kasparov (A, B)-bimodule and $\phi : B \longrightarrow C$ is a graded $*$-homomorphism, the interior tensor product of $\mathcal{E}$ and C with respect to ϕ, denoted by $\mathcal{E} \underset{\phi}{\otimes} C$, is the Kasparov (A, C)-bimodule defined as the triple

$$\mathcal{E} \underset{\phi}{\otimes} C := (H \underset{\phi}{\otimes} C, j(\pi), j(F)),$$

where $j : \mathcal{L}_B(H) \longrightarrow \mathcal{L}_C(H \underset{\phi}{\otimes} C)$ *is the canonical* $*$*-homomorphism such that* $j(T) = T \otimes id$, *for all* $T \in \mathcal{L}_B(H)$.

iii) (Pushout) Let A, B, C *be graded* C^**-algebras. If* $\mathcal{E} := (H, \pi, F)$ *is a Kasparov* (A, B)*-bimodule and* $\varphi : B \longrightarrow C$ *is a surjective graded* $*$*-homomorphism, the interior tensor product of* $\mathcal{E}$ *and* C *with respect to* φ, *denoted by* $\mathcal{E}_\varphi$, *is the Kasparov* (A, C)*-bimodule defined as the triple*

$$\mathcal{E}_\varphi := (H_\varphi, \pi_\varphi, F_\varphi),$$

where $j : \mathcal{L}_B(H) \longrightarrow \mathcal{L}_C(H_\varphi)$ *is the canonical* $*$*-homomorphism such that* $j(T)([\xi]) := T_\varphi([\xi]) = [T(\xi)]$, *for all* $T \in \mathcal{L}_B(H)$ *and all* $\xi \in H$.

A.3.23 Remark. The constructions of Hilbert modules stated in Theorem A.3.5 can be performed under G-equivariance assumptions for a locally compact group G in the obvious way. Consequently, the constructions of Kasparov triples stated in the preceding theorem can also be performed under G-equivariance assumptions.

Moreover, if we replace G by any Hopf C^*-algebra $\mathbb{S} := (S, \Delta)$, then the preceding Kasparov triple constructions are still valid with the obvious definitions.

For instance, let us describe the case for an exterior tensor product and for an interior tensor product.

i) Let (B, δ_B) be ($\mathbb{S}$-graded) $\mathbb{S}$-equivariant C^*-algebra and (C, δ_C) a ($\mathbb{P}$-graded) $\mathbb{P}$-equivariant C^*-algebra, where $\mathbb{S} := (S, \Delta_S)$ and $\mathbb{P} := (P, \Delta_P)$ are given Hopf C^*-algebras. If (H, δ_H) is a (graded) $\mathbb{S}$-equivariant Hilbert B-module and (E, δ_E) is a (graded) $\mathbb{P}$-equivariant Hilbert C-module, then the exterior tensor product $H \otimes E$ is a (graded) Hilbert $B \otimes C$-module equipped with an action of $\mathbb{S} \otimes \mathbb{P}$, $\delta : H \otimes E \longrightarrow \widetilde{M}(S \otimes P \otimes H \otimes E)$ such that

$$\delta(\xi, \eta) := (id_P \otimes \Sigma \otimes id_H) \circ (\delta_H(\xi) \otimes \delta_E(\eta)),$$

for all $\xi \in H$ and all $\eta \in E$.

ii) Let (A, δ_A) and (B, δ_B) be ($\mathbb{S}$-graded) $\mathbb{S}$-equivariant C^*-algebras, where $\mathbb{S} := (S, \Delta)$ is a given Hopf C^*-algebra. If (H, δ_H) is a (graded) $\mathbb{S}$-equivariant Hilbert A-module, (K, δ_K) is a (graded) $\mathbb{S}$-equivariant Hilbert B-module and $\phi : A \longrightarrow \mathcal{L}_B(K)$ is a (graded) $\mathbb{S}$-equivariant $*$-homomorphism, then the map

$$\widetilde{\delta}_\phi(\xi, \eta) := (\delta_H(\xi) \underset{id_S \otimes \phi}{\otimes} id) \circ \delta_K(\eta),$$

for all $\xi \in H, \eta \in K$ defines a linear map $\delta_\phi : H \underset{\phi}{\otimes} K \longrightarrow \widetilde{M}(S \otimes H \underset{\phi}{\otimes} K)$ so that $(H \underset{\phi}{\otimes} K, \delta_\phi)$ is a ($\mathbb{S}$-graded) $\mathbb{S}$-equivariant Hilbert B-module.

A.3.24 Remark. It is important to say that there doesn't exist a natural way of defining a tensor product of $\mathbb{S}$-C^*-algebras in order to obtain again a $\mathbb{S}$-C^*-algebra. For this reason, we have to consider two different Hopf C^*-algebras in order to give a sense to the equivariance of a exterior product of Hilbert modules.

In other words, if $\mathbb{G}$ denotes a locally compact quantum group, then the Kasparov category $\mathscr{KK}^{\mathbb{G}}$ is *not* equipped with a natural tensor product. In [140], R. Nest and C. Voigt introduced the *braided tensor product* $\boxtimes$ in order to solve this problem. This notion has been occasionally used in Chapter 3. We do not explain the construction of $\boxtimes$ here, but for completeness of the exposition we include here some elementary related properties. Let $\mathbb{G}$ be a locally compact quantum group.

- We denote by $\mathcal{D}(\mathbb{G})$ the *Drinfeld quantum double of* $\mathbb{G}$, which is a locally compact quantum group given by
$$C_0(\mathcal{D}(\mathbb{G})) := C_0(\mathbb{G}) \otimes C_r^*(\mathbb{G})$$
$$\Delta_{\mathcal{D}(\mathbb{G})} := (id \otimes \Sigma \otimes id) \circ (id \otimes Ad_{W_{\mathbb{G}}} \otimes id) \circ (\Delta \otimes \widehat{\Delta})$$
- A $\mathbb{G}$-YD-C^*-algebra (where YD stands for *Yetter-Drinfeld*) is a C^*-algebra A equipped with an action α of $\mathbb{G}$ and an action λ of $\widehat{\mathbb{G}}$ such that
$$(\Sigma \otimes id)(id \otimes \alpha)\lambda = (Ad_{W_{\mathbb{G}}} \otimes id)(id \otimes \lambda)\alpha$$
- It can be shown (see Proposition 3.2 in [140] for a proof) that the $\mathbb{G}$-YD-C^*-algebras are exactly the $\mathcal{D}(\mathbb{G})$-C^*-algebras.
- If (A, α, λ) is a $\mathbb{G}$-YD-C^*-algebra and (B, β) is a $\mathbb{G}$-C^*-algebra, then the braided tensor product $A \underset{\mathbb{G}}{\boxtimes} B$ is the C^*-subalgebra of $\mathcal{L}_{A\otimes B}(L^2(\mathbb{G}) \otimes A \otimes B)$ generated by all elements of the form $\lambda_{12}(a)\beta_{13}(b)$ with $a \in A$, $b \in B$ (see Definition 3.3 in [140] for more details).
- The braided tensor product $A \underset{\mathbb{G}}{\boxtimes} B$ becomes a $\mathbb{G}$-C^*-algebra in a canonical way whose action is denoted by $\alpha \boxtimes \beta$. In particular, if B is a *trivial* $\mathbb{G}$-C^*-algebra, then $A \underset{\mathbb{G}}{\boxtimes} B \cong A \otimes B$ as $\mathbb{G}$-C^*-algebras with $\alpha \otimes id$.
- Moreover, the braided tensor product defines a triangulated functor (see Proposition 4.8 in [140] for a proof)
$$A \underset{\mathbb{G}}{\boxtimes} \cdot : \mathscr{K}\mathscr{K}^{\mathbb{G}} \longrightarrow \mathscr{K}\mathscr{K}^{\mathbb{G}}$$
- If $\mathbb{H} < \mathbb{G}$ is a closed quantum subgroup and A' is a $\mathbb{H}$-YD-C^*-algebra, then $Ind_{\mathbb{H}}^{\mathbb{G}}(A')$ is a $\mathbb{G}$-YD-C^*-algebra (see Proposition 3.4 in [140] for a proof).
- If $\mathbb{H} < \mathbb{G}$ is a closed quantum subgroup, A' is a $\mathbb{H}$-YD-C^*-algebra and B is a $\mathbb{G}$-C^*-algebra, then there exists a natural $\mathbb{G}$-equivariant isomorphism (see Theorem 3.6 in [140] for a proof)
$$Ind_{\mathbb{H}}^{\mathbb{G}}\Big(A' \underset{\mathbb{H}}{\boxtimes} Res_{\mathbb{H}}^{\mathbb{G}}(B)\Big) \cong Ind_{\mathbb{H}}^{\mathbb{G}}(A') \underset{\mathbb{G}}{\boxtimes} B$$

A.4 Elements of multiplier algebras

Standard references for the necessary material on this subject are [137], [224], [117] (see as well [24] and [148]).

A.4.1 Definition. Let A be a C^*-algebra. The multiplier algebra of A, denoted by $M(A)$, is the unital C^*-algebra
$$M(A) := \mathcal{L}_A(A),$$
where A is regarded as a Hilbert A-module.

A.4.2 Remark. Let A and B two C^*-algebras. The notion of *non-degenerate $*$-homomorphism* for Hilbert modules (recall Definition A.3.7) can be applied to a given $*$-homomorphism $\varphi : A \longrightarrow M(B)$. Observe that a *surjective* $*$-homomorphism $\varphi : A \longrightarrow B$ is automatically non-degenerate. In addition, we shall equip $M(B) = \mathcal{L}_B(B)$ with the *strict topology* (recall Remark A.3.4).

It is important to say that this class of homomorphism turns out to be more flexible and so more appropriate than the usual class of $*$-homomorphism between C^*-algebras. For this reason, it is common to find in the literature that a homomorphism between C^*-algebras $\varphi : A \longrightarrow B$ is presupposed to be a non-degenerate $*$-homomorphism $\varphi : A \longrightarrow M(B)$.

A.4.3 Proposition (Main properties of multiplier algebras)**.** *i) If A is a C^*-algebra and H is a Hilbert A-module, then $M(\mathcal{K}_A(H)) = \mathcal{L}_A(H)$.*

ii) Let A and B two C^*-algebras. If $\varphi : A \longrightarrow M(B)$ is a non-degenerate $*$-homomorphism, then φ extends in a unique fashion into a unital $*$-homomorphism $\varphi : M(A) \longrightarrow M(B)$.

iii) If $(A_i)_{i\in I}$ is a family of C^*-algebras, then

$$M\Big(\bigoplus_{i\in I} A_i\Big) \cong \prod_{i\in I} M(A_i) \text{ and } M\Big(\prod_{i\in I} A_i\Big) \cong \prod_{i\in I} M(A_i)$$

iv) If A and B are two C^*-algebras, there always exists a unital injective $*$-homomorphism

$$M(A) \otimes M(B) \hookrightarrow M(A \otimes B)$$

v) If A is a C^*-algebra, then $\mathcal{M}_n(M(A)) \cong M(\mathcal{M}_n(A))$, for all $n \in \mathbb{N}$.

vi) Let A, A', B, B' be C^*-algebras. If $\varphi : A \longrightarrow M(B)$ and $\varphi' : A' \longrightarrow M(B')$ are two non-degenerate $*$-homomorphisms, then the tensor product $*$-homomorphism $\varphi \otimes \varphi' : A \otimes A' \longrightarrow M(B) \otimes M(B') \subset M(B \otimes B')$ is again a non-degenerate $*$-homomorphism. As a consequence, we have a unital $*$-homomorphism $\varphi \otimes \varphi' : M(A \otimes A') \longrightarrow M(B \otimes B')$.

vii) Let A and B two C^*-algebras. The flip $*$-isomorphism $\Sigma : A \otimes B \longrightarrow B \otimes A$ induces a $*$-isomorphism $\Sigma : M(A \otimes B) \longrightarrow M(B \otimes A)$.

viii) Let A be a C^*-algebra and $B \subset A$ a C^*-subalgebra. If A is σ-unital and B contains the corresponding strictly positive element, then there exists a unital injective $*$-homomorphism $M(B) \hookrightarrow M(A)$.

The following definitions are useful for the general treatment of quantum groups in the context of *Hopf C^*-algebras*. For more details we refer to [6] and [206].

A.4.4 Definition. Let S, A be C^*-algebras. We define the following C^*-subalgebra of $M(S \otimes A)$

$$\widetilde{M}(S \otimes A) := \{x \in M(S \otimes A) \mid x(S \otimes id_A) \subset S \otimes A \text{ and } (S \otimes id_A)x \subset S \otimes A\},$$

which contains $S \otimes M(A)$.

A.4.5 Definition. Let A be a C^*-algebra and H a Hilbert A-module. We define the multiplier algebra of H, denoted by $M(H)$, as the following Hilbert $M(A)$-module

$$M(H) := \mathcal{L}_A(A, H),$$

which contains canonically $H \cong \mathcal{K}(A, H)$.

A.4.6 Definition. Let S, A be C^*-algebra and H a Hilbert A-module. We define the following Hilbert $\widetilde{M}(S \otimes A)$-submodule of $M(S \otimes H)$

$$\widetilde{M}(S \otimes H) := \{X \in M(S \otimes H) \mid X(S \otimes id_A) \subset S \otimes H \text{ and } (S \otimes id_H)X \subset S \otimes H\},$$

which contains $M(S) \otimes H$.

APPENDIX B

Categories

In this appendix we want to collect the main definitions and results of the category theory for two reasons: *a*) the current formulation of the Baum-Connes property for quantum groups fit in a categorical framework *b*) quantum groups and related concepts admit a fruitful categorical perspective (for instance, categorification of the Yang-Baxter equation or the notion of monoidal equivalence). Hence, it is advisable to have in mind the categorical language.

Because of the well-known logical issue that appears as soon as we want to formalize the category theory within the Zermelo-Fraenkel set theory, we must be careful in the nomenclature. Recall that a category $\mathscr{C}$ is said to be *small* if both $Obj(\mathscr{C})$ and $Hom_{\mathscr{C}}(\,\cdot\,,\,\cdot\,)$ are sets. It is said to be *large* otherwise. It is said to be *locally small* if only $Hom_{\mathscr{C}}(\,\cdot\,,\,\cdot\,)$ are sets. Recall that a *class* is a collection of sets that can be well-defined by a property that all of its members have. The category of sets is denoted by Set and it is locally small. Remark that in the framework of this dissertation we work just with locally small categories (for instance, the equivariant Kasparov category with respect to a locally compact (quantum) group).

We don't give any proof in this appendix and we refer to standard references for all the details, [122], [4], [61], [139], [103].

B.1 Generalities

B.1.1 Definition. A locally small category is the data $\mathscr{C} = (Obj(\mathscr{C}), Hom_{\mathscr{C}}(\,\cdot\,,\,\cdot\,), id., \circ)$ where

i) $Obj(\mathscr{C})$ is a class called *class of objects of* $\mathscr{C}$,

ii) for every two objects $X, Y \in Obj(\mathscr{C})$, $Hom_{\mathscr{C}}(X, Y)$ is a set called *set of homomorphisms between* X *and* Y whose elements are represented as an arrow from X to Y, say $f : X \longrightarrow Y$.

