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# THÈSE DE DOCTORAT

de l'Université de recherche Paris Sciences et Lettres  
PSL Research University

Préparée à Université Paris Dauphine

Solutions Périodiques Symétriques dans le Problème de N-Vortex

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# Solutions Périodiques Symétriques dans le Problème de N-Vortex



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Une thèse soumise pour le diplôme de  
*Docteur en Sciences*

## 博士毕业致谢

2015年7月1日是巴黎1873年以来记录到的最热的7月1日。也就是在这一天我接到了Université Paris IX的数学系博士录取通知，由此历经了如同炎热夏天一般艰苦卓绝而有热情澎湃的三年。值此尘埃落定之时，我想要向在这三年中给予我无私帮助和鼓励的人们致以真诚的谢意。

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## Résumé

Cette thèse porte sur l'étude des solutions périodiques du problème des  $N$  tourbillons à vorticité positive. Ce problème, formulé par Helmholtz il y a plus de 160 ans, possède une histoire très riche et reste un domaine de recherche très actif. Pour un nombre quelconque de tourbillons et sans contrainte sur les vorticités, ce système n'est pas intégrable au sens de Liouville : on ne peut trouver de solution périodique non triviale par des méthodes explicites. Dans cette thèse, à l'aide de méthodes variationnelles, nous prouvons l'existence d'une infinité de solutions périodiques non triviales pour un système de  $N$  tourbillons à vorticités positives. De plus, lorsque les vorticités sont des nombres rationnels positifs, nous montrons qu'il n'existe qu'un nombre fini de niveaux d'énergie sur lesquels un équilibre relatif pourrait exister. Enfin, pour un système de  $N$  tourbillons identiques, nous montrons qu'il existe une infinité de chorégraphies simples.

**Mots clés:** système Hamiltonien, orbite périodique, N-Tourbillon, symétrie

## Abstract

This thesis focuses on the study of the periodic solutions of the  $N$ -vortex problem of positive vorticity. This problem was formulated by Helmholtz more than 160 years ago and remains an active research field. For an undetermined number of vortices and general vorticities the system is not Liouville integrable and periodic solutions cannot be determined explicitly, except for relative equilibria. By using variational methods, we prove the existence of infinitely many non-trivial periodic solutions for arbitrary  $N$  and arbitrary positive vorticities. Moreover, when the vorticities are positive rational numbers, we show that there exists only finitely many energy levels on which there might exist a relative equilibrium. Finally, for the identical  $N$ -vortex problem, we show that there exists infinitely many simple choreographies.

**Key words:** Hamiltonian System, periodic orbits, N-Vortex, symmetry

# Table of contents

<b>List of figures</b>	<b>vii</b>
<b>Nomenclature</b>	<b>ix</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Vortex Model: From Continuum to Discrete . . . . .	1
1.1.1 Vortices Model in Hydrodynamics: Euler's Equation . . . . .	3
1.1.2 Vortices Model in Quantum Mechanics: Gross-Pitaevskii Equation . . . . .	4
1.2 From Integrable System to Non-Integrable System . . . . .	5
1.2.1 Integrable Cases . . . . .	5
1.2.2 Non-Integrable Cases . . . . .	9
1.3 Periodic Solutions of the N-vortex Problem . . . . .	11
1.3.1 Equilibria . . . . .	11
1.3.2 Non-Equilibrium Solutions . . . . .	15
1.3.3 Variational Method: From Poincaré to the <i>Eight</i> . . . . .	17
1.4 Variational Methods in the N-Vortex Hamiltonian Systems . . . . .	22
1.5 Main Results . . . . .	24
1.5.1 Periodic Orbits of the Positive N-Vortex Problem . . . . .	24
1.5.2 Choreographies of the Identical N-Vortex Problem . . . . .	26
1.5.3 An Uniform Bound Estimate for Symmetric Periodic Orbits . . . . .	28
<b>2 Periodic Orbits of the Positive N-Vortex Problem</b>	<b>31</b>
2.1 Sparseness of Relative Equilibria . . . . .	32
2.1.1 Positive Vorticities . . . . .	32
2.1.2 Rational Positive Vorticities And Beyond . . . . .	34
2.2 Abundance of Non-Equilibrium Relative Periodic Solutions . . . . .	38
2.2.1 Symplectic Reduction and Relative Periodic Orbits in the Plane . . . . .	38
2.3 Periodic Orbits with Discrete $C_N$ Symmetry . . . . .	44

<b>3</b>	<b>Periodic Orbits of the Identical N-Vortex Problem</b>	<b>49</b>
3.1	Absolute and relative choreographies . . . . .	53
3.1.1	(Simple) Choreographic Loop . . . . .	53
3.1.2	Reduced Choreographic Loop in $\mathbb{C}\mathbb{P}^{N-1}$ . . . . .	54
3.1.3	Centred Reduced Choreographic Loop in $\mathbb{C}\mathbb{P}^{N-2}$ . . . . .	55
3.1.4	Relative choreographic loop in $\mathbb{R}^{2N}$ . . . . .	56
3.2	Choreographic Holomorphic Spheres in Reduced Phase space . . . . .	58
3.3	Choreographic Fiber Bundle . . . . .	59
3.3.1	Choreographic Fiber and Section . . . . .	60
3.4	Choreographic Hamiltonian Perturbation . . . . .	61
3.4.1	Invariant Hamiltonian Under Choreographic Symmetry . . . . .	61
3.5	Well Posedness Of Choreographic Holomorphic Sphere . . . . .	63
3.5.1	Well Posedness of Choreographic Holomorphic Sphere . . . . .	64
3.6	Simple Relative Choreographies Of Planar Interactive Hamiltonian System	73
3.6.1	Simple Relative Choreography . . . . .	73
3.6.2	A Sufficient Condition For Existence Of Symmetric Component . .	76
3.7	Application To Some Physical Models . . . . .	76
3.7.1	The Non-Linear Discrete Schrödinger Equation . . . . .	77
3.7.2	The N-Vortex Problem in Hydrodynamics . . . . .	77
3.7.3	The N-Vortex Problem in Bose-Einstein Condensation . . . . .	80
3.7.4	Comparison With Other Methods . . . . .	84
<b>4</b>	<b>Uniform Upper Bounds for Mutual Distances of Symmetric Periodic Solutions of N-Vortex Type Hamiltonian</b>	<b>87</b>
4.1	Upper Bounds of Mutual Distances . . . . .	87
4.1.1	The Group of Italian Symmetry . . . . .	88
4.2	Uniform bound for Italian Symmetric T-periodic solution . . . . .	89
4.3	Application to Identical N-vortex System . . . . .	93
4.3.1	N-vortex System as Hamiltonian System . . . . .	93
4.3.2	Reparametrization Of Time . . . . .	93
4.3.3	Compactness for solution space of Floer Equation . . . . .	95
<b>A</b>	<b>Some Elementary Results on the Hamiltonian System</b>	<b>101</b>
A.1	Poincaré-Melnikov Method . . . . .	101
A.2	Symplectic Reduction and Reduced Hamiltonian . . . . .	103



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<b>B A minimax Approach For Identical N-Vortex Problem</b>	<b>105</b>
B.1 Planar N-vortex Problem as Hamiltonian System . . . . .	105
B.1.1 Hamiltonian Structure and First Integrals . . . . .	105
B.1.2 Sketch of the proof . . . . .	107
B.2 The Existence of T-periodic solution for $H_2$ . . . . .	107
B.2.1 Commuted Hamiltonian flows and the induced T-periodic solution of the Hamiltonian $H_1$ . . . . .	112
B.3 Collision, Minimal Period and the Induced Periodic Solution of the Hamilto- nian $H_0$ . . . . .	113
B.4 Symmetry and Exclusion of Collision . . . . .	116
B.4.1 Simple choreography . . . . .	116
B.4.2 Simple choreography with a center . . . . .	117
B.5 Verification of Palais-Smale condition . . . . .	118
B.6 Palais' Principle and the Symmetry of choreography . . . . .	121
B.6.1 Symmetry of choreography . . . . .	121
<b>References</b>	<b>123</b>



## List of figures

1.1	The Point Vortex Model of Helmholtz . . . . .	2
1.2	The velocity of A due to B, both with positive vorticity . . . . .	2
1.3	Thomson configuration for 8 vortices which form an octagon . . . . .	13
1.4	Figure "8" of the 3-body problem (picture taken from [33]) . . . . .	20
2.1	A non trivial relative periodic (left) coming from a non-centred relative equilibrium in the original phase space (right) . . . . .	39
2.2	An example of a $M \times N$ -vortex configuration that is $C_N$ symmetric, with $M=3, N=4$ . . . . .	47
3.1	Two configurations of same $H$ and $I$ . . . . .	50
3.2	4-vortex problem in BEC restricted to $\mathcal{M}_\rho$ . . . . .	83
3.3	A 3-polygon (equilateral triangle) in a rotational frame . . . . .	83
3.4	The configuration changing with reduced energy level . . . . .	85
B.1	The critical value is taken as the inf-sup among all the surfaces modelled on $Q$ (left), thus is bounded above by sup on $Q$ (right) itself . . . . .	112



# Nomenclature

$\mathbf{O}(N, \mathbb{R})$  The Orthogonal Group

$\mathbf{SO}(N, \mathbb{R})$  The Special Orthogonal Group

$\mathbf{Sp}(N, \mathbb{R})$  The Symplectic Group

$\mathbf{U}(N, \mathbb{C})$  The Unitary Group

$H_M$  The Hamiltonian on the symplectic manifold  $M$

$\mathbf{X}_H$  The Hamiltonian vector field of  $H$

$\|\cdot\|$  The Norm

$\mathbf{J}$  The almost complex structure

$\{\cdot, \cdot\}$  The Poisson bracket

$\mathbb{C}\mathbb{P}^k$  The  $k$ -dimensional complex projective space

$\mathbb{R}^{2N}$  The  $2N$ -dimensional Euclidean space

$\mathbb{S}^k$  The sphere in  $k + 1$  dimensional space

$(M, \omega)$  The symplectic manifold  $M$  with symplectic structure  $\omega$

$\mathbf{H}^k$  The Hilbert Space  $\mathbf{W}^{k,2}$

$\mathbf{L}^p$  The Lebesgue's Space

$\mathbf{C}^\infty$  The Space of smooth functions

$\mathbf{W}^{k,\alpha}$  The Fractal Sobolev Space

$\mathbf{W}^{k,p}$  The Sobolev Space

- $\mathfrak{J}$  The generalised momentum map
- $\Gamma_i$  The vorticity of the  $i$ -th Vortex
- $\mathbf{z}$  The  $2N$ -dimensional vector describing the positions of  $N$  particles
- $z_i$  The 2-dimensional vector describing the position of the  $i$ -th particle

**Acronyms / Abbreviations**

- BEC* Bose Einstein Condensation
- NLDS* Non Linear Discrete Schrödinger Equation
- NNRPO* Nontrivial Normalised Relative Periodic Solution
- NRPO* Normalised Relative Periodic Solution
- ODE* Ordinary Differential Equation
- PDE* Partial Differential Equation
- RPO* Relative Periodic Solution
- RSC* Relative Simple Choreography

# Chapter 1

## Introduction

### 1.1 Vortex Model: From Continuum to Discrete

The study of vortex dynamics has up to now 160 years of history, whose birth is marked by Hermann von Helmholtz's seminal paper in hydrodynamics *Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen* [49], published in the year 1858.<sup>1</sup> In his famous paper, Helmholtz has developed the conservation laws of vorticity for Euler's model, which shows that the vorticity can neither be created or destroyed by any conservative forces. It is known today as the following theorems:

1. **Helmholtz's first theorem:** The total vorticity flux in a vorticity tube remains constant along the tube;
2. **Helmholtz's second theorem:** The total vorticity flux across any material surface remains constant in time<sup>2</sup>.

In chapter 5 of his 1858 paper, by using these theorems, Helmholtz considered the perpendicular section of infinitely thin, straight, parallel vortex filaments with constant vorticity with a plane, thus he had introduced the point vortex model, known today as the ***N*-vortex problem** in the plane.

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<sup>1</sup>It has been translated into English by Tait [86], and has shown considerable impact on the Victorian school of hydrodynamics, including the development of vortex atom theory by William Thomson (more frequently mentioned as Lord Kelvin) [107–109] during the period 1867-1878.

<sup>2</sup>This is equivalent to Lord Kelvin's famous circulation theorem

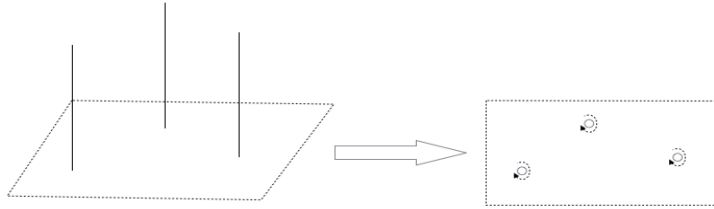


Fig. 1.1 The Point Vortex Model of Helmholtz

Given a system of  $N$  vortices, each vortex  $z_i = (x_i, y_i)$  with intensity  $\Gamma_i \in \mathbb{R} \setminus \{0\}$ , their dynamics are governed by the ODEs:

$$\dot{x}_i = -\frac{1}{2\pi} \sum_{j \neq i} \frac{\Gamma_j}{|z_i - z_j|^2} (y_i - y_j), \quad \dot{y}_i = \frac{1}{2\pi} \sum_{j \neq i} \frac{\Gamma_j}{|z_i - z_j|^2} (x_i - x_j) \quad (1.1)$$

It is Kirchhoff who first has shown the Hamiltonian nature of this system in his lecture notes

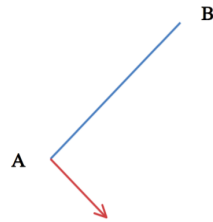


Fig. 1.2 The velocity of A due to B, both with positive vorticity

on mathematical physics in 1876 [54]. More precisely, he had shown that the system could be written as

$$\begin{aligned} \Gamma_i \frac{d}{dt} x_i &= \frac{\partial}{\partial y_i} H(z) \\ \Gamma_i \frac{d}{dt} y_i &= -\frac{\partial}{\partial x_i} H(z) \end{aligned}$$

where

$$H(z) = -\frac{1}{4\pi} \sum_{1 \leq i < j \leq N} \Gamma_i \Gamma_j \log |z_i - z_j|^2$$



Similar systems of point vortices have emerged from Bose-Einstein condensation to superconductivity, from evolution of stellar system, to the geographical ocean flow. In this section, we give a brief discussion on the procedure of passing from continuum model to discrete model. It allows one to study infinite dimensional problems through an efficient finite dimensional approximation, and retrieve essential information on various phenomena in physics from simplified models.

### 1.1.1 Vortices Model in Hydrodynamics: Euler's Equation

The motion of ideal incompressible flow is governed by the Euler' equation

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p \quad (1.2)$$

Here  $\mathbf{u} \in \mathbb{R}^3$  represents the velocity vector field of the ideal fluid. Letting

$$\boldsymbol{\omega} = \text{curl} \mathbf{u} = \nabla \wedge \mathbf{u} = (\partial_y u_z - \partial_z u_y, \partial_z u_x - \partial_x u_z, \partial_x u_y - \partial_y u_x), \quad (1.3)$$

equation (1.2) becomes

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}. \quad (1.4)$$

By considering a very thin layer, we may assume that  $z = 0$ . The system is then 2-dimensional. For regularity considerations, we turn the above equation into the following weak form. Define

$$\boldsymbol{\omega}_t(f) = \int_D f(z) \boldsymbol{\omega}_t(dz). \quad (1.5)$$

We look for solutions  $\boldsymbol{\omega}(z, t)$  s.t.

$$\begin{cases} \frac{d}{dt} \boldsymbol{\omega}_t(f) = \boldsymbol{\omega}_t(\mathbf{u} \cdot \nabla f) \\ \mathbf{u}(z, t) = \int_D \mathbf{J} \nabla G_D(z, v) \boldsymbol{\omega}(dv). \end{cases}$$

Here  $f : D \subset \mathbf{R}^2 \rightarrow \mathbf{R}^2$  is a bounded smooth function, and  $G_D$  is the Green function in domain  $D$ . We are interested in the evolution of point vortices in the whole plane. In this case,

- The initial vorticity function is the following point vorticity distribution:

$$\boldsymbol{\omega}_0(dz) = \sum_{1 \leq i \leq N} \Gamma_i \delta_{z_i}(dz), \Gamma_i \in \mathbf{R} \setminus \{0\}; \quad (1.6)$$

- $D = \mathbf{R}^2$ , and  $G_D(z, v) = -\frac{1}{2\pi} \log |z - v|$ .

Observe (by using a regularizing sequence if necessary) that a vortex is at rest under the action of its own field due to symmetry. We finally arrive at the following system:

$$\begin{cases} \Gamma_i \frac{d}{dt} x_i = \frac{\partial}{\partial y_i} H(z) \\ \Gamma_i \frac{d}{dt} y_i = -\frac{\partial}{\partial x_i} H(z), \quad 1 \leq i \leq N, \end{cases}$$

where

$$H_{\mathbb{R}^{2N}}(\mathbf{z}) = -\frac{1}{4\pi} \sum_{1 \leq i < j \leq N} \Gamma_i \Gamma_j \log |z_i - z_j|^2 \quad (1.7)$$

By taking

$$X_i = \sqrt{|\Gamma_i|} x_i, Y_i = \Gamma_i \sqrt{|\Gamma_i|} y_i \quad (1.8)$$

the above system becomes a standard Hamiltonian system.

In this thesis we will not focus on the impact of a boundary on the dynamical behavior, we only mention that in the presence of a boundary the Hamiltonian is more complicated, i.e.,

$$H_{\Omega}(\mathbf{z}) = -\frac{1}{4\pi} \sum_{1 \leq i < j \leq N} \Gamma_i \Gamma_j \log |z_i - z_j|^2 + \sum_{1 \leq i \leq N} R_{\Omega}(z_i) \quad (1.9)$$

It consists of two parts: the Kirchoff function  $H_{\mathbb{R}^{2N}}$ , which rules the interactions between vortices; and the Routh function  $R_D$ , which depends on the Green function of the domain  $\Omega$ , and which evaluates the interaction of each vortex individually with the boundary  $\partial\Omega$ . In some situations,  $R_{\Omega}$  could be found explicitly by using the so-called image method. For a general discussion, see Lim [62].

### 1.1.2 Vortices Model in Quantum Mechanics: Gross-Pitaevskii Equation

Consider a dilute gas of bosons that is cooled to an extremely low temperature near absolute zero. Normally, atoms will present different macroscopic wave functions. However in this extreme situation, all the atoms will present a single macroscopic wave function. This state of matter is called *Bose-Einstein Condensation* (BEC). The wave function  $\psi$  of the

cloud of atoms is described by a partial differential equation(PDE), i.e., the Gross-Pitaevskii (GP) equation:

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi + V(x,y)\psi + |\psi|^2\psi \quad (1.10)$$

Here  $V(x,y)$  is the function describing the artificially set external potential(magnetic and optic fields), which is used for confining the atoms. In practice,  $V(x,y)$  is taken to be isotropic about the origin, and is realized either via a harmonic trap [41] or via a hard wall container [3]. Note that when  $V = 0$ , it is the classical cubic Schrödinger equation. Again we could consider the interaction of straight vortex lines and write a ODE system as an approximation of this PDE system. It turns out that the governing Hamiltonian becomes

$$H(\mathbf{z}) = -\frac{1}{2}\left(\mu \sum_{i=1}^N \log \frac{1}{1-|z_i|^2} + \lambda \sum_{i<j} \log |z_i - z_j|^2\right) \quad (1.11)$$

In this case the vortices are confined in the unit disk. As in the bounded domain N-vortex problem in hydrodynamics, these vortices intersect pairwise with each other, and, simultaneously, individually with the boundary.

## 1.2 From Integrable System to Non-Integrable System

The Hamiltonian nature of the N-vortex problem opens the door to using symplectic methods, and naturally raises the question of integrability. Integrability is one of the first important qualitative features of a Hamiltonian system. It implies the existence of a regular invariant foliation, thus excludes the possibility of chaotic behavior. Moreover, integral curves may be found by means of quadratures and eliminations. To the contrary, the non-integrable Hamiltonian system is in general much harder to understand. In this chapter, we take N-vortex problem from hydrodynamics as our example, and review some known results about the integrability of the N-vortex problem. It turns out that for  $N \leq 3$  the system is completely integrable, while for  $N > 4$  it is in general non-integrable.

### 1.2.1 Integrable Cases

In this subsection, we recall the definition of Liouville integrability, and show that the 3-vortex problem from Euler's equation is an integrable Hamiltonian system. Similar analysis applies to 2-vortex problem from BEC.

### Liouville Integrability

Let  $(M, \omega)$  be a symplectic manifold, where

$$M = \mathbb{R}^{2N}, \quad \omega = \sum_{i=1}^N \frac{1}{\Gamma_i} dy_i \wedge dx_i$$

is the vorticity-weighted symplectic structure. The N-vortex problem could then be written as

$$\Gamma \dot{\mathbf{z}} = \mathbf{X}_H(\mathbf{z})$$

**Definition 1.2.1** (Poisson Bracket). *The Poisson Bracket of two functions  $F, G \in C^\infty(M, \mathbb{R})$  is defined as*

$$\{F, G\} = \omega(dF, dG) \quad (1.12)$$

In our case, in local coordinates the Poisson Bracket can also be interpreted as

$$\{F, G\} = \sum_{1 \leq i \leq N} \frac{1}{\Gamma_i} \left( \frac{dF}{dx_i} \frac{dG}{dy_i} - \frac{dF}{dy_i} \frac{dG}{dx_i} \right) \quad (1.13)$$

It is easy to check that the following properties holds for the Poisson Bracket

$$\begin{aligned} \{F, \mu_1 G + \mu_2 H\} &= \mu_1 \{F, G\} + \mu_2 \{F, H\}; && \text{(bi-linearity)} \\ \{F, G\} &= -\{G, F\}; && \text{(skew-symmetry)} \\ \{F, GH\} &= G\{F, H\} + H\{F, G\}; && \text{(Leibniz rule)} \\ \{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} &= 0 && \text{(Jacobi Identity)} \end{aligned}$$

**Definition 1.2.2** (First Integral). *A function  $F \in C^\infty(M, \mathbb{R})$  is called a first integral of the Hamiltonian system if  $\{F, H\} = 0$ .*

The following theorem on Liouville integrability is taken from [9].

**Theorem 1.2.1** (Integrable System). *Suppose that we are given  $N$  functions on a  $2N$ - dimensional symplectic manifold,  $h = (h_1, h_2, \dots, h_N) \in \mathbb{R}^N$ , and*

$$L_h = \{z \in M \mid F_i(z) = h_i, 1 \leq i \leq N\} \quad (1.14)$$

*If they satisfy moreover that*

- $F_i$  and  $F_j$  are in involution, i.e.,  $\{F_i, F_j\} = 0, \forall 1 \leq i < j \leq N$

- $F_i, 1 \leq i \leq N$  are independent on  $L_h$ , i.e.  $\det(\frac{dF}{dz}) \neq 0$  on  $L_h$

Then

1.  $L_h$  is an smooth manifold invariant under the flow  $\phi_H$  of the Hamiltonian.
2. If further more  $L_h$  is connected and compact, then  $L_h$  is diffeomorphic to  $\mathbb{T}^N$
3. There exists so-called action angle variables  $(I, \phi)$  s.t. under this symplectic transformation the flow of the Hamiltonian flow is quasi-periodic:

$$\dot{\phi} = \omega_h, \quad \omega_h = \omega(h) \in \mathbb{R}^N \quad (1.15)$$

4. The canonical Hamiltonian equation can be integrated by quadratures.

#### Integrability of N-Vortex Problem: $N \leq 3$

The first three integrals of the N-vortex problem have first been found explicitly by Henri Poincaré in [88]. Note that

- The system is invariant under translation, hence

$$P(\mathbf{z}(t)) = \sum_{1 \leq i \leq N} \Gamma_i x_i(t) = cst, \quad Q(\mathbf{z}(t)) = \sum_{1 \leq i \leq N} \Gamma_i y_i(t) = cst \quad (1.16)$$

- The system is invariant under rotation, hence

$$I(\mathbf{z}(t)) = \sum_{1 \leq i \leq N} \Gamma_i |z_i(t)|^2 = cst \quad (1.17)$$

It turns out that

$$\{H, I\} = \{H, P^2 + Q^2\} = \{P^2 + Q^2, I\} = 0 \quad (1.18)$$

As a result the 3-vortex problem is integrable and much about it has been understood since a long time. In 1877, Gröbli in his dissertation [44] has first introduced the relative coordinates represented by the mutual distances  $\rho_{12}, \rho_{23}, \rho_{13}$  between the three pairs of vortices. Using these coordinates, he has re-calculated the first integrals, and investigated in particular problems today known as the relative equilibria and the self-similar motions. In 1949, Sygne [105] has reinvestigated the problem using the same coordinates, and analyzed the stability of relative equilibria. He has also found different relative periodic solution configurations. Some

general observation on discrete symmetry of the system has also been discussed therein. Later on Novikov [78] has use the phase diagram technique to classify possible motion regimes for 3 identical vortices, followed by the generalisation to 3-vortex problem with arbitrary vorticities by Aref [5]. Poisson geometric aspect of 3-vortex problem is studied by Borisov et al in a series of papers [23, 20, 21].

### Symplectic Reduction and Reduced Hamiltonian

Before we enter into the discussion for periodic solutions of the N-vortex problem, let's first notice that closed orbits of N-vortex problem of hydrodynamics are not isolated. Indeed, if  $\mathbf{z}(t)$  is an orbit, then so are

- $(z_1(t) + c, \dots, z_N(t) + c), c \in \mathbf{R}^2$ ;
- $(e^{i\theta} z_1(t), \dots, e^{i\theta} z_N(t)), \theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ ;
- $\lambda^{\frac{1}{2}} \mathbf{z}(\frac{t}{\lambda}), \lambda > 0$ .

We wish not to distinguish such orbits, thus introducing the following definition.

**Definition 1.2.3.** We will call an orbit  $\mathbf{z}(t)$

- **centred** if it satisfies  $P(\mathbf{z}(t)) = Q(\mathbf{z}(t)) = 0$  ;
- **normalized** if it is centred and satisfies  $I(\mathbf{z}(t)) = 1$  ;
- **periodic** if  $\mathbf{z}(t) = \mathbf{z}(t + T)$  for some  $T > 0$  ;
- **relatively periodic orbit (RPO)** if  $\mathbf{z}(t) = g\mathbf{z}(t + T)$  for some  $T > 0$  and  $g \in \mathbb{E}(2)$ .

Thus, (NRPO) will stand for a **normalized relative periodic orbit**, and this is the object that we want to study. For the N-vortex problem in BEC, although the scaling and translation in general does not give new solutions, the system is still invariant under rotation while we are more interested in studying the deformation rather than the rotation of the configuration. For these purposes, we would like to study the projected flow of the system on some quotient manifold, which represents the truly deformation of the configuration. In Appendix A we have recalled briefly the theory for symplectic reduction and the reduced Hamiltonian, which serves exactly our need.

- **N-Vortex Problem of Hydrodynamics:**

The system is invariant under the action of the special Euclidean group  $\mathbf{SE}(2)$ , the phase space is  $\mathbb{C}\mathbb{P}^{N-2}$ , as is shown in the following diagram:

$$\begin{array}{ccccc}
 & & & & S^1 \\
 & & & & \downarrow \\
 \mathbb{R}^{2N} & \xleftarrow{p=q=0} & \mathbb{R}^{2N-2} & \xleftarrow{I=1} & S^{2N-3} \\
 & & & & \downarrow /SO(2) \\
 & & & & \mathbb{C}\mathbb{P}^{N-2}
 \end{array}$$

- **N-Vortex Problem of Bose-Einstein Condensation:**

The system is invariant under the action of the special orthogonal group  $\mathbf{SO}(2)$ , the phase space is  $\mathbb{C}\mathbb{P}^{N-1}$ , as is shown in the following diagram:

$$\begin{array}{ccc}
 & & S^1 \\
 & & \downarrow \\
 \mathbb{R}^{2N} & \xleftarrow{I=1} & S^{2N-1} \\
 & & \downarrow /SO(2) \\
 & & \mathbb{C}\mathbb{P}^{N-1}
 \end{array}$$

### 1.2.2 Non-Integrable Cases

Analysis of the N-vortex problem for  $N \geq 4$  is in general quite difficult, because there is not enough first integrals in involution to give a solution explicitly by quadratures. In this section, we investigate two aspects of the dynamical behavior of some special N-vortex problems, which could somehow be seen as nearly integrable Hamiltonian systems. On one hand, the application of Poincaré-Melnikov method reveals the chaotic behavior of the system; on the other hand, the application of Kolmogorov-Arnold-Moser theory ensures the stability of invariant tori.

#### Chaotic Behavior of N-vortex Problem: $N \geq 4$

In this subsection, we review the detection of chaotic behavior of N-vortex problems by the Poincaré Melnikov method.

There has been a couple of analytic proofs of the non-integrability of the 4-vortex problem based on the Poincaré-Melnikov method. In general, one assumes that one or more of the vortices have zero vorticity, hence they are particles under influence of the large vortices. This idea is somehow similar to the restricted 3-body problem in celestial mechanics. As the zero vorticity is turned into small but positive vorticity, the system will trigger the homoclinic chaos.

1. Ziglin's configuration

In 1980, Ziglin first proved the non-integrability of 4-vortex problem by considering a perturbation of the equilateral triangle configuration [121]. The configuration involves essentially a passive particle in the vector fields generated by a equilateral triangle formed by 3 identical vortices. Based on Ziglin's method later on Bagrets and Bagrets have proved the non-integrability of 4-vortex problem on the sphere [11].

2. Koiller and Carvalho's configuration

Koiller and Carvalho's proof for the non-integrability of the 4-vortex problem in 1989 [55] has chosen a different configuration. where two vortices with opposite vorticity  $\Gamma_1 = -\Gamma_2$  will have impact on the passive particles  $\Gamma_3 = \Gamma_4 = \varepsilon \ll 1$ .

3. Castilla, Moauro, Negrini, and Oliva's configuration

Castilla et al have considered another configuration to show the non-integrability of the 4-vortex problem in 1993 [27]. It consists of 3 identical vortices of vorticity 1 and a 4<sup>th</sup> passive vortex of small vorticity  $0 < \varepsilon \ll 1$ . Their configuration could be seen as a perturbation of the heteroclinic orbits of Euler's configurations between different permutations.

**Stable Behavior of N-vortex Problem:  $N \geq 4$**

We have already seen in the previous section that for an integrable Hamiltonian system, its phase space up to a symplectomorphism, is foliated by Lagrangian invariant tori. The dynamics on these tori are quasi-periodic. The Kolmogorov-Arnold-Moser theory deals with the stability of these tori: it implies that, under suitable assumptions, for the perturbed Hamiltonian system (which are nearly integrable Hamiltonian systems) these tori persist. For brief introduction of KAM theory, see J.B.Bost [24] and [40] for application to celestial mechanics. The first application of KAM theorem to N-vortex problem is given by Khanin in 1982 [53], who has shown that for general N-vortex with arbitrary vorticity  $\Gamma_i \in \mathbb{R} \setminus \{0\}$ ,  $1 \leq i \leq N$ , there exists a set of initial conditions of positive measure, for which the motion of vortices is quasi-periodic. While the existence result is established, little is known about



the size of perturbation admissible for such tori to survive. In 1988 Alessandra Celletti and Corrado Falcolini [28] has shown that a lower bound of perturbation size could be  $\epsilon_{KAM} = 7.81 \times 10^{-23}$  for a prescribed frequency  $\omega = \frac{\sqrt{5}-1}{2}$ . Lim [64] has studied the existence of KAM tori for vortex lattice. Blackmore and Knio [18] have studied various KAM type results for three coaxial vortex rings.

### 1.3 Periodic Solutions of the N-vortex Problem

As mentioned in the last section, the N-vortex problem is in general not integrable when  $N > 3$ . This is somehow similar to the case of 3-body problem in celestial mechanics, which serves as one of the main resources for the modern development of dynamical systems. The singularities at collision and at infinity which put considerable difficulties from the analytical point of view, could be overcome by the construction of periodic solutions. Moreover, in Poincaré's mind, these solutions are also building blocks of general motions of the 3-body problem, as he believes one can use them to approximate any solutions. Actually, Poincaré has pointed out in his revolutionary monograph of celestial mechanics the significance of (relative) periodic solutions:

*D'ailleurs, ce qui nous rend ces solutions si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable.*

We believe the same philosophy applies to the N-vortex problem too. Thus in this section, we will discuss some of the results in the study of periodic solutions for the N-vortex problem.

#### 1.3.1 Equilibria

##### Absolute Equilibria

Equilibria may appear either in an inertial frame or in some rotating frame. In the former case, these solutions are called *absolute* equilibria (fixed points), while in the later case they are called *relative* equilibria.

The 3-vortex problem cannot have any fixed point unless the following conditions are fulfilled simultaneously [44, 105]

$$\begin{aligned} \mathbf{z}_2 - \mathbf{z}_1 &= \frac{\Gamma_2}{\Gamma_3}(\mathbf{z}_1 - \mathbf{z}_3) \\ \Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 &= 0 \end{aligned} \tag{1.19}$$

Then O'Neil [79] has studied the general case and concluded that the corresponding necessary condition for the existence of an absolute equilibrium is that the total angular momentum vanishes:

$$L = \sum_{1 \leq i < j \leq N} \Gamma_i \Gamma_j = 0 \quad (1.20)$$

Moreover, the converse is almost true: given almost all choices  $\Gamma = (\Gamma_1, \Gamma_2, \dots, \Gamma_N)$  s.t.  $L = 0$ , there exists exactly  $(N - 2)!$  different absolute equilibria. Recently, Bartsch, Micheletti and Pistoia have studied the existence of fixed points for the planar N-vortex problem in a bounded domain, together with their non-degeneracy [15, 14]. In particular, they have shown that the Kirchhoff-Routh function being Morse is a generic property. Kuhl has shown under some technical assumption the existence of the collinear equilibria and possible symmetry [58, 57].

### Relative Equilibria

There exists much more intensive study for relative equilibria, especially those becoming fixed point in a rotating frame. Such notion exists in celestial mechanics. These configurations correspond to a larger category of configurations, i.e., the central configuration in celestial mechanics [70]. However due to the fact that for N-vortex problem the phase space coincides with the configuration space, the notion of central configurations and relative equilibria coincide in N-vortex problem. We assume that the total vorticity  $\sum_{1 \leq i \leq N} \Gamma_i \neq 0$ , as a result the vorticity center is finite. The relative equilibrium configurations in the N-vortex problem could be defined as the following:

**Definition 1.3.1.** *A periodic solution of the planar N-vortex problem is called a **relative equilibrium**, if it is of the form*

$$z_i(t) = e^{j\omega t} (z_i(0) - C) + C$$

where  $C$  is the vorticity center.

We list some properties that will be used frequently later on:

**Proposition 1.3.1.** *The following are equivalent:*

$$(1) \quad \mathbf{z} \in \mathcal{L}_1; \quad (1.21)$$

$$(2) \quad \nabla H(\mathbf{z}) = -\frac{L}{4\pi} \nabla I(\mathbf{z}) \quad (1.22)$$

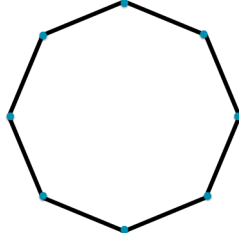


Fig. 1.3 Thomson configuration for 8 vortices which form an octagon

*Proof.* : (1) $\Rightarrow$ (2) : By definition of relative equilibrium,  $\mathbf{z}(t) \in \mathcal{Z}_1$  implies  $\exists \omega \in \mathbb{R}$  s.t.

$$\nabla H(\mathbf{z}(t)) = \frac{\omega}{2} \nabla I(\mathbf{z}(t))$$

taking inner product with  $\mathbf{z}(t)$  on both sides. Since  $I(\mathbf{z}) = 1$ , one sees that

$$-\frac{L}{2\pi} = \omega I(\mathbf{z}(t)) \Rightarrow \frac{\omega}{2} = -\frac{L}{4\pi}$$

Hence (2) is proved.

(2) $\Rightarrow$ (1) : If  $\mathbf{z}$  satisfies that  $\nabla H(\mathbf{z}) = -\frac{L}{4\pi} \nabla I(\mathbf{z})$ , then the flow passing through  $\mathbf{z}$  will be a relative equilibrium. We need to show that such a relative equilibrium is normalized. First, by considering  $(x, y) \in \mathbb{R}^2$  as a complex number  $x + iy \in \mathbb{C}$ , (3.18) implies that

$$-\frac{1}{2\pi} \sum_{j \neq i} \Gamma_i \Gamma_j \frac{\bar{z}_i - \bar{z}_j}{|z_i - z_j|^2} = -\frac{L}{4\pi} \Gamma_i \bar{z}_i, \quad \forall 1 \leq i \leq N$$

It follows that

$$0 = -\frac{1}{2\pi} \sum_{i=1}^N \sum_{j \neq i} \Gamma_j \Gamma_i \frac{\bar{z}_i - \bar{z}_j}{|z_i - z_j|^2} = -\sum_{i=1}^N \frac{L}{4\pi} \Gamma_i \bar{z}_i$$

Thus  $\sum_{i=1}^N \Gamma_i z_i = 0$ , and  $\mathbf{z}$  is centred. Next, multiply  $\mathbf{z}$  on both sides of (3.18), so that  $-\frac{L}{2\pi} = \nabla H(\mathbf{z}) \mathbf{z} = -\frac{L}{4\pi} \nabla I(\mathbf{z}) \mathbf{z} = -\frac{L}{2\pi} I(\mathbf{z})$ . Thus  $I(\mathbf{z}) = 1$ .  $\square$

The study of relative equilibria comprises various aspects, for instance the explicit construction of solutions, or the finiteness of configurations for given or generic vorticities, etc.

