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On the Constraint Equations in General Relativity

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Université Claude Bernard Lyon 1
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Autour des équations de contrainte en relativité générale

Thèse présentée en vue d'obtenir le diplôme de
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Caterina Vâlcu

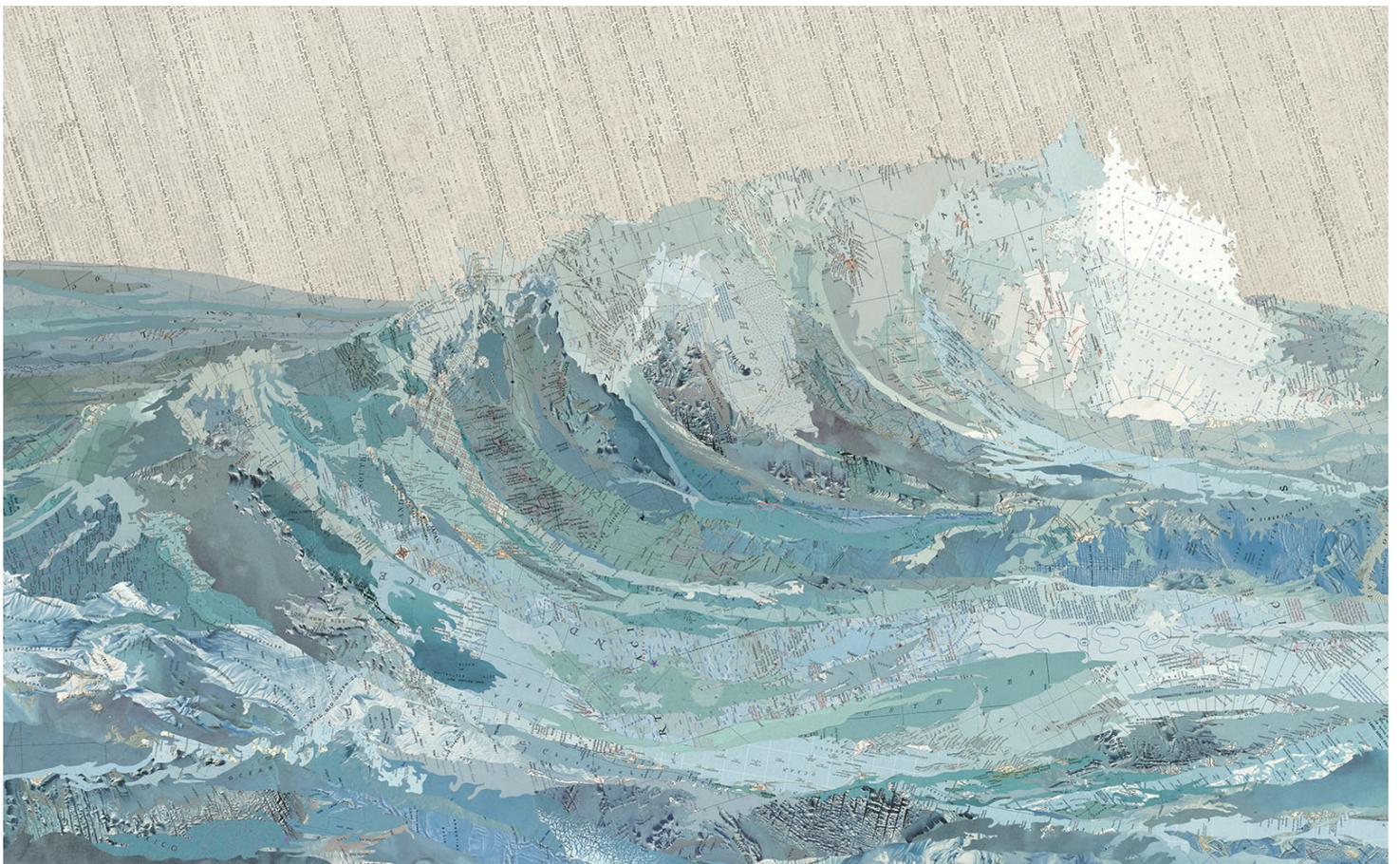
devant le jury composé de :

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Autour des équations de contrainte en relativité générale



Caterina VÂLCU
Thèse de doctorat

Remerciements

Much of writing mathematics is atemporal. The objective present tense is used to describe what we believe to be timeless truths. Here and there, an author might permit themselves to conjure a not-very-convincing future : “in the next section, we shall...” ; but for now, let us ground ourselves in the more common present. It is early autumn in Lyon, a city in France, twice-river-crossed, that I have grown to know. A few mounds of wilted leaves spot the sidewalks, caught in the speckled light of yellow, red and pale greens trees. I would like to speak of the three years I have spent here, and extended my most sincere gratitude to the people who have been by my side. Without you, none of this work would have ever been possible. Writing a thesis, I have found, is both deeply frustrating and a great joy, but ultimately meaningless were it not, as they say, for the wonderful people met along the way.

Here goes.

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Célia and Vincent, this past year has been magnificent! Thank you for the unreasonable, enriching experience that has been our time together, with its brunches, and picnics, and breakfasts on the window sill, with music and jams and so much more. You were my family in Lyon. And through you, I got the lucky chance to meet Thibault (whom I must particularly thank for the funniest, most informative introduction to France any foreigner could ever get), the both gentle and radical Estelle, and Loreena - baker, dancer, activist. You are all part of my heart.

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Introduction en français

La relativité générale est l'étude des espace-temps, qui sont des variétés lorentziennes $(\widetilde{M}, \widetilde{g})$ satisfaisant les équations d'Einstein

$$Ric_{\alpha\beta}(\widetilde{g}) - \frac{1}{2}R(\widetilde{g})\widetilde{g}_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

$\alpha, \beta = \overline{0, n}$. Ici, $R(\widetilde{g})$ désigne la courbure scalaire de \widetilde{g} , Ric est la courbure de Ricci, et $T_{\alpha\beta}$ est le tenseur d'énergie impulsion, qui décrit la présence de matière et d'énergie. En présence d'un champ scalaire, le tenseur d'énergie-impulsion devient

$$T_{\alpha\beta} = \nabla_{\alpha}\widetilde{\psi}\nabla_{\beta}\widetilde{\psi} - \left(\frac{1}{2}|\nabla\widetilde{\psi}|_{\widetilde{g}}^2 + V(\widetilde{\psi})\right)\widetilde{g}_{\alpha\beta},$$

où $\widetilde{\psi} \in \mathcal{C}^{\infty}(M)$ est le champ scalaire, et $V \in \mathcal{C}^{\infty}(\mathbb{R})$ est le potentiel. Par exemple, $V = \Lambda$ et $\widetilde{\psi} = 0$ modélise le vide avec Λ pour constante cosmologique, et $V = \frac{1}{2}m\widetilde{\psi}^2$, $m > 0$ correspond au cas de Klein-Gordon.

Dans les années 50 et 60, le champ de la relativité générale a connu une grande avancée. En effet, Choquet-Bruhat, et plus tard Choquet-Bruhat et Geroch ([CBG69], [FB52]) ont montré que les équations d'Einstein peuvent être envisagées comme un problème d'évolution. Dans le cas d'un champ scalaire, les conditions initiales sont la donnée de (M, g, K, ψ, π) , où (M, g) est une variété riemannienne de dimension n , K est un 2-tenseur symétrique correspondant à la seconde forme fondamentale, ψ représente le champ scalaire dans M , et π sa dérivée temporelle. En relativité générale, les conditions initiales sont solutions des équations de contraintes,

$$\begin{cases} R(g) + tr_g K^2 - |K|_g^2 = \pi^2 + |\nabla\psi|_g^2 + 2V(\psi) \\ \partial_i(tr_g K) - K_{i,j}^j = \pi\partial_i\psi. \end{cases} \quad (0.0.1)$$

Ce système est clairement sous-déterminé. Dans le cas $n = 3$, par exemple, il consiste en 4 équations de 14 inconnues. Ainsi, bien que l'on ne puisse pas choisir des conditions initiales données (M, g, K, ψ, π) quelconques, l'ensemble des données admissibles est riche – on peut intuitivement l'assimiler à une variété de dimension 10. En outre, chaque condition initiale admissible détermine de manière unique un espace-temps maximal, et tout espace-temps globalement hyperbolique est la solution d'un problème de Cauchy, ce qui justifie le bien-fondé de notre étude.

La méthode conforme est la première approche générale dans l'identification de conditions initiales (Lichnerowicz [Lic44]). Elle permet aux équations de contrainte (0.0.1) de devenir un système déterminé en fixant certains paramètres bien choisis. Pour illustrer cette méthode, revenons au cas $n = 3$. Le système (0.0.1) contient une équation scalaire et une équation vectorielle, soit un total de 4 équations scalaires. Les inconnues sont les 2-tenseurs symétriques \hat{g}_{ab} et \hat{K}_{ab} , où $a, b \in \overline{1, 3}$, ainsi que les fonctions $\hat{\pi}$ et $\hat{\psi}$, soit 14 inconnues. Le système comporte exactement 10 degrés de liberté. On commence donc par fixer un premier paramètre en imposant à \hat{g}_{ab} d'appartenir à une classe conforme qu'on écrit \mathbf{g} , ou $[\hat{g}_{ab}]$, où

$$\mathbf{g} = \{e^{\alpha}g, \quad \alpha \in \mathcal{C}^{\infty}\}.$$

Soit $\hat{g} = u^{q-2}g$, $u \in \mathcal{C}^\infty$, $u > 0$, où q est l'exposant de Sobolev critique, $q = \frac{2n}{n-2}$. Alors, la courbure scalaire, sous une transformation conforme, se transforme selon la formule suivante

$$R(\hat{g}) = u^{1-q} \left(\frac{4(n-1)}{n-2} \Delta_g u + R(g)u \right),$$

où $\Delta_g = -\text{div}_g \nabla$ est l'opérateur de Laplace-Beltrami. La méthode conforme consiste en un choix de paramètres qui se comportent bien sous les transformations conformes. Les tenseurs de trace et de divergence nulles en sont un autre exemple. Ils obéissent à la loi $\hat{U}_{ab} = u^{-2}U_{ab}$, ce qui signifie qu'on peut considérer la classe d'équivalence $\mathbf{U} = [\hat{g}_{ab}, \hat{U}_{ab}] = [g_{ab}, U_{ab}]$ comme un autre paramètre, où U_{ab} apparaît dans la décomposition de la partie de trace nulle de K , comme le montre le tableau suivant pour $n = 3$:

Données physiques	Paramètres	Dimensions	Inconnues
$\hat{g}_{ab} = u^{q-2}g_{ab}$	\mathbf{g}	5	1
$\hat{K}_{ab} = u^{-2}[(\mathcal{L}_g W)_{ab} + U_{ab}] + \frac{\tau}{n}u^{q-2}g_{ab}$	\mathbf{U}, τ	2 + 1	3
$\hat{\psi} = \psi$	ψ	1	0
$\hat{\pi} = u^{-q}\pi$	π	1	0

Pour une fonction u et un champ de vecteur W , lisses sur Σ , une variété fermée, le système en (u, W) , se simplifie en

$$\begin{aligned} \Delta_g u + \frac{n-2}{4(n-1)} (R(g) - |\nabla \psi|_g^2) u &= -\frac{n-2}{4(n-1)} \left(\frac{n-1}{n} \tau^2 - 2V(\psi) \right) u^{q-1} \\ &\quad + \frac{n-2}{4(n-1)} \frac{(|U + \mathcal{L}_g W|_g^2 + \pi^2)}{u^{q+1}}, \\ \vec{\Delta}_g W &= \frac{n-1}{n} u^q \nabla \tau - \pi \nabla \psi, \end{aligned}$$

où $\mathcal{L}_g W_{ij} = W_{i,j} + W_{j,i} - \frac{2}{n} \text{div}_g W g_{ij}$ est l'opérateur de Killing conforme, et $\vec{\Delta}_g W = -\text{div}_g \mathcal{L}_g W$ est l'opérateur de Lamé. Cette méthode s'avère particulièrement efficace pour trouver des solutions dans le cas d'une courbure moyenne constante ou presque constante. Cela est principalement dû au fait que le système devient, dans ce cas, seulement semi-couplé, ce qui rend son analyse plus aisée. À ce jour, il n'est pas sûr que cette méthode soit pertinente dans le cas général, puisque le système couplé reste très compliqué.

En 2011, David Maxwell a montré que les paramétrages donnés par la méthode conforme classique peuvent être dégénérés, en identifiant certains paramètres explicites qui correspondent à une infinité de solutions ([Max11], [Max15]). Cependant, cet échec de la théorie n'est pas nécessairement lié à l'existence de singularités dans l'espace des solutions, mais est peut-être conséquence d'un choix de paramétrage malheureux. Dans son article de 2014, Maxwell propose une variante de la méthode conforme standard [Max14b]. Dans le cas d'un champ scalaire, le système prend la forme suivante

$$\begin{aligned} \Delta_g u + \frac{n-2}{4(n-1)} (R(g) + |\nabla \psi|_g^2) u &= \frac{(n-2)|U + \mathcal{L}_g W|^2 + \pi^2}{4(n-1)u^{q+1}} \\ &\quad + \frac{n-2}{4(n-1)} \left[2V(\psi) - \frac{n-1}{n} \left(\tau^* + \frac{\text{div}_g(u^q \tilde{V})^2}{N_{g,\omega} u^{2q}} \right) \right] u^{q-1} \quad (0.0.2) \\ \text{div}_g \left(\frac{1}{2N_{g,\omega}} \mathcal{L}_g W \right) &= \frac{n-1}{n} u^q d \left(\frac{\text{div}_g(u^q \tilde{V})}{2N_{g,\omega} u^{2q}} \right) + \pi \nabla \psi = 0. \end{aligned}$$

où les inconnues sont la fonction strictement positive et lisse u et le champ vectoriel W^i , et où d est la dérivée extérieure, τ^* est la constante qui apparaît dans l'expression du moment

volumétrique, \tilde{V} est le champ vectoriel associé à la dérive inertielle (« drift momentum », une quantité introduite par Maxwell), et $N_{g,\omega}$ est la « densitized lapse ». Ici, nous avons supposé que g n'admet pas de champ de Killing conforme non trivial.

Dans [Vâl19], nous nous intéressons au cas d'un potentiel V positif ou nul dans le système de Maxwell. L'équation scalaire présente une non-linéarité critique, avec un signe positif. En d'autres termes, nous nous intéressons au cas focalisant. Le cas non focalisant a été traité par Holst, Maxwell et Mazzeo en 2018 [HMM18]. Notre problème traite de l'existence de solutions (u, W) , $u > 0$, à

$$\begin{aligned} \Delta_g u + hu &= fu^{q-1} + \frac{\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W|_g^2}{u^{q+1}} \\ &\quad - \frac{b}{u} - c \langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) - \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}} \quad (0.0.3) \\ \operatorname{div}_g(\rho_3 \mathcal{L}_g W) &= \mathcal{R}(u, \nabla u, \nabla^2 u), \end{aligned}$$

où $f > 0$, $h, b, c, d, \rho_1, \rho_2, \rho_3$ sont des fonctions lisses, Ψ, Y sont des champs vectoriels, et l'opérateur \mathcal{R} vérifie, pour une constante $C_{\mathcal{R}} > 0$,

$$\mathcal{R}(u, \nabla u, \nabla^2 u) \leq C_{\mathcal{R}} \left(1 + \frac{\|u\|_{\mathcal{C}^2}^2}{(\inf_M u)^2} \right)$$

Soient

$$\theta = \min(\inf_M \rho_1, \inf_M f), \quad (0.0.4)$$

et

$$T = \max(\|f\|_{\mathcal{C}^{1,\gamma}}, \|\rho_1\|_{\mathcal{C}^{0,\gamma}}, \|c\|_{\mathcal{C}^{0,\gamma}}, \|d\|_{\mathcal{C}^{0,\gamma}}, \|h\|_{\mathcal{C}^{0,\gamma}}). \quad (0.0.5)$$

Theorem 1 ([Vâl19]). *Soit $n = 3, 4$ ou 5 . Il existe une constante $C = C(n, h) > 0$ telle que si ρ_1 vérifie*

$$\|\rho_1\|_{L^1(M)} \leq C(n, h) \left(\max_M |f| \right)^{1-n}, \quad (0.0.6)$$

alors il existe une constante

$$\delta = \delta(\theta, T) > 0$$

telle que le système (0.0.3) admet une solution si

$$\|b\|_{\mathcal{C}^{0,\gamma}} + \|Y\|_{\mathcal{C}^{0,\gamma}} + \|\Psi\|_{\mathcal{C}^{0,\gamma}} + \|\rho_2\|_{\mathcal{C}^{0,\gamma}} + C_{\mathcal{R}} \leq \delta. \quad (0.0.7)$$

Pour obtenir ce résultat d'existence, nous imposons essentiellement deux conditions : les paramètres doivent être suffisamment petits, et la dimension aussi. La démonstration est inspirée du travail de Bruno Premoselli [Pre14], mais traite les difficultés supplémentaires liées à la présence de termes non-linéaires impliquant le gradient de u , et aussi à la structure non-variationnelle de la première équation. Nous utilisons un argument de point fixe. En bref, nous définissons un opérateur de « ping-pong » qui alterne entre les deux équations du système, fixant une inconnue pour résoudre en l'autre, et vice-versa. Un théorème du point fixe nous permet alors de conclure. Nous commençons par établir, grâce à une méthode itérative, l'existence de solutions à l'équation scalaire de Lichnerowicz. La solution s'avère minimale et stable.

Nous montrons aussi que les perturbations des paramètres de l'équation n'entraînent pas de perte de compacité dans les dimensions $n = 3, 4$, et 5 .

Theorem 2 ([Vâl19]). *Soit (M, g) une variété riemannienne fermée de dimension $n = 3, 4$ ou 5 . Soient $\frac{1}{2} < \eta < 1$ et $0 < \alpha < 1$. Soient a, b, c, d, f , et h des fonctions lisses sur M , et Y un champ vectoriel lisse sur M . Pour tout $0 < \theta < T$, il existe deux constantes $S_{\theta, T}$ et $\vartheta_{\theta, T}$ telle que toute solution strictement positive et lisse u de l'équation de Lichnerowicz vérifie $\|u\|_{\mathcal{C}^2} \leq S_{\theta, T}$, dès lors que les paramètres sont dans*

$$\mathcal{E}_{\theta, T} := \left\{ (f, a, b, c, d, h, Y), \quad f \geq \theta, \quad a \geq \theta, \right. \quad (0.0.8)$$

$$\left. \|f\|_{\mathcal{C}^{1, \eta}} \leq T, \quad \|a\|_{\mathcal{C}^{0, \alpha}}, \|b\|_{\mathcal{C}^{0, \alpha}}, \|c\|_{\mathcal{C}^{0, \alpha}}, \|d\|_{\mathcal{C}^{0, \alpha}}, \|h\|_{\mathcal{C}^{0, \alpha}}, \|Y\|_{\mathcal{C}^{0, \alpha}} \leq T \right\},$$

avec

$$\|Y\|_{\mathcal{C}^{0, \alpha}} \leq \vartheta_{\theta, T}, \quad (0.0.9)$$

Nous montrons la stabilité des solutions du système (0.0.3) : ce résultat est obtenu en dimension $3 \leq n \leq 5$, dans le cas où la métrique est localement conformément plate, et le drift est petit.

Theorem 3. *Soit (M, g) une variété riemannienne fermée de dimension $n = 3, 4$ ou 5 , où g est localement conformément plat. Soient $\frac{1}{2} < \eta < 1$ et $0 < \alpha < 1$. Soient $\frac{1}{2} < \eta < 1$ et $0 < \alpha < 1$. Soient $a, b, c, d, f, h, \rho_1, \rho_2, \psi, \pi$, et \tilde{N} des fonctions lisses sur M , et Y un champ vectoriel lisse sur M . Pour tout $0 < \theta < T$, il existe deux constantes $S_{\theta, T}$ et $\vartheta_{\theta, T}$ telle que, dès lors que les paramètres sont dans*

$$\begin{aligned} \mathcal{E}_{\theta, T} := & \left\{ (f, a, b, c, d, h, \rho_1, \rho_2, Y) \times (\tilde{N}, \tilde{V}, \psi, \pi), \quad f \geq \theta, \quad a \geq \theta, \quad \tilde{N} \geq \theta, \right. \\ & \|f\|_{\mathcal{C}^{1, \eta}} \leq T, \\ & \|a\|_{\mathcal{C}^{1, \alpha}}, \|b\|_{\mathcal{C}^{1, \alpha}}, \|c\|_{\mathcal{C}^{1, \alpha}}, \|d\|_{\mathcal{C}^{1, \alpha}}, \\ & \left. \|\rho_1\|_{\mathcal{C}^{1, \alpha}}, \|\rho_2\|_{\mathcal{C}^{1, \alpha}}, \|h\|_{\mathcal{C}^{1, \alpha}}, \|Y\|_{\mathcal{C}^{1, \alpha}} \leq T, \right. \\ & \left. \text{and } \|\tilde{N}\|_{\mathcal{C}^{2, \alpha}}, \|\tilde{V}\|_{\mathcal{C}^{2, \alpha}} \leq T \right\}, \end{aligned} \quad (0.0.10)$$

avec

$$\|Y\|_{\mathcal{C}^{1, \alpha}}, \|\tilde{V}\|_{\mathcal{C}^{2, \alpha}} \leq \vartheta_{\theta, T}, \quad (0.0.11)$$

toute solution (u, W) , avec u strictement positive et lisse, vérifie

$$\|u\|_{\mathcal{C}^{2, \alpha}} + \|\mathcal{L}_g W\|_{\mathcal{C}^{1, \alpha}} \leq S_{\theta, T}. \quad (0.0.12)$$

Le but à long terme est de trouver une alternative viable à la méthode conforme, qui nous permettrait de mieux comprendre la structure géométrique de l'espace des solutions des équations de contrainte. L'idée sera, par exemple, de montrer que, en dimension 3, l'ensemble des solutions est une variété lisse de dimension 10. Nous rappelons que les anciens paramétrages proposés sont souvent trop compliqués à analyser (comme, par exemple, dans le cas des systèmes qui correspondent aux solutions à courbure moyenne loin d'être constante), et qu'ils présentent des singularités.

L'avantage du système de Maxwell par rapport aux modèles plus classiques est la présence des paramètres supplémentaires. Une idée sera d'utiliser ces paramètres de plus pour incliner les axes des coordonnées (de dimension 10) dans le voisinage d'une singularité. Ainsi, nous montrerons que ces singularités ne sont que des fausses singularités, dues à de mauvais choix de paramétrage. Le prix à payer, par contre, sera la complexité analytique du système correspondant.

Many thanks to Simon Zugmeyer for his help in writing this section in French.

1	Introduction	1
1.1	Differential geometry	2
1.1.1	Manifolds and tensors	3
1.1.2	Metrics	3
1.1.3	Geodesics on Lorentzian manifolds. The exponential map	3
1.1.4	Pullbacks. Flow. The Lie derivative	4
1.1.5	Killing fields	5
1.1.6	Connections	5
1.1.7	Curvature	6
1.1.8	The second fundamental form	6
1.2	General relativity preliminaries	7
1.2.1	Space-times	7
1.2.2	Initial data	9
1.3	The conformal method	12
1.3.1	Toy model : the CMC conformal method	12
1.3.2	The “classical” conformal method	14
1.4	Solving the classical method system	15
1.4.1	Non-focusing case	17
1.4.2	Focusing case	18
1.4.3	The viability of conformal method models, examples and counter-examples	20
1.5	David Maxwell’s drift model	21
1.6	Is the drift model a better alternative?	23
1.7	Existence result for small data	24
1.7.1	Is the ping-pong operator well defined?	26
1.7.2	Is it a contraction?	26
1.8	The stability of the system.	31
2	Existence	35
2.1	Introduction	36
2.2	Existence of minimal solutions of the scalar equation	41
2.3	A priori estimates on solutions of the scalar equation in low dimensions	48
2.3.1	Concentration points	49
2.3.2	Local blow-up analysis	54
2.3.3	Proof of the stability theorem	66

2.4	Existence of solutions to the system	71
2.4.1	The proof of the main theorem	71
2.4.2	The case of a metric with conformal Killing fields	74
2.5	Annex	76
3	Stability	79
3.1	Introduction	80
3.1.1	The main result.	82
3.2	Conformal changes of coordinates.	85
3.3	Weak pointwise estimates	86
3.4	Asymptotic analysis	92
3.4.1	Harnack inequality	92
3.4.2	Strong estimate on v_α in $B_0(\mu_\alpha)$	93
3.4.3	The sphere of dominance around a blow-up point	94
3.4.4	Stability theorem proof	103
3.5	Annex	105
3.5.1	Standard elliptic theory for the Lamé operator	105
3.5.2	Limiting equation	107
	Bibliographie	109

CHAPITRE 1

Introduction

Introduction

Contents

1.1	Differential geometry	2
1.1.1	Manifolds and tensors	3
1.1.2	Metrics	3
1.1.3	Geodesics on Lorentzian manifolds. The exponential map	3
1.1.4	Pullbacks. Flow. The Lie derivative	4
1.1.5	Killing fields	5
1.1.6	Connections	5
1.1.7	Curvature	6
1.1.8	The second fundamental form	6
1.2	General relativity preliminaries	7
1.2.1	Space-times	7
1.2.2	Initial data	9
1.3	The conformal method	12
1.3.1	Toy model : the CMC conformal method	12
1.3.2	The “classical” conformal method	14
1.4	Solving the classical method system	15
1.4.1	Non-focusing case	17
1.4.2	Focusing case	18
1.4.3	The viability of conformal method models, examples and counter-examples	20
1.5	David Maxwell’s drift model	21
1.6	Is the drift model a better alternative?	23
1.7	Existence result for small data	24
1.7.1	Is the ping-pong operator well defined?	26
1.7.2	Is it a contraction?	26
1.8	The stability of the system.	31

1.1 Differential geometry

Before we properly begin, let us recall some notions from differential geometry that will be useful in the reading of the present text. For a more detailed introduction, we recommend the book of Sylvestre Gallot, Dominique Hulin and Jacques Lafontaine, together with that of Peter Peterson, and the course notes of Frédéric Paulin [GHL04, Pet16, Pau13].

For more background on general relativity, see the work of Robert Wald [Wal84]. A particularly lovely resource that functions as a great introduction to general relativity and also includes a chapter on differential geometry is the book of Yvonne Choquet-Bruhat [CB15].

1.1.1 Manifolds and tensors

Let M be a differential manifold; intuitively, a manifold may be seen as an object that can be locally identified with an Euclidean space. A topological n -dimensional manifold is defined as a set M , together with a collection of charts (U_I, φ_I) called an atlas. Here, I is an arbitrary set of indices. For any $i \in I$, we define $U_i \subset M$, and $\varphi_i : U_i \rightarrow V \subset \mathbb{R}^n$, $x \mapsto \varphi(x) \equiv (x^1, \dots, x^n)$, a one-to-one invertible mapping, where V is an open in \mathbb{R}^n . The numbers x^k , $k = \overline{1, n}$, are called local coordinates. The atlas we describe is the structure that allows for the local identification of M with \mathbb{R}^n . If the mappings $\varphi_i \circ \varphi_j^{-1}$ are homeomorphisms between open sets in \mathbb{R}^n for any two $i, j \in I$, then M is a topological manifold. If moreover they are \mathcal{C}^k , with $k \in \mathbb{N} \cup \{\infty\}$, then M is a \mathcal{C}^k differential manifold.

Finally, we ask that any two different points have non-intersecting neighbourhoods. This is the Hausdorff topology.

A differentiable submanifold $\Sigma \subset M$ is the image by a differentiable mapping $f : M' \rightarrow \Sigma \subset M$ on a manifold M' . We call Σ an embedded manifold if f is injective.

We denote by $T_x M$ the tangent space to M in a point x . Tangent vectors are geometrical objects (*i.e.* they are independent of changes of coordinates). The disjoint union TM of all $T_x M$, with $x \in M$, is called the tangent bundle to M . A vector field is a mapping $M \rightarrow TM$ which gives the identity when composed with the standard projection, also known as a section of the tangent bundle. The set of all vector fields is denoted as $\Gamma(TM)$.

Additionally, the cotangent space at x is the set of linear forms on $T_x M$ and we denote it by $(T_x M)^*$. The disjoint union of all $(T_x M)^*$ is called the cotangent bundle $(TM)^*$.

A tensor of type (p, q) is a multilinear form that is p -contravariant and q -covariant. The set of such tensors is denoted as

$$\underbrace{TM \otimes \dots \otimes TM}_p \otimes \underbrace{T^*M \otimes \dots \otimes T^*M}_q. \quad (1.1.1)$$

A tensor field of type (p, q) is a section of the respective bundle.

1.1.2 Metrics

A pseudo-Riemannian metric on a manifold M is a symmetric covariant 2-tensor field such that the quadratic form it defines on contravariant vectors is non-degenerate (in other words, if g is a metric, then $\det g$ does not vanish in any chart).

A metric is Riemannian if its quadratic form in each point is positive-definite. The Euclidean δ is such an example. If the signature of the quadratic form is $(-, +, \dots, +)$, however, the metric is called Lorentzian. The standard example in this case is the Minkowski metric, whose quadratic form reads

$$m = -(dx^0)^2 + (dx^1)^2 \dots + (dx^n)^2. \quad (1.1.2)$$

The volume form of the metric g is the exterior n -form that can be locally represented as

$$dV_g = \sqrt{|\det g|} dx^1 dx^2 \dots dx^n. \quad (1.1.3)$$

An isometry f of a pseudo-Riemannian manifold (M, g) is a diffeomorphism that leaves g invariant : $\forall x \in M, (f^*g)(x) = g(x)$.

1.1.3 Geodesics on Lorentzian manifolds. The exponential map

The length of a causal continuously differential curve γ between two points $\gamma(a)$ and $\gamma(b)$ is

$$l(\gamma) = \int_a^b \left(-g_{ab} \frac{d\gamma^\alpha}{d\lambda} \frac{d\gamma^\beta}{d\lambda} \right)^{\frac{1}{2}} d\lambda. \quad (1.1.4)$$

This is similar to the Riemannian case, up to a change of sign. Null curves have zero length.

In order to define geodesics, we ask that it is a critical point of the Lagrangian

$$\int_a^b g_{\alpha\beta}(x(\lambda)) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} d\lambda. \quad (1.1.5)$$

In coordinate form, its corresponding Euler equation takes the form

$$2g_{\alpha\beta} \frac{d^2 x^\beta}{d\lambda^2} + \left(2 \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \right) \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0. \quad (1.1.6)$$

Any geodesic minimizes the length of a path between two given points. The inverse does not hold. Let $X \in T_x M$ be a tangent vector. There exists a unique geodesic γ_X satisfying

$$\begin{aligned} \gamma_X(0) &= x \\ \gamma'_X(0) &= X. \end{aligned} \quad (1.1.7)$$

The exponential mapping is defined by $\exp_x(X) = \gamma_X(1)$.

1.1.4 Pullbacks. Flow. The Lie derivative

Let $\varphi : U \rightarrow V$ be a diffeomorphism between two open sets in M and let ω be a linear form defined on V ; the pullback of ω is defined on U as

$$\forall x \in U, \forall X \in T_x M, \quad \varphi^* \omega_x(X) = \omega_{\varphi(x)}(d\varphi(x), X), \quad (1.1.8)$$

whereas the pullback of a vector field X is

$$\varphi^* Y(x) = (d\varphi(x))^{-1}(Y(\varphi(x))). \quad (1.1.9)$$

These definitions may be extended to tensors.

Let I be an interval in \mathbb{R} . We call the flow associated with X on a differential manifold M the differential mapping $\varphi : I \times M \rightarrow M$ that verifies

$$\begin{aligned} \frac{d}{dt} \varphi_t(x) &= X(\varphi_t(x)), \\ \varphi_0(x) &= x. \end{aligned} \quad (1.1.10)$$

Here, φ_t is a local diffeomorphism.

We can now define the Lie derivative, which is a derivation operator. It maps a vector field T in the direction of another vector field X to

$$L_X T = \frac{d}{dt} (\varphi_{-t}^* T) |_{t=0}. \quad (1.1.11)$$

There is another way to view the Lie derivative,

$$L_X T = [X, T], \quad (1.1.12)$$

where $[X, Y]$ is a vector called the Lie bracket. Given $f \in \mathcal{C}^\infty$, the Lie bracket corresponds to

$$[X, Y]f = X(Yf) - Y(Xf). \quad (1.1.13)$$

where $Xf := df(X)$.

Note that the Lie derivative satisfies the Leibniz rule of multiplication. It is also \mathbb{R} linear with respect to T .

The definition of the Lie derivative can naturally be extended to p -tensors T , taking values in the set of p -tensors.

1.1.5 Killing fields

A vector field X is a Killing field if its associated flow is an isometry, or equivalently if

$$L_X g = 0. \quad (1.1.14)$$

We say that X is a conformal Killing field if

$$L_X g = fg, \quad (1.1.15)$$

where f is a positive scalar function.

Remark 1. *As an example, the Killing fields associated to the Minkowski space (\mathbb{R}^{1+n}, m) are*

- *translations* : $T_\alpha = \frac{\partial}{\partial x_\alpha}$,
- *rotations* : $\Omega_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$,
- *hyperbolic rotations* : $\Omega_{0i} = t \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial t}$.

We've taken $i \in \overline{1, n}$ and $t = x^0$.

1.1.6 Connections

We are interested in having a notion of a derivative that can be applied to tensors. By naively deriving the individual components of tensors, we do not necessarily obtain another tensor. At first glance, the Lie derivative seems like the perfect candidate. There is a drawback, however. When writing the Lie derivative $\mathcal{L}_X T$ of a tensor in coordinate form, we realize that not only does it depend on a neighbourhood of T (which seems natural), but also on a neighbourhood of X ; this clashes with our usual intuition of a derivation.

Connections are structures we may add to manifolds so that we can define a notion of derivative where this is not a problem.

Let $\tau \in \Gamma(E)$ be a section of a tensor bundle (TM or T^*M , for example) and a vector field of the tangent bundle $X \in \Gamma(TM)$. A covariant derivative or connection is an application $\nabla : \Gamma(TM) \times \Gamma(E)$ into 2-tensor fields, that satisfies the Leibniz rule for a product with a function,

$$\forall f \in \mathcal{C}^\infty(M), \forall \tau \in \Gamma(E), \quad \nabla_X(f\tau) = df(X)\sigma + f\nabla_X\sigma, \quad (1.1.16)$$

and that is \mathcal{C}^∞ linear in X and \mathbb{R} linear in τ . We call $\nabla_X\tau$ the derivative of τ with respect to X .

The Levi-Civita connection is a particular type of connection on TM associated to g such that

1. the covariant derivative of the metric is zero, $\nabla g = 0$, and
2. it has vanishing torsion, *i.e.* the second derivatives of scalar functions commute : $\forall f \in \mathcal{C}^\infty, \nabla_\alpha \partial_\beta f - \nabla_\beta \partial_\alpha f \equiv 0$.

It is unique. In coordinate form, the Levi-Civita connection associated to g reads

$$\nabla_X Y = X^i \frac{\partial Y^k}{\partial x^i} \frac{\partial}{\partial x_k} + X^i Y^k \Gamma_{ik}^j \frac{\partial}{\partial x_j}, \quad (1.1.17)$$

where Γ_{ik}^j are the Christoffel symbols associated to the Levi-Civita connection. They take the form

$$\Gamma_{ik}^j = \frac{1}{2}g^{jl}(\partial_i g_{kl} + \partial_k g_{il} - \partial_l g_{ik}). \quad (1.1.18)$$

Whenever we refer to a connection in the rest of the present text, we are referring to the Levi-Civita connection.

When applied to a scalar function, all the notions of derivatives coincide :

$$L_X f = \nabla_X f = df(X). \quad (1.1.19)$$

1.1.7 Curvature

In general, the covariant derivatives are non-commutative. The non-commutativity of covariant derivatives is a geometric property; it is measured through the 2-covariant, 1-contravariant curvature tensor R , which is defined as

$$\forall X, Y, Z \in \Gamma(TM), \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z, \quad (1.1.20)$$

and which, in coordinate form, reads as

$$R_{\alpha\beta}{}^{\gamma}{}_{\delta} v^{\delta} = (\nabla_{\alpha} \nabla_{\beta} - \nabla_{\beta} \nabla_{\alpha})v^{\gamma}. \quad (1.1.21)$$

As a function of the Christoffel symbols, the curvature tensors is written as

$$R_{\alpha\beta}{}^{\gamma}{}_{\delta} = \partial_{\alpha} \Gamma_{\beta\delta}^{\gamma} - \partial_{\beta} \Gamma_{\alpha\delta}^{\gamma} + \Gamma_{\alpha\mu}^{\gamma} \Gamma_{\beta\delta}^{\mu} - \Gamma_{\beta\mu}^{\gamma} \Gamma_{\alpha\delta}^{\mu}. \quad (1.1.22)$$

Moreover, we recall the definition of the Ricci operator

$$R_{\alpha\beta} = R_{\mu\alpha}{}^{\mu}{}_{\beta} \quad (1.1.23)$$

and of the scalar trace

$$R = g^{\alpha\beta} R_{\alpha\beta}. \quad (1.1.24)$$

The following properties are useful to keep in mind when working with the notion of the curvature :

1. $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$,
2. $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$,
3. the first Bianchi identity : $R_{\alpha\beta\gamma}{}^{\delta} + R_{\beta\gamma\alpha}{}^{\delta} + R_{\gamma\alpha\beta}{}^{\delta} = 0$,
4. the second Bianchi identity : $\nabla_{\alpha} R_{\beta\gamma\delta}{}^{\mu} + \nabla_{\beta} R_{\gamma\alpha\delta}{}^{\mu} + \nabla_{\gamma} R_{\alpha\beta\delta}{}^{\mu} = 0$.

1.1.8 The second fundamental form

Given Σ a hypersurface of M and T a normal vector field to Σ such that $g(T, T) = 1$ (or $g(T, T) = -1$). We consider both cases in order to account for hypersurfaces not only in Riemannian manifolds, but also Lorentzian ones. Let $\hat{\nabla}$ be the connection induced by ∇ to Σ . The second fundamental form K is defined as

$$\nabla_X Y = \hat{\nabla}_X Y - K(X, Y)T \quad (1.1.25)$$

Thus, if $g(T, T) = 1$, then

$$K(X, Y) = g(\nabla_X T, Y) \quad (1.1.26)$$

and if $g(T, T) = -1$, then

$$K(X, Y) = -g(\nabla_X T, Y). \quad (1.1.27)$$

Moreover, the second fundamental form can be expressed as

$$K = \frac{1}{2}(L_T g)_{T^* \Sigma \times T^* \Sigma} \quad (1.1.28)$$

if $g(T, T) = 1$, with the opposite sign if $g(T, T) = -1$.

1.2 General relativity preliminaries

1.2.1 Space-times

The theory of general relativity was developed by Einstein through a series of papers between 1907 and 1915. From a mathematical standpoint, the central object of study in general relativity is the space-time, which is defined as a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$ satisfying the Einstein field equations

$$Ric_{\alpha\beta}(\widetilde{g}) - \frac{1}{2}R(\widetilde{g})\widetilde{g}_{\alpha\beta} = 8\pi T_{\alpha\beta}, \quad \alpha, \beta = \overline{0, n}, \quad (1.2.1)$$

where $R(\widetilde{g})$ is the scalar curvature of \widetilde{g} , Ric the Ricci curvature and $T_{\alpha\beta}$ the stress-energy tensor describing the presence of matter and energy. Moreover, two space-times $(\widetilde{M}_1, \widetilde{g}_1)$ and $(\widetilde{M}_2, \widetilde{g}_2)$ are said to be equivalent if there exists a diffeomorphism

$$\varphi : \widetilde{M}_1 \rightarrow \widetilde{M}_2 \quad \text{such that} \quad \varphi^* \widetilde{g}_2 = \widetilde{g}_1. \quad (1.2.2)$$

This encompasses one of general relativity's postulates, namely the covariance principle, which states that physical laws are independent of any chosen coordinate system. Throughout this work, we consider Lorentzian manifolds with a $(-, +, \dots, +)$ signature.

In a given coordinate system, we generally denote by Latin indices any space-like directions and we have them take values between 1 and n . We use Greek indices for any general direction, with values between 0 and n . The causal structure on \widetilde{M} is defined as follows : a vector X in $T_p(\widetilde{M})$, $p \in \widetilde{M}$ is called

- time-like if $\widetilde{g}(X, X) < 0$,
- null if $\widetilde{g}(X, X) = 0$,
- space-like if $\widetilde{g}(X, X) > 0$.

Similarly, a vector field is timelike, null or spacelike if, at all $p \in \widetilde{M}$, it is a timelike, null or spacelike vector, respectively. The causal nature of a geodesic is given by that of its tangent vector field, and that of a surface by its normal vector field. An object is sometimes called causal if it is defined by timelike or null vectors.

In general, we use the indice 0 for local coordinates to refer to the time-like direction. The Einstein field equations may be rewritten as

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \quad (1.2.3)$$

where $G_{\alpha\beta}$ is called the gravitational tensor. Through its simplified form, (1.2.3) emphasizes a leading principle in general relativity, which is that the effect of gravitation is coupled to the presence of matter and energy within the space-time. Both $G_{\alpha\beta}$ and $T_{\alpha\beta}$ are symmetric 2-tensors. Let $\mathbf{R}(\widetilde{g})$ be the curvature tensor corresponding to \widetilde{g} and $\widetilde{\nabla}$ the

covariant derivative corresponding to \tilde{g} (see (1.1.17), (1.1.20)). From the second Bianchi identity,

$$\tilde{\nabla}_\nu \mathbf{R}(\tilde{g})^\alpha_{\beta\gamma\mu} + \tilde{\nabla}_\mu \mathbf{R}(\tilde{g})^\alpha_{\beta\nu\gamma} + \tilde{\nabla}_\gamma \mathbf{R}(\tilde{g})^\alpha_{\beta\mu\nu} = 0, \quad (1.2.4)$$

it follows that

$$\tilde{\nabla}^\alpha G_{\alpha\beta} = 0 \quad (1.2.5)$$

and thus,

$$\tilde{\nabla}^\alpha T_{\alpha\beta} = 0. \quad (1.2.6)$$

Equation (1.2.6) can be interpreted as a law of local conservation. Historically, this property came before the Einstein equations were ever written down, and even served as inspiration in defining the gravitational tensor. If the space-time contains non-trivial Killing fields, we may contract T against such fields to obtain conserved quantities in well-chosen regions of the space-time. The stress-energy tensor can present in a number of ways. We give a non-exhaustive list of possible stress-energy tensors.

- The vacuum corresponds to

$$T_{\alpha\beta} = 0. \quad (1.2.7)$$

- Electromagnetic field relative to a space-time observer at rest with respect to the initial manifold :

$$T_{\alpha\beta} = F_{\alpha\gamma} F_{\beta}^{\gamma} - \frac{1}{4} g_{\alpha\beta} F^{\mu\nu} F_{\mu\nu}, \quad (1.2.8)$$

where F is the electromagnetic field, represented by an anti-symmetric 2-tensor, that solves the Maxwell equations

$$\begin{aligned} \tilde{\nabla}_\alpha F_{\beta\gamma} + \tilde{\nabla}_\gamma F_{\alpha\beta} + \tilde{\nabla}_\beta F_{\gamma\alpha} &= 0 \\ \tilde{\nabla}^\alpha F_{\alpha\beta} &= j. \end{aligned} \quad (1.2.9)$$

Here, $j = (\rho, J)$, with ρ the charge density and J the current. As an aside, note that $T_{00} = \frac{1}{2} (|E|^2 + |B|^2)$, where E is the electric field and B is the magnetic field.

- Perfect fluid :

$$T_{\alpha\beta} = (\rho + P) u_\alpha u_\beta + p g_{\alpha\beta}, \quad (1.2.10)$$

where ρ is the mass density, P is the isotropic pressure and u is the velocity of the fluid.