The object X is called *domain (or source) of* f and denoted by $Dom(f)$ and the object Y is called *co-domain (or target) of* f and denoted by $Codom(f)$,

iii) for every object $X \in Obj(\mathscr{C})$, $id_X : X \longrightarrow X$ is a distinguished homomorphism from X to itself called *identity of* X,

iv) for every objects $X, Y, Z \in Obj(\mathscr{C})$,

$$\begin{array}{rccc} \circ : & Hom_{\mathscr{C}}(X,Z) \times Hom_{\mathscr{C}}(Z,Y) & \longrightarrow & Hom_{\mathscr{C}}(X,Y) \\ & (f,g) & \longmapsto & g \circ f \end{array}$$

is an operation called *composition* such that

- $\circ$ is *associative* meaning that for every homomorphisms $f \in Hom_{\mathscr{C}}(X,Z)$, $g \in Hom_{\mathscr{C}}(Z,Y)$ and $h \in Hom_{\mathscr{C}}(Y,W)$ with objects $X, Y, Z, W \in Obj(\mathscr{C})$ we have that

$$h \circ (g \circ f) = (h \circ g) \circ f$$

in $Hom_{\mathscr{C}}(X,W)$.

- $\circ$ has an *identity* meaning that for every objects $X, Y \in Obj(\mathscr{C})$ and every homomorphism $f \in Hom_{\mathscr{C}}(X,Y)$ we have that

$$id_Y \circ f = f = f \circ id_X$$

An object $Z \in Obj(\mathscr{C})$ is called

- initial if for every object $X \in Obj(\mathscr{C})$ there exists exactly one homomorphism $Z \longrightarrow X$,
- terminal if for every object $X \in Obj(\mathscr{C})$ there exists exactly one homomorphism $X \longrightarrow Z$,
- null if it is both initial and terminal.

B.1.2 Note. From now on, the word *category* will mean *locally small category.*

B.1.3 Remark. Given a category $\mathscr{C} = (Obj(\mathscr{C}), Hom_{\mathscr{C}}(\ \cdot\ ,\ \cdot\ ), id., \circ)$, we call *dual or opposite category of* $\mathscr{C}$ the category $\mathscr{C}^{op} = (Obj(\mathscr{C}^{op}), Hom_{\mathscr{C}^{op}}(\ \cdot\ ,\ \cdot\ ), id.^{op}, \circ^{op})$ where

- $Obj(\mathscr{C}^{op}) := Obj(\mathscr{C})$,
- $Hom_{\mathscr{C}^{op}}(X,Y) := Hom_{\mathscr{C}}(Y,X)$, for all $X, Y \in Obj(\mathscr{C}^{op})$,
- $id.^{op} = id.$,
- $g \circ^{op} f := f \circ g$, for all $f \in Hom_{\mathscr{C}^{op}}(X,Z)$ and all $g \in Hom_{\mathscr{C}^{op}}(Z,Y)$ with $X, Y, Z \in Obj(\mathscr{C}^{op})$.

B.1.4 Definition. Let $\mathscr{C}$ be a category. A category $\mathscr{S}$ is a subcategory of $\mathscr{C}$ if the following conditions hold

i) $Obj(\mathscr{S}) \subset Obj(\mathscr{C})$,

ii) $Hom_{\mathscr{S}}(X,Y) \subset Hom_{\mathscr{C}}(X,Y)$, for all objects $X, Y \in Obj(\mathscr{S})$,

iii) for all object $X \in Obj(\mathscr{S})$ the identity homomorphism $id_X \in Hom_{\mathscr{C}}(X,X)$ is an element of $Hom_{\mathscr{S}}(X,X)$,

iv) the composition operation on $Hom_{\mathscr{S}}(\,\cdot\,,\,\cdot\,)$ is the restriction of the composition operation on $Hom_{\mathscr{C}}(\,\cdot\,,\,\cdot\,)$.

We say that $\mathscr{S}$ is a *full subcategory of* $\mathscr{C}$ if $Hom_{\mathscr{S}}(X,Y) = Hom_{\mathscr{C}}(X,Y)$, for all objects $X, Y \in Obj(\mathscr{S})$. In that case we write $\mathscr{S} \subset \mathscr{C}$.

B.1.5 Definition. Let $\mathscr{C}$ be a category. Given two objects $X, Y \in Obj(\mathscr{C})$ consider a homomorphism $f : X \longrightarrow Y$.

- f is called isomorphism if there exists a (necessarily unique) homomorphism $g : Y \longrightarrow X$, called *inverse of* f, such that $g \circ f = id_X$ and $f \circ g = id_Y$. In that case we write $f^{-1} :=$ and we say that X and Y are *isomorphic*.

- f is called monomorphism if the following condition holds,

$$\forall\ g_1, g_2 : Z \longrightarrow X \text{ with } Z \in Obj(\mathscr{C}),\ f \circ g_1 = f \circ g_2 \ \Rightarrow g_1 = g_2$$

- f is called epimorphism if the following condition holds,

$$\forall\ g_1, g_2 : Y \longrightarrow Z \text{ with } Z \in Obj(\mathscr{C}),\ g_1 \circ f = g_2 \circ f \ \Rightarrow g_1 = g_2$$

- a section of f is a left inverse for f, that is, a homomorphism $s : Y \longrightarrow X$ such that $s \circ f = id_X$.

- a retraction of f is a right inverse for f, that is, a homomorphism $r : Y \longrightarrow X$ such that $f \circ r = id_Y$.

B.1.6 Remark. The following are straightforward observations from the definitions. Let $\mathscr{C}$ be a category.

- $(id_X)^{-1} = id_X$, for all object $X \in Obj(\mathscr{C})$.

- Given two objects $X, Y \in Obj(\mathscr{C})$ if $f : X \longrightarrow Y$ is an isomorphism, then its inverse $f^{-1} : Y \longrightarrow X$ is an isomorphism as well and $(f^{-1})^{-1} = f$.

- Given three objects $X, Y, Z \in Obj(\mathscr{C})$ if $f \in Hom_{\mathscr{C}}(X,Z)$ and $g \in Hom_{\mathscr{C}}(Z,Y)$ are isomorphisms, then their composition $g \circ f$ is an isomorphism as well and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

- The relation *"to be isomorphic"* establish an equivalence relation on $Obj(\mathscr{C})$ whenever $Obj(\mathscr{C})$ is a set.

B.1.7 Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A covariant (resp. contravariant) functor F from $\mathscr{C}$ to $\mathscr{D}$, denoted by $F : \mathscr{C} \longrightarrow \mathscr{D}$, is a rule that assigns to any object $X \in Obj(\mathscr{C})$ an object $F(X) \in Obj(\mathscr{D})$ and to any homomorphism $f \in Hom_{\mathscr{C}}(X,Y)$ a homomorphism $F(f) \in Hom_{\mathscr{D}}(F(X), F(Y))$ (resp. $F(f) \in Hom_{\mathscr{D}}(F(Y), F(X))$) with $X, Y \in Obj(\mathscr{C})$ such that

i) $F(g \circ f) = F(g) \circ F(f)$ (resp. $F(g \circ f) = F(f) \circ F(g)$), for all $f \in Hom_{\mathscr{C}}(X,Z)$ and all $g \in Hom_{\mathscr{C}}(Z,Y)$ with $X, Y, Z \in Obj(\mathscr{C})$,

ii) $F(id_X) = id_{F(X)}$, for all object $X \in Obj(\mathscr{C})$.

B.1.8 Remark. The following are straightforward observations from the definitions. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories.

- If $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a (covariant or contravariant) functor, then $F^{op} : \mathscr{C}^{op} \longrightarrow \mathscr{D}^{op}$ defined exactly as F is a (covariant or contravariant) functor between the dual categories.
- If $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a functor, then F preserves isomorphisms. In fact, F transforms commutative diagrams on $\mathscr{C}$ into commutative diagrams on $\mathscr{D}$.

Some typical examples of functors are the following,

i) *identity functor*: the obvious rule $id_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathscr{C}$ that assigns to any object and to any homomorphism the same object and the same homomorphism is a functor from $\mathscr{C}$ to itself called *identity functor*.

ii) *constant functor*: fix an object $X_0' \in Obj(\mathscr{D})$, a constant functor from $\mathscr{C}$ to $\mathscr{D}$ assigns to any object of $\mathscr{C}$ the object X_0' and to any homomorphism on $\mathscr{C}$ to the identity of X_0'.

iii) *forgetful functor*: a functor which simply *"forgets"* some or all the structure of an algebraic object is commonly called a *forgetful functor or underlying functor*.

iv) *homomorphism functors*: if we fix an object $X \in Obj(\mathscr{C})$, the left homomorphism functor is the following covariant functor

$$\begin{array}{rccl} Hom_{\mathscr{C}}(X,\ \cdot\ ): & \mathscr{C} & \longrightarrow & Set \\ & Y & \longmapsto & Hom_{\mathscr{C}}(X,Y) \\ & f : Y \longrightarrow Y' & \longmapsto & f_* := f \circ \cdot \end{array}$$

If we fix an object $Y \in Obj(\mathscr{C})$, the right homomorphism functor is the following contravariant functor

$$\begin{array}{rccl} Hom_{\mathscr{C}}(\ \cdot\ ,Y): & \mathscr{C} & \longrightarrow & Set \\ & X & \longmapsto & Hom_{\mathscr{C}}(X,Y) \\ & f : X \longrightarrow X' & \longmapsto & : f^* := \cdot \circ f \end{array}$$

B.1.9 Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is called

- isomorphism if there exists a (necessarily unique) functor $G : \mathscr{D} \longrightarrow \mathscr{C}$, called *inverse of* F, such that $G \circ F = id_{\mathscr{C}}$ and $F \circ G = id_{\mathscr{D}}$. In that case we write $F^{-1} := G$ and we say that $\mathscr{C}$ and $\mathscr{D}$ are *isomorphic*. If $\mathscr{C} = \mathscr{D}$, we that that F is an *automorphism*.
- faithful if for every objects $X, Y \in Obj(\mathscr{C})$ the map

$$F_{|Hom} : Hom_{\mathscr{C}}(X,Y) \longrightarrow Hom_{\mathscr{D}}(F(X),F(Y))$$

  is injective.
- full if for every objects $X, Y \in Obj(\mathscr{C})$ the map

$$F_{|Hom} : Hom_{\mathscr{C}}(X,Y) \longrightarrow Hom_{\mathscr{D}}(F(X),F(Y))$$

  is surjective.

- equivalence if F is full and faithful and for every object $X' \in Obj(\mathscr{D})$ there exists an object $X \in Obj(\mathscr{C})$ such that $F(X) \cong X'$ in $\mathscr{D}$. In that case we say that $\mathscr{C}$ and $\mathscr{D}$ are *equivalent*. If $\mathscr{C} = \mathscr{D}$, we say that F is an *auto-equivalence*.

B.1.10 Remarks. 1. Given a full and faithful functor $F : \mathscr{C} \longrightarrow \mathscr{D}$, it is straightforward to see that F is an isomorphism of categories if and only if the map $F_{|Obj} : Obj(\mathscr{C}) \longrightarrow Obj(\mathscr{D})$ is bijective. As a consequence, every full and faithful functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ yields an isomorphism between $\mathscr{C}$ and $F(\mathscr{D})$, where $F(\mathscr{D})$ is the image of $\mathscr{C}$ by F, which is a category in an obvious fashion (actually, it is a full subcategory of $\mathscr{D}$).

2. If $\mathscr{S}$ is a subcategory of a given category $\mathscr{C}$, then there exists an obvious *inclusion functor*. This functor is automatically faithful, so that $\mathscr{S}$ is a full subcategory of $\mathscr{C}$ if and only if the inclusion functor is full. As a result, in order to define a *full subcategory* we only need to define its class of objects.

B.1.11 Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ two functors. A natural transformation between F and G, denoted by $\eta : F \longrightarrow G$, is a collection of homomorphisms $\{\eta_X : F(X) \longrightarrow G(X)\}_{X \in Obj(\mathscr{C})}$ in $\mathscr{D}$ such that the following diagram commutes

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

for all $X, Y \in Obj(\mathscr{C})$ and all $f \in Hom_{\mathscr{C}}(X, Y)$.

If the collection $\{\eta_X : F(X) \longrightarrow G(X)\}_{X \in Obj(\mathscr{C})}$ is formed by isomorphisms, then we say that η is a *natural isomorphism*. In this case we say that F and G are *naturally isomorphic* and we write $F \overset{\eta}{\cong} G$.

B.1.12 Remark. We can show that a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is an equivalence if and only if there exists a functor $G : \mathscr{D} \longrightarrow \mathscr{C}$ such that $G \circ F \cong id_{\mathscr{C}}$ and $F \circ G \cong id_{\mathscr{D}}$.

B.1.13 Definition. Let $\mathscr{C}$ be a category and $R \in Obj(\mathscr{C})$ an object. A functor $F : \mathscr{C} \longrightarrow Set$ is called representable by R if

$$F \cong Hom_{\mathscr{C}}(R, \ \cdot \ )$$

B.1.14 Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ a functor. A left (resp. right) adjoint for F is a functor $F^{\vdash} : \mathscr{D} \longrightarrow \mathscr{C}$ (resp. $F^{\dashv} : \mathscr{D} \longrightarrow \mathscr{C}$) such that for every objects $X \in Obj(\mathscr{C})$ and $Y \in Obj(\mathscr{D})$ we have an isomorphism

$$Hom_{\mathscr{C}}(F^{\vdash}(Y), X) \cong Hom_{\mathscr{D}}(Y, F(X)) \text{ (resp. } Hom_{\mathscr{C}}(X, F^{\dashv}(Y)) \cong Hom_{\mathscr{D}}(F(X), Y))$$

such that $\psi : Hom_{\mathscr{C}}(F^{\vdash}(\cdot), \ \cdot \ ) \longrightarrow Hom_{\mathscr{D}}(\ \cdot \ , F(\cdot))$
(resp. $Hom_{\mathscr{C}}(\ \cdot \ , F^{\dashv}(\cdot)) \longrightarrow Hom_{\mathscr{D}}(F(\cdot), \ \cdot \ )$) is a natural isomorphism in each variable.

In order to prove that two functors are adjoints each other, it is advisable to have another characterization that allow manipulable formulas. Namely, we have the following result whose proof can be found in Theorem *IV*.1.2 of [122].