**Explicit construction of relative equilibria** Historically, the first such solution is the regular  $N$ -polygon rotating around its center. This configuration first appears in the work of J.J.Thomson [106] and is known as Thomson's configuration since then. Starting from this point, Havelock [48] has found the double vortex ring which is named after him too. Aref [6] and Koiller et al [56] has studied the case of relative equilibria with a center of symmetry, which is later on generalized by Lewis and Ratiu [60] for cases of sub-rings with different vorticity. For a comprehensive study of these vortex rings and multi-rings, one could turn to [7], which discussed not only such relative equilibria in the plane but also on the sphere, and even on various two dimensional manifolds. There exists relative equilibria which are not symmetric, as Aref and Vainchtein have shown by the method of continuation [8].

**Finiteness of relative equilibria** Relative equilibria of the  $N$ -vortex problem are in general not isolated due to the invariance under translation and rotation. After the normalisation, it turns out that the above defining equation represents a rather complicated system of algebraic equations, depending on the  $N$  vorticities. With the preassumed vorticities, the solution set of these equations is an algebraic subset of the product space of the phase space. This is quite similar to the situation of celestial mechanics, where the finiteness of central configuration of Newtonian gravitational systems, known as the Smale's 6<sup>th</sup> problem for 21st century [101], is only solved in the first simplest cases and remains as a challenge. O'Neil has shown in [79] that when  $\sum_{1 \leq i < j \leq N} \Gamma_i \Gamma_j \neq 0$ ,  $\sum_{1 \leq i \leq N} \Gamma_i \neq 0$ , there are no more than  $\frac{n!}{2}$  collinear relative equilibrium configurations. Hampton and Moeckel [47] have shown the finiteness of the number of relative equilibria configurations for  $N = 4$  is generic by using similar methods in their earlier work for 4-body problem [46]. More precisely, they have shown that

**Theorem 1.3.1** (Theorem 1.[46]). *Let  $L = \sum_{1 \leq i < j \leq N} \Gamma_i \Gamma_j$ ,  $\Gamma = \sum_{1 \leq i \leq N} \Gamma_i$ . If the vorticities  $\Gamma_i$  are nonzero, then the four-vortex problem has:*

- (1) *exactly 2 equilibria when the necessary condition  $L = 0$  holds;*
- (2) *at most 6 rigidly translating configurations when the necessary condition  $\Gamma = 0$  holds;*
- (3) *at most 12 collinear relative equilibria;*
- (4) *at most 14 strictly planar relative equilibria when  $\Gamma = 0$ ;*
- (5) *at most 74 strictly planar relative equilibria when  $\Gamma \neq 0$  provided  $\Gamma_i + \Gamma_j \neq 0$  and  $\Gamma_i + \Gamma_j + \Gamma_k \neq 0$  for all distinct indices  $i, j, k \in \{1, 2, 3, 4\}$ .*

O'Neil has used another formulation to show the finiteness of the number of relative configurations and has used Bezout's theorem to find an upper bound. More precisely

**Theorem 1.3.2** (Theorem 1.[79]). *Let  $L = \sum_{1 \leq i < j \leq N} \Gamma_i \Gamma_j, \Gamma = \sum_{1 \leq i \leq N} \Gamma_i$ . If the vorticities  $\Gamma_i$  are nonzero, moreover then the four-vortex problem has at most 56 planar relative equilibria when*

- (1)  $L \neq 0$  and  $\Gamma \neq 0$ ;
- (2)  $\Gamma_i + \Gamma_j \neq 0, 1 \leq i < j \leq 4$
- (3)  $\Gamma_1 + \Gamma_2 + \Gamma_j \neq 0, j = 3, 4$
- (4)  $\Gamma_1 \Gamma_2 + \Gamma_j(\Gamma_1 + \Gamma_2) \neq 0, j = 3, 4$
- (5)  $\Gamma_1 \Gamma_3 - \Gamma_2 \Gamma_4 \neq 0, \Gamma_1 \Gamma_4 - \Gamma_2 \Gamma_3 \neq 0$

For general  $N$ , Palmore has developed a Morse theoretical approach based on his earlier work in celestial mechanics [81–84]. He concluded in [85] that for generic choice of positive vorticity the equivalent classes (after taking quotient of translation and rotation) are non-degenerated critical point of the reduced Hamiltonian, and he used Morse type inequality to get lower bound for the number of different relative equilibria. This is fully justified in details for the case  $N = 4$  by Roberts [95].

### 1.3.2 Non-Equilibrium Solutions

The study for existence of relative periodic solutions that are not equilibria is in general more difficult, since after the reduction, the search of equilibria is a finite dimensional problem, which is not the case for non-equilibria relative periodic solutions. As a result much less is known in this direction. We mention two methods that have been used in the literature.

#### Symmetry Reduction

As we have seen, the main difficulty of the N-vortex problem is that its degree of freedom is in general too large to permit any efficient quantitative interpretation. On the other hand, the situation of 1-degree of freedom is extremely simple for the search of periodic solutions, since each compact regular component of the energy surface will be a periodic solution, which is ensured by the topological classification of 1-dimensional manifolds. Applications of this method consist in general in two steps: first, one focuses on configurations with some symmetry that allows reduction of the system to 1-degree of freedom; next, one tries to search for the compactness of the level of hyper-surface of the reduced Hamiltonian. If it happens to be compact and the flow is global, then one sees from the above topological classification, that the hyper-surface is homeomorphic to a disjoint union of circles and each of them corresponds to a periodic solution.

This idea has already appeared in the work of Aref [6] and Koiller et al [56]. Tokieda [110]

named these orbits as "tourbillons dansants"(dancing vortices) and has applied this idea to various 2-dimensional manifolds, which is further developed in Soulière and Tokieda [103], Montaldi, Soulière and Tokieda [72]. Laurent-Polz [59] has found many relative periodic solutions with respect to various symmetric groups by mixing the idea of symplectic reduction and such discret reduction. Borisov, Mamaev, and Kilin [22] used similar ideas to find relative periodic orbits in the plane and the sphere for 3 and 4 vortices.

### Continuation Methods

Another basic approach of finding periodic solution starts with a solution that is already known. Then with some assumption about the non-degeneracy, one can see that there exist periodic solutions with could be seen as a continuation of the original periodic solutions with respect to certain parameter. This idea is explored since the work of Poincaré and first sees its application in celestial mechanics [87]. In particular, we claim the following theorem about the center manifold, known as *Lyapunov center theorem* see [61, 98]. The following version is taken from the monograph of Meyer [69]:

**Theorem 1.3.3** (Lyapunov center theorem). *Assume that the system  $\dot{\mathbf{z}} = f(\mathbf{z})$  admits a nondegenerate integral and has an equilibrium point with exponents  $\pm i\omega, \lambda_3, \lambda_4, \dots, \lambda_m$ , where  $i\omega \neq 0$  is purely imaginary. If  $\frac{\lambda_j}{i\omega} \notin \mathbb{Z}$  for  $j = 3, \dots, m$ , then there exists a one-parameter family of periodic orbits emanating from the equilibrium point. Moreover, when approaching the equilibrium point along the family, the periods tend to  $\frac{2\pi}{\omega}$  and the nontrivial multipliers tend to  $\exp(\frac{2\pi\lambda_j}{\omega})$ ,  $j = 3, 4, \dots, m$ .*

This theorem could be seen as a special case of the *Weinstein-Moser theorem*, first studied by Alain Weinstein [115] for positive definite Hamiltonian case and by Moser [76] for a more general situation:

**Theorem 1.3.4** (Weinstein-Moser theorem). *Assume that the system possesses a fixed point  $\mathbf{z}_0 = 0$  and an integral  $\mathbf{G}(z) \in C^2$  and that  $\mathbb{R}^{2N} = \mathbf{E} + \mathbf{F}$ , where  $\mathbf{E}, \mathbf{F}$  are invariant subspaces of the linearized flow*

$$\dot{\mathbf{z}} = \mathbf{Cz}, \quad \mathbf{C} = \nabla^2 \mathbf{H}(\mathbf{0}) \tag{1.23}$$

such that

1. all solutions  $\mathbf{z}(t) \in \mathbf{E}$  share a common period  $T$ ;
2. none of the solutions  $\mathbf{z}(t) \in \mathbf{F} \setminus \{\mathbf{0}\}$  has  $T$  as its period.

3. *assume further more*

$$\nabla^2 \mathbf{G}(\mathbf{0})|_{\mathbf{E}} \geq 0 \quad (1.24)$$

Then for sufficiently small  $\varepsilon > 0$ , the hyper-surface  $\mathbf{G}^{-1}(\varepsilon + \mathbf{G}(\mathbf{0}))$  has at least one periodic solution whose period is close to  $T$ .

In the study of the N-vortex problem, Roberts [94] has discussed the stability of the linearized equation in details, in particular, the form of characteristic polynomial is calculated. The treatment of symplectic decomposition is similar to the work of Moeckel [71] in celestial mechanics. Borisov et al [22] have used the symplectic reduction techniques from their own construction of Lie-Poisson dynamics [19] and then applied the Lyapunov center theorem to the 4-vortex problem to find periodic solutions bifurcating from Goryachev's configuration; Carvalho and Cabral [26] have used a discrete Fourier transform to simplify the linearized equation and applied Lyapunov center theorem to the Thomson's configuration.

Recently, Bartsch and his collaborators find new periodic solutions by the superposition principle, where the analysis is based on the degree theory, and could be understood as a continuation method applied at singularity (collision). For example they have found periodic solutions by replacing a fixed point of Routh's function by N vortices; or by replacing 1-vortex on a level set near the boundary by 2 vortices very close to each other, whose vorticity center remains on the level set. See for example [12] [13] [36]. This approach is powerful in the sense that it applies to a large family of boundaries. It shares somehow similar spirit with the KAM approach for invariant tori as discussed earlier.

### 1.3.3 Variational Method: From Poincaré to the *Eight*

Variational method is versatile in mathematical physics in establishing existence results for solutions of a physical system. The N-body problem in celestial mechanics is not an exception neither.

let  $\mathbf{q} = (q_1, \dots, q_N) \in \mathbb{R}^d, d \in \{2, 3\}$  be positions of mass particles in either  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . Their interactions follows the Newtonian gravity, namely

$$m_i \ddot{q}_i(t) = \sum_{j \neq i} m_i m_j \frac{q_j(t) - q_i(t)}{\|q_j(t) - q_i(t)\|^3} \quad (1.25)$$

Let  $K$  and  $U$  be the kinetic energy and the potential energy respectively.

$$K(\dot{\mathbf{q}}(t)) = \frac{\|\dot{\mathbf{q}}(t)\|^2}{2} \quad (1.26)$$

$$U(\mathbf{q}(t)) = - \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\|} \quad (1.27)$$

Then the action functional is defined as

$$\mathcal{A}_U^T(\mathbf{q}) = \int_0^T K(\dot{\mathbf{q}}(t)) - U(\mathbf{q}(t)) dt = \int_0^T \frac{\|\dot{\mathbf{q}}(t)\|^2}{2} + \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\|} dt \quad (1.28)$$

Note that we have emphasized the dependence of  $U$  in the action functional, since one could take other forms of  $U$ , for example the so-called *strong force* or *weak force*, instead of picking the Newtonian potential. It is well-known that the natural function space associated to this functional is

$$\Lambda_T = H^1([0, T], \mathbb{R}^{2d}) \quad (1.29)$$

$$\Lambda_T^0 = \{\mathbf{q}(t) \in H^1([0, T], \mathbb{R}^{2d}), \mathbf{q}(0) = \mathbf{q}_0, \mathbf{q}(T) = \mathbf{q}_T\} \quad (1.30)$$

where  $\mathbf{q}_0, \mathbf{q}_T$  are prefixed configurations.

The attempt to apply variational method for proving the existence of periodic solutions began as early as Henri Poincaré at the end of 19<sup>th</sup> century. In a short note in 1896 [89], he has already mentioned the idea of searching (relative) periodic solutions by minimizing the Lagrangian action among all loops in a given homology class. However, in practice this is not so easy because of two reasons: the collision and the infinity.

The first reason is the singularity at collision. It is known that the action of the trajectories with collision(s) are still finite, as a result, the minimization does **not** necessarily give collision free orbit. The second reason is the singularity at infinity. In modern terminology, minimization of the Lagrangian functional involves two ingredients: lower semi-continuity and coercivity of the action functional. However, the action functional of the N-body problem is **not** coercive.

It turns out that the searching for collision free periodic orbit through unconstrained minimization is somehow hopeless (with an exception of the trivial solution at infinity s.t.  $\mathcal{A}_T(\infty) = 0$ ). As a result, one must put extra constraints in the optimization.



### Homology Constraints

The first method combines the strong force assumption and imposes special homology class constraints. Since the Newtonian potential is too weak, the collision does not blow up the action. As a result we suppose in our model that the potential is "stronger" than the classical Newtonian potential

**Definition 1.3.2** (Strong Force). *A potential  $U : \mathbb{R}^d \rightarrow (-\infty, 0]$  is said to satisfy the strong force condition if  $\exists c > 0$  s.t.*

$$U(|z|) \leq -\frac{c}{|z|^2} \quad \text{when } |z| \rightarrow 0 \quad (1.31)$$

Using the strong force condition, the action functional will be pushed to infinity when collision happens. Next, for singularity at infinity, one could focus on some special free homotopy class. In particular, the *tied* class, as is used in the work of Gordon [43], Montgomery [73] and other authors.

**Definition 1.3.3** (Tied Class). *Let  $M$  be a non-compact complete Riemannian manifold and  $\Delta$  be a non-compact sub-variety. Let  $\alpha$  be a free homotopy class, and  $c_n$  be a sequence of free loops in  $\alpha$ . We say that  $c_n \rightarrow \infty$  if we can pick a point sequence  $p_n \in c_n$  s.t.  $|p_n| \rightarrow \infty$ . Then we say  $\alpha$  is a tied class if for any  $p_n \rightarrow \infty$ , we have*

$$l(c_n) \rightarrow \infty \quad (1.32)$$

where  $l(c_n)$  is the length of the loop  $c_n$ .

Clearly this tied class will provide us the coercivity needed in variational methods. As a result Montgomery [73] has found many periodic orbits of various free homotopy classes by applying such method. It is interesting that when the strong force condition is dropped, this approach is still capable to give variational characterization for some well known solutions, for example Gordon [43] and Venturelli [111] have given characterization of Kepler solutions for planar 2 bodies and Lagrange equilateral configuration of spatial 3-body respectively. However due to the difficulty we discussed it fails to give many new solutions.

### Symmetry Constraints

Another recently emerged approach uses the symmetry constraint. The first well known symmetry is formulated by the Italian school and bears the name **Italian symmetry** [37, 119]. Following earlier numerical work of Moore [74] Using the symmetry, Chenciner and Montgomery proved analytically the existence of the *eight* curve for the 3-body problem

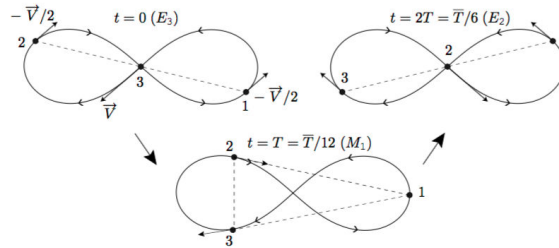


Fig. 1.4 Figure "8" of the 3-body problem (picture taken from [33])

in their seminal paper [33]. In this paper, by fixing an initial configuration and a terminal configuration, they get  $\frac{1}{12}$  of the whole orbit by minimization of the action functional. By comparing the value of action functional at with that evaluated at a collision, they showed that the orbit thus found is collision-free. Then the symmetry permits them to extend the orbit to get the complete *eight* curve. The proof contains a numerical part, and in 2001 Chen [29] has formulated an analytical proof for this part.

Later on Christian Marchal [66] has proved a general lemma that permits various constrained minimizations. The following version is taken from [30]:

**Theorem 1.3.5** (Marchal's Lemma). *A minimizer of  $\mathcal{A}_T$  in the space  $\Lambda_T^0(\mathbf{q}_0, \mathbf{q}_T)$  is collision-free in the whole open interval  $(0, T)$ .*

In other words, the collision cannot happen in any intermediate time spot (the two end configurations thus excluded). This theorem together with specific symmetries assigned to the problem will generate various symmetric periodic solutions that are collision free. See for example [30].

### Choreography

One special property of the *eight* curve is being a choreography. This is a special class of periodic orbits showing very symmetric behavior. More precisely:

**Definition 1.3.4** (Choreography). *Let  $\mathbf{z}(t)$  be a collision free  $T$ -periodic solution of the  $N$ -body problem. We say that  $\mathbf{z}(t)$  is a*

- *Simple Choreography, if all masses move on the same curve with constant time shift;*

- *Multiple Choreography, if all masses similarly move on several curves (with at least one fewer curves than masses), and those on the same curve move with constant time shift.*

In other words, a  $T$ -periodic solution  $\mathbf{z}(t)$  of the N-body problem is a simple choreography if and only if

$$\mathbf{z}_i\left(t + \frac{T}{N}\right) = \mathbf{z}_{i-1}(t)$$

**Example 1.3.1.** *We give some examples of simple choreographies:*

- *The Lagrangian triangle relative equilibrium is a simple choreography of the 3-body problem. More generally, the N-polygon relative equilibrium is a simple choreography of the N-body problem. These rigid motion type simple choreographies are called **trivial** ones, and they becomes a fixed point in a rotating frame;*
- *The eight curve in the previous section is a non-equilibrium simple choreography of the 3-body problem, which means that it is a simple choreography but not a rigid motion in any rotating frame.*
- *Barutello and Terracini [16] have studied the simple choreography by variational methods, by putting it into a rotating frame. It turns out that while for some values of the angular velocity minimizers are still relative-equilibria, for others the minima of the action are not anymore rigid motions.*

It is believed that as  $N$  increases, the number of (non-equilibrium) simple choreographies is increasing rapidly too. This is proved by Chenciner et al [32] for the strong force and by Yu [118] for the Newtonian case. One should note that the choreography is also a discrete symmetry, however, it does not decrease the degree of freedom of the original system and should not be confused with the symmetric reduction mentioned earlier. Rather, by looking for simple choreographies, one usually benefits from extra information about the action functional and from Palais' principle of symmetric criticality [80].

## 1.4 Variational Methods in the N-Vortex Hamiltonian Systems

As discussed in the previous chapter, the action functional associated with a loop  $\gamma(t) = (p(t), q(t)) \in C^\infty(S^1, \mathbb{R}^{2n})$  in the phase space is

$$\mathcal{A}_H(\mathbf{z}) = \int_{S^1} pdq - H(z)dt \quad (1.33)$$

Periodic solutions of the Hamiltonian system could be seen as critical points of this functional in some function space to be precised later on. Such an action functional is highly indefinite, since all of the critical points are of infinite Morse index. As a result, one cannot expect in general to find a critical point by minimization, and the application of variational methods (for example mountain pass) seems to be very difficult. On the other hand, the flow of such a system in most situations is global, hence it is natural to search solutions by global methods. In the rest of the thesis we will mainly use two approaches in this section: the minimax method of Rabinowitz that is based on the linking argument [90, 17, 93] (see also the development in [38, 91, 35]), and the Floer's Hamiltonian perturbation of  $J$ -holomorphic curves [45, 50, 42, 51], which is closely related to the Weinstein's conjecture [117, 116, 90]. Again, we will focus on autonomous Hamiltonian, and results for time-dependent Hamiltonian (for example the proof of Arnold's conjecture of fixed points) are omitted. We refer the reader to the book of Long [65] for detailed description of Maslov type index and its application to Hamiltonian system and the book of Abbondandolo [1], Audin [10] and the reference therein for a Morse theoretical approach to Hamiltonian systems. It should be aware that the variational method could be applied to find not only periodic solutions, but also other orbits, in particular the homoclinic and heteroclinic orbits. We refer to the work of Ekeland, Séré, Zelati, Rabinowitz [120, 96, 92] and the references therein.

Before we go further, it might worth comparing the variational formulation of the  $N$ -body problem and the  $N$ -vortex problem, and identify some difficulties for directly application of ideas from  $N$ -body problem to  $N$ -vortex problem.

Recall that

$$\mathcal{A}_U^T(\mathbf{q}) = \int_0^T \frac{\|\dot{q}(t)\|^2}{2} + \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{\|q_i(t) - q_j(t)\|} dt \quad (\text{N-body functional})$$

$$\mathcal{A}_H^T(\mathbf{z}) = \int_0^T y\Gamma dx - H(z)dt \quad (\text{N-vortex functional})$$

It turns out that

- In N-body functional, the momentum and the position are conjugate variables. They are separated into two terms, i.e., the kinetic energy and the potential energy, relatively ; while in N-vortex functional, the horizontal position and the vertical position are conjugate variables. They are mixed together.
- In N-body functional, the kinetic energy part is positive ; while in N-vortex functional, even for all positive vorticities, the  $\int_0^T y dx$  could be either positive or negative, hence it is difficult to consider the coercivity.
- In N-body functional, the  $m_i$  represents the mass of the particle, which is supposed to be positive ; while in N-vortex functional,  $\Gamma_i$  could either be positive or negative, which increased the difficulty.
- In N-body functional, the natural function space is  $H^1$ , which is embedded into the space of continuous functions ; while in N-vortex functional, the functional space under consideration is  $H^{\frac{1}{2}}$ , which is not embedded into the space of continuous functions. As a result, it is ambiguous for notions of homotopical constraints.

On the other hand, if we would like to consider the energy surface of the Hamiltonian, as is already noted in [13], there are also some difficulties, for example

- The energy surface is not compact, hence symplectic methods [52] in general cannot be applied directly ;
- The energy surface is not convex, hence convex methods [39] in general cannot be applied directly ;
- When  $N \geq 4$  it is difficult to verify whether the surface is of contact type or not.

As a result, we would like to focus on the normalized orbits. This does not lead to any essential loss: after all the  $N$ -vortex Hamiltonian from Euler's equation has some homogeneous property, hence once the normalized orbits are found, the behaviours of other orbits are immediately known, up to a re-scaling factor.

## 1.5 Main Results

Consider the system

$$\begin{aligned}\Gamma \dot{\mathbf{z}}(t) &= X_H(\mathbf{z}(t)) = \mathbf{J} \nabla H(\mathbf{z}(t)) \\ H(z) &= -\frac{1}{4\pi} \sum_{i,j=1, i < j}^N \Gamma_i \Gamma_j \log |z_i - z_j|^2\end{aligned}\quad (\text{System-I})$$

while the Poisson matrix  $\mathbf{J}$  and the vorticity matrix  $\Gamma$  are

$$\mathbf{J} = \begin{bmatrix} \mathbb{J} & & \\ & \ddots & \\ & & \mathbb{J} \end{bmatrix}, \quad \mathbb{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} \Gamma_1 & & & & \\ & \Gamma_1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \Gamma_N \\ & & & & & \Gamma_N \end{bmatrix}.$$

### 1.5.1 Periodic Orbits of the Positive N-Vortex Problem

In Chapter II, we always assume that all the vorticities are positive, i.e.,

$$\Gamma_i > 0, \quad 1 \leq i \leq N.$$

we show the existence of infinitely many non-trivial relative periodic solutions of H1 We define

$$\mathfrak{Z}_0(H) = \{\mathbf{z} | \mathbf{z} \text{ is a normalized orbit of H1}\}$$

$$\mathfrak{Z}_1(H) = \{\mathbf{z} | \mathbf{z} \text{ is a normalized relative equilibrium of H1}\}$$

$$\mathfrak{Z}_2(H) = \{\mathbf{z} | \mathbf{z} \text{ is a non-trivial normalized relative periodic orbit of H1}\}.$$

and correspondingly

$$\begin{aligned}\mathcal{H}_0 &= \{h \in \mathbb{R} | h = H(\mathbf{z}), \mathbf{z} \in \mathfrak{Z}_0(H)\} \\ \mathcal{H}_1 &= \{h \in \mathbb{R} | h = H(\mathbf{z}), \mathbf{z} \in \mathfrak{Z}_1(H)\} \\ \mathcal{H}_2 &= \{h \in \mathbb{R} | h = H(\mathbf{z}), \mathbf{z} \in \mathfrak{Z}_2(H)\}\end{aligned}$$

We would like to use the symplectic capacity theory (see for example [52]), which requires basically a regular and compact energy surface of the Hamiltonian. The main result of the chapter could be summarized has the following:

**Theorem A:**

For  $\Gamma_i \in \mathbb{R}_+^*$  (resp.  $\mathbb{Q}_+^*$ ),  $\forall 1 \leq i \leq N$ ,  $\mathcal{H}_1$  is a closed (resp. finite) set in  $\mathbb{R}$  and  $\mu(\mathcal{H}_1) = 0$ .

The proof of theorem A is summarized in lemma 2.1.1, theorem 2.1.1, and theorem 2.1.2.

One then verifies that in the reduced dynamic Hamiltonian system, the conditions for application of symplectic capacity theory are valid, as a result we can prove that:

**Theorem B:**

For  $\Gamma_i \in \mathbb{R}_+^*$ ,  $\mathcal{H}_2$  is dense in  $\mathcal{H}_0$ .

The proof of theorem B is summarized in lemma 2.2.1, theorem 2.2.1 and theorem 2.0.1.

Finally, motivated by the multiple vortex rings, we observe that when the vortices could be divided into  $M$  groups, and in each group the  $N$  vortices present the  $C_N$  symmetry, then the reduced phase space are still of finite symplectic capacity. More precisely,

**Definition 1.5.1.** Let  $M, N \in \mathbb{N}$ . We say a centred  $M \times N$ -vortex configuration is  $C_N$ -symmetric, if

$$\mathbf{z} = e^{\mathbf{J}_{M \times N} \frac{2\pi}{N} \mathbf{z}} \quad (1.34)$$

We say an orbit of the centred  $M \times N$ -vortex problem is  $C_N$  symmetric, if  $\mathbf{z}(t)$  is a  $C_N$  symmetric configuration for all  $t \in \mathbb{R}$ .

Thus we have that

**Theorem C:**

Consider the above symmetric  $M \times N$ -vortex problem with positive vorticities s.t.

$$\Gamma_{li} = \Gamma_{lj}, 1 \leq l \leq M, 1 \leq i < j \leq N.$$

Then here are infinitely many  $C_N$ -symmetric non-trivial normalized periodic solution of the original  $M \times N$ -vortex problem.

The proof of theorem C is summarized in theorem 2.3.1.

**1.5.2 Choreographies of the Identical N-Vortex Problem**

In chapter 3, we further more assume that all the vortices have identical vorticity. We can assume the common vorticity is 1 without loss of any generality. We would like to search for periodic orbits with some discrete symmetry. More precisely, we denote the set of  $2\pi$ -periodic continuous loops by

$$\Lambda = \{\mathbf{Z} \in \mathcal{C}(\mathbb{S}^1, \mathbb{R}^{2N}) \mid \mathbf{Z}(0) = \mathbf{Z}(2\pi)\}, \quad \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

$$\tau : \mathbb{S}^1 \rightarrow \mathbb{S}^1 \quad \tau(t) = \frac{2\pi}{n} + t \quad (1.35)$$

$$\tilde{\sigma} : \mathbb{R}^{2N} \rightarrow \mathbb{R}^{2N} \quad (z_1, z_2, \dots, z_{N-1}, z_N) \xrightarrow{\tilde{\sigma}} (z_N, z_1, \dots, z_{n-2}, z_{N-1}) \quad (1.36)$$

and

$$g : \Lambda \rightarrow \Lambda \quad (g\mathbf{Z})(t) = \tilde{\sigma}\mathbf{Z}(\tau^{-1}t)$$

We are interested in the fixed points of  $g$ , namely free loops satisfying

$$z_{i+1}\left(t + \frac{T}{N}\right) = z_i(t) \quad (1.37)$$

**Definition 1.5.2.** We call a loop  $\mathbf{Z} \in \Lambda$

- a *choreography*, if  $g\mathbf{Z} = \mathbf{Z}$ ;
- a *centred choreography*, if  $\mathbf{Z}(t)$  is a choreography and

$$P(\mathbf{Z}(t)) = Q(\mathbf{Z}(t)) = 0, \forall t \in [0, 2\pi] \quad (1.38)$$



Again we are interested in the reduced Hamiltonian system on the reduced manifold. By analogue to the relative periodic solutions studied in the previous chapter, we can thus define a relative choreography to be those orbits that become choreography in some appropriate rotating frame. We consider the general planar Hamiltonian system:

$$\dot{\mathbf{Z}}(t) = X_{H_{\mathbb{R}^{2N}}}(\mathbf{Z}(t)) = \mathbf{J}_{\mathbb{R}^{2N}} \nabla H_{\mathbb{R}^{2N}}(\mathbf{Z}(t)), \quad \mathbf{Z} = (z_1, z_2, \dots, z_N), \quad z_i = (x_i, y_i) \in \mathbb{R}^2$$

with

$$H_{\mathbb{R}^{2N}}(\mathbf{Z}) = \sum_{i=1}^n \alpha_i V(|z_i|^2) + \sum_{1 \leq i < j \leq N} \beta_{ij} F(|z_i - z_j|^2). \quad (\text{System-I})$$

**Hypothesis D:**

Assume that the reduced Hamiltonian  $H$  satisfies the following assumptions:

$H_{\mathbb{R}^{2N}}$  is smooth;

$$\alpha_i = \alpha_j, \quad \forall 1 \leq i < j \leq N., \quad \beta_{ij} = \beta_{mn}, \quad \forall (i, j) \neq (m, n);$$

$$H(A) < H(B), \text{ with } \tilde{\sigma}A = A, \tilde{\sigma}B = B.$$

Under these assumptions, we will develop a symmetric version of holomorphic spheres, and use it to prove the existence of relative choreographies. The main result is the following theorem:

**Theorem E:**

Let  $I = (H(A), H(B))$  be the open interval. Denote

$$\mathcal{D} = \{c \in I \mid S_c = H^{-1}(c) \text{ has a } \sigma\text{-invariant connected component } S_c^\sigma\}$$

$$\mathcal{G} = \{c \in I \mid S_c = H^{-1}(c) \text{ possesses a reduced simple choreography on it}\}$$

Then

$$\mu(\mathcal{G}) = \mu(\mathcal{D})$$

The theory of symmetric holomorphic sphere is developed in section 3.1-3.5. Then the proof of theorem E is carried out in theorem 3.6.1. As an application to the identical N-vortex problem, we can prove that

**Corollary F:**

Consider the Hamiltonian

$$H_{\mathbb{R}^{2N}}(\mathbf{Z}) = -\frac{1}{4\pi} \sum_{1 \leq i < j \leq N} \log |z_i - z_j|^2$$

Assume that  $N$  is even. Then there exist infinitely many non-trivial centred reduced relative choreographies.

The proof is summarized in theorem 3.7.2. The method is however applicable to other physical models, as is explained in theorem 3.7.1 and theorem 3.7.3.

We mention that we have also tried to apply the minimax method to find choreographies for the identical  $N$ -vortex problem. Unfortunately, we don't know if the solution thus found is a relative equilibrium or not, hence according to our insistence on non-equilibrium, this minimax method might fail to meet our criteria of being suitable<sup>3</sup>. As a result we only report it in the appendix B in order not to diverge from the main points in the thesis.

### 1.5.3 An Uniform Bound Estimate for Symmetric Periodic Orbits

Finally In chapter 4, we study a uniform bound for Hamiltonians of  $N$ -vortex type. We have already seen that in general it is hopeless to have a uniform bound for periodic solutions of fixed period  $T$ . However, if we can have some symmetric constraints imposed on the orbit, it gives some extra control of the orbits. More precisely, define

$$M(z) = \sup_{1 \leq i < j \leq N, t \in [0, T]} \log |z_i(t) - z_j(t)|^2 \quad (1.39)$$

$$\bar{M}(T, N) = \sup_{z \in \Lambda_T} M(z) \quad (1.40)$$

and  $\Lambda_T$  stands for all the absolute centred  $T$ -choreographies of the  $N$ -vortex problem. we prove the following theorem:

**Theorem G:**

Let  $z(t)$  be a  $T$ -periodic solution of an  $N$ -vortex system where the Hamiltonian is of the form

$$H(z) = -\frac{1}{4\pi} \sum_{1 \leq i < j \leq n} \Gamma_i \Gamma_j |z_i - z_j|^2$$

<sup>3</sup>Hilbert has posed his 19<sup>th</sup> problem in the International Mathematical Congress that "Has not every variational problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied, and provided also that if need be that the notion of solution be suitably extended?"

then

$$\bar{M}(T, N) < \infty$$

The proof is done in theorem 4.2.1 for orbits that are Italian symmetry, namely that  $\mathbf{z}(t + \frac{T}{2}) = -\mathbf{z}(t)$ . By similar argument the conclusion however holds for centred choreography too.

Next we would like to study the bound for the action. To this end we study the re-parametrised Hamiltonian

$$G = \exp\left(-\prod_{1 \leq i < j \leq N} |z_i - z_j|^2\right)$$

We are interested in studying the trajectory space

$$\begin{aligned} \mathcal{M}_{CH} = \{ & u : \mathbb{R} \times \mathbb{R} \setminus T\mathbb{Z} \rightarrow \mathbb{R}^{2N} \mid \forall s \in \mathbb{R}, t \in [0, T], \\ & \frac{\partial u}{\partial s} + \mathbf{J} \frac{\partial u}{\partial t} + \nabla G(u) = 0, E(u) < \infty \\ & u(s, \cdot) \text{ is an absolute centred choreography} \} \end{aligned}$$

By using theorem G, one can prove a version of Gromov compactness for  $\mathcal{M}_{CH}$ . More precisely,

**Theorem H:**

$\mathcal{M}_{CH}$  is compact.

This might serve as a starting step for the construction of Floer type theory of the N-vortex type Hamiltonian system.



## Chapter 2

# Periodic Orbits of the Positive N-Vortex Problem

### Abstract

In this chapter, we study the N-vortex problem in the plane with positive vorticities.

$$\Gamma \dot{\mathbf{z}}(t) = \mathbf{X}_H(\mathbf{z}(t)) = \mathbf{J} \nabla H(\mathbf{z}(t)), \quad \mathbf{z} = (z_1, z_2, \dots, z_N), \quad z_i = (x_i, y_i) \in \mathbb{R}^2 \quad (\text{H1})$$

where the Hamiltonian is

$$H(z) = -\frac{1}{4\pi} \sum_{1 \leq i < j \leq N} \Gamma_i \Gamma_j \log |z_i - z_j|^2 \quad (2.1)$$

After an investigation of some properties for normalized relative equilibria of the system, we use symplectic capacity theory to show that, there exist infinitely many normalized relative periodic orbits on a dense subset of all energy levels, which are neither fixed points nor relative equilibria. Let  $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$  be defined as in chapter 1 we study the N-vortex problem with positive vorticity. The main result is that:

**Theorem 2.0.1.** *If  $\Gamma_i > 0$  ( $\forall 1 \leq i \leq N$ ),  $\mathcal{H}_2$  is dense in  $\mathcal{H}_0$ .*

## 2.1 Sparseness of Relative Equilibria

Before we proceed to study NTNRP, we first need to have some preparation for properties of the normalized relative equilibria of  $H$ . In this section, we study the normalized relative equilibria of  $H$ , with an emphasis on their energy levels.

### 2.1.1 Positive Vorticities

First note that the mutual distances between vortices in a normalized relative equilibrium configuration cannot be too small. More precisely:

**Lemma 2.1.1.** *For  $\Gamma_i \in \mathbb{R}^+$ , there exists constant  $\varepsilon(\Gamma)$  which depends only on the vorticities  $\Gamma = (\Gamma_1, \Gamma_2, \dots, \Gamma_N)$ ,  $1 \leq i \leq N$ , s.t.*

$$\inf_{\substack{z \in \mathbb{R}^2 \\ 1 \leq i < j \leq N}} |z_i - z_j|^2 > \varepsilon > 0$$

**Remark 2.1.1.** *As the relative equilibria are rigid body motions, we have dropped the dependence of time of  $z$  to simplify the discussion.*

This result first appears in the work of O'Neil [79] and has been reproved recently by Roberts [95] using a renormalisation argument, followed by a detailed discussion on Morse index of relative equilibria. We here give an alternative proof by the observation that for a relative equilibrium, the vorticity center of a given cluster also rotates uniformly.