- Scalar field theory :

$$T_{\alpha\beta} = \tilde{\nabla}_\alpha \tilde{\psi} \tilde{\nabla}_\beta \tilde{\psi} - \left(\frac{1}{2} |\tilde{\nabla} \tilde{\psi}|_{\tilde{g}}^2 + V(\tilde{\psi}) \right) \tilde{g}_{\alpha\beta}$$

and models the existence of a scalar field $\tilde{\psi} \in \mathcal{C}^\infty(M)$ with potential $V \in \mathcal{C}^\infty(\mathbb{R})$,

$$\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \tilde{\psi} = V'(\tilde{\psi}). \quad (1.2.11)$$

To give a few concrete examples, $V = \Lambda$ and $\tilde{\psi} = 0$ models the vacuum with cosmological constant Λ ; if $V = \frac{1}{2} m \tilde{\psi}^2$, where $m > 0$, then we obtain the Klein-Gordon case. Finally, we recall that the vacuum corresponds to both $V \equiv 0$ and $\tilde{\psi} \equiv 0$.

Another way to look at the Einstein equations is the Euler-Lagrange equations corresponding to an appropriate Einstein-Hilbert action. In the vacuum case, the Lagrangian is

$$\mathcal{L}_G[\tilde{g}] = \underbrace{\int_{\tilde{M}} R(\tilde{g}) dV_{\tilde{g}}}_{\text{gravitational energy}} \quad (1.2.12)$$

for electromagnetism,

$$\mathcal{L}_{G,H}[\tilde{g}] = \int_{\tilde{M}} R(\tilde{g}) dV_{\tilde{g}} + \underbrace{\frac{1}{2} \int_{\tilde{M}} F^{\alpha\beta} F_{\alpha\beta} dV_{\tilde{g}}}_{\text{electromagnetic energy}} \quad (1.2.13)$$

where $F_{\alpha\beta}$ is an anti-symmetric 2-tensor, and for the scalar-field theory,

$$\begin{aligned} \mathcal{L}_{G,T}[\tilde{\psi}, \tilde{g}] = & \int_{\tilde{M}} R(\tilde{g}) dV_{\tilde{g}} + \underbrace{\frac{1}{2} \int_{\tilde{M}} |\tilde{\nabla}\tilde{\psi}|_{\tilde{g}}^2 dV_{\tilde{g}}}_{\text{kinetic energy}} \\ & + \underbrace{\int_{\tilde{M}} V(\tilde{\psi}) dV_{\tilde{g}}}_{\text{potential energy}} \end{aligned} \quad (1.2.14)$$

Here, $dV_{\tilde{g}}$ is the volume form corresponding to \tilde{g} , as defined in (1.1.3). Throughout this work, we focus almost exclusively on scalar-field theory.

1.2.2 Initial data

In this section, we explain how to construct a Cauchy problem in general relativity. For more details, we instruct the reader to consult the book by Yvonne-Choquet Bruhat [CB09] and the survey paper by Robert Bartnik and Jim Isenberg [BI04]. Consider a Lorentzian manifold that can be written as $M \times \mathbb{R}$, with $M_t = M \times \{t\}$ spacelike hypersurfaces. Let ∇ be the covariant derivative associated to g . We fix local coordinates given by a basis that reflects the product structure $(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_i})$. Locally, the metric takes the form

$$\tilde{g} = -N^2 dt \otimes dt + g_{ij} \theta^i \otimes \theta^j \quad (1.2.15)$$

where (dt, θ^i) is the dual basis associated to $(e_{\perp} = \frac{1}{N} (\frac{\partial}{\partial t} - X^j (\frac{\partial}{\partial x_j})), \frac{\partial}{\partial x_i})$, and g_{ij} is the Riemannian metric induced by \tilde{g} on a spacelike hypersurface \tilde{M}_t . The function N is the lapse, and X^j are the components of the shift; both depend on the local foliation of \tilde{M} and on the chosen time function. The vector field e_{\perp} is the normal to \tilde{M} , with $\tilde{g}(e_{\perp}, e_{\perp}) = -1$. We may locally define the second fundamental form

$$K_{ij} = -\langle \tilde{\nabla}_i e_{\perp}, e_j \rangle \quad (1.2.16)$$

with L the Lie derivative. A great leap forward in the field of general relativity took place in the 50's and 60's. The Einstein equations were shown to be well-posed as a Cauchy problem thanks to the work of Yvonne Choquet-Bruhat, and later Choquet-Bruhat and Robert Geroch [FB52, CBG69]. Intuitively, initial data describes the quantities

$$(\tilde{g}_{\alpha\beta})_{t=0} \quad \text{and} \quad (\partial_t \tilde{g}_{\alpha\beta})_{t=0} \quad (1.2.17)$$

They encode the geometry and matter distribution of a space-time at an instance in ‘‘coordinate time’’, together with their instantaneous rate of change. In the case of vacuum with a vanishing cosmological constant, initial data sets take the form :

- (M, g) is an n -dimensional Riemannian manifold,
- K is a symmetric 2-tensor corresponding to the second fundamental form.

The question we may ask at this point is whether or not the set of initial data may be freely specified. To better understand this, it is worth looking at the decomposition of the Ricci tensor $Ric_{\alpha\beta}(\tilde{g})$ with respect to the directions e_\perp and $\frac{\partial}{\partial x_i}$:

$$\begin{aligned} Ric_{ij}(\tilde{g}) &= Ric_{ij}(g) + K_{ij}K^{ij} - 2K_i^l K_{jl} - N^{-1}(L_{e_\perp}K_{ij} + \nabla_i \partial_j N), \\ Ric_{\perp j} &= N(\partial_j K_i^i - \nabla_i K_j^i), \\ Ric_{\perp\perp} &= N(\partial_\perp(K_i^i) - K_{ij}K^{ij} + \Delta N), \end{aligned} \tag{1.2.18}$$

and at the similar decomposition of the scalar curvature

$$R(\tilde{g}) = R(g) + (tr_g K)^2 + K^{ij}K_{ij} - 2N^{-1}\partial_\perp(K_i^i) - 2N^{-1}\Delta N. \tag{1.2.19}$$

Using (1.2.19), the $G_{\perp\perp}$ and $G_{\perp j}$ equations of (1.2.1), together with the $Ric_{\perp\perp}(\tilde{g})$ and $Ric_{\perp j}$ formulas (1.2.18), respectively, we obtain the vacuum constraint equations, which determine vacuum initial data (g, K) :

$$\begin{aligned} R(g) + (tr_g K)^2 - |K|_g^2 &= 0 \\ \partial_i(tr_g K) - div_g K_i &= 0. \end{aligned} \tag{1.2.20}$$

We recall that $tr_g K = g^{ab}K_{ab}$, $|K|_g^2 = (K_a^a)^2$ and $div_g K = K_{a,b}^a$. The first equation of the system is called the Hamiltonian constraint, whereas the second is the momentum constraint. Similarly, the G_{ij} components give the evolution equations

$$\frac{\partial K_{ij}}{\partial t} = N(Ric_{ij}(\tilde{g}) - 2K_{ik}K_j^k + (tr_g K)K_{ij}) - N^{-1}N_{,ij} + L_X K_{ij}. \tag{1.2.21}$$

We also obtain that the Riemannian metric verifies

$$\frac{\partial g_{ij}}{\partial t} = -2NK_{ij} + L_X g_{ij}. \tag{1.2.22}$$

The associated Cauchy problem therefore consists in determining the ambient space-time for the initial data that satisfies the Einstein equations, *i.e.* solving the evolution equations. It was first shown by Yvonne Choquet-Bruhat that, in the vacuum regime, there exists a solution for a given set of smooth initial data verifying the constraint equations, and moreover that the solution is unique, up to a change of coordinates [FB52]. This result was achieved by using the wave coordinates,

$$\tilde{\nabla}^\alpha \tilde{\nabla}_\alpha x_\beta = 0, \tag{1.2.23}$$

to write the vacuum Einstein equations

$$\tilde{\nabla}^\alpha \tilde{\nabla}_\alpha \tilde{g}_{\alpha\beta} = \mathcal{N}_{\alpha\beta}(\tilde{g}), \tag{1.2.24}$$

where \mathcal{N} is a non-linear operator that is quadratic with respect to the first order partial derivatives of \tilde{g} .

The subsequent joint paper of Choquet-Bruhat and Geroch states that if a set of initial data satisfies the constraint equations and admits a local solution, then there exists a

corresponding space-time which is maximal [CBG69]. Note that a maximal space-time for a given initial data set is roughly an extension of any other development obtained from the same data.

We should mention a caveat. The previous result refers only to globally hyperbolic space-times, which are space-times that admits an embedded space-like Cauchy surface. A Cauchy surface is any subspace of space-time intersected by every inextendible, causal curve exactly once.

To conclude, every valid initial data set uniquely determines a maximal space-time, and every globally hyperbolic space-time admits a Cauchy formulation, which further motivates us in our study.

Theorem 4 (Well-posedness Theorem). *Let (Σ, g, K) be a smooth vacuum initial data set. There exists a unique smooth vacuum Cauchy development $(\widetilde{\mathcal{M}}_1, \widetilde{g}_1)$ with the property that if $(\widetilde{\mathcal{M}}_2, \widetilde{g}_2)$ is any other vacuum Cauchy development, then there exists an isometric embedding $i : (\widetilde{\mathcal{M}}_2, \widetilde{g}_2) \rightarrow (\widetilde{\mathcal{M}}_1, \widetilde{g}_1)$ commuting with the embeddings of Σ .*

In the particular case of a scalar field theory, the Cauchy data takes the form (M, g, K, ψ, π) , where

- (M, g) is an n -dimensional Riemannian manifold,
- K is a symmetric 2-tensor corresponding to the second fundamental form,
- ψ represents the scalar field in M , and
- π is its derivative given by

$$\pi = N^{-1} \left(\frac{\partial \psi}{\partial t} - X^j \frac{\partial \psi}{\partial x_j} \right). \quad (1.2.25)$$

The constraint equations in turn take the form

$$\begin{aligned} R(g) + (tr_g K)^2 - |K|_g^2 &= \pi^2 + |\nabla \psi|_g^2 + 2V(\psi) \\ \partial_i (tr_g K) - div_g K_i &= \pi \partial_i \psi. \end{aligned} \quad (1.2.26)$$

Additionally, we ask that

$$\square_{\widetilde{g}} \widetilde{\psi} = \frac{dV}{d\widetilde{\psi}}, \quad (1.2.27)$$

which, in turn, is necessary in order for the local conservation of energy to hold true

$$div_{\widetilde{g}} T = 0. \quad (1.2.28)$$

The well-posedness result in the presence of a scalar field is a straightforward consequence of the vacuum case. Once initial data satisfy the constraint equations, we may find a (unique) maximal development.

At first glance, the fact that the system (1.2.26) is under-determined is immediately apparent. To illustrate this, consider the case $n = 3$. The system (1.2.26) contains one scalar equation and one vector equation, which is equivalent to a total of 4 scalar equations. We recall that the unknowns are the symmetric 2-tensors \hat{g}_{ab} and \hat{K}_{ab} , where $a, b \in \overline{1, 3}$, and the functions $\hat{\pi}$ and $\hat{\psi}$, which gives an overall count of 14 unknowns. This suggests that there is a considerable amount of freedom in choosing solutions of the form $(\hat{g}, \hat{K}, \hat{\psi}, \hat{\pi})$.

The question of finding solutions to the constraint equations has been approached in multiple ways. One example is provided by gluing techniques, see for example [CD03, CS06,

CIP05, CCI11, CS14]. Another approach consists of density and perturbation techniques [Hua09, Hua10]. Both use known solutions, for example Minkowski, Schwarzschild or Kerr, as springboards to create new, different solutions. The focus of the present work takes a slightly different approach, by asking the question : how do you study the *entire* space of solutions to the constraint equations? The conformal method is a way of trying to find a good mapping.

1.3 The conformal method

The conformal method is the first approach to identifying generic initial data. It allows for the constraint equations (1.2.26) to become a determined system by prescribing a number of well-chosen parameters. The set of parameters are therefore used to identify any given solution. In the familiar case $n = 3$, for example, the constraint equations (1.2.26) call for 10 such parameters, corresponding to as many degrees of freedom. Intuitively, the space of initial data with a scalar field source appears to form a manifold of dimension 10, and we are interested in finding a good mapping to describe this manifold.

The conformal method is based on the work of André Lichnerowicz from 1944, who provided a blueprint to construct solutions of null mean curvature [Lic44]. It was later extended by James W. York, Jr. to tackle constant mean curvature (CMC) solutions [Yor73]. The method was further developed together with Niall Ó Murchadha to include non constant mean curvature (non-CMC) solution [ÓMY74]. Two decades later, York introduced the conformal "thin-sandwich" (CTS) model as an alternative for treating the non-CMC case, followed by a variant in a joint paper with Harald P. Pfeiffer, called the Hamiltonian conformal "thin-sandwich" (CTS-H) method [Yor99, PY03]. The non-CMC methods were later all shown to be equivalent in a paper by David Maxwell, which also contains a detailed overview of the conformal method [Max14a].

1.3.1 Toy model : the CMC conformal method

To begin, let us consider the vacuum case. We consider a set of initial data (\hat{g}, \hat{K}) . Since the mean curvature $tr_{\hat{g}}\hat{K}$ is constant, the constraint equations take the form

$$\begin{aligned} R(\hat{g}) + \left(tr_{\hat{g}}\hat{K}\right)^2 - |\hat{K}|_{\hat{g}}^2 &= 0 \\ div_{\hat{g}}\hat{K} &= 0. \end{aligned} \tag{1.3.1}$$

We look at how the parameters are chosen and calculated in the case of CMC solutions $(\hat{g}_{ab}, \hat{K}_{ab})$. Lichnerowicz suggested one first parameter as the conformal class of \hat{g}_{ab} , denoted as \mathbf{g} or $[\hat{g}_{ab}]$, where

$$\mathbf{g} = \{e^{\alpha}g, \quad \alpha \in \mathcal{C}^{\infty}\}. \tag{1.3.2}$$

Let

$$\hat{g}_{ab} = u^{q-2}g_{ab}, \tag{1.3.3}$$

with $u \in \mathcal{C}^{\infty}$, $u > 0$, and let q be the critical Sobolev constant $q = \frac{2n}{n-2}$. In the literature, q is often denoted by 2^* . The scalar curvature changes conformally according to the formula

$$R(\hat{g}) = u^{1-q} \left(\frac{4(n-1)}{n-2} \Delta_g u + R(g)u \right),$$

where $\Delta_g = -div_g \nabla$ is the Laplace-Beltrami operator, defined with positive eigenvalues.

The second and third parameters come from the unique decomposition of \hat{K}_{ab} into its trace-free and pure-trace parts :

$$\hat{K}_{ab} = \hat{U}_{ab} + \frac{\tau_0}{n} \hat{g}_{ab}, \quad (1.3.4)$$

where $\tau_0 = \text{tr}_{\hat{g}} \hat{K}$ is the constant mean curvature and \hat{U}_{ab} is a symmetric trace-free 2-tensor. The decomposition (1.3.4) together with the momentum constraint $\text{div}_{\hat{g}} \hat{K} = 0$ imply that \hat{U}_{ab} is also divergence-free with respect to \hat{g}_{ab} . Tensors which are both trace-free and divergence-free are referred to as TT-tensors.

We define the conformal momentum \mathbf{U} as the equivalence class of pairs (g_{ab}, U_{ab}) , where g is a metric and U is a TT tensor with respect to g , where pairs are identified by

$$(g_{ab}, U_{ab}) \equiv (u^{q-2} g_{ab}, u^{-2} U_{ab}), \quad (1.3.5)$$

where u is an arbitrary positive smooth function. Every CMC solution $(\hat{g}_{ab}, \hat{K}_{ab})$ determines a unique conformal momentum $\mathbf{U} = (\hat{g}_{ab}, \hat{U}_{ab})$ and a unique constant τ_0 .

The triplet of parameters $(\mathbf{g}, \mathbf{U}, \tau_0)$ is called the CMC conformal data.

Conversely, we look at how to obtain a CMC solution $(\hat{g}_{ab}, \hat{K}_{ab})$, given a set of conformal data $(\mathbf{g}, \mathbf{U}, \tau_0)$.

1. Choose $g_{ab} \in \mathbf{g}$ an arbitrary representative of the conformal class.
2. Choose the *unique* TT-tensor $U_{ab} \in \mathbf{U}$ such that $(g_{ab}, U_{ab}) \in \mathbf{U}$.

The triplet (g_{ab}, U_{ab}, τ_0) is called a CMC representative data set. It is not unique. Since $g_{ab} \in \mathbf{g}$, there exists a smooth positive function u such that $\hat{g}_{ab} = u^{q-2} g_{ab}$. At this point, u is still unknown. We write

$$\begin{aligned} \hat{g}_{ab} &= u^{q-2} g_{ab} \\ \hat{K}_{ab} &= u^{-2} U_{ab} + \frac{\tau_0}{n} u^{q-2} g_{ab} \end{aligned} \quad (1.3.6)$$

and plug these quantities into the vacuum Hamiltonian constraint

$$R(\hat{g}) + \frac{n-1}{n} \tau_0^2 - |\hat{U}|_{\hat{g}}^2 = 0 \quad (1.3.7)$$

to obtain

$$\Delta_g u + \frac{n-2}{4(n-1)} R(g) u = -\frac{n-2}{4n} \tau_0^2 u^{q-1} + \frac{n-2}{4(n-1)} \frac{|U|_g^2}{u^{q+1}}. \quad (1.3.8)$$

Finding the initial data $(\hat{g}_{ab}, \hat{K}_{ab})$ reduces to solving the so-called Einstein-Lichnerowicz equation (1.3.8). The conformal method relies on the choice of parameters that change well under conformal transformations. In this case, for any two choices of representative data $(g_{ab}^1, U_{ab}^1, \tau_0^1)$ and $(g_{ab}^2, U_{ab}^2, \tau_0^2)$ for the same conformal data, there exists a smooth positive function v such that

$$(g_{ab}^2, U_{ab}^2, \tau_0^2) \equiv (v^{q-2} g_{ab}^1, v^{-2} U_{ab}^1, \tau_0^1), \quad (1.3.9)$$

This property is called conformal covariance. It implies that, given a conformal data set, any arbitrary choice of representative data leads to the same constraint equations solution $(\hat{g}_{ab}, \hat{K}_{ab})$.

1.3.2 The “classical” conformal method

We consider space-times with scalar field sources. We define the scalar field as a smooth function ψ and the time-coordinate partial derivative of the scalar field as the equivalence class of the pairs

$$(g_{ab}, \pi) \equiv (u^{q-2}g_{ab}, u^{-q}\pi), \quad (1.3.10)$$

with g a metric, π a smooth function and u a positive smooth function. This particular choice of representing source elements ensures that the constraint equations are rewritten in a way which makes them easier to approach, analytically, by having them decoupled in the case of τ constant. [CBY80, Yor73]

The choice of conformal momentum \mathbf{U} relies on the unique decomposition of \hat{K} into a pure-trace and a trace-free part, respectively (1.3.4). The trace-free part \hat{U}_{ab} is divergence-free as a consequence of the momentum constraint and the constant mean curvature. When τ is not constant, the decomposition does not provide a TT-tensor, which we need based on its good covariance properties. Let

$$\hat{K}_{ab} = \hat{A}_{ab} + \frac{\tau}{n}\hat{g}_{ab}, \quad (1.3.11)$$

where \hat{A}_{ab} is a unique trace-free symmetric $(0, 2)$ tensor. The next step is to split A_{ab} into a divergence-free part and a pure-divergence part. This decomposition, however, is not unique.

Let ω be a volume form on M . We define the densitized lapse \mathbf{N} as the equivalence class of pairs (g_{ab}, N) , where g is a metric and N a positive smooth function. The pairs are identified by

$$(g_{ab}, N) \equiv (u^{q-2}g_{ab}, u^qN), \quad (1.3.12)$$

where u is an arbitrary positive smooth function. One convenient way to define N is as a function of g and a volume gauge ω ,

$$N_{g,\omega} = \frac{dV_g}{\omega}. \quad (1.3.13)$$

Here, dV_g is the volume form corresponding to g (as in (1.1.3)) and ω an arbitrary volume form on the manifold. The corresponding lapse is denoted \mathbf{N}_ω . To choose a conformal momentum, a variation of the York splitting is used : given a symmetric trace-free $(0, 2)$ -tensor \hat{A}_{ab} and a positive function \hat{N} , there exists a unique TT-tensor \hat{U}_{ab} and a vector field W such that

$$\hat{A}_{ab} = \hat{U}_{ab} + \frac{1}{2\hat{N}}\mathcal{L}_{\hat{g}}W, \quad (1.3.14)$$

where

$$\mathcal{L}_gW_{ij} = W_{i,j} + W_{j,i} - \frac{2}{n}\text{div}_gWg_{ij} \quad (1.3.15)$$

is called the conformal Killing operator and W is uniquely determined up to a conformal Killing field. The conformal momentum \mathbf{U} as measured by ω corresponds to (g_{ab}, U_{ab}) . The conformal data set is $(\mathbf{g}, \mathbf{U}, \tau; \mathbf{N})$. Note that the densitized lapse \mathbf{N} is separated from the rest of the data, being an equivalent of the gauge in this context : given a representative metric g and a lapse N written as (1.3.13), we can deduce ω . In this sense, the choice of \mathbf{N} contains the choice of volume gauge, by which we measure the conformal momentum and the densitized lapse [Max14a]. The representative data is conformally covariant

$$(\hat{g}, \hat{U}, \hat{\tau}, \hat{\psi}, \hat{\pi}; \hat{N}) = (u^{q-2}g, u^{-2}U, \tau, \psi, u^{-q}\pi; u^qN), \quad (1.3.16)$$

so once ω is fixed, any set of representatives corresponds to the same initial data solutions.

The following steps show how to obtain a solution $(\hat{g}_{ab}, \hat{U}_{ab})$ from a set of conformal data given a lapse form ω .

1. Choose an arbitrary representative $g_{ab} \in \mathbf{g}$.
2. Choose the unique densitized lapse representative $N_{g,\omega}$.
3. Choose the unique TT-tensor U_{ab} such that $(g_{ab}, U_{ab}) = \mathbf{U}$, where \mathbf{U} is the conformal momentum measured by ω .
4. Choose the unique scalar field ψ .

At this point, both u and W are still unknown. We write

$$\begin{aligned}\hat{g}_{ab} &= u^{q-2}g_{ab} \\ \hat{K}_{ab} &= u^{-2}\left[\left(\frac{1}{2N}\mathcal{L}_gW\right)_{ab} + U_{ab}\right] + \frac{\tau}{n}u^{q-2}g_{ab}.\end{aligned}\tag{1.3.17}$$

and plug these quantities into the constraint equations to obtain

$$\begin{aligned}\Delta_g u + \frac{n-2}{4(n-1)}(R(g) - |\nabla\psi|_g^2)u &= -\frac{n-2}{4(n-1)}\left(\frac{n-1}{n}\tau^2 - 2V(\psi)\right)u^{q-1} \\ &\quad + \frac{n-2}{4(n-1)}\frac{(|U + \mathcal{L}_gW|_g^2 + \pi^2)}{u^{q+1}}, \\ \vec{\Delta}_g W &= \frac{n-1}{n}u^q\nabla\tau - \pi\nabla\psi,\end{aligned}\tag{1.3.18}$$

and

$$\vec{\Delta}_g W = -\text{div}_g \mathcal{L}_g W\tag{1.3.19}$$

is the Lamé operator. The method is particularly successful for finding solutions in the case of a constant or almost constant mean curvature. It remains unclear how well the method works otherwise.

The following table regroups, for $n = 3$, the conformal data and their dimensions (columns 2 and 3), the expressions of physical data as functions of representatives of conformal data (column 1) and the dimensions of the remaining unknowns (column 4).

Physical data	Parameters	Dimensions	Unknowns
$\hat{g} = u^{q-2}g$	\mathbf{g}	5	1
$\hat{K} = u^{-2}\left(\frac{1}{2N}\mathcal{L}_gW + U\right) + \frac{\tau}{n}u^{q-2}g$	$\mathbf{U}, \tau, \mathbf{N}$	2 + 1(+1)	3
$\hat{\psi} = \psi$	ψ	1	0
$\hat{\pi} = u^{-q}\pi$	π	1	0

(1.3.20)

We notice that we obtain 11 dimensions instead of the 10 we expected. Historically, the parameter \mathbf{N} was introduced quite late, within the CTS method. As we explained above, \mathbf{N} corresponds to the choice of gauge - once fixed, we locally obtain a “good” parametrization for the 10 manifold, just as in the CMC model.

1.4 Solving the classical method system

So far, we have described how to define a tuple of conformal data $(\mathbf{g}, \mathbf{U}, \tau; \mathbf{N})$ associated to an initial data set (\hat{g}, \hat{K}) . Additionally, we began to explain how to reconstruct (\hat{g}, \hat{K})

starting from a conformal data set : the first step is to choose a set of representative data $(g, U, \tau; N)$, and then to rewrite the constraint equations as a system (1.3.18). It remains to find the corresponding solutions (u, W) . Once this is done, we obtain (\hat{g}, \hat{K}) by (1.3.17). In this section, we list a number of results related to the existence of solutions (u, W) for the classical conformal method. We introduce the following notation : let

$$\begin{aligned}\Delta_g u + h_\psi u &= f_{\tau, \psi, V} u^{q-1} + \frac{a_{\pi, U}(\mathcal{L}_g W)}{u^{q+1}}, \\ \vec{\Delta}_g W &= X_\tau u^q + Y_{\psi, \pi},\end{aligned}\tag{1.4.1}$$

where

$$\begin{aligned}h_\psi &= \frac{n-2}{4(n-1)} (R(g) - |\nabla\psi|_g^2), \\ f_{\tau, \psi, V} &= -\frac{n-2}{4(n-1)} \left(\frac{n-1}{n} \tau^2 - 2V(\psi) \right), \quad \text{and} \\ a_{\pi, U}(\mathcal{L}_g W) &= \frac{n-2}{4(n-1)} (|U + \mathcal{L}_g W|_g^2 + \pi^2) \\ X_\tau &= \frac{n-1}{n} \nabla\tau \\ Y_{\psi, \pi} &= -\pi \nabla\psi.\end{aligned}\tag{1.4.2}$$

In the rest of this work, we assume that M is a closed manifold unless otherwise stated. By Stokes theorem, we see that

$$\vec{\Delta}_g W \equiv 0 \iff \mathcal{L}_g W \equiv 0.\tag{1.4.3}$$

We consider the following two classifications :

- positive : $\Delta_g + h_\psi$ is coercive ;
- null : $\Delta_g + h_\psi$ has a null first eigenvalue ; and
- negative : $\Delta_g + h_\psi$ admits a negative eigenvalue ;

and

- focusing : $f_{\tau, \psi, V} > 0$, and
- non-focusing : $f_{\tau, \psi, V} < 0$.

In the particular case of scalar field theory, the first classification was suggested by Choquet-Bruhat, Isenberg and Pollack in their survey of the conformal method on compact manifolds [CBIP06]. They introduce the Yamabe-scalar field conformal invariant

$$\mathcal{Y}_\psi(\mathbf{g}) = \inf_{u \in C^\infty(M), u \neq 0} \frac{\int_M \frac{4(n-1)}{n-2} |\nabla u|^2 + (R(g) - |\nabla\psi|_g^2) u^2 dV_g}{\left(\int_M u^{\frac{2n}{n-2}} dV_g \right)^{\frac{n-2}{n}}}\tag{1.4.4}$$

and prove that the following conditions are equivalent

- $\mathcal{Y}_\psi(\mathbf{g})$ is positive (respectively null, negative)
- there exists $\tilde{g} \in \mathbf{g}$ such that $(R(\tilde{g}) - |\nabla\psi|_{\tilde{g}}^2)$ is positive (respectively null, negative)
- For any metric $\tilde{g} \in \mathbf{g}$, the first eigenvalue λ_1 of the self-adjoint operator $\Delta + h_\psi$ is positive (respectively null, negative).

We begin by indicating the good properties of system (1.4.1). First of all, note that the second equation is linear in W . For a fixed scalar function u , the solvability of the momentum constraint is relatively straightforward, given that the kernel of the elliptic operator is known (1.4.3). In fact, in the CMC case, the vector field W does not depend on the conformal factor u . We can calculate W and plug it into the Hamiltonian constraint. All that needs to be verified is that $Y_{\psi,\pi}$ is orthogonal to the kernel of $\vec{\Delta}_g$, *i.e.* to the space of conformal Killing fields. This is not a strong demand, as conformal Killing fields are rare [BCS05]. Another good feature of the system is that the first equation is similar to the Lichnerowicz equation, as it appears in the Yamabe problem, which is well-understood.

We move on to the more difficult aspects. Note that the u^{q-1} term is critical (in the sense of the H_0^1 embeddings into Lebesgue spaces); this does not pose a problem in the non-focusing case, when the Laplacian and critical non-linearity do not act in competition, but it does lead to possible lack of compactness in the focusing regime. Moreover, the negative power of the u^{-q-1} term implies that a sequence of solutions u which approach 0 can induce a non-compactness result.

In the rest of the section, we look in more detail at a series of existing results for the constraint equations, categorized as either focusing or non-focusing.

1.4.1 Non-focusing case

Consider first the case of vacuum with null cosmological constant, *i.e.* $V \equiv 0$, $\psi \equiv 0$ and $\pi \equiv 0$. In this case

$$f_{\tau,\psi,V} = -\frac{n-2}{4(n-1)} \frac{n-1}{n} \tau^2 < 0, \quad (1.4.5)$$

so we are obviously in the non-focusing regime. We begin by analyzing the CMC solutions. Since $\nabla\tau \equiv 0$, the momentum constraint becomes

$$\vec{\Delta}_g W = 0, \quad (1.4.6)$$

solved by conformal Killing fields, while the Hamiltonian constraint is written as

$$\Delta u + \frac{n-2}{4(n-1)} R(g)u = -\frac{n-2}{4(n-1)} \frac{n-1}{n} \tau^2 + \frac{|U|_g^2}{u^{q+1}}. \quad (1.4.7)$$

In this case, Isenberg uses a sub and super solution method to show that the above equation admits solutions in most cases (see the case study in Section 1.4.3), and that they are unique [Ise95]. In fact, the result extend to non-trivial stress-energy tensors, including the electromagnetic regime.

It was in the 90's that a first result for non-CMC solutions was obtained by Isenberg and Moncrief [IM96]. They further demand that the parameters correspond to the negative case, by asking that $R(g) \equiv -1$. Moreover, they ask that the metric g_{ab} admits no non-trivial conformal Killing fields. The authors obtain existence and uniqueness when τ is smooth, doesn't change sign, and when $\nabla\tau$ verifies the smallness assumption

$$\frac{\max_M |\nabla\tau|_g^2}{\min_M \tau^2} < C_1(n, g, U) \quad \text{and} \quad |\nabla\tau|_g < C_2(n, g, U). \quad (1.4.8)$$

Under these conditions, one may use an iterative approach to construct a sequence of sub-solutions to the Hamiltonian constraint and ensure that it converges to a solution. These results were extended to the positive and null case, again for τ that does not change sign and under smallness assumptions on $\nabla\tau$, that is on the variation of the mean curvature. Solutions corresponding to systems that verify a condition like that in (1.4.8) are called

near-CMC solutions. Without this assumption, it becomes much more delicate to ascertain existence results.

The first case of a far-from CMC existence result comes from Michael Holst, Gabriel Nagy and Guantumur Tsogtgerel [HNT09]. The authors consider the positive case and demand the presence of a weak, non-trivial source. Instead of asking for $\nabla\tau$ to be small, they instead impose bounds on the size of $\|U\|_\infty$. A variety of techniques is used, including *a priori* estimates, barrier constructions and fixed-point arguments. David Maxwell extends this result to the vacuum, under the additional assumption that $U \neq 0$. A global super-solution construction is central to this proof. Yet another approach is apparent in the paper of Mattias Dahl, Romain Gicquaud and Emmanuel Humbert, one that doesn't explicitly call for smallness assumptions [DGH12]. Given that g admits no non-trivial conformal Killing fields, that τ does not change sign, that $U \neq 0$, they prove that the Einstein constraint equations admit solutions and the set of solutions is compact as soon as the equation

$$\vec{\Delta}_g W = -\alpha_0 \sqrt{\frac{n-1}{n}} |\mathcal{L}_g W|_g \frac{d\tau}{\tau} \quad (1.4.9)$$

does not admit any non-trivial solution for all $\alpha_0 \in (0, 1]$.

The authors of [DGH12] found examples when (1.4.9) admits non-trivial solutions; in these cases, the question of existence of solutions to the constraint equations remains open. On the other hand, they also prove the existence open sets of parameters g_{ab} and τ such that (1.4.9) is unsolvable, and thus such that the constraint equations admit a solution.

Note also that the above results hold in low regularity. But the types of fixed point techniques used in the paper do not allow any glimpses of possible uniqueness results.

As is apparent from the previous papers, the condition that g_{ab} does not admit non-trivial Killing fields often appears in the non-CMC regime. This is linked to the solvability of the momentum constraint. The assumption is not as strong as might seem at first glance, as suggested by the result of Robert Beig, Piotr Chruściel and Richard Schoen, which states that generic initial data have no symmetry [BCS05].

1.4.2 Focusing case

The focusing case has received more attention in recent years. We recall that it differs from the previous regime as the critical non-linearity fu^{q-1} does compete with the Laplace-Beltrami operator Δ_g , which allows for non compact behaviour. We do not generally expect the system to admit one and only one solution; this is often not the case. Rather, we can obtain multiple solutions (*e.g.* a one-parameter family of solutions). Moreover, the stability of the set of solutions necessitates careful analysis, as concrete examples of instability exist.

We define perturbations of system (1.4.1); they are written as

$$\begin{aligned} \Delta_g u + h_\alpha u &= f_\alpha u^{q-1} + (\rho_\alpha + |\Psi_\alpha + \mathcal{L}_g W|_g^2) u^{-q-1}, \\ \vec{\Delta}_g W &= X_\alpha u^q + Y_\alpha, \end{aligned} \quad (1.4.10)$$

for all α , where $(h_\alpha)_\alpha$, $(f_\alpha)_\alpha$, $(\rho_\alpha)_\alpha$ are sequences of smooth functions in M , $(X_\alpha)_\alpha$, $(Y_\alpha)_\alpha$ are sequences of smooth vector fields and $(\Psi_\alpha)_\alpha$ is a sequence of smooth $(0, 2)$ -tensors such that

$$(h_\alpha, f_\alpha, \rho_\alpha, X_\alpha, Y_\alpha, \Psi_\alpha) \xrightarrow{\mathcal{T}} (h, f, \rho, X, Y, \Psi), \quad (1.4.11)$$

with \mathcal{T} an appropriate topology. Given a sufficiently regular perturbation of the coefficients of system (1.4.10), one tries to obtain *a priori* bounds on solutions (u_α, W_α) .

We say that the system (1.4.10) is stable in a topology \mathcal{T} if, for any perturbation of the coefficients h, f, ρ, X, Y, Ψ , there exists a solution (u, W) of the limiting system such that,

up to a subsequence, (u_α, W_α) converges towards (u, W) in $\mathcal{C}^{1,\eta}(M)$. For further details on the notion of stability defined here, we refer the reader to a series of papers by Olivier Druet, Emmanuel Hebey, Frédéric Robert [Dru10, DHR04, Heb14].

The existence of solutions to the Hamiltonian constraint when $f > 0$ implies that we are in the positive case, so from now on we assume that $\Delta_g + h$ is coercive [Pre14][CBIP07].

The decoupled case

Assume first that $X \equiv 0$, which is true for CMC-solutions. The momentum constraint becomes

$$\vec{\Delta}_g W = Y \tag{1.4.12}$$

The solvability of (1.4.12) is equivalent to asking that Y is orthogonal to the kernel of the Lamé operator. When g admits no non-trivial Killing fields, W exists, is unique and depends continuously on Y . By plugging W into the first equation, the system (1.4.1) reduces to the Einstein-Lichnerowicz equation

$$\Delta_g u + hu = fu^{q-1} + \frac{a}{u^{q+1}}, \tag{1.4.13}$$

where $a \geq 0$, $a \neq 0$.

A versatile existence result for the above equation is due to Emmanuel Hebey, Frank Pacard and Daniel Pollack [HPP08] showed that, given a bound on a depending only on n , h and f , the equation (1.4.13) admits a positive, smooth solution. While f needs not be positive, the proof does however ask that its maximum be. We insist on the fact that this solution is not necessarily unique. The authors used variational techniques, sub-critical perturbations and an argument based on the mountain-pass theorem. The solution is a saddle-point, rather than a local minimum. By sub-critical perturbation, we understand a generalized perturbation of the equation (1.4.13) such that terms of the form $u^{q_\alpha-1}$ are considered in place of u^{q-1} , with $q_\alpha < q$. Thus, we obtain

$$\Delta_g u + h_\alpha u = f_\alpha u^{q_\alpha-1} + \frac{a_\alpha}{u^{q+1}}. \tag{1.4.14}$$

In 2015, Bruno Premoselli builds on the proof of Hebey, Pacard and Pollack and proves that, depending on the size of a , equation (1.4.13) admits either one, at least two, or no solutions [Pre15]. Sub-critical perturbations and variational methods are used to show that, when the equation admits solutions, it also admits stable, minimal solution. We also mention the work of Li Ma and Juncheng Wei. They proved multiplicity of solutions, assuming that a stable solution exists [MW13]. Furthermore, Quoc Anh Ngo and Xingwang Xu wrote a series of papers on the existence of solutions when f is allowed to change sign [NX12, NX14].

Regarding stability results for (1.4.13), we cite the paper of Emmanuel Hebey and Olivier Druet [DH09]; stability is established in dimensions $3 \leq n \leq 5$, a result which is optimal, as examples of instability are stated in dimension $n = 6$.

Coupled case

The first result in the fully coupled regime is due to Bruno Premoselli : for g that admits no non-trivial conformal Killing fields, $n \geq 3$, and under suitable smallness assumptions, the system (1.4.1) admits solutions [Pre14]. In regard to the physical system (1.3.18), the smallness assumptions translate to conditions on the coupling coefficients and on the source, $\|\nabla\tau\|_\infty \ll 1$, $\|U\|_\infty \ll 1$ and $\|\psi\|_\infty \ll 1$. Broadly, the proof consists of a fixed-point argument, a method which we explain in more detail in a subsequent section. Special care

is given to establishing that the fixed-point functional is well-defined, as the Hamiltonian equation can very well admit multiple solutions, and to establishing *a priori* bounds on solutions.

Known examples of instability exist when M is the standard sphere \mathbb{S}^3 and X is as small as we'd like, $X \neq 0$ [Pre16]. An interesting property of this particular set of examples is that, while the system is unstable for small $X \neq 0$, it is stable in the uncoupled case, when $X \equiv 0$. This suggests that the lack of compactness is not solely due to the Hamiltonian constraint and its positive critical non-linearity, but also to the momentum constraint. Stability results when the metric g is locally conformally flat are given by Druet and Premoselli [DP15] and Premoselli [Pre16]; it holds for $3 \leq n \leq 5$, $|X|_g > 0$ and $\rho \neq 0$. Moreover, the examples of instability found in [Pre16] and [PW16] show optimality.

1.4.3 The viability of conformal method models, examples and counter-examples

The conformal method essentially provides a mapping from the set of conformal data representatives to the set of initial data,

$$\text{Conformal data representatives} \rightarrow \text{Initial data.} \quad (1.4.15)$$

More precisely, in the case of the classical conformal method, given a volume gauge ω , the mapping presents as

$$(g_{ab}, U_{ab}, \tau; N) \xrightarrow{\text{solve}(u, W)} (\hat{g}, \hat{K}). \quad (1.4.16)$$

By the nature of the conformal method, the mapping is onto : from any set of initial data, one can calculate a set of corresponding conformal data representatives. We list a number of criteria by which the strength of a conformal method may be judged.

1) Is the mapping a bijection ?

Ideally, to any set of conformal data representatives there corresponds one and only one set of initial data. Thus, the set of all possible initial data is completely characterized by the conformal method.

In lieu of such a strong result, one may ask :

- Where is the mapping well-defined (in the sense that there exists (\hat{g}, \hat{K}) corresponding to a fixed set of conformal data representatives) ? As long as we properly identify the problem sets, we can simply remove them from the domain.
- Where is the mapping one-to-one ? If we obtain multiple solutions, where does this happen ?

2) Is the mapping continuous ?

This question tests that the mapping is, in some sense, physically relevant.

Case study : the constant mean curvature

Let us consider a concrete example. We restrict our attention to the set of solutions with constant mean curvature, *i.e.* let τ be a constant. James Isenberg gathered the results of multiple authors in the compact case and showed that the mapping is generally a bijection outside of the following known cases [Ise95]. Let Y_g be the sign of the Yamabe invariant of g .

- Positive case and $\mathbf{U} = 0$, then there is no corresponding solution ;
- Negative case and $\tau = 0$, then there is no corresponding solution ;
- Null case and either $U = 0$ or $\tau = 0$, then there is no corresponding solution ;
- Null case and both $U = 0$ and $\tau = 0$, then there is a homothety family of solutions.

The same results can to some extent be generalized to the non-CMC regime, see papers by Allen, Clausen, Isenberg, Maxwell, Moncrief and Ó Murchadha [IM96, ACI08, Max14a, IÓM04]. The main caveat is that we additionally require that g admit no non-trivial conformal Killing fields ; otherwise, the question remains open.

What happens in the regime of far-from-constant-CMC is still very unclear. One known result is due to Michael Holst, Gabriel Nagy and Gantumur Tsogtgerel [HNT09, Max09], which states that when $Y_g > 0$ and $U \neq 0$ is close to zero, then there exist solutions associated to the given parameter set, but we no longer necessarily have uniqueness.

Maxwell's counter-examples

So far, we have looked at examples of good behaviours of the conformal mapping. David Maxwell identifies two negative examples in the far-from-CMC regime. They show that issues with the mapping are not just theoretical. This failure, however, does not necessarily stem from the existence of singularities in the space of solutions, but may instead follow from an unlucky choice of mapping.

The first case refers to low regularity data, specifically to L^∞ bounds on the mean curvature [Max11]. Maxwell considers a family of symmetric conformal data on the torus. There exist multiple solutions when U is small, no solutions when U is large, and a one-parameter family of solutions in certain rare cases.

The second example deals with smooth mean curvatures [Max15]. It concerns the parametrization $(\mathbf{g}, \mu U^b, \tau, N)$, where U^b is a fixed conformal momentum, μ is a constant, τ and N are scalar functions. If both $\tau^* = 0$ and $\mu = 0$, then there exists a one parameter family of corresponding solutions. We draw the reader's attention to the fact that the volumetric momentum τ^* , as defined below in (1.5.3), plays a central role throughout the present text. Two new conformal models are suggested in the same paper, including an early version of the drift model.

1.5 David Maxwell's drift model

In order to have a better understanding of the drift method, we recall a basic fact of differential geometry : any metric is uniquely identified by its conformal class together with its volume form. In fact,

$$\mathcal{M} = \mathcal{C} \times \mathcal{V}, \tag{1.5.1}$$

where \mathcal{M} is the space of metrics, \mathcal{V} is the space of volume forms and \mathcal{C} is the space of conformal classes. In the context of the Einstein equations, it makes sense to consider \mathcal{M} , \mathcal{C} and \mathcal{V} modulo diffeomorphisms \mathcal{D}_0 , with \mathcal{D}_0 the connected component of the identity in the diffeomorphism group.

In his papers, Maxwell describes in great detail how the spaces \mathcal{M} , \mathcal{C} and \mathcal{V} , together with their tangent, cotangent and quotient spaces, are represented within the choice of parameters [Max14a, Max14b]. The conformal momentum \mathbf{U} , for example, is shown to be an element of $T_{\mathbf{g}}(\mathcal{C} \setminus \mathcal{D}_0)$. By this interpretation, it becomes clear that \mathcal{C} is prioritized over \mathcal{V} when it comes to choices of parameters.

In a 2014 paper, Maxwell introduces a variant to the standard conformal method [Max14b]. Very succinctly, the drift model differs from its predecessor in that it replaces the mean curvature τ with two new conformal data, a volumetric momentum and a drift. These new quantities are defined by the volumetric equivalent to the York splitting [Max15] :

$$\tau = \tau^* + \frac{1}{N_{\hat{g},\omega}} \text{div}(V + Q) \quad (1.5.2)$$

where $\tau^* \in \mathbb{R}$, V is a smooth vector field and Q is a conformal Killing field. The volumetric momentum τ^* as measured by ω is uniquely determined and can be rewritten as

$$\tau^* = \frac{\int_M N_{\hat{g},\omega} \tau \, dV_{\hat{g}}}{\int_M N_{\hat{g},\omega} \, dV_{\hat{g}}}. \quad (1.5.3)$$

The vector field V is uniquely determined up to a \hat{g} divergence-free vector field.