B.1.15 Theorem. *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ a functor. The following assertions are equivalent,*

i) $F^{\vdash} : \mathscr{D} \longrightarrow \mathscr{C}$ is a left adjoint for F with natural isomorphism between the homomorphisms sets ψ.

ii) There exist a functor $S : \mathscr{D} \longrightarrow \mathscr{C}$ and natural transformations $\eta : Id_{\mathscr{D}} \longrightarrow F \circ S$, $\varepsilon : S \circ F \longrightarrow Id_{\mathscr{C}}$ such that

- *the composition $F \xrightarrow{\eta \circ id_F} F \circ S \circ F \xrightarrow{id_F \circ \varepsilon} F$ is the identity transformation of F,*
- *the composition $S \xrightarrow{id_S \circ \eta} S \circ F \circ S \xrightarrow{\varepsilon \circ id_S} S$ is the identity transformation of S.*

The natural transformations η and ε are called unit *and* co-unit *of the adjunction, respectively.*

The correspondence above is realized by means of the following formulas

$$S := F^{\vdash},\ \eta_Y := \psi_{Y,F^{\vdash}(Y)}(id_{F^{\vdash}(Y)}) \text{ and } \varepsilon_X := \psi^{-1}_{F(X),X}(id_{F(X)})$$

$$F^{\vdash} := S,\ \psi(f) := S(f) \circ \eta_Y \text{ and } \psi^{-1}(g) := \varepsilon_X \circ S(g),$$

for all $X \in Obj(\mathscr{C})$, $Y \in Obj(\mathscr{D})$, $f \in Hom_{\mathscr{C}}(F^{\vdash}(Y), X)$ and $g \in Hom_{\mathscr{D}}(Y, F(X))$.

Moreover, if $\widetilde{F}^{\vdash}$ is another left adjoint for F, then $\widetilde{F}^{\vdash}$ and $F^{\vdash}$ are naturally isomorphic.

Of course, given any functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ it is not at all guarantee the existence of its (left or right) adjoint. Actually, the existence of such an adjoint is related to the *continuity* of the functor F in the sense that it commutes with limits. This characterization is due to P. J. Freyd and we refer to Theorem V.6.2 of [122] or Theorem 9.28 of [4] for a proof and more precisions. It is important to remark that in the above definition we require the domain of the adjoint functor to be the whole category $\mathscr{D}$, which is the arrival category of the starting functor F. We can be more flexible and define $F^{\vdash}$ just on some appropriated class of objects of $\mathscr{D}$. In the context of triangulated categories this is advisable in order to construct complementary pair of subcategories (see Definition 1.2.2.18 and Theorem 1.2.2.20).

Finally, we shall recall some standard constructions of categories, namely the product category, the quotient category and the localization category.

B.1.16 Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. The product category of $\mathscr{C}$ and $\mathscr{D}$, denoted by $\mathscr{C} \times \mathscr{D}$, is the category defined in the following way

- Objects of $\mathscr{C} \times \mathscr{D}$ are pairs of objects of $\mathscr{C}$ and $\mathscr{D}$, which is indicated by $Obj(\mathscr{C} \times \mathscr{D}) := Obj(\mathscr{C}) \times Obj(\mathscr{D})$.
- Given two objects $U := (X, Y), V := (X', Y') \in Obj(\mathscr{C} \times \mathscr{D})$, a homomorphism $h \in Hom_{\mathscr{C} \times \mathscr{D}}(U, V)$ is a pair of homomorphisms $(f, g) \in Hom_{\mathscr{C}}(X, X') \times Hom_{\mathscr{D}}(Y, Y')$.
- The composition operation on $Hom_{\mathscr{C} \times \mathscr{D}}(\ \cdot\ ,\ \cdot\ )$ is defined component-wise.

B.1.17 Remarks. 1. Observe that the product category $\mathscr{C} \times \mathscr{D}$ enjoys the following universal property. There exist two obvious functors

$$P : \mathscr{C} \times \mathscr{D} \longrightarrow \mathscr{C} \text{ and } Q : \mathscr{C} \times \mathscr{D} \longrightarrow \mathscr{D}$$

called *projections* such that given any category $\mathscr{E}$ with two functors $T : \mathscr{E} \longrightarrow \mathscr{C}$ and $S : \mathscr{E} \longrightarrow \mathscr{D}$, there exists a unique functor $F : \mathscr{E} \longrightarrow \mathscr{C} \times \mathscr{D}$ such that $P \circ F = T$ and $Q \circ F = S$.

2. Given categories $\mathscr{C}, \mathscr{D}, \mathscr{C}', \mathscr{D}'$ and functors $F : \mathscr{C} \longrightarrow \mathscr{D}$, $F' : \mathscr{C}' \longrightarrow \mathscr{D}'$, we can define the product functor by component-wise

$$F \times F' : \mathscr{C} \times \mathscr{D} \longrightarrow \mathscr{C}' \times \mathscr{D}',$$

which is, by the way, compatible with the canonical projection functors.

3. We can define the product category for an arbitrary family of categories, say $\{\mathscr{C}_i\}_{i \in I}$, and it is denoted by $\prod_{i \in I} \mathscr{C}_i$.

B.1.18 Theorem-Definition. *Let $\mathscr{C}$ be a category and $\mathscr{R}$ a rule that assigns to any pair of objects $X, Y \in Obj(\mathscr{C})$ a binary relation $\mathscr{R}_{X,Y}$ on $Hom_{\mathscr{C}}(X, Y)$.*

The quotient category of $\mathscr{C}$ by the rule $\mathscr{R}$ is the category denoted by $\mathscr{C}/\mathscr{R}$ endowed with a functor $Q : \mathscr{C} \longrightarrow \mathscr{C}/\mathscr{R}$, which is a bijection on objects, such that

i) if $f, g : X \longrightarrow Y$ are homomorphisms in $\mathscr{C}$ such that $(f, g) \in \mathscr{R}_{X,Y}$ for some objects $X, Y \in Obj(\mathscr{C})$, then $Q(f) = Q(g)$,

ii) if $\mathscr{C}'$ is another category and $F : \mathscr{C} \longrightarrow \mathscr{C}'$ is a functor such that $F(f) = F(g)$ whenever $f, g : X \longrightarrow Y$ are homomorphisms in $\mathscr{C}$ such that $(f, g) \in \mathscr{R}_{X,Y}$ for some objects $X, Y \in Obj(\mathscr{C})$, then there exists a unique functor $H : \mathscr{C}/\mathscr{R} \longrightarrow \mathscr{C}'$ such that $H \circ Q \cong F$.

There is, however, a quotient category in the sense of Verdier which is more interesting for this dissertation. The basic idea for the Verdier quotient category is the localization process in ring theory. Namely we want to transform a class of homomorphisms into isomorphisms in a bigger category. To this end, we can develop a theory of *calculus of fractions*, which imitates the classical ring localization. In addition, we would like to endow such a category with a triangulated structure whenever the starting category is a triangulated one. This is a non trivial problem solved by J. L. Verdier in [205]. For more details we refer to [138] or [69].

B.1.19 Definition. Let $\mathscr{C}$ be a category and $\mathcal{S} \subset Hom_{\mathscr{C}}(\,\cdot\,,\,\cdot\,)$ a class of homomorphisms. The localization of $\mathscr{C}$ with respect to $\mathcal{S}$ is the category denoted by $\mathscr{C}\left[\mathcal{S}^{-1}\right]$ endowed with a functor $Q : \mathscr{C} \longrightarrow \mathscr{C}\left[\mathcal{S}^{-1}\right]$ such that

i) $Q(s)$ is an isomorphism in $\mathscr{C}\left[\mathcal{S}^{-1}\right]$, for every homomorphism s in $\mathcal{S}$,

ii) if $\mathscr{C}'$ is another category and $F : \mathscr{C} \longrightarrow \mathscr{C}'$ is a functor such that $F(s)$ is an isomorphism in $\mathscr{C}'$ for all homomorphism s in $\mathcal{S}$, then there exists a unique factorization

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "F"'] & \mathscr{C}\left[\mathcal{S}^{-1}\right] \arrow[dl, dashed] \\ \mathscr{C}' & \end{tikzcd}$$

Let us consider a specific localization in the context of triangulated categories following the preceding definition. Let $\mathcal{T}$ be a triangulated category and let $\mathcal{N} \subset \mathcal{T}$ be a triangulated (resp. thick) subcategory.

A homomorphism $f \in Hom_{\mathcal{T}}$ is called *$\mathcal{N}$-quasi-isomorphism* if $C_f \in Obj(\mathcal{N})$. In this case, the localization of $\mathcal{T}$ with respect to the class of $\mathcal{N}$-quasi-isomorphisms is denoted by

$$\mathcal{T}\Big[\big(\mathcal{N}\text{-quasi-isomorphisms}\big)^{-1}\Big] =: \mathcal{T}/\mathcal{N}$$

B.1.20 Theorem. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{N} \subset \mathcal{T}$ a triangulated (resp. thick) subcategory.*

The localization category $\mathcal{T}/\mathcal{N}$ is a triangulated category endowed with a triangulated localization functor $Q : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{N}$ such that

i) $\mathcal{N} \subset \ker(Q)$ (resp. $\mathcal{N} = \ker(Q)$),

ii) if $\mathcal{T}'$ is another triangulated category and $F : \mathcal{T} \longrightarrow \mathcal{T}'$ is a triangulated functor such that $\mathcal{N} \subset \ker(F)$ (resp. $\mathcal{N} = \ker(F)$), then there exists a unique factorization

$$\begin{tikzcd} \mathcal{T} \arrow[r, "Q"] \arrow[d, "F"'] & \mathcal{T}/\mathcal{N} \arrow[dl, dashed] \\ \mathcal{T}' & \end{tikzcd}$$

The category $\mathcal{T}/\mathcal{N}$ is called Verdier quotient of $\mathcal{T}$ by $\mathcal{N}$ *and the functor Q is called* Verdier localization functor.

B.2 Abelian categories

B.2.1 Definition. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is additive if

i) for every objects $X, Y \in Obj(\mathscr{C})$, the corresponding set of homomorphisms $Hom_{\mathscr{C}}(X, Y)$ is an additive abelian group such that the composition operation $\circ$ is bilinear with respect to the group additive law,

ii) there exists a distinguished object in $\mathscr{C}$ denoted by $0 \in Obj(\mathscr{C})$ such that $Hom_{\mathscr{C}}(0, 0) = 0$. It is called *zero object*,

iii) for every objects $X, Y \in Obj(\mathscr{C})$, there exists an object $Z \in Obj(\mathscr{C})$ with homomorphisms

$$\begin{tikzcd} X \arrow[r, shift left, "\iota"] & Z \arrow[l, shift left, "p"] \arrow[r, shift left, "q"] & Y \arrow[l, shift left, "j"] \end{tikzcd}$$

such that

$$p \circ \iota = id_X,\ q \circ j = id_Y \text{ and } \iota \circ p + j \circ q = id_Z$$

The object Z is called *direct sum of X and Y* and it is denoted by $Z := X \oplus Y$. The homomorphisms ι and j are called *canonical injections of $X \oplus Y$* and the homomorphisms p and q are called *canonical projections of $X \oplus Y$*.

B.2.2 Remark. If $\mathscr{C}$ is a pre-additive category (that is, a category $\mathscr{C}$ satisfying only the axiom (i) of the above definition) and $Z \in Obj(\mathscr{C})$ is any object, it is straightforward to show that the following assertions are equivalent

i) Z is initial.

ii) Z is terminal.

iii) $id_Z = 0$.

iv) $Hom_{\mathscr{C}}(Z, Z) = 0$.

In particular, any initial or any terminal object in $\mathscr{C}$ is a null object. Hence in an additive category, the zero object is a null object and it is unique up to isomorphism.

B.2.3 Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be two additive categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ a functor. We say that F is additive if for every objects $X, Y \in Obj(\mathscr{C})$ the map

$$F_{|Hom} : Hom_{\mathscr{C}}(X, Y) \longrightarrow Hom_{\mathscr{D}}(F(X), F(Y))$$

is a group homomorphism.

B.2.4 Remarks. 1. If $X, Y \in Obj(\mathscr{C})$ are two objects in an additive category $\mathscr{C}$, then it is straightforward to deduce that

$$p \circ j = 0 \text{ and } q \circ \iota = 0$$

2. If $X, Y \in Obj(\mathscr{C})$ are two objects in an additive category $\mathscr{C}$, then their direct sum object $X \oplus Y$ together with the canonical injections and projections is uniquely determined up to an isomorphism of $X \oplus Y$. If all such direct sums exist, then a choice of the object $X \oplus Y$ for each pair (X, Y) in $\mathscr{C}$ defines a bifunctor

$$\oplus : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$$

whose definition on homomorphisms is the following: given objects $X, X', Y, Y' \in Obj(\mathscr{C})$ and homomorphisms $f \in Hom_{\mathscr{C}}(X, X')$ and $g \in Hom_{\mathscr{C}}(Y, Y')$, then $f \oplus g \in Hom_{\mathscr{C}}(X \oplus Y, X' \oplus Y')$ is defined by

$$f \oplus g := \iota' \circ f \circ p + j' \circ g \circ q,$$

where (ι, p, j, q) and (ι', p', j', q') are the canonical homomorphisms associated to the direct sums $X \oplus Y$ and $X' \oplus Y'$, respectively.

3. A straightforward iterated argument yields the existence of a direct sum object for any finite collection of objects in the category. Namely, if $X_1, \ldots, X_n \in Obj(\mathscr{C})$ are objects in an additive category $\mathscr{C}$, then their direct sum $X_1 \oplus \ldots \oplus X_n$ is defined, uniquely up to isomorphism, by the equations

$$\iota_1 \circ p_1 + \ldots + \iota_n \circ p_n = id \text{ and } p_k \circ \iota_k = id_{X_k} \ \forall k = 1, \ldots, n$$

Moreover, if $Y_1, \ldots, Y_n \in Obj(\mathscr{C})$ are other objects in $\mathscr{C}$, then we have a isomorphism of abelian groups

$$Hom_{\mathscr{C}}\Big(\bigoplus_{k=1}^{n} X_k, \bigoplus_{k=1}^{n} Y_k\Big) \cong \bigoplus_{k=1}^{n} Hom_{\mathscr{C}}(X_k, Y_k)$$

4. If $F : \mathscr{C} \longrightarrow \mathscr{D}$ is an additive functor between additive categories, it is clear that F transforms the zero object in $\mathscr{C}$ into the zero object in $\mathscr{D}$. Moreover, F is compatible with direct sums, that is, $F(X \oplus Y) \cong F(X) \oplus F(Y)$, for every objects $X, Y \in Obj(\mathscr{C})$. Finally, if F is in addition an equivalence, then the corresponding inverse equivalence is always an additive functor.

5. It is important to observe the following: given two objects $X, Y \in Obj(\mathscr{C})$ in an additive category $\mathscr{C}$, then their direct sum object $X \oplus Y$ as defined above corresponds to both a *product* and a *co-product* of X and Y (denoted usually by $X \times Y$ and by $X \sqcup Y$, respectively). In this sense, we say that $X \oplus Y$ is a *bi-product*. In fact, we can show that in a pre-additive category two objects have a product if and only if they have a co-product so that the corresponding bi-product is characterized as in the definition above (see Theorem $VIII.2$ in [122] for a proof). In other words, finite product and finite co-product objects are isomorphic in an additive category.

   Notice that the product and co-product can be defined for an arbitrary family of objects $\{X_i\}_{i \in I}$ in $\mathscr{C}$. However, they are not isomorphic in general. In this way, an additive category always has finite bi-products by definition but not infinite ones in general.

   For the appropriate development of the homological algebra on a triangulated category, we have to require the existence of *countable bi-products*, that is, *countable direct sums* in the terminology above. This condition is automatically fulfilled for our main example of triangulated category: the equivariant Kasparov category with respect to a locally compact (quantum) group.