*Proof.* : Denote

$$m(\mathbf{z}) = \inf_{1 \leq i < j \leq N} |z_i - z_j|^2$$

Suppose to the contrary that  $\mathbf{z}^k$  is a sequence of relative equilibria whose mutual distances s.t.  $\lim_{k \rightarrow \infty} m(\mathbf{z}^k) = 0$ . Then by consecutively passing to subsequence if necessary, we may suppose that there exists a sub-index set  $V \subset \{1, 2, \dots, N\}$  s.t.  $z_i^k \rightarrow z_i^*$ ,  $\forall i \in V$ . Denote  $\mathbf{z}_V$  as the vector of vortices with index in  $V$ . The Hamiltonian could be separated into two parts, the interactions between vortices in  $V$  and otherwise. Let  $H(\mathbf{z}) = H_V(\mathbf{z}) + H_{V^c}(\mathbf{z})$ , where

$$H_V(\mathbf{z}) = -\frac{1}{4\pi} \sum_{\substack{i < j \\ i, j \in V}} \Gamma_i \Gamma_j \log |z_i - z_j|^2 \quad (2.2)$$

$$H_{V^c}(\mathbf{z}) = -\frac{1}{4\pi} \sum_{\substack{i < j \\ (i, j) \notin V \times V}} \Gamma_i \Gamma_j \log |z_i - z_j|^2 \quad (2.3)$$

It follows that  $\nabla H(\mathbf{z}^k)\mathbf{z}^k = -\frac{1}{2\pi}L$ , while  $\nabla H_V(\mathbf{z}_V^k)\mathbf{z}_V^k = -\frac{1}{2\pi}L_V$ . Observe that  $c_V^k$ , the vorticity centre of  $\mathbf{z}_V^k$ , also follows a uniform rotation with the vortices. As a result,

$$c_V^k = \frac{\sum_{i \in V} \Gamma_i z_i^k}{\sum_{i \in V} \Gamma_i} = \mathbb{J} \frac{\omega}{2} c_V^k \quad (2.4)$$

$$\Gamma_i z_i^k = \mathbb{J}(\nabla_{z_i} H_V(\mathbf{z}) + \nabla_{z_i} H_{V^c}(\mathbf{z})) = \mathbb{J} \Gamma_i \frac{\omega}{2} z_i^k, \quad i \in V \quad (2.5)$$

Since  $\lim_{k \rightarrow \infty} c_V^k = \lim_{k \rightarrow \infty} z_i^k = z^*, \forall i \in V$ , We see that  $\lim_{k \rightarrow \infty} \nabla H_V(\mathbf{z}_V^k) = \mathbf{0}$ . But we know already that  $\nabla H_V(\mathbf{z}_V^k)\mathbf{z}_V^k = -\frac{1}{2\pi}L_V$ . As  $|z_V^k|$  is bounded (since  $\mathbf{z}^k \in \mathfrak{Z}_1(H)$ ), this implies that  $L_V = 0$ , which contradicts the fact that  $\Gamma_i > 0, \forall i \in V$ . As a result, such sequence  $\mathbf{z}^k$  does not exist. The lemma is proved.  $\square$

Lemma 2.1.1 tells us that the relative equilibria are isolated from the diagonals, where collision happens and singularity rises. With this result in hand, we will study the distribution of energy levels on which normalized relative equilibria exist. For a subset  $\mathcal{A} \subset \mathbb{R}$ , we denote by  $\mu(\mathcal{A})$  its Lebesgue measure. Roughly speaking, we show that  $\mathcal{H}_1$  is somehow a small subset of  $\mathbb{R}$ .

**Theorem 2.1.1.** *For  $\Gamma_i \in \mathbb{R}^+, \forall 1 \leq i \leq N$ ,  $\mathcal{H}_1$  is a closed set in  $\mathbb{R}$ . Moreover  $\mu(\mathcal{H}_1) = 0$ .*

*Proof.* : Suppose given a sequence of real numbers  $h^k \in \mathcal{H}_1$  s.t.  $\lim_{k \rightarrow \infty} h^k \rightarrow h^* \in \mathbb{R}$ . Then by definition of  $\mathcal{H}_1$ , there exists a sequence of normalized relative equilibria  $\mathbf{z}^k \in \mathfrak{Z}_1$  s.t.

$$H(\mathbf{z}^k) = h^k \rightarrow h^* \quad (2.6)$$

Since  $I(\mathbf{z}^k) = 1, \mathbf{z}^k \in \mathbb{R}^{2N}$  is a bounded sequence, hence  $\mathbf{z}^k \xrightarrow{k \rightarrow \infty} \mathbf{z}^*$ . Thanks to lemma 2.1.1, we see that points in  $\mathfrak{Z}_1$  are isolated from collision, hence  $H$  is smooth at these points. As a result

$$\nabla H(\mathbf{z}^*) = \lim_{k \rightarrow \infty} \nabla H(\mathbf{z}^k) = \lim_{k \rightarrow \infty} -\frac{L}{4\pi} \nabla I(\mathbf{z}^k(t)) = -\frac{L}{4\pi} \nabla I(\mathbf{z}^*) \quad (2.7)$$

$$I(\mathbf{z}^*) = \lim_{k \rightarrow \infty} I(\mathbf{z}^k) = 1, \quad (2.8)$$

$$H(\mathbf{z}^*) = \lim_{k \rightarrow \infty} H(\mathbf{z}^k) = \lim_{k \rightarrow \infty} h^k = h^* \quad (2.9)$$

In other words,  $z^* \in \mathfrak{Z}_1$  and  $H(z^*) = h^*$ . Hence  $\mathcal{H}_1$  is a closed set.

Next, consider the function

$$f : \mathbb{R}^{2N} \rightarrow \mathbb{R}$$

$$f(\mathbf{z}) = 2H(\mathbf{z}) + \frac{L}{2\pi}I(\mathbf{z})$$

Now by proposition 1.3.1  $\nabla f(\mathbf{z}) = 0$  implies that  $\mathbf{z} \in \mathfrak{Z}_1$ , which is isolated from collision. Hence Sard's theorem applies and  $f(\mathfrak{Z}_1)$  is a null set. But on  $\mathfrak{Z}_1$ , one has  $I(\mathbf{z}) = 1$ , hence  $\mathcal{H}_1 = H(\mathfrak{Z}_1)$  is a null set too. The theorem is thus proved.  $\square$

One important consequence of theorem 2 is the following corollary:

**Corollary 2.1.1.**  $\mathcal{H}_0 \setminus \mathcal{H}_1$  is an open dense subset of  $\mathcal{H}_0$ .

*Proof.* : Immediately from theorem 2.1.1.  $\square$

## 2.1.2 Rational Positive Vorticities And Beyond

So far corollary 2.1.1 is sufficient for our further need. But when vorticities are positive rational numbers we can do even more. Actually, if  $\Gamma_i \in \mathbb{Q}^+$ , we can even prove that there are only finitely many energy levels on which a normalized equilibrium exists.

**Theorem 2.1.2.** *If  $\Gamma_i \in \mathbb{Q}^+$ ,  $1 \leq i \leq N$ , then  $\mathcal{H}_1$  is a finite set.*

First we give some definitions as preparation.

**Definition 2.1.1.** *A **closed algebraic set** is the locus of zeros of a collection of polynomials.*

The following lemma is taken from Albouy and Kaloshin[4]:

**Lemma 2.1.2.** *([4, page 540]) Let  $X$  be a closed algebraic subset of  $\mathbb{C}^N$  and  $f : \mathbb{C}^N \rightarrow \mathbb{C}$  be a polynomial. Either the image  $f(X) \subset \mathbb{C}$  is a finite set, or it is the complement of a finite set. In the second case one says that  $f$  is **dominating**.*

A necessary condition for a polynomial to be dominating is the following condition:

**Lemma 2.1.3.** *([77, page 42]) A dominating polynomial  $f$  on a closed algebraic subset possesses **smooth points**, i.e., points where the dimension of the tangent space is minimal and where  $df \neq 0$ .*



Now back to our subject. Consider the Hamiltonian system

$$\Gamma \dot{\mathbf{z}}(t) = \mathbf{X}_G(\mathbf{z}(t)) = \mathbf{J}\nabla G(\mathbf{z}(t)) \quad \dot{\mathbf{z}} = (z_1, z_2, \dots, z_N), \quad z_i \in \mathbb{R}^2 \quad (\text{G1})$$

$$G(\mathbf{z}) = \prod_{1 \leq i < j \leq N} |z_i - z_j|^{\Gamma_i \Gamma_j}$$

The relation between the Hamiltonian  $G$  and Hamiltonian  $H$  is justified by the relation  $G(\mathbf{z}) = \exp\{-2\pi H(\mathbf{z})\}$ . The dynamic interpretation of this reparametrisation is that, in case of no collision, we re-parametrise the orbit; while when ever collision happens, we replace the collision orbit by a fixed point. We define

$$\begin{aligned} \mathfrak{Z}_1(G) &= \{\mathbf{z} \in \mathbb{R}^{2N} \mid \mathbf{z} \text{ is a normalized relative equilibrium of the system (G1)}\} \\ \mathfrak{Z}^{2\pi}(G) &= \{\mathbf{z} \in \mathbb{R}^{2N} \mid \mathbf{z} \text{ is a relative equilibrium of the system (G1),} \\ &\quad \text{with minimal period } T = 2\pi\} \\ \mathcal{G}_1 &= \{g \in \mathbb{R} \mid g = G(\mathbf{z}), \mathbf{z} \in \mathfrak{Z}_1(G)\} \\ \mathcal{G}^{2\pi} &= \{g \in \mathbb{R} \mid g = G(\mathbf{z}), \mathbf{z} \in \mathfrak{Z}^{2\pi}(G)\} \end{aligned}$$

Note that for all relative equilibrium in  $\mathfrak{Z}^{2\pi}(G)$  the angular velocity  $\omega = \frac{2\pi}{T}$  is fixed to be 1. The first observation is the following re-scaling property. Recall that  $L = \sum_{1 \leq i < j \leq N} \Gamma_i \Gamma_j$ .

**Lemma 2.1.4.** *Suppose  $z(t)$  is an orbit of (G1). Then for  $\lambda > 0$ ,  $\bar{z}(t) = \lambda z(\lambda^{L-2}t)$  is also an orbit of (G1).*

*Proof.* : This can be verified directly. Let  $\bar{z}(t) = \alpha z(\beta t)$ . Since  $z(t)$  is an orbit, we have

$$\dot{z}_i(t) = \mathbb{J} \nabla_{z_i} G(\mathbf{z}(t)) = \mathbb{J} \sum_{i \neq j} \Gamma_i \Gamma_j \left( \frac{G(\mathbf{z}(t))}{|z_i(t) - z_j(t)|^{\Gamma_i \Gamma_j}} |z_i(t) - z_j(t)|^{\Gamma_i \Gamma_j - 2} (z_i(t) - z_j(t)) \right) \quad (2.10)$$

As a result, we have

$$\begin{aligned}
\dot{z}_i(t) &= \mathbb{J} \alpha \beta \nabla_{z_i} G(\mathbf{z}(\beta t)) \\
&= \mathbb{J} \alpha \beta \sum_{i \neq j} \Gamma_i \Gamma_j \left( \frac{G(\mathbf{z}(\beta t))}{|z_i(\beta t) - z_j(\beta t)|^{\Gamma_i \Gamma_j}} |z_i(\beta t) - z_j(\beta t)|^{\Gamma_i \Gamma_j - 2} (z_i(\beta t) - z_j(\beta t)) \right) \\
&= \mathbb{J} \alpha^{2-L} \beta \sum_{i \neq j} \Gamma_i \Gamma_j \left( \frac{G(\alpha \mathbf{z}(\beta t))}{|\alpha z_i(\beta t) - \alpha z_j(\beta t)|^{\Gamma_i \Gamma_j}} |\alpha z_i(\beta t) - \alpha z_j(\beta t)|^{\Gamma_i \Gamma_j - 2} (\alpha z_i(\beta t) - \alpha z_j(\beta t)) \right) \\
&= \mathbb{J} \alpha^{2-L} \beta \nabla_{z_i} G(\bar{\mathbf{z}}(t))
\end{aligned}$$

Let  $\alpha = \lambda, \beta = \lambda^{L-2}$ , the result follows.  $\square$

For a centred relative equilibrium of (G1), the energy, the angular velocity and the angular momentum are closely related by the following lemma:

**Lemma 2.1.5.** *Suppose now that  $\mathbf{z}$  is a centred relative equilibrium of (G1), with angular velocity  $\omega$  and angular momentum  $I(\mathbf{z})$ . Then*

1.  $\nabla G(\mathbf{z}) = \frac{\omega}{2} \nabla I(\mathbf{z}(t))$
2.  $\omega = \frac{LG}{I}$

*Proof.* : 1. This is direct consequence by the definition of the centred relative equilibrium.  
2. Given that  $\nabla G(\mathbf{z}) = \frac{\omega}{2} \nabla I(\mathbf{z}(t))$ , we take inner product with  $\mathbf{z}$  on both sides and the result follows.  $\square$

**Lemma 2.1.6.** *If  $\Gamma_i \in \mathbb{N}^+$  and  $\Gamma_i \geq 2$ , then  $\mathcal{G}^{2\pi}$  is a finite set.*

*Proof.* : Consider  $\mathbf{z} \in \mathfrak{Z}^{2\pi}(G)$ , it satisfies the following algebraic systems

$$\begin{aligned}
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= \sum_{1 \neq i} \Gamma_i \delta_{i1} \begin{pmatrix} x_{1i} \\ y_{1i} \end{pmatrix} \\
\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \sum_{2 \neq i} \Gamma_i \delta_{i2} \begin{pmatrix} x_{2i} \\ y_{2i} \end{pmatrix} \\
&\vdots \\
\begin{pmatrix} x_N \\ y_N \end{pmatrix} &= \sum_{N \neq i} \Gamma_i \delta_{iN} \begin{pmatrix} x_{Ni} \\ y_{Ni} \end{pmatrix}
\end{aligned} \tag{P}$$

where  $x_{ij} = x_j - x_i$ ,  $y_{ij} = y_j - y_i$ , and  $\delta_{ij} = G(\mathbf{z}) = \left( \prod_{\substack{1 \leq p < q \leq N \\ (p,q) \neq (i,j)}} |z_p - z_q|^{\Gamma_p \Gamma_q} \right) |z_i - z_j|^{\Gamma_i \Gamma_j - 2}$ . If

we consider  $x_i$ ,  $y_i$  and  $\delta_{ij}$  as complex numbers, the system (P) is a polynomial system in  $\mathbb{C}^{2N}$ . This system then defines a closed algebraic subset  $\mathcal{A} \subset \mathbb{C}^{2N}$ .

On the other hand, by lemma 2 in section 2, we see that  $\nabla G(\mathbf{z}) = \frac{\omega}{2} \nabla I(\mathbf{z}(t))$  while  $\omega = \frac{LG}{I}$ . Taking  $\omega = 1$ , it turns out that for any  $z \in \mathcal{G}^{2\pi}$ , it satisfies

$$2\nabla G(\mathbf{z}) = \nabla I(\mathbf{z}(t)), \quad I = LG \quad (2.11)$$

Consider the function  $g = 2G + I$  as a polynomial on  $\mathcal{A}$ . Since  $dg = 0$  on  $\mathcal{A}$ ,  $g$  does not possess any smooth point on  $\mathcal{A}$ . As a result  $g$  is not a dominating polynomial due to lemma 2.1.3. Thus according to lemma 2.1.2,  $g(\mathcal{A})$  contains only finitely many values in  $\mathbb{C}$ . But on  $\mathcal{A}$ , we must have  $g = 2G + I = (L+2)G$ . Since  $L > 0$  is a constant, we thus conclude that  $G$  itself only gain finitely many values on  $\mathcal{A}$ . In other words,  $\mathcal{G}^{2\pi}$  is a finite set.  $\square$

We have proved that relative equilibrium with fixed angular velocity only possess finitely many energy levels. This however implies that relative equilibrium with fixed angular possess only finitely many energy levels too.

**Lemma 2.1.7.** *If  $\Gamma_i \in \mathbb{Q}^+$ , then  $\mathcal{G}_1$  is a finite set.*

*Proof.* : First, we assume that  $\Gamma_i \in \mathbb{N}^+$  and  $\Gamma_i \geq 2$ . In this case, Suppose to the contrary that  $\{\mathbf{z}^k\}_{k \in \mathbb{N}} \in \mathfrak{Z}_1(G)$  s.t.

$$0 < G(\mathbf{z}^1) < G(\mathbf{z}^2) < G(\mathbf{z}^3) \dots < G(\mathbf{z}^k) < \dots \quad (2.12)$$

by lemma 2.1.5 their frequencies satisfy  $\omega_k = \frac{LG(\mathbf{z}^k)}{I(\mathbf{z}^k)} = LG(\mathbf{z}^k) > 0$ , moreover (2.12) implies

$$0 < \omega_1 < \omega_2 < \dots < \omega_k < \dots \quad (2.13)$$

Now define  $\tilde{\mathbf{z}}^k(t) = (\omega_k)^{\frac{1}{2-L}} \mathbf{z}^k \left( \frac{1}{\omega_k} t \right)$ , by lemma 2.1.4,  $\tilde{\mathbf{z}}^k \in \mathcal{G}^{2\pi}$ . Then by lemma 2.1.6,  $G(\tilde{\mathbf{z}}^k)$  has only finite values. Again by lemma 2.1.5,  $I(\tilde{\mathbf{z}}^k) = LG(\tilde{\mathbf{z}}^k)$ . Thus  $I(\tilde{\mathbf{z}}^k) = (\omega_k)^{\frac{2}{2-L}}$  has only finite values. By (2.13) this leads to a contradiction. As a result, the lemma is proved. Now for general case, suppose that  $\Gamma_i = \frac{p_i}{q_i} \in \mathbb{Q}^+$ . let  $K = lcm(q_1, q_2, \dots, q_n)$  be the least common multiple of  $q_1, q_2, \dots, q_n$ . Consider now the new Hamiltonian

$$\tilde{G} = \prod_{1 \leq i < j \leq N} |z_i - z_j|^{\tilde{\Gamma}_i \tilde{\Gamma}_j} \quad (G2)$$

with  $\tilde{\Gamma}_i = 2K\Gamma_i, 1 \leq i \leq N$ . Now  $\tilde{\Gamma}_i \in \mathbb{N}^+$  and  $\tilde{\Gamma}_i \geq 2$ , thus we are back to previous situation. As a result  $\tilde{\mathcal{G}}_1$  is a finite set. But note that  $\tilde{G}(\mathbf{z}) = (G(\mathbf{z}))^{4K^2}$  and  $\mathfrak{Z}_1(\tilde{G}) = \mathfrak{Z}_1(G)$ , hence  $\mathcal{G}_1$  itself is also a finite set and the lemma is proved.  $\square$

Now it is easy to prove Theorem 2.1.2:

*Proof. (proof of Theorem 2.1.2):* Clearly  $\mathfrak{Z}_1(H) = \mathfrak{Z}_1(G)$ , and  $G(\mathbf{z}) = \exp(-2\pi H(\mathbf{z}))$ . Since  $\mathcal{G}_1$  is a finite set,  $\mathcal{H}_1$  is a finite set too.  $\square$

We have thus proved Theorem 1 under the assumption that  $\Gamma_i \in \mathbb{Q}^+$ . Some remarks might be useful:

**Remark 2.1.2.** *Note that we have only proved the finiteness of energy surface for normalized relative equilibria, not the finiteness for normalized relative equilibria.*

**Remark 2.1.3.** *The switching from logarithm to polynomial serves to provide a linear relation between  $G(\mathbf{z})$  and  $I(\mathbf{z})$  when  $\mathbf{z}$  is a relative equilibrium. Actually, if we work directly with  $H$ , one verifies that  $\nabla H(\mathbf{z})\mathbf{z} = -\frac{L}{2\pi}$  for any orbit  $\mathbf{z}$ , with  $L$  is a constant and we cannot benefit from any homogeneous condition.*

Theorem 2.1.2 is interesting in its own right, although we still do not know whether the number of normalized relative equilibria configurations are finite or not. Actually, from the proof, we see that  $\Gamma_i \in \mathbb{Q}^+$  is sufficient but not necessary. More generally, if  $\frac{\Gamma_i}{\Gamma_j} \in \mathbb{Q}^+, \forall 1 \leq i < j \leq N$ , the result will hold. In particular, this is case for identical vorticities:

**Corollary 2.1.2.** *If  $\Gamma_i = c \in \mathbb{R} \setminus \{0\}, 1 \leq i \leq N$ , then  $\mathcal{H}_1$  is a finite set.*

## 2.2 Abundance of Non-Equilibrium Relative Periodic Solutions

### 2.2.1 Symplectic Reduction and Relative Periodic Orbits in the Plane

In this section, we will use standard symplectic reduction to study the Hamiltonian in a reduced phase space. In the first section, we give some properties for the generalized Jacobi variable introduced by Lim [63]. The main result is the compactness of energy surface of the reduced Hamiltonian in the reduced phase space. We do not give explicit calculation for coordinates transformations in this section. Instead, a detailed example of the 5-vortex problem is studied with explicit coordinate transformation in Appendix B of [114].

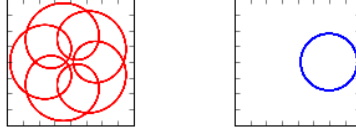


Fig. 2.1 A non trivial relative periodic (left) coming from a non-centred relative equilibrium in the original phase space (right)

### Lim's generalized Jacobi coordinates

We would like to fix the center of vorticity to the origin thus study only centred orbits. The reason is that, any non-centred relative equilibrium, when putting into a rotating framework around the origin, might automatically become a relative periodic solution that is not a relative equilibrium. This situation is illustrated in figure 2.1. However, this kind of solution (orbits in red color in the left of figure 2.1) is not the solution that we are searching for. Because it does not give any further insights about our dynamic system. As a result, we should insist on centred orbits, and we need some transformation to fix the vorticity centre to the origin.

The usual tool in celestial mechanics is the so called Jacobi coordinates. However, the usual Jacobi coordinates are not suitable for the  $N$ -vortex problems. This is because the conjugate variables  $(q, p)$  are separated in the Hamiltonian for Newtonian gravitation  $N$ -body problem, i.e.,

$$H(q, p) = \frac{|p|^2}{2} + U(q) \quad (\text{N-Body})$$

while in  $N$ -vortex problem the conjugate variables  $(x, y)$  are mixed

$$H(x, y) = -\frac{1}{4\pi} \sum_{i,j=1, i < j}^N \Gamma_i \Gamma_j \log |z_i - z_j|^2 \quad (\text{N-Vortex})$$

Hence if we perform a normal Jacobi transformation, we can fix the center of vorticity, but the resulting new Hamiltonian might be no longer invariant under rotation. There has been some study on symplectic transformations adapted to the  $N$ -vortex problem. For example [53, 23, 63] and so on. In particular, Lim's method in [63] has introduced a canonical transformation for the  $N$ -vortex Hamiltonian based on graph theory. This transformation works particularly well when all the vorticities are positive, and is quite ideal for our purpose of evaluating the energy surfaces. We hence apply Lim's generalized Jacobi coordinates to simplify our  $N$ -vortex system.

First, we make the change of variable

$$Z_i = (X_i, Y_i) = (\sqrt{\Gamma_i}x_i, \sqrt{\Gamma_i}y_i) \quad (2.14)$$

It turns out that  $\mathbf{Z} = (Z_1, Z_2, \dots, Z_N)$  follows the usual Hamiltonian system

$$\dot{\mathbf{Z}}(t) = X_{\hat{H}}(\mathbf{Z}(t)) = \mathbf{J}\nabla\hat{H}(\mathbf{Z}(t)) \quad \mathbf{Z} = (Z_1, Z_2, \dots, Z_N), \quad Z_i \in \mathbb{R}^2 \quad (\text{H2})$$

where

$$\hat{H}(\mathbf{Z}) = -\frac{1}{4\pi} \sum_{i,j=1, i < j}^N \Gamma_i \Gamma_j \log \left| \frac{Z_i}{\sqrt{\Gamma_i}} - \frac{Z_j}{\sqrt{\Gamma_j}} \right|^2$$

Then for the new variables,

$$\hat{P}(\mathbf{Z}(t)) = \sum_{1 \leq i \leq N} \sqrt{\Gamma_i} X_i(t), \quad \hat{Q}(\mathbf{Z}(t)) = \sum_{1 \leq i \leq N} \sqrt{\Gamma_i} Y_i(t), \quad \hat{I}(\mathbf{Z}(t)) = \sum_{1 \leq i \leq N} |Z_i(t)|^2$$

are first integrals. We identify till the end of this section the coordinate in  $Z_k = (X_k, Y_k) \in \mathbb{R}^2$  to the complex number  $Z_k = X_k + iY_k$ . A transformation from  $\mathbb{C}^N$  to  $\mathbb{C}^N$  will also be considered as a transformation from  $\mathbb{R}^{2N}$  to  $\mathbb{R}^{2N}$ .

**Proposition 2.2.1.** ([63, page 263]) *There exists a linear transformation for the positive planar N-vortex problem*

$$\begin{aligned} \phi: \quad \mathbb{C}^N &\rightarrow \mathbb{C}^N \\ Z = (X, Y) &\xrightarrow{\phi} W = (q, p) \end{aligned}$$

s.t.

1.  $\phi$  is unitary;
2. In the new coordinate  $W = (q, p)$ , one has

$$\begin{cases} q_N = \frac{\sum_{1 \leq i \leq N} \sqrt{\Gamma_i} X_i}{\sum_{1 \leq i \leq N} \Gamma_i} \\ p_N = \frac{\sum_{1 \leq i \leq N} \sqrt{\Gamma_i} Y_i}{\sum_{1 \leq i \leq N} \Gamma_i} \end{cases} \quad (2.15)$$

Since  $\mathbf{U}(N) = \mathbf{O}(2N) \cap \mathbf{Sp}(2N)$ , the transformation  $\phi$ , seen as a transformation  $\mathbb{R}^{2N} \xrightarrow{\phi} \mathbb{R}^{2N}$ , is thus a real linear symplectic transformation. As a result, we see that  $q_N$  is a first integral and  $p_N$  as its conjugate variable is cyclic. We can thus fix  $q_N = p_N = 0$ , and get a reduced Hamiltonian on  $\mathbb{R}^{2N-2}$ :

$$\bar{H}(q_1, p_1, q_2, p_2, \dots, q_{N-1}, p_{N-1}; q_N = p_N = 0) = \bar{H}(\mathbf{W}; W_N = 0) \quad (2.16)$$

Consider the dynamic system

$$\dot{\mathbf{W}}(t) = \mathbb{X}_{\bar{H}}(\mathbf{W}(t)) \quad (\text{H3})$$

We resume some properties of the new Hamiltonian  $\bar{H}$ :

**Proposition 2.2.2.** *Consider the Hamiltonian system (H3) and the original Hamiltonian system (H1) and (H2). Then:*

1. Any orbit of  $\bar{H}$  is a centred orbit of  $H$ ;
2. The system (H3) is invariant under rotation;
3. Define

$$\bar{I}(W) = \sum_{1 \leq i \leq N-1} (p_i^2 + q_i^2) \quad (2.17)$$

Then  $\bar{I}(\mathbf{W}) = \hat{I}(\mathbf{Z})$ .

*Proof.* : These propositions are direct consequences of the special symplectic transformation  $\phi$ .

1.  $(q_N, p_N)$  corresponds to the vorticity centre in the original Hamiltonian and they are fixed at 0. Hence all the orbits of  $\bar{H}$  are centred orbit of  $H$ .
2.  $\phi$  is a linear transformation  $\mathbb{C}^N \xrightarrow{\phi} \mathbb{C}^N$ . The term  $\log \left| \frac{Z_i}{\Gamma_i} - \frac{Z_j}{\Gamma_j} \right|^2$  under the transformation  $\phi$  now becomes

$$\log \left| \frac{Z_i}{\sqrt{\Gamma_i}} - \frac{Z_j}{\sqrt{\Gamma_j}} \right|^2 = \log \left| \frac{\sum_{1 \leq k \leq N-1} c_{ki} W_i}{\sqrt{\Gamma_i}} - \frac{\sum_{1 \leq k \leq N-1} c_{kj} W_j}{\sqrt{\Gamma_j}} \right|^2 \quad (2.18)$$

where the coefficients  $c_{ki}$  and  $c_{kj}$  are decided by  $\phi$ . It is clearly still invariant under rotation.

3. We know that  $I(\mathbf{z})$  is a first integral for system (H1), hence  $\hat{I}(\mathbf{Z}) = \sum_{1 \leq i \leq N} |Z_i|^2$  is a first

integral for system (H2). Now that  $\phi$  is orthogonal, we have  $\sum_{1 \leq i \leq N} |Z_i|^2 = \sum_{1 \leq i \leq N} |W_i|^2$ , while  $W_N = (q_N, p_N) = 0$ , we see that actually  $\sum_{1 \leq i \leq N} |W_i|^2 = \sum_{1 \leq i \leq N-1} |W_i|^2$ . In other words,  $\bar{I}(\mathbf{W}) = \hat{I}(\mathbf{Z})$ .  $\square$

Recall we are interested in normalized orbits of the original Hamiltonian system (H1). According to results in the previous proposition, they can be characterized by the new coordinates, i.e.:

**Proposition 2.2.3.** *The orbits of system (H3) which satisfies  $\bar{I}(\mathbf{W}) = 1$  are the normalized orbits of the system (H1).*

### Energy Surface in Reduced Phase Space

The Hamiltonian system (H3) with  $\bar{H}(\mathbf{W}; W_N = 0) : \mathbb{R}^{2N-2} \rightarrow \mathbb{R}$  is invariant under rotation, and  $\bar{I}(\mathbf{W})$  is the first integral. By the theory of the standard symplectic reduction, we can fix  $\bar{I} = 1$  and apply Hopf-fibration, it turns out that (H3) canonically induces a Hamiltonian system

$$\dot{\tilde{\mathbf{W}}} = \mathbb{X}_{\tilde{H}}(\tilde{\mathbf{W}}) = \tilde{\mathbf{J}}(\tilde{\mathbf{W}}) \nabla \tilde{H}(\tilde{\mathbf{W}}) \quad (\text{H4})$$

on  $\mathbb{C}\mathbb{P}^{N-2}$  [2]. Each point in  $\mathbb{C}\mathbb{P}^{N-2}$  represents a equivalent class of configurations up to the translation (by fixing  $q_N = p_N = 0$ ) the rotation (by taking quotient of  $\mathbb{S}\mathbb{O}(2)$ ), and the homothety (by fixing  $\bar{I}(\mathbf{W}) = 1$ , thus  $\nabla \bar{I}(\mathbf{W}) \neq 0$ ). By Proposition 2.2.3, each orbit on  $\mathbb{C}\mathbb{P}^{N-2}$  stands for a relative normalized orbit of system (H1). We summarized the whole reduction process in the following diagram:

$$\begin{array}{ccc} & & \mathbb{S}^1 \\ & & \downarrow \\ \mathbb{R}^{2N} \xleftarrow{q_N=p_N=0} \mathbb{R}^{2N-2} & \xleftarrow{\bar{I}=1} & \mathbb{S}^{2N-3} \\ & & \downarrow / \mathbb{S}\mathbb{O}(2) \\ & & \mathbb{C}\mathbb{P}^{N-2} \end{array}$$

Although the energy surfaces for original Hamiltonian is not even bounded, due to the invariance under translation and the mixed singularities ( $\infty$  and  $-\infty$ ) in logarithm function, the energy surface of the reduced Hamiltonian is indeed compact.

**Remark 2.2.1.** *Strictly speaking the reduced dynamics is only defined on  $\mathbb{C}\mathbb{P}^{N-2} \setminus \tilde{\Delta}$ . Here  $\tilde{\Delta}$  is projection of the generalized diagonal  $\Delta$  where collision (of two or multiple vortices) happens, i.e.,*

$$\Delta = \{\mathbf{z} \in \mathbb{R}^{2N} \mid z_i = z_j \text{ for some } 1 \leq i < j \leq N\}$$



Fortunately, as we see in lemma 2.2.1 that the energy surface  $\tilde{S}_c$  is bounded away from  $\tilde{\Delta}$ , this subtlety thus does not have impact on our proof.

**Lemma 2.2.1.** *Let  $c \in \mathbb{R}$ . Consider the hyper-surface  $S_c = \tilde{H}^{-1}(c) \subset \mathbb{C}\mathbb{P}^{N-2}$ . If  $S_c \neq \emptyset$ , then  $S_c$  is compact.*

*Proof.* : Consider the set  $\bar{S}_c = \tilde{H}^{-1}(c) \cap \bar{I}^{-1}(1)$ , which is the lifted set of  $S_c$  from  $\mathbb{C}\mathbb{P}^{N-2}$  to  $\mathbb{S}^{2N-3}$ . If  $\bar{S}_c$  is compact, then  $S_c$  will be compact by quotient topology. First,  $\mathbb{S}^{2N-3}$  is a bounded manifold, hence the boundedness of  $\bar{S}_c$ . Next, recall that  $\bar{I}(\mathbf{W}) = 1$  for all points in  $\bar{S}_c$ , which implies that all the mutual distances are bounded from above, since each squared mutual distance is a quadratic functions of  $\mathbf{W}$ , as is shown in (2.18). In other word, by the fact that  $\tilde{H}$  and  $\bar{I}$  are preserved by the lifted flow of  $\phi_{\tilde{H}}$ , the mutual distances cannot be too small. As a result, the energy surface  $\bar{S}_c$  is isolated from singularity. But then the pre-image of a closed set must be closed, hence  $\bar{S}_c$  is closed. Hence  $S_c$  is compact. So is  $S_c$ .  $\square$

### Symplectic Capacity and Existence of normalized Non-Trivial Relative Periodic Orbits

We are now ready to prove the theorem concerning the existence of NTNRPPOs of system (H1). Our main tool is the so called symplectic capacity, in particular the Hofer-Zehnder capacity  $c_0$ [52], which links periodic solution of Hamiltonian system to symplectic invariant. It is closely related to the searching of periodic orbits on a prescribed energy surface, initially studied by Rabinowitz [90] and Weinstein [116]. For general introduction to symplectic capacity theory one could turn to [113, 52] and the references therein.

**Theorem 2.2.1.** *Suppose that  $S_c = \tilde{H}^{-1}(c)$  is a non-empty regular hyper-surface, then there exists a non-constant sequence  $\lambda_k \rightarrow c$  and a sequence of normalized non-trivial relative periodic orbits  $\mathbf{z}^k(t)$  of system (H1) s.t.  $H(\mathbf{z}^k) = \lambda_k$ .*

*Proof.* : Since the hyper-surface  $S_c$  is regular, and by Lemma 2.2.1 it is compact. In other words, the vector field  $\dot{W} = \frac{\nabla \tilde{H}(\tilde{W})}{|\nabla \tilde{H}(\tilde{W})|^2}$  is locally well defined. By consequence we can almost surely extend  $S_c$  to a 1-parameter family of regular energy surfaces  $S(\delta)$ , with  $-\varepsilon < \delta < \varepsilon$  and  $S(\delta) = S_{c+\delta}$ . Define

$$U_\varepsilon = \bigcup_{\delta \in (-\varepsilon, \varepsilon)} S(\delta)$$

Let  $c_0(\mathbb{C}\mathbb{P}^{N-2}, \omega)$  be the symplectic capacity, where  $\omega = \text{Im}(g)$  and  $g$  is the induced Kähler metric by the standard Hermitian, then  $c_0(\mathbb{C}\mathbb{P}^{N-2}, \omega) = \pi < \infty$  ([51, Corollary 1.5]), thus *a fortiori*,  $c_0(U, \omega) < \infty$ . Classical result of almost existence ([52, Theorem 4.1]) now

implies the existence of infinitely many **non-constant** periodic solutions  $\{\tilde{\mathbf{W}}^k\}_{k \in \mathbb{N}}$  of the Hamiltonian system (H4) and a corresponding non-constant sequence  $\{\lambda_k\}_{k \in \mathbb{N}}$ , which satisfy that  $\tilde{H}(\tilde{\mathbf{W}}^k) = \lambda_k \rightarrow c$ .

Now given a non-constant periodic orbit  $\tilde{\mathbf{W}}^k(t) = \phi_{\tilde{H}}(t) \subset \mathbb{C}\mathbb{P}^{N-2}$  of system (H4), its lifted orbit  $\mathbf{z}^k = \phi_H(t) \subset \mathbb{R}^{2N}$  is a normalized relative periodic solution of the original Hamiltonian system (H1). We show that  $\mathbf{z}^k$  is not a relative equilibrium. Recall that by our construction of the reduced phase space, the vortex center of  $\mathbf{z}^k(t)$  is fixed at 0. If  $\mathbf{z}^k(t)$  is a relative equilibrium, then  $\tilde{\mathbf{W}}^k(t)$  is a fixed point in the reduced space, which contradicts the fact that  $\tilde{\mathbf{W}}^k(t)$  is a non-constant periodic solution. The theorem is thus proved.  $\square$

We have seen that the existence of infinitely many NTNRPPOs depends on the existence of a regular energy surface of the reduced Hamiltonian. Since fixed points of the reduced Hamiltonian  $\tilde{H}$  lift to normalized relative equilibria of the original Hamiltonian  $H$ . Thus to understand where are these NTNRPPOs, we must have some information about the distribution of the set  $\mathcal{H}_1$  in the set  $\mathcal{H}_0$ . But this has already been answered by theorem 2.1.1 and corollary 2.1.1. We resume all the discussion above and theorem 2.0.1 is thus proved:

**of theorem 2.0.1.** : By combining theorem 2.2.1 and corollary 2.2.1 implies that  $\mathcal{H}_2$  is dense in  $\mathcal{H}_0 \setminus \mathcal{H}_1$ . Corollary 2.1.1 implies that  $\mathcal{H}_0 \setminus \mathcal{H}_1$  is dense in  $\mathcal{H}_0$ . As a result  $\mathcal{H}_2$  is dense in  $\mathcal{H}_0$ .  $\square$

**Remark 2.2.2.** *To know if there exists a periodic solution exactly on the prescribed energy surface, we need in general more condition, for example being of a contact type, see [112].*

### 2.3 Periodic Orbits with Discrete $C_N$ Symmetry

So far we have only considered the continuous symmetry, and have used the symplectic reduction to work in the reduced phase space. The factors that allowed us to find NTNRPPOs are essentially:

1. The unitary change of variable;
2. Existence of regular and compact energy surface;
3. The finite symplectic capacity of the reduced spaces.

On the other hand, one could alternatively impose discrete symmetry constraints on the orbits, which will largely reduce the degree of freedom until the reduced phase space is simple enough for explicit investigation.