As we explain above, $\tau^* = 0$ seems to be a common property of the known non-CMC cases of an infinity of solutions corresponding to the same data set. The drawback of the classical conformal method is that the value of τ^* cannot be calculated *a priori* from a choice of representatives. One needs to first solve the corresponding system, as

$$\tau_* = \tau_*(g, u) = \frac{\int_M u^{2q} N_{g,\omega} \tau \, dV_g}{\int_M u^{2q} N_{g,\omega} \, dV_g}. \quad (1.5.4)$$

Coming back to (\mathcal{Q}_1) , this is an argument against the classical conformal model.

The volumetric momentum $[g, \tau]_\alpha$ as measured by ω is $-2\frac{n-1}{n}\tau_*$. A drift $[V]_g^{\text{drift}}$ at g is the equivalence class of V , modulo $\text{Ker } \mathcal{L}_g$ and $\text{Ker } \text{div}_g$. The space of drifts at g is denoted as Drift_g . David Maxwell introduces the concept of drift as an infinitesimal motion in the space of metrics, modulo diffeomorphisms, that preserves conformal class, up to a diffeomorphism, and the volume form, also up to a diffeomorphism.

In this section, we assume that \mathbf{g} admits no non-trivial conformal Killing field and therefore that $Q \equiv 0$.

We show how to obtain the initial data $(\hat{g}_{ab}, \hat{K}_{ab})$ from a conformal data set, given a gauge ω .

1. Choose an arbitrary representative $g_{ab} \in \mathbf{g}$.
2. Choose the unique densitized lapse $N_{g,\omega}$.
3. Choose the unique TT-tensor U_{ab} such that $(g_{ab}, U_{ab}) = \mathbf{U}$, where \mathbf{U} is the conformal momentum as measured by ω .
4. Choose a vector field \tilde{V} , unique up to a conformal Killing field, such that $(g_{ab}, \tilde{V}^a) = \mathbf{V}$, where \mathbf{V} is the volumetric drift measured by ω . We use the tilde to differentiate the drift from the potential, while still staying true to Maxwell's initial notation.

Both u and W are unknown. We write

$$\begin{aligned} \hat{g}_{ab} &= u^{q-2} g_{ab} \\ \hat{K}_{ab} &= u^{-2} \left[\frac{1}{2N_{g,\omega}} (\mathcal{L}_g W)_{ab} + U_{ab} \right] + \frac{1}{n} u^{q-2} g_{ab} \left(\tau^* + \frac{1}{N_{g,\omega}} \text{div}(\tilde{V}) \right). \end{aligned} \quad (1.5.5)$$

Plug these quantities into the constraint equations to obtain

$$\begin{aligned} \Delta_g u + \frac{n-2}{4(n-1)} (R(g) + |\nabla \psi|_g^2) u &= \frac{(n-2)|U + \mathcal{L}_g W|^2 + \pi^2}{4(n-1)u^{q+1}} \\ &\quad + \frac{n-2}{4(n-1)} \left[2V(\psi) - \frac{n-1}{n} \left(\tau^* + \frac{\text{div}_g(u^q \tilde{V})}{N_{g,\omega} u^{2q}} \right) \right] u^{q-1} \quad (1.5.6) \\ \text{div}_g \left(\frac{1}{2N_{g,\omega}} \mathcal{L}_g W \right) &= \frac{n-1}{n} u^q d \left(\frac{\text{div}_g(u^q \tilde{V})}{2N_{g,\omega} u^{2q}} \right) + \pi \nabla \psi = 0. \end{aligned}$$

The following table regroups for $n = 3$ the conformal data and their dimensions (columns 2 and 3), the expressions of physical data as functions of representatives of conformal data (column 1) and the dimensions of the remaining unknowns (column 4).

Physical data	Parameters	Dimensions	Unknowns
$\hat{g} = u^{q-2}g$	\mathbf{g}	5	1
$\hat{K}_{ab} = u^{-2}[\frac{1}{2N_{g,\omega}}(\mathcal{L}_g W)_{ab} + U_{ab}]$ $+ \frac{1}{n}u^{q-2}g_{ab}(\tau^* + \frac{1}{N_{g,\omega}}div(V))$	$\mathbf{U}, \tau^*, \mathbf{N}, \mathbf{V}$	2 + 1 + 3	3
$\hat{\psi} = \psi$	ψ	1	0
$\hat{\pi} = u^{-q}\pi$	π	1	0

(1.5.7)

This time, we obtain additional parameters. More on this in the following section.

1.6 Is the drift model a better alternative ?

We recall that not much is known about far-from-CMC solutions. The classical conformal method seems to display a number of singularities, and these singularities are sometimes difficult to find *a priori* without first solving the corresponding conformal system (1.3.18) [Max11, Max15].

As we've discussed in the previous section, an advantage of the drift model is that the singularities identified by Maxwell can be found in *a priori* known conformal data sets - *i.e* when the volumetric momentum is null.

Apart from being more natural from a physical and geometrical point of view, another feature of Maxwell's model is that it prescribes more than 10 parameters. At first glance, it "over-describes" the initial data. An important idea underlying the works presented in the sequel is the hope to use these four additional parameters to "tilt" the coordinate system (the other ten parameters) in the neighbourhood of a singularity. Another way to think about this is that the 10-dimensional manifold of initial data cannot accurately be covered by only one chart; by changing the additional drift parameters whenever we approach of singularity, we essentially switch to a different chart. In this way, we might prove that the set of solutions to the constraint equations does not possess any real singularity, but only ones due to the choice of coordinates. Naively, one might think of a curve having a vertical tangent which is not well parametrized by its x-axis. The price we pay is that the drift system is analytically much more complicated than the classical one.

The goal is to find a viable alternative to the conformal method that gives insight into the structure of the set of solutions of the constraint equations. The drift method proposed by Maxwell provides a promising way forward. The following steps are necessary in order to achieve this :

- a. *Existence for small data.* We essentially verify that Maxwell's system is reasonable : it can be solved even in the case of focusing non linearities. An immediate consequence is that the set of solutions is non-empty. This is the purpose of chapter 2 of the present manuscript.
- b. *Stability.* We check that, given a perturbation of the coefficients, then the solutions to the perturbed system converge to a solution of the limiting system. This is the purpose of chapter 3.

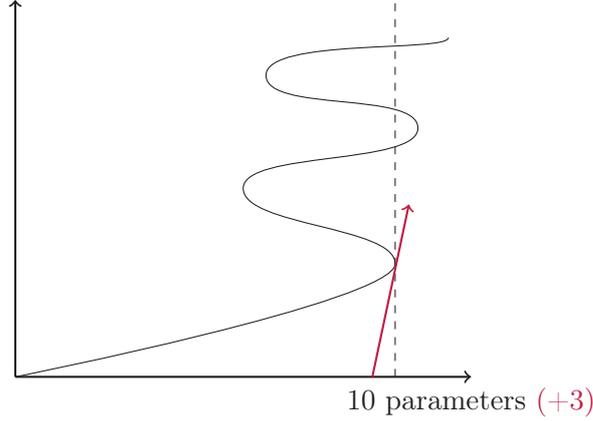


FIGURE 1.6.1 : Initial data manifold, parametrized by the drift method.

- c. *The study of bifurcations.* This is where the extra parameters of Maxwell’s method might come into play, by allowing us the freedom to continuously change our mapping as needed. Indeed, as proved by Premoselli [Pre15], there is no hope that a single choice of N and V lead to a nice smooth parametrization of the set of solutions (since his study is a subcase of ours). Bifurcations must occur. Even in the defocusing case, such bifurcations can occur, as shown by James Dilts, Michael Holst and David Maxwell [DHKM17]. Thus, tilting the coordinates (the parameters) in a neighbourhood of these bifurcations is a way to understand them and the extra parameters give an opportunity to do so. This part of the program is not treated here.

We summarize this program with the help of the following figure. Point a. allows us to start the process of proving that solutions do exist, for small parameters. Point b. roughly says that the only problem could come from bifurcations corresponding to folding (at least for the parameters for which stability holds). We rule out vertically asymptotic branches. Part c. consists intuitively in tilting the coordinates with the four added parameters, as shown below. These three steps should permit to obtain a nice smooth description of the set of solutions. In the remaining sections, we explain how to get existence for small data and some stability results.

1.7 Existence result for small data

The non-focusing case was addressed by Michael Holst, David Maxwell and Rafe Mazzeo in 2018 [HMM18]. They proved existence under suitable smallness hypotheses on the coefficients. Moreover, they improve previous known results, by applying their result to metrics that admit conformal Killing fields.

The first result included in the present text addresses the focusing case; it corresponds to the presence of a scalar field source [Val19]. We study the existence of solutions (u, W) , $u > 0$, to

$$\begin{aligned} \Delta_g u + hu = & f u^{q-1} + \frac{\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W|_g^2}{u^{q+1}} \\ & - \frac{b}{u} - c \langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) - \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}} \end{aligned} \quad (1.7.1)$$

$$\operatorname{div}_g(\rho_3 \mathcal{L}_g W) = \mathcal{R}(u, \nabla u, \nabla^2 u),$$

where $f > 0$, $h, b, c, d, \rho_1 > 0$, ρ_2, ρ_3 are smooth functions, $|\nabla \rho_3| < (2C)^{-1}$, with C_1 a

dimensional constant as in section 2 of [IÓM04], and Ψ, Y vector fields. Here, \mathcal{R} is an operator verifying

$$\begin{aligned} \mathcal{R}(u, \nabla u, \nabla^2 u) &\leq C_{\mathcal{R}} \left(1 + \frac{\|u\|_{\mathcal{C}^2}^2}{(\inf_M u)^2} \right) \\ &\leq C'_{\mathcal{R}} \left(1 + \left\| \frac{\nabla u}{u} \right\|_{L^\infty} + \left\| \frac{\nabla u}{u} \right\|_{L^\infty}^2 + \left\| \frac{\nabla^2 u}{u} \right\|_{L^\infty} \right) \end{aligned} \quad (1.7.2)$$

for a constant $C_{\mathcal{R}} > 0$ and $C'_{\mathcal{R}} = C'_{\mathcal{R}}(C_R, \inf_M u)$. We fix

$$\theta = \min(\inf_M \rho_1, \inf_M f), \quad (1.7.3)$$

and

$$T = \max(\|f\|_{\mathcal{C}^{1,\gamma}}, \|\rho_1\|_{\mathcal{C}^{0,\gamma}}, \|c\|_{\mathcal{C}^{0,\gamma}}, \|d\|_{\mathcal{C}^{0,\gamma}}, \|h\|_{\mathcal{C}^{0,\gamma}}). \quad (1.7.4)$$

Theorem 5 ([Vâl19]). *Let $n = 3, 4$, or 5 . There exists a constant $C = C(n, h)$, $C > 0$ such that if ρ_1 verifies*

$$\|\rho_1\|_{L^1(M)} \leq C(n, h) \left(\max_M |f| \right)^{1-n}, \quad (1.7.5)$$

then there exists a constant

$$\delta = \delta(\theta, T) > 0, \quad (1.7.6)$$

which allows the previous system to admit a solution whenever its coefficients satisfy the following smallness condition :

$$\|b\|_{\mathcal{C}^{0,\gamma}} + \|Y\|_{\mathcal{C}^{0,\gamma}} + \|\Psi\|_{\mathcal{C}^{0,\gamma}} + \|\rho_2\|_{\mathcal{C}^{0,\gamma}} + C_{\mathcal{R}} \leq \delta. \quad (1.7.7)$$

In order to obtain the desired existence result, we essentially impose two conditions : for the free data to be suitably small and for the dimension to be low. The proof builds on the work of Bruno Premoselli [Pre14], with added difficulties arising from the presence of non-linear terms involving the gradient of u and from the non-variational structure of the first equation.

We make use of a fixed point argument. We define a ‘‘ping-pong’’ operator which moves between the two equations of the system, alternatively fixing an unknown and solving for the remaining one. We define an operator

$$\Phi : \mathcal{C}^2(M) \rightarrow \mathcal{C}^2(M), \quad (1.7.8)$$

with

$$\Phi : \varphi \rightarrow u(\mathcal{L}_g W(\varphi)). \quad (1.7.9)$$

More precisely,

- $W(\varphi)$ solves the second equation for a fixed φ and
- $u(\mathcal{L}_g W(\varphi))$ is a solution for the scalar equation with fixed $W(\varphi)$.

In order for the resulting sequence to converge to a solution of the system (1.7.1), we must check

1. that Φ is well defined,
2. that there is a constant $M > 0$ such that $\Phi : B_M \rightarrow B_M$, $B_M := \{\varphi \in \mathcal{C}_+^2(\Sigma), \|\varphi\|_{\mathcal{C}^2} \leq M\}$,
3. that Φ is continuous
4. that Φ is compact.

In the following subsections, we briefly explain the main steps for obtaining the existence result of Theorem 5. The full proof is found in chapter 2.

1.7.1 Is the ping-pong operator well defined ?

We begin by establishing the existence of solutions to the Lichnerowicz-type scalar equation that can be precisely identified.

First, we see that lower bounds on positive solutions u are immediate. Given an equation of the type

$$\begin{aligned} \Delta_g u + hu &= fu^{q-1} + \frac{a}{u^{q+1}} \\ &\quad - \frac{b}{u} - c\langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) - \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}}, \end{aligned} \quad (1.7.10)$$

there exists $\varepsilon > 0$ such that $u \geq \varepsilon$. By looking at the equation at a minimum x_0 of u , where $\nabla u(x_0) = 0$, we obtain

$$h(x_0)u(x_0)^{q+2} \geq a(x_0) - b(x_0)u(x_0)^q. \quad (1.7.11)$$

Here, we use the fact that $f \geq 0$. The positive lower bound on u follows from the fact that $a > 0$ on M . Additionally, it becomes clear that $a > 0$ implies that the negative power terms do not explode for perturbations that allow u_α to approach 0.

We use an iterative construction to create a sequence of subsolutions which are bounded from both above and below. A supersolution (respectively subsolution) of the first equation verifies

$$\begin{aligned} \Delta_g \psi + h\psi &\stackrel{(\leq)}{\geq} f\psi^{q-1} + \frac{a}{\psi^{q+1}} \\ &\quad - \frac{b}{\psi} - c\langle \nabla \psi, Y \rangle \left(\frac{d}{\psi^2} + \frac{1}{\psi^{q+2}} \right) - \frac{\langle \nabla \psi, Y \rangle^2}{\psi^{q+3}}, \end{aligned} \quad (1.7.12)$$

For $i \in \mathbb{N}^*$, we find the unique solutions u_i of equations (E_i) :

$$\begin{aligned} \Delta_g u_i + (h + K)u_i &= fu_{i-1}^{q-1} + \frac{a}{u_{i-1}^{q+1}} \\ &\quad - \frac{b}{u_{i-1}} - c\langle \nabla u_i, Y \rangle \left(\frac{d}{u_{i-1}^2} + \frac{1}{u_{i-1}^{q+2}} \right) - \frac{\langle \nabla u_i, Y \rangle^2}{u_{i-1}^{q+3}}, \end{aligned} \quad (1.7.13)$$

Here, u_0 is chosen to be the constant ε given above. By induction, we show that

- u_i is a subsolution of E_{i+1} , and therefore a subsolution of the Lichnerowicz equation, and
- $u_i \leq \psi$.

By a comparison principle, we obtain the sequence u_i which is pointwise increasing and bounded from above by ψ . In fact, the construction is independent of the choice of ψ - therefore, u is the smallest supersolution (and therefore solution) larger than ε . The limit is thus shown to be a minimal solution.

1.7.2 Is it a contraction ?

We show that perturbations of first equation's coefficients do not lead to loss of compactness in dimensions $n = 3, 4$, or 5 .

Theorem 6 ([Vál19]). *Let (M, g) be a closed Riemannian manifold of dimension $n = 3, 4, 5$. Let $\frac{1}{2} < \eta < 1$ and $0 < \alpha < 1$. Let a, b, c, d, f, h be smooth functions on M , let Y be a smooth vector field on M . For any $0 < \theta < T$, there exists $S_{\theta, T}$ and $\vartheta_{\theta, T}$ such that, given any parameters within*

$$\begin{aligned} \mathcal{E}_{\theta, T} &:= \left\{ (f, a, b, c, d, h, Y), \quad f \geq \theta, \quad a \geq \theta, \right. \\ &\quad \left. \|f\|_{\mathcal{C}^{1, \eta}} \leq T, \quad \|a\|_{\mathcal{C}^{0, \alpha}}, \|b\|_{\mathcal{C}^{0, \alpha}}, \|c\|_{\mathcal{C}^{0, \alpha}}, \|d\|_{\mathcal{C}^{0, \alpha}}, \|h\|_{\mathcal{C}^{0, \alpha}}, \|Y\|_{\mathcal{C}^{0, \alpha}} \leq T \right\}, \end{aligned} \quad (1.7.14)$$

with

$$\|Y\|_{C^{0,\alpha}} \leq \vartheta_{\theta,T}, \quad (1.7.15)$$

then any smooth positive solution u of the Lichnerowicz-type equation satisfies

$$\|u\|_{C^2} \leq S_{\theta,T}. \quad (1.7.16)$$

Solutions risk exploding both in $\|u_\alpha\|_{L^\infty}$ and in $\|\nabla u_\alpha\|_{L^\infty}$. The analytical techniques involved in this section are subtle (see for example Druet, Hebey [DH09], [DHR04] or the survey [Dru10]).

The notion of concentration point is useful for studying stability. Its definition depends on the equation or system of equations to which it is applied. In the case of (1.7.1), a concentration point is the limit in M of any sequence $(x_\alpha)_\alpha$ where

$$u_\alpha(x_\alpha) + \nabla u_\alpha(x_\alpha) \rightarrow \infty. \quad (1.7.17)$$

When (1.7.17) holds, the system (1.7.1) is unstable (in the sense of [DH09]). When it does not, the system is stable.

We briefly describe the strategy of obtaining *a priori* bounds on solutions to (1.7.1), which is a proof by contradiction. We first assume that concentration points exist and we study the pointwise asymptotic blow-up profiles they display. Naively, we look at the shape of solutions around points where they blow-up. The next step is to show that these particular blow-up profiles cannot actually exist whenever $3 \leq n \leq 5$.

Identifying a “first” potential concentration point

Assuming that concentration points exist, the first concentration point we identify is the one that corresponds to the biggest blow-up. To do this, we rescale the solutions such that the dominant terms, $\Delta_g u$ and $f u^{q-2}$, are preserved. This technique was initially developed by Michael Struwe [Str84] and Richard Schoen [Sch89]. Instead of using the characterization (1.7.17), we use an equivalent form,

$$u_\alpha(x_\alpha) + \left| \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right|^{\frac{n-2}{2}} \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty, \quad (1.7.18)$$

which makes calculations easier when taking the limit of perturbations. We suppose that there exists a sequence $(x_\alpha)_\alpha$ such that (1.7.18) holds, where

$$u_\alpha(x_\alpha) + \left| \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right|^{\frac{n-2}{2}} = \sup_{x \in M} \left(u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right). \quad (1.7.19)$$

We denote

$$\mu_\alpha^{1-\frac{n}{2}} := u_\alpha(x_\alpha) + \left| \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right|^{\frac{n-2}{2}}. \quad (1.7.20)$$

We consider the rescaled quantities

$$v_\alpha(x) := \mu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_\alpha}(\mu_\alpha x)) \quad \text{et} \quad g_\alpha(x) := (\exp_{x_\alpha}^* g)(\mu_\alpha x) \quad (1.7.21)$$

defined on progressively smaller balls, $B_0\left(\frac{\delta}{\mu_\alpha}\right)$, where $0 < \delta < \frac{1}{2}i_g(M)$. Here, $i_g(M)$ is the injectivity radius of M . Thus,

$$\limsup_{\alpha \rightarrow \infty} \sup_{B_0(R)} \left(v_\alpha(x) + \left| \frac{\nabla v_\alpha(x)}{v_\alpha(x)} \right|^{\frac{n-2}{2}} \right) = 1. \quad (1.7.22)$$

Additionally, in order to treat the potential case when both u_α and ∇u_α blow-up around the same concentration points, we need to consider a further rescaling. Let

$$w_\alpha(x) := \frac{\hat{u}_\alpha(x)}{\hat{u}_\alpha(0)} = \frac{u_\alpha(\exp_{x_\alpha}(\mu_\alpha x))}{u_\alpha(x_\alpha)}. \quad (1.7.23)$$

Up to a subsequence, we denote

$$\begin{aligned} \hat{l}_\alpha &= \hat{u}_\alpha(0), & \text{with } \lim_{\alpha \rightarrow \infty} \hat{l}_\alpha &=: \hat{l} \in [0, 1], \\ l_\alpha &= u_\alpha^{-1}(x_\alpha), & \text{with } \lim_{\alpha \rightarrow \infty} l_\alpha &=: l \in [0, \varepsilon^{-1}] \end{aligned} \quad (1.7.24)$$

which follows from (1.7.21) in the case of the first limit, and from (1.7.11) for the second. Furthermore, (1.7.20) implies that

$$\hat{l} = \lim_{\alpha \rightarrow \infty} \mu_\alpha^{\frac{n-2}{2}} = 0. \quad (1.7.25)$$

By standard elliptic theory, we find that there exists $W := \lim_{\alpha \rightarrow \infty} w_\alpha$ in $C_{\text{loc}}^{1,\eta}(\mathbb{R}^n)$ solving :

$$\Delta W = f(x_0)W^{q-1}\hat{l}^{q-2} - \frac{\langle \nabla W, Y(x_0) \rangle^2}{W^{q+3}}l^{q+2}, \quad x \in \mathbb{R}^n. \quad (1.7.26)$$

We consider three separate cases. Let

$$l = 0 \quad \text{and} \quad \hat{l} \neq 0. \quad (1.7.27)$$

By passing to the limit in the equation of v_α , we obtain

$$\Delta V = f(x_0)V^{q-1} \quad (1.7.28)$$

in \mathbb{R}^n . The exact form of non-trivial solutions is due to Caffarelli, Gidas and Spruck [CGS89]

$$V(x) = \left(1 + \frac{f(x_0)|x - y_0|^2}{n(n-2)} \right)^{1-\frac{n}{2}}. \quad (1.7.29)$$

In the remaining cases, we consider the limit equation (1.7.26). If

$$l \neq 0 \quad \text{and} \quad \hat{l} = 0, \quad (1.7.30)$$

we obtain

$$\Delta W = -\frac{\langle \nabla W, Y(x_0) \rangle^2}{W^{q+3}}l^{q+2}, \quad (1.7.31)$$

and if

$$l = 0 \quad \text{and} \quad \hat{l} = 0, \quad (1.7.32)$$

we see

$$\Delta W = 0. \quad (1.7.33)$$

These three cases encompass all potential blow-up behaviours of solutions u to the first constraint equation. In order to get the *a priori* estimates we need, we must show that any of these blow-up profiles leads to a contradiction.

The next step is to identify all possible concentration points. Naively, it may seem unusual to try to characterize a set of blow-up points that we then want to show do not actually exist. In fact, we're ultimately interested in how concentration points could potentially be distributed within M , either clustered or isolated, and how the sizes of the blow-ups compare to each other.

Other potential concentration points

In order to find a second potential concentration point, we construct a test operator

$$\Phi_\alpha(x) = d_g(x_\alpha, x)^{\frac{n-2}{2}} \left(u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right). \quad (1.7.34)$$

Note that

$$\Phi_\alpha(x_\alpha) = 0. \quad (1.7.35)$$

However, if $\sup_{x \in M} \Phi_\alpha(x) \rightarrow \infty$, then

$$\exists y_\alpha \in M, d_g(x_\alpha, y_\alpha)^{\frac{n-2}{2}} \left(u_\alpha(y_\alpha) + \left| \frac{\nabla u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \right|^{\frac{n-2}{2}} \right) \rightarrow \infty, \quad (1.7.36)$$

so, since M is compact,

$$u_\alpha(y_\alpha) + \left| \frac{\nabla u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \right|^{\frac{n-2}{2}} \rightarrow \infty. \quad (1.7.37)$$

Thus, if Φ_α blows up, it indicates the presence of a second concentration point. The process can be generalized so that if k potential concentration points have been identified, one might define an operator to identify a $k + 1$ th.

Weak pointwise estimate on M

The first result we might get is the following pointwise estimate on solutions u_α . This is a more or less immediate consequence of the discussion above on concentration points and possible blow-up profiles. It essentially states that, away from the points we just identified, the solutions are bounded. Moreover, it sets a "weak" bound on the size of the explosions we can expect to have.

Lemma 1. *There exists $N_\alpha \in \mathbb{N}^*$ and $\mathcal{S}_\alpha := (x_{1,\alpha}, \dots, x_{N_\alpha,\alpha})$ a set of critical points of $(u_\alpha)_\alpha$ such that*

$$d_g(x_{i,\alpha}, x_{j,\alpha})^{\frac{n-2}{2}} u_\alpha(x_{i,\alpha}) \geq 1$$

for all $i, j \in \{1, \dots, N_\alpha\}$, $i \neq j$, and

$$\left(\min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x) \right)^{\frac{n-2}{2}} u_\alpha(x) \leq 1$$

for all critical points of u_α and such that there exists $C_1 > 0$ such that

$$\left(\min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x) \right)^{\frac{n-2}{2}} \left(u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right) \leq C_1$$

for all $x \in M$ and all $\alpha \in \mathbb{N}$.

Fine asymptotic analysis around potential concentration points

Better estimates may be obtained through fine analysis. While the objects we use are not complicated (Green representation theorems, maximum principles), their application is somewhat arduous. Consider points (x_α) where $u(x_\alpha)_\alpha$ explodes. First of all, we get that in $B_{x_\alpha}(\mu_\alpha)$,

- $\mu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_\alpha}(\mu_\alpha x)) \rightarrow \left(1 + \frac{f(x_0)|x|^2}{n(n-2)} \right)^{1-\frac{n}{2}}$ in $\mathcal{C}_{loc}^1(\mathbb{R}^n)$.

Next, we look at what happens in a larger ball $B_{x_\alpha}(\rho_\alpha)$. We're especially interested in domains where, at the boundary, the influence of neighbouring blow-up's begins to be felt in a non-negligible fashion.

- $u_\alpha(x) \leq C\mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha, x)^{2-n}$ and
- $|\nabla u_\alpha(x)| \leq C\mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha, x)^{1-n}$,

and finally, on the boundary of this ball, we have

- $u_\alpha(x_\alpha)\rho_\alpha^{n-2}u_\alpha(\exp_\alpha(\rho_\alpha x)) \rightarrow \frac{\lambda_\alpha}{|x|^{n-2}} + H(x)$,

where H is a harmonic function on $B_{x_\alpha}(\rho_\alpha)$.

Contradictions

In order to rule out blow-ups, we need to obtain contradictions, one by one, for all possible distributions of explosions :

1. isolated
2. clustered, where the explosions are all of the same rate, and
3. clustered, where the bubbles explode at different rates.

We must also keep in mind that the profiles we must rule out roughly correspond to situations where

1. only u explodes,
2. only ∇u explodes,
3. both u and ∇u explode.

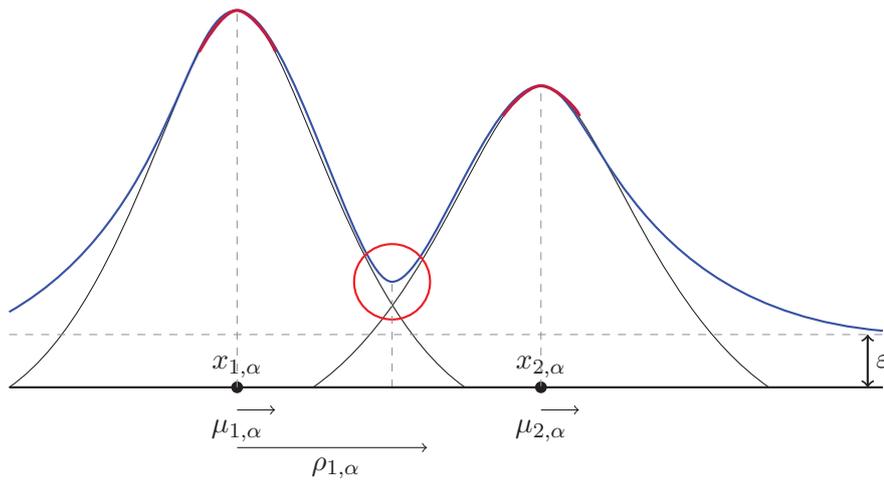


FIGURE 1.7.1 : A cluster with two similar bubbles

Dimension plays a key role in this discussion. Naively, the higher the dimension, the more lax the constrictions on possible bubble interactions. To illustrate this, we consider the case of a cluster with two bubbles of comparable sizes. Let them be centered at $x_{1,\alpha}$ and $x_{2,\alpha}$.

Their profiles on $B_{x_{1,\alpha}}(\mu_{1,\alpha})$ and $B_{x_{2,\alpha}}(\mu_{2,\alpha})$ are very close to $B_{1,\alpha}$ and $B_{2,\alpha}$, respectively. Consider p_α a point close to $x_{1,\alpha}$. Formally, we define

$$\begin{aligned} L_{g,\alpha}(u) &= \Delta_g u + h_\alpha u - f_\alpha u^{q-1} - \frac{a_\alpha}{u^{q+1}} \\ &\quad + \frac{b_\alpha}{u} + c_\alpha \langle \nabla u, Y_\alpha \rangle \left(\frac{d_\alpha}{u^2} + \frac{1}{u^{q+2}} \right) - \frac{\langle \nabla u, Y_\alpha \rangle^2}{u^{q+3}} \end{aligned} \quad (1.7.38)$$

Here, solutions of $L_{g,\alpha}(u_\alpha) = 0$ solve the perturbed equations.

We give a heuristic argument to explain why and how interaction between bubbles is constrained. At p_α , $u_\alpha \approx B_{1,\alpha}$. We rewrite it as $u_\alpha = B_{1,\alpha} + R_\alpha$. Moreover, let $L_{g,\alpha}(B_{1,\alpha}) =: F_\alpha$, which we can explicitly calculate. Again, formally we may obtain

$$0 = L_{g,\alpha}(B_{1,\alpha} + R_\alpha) = L_{g,\alpha}(B_{1,\alpha}) + L'_\alpha(B_{1,\alpha})(R_\alpha) + O(|R_\alpha|^2) \quad (1.7.39)$$

i.e.

$$L'_\alpha(B_{1,\alpha})(R_\alpha) = -F_\alpha + O(|R_\alpha|^2). \quad (1.7.40)$$

We also take into account the error terms that might arise from the geometry,

$$L'_{g,\alpha}(B_{1,\alpha}) \approx L'_{\xi,\alpha}(B_{1,\alpha}) + g \text{ error terms.} \quad (1.7.41)$$

which are small, as locally at a microscopic scale, $g \approx \xi$.

The kernel in the Euclidean is known and is given by the symmetries of the solutions, be it by scaling or by translation. For any $\varphi \in \text{Ker } L'_{\xi,\alpha}(B_{1,\alpha})$, we need

$$\varphi \perp -F_\alpha + O(|R_\alpha|^2) + g \text{ error terms,} \quad (1.7.42)$$

so $\varphi \perp L'_\alpha(B_{1,\alpha})(R_\alpha)$. These give us estimates on the error term R_α , which, in low dimension, are shown to lead to a contradiction at distance ρ_1 ; there, by definition, $R_\alpha \approx B_{2,\alpha}$. All this can be otherwise obtained through a Pohozaev identity [Poh65], which does the same, but in a more rigorous and direct way (more in the next chapter). See also the note by Jesse Ratzkin for more information on Pohozaev-type identities [Rat09].

1.8 The stability of the system.

The final chapter of this work details a compactness result obtained for Maxwell's system.

For convenience, we recall that the scalar equation of the general system (1.7.1) is

$$\begin{aligned} \Delta_g u + hu &= fu^{q-1} + \frac{\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W|_g^2}{u^{q+1}} \\ &\quad - \frac{b}{u} - c \langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) - \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}} \end{aligned} \quad (1.8.1)$$

and that the second equation of the physical system (1.5.6) writes as

$$\begin{aligned} \vec{\Delta}_g W &= \langle \nabla \ln \tilde{N}, \mathcal{L}_g W \rangle + 2 \frac{n-1}{n-2} \left(\frac{3n-2}{n-2} \frac{\langle \nabla u, \tilde{V} \rangle \nabla u}{u^2} - \frac{\langle \nabla^2 u, \tilde{V} \rangle}{u} \right) \\ &\quad + 2 \frac{n-1}{n-2} \left(-\frac{\langle \nabla u, \tilde{V} \rangle}{u} \nabla \ln \tilde{N} + \text{div}_g \tilde{V} \frac{\nabla u}{u} - \frac{\langle \nabla \tilde{V}, \nabla u \rangle}{u} \right) \\ &\quad - \frac{n-1}{n} \left(\text{div}_g \tilde{V} \nabla \ln \tilde{N} + \nabla \text{div}_g \tilde{V} \right) - 2 \tilde{N}^{-1} \pi \nabla \psi. \end{aligned} \quad (1.8.2)$$

We would like to prove the *a priori* estimate

$$\|u_\alpha\|_{C^{2,\eta}} + \|W_\alpha\|_{C^{1,\eta}} \leq C. \quad (1.8.3)$$

If this is true, then by standard elliptic theory there exists, up to a subsequence, a $\mathcal{C}^{2,\eta}$ limit of (u_α, W_α) solving the limiting system (1.7.1). In effect, since the system (1.7.1) is invariant by the addition of conformal Killing fields, it suffices to show that

$$\|u_\alpha\|_{L^\infty} + \|\nabla u_\alpha\|_{L^\infty} + \|\nabla^2 u_\alpha\|_{L^\infty} + \|\mathcal{L}_g W_\alpha\|_{L^\infty} \leq C. \quad (1.8.4)$$

The proof follows by contradiction. We assume instead that there exists a sequence of solutions (u_α, W_α) of the perturbed system such that

$$\|u_\alpha\|_{L^\infty} + \|\nabla u_\alpha\|_{L^\infty} + \|\nabla^2 u_\alpha\|_{L^\infty} + \|\mathcal{L}_g W_\alpha\|_{L^\infty} \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty. \quad (1.8.5)$$

Theorem 7. *Let (M, g) be a closed Riemannian manifold of dimension $n = 3, 4, 5$, where g is locally conformally flat. Let $\frac{1}{2} < \eta < 1$ and $0 < \alpha < 1$. Let $a, b, c, d, f, h, \rho_1, \rho_2, \psi, \pi, \tilde{N}$ be smooth functions on M , let \tilde{V} and Y be smooth vector field on M . For any $0 < \theta < T$, there exists $S_{\theta,T}$ and $\vartheta_{\theta,T}$ such that, given any parameters within*

$$\begin{aligned} \mathcal{E}_{\theta,T} := & \left\{ (f, a, b, c, d, h, \rho_1, \rho_2, Y) \times (\tilde{N}, \tilde{V}, \psi, \pi), \quad f \geq \theta, \quad a \geq \theta, \quad \tilde{N} \geq \theta, \right. \\ & \|f\|_{\mathcal{C}^{1,\eta}} \leq T, \\ & \|a\|_{\mathcal{C}^{1,\alpha}}, \|b\|_{\mathcal{C}^{1,\alpha}}, \|c\|_{\mathcal{C}^{1,\alpha}}, \|d\|_{\mathcal{C}^{1,\alpha}}, \\ & \|\rho_1\|_{\mathcal{C}^{1,\alpha}}, \|\rho_2\|_{\mathcal{C}^{1,\alpha}}, \|h\|_{\mathcal{C}^{1,\alpha}}, \|Y\|_{\mathcal{C}^{1,\alpha}} \leq T, \\ & \left. \text{and } \|\tilde{N}\|_{\mathcal{C}^{2,\alpha}}, \|\tilde{V}\|_{\mathcal{C}^{2,\alpha}} \leq T \right\}, \end{aligned} \quad (1.8.6)$$

with

$$\|Y\|_{\mathcal{C}^{1,\alpha}}, \|\tilde{V}\|_{\mathcal{C}^{2,\alpha}} \leq \vartheta_{\theta,T}, \quad (1.8.7)$$

then any smooth solution (u, W) , with $u > 0$, satisfies

$$\|u\|_{\mathcal{C}^{2,\alpha}} + \|\mathcal{L}_g W\|_{\mathcal{C}^{1,\alpha}} \leq S_{\theta,T}. \quad (1.8.8)$$

The proof strategy is very similar to the one used in obtaining *a priori* estimates to solutions of the Lichnerowicz-type equation. However, this time we must take into account possible lack of compactness stemming from the $\mathcal{L}_g W_\alpha$ term. The weak pointwise estimate takes the form

$$\begin{aligned} & \left(\min_{1 \leq i \leq N_\alpha} d_g(x_{i,\alpha}, x) \right)^n \\ & \times \left(u_\alpha^q(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^n + \left| \frac{\nabla^2 u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n}{2}} + |\mathcal{L}_g W_\alpha|_g(x) \right) \leq C. \end{aligned} \quad (1.8.9)$$

Most of the profiles have already been treated in [Vâl19]. The main difference comes from the limiting system

$$\begin{aligned} \Delta w &= \frac{n-2}{4(n-1)} (2V(\psi(x_0)) - \frac{n-1}{n} \tau^{*2}) w^{q-1} \hat{l}^{q-2} + \frac{n-2}{16(n-1)} \frac{\tilde{N}^2(x_0) |\mathcal{L}_\xi Z|_\xi^2}{w^{q+1}} \\ & \quad - \frac{n}{n-2} \frac{\langle \nabla w, \tilde{N}(x_0) \tilde{V}(x_0) \rangle^2}{w^{q+3}} l^{q+2} \\ \vec{\Delta}_\xi Z &= -2 \frac{n-1}{n+1} \frac{\langle \tilde{V}(x_0), \nabla w \rangle \nabla w}{w^2} l^{\frac{q+2}{2}} - \frac{n-1}{n} \langle \tilde{V}(x_0), \frac{\nabla^2 w}{w} \rangle l^{\frac{q+2}{2}}, \end{aligned} \quad (1.8.10)$$

with l and \hat{l} as in (1.7.24), which results from both $\mathcal{L}_g W$ and ∇u_α or $\nabla^2 u_\alpha$ simultaneously. The additional smallness condition on \tilde{V}_α ensures that the above system reduces to the classical equation

$$\Delta w = \frac{n-2}{4(n-1)} \left(2V(\psi(x_0)) - \frac{n-1}{n} \tau^{*2} \right) w^{q-1} \hat{l}^{q-2}. \quad (1.8.11)$$

In the future, it might prove fruitful to further examine the possible solutions of the second equation, together with the exact coupling of the system, in order to not have to impose additional limits on the drift term \tilde{V} .

Finally, we'd like to discuss the condition that g be locally conformally flat. The same holds true for the stability of the classical system, also in the focusing case, which was treated in [Pre16]. In order to obtain the improved weak estimates on $\mathcal{L}_g W_\alpha$ that we need, the Green representation formula is applied on balls of diminishing radius $B_{x_\alpha}(\delta_\alpha)$, $\delta_\alpha \rightarrow 0$, where x_α is a concentration point. The bounds needs to be uniform with respect to α , which is why we need the kernel of $\vec{\Delta}_{g_\alpha}$ to have the same dimension as that of $\vec{\Delta}_g$, with $g_\alpha = \exp_{x_\alpha}^*(\delta_\alpha \cdot)$.

The chapter is dedicated to the establishment of the existence in low dimensions of solutions to the constraint equations in the case of the conformal system proposed by David Maxwell [Max14b], with the added presence of a scalar field and under suitable smallness assumptions on its parameters. This chapter contains a reproduction of [Vâl19] :

Caterina Vâlcu, *The Constraint Equations in the Presence of a Scalar Field : The Case of the Conformal Method with Volumetric Drift*, Comm. Math. Phys., (2019), arXiv:1809.07218, HAL:hal-02022713, version 1.

The constraint equations in the presence of a scalar field - the case of the conformal method with volumetric drift

Contents

2.1	Introduction	36
2.2	Existence of minimal solutions of the scalar equation	41
2.3	A priori estimates on solutions of the scalar equation in low dimensions	48
2.3.1	Concentration points	49
2.3.2	Local blow-up analysis	54
2.3.3	Proof of the stability theorem	66
2.4	Existence of solutions to the system	71
2.4.1	The proof of the main theorem	71
2.4.2	The case of a metric with conformal Killing fields	74
2.5	Annex	76

2.1 Introduction

The field of general relativity deals with the study of spacetime, an object defined as the equivalence class, up to an isometry, of Lorentzian manifolds (\tilde{M}, \tilde{g}) of dimension $n + 1$ satisfying the Einstein field equations

$$Ric_{\alpha\beta}(\tilde{g}) - \frac{1}{2}R(\tilde{g})\tilde{g}_{\alpha\beta} = 8\pi T_{\alpha\beta}, \quad (2.1.1)$$

$\alpha, \beta = \overline{1, n+1}$. Here, $R(\tilde{g})$ is the scalar curvature of \tilde{g} , Ric the Ricci curvature and $T_{\alpha\beta}$ the stress-energy tensor describing the presence of matter and energy. For example, $T_{\alpha\beta} = 0$ describes vacuum. Our interest focuses on the more general case

$$T_{\alpha\beta} = \tilde{\nabla}_\alpha \tilde{\psi} \tilde{\nabla}_\beta \tilde{\psi} - \left(\frac{1}{2} |\tilde{\nabla} \tilde{\psi}|_{\tilde{g}}^2 + V(\tilde{\psi}) \right) \tilde{g}_{\alpha\beta}, \quad (2.1.2)$$

which models the existence within the spacetime of a scalar field $\tilde{\psi} \in \mathcal{C}^\infty(M)$ having potential $V \in \mathcal{C}^\infty(\mathbb{R})$. Thus, $\tilde{\psi} = 0$ and $V = \Lambda$ yield the vacuum with cosmological constant Λ , while $V = \frac{1}{2}m\tilde{\psi}^2$ corresponds to the Einstein-Klein-Gordon setting.

For a globally hyperbolic spacetime, we define its initial data $(M, \hat{g}, \hat{K}, \hat{\psi}, \hat{\pi})$. They consist of an n -dimensional Riemannian manifold (M, \hat{g}) , which models the spacetime at a particular moment in time, a symmetric 2-tensor \hat{K} , corresponding to its second fundamental form, the scalar field $\hat{\psi}$ in M , and its temporal derivative $\hat{\pi}$. The associated spacetime development takes the form $(M \times \mathbb{R}, \tilde{g}, \tilde{\psi})$, where \tilde{g} is a Lorentzian metric that verifies $\tilde{g}|_M = \hat{g}$ and $\tilde{\psi}$ is a scalar field such that $\psi|_M = \hat{\psi}$ and $\partial_t \tilde{\psi}|_M = \hat{\pi}$.

Initial data in general relativity may not be freely specified, unlike their Newtonian counterparts. Instead, they must verify the Gauss and Codazzi equations,

$$\begin{aligned} R(\hat{g}) + (tr_{\hat{g}} \hat{K})^2 - |\hat{K}|_{\hat{g}}^2 &= \hat{\pi}^2 + |\hat{\nabla} \hat{\psi}|_{\hat{g}}^2 + 2V(\hat{\psi}) \\ \partial_i (tr_{\hat{g}} \hat{K}) - \hat{K}_{i,j}^j &= \hat{\pi} \partial_i \hat{\psi}, \end{aligned} \quad (2.1.3)$$

which are referred to as the *constraint equations*. The work of Choquet-Bruhat [FB52] establishes, once and for all, that the constraint equations are not only necessary but sufficient conditions for the (local) existence of a solution. Later, Choquet-Bruhat and Geroch [CBG69] prove that the maximal development of initial data is unique, up to an isometry. Globally hyperbolic spacetimes may rigorously be studied in the context of mathematical analysis as the result of an evolution problem. The above system is clearly under-determined, which allows for considerable freedom in choosing a solution $(\hat{g}, \hat{K}, \hat{\psi}, \hat{\pi})$.