B.2.5 Definition. Let $\mathscr{C}$ be an additive category. Consider a homomorphism $f \in Hom_{\mathscr{C}}(X, Y)$ with $X, Y \in Obj(\mathscr{C})$.

- A kernel for f is an object $K \in Obj(\mathscr{C})$ and a homomorphism $k : K \longrightarrow X$ such that

  i) $f \circ k = 0$,

  ii) if $K' \in Obj(\mathscr{C})$ is another object with a homomorphism $k' : K' \longrightarrow X$ such that $f \circ k' = 0$, then there exists a unique homomorphism $u : K' \longrightarrow K$ such that $k' = k \circ u$.

  In this case we write $\ker(f) := (K, k)$. By abuse of notation we write $\ker(f) = K$ as well.

- A co-kernel for f is an object $Q \in Obj(\mathscr{C})$ and a homomorphism $q : Y \longrightarrow Q$ such that

  i) $q \circ f = 0$,

  ii) if $Q' \in Obj(\mathscr{C})$ is another object with a homomorphism $q' : Y \longrightarrow Q'$ such that $q' \circ f = 0$, then there exists a unique homomorphism $u : Q \longrightarrow Q'$ such that $q' = u \circ q$.

  In this case we write $coker(f) := (Q, q)$. By abuse of notation we write $coker(f) = Q$ as well.

- If f has a co-kernel (Q, q) and p has a kernel, then we define the image of f as $Im(f) := \ker(q) := (Im(f), u)$.

- If f has a kernel (K, k) and k has a co-kernel, then we define the co-image of f as $coim(f) := coker(k) := (coim(f), v)$.

B.2.6 Remark. If (K, k) is a kernel for a homomorphism f, then the object K is unique up to isomorphism and the homomorphism k is necessarily a monomorphism. If (Q, q) is a co-kernel for f, then the object Q is unique up to isomorphism and the homomorphism q is necessarily an epimorphism.

B.2.7 Definition. Let $\mathscr{C}$ be an additive category. We say that $\mathscr{C}$ is abelian if

i) every homomorphism in $\mathscr{C}$ has a kernel and a co-kernel,

ii) every monomorphism is a kernel for some homomorphism in $\mathscr{C}$ and every epimorphism is a co-kernel for some homomorphism in $\mathscr{C}$.

B.2.8 Remark. Observe that the axiom (ii) in the previous definition implies that any homomorphism in an abelian category which is both a monomorphism and an epimorphism is automatically an isomorphism.

B.2.9 Definition. Let $\mathscr{C}$ be an abelian category. Given objects $X, Y, Z \in Obj(\mathscr{C})$ and composable homomorphisms $f \in Hom_{\mathscr{C}}(X, Z)$ and $g \in Hom_{\mathscr{C}}(Z, Y)$, a sequence

$$X \xrightarrow{f} Z \xrightarrow{g} Y$$

is called

- exact at Z if $Im(f) = \ker(g)$. In this case we write $X \xrightarrow{f} Z \xrightarrow{g} Y$,
- left (resp. right) exact if it is exact at X (resp. at Y) and at Z, that is, if f is a monomorphism (resp. g is an epimorphism) and $Im(f) = \ker(g)$. In that case we write $0 \longrightarrow X \xrightarrow{f} Z \xrightarrow{g} Y$ (resp. $X \xrightarrow{f} Z \xrightarrow{g} Y \longrightarrow 0$),
- exact if it is left and right exact. In this case we write $0 \longrightarrow X \xrightarrow{f} Z \xrightarrow{g} Y \longrightarrow 0$ and it is called *short exact sequence*.

B.2.10 Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be two abelian categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ an additive functor. We say that F is exact if it transforms short exact sequences in $\mathscr{C}$ into short exact sequences in $\mathscr{D}$.

It is well known that abelian categories have enough good properties so that we can develop a fruitful homological algebra, which is very useful in lots of different branches of mathematics. Of course, this classical homological algebra is presupposed for this dissertation and standard references are [225], [33], [123], [85]. On the contrary, the concept of *triangulated category* introduced in Section 1.2 generalizes in a natural fashion the structure of abelian categories in such a way that we can also develop an analogous homological algebra. We have included a detailed presentation in Section 1.2.2 and Section 1.2.4 and it shall serve as a comparison with the classical one.

For instance, the paradigmatic example of a triangulated category is the homotopy category related to a given additive category. This is the starting point to the development of the celebrated derived category of A. Grothendieck and J. L. Verdier. We recall here its definition.

B.2.11 Theorem-Definition. *Let $\mathscr{C}$ be an additive category. The homotopy category related to $\mathscr{C}$, denoted by $\mathscr{H}(\mathscr{C})$, is the additive category defined in the following way*

- *objects of $\mathscr{H}(\mathscr{C})$ are chain complexes on $\mathscr{C}$,*
- *homomorphisms of $\mathscr{H}(\mathscr{C})$ are homotopy classes of chain complex homomorphisms.*

Moreover, $\mathscr{H}(\mathscr{C})$ is not an abelian category in general, but it is always a triangulated one. Precisely,

- *the suspension functor $\Sigma =: [1] : \mathscr{H}(\mathscr{C}) \longrightarrow \mathscr{H}(\mathscr{C})$ is defined on objects by,*

$$\Sigma\big(((X_\bullet, d_\bullet)\big) := (X_\bullet\,[1]\,, d_\bullet\,[1]), \text{ for all } (X_\bullet, d_\bullet) \in Obj(\mathscr{H}(\mathscr{C})),$$

  where $\big(X_\bullet\,[1]\,\big)_n := \big(X_\bullet\big)_{n-1}$ and $\big(d_\bullet\,[1]\,\big)_n := -\big(d_\bullet\big)_{n-1}$, for all $n \in \mathbb{Z}$; and on homomorphisms by,

$$\Sigma(f) := f\,[1]\,, \text{ for all } f \in Hom_{\mathscr{H}(\mathscr{C})}(X_\bullet, Y_\bullet) \text{ with } X_\bullet, Y_\bullet \in Obj(\mathscr{H}(\mathscr{C}))$$

  where $\big(f\,[1]\,\big)_n := f_{n-1}$, for all $n \in \mathbb{Z}$.

- *given a chain complex homomorphism $f \in Hom_{\mathscr{H}(\mathscr{C})}(X_\bullet, Y_\bullet)$ with $(X_\bullet, d_{X\bullet}), (Y_\bullet, d_{Y\bullet}) \in Obj(\mathscr{H}(\mathscr{C}))$, define its cone, denoted by $(C_{f\bullet}, d_{f\bullet})$, as the following chain complex*

$$\big(C_{f\bullet}\big)_n := \big(X_\bullet\big)_{n-1} \oplus \big(Y_\bullet\big)_n, \text{ for all } n \in \mathbb{Z}$$

$$\big(d_{f\bullet}\big)_n := \begin{pmatrix} -\big(d_{X\bullet}\big)_{n-1} & 0 \\ f_{n-1} & \big(d_{Y\bullet}\big)_n \end{pmatrix}, \text{ for all } n \in \mathbb{Z}$$

  Hence, a mapping cone triangle *is a triangle in $\mathscr{H}(\mathscr{C})$ of the form*

$$X_\bullet \xrightarrow{f} Y_\bullet \longrightarrow C_{f\bullet} \longrightarrow \Sigma(X_\bullet)$$

  The class of distinguished triangles on $\mathscr{H}(\mathscr{C})$, denoted by Δ_Σ, is given by any triangle isomorphic to a mapping cone triangle.

B.3 C^*-tensor categories. Categorical picture of Quantum Groups

B.3.1 Definition. A tensor category is the data $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, l, r)$ where

- $\mathscr{C}$ is a category,
- $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ is a functor, called *tensor product on $\mathscr{C}$*,
- $\mathbb{1} \in Obj(\mathscr{C})$ is a distinguished object of the category $\mathscr{C}$, called *unit object*,

- $\alpha : \otimes \circ (\otimes \times id) \longrightarrow \otimes \circ (id \times \otimes)$ is a natural equivalence, called *associativity constraint for* $\otimes$,
- $l : \otimes \circ (\mathbb{1} \times id) \longrightarrow id$ is a natural equivalence, called *left unit constraint for* $\otimes$,
- $r : \otimes \circ (id \times \mathbb{1}) \longrightarrow id)$ is a natural equivalence, called *right unit constraint for* $\otimes$;

such that

i) α satisfies the pentagon axiom meaning that the diagram

$$\begin{tikzcd}
 & (X \otimes Y) \otimes (Z \otimes W) \arrow[dr, "\alpha_{X,Y,Z\otimes W}"] & \\
\big((X \otimes Y) \otimes Z\big) \otimes W \arrow[ur, "\alpha_{X\otimes Y,Z,W}"] \arrow[d, "\alpha_{X,Y,Z} \otimes id_W"'] & & X \otimes \big(Y \otimes (Z \otimes W)\big) \\
\big(X \otimes (Y \otimes Z)\big) \otimes W \arrow[rr, "\alpha_{X,Y\otimes Z,W}"'] & & X \otimes \big((Y \otimes Z) \otimes W\big) \arrow[u, "id_X \otimes \alpha_{Y,Z,W}"']
\end{tikzcd}$$

is commutative for all objects $X, Y, Z, W \in Obj(\mathscr{C})$.

ii) α, l and r satisfy the triangle axiom meaning that the diagram

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbb{1},Y}"] \arrow[dr, "r_A \otimes id_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "id_X \otimes l_Y"] \\
 & X \otimes Y &
\end{tikzcd}$$

is commutative for all objects $X, Y \in Obj(\mathscr{C})$.

We say that $\mathscr{C}$ is *strict* if α, l and r are identities.

B.3.2 Definition. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is a C^*-category if

i) for any pair of objects $X, Y \in Obj(\mathscr{C})$, the set of the corresponding homomorphisms $Hom_{\mathscr{C}}(X, Y)$ is a Banach space and the composition map

$$\begin{array}{ccc} Hom_{\mathscr{C}}(Y, Z) \times Hom_{\mathscr{C}}(X, Y) & \longrightarrow & Hom_{\mathscr{C}}(X, Z) \\ (S, T) & \longmapsto & S \circ T \end{array}$$

is bilinear for all object $Z \in Obj(\mathscr{C})$ such that $||S \circ T|| \leqslant ||S|| \cdot ||T||$,

ii) there exists an antilinear contravariant functor $* : \mathscr{C} \longrightarrow \mathscr{C}$ such that

- $*_{|\ Obj(\mathscr{C})} = id$,
- given objects $X, Y \in Obj(\mathscr{C})$ and a homomorphism $T \in Hom_{\mathscr{C}}(X, Y)$, then

$$T^* \in Hom_{\mathscr{C}}(Y, X),\ T^{**} = T,\ ||T^*T|| = ||T||^2$$

B.3.3 Definition. A C^*-tensor category is the data $(\mathscr{C}, *, \otimes, \mathbb{1}, \alpha, l, r)$ where $(\mathscr{C}, *)$ is a C^*-category and $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, l, r)$ is a tensor category such that both structures are compatibles in the following sense

$$(T \otimes S)^* = T^* \otimes S^*,$$

for all $T \in Hom_{\mathscr{C}}(X, Y), S \in Hom_{\mathscr{C}}(X', Y')$ with $X, X', Y, Y' \in Obj(\mathscr{C})$.

B.3.4 Remark. The functor $*$ of a C^*-category allows to define the notions of *unitary*, *projection*, etc. for homomorphisms imitating the usual case of C^*-algebras. Moreover, we observe that for every object $X \in Obj(\mathscr{C})$, $End_{\mathscr{C}}(X)$ is a C^*-algebra.

For a more successful development of the theory, it is advisable to make also the following assumptions in our C^*-tensor categories.

i) $\mathscr{C}$ has finite direct sums. More precisely, given objects $X_1, \ldots, X_n \in Obj(\mathscr{C})$, there exists an object $S \in Obj(\mathscr{C})$ and isometries $u_i \in Hom_{\mathscr{C}}(X_i, S)$ for each $i = 1, \ldots, n$ such that

$$\sum_{i=1}^{n} u_i u_i^* = id_S \text{ and } u_i u_j^* = \delta_{ij}\ \forall i, j = 1, \ldots, n$$

ii) $\mathscr{C}$ has subobjects. More precisely, for any object $X \in Obj(\mathscr{C})$ and for any projection $p \in End_{\mathscr{C}}(X)$, there exists an object $Y \in Obj(\mathscr{C})$ and an isometry $u \in Hom_{\mathscr{C}}(Y, X)$ such that $p = uu^*$.

In particular, $\mathscr{C}$ has a zero object. Namely, the object defined by the zero projection.

iii) The unit object $\mathbb{1}$ is simple (or irreducible), that is, $End_{\mathscr{C}}(\mathbb{1}) = \mathbb{C}$.

iv) $\mathscr{C}$ is small, that is, the class of objects is a set.

B.3.5 Definition. Let $(\mathscr{C}, *, \otimes, \mathbb{1}_{\mathscr{C}}, \alpha, l, r)$ and $(\mathscr{D}, *, \otimes, \mathbb{1}_{\mathscr{D}}, \alpha', l', r')$ be two C^*-tensor categories. A C^*-tensor functor between $\mathscr{C}$ and $\mathscr{D}$ is the data $(F, \varphi_0, \varphi_2)$ where $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a functor, $\varphi_0 : \mathbb{1}_{\mathscr{D}} \longrightarrow F(\mathbb{1}_{\mathscr{C}})$ is an isomorphism and $\varphi_2 : \otimes \circ (F \times F) \longrightarrow F \circ \otimes$ is a natural equivalence such that

i) F is linear on homomorphisms,

ii) for every objects $X, Y, Z \in Obj(\mathscr{C})$ we have

$$\begin{tikzcd}
\big(F(X)\otimes F(Y)\big)\otimes F(Z) \arrow[r, "\alpha'_{F(X),F(Y),F(Z)}"] \arrow[d, "\varphi_2(X,Y)\otimes id_{F(Z)}"'] & F(X)\otimes\big(F(Y)\otimes F(Z)\big) \arrow[d, "id_{F(X)}\otimes\varphi_2(Y,Z)"] \\
F(X\otimes Y)\otimes F(Z) \arrow[d, "\varphi_2(X\otimes Y,Z)"'] & F(X)\otimes F(Y\otimes Z) \arrow[d, "\varphi_2(X,Y\otimes Z)"] \\
F\big((X\otimes Y)\otimes Z\big) \arrow[r, "F(\alpha_{X,Y,Z})"'] & F\big(X\otimes(Y\otimes Z)\big)
\end{tikzcd}$$

iii) for every object $X \in Obj(\mathscr{C})$ the following diagram is commutative

$$\begin{tikzcd}
\mathbb{1}_{\mathscr{D}}\otimes F(X) \arrow[r, "l'_{F(X)}"] \arrow[d, "\varphi_0\otimes id_{F(X)}"'] & F(X) \\
F(\mathbb{1}_{\mathscr{C}})\otimes F(X) \arrow[r, "\varphi_2(\mathbb{1}_{\mathscr{C}},X)"'] & F(\mathbb{1}_{\mathscr{C}}\otimes X) \arrow[u, "F(l_X)"']
\end{tikzcd}$$

iv) for every object $X \in Obj(\mathscr{C})$ the following diagram is commutative

$$\begin{tikzcd}
F(X)\otimes\mathbb{1}_{\mathscr{D}} \arrow[r, "r'_{F(X)}"] \arrow[d, "id_{F(X)}\otimes\varphi_0"'] & F(X) \\
F(X)\otimes F(\mathbb{1}_{\mathscr{C}}) \arrow[r, "\varphi_2(X,\mathbb{1}_{\mathscr{C}})"'] & F(X\otimes\mathbb{1}_{\mathscr{C}}) \arrow[u, "F(r_X)"']
\end{tikzcd}$$

We say that F is *strict* if the φ_0, φ_2 are identities in $\mathscr{D}$.
We say that F is *unitary* if for every objects $X, Y \in Obj(\mathscr{C})$ we have that

$$F(T)^* = F(T^*),\ \varphi_2(X,Y) \text{ is unitary, } \varphi_0 \text{ is unitary,}$$

for all $T \in Hom_{\mathscr{C}}(X,Y)$.