The systematic investigation of this direction starts with Aref [6], where the double alternate ring configurations are studied in details. Then Koiller et al.[56] studied two and three vortex rings together with their bifurcations. One could turn to [7] for the generalisation of previous results to various 2-dimensional manifolds. Later on, Tokieda, Soulière, Montaldi and Laurent-Polz, among others, further generalized this method to find non-equilibrium (relative) periodic solutions of the so called "dansing vortices" on spheres and other manifolds under different symmetric group actions [110, 103, 72, 59]. Essentially these existence results are proved in two steps. In the first step, discrete symmetric reductions are carefully chosen to reduce the phase space to be 2-dimensional. Next, by fixing a regular energy level, one gets a 1-parameter curve in 1-dimensional compact space, which is diffeomorphic to a circle. As a result the (relative) periodic solutions are found.

In this section, we explain how to mix symplectic reduction and center symmetric reduction to get plenty of normalized non-trivial relative periodic solution with a center of symmetry. The whole idea is illustrated by the following example:

**Example 2.3.1.** *Let  $a_1, a_2, b_1, b_2$  be 4 vortices of positive vorticity. Moreover, the vorticities of  $a_i$  and that of  $b_i$  are the same, denoted by  $\Gamma_i, i = 1, 2$ . Consider that at time 0,  $a_i(0) = -b_i(0)$ . Then by symmetry of the Hamiltonian, we see that  $a_i(t) = -b_i(t), \forall t \in \mathbb{R}$ . As a result, the Hamiltonian  $H(a_1, a_2, b_1, b_2)$  could be considered as a system of 2 vortices:*

$$H^{sym}(z) = -\frac{1}{4\pi}(2\Gamma_1\Gamma_2(\log|a_1 - a_2|^2 + \log|a_1 + a_2|^2) + \sum_{i=1}^2 \Gamma_i^2 \log|2a_i|^2)$$

*If we can find a relative periodic solution of this modified 2-vortex problem, we then will have actually found a symmetric relative periodic solution of the original 4-vortex problem. In particular, the above simplified Hamiltonian is still invariant under rotation. It turns out that, by mixing the discrete symmetry reduction with the symplectic reduction, the reduced phase space is*

$$\begin{array}{ccccc} & & & & \mathbb{S}^1 \\ & & & & \downarrow \\ \mathbb{R}^8 & \xleftarrow{z=-z} & \mathbb{R}^4 & \xleftarrow{I=\frac{1}{2}} & \mathbb{S}^3 \\ & & & & \downarrow / \mathbb{S}\mathbb{O}(2) \\ & & & & \mathbb{C}\mathbb{P}^1 \end{array}$$

*Now that each term in the logarithm is a quadratic function, and  $I = 1$ , we conclude that the nonempty energy hyper-surfaces are compact. Moreover  $a_1, a_2$  forms a relative equilibrium, if and only if  $a_1, a_2, b_1, b_2$  also forms a symmetric relative equilibrium.*

We claim the result more precisely:

**Definition 2.3.1.** Let  $M, N \in \mathbb{N}$ . We say a centred  $M \times N$ -vortex configuration is  $C_N$ -symmetric, if

$$\mathbf{z} = e^{\mathbf{J}_{M \times N} \frac{2\pi}{N} \mathbf{z}} \quad (2.19)$$

We say a centred  $M \times N$ -vortex problem orbit is a  $C_N$  symmetric orbit, if  $\mathbf{z}(t)$  is a  $C_N$  symmetric configuration for all  $t \in \mathbb{R}$ .

**Example 2.3.2.** Let  $M = 3$  and  $N = 4$ , figure 2.2 shows roughly how these vortices are arranged at time 0.

**Remark 2.3.1.** A  $C_N$  symmetric orbit is automatically a centred orbit.

Now consider a  $M \times N$ -vortex problem, with  $M$  groups of vortices, and each group  $M_l$  contains  $N$  vortices of the same vorticity  $\Gamma_l > 0$ . At time 0, we put each group  $M_l$  into a  $C_N$  symmetric configuration, i.e.,  $\forall 1 \leq i \leq N, 1 \leq l \leq M$

$$z_{li} = e^{\mathbb{J} \frac{2\pi(i-1)}{N}} z_{l1} \quad (2.20)$$

Then by symmetry of the Hamiltonian, we will have an orbit s.t. each vortices in each group  $M_i, 1 \leq i \leq M$  follow a  $C_N$  symmetric orbit. We only need to study the Hamiltonian taking the  $C_N$  symmetry into account. Denote  $w_l = z_{l1}, 1 \leq l \leq M$  for short, which serves as a representative of the  $N$  vortices in the  $l$ -th group  $M_l$ . We then consider the simplified Hamiltonian system

$$\Gamma \dot{\mathbf{w}}(t) = \mathbb{X}_{H^{sym}}(\mathbf{w}(t)) = \mathbf{J}_M \nabla H^{sym}(\mathbf{w}(t)) \quad \mathbf{w} = (w_1, w_2, \dots, w_M), \quad w_i \in \mathbb{R}^2 \quad (\text{H-Sym})$$

where

$$H^{sym}(\mathbf{w}) = -\frac{1}{4\pi} \sum_{\substack{1 \leq p, q \leq M \\ 1 \leq i, j \leq N \\ (p,i) \neq (q,j)}} \Gamma_p \Gamma_q \log |e^{\mathbb{J} \frac{2\pi i}{N}} w_p - e^{\mathbb{J} \frac{2\pi j}{N}} w_q|^2$$

Clearly each periodic solution of the system (H-Sym) will imply a  $C_N$  symmetric periodic solution of the original  $M \times N$ -vortex problem as in system (H1). If we further more require that  $I(\mathbf{w}) = \frac{1}{N}$ , then it corresponds to a normalized  $C_N$ -symmetric periodic solution of the original  $M \times N$ -vortex problem as in system (H1).

We resume the above discussion in the following theorem:

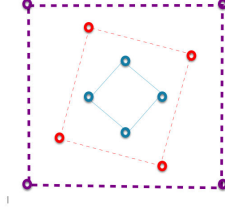


Fig. 2.2 An example of a  $M \times N$ -vortex configuration that is  $C_N$  symmetric, with  $M=3$ ,  $N=4$

**Theorem 2.3.1.** Consider the above symmetric  $M \times N$ -vortex problem with positive vorticities s.t.  $\Gamma_{li} = \Gamma_{lj}$ ,  $1 \leq l \leq M$ ,  $1 \leq i < j \leq N$ . Let

$$\mathfrak{Z}_0^{\text{sym}} = \{\mathbf{w} | \mathbf{w} \text{ is a normalized orbit of the system (H-Sym)}\}$$

$$\mathfrak{Z}_2^{\text{sym}} = \{\mathbf{w} | \mathbf{w} \text{ is a NTNRPPO of the system (H-Sym)}\}$$

$$\mathcal{H}_0^{\text{sym}} = \{h \in \mathbb{R} | h = H^{\text{sym}}(\mathbf{w}), \mathbf{z} \in \mathfrak{Z}_0^{\text{sym}}\}$$

$$\mathcal{H}_2^{\text{sym}} = \{h \in \mathbb{R} | h = H^{\text{sym}}(\mathbf{w}), \mathbf{z} \in \mathfrak{Z}_2^{\text{sym}}\}$$

Then  $\mathcal{H}_2^{\text{sym}}$  is dense in  $\mathcal{H}_0^{\text{sym}}$ . In otherwords, there are infinitely many  $C_N$ -symmetric non-trivial normalized periodic solution of the original  $M \times N$ -vortex problem in system (H1).

*Proof.* : Similar as the discussion in theorem 2.0.1. □

**Remark 2.3.2.** Again, since one doesn't need to worry about the degree of freedom, we can take  $M$  to be any positive integer, as long as there exists regular and compact energy surface in the (symplectically and symmetrically) reduced phase spaces.



## Chapter 3

# Periodic Orbits of the Identical N-Vortex Problem

In the previous chapter, periodic orbits for the  $N$ -vortex problem in the plane have been found. Unfortunately, it is a difficult problem to distinguish periodic orbits on a given energy level. After all these orbits are determined through implicit methods, instead of explicit constructions. Let us consider the following simple example of RPO for the BEC identical 4-vortex problem (**Figure 3.1**). In the left (1234) configuration, the distances of the four vortices are, roughly speaking, of the same scale. As a result, the motion will be that the four vortices confine themselves in a relatively small cluster and chase each other therein, while the cluster as an entity rotates together around the origin  $\mathbf{O}$ ; However in the right (123)(4) configuration, the 4<sup>th</sup> vortex is relatively far away from the other, hence the behavior will be that the three vortices form their own cluster, thus this cluster and the 4<sup>th</sup> vortex rotate as two clusters around the origin  $\mathbf{O}$ . Note that we can adjust the distances to make them of same energy level  $H$  and of same angular momentum  $I$ . So, a constant issue we have is the triviality issue: one has to show that the periodic orbits we find, absolute or relative, are distinct from well known ones (and in particular that their reduction is not a fixed point). More precisely, we could ask the following questions:

- Can one find an orbit that looks like (1234), instead of (123)(4)?
- Further more, suppose an orbit looks like (1234) has been found, can one distinguish this orbit from relative equilibria, i.e., a square configuration rotating around its center of vorticity in certain rotating frame?

The above example explains our motivation in this work: since the dynamical systems we study are in general non-integrable (this non-integrability is rarely trivial and often requires



Fig. 3.1 Two configurations of same  $H$  and  $I$

special arguments, which we will not develop here), it is hopeless to characterise orbits by quadratures and eliminations. Nevertheless, we claim that it is possible to find non-trivial periodic orbits of the N-vortex problem with some abstract, variational methods, and even some more specific classes of orbits, displaying a rich discrete symmetry group.

Again, the study of N-body problem in the plane sheds some light on our problem. In [89], Poincaré had understood the difficulty of minimizing for the Lagrangian action functional in a given homotopy class, due to the possibility of collisions. Since then there have been at least two perspectives to add topological constraints. These constraints serve not only as the guarantee of coercivity, but also as ways to distinguish different orbits such like the ones we see in **Figure 3.1**. More precisely, we may consider:

- **Homotopic Constraint:** it is requested that the orbit fall in a special free homotopic class [43, 73, 111];
- **Symmetry Constraint:** it is requested that the orbit be invariant under the action of a special symmetry group [37, 119, 33, 30, 31].

If the orbits found meet these constraints, then we will have gained qualitative insight of them. In this chapter, for the sake of simplicity we will focus on a special symmetry constraint, namely the *simple choreographic symmetry*, and study the existence of relative periodic orbits with such symmetry.

Let us consider a class of Hamiltonian systems in  $\mathbb{R}^{2N}$  of the form

$$\dot{\mathbf{Z}}(t) = X_{H_{\mathbb{R}^{2N}}}(\mathbf{Z}(t)) = \mathbf{J}_{\mathbb{R}^{2N}} \nabla H_{\mathbb{R}^{2N}}(\mathbf{Z}(t)), \quad \mathbf{Z} = (z_1, z_2, \dots, z_N), \quad z_i = (x_i, y_i) \in \mathbb{R}^2$$

with

$$H_{\mathbb{R}^{2N}}(\mathbf{Z}) = \sum_{i=1}^n \alpha_i V(|z_i|^2) + \sum_{1 \leq i < j \leq N} \beta_{ij} F(|z_i - z_j|^2). \quad (\text{System-I})$$



Here

- $z_i = (x_i, y_i)$  is the position of the  $i^{\text{th}}$  particle in the plane
- $\nabla H_{\mathbb{R}^{2N}}$  is the gradient of  $H$
- $\mathbf{J}_{\mathbb{R}^{2N}}$  is the standard complex structure

$$\mathbf{J}_{\mathbb{R}^{2N}} = \begin{bmatrix} \mathbb{J} & & \\ & \ddots & \\ & & \mathbb{J} \end{bmatrix}, \quad \mathbb{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- $F$  is smooth in  $\mathbb{R}^{2n} \setminus \Delta$ , where

$$\Delta = \bigcup_{1 \leq i < j \leq n} \{\mathbf{Z} \in \mathbb{R}^{2n} \mid z_i = z_j\}$$

Such a system describes the motion of  $N$  particles in the plane, driven by a radial potential  $V$  and an interaction function  $F$  depending on only the mutual distance of each pair of particles. Here  $\alpha_i$  and  $\beta_{ij}$  are parameters (mass, vorticity, charge...), which might vary with indices  $i$  and  $j$ .

**Example 3.0.1** (The identical N-Vortex problem of hydrodynamics). *Let*

$$V = 0, \beta_{ij} = -\frac{1}{4\pi}, 1 \leq i < j \leq N, F(\eta) = \log |\eta|^2$$

*In this case (System-I) becomes*

$$H_{\mathbb{R}^{2N}}(\mathbf{Z}) = -\frac{1}{4\pi} \sum_{1 \leq i < j \leq N} \log |z_i - z_j|^2 \quad (\text{N-Vortex Euler})$$

*This is the Hamiltonian function for N-vortex problem coming from Euler Equation*

**Example 3.0.2** (The Discrete Nonlinear Schrödinger Equation). *Let*

$$\alpha_i = -\frac{1}{4}, 1 \leq i \leq N, V(\eta) = |\eta|^4, \beta_{ij} = -\frac{1}{2} \delta_{ij}, 1 \leq i < j \leq N, F(\eta) = \log |\eta|^2$$

*where*

$$\delta_{ij} = \begin{cases} 0 & \text{if } i - j > 1 \pmod{N}; \\ 1 & \text{if } i - j = 1 \pmod{N}. \end{cases}$$

In this case (System-I) becomes

$$H_{\mathbb{R}^{2N}}(\mathbf{Z}) = \frac{1}{2} \sum_{j=1}^n \left( \frac{1}{2} |z_j|^4 - |z_{j+1} - z_j|^2 \right) \quad (\text{N-Sites NLS})$$

This Hamiltonian system describes a simplified model for a lattice of coupled harmonic oscillators. Here  $z_j = z_j(t)$  is the complex mode amplitude of the oscillator at site  $j$ . This system can be seen as a standard finite difference approximation to the continuous Schrödinger equation:

$$i\mathbf{Z}_t + |\mathbf{Z}|^2 \mathbf{Z} + \mathbf{Z}_{xx} = 0 \quad (3.1)$$

For more details, see [34].

**Example 3.0.3** (The Identical N-Vortex Problem in Bose-Einstein Condensation). *Let*

$$\alpha_i = -\frac{1}{2}\mu, 1 \leq i \leq N, V(\eta) = \log \frac{1}{1-|\eta|^2}, \beta_{ij} = -\frac{1}{2}\lambda, 1 \leq i < j \leq N, F(\eta) = \log |\eta|^2$$

$$H_{\mathbb{R}^{2N}}(\mathbf{Z}) = -\frac{1}{2} \left( \mu \sum_{i=1}^N \log \frac{1}{1-|z_i|^2} + \lambda \sum_{i < j} \log |z_i - z_j|^2 \right) \quad (\text{N-Vortex BEC})$$

This Hamiltonian system describes the motion of vortices in Bose Einstein condensation (BEC). It can be observed by experiments, either via a harmonical trap [41] or via a hard wall container[3]. This system is a 2D reduction of the Gross-Pitaevskii partial differential equation concerning the ground state of a quantum system of identical bosons. Here the topological charge of each vortex is fixed to be 1,  $\mu > 0$  is the precession of trap center, and  $\lambda > 0$  is the interaction strength. The case  $\mu = 0$  corresponds to the classical identical N-vortex problem in hydrodynamics given in example 3.0.1.

We will be primarily interested in vortex-like systems, but part of the coming study holds in larger generality. The Hamiltonian descends to a Hamiltonian  $H$  on the quotient, thus defining a reduced Hamiltonian vector field

$$\begin{aligned} \dot{\mathbf{z}}(t) &= X_H(\mathbf{z}(t)), \quad \mathbf{z} \in \mathbb{C}\mathbb{P}^{N-1}, \text{ when } V \text{ is not a constant} \\ \dot{\mathbf{z}}(t) &= X_H(\mathbf{z}(t)), \quad \mathbf{z} \in \mathbb{C}\mathbb{P}^{N-2}, \text{ when } V \text{ is constant} \end{aligned} \quad (\text{System-II})$$

It is these vector fields that will be the primary source of interest in our study, aimed at finding symmetric periodic orbits.

## 3.1 Absolute and relative choreographies

### 3.1.1 (Simple) Choreographic Loop

We are interested in relative periodic solutions of the system (System-I) that satisfy some symmetry condition, namely the choreographic symmetry. The study of choreographies begins with the seminal paper of Chenciner and Montgomery [33] on the proof of existence of the figure-eight solution for the 3-body problem, following the earlier numerical experiment of [74].

We denote the set of  $2\pi$ -periodic continuous loops by

$$\Lambda = \{\mathbf{Z} \in \mathcal{C}(\mathbb{S}^1, \mathbb{R}^{2N}) \mid \mathbf{Z}(0) = \mathbf{Z}(2\pi)\}, \quad \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

$$\tau : \mathbb{S}^1 \rightarrow \mathbb{S}^1 \quad \tau(t) = \frac{2\pi}{n} + t \quad (3.2)$$

$$\tilde{\sigma} : \mathbb{R}^{2N} \rightarrow \mathbb{R}^{2N} \quad (z_1, z_2, \dots, z_{N-1}, z_N) \xrightarrow{\tilde{\sigma}} (z_N, z_1, \dots, z_{n-2}, z_{N-1}) \quad (3.3)$$

and

$$g : \Lambda \rightarrow \Lambda \quad (g\mathbf{Z})(t) = \tilde{\sigma}\mathbf{Z}(\tau^{-1}t)$$

We are interested in the fixed points of  $g$ , namely free loops satisfying

$$z_{i+1}\left(t + \frac{T}{N}\right) = z_i(t) \quad (3.4)$$

**Definition 3.1.1.** We call a loop  $\mathbf{Z} \in \Lambda$

- a *choreography*, if  $g\mathbf{Z} = \mathbf{Z}$ ;
- a *centred choreography*, if  $\mathbf{Z}(t)$  is a choreography and

$$P(\mathbf{Z}(t)) = Q(\mathbf{Z}(t)) = 0, \forall t \in [0, 2\pi] \quad (3.5)$$

This choreographic symmetry means that particles describe the same orbit in the plane, and are merely separated by a fixed amount of time. One may define more complicated kinds of choreographies, corresponding to permutations  $\tilde{\sigma}$  splitting into several cycles, but we will not consider such so-called multiple choreographies. We will thus have omitted the adjective "simple" in this article.

The simplest choreography is the regular  $N$ -gon relative equilibrium, namely the motion along which the  $N$  particles sit on the  $N$  vertices of a regular  $N$ -gon, and rotate uniformly. A direct elementary computation shows that such solutions exist in the identical  $N$ -vortex problem ("Thomson configuration") or in the identical  $N$ -body problem (the bodies should additionally then be given the right velocities, without which the motion is homographic). The Trojan satellites in the Solar system are close to an equilateral configuration with Jupiter and the Sun.

### 3.1.2 Reduced Choreographic Loop in $\mathbb{C}\mathbb{P}^{N-1}$

Similarly to when we weakened the notion of periodic orbit by introducing the idea of reduced or relative periodic orbits, it is natural to consider solutions which are choreographic for the reduced dynamics, in the sense which follows, and which primarily uses the existence of an action of the symmetric group on  $\mathbb{C}\mathbb{P}^{N-1}$ .

Denote the set of  $2\pi$ -parameterised continuous loops in  $\mathbb{C}\mathbb{P}^{N-1}$  by

$$\Lambda_{N-1} = \{\mathbf{z} \in \mathcal{C}(\mathbb{S}^1, \mathbb{C}\mathbb{P}^{N-1}) \mid \mathbf{z}(0) = \mathbf{z}(2\pi)\}$$

As earlier, we write  $\mathbf{Z} = (z_1, z_2, \dots, z_N) \in \mathbb{S}^{2N-1}$ , and  $\mathbf{z} = [z_1 : z_2 : \dots : z_N] \in \mathbb{C}\mathbb{P}^{N-1}$ . The restriction of  $\tilde{\sigma}$  to  $\mathbb{S}^{2N-1}$  induces a natural symmetry on  $\mathbb{C}\mathbb{P}^{N-1}$ . The above circular permutation  $\tilde{\sigma}$  induces a map

$$\sigma_1 : \mathbb{C}\mathbb{P}^{N-1} \rightarrow \mathbb{C}\mathbb{P}^{N-1}, \quad [z_1 : z_2 : \dots : z_n] \xrightarrow{\sigma_1} [z_n : z_1 : \dots : z_{n-2} : z_{n-1}], \quad (3.6)$$

letting the following diagram commute:

$$\begin{array}{ccc} \mathbb{S}^{2N-1} & \xrightarrow{\tilde{\sigma}} & \mathbb{S}^{2N-1} \\ \pi \downarrow & & \pi \downarrow \\ \mathbb{C}\mathbb{P}^{N-1} & \xrightarrow{\sigma_1} & \mathbb{C}\mathbb{P}^{N-1}. \end{array}$$

Here  $\mathbb{S}^{2N-1} \rightarrow \mathbb{C}\mathbb{P}^{N-1}$  is the Hopf fibration. We can then define the loop transformation

$$g_1 : \Lambda_{N-1} \rightarrow \Lambda_{N-1}, \quad (g\mathbf{z})(t) = \sigma_1 \mathbf{z}(\tau^{-1}t)$$

**Definition 3.1.2.** We call a loop  $\mathbf{z}(t) \in \Lambda_{N-1}$  a *reduced choreographic loop*, if

$$g_1 \mathbf{z} = \mathbf{z}, \text{ i.e. } \mathbf{z}\left(t + \frac{2\pi}{N}\right) = \sigma_1 \mathbf{z}(t) \quad \forall t \in [0, 2\pi] \quad (3.7)$$

A reduced simple choreographic loop is called *nontrivial* if it is not a constant in  $\mathbb{C}\mathbb{P}^{N-1}$ .

### 3.1.3 Centred Reduced Choreographic Loop in $\mathbb{C}\mathbb{P}^{N-2}$

It is also possible to define an induced choreographic symmetry in loop space of  $\mathbb{C}\mathbb{P}^{N-2}$ . However,  $\mathbb{C}\mathbb{P}^{N-2}$  does not lift to  $\mathbb{S}^{2N-1}$  directly. Due to this reason, we first define the permutation on  $\mathbb{S}^{2N-3}$ . Let

$$\begin{aligned} i_N : \mathbb{C}^{N-1} &\rightarrow \mathbb{C}^N, & (w_1, w_2, \dots, w_{N-1}) &\xrightarrow{i_N} (w_1, w_2, \dots, w_{N-1}, 0); \\ \pi_N : \mathbb{C}^N &\rightarrow \mathbb{C}, & (w_1, w_2, \dots, w_{N-1}, w_N) &\xrightarrow{\pi_N} w_N; \\ L : \mathbb{C}^N &\rightarrow \mathbb{C}^N & \text{is a unitary linear transformation s.t. } \mathbf{Z} &\xrightarrow{L} \mathbf{W} \text{ with } w_N = \frac{1}{N} \sum_{1 \leq i \leq N} z_i. \end{aligned}$$

Here  $L$  is known as Lim's transformation [63]. It is a generalized Jacobi coordinate obtained by graph theory and is a suitable canonical transformation for the N-vortex type problems.

**Lemma 3.1.1.** *Suppose that  $\mathbf{W} = (w_1, w_2, \dots, w_{N-1}) \in \mathbb{S}^{2N-3}$ . Then*

$$\pi_N \circ L \circ \tilde{\sigma} \circ L^{-1} \circ i_N(\mathbf{W}) = 0$$

*Proof.* Let  $\mathbf{W} = (w_1, w_2, \dots, w_{N-1}) \in \mathbb{S}^{2N-3}$ , thus  $|\mathbf{W}|^2 = \sum_{i=1}^{N-1} |w_i|^2 = 1$ . Now let  $\hat{\mathbf{W}} = (w_1, w_2, \dots, w_{N-1}, 0) = i_N(\mathbf{W})$ , one has that

$$|\hat{\mathbf{W}}|^2 = \sum_{i=1}^{N-1} |w_i|^2 + 0 = 1$$

As a result,  $\hat{\mathbf{W}} \in \mathbb{S}^{2N-1} \subset \mathbb{C}^N$ . Next let  $\hat{\mathbf{Z}} = (\hat{z}_1, \hat{z}_2, \hat{z}_3, \dots, \hat{z}_N) = L^{-1}(\hat{\mathbf{W}})$ . Since  $L$  is unitary, it follows that  $|\hat{\mathbf{Z}}|^2 = 1$  and  $\sum_{i=1}^N \hat{z}_i = 0$ . As a result, Let  $\mathbf{Z} = (z_1, z_2, \dots, z_N) = \tilde{\sigma}\hat{\mathbf{Z}}$ , one has that  $\mathbf{Z} \in \mathbb{S}^{2N-1}$  with  $\sum_{i=1}^N z_i = 0$ . In other words it implies that if we denote  $\tilde{\mathbf{W}} = (\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_N) = L(\mathbf{Z})$ , then  $\tilde{w}_N = 0$ . To summarize, we have thus proved that  $\pi_N \circ L \circ \tilde{\sigma} \circ L^{-1} \circ i_N(\mathbf{W}) = 0$ .  $\square$

Lemma 3.1.1 implies that there is a well defined transformation  $\hat{\sigma} : \mathbb{S}^{2N-3} \rightarrow \mathbb{S}^{2N-3}$  s.t. the following diagram commutes:

$$\begin{array}{ccc} \mathbb{S}^{2N-3} & \xrightarrow{\hat{\sigma}} & \mathbb{S}^{2N-3} \\ \downarrow i_N & & \downarrow i_N \\ \mathbb{S}^{2N-1} & \xrightarrow{L \circ \tilde{\sigma} \circ L^{-1}} & \mathbb{S}^{2N-1} \end{array}$$

Now we can define  $\sigma_2 : \mathbb{C}\mathbb{P}^{N-2} \rightarrow \mathbb{C}\mathbb{P}^{N-2}$  as s.t. the following diagram commutes

$$\begin{array}{ccc} \mathbb{S}^{2N-3} & \xrightarrow{\hat{\sigma}} & \mathbb{S}^{2N-3} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{C}\mathbb{P}^{N-2} & \xrightarrow{\sigma_2} & \mathbb{C}\mathbb{P}^{N-2} \end{array}$$

**Remark 3.1.1.** Let  $R = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 1 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$  be the permutation matrix that corresponds to  $\tilde{\sigma}$ . Note that in general  $L \circ R \neq R \circ L$ .

Similarly we define the loop transformation

$$g_2 : \Lambda_{N-2} \rightarrow \Lambda_{N-2}, \quad (g_2 \mathbf{w})(t) = \sigma_2 \mathbf{w}(\tau^{-1}t)$$

We call a loop  $\mathbf{w}(t) \in \Lambda_{N-2}$  a **centred reduced choreographic loop**, if

$$g_2 \mathbf{w} = \mathbf{w}, \text{ i.e. } \mathbf{w}\left(t + \frac{2\pi}{N}\right) = \sigma_2 \mathbf{w}(t) \quad \forall t \in [0, 2\pi] \quad (3.8)$$

A reduced centred choreographic loop is called **nontrivial** if it is not a constant in  $\mathbb{C}\mathbb{P}^{N-2}$ .

### 3.1.4 Relative choreographic loop in $\mathbb{R}^{2N}$

The reduced choreographic loops defined in the last subsection could lift to orbits in the original phase space. If  $\mathbf{z}(t) \in \Lambda_{N-1}$  is a reduced choreographic loop and let  $\mathbf{Z}$  be its lifting to  $\Lambda$ . Then there exists a rotation  $g \in SO(2)$  of angle  $\alpha$  s.t.  $\mathbf{Z}(0) = g \tilde{\sigma} \mathbf{Z}(\frac{2\pi}{N})$ . Take a frame of reference which rotates continuously (possibly non-uniformly) by the angle  $\alpha$  during a time interval of length  $2\pi/N$ , and then continue the rotation of the frame by making its rotation velocity  $2\pi/N$ -periodic. Then  $Z$  is simple choreographic in this frame, thanks to the rotational invariance of the Hamiltonian and to the uniqueness of integral curves through a point. We thus define the following objects:

**Definition 3.1.3.** We call a curve  $\mathbf{Z}(t) \in \mathcal{C}([0, 2\pi], \mathbb{R}^{2N})$

- a **relative choreographic loop**, if  $\mathbf{Z}(t)$  is a lifting of a reduced choreographic loop  $\mathbf{z}(t) \in \Lambda_{N-1}$ .  $\mathbf{Z}(t)$  is **non-trivial** if  $\mathbf{z}(t)$  is non-trivial;

- a **centred relative choreographic loop**, if  $\mathbf{Z}(t)$  is a lifting of a reduced choreographic loop  $\mathbf{w}(t) \in \Lambda_{N-2}$ .  $\mathbf{Z}(t)$  is **non-trivial** if  $\mathbf{w}(t)$  is non-trivial;

From now on, to simplify the symbols and discussion, we make the following convention. Let  $k \in \{N-1, N-2\}$  and  $\sigma : \mathbb{C}\mathbb{P}^k \rightarrow \mathbb{C}\mathbb{P}^k$  s.t.

$$\sigma \mathbf{z} := \sigma_1 \mathbf{z}, \quad \text{if } \mathbf{z} \in \mathbb{C}\mathbb{P}^{N-1}; \quad (3.9)$$

$$\sigma \mathbf{z} := \sigma_2 \mathbf{z}, \quad \text{if } \mathbf{z} \in \mathbb{C}\mathbb{P}^{N-2} \quad (3.10)$$

By considering the standard symplectic structure  $\Omega = \sum_{i=1}^N dp_i \wedge dq_i$ , we see clearly that  $\tilde{\sigma} : \mathbb{R}^{2N} \rightarrow \mathbb{R}^{2N}$  is a symplectic transformation, i.e.  $\tilde{\sigma}^* \Omega = \Omega$ . Now consider the natural symplectic form  $\omega$  induced on  $\mathbb{C}\mathbb{P}^k$ .

**Lemma 3.1.2.** *The map  $\sigma : \mathbb{C}\mathbb{P}^k \rightarrow \mathbb{C}\mathbb{P}^k$  is both holomorphic and symplectic.*

*Proof.* We prove that  $\sigma$  is holomorphic and symplectic in details for  $k = N-1$ . Similar argument works for  $k = N-2$ .

First,  $\tilde{\sigma}$  could be seen as an invertible linear transformation of  $\mathbb{C}^N$ , hence  $\sigma$  is holomorphic. Next we show that  $\sigma$  is a symplectic transformation. Consider

$$\mathbb{C}\mathbb{P}^{N-1} \xleftarrow{\pi} \mathbb{S}^{2N-1} \xrightarrow{i} \mathbb{R}^{2N}$$

The symplectic form  $\omega$  is defined by  $\pi^* \omega = i^* \Omega$ , where  $\pi$  is natural projection and  $i$  the natural inclusion. Now consider  $v_1, v_2 \in T_z \mathbb{C}\mathbb{P}^{N-1}$ , which are equivalent classes of  $T\mathbb{S}^{2N-1}$  taking quotient of the symmetry. Taking thus  $\mathbf{Z} \in \mathbb{S}^{2N-1}$  s.t.  $\pi(\mathbf{Z}) = \mathbf{z}$  and  $V_1, V_2 \in T_{\mathbf{Z}} \mathbb{S}^{2N-1}$  be their representatives. It follows from the defining equation of  $\omega$  that there exists  $g \in \mathbf{SO}(2)$  s.t.

$$\sigma^* \omega(v_1, v_2) = \omega(\sigma_* v_1, \sigma_* v_2) = \Omega(\tilde{\sigma}_* g V_1, \tilde{\sigma}_* g V_2) = \tilde{\sigma}^* \Omega(V_1, V_2) = \Omega(V_1, V_2) = \omega(v_1, v_2)$$

The action of  $g$  is a diagonal action, and the second equality is due to the diagram (3.1.2) while the third equality is true because the action of  $g$  is in fact a symplectic transformation. The proof for the case  $k = N-2$  is similar, by using the above argument and taking into account that the Lim transformation  $f : \mathbb{C}^N \rightarrow \mathbb{C}^N$  is linear and symplectic (so is  $f^{-1}$ ).  $\square$

### 3.2 Choreographic Holomorphic Spheres in Reduced Phase space

Our aim is to find non-trivial reduced choreography which are integral curves of the System-II. Such a loop  $\mathbf{z}(t) \in \mathbb{C}\mathbb{P}^{n-1}$  (resp.  $\mathbf{w}(t) \in \mathbb{C}\mathbb{P}^{n-2}$ ) possesses lifts  $\mathbf{Z}(t)$  solving the original System-I; such lifts are obtained by mere quadrature, as can be checked by switching to local coordinate systems in  $\mathbb{R}^2$  which are adapted to the reduction by rotations( resp. rotations and translations). These lifted orbits  $\mathbf{Z}(t)$  are non-trivial relative choreographies (resp. non-trivial centred relative choreographies) of the original System-I.

Searching non-constant periodic solutions on a hyper-surface is closely related to the conjecture of Weinstein. The proof of this conjecture when the underlying symplectic manifold is complex projective space been done by Hofer and Viterbo [51]. They studied the Hamiltonian perturbed  $J$ -holomorphic spheres, which satisfy a nonlinear partial differential equation (PDE) of Cauchy-Riemann type. This PDE could be seen as a zero section of a fiber bundle. Now, our original Hamiltonian is symmetric with respect to permutation of particles, and this symmetry will be heritaged by the PDE. Our aim is to take the reduced choreographic symmetry into the construction of the fiber bundle. Once this is done, the fact that the Riemannian metric on the fiber bundle is invariant under the induced symmetry implies that the PDE has a symmetric weak solution, and the elliptic regularity applies to show it is a classical solution. From that point, one can continue with the analysis given in [51] and conclude the existence of a reduced choreography for the Hamiltonian system. To this end, we will define and study holomorphic spheres having a choreographic symmetry. For a systematic investigation of  $J$ -holomorphic curves, we refer to [45, 68, 10].

Let  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$  be the the Riemann sphere and  $(M, \omega)$  be a symplectic manifold. Let  $J$  be an almost complex structure calibrated by  $\omega$  ( $J$  and  $\omega$  are also said to be compatible), meaning that the symplectic structure twisted by  $J$ ,

$$(x, y) \mapsto \omega(x, Jy),$$

is a Riemannian metric. A holomorphic sphere in  $M$  is a smooth map  $u : \hat{\mathbb{C}} \rightarrow M$  s.t.

$$J \circ Tu = Tu \circ i \tag{3.11}$$

Now in particular let  $M$  be  $\mathbb{C}\mathbb{P}^k$ . This is indeed a complex manifold with standard complex structure  $i$ . We denote by  $J_0$  the regular almost complex structure induced by  $i$ . Note that by a reparametrisation of the augmented complex plane, a holomorphic sphere, after



taking a cylinder parametrisation of  $\hat{\mathbb{C}}$ , can be written as a map  $v(s, t) = u \circ \phi$ , where  $\phi(s, t) = \exp(s + it)$ ,  $-\infty \leq s \leq \infty, 0 \leq t < 2\pi$ . Let  $(\tau u)(z) = u \circ \phi(s, t + \frac{2\pi}{N})$ . Note that  $t + \frac{2\pi}{N}$  is to be understood as  $t + \frac{2\pi}{N} \pmod{2\pi}$ . Sometimes we also denote  $\tau$  by letting

$$\tau : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}, \quad z \xrightarrow{\tau} e^{i\frac{2\pi}{N}} z \quad (3.12)$$

The somehow abused notion  $\tau$  should not bring any ambiguity. It is to be understood as a translation of time for  $t$  variable in our cylinder parametrisation, thus coincides with the definition before.

**Definition 3.2.1.** *A holomorphic sphere  $u$  in  $\mathbb{C}\mathbb{P}^k$  is **choreographic** if*

$$u \circ \tau = \sigma \circ u.$$

In other words, if  $u$  is a choreographic holomorphic sphere in  $\mathbb{C}\mathbb{P}^{N-1}$  (resp.  $\mathbb{C}\mathbb{P}^{N-2}$ ), then for each fixed  $s \in \mathbb{R}$ ,  $z(t) := u(s, t)$  is a reduced choreographic loop (resp. a centred reduced choreographic loop).

### 3.3 Choreographic Fiber Bundle

#### Base Manifold

Next given  $\alpha$  a  $\omega$ -minimal free homotopy class<sup>1</sup>, we consider the Hilbert Manifold  $\mathcal{B}$

$$\mathcal{B} = \{u \in H^{2,2}(\hat{\mathbb{C}}, \mathbb{C}\mathbb{P}^k) \mid [u] = \alpha, u(0) = P_0, u(\infty) = P_\infty, \int_{\|z\| \leq 1} u^* \omega = \frac{1}{2} \langle \omega, \alpha \rangle\} \quad (3.13)$$

**Proposition 3.3.1.** *Let  $G = \langle g \rangle$  be the cyclic group generated by  $g$ , where  $gu = (\sigma \circ \tau^{-1})u$  and let  $\mathcal{B}_G = \text{Fix}_G(\mathcal{B})$  be the  $G$ -invariant subset of  $\mathcal{B}$ . If  $\mathcal{B}_G \neq \emptyset$ , then  $\mathcal{B}_G$  is itself a (totally geodesic) Hilbert sub-manifold.*

*Proof.* According to lemma 3.1.2  $\sigma$  is a symplectic transformation, and  $\mathbb{C}\mathbb{P}^k$  is a Kähler manifold, hence  $g$  induces an isometry in the Hilbert manifold  $\mathcal{B}$  and by applying Palais' principle [80] we see that  $\mathcal{B}_G$  is a totally geodesic Hilbert sub-manifold.  $\square$

<sup>1</sup>This means that

$$0 < \langle \omega, \alpha \rangle = \inf\{\langle \omega, [u] \rangle \mid u \text{ is a non-constant } J_0\text{-holomorphic sphere}\}.$$

Actually, for the case of  $\mathbb{C}\mathbb{P}^k$ , it is easy to see that this class is 1.