Using the conformal method introduced by Lichnerowicz [Lic44], the constraint equations may be transformed into a determined system of equations by fixing well-chosen quantities (see Choquet-Bruhat, Isenberg and Pollack [CBIP07]). The appeal of such a method lies in that it provides a characterisation of the resulting initial data by fixed quantities. Essentially, it maps a space of parameters to the space of solutions.

Given an initial data set $(\hat{g}, \hat{K}, \hat{\psi}, \hat{\pi})$, the classical choice of parameters is $(\mathbf{g}, \mathbf{U}, \tau, \psi, \pi; \alpha)$: in this case, the conformal class \mathbf{g} is represented by a Riemannian metric g , the smooth function $\tau = \hat{g}^{ab} \hat{K}_{ab}$ is a mean curvature and the conformal momentum \mathbf{U} measured by a volume form α (volume gauge) is a 2-tensor that is both trace-free and divergence-free with respect to g (a transverse-traceless tensor). We sometimes prefer to indicate the volume gauge by the densitized lapse

$$\tilde{N}_{g,\alpha} := \frac{\alpha}{dV_g}. \quad (2.1.4)$$

Note that this quantity depends on the choice of representative g , unlike the volume gauge α which does not. The standard conformal method implicitly fixes $\tilde{N}_{g,\alpha} = 2$; in the present paper, we prefer to make use of the freedom of choosing $\tilde{N}_{g,\alpha}$ as needed. We often refer to a parameter set by indicating the representative metric g and the corresponding densitized lapse $\tilde{N}_{g,\alpha}$ instead of giving the conformal class and volume gauge. However, these quantities can immediately be reconstructed from our data. We refer to Maxwell [Max14b] for an introduction to the conformal method in our context.

Starting from the parameter set $(g, U, \tau, \psi, \pi; \tilde{N})$, the corresponding (physical) initial data is pinpointed by solving a resulting system, comprising the Lichnerowicz-type equation and the momentum constraints, for a smooth positive function (or conformal factor) u and a smooth vector field W in M ,

$$\begin{aligned} \Delta_g u + \mathcal{R}_\psi &= -\mathcal{B}_{\tau,\psi,V} u^{q-1} + \frac{\mathcal{A}_{\pi,U}(W)}{u^{q+1}}, \\ \vec{\Delta}_g W &= \frac{n-1}{n} u^q \nabla \tau + \pi \nabla \psi, \end{aligned} \quad (2.1.5)$$

where

$$\begin{aligned} \mathcal{R}_\psi &= \frac{n-2}{4(n-1)} (R(g) - |\nabla \psi|_g^2), \\ \mathcal{B}_{\tau,\psi,V} &= \frac{n-2}{4(n-1)} \left(\frac{n-1}{n} \tau^2 - 2V(\psi) \right), \\ \mathcal{A}_{\pi,U}(W) &= \frac{n-2}{4(n-1)} (|U + \mathcal{L}_g W|_g^2 + \pi^2). \end{aligned} \quad (2.1.6)$$

If (u, W) solves the above system, then the initial data we've been searching for are

$$\hat{g} = u^{q-2} g, \quad \hat{K} = u^{-2} \left(U + \frac{\tilde{N}}{2} \mathcal{L}_g W \right) + \frac{\tau}{n} \hat{g}, \quad \hat{\psi} = \psi, \quad \hat{\pi} = u^{-q} \pi. \quad (2.1.7)$$

Note also that the solutions generated by $(g, U, \tau; \tilde{N})$ and

$$(\varphi^{q-2} g, \varphi^{-2} U, \tau, \psi, \varphi^{-q} \pi; \varphi^q \tilde{N}) \quad (2.1.8)$$

are the same, where φ is a smooth positive function. The notations above are similar to those of Choquet-Bruhat, Isenberg and Pollack [CBIP07]. The following quantities often appear throughout the present paper: $q = \frac{2n}{n-2}$ is the critical Sobolev exponent for the embedding of H^1 in Lebesgue spaces, $\Delta_g = -\text{div}_g \nabla$ denotes the Laplace-Beltrami operator taken with non-negative eigenvalues, $\overrightarrow{\Delta}_g W = -\text{div}_g(\mathcal{L}_g W)$ is the Lamé operator and \mathcal{L}_g is the conformal Killing operator with respect to g ,

$$\mathcal{L}_g W_{ij} = W_{i,j} + W_{j,i} - \frac{2}{n} \text{div}_g W g_{ij}. \quad (2.1.9)$$

Conformal Killing fields are defined as vector fields in the kernel of \mathcal{L}_g .

The conformal method is particularly successful in finding solutions when the mean curvature τ is constant as the system (2.1.5) becomes uncoupled, but it is unclear how well the method functions when the mean curvature is far from being constant: see Maxwell [Max11] and [Max15], where a given set of parameters point to no or to an infinite number of solutions. We emphasize that any failing of the system does not necessarily translate to a singularity in the space of solutions to the constraints system, but may instead derive from a poor choice of mapping. This motivates the study of variations to standard conformal methods.

The drift method introduced by Maxwell replaces the mean curvature τ as a parameter by a pair (τ^*, \tilde{V}) , where τ^* is a unique constant called volumetric momentum and \tilde{V} a vector field related to the drift. They verify an analogue of York splitting, namely

$$\tau = \tau^* + \tilde{N}_{\hat{g},\alpha} \text{div}_{\hat{g}} \tilde{V} = \tau^* + \frac{\tilde{N}_{g,\alpha}}{u^{2q}} \text{div}_g(u^q \tilde{V}), \quad (2.1.10)$$

the notation \tilde{V} being specific to this paper in order to avoid confusion with the potential V . Interestingly, $\tau^* = 0$ holds true for all counterexamples found by Maxwell [Max11, Max15]. This suggests that the volumetric momentum may play an important role in characterizing the space of initial data. Ideally, we would like to know as soon as we fix a set of parameters $(g, U, \tau, \psi, \pi; \tilde{N})$ if we find ourselves in the case $\tau^* = 0$. However, τ^* cannot be directly calculated by (2.1.10) without first solving (2.1.5), which somewhat defeats the purpose. This motivates a new choice of parameters, even at the risk of working with an analytically more complicated system. The idea of Maxwell [Max11, Max15] is thus to choose τ^* as an additional parameter that is to be fixed in the place of τ : the hope is thus to avoid the aforementioned problem. As well as τ^* , Maxwell added \tilde{N} and \tilde{V} as parameters, for geometric and physical reasons. Therefore, instead of fixing τ , we fix τ^* , \tilde{N} and \tilde{V} .

Intuitively, the drift is a geometric quantity describing infinitesimal motion in the space of metrics modulo the group of diffeomorphisms connected to the identity such that the conformal class and volume are preserved. For any given drift, the choice of a representative vector field \tilde{V} is unique up to conformal Killing fields and vector fields which are divergence-free with respect to the initial metric \hat{g} . Given g an arbitrary representative of the conformal class, it is not clear whether two vector fields are indicative of the same drift class defined for \hat{g} ; this problem is discussed at length in the paper of Mike Holst, David Maxwell and Rafe Mazzeo [HMM18] for conformal systems where the critical non-linearity is non-focusing, or negative. Our analysis treats systems with focusing (that is to say positive) non-linearities stemming from the presence of a scalar field with positive potential.

The following system corresponds to Problem 12.1 of [Max14b] in the presence of a

scalar field, where g admits no non-trivial conformal Killing fields:

$$\begin{aligned} \Delta_g u + \frac{n-2}{4(n-1)}(R(g) - |\nabla\psi|_g^2)u - \frac{(n-2)|U + \mathcal{L}_g W|^2 + \pi^2}{4(n-1)u^{q+1}} \\ - \frac{n-2}{4(n-1)} \left[2V(\psi) - \frac{n-1}{n} \left(\tau^* + \frac{\tilde{N} \operatorname{div}_g(u^q \tilde{V})}{u^{2q}} \right)^2 \right] u^{q-1} = 0 \end{aligned} \quad (2.1.11)$$

$$\operatorname{div}_g \left(\frac{\tilde{N}}{2} \mathcal{L}_g W \right) - \frac{n-1}{n} u^q \mathbf{d} \left(\frac{\tilde{N} \operatorname{div}_g(u^q \tilde{V})}{2u^{2q}} \right) - \pi \nabla \psi = 0.$$

We denote the exterior derivative by \mathbf{d} . The unknowns are a smooth positive scalar function u defined on M and a smooth vector field W on M . The parameters are $(g, U, \tau^*, \tilde{V}, \psi, \pi; \tilde{N})$. Maxwell's new set of parameters include τ^* , which could not be calculated *a priori* in the classical method. The initial data of the constraint equations verify

$$\begin{aligned} \hat{g} = u^{q-2}g, \quad \hat{K} = u^{-2} \left(U + \frac{\tilde{N}}{2} \mathcal{L}_g W \right) + \frac{1}{n} \left(\tau^* + \frac{\tilde{N}}{u^{2q}} \operatorname{div}(u^q \tilde{V}) \right) \hat{g}, \\ \hat{\psi} = \psi, \quad \hat{\pi} = u^{-q}\pi. \end{aligned} \quad (2.1.12)$$

The following is a more general system than (2.1.11). The central result of the paper consists in showing that it admits solutions. Let (M, g) be a closed Riemannian manifold of dimension $n \in \{3, 4, 5\}$, and g has no non-trivial conformal Killing fields. Let $b, c, d, f, h, \rho_1, \rho_2, \rho_3$ be smooth functions on M and let Y and Ψ be smooth vector fields defined on M . Let $0 < \gamma < 1$. Assume that $\Delta_g + h$ is coercive, in the sense that its first eigenvalue is positive. Assume that $f > 0, \rho_1 > 0$ and $|\nabla \rho_3| < (2C_1)^{-1}$, where C_1 is a dimensional constant - see (2.4.1). Consider the system

$$\begin{aligned} \Delta_g u + hu &= fu^{q-1} + \frac{\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W|_g^2}{u^{q+1}} \\ &\quad - \frac{b}{u} - c \langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) - \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}} \\ \vec{\Delta}_g W &= \rho_3 \mathcal{L}_g W + \mathcal{R}(u, \nabla u, \nabla^2 u). \end{aligned} \quad (2.1.13)$$

Here \mathcal{R} is an operator verifying

$$\mathcal{R}(u, \nabla u, \nabla^2 u) \leq C_{\mathcal{R}} \left(1 + \frac{\|u\|_{C^2}^2}{(\inf_M u)^2} \right) \quad (2.1.14)$$

for a constant $C_{\mathcal{R}} > 0$.

A supersolution of the Lichnerowicz-type equation is a smooth function u verifying that

$$\begin{aligned} \Delta_g u + hu \geq fu^{q-1} + \frac{\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W|_g^2}{u^{q+1}} - \frac{b}{u} \\ - c \langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) - \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}}. \end{aligned} \quad (2.1.15)$$

Similarly, a subsolution satisfies an inequality of opposite sign. Whenever the inequality is strict, we say u is a strict subsolution or a strict supersolution respectively.

We fix

$$\theta = \min(\inf_M \rho_1, \inf_M f), \quad (2.1.16)$$

and

$$T = \max(\|f\|_{C^{1,\gamma}}, \|\rho_1\|_{C^{0,\gamma}}, \|c\|_{C^{0,\gamma}}, \|d\|_{C^{0,\gamma}}, \|h\|_{C^{0,\gamma}}). \quad (2.1.17)$$

Here is the main result of our paper:

Theorem 8. *Let $n = 3, 4,$ or 5 . There exists a constant $C = C(n, h)$, $C > 0$ such that if ρ_1 verifies*

$$\|\rho_1\|_{L^1(M)} \leq C(n, h) \left(\max_M |f| \right)^{1-n}, \quad (2.1.18)$$

then there exists a constant

$$\delta = \delta(\theta, T) > 0, \quad (2.1.19)$$

which allows the system (2.1.13) to admit a solution whenever its coefficients satisfy the following smallness condition:

$$\|b\|_{C^{0,\gamma}} + \|Y\|_{C^{0,\gamma}} + \|\Psi\|_{C^{0,\gamma}} + \|\rho_2\|_{C^{0,\gamma}} + C_{\mathcal{R}} \leq \delta. \quad (2.1.20)$$

Remark 2. *For a slightly more detailed expression of the smallness assumptions, see Section 2.4. The constant $C(n, h) = \frac{C(n)}{S_h^{n-1}}$ appears explicitly in a paper by Hebey, Pacard and Pollack ([HPP08], Corollary 3.1). By S_h we understand the Sobolev constant which is defined as the smallest constant $S_h > 0$ such that*

$$\int_M |v|^q dv_g \leq S_h \left(\int_M (|\nabla v|^2 + hv^2) dv_g \right)^{\frac{q}{2}} \quad (2.1.21)$$

for all $v \in H^1(M)$.

The following corollary deals with the existence of solutions to the conformal system. It suffices to take

$$\begin{aligned} h &= \frac{n-2}{4(n-1)} (\mathcal{R}_g - |\nabla\psi|_g^2), \quad f = \frac{n-2}{4(n-1)} [2V(\psi) - \frac{n-1}{n}(\tau^*)^2], \\ \rho_1 &= \frac{n-2}{4(n-1)} \left(\pi - \frac{n-1}{n}(\tilde{N})^2 \operatorname{div}_g(\tilde{V}) \right), \quad \rho_2 = \sqrt{\frac{n-2}{(n-1)} \frac{\tilde{N}}{4}}, \\ \Psi &= \sqrt{\frac{n-2}{4(n-1)}} U, \\ b &= \frac{n-2}{2n} \tau^* \tilde{N} \operatorname{div}_g(\tilde{V}), \quad c = \sqrt{\frac{n-2}{n}}, \quad d = \tau^* \\ Y &= \sqrt{\frac{n}{n-2}} \tilde{N} \tilde{V}, \quad \rho_3 = \ln \tilde{N}, \end{aligned} \quad (2.1.22)$$

and

$$\begin{aligned} \mathcal{R} &= \frac{n-1}{n} \operatorname{div}_g(\tilde{V}) \nabla \ln \tilde{N} + \frac{n-1}{n} \nabla(\operatorname{div}_g(\tilde{V})) + \frac{\pi \delta_i \psi}{\tilde{N}} \\ &\quad + 2 \langle \tilde{V}, \frac{\nabla u}{u} \rangle \nabla \ln \tilde{N} - 2 \frac{n-1}{n+1} \frac{\langle \tilde{V}, \nabla u \rangle \nabla u}{u^2} - \frac{n-1}{n} \langle \tilde{V}, \frac{\Delta_g u}{u} \rangle \end{aligned} \quad (2.1.23)$$

in (2.1.13). It is a direct application of Theorem 8.

Corollary 1. *Let $\Delta_g + \frac{n-2}{4(n-1)} (\mathcal{R}_g - |\nabla\psi|_g^2)$ be a coercive operator. Assume that*

$$2V(\psi) > \frac{n-1}{n} (\tau^*)^2, \quad \pi > \frac{n-1}{n} (\tilde{N})^2 \operatorname{div}_g(\tilde{V}) \quad \text{and} \quad |\nabla \ln \tilde{N}| < C_1^{-1}, \quad (2.1.24)$$

where C_1 depends on n and g (see (2.4.1) for more details). Moreover, assume that

$$\left\| \pi - \frac{n-1}{n} (\tilde{N})^2 \operatorname{div}_g(\tilde{V}) \right\|_{L^1} \leq C(n, g, h) \left\| 2V(\psi) - \frac{n-1}{n} (\tau^*)^2 \right\|_{L^\infty}^{1-n}. \quad (2.1.25)$$

Then there exists a positive constant

$$\begin{aligned} \delta = \delta \left(\inf_M \frac{n-2}{4(n-1)} [2V(\psi) - \frac{n-1}{n}(\tau^*)^2], \right. \\ \left. \inf_M \frac{n-2}{4(n-1)} \left(\pi - \frac{n-1}{n}(\tilde{N})^2 \operatorname{div}_g(\tilde{V}) \right), \right. \\ \left. \tau^*, \|\pi\|_{\mathcal{C}^{0,\gamma}}, \|\mathcal{R}_g - |\nabla\psi|_g^2\|_{\mathcal{C}^{0,\gamma}}, \|2V(\psi)\|_{\mathcal{C}^{1,\gamma}} \right) \end{aligned} \quad (2.1.26)$$

such that, if

$$\|U\|_{\mathcal{C}^{0,\gamma}} + \|\pi\|_{\mathcal{C}^{0,\gamma}} + \|\nabla\psi\|_{\mathcal{C}^{0,\gamma}} + \|\ln \tilde{N}\|_{\mathcal{C}^{1,\gamma}} + \|\tilde{V}\|_{\mathcal{C}^{1,\gamma}} \leq \delta, \quad (2.1.27)$$

then (2.1.11) admits a solution (u, W) , where u is a smooth positive function on M and W a smooth vector field on M .

A few remarks on the results of the present paper. The classical system of constraint equations obtained by the conformal method (without the modifications proposed by Maxwell [Max14b]) was studied by Bruno Premoselli [Pre14, Pre15] in the presence of a scalar field. Second, the above system is the subject of a paper by Mike Holst, David Maxwell and Rafe Mazzeo [HMM18] - in their case, certain conditions are imposed on the presence of the matter field. We treat the separate and delicate case wherein the dominant non linearity is focusing and leads to possible loss of compactness. It is interesting to note that the size of n plays a role; as Premoselli proves in his paper, while the scalar equation is stable in low dimensions ($3 \leq n \leq 5$), it most certainly fails to be so in higher dimensions ($n \geq 6$); therefore, the techniques used in the present proofs cannot be applied. Even if our results are similar to those of Premoselli, they are considerably more difficult to obtain. This is mainly due to the presence of a $|\nabla u|^2$ term in the scalar equation, a term which is not compact a priori.

Outline of the paper. Section 2 is devoted to the study of the first equation in (2.1.13), the so-called Lichnerowicz equation. We prove the existence of stable solutions under suitable assumptions.

Section 3 deals with a priori estimates for solutions of the Lichnerowicz equation. A careful blow-up analysis is carried out. As already mentioned, the term $|\nabla u|^2$ poses additional difficulty: blow-up can occur at the \mathcal{C}^1 level, even if the solution is bounded in L^∞ .

Section 4 is devoted to the proof of Theorem 8 and Corollary 1, which relies heavily on the a priori estimates obtained in Section 3. At the end of Section 4, we also explain how to extend Corollary 1 in the presence of conformal Killing vector fields.

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2.2 Existence of minimal solutions of the scalar equation

We study the Lichnerowicz-type scalar equation in (2.1.13). The following theorem states that, given the existence of supersolutions, one may use an iterative procedure to obtain a sequence which converges in \mathcal{C}^1 norm to a solution. We draw the reader's attention to the fact that this solution is uniquely determined by its construction. The proof contains some similarities with that of Premoselli [Pre14], but some new difficulties appear. The main difference here comes from the presence of non-linearities containing gradient terms, which force us to further refine the analysis. These gradient terms lead to difficulties in obtaining a priori estimates on solutions of the equation, which in turn lead to problems of stability. The existence result we prove in this section reads as follows:

Theorem 9. *Let (M, g) be a closed Riemannian manifold. Let a, b, c, d, f, h be smooth functions on M and Y be a smooth vector field on M . Assume that $a > 0$ and $f > 0$. The equation*

$$\Delta_g u + hu - fu^{q-1} - \frac{a}{u^{q+1}} + \frac{b}{u} + \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}} + c \langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) = 0 \quad (2.2.1)$$

admits a smooth positive solution u as soon as it admits a supersolution.

Remark 3. *For a reference on the existence of supersolutions to Lichnerowicz-type equations, see [HPP08].*

Remark 4. *The solution obtained by the construction below is unique. Moreover, it is stable (see Lemma 2 at the end of this section.)*

Proof of Theorem 9: We begin by fixing a supersolution and a subsolution to serve as upper and lower bounds respectively for the iterative process. Let ψ be a positive supersolution of (2.2.1). Let $\varepsilon_0 > 0$ be a small constant such that

$$\varepsilon_0 < \inf_M \psi, \quad \left(\sup_M h \right) \varepsilon_0^{q+2} < \frac{\inf_M a}{2} \quad \text{and} \quad \left(\sup_M b \right) \varepsilon_0^q < \frac{\inf_M a}{2}. \quad (2.2.2)$$

The last two bounds ensure that $u_0 = \varepsilon_0$ is a strict subsolution of (2.2.1) since $f > 0$. We let

$$F_K(t, x) = -f(x)t^{q-1} - \frac{a(x)}{t^{q+1}} + \frac{b(x)}{t} - Kt \quad (2.2.3)$$

for $x \in M$ and $t \in [\varepsilon_0, \sup_M \psi]$. Here, we fix $K > 0$ large enough such that it verifies the following three conditions, which are necessary for the proof below.

First, let us define

$$\tilde{F}_K(t, x, A(x)) = F_K(t, x) + \frac{A(x)^2}{t^{q+3}} + c(x)A(x) \left(\frac{d(x)}{t^2} + \frac{1}{t^{q+2}} \right) \quad (2.2.4)$$

where $x \in M$, $\varepsilon_0 \leq t \leq \sup_M \psi$ and $A \in \mathcal{F}(M; \mathbb{R})$, with $\mathcal{F}(M; \mathbb{R})$ the space of real-valued functions on M . The first thing we ask is that

$$\frac{\partial}{\partial t} \tilde{F}_K(t, x, A(x)) \leq 0 \quad \text{for any } (x, t, A) \in M \times [\varepsilon_0, \sup_M \psi] \times \mathcal{F}(M; \mathbb{R}). \quad (2.2.5)$$

If t and x are seen as fixed quantities, the operator \tilde{F} becomes a polynomial of degree 2 in \mathbb{R} , calculated in $A(x)$.

Keeping this in mind, it suffices to take

$$K \leq \sup_{x \in M, t \in [\varepsilon_0, \sup_M \psi]} \left[- (q-1)f(x)t^{q-2} + (q+1)\frac{a(x)}{t^{q+2}} - \frac{b(x)}{t^2} + \frac{c(x)^2 \left(\frac{2d(x)}{t^3} + \frac{d+2}{t^{q+3}} \right)^2 t^{q+4}}{4(q+3)} \right] \quad (2.2.6)$$

for (2.2.5) to hold. Note that the right-hand side in the above inequality is bounded, since it is the supremum of a continuous function defined on a compact set $M \times [\varepsilon_0, \sup_M \psi]$.

The second condition we impose on K is that $h + K > 0$, which ensures that the operator $\Delta_g + (h + K)$ to be coercive.

Finally, we would like $F_K(t, x)$ to be negative. This reduces to choosing K large with respect to $\sup_M b$ and ε_0 .

We shall now consider a sequence $(u_i)_{i \in \mathbb{N}}$ defined by induction. Let $u_0 \equiv \varepsilon_0$. We denote by (E_i) the following equations on M :

$$(E_i) : \quad \Delta_g v(x) + (h(x) + K)v(x) + F_K(u_{i-1}(x), x) + \frac{\langle \nabla v(x), Y(x) \rangle^2}{u_{i-1}^{q+3}(x)} \\ + c \langle \nabla v(x), Y(x) \rangle \left(\frac{d(x)}{u_{i-1}^2(x)} + \frac{1}{u_{i-1}^{q+2}(x)} \right) = 0. \quad (2.2.7)$$

and we ask that u_i solve (E_i) for $i \in \mathbb{N}^*$.

We prove in Step 1 below that the sequence is well defined. In Step 2, we prove that the sequence is pointwise increasing and uniformly bounded. At last, Step 3 is devoted to the proof that the sequence (u_i) converges to a solution of (2.2.1).

Step 1: We prove that (u_i) is well defined. We consider the more general equation

$$\Delta_g u + H u + \theta_1 \langle \nabla u, Z \rangle^2 + \theta_2 \langle \nabla u, Z \rangle + \theta_3 = 0, \quad (2.2.8)$$

with $H, \theta_1, \theta_2, \theta_3$ smooth functions on M and Z a smooth vector field on M such that $\theta_1 > 0, H > 0, \theta_3 < 0$. We claim that (2.2.8) admits a unique smooth positive solution.

Proof of Step 1: We shall use Schaefer's fixed point theorem as stated in Evans [Eva10], Section 9.2.2, Theorem 4.

Let us define the operator $T : \mathcal{C}^{1,\gamma}(M) \rightarrow \mathcal{C}^{1,\gamma}(M)$ such that

$$\Delta_g T(u) + H T(u) + \theta_1 \langle \nabla u, Z \rangle^2 + \theta_2 \langle \nabla u, Z \rangle + \theta_3 = 0. \quad (2.2.9)$$

Another way to express the operator T is as

$$T = (\Delta_g + H)^{-1} \circ (-\theta_1 \langle \nabla \cdot, Z \rangle^2 - \theta_2 \langle \nabla \cdot, Z \rangle - \theta_3) \quad (2.2.10)$$

where $(\Delta_g + H)^{-1} : \mathcal{C}^{k,\gamma} \mapsto \mathcal{C}^{k+2,\gamma}$ and $\theta_1 \langle \nabla \cdot, Z \rangle^2 + \theta_2 \langle \nabla \cdot, Z \rangle + \theta_3 : \mathcal{C}^{l+1,\gamma} \mapsto \mathcal{C}^{l,\gamma}$, for $k, l \in \mathbb{N}$, are two continuous operators. Moreover, their composition is compact: if (u_i) is bounded in $\mathcal{C}^{1,\gamma}$, then $\theta_1 \langle \nabla \cdot, Z \rangle^2 + \theta_2 \langle \nabla \cdot, Z \rangle + \theta_3$ is bounded in $\mathcal{C}^{0,\gamma}$. This means that $T(u_i)$ is bounded in $\mathcal{C}^{2,\gamma}$, and therefore that it is pre-compact in $\mathcal{C}^{1,\gamma}$.

Consequently, if we can prove that there exists $C > 0$ such that

$$\forall \quad 0 \leq \tau \leq 1, \quad w = \tau T(w) \quad \Rightarrow \quad \|w\|_{\mathcal{C}^{1,\gamma}(M)} \leq C, \quad (2.2.11)$$

then the operator T will have a fixed point, leading to a solution of (2.2.8).

Note that this solution will be unique. Indeed, assume that w_1 and w_2 are two solutions of (2.2.8), then at a point of maximum x_0 of $w_1 - w_2$, we have that $\nabla w_1(x_0) = \nabla w_2(x_0)$ and $\Delta_g w_1(x_0) \geq \Delta_g w_2(x_0)$ so that (2.2.8) gives

$$H(x_0) (w_1(x_0) - w_2(x_0)) \leq 0. \quad (2.2.12)$$

Since $H > 0$, this leads to $w_1 \leq w_2$. By symmetry, uniqueness is proved. The fixed point of T is smooth and positive by the standard regularity theory and the maximum principle.

Thus we are left with the proof of (2.2.11). Let $0 \leq \sigma_m \leq 1$ and let $w_m \in \mathcal{C}^{1,\gamma}(M)$ be such that

$$w_m = \sigma_m T(w_m). \quad (2.2.13)$$

Multiplying (2.2.9) by σ_m , we obtain that

$$\Delta_g w_m + H w_m + \sigma_m \theta_1 \langle \nabla w_m, Z \rangle^2 + \sigma_m \theta_2 \langle \nabla w_m, Z \rangle + \sigma_m \theta_3 = 0. \quad (2.2.14)$$

First, the L^∞ bounds on w_m exist *a priori*. Indeed, consider $x_0 \in M$ a minimum of w_m . Since $\Delta_g w_m(x_0) \leq 0$ and $\nabla w_m(x_0) = 0$, which holds true for all minima, then $Hw_m(x_0) \geq -\sigma_m \theta_3(x_0)$. By applying the same procedure to the study of maxima, we obtain that

$$\inf_M \frac{-\sigma_m \theta_3}{H} \leq w_m \leq \sup_M \frac{-\sigma_m \theta_3}{H}. \quad (2.2.15)$$

Assume now that $\|\nabla w_m\|_{L^\infty(M)} \rightarrow \infty$. Let

$$\mu_m := \frac{1}{\|\nabla w_m\|_{L^\infty(M)}} \rightarrow 0 \quad \text{as } m \rightarrow \infty, \quad (2.2.16)$$

and $(x_m)_m \subset M$ be such that

$$\|\nabla w_m\|_{L^\infty(M)} = |\nabla w_m(x_m)|. \quad (2.2.17)$$

Consider the domains $\Omega_m := B_0\left(\frac{i_g(M)}{2\mu_m}\right) \subset T_{x_m}(M)$, where $x_m \in M$ and $i_g(M)$ is the injectivity radius of M . We work with the rescaled quantities

$$v_m(x) := w_m(\exp_{x_m}(\mu_m x)) \quad \text{and} \quad g_m(x) := (\exp_{x_m}^* g)(\mu_m x), \quad (2.2.18)$$

where $\mu_m \in \mathbb{R}$ and $x \in \Omega_m$. Clearly, $\|\nabla v_m\|_{L^\infty} \leq 1$ and $|\nabla v_m(0)| = 1$. The L^∞ bounds remain unchanged. In $(\Omega_m)_{m \geq 1}$, we have that

$$\begin{aligned} & \Delta_{g_m} v_m + \mu_m^2 H(\exp_{x_m}(\mu_m \cdot)) v_m + \sigma_m \mu_m^2 \theta_3(\exp_{x_m}(\mu_m \cdot)) \\ & + \sigma_m \theta_1(\exp_{x_m}(\mu_m \cdot)) \langle \nabla v_m, Z(\exp_{x_m}(\mu_m \cdot)) \rangle^2 \\ & + \mu_m \sigma_m \theta_2(\exp_{x_m}(\mu_m \cdot)) \langle \nabla v_m, Z(\exp_{x_m}(\mu_m \cdot)) \rangle = 0 \end{aligned} \quad (2.2.19)$$

A few words on the use of the exponential coordinates when doing local analysis on a Riemannian manifold M might be welcome. In general, for any point $p \in M$, one might find an open set $U_p \subset T_p M$ and, with the help of geodesics, define $\exp_p : U_p \rightarrow M$. It is a diffeomorphism.

For all $m \in \mathbb{N}$, $\exp_{x_m}^* g_m(0) = Id$ and there exists a constant C independent of m such that $|\nabla^k \exp_{x_m}^* g_m| \leq C$, for any $k \in \mathbb{N}$. Consequently, $g_m \rightarrow \xi$ in \mathcal{C}_{loc}^k .

The argument that follows comes from standard elliptic theory; we often use similar reasonings throughout this text. So far, we have concluded that $(v_m)_m$ is uniformly bounded in $W^{1,\infty}(\Omega_m)$, and thus in $W^{1,p}(\Omega_m)$ for some $p > \frac{n}{2}$. The regularizing effect of Δ_{g_m} applied to (2.2.19) implies that v_m is bounded in $W^{2,p}(\Omega_m)$; we continue by using a Sobolev embedding theorem, according to which $\mathcal{C}_0^{1,\eta}(\Omega_m)$ is compactly embedded in $W^{2,p}(\Omega_m)$ for some $\eta \in (0,1)$. We apply a diagonalization argument (Arzelà-Ascoli) to conclude that there exists a subsequence that converges uniformly in \mathcal{C}_{loc}^1 . That is, one can find a \mathcal{C}_{loc}^1 -limit v_∞ such that $v_\infty = \lim_{m \rightarrow \infty} v_m$ in \mathcal{C}_{loc}^1 norm, where $x_\infty = \lim_{m \rightarrow \infty} x_m$ and $\sigma_\infty := \lim_{m \rightarrow \infty} \sigma_m$. From this it follows that $\|\nabla v_\infty\|_{L^\infty} = 1$ and that the *a priori* bounds (2.2.15) become

$$\inf_M \frac{-\sigma_\infty \theta_3}{H} \leq v_\infty \leq \sup_M \frac{-\sigma_\infty \theta_3}{H}. \quad (2.2.20)$$

Moreover, v_∞ solves the limit equation

$$\Delta v_\infty + (\partial_1 v_\infty)^2 = 0 \quad (2.2.21)$$

in \mathbb{R}^n , where we have let

$$\partial_1 v_\infty := \sqrt{\sigma_\infty \theta_1(0)} \nabla v_\infty \cdot Z(x_0). \quad (2.2.22)$$

If $\sigma_\infty = 0$, then v_∞ is a bounded harmonic function, and thus a constant. Let us assume that $\sigma_\infty \neq 0$. Note that, for $\alpha \in \mathbb{R}$,

$$\begin{aligned} \Delta v_\infty^{-\alpha} &= -\frac{\alpha}{v_\infty^{\alpha+1}} \Delta v_\infty - \frac{\alpha(\alpha+1)|\nabla v_\infty|^2}{v_\infty^{\alpha+2}} \\ &\leq \frac{\alpha|\nabla v_\infty|^2}{v_\infty^{\alpha+1}} \left(\sigma_\infty \theta_1(0) |Z(0)|^2 - \frac{\alpha+1}{v_\infty} \right). \end{aligned} \quad (2.2.23)$$

This and (2.2.20) imply that, for α sufficiently large, $v_\infty^{-\alpha}$ is subharmonic. We then apply Lemma 12 (see annex) to get that v_∞ must be constant. Whichever the case, $\nabla v_\infty \equiv 0$ leads to a contradiction. The $\mathcal{C}^{1,\gamma}$ bound we need for (2.2.11) follows from a similar elliptic regularity argument as the one used above. This ends the proof of Step 1. \square

Step 2: We claim that

$$\varepsilon_0 \leq u_i(x) \leq u_{i+1}(x) \leq \psi(x) \quad (2.2.24)$$

for all $x \in M$ and all $i \geq 0$.

Proof of Step 2: We proceed by induction. We prove first that

$$\forall i \geq 0, \quad u_i \text{ is a subsolution of } (E_{i+1}) \text{ and } u_i \leq \psi. \quad (2.2.25)$$

Note that

$$(2.2.25) \Rightarrow u_i \leq u_{i+1}. \quad (2.2.26)$$

Indeed, let $x_0 \in M$ be a maximum point of $u_i - u_{i+1}$. Then $\nabla u_i(x_0) = \nabla u_{i+1}(x_0)$ and we can use the fact that u_i is a subsolution of (E_{i+1}) and u_{i+1} a solution of (E_{i+1}) to write that

$$\Delta_g(u_i - u_{i+1})(x_0) + (h(x_0) + K)(u_i - u_{i+1})(x_0) \leq 0 \quad (2.2.27)$$

which implies that $u_i(x_0) \leq u_{i+1}(x_0)$ since $\Delta_g(u_i - u_{i+1})(x_0) \geq 0$ and $h + K > 0$. This proves (2.2.26).

We now prove (2.2.25) by induction. For $i = 0$, it follows from the choice of ε_0 we made. Assume that (2.2.25) holds for some $i \geq 0$. We need to prove that u_{i+1} is a subsolution of (E_{i+2}) , which means we have to show that

$$\begin{aligned} \Delta_g u_{i+1}(x) + (h(x) + K)u_{i+1}(x) + F_K(u_{i+1}(x), x) + \frac{\langle \nabla u_{i+1}(x), Y(x) \rangle^2}{u_{i+1}^{q+3}(x)} \\ + c \langle \nabla u_{i+1}(x), Y(x) \rangle \left(\frac{d(x)}{u_{i+1}^2(x)} + \frac{1}{u_{i+1}^{q+2}(x)} \right) \leq 0. \end{aligned} \quad (2.2.28)$$

We know that

$$\begin{aligned} F_K(u_{i+1}(x), x) + c(x) \langle \nabla u_{i+1}(x), Y(x) \rangle \left(\frac{d(x)}{u_{i+1}^2(x)} + \frac{1}{u_{i+1}^{q+2}(x)} \right) + \frac{\langle \nabla u_{i+1}(x), Y(x) \rangle^2}{u_{i+1}^{q+3}(x)} \\ \leq F_K(u_i(x), x) + c(x) \langle \nabla u_{i+1}(x), Y(x) \rangle \left(\frac{d(x)}{u_i^2(x)} + \frac{1}{u_i^{q+2}(x)} \right) + \frac{\langle \nabla u_{i+1}(x), Y(x) \rangle^2}{u_i^{q+3}(x)}. \end{aligned} \quad (2.2.29)$$

This is true because of (2.2.5), with $A(x) = \langle \nabla u_{i+1}(x), Y(x) \rangle$, and since (2.2.26) implies that $u_{i+1} \geq u_i$ by the induction hypothesis.

Given that u_{i+1} is defined as a solution of (E_{i+1}) , we know that

$$\begin{aligned} \Delta_g u_{i+1}(x) + (h(x) + K)u_{i+1}(x) + F_K(u_{i-1}(x), x) + \frac{\langle \nabla u_{i+1}(x), Y(x) \rangle^2}{u_{i-1}^{q+3}(x)} \\ + c \langle \nabla u_{i+1}(x), Y(x) \rangle \left(\frac{d(x)}{u_{i-1}^2(x)} + \frac{1}{u_{i-1}^{q+2}(x)} \right) = 0. \end{aligned} \quad (2.2.30)$$

By combining (2.2.29) and (2.2.30), we see that u_{i+1} is a subsolution of (E_{i+2}) .

Finally, so as to check the last point, assume there exists $x_0 \in M$ such that $u_{i+1}(x_0) > \psi(x_0)$ and that it corresponds to $\max_M (u_{i+1}(x) - \psi(x))$. Since $\nabla u_{i+1}(x_0) = \nabla \psi(x_0)$ and $\Delta_g u_{i+1}(x_0) \geq \Delta_g \psi(x_0)$, we obtain that

$$\Delta_g (u_{i+1} - \psi)(x_0) + (h + K)(u_{i+1} - \psi)(x_0) > 0. \quad (2.2.31)$$

But ψ is a supersolution for (2.2.1), so we get that

$$\begin{aligned} 0 < \Delta_g (u_{i+1} - \psi)(x_0) + (h(x_0) + K)(u_{i+1} - \psi)(x_0) \\ \leq F_K(\psi(x_0), x_0) - F_K(u_i(x_0), x_0) - \langle \nabla \psi(x_0), Y(x_0) \rangle^2 \left(\frac{1}{u_i^{q+3}} - \frac{1}{\psi^{q+3}} \right) (x_0) \\ - \langle \nabla \psi(x_0), Y(x_0) \rangle \left(\frac{d}{u_i^2} - \frac{d}{\psi^2} + \frac{1}{u_i^{q+2}} - \frac{1}{\psi^{q+2}} \right) (x_0). \end{aligned} \quad (2.2.32)$$

Thanks to (2.2.5) with $A(x) = \langle \nabla \psi(x), Y(x) \rangle$ and to the induction hypothesis which says that $u_i \leq \psi$, we obtain a contradiction. This wraps up the induction argument and the proof of Step 2. \square

Step 3: The sequence $(u_i)_{i \in \mathbb{N}}$ is uniformly bounded in $\mathcal{C}^1(M)$.

Proof of Step 3: Thanks to Step 2, we know that $(u_i)_{i \in \mathbb{N}}$ is an increasing sequence bounded by ψ . Thus, (u_i) is bounded in $\mathcal{C}^0(M)$.

Assume by contradiction that exists a subsequence $(u_{\phi(m)})_{m \in \mathbb{N}}$ such that $\|\nabla u_{\phi(m)}\|_{L^\infty} \rightarrow \infty$. Moreover, up to choosing a subsequence, we might find that $\|\nabla u_{\phi(m)-1}\|_{L^\infty} \leq \|\nabla u_{\phi(m)}\|_{L^\infty}$, $\forall m \in \mathbb{N}^*$. Let

$$\mu_m := \frac{1}{\|\nabla u_{\phi(m)}\|_{L^\infty}} \quad (2.2.33)$$

and let $(x_m)_m \subset M$ be such that

$$|\nabla u_{\phi(m)}(x_m)| = \|\nabla u_{\phi(m)}\|_{L^\infty}. \quad (2.2.34)$$

Consider the domains $\Omega_m = B_0 \left(\frac{i_g M}{2\mu_m} \right) \subset T_{x_m} M$ and the rescaled quantities

$$v_m(x) := u_{\phi(m)}(\exp_{x_m}(\mu_m x)) \quad \text{and} \quad g_m(x) := (\exp_{x_m}^* g)(\mu_m x) \quad (2.2.35)$$

in Ω_m . We get

$$\begin{aligned} \Delta_{g_m} v_m + \mu_m^2 (h(\exp_{x_m}(\mu_m \cdot)) + K) v_m(\exp_{x_m}(\mu_m \cdot)) \\ + \mu_m^2 F_K(u_{\phi(m)-1}(\exp_{x_m}(\mu_m \cdot))) + \frac{\langle \nabla v_m, Y(\exp_{x_m}(\mu_m \cdot)) \rangle^2}{u_{\phi(m)-1}^{q+3}(\exp_{x_m}(\mu_m \cdot))} \\ + \mu_m \langle \nabla v_m, Y(\exp_{x_m}(\mu_m \cdot)) \rangle c(\exp_{x_m}(\mu_m \cdot)) \left[\frac{d(\exp_{x_m}(\mu_m \cdot))}{u_{\phi(m)-1}^2(\exp_{x_m}(\mu_m \cdot))} \right. \\ \left. + \frac{1}{u_{\phi(m)-1}^{q+2}(\exp_{x_m}(\mu_m \cdot))} \right] = 0, \end{aligned} \quad (2.2.36)$$

with $(v_m)_{m \in N}$ bounded in L^∞ , $\|\nabla v_m\|_{L^\infty} = 1$ and $\varepsilon_0 \leq v_m$.

Let $w_m(x) := u_{\phi(m)-1}(\exp_{x_m}(\mu_m x))$ be defined on Ω_m . Since, by our choice of subsequence $\psi(m)$, we have

$$\begin{aligned} \nabla w_m(x) &= \mu_m \nabla u_{\phi(m)-1}(\exp_{x_m}(\mu_m x)) \\ &= \frac{\nabla u_{\phi(m)-1}(\exp_{x_m}(\mu_m x))}{\|\nabla u_{\phi(m)-1}(\exp_{x_m}(\mu_m x))\|_{L^\infty}} \\ &\leq 1 \end{aligned} \tag{2.2.37}$$

we know that w_m is uniformly bounded in \mathcal{C}^1 .

By the Sobolev embedding theorem and standard elliptic regularity, there exists a smooth positive \mathcal{C}^1 -limit v_∞ of $(v_m)_{m \in N}$, up to a subsequence. There also exists a positive function w such that $w_n \rightarrow w$ in \mathcal{C}^0 as $m \rightarrow \infty$.

By taking $m \rightarrow \infty$ in (2.2.36), we obtain

$$\Delta v_\infty + \frac{(\nabla v_\infty \cdot Y(0))^2}{w^{q+3}} = 0. \tag{2.2.38}$$

Note also that

$$\Delta v_\infty^{-\alpha} \leq \frac{\alpha |\nabla v_\infty|^2}{w^{\alpha+2}} \left(\frac{|Y(0)|^2}{w^{q+2}} - (\alpha + 1) \right). \tag{2.2.39}$$

For α large enough, $v_\infty^{-\alpha}$ is subharmonic. Using Lemma 12 (see Annex), we find that v_∞ is constant, which contradicts the fact that $\|\nabla v_\infty\|_{L^\infty} = 1$. This ends the proof of Step 3. \square

Since $(u_i)_{i \in N}$ is uniformly bounded in \mathcal{C}^1 , we conclude by standard elliptic theory that its limit u is a positive smooth function solving equation (2.2.1). This ends the proof of the theorem. \square

The solution constructed in the previous proof is uniquely determined as the pointwise limit of $(u_i)_{i \in N}$, where each u_i is the unique solution of (2.2.7). Furthermore, the solution is minimal among all supersolutions (including solutions) of (2.2.1) with values between ε_0 and $\sup_M \psi$. These bounds were explicitly used in the inductive argument. By construction, $u \leq \psi$, where ψ is the supersolution fixed at the very beginning. Note that the constant K appearing in (2.2.7) depend on $\sup_M \psi$ and ε_0 . We would obtain the same iteration were we to use another supersolution $\tilde{\psi}$ and the same K , given that $\varepsilon_0 < \tilde{\psi} < \sup_M \psi$. Therefore, u is smaller than any supersolution between ε_0 and $\sup_M \psi$.