B.3.6 Definition. Let $(\mathscr{C}, *, \otimes, \mathbb{1}_{\mathscr{C}}, \alpha, l, r)$ and $(\mathscr{D}, *, \otimes, \mathbb{1}_{\mathscr{D}}, \alpha', l', r')$ be two C^*-tensor categories. Suppose that $(F, \varphi_0, \varphi_2)$ and $(G, \varphi'_0, \varphi'_2)$ are two C^*-tensor functors between $\mathscr{C}$ and $\mathscr{D}$. A natural tensor transformation between $(F, \varphi_0, \varphi_2)$ and $(G, \varphi'_0, \varphi'_2)$ is a natural transformation $\eta : F \longrightarrow G$ such that

i) the following diagram is commutative

$$\begin{tikzcd} & \mathbb{1}_{\mathscr{D}} \arrow[dl, "\varphi_0"'] \arrow[dr, "\varphi_0'"] & \\ F(\mathbb{1}_{\mathscr{C}}) \arrow[rr, "\eta"'] & & G(\mathbb{1}_{\mathscr{C}}) \end{tikzcd}$$

ii) for every objects $X, Y \in Obj(\mathscr{C})$ the following diagram is commutative

$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "\varphi_2"] \arrow[d, "\eta \otimes \eta"'] & F(X \otimes Y) \arrow[d, "\eta"] \\ G(X) \otimes G(Y) \arrow[r, "\varphi_2'(X,Y)"'] & F(X \otimes Y) \end{tikzcd}$$

We say that η is a *monoidal equivalence* if η is a natural equivalence.

B.3.7 Definition. Two C^*-tensor categories $(\mathscr{C}, *, \otimes, \mathbb{1}_{\mathscr{C}}, \alpha, l, r)$ and $(\mathscr{D}, *, \otimes, \mathbb{1}_{\mathscr{D}}, \alpha', l', r')$ are said to be (unitarily) monoidally equivalent if there exist C^*-tensor (unitary) functors $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{C}$ such that

i) $G \circ F$ is naturally monoidally (unitary) equivalent to $id_{\mathscr{C}}$ in $\mathscr{C}$,

ii) $F \circ G$ is naturally monoidally (unitary) equivalent to $id_{\mathscr{D}}$ in $\mathscr{D}$

B.3.8 Remark. It is important to make the following observation concerning the *strict* tensor categories.

Roughly speaking, a tensor category is a category equipped with a "tensor product" operation. This operation must be, at least, associative and must verify some obvious axioms with respect a unit object (actually, It is an abstraction of the notion of *monoid*). However, this associativity holds *up to isomorphisms*. Sometimes it is advisable to have true equalities for the tensor product associativity.

Mac Lane has proven that *"every C^*-tensor category is unitarily monoidally equivalent to a strict C^*-tensor category"* (see Theorem $XI.5.3$ in [103] for a proof).

Thus, for the general development of the theory of tensor categories, we may assume that they are strict. Nevertheless, the *strictification* of a tensor category insert new objects to the category that are isomorphic to the original ones, so that a *non-strictified* tensor category may give a better picture than the strictified one in some cases.

Fortunately, the main example of tensor category for the present dissertation, specifically the category of finite dimensional unitary representations of a compact quantum group, is strict.

B.3.9 Definition. Let $\mathscr{C}$ be a strict C^*-tensor category. Given an object $X \in Obj(\mathscr{C})$, an object $\overline{X} \in Obj(\mathscr{C})$ is called conjugate of X in $\mathscr{C}$ if there exist homomorphisms

$$R : \mathbb{1} \longrightarrow \overline{X} \otimes X \text{ and } \overline{R} : \mathbb{1} \longrightarrow X \otimes \overline{X}$$

such that the compositions

$$X \xrightarrow{id\otimes R} X\otimes\overline{X}\otimes X \xrightarrow{\overline{R^*}\otimes id} X \text{ and } \overline{X} \xrightarrow{id\otimes\overline{R}} \overline{X}\otimes X\otimes\overline{X} \xrightarrow{R^*\otimes id} \overline{X}$$

are identities in $\mathscr{C}$.

In this case, the identities $(\overline{R^*}\otimes id)\circ(id\otimes R) = id_X$ and $(R^*\otimes id)\circ(id\otimes\overline{R}) = id_{\overline{X}}$ are called *conjugate equations*.

We say that $\mathscr{C}$ is *rigid* if any object of $\mathscr{C}$ admits a conjugate object.

Conjugate objects are also called *dual objects* in the literature and they can be defined and studied for a general tensor category (see [61] for the details). In the context of the present dissertation, it is convenient to restrict ourselves to C^*-tensor categories and to use the above terminology. There exist some results of interest concerning conjugate objects that can be found in Chapter 2 of [139], for instance. Let us state some of them.

B.3.10 Theorem (Frobenius reciprocity)**.** *Let $\mathscr{C}$ be a strict C^*-tensor category. If $X\in Obj(\mathscr{C})$ is an object with conjugate $\overline{X}\in Obj(\mathscr{C})$, then the homomorphism*

$$\begin{array}{ccc} Hom_{\mathscr{C}}(X\otimes Y, Z) & \longrightarrow & Hom_{\mathscr{C}}(Y, \overline{X}\otimes Z) \\ T & \longmapsto & (id_{\overline{X}}\otimes T)\circ(R\otimes id_Y) \end{array}$$

is an isomorphism of vector spaces for all objects $Y, Z\in Obj(\mathscr{C})$, where $(R,\overline{R})$ are some solutions of the conjugate equations for X and $\overline{X}$.

Moreover, if X is a simple object, then $\overline{X}$ is simple as well and the spaces $Hom_{\mathscr{C}}(\mathbb{1}, \overline{X}\otimes X)$, $Hom_{\mathscr{C}}(\mathbb{1}, X\otimes\overline{X})$ are one dimensional.

B.3.11 Proposition. *Let $\mathscr{C}$ be a strict C^*-tensor category. If $X\in Obj(\mathscr{C})$ is an object that admits a conjugate $\overline{X}\in Obj(\mathscr{C})$, then $\overline{X}$ is unique up to isomorphism. More precisely, if $(R,\overline{R})$ are the solutions of the conjugate equations for X and $\overline{X}$ and $(R',\overline{R'})$ are the solutions for the conjugate equations for X and $\overline{X}'$, then the homomorphism*

$$T := (id_{\overline{X}}\otimes\overline{R'}^*)\circ(R\otimes id_{\overline{X}'}) \in Hom_{\mathscr{C}}(\overline{X}', \overline{X})$$

is invertible with inverse

$$S := (id_{\overline{X}'}\otimes\overline{R}^*)\circ(R'\otimes id_{\overline{X}}) \in Hom_{\mathscr{C}}(\overline{X}, \overline{X}')$$

Moreover, we have

$$R' = (T^{-1}\otimes id)\circ R \text{ and } \overline{R'} = (id\otimes T^*)\circ\overline{R}$$

In particular, if X is simple, then there exists $\lambda\in\mathbb{C}^*$ such that

$$R' = \overline{\lambda}R \text{ and } \overline{R'} = \lambda^{-1}\overline{R}$$

B.3.12 Proposition. *Let $\mathscr{C}$ be a strict C^*-tensor category. If $X\in Obj(\mathscr{C})$ is an object with conjugate $\overline{X}\in Obj(\mathscr{C})$, then $End_{\mathscr{C}}(X)$ is finite dimensional.*

As a consequence, every object with a conjugate decomposes into a finite direct sum of simple objects. In other words, in a rigid C^*-tensor category every object is semi-simple.

B.3.13 Definition. Let $\mathscr{C}$ be a strict rigid C^*-tensor category. If $X \in Obj(\mathscr{C})$ is a simple object with conjugate $\overline{X} \in Obj(\mathscr{C})$, we call intrinsic dimension of X the number

$$d_i(X) := ||R|| \cdot ||\overline{R}||,$$

where $(R, \overline{R})$ are the solution to the conjugate equations for X and $\overline{X}$.

B.3.14 Remarks.

1. Notice that the intrinsic dimension of a simple object is independent of the choice of the solutions for the corresponding conjugate equations by virtue of Proposition B.3.11 above.

2. If $X \in Obj(\mathscr{C})$ is any object (not necessarily simple), then we decompose it into a finite direct sum of simple objects, say $X = \bigoplus_{k=1}^{n} X_k$ by virtue of Proposition B.3.12 above. In this case, we define the intrinsic dimension of X to be

$$d_i(X) := \sum_{k=1}^{n} d_i(X_k)$$

3. Observe that we always have $d_i(\mathbb{1}) = 1$.

Next, let us describe the categorical picture for quantum groups. To this end, we give the two main examples of C^*-tensor categories for the present dissertation. Namely, the C^*-tensor category of Hilbert spaces and the C^*-tensor category of finite dimensional unitary representations of a compact quantum group. Both of them are crucial in order to achieve the celebrated *Tannaka-Krein-Woronowicz duality*. For this reason, we wish to illustrate the preceding theory with these two examples. For the details in the computations and in the arguments we refer to [139].

The C^*-tensor category of Hilbert Spaces

We denote by Hilb_f the category of finite dimensional Hilbert spaces.

- Choosing a large enough *set* of finite dimensional Hilbert spaces to fit all constructions of interest, we can assume that Hilb_f is *small*.

- The usual tensor product of Hilbert spaces yields a tensor functor on Hilb_f.

- The unit object on Hilb_f is given by $\mathbb{C}$.

- We can assume without loose of generality that Hilb_f is *strict*.

- Given two finite dimensional Hilbert spaces $H, H' \in Obj(\mathrm{Hilb}_f)$, the homomorphisms between them are given by the linear operators

$$Hom_{\mathrm{Hilb}_f}(H, H') := \mathcal{B}(H, H'),$$

  so that it is a Banach space with bilinear composition map satisfying the norm condition of Definition B.3.2.

- We define an antilinear contravariant functor $* : \mathrm{Hilb}_f \longrightarrow \mathrm{Hilb}_f$ to be the identity on objects and the adjoint operator on homomorphisms. It satisfies the conditions of Definition B.3.2 and Definition B.3.3 by virtue of the well-known theory of operators on Hilbert spaces.

- Hilb_f is a *rigid* C^*-tensor category. Indeed, given a finite dimensional Hilbert space $H \in Obj(\mathrm{Hilb}_f)$, fix an orthonormal basis of H, say $\{\xi_1, \ldots, \xi_n\}$ where $n := dim(H)$.

  The conjugate object of H is simply its dual space $\overline{H}$. One possible pair of solutions to the corresponding conjugate equations are given by

$$\begin{array}{rccl} R: & \mathbb{C} & \longrightarrow & \overline{H} \otimes H \\ & 1 & \longmapsto & R(1) := \sum_{i=1}^{n} \overline{\xi}_i \otimes \xi_i \end{array} \qquad \begin{array}{rccl} \overline{R}: & \mathbb{C} & \longrightarrow & H \otimes \overline{H} \\ & 1 & \longmapsto & \overline{R}(1) := \sum_{i=1}^{n} \xi_i \otimes \overline{\xi}_i \end{array}$$

- Notice that given a finite dimensional Hilbert space $H \in Obj(\mathrm{Hilb}_f)$ and fixing an orthonormal basis of H, we can always write the following direct sum decomposition $H \cong \bigoplus_{i=1}^{n} \mathbb{C}$, where $n = dim(H)$. As a consequence, for every object $H \in Obj(\mathrm{Hilb}_f)$ its intrinsic dimension is given simply by the dimension of the corresponding Hilbert space,

$$d_i(H) = dim(H)$$

The C^*-tensor category of a C.Q.G.

Consider a compact quantum group $\mathbb{G}$ in the sense of Woronowicz. We denote by $\mathscr{R}ep(\mathbb{G})$ the category of finite dimensional unitary representations of $\mathbb{G}$. $\mathscr{R}ep(\mathbb{G})$ is called *representation category of* $\mathbb{G}$.

- Assuming that the Hilbert spaces of the representations of $\mathbb{G}$ are those of the set considered above to define Hilb_f, we can assume that $\mathscr{R}ep(\mathbb{G})$ is *small*.

- The tensor product of representations (recall Definition 1.3.1.11) yields a tensor functor on $\mathscr{R}ep(\mathbb{G})$.

- The unit object on $\mathscr{R}ep(\mathbb{G})$ is given by the trivial representation ϵ.

- We can assume without loose of generality that $\mathscr{R}ep(\mathbb{G})$ is *strict*.

- Given two finite dimensional unitary representations of $\mathbb{G}$, say $w, v \in Obj(\mathscr{R}ep(\mathbb{G}))$, the homomorphisms between them are given by the corresponding space of intertwiners

$$Hom_{\mathscr{R}ep(\mathbb{G})}(w, v) := Mor(w, v),$$

  so that it is a Banach space with bilinear composition map satisfying the norm condition of Definition B.3.2 (recall Definition 1.3.1.13 and Remark 1.3.1.16)

- We define an antilinear contravariant functor $* : \mathscr{R}ep(\mathbb{G}) \longrightarrow \mathscr{R}ep(\mathbb{G})$ to be the identity on objects and the adjoint operator on homomorphisms. It satisfies the conditions of Definition B.3.2 and Definition B.3.3 by virtue of the well-known theory of operators on Hilbert spaces and Remark 1.3.1.16.

- $\mathscr{R}ep(\mathbb{G})$ is a *rigid* C^*-tensor category. Indeed, given a finite dimensional unitary representations of $\mathbb{G}$, say $w \in Obj(\mathscr{R}ep(\mathbb{G}))$, suppose that the representation space of w is $H \in Obj(\mathrm{Hilb}_f)$.

  The conjugate object of w is simply its contragredient representation $\overline{w}$, which is a finite dimensional unitary representation of $\mathbb{G}$ on $\overline{H}$ (recall Definition 1.3.1.18).