**Remark 3.3.1.** *The normalization condition is satisfied because  $\int_{\|z\|\leq 1} u^* \omega = \int_{\|z\|\leq 1} (gu)^* \omega$ . By passing  $s \rightarrow \pm\infty$  in the cylinder parametrisation, one sees from the definition of  $\mathcal{B}$  that a necessary condition for  $\mathcal{B}_G \neq \emptyset$  is that*

$$\sigma P_0 = P_0, \quad \sigma P_\infty = P_\infty. \quad (3.14)$$

Later on in lemma 3.5.1 and 3.5.2 it will turn out that this condition is somehow sufficient too.

We will take  $\mathcal{B}_G$  as our base space and construct a fiber bundle on it in the usual way while take the choreographic symmetry into the frame.

### 3.3.1 Choreographic Fiber and Section

Let  $X_{J_0}$  contains all the complex anti-linear map  $\phi : T_z \hat{\mathbb{C}} \rightarrow T_v \mathbb{C}\mathbb{P}^k$ , i.e.

$$\begin{array}{ccc} T_z \hat{\mathbb{C}} & \xrightarrow{-i} & T_z \hat{\mathbb{C}} \\ \phi \downarrow & & \phi \downarrow \\ T_v \mathbb{C}\mathbb{P}^k & \xrightarrow{J_0} & T_v \mathbb{C}\mathbb{P}^k \end{array} \quad (D1)$$

Denote  $X_{J_0}^G \subset X_{J_0}$  the subset that furthermore satisfies the condition

$$\begin{array}{ccc} T_z \hat{\mathbb{C}} & \xrightarrow{d\tau} & T_{\tau z} \hat{\mathbb{C}} \\ \phi \downarrow & & \phi \downarrow \\ T_v \mathbb{C}\mathbb{P}^k & \xrightarrow{d\sigma} & T_{\sigma v} \mathbb{C}\mathbb{P}^k \end{array} \quad (D2)$$

Here  $d\tau$  and  $d\sigma$  are the push forward of tangent vector, and the commuted diagram (D2) is for being consistent with simple choreography.

For  $\forall u \in \mathcal{B}$ , consider the pull back fiber bundle induced by the graph map  $\bar{u}(z) = (z, u(z))$ , i.e.,

$$\begin{array}{ccc} \bar{u}^* X_{J_0} & \longrightarrow & X_{J_0} \\ \pi \downarrow & & \pi \downarrow \\ \hat{\mathbb{C}} & \xrightarrow{z \rightarrow (z, u(z))} & \hat{\mathbb{C}} \times \mathbb{C}\mathbb{P}^k \end{array}$$

Finally define the symmetric fiber bundle  $\mathcal{E} \rightarrow \mathcal{B}$

$$\mathcal{E} = \bigcup_{u \in \mathcal{B}} \{u\} \times H^{1,2}(\bar{u}^* X_{J_0}) \quad (3.15)$$

similarly define

$$\mathcal{E}^G = \bigcup_{u \in \mathcal{B}_G} \{u\} \times H^{1,2}(\bar{u}^* X_{J_0}^G) \quad (3.16)$$

**Lemma 3.3.1.**  $\bar{\partial}_{J_0} u = du + J_0 \circ du \circ i$  is a smooth section of  $\mathcal{E}^G \rightarrow \mathcal{B}_G$

*Proof.* It is well known that  $\bar{\partial}_{J_0} u$  is smooth section seen as  $\mathcal{E} \rightarrow \mathcal{B}$ . We only need to verify that the diagram (D2) commutes when  $\phi = \bar{\partial}_{J_0} u$ . Actually, since  $u(\tau z) = \sigma u(z)$ , one sees that for  $\eta \in T_z \hat{\mathbb{C}}$

$$d_{\tau z} u \circ d_z \tau(\eta) = d_{u(z)} \sigma \circ d_z u(\eta) \quad (3.17)$$

Now since  $\tau : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  and  $\sigma : \mathbb{C}\mathbb{P}^k \rightarrow \mathbb{C}\mathbb{P}^k$  are holomorphic maps,

$$J_0 \circ d_{\tau z} u \circ i \circ d_z \tau(\eta) = J_0 \circ d_{\tau z} u \circ d_z \tau \circ i(\eta) \quad (3.18)$$

$$d_{u(z)} \sigma \circ J_0 \circ d_z u \circ i(\eta) = J_0 \circ d_{u(z)} \sigma \circ d_z u \circ i(\eta) \quad (3.19)$$

Putting (3.17) into right hand side of (3.18) and (3.19), one sees that  $d\sigma \circ \bar{\partial}_{J_0} u = \bar{\partial}_{J_0} u \circ d\tau$ .  $\square$

This lemma justifies in particular that the zero section corresponds to the class of choreographic holomorphic spheres in our setting.

## 3.4 Choreographic Hamiltonian Perturbation

### 3.4.1 Invariant Hamiltonian Under Choreographic Symmetry

Having defined the action of  $\sigma : \mathbb{C}\mathbb{P}^k \rightarrow \mathbb{C}\mathbb{P}^k$ , in this subsection, we first show that if the Hamiltonian is in System-I is symmetric with the permutation of  $\tilde{\sigma}$ , then the reduced Hamiltonian system is invariant under relative choreographic symmetry. Note that in the case of the N-vortex problem (either from Euler equation or from Gross-Pitaevskii equation), this is indeed true when all the vorticities are identical.

**Lemma 3.4.1.** *If  $H_{\mathbb{R}^{2N}}$  is invariant under  $\tilde{\sigma}$ , then  $H$  is invariant under  $\sigma$ .*

*Proof.* According to the diagram (3.1.2),  $\exists \mathbf{Z} \in \mathbb{S}^{2N-1}$  s.t.

$$H(\mathbf{z}) = H_{\mathbb{R}^{2N}}(\mathbf{Z}) = H_{\mathbb{R}^{2N}}(\tilde{\sigma}\mathbf{Z}) = H(\sigma\mathbf{z}) \quad (3.20)$$

□

Now since both the reduced Hamiltonian and the symplectic form on  $\mathbb{C}\mathbb{P}^k$  are invariant under the action of  $\sigma$ , we have proved actually that

**Proposition 3.4.1.** *Suppose that  $H_{\mathbb{R}^{2N}} : \mathbb{R}^{2N} \rightarrow \mathbb{R}$  is a  $\tilde{\sigma}$ -invariant Hamiltonian, meaning that  $H_{\mathbb{R}^{2N}}(\tilde{\sigma}\mathbf{Z}) = H_{\mathbb{R}^{2N}}(\mathbf{Z}), \forall \mathbf{Z} \in \mathbb{R}^{2N}$ . Then the flow  $\phi_H(t)$  of the reduced Hamiltonian on  $\mathbb{C}\mathbb{P}^k$  is  $\sigma$ -invariant, i.e.,*

$$\phi_H^t(\sigma\mathbf{z}) = \sigma\phi_H^t(\mathbf{z}), \forall \mathbf{z} \in \mathbb{C}\mathbb{P}^k.$$

*Proof.* Direct consequence of lemma 3.4.1 and lemma 3.1.2. □

Now let  $H : \mathbb{C}\mathbb{P}^k \rightarrow \mathbb{R}$  be a smooth map satisfying

**Hypothesis 3.4.1.**

$$\begin{aligned} H(\sigma z) &= H(z), \forall z \in \mathbb{C}\mathbb{P}^k \\ H|_{\mathcal{U}(\Sigma_0)} &= h_0 \in \mathbb{R}, H|_{\mathcal{U}(\Sigma_\infty)} = h_\infty \in \mathbb{R} \\ h_0 < h_\infty, \quad h_0 \leq H \leq h_\infty \end{aligned}$$

where  $\mathcal{U}(\Sigma_0)$  and  $\mathcal{U}(\Sigma_\infty)$  are  $\sigma$ -invariant open neighborhood of  $\Sigma_0$  and  $\Sigma_\infty$ , respectively.

**Remark 3.4.1.**  $\mathcal{U}(\Sigma_0)$  and  $\mathcal{U}(\Sigma_\infty)$  can be assumed to be  $\sigma$ -invariant because  $H$  is  $\sigma$ -invariant.

We define  $\bar{h}(z, v) := \phi$  be the unique complex anti-linear map

$$\phi : T_z\hat{\mathbb{C}} \rightarrow T_v\mathbb{C}\mathbb{P}^k, \phi(z) = \begin{cases} 0, z \in \{0, \infty\} \\ \frac{1}{2\pi}H'(v) \end{cases}$$

Let  $h(u)(z) = \bar{h}(z, u(z))$ . The following lemma shows that, if in particular  $u \in \mathcal{B}_G$ , then  $h(u)(z)$  will respect the choreographic symmetry

**Lemma 3.4.2.**  $h(u)$  is a section from  $\mathcal{B}_G$  to  $\mathcal{E}^G$ .

*Proof.* Clearly  $h(u)(z)$  is in  $\mathcal{E}$ . Now for  $z$ , since  $u$  is a choreographic holomorphic sphere and that  $H(u) = H(\sigma u)$ . Suppose  $\eta \in T_z\hat{\mathbb{C}}$ , then there exists a unique  $\lambda \in \mathbb{C}$  s.t.  $\eta = \lambda z$ . Now we see that

$$\phi_{\tau z}(\tau(\eta)) = \phi_{\tau z}(\tau(\lambda z)) = \phi_{\tau z}((\lambda \tau z)) = \bar{\lambda}\phi_{\tau z}(\tau z) = \bar{\lambda}d\sigma(\phi_z(z)) = d\sigma(\bar{\lambda}\phi_z(z)) = d\sigma(\phi_z(\lambda z))$$

where the fourth equality is due to proposition 3.4.1, i.e.,

$$\phi_{\tau z}(\tau z) = \frac{1}{2\pi} \nabla H(u(\tau z)) = \frac{1}{2\pi} \nabla H(\sigma u(z)) = \frac{1}{2\pi} d\sigma \nabla H(u(z)) = d\sigma(\phi_z(z))$$

In other words, we have verified that if  $u \in \mathcal{B}_G$  then  $h(u) \in \mathcal{E}^G$ .  $\square$

Let

$$f_\lambda(u) = \bar{\partial}_{J_0} u + \lambda h(u)$$

Our aim is to study the parameter depending family of smooth sections  $f_\lambda^G(u) : \mathcal{E}^G \rightarrow \mathcal{B}_G$  defined by

$$f_\lambda^G(u) = f_\lambda(u)|_{\mathbb{R}_{\geq 0} \times \mathcal{B}_G}$$

Note that in general, for  $u \in \mathcal{B}$  or  $H$  that is not  $\sigma$ -invariant (hence  $h(u)$  is no longer a section from  $\mathcal{B}_G$  to  $\mathcal{E}^G$ .)  $f_\lambda(u)$  can still be seen a section  $\mathcal{E} \rightarrow \mathcal{B}$ . We define moreover the sets of pairs

$$\mathcal{C} = \{(\lambda, u) \in \mathbb{R}_{\geq 0} \times \mathcal{B} \mid f_\lambda(u) = 0\} \quad (3.21)$$

$$\mathcal{C}_G = \{(\lambda, u) \in \mathbb{R}_{\geq 0} \times \mathcal{B}_G \mid f_\lambda^G(u) = 0\} \quad (3.22)$$

We will also denote by  $\mathcal{C}(\lambda)$  a slice of  $\mathcal{C}$ , and  $\mathcal{C}_G(\lambda)$  a slice of  $\mathcal{C}_G$ , i.e.

$$\mathcal{C}(\lambda) = \{u \in \mathcal{B} \mid f_\lambda(u) = 0\} \quad \mathcal{C}_G(\lambda) = \{u \in \mathcal{B}_G \mid f_\lambda^G(u) = 0\} \quad (3.23)$$

In particular,  $\mathcal{C}(0)$  is the set of normalized holomorphic spheres of homotopy class  $\alpha$  with two ends in  $\Sigma_0$  and  $\Sigma_\infty$ . By assuming that the homotopy class  $\alpha$  is  $\sigma$ -minimal, we show in the next section that when  $\Sigma_0$  and  $\Sigma_\infty$  are chosen to be two special points, one has  $\mathcal{C}(0) = \mathcal{C}_G(0)$ .

### 3.5 Well Posedness Of Choreographic Holomorphic Sphere

So far we have constructed  $\mathcal{B}_G, \mathcal{C}_G$  in an abstract manner, yet we have not answered some essential questions. For example, are there non-empty choreographic holomorphic spheres, i.e., whether  $\mathcal{C}_G(0)$  is not empty? If yes, then does there exists  $\lambda > 0$  s.t.  $\mathcal{C}_G(\lambda)$  is not empty? In this subsection, we justify the well posedness of these notions. First we will study

some special configurations. Then we will show that  $\mathcal{C}_G(0)$  is a compact manifold while  $\mathcal{C}_G$  is not compact. We distinguish the case when  $k = N - 1$  and  $k = N - 2$ .

### 3.5.1 Well Posedness of Choreographic Holomorphic Sphere

So far we have constructed  $\mathcal{B}_G, \mathcal{C}_G$  in an abstract manner, yet we have not answered some essential questions. For example, are there non-empty choreographic holomorphic spheres, i.e., whether  $\mathcal{C}_G(0)$  is not empty? In this sub-section we distinguish the two cases when  $k = N - 1$  and  $k = N - 2$  relatively and we justify the well posedness of these notions by explicit calculation. It has already been mentioned in remark 3.3.1 that the two ends must be carefully chosen. It turns out that this is actually enough.

#### Special configurations in $\mathbb{C}\mathbb{P}^{N-1}$

Let us consider two configurations in  $\mathbb{C}\mathbb{P}^{N-1}$ , denoted by  $A$  and  $B$  respectively, such that

$$\begin{aligned} A &= [1 : 1 : 1 : \dots, 1 : 1] && \text{(total collision)} \\ B &= [e^{i\frac{2\pi}{N}} : e^{i\frac{4\pi}{N}} : e^{i\frac{6\pi}{N}} : \dots : e^{i\frac{2\pi}{N}(N-1)} : 1] && \text{(N-polygon)} \end{aligned}$$

We call  $A$  the **total collision configuration**, and  $B$  the **N-polygon configuration**. Note that they are both  $\sigma$ -invariant. Assume that  $P_0 = B$  and  $P_\infty = A$ ,

**Lemma 3.5.1.** *All the simple holomorphic spheres  $u : \hat{\mathbb{C}} \rightarrow \mathbb{C}\mathbb{P}^{N-1}$  s.t.  $u(0) = B$  the N-polygon configuration and  $u(\infty) = A$  the total collision configuration are choreographic holomorphic spheres.*

*Proof.* Consider  $\hat{\mathbb{C}}$  with the complex projective line  $\mathbb{C}\mathbb{P}^1$  by identifying  $z \in \hat{\mathbb{C}}$  with  $[z : 1] \in \mathbb{C}\mathbb{P}^1$ . Suppose that  $[\eta_A : \eta_B] = [z : 1]$ , and define a holomorphic sphere  $u : \hat{\mathbb{C}} \rightarrow \mathbb{C}\mathbb{P}^{N-1}$  by

$$u(z) = u([\eta_A : \eta_B]) = [\eta_A + \eta_B e^{i\frac{2\pi}{N}} : \eta_A + \eta_B e^{i\frac{4\pi}{N}} : \eta_A + \eta_B e^{i\frac{6\pi}{N}} : \dots : \eta_A + \eta_B] \quad (3.24)$$

By explicit calculation

$$\begin{cases} u(0) = u([0 : 1]) = B \\ u(\infty) = u([1 : 0]) = A. \end{cases}$$

Then for  $-\infty < r < \infty$ ,

$$\begin{aligned} u(\tau z) &= u(\exp(r + i(t + \frac{2\pi}{N}))z) = u(\exp(i\frac{2\pi}{N})z) = u([e^{i\frac{2\pi}{N}}\eta_A : \eta_B]) \\ &= [\eta_A e^{i\frac{2\pi}{N}} + \eta_B e^{i\frac{2\pi}{N}} : \eta_A e^{i\frac{2\pi}{N}} + \eta_B e^{i\frac{4\pi}{N}} : \eta_A e^{i\frac{2\pi}{N}} + \eta_B e^{i\frac{6\pi}{N}} : \dots : \eta_A e^{i\frac{2\pi}{N}} + \eta_B] \\ &= \sigma u(z) \end{aligned}$$

As a result,  $gu = u$ . Next, suppose that  $v : \hat{\mathbb{C}} \rightarrow \mathbb{C}\mathbb{P}^{N-1}$  is another simple holomorphic sphere running through  $A$  and  $B$  of the same homotopy class. By calculate the Gromov-Witten invariant if necessary (see for example [68, chapter 7]), one sees that  $v(\hat{\mathbb{C}}) = u(\hat{\mathbb{C}})$ , as a result there exists then a Möbius transformation  $\phi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  s.t.  $v(z) = u(\phi(z))$  and  $v(0) = B, v(\infty) = A$ , it follows that  $v(z) = u(\zeta z)$  for some non-zero  $\zeta \in \mathbb{C}$ . This implies

$$\tau v(z) = \tau u(\zeta z) = u(\tau \zeta z) = \sigma u(\zeta z) = \sigma v(z) \quad (3.25)$$

Hence  $v$  is clearly choreographic.  $\square$

### Special Configurations in $\mathbb{C}\mathbb{P}^{N-2}$

When it comes to the case  $V = cst$  in System-I, the reduced phase space is  $\mathbb{C}\mathbb{P}^{N-2}$ . The situation is slightly more complicated. We cannot use the total collision point any longer, because  $P(\mathbf{Z}) = Q(\mathbf{Z}) = 0$  and  $z_i = z_j, 1 \leq i < j \leq N$  implies that  $\mathbf{Z} = 0$ . Thus the total collision configuration does not exist on  $\mathbb{C}\mathbb{P}^{N-2}$ . On the other hand, if we give up the reduction of translation, we cannot exclude the triviality later on (this point will become more clear in section 3.6).

In this sub-section we make an extra assumption that  $N = 2m$  is an even integer. Let us consider two points  $\mathbf{Z}_A$  and  $\mathbf{Z}_B$  in  $\mathbb{R}^{2N}$  s.t.

$$\mathbf{Z}_A = (e^{i\frac{2\pi}{m}}, e^{i\frac{4\pi}{m}}, \dots, 1, e^{i\frac{2\pi}{m}}, e^{i\frac{4\pi}{m}}, \dots, 1) \quad (3.26)$$

$$\mathbf{Z}_B = (e^{i\frac{2\pi}{N}}, e^{i\frac{4\pi}{N}}, e^{i\frac{6\pi}{N}}, \dots, e^{i\frac{2\pi(N-1)}{N}}, 1) \quad (3.27)$$

Note that these two points are centred, hence after Lim's coordinate transformation  $\mathbf{W} = f(\mathbf{Z})$ , they become two points

$$\mathbf{W}_A = (w_1(\mathbf{Z}_A), w_2(\mathbf{Z}_A), \dots, w_{N-1}(\mathbf{Z}_A), (0, 0)) \in \mathbb{R}^{2N} \quad (3.28)$$

$$\mathbf{W}_B = (w_1(\mathbf{Z}_B), w_2(\mathbf{Z}_B), \dots, w_{N-1}(\mathbf{Z}_B), (0, 0)) \in \mathbb{R}^{2N} \quad (3.29)$$

They thus pass to two configurations in  $\mathbb{C}\mathbb{P}^{N-2}$ , denoted as  $A$  and  $B$

$$\begin{aligned} A &= [w_1(\mathbf{Z}_A) : w_2(\mathbf{Z}_A) : \dots : w_{N-1}(\mathbf{Z}_A)] && \text{(binary total collision)} \\ B &= [w_1(\mathbf{Z}_B) : w_2(\mathbf{Z}_B) : \dots : w_{N-1}(\mathbf{Z}_B)] && \text{(N-polygon)} \end{aligned}$$

We call  $A$  the **binary total collision configuration**, and  $B$  the **N-polygon configuration**. Assume that  $P_0 = A$  and  $P_\infty = B$ .

**Lemma 3.5.2.** *All the simple holomorphic spheres  $u : \hat{\mathbb{C}} \rightarrow \mathbb{C}\mathbb{P}^{N-2}$  s.t.  $u(0) = A$  the binary collision configuration and  $u(\infty) = B$  the N-polygon configuration are choreographic holomorphic spheres.*

*Proof.* Consider  $\hat{\mathbb{C}}$  with the complex projective line  $\mathbb{C}\mathbb{P}^1$  by identifying  $z \in \hat{\mathbb{C}}$  with  $[z : 1] \in \mathbb{C}\mathbb{P}^1$ . Suppose that  $[\eta_B : \eta_A] = [z : 1]$ , and define a holomorphic sphere  $u : \hat{\mathbb{C}} \rightarrow \mathbb{C}\mathbb{P}^{N-2}$  by

$$\begin{aligned} u(z) = u([\eta_B : \eta_A]) &= [\eta_B w_1(\mathbf{Z}_B) + \eta_A w_1(\mathbf{Z}_A) : \\ &\quad \eta_B w_2(\mathbf{Z}_B) + \eta_A w_2(\mathbf{Z}_A) : \\ &\quad \eta_B w_3(\mathbf{Z}_B) + \eta_A w_3(\mathbf{Z}_A) : \dots : \\ &\quad \eta_B w_{N-1}(\mathbf{Z}_B) + \eta_A w_{N-1}(\mathbf{Z}_A)] \end{aligned} \quad (3.30)$$

By the definition of  $\sigma$ , we see that

$$u(\tau z) = u([e^{i\frac{2\pi}{N}} \eta_B : \eta_A]) \quad (3.31)$$

$$\begin{aligned} &= [e^{i\frac{2\pi}{N}} \eta_B w_1(\mathbf{Z}_B) + \eta_A w_1(\mathbf{Z}_A) : \dots : e^{i\frac{2\pi}{N}} \eta_B w_{N-1}(\mathbf{Z}_B) + \eta_A w_{N-1}(\mathbf{Z}_A)] \\ &= [w_1(e^{i\frac{2\pi}{N}} \eta_B \mathbf{Z}_B + \eta_A \mathbf{Z}_A) : \dots : w_{N-1}(e^{i\frac{2\pi}{N}} \eta_B \mathbf{Z}_B + \eta_A \mathbf{Z}_A)] \end{aligned} \quad (3.32)$$

Now one verifies easily that

$$\begin{aligned} e^{i\frac{2\pi}{N}} \eta_B \mathbf{Z}_B + \eta_A \mathbf{Z}_A &= (\eta_B e^{i\frac{4\pi}{N}} + \eta_A e^{i\frac{2\pi}{m}}, \eta_B e^{i\frac{6\pi}{N}} + \eta_A e^{i\frac{4\pi}{m}}, \\ &\quad \dots, \eta_B e^{i\frac{2(m+1)\pi}{N}} + \eta_A, \eta_B e^{i\frac{2(m+2)\pi}{N}} + \eta_A e^{i\frac{2\pi}{m}}, \\ &\quad \dots, \eta_B + \eta_A e^{i\frac{(N-1)\pi}{m}}, e^{i\frac{2\pi}{N}} \eta_B + \eta_A) \\ &= e^{i\frac{2\pi}{m}} \tilde{\sigma}(\eta_B \mathbf{Z}_B + \eta_A \mathbf{Z}_A) \end{aligned}$$

Thus

$$\begin{aligned} u(\tau z) &= [w_1(e^{i\frac{2\pi}{m}} \tilde{\sigma}(\eta_B \mathbf{Z}_B + \eta_A \mathbf{Z}_A)) : \dots : w_{N-1}(e^{i\frac{2\pi}{m}} \tilde{\sigma}(\eta_B \mathbf{Z}_B + \eta_A \mathbf{Z}_A))] \\ &= [w_1(\tilde{\sigma}(\eta_B \mathbf{Z}_B + \eta_A \mathbf{Z}_A)) : \dots : w_{N-1}(\tilde{\sigma}(\eta_B \mathbf{Z}_B + \eta_A \mathbf{Z}_A))] \end{aligned} \quad (3.33)$$



Now by the definition of the action  $\sigma$  for  $\mathbb{C}\mathbb{P}^{N-2}$  (see diagram (3.1.3)), one sees that

$$u(\tau z) = \sigma u(z) \quad (3.34)$$

The rest of the proof is the same as that in lemma 3.5.1.  $\square$

### The Compactness of $\mathcal{C}_G(0)$

Let  $\alpha$  be the  $\omega$ -minimal class, and

$$\begin{aligned} \mathcal{H}(\alpha, J_0, P_0, P_\infty) = \{u \in \mathcal{C}^\infty[\hat{\mathbb{C}}, \mathbb{C}\mathbb{P}^k] \mid \\ [u] = \alpha, \\ u(0) = P_0 \in \mathbb{C}\mathbb{P}^k, \\ u(\infty) = P_\infty \in \mathbb{C}\mathbb{P}^k, \\ \int_{\|z\| \leq 1} u^* \omega = \frac{1}{2} \langle \omega, \alpha \rangle, \\ \bar{\partial}_{J_0} u = 0 \} \end{aligned} \quad (3.35)$$

Then lemma 3.5.1 and 3.5.2 actually imply that

**Proposition 3.5.1.** *Let  $P_0, P_\infty$  be chosen as in lemma 3.5.1 and in lemma 3.5.2 respectively, and let  $\mathcal{H}(\alpha, J_0, P_0, P_\infty)$  be defined as above. Then  $\mathcal{C}_G(0)$  is a  $\mathbb{S}^1$ -invariant compact manifold.*

*Proof.* By lemma 3.5.1 and lemma 3.5.2 one sees that for such specially chosen configurations

$$\mathcal{C}(0) = \mathcal{C}_G(0) = \mathcal{H}(\alpha, J_0, P_0, P_\infty) \quad (3.36)$$

It is well known that  $J_0$  is regular and  $\mathcal{C}(0)$  is a  $\mathbb{S}^1$ -invariant compact manifold for arbitrary  $P_0 \neq P_\infty$ . The consequence follows.  $\square$

**Remark 3.5.1.** *Actually from the previous discussion, we see that the simple holomorphic spheres connecting the two points with the normalization condition are homeomorphic to the circle  $\mathbb{S}^1$ , which is of one dimension. This gives information later on when we calculate the index of the Fredholm operator.*

### The non-compactness of $\mathcal{C}_G$

Perhaps the most crucial observation in the work of Hofer and Viterbo in [51] is the non-compactness of  $\mathcal{C}$ , if  $[\mathcal{H}]$ , the  $\mathbb{S}^1$ -free cobordism class of the manifold  $\mathcal{H}$ , were not empty. This together with some asymptotic estimation and the Gromov compactness will permit one to find a periodic solution of System-II, although not necessarily a choreography. More precisely, in [51] the following proposition is proved:

**Proposition 3.5.2.** [51, proposition 2.7] *Let  $(V, \omega)$  be a complex symplectic manifold and  $J$  be a regular almost complex structure calibrated by  $\omega$ ,  $\alpha$  be a  $\omega$ -minimal free homotopy class, and  $\Sigma_0, \Sigma_\infty$  are disjoint closed sub-manifold and  $\mathcal{C}$  is defined as in (3.21). If  $\mathcal{C}$  is compact, then  $[\mathcal{C}(0)] = [\emptyset]$ .*

By adapting ourselves with the symmetric constraints, we would like to show the existence of a reduced simple choreography. To this end we need some non-compactness for the 1-parameter trajectory space. We would like to prove a choreographic symmetric version of this proposition, namely

**Proposition 3.5.3.** *Let  $(\mathbb{C}\mathbb{P}^k, \omega)$  be the standard complex projective space and  $J_0$  be the regular almost complex structure induced by  $i$ . Let  $P_0, P_\infty$  be chosen as in lemma 3.5.1 and 3.5.2,  $\alpha$  be the  $\omega$ -minimal free homotopy class, and  $\mathcal{C}_G$  is defined as in (3.22). If  $\mathcal{C}_G$  is compact, then  $[\mathcal{C}_G(0)] = [\emptyset]$ .*

Postponing the proof for proposition 3.5.3 until the end of this section, first we note that  $f_\lambda^G(u)$ , the restriction of the Fredholm section  $f_\lambda(u)$  on  $\mathcal{C}_G$ , is still a Fredholm section.

**Lemma 3.5.3.**  *$f_\lambda^G(u)$  is a Fredholm section of  $\mathcal{E}^G \rightarrow \mathbb{R}_{\geq 0} \times \mathcal{B}_G$ , meaning that if  $u_\lambda := (\lambda, u) \in \mathcal{C}_G$ , then*

$$df_\lambda^G(u) : T_\lambda \mathbb{R} \times T_u \mathcal{B}_G \rightarrow \mathcal{E}_u^G \quad (3.37)$$

is a Fredholm operator.

*Proof.* It is known that  $df_\lambda(u)$  seen as  $T_\lambda \mathbb{R} \times T_u \mathcal{B} \xrightarrow{df_\lambda(u)} \mathcal{E}_u$  is a Fredholm operator. We can show actually that

$$\text{Ker}(df_\lambda^G(u)) \subset \text{Ker}(df_\lambda(u)) \quad (3.38)$$

To this end, suppose that

$$\begin{aligned} u_\lambda &= (u, \lambda) \in \mathcal{C}_G; \\ \eta_\alpha &= (\alpha, \eta) \in \mathbb{R} \times T_u \mathcal{B}_G \subset \mathbb{R} \times T_u \mathcal{B} \end{aligned}$$

is a tangent vector, then the linearisation  $df_{u_\lambda}^G$  is indeed

$$df_{u_\lambda}^G \eta_\alpha = \frac{d}{d\theta} \mathcal{F}_{u_\lambda}(\theta \eta_\alpha) \quad (3.39)$$

where

$$\mathcal{F}_{u_\lambda}(\eta_\alpha) = \Phi_{u_\lambda}(\eta_\alpha)^{-1} f_{\lambda^*}(u^*)$$

with

$$u_{\lambda^*}^* := (\lambda^*, u^*) = \exp_{u_\lambda}(\eta_\alpha) \text{ is the geodesics on } \mathcal{B}_G$$

$$\Phi_{u_\lambda}(\eta_\alpha) \text{ is the parallel transport of } \eta_\alpha \text{ along the geodesics}$$

However, as we have seen in proposition 3.3.1,  $\mathcal{B}_G$  is a totally geodesic sub-manifold. It turns out that  $u_{\lambda^*}^*$  is also a geodesic on  $\mathcal{B}$ . As a result, if  $u_\lambda \in \mathcal{C}_G$ , then  $df_{u_\lambda}^G$  is the restriction of  $df_\lambda(u) : T_\lambda \mathbb{R} \times T_u \mathcal{B} \rightarrow \mathcal{E}_u$  on  $T_\lambda \mathbb{R} \times T_u \mathcal{B}_G$ . Thus  $\text{Ker}(df_\lambda^G(u)) \subset \text{Ker}(df_\lambda(u))$ . Since  $\text{Ker}(df_\lambda(u))$  is of finite dimension, so is  $\text{Ker}(df_\lambda^G(u))$ .

Next, we show that the  $\text{CoKer}(df_\lambda^G(u))$  is of finite dimensional. To this end, we show that

$$\text{CoKer}(df_\lambda^G(u)) \subset \text{CoKer}(df_\lambda(u))$$

We only needs to show that

$$df_{u_\lambda}^G(\mathbb{R} \times T_u \mathcal{B}_G) = df_\lambda(u)(\mathbb{R} \times T_u \mathcal{B}_G) \cap \mathcal{E}_{u_\lambda}^G \quad (3.40)$$

It is clear that the left hand side is included in the right hand side. We show the other direction. Take a smooth  $\zeta \in df_\lambda(u)(\mathbb{R} \times T_u \mathcal{B}_G) \cap \mathcal{E}_{u_\lambda}^G$ , s.t. there exists  $\eta_\alpha \in \mathbb{R} \times T_u \mathcal{B}_G$  with  $df_\lambda(u)(\eta_\alpha) = \zeta$ . By Sobolev embedding we know that  $\eta_\alpha$  is also continuous. Recall  $gu = \sigma \circ u(\tau^{-1}z)$ . Since  $\zeta \in \mathcal{E}_{u_\lambda}^G$ , one sees that  $dg \circ \zeta = \zeta$ . Moreover, since  $u_\lambda \in \mathcal{B}_G$ , we have  $df_\lambda^G(u)(dg \circ \eta_\alpha) = dg \circ \zeta = \zeta$ . It turns out that by setting  $\bar{\eta}_\alpha = \frac{1}{n} \sum_{l=0}^{n-1} (dg)^l \eta_\alpha \in \mathbb{R} \times T_u \mathcal{B}_G$ , one sees that  $df_\lambda^G(u)(\bar{\eta}_\alpha) = \zeta$ . The lemma is thus proved.  $\square$

Let  $W$  be the Banach space as the completion of  $C_G$  with respect to the  $H^{1,2}$  norm. As before  $u_\lambda := (\lambda, u) \in W$  is the abbreviation for a pair in  $W$ . By the definition of  $W$ , one sees that  $\forall (\lambda, u) \in W, f_\lambda(u) = 0$ . The idea of the proof is to interpret  $\mathcal{C}_G(0)$  as the boundary of a compact manifold. This compact manifold turns out to be the solutions for the perturbed Floer equation (Note that Floer equation is itself a perturbed equation of the nonlinear Cauchy Riemann equation). There are in general two ways to perturb the Floer equation, either on

the almost complex structure or on the Hamiltonian, see for example [68] and [10]. In our case, we will keep the almost complex structure  $J_0$  and we need to make the perturbation coherent with the symmetric constraints. More precisely, we define

**Definition 3.5.1.** *Given the projection map:*

$$P : [0, \lambda_\infty + 1] \times \hat{\mathbb{C}} \times \mathbb{C}\mathbb{P}^k \rightarrow \hat{\mathbb{C}} \times \mathbb{C}\mathbb{P}^k, \quad P(\lambda, z, v) = (z, v)$$

and the pull-back bundle  $P^*X_J^G \rightarrow [0, \lambda_\infty + 1] \times \hat{\mathbb{C}} \times \mathbb{C}\mathbb{P}^k$ . Consider vector spaces  $\mathcal{A}$  of all smooth section  $r(\lambda, z, v)$  of this bundle.

- (1)  $r(\lambda, z, v) = 0$  if  $\lambda$  is close to 0 or  $z$  is close to either 0 or  $\infty$ ;
- (2)  $r(\lambda, z, v) = r(\lambda, \zeta z, v)$ ,  $\forall \zeta \in \mathbb{S}^1$ .

The admissible perturbation space is defined by  $\mathcal{G}_k^G$ , which is the completion of  $\mathcal{A}$  in some Sobolev norm  $\|\cdot\|_{W^{k,2}}$  for some  $k \in \mathbb{N}$  large enough. Define moreover

$$\hat{r} : \mathcal{G}_k^G \times W \rightarrow E^G \tag{3.41}$$

$$\hat{r}_\lambda(u)(z) = r(\lambda, z, u(z)). \tag{3.42}$$

**Remark 3.5.2.** *Note that then  $k$  is large enough  $\mathcal{G}_k$  is embedded in to continuous sections, thus the symmetric constraint is well defined for  $\mathcal{G}_k$ .*

We first give some lemmas. Suppose that  $\mathcal{C}_G$  is compact. Let  $u_{\lambda_k} \in W$  s.t.  $f_{\lambda_k}(u_k) \rightarrow 0$  in  $W$ . Since  $f$  is locally proper, there exists  $(\lambda, u)$  s.t.  $f_\lambda(u) = 0$ . Since in our case the spheres are simple (due to the  $\omega$ -minimal constraints on the free homotopy group), we have the following ‘‘somewhere’’ injectivity and the unique continuation properties

**Lemma 3.5.4.** *Let  $(\lambda, u) \in W$ , then there exists  $\varepsilon$  small enough (depending on  $u$ ) and  $z_0 \in \hat{\mathbb{C}}$ , s.t.*

$$\begin{aligned} |z_0| &= \varepsilon \\ Tu(z_0) &\rightarrow 0 \\ u^{-1}(u(z_0)) \cap \{|z| = \varepsilon\} &= \{z_0\} \end{aligned}$$

For proofs and more details of these properties, one could turn to [51, 68, 10]. Next, we consider the perturbed Floer equation:

$$F : \mathcal{G}_{k,\delta}^G \times W \rightarrow \mathcal{E}^G, \quad F(r, \lambda, u) = f_\lambda^G(u) + \hat{r}_\lambda(u). \tag{3.43}$$

The following lemma aims to show that  $F^{-1}(0)$  could be equipped with the structure of a manifold.

**Lemma 3.5.5.**  $dF|_{(0,\lambda,u)}$  is onto.

*Proof.* First we show that its image is dense.

$$dF|_{(0,\lambda,u)}(a,0,b) = df_\lambda(u)b + \hat{a}_\lambda(u) \quad (3.44)$$

Suppose  $c \neq 0$ , we discuss two possibilities.