As an immediate consequence of the minimality discussed above, the solutions we found corresponding to different functions a are ordered. Let $0 < a < \tilde{a}$ be two functions, and assume that the equation associated to \tilde{a} admits a solution \tilde{u} . Then \tilde{u} is a supersolution for (2.2.1) corresponding to a , and by the previous proof we find a solution $u \leq \tilde{u}$. Moreover, given that \tilde{u} may be viewed as a supersolution to all (2.2.1) with $a \leq \tilde{a}$, we obtain a monotonicity of u in a : for $a_1 \leq a_2 \leq \tilde{a}$, then $u_1 \leq u_2 \leq \tilde{u}$.

Finally, the solution u is stable, as defined in the following lemma.

Lemma 2. *The operator L resulting from the linearization of (2.2.1) at the minimal so-*

lution u admits a real, simple eigenvalue $\lambda_0 \geq 0$ such that

$$\begin{aligned} L\varphi_0 &= \Delta_g \varphi_0 + \left[h - (q-1)fu^{q-2} + (q+1)au^{-q-2} - bu^{-2} \right. \\ &\quad \left. - (q+3)\frac{\langle \nabla u, Y \rangle^2}{u^{q+4}} - c\langle \nabla u, Y \rangle \left(\frac{2d}{u^3} + \frac{q+2}{u^{q+3}} \right) \right] \varphi_0 \\ &\quad + \langle \nabla \varphi_0, Y \rangle \left[c \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) + \frac{2\langle \nabla u, Y \rangle}{u^{q+3}} \right] \\ &= \lambda_0 \varphi_0, \end{aligned}$$

where φ_0 is the corresponding positive eigenfunction. Furthermore, if $\lambda \in \mathbb{C}$ is any other eigenvalue, then $\operatorname{Re}(\lambda) \geq \lambda_0$.

Proof of Lemma 2: Notice that L is nonsymmetric; moreover, one may find a large enough constant K such that

$$\begin{aligned} h - (q-1)fu^{q-2} + (q+1)au^{-q-2} - bu^{-2} - (q+3)\frac{\langle \nabla u, Y \rangle^2}{u^{q+4}} \\ - c\langle \nabla u, Y \rangle \left(\frac{2d}{u^3} + \frac{q+2}{u^{q+3}} \right) + K \geq 0. \end{aligned} \quad (2.2.40)$$

According to ([Eva10], Section 6.5, Theorem 1) there exists a real, positive eigenvalue $\lambda_K > 0$ of $L + K$, such that any other complex eigenvalue of $L + K$ has a greater real part. Consequently, the operator L admits a minimal real eigenvalue $\lambda_0 > -K$. We now assume that $\lambda_0 < 0$. Let $u_\delta := u_0 - \delta\varphi_0$, $\delta > 0$. By taking δ small enough, we may ensure that $\varepsilon_0 < u_\delta$. Then

$$\begin{aligned} \Delta_g u_\delta + h u_\delta - f u_\delta^{q-1} - \frac{a}{u_\delta^{q+1}} + \frac{b}{u_\delta} + \frac{\langle \nabla u_\delta, Y \rangle^2}{u_\delta^{q+3}} + c\langle \nabla u_\delta, Y \rangle \left(\frac{d}{u_\delta^2} + \frac{1}{u_\delta^{q+2}} \right) \\ = -\delta\lambda_0\varphi_0 + o(\delta). \end{aligned} \quad (2.2.41)$$

This implies that $\varepsilon_0 < u_\delta < u$ is a supersolution of (2.2.1), which cannot be the case, as discussed above. Thus, $\lambda_0 \geq 0$. \square

2.3 A priori estimates on solutions of the scalar equation in low dimensions

The estimates obtained in this section will play a crucial role in the proof of Theorem 8, which is based on a fixed-point argument. This section is devoted to the proof of the following theorem:

Theorem 10. *Let (M, g) be a closed Riemannian manifold of dimension $n = 3, 4, 5$. Let $\frac{1}{2} < \eta < 1$ and $0 < \alpha < 1$. Let a, b, c, d, f, h be smooth functions on M , let Y be a smooth vector field on M . For any $0 < \theta < T$, there exists $S_{\theta, T}$ and $\vartheta_{\theta, T}$ such that, given any parameters within*

$$\mathcal{E}_{\theta, T} := \left\{ (f, a, b, c, d, h, Y), \quad f \geq \theta, \quad a \geq \theta, \right. \quad (2.3.1)$$

$$\left. \|f\|_{C^{1, \eta}} \leq T, \quad \|a\|_{C^{0, \alpha}}, \|b\|_{C^{0, \alpha}}, \|c\|_{C^{0, \alpha}}, \|d\|_{C^{0, \alpha}}, \|h\|_{C^{0, \alpha}}, \|Y\|_{C^{0, \alpha}} \leq T \right\},$$

with

$$\|Y\|_{C^{0, \alpha}} \leq \vartheta_{\theta, T}, \quad (2.3.2)$$

then any smooth positive solution u of (2.2.1) satisfies $\|u\|_{C^2} \leq S_{\theta, T}$.

Remark 5. For the sake of clarity, we've taken the bounds on the parameters to be of the form θ and T . They can of course be individually specified.

We proceed by contradiction. Let $(a_\alpha, b_\alpha, c_\alpha, d_\alpha, f_\alpha, h_\alpha, Y_\alpha)$ be a set of parameters in $\mathcal{E}_{\theta, T}$. We assume the existence of a sequence

$$\|Y_\alpha\|_{C^{0, \alpha}} \rightarrow 0 \quad (2.3.3)$$

such that one might find a corresponding sequence $(u_\alpha)_{\alpha \in \mathbb{N}}$ of smooth positive solutions of equations (EL_α)

$$\begin{aligned} \Delta_g u_\alpha + h_\alpha u_\alpha - f_\alpha u_\alpha^{q-1} - \frac{a_\alpha}{u_\alpha^{q+1}} + \frac{b_\alpha}{u_\alpha} + \frac{\langle \nabla u_\alpha, Y_\alpha \rangle^2}{u_\alpha^{q+3}} \\ + c_\alpha \langle \nabla u_\alpha, Y_\alpha \rangle \left[\frac{d_\alpha}{u_\alpha^2} + \frac{1}{u_\alpha^{q+2}} \right] = 0 \end{aligned} \quad (2.3.4)$$

with

$$\|u_\alpha\|_{C^1(M)} \rightarrow \infty, \quad \text{as } \alpha \rightarrow \infty. \quad (2.3.5)$$

Remark 6. The condition (2.3.2) is only used later in the proof. For now, we proceed as if Y_α is simply in $\mathcal{E}_{\theta, T}$. We emphasize when (2.3.3) becomes necessary.

A *concentration point* is the limit in M of any sequence $(x_\alpha)_\alpha$ where (2.3.5) holds. Note that a C^1 -bound on $(u_\alpha)_\alpha$ automatically gives a C^2 -bound by elliptic theory. Note also that, up to a subsequence, all parameters converge in $C^0(M)$.

Let $m_\alpha = \min_{x \in M} u_\alpha(x) = u_\alpha(x_\alpha) > 0$. Since $\nabla u_\alpha(x_\alpha) = 0$ and since $\Delta_g u_\alpha(x_\alpha) \leq 0$, we have thanks to (2.3.4) that

$$h_\alpha(x_\alpha)m_\alpha - f_\alpha(x_\alpha)m_\alpha^{q-1} - \frac{a_\alpha(x_\alpha)}{m_\alpha^{q+1}} + \frac{b_\alpha(x_\alpha)}{m_\alpha} \geq 0. \quad (2.3.6)$$

Thanks to the definition of $\mathcal{E}_{\theta, T}$, it follows that

$$\frac{\theta}{m_\alpha^{q+1}} \leq T(m_\alpha + \frac{1}{m_\alpha}). \quad (2.3.7)$$

Then there exists $\varepsilon = \varepsilon(\theta, T, n) > 0$ such that $m_\alpha \geq \varepsilon$, meaning that

$$u_\alpha > \varepsilon > 0 \quad \text{for all } x \in M \text{ and all } \alpha. \quad (2.3.8)$$

The scheme of the proof follows the work of Druet and Hebey [DH09], with the added difficulty consisting in the gradient terms in (2.3.4).

2.3.1 Concentration points

The first step in finding *a priori* estimates for $(u_\alpha)_\alpha$ is to find all potential concentration points.

Lemma 3. *There exists $N_\alpha \in \mathbb{N}^*$ and $\mathcal{S}_\alpha := (x_{1, \alpha}, \dots, x_{N_\alpha, \alpha})$ a set of critical points of $(u_\alpha)_\alpha$ such that*

$$d_g(x_{i, \alpha}, x_{j, \alpha})^{\frac{n-2}{2}} u_\alpha(x_{i, \alpha}) \geq 1 \quad (2.3.9)$$

for all $i, j \in \{1, \dots, N_\alpha\}$, $i \neq j$, and

$$\left(\min_{i=1, \dots, N_\alpha} d_g(x_{i, \alpha}, x) \right)^{\frac{n-2}{2}} u_\alpha(x) \leq 1 \quad (2.3.10)$$

for all critical points of u_α and such that there exists $C_1 > 0$ such that

$$\left(\min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x) \right)^{\frac{n-2}{2}} \left(u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right) \leq C_1 \quad (2.3.11)$$

for all $x \in M$ and all $\alpha \in \mathbb{N}$.

Remark 7. Estimate (2.3.11) implies that any concentration point of $(u_\alpha)_{\alpha \in \mathbb{N}}$ calls for the existence of a sequence $(x_\alpha)_\alpha \subset (\mathcal{S}_\alpha)_\alpha$ converging to it. We shall focus our analysis in the neighbourhood of S_α as $\alpha \rightarrow \infty$ to find concentration points.

Proof of Lemma 3: In order to choose $(S_\alpha)_\alpha$, we make use of a simple result describing any sufficiently regular function on a compact manifold.

Lemma 4. Let u be a positive real-valued \mathcal{C}^1 function defined in a compact manifold M . Then there exists $N \in \mathbb{N}^*$ and (x_1, x_2, \dots, x_N) a set of critical points of u such that

$$d_g(x_i, x_j)^{\frac{n-2}{2}} u(x_i) \geq 1 \quad (2.3.12)$$

for all $i, j \in \{1, \dots, N\}$, $i \neq j$, and

$$\left(\min_{i=1, \dots, N} d_g(x_i, x) \right)^{\frac{n-2}{2}} u(x) \leq 1 \quad (2.3.13)$$

for all critical points x of u .

Sketch of proof. The lemma and its proof may be found in Druet and Hebey's paper (Lemma 1.1) [DH09]. Briefly, the authors begin with the set K_0 , the closed set of critical points of u , and construct $x_1 \in K_0$ and $K_1 \subset K_0$ such that

$$u(x_1) = \max_{K_0} u,$$

$$K_1 = \{x \in K_0 \mid d_g(x_1, x)^{\frac{n-2}{2}} u(x) \geq 1\},$$

and subsequently recurrently build $x_p \in K_{p-1}$ and $K_p \subset K_{p-1}$ such that

$$u(x_p) = \max_{K_{p-1}} u,$$

$$K_p = \{x \in K_{p-1} \mid d_g(x_p, x)^{\frac{n-2}{2}} u(x) \geq 1, \min_{i=1, \dots, p} d_g(x_i, x)^{\frac{n-2}{2}} u(x) \geq 1\}.$$

As M is compact, one quickly sees that the sequence of sets thus constructed is finite: there exists $N \geq 1$ such that

$$\emptyset = K_N \subset K_{N-1} \subset \dots \subset K_0.$$

The set we are interested in is $\{x_1, x_2, \dots, x_N\}$. □

Applying this lemma to (u_α) gives N_α and \mathcal{S}_α as in Lemma 3 such that (2.3.9) and (2.3.10) hold. We need to prove (2.3.11). Proceeding by contradiction, assume that there exists a sequence $(x_\alpha)_\alpha$ such that

$$\left(\min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x_\alpha) \right)^{\frac{n-2}{2}} \left(u_\alpha(x_\alpha) + \left| \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right|^{\frac{n-2}{2}} \right) \rightarrow \infty \quad (2.3.14)$$

as $\alpha \rightarrow \infty$, where

$$\begin{aligned} & \left(\min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x_\alpha) \right)^{\frac{n-2}{2}} \left(u_\alpha(x_\alpha) + \left| \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right|^{\frac{n-2}{2}} \right) \\ &= \sup_{x \in M} \left(\min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x) \right)^{\frac{n-2}{2}} \left(u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right). \end{aligned} \quad (2.3.15)$$

Denote

$$\nu_\alpha^{1-\frac{n}{2}} := u_\alpha(x_\alpha) + \left| \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right|^{\frac{n-2}{2}} \quad (2.3.16)$$

and see that (2.3.14) translates to

$$\frac{d_g(x_\alpha, \mathcal{S}_\alpha)}{\nu_\alpha} \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty. \quad (2.3.17)$$

Also, since M is compact,

$$\nu_\alpha \rightarrow 0 \quad \text{as } \alpha \rightarrow \infty. \quad (2.3.18)$$

Consider the rescaled quantities

$$v_\alpha(x) := \nu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_\alpha}(\nu_\alpha x)) \quad \text{and} \quad g_\alpha(x) := (\exp_{x_\alpha}^* g)(\nu_\alpha x) \quad (2.3.19)$$

defined in $\Omega_\alpha := B_0\left(\frac{\delta}{\nu_\alpha}\right)$, with $0 < \delta < \frac{1}{2}i_g(M)$. We emphasize that, for any $R > 0$,

$$\limsup_{\alpha \rightarrow \infty} \sup_{B_0(R)} \left(v_\alpha + \left| \frac{\nabla v_\alpha}{v_\alpha} \right|^{\frac{n-2}{2}} \right) = 1 \quad (2.3.20)$$

thanks to (2.3.15) and (2.3.17). However, unlike $(u_\alpha)_\alpha$, the sequence $(v_\alpha)_\alpha$ is not necessarily bounded from below by a small positive constant ε . Instead, we deduce from (2.3.20) that

$$|\nabla \ln v_\alpha| \leq 1 + o(1) \quad \text{in } B_0(R) \quad (2.3.21)$$

for all $R > 0$ so that

$$v_\alpha(0)e^{-2|x|} \leq v_\alpha(x) \leq v_\alpha(0)e^{2|x|} \quad \text{in } B_0(R) \quad (2.3.22)$$

for all $R > 0$ as soon as α is large enough. We rewrite (2.3.4) in Ω_α as

$$\begin{aligned} \Delta_{g_\alpha} v_\alpha &= f_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) v_\alpha^{q-1} + \nu_\alpha^{\frac{n+2}{2}} \frac{a_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha^{q+1}(\exp_{x_\alpha}(\nu_\alpha \cdot))} \\ &\quad - \nu_\alpha^2 h_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) v_\alpha - \nu_\alpha^{\frac{n+2}{2}} \frac{b_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))} \\ &\quad - \frac{\langle \nabla v_\alpha, Y_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) \rangle^2}{v_\alpha} \frac{1}{u_\alpha^{q+2}(\exp_{x_\alpha}(\nu_\alpha \cdot))} \\ &\quad - \nu_\alpha c_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) \langle \nabla v_\alpha, Y_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) \rangle \left[\frac{d_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha^2(\exp_{x_\alpha}(\nu_\alpha \cdot))} \right. \\ &\quad \left. + \frac{1}{u_\alpha^{q+2}(\exp_{x_\alpha}(\nu_\alpha \cdot))} \right] \end{aligned} \quad (2.3.23)$$

Note that the metrics $g_\alpha \rightarrow \xi$ in \mathcal{C}_{loc}^2 as $\alpha \rightarrow \infty$. Because of (2.3.8) and (2.3.20), the right hand side is bounded, so by standard elliptic theory there exists up to a subsequence a \mathcal{C}^1 limit $U := \lim_{\alpha \rightarrow \infty} v_\alpha$ and $x_0 := \lim_{\alpha \rightarrow \infty} x_\alpha$. Let us set

$$w_\alpha(x) := \frac{v_\alpha(x)}{v_\alpha(0)} = \frac{u_\alpha(\exp_{x_\alpha}(\nu_\alpha x))}{u_\alpha(x_\alpha)}. \quad (2.3.24)$$

It follows that

$$w_\alpha(0) = 1 \quad \text{and} \quad \lim_{\alpha \rightarrow \infty} \left| \frac{\nabla w_\alpha(0)}{w_\alpha(0)} \right| \leq 1. \quad (2.3.25)$$

Moreover, (2.3.22) implies that

$$e^{-2|x|} \leq w_\alpha(x) \leq e^{2|x|} \quad (2.3.26)$$

in $B_0(R)$ for α large. Multiply (2.3.23) by $v_\alpha(0)^{-1}$ to get

$$\begin{aligned} \Delta_g w_\alpha = & f_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) w_\alpha^{q-1} v_\alpha^{q-2}(0) + \nu_\alpha^2 \frac{a(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha^{q+1}(\exp_{x_\alpha}(\nu_\alpha \cdot)) u_\alpha(x_\alpha)} \\ & - \nu_\alpha^2 h_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) w_\alpha - \nu_\alpha^2 \frac{b_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) u_\alpha(x_\alpha)} \\ & - \frac{\langle \nabla w_\alpha, Y_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) \rangle^2}{w_\alpha^{q+2}} \frac{1}{u_\alpha(x_\alpha)^{q+2}} \\ & - \nu_\alpha c_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) \langle \nabla w_\alpha, Y_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) \rangle \left[\frac{d_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha^2(\exp_{x_\alpha}(\nu_\alpha \cdot))} \right. \\ & \left. + \frac{1}{u_\alpha^{q+2}(\exp_{x_\alpha}(\nu_\alpha \cdot))} \right]. \end{aligned} \quad (2.3.27)$$

Up to a subsequence, let

$$\begin{aligned} \hat{l}_\alpha = v_\alpha(0), \quad \text{with} \quad \lim_{\alpha \rightarrow \infty} \hat{l}_\alpha =: \hat{l} \in [0, 1], \\ l_\alpha = u_\alpha^{-1}(x_\alpha), \quad \text{with} \quad \lim_{\alpha \rightarrow \infty} l_\alpha =: l \in [0, \varepsilon^{-1}]. \end{aligned} \quad (2.3.28)$$

The above limits hold thanks to (2.3.8), (2.3.19), and (2.3.20). By our hypothesis (2.3.18), we obtain

$$\hat{l} = \lim_{\alpha \rightarrow \infty} \nu_\alpha^{\frac{n-2}{2}} = 0. \quad (2.3.29)$$

Note that, if the above limit is different from zero, then the sequence $(u_i)_{i \in \mathbb{N}}$ is already *a priori* bounded.

By standard elliptic theory, we find that there exists $w := \lim_{\alpha \rightarrow \infty} w_\alpha$ in \mathcal{C}^1 solving:

$$\Delta w = f(x_0) w^{q-1} \hat{l}^{q-2} - \frac{\langle \nabla w, Y(x_0) \rangle^2}{w^{q+3}} \hat{l}^{q+2}, \quad x \in \mathbb{R}^n. \quad (2.3.30)$$

Based on (2.3.28), we consider three separate cases.

First case: Let

$$l = 0 \quad \text{and} \quad \hat{l} \neq 0. \quad (2.3.31)$$

By passing to the limit in the first equation (2.3.23), we get

$$\Delta U = f(x_0) U^{q-1} \quad (2.3.32)$$

in \mathbb{R}^n . The exact form of these solutions is found in a paper by Caffarelli, Gidas and Spruck [CGS89]:

$$U(x) = \left(1 + \frac{f(x_0)|x - y_0|^2}{n(n-2)} \right)^{1-\frac{n}{2}} \quad \text{or} \quad U \equiv 0. \quad (2.3.33)$$

If U is non-trivial, then there exists $y_0 \in \mathbb{R}^n$ its unique maximum point, and therefore one might also find $(y_\alpha)_\alpha$ local maxima of $(u_\alpha)_\alpha$ such that

$$d_g(x_\alpha, y_\alpha) = O(\nu_\alpha) \quad (2.3.34)$$

and

$$\nu_\alpha^{\frac{n-2}{2}} u_\alpha(y_\alpha) \rightarrow 1 \quad \text{as } \alpha \rightarrow \infty. \quad (2.3.35)$$

Since $(y_\alpha)_\alpha$ are critical points, (2.3.10) implies that

$$d_g(\mathcal{S}_\alpha, y_\alpha)^{\frac{n-2}{2}} u_\alpha(y_\alpha) \leq 1 \quad (2.3.36)$$

for all $\alpha \in \mathbb{N}$, so by (2.3.35), $d_g(\mathcal{S}_\alpha, y_\alpha) = O(\nu_\alpha)$; together with (2.3.34), this leads to $d_g(\mathcal{S}_\alpha, x_\alpha) = O(\nu_\alpha)$, which contradicts (2.3.17).

If $U \equiv 0$, then

$$\lim_{\alpha \rightarrow \infty} v_\alpha(0) = 0, \quad (2.3.37)$$

which contradicts (2.3.31).

Second case: Let

$$l \neq 0 \quad \text{and } \hat{l} = 0. \quad (2.3.38)$$

As $l \neq 0$ and (2.3.8),

$$w_\alpha(x) = \frac{u_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha(x_\alpha)} \geq \frac{\varepsilon}{l} + o(1) \quad (2.3.39)$$

in $B_0(R)$ for all $R > 0$. Note that, since $w \geq \frac{\varepsilon}{l}$,

$$\Delta w^{-\alpha} \leq \alpha \frac{|\nabla w|^2}{w^{\alpha+2}} \left[\frac{|Y(x_0)|^2}{\varepsilon^{q+2}} - (\alpha + 1) \right], \quad (2.3.40)$$

so $w^{-\alpha}$ is subharmonic for α large. By applying Lemma 12 (see the Annex), we deduce that w is constant, in contradiction with (2.3.25).

Third case: Let

$$l = 0 \quad \text{and } \hat{l} = 0, \quad (2.3.41)$$

then w is a non-negative harmonic function on \mathbb{R}^n . Thus, $w = cst$ and, by (2.3.25), $w \equiv 1$. Since $\hat{l} = 0$ and (2.3.21) imply that

$$\left| \frac{\nabla w(0)}{w(0)} \right| = 1, \quad (2.3.42)$$

which leads to a contradiction. \square

The following is a Harnack-type inequality. It holds whenever an estimate like (2.3.11) is verified, that is when there exists a constant C_2 and a sequence $(x_\alpha, \rho_\alpha)_\alpha$ such that

$$d_g(x_\alpha, x)^{\frac{n-2}{2}} \left[u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right] \leq C_2, \quad \forall x \in B_{x_\alpha}(7\rho_\alpha). \quad (2.3.43)$$

Lemma 5. *Let $(x_\alpha, \rho_\alpha)_\alpha$ be a sequence such that (2.3.43) holds. Then there exists a constant $C_3 > 1$ such that for any sequence $0 < s_\alpha \leq \rho_\alpha$, we get*

$$s_\alpha \|\nabla u_\alpha\|_{L^\infty(\Omega_\alpha)} \leq C_3 \sup_{\Omega_\alpha} u_\alpha \leq C_3^2 \inf_{\Omega_\alpha} u_\alpha, \quad (2.3.44)$$

where $\Omega_\alpha = B_{x_\alpha}(6s_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}s_\alpha)$.

Proof of Lemma 5: Estimate (2.3.43) implies that

$$\left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right| \leq C_2 d_g(x_\alpha, x)^{-1} \quad (2.3.45)$$

in Ω_α , and therefore

$$s_\alpha |\nabla \ln u_\alpha(x)| \leq 6C_2 \quad (2.3.46)$$

in Ω_α . Taking $C_3 \geq 6C_2$, we get the first inequality from (2.3.45). Then, from (2.3.46) and from the fact that the domain is an annulus $\Omega_\alpha = B_{x_\alpha}(6s_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}s_\alpha)$, we estimate that

$$\sup_{\Omega_\alpha} \ln u_\alpha - \inf_{\Omega_\alpha} \ln u_\alpha \leq l_\alpha(\Omega_\alpha) \|\nabla \ln u_\alpha\|_{L^\infty(\Omega_\alpha)} \leq 42C_2, \quad (2.3.47)$$

where $l_\alpha(\Omega_\alpha)$ is the infimum of the length of a curve in Ω_α drawn between a maximum and a minimum of u_α . Equivalently

$$\sup_{\Omega_\alpha} u_\alpha \leq e^{42C_2} \inf_{\Omega_\alpha} u_\alpha, \quad (2.3.48)$$

so it suffices to take $C_3 = e^{42C_2}$. \square

2.3.2 Local blow-up analysis

In order to show that u_α is bounded in \mathcal{C}^1 , we define a *blow-up sequence* $(x_\alpha)_\alpha$ with $(\rho_\alpha)_\alpha$ as follows: let $(x_\alpha)_\alpha$ be critical points of $(u_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ positive numbers such that they verify the following three conditions:

$$0 < \rho_\alpha < \frac{1}{7} i_g(M), \quad (2.3.49)$$

$$\rho_\alpha^{\frac{n-2}{2}} \sup_{B_{x_\alpha}(6\rho_\alpha)} u_\alpha \rightarrow \infty, \quad (2.3.50)$$

and

$$d_g(x_\alpha, x)^{\frac{n-2}{2}} \left[u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right] \leq C_2 \quad \forall x \in B_{x_\alpha}(7\rho_\alpha), \quad (2.3.51)$$

where C_2 is a constant. In the rest of the section, we denote

$$\mu_\alpha^{1-\frac{n}{2}} := u_\alpha(x_\alpha). \quad (2.3.52)$$

Remark 8. *The limit as $\alpha \rightarrow \infty$ of a blow-up sequence is a concentration point, as seen from (2.3.50).*

Remark 9. *Any sequence $(x_\alpha)_\alpha \subset (S_\alpha)_\alpha$ qualifies as a blow-up sequence as soon as (2.3.50) is verified. In this case, $(\rho_\alpha)_\alpha$ can be chosen as*

$$\rho_\alpha := \min \left(\frac{1}{7} i_g(M), \frac{1}{2} \min_{1 \leq i < j \leq N_\alpha} d_g(x_{i,\alpha}, x_{j,\alpha}) \right) \quad (2.3.53)$$

and $C_2 = C_1$.

Given any blow-up sequence, the following proposition gathers the central results of our local analysis for the reader's convenience: namely, it states the exact asymptotic profile of $(u_\alpha)_\alpha$ at distance $(\rho_\alpha)_\alpha$ of $(x_\alpha)_\alpha$ and it gives sharp pointwise asymptotic estimates on balls of radius ρ_α . This is the result we shall point to whenever we want to describe the local asymptotic behaviour of $(u_\alpha)_\alpha$ around a concentration point corresponding to local maximum points x_α . Note that in this case $\nabla u_\alpha(x_\alpha) = 0$.

Proposition 1. *Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be a blow-up sequence. Then there exists $C_4 > 0$ such that*

$$u_\alpha(x) + d_g(x_\alpha, x)|\nabla u_\alpha(x)| \leq C_4 \mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha, x)^{2-n} \quad (2.3.54)$$

for all $x \in B_{x_\alpha}(6\rho_\alpha) \setminus \{x_\alpha\}$. Moreover, we see that up to a subsequence, the asymptotic profile of $(u_\alpha)_\alpha$ is

$$u_\alpha(x_\alpha) \rho_\alpha^{n-2} u_\alpha(\exp_{x_\alpha}(\rho_\alpha x)) \rightarrow \frac{R_0^{n-2}}{|x|^{n-2}} + H(x) \quad (2.3.55)$$

in $C_{loc}^2(B_0(5) \setminus \{0\})$, where H is some harmonic function in $B_0(5)$ satisfying $H(0) = 0$. Here $R_0^2 = \frac{n(n-2)}{f(x_0)}$ where $x_0 = \lim_{\alpha \rightarrow \infty} x_\alpha$.

The proof of this proposition is the subject of this section. It will follow from Lemma 8 and Lemma 9 below. We first describe the asymptotic profile at distance $(\mu_\alpha)_\alpha$ of $(x_\alpha)_\alpha$ as $\alpha \rightarrow \infty$ of any blow-up sequence.

Lemma 6. *Let $(x_\alpha)_\alpha$ with $(\rho_\alpha)_\alpha$ be a blow-up sequence. We have*

$$\mu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_\alpha}(\mu_\alpha x)) \rightarrow \left(1 + \frac{f(x_0)|x|^2}{n(n-2)}\right)^{1-\frac{n}{2}} \quad (2.3.56)$$

in $C_{loc}^1(\mathbb{R}^n)$ as $\alpha \rightarrow \infty$, with $\mu_\alpha^{1-\frac{n}{2}} := u_\alpha(x_\alpha)$ and $x_0 := \lim_{\alpha \rightarrow \infty} x_\alpha$, up to a subsequence.

Proof of Lemma 6: The proof involves similar arguments to the ones used for Lemma 3. Let $y_\alpha \in B_{x_\alpha}(6\rho_\alpha)$ be such that

$$u_\alpha(y_\alpha) + \left| \frac{\nabla u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \right|^{\frac{n-2}{2}} = \sup_{B_{x_\alpha}(6\rho_\alpha)} \left(u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right)$$

and let

$$\nu_\alpha^{1-\frac{n}{2}} := u_\alpha(y_\alpha) + \left| \frac{\nabla u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \right|^{\frac{n-2}{2}}.$$

Conditions (2.3.50) and (2.3.51) imply that

$$\frac{\rho_\alpha}{\nu_\alpha} \rightarrow \infty$$

and

$$d_g(x_\alpha, y_\alpha) \leq C_2^{\frac{2}{n-2}} \nu_\alpha.$$

It follows that the coordinates of y_α in the exponential chart around x_α defined as $\tilde{y}_\alpha := \nu_\alpha^{-1} \exp_{x_\alpha}^{-1}(y_\alpha)$ are bounded by $C_2^{\frac{2}{n-2}}$. Up to a subsequence, we may choose a finite limit $\tilde{y}_0 := \lim_{\alpha \rightarrow \infty} \tilde{y}_\alpha$. We denote

$$v_\alpha(x) = \nu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_\alpha}(\nu_\alpha x)) \quad \text{and} \quad g_\alpha(x) = (\exp_{x_\alpha}^* g)(\exp_{x_\alpha}(\nu_\alpha x))$$

for $x \in \Omega_\alpha := B_0\left(\frac{\rho_\alpha}{\nu_\alpha}\right)$. As before, $g_\alpha \rightarrow \xi$ in \mathcal{C}_{loc}^2 , $v_\alpha = O(1)$, and $\left| \frac{\nabla v_\alpha}{v_\alpha} \right| = O(1)$. By applying the same analysis as in the proof of Lemma 3, we get that, up to passing to a subsequence, there exists $U := \lim_{\alpha \rightarrow \infty} v_\alpha$ in $\mathcal{C}_{loc}^1(\mathbb{R}^n)$, with $x_0 := \lim_{\alpha \rightarrow \infty} x_\alpha$,

$$\Delta U = f(x_0)U^{q-1}.$$

where

$$U(x) = \left(1 + \frac{f(x_0)|x - \tilde{y}_0|^2}{n(n-2)}\right)^{1-\frac{n}{2}}.$$

We know that x_α are local maxima for u_α , so both 0 and \tilde{y}_0 are maxima of U . However, since U admits a unique maximum, we conclude that $\tilde{y}_0 = 0$. \square

We recall the aim of this section is to show that concentration points do not exist for the system (2.1.13). So far, we have obtained a pointwise estimate (2.3.11) that holds everywhere on M and an asymptotic profile in the neighbourhood of x_α , a blow-up sequence; we aim to also find estimates around x_α . We defined $(\rho_\alpha)_\alpha$ as the quantity describing the sphere of dominance of the blow-up sequence $(x_\alpha)_\alpha$. However, the influence of other blow-up sequences may be felt earlier. Let $\varphi_\alpha : (0, \rho_\alpha) \rightarrow \mathbb{R}^+$ be the average of u_α defined as

$$\varphi_\alpha(r) := \frac{1}{|\partial B_{x_\alpha}(r)|_g} \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g.$$

It follows from Lemma 6 that

$$(\mu_\alpha r)^{\frac{n-2}{2}} \varphi_\alpha(\mu_\alpha r) \rightarrow r^{\frac{n-2}{2}} \left(1 + \frac{f(x_0)r^2}{n(n-2)}\right)^{1-\frac{n}{2}} \quad (2.3.57)$$

in $\mathcal{C}_{loc}^1([0, +\infty))$. We define

$$r_\alpha := \sup_{r \in (2R_0\mu_\alpha, \rho_\alpha)} \left\{ s^{\frac{n-2}{2}} \varphi_\alpha(s) \text{ is non-increasing in } (2R_0\mu_\alpha, r) \right\} \quad (2.3.58)$$

with

$$R_0^2 := \frac{n(n-2)}{f(x_0)}.$$

Note that

$$\text{if } r_\alpha < \rho_\alpha, \quad \text{then } \left(r^{\frac{n-2}{2}} \varphi_\alpha(r) \right)'(r_\alpha) = 0.$$

Since $r^{\frac{n-2}{2}} U$ is non-increasing in $[2R_0, \infty)$, then (2.3.57) implies

$$\frac{r_\alpha}{\mu_\alpha} \rightarrow \infty. \quad (2.3.59)$$

so $\mu_\alpha = o(r_\alpha)$.

Remark 10. *Considering that the asymptotic profile of blow-up sequences $(x_\alpha)_\alpha$, which is a bump function with a unique maximum, the quantity r_α is an indicator of the beginning of the influence of neighbouring blow-up sequences within the sphere of dominance, as the average of u_α is no longer decreasing.*

Let

$$\eta_\alpha := \sup_{B_{x_\alpha}(6r_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}r_\alpha)} u_\alpha. \quad (2.3.60)$$

Note that, by Lemma 5,

$$\frac{1}{C_3} \sup_{B_{x_\alpha}(6s_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}s_\alpha)} u_\alpha \leq \varphi_\alpha(s_\alpha) \leq C_3 \inf_{B_{x_\alpha}(6s_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}s_\alpha)} u_\alpha$$

for $0 < s_\alpha \leq r_\alpha$ and all α . By (2.3.57), we obtain the estimate

$$\lim_{R \rightarrow \infty} \limsup_{\alpha \rightarrow \infty} \sup_{B_{x_\alpha}(6r_\alpha) \setminus B_{x_\alpha}(R\mu_\alpha)} d_g(x_\alpha, x)^{\frac{n-2}{2}} u_\alpha = 0. \quad (2.3.61)$$

Thus,

$$r_\alpha^2 \eta_\alpha^{q-2} \rightarrow 0 \quad (2.3.62)$$

as $\alpha \rightarrow \infty$. It is important to note that this implies that

$$r_\alpha \rightarrow 0 \quad (2.3.63)$$

as $\alpha \rightarrow \infty$ since $u_\alpha \geq \varepsilon$ by (2.3.8). We now prove a pointwise asymptotic estimate for u_α in $B_{x_\alpha}(6r_\alpha) \setminus \{x_\alpha\}$.

Lemma 7. Let $(x_\alpha)_\alpha$ with $(\rho_\alpha)_\alpha$ be a blow-up sequence. Then, for any $0 < \varepsilon < \frac{1}{2}$, there exists $C_\varepsilon > 0$ such that

$$u_\alpha(x) \leq C_\varepsilon \left(\mu_\alpha^{\frac{n-2}{2}(1-2\varepsilon)} d_g(x_\alpha, x)^{(n-2)(1-\varepsilon)} + \eta_\alpha \left(\frac{r_\alpha}{d_g(x_\alpha, x)} \right)^{(n-2)\varepsilon} \right)$$

for all $x \in B_{x_\alpha}(6r_\alpha) \setminus \{x_\alpha\}$.

Proof of Lemma 7: Let G be a Green function for the Laplace operator Δ_g on M with $G > 0$. Recall the following estimates, that can be found in Aubin [Aub82]:

$$\begin{aligned} \left| d_g(x_\alpha, y)^{n-2} G(x, y) - \frac{1}{(n-2)\omega_{n-1}} \right| &\leq \tau(d_g(x, y)) \\ \left| d_g(x_\alpha, y)^{n-1} |\nabla G(x, y)| - \frac{1}{\omega_{n-1}} \right| &\leq \tau(d_g(x, y)) \end{aligned} \quad (2.3.64)$$

where $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function satisfying $\tau(0) = 0$. For a fixed ε , let

$$\Phi_\alpha^\varepsilon(x) := \mu_\alpha^{\frac{n-2}{2}(1-2\varepsilon)} G(x_\alpha, x)^{1-\varepsilon} + \eta_\alpha r_\alpha^{(n-2)\varepsilon} G(x_\alpha, x)^\varepsilon \quad (2.3.65)$$

and let $y_\alpha \in \overline{B_{x_\alpha}(6r_\alpha)} \setminus \{x_\alpha\}$ be such that

$$\sup_{B_{x_\alpha}(6r_\alpha)} \frac{u_\alpha}{\Phi_\alpha^\varepsilon} = \frac{u_\alpha(y_\alpha)}{\Phi_\alpha^\varepsilon(y_\alpha)}, \quad (2.3.66)$$

We continue by studying the following two cases, separately.

First case: Assume that the relative size of $d_g(x_\alpha, y_\alpha)$ with respect to μ_α is

$$R := \lim_{\alpha \rightarrow \infty} \frac{d_g(x_\alpha, y_\alpha)}{\mu_\alpha} \quad \text{with } R \in [0, \infty). \quad (2.3.67)$$

Thanks to Lemma 6,

$$\mu_\alpha^{\frac{n-2}{2}} u_\alpha(y_\alpha) = \left(1 + \frac{R^2}{R_0^2} \right)^{1-\frac{n}{2}} + o(1),$$

so that, whenever $R \in [0, \infty)$, using (2.3.59), (2.3.64) and (2.3.67), it is easily shown that

$$\frac{u_\alpha(y_\alpha)}{\Phi_\alpha^\varepsilon(y_\alpha)} \rightarrow ((n-2)\omega_{n-1})^{1-\varepsilon} R^{(n-2)(1-\varepsilon)} \left(1 + \frac{R^2}{R_0^2} \right)^{1-\frac{n}{2}}$$

as $\alpha \rightarrow \infty$.

Second case: It remains to study the case

$$\lim_{\alpha \rightarrow \infty} \frac{d_g(x_\alpha, y_\alpha)}{\mu_\alpha} \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty.$$

If $(y_\alpha)_\alpha$ sits on the outer boundary $\partial B_{x_\alpha}(6r_\alpha)$, then by (2.3.63), (2.3.64), and (2.3.65),

$$\frac{u_\alpha(y_\alpha)}{\Phi_\alpha^\varepsilon(y_\alpha)} \leq (6^{n-2}(n-2)\omega_{n-1})^\varepsilon + o(1).$$

Otherwise, if up to a subsequence $y_\alpha \in B_{x_\alpha}(6r_\alpha)$, then

$$\frac{\Delta_g u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \geq \frac{\Delta_g \Phi_\alpha^\varepsilon(y_\alpha)}{\Phi_\alpha^\varepsilon(y_\alpha)}$$

as a consequence of the fact that y_α is the maximum of $\frac{u_\alpha}{\Phi_\alpha^\varepsilon}$. On the other hand, taking note of the sign of the dominant gradient term in equations (2.3.4), we see that

$$\begin{aligned}\Delta_g u_\alpha &= -h_\alpha u_\alpha + f_\alpha u_\alpha^{q-1} + \frac{a_\alpha}{u_\alpha^{q+1}} - \frac{b_\alpha}{u_\alpha} - \frac{\langle \nabla u_\alpha, Y_\alpha \rangle^2}{u_\alpha^{q+3}} \\ &\quad - c_\alpha \langle \nabla u_\alpha, Y_\alpha \rangle \left[\frac{d_\alpha}{u_\alpha^2} + \frac{1}{u_\alpha^{q+2}} \right] \\ &\leq C u_\alpha^{q-1},\end{aligned}\tag{2.3.68}$$

where C is a constant depending on θ and T . Here we used (2.3.8). Finally, thanks to (2.3.61),

$$d_g(x_\alpha, y_\alpha)^2 \frac{\Delta_g u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \leq C d_g(x_\alpha, y_\alpha)^2 u_\alpha^{q-2}(y_\alpha) \rightarrow 0.$$

To conclude, (2.3.63) and (2.3.64) imply that

$$d_g(x_\alpha, y_\alpha)^2 \frac{\Delta_g \Phi_\alpha^\varepsilon(y_\alpha)}{\Phi_\alpha^\varepsilon(y_\alpha)} = \varepsilon(1 - \varepsilon)(n - 2)^2 + o(1).$$

We deduce that $u_\alpha(y_\alpha) = O(\Phi_\alpha^\varepsilon(y_\alpha))$. The study of the previous two cases ends the proof of the lemma. \square

The following lemma improves the estimate we've just obtained and gives a very important bound on the size of r_α .

Lemma 8. *Let $(x_\alpha)_\alpha$ with $(\rho_\alpha)_\alpha$ be a blow-up sequence. Then there exists $C_4 > 0$ such that*

$$u_\alpha(x) + d_g(x_\alpha, x) |\nabla u_\alpha(x)| \leq C_4 \mu_\alpha^{\frac{n-2}{2}} (d_g(x_\alpha, x) + \mu_\alpha)^{2-n}\tag{2.3.69}$$

for all $x \in B_{x_\alpha}(6r_\alpha) \setminus \{x_\alpha\}$. Moreover, $r_\alpha^2 = O(\mu_\alpha)$.

Proof of Lemma 8: It suffices to prove the estimate for u_α ; the rest follows as an immediate consequence of Lemma 5. We start by showing that for any sequence $z_\alpha \in \overline{B_{x_\alpha}(6r_\alpha)} \setminus \{x_\alpha\}$, there holds

$$u_\alpha(z_\alpha) = O\left(\mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha, z_\alpha)^{2-n} + \eta_\alpha\right).\tag{2.3.70}$$

First, if $d_g(x_\alpha, z_\alpha) = O(\mu_\alpha)$, it falls within the range described in Lemma 6. On the other hand, when $r_\alpha = O(d_g(x_\alpha, z_\alpha))$, we use Lemma 5 together with (2.3.60). It remains to consider the intermediary case:

$$\frac{d_g(x_\alpha, z_\alpha)}{\mu_\alpha} \rightarrow \infty \quad \text{and} \quad \frac{d_g(x_\alpha, z_\alpha)}{r_\alpha} \rightarrow 0 \quad \text{at } \alpha \rightarrow \infty.$$

According to the Green representation formula,

$$u_\alpha(z_\alpha) = O\left(\int_{B_{x_\alpha}(6r_\alpha)} d_g(z_\alpha, x)^{2-n} \Delta_g u_\alpha(x) dv_g\right) + O(\eta_\alpha),$$

where the second term corresponds to the boundary element. Recall that

$$\Delta_g u_\alpha \leq C u_\alpha^{q-1}$$

because of the sign of the dominant gradient term, see (2.3.68). Using (2.3.8), (2.3.62) and Lemma 7, we can write that

$$\begin{aligned}
& \int_{B_{x_\alpha}(6r_\alpha)} d_g(z_\alpha, x)^{2-n} u_\alpha^{q-1}(x) dv_g \\
&= O\left(\mu_\alpha^{\frac{n}{2}-1} d_g(x_\alpha, z_\alpha)^{2-n}\right) \\
&+ O\left(\mu_\alpha^{\frac{n+2}{2}(1-2\varepsilon)} \int_{B_{x_\alpha}(6r_\alpha) \setminus B_{x_\alpha}(\mu_\alpha)} d_g(z_\alpha, x)^{2-n} d_g(x_\alpha, x)^{-(n+2)(1-\varepsilon)} dv_g\right) \\
&+ O\left(\eta_\alpha^{q-1} r_\alpha^{(n+2)\varepsilon} \int_{B_{x_\alpha}(6r_\alpha) \setminus B_{x_\alpha}(\mu_\alpha)} d_g(z_\alpha, x)^{2-n} d_g(x_\alpha, x)^{-(n+2)\varepsilon} dv_g\right) \\
&= O\left(\mu_\alpha^{\frac{n}{2}-1} d_g(x_\alpha, z_\alpha)^{2-n}\right) + O\left(\eta_\alpha^{q-1} r_\alpha^2\right) \\
&= O\left(\mu_\alpha^{\frac{n}{2}-1} d_g(x_\alpha, z_\alpha)^{2-n}\right) + O(\eta_\alpha).
\end{aligned}$$

In order to get estimate (2.3.69), it suffices to show that

$$\eta_\alpha = O\left(\mu_\alpha^{\frac{n-2}{2}} r_\alpha^{2-n}\right). \quad (2.3.71)$$

For any fixed $0 < \delta < 1$, taking α large enough, then the monotonicity of $r^{\frac{n-2}{2}} \varphi_\alpha(r)$ expressed in the definition of r_α , see (2.3.58), and the fact that $\mu_\alpha = o(r_\alpha)$, see (2.3.59), imply that

$$r_\alpha^{\frac{n-2}{2}} \varphi_\alpha(r_\alpha) \leq (\delta r_\alpha)^{\frac{n-2}{2}} \varphi_\alpha(\delta r_\alpha) \quad \text{for all } 0 < \delta < 1,$$

so by Lemma 5,

$$\frac{1}{C_3} \eta_\alpha \leq \delta^{\frac{n-2}{2}} \sup_{\partial B_{x_\alpha}(\delta r_\alpha)} u_\alpha.$$

According to estimate (2.3.70), this leads to

$$\eta_\alpha \leq C \left(\mu_\alpha^{\frac{n-2}{2}} \delta^{2-n} r_\alpha^{2-n} + \eta_\alpha \right) \delta^{\frac{n-2}{2}},$$

where C is independent of δ and α . Choosing δ small enough leads to (2.3.71).