  A straightforward computation shows that if $(R, \overline{R})$ is the pair of solutions to the conjugate equations for H and $\overline{H}$ from the above example, then $R \in Mor(\epsilon, \overline{w} \oplus w)$ and $\overline{R} \in Mor(\epsilon, w \oplus \overline{w})$. One possible pair of solutions to the conjugate equations for w and $\overline{w}$ are given by

  $$R_{\mathbb{G}} := (id \otimes Q_w^{-1/2}) \circ R \text{ and } \overline{R_{\mathbb{G}}} := (Q_w^{1/2} \otimes id) \circ \overline{R},$$

  where Q_w is the invertible positive self-adjoint operator of Theorem 1.3.1.24 defining the quantum dimension of w.

- Let $w \in Obj(\mathscr{R}ep(\mathbb{G}))$ be an *irreducible* unitary representation of $\mathbb{G}$. Given the solutions $(R_{\mathbb{G}}, \overline{R_{\mathbb{G}}})$ to the conjugate equations for w and $\overline{w}$ as above, a straightforward computations yields that

  $$d_i(w) = dim_q(w)$$

The Tannaka-Krein duality for compact quantum groups in the sense of Woronowicz can be successfully established using the language and the formalism of C^*-tensor categories. We refer to Section 2.3 of [139] for a detailed proof.

B.3.15 Definition. Let $\mathscr{C}$ be a C^*-tensor category. A C^*-tensor functor $F : \mathscr{C} \longrightarrow \mathrm{Hilb}_f$ is called fiber functor if it is faithful and exact.

Given a compact quantum group $\mathbb{G}$, there exists a canonical fiber functor given simply by the obvious forgetful functor

$$\mathscr{R}ep(\mathbb{G}) \longrightarrow \mathrm{Hilb}_f,$$

which send to any finite dimensional unitary representation of $\mathbb{G}$ to its corresponding representation Hilbert space.

B.3.16 Theorem (Tannaka-Krein-Woronowicz's duality)**.** *Let $\mathscr{C}$ be a rigid C^*-tensor category and $F : \mathscr{C} \longrightarrow Hilb_f$ a unitary fiber functor. Then there exist a compact quantum group $\mathbb{G}$ and a unitary monoidal equivalence $E : \mathscr{C} \longrightarrow \mathscr{R}ep(\mathbb{G})$ such that F is naturally unitarily monoidally isomorphic to the composition $\mathscr{C} \xrightarrow{E} \mathscr{R}ep(\mathbb{G}) \longrightarrow Hilb_f$.*

Besides, the $*$-Hopf algebra $(Pol(\mathbb{G}), \Delta, \varepsilon, S)$ associated to $\mathbb{G}$ is uniquely determined up to isomorphism.

B.3.17 Remark. Observe that a discrete quantum group $\widehat{\mathbb{G}}$ is completely defined by the corresponding compact quantum group $\mathbb{G}$ (and *vice versa*) by means of the $*$-Hopf algebra $Pol(\mathbb{G})$ (recall Theorem 1.3.1.36). Moreover, Tannaka-Krein-Woronowicz duality allows to construct discrete quantum subgroups by means of purely algebraic and and categorical methods as we have already pointed out in Proposition 1.4.3.4.

Namely, if $(\mathcal{A}, \widehat{\Delta}, \widehat{\varepsilon}, \widehat{S})$ is any $*$-Hopf algebra, we can easily show that the category $\mathscr{R}ep(\mathcal{A})$ of its finite dimensional unital $*$-representations is a C^*-tensor category. Consider now a *rigid* C^*-tensor subcategory of $\mathscr{R}ep(\mathcal{A})$, say $\mathscr{C}$. Assume the following properties.

i) $\mathscr{C}$ is full,

ii) the class of representations in $\mathscr{C}$ is closed (up to isomorphism) under taking direct sums, tensor product and contragredient representations,

iii) $\mathscr{C}$ contains the trivial representation $\mathbb{1} = \widehat{\varepsilon}$.

In this situation, the dual space $\mathcal{A}^*$ gives raise to a new $*$-Hopf algebra, say $(\mathcal{B}, \Delta, \varepsilon, S)$ where $\mathcal{B}$ is the subspace of $\mathcal{A}^*$ spanned by the matrix coefficients of all representations in $\mathscr{C}$.

Furthermore, there exists a compact quantum group $\mathbb{G}$ such that

i) $(\mathcal{B}, \Delta, \varepsilon, S) = (Pol(\mathbb{G}), \Delta, \varepsilon, S)$,

ii) $\mathscr{C}$ is unitarily monoidally equivalent to $\mathscr{R}ep(\mathbb{G})$.

For the details of these constructions we refer to Theorem 2.3.13 of [139].

As an application of the preceding construction we observe the following (see Proposition 6.1 in [206] for more details). Given a compact quantum group $\mathbb{G}$ and any subset $\mathcal{S} \subset Irr(\mathbb{G})$ of equivalence classes of irreducible representations of $\mathbb{G}$, denote by $\mathscr{C} := \langle \mathcal{S} \rangle$ the smallest full subcategory of $\mathscr{R}ep(\mathbb{G})$ containing $\mathcal{S}$. If, in addition, $\mathscr{C}$ contains the trivial representation and it is closed under taking tensor product and contragredient representations, then Tannaka-Krein-Woronowicz duality and the above discussion guarantee that there is an associated C^*-subalgebra $C(\mathbb{H})$ such that restricting the co-product of $\mathbb{G}$ to $C(\mathbb{H})$ endows it with the structure of compact quantum group $\mathbb{H}$. By definition (recall Proposition 1.4.3.4), $\widehat{\mathbb{H}}$ is a discrete quantum subgroup of $\widehat{\mathbb{G}}$.

Moreover, $\mathscr{R}ep(\mathbb{H})$ naturally identifies with $\mathscr{C}$ and we say that $\widehat{\mathbb{H}}$ *is the discrete quantum subgroup of* $\widehat{\mathbb{G}}$ *generated by* $\mathcal{S}$. Sometimes, by abuse of language, we say as well that $\mathbb{H}$ *is generated by* $\mathcal{S}$.

By virtue of the Tannaka-Krein-Woronowicz's duality, any compact quantum group is completely determined by the category of its finite dimensional unitary representations. However, a rigid C^*-tensor category can also have fiber functors producing non-isomorphic compact quantum groups (for instance, see Example 2.3.9 in [139]). In this sense, we have the following definition.

B.3.18 Definition. Two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ are called monoidally equivalent if $\mathscr{R}ep(\mathbb{G}_1)$ and $\mathscr{R}ep(\mathbb{G}_2)$ are unitarily monoidally equivalent.

This notion is important for the present dissertation. Indeed, monoidal equivalences have been turned out to be very useful in order to obtain the Baum-Connes property for some quantum groups and also for K-theory computations. In this sense, there are some important results that we have to mention.

On the one hand, J. Bichon, A. De Rijdt and S. Vaes give a very explicit description of the notion of monoidal equivalence [23]. They show the monoidal equivalence between the orthogonal quantum groups and the one between the unitary quantum groups (using the earlier work by T. Banica [9], [10]). They provide as well a tool to produce new examples of ergodic actions coming from unitary fiber functors in order to study the spectral theory of compact quantum groups.

On the other hand, A. De Rijdt and N. V. Vennet obtain, in a concrete and constructive way, a bijective correspondence between actions (not necessarily ergodic) of monoidally equivalent compact

quantum groups on unital C^*-algebras [163]. Moreover, this correspondence is such that spectral subspaces of the actions are preserved. These results have been extended by C. Voigt [210] at the level of the equivariant Kasparov category of discrete quantum groups, which allows to study the Baum-Connes property and the K-theory for quantum groups. In particular, the equivariant Morita equivalence classes of torsion actions are in one-to-one correspondence between monoidally equivalent compact quantum groups. Recently, these results have been generalized for *regular* locally compact quantum groups by S. Baaj and J. Crespo [5].

B.3.19 Theorem. *i) (J. Bichon, A. De Rijdt and S. Vaes, [23]) Fix a natural number $n \in \mathbb{N}$. Let $Q, P \in GL_n(\mathbb{C})$ be two invertible matrices such that $Tr(Q^*Q) = Tr\big((Q^*Q)^{-1}\big)$ and $Tr(P^*P) = Tr\big((P^*P)^{-1}\big)$.*

The free unitary quantum groups $U^+(Q)$ and $U^+(P)$ are monoidally equivalent if and only if

$$Tr(Q^*Q) = Tr(P^*P)$$

ii) (J. Bichon, A. De Rijdt and S. Vaes, [23]) Fix a natural number $n \in \mathbb{N}$. Let $Q, P \in GL_n(\mathbb{C})$ be two invertible matrices such that $Q\overline{Q} = \lambda Id$ and $P\overline{P} = \mu Id$, for some $\lambda, \mu \in \mathbb{R}$.

The free orthogonal quantum groups $O^+(Q)$ and $O^+(P)$ are monoidally equivalent if and only if

$$\frac{\lambda}{Tr(Q^*Q)} = \frac{\mu}{Tr(P^*P)}$$

In particular, for any $Q \in GL_n(\mathbb{C})$ such that $Q\overline{Q} = \pm 1$, $SU_q(2)$ is monoidally equivalent to $O^+(Q)$ for a unique $q \in [-1, 1]\backslash\{0\}$.

iii) (A. D. Rijdt and N. V. Vennet, [163]) Let A and B two finite dimensional C^*-algebras with $dim(A) \geqslant 4$ and $dim(B) \geqslant 4$. Let $\omega \in A^*$ be a continuous δ-form and let $\omega' \in B^*$ be a continuous δ'-form, for some $\delta, \delta' > 0$.

The quantum automorphism groups $Qut(A, \omega)$ and $Qut(B, \omega')$ are monoidally equivalent if and only if $\delta = \delta'$.

As a consequence, $S_N^+ = Qut(\mathbb{C}^N)$ with $N \in \mathbb{N}$ are pairwise monoidally inequivalent and every quantum automorphism group is monoidally equivalent to $SO_q(3)$ for some $q \in (0, 1]$.

iv) (C. Voigt, [210]) Let $\widehat{\mathbb{G}}_1$ and $\widehat{\mathbb{G}}_2$ be two discrete quantum groups. If $\widehat{\mathbb{G}}_1$ is monoidally equivalent to $\widehat{\mathbb{G}}_2$, then $\widehat{\mathbb{G}}_1$ is torsion-free if and only if $\widehat{\mathbb{G}}_2$ is torsion-free.

v) (C. Voigt, [210]) Let $\widehat{\mathbb{G}}_1$ and $\widehat{\mathbb{G}}_2$ be two discrete quantum groups. If $\widehat{\mathbb{G}}_1$ is monoidally equivalent to $\widehat{\mathbb{G}}_2$, then $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}_1}$ is equivalent to $\mathscr{K}\mathscr{K}^{\widehat{\mathbb{G}}_2}$ as triangulated categories.

vi) (C. Voigt, [210]) Let $\widehat{\mathbb{G}}_1$ and $\widehat{\mathbb{G}}_2$ be two torsion-free discrete quantum groups. If $\widehat{\mathbb{G}}_1$ is monoidally equivalent to $\widehat{\mathbb{G}}_2$, then $\widehat{\mathbb{G}}_1$ satisfies the strong Baum-Connes property if and only if $\widehat{\mathbb{G}}_2$ satisfies the strong Baum-Connes property.

Bibliography

[1] A. Gaal, S. *Linear Analysis and Representation Theory.* Springer-Verlag, 2012.

[2] Akermann, C. A., Pedersen, G. K., and Tomiyama, J. Multiplier of C^*-Algebras. *Jouenal of Functional Analysis*, 13 (1973), 277–301.

[3] Arano, Y., and De Commer, K. Torsion-freeness for fusion rings and tensor C^*-categories. *arXiv* (2015).

[4] Awodey, S. *Category Theory.* Oxford University Press, 2006.

[5] Baaj, S., and Crespo, J. Equivalence monoidale de groupes quantiques et K-théorie bivariante. *arXiv* (2015).

[6] Baaj, S., and Skandalis, G. C^*-algèbres de Hopf et théorie de Kasparov équivariante. *K-theory 2*, 6 (1989), 683–721.

[7] Baaj, S., and Skandalis, G. Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres. *Annales Scientifiques de l'Ecole Normale Supérieure 26*, 4 (1993), 425–488.

[8] Banach, S., and Tarski, A. Sur la décomposition des ensembles des points en parties respectivement congruentes. *Fundamenta Mathematicae 6* (1924), 244–277.

[9] Banica, T. Théorie des représentations du groupe quantique compact libre $o(n)$. *Acad. Sci. Paris Sér. I Math. 322* (1996), 241–244.

[10] Banica, T. Le groupe quantique compact libre $U(n)$. *Comm. Math. Phys. 190* (1997), 143–172.

[11] Banica, T. Fusion rules for representations of compact quantum groups. *Exposition Math. 17* (1999), 313–337.

[12] Banica, T. Representations of compact quantum groups and subfactors. *J. Reine Angew. Math. 509* (1999), 167–198.

[13] Banica, T. Symmetries of a generic coaction. *Math. Ann. 314* (1999), 763–780.

[14] Banica, T. Quantum Groups and Fuss-Catalan Algebras. *Communications in mathematical physics 226*, 1 (2002), 221–232.

[15] Banica, T., and Bichon, J. Quantum groups acting on 4 points. *J. Reine Angew. Math 626* (2009), 75–114.

[16] Banica, T., Bichon, J., and Collins, B. Quantum permutation groups: a survey. *arXiv* (2006).

[17] Banica, T., and Patri, I. Maximal torus theory for compact quantum groups. *arXiv* (2016).

[18] Baum, P., Connes, A., and Higson, N. Classifying space for proper actions and K-theory of group C^*-algebras. *Contemporary Mathematics 167* (1994), 240–291.

[19] Beligiannis, A. Relative homological algebra and purity in triangulated categories. *J. Algebra 227*, 1 (2000), 268–361.

[20] Bhowmick, J. Quantum isometry groups of the n-tori. *Proceedings of the American Mathematical Society 137*, 9 (2009), 3155–3161.

[21] Bichon, J. Homological invariants of discrete quantum groups. *Lecture Notes for a mini-course at Seoul*.

[22] Bichon, J. Free wreath product by the quantum permutation group. *Algebras and representation theory 7*, 4 (2004), 343–362.

[23] Bichon, J., De Rijdt, A., and Vaes, S. Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups. *Comm. Math. Phys 262* (2006), 703–728.

[24] Blackadar, B. *K-theory for operator algebras*, vol. 5. Cambridge University Press, 1998.

[25] Boca, F. Ergodic actions of compact matrix pseudogroups on C^*-algebras. *Recent Advances in Operator Algebras, Astérisque 232* (1995), 93–110.

[26] Brannan, M. Approximation properties for free orthogonal and free unitary quantum groups. *Journal für die reine und angewandte Mathematik (Crelles Journal) 2012*, 672 (2012), 223–251.

[27] Brannan, M. Reduced operator algebras of trace-preserving quantum automorphism groups. *Documenta Mathematica 18* (2013), 1349–1402.

[28] Brannan, M. Approximation properties for locally compact quantum groups. *arXiv* (2016).