1. if  $c \in df_\lambda^G(T_u\mathcal{B}_G)$ , then setting  $a = 0$ , we see that  $\exists b \in \mathcal{B}_G$  s.t.  $\langle df_\lambda^G(u)b, c \rangle \neq 0$ .
2. if  $c \in \text{Coker}(f_\lambda)$ , then since  $df_\lambda(u)$  is Fredholm, its cokernel is of finite dimension. As a result, take  $c \in H^{1,2}(u^*X_{j_0}^G)$ , s.t.

$$\langle df_\lambda(u)b, c \rangle_{L^2} = 0, \forall b \in T_u\mathcal{B}_G \quad (3.45)$$

Then by using symmetry and the ellipticity of the adjoint operator, one sees that  $c$  will not vanish on some nonempty open set of  $\mathbb{S}^2$ . We can then choose  $a \in \mathcal{G}_k$  s.t.  $\langle \hat{a}_\lambda(u), c \rangle_{L^2} \neq 0$

We have actually shown that

$$\langle v, c \rangle_{L^2} = 0 \quad \forall v \in dF|_{(0,\lambda,u)}(\mathcal{G}_k^G \times TW) \Rightarrow c = 0 \quad (3.46)$$

It turns out that the image of  $dF|_{(0,\lambda,u)}$  is dense in  $H^{1,2}(u^*X_{j_0}^G)$ . The image is also closed by the form of  $dF$  and the fact that  $f_\lambda : \mathbb{R} \times \mathcal{B}_G \rightarrow \mathcal{E}^G$  is a Fredholm section.  $\square$

Lemma 3.5.5 implies immediately, that

**Corollary 3.5.1.** For small  $\delta$  in  $\mathcal{G}_{\delta,k}^G$ ,  $F^{-1}(0)$  is a  $C^l$  sub-manifold of  $\mathcal{G}_{\delta,k}^G \times W$ , where  $l$  depends on  $k$ .

*Proof.* This is due to the implicit function theorem and the Sobolev embedding theorem.  $\square$

Now we are ready to prove the proposition 3.5.3, with the help of Sard-Smale theorem:

**Theorem 3.5.1 (Sard-Smale, [102]).** Let  $M, N$  be Banach manifold and  $f : M \rightarrow N$  be a  $C^l$  Fredholm map with

$$l > \max\{\text{index}(f), 0\} \quad (3.47)$$

Then the set of regular values of  $f$  is residual.

*Proof.* (of **proposition 3.5.3**) Suppose that  $\mathcal{C}_G$  is compact. Consider the projection map

$$\Pi_1 : F^{-1}(0) \rightarrow a \in \mathcal{G}_{k,\delta}^G \quad (3.48)$$

$$\Pi_2 : F^{-1}(0) \rightarrow a \in W \quad (3.49)$$

Note that the kernel of  $d\Pi_1$  is the kernel of  $df_\lambda(u)$  in  $\mathcal{B}_G$ . Moreover since we have assumed that  $C_G$  is compact,  $\forall a \in \mathcal{G}_{k,\delta}^G$ ,  $\pi_2 \circ \pi_1^{-1}(a)$  is a compact set. We conclude that  $\Pi_1$  is a Fredholm map. By taking  $k$  large enough and  $\delta$  small enough, we can take a regular value of  $\Pi_1$ , namely  $a_0$  in  $\mathcal{G}_{k,\delta}^G$ , thanks to the Sard-Smale theorem. Finally, replace  $a_0$  by an  $a_1$  in  $A$ , we see that

$$M = \{(\lambda, u) \in W | F(a_1, \lambda, u) = 0\} \quad (3.50)$$

is a compact manifold, with  $\mathcal{C}_G(0) = \partial M$ . In other words,  $[\mathcal{C}_G(0)] = [\emptyset]$ .  $\square$

Now proposition 3.5.3 and proposition 3.5.1 together indicate the non-compactness we are looking for:

**Proposition 3.5.4.**  $\mathcal{C}_G$  is not compact.

*Proof.* By proposition 3.5.1, we have seen that if we choose  $P_0$  and  $P_\infty$  in such a special way, then  $\mathcal{C}(0) = \mathcal{C}_G(0)$ . As a result,  $[\mathbb{S}^1] = [\mathcal{C}(0)] = [\mathcal{C}_G(0)]$ . It follows by proposition 3.5.3 that  $\mathcal{C}_G$  is not compact.  $\square$

### Existence of Choreography for Special Hamiltonian

Finally, once a solution for the symmetric invariant manifold is found, after using the elliptic regularity, we see that these solutions are all smooth and they become solutions in classical sense. The estimate for asymptotic behavior of the action functional around  $P_0$  and  $P_\infty$  and the Gromov compactness are thus still valid. In particular we have actually achieved the following result, which is a choreographic analogue of [51, Theorem 1.1] for some special Hamiltonian functions:

**Theorem 3.5.2.** Let  $H : \mathbb{C}\mathbb{P}^k \rightarrow \mathbb{R}$  be a smooth Hamiltonian satisfying:

1.  $H(\sigma z) = H(z), \forall z \in \mathbb{C}\mathbb{P}^k$
2. There exist  $\sigma$ -invariant open neighborhoods  $\mathcal{U}(P_0)$  and  $\mathcal{U}(P_\infty)$  s.t.

$$H|_{\mathcal{U}(P_0)} = h_0 \in \mathbb{R}, H|_{\mathcal{U}(P_\infty)} = h_\infty \in \mathbb{R}$$

3.  $h_0 < h_\infty$ ,  $h_0 \leq H \leq h_\infty$

Then the Hamiltonian system  $\dot{z} = \mathbf{X}_H(z)$  possesses a non-constant  $T$ -periodic reduced choreography  $z^*$ , satisfying

$$h_0 < H(z^*) < h_\infty, \quad T(h_\infty - h_0) < \pi$$

*Proof.* See [51, Theorem 1.1]. □

## 3.6 Simple Relative Choreographies Of Planar Interactive Hamiltonian System

### 3.6.1 Simple Relative Choreography

In this section let us consider the induced Hamiltonian system (System-II) on  $\mathbb{C}\mathbb{P}^k$ . When  $k = N - 2$ , we will assume in addition that  $N$  is even. Our aim is to show that, under mild conditions, the energy levels on which there exist reduced simple choreography form a set of positive Lebesgue measure.

$$\dot{\mathbf{z}}(t) = X_H(\mathbf{z}(t)) = \mathbf{J}\nabla H(\mathbf{z}(t)), \quad \mathbf{z} \in \mathbb{C}\mathbb{P}^k$$

First we show the following simple yet useful lemma on the existence of a  $\sigma$ -invariant component of the energy surface  $S_c = H^{-1}(c)$ :

**Lemma 3.6.1.** *Suppose there is a connected subset  $U \subset S_c$  s.t.  $U$  is  $\sigma$ -invariant, i.e.,  $\sigma U = U$ . If  $H$  is  $\sigma$ -invariant, then  $U$  is contained in a  $\sigma$ -invariant component of  $S_c$ .*

*Proof.* Since  $U$  is connected,  $U$  is included in a component  $S_c^\sigma$  of  $S_c$ . We only need to show this component itself is  $\sigma$ -invariant. To this end, let  $\mathbf{z} \in S_c^\sigma$ . Then  $\exists \mathbf{u} \in U$  and a continuous function  $f: [0, 1] \rightarrow \mathbb{C}\mathbb{P}^k$  s.t.  $f(0) = \mathbf{u}, f(1) = \mathbf{z}$ . As a result, let  $g: [0, 1] \rightarrow \mathbb{C}\mathbb{P}^k$  defined by  $g(t) = \sigma f(t)$ . Clearly  $g$  is a continuous function satisfies that  $g(0) = \sigma \mathbf{u}, g(1) = \sigma \mathbf{z}$ , and  $\forall t \in [0, 1]$ , we have  $H(g(t)) = H(\sigma f(t)) = H(f(t)) = c$ , hence  $g(t) \in S_c, \forall t \in [0, 1]$ . We have thus shown that  $\sigma \mathbf{u}$  and  $\sigma \mathbf{z}$  are connected, hence  $\sigma \mathbf{z} \in S_c^\sigma$  too. We conclude that  $S_c^\sigma$  is the  $\sigma$ -invariant component and the lemma is proved. □

**Hypothesis 3.6.1.** *Assume that the reduced Hamiltonian  $H$  satisfies the following assumptions:*

$$H \text{ is smooth;} \tag{V0}$$

$$H \text{ is } \tilde{\sigma}\text{-invariant, i.e. } H(\mathbf{z}) = H(\tilde{\sigma}\mathbf{z}), \forall \mathbf{z} \in \mathbb{R}^{2N}; \tag{V1}$$

$$H(A) < H(B) \tag{V2}$$

Note that the assumption (V2) does not lose any generality. Because we can otherwise consider  $-H$  instead. Let  $\mu$  be the Lebesgue measure on  $\mathbb{R}$ . As an application of Theorem 3.5.2 we prove the following results:

**Theorem 3.6.1.** *Suppose that  $H$  satisfies (V0)-(V2). Let  $\mathcal{I} = (H(A), H(B))$  be the open interval. Denote*

$$\mathcal{D} = \{c \in \mathcal{I} \mid S_c = H^{-1}(c) \text{ has a } \sigma\text{-invariant connected component } S_c^\sigma\}$$

$$\mathcal{G} = \{c \in \mathcal{I} \mid S_c = H^{-1}(c) \text{ possesses a reduced simple choreography on it}\}$$

Then

$$\mu(\mathcal{G}) = \mu(\mathcal{D})$$

*Proof.* First, note that if  $S_c$  supports a reduced simple choreography  $\mathbf{z}^c$ , then  $S_c$  must have a  $\sigma$ -invariant component, because  $\mathbf{z}^c$  is  $\sigma$ -invariant. As a result,  $\mathcal{G} \subset \mathcal{D}$ . Hence  $\mu(\mathcal{G}) \leq \mu(\mathcal{D})$ . If  $\mu(\mathcal{D}) = 0$ , then  $\mu(\mathcal{G}) = 0$ . In this simple case we are done.

From now on suppose that  $\mu(\mathcal{D}) > 0$ . By Sard-Smale theorem, the regular value  $\mathcal{R}$  form a full measure subset in  $\mathcal{I}$ , i.e.,  $\mu(\mathcal{R} \cap \mathcal{D}) = \mu(\mathcal{D}) > 0$ . Let  $\mathcal{D}^* = \mathcal{R} \cap \mathcal{D}$ . We prove next that  $\mu(\mathcal{D}^*) = \mu(\mathcal{G})$ .

Take a number  $c \in \mathcal{D}^*$  and consider  $S_c = H^{-1}(c)$ . Since  $\mathbb{C}\mathbb{P}^k$  is a compact manifold and  $\mathbb{R}$  is Hausdorff, (V0) implies that  $H$  is a proper map. As a result,  $S_c$  is compact, so is  $S_c^\sigma$ . We can construct for small  $\varepsilon > 0$  a one parameter family of the form

$$U = \bigcup_{\delta \in (-\varepsilon, \varepsilon)} S_{c+\delta}^\sigma$$

s.t.  $U$  is diffeomorphic to a submanifold of  $\mathbb{C}\mathbb{P}^k$ , moreover  $\sigma U = U$  because  $S_{c+\delta}^\sigma$  are all  $\sigma$ -invariant. Note also that  $U$  separate  $CP$  into two disjoint component  $U_A$  and  $U_B$ , s.t.  $A \in U_A$  and  $B \in U_B$ . This is because  $\mathbb{C}\mathbb{P}^k$  is a complex manifold and hence orientable, thus the Alexander duality works.



Now by choosing a smooth function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  s.t.

$$\phi(s) = \begin{cases} 0 & \text{if } s \leq -\frac{1}{2}; \\ 1 & \text{if } s \geq \frac{1}{2} \end{cases} \quad (3.51)$$

$$\phi'(s) > 0, \text{ for } s \in \left(-\frac{1}{2}, \frac{1}{2}\right) \quad (3.52)$$

and let

$$F(\mathbf{z}) = \begin{cases} \phi\left(\frac{\delta}{\varepsilon}\right) & \text{if } \mathbf{z} \in S_{c+\delta}^\sigma; \\ 0 & \text{if } \mathbf{z} \in U_A \setminus S_{c+\delta}^\sigma \\ 1 & \text{if } \mathbf{z} \in U_B \setminus S_{c+\delta}^\sigma \end{cases} \quad (3.53)$$

$$(3.54)$$

One verifies that  $F(\mathbf{z})$  satisfies the condition of theorem 3.5.2, by taking  $U(A) = U_A, U(B) = U_B, h_0 = 0, h_\infty = 1$ . Theorem 3.5.2 then implies that  $F(\mathbf{z})$  has a periodic solution  $\mathbf{z}^*$ , which is, after a reparametrisation of time, a reduced simple choreography of system (System-II) and satisfies that  $|H(\mathbf{z}^*) - c| < \frac{\varepsilon}{2}$ .

Since one has the right to choose  $\varepsilon$  arbitrarily small, we have actually shown that, given  $c \in \mathcal{D}^*$ , there exists a sequence of reduced simple choreographies of  $\{\mathbf{z}^m(t)\}_{m \in \mathbb{N}}$  s.t.  $H(\mathbf{z}^m) \rightarrow c$ . Moreover, again by using Alexander duality, we see that  $S_c^\sigma$  bounds a symplectic (sub)manifold. As a result, for a fixed  $\varepsilon$ ,

$$\mu(\mathcal{G} \cap (c - \varepsilon, c + \varepsilon)) = \mu((c - \varepsilon, c + \varepsilon)) \quad (3.55)$$

Finally, since  $\mathcal{D}^*$  is open, it is union of disjoint intervals, i.e.,

$$\mathcal{D}^* = \bigcup_{1 \leq m \leq M} \mathcal{D}_m$$

By using the local result (3.55), we see that  $\mu(\mathcal{D}_m) = \mu(\mathcal{D}_m \cap \mathcal{G})$  As a result

$$\mu(\mathcal{D}^*) = \sum_{1 \leq m \leq M} \mu(\mathcal{D}_m) = \sum_{1 \leq m \leq M} \mu(\mathcal{D}_m \cap \mathcal{G}) = \mu(\mathcal{D}^* \cap \mathcal{G}) \quad (3.56)$$

The theorem is thus proved.  $\square$

### 3.6.2 A Sufficient Condition For Existence Of Symmetric Component

Before we go to the application, we state another useful criteria for showing that there is a symmetric component on some prescribed energy level of the reduced Hamiltonian.

**Lemma 3.6.2.** *Suppose that the N-polygon configuration B is a non-degenerate maximum of H restricted to the manifold*

$$\mathcal{M}_\rho = \{|z_1|^2 = |z_2|^2 = \dots = |z_N|^2 = \rho\}, \rho > 0 \quad (3.57)$$

*Let  $H(B) > c > H(B) - \varepsilon$  for small  $\varepsilon > 0$ , then  $S_c$  has a  $\sigma$ -invariant component.*

*Proof.* We see that the N-polygon configuration is a maximum for  $H(z)|_{\mathcal{M}_\rho}$ . Since it is a non-degenerate critical point, there is no other critical point nearby. As a result, the set  $M_c = H_{\mathcal{M}_\rho}^{-1}(c) = H^{-1}(c) \cap \mathcal{M}_\rho$  has a connected component, denoted as  $M_c^\sigma$  that is  $\sigma$ -invariant. It is then included in a  $\sigma$ -invariant component  $S_c^\sigma$ , due to lemma 3.6.1.  $\square$

We see from the above lemma immediately that :

**Corollary 3.6.1.** *Suppose that the N-polygon configuration B is a non-degenerate minimum of H restricted to the manifold*

$$\mathcal{M}_\rho = \{|z_1|^2 = |z_2|^2 = \dots = |z_N|^2 = \rho\}, \rho > 0 \quad (3.58)$$

*Let  $H(B) < c < H(B) - \varepsilon$  for small  $\varepsilon > 0$ , then  $S_c$  has a  $\sigma$ -invariant component.*

*Proof.* By considering  $-H$  and applying lemma 3.6.2  $\square$

We will see in the next chapter that this property is useful when one wants to look for  $\sigma$ -invariant component of energy surface of the reduced Hamiltonian.

## 3.7 Application To Some Physical Models

In this section we discuss how to apply the theorems proved in the last section to examples raised from condensed matter physics.

### 3.7.1 The Non-Linear Discrete Schrödinger Equation

First let us consider the Hamiltonian system

$$H_{\mathbb{R}^{2N}}(\mathbf{Z}) = \frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2} |z_i|^4 - |z_{i+1} - z_i|^2 \right)$$

We can thus fix  $I(\mathbf{Z}) = N\rho$  for some constant  $\rho > 0$  and pass to reduced system (System-II) with the induced Hamiltonian  $H$ . Since this system does not have any singularity, it is not difficult to verify directly that all assumptions of theorem 3.6.1 holds here.

**Lemma 3.7.1.** *There exists an open interval  $K$  s.t.  $S_c$  is compact, regular, and has a  $\sigma$ -invariant connected component.*

*Proof.* It is direct to see that there exists an open interval  $K$  s.t.  $\forall c \in K, S_c$  is compact and regular. The compactness follows the fact that  $\mathbb{C}\mathbb{P}^{N-1}$  itself is a compact manifold, while the regular value follows the application of Sard-Smale theorem. Moreover, the total collision configuration is evidently the absolute maximum of  $H|_{\mathcal{M}_\rho}$ . The rest of the proof follows the same lines as thoses in Lemma 3.7.4.  $\square$

With the lemma 3.7.1, the existence of simple relative choreographies follows immediately:

**Theorem 3.7.1.** *Consider the System-I with the Hamiltonian*

$$H_{\mathbb{R}^{2N}}(\mathbf{Z}) = \frac{1}{2} \sum_{i=1}^n \left( \frac{1}{2} |z_i|^4 - |z_{i+1} - z_i|^2 \right)$$

*Then there exist infinitely many relative choreography*

*Proof.* By combining lemma 3.7.1 and theorem 3.6.1.  $\square$

### 3.7.2 The N-Vortex Problem in Hydrodynamics

Next let us consider the Hamiltonian system

$$H_{\mathbb{R}^{2N}}(\mathbf{Z}) = -\frac{1}{4\pi} \sum_{1 \leq i < j \leq N} \log |z_i - z_j|^2$$

This system comes from the Euler equation that describes the interaction of  $N$  identical vortices in the plane without boundary. Since there is no boundary, there is no potential part due to vortex-boundary interaction. As a result the system is invariant under the diagonal

action of Euclidean group  $SE(2)$ , i.e., rotation and translation. Now by the discussion in previous sections, the reduced phase space is  $\mathbb{C}\mathbb{P}^{N-2}$ . To show the existence centred relative choreographies, we will need to study the regularity, the compactness and the existence of a choreographically symmetric component of its energy surfaces. The regularity and compactness has already been verified in an earlier work:

**Lemma 3.7.2.** *Let  $S_c = H^{-1}(c)$  be energy surface of the reduced Hamiltonian on  $\mathbb{C}\mathbb{P}^{N-2}$ . Then*

- (1)  $S_c$  is compact
- (2)  $S_c$  is regular except for at most finitely many  $c$

*Proof.* For proof, see [114, theorem 2.2 and lemma 3.1].  $\square$

We are left to prove the existence of a  $\sigma$ -invariant component on energy levels near  $H(B)$ , where  $B$  represents as before the projection of N-polygon configuration on  $\mathbb{C}\mathbb{P}^{N-2}$ .

To this end, consider the following problem: given  $N$  points  $\mathbf{A} = (A_1, A_2, \dots, A_N)$  on the unit circle, none of them overlaps, i.e.,  $A_i \neq A_j, \forall 1 \leq i < j \leq N$ . Denotes  $l_{ij} = \|A_i A_j\|$  to be length of the segment between  $A_i$  and  $A_j$ . We would like to consider

$$F(\mathbf{A}) = \prod_{1 \leq i < j \leq N} \log l_{ij} \quad (3.59)$$

**Lemma 3.7.3.**  *$F(\mathbf{A})$  achieves its maximum when  $\mathbf{A} = (A_1, A_2, \dots, A_N)$  form a N-polygon inscribed to the unit circle.*

*Proof.* Without loss of generality, we can assume that the index  $j$  of  $A_j$  increases along the clockwise direction. We can then denote by  $\theta_j$  the angle between  $OA_j$  and  $OA_{j+1}, \forall 1 \leq j \leq N-1$  while  $\theta_N$  is the angle between  $OA_N$  and  $OA_1$  (see figure 3.2). Now by the sine formulæ of chord length, we have that

$$l_{ij} = 2 \sin \frac{\theta_i + \theta_{i+1} + \dots + \theta_{j-1}}{2}, j > i \quad (3.60)$$

Note that if  $j > N$ ,  $A_j$  is to be considered as  $A_{j'}$ , where  $j' = j \bmod N$ . In this way, we regroup the items in the product  $F(\mathbf{A})$ , such that in each subset the the difference  $j - i$  is fixed. i.e., denote

$$F(\mathbf{A}) = \prod_{1 \leq i < j \leq N} \log l_{ij} = \prod_{1 \leq k \leq \lfloor \frac{N}{2} \rfloor} B_k, \quad B_k = \prod_{\substack{1 \leq i \leq N \\ j-i=k}} \log l_{ij} \quad (3.61)$$

One verifies explicitly that  $f(\theta) = \sin \theta, 0 < \theta < \pi$  is concave, hence

$$\begin{aligned}
B_k &= \prod_{\substack{1 \leq i \leq N \\ j-i=k}} \log l_{ij} = N \log 2 + \prod_{1 \leq i \leq N} \log \sin \frac{\theta_i + \theta_{i+1} + \dots + \theta_{i+k-1}}{2} \\
&\leq N \log 2 + N \log \sin \sum_{1 \leq i \leq N} \frac{\theta_i + \theta_{i+1} + \dots + \theta_{i+k-1}}{2N} \quad (\text{Jensen's Inequality}) \\
&= N \log 2 + N \log \sin \left( k \sum_{1 \leq i \leq N} \frac{\theta_i}{2N} \right) \\
&= N \log 2 + N \log \sin \left( \frac{k\pi}{N} \right) \quad (3.62)
\end{aligned}$$

As a result

$$F(\mathbf{A}) \leq \prod_{1 \leq k \leq \lfloor \frac{N}{2} \rfloor} (N \log 2 + N \log \sin(\frac{k\pi}{N})) \quad (3.63)$$

Since  $k \in \mathbb{N}$  and  $0 < k < \lfloor \frac{N}{2} \rfloor$ , we see that the inequality in (3.63) becomes the equality if and only if

$$\theta_1 = \theta_2 = \dots = \theta_N = \frac{2\pi}{N} \quad (3.64)$$

In other words,  $F(\mathbf{A})$  achieves its maximum when  $\mathbf{A} = (A_1, A_2, \dots, A_N)$  form a  $N$ -polygon inscribed to the unit circle.  $\square$

**Lemma 3.7.4.** *There exists  $\varepsilon > 0$  s.t. for  $H(B) < c < H(B) + \varepsilon$ , the energy surface  $S_c$  of reduced Hamiltonian  $H$  has  $\sigma$ -invariant component.*

*Proof.* This is a consequence of corollary 3.6.1 and lemma 3.7.3.  $\square$

We now apply theorem 3.6.1 and conclude that

**Theorem 3.7.2.** *Consider the system (System-I) with the Hamiltonian*

$$H_{\mathbb{R}^{2N}}(\mathbf{Z}) = -\frac{1}{4\pi} \sum_{1 \leq i < j \leq N} \log |z_i - z_j|^2$$

*Assume that  $N$  is even. Then there exist infinitely many centred relative choreography.*

### 3.7.3 The N-Vortex Problem in Bose-Einstein Condensation

Next let us consider the Hamiltonian

$$H_{\mathbb{R}^{2N}}(\mathbf{Z}) = -\frac{1}{2}(\mu \sum_{i=1}^N \log \frac{1}{1-|z_i|^2} + \lambda \sum_{i<j} \log |z_i - z_j|^2)$$

describing  $N$  identical vortices in the Bose Einstein Condensation. We would like to show that there exist many relative simple choreographies. However, the Hamiltonian is not as simple as the previous one due to the singularities at collision. Here is the syllabus of our strategy:

1. **For Being Compact:** We isolate vortices away from the boundary by choosing  $I(z) = N\rho$  for  $\rho$  not too big. The trouble from singularity due to the boundary thus disappeared. In particular this gives us the compactness;
2. **For Being Regular:** We prove a version of Shub's lemma for the vortex system to see that  $H$  does not have critical points accumulating to the generalized diagonal; This together with Sard-Smale theorem will ensure the set of regular values of  $H$  to be open and dense;
3. **For Being Connected:** We show that the reduced Hamiltonian  $H(B)$  is a minimum when further restricted to a smaller manifold. This will imply at least that for  $c$  near  $H(B)$  s.t  $S_c$  contains  $\sigma$ -invariant component.

Once all these preparations are done, we simply apply the theorem developed in section 3.6 on these regular  $\sigma$ -invariant connected energy surface to show the abundance of simple relative choreographies.

We fix a level of  $I(\mathbf{z}) = \sum_{1 \leq i \leq N} |z_i|^2 = N\rho$ . This will induce a reduced Hamiltonian on  $\mathbb{C}\mathbb{P}^{N-1}$ . We focus on the energy hyper-surface  $S_c$  of the reduced Hamiltonian  $H$  on  $\mathbb{C}\mathbb{P}^{N-1}$ .

#### Existence Of Compact $S_c$

**Lemma 3.7.5.** *For  $\rho < \frac{1}{N}$ , the energy surface  $S_c$ , if non-empty, is compact.*

*Proof.* Since  $\rho < \frac{1}{N}$ ,  $I(z) < 1$ . In particular,  $|z_i|^2 < 1 \forall 1 \leq i \leq N$ . Thus the vortices are isolated from the boundary. Let  $S_c$  be an energy surface that is non-empty. Since  $I(z) < 1$  the mutual distances are bounded above uniformly, hence they are also bounded below uniformly. This implies in particular that  $S_c$  is isolated not only from the boundary but also from the generalized diagonal  $\Delta$ (where collisions happen). As a result  $S_c$  is compact.  $\square$

**Existence Of Regular  $S_c$** 

Next we prove a lemma that claims for positive vorticities, the relative equilibria of  $H$  cannot accumulated into  $\Delta$ . This is a version of Shub's lemma [97] from celestial mechanics. The analogues in vortex problems without the harmonic trap are studied by [79, 95]. The following lemma is proved using the similar argument as that in [114].

**Lemma 3.7.6.** *Suppose that  $\mathbf{z}$  is a relative equilibrium s.t.  $I(\mathbf{z}(t)) = \alpha \leq \sum_{i=1}^N \Gamma_i$ . Denote*

$$m(\mathbf{z}) = \inf_{1 \leq i < j \leq N} |z_i - z_j|^2$$

*Then  $\forall \alpha < \sum_{i=1}^N \Gamma_i$ , there exists a constant  $\varepsilon(\alpha, \Gamma)$  s.t.*

$$m(\mathbf{z}) > \varepsilon$$

*Proof.* Suppose to the contrary that  $\mathbf{z}^k$  is a sequence of relative equilibria whose mutual distances s.t.  $\lim_{k \rightarrow \infty} m(\mathbf{z}^k) = 0$ . Then by consecutively passing to subsequence if necessary, we may suppose that there exists an sub-index set  $V \subset \{1, 2, \dots, N\}$  s.t.  $z_i^k \rightarrow z^*, \forall i \in V$ . Denote  $\mathbf{z}_V$  as the vector of vortices with index in  $V$ . As before let  $L = \sum_{1 \leq i < j \leq N} \Gamma_i \Gamma_j$  and define moreover  $L_V = \sum_{\substack{i < j \\ i, j \in V}} \Gamma_i \Gamma_j$ .

First, we show that  $z^*$  cannot be an interior point inside the potential well(which in our case is the unit circle). Actually, observe that  $c_V^k = \frac{\sum_{i \in V} \Gamma_i z_i^k}{\sum_{i \in V} \Gamma_i}$ , the vorticity centre of  $\mathbf{z}_V^k$ , also follows a uniform rotation with the vortices. Denote the angular speed to be  $\nu$ , then

$$\dot{c}_V^k = \frac{\sum_{i \in V} \Gamma_i \dot{z}_i^k}{\sum_{i \in V} \Gamma_i} = \mathbb{J} \frac{\nu}{2} c_V^k \rightarrow \mathbb{J} \frac{\nu}{2} z^* \quad (3.65)$$

$$\Gamma_i \dot{z}_i = \mathbb{J}(\nabla_{z_i} H_V(\mathbf{z}) + \nabla_{z_i} H_{V^c}(\mathbf{z})) = \mathbb{J} \Gamma_i \frac{\nu}{2} z_i^k \rightarrow \mathbb{J} \Gamma_i \frac{\nu}{2} z^*, \quad i \in V \quad (3.66)$$

Define the vector  $p = \lambda \sum_{j \in V^c} \Gamma_j \frac{z^* - z_j}{\|z^* - z_j\|^2}$ ,  $q = -\mu \frac{z^*}{1 - |z^*|^2}$ . Thus we have

$$\begin{aligned} \dot{c}_V^k &\rightarrow \frac{\sum_{i \in V} \Gamma_i^2}{\sum_{i \in V} \Gamma_i} q + p \\ \dot{z}_i^k &\rightarrow \Gamma_i q + p + \nabla_i H_V(\mathbf{z}^k) \end{aligned}$$

As  $c_V^k - z_i^k \rightarrow 0$ , it turns out that

$$\Gamma_i \left( \Gamma_i - \frac{\sum_{i \in V} \Gamma_i^2}{\sum_{i \in V} \Gamma_i} \right) q \sim \nabla_i H_V(z_k)$$

Hence

$$-\lambda L_V = \nabla H_V(z_k) z^k \rightarrow -\mu \frac{|z^*|^2}{1 - |z^*|^2} \sum_{i \in V} \Gamma_i \left( \Gamma_i - \frac{\sum_{i \in V} \Gamma_i^2}{\sum_{i \in V} \Gamma_i} \right) = 0$$

This is impossible. As a result,  $z^*$  must be a point on the boundary if it exists.

Now suppose  $z^*$  is on the boundary, then by considering the dynamics of  $c_V^k$ , clearly it becomes infinity. But we can also consider the centre of other cluster and we see that all the other vortices must also accumulate into boundary, hence  $I = \sum_{i=1}^N \Gamma_i$ .  $\square$

The above lemma implies that for positive BEC N-vortex system all the relative equilibria are bounded away uniformly from the generalized diagonal set  $\Delta$ . Equivalently, it means that the fixed points of the System-II on  $\mathbb{C}\mathbb{P}^{N-1}$  cannot accumulate in to  $\tilde{\Delta}$ . We see that

**Lemma 3.7.7.** *Define*

$$\mathcal{R} = \{c \in \mathbb{R} \mid S_c \text{ is regular}\} \quad (3.67)$$

*Then  $\mathcal{R}$  is a disjoint union of open intervals, and the complement of  $\mathcal{R}$  has null Lebesgue measure.*

*Proof.* By Sard-Smale theorem together with lemma 3.7.6, we see that  $\mathcal{R}$  is an open dense subset of  $\mathbb{R}$ . An open set of  $\mathbb{R}$  is a disjoint union of open intervals, hence the result.  $\square$

### Existence of Connected $S_c$

To show the existence of connected component  $S_c$  that has choreographic symmetry, we again focus on the set  $\mathcal{M}_\rho$ . Note that on this set the potential becomes constant and one only needs to study the behavior of the interactive terms, which is exactly the case of N-vortex of hydrodynamics.

### Application Of Theorem

Now we have actually proved the following theorem:



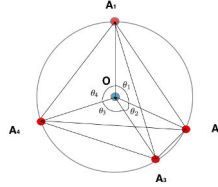


Fig. 3.2 4-vortex problem in BEC restricted to  $\mathcal{M}_p$

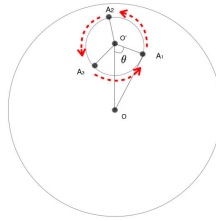


Fig. 3.3 A 3-polygon (equilateral triangle) in a rotational frame

**Theorem 3.7.3.** *Consider the System-I with the Hamiltonian*

$$H_{\mathbb{R}^{2N}}(\mathbf{Z}) = -\frac{1}{2}(\mu \sum_{i=1}^N \log \frac{1}{1 - |z_i|^2} + \lambda \sum_{i < j} \log |z_i - z_j|^2)$$

*Then for any  $N \in \mathbb{N}^+$  there exist infinitely many relative choreographies.*

*Proof.* By combining lemma 3.7.5, lemma 3.7.7 and lemma 3.7.4 we see that there exists an open interval  $K$  s.t.  $\forall c \in K$ ,  $S_c$  has a  $\sigma$ -invariant component that is compact, regular, and connected. Then we apply theorem 3.6.1 to establish the existence of a non-constant simple relative choreography on this component.  $\square$

To convince ourselves that these orbits are not N-polygon put in a rotational frame, we argue by contradiction. Suppose that the solution thus found is an equilateral configuration in a rotational frame, then it looks like in figure 3.3. Let  $r_1 = \|OO'\|$ ,  $r_2 = \|O'A\| = \|O'B\| =$

$\|O'C\|$  be constant, then by the cosine formulae

$$\begin{aligned}\|OA_1\|^2 &= r_1^2 + r_2^2 - 2r_1r_2 \cos \theta \\ \|OA_2\|^2 &= r_1^2 + r_2^2 - 2r_1r_2 \cos \left(\theta + \frac{2\pi}{N}\right) \\ &\dots \\ \|OA_N\|^2 &= r_1^2 + r_2^2 - 2r_1r_2 \cos \left(\theta + \frac{2(N-1)\pi}{N}\right)\end{aligned}$$

By the assumption,  $\sum_{1 \leq i \leq N} \log(1 - |z_i|^2)$  is a constant, which implies, by denoting  $\alpha = 1 - r_1^2 - r_2^2, \beta = 2r_1r_2$ , that the following quantity is a constant too.

$$\begin{aligned}CST &= (1 - \|OA_1\|^2)(1 - \|OA_2\|^2) \dots (1 - \|OA_N\|^2) \\ &= (\alpha + \beta \cos \theta)(\alpha + \beta \cos \left(\theta + \frac{2\pi}{N}\right))(\alpha + \beta \cos \left(\theta + \frac{2(N-1)\pi}{N}\right)) \\ &= \sum_{k=1}^N (a_k \cos k\theta + b_k \sin k\theta) + \alpha^N\end{aligned}$$

This is a trigonometric polynomial. In particular, explicit calculation shows that  $a_N = \beta^N$ . To make the above trigonometric polynomial a constant, we thus need that  $\beta = 0$ . In other words, either  $r_1 = 0$  or  $r_2 = 0$  (they cannot be both 0 because otherwise it corresponds to no point in  $\mathbb{C}\mathbb{P}^{N-1}$ ). However,

- $r_1 = 0$ : in this case the orbit corresponds to the centred N-polygon configuration;
- $r_2 = 0$ : in this case the orbit corresponds to the total collision configuration.

In either case, it becomes a fixed point in the reduced dynamics. As a result, it contradicts theorem 3.6.1, where the solutions found corresponds to non-constant reduced simple choreographies in the reduced phase space. We conclude thus the orbits we found for the N-vortex problem in BEC are **not** N-polygon in a rotational frame.

### 3.7.4 Comparison With Other Methods

Finally we give some heuristic remarks about the solutions which could be found using perturbative methods. Let us take the 4-vortex problem to illustrate the idea.

As mentioned before, the 4 vortex problem is non-integrable. As a result it seems hopeless to try to describe the complete bifurcation diagram of periodic orbits. Let us consider the reduced energy Hamiltonian  $H_{\mathbb{C}\mathbb{P}^{N-2}}$ , and denote by  $c$  the reduced energy level. There are at least two places where one might locally construct relative choreographies.



Fig. 3.4 The configuration changing with reduced energy level

**Bifurcation from the equilateral triangle:** Recall that the minimum of  $H_{\mathbb{C}P^{N-2}}$  is achieved when the 4 vortices form the centred square (4-polygon). Thus the Moser-Weinstein theorem should show the existence of relative periodic solutions of short period, bifurcating from the square. [25, 22]

**Bifurcation from the simultaneous pair of double collisions:** To the contrary, when the reduced energy tends to infinity, there is a pair of vortices that become close to one another. Now, consider two vortices of vorticity 2, located respectively at  $(\pm 1, 0)$  (thus forming a relative equilibrium). Next, consider replace each such vortex by a pair of close vortices of vorticity 1, that chase one another in the cluster. At the same time the two clusters will rotate approximately as two vortices would. As another illustration of the superposition principle (see the periodic orbits of Bartsch et al. [13] and the KAM tori of Khanin [53]), this should prove the existence of relative periodic orbits bifurcating out of the simultaneous pair of double collisions.

As a result, we believe our global approach can be seen as producing solutions of similar interests by both perturbation around the 4-polygon and around pairs of binary collisions (See figure 3.4).



## Chapter 4

# Uniform Upper Bounds for Mutual Distances of Symmetric Periodic Solutions of N-Vortex Type Hamiltonian

In this chapter, we study the mutual distance of symmetric periodic solutions. As mentioned in earlier chapters, although the collision could be excluded for positive N-vortex problem, the mutual distance in general is not bounded from above on a prescribed energy surface. Putting the existence aside, we will show in this chapter that given a N-vortex problem, if the orbit has certain discrete symmetry (Thus a periodic orbit *a priori*), then there is an upper bound of mutual distances, depends only on  $T$  and is uniform for all such symmetric orbits.