Estimates (2.3.8) and (2.3.71) imply that

$$r_\alpha^2 = O(\mu_\alpha). \quad (2.3.72)$$

This ends the proof of the lemma. \square

Given a blow-up sequence $(x_\alpha)_\alpha$ with $(\rho_\alpha)_\alpha$, the following lemma gives the exact asymptotic profile of $(u_\alpha)_\alpha$ at distance $(\rho_\alpha)_\alpha$ of $(x_\alpha)_\alpha$.

Lemma 9. *Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be a blow-up sequence. Then we have that $r_\alpha = \rho_\alpha$, where r_α is as in (2.3.58). Up to a subsequence, we have*

$$u_\alpha(x_\alpha) \rho_\alpha^{n-2} u_\alpha(\exp_{x_\alpha}(\rho_\alpha x)) \rightarrow \frac{R_0^{n-2}}{|x|^{n-2}} + H(x) \quad (2.3.73)$$

in $C_{loc}^2(B_0(5) \setminus \{0\})$, where H is some harmonic function in $B_0(5)$ satisfying $H(0) = 0$.

Proof of Lemma 9: First, we prove that, up to a subsequence,

$$u_\alpha(x_\alpha) r_\alpha^{n-2} u_\alpha(\exp_{x_\alpha}(r_\alpha x)) \rightarrow \frac{R_0^{n-2}}{|x|^{n-2}} + H(x). \quad (2.3.74)$$

Let us define the following rescaled quantities:

$$\hat{u}_\alpha(x) = \mu_\alpha^{1-\frac{n}{2}} r_\alpha^{n-2} u_\alpha(\exp_{x_\alpha}(r_\alpha x)) \quad \text{and} \quad \hat{g}_\alpha(x) = \exp_{x_\alpha}^* g(\exp_{x_\alpha}(r_\alpha x)).$$

Then

$$\Delta_{\hat{g}_\alpha} \hat{u}_\alpha = \hat{F}_\alpha,$$

in $B_0(\delta r_\alpha^{-1})$ for some $\delta > 0$ small enough, with

$$\begin{aligned} \hat{F}_\alpha = & -\mu_\alpha^{1-\frac{n}{2}} r_\alpha^n h_\alpha(\exp_{x_\alpha}(r_\alpha x)) u_\alpha(\exp_{x_\alpha}(r_\alpha x)) \\ & + \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n f_\alpha(\exp_{x_\alpha}(r_\alpha x)) u_\alpha^{q-1}(\exp_{x_\alpha}(r_\alpha x)) \\ & + \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n \frac{a_\alpha(\exp_{x_\alpha}(r_\alpha x))}{u_\alpha^{q+1}(\exp_{x_\alpha}(r_\alpha x))} - \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n \frac{b_\alpha(\exp_{x_\alpha}(r_\alpha x))}{u_\alpha(\exp_{x_\alpha}(r_\alpha x))} \\ & - \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n \frac{\langle \nabla u_\alpha(\exp_{x_\alpha}(r_\alpha x)), Y_\alpha(\exp_{x_\alpha}(r_\alpha x)) \rangle^2}{u_\alpha^{q+3}(\exp_{x_\alpha}(r_\alpha x))} \\ & - \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n c_\alpha(\exp_{x_\alpha}(r_\alpha x)) \langle \nabla u_\alpha, Y_\alpha \rangle(\exp_{x_\alpha}(r_\alpha x)) \left[\frac{d_\alpha(\exp_{x_\alpha}(r_\alpha x))}{u_\alpha(\exp_{x_\alpha}(r_\alpha x))} \right. \\ & \left. + \frac{1}{u_\alpha(\exp_{x_\alpha}(r_\alpha x))^{q+2}} \right] \end{aligned} \quad (2.3.75)$$

Thanks to Lemma 8, we know that $|\hat{u}_\alpha| \leq C_K$ and $|\hat{F}_\alpha| = O(1)$ on any compact $K \subset B_0(5) \setminus \{0\}$. The above system may be rewritten as

$$\Delta_{\hat{g}_\alpha} \hat{u}_\alpha = o(1) - O(1) \frac{\langle \nabla \hat{u}_\alpha, Y_\alpha \rangle^2}{\hat{u}_\alpha^{q+3}}. \quad (2.3.76)$$

Remark 11. *It is here that we use the additional smallness condition we impose on Y , as seen in (2.3.3).*

By standard elliptic theory, up to a subsequence,

$$\hat{u}_\alpha \rightarrow \hat{U} \quad \text{in } \mathcal{C}_{loc}^1(B_0(5) \setminus \{0\})$$

with

$$\Delta_\xi \hat{U} = 0 \quad \text{in } B_0(5) \setminus \{0\}.$$

Separate \hat{U} into the sum of a regular harmonic function and a singular part

$$\hat{U} = \frac{\lambda}{|x|^{n-2}} + H(x),$$

where $\lambda \geq 0$.

To get (2.3.74), it remains to show that $\lambda = R_0^{n-2}$. For any $\delta > 0$:

$$\int_{B_0(\delta)} \hat{F}_\alpha dv_{\hat{g}_\alpha} = - \int_{\partial B_0(\delta)} \partial_\nu \hat{u}_\alpha d\sigma_{\hat{g}_\alpha}. \quad (2.3.77)$$

Using the equation (2.3.75), we estimate the left hand side of (2.3.77). In particular,

$$\begin{aligned} & \int_{B_0(\delta)} f_\alpha(\exp_{x_\alpha}(r_\alpha x)) \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n u_\alpha^{q-1}(\exp_{x_\alpha}(r_\alpha x)) dx \\ & = \int_{B_0(\delta \frac{r_\alpha}{\mu_\alpha})} f_\alpha(\exp_{x_\alpha}(\mu_\alpha z)) \mu_\alpha^{\frac{n+2}{2}} u_\alpha^{q-1}(\exp_{x_\alpha}(\mu_\alpha z)) dz \end{aligned}$$

where $z = \frac{r_\alpha}{\mu_\alpha}x$, and by Lemma 6 and Lemma 8,

$$\begin{aligned} & \lim_{\alpha \rightarrow \infty} \int_{B_0(\delta)} f_\alpha(\exp_{x_\alpha}(r_\alpha x)) \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n u_\alpha^{q-1}(\exp_{x_\alpha}(r_\alpha x)) dx \\ &= f(x_0) \int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{R_0^2}\right)^{-1-\frac{n}{2}} dx. \end{aligned}$$

The gradient terms are controlled with the estimate (2.3.11), and together with (2.3.8), we obtain that the dominant gradient term of (2.3.75) verifies

$$\begin{aligned} \int_{B_0(\delta)} \frac{|\nabla u_\alpha|^2}{u_\alpha^{q+3}} (\exp_{x_\alpha}(r_\alpha x)) \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n dx &\leq C \mu_\alpha^{1-\frac{n}{2}} r_\alpha^{n-2} \int_{B_0(\delta)} |x|^{-2} dx \\ &\leq C \omega_{n-1} \mu_\alpha^{1-\frac{n}{2}} r_\alpha^{n-2} \delta^{n-2}. \end{aligned} \tag{2.3.78}$$

As $\mu_\alpha^{1-\frac{n}{2}} r_\alpha^{n-2} = O(1)$, the integral does not vanish as $\alpha \rightarrow \infty$; its size depends on δ . The remaining terms in (2.3.75) are negligible. Thus

$$\int_{B_0(\delta)} \hat{F}_\alpha dv_{\hat{g}_\alpha} = f(x_0) \int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{R_0^2}\right)^{-1-\frac{n}{2}} dx + o(1) + O(\delta^{n-2})$$

for any $\delta > 0$. It follows that

$$f(x_0) \int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{R_0^2}\right)^{-1-\frac{n}{2}} dx = (n-2)\omega_{n-1}R_0^{n-2}.$$

Note also that the right hand side of (2.3.77) verifies

$$\begin{aligned} - \int_{\partial B_0(\delta)} \partial_\nu \hat{u}_\alpha d\sigma_{\hat{g}_\alpha} &= - \int_{\partial B_0(\delta)} \partial_\nu \hat{U} + o(1) \\ &= \lambda(n-2)\omega_{n-1} + o(1). \end{aligned}$$

since H is smooth and harmonic. Since

$$\lambda(n-2)\omega_{n-1} = R_0^{n-2}(n-2)\omega_{n-1} + O(\delta^{n-2}) + o(1),$$

for any $\delta > 0$, we get that $\lambda = R_0^{n-2}$.

Finally, let us prove that $H(0) = 0$. The equation's dominant terms are invariant by rescaling, which leads us to use a Pohozaev identity to obtain new estimates for the remaining terms. Let Ω_α correspond to $B_0(\delta r_\alpha)$ in the exponential chart at $x_\alpha \in M$ and let $X_\alpha = \frac{1}{2}\nabla d_g(x_\alpha, x)^2$ be the vector field of coordinates. Using integration by parts,

$$\begin{aligned} \int_{\Omega_\alpha} \nabla u_\alpha(X_\alpha) \Delta_g u_\alpha dv_g &= \int_{\Omega_\alpha} \langle \nabla(\nabla u_\alpha(X_\alpha)), \nabla u_\alpha \rangle dv_g \\ &\quad - \int_{\partial\Omega_\alpha} \nabla u_\alpha(X_\alpha) \partial_\nu u_\alpha d\sigma_g \\ &= \int_{\Omega_\alpha} \nabla^\# \nabla u_\alpha(X_\alpha, \nabla u_\alpha) + \nabla^\# X_\alpha(\nabla u_\alpha, \nabla u_\alpha) dv_g \\ &\quad - \int_{\partial\Omega_\alpha} \nabla u_\alpha(X_\alpha) \partial_\nu u_\alpha d\sigma_g, \end{aligned}$$

where $(\nabla^\# X_\alpha) = (\nabla^i X_\alpha)^j$. Since

$$\begin{aligned} & \int_{\Omega_\alpha} |\nabla u_\alpha|^2 \operatorname{div}_g X_\alpha \, dv_g + \int_{\Omega_\alpha} \langle \nabla (|\nabla u_\alpha|^2), X_\alpha \rangle \, dv_g \\ &= \int_{\partial\Omega_\alpha} |\nabla u_\alpha|^2 \langle X_\alpha, \nu \rangle \, d\sigma_g, \end{aligned}$$

we can write that

$$\begin{aligned} & \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) \Delta_g u_\alpha \, dv_g \\ &= \int_{\Omega_\alpha} (\nabla^\# X_\alpha (\nabla u_\alpha, \nabla u_\alpha) - \frac{1}{2} (\operatorname{div}_g X_\alpha) |\nabla u_\alpha|^2) \, dv_g \\ &+ \int_{\partial\Omega_\alpha} \left(\frac{1}{2} \langle X_\alpha, \nu \rangle |\nabla u_\alpha|^2 - \nabla u_\alpha(X_\alpha) \partial_\nu u_\alpha - \frac{n-2}{2} u_\alpha \partial_\nu u_\alpha \right) \, d\sigma_g. \end{aligned} \tag{2.3.79}$$

We begin by analyzing the right-hand side of (2.3.79). By our choice of X_α , $(\nabla^\# X_\alpha)^{ij} = g^{ij} + O(d_g(x_\alpha, x)^2)$, and consequently

$$\begin{aligned} & \int_{\Omega_\alpha} \left(\nabla^\# X_\alpha (\nabla u_\alpha, \nabla u_\alpha) - \frac{1}{2} (\operatorname{div}_g X_\alpha) |\nabla u_\alpha|^2 \right) \, dv_g \\ &= O\left(\int_{\Omega_\alpha} d_g(x_\alpha, x)^2 |\nabla u_\alpha|^2 \, dv_g \right). \end{aligned}$$

According to (2.3.69),

$$\int_{\Omega_\alpha} d_g(x_\alpha, x)^2 |\nabla u_\alpha|^2 \, dv_g \leq C \int_{\Omega_\alpha} \mu_\alpha^{n-2} (d_g(x_\alpha, x) + \mu_\alpha)^{4-2n} \, dv_g,$$

so

$$\int_{\Omega_\alpha} d_g(x_\alpha, x)^2 |\nabla u_\alpha|^2 \, dv_g \leq \begin{cases} O(\mu_\alpha r_\alpha) & \text{if } n = 3 \\ O\left(\mu_\alpha^2 \ln \frac{1}{\mu_\alpha}\right) & \text{if } n = 4 \\ O(\mu_\alpha^2) & \text{if } n = 5 \end{cases}$$

In all these three cases, thanks to (2.3.72), the integral is of the order $o(\mu_\alpha^{n-2} r_\alpha^{2-n})$. From (2.3.73),

$$\begin{aligned} & \int_{\partial\Omega_\alpha} \left(\frac{1}{2} \langle X_\alpha, \nu \rangle |\nabla u_\alpha|^2 - \nabla u_\alpha(X_\alpha) \partial_\nu u_\alpha - \frac{n-2}{2} u_\alpha \partial_\nu u_\alpha \right) \, d\sigma_g \\ &= \left(\frac{(n-2)^2}{2} \omega_{n-1} R_0^{n-2} H(0) + o(1) \right) \mu_\alpha^{n-2} r_\alpha^{2-n}. \end{aligned}$$

Note that the boundary term does not depend on δ , and as a result

$$\begin{aligned} & \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) \Delta_g u_\alpha \, dv_g \\ &= \left(\frac{(n-2)^2}{2} \omega_{n-1} R_0^{n-2} H(0) + o(1) \right) \mu_\alpha^{n-2} r_\alpha^{2-n} \end{aligned} \tag{2.3.80}$$

We now analyse the right hand side of (2.3.79) by using (2.3.4):

$$\begin{aligned}
& \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) \Delta_g u_\alpha dv_g \\
&= \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) f_\alpha u_\alpha^{q-1} dv_g \\
&\quad - \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) \langle \nabla u_\alpha, Y_\alpha \rangle^2 u_\alpha^{-q-3} dv_g \\
&\quad - \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) c_\alpha \langle \nabla u_\alpha, Y_\alpha \rangle (d_\alpha u_\alpha^{-2} + u_\alpha^{-q-2}) dv_g \\
&\quad + \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) \left(a_\alpha u_\alpha^{-q-1} - b_\alpha u_\alpha^{-1} - h_\alpha u_\alpha \right) dv_g
\end{aligned} \tag{2.3.81}$$

and we look at each term in turn. By the estimates (2.3.8) and (2.3.11), we get

$$\begin{aligned}
\left| \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) \langle \nabla u_\alpha, Y_\alpha \rangle^2 u_\alpha^{-q-3} dv_g \right| &\leq C \int_{\Omega_\alpha} d_g(x_\alpha, x)^{-2} dv_g \\
&\leq C(\delta r_\alpha)^{n-2}
\end{aligned} \tag{2.3.82}$$

and, similarly,

$$\begin{aligned}
\left| \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) c_\alpha \langle \nabla u_\alpha, Y_\alpha \rangle (d_\alpha u_\alpha^{-2} + u_\alpha^{-q-2}) dv_g \right| \\
\leq C(\delta r_\alpha)^{n-1}.
\end{aligned} \tag{2.3.83}$$

We also have that

$$\left| \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha \right) (a_\alpha u_\alpha^{-q-1} - b_\alpha u_\alpha^{-1}) \right| \leq C r_\alpha^n. \tag{2.3.84}$$

From (2.3.69), we obtain that

$$\begin{aligned}
& \left| \int_{\Omega_\alpha} h_\alpha (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) u_\alpha dv_g \right| \\
&= O \left(\int_{\Omega_\alpha} \mu_\alpha^{n-2} (\mu_\alpha + d_g(x_\alpha, x))^{4-2n} dx \right) \\
&= o(\mu_\alpha^{n-2} r_\alpha^{2-n})
\end{aligned} \tag{2.3.85}$$

for $3 \leq n \leq 5$. Using integration by parts,

$$\begin{aligned}
\int_{\Omega_\alpha} \nabla u_\alpha(X_\alpha) f_\alpha u_\alpha^{q-1} dv_g &= \frac{1}{q} \int_{\partial\Omega_\alpha} f_\alpha r_\alpha u_\alpha^q d\sigma_g \\
&\quad - \frac{1}{q} \int_{\Omega_\alpha} \operatorname{div}_g X_\alpha f_\alpha u_\alpha^q dv_g \\
&\quad - \frac{1}{q} \int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g.
\end{aligned} \tag{2.3.86}$$

Thus we can write that

$$\begin{aligned}
\int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) f_\alpha u_\alpha^{q-1} dv_g &= \frac{1}{q} r_\alpha \int_{\partial\Omega_\alpha} f_\alpha u_\alpha^q d\sigma_g \\
&\quad + \int_{\Omega_\alpha} \left(-\frac{1}{q} \operatorname{div}_g(X_\alpha) + \frac{n-2}{2} \right) f_\alpha u_\alpha^q dv_g \\
&\quad - \frac{1}{q} \int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g.
\end{aligned}$$

Since $\operatorname{div}_g(X_\alpha) = n + O(d_g(x_\alpha, x)^2)$, this leads to

$$\begin{aligned} \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) f_\alpha u_\alpha^{q-1} dv_g &= \frac{1}{q} r_\alpha \int_{\partial\Omega_\alpha} f_\alpha u_\alpha^q d\sigma_g \\ &+ O\left(\int_{\Omega_\alpha} d_g(x_\alpha, x)^2 u_\alpha^q dv_g\right) \\ &- \frac{1}{q} \int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g. \end{aligned}$$

Using lemmas 6 and 8, this leads to

$$\begin{aligned} \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) f_\alpha u_\alpha^{q-1} dv_g &= O(\mu_\alpha^n r_\alpha^{-n}) + O(\mu_\alpha^2) \\ &- \frac{1}{q} \int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g \end{aligned}$$

so that, thanks to (2.3.72),

$$\begin{aligned} \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) f_\alpha u_\alpha^{q-1} dv_g &= o(\mu_\alpha^{n-2} r_\alpha^{2-n}) \\ &- \frac{1}{q} \int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g \end{aligned} \quad (2.3.87)$$

if $3 \leq n \leq 5$.

We claim that

$$\int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g = o(\mu_\alpha^{n-2} r_\alpha^{2-n}). \quad (2.3.88)$$

Thanks to (2.3.80), (2.3.82), (2.3.83), (2.3.84), (2.3.85), (2.3.87) and (2.3.88), we see that

$$H(0) = o(1) + \delta^4 \quad (2.3.89)$$

for any $\delta > 0$, so by taking $\delta \rightarrow 0$ we see that $H(0) = 0$.

In order to prove (2.3.88), we can first use Lemma 8 to write that

$$\int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g = O(\mu_\alpha)$$

which leads to (2.3.88) if $n = 3, 4$ thanks to (2.3.72), but is not enough for $n = 5$. In order to improve the estimate in the case $n = 5$, note that

$$\begin{aligned} \int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g &= \partial_i f(x_\alpha) \int_{\Omega_\alpha} x^i u_\alpha^q dv_g \\ &+ O\left(\int_{\Omega_\alpha} d_g(x_\alpha, x)^{1+\eta} u_\alpha^q dv_g\right) \\ &= o(\mu_\alpha |\nabla f_\alpha(x_\alpha)|) + O(\mu_\alpha^{1+\eta}) \\ &= o(\mu_\alpha |\nabla f_\alpha(x_\alpha)|) + o(\mu_\alpha^3 r_\alpha^{-3}). \end{aligned}$$

with $\eta > \frac{1}{2}$. Thus it remains to prove that

$$|\nabla f_\alpha(x_\alpha)| = O(\mu_\alpha^2 r_\alpha^{-3}). \quad (2.3.90)$$

As before, we use a Pohozaev-type identity. We make use of the equation's symmetry by translation, with $Z = Z^i$ a constant vector field in the exponential chart of x_α . We can write that

$$\int_{\Omega_\alpha} \nabla u_\alpha(Z_\alpha) \Delta_g u_\alpha dv_g = O\left(\int_{\Omega_\alpha} d_g(x_\alpha, x) |\nabla u_\alpha|^2 dv_g + \int_{\partial\Omega_\alpha} |\nabla u_\alpha|^2 d\sigma_g\right), \quad (2.3.91)$$

which is $o(\mu_\alpha^2 r_\alpha^{-3})$. On the left-hand side, we use (2.3.4). Lemma 5 and (2.3.8) imply that

$$\begin{aligned} \int_{\Omega_\alpha} \nabla u_\alpha(Z_\alpha) \frac{\langle \nabla u_\alpha, Y_\alpha \rangle^2}{u_\alpha^{q+3}} dv_g &\leq \frac{1}{\varepsilon^q} \int_{\Omega_\alpha} \left| \frac{\nabla u_\alpha}{u_\alpha} \right|^3 dv_g \\ &\leq \frac{1}{\varepsilon^q} \int_{\Omega_\alpha} d_g(x_\alpha, x)^{-3} dv_g \\ &= O(r_\alpha^2) = o(\mu_\alpha^2 r_\alpha^{-3}). \end{aligned} \quad (2.3.92)$$

We see that

$$\int_{\Omega_\alpha} \nabla u_\alpha(Z_\alpha) c_\alpha \langle \nabla u_\alpha, Y_\alpha \rangle \left(\frac{d_\alpha}{u_\alpha^2} + \frac{1}{u_\alpha^{q+2}} \right) dv_g = O(r_\alpha^2) = o(\mu_\alpha^2 r_\alpha^{-3}), \quad (2.3.93)$$

and that the same holds for the terms corresponding to h_α , b_α and c_α . So (2.3.91), (2.3.92) and (2.3.93) imply that

$$\int_{\Omega_\alpha} \nabla u_\alpha(Z_\alpha) f_\alpha u_\alpha^{q-1} dv_g = o(\mu_\alpha^2 r_\alpha^{-3}).$$

Furthermore,

$$\begin{aligned} \int_{\Omega_\alpha} \nabla u_\alpha(Z_\alpha) f_\alpha u_\alpha^{q-1} dv_g &= O\left(\int_{\partial\Omega_\alpha} u_\alpha^q d\sigma_g\right) \\ &\quad - \frac{1}{q} \int_{\Omega_\alpha} \operatorname{div}_g(Z_\alpha) f_\alpha u_\alpha^q dv_g \\ &\quad - \frac{1}{q} \nabla f_\alpha(Z_\alpha) \int_{\Omega_\alpha} u_\alpha^q dv_g, \end{aligned}$$

which leads us to conclude the claim in (2.3.88). Note that

$$\operatorname{div}_g(Z_\alpha) = O(d_g(x_\alpha, x)^\eta)$$

and

$$\int_{\Omega_\alpha} u_\alpha^q dv_g \rightarrow \int_{R^n} \left(1 + \frac{|x|^2}{R_0^2}\right)^{-5} dx < \infty.$$

Finally, we are in the position to remark that $\rho_\alpha = r_\alpha$. Remember that

$$\varphi(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_0(r)} \hat{U} = \left(\frac{R_0}{r}\right)^{n-2} + H(0)$$

and that $\left(r^{\frac{n-2}{2}} \varphi(r)\right)'(1) = 0$, so if $r_\alpha < \rho_\alpha$, then $H(0) = R_0^{n-2}$, which contradicts (2.3.89). Thus (2.3.74) implies (2.3.73), and this wraps up the proof of the lemma. \square

Moreover, $\rho_\alpha = r_\alpha$ means that $\rho_\alpha \rightarrow 0$ because $r_\alpha = O\left(\mu_\alpha^{\frac{1}{2}}\right)$ thanks to (2.3.72). As an important consequence, there do not exist any isolated bubbles. Otherwise, if a bubble were isolated, then we could choose a blow-up sequence with $0 < \delta < \rho_\alpha$, contradicting the previous result.

2.3.3 Proof of the stability theorem

We are now in the position to prove Theorem 10. Let

$$\delta_\alpha := \min_{1 \leq i < j \leq N_\alpha} d_g(x_{i,\alpha}, x_{j,\alpha}).$$

For any $R > 0$, let $1 \leq M_{R,\alpha}$ be such that

$$\begin{aligned} d_g(x_{1,\alpha}, x_{i_\alpha,\alpha}) &\leq R\delta_\alpha \quad \text{for } i_\alpha \in \{1, \dots, M_{R,\alpha}\}, \text{ and} \\ d_g(x_{1,\alpha}, x_{j_\alpha,\alpha}) &> R\delta_\alpha \quad \text{for } j_\alpha \in \{M_{R,\alpha} + 1, \dots, N_\alpha\}. \end{aligned}$$

We consider the rescaled quantities

$$\check{u}_\alpha(x) := \delta_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_{1,\alpha}}(\delta_\alpha x)) \quad \text{and} \quad \check{g}_\alpha(x) := \left(\exp_{x_{1,\alpha}}^* g\right)(\delta_\alpha x)$$

and the coordinates $\check{x}_{i,\alpha} := \delta_\alpha^{-1} \exp_{x_{1,\alpha}}^{-1}(x_{i,\alpha})$ in the exponential chart. It's obvious that $|\check{x}_{2,\alpha}| = 1$ and $|\check{x}_{i,\alpha}| \geq 1$.

The following lemma is a direct consequence of Lemma 5.

Lemma 10. *For all $R > 0$, there exists $C_R > 0$ such that the Harnack-type inequality*

$$\|\nabla \check{u}_\alpha\|_{L^\infty(\Omega_R)} \leq C_R \sup_{\Omega_R} \check{u}_\alpha \leq C_R^2 \inf_{\Omega_R} \check{u}_\alpha$$

holds, where $\Omega_R = B_0(R) \setminus \bigcup_{i=1}^{M_{2R,\alpha}} B_{\check{x}_{i,\alpha}}\left(\frac{1}{R}\right)$.

Note that, for $1 \leq i < j \leq M_{R,\alpha}$, $B_{x_{i,\alpha}}\left(\frac{\delta_\alpha}{4}\right)$ and $B_{x_{j,\alpha}}\left(\frac{\delta_\alpha}{4}\right)$ are disjoint, which is equivalent to saying that $B_{\check{x}_{i,\alpha}}\left(\frac{1}{4}\right)$ and $B_{\check{x}_{j,\alpha}}\left(\frac{1}{4}\right)$ are also disjoint.

At this point, we are finally able to prove Theorem 10, which we stated at the very beginning of this section. We define two possible types of concentration points, according to how \check{u}_α explodes. We prove that, within a cluster, we can only find one type or the other, but never both. Finally, we see that the existence of either type leads to contradictions, which implies that \check{u}_α admits no concentration points whatsoever.

Proof of Theorem 10: Consider the cluster around $(x_{1,\alpha})_\alpha$, for some $R > 0$. There are two possible cases. The first type of concentration point corresponds to

$$\sup_{B_{\check{x}_{i,\alpha}}\left(\frac{1}{2}\right)} \left(\check{u}_\alpha(x) + \left| \frac{\nabla \check{u}_\alpha(x)}{\check{u}_\alpha(x)} \right|^{\frac{n-2}{2}} \right) = O(1). \quad (2.3.94)$$

In this case, note that $(\check{u}_\alpha)_\alpha$ is uniformly bounded in \mathcal{C}_{loc}^1 . Moreover, we find a lower bound, as by (2.3.9) from Lemma 3,

$$|\check{x}_{i,\alpha}|^{\frac{n-2}{2}} \check{u}_\alpha(\check{x}_{i,\alpha}) \geq 1.$$

There exists $\delta_i > 0$ such that

$$\inf_{B_{\check{x}_{i,\alpha}}(\delta_i)} \check{u}_\alpha \geq \frac{1}{2} |\check{x}_{i,\alpha}|^{1-\frac{n}{2}},$$

which leads to the existence of $\delta_0 > 0$ where

$$\inf_{B_{\tilde{x}_i, \alpha}(\delta_0)} \check{u}_\alpha \geq \frac{1}{2}. \quad (2.3.95)$$

The second type is defined by

$$\sup_{B_{\tilde{x}_i, \alpha}(\frac{1}{2})} \left(\check{u}_\alpha(x) + \left| \frac{\nabla \check{u}_\alpha(x)}{\check{u}_\alpha(x)} \right|^{\frac{n-2}{2}} \right) \rightarrow \infty. \quad (2.3.96)$$

In this case, either

$$\sup_{B_{\tilde{x}_i, \alpha}(\frac{1}{2})} \check{u}_\alpha(x) \leq M \quad \text{and} \quad \sup_{B_{\tilde{x}_i, \alpha}(\frac{1}{2})} |\nabla \check{u}_\alpha(x)| \rightarrow \infty, \quad (2.3.97)$$

or

$$\sup_{B_{\tilde{x}_i, \alpha}(\frac{1}{2})} \check{u}_\alpha(x) \rightarrow \infty. \quad (2.3.98)$$

We show (2.3.97) is not actually possible. Assume it holds true. Then there exist $(\check{x}_\alpha)_\alpha \subset (B_{\tilde{x}_i, \alpha}(\frac{1}{2}))_\alpha$ and $(\check{\nu}_\alpha)_\alpha$ such that

$$\check{\nu}_\alpha^{1-\frac{n}{2}} := \check{u}_\alpha(\check{x}_\alpha) + \left| \frac{\nabla \check{u}_\alpha(\check{x}_\alpha)}{\check{u}_\alpha(\check{x}_\alpha)} \right|^{\frac{n-2}{2}} = \sup_{x \in B_{\tilde{x}_i, \alpha}(\frac{1}{2})} \left(\check{u}_\alpha(x) + \left| \frac{\nabla \check{u}_\alpha(x)}{\check{u}_\alpha(x)} \right|^{\frac{n-2}{2}} \right)$$

with

$$\check{\nu}_\alpha \rightarrow 0. \quad (2.3.99)$$

We define the rescaled quantities

$$\check{v}_\alpha(x) := \check{\nu}_\alpha^{\frac{n-2}{2}} \check{u}_\alpha(\exp_{\check{x}_\alpha}(\check{\nu}_\alpha x)) \quad \text{and} \quad \check{h}_\alpha(x) := (\exp_{\check{x}_\alpha}^* \check{g})(\check{\nu}_\alpha x)$$

respectively, defined in $\Omega_\alpha := B_0(\frac{1}{2\check{\nu}_\alpha})$. For any $R > 0$ and α large enough so that $R < \frac{1}{2\check{\nu}_\alpha}$,

$$\limsup_{\alpha \rightarrow \infty} \sup_{B_0(R)} \left(\check{v}_\alpha + \left| \frac{\nabla \check{v}_\alpha}{\check{v}_\alpha} \right| \right) = 1. \quad (2.3.100)$$

Thus

$$|\nabla \ln \check{v}_\alpha| \leq 1$$

and

$$\check{v}_\alpha(0)e^{-x} \leq \check{v}_\alpha(x) \leq \check{v}_\alpha(0)e^x. \quad (2.3.101)$$

Note that the metrics $\check{h}_\alpha \rightarrow \xi$ in \mathcal{C}_{loc}^2 as $\alpha \rightarrow \infty$. Assume that, up to a subsequence, $u_\alpha(\check{x}_\alpha) \rightarrow l < \infty$. We also deduce that $\check{v}_\alpha(0) \rightarrow 0$. Let $\check{x}_0 := \lim_{\alpha \rightarrow \infty} \check{x}_\alpha$ and let us denote

$$\check{w}_\alpha(x) := \frac{\check{v}_\alpha(x)}{\check{v}_\alpha(0)}.$$

These functions are bounded from below,

$$\check{w}_\alpha(x) \geq \frac{\varepsilon}{l} + o(1) > 0. \quad (2.3.102)$$

Moreover,

$$\check{w}_\alpha(x) \leq e^{|x|}.$$

By standard elliptic theory, we find that there exists $\check{w} := \lim_{\alpha \rightarrow \infty} \check{w}_\alpha$ in \mathcal{C}^1 solving:

$$\Delta \check{w} = -\frac{1}{l^{q+2}} \frac{\langle \nabla \check{w}, Y(\check{x}_0) \rangle^2}{\check{w}^{q+3}}.$$

Note that

$$\Delta \check{w}^{-\alpha} \leq \alpha \frac{|\nabla \check{w}|^2}{\check{w}^{\alpha+2}} \left[\frac{\|Y(\check{x}_0)\|_{L^\infty}^2}{\varepsilon^{q+2}} - (\alpha + 1) \right],$$

so $\check{w}^{-\alpha}$ is subharmonic for α large, and so Lemma 12 (see the Annex) implies that \check{w} is constant, which in turn implies that $\check{U} = 0$ and $\nabla \check{U} = 0$, which is false (see proof of Lemma 4). Therefore, the second subcase cannot be true. This essentially means that when a concentration point is of the second type, then

$$\sup_{B_{\check{x}_i, \alpha}(\frac{1}{2})} \check{u}_\alpha(x) \rightarrow \infty$$

and so

$$\check{u}_\alpha(\check{x}_{i, \alpha}) \rightarrow \infty.$$

Let us denote $\check{x}_i := \lim_{\alpha \rightarrow \infty} \check{x}_{i, \alpha}$ up to a subsequence. According to Proposition 1,

$$\check{u}_\alpha(\check{x}_{i, \alpha}) \check{u}_\alpha(x) \mapsto \frac{\lambda_i}{|x - \check{x}_i|^{n-2}} + H_i(x) \quad (2.3.103)$$

in \mathcal{C}^1 in $B_{\check{x}_i}(\frac{1}{2}) \setminus \{\check{x}_i\}$, with $\lambda_i > 0$, where H_i is a harmonic function in $B_{\check{x}_i, \alpha}(\frac{1}{2})$, $H(\check{x}_i) = 0$.

Let U be a connected open set of \mathbb{R}^n , $U_R \subset B_0(R+1)$, containing no other point of the cluster apart from \check{x}_i and \check{x}_j . For any $0 < r < \frac{1}{8}$, we set

$$V_{r, R} = U_R \setminus \left(\overline{B_{\check{x}_i}(r)} \cup \overline{B_{\check{x}_j}(r)} \right).$$

For a fixed $x \in B_{\check{x}_i}(\frac{1}{4}) \setminus V_{r, R}$, (2.3.103) implies that $\check{u}_\alpha(x) \rightarrow 0$ as $\alpha \rightarrow \infty$. It follows from Lemma 10 and (2.3.95) that all points of a cluster must be of the same type.

Assuming all points in the cluster are of the first type, then

$$\check{u}_\alpha(0) + \left| \frac{\nabla \check{u}_\alpha(0)}{\check{u}_\alpha(0)} \right|^{\frac{n-2}{2}} = O(1),$$

then by standard elliptic theory there exists $\check{u} := \lim_{\alpha \rightarrow \infty} \check{u}_\alpha$ in $\mathcal{C}^1(B_0(R))$, $R > 0$. Repeating the reasoning of Lemma 3 or Lemma 6, we know that

$$\Delta_\xi \check{u} = f(x_1) \check{u}^{q-1}.$$

However, \check{u} must have at least two separate maxima, at 0 and \check{x}_2 , which leads to a contradiction by the classification result of Caffarelli, Gidas and Spruck [CGS89].

Therefore $\check{u}_\alpha(\check{x}_{i, \alpha}) \rightarrow \infty$, for any $i = \overline{1, M_{2R, \alpha}}$. Up to a subsequence

$$\frac{\check{u}_\alpha(\check{x}_{i, \alpha})}{\check{u}_\alpha(0)} \rightarrow \mu_i > 0 \quad \text{as } \alpha \rightarrow \infty.$$

We fix $R > 0$ and assume, without loss of generality, that $(M_{2R, \alpha})_\alpha$ is a constant denoted by M_{2R} . Using Lemma 10 and standard elliptic theory, we pass to a subsequence and get

$$\check{u}_\alpha(0) \check{u}_\alpha(x) \rightarrow \check{G}(x)$$

in $\mathcal{C}_{loc}^1\left(B_0(R)\setminus\{\check{x}_i\}_{i=1,N_{2R}}\right)$ for $\alpha \rightarrow \infty$, with

$$\begin{aligned}\check{G}(x) &= \sum_{i=1}^p \frac{\lambda_i}{\mu_i|x-\check{x}_i|^{n-2}} + \check{H}(x) \\ &= \frac{\lambda_1}{|x|^{n-2}} + \left(\sum_{i=2}^p \frac{\lambda_i}{\mu_i|x-\check{x}_i|^{n-2}} + \check{H}(x)\right)\end{aligned}$$

Here, \check{H} is harmonic on $B_0(R)$, and $2 \leq p \leq M_{2R}$ such that $|\check{x}_p| \leq R$ as $|\check{x}_{p+1}| > R$. If we apply Proposition 1 to the blow-up sequence $x_\alpha = x_{1,\alpha}$ with $\rho_\alpha = \frac{1}{16}d_\alpha$, we obtain

$$\hat{H}(0) := \sum_{i=2}^p \frac{\lambda_i}{\mu_i|\check{x}_i|^{n-2}} + \check{H}(0) = 0$$

Since $\hat{H}(x) - \frac{\lambda_2}{\mu_2|x-\check{x}_2|} = \check{G}(x) - \frac{\lambda_1}{|x|^{n-2}} - \frac{\lambda_2}{\mu_2|x-\check{x}_2|^{n-2}}$ is harmonic in the ball $B_0(R)\setminus\{\check{x}_i\}_{i \in \{2, N_{2R}\}}$ and $\check{G} \geq 0$, then as a consequence of the maximum principle, by considering a minimum on $\partial B_0(R)$, we see that

$$\hat{H}(0) \geq \frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{R^{n-2}} - \frac{\lambda_2}{\mu_2(R-1)^{n-2}}.$$

Choosing $R > 0$ large enough, we ensure that $\hat{H} > 0$, which contradicts Theorem 9. Consequently, u_α admits no concentration points and is therefore uniformly bounded in \mathcal{C}^1 . \square

Lemma 11. *Assuming equation (2.2.1) associated to \tilde{a} admits a supersolution and that $\Delta_g + h$ is coercive, then for any $0 < T < \inf_M \tilde{a}$ and any equation with parameters in $\mathcal{E}_{\theta,T}$ (as in Theorem 10), there exists a constant $C_{\theta,T} = C(n, \theta, T) > 0$ such that, for any $\|Y\|_{L^\infty} \leq C_{\theta,T}$ and $\|b\|_{L^\infty} \leq C_{\theta,T}$, we may find a smallest real eigenvalue $\lambda_0 > 0$, where λ_0 is as in Lemma 2.*

Proof. Given any parameters (f, a, b, c, d, h, Y) in $\mathcal{E}_{\theta,T}$ and additionally asking for Y and b to be sufficiently small in L^∞ norm with respect to θ and T , we aim to prove that minimal solutions to the Lichnerowicz-type equation change continuously with their parameters. In order to do this, we study the sign of the smallest real eigenvalue associated to the linearisation around a minimal solution and show that it is positive by comparing it to the smallest real eigenvalue at $b = 0$ and $Y = 0$. Indeed, let $s > 0$ a real number and E_s the equation

$$\begin{aligned}E_s(u_s) := \Delta_g u_s + h u_s - f u_s^{q-1} - \frac{a}{u_s^{q+1}} + \frac{sb}{u_s} + c \langle \nabla u_s, sY \rangle \left(\frac{d}{u_s^2} + \frac{1}{u_s^{q+2}} \right) \\ + \frac{\langle \nabla u_s, sY \rangle^2}{u_s^{q+3}} = 0,\end{aligned}\tag{2.3.104}$$

with u_s its minimal solution. Let L_s be the linearisation of E_s around u_s ,

$$\begin{aligned}\Delta_g \varphi_s + \left[h - (q-1) f u_s^{q-2} + (q+1) \frac{a}{u_s^{q+2}} - \frac{sb}{u_s} - c \langle \nabla u_s, sY \rangle \left(\frac{2d}{u_s^3} + \frac{q+2}{u_s^{q+3}} \right) \right. \\ \left. - (q+3) \frac{\langle \nabla u_s, sY \rangle^2}{u_s^{q+4}} \right] \varphi_s + \langle \nabla \varphi_s, sY \rangle \left[c \left(\frac{d}{u_s^2} + \frac{1}{u_s^{q+2}} \right) + \frac{2 \langle \nabla u_s, sY \rangle}{u_s^{q+3}} \right] = \lambda_s \varphi_s,\end{aligned}\tag{2.3.105}$$

with $\lambda_s \geq 0$ the smallest real eigenvalue, $\varphi_s > 0$ the associated eigenfunction, normalised such that $\|\varphi_s\|_{L^2} = 1$. Note that the linear equations L_s are stable, in the sense that φ_s is *a priori* uniformly bounded in \mathcal{C}^1 . This follows from the fact that the u_s is uniformly bounded. We may also suppose that λ_s is uniformly bounded, because if $\lambda_s \rightarrow \infty$, then it is clear that $\lambda_s > 0$.

As Premoselli proved by way of a variational argument [Pre14], the equation E_0 is strictly stable, in the sense that its corresponding smallest real eigenvalue is positive. It uses the coerciveness of $\Delta_g + h$. We emphasize that his argument makes use of the fact that E_0 is symmetric, which is not the case for our more general equations. The strict stability implies continuity, *i.e.* that $u_s \rightarrow u_0$, with u_0 the minimal value. Indeed, let $u_s \rightarrow \tilde{u}$ another solution of E_0 . Clearly, $\tilde{u} > u_0$. Let $\tilde{u}_\delta = u_0 + \delta\varphi_0$. Note that

$$\begin{aligned} E_s(\tilde{u}_\delta) &= E_0(\tilde{u}_\delta) + \frac{sb}{u_\delta} + c\langle \nabla u_\delta, sY \rangle \left(\frac{d}{u_\delta^2} + \frac{1}{u_\delta^{q+2}} \right) + \frac{\langle \nabla u_\delta, sY \rangle^2}{u_\delta^{q+3}} \\ &= E_0(u_0) + \lambda_0\delta\varphi_0 + o(\delta) + \frac{sb}{u_\delta} + sc\langle \nabla u_\delta, Y \rangle \left(\frac{d}{u_\delta^2} + \frac{1}{u_\delta^{q+2}} \right) \\ &\quad + s^2 \frac{\langle \nabla u_\delta, Y \rangle^2}{u_\delta^{q+3}} \end{aligned}$$

If we fix $\delta > 0$ sufficiently small, the error terms $|o(\delta)| \leq \frac{\lambda_0\delta\varphi_0}{3}$ and $\tilde{u}_\delta < \tilde{u} \leq u_s$, $\forall s$. Then, by taking s sufficiently close to 0, we get that the rest of the terms are also smaller in absolute size than $\frac{\lambda_0\delta\varphi_0}{3}$. Consequently, $E_s(\tilde{u}_\delta) > 0$, so \tilde{u}_δ is a supersolution of E_s that is smaller than the minimal solution u_s .

Since $u_s \rightarrow u_0$, we also get that $\lambda_s \rightarrow \lambda_0$, so for s small, the first eigenvalue $\lambda_s > 0$. We would like to obtain that there exists $s_{\theta,T} > 0$ such that, for any $0 \leq s < s_{\theta,T}$, the minimal eigenvalue corresponding to L_s is positive, where $(a, b, c, d, f, h, Y) \in \mathcal{E}_{\theta,T}$. In other words, we attempt to set a size for Y and b , depending on θ and T (and n), such that the resulting equations are strictly stable.