[29] Brown, N. P., and Ozawa, N. *C^*-algebras and finite-dimensional approximations*, vol. 88. American Mathematical Soc., 2008.

[30] Buachalla, R. Noncomutative complex structures on quantum homogeneous spaces. *Journal of Geometry and Physics 99* (2016), 154–173.

[31] Bédos, E., Murphy, G. J., and Tuset, L. Co-amenability of compact quantum groups. *Journal of Geometry and Physics 40*, 2 (2001), 129–153.

[32] Bédos, E., and Tuset, L. Amenability and co-amenability for locally compact quantum groups. *International Journal of Mathematics 14*, 08 (2003), 865–884.

[33] Cartan, H., and Eilenberg, S. *Homological algebra*. Princeton University Press, 1999.

[34] Chabert, J. Baum–Connes conjecture for some semi-direct products. *J. reine angew. 521* (2000), 161–184.

[35] Chabert, J., and Echterhoff, S. Permanence properties of the Baum-Connes conjecture. *Documenta Mathematica 6* (2001), 127–183.

[36] Chabert, J., Echterhoff, S., and Meyer, R. Deux remarques sur l'application de Baum-Connes. No. 332, pp. 607–610.

[37] Chabert, J., Oyono-Oyono, H., and Echterhoff, S. Going-down functors, the Künneth formula, and the Baum-Connes conjecture. *Geometric and functional analysis 14*, 3 (2004), 491–528.

[38] Cheng, C. A character theory for projective representations of finite groups. *Linear Algebra and its applications 469* (2015), 230–242.

[39] Choi, M. D., and Effros, E. G. Nuclear C^*-algebras and the approximation property. *Amer.. J. Math. 100*, 1 (1978), 61–79.

[40] Christensen, J. D. Ideals in triangulated categories: phantoms, ghosts and skeleta. *Adv. Math. 136*, 2 (1998), 284–339.

[41] Connes, A. *Géométrie non commutative*. InterEditions, 1990.

[42] Cornulier, Y., Stalder, Y., and Valette, A. Proper actions of wreath products and generalizations. *Transactions of the American Mathematical Society 364*, 6 (2012), 3159–3184.

[43] Cuntz, J. The K-groups for free products of C^*-algebras. *Operator algebras and applications, Part I (Kingston, Ont., 1980) 38* (1982), 81–84.

[44] Cuntz, J. K-theoretic amenability for discrete groups. *J. Reine Angew. Math*, 344 (1983), 180–195.

[45] Cuntz, J. A new look at KK-theory. *K-theory 1* (1987), 31–51.

[46] D'Andrea, F., and Dąbrowski, L. Dirac operators on quantum projective spaces. *Communications in Mathematical Physics 295*, 3 (2010), 731–790.

[47] Daws, M., Fima, P., Skalski, A., and White, S. The Haagerup property for locally compact quantum groups. *Journal für die reine und angewandte Mathematik (Crelles Journal)* (2014).

[48] Daws, M., Kasprzak, P., Skalski, A., and Sołtan, P. M. Closed quantum subgroups of locally compact quantum groups. *Advances in Mathematics 231*, 6 (2012), 3473–3501.

[49] De Commer, K. Actions of compact quantum groups. *arXiv* (2015).

[50] De Commer, K., and Yamashita, M. Tannaka-Krein duality for compact quantum homogeneous spaces I. General theory. *Theory and Applications of Categories 28*, 31 (2013), 1099–1138.

[51] De Commer, K., and Yamashita, M. Tannaka-Krein duality for compact quantum homogeneous spaces II. Classification of quantum homogeneous spaces for quantum $SU(2)$. *Journal für die reine und angewandte Mathematik (Crelles Journal) 2015*, 708 (2015), 143–171.

[52] Dijkhuizen, M. S., and Koornwinder, T. H. Quantum homogeneous spaces, duality, and quantum 2-spheres. *Geometriae dedicata 52*, 3 (1994), 291–315.

[53] Dixmier, J. *Les C^*-algebras et leurs représentations.* Cahiers Scientifiques, Fasc. XXIX, Gauthier-Villars, 1964.

[54] Dixmier, J. *Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann).* Cahiers Scientifiques, Fasc. XXV, Gauthier-Villars, 1969.

[55] Drinfeld, V. G. Quantum groups. *Proceedings of the International Congress of Mathematicians 1*, 2 (1987), 798–820.

[56] Edwin, H., and Kenneth A., R. *Abstract Harmonic Analysis I*, vol. 1. Springer, 1970.

[57] Eilenberg, S., J. C. M. Foundations of relative homological algebra. *Mem. Am. Math. Soc. 55*, 39 (1965).

[58] Enock, M., and Schwartz, J.-M. Une dualité dans les algèbres de von Neumann. *C. R. Acad. Sc. Paris 277* (1973), 683–685.

[59] Enock, M., and Schwartz, J.-M. Une dualité dans les algèbres de von Neumann. *Supp. Bull. Soc. Math. France 44* (1975), 1–144.

[60] Enock, M., and Vallin, J.-M. C^*-algèbres de Hopf et C^*-algèbres de Kac. *Proc. London. Math. Soc. 3*, 3 (1993), 619–650.

[61] Etingof, P. I., Gelaki, S., Nikshych, D., and Ostrik, V. *Tensor Categories*, vol. 205. American Mathematical Society, 2015.

[62] Fima, P. Kazhdan's property T for discrete quantum groups. *International Journal of Mathematics 21*, 01 (2010), 47–65.

[63] Fima, P. K-amenability of HNN extensions of amenable discrete quantum groups. *J. Functional Analysis 265*, 4 (2013), 507–519.

[64] Fima, P., and Freslon, A. Graphs of quantum groups and K-amenability. *Advances in Mathematics 260* (2014), 233–280.

[65] Fima, P., Mukherjee, K., and Patri, I. On compact bicrossed products. *arXiv* (2015).

[66] Fima, P., and Pittau, L. The free wreath product of a compact quantum group by a quantum automorphism group. *Journal of Functional Analysis* (2016).

[67] Flores, R., Pooya, S., and Valette, A. K-homology and K-theory for the lamplighter groups of finite groups. *arXiv* (2016).

[68] Freslon, A., and Weber, M. On the representation theory of easy quantum groups. *To appear in J. Reine Angew. Math.* (2014).

[69] Gabriel, P., and Zisman, M. *Calculus of fractions and homotopy theory*, vol. 35. Springer-Verlag, 1967.

[70] Germain, E. KK-theory of reduced free-product C^*-algebras. *Duke Math. J. 82*, 3 (1996), 707–724.

[71] Germain, E. Amalgamated free product C^*-algebras and KK-theory. *Fields Inst. Commun 12* (1997), 89–103.

[72] Goswami, D. Quantum Group of isometries in Classical and Non Commutative Geometry. *Comm. Math. Phys. 285*, 1 (2009), 141–160.

[73] Goswami, D., and Bhowmick, J. Quantum group of orientation preserving Riemannian Isometries. *Comm. Math. Phys. 257*, 8 (2009), 2530–2572.

[74] Goswami, D., and Bhowmick, J. Quantum isometry groups: examples and computations. *Comm. Math. Phys. 285*, 2 (2009), 421–444.

[75] Goswami, D., and Bhowmick, J. Quantum isometry groups of the Podleś spheres. *Journal of Functional Analysis 258*, 9 (2010), 2927–2960.

[76] Grothendieck, A. Sur une notion de produit tensoriel topologique d'espaces vectoriels topologiques, et une classe remarquable d'espaces vectoriels liées à cette notion. *C. R. Acad. Sci. Paris 233* (1951), 1556–1558.

[77] Grothendieck, A. Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc. 16* (1955).

[78] Gómez Aparicio, M. P. *Propiété (T) et morphisme de Baum-Connes tordus par une représentation non unitaire.* PhD thesis, Université Paris VII, 2007.

[79] Haageru, U., and Cowling, M. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Ivent. Math. 96*, 3 (1989), 507–549.

[80] Haageru, U., and de Cannière, J. Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. *American J. Math. 107*, 2 (1985), 455–500.

[81] Haagerup, U. An example of a non nuclear C^*-algebra, which has the metric approximation property. *Ivent. Math. 50*, 3 (1978), 279–293.

[82] Higson, N., and Kasparov, G. E-theory and KK-theory for groups which act properly and isometrically on hilbert space. *Inventiones mathematicae 144*, 1 (2001), 23–74.

[83] Higson, N., Lafforgue, V., and Skandalis, G. Counter-examples to the Baum-Connes conjecture. *Geom. Funct. Anal. 12*, 2 (2002), 330–354.

[84] Høegh-Krohn, R., Landstad, M. B., and Størmer, E. Compact ergodic groups of automorphisms. *Ann. of Math. 2*, 114 (1981), 75–86.

[85] Holm, T., Jorgensen, P., and Rouquier, R. *Triangulated Categories.* Cambridge University Press, 2010.

[86] Jensen, K. K., and Thomsen, K. *Elements of KK-theory.* Springer Science & Business Media, 2012.

[87] Jimbo, M. A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation. *Lett. Math. Phys. 10*, 1 (1985), 63–69.

[88] Jones, V. Planar algebras I. *arXiv* (1999).

[89] Jones, V., and Sunder, V. S. *Introduction to subfactors*, vol. 234. Cambridge University Press, 1997.

[90] Julg, P., and Valette, A. K-theoretic amenability for $SL2(Qp)$, and the action on the associated tree. *Journal of functional analysis 58*, 2 (1984), 194–215.

[91] Kac, G. I. Generalization of the group principle of duality. *Soviet Math. Dokl. 2* (1961), 581–584.

[92] Kac, G. I. Ring groups and the principle of duality I. *Trans. Moscow Math. Soc. 13* (1965), 291–339.

[93] Kac, G. I. Ring groups and the principle of duality II. *Trans. Moscow Math. Soc. 13* (1965), 94–126.

[94] Kac, G. I. Extensions of groups to ring groups. *Math. USSR Sbornik 5* (1968), 451–474.

[95] Kasparov, G. The operator K-functor and extensions of C^*-algebras. *Izv. Akad. Nauk 44*, 3 (1980), 571–636.

[96] Kasparov, G. The operator K-functor and extensions of C^*-algebras. *Math. USSR Izv. 16* (1981), 513–672.

[97] Kasparov, G. K-theory, group C^*-algebras, and higher signatures (conspectus). *Novikov conjectures, index theorems and rigidity, London Math. Soc. lecture notes series 226* (1995), 101–146.

[98] Kasparov, G. Equivariant KK-theory and the Novikov conjecture. *Inventiones mathematicae 91* (1998), 147–201.

[99] Kasparov, G., and Skandalis, G. Groups acting on buildings, operator K-theory and Novikov's conjecture. *K-theory 4*, 4 (1991), 303–337.

[100] Kasparov, G., and Skandalis, G. Groups "boliques" et conjecture de Novikov. *C. R. Acad. Sci. Paris Sér. I Math. 319*, 8 (1994), 815–820.

[101] Kasparov, G. G. Hilbert C^*-modules: theorems of Stinespring and Voiculescu. *J. Oper. Theory 4* (1980), 133–150.

[102] Kasprzak, P. On a certain approach to quantum homogeneous spaces. *Communications in Mathematical Physics* (2012), 1–19.

[103] Kassel, C. *Quantum groups*, vol. 155. Springer Science & Business Media, 2012.

[104] Kazhdan, D. On connection of the dual space of a group with the structure of its closed subgroups. *Functional Analysis and its Applications 1*, 1 (1967), 63–65.

[105] Khalkhali, M., Landi, G., and van Suijlekom, W. Holomorphic structures on the quantum projective line. *Int. Math. Res. Not. IMRN* (2010), 851–884.

[106] Khalkhali, M., and Moatadelro, A. Noncommutative complex geometry of the quantum projective plane. *J. Geom. Phys. 61* (2011), 2436–2452.

[107] Khalkhali, M., and Moatadelro, A. The homogeneous coordinate ring of the quantum projective plane. *J. Geom. Phys. 61* (2011), 276–289.

[108] Klein, F. Vergleichende Betrachtungen über neuere geometrische Forschungen. *Gesammelte mathematische Abhandlungen 1* (1872), 460–497.

[109] Krein, M. G. Hermitian-positive kernels on homogeneous spaces I. *Amer. Math. Soc. Transl. 34* (1963), 69–108.

[110] Krein, M. G. Hermitian-positive kernels on homogeneous spaces I. *Amer. Math. Soc. Transl. 34* (1963), 109–164.

[111] Kustermans, J. Locally compact quantum groups in the universal setting. *International Journal of Mathematics 12*, 3 (2001), 289–338.

[112] Kustermans, J., and Vaes, S. A simple definition for locally compact quantum groups. *C. R. Acad. Sci. Paris Sér. I Math. 328*, 10 (1999), 871–876.

[113] Kustermans, J., and Vaes, S. Locally compact quantum groups. *Annales Scientifiques de l'Ecole Normale Supérieure 33* (2000), 837–934.

[114] Lafforgue, V. *K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes*. PhD thesis, Université de Paris-Sud, 1999.

[115] Lafforgue, V. La conjecture de Baum-Connes à coefficients pour les groupes hyperboliques. *J. Noncommutative Geometry* (2012), 1–197.

[116] Lance, C. On nuclear C^*-algebras. *J. Funct. Anal. 12*, 2 (1973), 157–176.

[117] Lance, E. C. *Hilbert C^*-modules: a toolkit for operator algebraists*, vol. 210. Cambridge University Press, 1995.

[118] Landstad, M. Semisimplicity of crossed products from ergodic actions of compact matrix pseudogroups. *Recent Advances in Operator Algebras, Astérisque 232* (1995), 111–114.

[119] Lemeux, F. Haagerup approximation property for quantum reflection groups. *Proceedings of the American Mathematical Society 143*, 5 (2015), 2017–2031.

[120] Lemeux, F., and Tarrago, P. Free wreath product quantum groups: the monoidal category, approximation properties and free probability. *arXiv* (2014).

[121] Longo, R., and Roberts, J. A theory of dimension. *K-theory 11* (1997), 103–159.

[122] Mac Lane, S. *Categories for the working mathematician*, vol. 5. Springer Science & Business Media, 1978.

[123] Mac Lane, S. *Homology*. Springer-Verlag, 1995.

[124] Maes, A., and Van Daele, A. *Notes on Compact Quantum Groups*. Nieuw Arch. Wisk, 1998.

[125] Majid, S. Noncommutative Riemannian and spin geometry of the standard q-sphere. *Ccomm. Math. Phys. 256* (2005), 255–285.

[126] Marciniak, M. Quantum symmetries in noncommutative C^*-systems. *Quantum probability Banach Center Publications* (1998), 297–307.

[127] Martos, R., and Freslon, A. Torsion and K-theory for some free wreath products. *arXiv* (2017).

[128] Masuda, T., Nakagami, Y., and Watanabe, J. Noncommutative differential geometry on the quantum $SU(2)$. *K-theory 4*, 2 (1990), 157–180.

[129] McLeod, J. The Künneth formula in equivariant K-theory. *Algebraic Topology Waterloo 1978* (1979), 316–333.