### 4.1 Upper Bounds of Mutual Distances

We assume that we are interested in the following Hamiltonian for  $\mathbb{R}^{2N}$ :

$$\dot{z}(t) = X_H(z(t)) = \mathbf{J}\nabla H(z(t))$$

where  $z(t) = (z_1(t), z_2(t), \dots, z_N(t))$  and  $z_i(t) = (x_i(t), y_i(t))$  describes the position of the  $i^{\text{th}}$  particle in  $\mathbb{R}^2$ . We suppose the Hamiltonian  $H(z)$  is of the form

$$H(z) = \sum_{1 \leq i < j \leq n} f(|z_i - z_j|^2) \tag{H}$$

Note that such a system is both invariant under translation and rotation. As a result the following quantities

$$P = \sum_{1 \leq i \leq n} x_i, \quad Q = \sum_{1 \leq i \leq n} y_i, \quad I = \sum_{1 \leq i \leq n} |z_i|^2$$

are first integrals. Define the set of all T-periodic solutions (with quotient of translation) of the Hamiltonian system to be

$$\mathcal{O}_H = \{\dot{z}(t) = X_H(z(t)) | z(0) = z(T)\}$$

We would like to know if the mutual distances  $l_{ij}(t) = |z_i(t) - z_j(t)|$  will stay uniformly bounded. In other words, define for  $z \in \mathcal{O}_H$  the quantity

$$M(z) = \sup_{1 \leq i < j \leq N, t \in [0, T]} l_{ij}(t)$$

we would like to know if

$$M(T, N) = \sup_{z \in \mathcal{O}_T} M(z) < +\infty$$

is finite, which depends only on  $T$  and  $N$ . It is easy to see that in general  $M(T, N)$  is **NOT** always finite. To this end, we put some symmetric constraints.

### 4.1.1 The Group of Italian Symmetry

Let  $\Lambda$  be T-periodic loops in the configuration space of our N-particle system. Let  $g = (\tau, \sigma, \rho) \in G$  acts on  $z(t) = (z_1(t), z_2(t), \dots, z_n(t)) \in \mathcal{O}_T$  be such that:

$$gz_i(t) = \rho y_{\sigma^{-1}(j)}(\tau^{-1}(t))$$

**Definition 4.1.1.** *The group of Italian symmetry is defined to be*

$$G = \{g = (\rho, \sigma, \tau) | \rho = e^{\pi i}, \sigma(i) = I, \tau(t) = t - \frac{T}{2}\}$$

We say a T-periodic solution of the Hamiltonian system if  $z(t) \in \mathcal{O}_T$  is **Italian symmetric** if it is G-invariant, i.e.,

$$z \in \Lambda_H = \{z \in \mathcal{O}_H | z(t + \frac{T}{2}) = -z(t), \forall t \in [0, T]\}$$

A quick observation is that T-periodic solutions of Italian symmetry is a centred solution. This is claimed by the following lemma:

**Lemma 4.1.1.** *If  $z \in \Lambda_H$ , then*

$$\sum_{1 \leq i \leq N} z_i(t) \equiv 0$$

*Proof.* We have seen that  $\sum_{1 \leq i \leq N} z_i(t) = (P(t), Q(t))$  is preserved. Moreover, the Italian symmetry implies that

$$(P(t), Q(t)) = (P(t + \frac{T}{2}), Q(t + \frac{T}{2})) = \sum_{1 \leq i \leq N} z_i(t + \frac{T}{2}) = - \sum_{1 \leq i \leq N} z_i(t) = -(P(t), Q(t))$$

Hence

$$\sum_{1 \leq i \leq N} z_i(t) = (P(t), Q(t)) \equiv 0$$

□

An advantage of being a centred orbit is that there is a simple relation between the mutual distances and the angular momentum. Actually, let  $z(t)$  be any solution of the Hamiltonian system of type (H) . We can fix the center of vorticity at the origin. Let

$$L = \sum_{1 \leq i < j \leq N} l_{ij}^2$$

$$I = \sum_{1 \leq i \leq N} |z_i|^2$$

Then  $L = NI$ .

## 4.2 Uniform bound for Italian Symmetric T-periodic solution

We assume from now on that  $f$  decays asymptotically, s.t. the following condition is satisfied:

$$\exists R, M > 0, \quad \text{s.t.} \quad \forall |x| > R, |f'(x)x| < M \quad (\text{f1})$$

Since we are interested in T-periodic solutions of the Hamiltonian system with Italian symmetry, we define

$$\bar{M}(T, N) = \sup_{z \in \Lambda_T} M(z)$$

**Theorem 4.2.1.** *Let  $z(t)$  be a  $T$ -periodic solution of an  $N$ -particle system where the Hamiltonian is of the form*

$$H(z) = \sum_{1 \leq i < j \leq n} f(|z_i - z_j|^2)$$

*Suppose that  $f$  satisfies (f1), then*

$$\bar{M}(T, N) < \infty$$

*Proof.* We prove this result by three steps.

**First step: Construction of Clusters** Given  $z(t)$ , we denote by  $l_{ij}(t)$  the distance between  $z_i$  and  $z_j$  at time  $t$ . We can suppose that w.l.o.g

$$M(z) = \max_{1 \leq i < j \leq n, t \in [0, T]} l_{ij}(t)$$

is achieved at  $t = 0$  for some pair  $z_i, z_j$ , otherwise we can simply translate the time to make this true.

Suppose now that there exists a sequence of Italian symmetric  $T$ -periodic orbit  $\mathbf{z}^k = (z_1^k, z_2^k, \dots, z_n^k)$  such that

$$\sum_{1 \leq i < j \leq n} l_{ij}^k(0) \xrightarrow{k \rightarrow \infty} \infty$$

Clearly at least one pair of distance satisfies that

$$\limsup_k l_{ij}^k(0) = \infty$$

. Again without generality we can suppose that

$$\limsup_k l_{12}^k(0) = \infty$$

Next, take a subsequence, still denoted as  $\mathbf{z}^k$ , s.t.  $l_{13}^k(0) \xrightarrow{k \rightarrow \infty} \limsup l_{13}(0)$ . Repeat the process to iterate all pairs of particles and, by consecutively passing to subsequence if necessary, we have that

$$l_{ij}^k(0) \xrightarrow{k \rightarrow \infty} l_{ij}^* \in [0, \infty]$$

$$\sum_{1 \leq i < j \leq n} l_{ij}^* = \infty$$



Now define an equivalent relation " $\sim$ " between the vortices index  $\{i, j\}$ , s.t.

$$i \sim j \Leftrightarrow l_{ij}^* < \infty$$

Since

1.  $l_{ii}^k(0) = 0$  by convention,  $l_{ii}^* = 0$ ;
2.  $l_{ij}^k = l_{ji}^k$  implies that  $l_{ij}^* = l_{ji}^*$ ;
3. By triangle inequality,  $l_{pq}^k + l_{qr}^k > l_{rp}^k, \forall k$ . Passing to the limit we see that  $l_{pq}^* + l_{qr}^* \geq l_{rp}^*$ , the equivalent class is thus well defined. By the construction, particles in the same equivalent class will tend to accumulate in a cluster. Denote the clusters by  $V_1, V_2, \dots, V_r$

**Second step: Estimate of Cluster Size** Now consider particles in the cluster  $V_1$ , which contains vortices  $i_1, i_2, \dots, i_{|V_1|}$ . For the moment we ignore other particles out of this cluster, and fix the centre of  $V_1$  to be  $C_1 = O$ . By the previous lemma we see that  $z_{i_1}^k, z_{i_2}^k, \dots, z_{i_{|V_1|}}^k$  at time 0 are located in the  $B_{C_1}(\sqrt{I_{V_1}^k})$ . However  $I_{V_1}^k$  is conserved under flow of the Hamiltonian

$$H_{V_1}^k = \sum_{i,j \in V_1, i < j} f(l_{i,j}^k)^2$$

It follows that  $z_{i_1}^k, z_{i_2}^k, \dots, z_{i_{|V_1|}}^k$  will stay in  $B_{C_1}(\sqrt{I_{V_1}^k})$  all time long under the Hamiltonian  $H_{V_1}^k$ . Since  $I_{V_1}^k \xrightarrow{k \rightarrow \infty} I_{V_1}$ , we see that for large  $k$ , the flow under Hamiltonian  $H_{V_1}^k$  will stay in the disc  $B_{C_1}(r_1), r_1 = \sqrt{2I_{V_1}}$ . Similar analysis is true for other equivalent classes.

**Third Step: Original Hamiltonian As Perturbation** Again by construction of the equivalence class,  $\|C_i - C_j\| \rightarrow \infty$  if  $i \neq j$ . Otherwise  $V_i$  and  $V_j$  will fall into the same equivalent class.

Define the following stopping time:

$$\begin{aligned} \tau_i &= \inf_{t \in [0, T]} \{z_i^k(t) \in \partial B_{C_i}((TMN + 1)r_i)\} \\ \tau &= \min\{\tau_i\} \end{aligned}$$

w.l.o.g we may assume that  $\tau$  is activated by particle(s) in  $V_1$ . We consider the trajectory of the particles in  $V_1$  under the equations:

$$\begin{aligned}\dot{z}^k(t) &= \mathbf{J}\nabla H_{V_1}^k(z^k(t)) \\ \dot{\tilde{z}}^k(t) &= \mathbf{J}\nabla H^k(z^k(t))\end{aligned}$$

Define  $g(t) = |z(t)|^2 - |\tilde{z}(t)|^2$ . It follows that

$$\begin{aligned}\dot{g}(t) &= z^k(t)\dot{z}^k(t) - \tilde{z}^k(t)\dot{\tilde{z}}^k(t) \\ &= \sum_{i \in V_1, j \in V/V_1} f'(|z_i^k - z_j^k|^2) z_i^k \mathbf{J}(z_i^k - z_j^k)\end{aligned}$$

As a result, for large  $k$ , in the time interval  $[0, \tau]$

$$|\dot{g}(t)| < NM r_1 \Rightarrow g(t) \leq TNM r_1$$

In other words, particles in the cluster  $V_i$  will stay in the  $B_{C_i}((TMN + 1)r_i)$  for large  $k$ . This contradicts the definition of  $\tau$  under the assumption of Italian symmetry. The theorem is thus proved.  $\square$

**Remark 4.2.1.** *Suppose we are in the following case where the pairwise interaction function is not identical, i.e.,*

$$H(z) = \sum_{1 \leq i < j \leq n} f_{ij}(|z_i - z_j|^2)$$

Suppose that

$$\exists R, M > 0, \quad s.t. \quad \forall |x| > R, |f'_{ij}(x)x| < M_{ij} \quad (f1')$$

Then it is clear that the above argument in the proof of theorem 1 is still valid. Thus the Italian Symmetric orbit under this Hamiltonian is still uniformly bounded.

## 4.3 Application to Identical N-vortex System

### 4.3.1 N-vortex System as Hamiltonian System

Given a system of  $N$  vortices, each vortex  $z_i = (x_i, y_i)$  with intensity  $\Gamma_i$ , their dynamics follow the Hamiltonian System (HS)

$$\begin{aligned}\Gamma_i \frac{d}{dt} x_i &= \frac{\partial}{\partial y_i} H(z) \\ \Gamma_i \frac{d}{dt} y_i &= -\frac{\partial}{\partial x_i} H(z)\end{aligned}$$

for  $i = 1, 2, \dots, N$ , or in a more concise way,

$$\Gamma \dot{z}(t) = X_H(z(t)) = \mathbf{J} \nabla H(z(t))$$

where

$$\begin{aligned}H(z) &= -\frac{1}{4\pi} \sum_{i,j=1, i<j}^N \Gamma_i \Gamma_j \log |z_i - z_j|^2 \\ \Gamma &= \text{diag}[\Gamma_1, \Gamma_1, \Gamma_2, \Gamma_2, \dots, \Gamma_N, \Gamma_N]\end{aligned}$$

**Remark 4.3.1.** *We can also adapt our argument to the case where  $G$  is the group of choreography. Actually the essential point is that*

$$\sup_{t \in [0, T]} \|Gz(t) - z(t)\| \xrightarrow{M(z) \rightarrow \infty} \infty$$

**Corollary 4.3.1.** *Let  $z(t)$  be a centered  $T$ -periodic solution of an  $N$ -vortex system where the Hamiltonian. Then*

$$\bar{M}(T, N) < \infty$$

### 4.3.2 Reparametrization Of Time

We would like to consider the space of all  $T$ -periodic solution of the  $N$ -vortex Hamiltonian. However, even if we equip this space with the topology  $\mathcal{C}^0(S^1, \mathbb{R}^{2n})$  topology, very few results could be drawn due to two difficulties.

1. The manifold  $W$  (in our case  $\mathbb{R}^{2N}$ ) is not compact, thus no uniform boundedness

2. The Hamiltonian vector field is not bounded, thus no equi-continuity.

To overcome the first obstacle, we can consider the solutions with Italian Symmetry, while for the second obstacle, we would like to use a reparametrization of time, which gives us roughly the same behavior as in the original system. We consider the following Hamiltonian:

$$\begin{aligned} H &= \sum_{1 \leq i < j \leq N} \log |z_i - z_j|^2 \\ K &= \exp(H) = \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 \\ G &= \exp(-K) = \exp\left(-\prod_{1 \leq i < j \leq N} |z_i - z_j|^2\right) \end{aligned}$$

We consider  $K$  as an intermediate change of variable, and we would like to study the relation between periodic solutions of  $H$  and those of  $G$ , with special focus given on their corresponding period. It is resumed in the following two propositions:

**Proposition 4.3.1.** *Every periodic solution of  $H$  period  $T_H$  is a periodic solution of  $G$  with period  $T_G$ ; Moreover,  $T_H < T_G$ .*

*Proof.* Given a  $\mathbf{z}_H(t)$  a  $T_H$ -periodic orbit of  $H$ , i.e., it is a solution for the dynamic system  $\dot{\mathbf{z}}_H(t) = X_H(\mathbf{z}_H(t)) = \mathbf{J}\nabla H(\mathbf{z}_H(t))$ . We see that there is no collision because all vorticities are of the same sign. As a result,  $K = K(\mathbf{z}_H) = \exp H(\mathbf{z}_H) > 0$ , and  $\frac{1}{K}$  is thus well defined. Let  $\mathbf{z}_K(t) = \mathbf{z}_H(Kt)$ . Clearly  $\mathbf{z}_K$  satisfies the system

$$\dot{\mathbf{z}}_K(t) = KX_H(\mathbf{z}_K(t)) = \mathbf{J}K\nabla H(\mathbf{z}_K(t)) = \mathbf{J}\nabla K(\mathbf{z}_K(t))$$

As a result,  $\mathbf{z}_K(t)$  is a  $T_K$  periodic solution of the Hamiltonian  $K$ , with

$$T_K = \frac{1}{K}T_H$$

Similarly, after another reparametrization by letting  $\mathbf{z}_G(t) = \mathbf{z}_K(-\exp(-K)t)$ . Again this is well defined because  $K > 0$ , it follows that

$$\dot{\mathbf{z}}_G(t) = -\exp(-K)X_K(\mathbf{z}_G(t)) = \mathbf{J}(-\exp(-K))\nabla K(\mathbf{z}_G(t)) = \mathbf{J}\nabla G(\mathbf{z}_G(t))$$

Thus  $\mathbf{z}_G(t)$  is a  $T_G$  periodic solution of the Hamiltonian  $K$ , with

$$T_G = \frac{1}{\exp(-K)}T_K = \frac{1}{K \exp(-K)}T_H$$

But the denominator is bounded above and is achieved, i.e.,

$$\forall K > 0, K \exp(-K) \leq e^{-1} < \frac{1}{2}$$

Thus  $T_H < \frac{1}{2}T_G$  □

The contrary is not true: Following direct calculation, all the collision orbits for K and G will become fixed point, thus are constant periodic orbit for any prescribed period. However they corresponds to singular points of H and are not periodic solutions of H. However, it is almost true if we restrict ourselves to solutions with Italian symmetry:

**Proposition 4.3.2.** *Every nontrivial Italian symmetric  $T_G$ -periodic solution  $\mathbf{z}_G(t) \neq 0$  of G is a  $T_H$ -periodic solution  $\mathbf{z}_H(t)$  of H; Moreover,  $T_H < T_G$ .*

*Proof.* Suppose that  $\mathbf{z}_G$  is Italian symmetric and has a collision, then it becomes a fixed point  $C_G \in \mathbb{R}^{2N}$ . As a result the Italian symmetry implies that  $C_G = 0$ . Hence if  $\mathbf{z}_G(t) \neq 0$ , then it corresponds to a periodic solution  $\mathbf{z}_H(t)$  of the Hamiltonian H. Moreover, from the previous proposition we have seen that  $T_H < T_G$ . Since all the reparametrization of variables here are linear, the Italian symmetry is preserved, i.e.,  $\mathbf{z}_H(t + \frac{T_H}{2}) = -\mathbf{z}_H(t)$ . □

It turns out that, except the constant solution at origin, the Italian symmetric solutions corresponds to the Italian symmetric solutions of G. We thus **fix a prescribed period T** and define hence as before

$$\Lambda_G = \{z(t) = X_G(z(t)) | z(0) = z(T), z(t + \frac{T}{2}) = -z(t)\}$$

Note that

**Proposition 4.3.3.** *If  $z \in \Lambda_G$ , then the vorticity center is fixed at 0.*

*Proof.* We know that  $\sum_{1 \leq i \leq N} z^i(t) = C$  is a first integral due to the translation invariance of Hamiltonian. Now z is further more Italian symmetric, hence  $\sum_{1 \leq i \leq N} z^i(t + \frac{T}{2}) = C = -\sum_{1 \leq i \leq N} z^i(t) = -C$  Thus  $C = 0$  and the vorticity center is fixed at the origin. □

### 4.3.3 Compactness for solution space of Floer Equation

#### Solution Space of Floer Equation with Italian Symmetry

Consider  $\mathbb{R}^{2N}$  as a symplectic vector space, with the canonical almost complex structure **J**. Define the space of contractible smooth loops:

$$\mathcal{L} = \{z \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}^{2N})\}$$

For  $z = (x, y) \in \mathcal{L}$ , consider the physicists' action functional

$$\mathcal{A}_G(z) = \int_0^T (G(z)dt - ydx)$$

It is well known that the gradient of  $\mathcal{A}_G$  is

$$-\mathbf{X}_G(t) = \mathbf{J}(\dot{z}(t)) + \nabla G(z)$$

If we consider a trajectory

$$u : \mathbb{R} \rightarrow \mathcal{L}, u(s, \cdot) = z \in \mathcal{L}$$

of the vector field  $\mathbf{X}_G$ , then it satisfies the Floer equation:

$$\frac{\partial u}{\partial s} + \mathbf{J} \frac{\partial u}{\partial t} + \nabla G(u) = 0$$

For such a solution  $u$ , define its energy  $E(u) \in [0, \infty]$  to be

$$E(u) = - \int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}_G(u(s)) ds = \int_{\mathbb{R} \times \mathbb{S}^1} \left| \frac{\partial u}{\partial s} \right|^2 ds dt$$

Note that we can introduce an Hilbert manifold structure on  $L$ . Now that  $\mathcal{A}_G$  is invariant under the action of Italian symmetry, and the Italian symmetry induces a Riemannian isometry on the Hilbert manifold, we can thus define the following Italian symmetric solution space:

$$\begin{aligned} \mathcal{M}_G = \{ & u : \mathbb{R} \times \mathbb{S}^1 \rightarrow \mathbb{R}^{2N} \mid \forall s \in \mathbb{R}, t \in [0, T], \\ & \frac{\partial u}{\partial s} + \mathbf{J} \frac{\partial u}{\partial t} + \nabla G(u) = 0, E(u) < \infty \\ & u(s, t + \frac{T}{2}) = u(s, t) \} \end{aligned}$$

We first show that, following (either forward or backward) the gradient flow defined by Floer equation, the functional  $A_G$  will always approaching to its critical values. We begin with the following lemma:

**Lemma 4.3.1.** *For any sequence  $s_k \nearrow +\infty$ , let  $z_k(t) = u(s_k, t)$  be a sequence of  $T$ -loops taken from the solution  $u(s, t) \in \mathcal{M}_G$ , s.t.*

$$\lim_{k \rightarrow \infty} \|\dot{z}_k(t) - X_G(z_k(t))\|_{L^2_T(\mathbb{S}^1, \mathbb{R}^{2n})} = 0$$

Then  $\{u_k\}_{k \in \mathbb{N}}$  is uniformly bounded.

*Proof.* If  $I(z_k) = 0$  for any  $k$ , then  $z_k$  is the origin and become automatically a fixed point. As a result  $u(s, t) \equiv 0$  and the consequence is obvious. Now suppose  $I(z_k) \neq 0$  for all  $k$ . Let  $\dot{z}_k(t) - X_G(z_k(t)) = f_k(t)$ , then  $\|f_k\|_{L^2_T(S^1, \mathbb{R}^{2n})} \rightarrow 0$  Consider  $I(z_k(t))$ . Clearly it is in general not preserved. However,

$$\begin{aligned} \frac{dI^{\frac{1}{2}}(z_k(t))}{dt} &= \langle I^{-\frac{1}{2}}z_k(t), \dot{z}_k(t) \rangle \\ &= \langle I^{-\frac{1}{2}}z_k(t), X_G + f_k(t) \rangle \\ &= \langle I^{-\frac{1}{2}}z_k(t), f_k(t) \rangle \\ &\leq I^{\frac{1}{2}}(z_k(t)) \|f_k(t)\| \end{aligned}$$

Then Gronwall's inequality and Hölder inequality imply that

$$\begin{aligned} I^{\frac{1}{2}}(z_k(t)) &\leq I^{\frac{1}{2}}(z_k(0)) \exp\left\{\left(\int_0^T \|f_k(t)\|\right)\right\} \\ &= I(z_k(0)) \exp\left\{\sqrt{T} \|f_k(t)\|_{L^2_T(S^1, \mathbb{R}^{2n})}\right\} \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{dP(z_k(t))}{dt} &= \langle [\mathbf{1}, \mathbf{0}], \dot{z}_k(t) \rangle \\ &= \langle [\mathbf{1}, \mathbf{0}], X_G + f_k(t) \rangle \\ &= \langle [\mathbf{1}, \mathbf{0}], f_k(t) \rangle \\ &\leq \|f_k\| \end{aligned}$$

Thus

$$|P(z_k(t)) - P(z_k(0))| = \int_0^t \frac{dP(z_k(s))}{ds} ds \leq \int_0^T \|f_k\| ds \leq \sqrt{T} \|f_k(t)\|_{L^2_T(S^1, \mathbb{R}^{2n})}$$

same result for  $Q$  To summarize, together with the Italian Symmetry, these implies that

- The angular momentum  $I(z_k(t))$  is bounded above for each  $k$ , where the bound depends continuously on the initial position and approaches to  $I(z_k(0))$  as  $k \rightarrow \infty$
- The vorticity center is approaching 0 and stays in a neighborhood of 0. The neighborhood itself retracts to 0 as  $k \rightarrow \infty$
- $f_k$  is small in average (which means its  $L^2_T$  norm is small).

As a result, the argument in theorem 4.2.1 still works. Thus we conclude thus the sequence is uniformly bounded.

**Theorem 4.3.1.** *Let  $u \in \mathcal{M}_G$ . There exists two Italian symmetric critical points  $\phi$  and  $\psi$  of  $\mathcal{A}_G$  s.t.*

$$\begin{aligned} \lim_{s \rightarrow \infty} &= \mathcal{A}_G(\phi) \\ \lim_{s \rightarrow -\infty} &= \mathcal{A}_G(\psi) \end{aligned}$$

We only need to show the case  $s \rightarrow \infty$ . The case  $s \rightarrow -\infty$  is quite similar. Since  $u \in \mathcal{M}_G$ , we see that, by letting  $s_k \nearrow \infty, z_k(t) = u(s_k, t)$

$$\lim_{k \rightarrow \infty} \|\dot{z}_k(t) - X_G(z_k(t))\|_{L^2_+(S^1, \mathbb{R}^{2n})} = 0$$

Since  $G = \exp(-\prod_{1 \leq i < j \leq N} |z^i - z^j|^2)$ , we see that  $G$  together with all its derivatives are bounded. In particular,

$$\exists B > 0 \quad \text{s.t.} \quad \|\dot{z}_k\|_{L^2_+(S^1, \mathbb{R}^{2n})} < B$$

This implies the family  $\{z_k\}$  are equi-continuous. Moreover, by the previous lemma, the family  $\{z_k\}$  are uniformly bounded. As a result, according to the theorem of Ascoli-Arzelà, we conclude that, up to a subsequence if necessary, the limit exists in  $\mathcal{C}^0(S^1, \mathbb{R}^{2n})$ . The same argument applies to the case  $s_k \rightarrow -\infty$ . To summarize, we have proved that  $\exists \phi, \psi \in \mathcal{C}^0(S^1, \mathbb{R}^{2n})$ , s.t.

$$\begin{aligned} \lim_{k \rightarrow \infty} z_k &\xrightarrow{\mathcal{C}^0((S^1, \mathbb{R}^{2n}))} \phi \\ \lim_{k \rightarrow -\infty} z_k &\xrightarrow{\mathcal{C}^0((S^1, \mathbb{R}^{2n}))} \psi \end{aligned}$$

Note that  $z_k$  are all Italian symmetric, so is  $\phi$  as its point-wise limit. Moreover, since  $X_G$  is globally Lipschitz continuous, the classical bootstrapping argument applies. As a result,  $\phi, \psi \in \Lambda_G$ . Finally the image of the trajectories are in  $\mathbb{R}^{2N}$ , hence the symplectic form is exact. Thus

$$\begin{aligned} \lim_{s \rightarrow \infty} &= \mathcal{A}_G(\phi) \\ \lim_{s \rightarrow -\infty} &= \mathcal{A}_G(\psi) \end{aligned}$$



□

To conclude, if we restrict ourselves in  $\Lambda_G$ , the regularity of  $G$  and the previous theorem of uniform boundedness of  $\Lambda_G$  will in turn give us the following result:

**Proposition 4.3.4.**  $\exists C > 0$ , s.t.

$$\forall z \in \Lambda_G, \quad |\mathcal{A}_G(z)| \leq C$$

*Proof.* Similar to the previous theorem, we see that  $\Lambda_G$  is compact in  $\mathcal{C}^1(S^1, \mathbb{R}^{2n})$  topology, which is a consequence of the theorem of Ascoli-Arzelà and the regularity of  $X_G$ , and the bootstrapping. Now  $\mathcal{A}_G$  is continuous functional on  $\mathcal{C}^1(S^1, \mathbb{R}^{2n})$ , as a result it is bounded on the compact subset  $\Lambda_G$  □

Now we are ready to prove our main result, which is a variation of the Gromov compactness theorem adapted to our choice of symmetric solution space:

**Corollary 4.3.2.**  $\mathcal{M}_G$  is compact.

*Proof.* This follows the same line as in for example [10]. □

As discussed in remark 4.3.1, the above argument works for more general symmetric orbits, in particular the centered choreography. Thus by repeating the same reasoning we see that

**Corollary 4.3.3.** Define

$$\begin{aligned} \mathcal{M}_{CH} = \{ & u : \mathbb{R} \times S^1 \rightarrow \mathbb{R}^{2N} \mid \forall s \in \mathbb{R}, t \in [0, T], \\ & \frac{\partial u}{\partial s} + \mathbf{J} \frac{\partial u}{\partial t} + \nabla G(u) = 0, E(u) < \infty \\ & u \text{ is a centered simple choreography} \} \end{aligned}$$

Then  $\mathcal{M}_{CH}$  is compact.



# Appendix A

## Some Elementary Results on the Hamiltonian System

In this appendix we recall some elementary notions and results about the integrability.

### A.1 Poincaré-Melnikov Method

Suppose we are given an original integrable dynamical system

$$\dot{\mathbf{z}} = X_{H_0}(\mathbf{z}) \tag{A.1}$$

and the nearly integrable system

$$\dot{\mathbf{z}} = X_{H_1}(\mathbf{z}) \tag{A.2}$$

Here

$$H_1(\mathbf{z}) = H_0(\mathbf{z}) + \varepsilon G(\mathbf{z}, t), 0 < \varepsilon \ll 1 \tag{A.3}$$

$$G(\mathbf{z}, t) = G(\mathbf{z}, t + T) \text{ for some } T > 0, \quad \forall \mathbf{z} \in \mathbb{R}^2 \tag{A.4}$$

**Definition A.1.1** (Hyperbolic Fixed Point). *We say that  $\mathbf{z}_0$  is a hyperbolic fixed point of the integrable system if none of eigenvalues of the linearized system around  $\mathbf{z}_0$  are purely imaginary.*

**Definition A.1.2** (Homoclinic/Heteroclinic orbit). *A nonconstant orbit  $\mathbf{z}(t)$  is called*

- *homoclinic if there exists a hyperbolic fixed point  $\mathbf{z}_0$  s.t.*

$$\lim_{t \rightarrow \pm\infty} \mathbf{z}(t) = \mathbf{z}_0 \quad (\text{A.5})$$

- *heteroclinic if there exists two hyperbolic fixed point  $\mathbf{z}_0, \mathbf{z}_1$  s.t.*

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbf{z}(t) &= \mathbf{z}_0 \\ \lim_{t \rightarrow -\infty} \mathbf{z}(t) &= \mathbf{z}_1 \end{aligned}$$

**Definition A.1.3** (Melnikov Integral). *For a homoclinic orbit  $\mathbf{z}(t)$  of the unperturbed system, define*

$$M(t_0) = \int_{-\infty}^{\infty} \{H_0(\mathbf{z}(t-t_0)), G_0(\mathbf{z}(t-t_0), y)\} dt$$

*The function  $M(t_0)$  is called the Melnikov integral.*

We now consider the suspended system

$$\dot{\mathbf{z}} = X_{H_1}(\mathbf{z}) \quad (\text{A.6})$$

$$\dot{t} = 1 \quad (\text{A.7})$$

Define an augmented Poincaré map

$$\Sigma^{t_0} = \{(\mathbf{z}, t) | t = t_0 \in [0, T]\} \subset \mathbb{R}^2 \times S^1$$

be the global cross section at time  $t_0$  of the suspended system. Under mild assumptions, one can apply the implicit function theorem to guarantee the a unique hyperbolic periodic orbit  $\mathbf{z}_\varepsilon(t) = \mathbf{z}_0 + \mathcal{O}(\varepsilon)$ , and the augmented Poincaré map  $f_\varepsilon^{t_0}$  has a unique hyperbolic saddle point  $\mathbf{z}_\varepsilon^{t_0} = \mathbf{z}_0 + \mathcal{O}(\varepsilon)$ .

**Theorem A.1.1.** *If  $M(t_0)$  has simple zeros and is independent of  $\varepsilon$ , then for  $\varepsilon > 0$  sufficiently small,  $W^u(\mathbf{z}_\varepsilon^{t_0})$  and  $W^s(\mathbf{z}_\varepsilon^{t_0})$  intersect transversely. If  $M(t_0)$  remains away from zero then  $W^u(p_\varepsilon^{t_0}) \cap W^u(p_\varepsilon^{t_0}) = \emptyset$ .*

**Theorem A.1.2** (Smale-Birkhoff). *Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$  be a diffeomorphism such that  $\mathbf{z}$  is a hyperbolic fixed point and there exists a point  $\mathbf{v} \neq \mathbf{z}$  of transversal intersection between  $W^s(\mathbf{v})$  and  $W^u(\mathbf{v})$ . Then  $f$  has a hyperbolic invariant set  $\Lambda$  on which  $f$  is topologically equivalent to a subshift of finite type.*

It turns out that, by replacing  $f$  by the augmented Poincaré map, the Melnikov integral gives us in practice a way to detect the so-called Smale horseshoes, which consist of:

- a countable set of periodic orbits of arbitrarily long periods;
- an uncountable set of bounded nonperiodic motions;
- a dense orbit.

In particular, the presence of a dense orbit closes the door for the search of global analytic first integrals.

We mention that there exists other ways to investigate the integrability, for example the Morales-Ramis theory, see [75].

## A.2 Symplectic Reduction and Reduced Hamiltonian

The symmetry under the symplectic action of some continuous group implies the possibility of considering a Hamiltonian system on a reduced symplectic manifold. We briefly discuss the idea of symplectic reduction. The program of using the symmetry to construct the so-called *generalised momentum map* and using it to simplify the Hamiltonian system has been systematically established in the work of Smale [99, 100] or Marsden and Weinstein [67]. One could turn to [2] for detailed exposition of this theory.

Assume that

$$- (M, \omega) \text{ is a symplectic manifold} \quad (\text{A.8})$$

$$- \Phi : G \times M \rightarrow M \text{ is a symplectic action} \quad (\text{A.9})$$

$$- \mathfrak{J} : M \rightarrow \mathfrak{g}^* \text{ is a } Ad^* \text{-equivariant momentum map} \quad (\text{A.10})$$

$$- G_\mu = \{g \in G \mid Ad_g^* \mu = \mu\} \text{ is the isotropy subgroup of } G \text{ under the co-adjoint action } Ad^* \quad (\text{A.11})$$

Then the orbit space

$$M_\mu = \frac{\mathfrak{J}^{-1}(\mu)}{G_\mu} \quad (\text{A.12})$$

is well defined and is called the *reduced phase space*. The following theorem guarantees that  $M_\mu$  is actually a symplectic manifold:

**Theorem A.2.1.** *Suppose that*

$$(1) \quad \mu \text{ is a regular value of } \mathfrak{J}; \quad (\text{A.13})$$

$$(2) \quad G_\mu \text{ acts freely and properly on } \mathfrak{J}^{-1}(\mu) \quad (\text{A.14})$$

Then  $M_\mu$  admits a unique symplectic form  $\omega_\mu$  s.t.

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega \quad (\text{A.15})$$

where  $\pi_\mu : \mathfrak{J}^{-1}(\mu) \rightarrow M_\mu$  is the canonical projection and  $i_\mu : \mathfrak{J}^{-1}(\mu) \rightarrow M$  is the inclusion.

So far we have talked about the reduction of a symplectic manifold. This is only half of the story, as we hope that the original Hamiltonian system could be reduced to another Hamiltonian system on this reduced manifold. The following theorem answered this need:

**Theorem A.2.2.** *Under the above hypothesis, if  $H : M \rightarrow \mathbb{R}$  is invariant under the action of  $G$ , i.e.  $H(\mathbf{z}) = H(g\mathbf{z}), \forall \mathbf{z} \in M, g \in G$  then the flow  $\phi_H^t$  of the Hamiltonian vector field  $X_H$  leaves  $\mathfrak{J}^{-1}(\mu)$  invariant, and commutes with the action of  $G_\mu$  on  $\mathfrak{J}^{-1}(\mu)$ . So it induces canonically a flow  $\phi_{H_\mu}^t$  on  $M_\mu$ , satisfying*

$$\pi_\mu \circ \phi_H^t = \phi_{H_\mu}^t \circ \pi_\mu \quad (\text{A.16})$$

This flow is a Hamiltonian flow on  $M_\mu$  with a Hamiltonian  $H_\mu$  which satisfies

$$H \circ i_\mu = H_\mu \circ \pi_\mu \quad (\text{A.17})$$

We call  $H_\mu$  the reduced Hamiltonian.

In the context of N-vortex problem, the reduced phase space is an easy application of the above abstract methods. Here the generalised momentum map is

$$\mathfrak{J} = \sum_1^N \Gamma_i |z_i|^2$$

$$G_\mu = \mathbf{SO}(2)$$

Hence the reduced manifold is either  $\mathbb{C}\mathbb{P}^{N-1}$  or  $\mathbb{C}\mathbb{P}^{N-2}$ , depends on whether the system is invariant under translation or not.

## Appendix B

# A minimax Approach For Identical N-Vortex Problem

In this chapter we will try to make some attempts in searching relative periodic solutions for identical N-vortex problem in the plane, in the sense that a minimax solution of certain symmetry is proved to exist by using the variational method of Rabinowitz and Palais' principle of symmetric criticality. The rest of the paper is organized in the following structure: In chapter 2, we recall some preliminaries in the Hamiltonian Structure of N-vortex system, together with the variational setting for general Hamiltonian systems; In chapter 3, a classical topological linking theorem will be applied to a modified Hamiltonian with the linking structure carefully chosen. This gives a periodic solution, possibly with collision, of the modified Hamiltonian. In chapter 4, we show that the non-collision is closely related to the minimal period problem. In chapter 5, by using some discrete group of symmetry, we will guarantee the existence of a collision-free relative periodic solution of the original system.