First, there exists $\delta_{\theta,T} > 0$ such that if $Y = 0$, $b = 0$ and the equation's parameters are found in $\mathcal{E}_{\theta,T}$, then $\lambda_0 > \delta_{\theta,T}$. We let $u_s = u_0 + \varepsilon_s v_s$ such that $\|v_s\|_{L^2} = 1$, $\varepsilon_s \in \mathbb{R}$. Note that $\varepsilon_s \rightarrow 0$ as $s \rightarrow 0$.

We begin by analyzing the difference in size between ε_s and s , or equivalently between $\|u_s - u_0\|_{L^\infty}$ and s . Let

$$E_s = E_0 + sM_s,$$

where

$$M_s(u_s) = \frac{b}{u_s} + c\langle \nabla u_s, Y \rangle \left(\frac{d}{u_s^2} + \frac{1}{u_s^{q+2}} \right) + s \frac{\langle \nabla u_s, Y \rangle^2}{u_s^{q+3}}.$$

Recall that

$$E_0(u_s) = -sM_s(u_s) \tag{2.3.106}$$

where

$$E_0(u_s) = L_0(u_s - u_0) + O(\|u_s - u_0\|^2).$$

Since u_0 is a solution of E_0 and the operator L_0 is coercive, with minimal eigenvalue λ_0 , then by testing (2.3.106) against $(u_s - u_0)$, we see that

$$\lambda_0(1 + o(1))\|u_s - u_0\|_{L^2}^2 \leq -s \int_M M_s(u_s)(u_s - u_0) \leq s\|M_s(u_s)\|_{L^2}\|u_s - u_0\|_{L^2}.$$

The size of $M_s(u_s)$ is determined by a constant depending on θ and T . Therefore, we may write

$$(1 + o(1))\varepsilon_s = (1 + o(1))\|u_s - u_0\|_{L^2} \leq s \frac{C}{\lambda_0}$$

Finally, in order to compare λ_s to λ_0 , extract the terms of order s from the quantity

$$\begin{aligned}
& \int_M \varphi_0 L_s(\varphi_s) - \varphi_s L_0(\varphi_0), \\
& - \int_M \frac{sb}{u_0} \varphi_0 \varphi_s - \int_M c \langle \nabla u_0, sY \rangle \left(\frac{2d}{u_0^3} + \frac{q+3}{u_0^{q+3}} \right) \varphi_s \varphi_0 - \int_M (q+3) \frac{\langle \nabla u_s, sY \rangle^2}{u_s^{q+4}} \varphi_s \varphi_0 \\
& \quad \varepsilon_s \int_M \left[(q-1)(q-2) f u_0^{q-3} - (q+1)(q+2) \frac{a}{u_0^{q+3}} \right] v_s \varphi_s \varphi_0 \\
& \quad + O(s^2) = (\lambda_\theta - \lambda_0) \int_M \varphi_s \varphi_0,
\end{aligned} \tag{2.3.107}$$

so there exists a constant C depending on θ and T such that

$$\lambda_s - \lambda_0 \leq sC \left(\int_M \varphi_s \varphi_0 \right)^{-1}. \tag{2.3.108}$$

As $\varphi_s = \varphi_0 + o(1)$ and the L^2 norm of φ_0 is 1, we may choose s small enough so that $|\lambda_s - \lambda_0| \geq \frac{\delta_{\theta, T}}{2}$, and thus $\lambda_s > 0$. \square

2.4 Existence of solutions to the system

2.4.1 The proof of the main theorem

The following is a useful estimate we can find in [IÓM04]; it plays a crucial role in ensuring the necessary compactness of the sequence W_α in the main theorem.

Proposition 2. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ such that g has no conformal Killing fields. Let X be a smooth vector field in M . Then there exists a unique solution W of*

$$\Delta_{g, \text{conf}} W = X.$$

Also, for $0 < \gamma < 1$, there exists a constant $C_0 > 0$ that depends only on n and g such that

$$\|W\|_{C^{1, \gamma}} \leq C_0 \|X\|_\infty.$$

Remark 12. *As a consequence, there exists a constant $C_1 = C_1(n, g)$ such that*

$$\|\mathcal{L}_g W\|_{C^{0, \gamma}} \leq C_1 \|X\|_\infty \tag{2.4.1}$$

Let (M, g) be a closed Riemannian manifold of dimension $n \in \{3, 4, 5\}$ such that g has no conformal Killing fields. Let $b, c, d, f, h, \rho_1, \rho_2, \rho_3$ be smooth functions on M and let Y and Ψ be smooth vector fields defined on M . Let $0 < \gamma < 1$.

Assume that $\Delta_g + h$ is coercive. Assume that $f > 0, \rho_1 > 0$ and $|\nabla \rho_3| < (2C_1)^{-1}$, where C_1 is defined in (2.4.1).

Consider the coupled system

$$\begin{cases}
\Delta_g u + hu & = f u^{q-1} + \frac{\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W|_g^2}{u^{q+1}} \\
-\frac{b}{u} - c \langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) - \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}} \\
\text{div}_g (\rho_3 \mathcal{L}_g W) & = \mathcal{R}(u),
\end{cases} \tag{2.4.2}$$

where \mathcal{R} is an operator verifying

$$\mathcal{R}(u) \leq C_{\mathcal{R}} \left(1 + \frac{\|u\|_{C^2}^2}{(\inf_M u)^2} \right) \tag{2.4.3}$$

for a constant $C_{\mathcal{R}} > 0$.

We fix

$$\theta = \min(\inf_M \rho_1, \inf_M f), \quad (2.4.4)$$

and

$$T = \max(\|f\|_{\mathcal{C}^{1,\eta}}, \|\rho_1\|_{\mathcal{C}^{0,\gamma}}, \|c\|_{\mathcal{C}^{0,\gamma}}, \|d\|_{\mathcal{C}^{0,\gamma}}, \|h\|_{\mathcal{C}^{0,\gamma}}). \quad (2.4.5)$$

Let

$$M = \ln S_{\theta, 2T}, \quad (2.4.6)$$

with $S_{\theta, 2T}$ a constant as in Theorem 10. The following theorem is the main result of the present paper.

Theorem 11. *Assume there exists a smooth positive function \tilde{a} for which*

$$\Delta_g \tilde{u} + h\tilde{u} = f\tilde{u}^{q-1} + \frac{\tilde{a}}{\tilde{u}^{q+1}} \quad (2.4.7)$$

admits a positive supersolution \tilde{u} . Assume that $\rho_1 < \tilde{a}$ and let $\omega = \inf_M(\tilde{a} - \rho_1)$. Then there exists

$$\delta = \delta(\omega, \theta, T) \quad (2.4.8)$$

such that if

$$\|b\|_{\mathcal{C}^{0,\gamma}} + \|Y\|_{\mathcal{C}^{0,\gamma}} + \|\Psi\|_{\mathcal{C}^{0,\gamma}} + \|\rho_2\|_{\mathcal{C}^{0,\gamma}} + C_{\mathcal{R}} \leq \delta, \quad (2.4.9)$$

the system (2.4.2) admits a solution (u, W) , with u a smooth positive function and W a smooth vector field.

Remark 13. *We can use a result by Hebey, Pacard and Pollack ([HPP08], Corollary 3.1) in order to ensure the existence of a supersolution \tilde{u} . There exists a constant $C = C(n, h)$, $C > 0$ such that if \tilde{a} is a smooth positive function verifying*

$$\|\tilde{a}\|_{L^1(M)} \leq C(n, h) \left(\max_M |f| \right)^{1-n}, \quad (2.4.10)$$

then (2.4.7) accepts a smooth positive solution.

Proof of Theorem 11: The proof of the theorem consists of a fixed-point argument. Formally, we define the operator

$$\Phi : \varphi \rightarrow \ln u(\mathcal{L}_g W(e^\varphi)),$$

where $W(e^\varphi)$ solves the second equation of (2.4.2) for a fixed $u = e^\varphi$ and where $u(\mathcal{L}_g W(e^\varphi))$ is the solution of the scalar equation of (2.4.2) constructed in Section 2 for a fixed $W(e^\varphi)$. In order to apply Schauder's fixed point theorem, we show that $\Phi : B_M \rightarrow B_M$, $B_M := \{\varphi \in \mathcal{C}^2(M), \|\varphi\|_{\mathcal{C}^2} \leq M\}$, where M is defined as in (2.4.6), and that Φ is continuous and compact.

We first want to prove that

$$\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2 < \tilde{a} \quad (2.4.11)$$

to ensure that $\Phi(\varphi)$ is well defined, with \tilde{u} from (2.4.7) a supersolution. By (2.4.1), we have

$$\|\mathcal{L}_g W(e^\varphi)\|_{\mathcal{C}^{0,\gamma}} \leq C_1 (\|\nabla \rho_3\|_{L^\infty} \|\mathcal{L}_g W(e^\varphi)\|_{L^\infty} + \|\mathcal{R}(e^\varphi)\|_{L^\infty}) \quad (2.4.12)$$

and thanks to (2.4.3) we see that

$$\begin{aligned} \rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2 &\leq \rho_1 + 2\|\Psi\|_{L^\infty}^2 \\ &+ 2 \left(\frac{C_1 C_{\mathcal{R}} \|\rho_2\|_{L^\infty}}{1 - C_1 \|\nabla \rho_3\|_{L^\infty}} \right)^2 \left(1 + \frac{M^2 e^{2M}}{e^{2\varepsilon}} \right)^2, \end{aligned} \quad (2.4.13)$$

where ε is the lower bound of any solution corresponding to $\mathcal{E}_{\theta, 2T}$ from Theorem 10. There exists

$$\delta_1 = \delta_1(\omega, \theta, T) \quad (2.4.14)$$

such that if

$$\|\Psi\|_{C^{0,\gamma}} + \|\rho_2\|_{C^{0,\gamma}} + C_{\mathcal{R}} \leq \delta_1, \quad (2.4.15)$$

then (2.4.11) holds.

In order to use the *a priori* estimate of Section 3 to see that $\Phi : B_M \rightarrow B_M$, we need to prove that

$$\theta \leq \rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2 \quad \text{and} \quad \|\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2\|_{C^{0,\gamma}} \leq 2T. \quad (2.4.16)$$

From (2.4.4) we deduce that

$$\theta \leq \rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2. \quad (2.4.17)$$

and thanks to (2.4.12) we see that

$$\begin{aligned} \|\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2\|_{C^{0,\gamma}} &\leq \|\rho_1\|_{C^{0,\gamma}} + 2\|\Psi\|_{C^{0,\gamma}}^2 \\ &+ 2 \left(\frac{C_1 C_{\mathcal{R}} \|\rho_2\|_{C^{0,\gamma}}}{1 - C_1 \|\nabla \rho_3\|_{L^\infty}} \right)^2 \left(1 + \frac{M^2 e^{2M}}{e^{2\varepsilon}} \right)^2. \end{aligned} \quad (2.4.18)$$

There exists

$$\delta_2 = \delta_2(\omega, \theta, T) \quad (2.4.19)$$

such that if

$$\|\Psi\|_{C^{0,\gamma}} + \|\rho_2\|_{C^{0,\gamma}} + C_{\mathcal{R}} \leq \delta_2, \quad (2.4.20)$$

then

$$\|\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2\|_{C^{0,\gamma}} \leq 2T. \quad (2.4.21)$$

Thanks to (2.4.17) and (2.4.21), the *a priori* estimates in Section 3 imply that

$$\|u(\mathcal{L}_g W(e^\varphi))\|_{C^2} \leq S_{\theta, 2T},$$

so

$$\Phi(\varphi) \leq M,$$

where M is as in (2.4.6). We have thus proved that Φ is well-defined and that $\Phi : B_M \rightarrow B_M$.

In order to show that Φ is continuous, we want to check that it holds true for $a \mapsto u(a)$, where $u(a)$ is the minimal solution constructed in Section 2. For all $a < \tilde{a}$, we've established monotony, which ensures that the minimal solutions exist. For $t > 0$ small, let us denote by u_t the solutions corresponding to $a(1+t) < \tilde{a}$. Let u_0 be the limit of u_t as $t \rightarrow 0$; it is also a solution of the Lichnerowicz-type equation associated to a . If $u_0 \neq u$, then $u < u_0$. According to Section 3, there exists $C_{\theta, 2T}$ such that

$$\|b\|_{C^{0,\gamma}} + \|Y\|_{C^{0,\gamma}} \leq C_{\theta, 2T} \quad (2.4.22)$$

implies that u is strictly stable. We ask that

$$\delta \leq \min(\delta_1, \delta_2, C_{\theta, 2T}) \quad (2.4.23)$$

where δ_1 is defined in (2.4.14) and δ_2 is defined in (2.4.19). We choose $\mu > 0$ small enough such that $u < \hat{u}_\mu < u_0$, where $\hat{u}_\mu := u + \mu\psi$, ψ a positive eigenfunction at u corresponding to the smallest real eigenvalue. But \hat{u}_μ is a supersolution for $a(1 + \epsilon)$, $\epsilon > 0$ small, which contradicts the monotonicity. Therefore, Φ is continuous.

Lastly, B_M being a closed convex set in \mathcal{C}^2 , it remains to show that $\Phi(B_M)$ is compact to conclude. From the previous discussion, $\Phi(B_M) \subset B_M$, and is thus bounded in \mathcal{C}^2 . By standard elliptic theory, we conclude the proof of Theorem 8. \square

2.4.2 The case of a metric with conformal Killing fields

Let us consider the case of a metric g with non-trivial conformal Killing fields associated to it. For \tilde{V} a representative of the drift, the equation

$$\operatorname{div}_{\tilde{g}} \left(\frac{\tilde{N}}{2} \mathcal{L}_g W \right) = \frac{n-1}{n} u^q \mathbf{d} \left(u^{-2q} \tilde{N} \operatorname{div}_g(u^q \tilde{V}) \right) + \pi \nabla \psi \quad (2.4.24)$$

admits a solution W if and only if

$$\frac{n-1}{n} \int_M u^{-2q} \tilde{N} \operatorname{div}_g(u^q \tilde{V}) \operatorname{div}_g(u^q P) = \int_M \langle \pi \nabla \psi, P \rangle \quad (2.4.25)$$

for all P conformal Killing fields. Moreover, the solution W is unique up to the addition of a conformal Killing field. Note that the drift is defined modulo conformal Killing fields, so \tilde{V} and $\tilde{V} + P$ are representatives of the same drift for all P conformal Killing fields. We claim that given a vector field \tilde{V} there exists a conformal Killing field \tilde{Q} which is unique up to a true Killing field and such that

$$\frac{n-1}{n} \int_M u^{-2q} \tilde{N} \operatorname{div}_g \left(u^q (\tilde{V} + \tilde{Q}) \right) \operatorname{div}_g(u^q P) = \int_M \langle \pi \nabla \psi, P \rangle. \quad (2.4.26)$$

By analyzing the homogeneous operator associated to the equation above,

$$\int_M u^{-2q} \tilde{N} \operatorname{div}_g \left(u^q (\tilde{V} + \tilde{Q}') \right) \operatorname{div}_g(u^q P) = 0, \quad (2.4.27)$$

we check that it is positive definite, thus invertible. Consider the functional

$$F(P) = \int_M u^{-2q} \tilde{N} \operatorname{div}_g \left(u^q (\tilde{V} + P) \right)^2 dv_g \quad (2.4.28)$$

on the finite-dimensional space of conformal Killing fields and note that \tilde{Q}' is stationary for F . Since F is quadratic and non-negative definite, stationary points are associated to minimizers. If \tilde{g} does not admit any nontrivial true Killing fields, then every conformal Killing field P satisfies $\operatorname{div} P \neq 0$ and the quadratic term of F is positive definite. On the other hand, if g admits proper Killing fields, then F descends to a functional on the quotient space and its quadratic order term is again positive definite. So the minimum of F is unique up to a true Killing field.

The conformal system proposed by Maxwell [Max14b] becomes in this framework

$$\begin{cases} \Delta_g u + \frac{n-2}{4(n-1)}(R(g) - |\nabla\psi|_g^2)u - \frac{(n-2)}{4(n-1)} \frac{|U + \mathcal{L}_g W|^2 + \pi^2}{u^{q+1}} \\ - \frac{n-2}{4(n-1)} \left[2V(\psi) - \frac{n-1}{n} \left(\tau^* + \frac{\tilde{N} \operatorname{div}_g(u^q(\tilde{V} + \tilde{Q}))}{u^{2q}} \right) \right]^2 u^{q-1} = 0 \\ \operatorname{div}_g \left(\frac{\tilde{N}}{2} \mathcal{L}_g W \right) = \frac{n-1}{n} u^q \mathbf{d} \left(\frac{\tilde{N} \operatorname{div}_g(u^q(\tilde{V} + \tilde{Q}))}{2u^{2q}} \right) + \pi \nabla \psi, \end{cases} \quad (2.4.29)$$

whose solution (u, W, \tilde{Q}) is a smooth positive function u , a smooth vector field W , defined up to a conformal Killing field, and \tilde{Q} a conformal Killing field defined up to a true Killing field.

The existence of solutions to (2.4.29) follows from Theorem 8 and is similar to Corollary 1, with slight modifications. Here,

$$\begin{aligned} \rho_1 &= \frac{n-2}{4(n-1)} \left[\pi - \frac{n-1}{n} \tilde{N}^2 \operatorname{div}_g(\tilde{Q} + \tilde{V}) \right], \quad b = -\tau^* \tilde{N} \operatorname{div}_g(\tilde{V} + \tilde{Q}), \\ Y &= \sqrt{\frac{n}{n-2}} \tilde{N}(\tilde{V} + \tilde{Q}) \end{aligned} \quad (2.4.30)$$

and

$$C_{\mathcal{R}} = C_{\mathcal{R}}(\|\tilde{Q}\|_{C^2}). \quad (2.4.31)$$

Moreover, we define θ and T as in (2.1.16) and (2.1.17) respectively, but without the dependency on $\rho_1 = \rho_1(\tilde{Q})$, *i.e.*

$$\theta = \min(\inf_M f), \quad (2.4.32)$$

and

$$T = \max(\|f\|_{C^{1,\eta}}, \|c\|_{C^{0,\gamma}}, \|d\|_{C^{0,\gamma}}, \|h\|_{C^{0,\gamma}}). \quad (2.4.33)$$

First of all, the stability of the first equation still holds, as in Lemma 2 and Lemma 11. In order to apply the last theorem, we need to check that: $\rho_1(\tilde{Q}) > \theta$, $\|\rho_1(\tilde{Q})\|_{C^{0,\gamma}} < 2T$, $\|b\|_{C^{0,\gamma}} \leq C_{\theta,2T}$ and $\|Y\|_{C^{0,\gamma}} \leq C_{\theta,2T}$. This translates to

$$\operatorname{div}_g \tilde{Q} < \left(\frac{n-2}{4(n-1)} \pi - \theta \right) \frac{n}{n-1} \tilde{N}^{-2} - \operatorname{div}_g \tilde{V}, \quad (2.4.34)$$

$$\frac{n-2}{4(n-1)} \|\pi\|_{C^{0,\gamma}} + \frac{n-1}{n} \|\tilde{N} \operatorname{div}_g(\tilde{V} + \tilde{Q})\|_{C^{0,\gamma}} \leq 2T, \quad (2.4.35)$$

$$\|\tau^* \tilde{N} \operatorname{div}_g(\tilde{V} + \tilde{Q})\|_{C^{0,\gamma}} \leq C_{\theta,2T}, \quad (2.4.36)$$

and

$$\sqrt{\frac{n}{n-2}} \|\tilde{N}(\tilde{V} + \tilde{Q})\| \leq C_{\theta,2T}. \quad (2.4.37)$$

We find bounds on \tilde{Q} depending on $\pi, \psi, \tilde{N}, \tilde{V}$ from (2.4.26), thereby proving the necessary compactness. Finally, the continuity $(a, b, Y) \rightarrow u_{a,b,Y}$ doesn't pose any problem, and the proof mirrors our previous argument for the continuity of $a \rightarrow u(a)$. This shows the existence of solutions (u, W, \tilde{Q}) .

2.5 Annex

We used the following result repeatedly throughout the paper.

Lemma 12. *Let u be a bounded subharmonic function defined on \mathbb{R}^n . If there exists $0 < \varepsilon \leq u$ which bounds u from below and $\alpha > 0$ such that $u^{-\alpha}$ is a subharmonic function, then u is a constant.*

Proof of Lemma 12: Let us denote

$$\bar{u}_x(R) := \frac{1}{\omega_{n-1}R^{n-1}} \int_{\partial B_x(R)} u(y) dy$$

the average of a smooth function u over the sphere $\partial B_x(R)$. We will sometimes use the simplified notation $\bar{u}(R)$. Recall that, given any subharmonic function u , $x \in \mathbb{R}^n$ and for any two radii $R \leq \tilde{R}$, then

$$\bar{u}_x(R) \leq \bar{u}_x(\tilde{R}). \quad (2.5.1)$$

This follows from

$$r^{n-1} \bar{u}'(r) = \frac{1}{\omega_{n-1}} \int_{\partial B_x(r)} \partial_\nu u(y) dy = -\frac{1}{\omega_{n-1}} \int_{B_x(r)} \Delta u(y) dy \geq 0$$

where $r > 0$ and ν is the exterior normal.

Note that $u^{-\alpha} \leq \varepsilon^{-\alpha}$ implies that the average of $u^{-\alpha}$ on arbitrary subsets is uniformly bounded. Let us fix $x \in \mathbb{R}^n$. Since $u^{-\alpha}$ is bounded, there exists a constant $M > 0$ and a sequence of radii $R_i \rightarrow \infty$ as $i \rightarrow \infty$ such that

$$M^{-\alpha} := \lim_{i \rightarrow \infty} \overline{u^{-\alpha}}_x(R_i). \quad (2.5.2)$$

In fact, because the averages are increasing (2.5.1), any sequence $R \rightarrow \infty$ around any point in \mathbb{R}^n leads to the same limit M , since one may always find a subsequence of R_i such that $B_x(R_i)$ includes the new sequence.

As $u^{-\alpha}$ is subharmonic,

$$u^{-\alpha}(x) \leq \overline{u^{-\alpha}}_x(R)$$

and therefore $u^{-\alpha}(x) \leq M^{-\alpha}$, or equivalently

$$M \leq u(x). \quad (2.5.3)$$

For $z \in \mathbb{R}^n$, let $R := |z - x|$ and $\tilde{R} > R$. By Green's representation theorem, we get

$$\begin{aligned} u(z) &\leq \int_{\partial B_x(\tilde{R})} u(y) \frac{\tilde{R}^2 - R^2}{\omega_{n-1} \tilde{R} |z - y|^n} dy \\ &\leq \frac{(\tilde{R} + R) \tilde{R}^{n-2}}{(\tilde{R} - R)^{n-1}} \bar{u}_x(\tilde{R}). \end{aligned} \quad (2.5.4)$$

For $\delta > 0$, we denote

$$\Omega_{\delta,R} := \{z \in \partial B_x(R), u(z) \geq M + \delta\}$$

a subset of $\partial B_x(R)$ and let

$$\theta_{\delta,R} := \frac{|\Omega_{\delta,R}|}{|\partial B_x(R)|} \in [0, 1]$$

be the corresponding relative size of its volume. Note that $\theta_{\delta,R} \rightarrow 0$ as $R \rightarrow \infty$. Otherwise, if there exists $\varepsilon \in (0, 1]$ such that

$$\limsup_{R \rightarrow \infty} \frac{|\{z \in \partial B_x(R), u(z) \geq M + \delta\}|}{|\partial B_x(R)|} = \varepsilon$$

then

$$\limsup_{R \rightarrow \infty} \overline{u^{-\alpha}_x}(R) \leq \varepsilon(M + \delta)^{-\alpha} + (1 - \varepsilon)M^{-\alpha} < M^{-\alpha}$$

which contradicts our definition (2.5.2) of M .

By choosing R large, $\theta_{\delta,R} \leq \delta$. Let

$$\lambda_{\delta,i} := \bar{u}_x(2^i R)$$

Note that, by (2.5.4), $\lambda_{\delta,i} \leq 3 \times 2^{n-2} \lambda_{\delta,i+1}$. Since

$$u(x) \leq \lambda_{\delta,i} \leq (M + \delta)(1 - \theta_{\delta,2^i R}) + \lambda_{\delta,i+1} \times \theta_{\delta,2^i R}$$

then, by induction,

$$u(x) \leq (M + \delta) \frac{1 - \delta^l}{1 - \delta} + \lambda_l \delta^l$$

for all $l \in \mathbb{N}$. As we take $l \rightarrow \infty$,

$$u(x) \leq (M + \delta) \frac{1}{1 - \delta}$$

for any $\delta > 0$, and therefore $u(x) \leq M$. By (2.5.3), $u(x) \equiv M$.

We may apply the same argument to any other $\tilde{x} \in \mathbb{R}^n$ and obtain the same value $u(\tilde{x}) = M$. Indeed, assuming that

$$\widetilde{M}^{-\alpha} := \lim_{\tilde{R} \rightarrow \infty} \overline{u^{-\alpha}_{\tilde{x}}}(\tilde{R})$$

so that $\widetilde{M}^{-\alpha} \geq M^{-\alpha}$, then for \tilde{R} large, $\overline{u^{-\alpha}_{\tilde{x}}}(\tilde{R}) \geq M^{-\alpha}$. But, at the same time, given any fixed \tilde{R} , then for R sufficiently large, by (2.5.4), $\overline{u^{-\alpha}_{\tilde{x}}}(\tilde{R}) \leq \overline{u^{-\alpha}_x}(R)$. Thus we obtain that $u \equiv M$ in \mathbb{R}^n . \square

CHAPITRE 3

Stability

We study the stability in low dimensions of solutions to the constraint equations, where the Riemannian metric of the initial data is conformally flat, and in the presence of a scalar field.

Caterina Vălcu, *Stability paper*

Stability paper

Contents

3.1	Introduction	80
3.1.1	The main result.	82
3.2	Conformal changes of coordinates.	85
3.3	Weak pointwise estimates	86
3.4	Asymptotic analysis	92
3.4.1	Harnack inequality	92
3.4.2	Strong estimate on v_α in $B_0(\mu_\alpha)$	93
3.4.3	The sphere of dominance around a blow-up point	94
3.4.4	Stability theorem proof	103
3.5	Annex	105
3.5.1	Standard elliptic theory for the Lamé operator	105
3.5.2	Limiting equation	107

3.1 Introduction

A spacetime is defined as the equivalence class, up to an isometry, of Lorentzian manifolds (\tilde{M}, \tilde{g}) of dimension $n + 1$, which satisfy the Einstein field equations

$$Ric_{\alpha\beta}(\tilde{g}) - \frac{1}{2}R(\tilde{g})\tilde{g}_{\alpha\beta} = 8\pi T_{\alpha\beta}, \quad (3.1.1)$$

$\alpha, \beta = \overline{1, n+1}$. We have used the following notation : $R(\tilde{g})$ is the scalar curvature of \tilde{g} , Ric the Ricci curvature and $T_{\alpha\beta}$ the stress-energy tensor. If $T_{\alpha\beta} = 0$, we describe the vacuum. If

$$T_{\alpha\beta} = \tilde{\nabla}_\alpha \tilde{\psi} \tilde{\nabla}_\beta \tilde{\psi} - \left(\frac{1}{2} |\tilde{\nabla} \tilde{\psi}|_{\tilde{g}}^2 + V(\tilde{\psi}) \right) \tilde{g}_{\alpha\beta}, \quad (3.1.2)$$

the model corresponds to the existence of a scalar field $\tilde{\psi} \in \mathcal{C}^\infty(M)$ having potential $V \in \mathcal{C}^\infty(\mathbb{R})$. By correctly choosing ψ and V , we can describe the vacuum with cosmological constant and the Einstein-Klein-Gordon setting.

A globally hyperbolic spacetime accepts initial data (M, g, K, ψ, π) , where

- (M, g) is an n -dimensional Riemannian manifold,
- K is a symmetric 2-tensor corresponding to the second fundamental form,
- ψ represents the scalar field in M , and
- π is its derivative.

The associated spacetime development takes the form $(M \times \mathbb{R}, \tilde{g}, \tilde{\psi})$, where \tilde{g} is a Lorentzian metric that verifies $\tilde{g}|_M = \hat{g}$ and $\tilde{\psi}$ is a scalar field such that $\psi|_M = \hat{\psi}$ and $\partial_t \tilde{\psi}|_M = \hat{\pi}$.

Through the work of Choquet-Bruhat and Geroch, having the initial data verify the constraint equations is proved to be not only a necessary, but a sufficient condition for the development of a maximal, globally hyperbolic space-time [FB52, CBG69] :

$$\begin{aligned} R(\hat{g}) + (tr_{\hat{g}} \hat{K})^2 - |\hat{K}|_{\hat{g}}^2 &= \hat{\pi}^2 + |\hat{\nabla} \hat{\psi}|_{\hat{g}}^2 + 2V(\hat{\psi}) \\ \partial_i (tr_{\hat{g}} \hat{K}) - \hat{K}_{i,j}^j &= \hat{\pi} \partial_i \hat{\psi}, \end{aligned} \quad (3.1.3)$$

The above system is clearly under-determined, which allows for a good amount of freedom in choosing $(\hat{g}, \hat{K}, \hat{\psi}, \hat{\pi})$.

The conformal method began with Lichnerowicz [Lic44], and was later developed by York, Jr., Ó Murchadha and Pfeiffer, [Yor73, ÓMY74, Yor99, PY03]. We allow for the constraint equations to be transformed into a determined system of equations by fixing well-chosen quantities (see Choquet-Bruhat, Isenberg and Pollack [CBIP07]). Essentially, the technique maps a space of parameters to the space of solutions.

Given an initial data set $(\hat{g}, \hat{K}, \hat{\psi}, \hat{\pi})$, the classical choice of parameters is $(\mathbf{g}, \mathbf{U}, \tau, \psi, \pi; \alpha)$: in this case, the conformal class \mathbf{g} is represented by a Riemannian metric g , the smooth function $\tau = \hat{g}^{ab} \hat{K}_{ab}$ is a mean curvature and the conformal momentum \mathbf{U} measured by a volume form α (volume gauge) is a 2-tensor that is both trace-free and divergence-free with respect to g (a transverse-traceless tensor). We sometimes prefer to indicate the volume gauge by the densitized lapse

$$\tilde{N}_{g,\alpha} := \frac{\alpha}{dV_g}. \quad (3.1.4)$$

In 2014, Maxwell introduces a variant to the standard conformal method called "the drift method" [Max14b]. Very succinctly, it differs from its predecessor in that it replaces the mean curvature τ with two new conformal data, a volumetric momentum and a drift. These new quantities are defined by the volumetric equivalent to the York splitting [Max15]:

$$\tau = \tau^* + N_{g,\omega} \operatorname{div}_g(V + Q) \quad (3.1.5)$$

where $\tau^* \in \mathbb{R}$, V is a smooth vector field and Q is a conformal Killing field.

The conformal method essentially provides a mapping from the set of conformal data representatives to the set of initial data,

$$\text{Conformal data representatives} \rightarrow \text{Initial data.} \quad (3.1.6)$$

Given a gauge ω , one might choose

1. an arbitrary representative $g_{ab} \in \mathbf{g}$,
2. the unique densitized lapse $\tilde{N}_{g,\omega}$,
3. the unique TT-tensor U_{ab} such that $(g_{ab}, U_{ab}) = \mathbf{U}$, where \mathbf{U} is the conformal momentum as measured by ω , and
4. a vector field \tilde{V} , unique up to a conformal Killing field, such that $(g_{ab}, \tilde{V}_{ab}) = \mathbf{V}$, where \mathbf{V} is the volumetric drift measured by ω .

We denote by

$$q := \frac{2n}{n-2} \quad (3.1.7)$$

the critical Sobolev constant corresponding to the embedding of H^1 into the Lebesgue spaces. Let

$$\mathcal{L}_g W_{ij} = W_{i,j} + W_{j,i} - \frac{2}{n} \operatorname{div}_g W g_{ij} \quad (3.1.8)$$

be the conformal Killing operator with respect to g . We use the decompositions

$$\begin{aligned} \hat{g}_{ab} &= u^{q-2} g_{ab} \\ \hat{K}_{ab} &= u^{-2} \left[\frac{\tilde{N}}{2} (\mathcal{L}_g W)_{ab} + U_{ab} \right] + \frac{1}{n} u^{q-2} g_{ab} \left(\tau^* + \tilde{N} \operatorname{div}_g(V + Q) \right). \end{aligned} \quad (3.1.9)$$

where u is a scalar function and W and Q are vector fields, all unknown. More precisely, the mapping presents as

$$(g_{ab}, U_{ab}, \tau_*, \tilde{V}; \tilde{N}) \xrightarrow{\text{solve}(u, W, Q)} (\hat{g}, \hat{K}). \quad (3.1.10)$$

The question we aim to tackle in this paper is as follows :

(Q) Is the mapping continuous ?

We test, in some sense, that the mapping is physically relevant.

3.1.1 The main result.

Let (M, g) be a closed locally conformally flat manifold of dimension n , which can be 3, 4 and 5. Let

$$\Delta_g = -\text{div}_g \nabla \quad (3.1.11)$$

be the Laplace-Beltrami operator with non-negative eigenvalues. Similarly, let

$$\vec{\Delta}_g W_i = -\text{div}_g(\mathcal{L}_g W)_i = -(\mathcal{L}_g W)_{ij}{}^j \quad (3.1.12)$$

be the corresponding Lamé operator. The volumetric drift model proposed by Maxwell leads to the reworking of the Einstein constraint equations as

$$\begin{aligned} \Delta_g u + \frac{n-2}{4(n-1)}(R(g) + |\nabla \psi|_g^2)u &= \frac{(n-2)|U + \mathcal{L}_g W|^2 + \pi^2}{4(n-1)u^{q+1}} \\ &+ \frac{n-2}{4(n-1)} \left(2V(\psi) - \frac{n-1}{n} \tau^* \right. \\ &\left. + \frac{n-1}{n} \frac{\tilde{N} \text{div}_g(u^q \tilde{V})^2}{u^{2q}} \right) u^{q-1} \end{aligned} \quad (3.1.13)$$

$$\text{div}_g \left(\frac{\tilde{N}}{2} \mathcal{L}_g W \right) = \frac{n-1}{n} u^q \mathbf{d} \left(\frac{\tilde{N} \text{div}_g(u^q \tilde{V})}{2u^{2q}} \right) + \pi \nabla \psi.$$

The existence of solutions to this system was treated in [HMM18] in the non-focusing case, and in [Vál19] for the focusing case. The classical conformal method, also in the focusing regime, is treated in [Pre14]. See [DH09] for the precursor of the asymptotic techniques used in the existence proofs.

The second equation may be rewritten as :

$$\begin{aligned} \vec{\Delta}_g W &= \langle \nabla \ln \tilde{N}, \mathcal{L}_g W \rangle + 2 \frac{n-1}{n-2} \left(\frac{3n-2}{n-2} \frac{\langle \nabla u, \tilde{V} \rangle \nabla u}{u^2} - \frac{\langle \nabla^2 u, \tilde{V} \rangle}{u} \right) \\ &+ 2 \frac{n-1}{n-2} \left(-\frac{\langle \nabla u, \tilde{V} \rangle}{u} \nabla \ln \tilde{N} + \text{div}_g \tilde{V} \frac{\nabla u}{u} - \frac{\langle \nabla \tilde{V}, \nabla u \rangle}{u} \right) \\ &- \frac{n-1}{n} \left(\text{div}_g \tilde{V} \nabla \ln \tilde{N} + \nabla \text{div}_g \tilde{V} \right) - 2 \tilde{N}^{-1} \pi \nabla \psi. \end{aligned} \quad (3.1.14)$$

In the present paper, it sometimes proves useful to work with the more general equation

$$\begin{aligned} \Delta_g u + hu &= f u^{q-1} + \frac{\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W|_g^2}{u^{q+1}} \\ &- \frac{b}{u} - c \langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) - \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}} \end{aligned} \quad (3.1.15)$$

where we make the following substitutions :

$$\begin{aligned}
h &= \frac{n-2}{4(n-1)} (\mathcal{R}_g - |\nabla\psi|_g^2), & f &= \frac{n-2}{4(n-1)} (2V(\psi) - \frac{n-1}{n}\tau^{*2}), \\
\rho_1 &= \frac{n-2}{4(n-1)} \left(\pi - \frac{n-1}{n}\tilde{N}^2 \operatorname{div}_g \tilde{V} \right), & \rho_2 &= \sqrt{\frac{n-2}{n-1}} \frac{\tilde{N}}{4}, & \Psi &= \sqrt{\frac{n-2}{n-1}} \frac{U}{2}, \\
b &= \frac{n-2}{2n} \tau^* \tilde{N} \operatorname{div}_g \tilde{V}, & c &= \sqrt{\frac{n-2}{n}}, & d &= \tau^*, & Y &= \sqrt{\frac{n}{n-2}} \tilde{N} \tilde{V}.
\end{aligned} \tag{3.1.16}$$

Consider $(u_\alpha, W_\alpha)_{\alpha \in \mathbb{N}}$ a sequence of smooth solutions of perturbations of the system (3.1.15),

$$\begin{aligned}
\Delta_g u_\alpha + h_\alpha u_\alpha &= f_\alpha u_\alpha^{q-1} + \frac{\rho_{1,\alpha} + |\Psi_\alpha + \rho_{2,\alpha} \mathcal{L}_g W_\alpha|_g^2}{u_\alpha^{q+1}} \\
&\quad - \frac{b_\alpha}{u_\alpha} - c_\alpha \langle \nabla u_\alpha, Y_\alpha \rangle \left(\frac{d_\alpha}{u_\alpha^2} + \frac{1}{u_\alpha^{q+2}} \right) - \frac{\langle \nabla u_\alpha, Y_\alpha \rangle^2}{u_\alpha^{q+3}} \\
\vec{\Delta}_g W_\alpha &= \mathcal{R}_\alpha(u_\alpha, \nabla u_\alpha, \nabla^2 u_\alpha, \mathcal{L}_g W_\alpha).
\end{aligned} \tag{3.1.17}$$

Here, we ask that the perturbed coefficients converge towards the initial ones in a sufficiently regular way, *e.g.* in $\mathcal{C}^{2,\eta}$ norm. The scalar solutions u_α are positive as long as ρ_1 which is positive. To see this, let $m_\alpha = \min_{x \in M} u_\alpha(x) = u_\alpha(x_\alpha) > 0$ and let

$$a_\alpha = \rho_{1,\alpha} + |\Psi_\alpha + \rho_{2,\alpha} \mathcal{L}_g W_\alpha|_g^2. \tag{3.1.18}$$

Since $\nabla u_\alpha(x_\alpha) = 0$ and since $\Delta_g u_\alpha(x_\alpha) \leq 0$, we have

$$h_\alpha(x_\alpha) m_\alpha - f_\alpha(x_\alpha) m_\alpha^{q-1} - \frac{a_\alpha(x_\alpha)}{m_\alpha^{q+1}} + \frac{b_\alpha(x_\alpha)}{m_\alpha} \geq 0.$$

Since $a_\alpha \rightarrow a$ in $\mathcal{C}^0(M)$ as $\alpha \rightarrow +\infty$ and $a > 0$ in M , there exists $\varepsilon > 0$ such that $m_\alpha \geq \varepsilon$, meaning that

$$u_\alpha \geq \varepsilon > 0 \quad \text{for all } x \in M \text{ and all } \alpha. \tag{3.1.19}$$

We would like to prove the *a priori* estimate

$$\|u_\alpha\|_{\mathcal{C}^{2,\eta}} + \|W_\alpha\|_{\mathcal{C}^{1,\eta}} \leq C. \tag{3.1.20}$$

If this is true, then by standard elliptic theory there exists, up to a subsequence, a $\mathcal{C}^{2,\eta}$ limit of (u_α, W_α) solving the limiting system (3.1.15). In effect, since the system (3.1.15) is invariant by the addition of conformal Killing fields, it suffices to show that

$$\|u_\alpha\|_{L^\infty} + \|\nabla u_\alpha\|_{L^\infty} + \|\nabla^2 u_\alpha\|_{L^\infty} + \|\mathcal{L}_g W_\alpha\|_{L^\infty} \leq C. \tag{3.1.21}$$

The proof follows by contradiction. We assume instead that there exists a sequence of solutions (u_α, W_α) of the perturbed system such that

$$\|u_\alpha\|_{L^\infty} + \|\nabla u_\alpha\|_{L^\infty} + \|\nabla^2 u_\alpha\|_{L^\infty} + \|\mathcal{L}_g W_\alpha\|_{L^\infty} \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty. \tag{3.1.22}$$

The main theorem in this paper is the following.

Theorem 12. *Let (M, g) be a closed Riemannian manifold of dimension $n = 3, 4, 5$, where g is locally conformally flat. Let $\frac{1}{2} < \eta < 1$ and $0 < \alpha < 1$. Let $a, b, c, d, f, h, \rho_1, \rho_2$,*

ψ, π, \tilde{N} be smooth functions on M , let \tilde{V} and Y be smooth vector field on M . For any $0 < \theta < T$, there exists $S_{\theta,T}$ and $\vartheta_{\theta,T}$ such that, given any parameters within

$$\mathcal{E}_{\theta,T} := \left\{ (f, a, b, c, d, h, \rho_1, \rho_2, Y) \times (\tilde{N}, \tilde{V}, \psi, \pi), \quad f \geq \theta, \quad a \geq \theta, \quad \tilde{N} \geq \theta, \right.$$

$$\|f\|_{\mathcal{C}^{1,\eta}} \leq T,$$

$$\|a\|_{\mathcal{C}^{1,\alpha}}, \|b\|_{\mathcal{C}^{1,\alpha}}, \|c\|_{\mathcal{C}^{1,\alpha}}, \|d\|_{\mathcal{C}^{1,\alpha}},$$

$$\|\rho_1\|_{\mathcal{C}^{1,\alpha}}, \|\rho_2\|_{\mathcal{C}^{1,\alpha}}, \|h\|_{\mathcal{C}^{1,\alpha}}, \|Y\|_{\mathcal{C}^{1,\alpha}} \leq T,$$

$$\text{and } \left. \|\tilde{N}\|_{\mathcal{C}^{2,\alpha}}, \|\tilde{V}\|_{\mathcal{C}^{2,\alpha}} \leq T \right\}, \quad (3.1.23)$$

with

$$\|Y\|_{\mathcal{C}^{1,\alpha}}, \|\tilde{V}\|_{\mathcal{C}^{2,\alpha}} \leq \vartheta_{\theta,T}, \quad (3.1.24)$$

then any smooth solution (u, W) (3.1.13), with $u > 0$, satisfies

$$\|u\|_{\mathcal{C}^{2,\alpha}} + \|\mathcal{L}_g W\|_{\mathcal{C}^{1,\alpha}} \leq S_{\theta,T}. \quad (3.1.25)$$

A few remarks are in order at this point. We have taken the decision to write the theorem using the physical coordinates for the second equation, and the general coefficients for the first. The same is true for the ensuing proof. This forcibly leads to some redundancies. We recall that $Y = \sqrt{\frac{n}{n-2}} \tilde{N} \tilde{V}$, so asking for bounds on \tilde{N} and \tilde{V} imply bounds on Y . The reasons why we still choose this writing are as follows :

1. The general notation of the first equation is the same as the ones used in the paper proving the existence of solutions to the system, and are more readable than the physical coordinates one. Moreover, they more accurately capture the nature of the scalar equation and make it easier to handle, since one can follow each of the different non-linear terms separately.
2. Writing the second equation in more general terms can prove counterproductive. For one, introducing new coefficients would actually burden the notation in this particular case.
3. Secondly, and most importantly, one hopes that there is a better way to treat potential blow-ups caused by $\mathcal{L}_g W_\alpha$. This could follow from a more detailed analysis of the second equation, where even the exact size of each of the dimensional constants can potentially play a role, given the coupling of the system.