[130] Meyer, R. Equivariant Kasparov theory and generalized homomorphisms. *K-theory*, 21 (2000), 201–228.

[131] Meyer, R. Homological algebra in bivariant K-theory and other triangulated categories II. *Tbilisi Mathematical Journal 1* (2008), 165–210.

[132] Meyer, R., and Nest, R. The Baum–Connes conjecture via localisation of categories. *Topology 45*, 2 (2006), 209–259.

[133] Meyer, R., and Nest, R. An analogue of the Baum-Connes isomorphism for coactions of compact groups. *Mathematica Scandinavica 100*, 2 (2007), 301–316.

[134] Meyer, R., and Nest, R. Homological algebra in bivariant K-theory and other triangulated categories I. *London Mathematical Society Lecture Notes 375* (2010), 236–290.

[135] Meyer, R., Roy, S., and Woronowicz, S. Homomorphisms of quantum groups. *Münster journal of Mathematics* (2012).

[136] Mislin, G., and Valette, A. *Proper Group Actions and the Baum-Connes Conjecture*. Springer Basel AG, 2003.

[137] Murphy, G. J. *C*-algebras and operator theory*. Academic press, 2014.

[138] Neeman, A. *Triangulated Categories*. Princeton University Press, 2014.

[139] Neshveyev, S., and Tuset, L. *Compact quantum groups and their representation categories*. Société mathématique de France, 2013.

[140] Nest, R., and Voigt, C. Equivariant Poincaré duality for quantum group actions. *Journal of Functional Analysis 258*, 5 (2010), 1466–1503.

[141] Oyono-Oyono, H. *Conjecture de Baum-Connes pour les groupes agissant sur les arbres.* PhD thesis, Université Claude Bernard-Lyon I, 1997.

[142] Oyono-Oyono, H. Conjecture de Baum-Connes et actions de groupes sur des arbres. *C. R. A. S. Série I Math. 326*, 7 (1998), 799–804.

[143] Oyono-Oyono, H. Baum-Connes conjecture and extensions. *J. reine angew. Math 532* (2001), 133–149.

[144] Oyono-Oyono, H. Baum-Connes Conjecture and group actions on trees. *K-theory 24*, 2 (2001), 115–134.

[145] Ozawa, N., and Popa, S. On a class of II_1 factors with at most one Cartan subalgebra. *Ann. of Math. 172*, 1 (2010), 713–749.

[146] Ozawa, N., and Popa, S. On a class of II_1 factors with at most one Cartan subalgebra. *American J. Math. 132*, 3 (2010), 841–866.

[147] P. Williams, D. *Crossed products of C^*-algebras*, vol. 134. American Mathematical Soc., 2007.

[148] Pedersen, G. K. *C^*-algebras and their automorphism groups*, vol. 14. London Mathematical Soc., 1979.

[149] Peter, F., and Weyl, H. Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe. *Math. Ann. 97*, 1 (1927), 737–755.

[150] Pimsner, M., and Popa, S. Entropy and index for subfactors. *Annales scientifiques de l'Ecole normale supérieure 19*, 1 (1986), 57–106.

[151] Pimsner, M., and Voiculescu, D. Exact sequences for K-groups and Ext-groups of certain cross-products C^*-algebras. *J. Operator Theory 4*, 1 (1980), 93–118.

[152] Pimsner, M., and Voiculescu, D. K-groups of reduced crossed products by free groups. *J. Operator Theory 8*, 1 (1982), 131–156.

[153] Pinzari, C., and Roberts, J. E. A duality theorem for ergodic actions of compact quantum groups on C^*-algebras. *Comm. Math. Phys. 277*, 2 (2008), 385–421.

[154] Pisier, G. *Introduction to operator space theory.* Cambridge Univ. Pr., 2003.

[155] Podleś, P. Quantum spheres. *Lett. Math. Phys. 14* (1987), 193–202.

[156] Podleś, P. *Przestrzenie kwantowe i ich grupy symetrii (Quantum spaces and their symmetry groups).* PhD thesis, Warsaw University, 1989.

[157] Podleś, P. Symmetries of Quantum Spaces. Subgroups and Quotient Spaces of Quantum $SU(2)$ and $SO(3)$ Groups. *Commun. Math. Phys. 170* (1995), 1–20.

[158] Polishchuk, A., and Schwarz, A. Categories of holomorphic vector bundles on noncommutative two-tori. *Comm. Math. Phys. 236* (2003), 135–159.

[159] Pontryagin, L. S. Theory of topological commutative groups. *Uspekhi Matematicheskikh Nauk*, 2 (1936), 177–195.

[160] Pooya, S. K-theory and K-homology of the wreath products of finite with free groups. *arXiv* (2017).

[161] Raeburn, I., and P. Williams, D. *Morita equivalence and continuous-trace C^*-algebras*, vol. 60. American Mathematical Soc., 1998.

[162] Rieffel, M. A. Morita equivalence for $C*$-algebras and $W*$-algebras. *Journal of pure and applied algebra 5*, 1 (1974), 51–96.

[163] Rijdt, A. D., and Vennet Vander, N. Actions of monoidally equivalent compact quantum groups and applications to probabilistic boundaries. *Ann. Inst. Fourier (Grenoble) 60*, 1 (2010), 169–216.

[164] Rørdam, M., Larsen, F., and Laustsen, N. *An introduction to K-theory for C^*-algebras*, vol. 49. Cambridge University Press, 2000.

[165] Rosenberg, J., and Schochet, C. The Künneth theorem and the universal coefficient theorem for equivariant K-theory and KK-theory. *American Mathematical Soc. 348* (1986).

[166] Rosso, M. Algèbres enveloppantes quantifiées, groupes quantiques compacts de matrices et calcul différentiel non commutatif. *Duke Math. J. 61*, 1 (1990), 11–40.

[167] Sakai, S. *C^*-algebras and W^*-algebras*, vol. 60. Springer-Verlag, 1971.

[168] Schochet, C. Topological methods for C^*-algebras II. Geometry resolutions and the Künneth formula. *Pacific Journal of Mathematics 98*, 2 (1982), 443–458.

[169] Serre, J.-P. *Trees*. Springer-Verlag, 1980.

[170] Skalski, A., and Bhowmick, J. Quantum isometry groups of noncommutative manifolds associated to group C^*-algebras. *Journal of Geometry and Physics 60*, 10 (2010), 1474–1489.

[171] Skandalis, G. Some remarks on Kasparov theory. *Journal of functional analysis 56*, 3 (1984), 337–347.

[172] Skandalis, G. Une notion de nucléarité en K-théorie (d'après J. Cuntz). *K-theory 1*, 6 (1988), 549–573.

[173] Soltan, P. M. Quantum $SO(3)$ groups and quantum group actions on M_2. *J. Noncommut. Geom. 4*, 1 (2010), 1–28.

[174] Soltan, P. M. On actions of compact quantum groups. *Illinois J. Math 55*, 3 (2011), 953–962.

[175] Soltan, P. M., and Woronowicz, S. L. A remark on manageable multiplicative unitaries. *Lett. Math. Phys. 57* (2001), 239–252.

[176] Speicher, R., and Banica, T., C. S. Classification results for easy quantum groups. *Pac. J. Math. 247*, 1 (2010), 1–26.

[177] Speicher, R., and Banica, T., C. S. Stochastic aspects of easy quantum groups. *Probab. Theory Related Fields 149* (2011), 435–462.

[178] Speicher, R., and Banica, T. Liberation of orthogonal Lie groups. *Adv. Math. 222*, 4 (2009), 1461–1501.

[179] Speicher, R., and Nica, A. *Lectures on the combinatorics of free probability.* Cambridge University Press, 2006.

[180] Steinberg, R. On a theorem of Pittie. *Topology 14*, 2 (1975), 173–177.

[181] Stroppel, M. *Locally Compact Groups.* European Mathematical Society, 2006.

[182] Takesaki, M. *Tomita's theory of modular Hilbert algebras and its applications*, vol. 128. Springer-Verlag, 1970.

[183] Takesaki, M. *Theory of operator algebras I*, vol. 124. Encyclopaedia Math. Sci. Springer-Verlag, 2002.

[184] Takesaki, M. *Theory of operator algebras II*, vol. 125. Encyclopaedia Math. Sci. Springer-Verlag, 2003.

[185] Tannaka, T. Über den Dualitätssatz der nichtkommutativen topologischen Gruppen. *Tôhoku Math. J. 45* (1939), 1–12.

[186] Tarrago, P., and Wahl, J. Free wreath product quantum groups and standard invariants of subfactors. *arXiv* (2016).

[187] Thomsen, K. Equivariant KK-theory and C^*-extensions. *K-theory*, 19 (2000), 219–249.

[188] Timmermann, T. *An invitation to quantum groups and duality: From Hopf algebras to multiplicative unitaries and beyond.* European Mathematical Society, 2008.

[189] Tomatsu, R. Amenable discrete quantum groups. *Journal of the Mathematical Society of Japan 58*, 4 (2006), 949–964.

[190] Tomatsu, R. Compact quantum ergodic systems. *J. Funct. Anal. 254* (2008), 1–83.

[191] Tu, J.-L. *La conjecture de Baum–Connes pour les feuilletages hyperboliques.* PhD thesis, Université Paris VII, 1996.

[192] Tu, J.-L. La conjecture de Baum–Connes pour les feuilletages moyennables. *K-theory 17*, 3 (1999), 215–264.

[193] Tu, J.-L. The Baum-Connes conjecture and discrete group actions on trees. *K-theory 17*, 4 (1999), 303–318.

[194] Vaes, S. A new approach to induction and imprimitivity results. *Journal of Functional Analysis 229*, 2 (2005), 317–374.

[195] Vaes, S., and Popa, S. Unique Cartan decomposition for II_1 factors arising from arbitrary actions of hyperbolic groups. *J. Reine Angew. Math.* (2013).

[196] Vaes, S., and Vainerman, L. Extensions of locally compact quantum groups and the bicrossed product construction. *Advances in Mathematics 175* (2003), 1–101.

[197] Vaes, S., and Vergnioux, R. The boundary of universal discrete quantum groups, exactness, and factoriality. *Duke Mathematical Journal 140*, 1 (2007), 35–84.

[198] Vainerman, L. I., and Kac, G. V. Nonunimodular ring-groups and Hopf-von Neumann algebras. *Soviet Math. Dokl. 14* (1973), 1144–1148.

[199] Vainerman, L. I., and Kac, G. V. Nonunimodular ring-groups and Hopf-von Neumann algebras. *Math. USSR, Sbornik 23* (1974), 185–214.

[200] Vallin, J.-M. C^*-algèbres de Hopf et C^*-algèbres de Kac. *Proc. London. Math. Soc. 50*, 1 (1985), 131–174.

[201] van Daele, A. Multiplier Hopf algebras. *Trans. Amer. Math. Soc. 342* (1994), 917–932.

[202] van Daele, A. An algebraic framework for group duality. *Adv. Math. 140* (1998), 323–366.

[203] van Daele, A. Tools for working with multiplier Hopf algebras. *Arabian Journal for Science and Engineering 33* (2008).

[204] van Daele, A., and Wang, S. Universal quantum groups. *International Journal of Mathematics 7*, 02 (1996), 255–263.

[205] Verdier, J.-L. *Des Catégories Dérivées des Catégories Abéliennes*. PhD thesis, Université Paris, 1967.

[206] Vergnioux, R. *KK-théorie équivariante et opérateur de Julg-Valette pour les groupes quantiques*. PhD thesis, Université Paris VII, 2002.

[207] Vergnioux, R. K-amenability for amalgamated free products of amenable discrete quantum groups. *J. Funct. Anal. 212*, 1 (2004), 206–221.

[208] Vergnioux, R., and Voigt, C. The K-theory of free quantum groups. *Mathematische Annalen 357*, 1 (2013), 355–400.

[209] Voiculescu, D., Dykema, K., and Nica, A. Free Random Variables. *CRM Monograph Series American Mathematical Society 1* (1992).

[210] Voigt, C. The Baum–Connes conjecture for free orthogonal quantum groups. *Advances in Mathematics 227*, 5 (2011), 1873–1913.

[211] Voigt, C. Quantum SU (2) and the Baum-Connes conjecture. *arXiv* (2012).

[212] Voigt, C. On the structure of quantum automorphism groups. *Journal für die reine und angewandte Mathematik (Crelles Journal)* (2015).

[213] von Dyck, W. Gruppentheoretische Studien. *Mathematische Annalen 20*, 1 (1882), 1–44.

[214] von Neumann, J. Zur allgemeinen Theorie des Masses. *Fund. Math. 13* (1930), 73–116.

[215] Wang, S. Free product of compact quantum groups. *Communications in Mathematical Physics 167* (1995), 671–692.

[216] Wang, S. Tensor products and crossed products of compact quantum groups. *Proc. London Math. Soc. 71* (1995), 695–720.

[217] Wang, S. Quantum symmetry groups of finite spaces. *Comm. Math. Phys. 195* (1998), 195–211.

[218] Wang, S. Structure and Isomorphism Classification of Compact Quantum Groups $A_u(Q)$ and $B_u(Q)$. *Journal of Operator Theory 48*, 3 (2002), 573–583.

[219] Wang, S. Simple compact quantum groups. *I. J. Funct. Anal. 256*, 10 (2009), 3313–3341.

[220] Wassermann, A. Equivariant K-theory II: Hodgnkin's spectral sequence in Kasparov theory. *Preprint* (1983).

[221] Wassermann, A. Ergodic actions of compact groups on operator algebras II. Classification of full multiplicity ergodic actions. *Canad. J. Math. 40* (1988), 1482–1527.

[222] Wassermann, A. Ergodic actions of compact groups on operator algebras III. Classification for $SU(2)$. *Invent. Math. 93* (1988), 309–354.

[223] Wassermann, A. Ergodic actions of compact groups on operator algebras I. General Theory. *Ann. of Math. 2*, 130 (1989), 273–319.

[224] Wegge-Olsen, N. E. *K-theory and C^*-algebras. A friendly approach.* Oxford University Press, 1993.

[225] Weibel, C. A. *An introduction to homological algebra*, vol. 38. Cambridge University Press, 1994.

[226] Woronowicz, S. L. Compact matrix pseudogroups. *Comm. Math. Phys. 111*, 4 (1987), 613–665.

[227] Woronowicz, S. L. Twisted $SU(2)$ group. An example of a noncommutative differential calculus. *Publ. Res. Inst. Math. Sci. 23* (1987), 117–181.

[228] Woronowicz, S. L. Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups. *Invent. math. 93*, 1 (1988), 35–76.

[229] Woronowicz, S. L. Differential calculus on compact matrix pseudogroups (quantum groups). *Comm. Math. Phys. 122* (1989), 125–170.

[230] Woronowicz, S. L. From multiplicative unitaries to quantum groups. *Internat. J. Math. 7* (1996), 127–149.

[231] Woronowicz, S. L. Compact quantum groups. *Symétries quantiques* (1998), 845–884.

[232] Woronowicz, S. L., and Podleś, P. Quantum deformation of Lorentz group. *Commun. Math. Phys. 130* (1991), 381–431.

"*Ce livre est terminé, mais l'histoire continue...*"

Saadi.