### B.1 Planar N-vortex Problem as Hamiltonian System

#### B.1.1 Hamiltonian Structure and First Integrals

Given a system of N vortices, each vortex  $z^i = (x^i, y^i)$  with intensity  $\Gamma_i$ , their dynamics follow the Hamiltonian System (HS)

$$\begin{aligned}\Gamma_i \frac{d}{dt} x^i &= \frac{\partial}{\partial y^i} H(z) \\ \Gamma_i \frac{d}{dt} y^i &= -\frac{\partial}{\partial x^i} H(z)\end{aligned}$$

for  $i = 1, 2, \dots, N$ , or in a more concise way,

$$\Gamma \dot{\mathbf{z}}(t) = X_H(\mathbf{z}(t)) = \mathbf{J} \nabla H(\mathbf{z}(t))$$

where

$$H(z) = -\frac{1}{4\pi} \sum_{i,j=1, i < j}^N \Gamma_i \Gamma_j \log |z^i - z^j|^2$$

$$\Gamma = \text{diag}[\Gamma_1, \Gamma_1, \Gamma_2, \Gamma_2, \dots, \Gamma_N, \Gamma_N]$$

Note that such a system is both invariant under translation and rotation. As a result the following quantities

$$P = \sum_{1 \leq i \leq N} \Gamma_i x^i(t), \quad Q = \sum_{1 \leq i \leq N} \Gamma_i y^i(t), \quad I = \sum_{1 \leq i \leq N} \Gamma_i |z^i(t)|^2$$

are first integrals. Define the quantity *total vortex angular momentum* to be

$$L = \sum_{i < j} \Gamma_i \Gamma_j$$

**In the rest of the paper, we will always assume  $\Gamma_i = 1, \forall 1 \leq i \leq N$ .** This will make our description easier without changing the natural of our result. From a variational point of view, to find a T-periodic solution for the system  $\dot{z}(t) = \mathbf{J} \nabla H(z(t))$ , one could instead look for a critical point of the functional  $I_H$  in the space  $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$ . Once this is done, some standard postiori estimate will show that, (assuming the square integrability of  $\nabla H$ ), the critical point indeed possesses enough regularity, and is a classical T-periodic solution of the Hamiltonian system. We focus on the following Hamiltonians:

$$H_0 = \sum_{i,j=1, i < j}^N \log |z^i - z^j|^2$$

$$H_1 = \prod_{i,j=1, i < j}^N |z^i - z^j|^2$$

$$H_2 = \prod_{i,j=1, i < j}^N |z^i - z^j|^2 + f(I(z))$$



where  $f(\lambda) = \mu\lambda^k$ , for an integer  $k > 0$  fixed large enough whose value is to be precised later on, while

$$\mu = \frac{\alpha}{kT}, \quad \alpha < 2\pi$$

The main result is the following:

**Theorem B.1.1.** *For every  $N > 0$ , the identical  $N$ -vortex system has a relative periodic solution, which is a choreography.*

### B.1.2 Scatch of the proof

The main lines of the proof are as the following:

1. We show that  $I_{H_2}$  possesses a critical point  $\mathbf{z}_{H_2}$  in  $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$  by the construction of topological linking;
2. Standard argument then shows that this critical point is indeed a classical solution  $\mathbf{z}_{H_2}$  of the Hamltonian  $H_2$ ;
3. By the fact that flows of Hamiltonians in involution commute, we show that,  $\mathbf{z}_{H_2}$  will induce a relative T-periodic solution  $\mathbf{z}_{H_1}$  of the Hamiltonian  $H_1$  ;
4. We will excludre the possibility of collision in  $\mathbf{z}_{H_1}$  , thus  $H_1 \neq 0$ ;
5. Now by taking logarithm of  $H_1$  (which is a legal operation when  $H_1 \neq 0$ ),  $\mathbf{z}_{H_1}$  will become, after a reparametrization of time, a relative periodic solution  $\mathbf{z}_{H_0}$  for  $H_0$ , the theorem is thus proved.

## B.2 The Existence of T-periodic solution for $H_2$

In this section we aim at proving the existence of a  $2\pi$ -periodic solution for the Hamiltonian

$$H_2 = \prod_{i,j=1, i < j}^N |z^i - z^j|^2 + f(I(z))$$

by construction of topological linking for the functional  $I_{H_2}$ . We use the terminology in chapter 1. In our situation, we can take

$$\begin{aligned} E &= H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N}), E_1 = E^+, E_2 = E^- \oplus E^0 \\ I &= I_{H_2} \\ \mathbf{S} &= \partial B_\rho \cap E_1 \\ \mathcal{Q} &= \{re | r \in [0, r_1]\} \oplus (B_{r_2} \cap E_2) \end{aligned}$$

Here  $r_1 > \rho > 0, r_2 > 0$ . Of course we need to specify the element  $e$ . We choose

$$e = \frac{1}{\sqrt{N\pi}} \left( \sin\left(\frac{2\pi}{T}\left(t + \frac{k-1}{N}T\right)\right) e_k, \cos\left(\frac{2\pi}{T}\left(t + \frac{k-1}{N}T\right)\right) e_{k+N} \right) \quad (\text{B.1})$$

Clearly  $e \in E_1$ . For later use, we calculate directly that

$$\begin{aligned} \|e\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^2 &= \frac{1}{N\pi} N \int_0^T \cos^2\left(\frac{2\pi}{T}t\right) dt + \frac{1}{N\pi} N \int_0^T \sin^2\left(\frac{2\pi}{T}t\right) dt = 1 \\ \|e\|_{L_T^2(\mathbb{S}^1, \mathbb{R}^{2N})}^2 &= \frac{1}{N\pi} N \int_0^T \cos^2\left(\frac{2\pi}{T}t\right) dt + \frac{1}{N\pi} N \int_0^T \sin^2\left(\frac{2\pi}{T}t\right) dt = \frac{T}{\pi} \end{aligned}$$

We need to find next appropriate  $\rho, r_1, r_2$  and make the hypothesis in the theorem hold. We will prove it step by step, while emphasizing the choice of  $r_1$  in the construction.

**Lemma B.2.1.**  $\exists \beta_1 > 0, \rho < \bar{r}$ , s.t.

$$I_{H_2}|_{\mathbf{S}} > \beta_1$$

*Proof.* First of all, Note that  $\forall 1 \leq i < j \leq N, |z_i - z_j|^2 \leq 2(|z_i|^2 + |z_j|^2)$ . As a result,

$$\sum_{i,j=1, i < j}^N |z_i - z_j|^2 \leq 2(N-1) \sum_{i=1}^N |z_i|^2 = 2(N-1)|z|^2$$

by the inequality of arithmetic average and geometric average, we see that

$$\prod_{i,j=1, i < j}^N |z_i - z_j|^2 \leq \left( \frac{\sum_{i,j=1, i < j}^N |z_i - z_j|^2}{\frac{N(N-1)}{2}} \right)^{\frac{N(N-1)}{2}} \leq a_1(N) |z|^{N(N-1)}$$

This implies that

$$H_2(z) = \prod_{i,j=1, i < j}^N |z_i - z_j|^2 + f(I(z)) \leq a_1(N) |z|^{N(N-1)} + \mu |z|^{2k}$$

Thus for  $z \in E_1$ ,

$$\begin{aligned}
I_{H_2}(z) &\geq \|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^2 - \int_0^T (a_1(N)|z|^{N(N-1)} + \mu|z|^{2k}) \\
&= \|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^2 - a_1(N)\|z\|_{L_T^{N(N-1)}}^{N(N-1)} - \mu\|z\|_{L_T^{2k}}^{2k} \\
&\geq \|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^2 - a_2(N)\|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^{N(N-1)} - a_3(k)\mu\|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^{2k} \\
&= (1 - a_2(N)\|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^{N(N-1)-2} - a_3(k)\mu\|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^{2k-2})\|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^2
\end{aligned}$$

It turns out that there exists  $\rho_0(N, k) > 0$ , s.t. for  $\rho < \rho_0$ ,  $(1 - a_2\rho^{N(N-1)-2} - a_3\rho^{2k-2}) \geq \frac{1}{2}$ .  
Taking  $\rho = \min\{\rho_0, 1\}$  and  $\beta_1 = \frac{1}{2}\rho^2 > 0$

$$I_{H_2}(z) \geq \beta_1, \forall z \in S$$

□

**Lemma B.2.2.**  $\exists r_1 > 0, r_2 > 0$ , s.t.

$$I_{H_2}|_{\partial Q} \leq 0$$

*Proof.* Recall that

$$H_2(z) \geq f(I(z)) = \mu|z|^{2k}, \quad \mu = \frac{\alpha}{kT}$$

As a result, for  $z \in B_{r_2} \cap E_2, z = z^- + z^0$  By Hölder inequality, the following embedding inequality holds:

$$\|z + re\|_{L_T^p} \leq T^{\frac{1}{p} - \frac{1}{q}} \|z + re\|_{L_T^q}, \quad 1 \leq p \leq q \leq \infty$$

Applying to  $p = 2$  and  $q = 2k$ , and taking  $e$  as chosen in (B.1) we see that

$$\begin{aligned}
\int_0^T H_2(z+re)dt &\geq \mu \int_0^T |z+re|^{2k} dt \\
&\geq \mu T^{1-k} \left( \int_0^T |z+re|^2 dt \right)^k \\
&= \mu T^{1-k} \left( \int_0^T |z^0|^2 + |z^-|^2 + r^2 e^2 dt \right)^k \\
&\geq \mu T^{1-k} T^k \left( |z^0|^{2k} + \frac{r^{2k}}{\pi^k} \right) \\
&= \mu T \left( |z^0|^{2k} + \frac{r^{2k}}{\pi^k} \right)
\end{aligned}$$

As a result,

$$I_{H_2}(z+re) \leq r^2 - \|z^-\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^2 - \mu T \left( |z^0|^{2k} + \frac{r^{2k}}{\pi^k} \right)$$

We only need to choose  $r_1$  so that  $M(r) = r^2 - \mu T \frac{r^{2k}}{\pi^k} < 0$  for all  $r > r_1$ . Choose  $r_1 = (1 + \varepsilon)\sqrt{\pi}$ ,

$$\left( (1 + \varepsilon)\sqrt{\pi} \right)^2 - \mu T \frac{\left( (1 + \varepsilon)\sqrt{\pi} \right)^{2k}}{\pi^k} = \pi \left( (1 + \varepsilon)^2 - \frac{\alpha}{k\pi} (1 + \varepsilon)^{2k} \right)$$

For any given  $k$ , there exists a  $\varepsilon_k > 0$  s.t.

$$\forall \varepsilon > \varepsilon_k, \quad (1 + \varepsilon)^2 - \frac{\alpha}{k\pi} (1 + \varepsilon)^{2k} < 0$$

when  $\varepsilon > \varepsilon_k$ . Moreover,  $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ . As consequence, for  $k$  large enough we can choose  $r_1 = (1 + \varepsilon_k)\sqrt{\pi}$ . Finally, as

$$\lim_{\|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} \rightarrow \infty} \left( \|z^-\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^2 + \mu T |z^0|^{2k} \right) = \infty$$

, choose  $r_2 > 0$  large enough s.t.

$$-\left( \|z^-\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^2 + \mu T |z^0|^{2k} \right) + \sup_{r \in [0, r_1]} M(r) < 0$$

we see that  $I_{H_2}(z+re) \leq 0, \forall z+re \in \partial Q$ . The lemma is thus proved.  $\square$

**Lemma B.2.3.** *S and  $\partial Q$  link*

*Proof.* We only need to prove that  $r_1 > \rho$ . Since  $r_1 = (1 + \varepsilon)\sqrt{\pi} > 1 > \rho$ , this is a typical linking situation, see for example [93] or [104] for details.  $\square$

Since  $H_2$  is regular and is bounded by polynomial growth, the Palais-Smale condition holds by standard argument (see in the appendix). As a result, we conclude that

**Theorem B.2.1.**  $I_{H_2}$  has a critical point in  $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$

From now on we denote this critical point as  $\mathbf{z}_{H_2}$ .

**Proposition B.2.1.**  $\mathbf{z}_{H_2}$  is a critical point of  $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$ , and is actually a classical T-periodic solution of the Hamiltonian  $H_2$ .

*Proof.*  $\nabla H_2$  is of polynomial growth, and  $\mathbf{z}_{H_2} \in H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$  implies  $\nabla H_2(\mathbf{z}_{H_2}) \in L_T^2$ , the rest follows the standard regularity argument.  $\square$

We see that  $z = 0$  is a natural candidate for our solution which is not of great interest to us. The following proposition shows that luckily the variational method have provided us a somehow non-trivial solution.

**Proposition B.2.2.**  $\mathbf{z}_{H_2} \neq 0$

*Proof.* if  $\mathbf{z}_{H_2} = 0$ , then the critical value should be

$$I_{H_2}(\mathbf{z}_{H_2}) = I_{H_2}(0) = 0 - TH(0) = 0$$

, which contradicts the fact that  $I_{H_2}(\mathbf{z}_{H_2}) = c \geq \beta_1 > 0$   $\square$

Before we go on to the next section, we first give an upper bound for the critical value  $c = H_2(\mathbf{z}_{H_2})$  found by the variational method.

**Proposition B.2.3.**  $c = H_2(\mathbf{z}_{H_2}) \leq (1 + \varepsilon_k)^2 \pi$

*Proof.* the critical value  $c$  is taken as the minimax among all surfaces modelled on  $\mathcal{Q}$ , which are described by a special class of homeomorphisms who fix the boundary  $\partial Q$ . On particular if we take the homeomorphism on  $\mathcal{Q}$  to be identity, it then provides a candidate surface  $\mathcal{Q}_{Id} = Q = \{re | r \in [0, r_1]\} \oplus (B_{r_2} \cap E_2)$ . Note moreover that  $H_2 \geq 0$ , we see that  $\forall z = re + z^- + z^0 \in \mathcal{Q}_{Id}$ ,

$$I_{H_2}(re + z) = \mathcal{A}(re) + \mathcal{A}(z^-) - \int_0^T H_2(re + z^- + z^0) dt \leq r^2 \leq r_1^2$$

This implies that



Fig. B.1 The critical value is taken as the inf-sup among all the surfaces modelled on Q(left), thus is bounded above by sup on Q (right) itself

$$c \leq \sup_{z \in Q_{td}} I_{H_2}(z) \leq r_1^2 = (1 + \varepsilon_k)^2 \pi \quad (c1)$$

The proposition is thus proved.  $\square$

### B.2.1 Commuted Hamiltonian flows and the induced T-periodic solution of the Hamiltonian $H_1$

To show that  $\mathbf{z}_{H_2}$  induces a T-periodic solution for  $H_1$ , we use some properties of commuted Hamiltonian flows. Given a symplectic manifold  $M$  and a Hamiltonian  $H : M \rightarrow \mathbb{R}$ , we use the following notations:

- $X_H$  : The Hamiltonian vector field generated by  $H$
- $\phi_H^t$  : The Hamiltonian flow at time  $t$

Now suppose that  $H$  possesses a  $T$ -periodic solution, and  $I$  is a first integral of the flow  $\phi_H$ . In other words,  $\{H, I\} = 0$ , i.e., they commute with each other.

We consider a special case where  $M = \mathbb{R}^{2n}$ ,  $I(z) = |z|^2 = \sum |z_i|^2$ . Note that  $I$  is a quadratic Hamiltonian which describes the oscillation with a uniform frequency, hence its solution is the uniform rotation of period  $T = 2\pi$ . Now let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function. We have

$$X_{H+f(I)} = \mathbb{J}\nabla(H + f(I)) = \mathbb{J}(\nabla H + \nabla f(I)) = X_H + X_{f(I)}$$

The two flows  $\phi_H^t$  and  $\phi_{f(I)}^t$  are groups of symplectomorphisms whose Lie algebras are Hamiltonian vector fields  $X_H$  and  $X_{f(I)}$  relatively. Since  $I$  and  $H$  are first integrals in involution, so are  $f(I)$  and  $H$ , i.e.  $\{f(I), H\} = 0$ , as a result  $[X_H, X_{f(I)}] = 0$ . It turns out that,

### B.3 Collision, Minimal Period and the Induced Periodic Solution of the Hamiltonian $H_0$ 13

according to the Zassenhaus formula (the dual of Baker-Campbell-Hausdorff formula)

$$\exp_{H+f(t)} = e^{(X_H+X_{f(t)})T} = e^{X_H T} \circ e^{X_{f(t)} T} = e^{X_{f(t)} T} \circ e^{X_H T} = \exp_{f(t)} \circ \exp_H$$

In other words the following diagram commutes

$$\begin{array}{ccc} x & & \\ \downarrow \exp_H & \searrow \exp_{H+f(t)} & \\ \phi_H^T(x) & \xrightarrow{\exp_{f(t)}} & \phi_{H+f(t)}^T(x) \end{array} \quad (\text{B.2})$$

we have actually shown that:

**Lemma B.2.4.**  $\mathbf{z}_{H_2}$  induces a relative  $T$ -periodic solution  $\mathbf{z}_{H_1}$  for  $H_1$

### B.3 Collision, Minimal Period and the Induced Periodic Solution of the Hamiltonian $H_0$

To finally construct the solution for  $H_0$ , the following observation is immediate:

**Proposition B.3.1.** *If  $\mathbf{z}_{H_1}$  does not have any collision, then it is, up to a reparametrization of time, a relative  $T_0$ -periodic solution  $\mathbf{z}_{H_0}$  of the Hamiltonian  $H_0$ .*

*Proof.* If  $\mathbf{z}_{H_1}$  does not have any collision, then  $H_1(\mathbf{z}_{H_1}) > 0$ . Let

$$\mathbf{z}_{H_0}(s) = \mathbf{z}_{H_1}(t(s)) = \mathbf{z}_{H_1}\left(\frac{1}{H_1(\mathbf{z}_{H_1})}s\right)$$

It is clearly a relative periodic orbit. Moreover, it could be verified directly that,

$$\begin{aligned} \frac{d\mathbf{z}_{H_0}(s)}{ds} &= \frac{\mathbf{z}_{H_1}(t(s))}{dt} \frac{dt(s)}{ds} = \mathbf{J} \frac{\nabla H_1(\mathbf{z}_{H_1}(t(s)))}{H_1(\mathbf{z}_{H_1})} \\ &= \mathbf{J} \nabla \log(H_1(\mathbf{z}_{H_1}(t(s)))) \\ &= \mathbf{J} \nabla H_0(\mathbf{z}_{H_0}(s)) \end{aligned}$$

Hence  $\mathbf{z}_{H_0}(s)$  is a flow of  $H_0$  □

Thus the final task is to show that  $\mathbf{z}_{H_1}$  does not have any collision. Since  $\mathbf{z}_{H_2}$  could be seen as  $\mathbf{z}_{H_1}$  in a rotating framework, we only need to exclude the possibility of collision in  $\mathbf{z}_{H_2}$

**Proposition B.3.2.** *If the solution  $\mathbf{z}_{H_2}$  has any collision, then it is a uniform rotation.*

*Proof.* Suppose that there is a collision, s.t. □

$$\exists 1 \leq i < j \leq N, t_0 \in [0, T], \quad \text{s.t.} \quad z_{H_2}^i(t_0) = z_{H_2}^j(t_0)$$

Then  $H_1(\mathbf{z}_{H_2}(t_0)) = 0$ . Moreover, since  $H_1(z) = H_2(z) - f(I(z))$ , and  $I(z)$  (hence  $f(I(z))$ ) is first integral of  $H(z)$ , it follows that (by explicit calculation)

$$\begin{aligned} H_1(\mathbf{z}_{H_2}(t)) &= 0, \forall t \in [0, T] \\ \nabla H_1(\mathbf{z}_{H_2}(t)) &= 0, \quad \forall t \in [0, T] \end{aligned}$$

Now recall that  $\mathbf{z}_{H_2}$  solves the Hamiltonian system  $\dot{z}_{H_2}(t) = \mathbf{J}\nabla H_2(\mathbf{z}_{H_2}(t))$ , which is equivalent to say that

$$\dot{z}_{H_2}(t) = \mathbf{J}(\nabla H_1(\mathbf{z}_{H_2}(t))) + \nabla f(I(\mathbf{z}_{H_2}(t))) = \mathbf{J}\nabla f(I(\mathbf{z}_{H_2}(t)))$$

Trajectories of this dynamic system are relative-equilibria. Moreover, by the form of  $f(z)$ ,  $\mathbf{z}_{H_2}$  is a relative equilibrium as a fixed configuration rotating clockwise around the origin, with the constant angular velocity

$$|\omega| = 2 \frac{df}{dI}(I(\mathbf{z}_{H_2}(t)))$$

Each vortex  $z_{H_2}^i$  will stay on the centred circle with radius  $|z_{H_2}^i|$ . The following lemma shows that  $I(\mathbf{z}_{H_2})$  cannot be "too small".

**Lemma B.3.1.** *If  $I(\mathbf{z}_{H_2}) \leq 1$ , then  $\mathbf{z}_{H_2}$  cannot have any collision.*

*Proof.* Suppose to the contrary that  $I(\mathbf{z}_{H_2}) < 1$ , and there is a collision, then by the previous proposition, the orbit becomes an uniform rotation with speed

$$\begin{aligned} |\omega| &= 2 \frac{df}{dI}(I(\mathbf{z}_{H_2})) = \mu k I^{k-1} \\ &\leq \frac{\alpha}{T} < \frac{2\pi}{T} \end{aligned}$$

Recall now that,  $\mathbf{z}_{H_2} \neq 0$  (due to Proposition 2.2) is a T-periodic orbit of  $H_2$  with collision. This leads to a contradiction. As a result the lemma is proved. □

Now we only need to study the situation when  $I(\mathbf{z}_{H_2}) > 1$ . We first prove a lemma:



### B.3 Collision, Minimal Period and the Induced Periodic Solution of the Hamiltonian $H_0$ 115

**Lemma B.3.2.** *Suppose  $I(\mathbf{z}_{H_2}) > 1$ , and the minimum period  $T^*$  of this solution is  $T^* = \frac{T}{m}$  for some  $m \in \mathbb{N}$ . Then  $c \geq m\pi(1 - \frac{1}{k})$*

*Proof.* We know that integral of the form  $\int_0^T y dx$  does have a geometric meaning: it describes the sum of projected area for each two dimensional subspace  $(x^i, y^i)$ . Now along  $\mathbf{z}_{H_2}$  each component  $(x^i, y^i)$  rotates with constant angular velocity  $\omega$  and radius  $|z^i|$  till time T, this integral could in this case be estimated explicitly:

$$\mathcal{A}(\mathbf{z}_{H_2}) = \frac{T\omega}{2\pi} \sum_{i=1}^N \pi |z_{H_2}^i|^2 = \frac{T\omega}{2} I(\mathbf{z}_{H_2})$$

It follows that the critical value  $c$  is forced to be "big" when  $I(\mathbf{z}_{H_2})$  is "big". Actually, let  $\lambda = I(\mathbf{z}_{H_2})$  we see that:

$$\begin{aligned} c = I_{H_2}(\mathbf{z}_{H_2}) &= \int_0^T y dx - H_2(\mathbf{z}_{H_2}) dt \\ &= \frac{T\omega}{2} \lambda - T f(\lambda) \\ &= \frac{T\omega}{2} \lambda - T \frac{1}{k} f'(\lambda) \lambda \\ &= \frac{T\omega}{2} \lambda - \frac{1}{k} \frac{T\omega}{2} \lambda \\ &= \frac{T\omega}{2} \lambda (1 - \frac{1}{k}) \\ &= m\pi \lambda (1 - \frac{1}{k}) \\ &\geq m\pi (1 - \frac{1}{k}) \end{aligned} \tag{c2}$$

since  $\lambda > 1$ . □

By comparing (c1) and (c2), we see the following proposition:

**Theorem B.3.1.** *Suppose that the solution we have found does have a collision, then this solution must verify that  $T$  is its minimal period.*

*Proof.* Suppose that  $T$  is not the minimal period, then  $T^* = \frac{T}{m}$  for some  $m \in \mathbb{N}$ . According to (c1) and (c2) we see that

$$2\pi(1 - \frac{1}{k}) \leq m\pi(1 - \frac{1}{k}) \leq c \leq (1 + \varepsilon_k)^2 \pi$$

Since  $\pi > 0$ , this leads to

$$2\left(1 - \frac{1}{k}\right) \leq (1 + \varepsilon_k)^2$$

This is not true for large  $k$

□

## B.4 Symmetry and Exclusion of Collision

We have seen that the key argument in the exclusion of collision relies on the minimal period. If under more conditions we can show that the solution does not verify Rabinowitz conjecture in case of collision, we will have a contradiction w.r.t. the theorem 4 above. As consequence the solution must be collision free.

In celestial mechanics, more constraints could be posed for topological or symmetrical consideration, see[30] for detailed discussion. Here we will try the symmetrical consideration. The reader could find in appendix B a brief introduction of discrete symmetry and Palais' Principle, and verification of various symmetries.

### B.4.1 Simple choreography

Consider the simple choreography of  $N$  vortices

$$z_i\left(t + \frac{T}{N}\right) = z_{i-1}(t), \quad i = 1, 2, \dots, N$$

This gives us a solution  $\mathbf{z}_{H_2}$  that is a simple choreography

**Proposition B.4.1.** *Under this symmetric constraint,  $\mathbf{z}_{H_2}$  is a collision free solution with simple choreographic symmetry*

*Proof.* Suppose to the contrary that  $\mathbf{z}_{H_2}$  has a collision. Then it becomes a uniform rotation with  $T^* = T$ . Moreover, Without loss of generality we could assume the collision involves  $z_{H_2}^1$ , i.e.,

$$z_{H_2}^i(t) = z_{H_2}^1(t), \forall 1 \leq i \leq N$$

Now by the definition of choreography again, we see that  $\forall t \in [0, T]$

$$z_{H_2}^{2i-1}\left(t + \frac{T(i-1)}{N}\right) = z_{H_2}^i(t) = z_{H_2}^1(t) = z_{H_2}^i\left(t + \frac{T(i-1)}{N}\right)$$

It turns out that

$$z_{H_2}^{2i-1}(t) = z_{H_2}^i(t) = z_{H_2}^1(t)$$

It is clear how we can define an equivalent class for vortices collided in this way. The index of vortices in one equivalent class will be a subgroup of the cyclic group  $S^N$ , thus each equivalent will at least have two elements. Dividing  $\mathbb{S}_1$  parameterized by  $[0, T]$  into two equal parts  $[0, \frac{T}{2}]$  and  $[\frac{T}{2}, T)$ . Now by Pigeonhole principle there must be at least two elements falling into the same part, i.e., the time gap is less or equal to  $\frac{T}{2}$ . In other words, any collision will imply that

$$T^* \leq \frac{T}{2}$$

This is a contradiction. As a result the proposition is proved.  $\square$

#### B.4.2 Simple choreography with a center

Consider the simple choreography of  $N$  vortices with an extra center

$$z_i(t + \frac{T}{N}) = z_{i-1}(t), \forall 1 \leq i \leq N, \quad w(t + \frac{T}{N}) = w(t)$$

**Proposition B.4.2.** *Under this symmetric constraint,  $\mathbf{z}_{H_2}$  is a collision free solution with simple choreographic symmetry and a center.*

*Proof.* If there is any collision, then we can suppose without generality that  $\mathbf{z}_{H_2}^1$  collides with  $\mathbf{z}_{H_2}^j$  or  $\mathbf{z}_{H_2}^1$  collides with  $w$

**Case I:**

$$z_{H_2}^i(t) = z_{H_2}^1(t), \forall 1 \leq i \leq N$$

This situation leads to a contradiction as in the previous simple choreographic symmetry case.

**Case II:**

$$z_{H_2}^i(t) = w(t)$$

In this case, the symmetry implies that all the points collides to  $w$ . Since  $w$  has period  $\frac{T}{N}$ , we conclude that

$$T^* \leq \frac{T}{N}$$

This is again a contradiction. As a result the proposition is proved.  $\square$

### Conclusion and future works

By adapting Palais' principle to the modified Hamiltonian, we have shown the existence of N-choreography and (N+1)-choreography of the vortex system as minimax of the action functional. These orbits are obtained via topological linking method. Unfortunately, little information is known about their concrete configuration. In particular, we don't not know whether these solutions correspond to Thomson's configuration. From a practical point of view, it will also be interesting if numerical methods based on the variational principle could be developed. We will explore these possibilities in the future.

## B.5 Verification of Palais-Smale condition

We follow the same line in [93] to show that the functional  $I_{H_2}$  verifies the Palais-Smale condition.

**Definition B.5.1.** *Given a Banach space  $E$  and a functional  $I \in \mathcal{C}(E, \mathbb{R})$ . We say  $z^m$  is a Palais Smale in  $E$ , if they satisfy*

$$\begin{aligned} |I_{H_2}(z^m)| &\leq M \\ |DI_{H_2}(z^m)| &\rightarrow 0 \end{aligned}$$

**Definition B.5.2.** *The functional  $I$  is said to satisfy Palais-Smale condition if every Palais-Smale sequence has a convergent subsequence.*

**Lemma B.5.1.**  *$I_{H_2}$  satisfies the Palais-Smale condition on  $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$*

*Proof.* For large  $m$ , by taking  $z = z^m$

$$\begin{aligned}
M + \|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} &\geq I_{H_2}(z) - \frac{1}{2}DI_{H_2}(z)z = \int_0^T \frac{1}{2}zDH_2(z) - H_2(z)dt \\
&= \int_0^T \mathcal{C}_N^2 H_1(z) + g'(|z|^2)|z|^{2k+2} + kg(|z|^2)|z|^{2k} - H_1(z) - g(|z|^2)|z|^{2k} dt \\
&\geq \int_0^T (k-1)g(|z|^2)|z|^{2k} dt \\
&\geq \mu(k-1)\|z\|_{L_T^{2k}}^{2k}
\end{aligned}$$

Next, as  $k$  is large, since we have that (due to (g4))

$$|DH_2(z)| \leq m(|z|^{2k-1} + 1)$$

by Holder's inequality for  $p = \frac{2k}{2k-1}, q = 2k$ , we have

$$\begin{aligned}
2\|z^+\|^2_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} &\leq \left| \int_0^T DH_2(z)z^+ \right| + \|z^+\| \leq \int_0^T a_3(|z|^{2k-1} + 1)|z^+| + \|z^+\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} \\
&\leq a_3 \left( \int_0^T |z|^{(2k-1)p} \right)^{\frac{1}{p}} \|z^+\|_{L^q} + a_4 \|z^+\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} \\
&\leq a_5 (\|z\|_{L^{2k}}^{2k-1} + 1) \|z^+\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}
\end{aligned}$$

As a result,

$$\|z^+\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} \leq a_5 (\|z\|_{L^{2k}}^{2k-1} + 1) \leq a_6 \left( \|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^{\frac{2k-1}{2k}} + 1 \right)$$

Similar analysis shows that  $z^-$  is also bounded in the same topology. Finally for the part in  $E^0$ , we have

$$\begin{aligned}
M + \|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} &\geq I_{H_2}(z) - \frac{1}{2}DI_{H_2}(z)z = \int_0^T \frac{1}{2}zDH_2(z) - H_2(z)dt \\
&\geq \mu(k-1)\|z\|_{L_T^{2k}}^{2k} \\
&\geq \mu(k-1)|z^0|^{2k}
\end{aligned}$$

Together we have shown

$$\|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} \leq M \left( 1 + \|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^{\frac{2k-1}{2k}} + \|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^{\frac{1}{2k}} \right)$$

As a result, put (51) into (53)

$$\|z^+\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} \leq a_5(\|z\|_{L^{2k}}^{2k-1} + 1) \leq a_6(\|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^{\frac{2k-1}{2k}} + 1)$$

Similar analysis shows that  $z^-$  is also bounded in the same topology. Finally for the part in  $E^0$ , we have

$$\begin{aligned} M + \|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} &\geq I_{H_2}(z) - \frac{1}{2}DI_{H_2}(z)z = \int_0^T \frac{1}{2}zDH_2(z) - H_2(z)dt \\ &\geq \mu(k-1)\|z\|_{L_T^{2k}}^{2k} \\ &\geq \mu(k-1)|z^0|^{2k} \end{aligned}$$

Together we have shown

$$\|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} \leq M(1 + \|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^{\frac{2k-1}{2k}} + \|z\|_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})}^{\frac{1}{2k}})$$

By comparing the powers on both sides we see  $z^m$  is bounded in the  $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$  topology. It implies that they three parts  $z^{m+}$ ,  $z^{m-}$ , and  $z^{m0}$  in the decomposition  $z^m = z^{m+} + z^{m-} + z^{m0}$  are all bounded. There only leaves to show the compactity.

1. compactity of  $z^0$ :  $E^0$  is a finite dimensional Banach Space. As a result the boundedness implies the compactity.
2. compactity of  $z^\pm$ : Let  $P^\pm$  be the projection of  $z \in H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$  to  $z^\pm \in E^\pm$  relatively. On one hand, note that  $z^m$  is a Palais-Smale sequence, hence  $DI_{\mathcal{H}_\varepsilon}(z_m) \rightarrow 0$ ; On the other side,  $H_2(z) = H_1(z) + f(I(z))$  is of polynomial growth (because of (g4)), as a result

$$\nabla H_2 : H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N}) \rightarrow (H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N}))', z \rightarrow \nabla H_2(z)$$

is a compact operator. It follows that

$$\pm P^\pm(DI_{H_2})(z^m) = z^\pm + P^\pm \nabla H_2(z^m)$$

hence

$$z^{m\pm} = \pm P^\pm(DI_{H_2})(z^m) - P^\pm \nabla H_2(z^m)$$

are compact.

We conclude that the Palais-Smale condition holds for  $I_{H_2}$   $\square$

## B.6 Palais' Principle and the Symmetry of choreography

### B.6.1 Symmetry of choreography

we define a natural finite group action on the loop spaces. We use the notation in [31]. Let  $G$  be a finite subgroup of  $O(2) \times \Sigma_N \times O(2)$ . Let  $\Lambda$  be  $T$ -periodic loops in the configuration space of our vortex system (Note that for the vortex problem, the configuration space coincides with the phase space). Let  $g = (\tau, \sigma, \rho) \in G$  acts on  $z(t) = (z_1(t), z_2(t), \dots, z_n(t)) \in \Lambda$  be such that:

$$gz_i(t) = \rho y_{\sigma^{-1}(j)}(\tau^{-1}(t))$$

In the special case, let  $\rho = I$ ,  $\sigma^{-1}(j) = j - 1$ , with the convention that  $z_n = z_{0-1}$   $\tau^{-1}(t) = t - \frac{T}{n}$ , then the group thus generated is called the **group of choreography**, noted as  $G_c$ . Finally we denote the stabilizer of  $\bar{E}$  under action  $G_c$  to be the space of choreography. We note it as  $E_c$ , i.e.,

$$CH_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2n}) = \text{Fix}_{G_c} = \{z \in H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N}) \mid gz = z, \forall g \in G_c\}$$

We would like to use the principle of symmetric criticality. The following theorem is due to Palais:

**Theorem B.6.1.** (*Palais' principle*) *Let  $G$  be a group of isometries of a Riemannian manifold  $M$  and let  $f : M \rightarrow \mathbf{R}$  be a  $\mathcal{C}^1$  function invariant under  $G$ . Then the set  $\Gamma$  of stationary points of  $M$  under the action of  $G$  is a totally geodesic smooth submanifold of  $M$ , and if  $p \in \Gamma$  is a critical point of  $f|_{\Gamma}$  then  $p$  is in fact a critical point of  $f$*

It remains to verify these hypothesis.

**Lemma B.6.1.**  $G_c$  is an isometric action on  $E$

*Proof.* Since the choreography is a cyclic group, we only need to show that the inner product of  $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$  is preserved under the action of the generator. Take arbitrary elements  $w = (\alpha, \beta), v = (\phi, \psi) \in H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$ . For the inner product, define the bilinear form

$$B(w, v) = \int_0^T \alpha d\psi + \int_0^T \phi d\beta \quad (\text{B.3})$$

We see that

$$\langle gw, gv \rangle_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} = B(gw^+, gv^+) - B(gw^-, gv^-) + \langle gw^0, gv^0 \rangle \quad (\text{B.4})$$

$$= \langle w^+, v^+ \rangle - B(w^-, v^-) + \langle w^0, v^0 \rangle \quad (\text{B.5})$$

$$= \langle w, v \rangle_{H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})} \quad (\text{B.6})$$

As a result we conclude that  $G_c$  is an isometric action on  $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$  □

Next we show the invariance of the functional under the group action of  $G_c$

**Lemma B.6.2.**  $I_{H_2}$  is invariant under the action of  $G_c$ .

*Proof.* As before we only need to verify the invariance for the generator of cyclic group. Actually,

$$\begin{aligned} I_{H_2}(gz) &= \sum_{1 \leq i \leq N} \int_0^T y_{i+1}(t + \frac{T}{N}) dx_{i+1}(t + \frac{T}{N}) - H_2(gz) \\ &= \sum_{1 \leq i \leq N} \int_0^T y_i(t) dx_i(t) - H_2(z) \\ &= I_{H_2}(z) \end{aligned}$$

□

By the above lemmas, we can apply Palais' symmetric principle to our analysis, and conclude that:

**Proposition B.6.1.** *If  $z$  is a critical point restricted on  $CH_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2n})$ , then  $z$  is a critical point in  $H_T^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^{2N})$  which is itself a choreographic orbit.*

The validity for Palais' principle for choreographic symmetry with a center is verified in a similar way.



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## Résumé

Cette thèse porte sur l'étude des solutions périodiques du problème des  $N$  tourbillons à vortacité positive. Ce problème, formulé par Helmholtz il y a plus de 160 ans, possède une histoire très riche et reste un domaine de recherche très actif. Pour un nombre quelconque de tourbillons et sans contrainte sur les vorticités, ce système n'est pas intégrable au sens de Liouville : on ne peut pas trouver de solution périodique non triviale par des méthodes explicites. Dans cette thèse, à l'aide de méthodes variationnelles, nous prouvons l'existence d'une infinité de solutions périodiques non triviales pour un système de  $N$  tourbillons à vorticités positives. De plus, lorsque les vorticités sont des nombres rationnels positifs, nous montrons qu'il n'existe qu'un nombre fini de niveaux d'énergie sur lesquels un équilibre relatif pourrait exister. Enfin, pour un système de  $N$  tourbillons identiques, nous montrons qu'il existe une infinité de choréographies simples.

## Mots Clés

système Hamiltonien, orbite périodique,  $N$ -Tourbillon, symétrie.

## Abstract

This thesis focuses on the study of the periodic solutions of the  $N$ -vortex problem of positive vorticity. This problem was formulated by Helmholtz more than 160 years ago and remains an active research field. For an undetermined number of vortices and general vorticities the system is not Liouville integrable and periodic solutions cannot be determined explicitly, except for relative equilibria. By using variational methods, we prove the existence of infinitely many non-trivial periodic solutions for arbitrary  $N$  and arbitrary positive vorticities. Moreover, when the vorticities are positive rational numbers, we show that there exists only finitely many energy levels on which there might exist a relative equilibrium. Finally, for the identical  $N$ -vortex problem, we show that there exist infinitely many simple choreographies.

## Keywords

Hamiltonian System, periodic orbits,  $N$ -Vortex, symmetry.