In the proof, we use the smallness of Y (and thus, \tilde{V} , since $\tilde{N} \geq \theta$) as sparsely as possible, and we take care to emphasize it each time. It follows that, in the argument by contradiction, we are working with

$$Y_\alpha \rightarrow 0 \quad \text{in } \mathcal{C}^{1,\alpha} \quad \text{and} \quad \tilde{V}_\alpha \rightarrow 0 \quad \text{in } \mathcal{C}^{2,\alpha}. \quad (3.1.26)$$

The fact that g is locally conformally flat is a condition we impose to get the improved estimates on $\mathcal{L}_g W_\alpha$ that we need. We briefly explain the reasoning. The Green representation formula is applied on balls of diminishing radius $B_{x_\alpha}(\delta_\alpha)$, $\delta_\alpha \rightarrow 0$, where x_α is a

concentration point. Moreover, we impose Neumann boundary conditions, so that there is no dependency on W_α , but just $\mathcal{L}_{g_\alpha} W_\alpha$. The bounds needs to be uniform with respect to α , which is why we need the kernel of $\vec{\Delta}_{g_\alpha}$ to have the same dimension as that of $\vec{\Delta}_\xi$, with $g_\alpha = \exp_{x_\alpha}^*(\delta_\alpha \cdot)$.

The stability of the classical system, also in the focusing case, was treated in [Pre16].

The proof is structured as follows. In **Section 2**, we conformally change (u_α, W_α) on (M, g) to (v_α, Z_α) defined in a Euclidean domain. In **Section 3**, we begin by obtaining pointwise estimates on both v_α , ∇v_α , $\nabla^2 v_\alpha$ and $\mathcal{L}_g Z_\alpha$. **Section 4** begins with an immediate consequence of the aforementioned bounds : they yield a Harnack inequality on v_α . Green's representation theory, applied to the elliptic operators of both the first and second equation, plays a central role in both obtaining and improving the aforementioned weak bounds on $\mathcal{L}_\xi Z_\alpha$. The next step consists of using the techniques of asymptotic analysis to describe potential blow-up behaviour, and their interactions. All leads to a contradiction.

3.2 Conformal changes of coordinates.

Since (M, g) is assumed to be locally conformally flat, for any sequence $x_\alpha \in M$ with $x_\alpha \rightarrow x$ as $\alpha \rightarrow +\infty$ and for any $\delta > 0$ small enough, there exist smooth diffeomorphisms

$$\Phi_\alpha : U_\alpha \subset M \mapsto B_0(\delta) \subset \mathbb{R}^n \quad (3.2.1)$$

and $\varphi_\alpha \in \mathcal{C}^\infty(B_0(\delta))$ where U_α is some neighbourhood of x_α in M such that

$$(\Phi_\alpha^{-1})^* g = \varphi_\alpha(x)^{q-2} \xi \quad (3.2.2)$$

where ξ is the Euclidean metric. Moreover we can choose the diffeomorphisms Φ_α and the functions φ_α to be uniformly bounded in any \mathcal{C}^k for $k \leq m$, m fixed as we want. Note that we can also choose $\varphi_\alpha(0) = 1$ and $\nabla \varphi_\alpha(0) = 0$. For $x \in B_0(\delta)$, consider the change of functions

$$v_\alpha(x) = \varphi_\alpha(x) u_\alpha \circ \Phi_\alpha^{-1}(x) \quad \text{and} \quad Z_\alpha(x) = \varphi_\alpha(x)^{2-q} (\Phi_\alpha)_* W_\alpha(x). \quad (3.2.3)$$

This change of functions will be used repeatedly in the sequel. First of all, note that, by (3.1.19), there exists $\varepsilon' > 0$ such that

$$v_\alpha \geq \varepsilon'. \quad (3.2.4)$$

Then it's convenient to recall the following formulas. Given that $(\Phi_\alpha^{-1})^* g_{ij} = \varphi_\alpha^{q-2} \xi_{ij}$, we see that the Laplace-Beltrami operator becomes

$$\begin{aligned} \Delta_\xi v_\alpha &= \Delta_\xi (\varphi_\alpha u_\alpha \circ \Phi_\alpha^{-1}) \\ &= \varphi_\alpha^{q-1}(x) \left(\Delta_g u_\alpha + \frac{n-2}{4(n-1)} R(g) u_\alpha \right) (\Phi_\alpha^{-1}) \end{aligned} \quad (3.2.5)$$

and that

$$\varphi_\alpha^{q-2} \mathcal{L}_\xi Z_\alpha = (\Phi_\alpha)_* (\mathcal{L}_g W_\alpha). \quad (3.2.6)$$

At last, the Lamé type operator transforms as

$$\vec{\Delta}_\xi (\varphi_\alpha^{2-q} (\Phi_\alpha)_* W_\alpha)_i - q \xi^{kl} \partial_k (\ln \varphi_\alpha) \mathcal{L}_\xi (\varphi_\alpha^{2-q} (\Phi_\alpha)_* W_\alpha)_{il} = (\Phi_\alpha)_* \left(\vec{\Delta}_g W_\alpha \right)_i, \quad (3.2.7)$$

so

$$\left(\vec{\Delta}_\xi Z_\alpha \right)_i - q \langle \nabla \ln \varphi_\alpha, \mathcal{L}_\xi Z_\alpha \rangle_i = (\Phi_\alpha)_* \left(\vec{\Delta}_g W_\alpha \right)_i. \quad (3.2.8)$$

Simple but tedious computations lead then to the transformation of the system (3.1.17) into

$$\begin{aligned}
\Delta_\xi v_\alpha(x) + \tilde{h}_\alpha(x)v_\alpha(x) &= \tilde{f}_\alpha(x)v_\alpha^{q-1}(x) + \frac{\tilde{a}_\alpha(x)}{v_\alpha^{q+1}(x)} - \frac{\tilde{b}_\alpha(x)}{v_\alpha(x)} \\
&\quad - \langle \nabla v_\alpha(x), \tilde{Y}_\alpha(x) \rangle \left(\frac{\tilde{c}_\alpha(x)}{v_\alpha^2(x)} + \frac{\tilde{d}_\alpha(x)}{v_\alpha^{q+2}(x)} \right) \\
&\quad - \frac{\langle \nabla v_\alpha(x), \tilde{Y}_\alpha(x) \rangle^2}{v_\alpha^{q+3}(x)} \\
\left(\vec{\Delta}_\xi Z_\alpha \right)_i &= q \langle \nabla \ln \varphi_\alpha, \mathcal{L}_\xi Z_\alpha \rangle_i + \tilde{\mathcal{R}}_\alpha(v_\alpha, \nabla v_\alpha, \nabla^2 v_\alpha, \mathcal{L}_\xi Z_\alpha)_i
\end{aligned} \tag{3.2.9}$$

where

$$\begin{aligned}
\tilde{Y}_\alpha &= \varphi_\alpha^2(\Phi_\alpha)_* Y_\alpha, \quad \tilde{f}_\alpha = f_\alpha \circ \Phi_\alpha^{-1}, \\
\tilde{h}_\alpha &= \varphi_\alpha^{q-2} \left(h_\alpha - \frac{n-2}{4(n-1)} R(g) \right) \circ \Phi_\alpha^{-1}, \\
\tilde{b}_\alpha &= \varphi_\alpha^q b_\alpha \circ \Phi_\alpha^{-1} - \varphi_\alpha \langle \nabla \varphi_\alpha, (\Phi_\alpha)_* Y_\alpha \rangle c_\alpha \circ \Phi_\alpha^{-1}, \\
\tilde{c}_\alpha &= c_\alpha \circ \Phi_\alpha^{-1}, \quad \tilde{d}_\alpha = 2\varphi_\alpha \langle \nabla \varphi_\alpha, (\Phi_\alpha)_* Y_\alpha \rangle + \varphi_\alpha^q d_\alpha \circ \Phi_\alpha^{-1}, \\
\tilde{a}_\alpha &= \tilde{\rho}_{1,\alpha} + \left| \tilde{\Psi}_\alpha + \tilde{\rho}_{2,\alpha} \mathcal{L}_\xi Z_\alpha \right|_\xi^2 \\
\tilde{\rho}_{1,\alpha} &= \varphi_\alpha^{2q} \rho_{1,\alpha} \circ \Phi_\alpha^{-1} + \varphi_\alpha^{q+1} \langle \nabla \varphi_\alpha, (\Phi_\alpha)_* Y_\alpha \rangle d_\alpha \circ \Phi_\alpha^{-1} - \varphi_\alpha^2 \langle \nabla \varphi_\alpha, (\Phi_\alpha)_* Y_\alpha \rangle^2, \\
\tilde{\rho}_{2,\alpha} &= \varphi_\alpha^q \rho_{2,\alpha} \circ \Phi_\alpha^{-1}, \quad \tilde{\Psi}_\alpha = \varphi_\alpha^2 (\Phi_\alpha)_* \Psi_\alpha
\end{aligned} \tag{3.2.10}$$

and

$$\begin{aligned}
\tilde{\mathcal{R}}_\alpha(v_\alpha, \nabla v_\alpha, \nabla^2 v_\alpha, \mathcal{L}_\xi Z_\alpha) &= (\Phi_\alpha)_* \mathcal{R}_\alpha(u_\alpha, \nabla u_\alpha, \nabla^2 u_\alpha, \mathcal{L}_g W_\alpha) \\
&= \langle (\Phi_\alpha)_* \nabla \ln \tilde{N}, \mathcal{L}_\xi Z_\alpha \rangle \\
&\quad + 2 \frac{n-1}{n-2} \varphi_\alpha^{2-q} \left(\frac{3n-2}{n-2} \frac{\langle \nabla v_\alpha, (\Phi_\alpha)_* \tilde{V}_\alpha \rangle \nabla v_\alpha}{v_\alpha^2} \right. \\
&\quad \quad \quad \left. - \frac{\langle \nabla^2 v_\alpha, (\Phi_\alpha)_* \tilde{V}_\alpha \rangle}{v_\alpha} \right) \\
&\quad + \tilde{\mathcal{T}}_\alpha(v_\alpha, \nabla v_\alpha, \nabla^2 v_\alpha).
\end{aligned} \tag{3.2.11}$$

Here, $\tilde{\mathcal{T}}_\alpha$ denotes the lower order terms of the second equation. It is clear that we have

$$|\tilde{\mathcal{T}}_\alpha(v_\alpha, \nabla v_\alpha, \nabla^2 v_\alpha)| \leq C \left(1 + \|(\Phi_\alpha)_* \tilde{V}_\alpha\|_{C^1} \left| \frac{\nabla v_\alpha}{v_\alpha} \right| \right). \tag{3.2.12}$$

3.3 Weak pointwise estimates

The following result describes a pointwise estimate that holds everywhere on M . It provides a way to identify a set of points \mathcal{S}_α where u_α or $\mathcal{L}_g W_\alpha$ can potentially explode.

Lemma 13. *Let (u_α, W_α) be a sequence of solutions of the perturbed system (3.1.17), verifying the non-compactness hypothesis (3.1.22). There exists an integer $N_\alpha \in \mathbb{N}^*$ and a set of critical points $\mathcal{S}_\alpha = (x_{1,\alpha}, \dots, x_{N_\alpha,\alpha})$ of u_α such that*

$$d_g(x_{i,\alpha}, x_{j,\alpha})^n u_\alpha(x_{i,\alpha})^q \geq 1, \tag{3.3.1}$$

for all $1 \leq i, j \leq N_\alpha$, $i \neq j$, and

$$\left(\min_{1 \leq i \leq N_\alpha} d_g(x_{i,\alpha}, x) \right)^n u_\alpha(x) \leq 1 \quad (3.3.2)$$

for any x critical point of u_α in M , and

$$\begin{aligned} & \left(\min_{1 \leq i \leq N_\alpha} d_g(x_{i,\alpha}, x) \right)^n \\ & \times \left(u_\alpha^q(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^n + \left| \frac{\nabla^2 u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n}{2}} + |\mathcal{L}_g W_\alpha|_g(x) \right) \leq C. \end{aligned} \quad (3.3.3)$$

Démonstration. Step 1 : Setting up the proof by contradiction. For every $\alpha \in \mathbb{N}^*$, we may define the integer $N_\alpha \in \mathbb{N}^*$ and the set of critical points

$$\mathcal{S}_\alpha = (x_{1,\alpha}, \dots, x_{N_\alpha,\alpha})$$

of u_α by the following lemma, which holds very generally for any sufficiently regular function.

Lemma 14. *Let u be a positive real-valued \mathcal{C}^2 function defined in a compact manifold M . Then there exists $N \in \mathbb{N}^*$ and (x_1, x_2, \dots, x_N) a set of critical points of u such that*

$$d_g(x_i, x_j)^{\frac{n-2}{2}} u(x_i) \geq 1 \quad (3.3.4)$$

for all $i, j \in \{1, \dots, N\}$, $i \neq j$, and

$$\left(\min_{i=1, \dots, N} d_g(x_i, x) \right)^{\frac{n-2}{2}} u(x) \leq 1 \quad (3.3.5)$$

for all critical points x of u .

The lemma and its proof may be found in Druet and Hebey's paper [DH09]. Let

$$\begin{aligned} \Psi_\alpha(x) = & \left(\min_{1 \leq i \leq N_\alpha} d_g(x_{i,\alpha}, x) \right)^n \\ & \times \left(u_\alpha^q(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^n + \left| \frac{\nabla^2 u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n}{2}} + |\mathcal{L}_g W_\alpha|_g(x) \right) \end{aligned} \quad (3.3.6)$$

for $x \in M$. Let $(x_\alpha)_\alpha \in M$ be such that

$$\Psi_\alpha(x_\alpha) = \sup_M \Psi_\alpha \rightarrow \infty \quad (3.3.7)$$

as $\alpha \rightarrow \infty$.

Step 2 : Rescaling. We denote the injectivity radius of M by $i_g(M)$. Let

$$0 < \delta_\alpha < \frac{1}{2} i_g(M) \quad (3.3.8)$$

be radii around x_α . Since (M, g) is conformally flat, let φ_α and Φ_α be as in previous section so that $(\Phi_\alpha^{-1})^* g_{ij} = \varphi_\alpha^{q-2} \xi_{ij}$, $\varphi_\alpha(0) = 1$ and $\nabla \varphi_\alpha(0) = 0$. In fact, these conformal factors can be chosen to be uniformly bounded up in \mathcal{C}^k , up to an arbitrary $k > 0$. Consider the following rescalings of the conformal factors :

$$\begin{aligned} \hat{v}_\alpha(x) &= \mu_\alpha^{\frac{n-2}{2}} \varphi_\alpha(\mu_\alpha x) u_\alpha \circ \Phi_\alpha^{-1}(\mu_\alpha x) \\ \hat{Z}_\alpha(x) &= \mu_\alpha^{n-1} \varphi_\alpha(\mu_\alpha x)^{2-q} (\Phi_\alpha)_* W_\alpha(\mu_\alpha x), \end{aligned} \quad (3.3.9)$$

where $x \in \Omega_\alpha$, with $\Omega_\alpha := B_{x_\alpha} \left(\frac{\delta}{\mu_\alpha} \right)$ and

$$\mu_\alpha^{-n} := u_\alpha(x_\alpha)^q + \left| \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right|^n + \left| \frac{\nabla^2 u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right|^{\frac{n}{2}} + |\mathcal{L}_g W_\alpha|_g(x_\alpha). \quad (3.3.10)$$

Moreover, because M is compact, and by (3.3.6) and (3.3.7),

$$\frac{d_g(x_\alpha, \mathcal{S}_\alpha)}{\mu_\alpha} \rightarrow \infty \quad \text{and} \quad \mu_\alpha \rightarrow 0. \quad (3.3.11)$$

We consider the rescaled perturbed system corresponding to (3.3.9),

$$\begin{aligned} \Delta_\xi \hat{v}_\alpha &= -\mu_\alpha^2 \hat{h}_\alpha \hat{v}_\alpha + \hat{f}_\alpha \hat{v}_\alpha^{q-1} + \frac{\hat{a}_\alpha}{\hat{v}_\alpha^{q+1}} - \mu_\alpha^{\frac{n+2}{2}} \frac{\hat{b}_\alpha}{v_\alpha(\mu_\alpha \cdot)} \\ &\quad - \mu_\alpha^{\frac{n}{2}} \frac{\langle \nabla \hat{v}_\alpha, \hat{Y}_\alpha \rangle}{\hat{v}_\alpha} \left(\frac{\hat{d}_\alpha}{v_\alpha(\mu_\alpha \cdot)} + \frac{\hat{c}_\alpha}{v_\alpha^{q+1}(\mu_\alpha \cdot)} \right) \\ &\quad - \mu_\alpha^{\frac{n-2}{2}} \frac{\langle \nabla \hat{v}_\alpha, \hat{Y}_\alpha \rangle^2}{\hat{v}_\alpha^2} \frac{1}{v_\alpha^{q+1}(\mu_\alpha \cdot)} \\ \vec{\Delta}_\xi \hat{Z}_\alpha &= q \mu_\alpha \xi^{kl} \partial_k (\ln \varphi_\alpha)(\mu_\alpha \cdot) \mathcal{L}_\xi \hat{Z}_{\alpha l} \\ &\quad + \hat{\mathcal{R}}_\alpha(\hat{v}_\alpha, \nabla \hat{v}_\alpha, \nabla^2 \hat{v}_\alpha, \mathcal{L}_\xi \hat{Z}_\alpha), \end{aligned} \quad (3.3.12)$$

where

$$\begin{aligned} \hat{h}_\alpha(x) &= \tilde{h}_\alpha(\mu_\alpha x), \quad \hat{f}_\alpha(x) = \tilde{f}_\alpha(\mu_\alpha x), \\ \hat{\rho}_{1,\alpha}(x) &= \tilde{\rho}_{1,\alpha}(\mu_\alpha x), \quad \hat{\rho}_{2,\alpha}(x) = \tilde{\rho}_{2,\alpha}(\mu_\alpha x), \\ \hat{\Psi}_\alpha(x) &= \tilde{\Psi}_\alpha(\mu_\alpha x), \quad \hat{Y}_\alpha(x) = \tilde{Y}_\alpha(\mu_\alpha x), \\ \hat{a}_\alpha(x) &= \tilde{a}_\alpha(\mu_\alpha x), \\ \hat{c}_\alpha(x) &= \tilde{c}_\alpha(\mu_\alpha x), \quad \hat{d}_\alpha(x) = \tilde{d}_\alpha(\mu_\alpha x). \end{aligned} \quad (3.3.13)$$

and

$$\begin{aligned} \hat{\mathcal{R}}(\hat{v}_\alpha, \nabla \hat{v}_\alpha, \nabla^2 \hat{v}_\alpha, \mathcal{L}_\xi \hat{Z}_\alpha) &\leq C'_\mathcal{R} \left(\mu_\alpha^{n+1} + \mu_\alpha^n \left| \frac{\nabla \hat{v}_\alpha}{\hat{v}_\alpha} \right| + \mu_\alpha^{n-1} \left| \frac{\nabla^2 \hat{v}_\alpha}{\hat{v}_\alpha} \right|^2 \right. \\ &\quad \left. + \mu_\alpha^{n-1} \left| \frac{\nabla^2 \hat{v}_\alpha}{\hat{v}_\alpha} \right| + \mu_\alpha |\mathcal{L}_\xi \hat{Z}_\alpha| \right). \end{aligned} \quad (3.3.14)$$

where $C'_\mathcal{R}$ is a constant.

By the definition (3.3.10),

$$\hat{v}_\alpha(0)^q + \left| \frac{\nabla \hat{v}_\alpha(0)}{\hat{v}_\alpha(0)} \right|^n + \left| \frac{\nabla^2 \hat{v}_\alpha(0)}{\hat{v}_\alpha(0)} \right|^{\frac{n}{2}} + |\mathcal{L}_\xi \hat{Z}_\alpha|_\xi(0) = 1 \quad (3.3.15)$$

and for any $R > 0$,

$$\sup_{x \in B_0(R)} \left(\hat{v}_\alpha^q(x) + \left| \frac{\nabla \hat{v}_\alpha(x)}{\hat{v}_\alpha(x)} \right|^n + \left| \frac{\nabla^2 \hat{v}_\alpha(x)}{\hat{v}_\alpha(x)} \right|^{\frac{n}{2}} + \left| \mathcal{L}_\xi \hat{Z}_\alpha(x) \right|_\xi \right) \leq 1 + o(1) \quad (3.3.16)$$

and thereby

$$\sup_{B_0(R)} |\nabla \ln \hat{v}_\alpha(x)| \leq 1 + o(1) \quad (3.3.17)$$

for all $R > 0$. As a consequence,

$$\hat{v}_\alpha(0)e^{-2|x|} \leq \hat{v}_\alpha(x) \leq \hat{v}_\alpha(0)e^{2|x|}. \quad (3.3.18)$$

Step 3 : $|\mathcal{L}_\xi \hat{Z}_\alpha|$ converges to zero. By Green's representation formula applied to the first equation of (3.3.12) on $B_x(3R)$, we get

$$\begin{aligned} \hat{v}_\alpha(x) \geq & \int_{B_x(3R)} \mathcal{G}_{3R}(x, y) \left[\frac{\hat{a}_\alpha(y)}{\hat{v}_\alpha^{q+1}(y)} - \mu_\alpha^2 \hat{h}_\alpha(y) \hat{v}_\alpha(y) - \mu_\alpha^{\frac{n+2}{2}} \frac{\hat{b}_\alpha(y)}{v_\alpha(\mu_\alpha y)} \right. \\ & - \mu_\alpha^{\frac{n}{2}} \frac{\langle \nabla \hat{v}_\alpha(y), \hat{Y}_\alpha(y) \rangle}{\hat{v}_\alpha(y)} \left(\frac{\hat{d}_\alpha(y)}{v_\alpha(\mu_\alpha y)} + \frac{\hat{c}_\alpha(y)}{v_\alpha^{q+1}(\mu_\alpha y)} \right) \\ & \left. - \mu_\alpha^{\frac{n-2}{2}} \frac{\langle \nabla \hat{v}_\alpha(y), \hat{Y}_\alpha(y) \rangle^2}{\hat{v}_\alpha(y)} \frac{1}{v_\alpha^{q+1}(\mu_\alpha y)} \right] dy. \end{aligned} \quad (3.3.19)$$

Here, $\mathcal{G}_{3R}(x, y) := \frac{1}{(n-2)\omega_{n-1}} (|x-y|^{2-n} - (3R)^{2-n})$. By taking α large, we get the bulk integral estimate

$$\int_{B_x(2R)} |x-y|^{2-n} |\mathcal{L}_\xi \hat{Z}_\alpha|_\xi^2(y) dy \leq C, \quad (3.3.20)$$

where C a positive constant independent of R or α . Therefore, we may find $s_\alpha \in (\frac{3}{2}R, 2R)$ such that the boundary estimate

$$\int_{\partial B_0(s_\alpha)} |\mathcal{L}_\xi \hat{Z}_\alpha|_\xi^2(y) d\sigma(y) \leq CR^{n-3} \quad (3.3.21)$$

holds. Moreover,

$$\left| q\mu_\alpha \xi^{kl} \partial_k (\ln \varphi_\alpha)(\mu_\alpha \cdot) \left(\mathcal{L}_\xi \hat{Z}_\alpha \right)_{li} \right| \leq C\mu_\alpha^2 |y| \left| \mathcal{L}_\xi \hat{Z}_\alpha \right|_\xi. \quad (3.3.22)$$

Turning to the second equation of (3.3.12), we use the Green representation formula for the Lamé type operator $\overrightarrow{\Delta}_\xi$ in $B_0(2R)$. This yields

$$\begin{aligned} \left| \mathcal{L}_\xi \hat{Z}_\alpha \right|_\xi(x) & \leq C \int_{B_0(s_\alpha)} |x-y|^{1-n} \left| \overrightarrow{\Delta}_\xi \hat{Z}_\alpha \right| dy \\ & \quad + C \int_{\partial B_0(s_\alpha)} |x-y|^{1-n} \left| \mathcal{L}_\xi \hat{Z}_\alpha \right|_\xi(y) d\sigma(y) \\ & \leq C'R\mu_\alpha + \frac{C'}{R}. \end{aligned} \quad (3.3.23)$$

for positive constants C and C' . We therefore get an improvement on the pointwise estimate of the rescaled \hat{W}_α from (3.3.16) :

$$\left| \mathcal{L}_\xi \hat{Z}_\alpha \right|_\xi \rightarrow 0 \quad (3.3.24)$$

and

$$\hat{a}_\alpha \rightarrow 0 \quad (3.3.25)$$

in $\mathcal{C}_{loc}^0(\mathbb{R}^n)$ as $\alpha \rightarrow \infty$.

Step 4 : The study of potential blow-up profiles. We turn to the study of the remaining terms of (3.3.10). From (3.3.15) and (3.3.24), we deduce that

$$\lim_{\alpha \rightarrow \infty} \left(\hat{v}_\alpha^q(0) + \left| \frac{\nabla \hat{v}_\alpha(0)}{\hat{v}_\alpha(0)} \right|^n + \left| \frac{\nabla^2 \hat{v}_\alpha(0)}{\hat{v}_\alpha(0)} \right|^{\frac{n}{2}} \right) = 1. \quad (3.3.26)$$

Let us set

$$w_\alpha(x) := \frac{\hat{v}_\alpha(x)}{\hat{v}_\alpha(0)} = \frac{u_\alpha(\exp_{x_\alpha}(\mu_\alpha x))}{u_\alpha(x_\alpha)}. \quad (3.3.27)$$

It follows that

$$w_\alpha(0) = 1 \quad \text{and} \quad \lim_{\alpha \rightarrow \infty} \left(\left| \frac{\nabla w_\alpha(0)}{w_\alpha(0)} \right|^n + \left| \frac{\nabla^2 w_\alpha(0)}{w_\alpha(0)} \right|^{\frac{n}{2}} \right) \leq 1 \quad (3.3.28)$$

and therefore

$$e^{-2|x|} \leq w_\alpha(x) \leq e^{2|x|} \quad (3.3.29)$$

We divide the first equation of system (3.3.12) by $\hat{u}_\alpha(0)$ and obtain

$$\begin{aligned} \Delta_\xi w_\alpha = & -\mu_\alpha^2 \hat{h}_\alpha w_\alpha + \hat{f}_\alpha w_\alpha^{q-1} \hat{v}_\alpha^{q-2}(0) \\ & + \frac{\mu_\alpha^{\frac{n+2}{2}} \hat{\rho}_{1,\alpha}(x)}{\hat{v}_\alpha^{q+1}(x) v_\alpha(x_\alpha)} + \frac{|\mu_\alpha^{\frac{n+2}{4}} \hat{\Psi}_\alpha(x) + \hat{\rho}_{2,\alpha}(x) \mathcal{L}_\xi \hat{Z}_\alpha(x)|_\xi^2}{\hat{v}_\alpha^{q+1}(x) v_\alpha(x_\alpha)} \\ & - \mu_\alpha^2 \frac{\hat{b}_\alpha}{v_\alpha(\exp_{x_\alpha}(\mu_\alpha \cdot)) v_\alpha(x_\alpha)} - \frac{\langle \nabla w_\alpha, \hat{Y}_\alpha \rangle^2}{w_\alpha^2} \frac{1}{v_\alpha(x_\alpha) v_\alpha^{q+1}(\mu_\alpha \cdot)} \\ & - \mu_\alpha \langle \nabla w_\alpha, \hat{Y}_\alpha \rangle \left(\frac{\hat{d}_\alpha}{v_\alpha(\exp_{x_\alpha}(\mu_\alpha \cdot)) v_\alpha(x_\alpha)} + \frac{\hat{c}_\alpha}{v_\alpha^{q+1}(\exp_{x_\alpha}(\mu_\alpha \cdot)) v_\alpha(x_\alpha)} \right). \end{aligned}$$

Up to a subsequence, we denote

$$\begin{aligned} \hat{l}_\alpha &= \hat{v}_\alpha(0), \quad \text{with} \quad \lim_{\alpha \rightarrow \infty} \hat{l}_\alpha =: \hat{l} \in [0, 1], \\ l_\alpha &= v_\alpha^{-1}(x_\alpha), \quad \text{with} \quad \lim_{\alpha \rightarrow \infty} l_\alpha =: l \in [0, \varepsilon^{-1}] \end{aligned} \quad (3.3.30)$$

which follows from (3.3.9) and (3.3.26) in the case of the first limit, and from (3.1.19) for the second. Furthermore, (3.3.10) implies that

$$\hat{l} \hat{l} = \lim_{\alpha \rightarrow \infty} \mu_\alpha^{\frac{n-2}{2}} = 0. \quad (3.3.31)$$

Remark 14. *It is here that we use the hypothesis $V_\alpha(x_\alpha) \rightarrow 0$.*

We denote

$$\frac{\mathcal{L}_\xi \hat{Z}_\alpha(x)}{\hat{l}_\alpha^{\frac{q+2}{2}}} \rightarrow \mathcal{L}_\xi Z. \quad (3.3.32)$$

By standard elliptic theory, we find that there exists $w := \lim_{\alpha \rightarrow \infty} w_\alpha$ in $C_{loc}^{1,\eta}(\mathbb{R}^n)$, and by dividing the first equation by \hat{l} , we obtain

$$\begin{aligned} \Delta w &= \frac{n-2}{4(n-1)} (2V(\psi(x_0)) - \frac{n-1}{n} \tau^{*2}) w^{q-1} \hat{l}^{q-2} + \frac{n-2}{16(n-1)} \frac{\tilde{N}^2(x_0) |\mathcal{L}_\xi Z|_\xi^2}{w^{q+1}} \\ &\quad - \frac{n}{n-2} \frac{\langle \nabla w, \tilde{N}(x_0) \tilde{V}(x_0) \rangle^2}{w^{q+3}} \hat{l}^{q+2} \\ \vec{\Delta}_\xi Z &= -2 \frac{n-1}{n+1} \frac{\langle \tilde{V}(x_0), \nabla w \rangle \nabla w}{w^2} \hat{l}^{\frac{q+2}{2}} - \frac{n-1}{n} \langle \tilde{V}(x_0), \frac{\nabla^2 w}{w} \rangle \hat{l}^{\frac{q+2}{2}}. \end{aligned} \quad (3.3.33)$$

Since $\tilde{V}(x_0) = 0$, we obtain $\mathcal{L}_\xi Z = 0$. Had we not imposed this hypothesis, the next step would have been to classify the solutions of the second equation. To our knowledge, this is an open problem.

Therefore, the limit equation becomes :

$$\Delta w = f(x_0) w^{q-1} \hat{l}^{q-2}.$$

In fact, we can easily tackle the slightly more general equation

$$\Delta w = f(x_0) w^{q-1} \hat{l}^{q-2} - \frac{\langle \nabla w, Y(x_0) \rangle^2}{w^{q+3}} \hat{l}^{q+2}, \quad x \in \mathbb{R}^n, \quad (3.3.34)$$

even if $Y(x_0) \neq 0$. Based on the observation (3.3.31), we consider three separate cases.

First case. Let

$$l = 0 \quad \text{and} \quad \hat{l} \neq 0. \quad (3.3.35)$$

Then by passing to the limit in the first equation of (3.3.12), we obtain

$$\Delta U = f(x_0)U^{q-1} \quad (3.3.36)$$

in \mathbb{R}^n . The exact form of the solutions of this equation is known, thanks to the work of Caffarelli, Gidas and Spruck [CGS89] :

$$U(x) = \left(1 + \frac{f(x_0)|x - y_0|^2}{n(n-2)}\right)^{1-\frac{n}{2}} \quad \text{or} \quad U \equiv 0, \quad (3.3.37)$$

If U is non-trivial, with $y_0 \in \mathbb{R}^n$ the unique maximum point, there exist $(y_\alpha)_\alpha$ local maxima of $(v_\alpha)_\alpha$ approaching y_0 such that

$$d_g(x_\alpha, y_\alpha) = O(\mu_\alpha) \quad (3.3.38)$$

and

$$\mu_\alpha^{\frac{n-2}{2}} v_\alpha(y_\alpha) \rightarrow 1 \quad \text{as} \quad \alpha \rightarrow \infty. \quad (3.3.39)$$

Since $(y_\alpha)_\alpha$ are critical points, the hypothesis (3.3.5) implies that

$$d_g(\mathcal{S}_\alpha, y_\alpha)^{\frac{n-2}{2}} v_\alpha(y_\alpha) \leq 1$$

for all $\alpha \in \mathbb{N}$, so by (3.3.39), $d_g(\mathcal{S}_\alpha, y_\alpha) = O(\mu_\alpha)$; together with (3.3.38), the triangle inequality implies $d_g(\mathcal{S}_\alpha, x_\alpha) = O(\mu_\alpha)$, which contradicts (3.3.11).

If $U \equiv 0$, then

$$\lim_{\alpha \rightarrow \infty} \hat{v}_\alpha(0) = 0, \quad (3.3.40)$$

which contradicts (3.3.35).

Second case. Let

$$l \neq 0 \quad \text{and} \quad \hat{l} = 0. \quad (3.3.41)$$

Since $l \neq 0$, thanks to (3.1.19) and (3.3.27), w is bounded from below by a constant,

$$w \geq \varepsilon l. \quad (3.3.42)$$

Note also that (3.3.34) implies that w is subharmonic and that

$$\Delta w^{-\alpha} \leq \alpha \frac{|\nabla w|^2}{w^{\alpha+2}} \left[\frac{|Y(x_0)|^2}{\varepsilon^{q+2}} - (\alpha + 1) \right], \quad (3.3.43)$$

so $w^{-\alpha}$ is subharmonic for α large. By applying Lemma 21 (see the Annex), we deduce that w is constant, in contradiction with (3.3.28).

Third case. Let

$$l = 0 \quad \text{and} \quad \hat{l} = 0, \quad (3.3.44)$$

then w is a non-negative harmonic function on \mathbb{R}^n . Thus, $w = cst$ and furthermore, by (3.3.28), $w \equiv 1$. But $\hat{l} = 0$ implies that

$$\left| \frac{\nabla w(0)}{w(0)} \right|^n + \left| \frac{\nabla^2 w(0)}{w(0)} \right|^{\frac{n}{2}} = 1. \quad (3.3.45)$$

which leads to a contradiction. □

3.4 Asymptotic analysis

In this section, we assume that $(u_\alpha, W_\alpha)_{\alpha \in \mathbb{N}}$ is an L^∞ blow-up sequence, *i.e.* we ask that there exist a sequence $(x_\alpha)_{\alpha \in \mathbb{N}}$ of critical points of $(u_\alpha)_{\alpha \in \mathbb{N}}$ and a series of positive real numbers $(\rho_\alpha)_{\alpha \in \mathbb{N}}$, where

$$0 < \rho_\alpha < \frac{1}{16}i_g(M), \quad (3.4.1)$$

such that

$$\rho_\alpha^n \sup_{B_{x_\alpha}(8)} u_\alpha^q(\rho_\alpha x) \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty, \quad (3.4.2)$$

and moreover we ask that

$$d_g(x_\alpha, x)^n \left(u_\alpha^q(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^n + \left| \frac{\nabla^2 u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n}{2}} + |\mathcal{L}_g W_\alpha|_g(x) \right) \leq C, \quad (3.4.3)$$

$$x \in B_{x_\alpha}(8\rho_\alpha).$$

In the reminder of this section, we assume that $(u_\alpha, W_\alpha)_{\alpha \in \mathbb{N}}$ is a blow-up sequence, and we look at the kind of asymptotic profiles we can potentially obtain. At the very end, we rule all of them out, and thus obtain our compactness result. Note that, if we were to assume that (3.4.2) holds for a sequence $(x_\alpha)_\alpha$ in \mathcal{S}_α , with ρ_α smaller than the distance of x_α to any other point in \mathcal{S}_α , then (3.4.3) holds as well.

3.4.1 Harnack inequality

The following is a Harnack-type inequality. It is a direct consequence of the weak estimate and it plays a key role in ruling out clusters of bubbles where some are much larger than others.

Lemma 15. *Let $(u_\alpha, \rho_\alpha)_\alpha$ be a blow-up sequence such that (3.4.2) and (3.4.3) hold. Then there exists a constant $C_3 > 1$ such that for any sequence $0 < s_\alpha \leq \rho_\alpha$, we get*

$$s_\alpha^2 \|\nabla^2 u_\alpha\|_{L^\infty(\Omega_\alpha)} + s_\alpha \|\nabla u_\alpha\|_{L^\infty(\Omega_\alpha)} \leq C_3 \sup_{\Omega_\alpha} u_\alpha \leq C_3^2 \inf_{\Omega_\alpha} u_\alpha, \quad (3.4.4)$$

where $\Omega_\alpha = B_{x_\alpha}(6s_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}s_\alpha)$.

Remark 15. *When considering a rescaling of the type*

$$\bar{u}(x) = s_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_\alpha}(s_\alpha x)), \quad (3.4.5)$$

and $\bar{\Omega}_\alpha = B_0(6) \setminus B_0(\frac{1}{6})$, then the above lemma gives

$$\|\nabla^2 \bar{u}_\alpha\|_{L^\infty(\bar{\Omega}_\alpha)} + \|\nabla \bar{u}_\alpha\|_{L^\infty(\bar{\Omega}_\alpha)} \leq C_3 \sup_{\bar{\Omega}_\alpha} \bar{u}_\alpha \leq C_3^2 \inf_{\bar{\Omega}_\alpha} \bar{u}_\alpha. \quad (3.4.6)$$

Proof of Lemma (15) : Estimate (3.3.10) implies that

$$\left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right| \leq C_2 d_g(x_\alpha, x)^{-1} \quad \text{in } \Omega_\alpha, \quad (3.4.7)$$

and therefore

$$s_\alpha |\nabla \ln u_\alpha(x)| \leq 6C_2 \quad \text{in } \Omega_\alpha. \quad (3.4.8)$$

Similarly, it holds true that

$$s_\alpha^2 |\nabla^2 \ln u_\alpha(x)| \leq 6C_2 \quad \text{in } \Omega_\alpha. \quad (3.4.9)$$

Taking $C_3 \geq 6C_2$, we get the first inequality from (3.4.6). Then, from (2.3.46) and from the fact that the domain is an annulus $\Omega_\alpha = B_{x_\alpha}(6s_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}s_\alpha)$, we estimate that

$$\sup_{\Omega_\alpha} \ln u_\alpha - \inf_{\Omega_\alpha} \ln u_\alpha \leq l_\alpha(\Omega_\alpha) \|\nabla \ln u_\alpha\|_{L^\infty(\Omega_\alpha)} \leq 42C_2,$$

where $l_\alpha(\Omega_\alpha)$ is the infimum of the length of a curve in Ω_α drawn between a maximum and a minimum of u_α . Equivalently

$$\sup_{\Omega_\alpha} u_\alpha \leq e^{42C_2} \inf_{\Omega_\alpha} u_\alpha,$$

so it suffices to take $C_3 = e^{42C_2}$. \square

Let $(B_{x_\alpha}(16), \Phi_\alpha)$ be a conformal chart around x_α . We study the blow-up sequence in a Euclidean framework through these charts. By the properties we've imposed on φ_α ,

$$\left| q\xi^{kl} \partial_k (\ln \varphi_\alpha) (\mathcal{L}_\xi Z_\alpha)_{li} \right| \leq C |y| |\mathcal{L}_\xi Z_\alpha|_\xi, \quad \text{on } B_0(8\rho_\alpha). \quad (3.4.10)$$

By the definition of a blow-up sequence, we also get that

$$|x|^n \left(v_\alpha^q(x) + \left| \frac{\nabla v_\alpha(x)}{v_\alpha(x)} \right|^n + \left| \frac{\nabla^2 v_\alpha(x)}{v_\alpha(x)} \right|^{\frac{n}{2}} + |\mathcal{L}_\xi Z_\alpha(x)| \right) \leq C. \quad (3.4.11)$$

3.4.2 Strong estimate on v_α in $B_0(\mu_\alpha)$

The following result is a strong estimate on the size of a blow-up sequence in a very small ball $B_0(\mu_\alpha)$.

Lemma 16. *Let $(u_\alpha, W_\alpha)_{\alpha \in \mathbb{N}}$ be a blow-up sequence. Let*

$$\mu_\alpha^{1-\frac{n}{2}} := u_\alpha(x_\alpha) = v_\alpha(0). \quad (3.4.12)$$

Up to a subsequence, we have

$$\mu_\alpha \rightarrow 0 \quad \text{and} \quad \frac{\rho_\alpha}{\mu_\alpha} \rightarrow \infty. \quad (3.4.13)$$

Moreover, we see that

$$\mu_\alpha^{\frac{n-2}{2}} v_\alpha(\mu_\alpha x) \rightarrow U(x) \quad \text{in } \mathcal{C}_{loc}^{2,\eta}(\mathbb{R}^n) \quad \text{as } \alpha \rightarrow \infty \quad (3.4.14)$$

and

$$\mu_\alpha^n |\mathcal{L}_\xi Z_\alpha|_\xi(\mu_\alpha x) \rightarrow 0 \quad \text{and } \mathcal{C}_{loc}^0(\mathbb{R}^n) \quad \text{as } \alpha \rightarrow \infty. \quad (3.4.15)$$

We have denoted

$$x_0 = \lim_{\alpha \rightarrow \infty} x_\alpha \quad (3.4.16)$$

and

$$U(x) = \left(1 + \frac{f_0(x_0)}{n(n-2)} |x|^2 \right)^{1-\frac{n}{2}}. \quad (3.4.17)$$

Démonstration. The proof involves similar arguments to the ones used for Lemma (13). Let $y_\alpha \in B_{x_\alpha}(8\rho_\alpha)$ be such that

$$\begin{aligned} u_\alpha^q(y_\alpha) + \left| \frac{\nabla u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \right|^n + \left| \frac{\nabla^2 u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \right|^{\frac{n}{2}} + |\mathcal{L}_g W_\alpha(y_\alpha)| \\ = \sup_{B_{x_\alpha}(8\rho_\alpha)} \left(u_\alpha^q(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^n + \left| \frac{\nabla^2 u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n}{2}} + |\mathcal{L}_g W_\alpha(x)| \right) \end{aligned} \quad (3.4.18)$$

and let

$$\nu_\alpha^{-n} := u_\alpha^q(y_\alpha) + \left| \frac{\nabla u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \right|^n + \left| \frac{\nabla^2 u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \right|^{\frac{n}{2}} + |\mathcal{L}_g W_\alpha(y_\alpha)|. \quad (3.4.19)$$

Conditions (3.4.2) and (3.4.3) imply that

$$\frac{\rho_\alpha}{\nu_\alpha} \rightarrow \infty \quad \text{and} \quad \nu_\alpha \rightarrow 0 \quad \text{as} \quad \alpha \rightarrow \infty. \quad (3.4.20)$$

Moreover,

$$d_g(x_\alpha, y_\alpha) \leq C_2^{\frac{1}{n}} \nu_\alpha, \quad (3.4.21)$$

which implies that the coordinates of y_α in the exponential chart around x_α , defined as $\tilde{y}_\alpha := \nu^{-1} \exp_{x_\alpha}^{-1}(y_\alpha)$, are bounded by $C_2^{\frac{1}{n}}$. Up to a subsequence, we may choose a finite limit $\tilde{y}_0 := \lim_{\alpha \rightarrow \infty} \tilde{y}_\alpha$. We denote

$$\hat{v}_\alpha(x) = \nu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_\alpha}(\nu_\alpha x)) \quad \text{and} \quad \hat{Z}_\alpha(x) = \nu_\alpha^{n-1} Z_\alpha(\exp_{x_\alpha}(\nu_\alpha x)) \quad (3.4.22)$$

for $x \in \Omega_\alpha := B_0\left(\frac{8\rho_\alpha}{\nu_\alpha}\right)$. As before,

$$\hat{v}_\alpha(x) = O(1), \quad \left| \frac{\nabla \hat{v}_\alpha(x)}{\hat{v}_\alpha(x)} \right| = O(1), \quad \left| \frac{\nabla^2 \hat{v}_\alpha(x)}{\hat{v}_\alpha(x)} \right| = O(1) \quad (3.4.23)$$

and

$$|\mathcal{L}_\xi \hat{Z}_\alpha(x)|_\xi \rightarrow 0. \quad (3.4.24)$$

This implies that

$$\hat{v}_\alpha^q(\tilde{y}_\alpha) + \left| \frac{\nabla \hat{v}_\alpha(\tilde{y}_\alpha)}{\hat{v}_\alpha(\tilde{y}_\alpha)} \right|^n + \left| \frac{\nabla^2 \hat{v}_\alpha(\tilde{y}_\alpha)}{\hat{v}_\alpha(\tilde{y}_\alpha)} \right|^{\frac{n}{2}} = 1. \quad (3.4.25)$$

By applying the same analysis as in the proof of Lemma 13, we get that, up to passing to a subsequence, there exists $U_\lambda := \lim_{\alpha \rightarrow \infty} v_\alpha$ in $\mathcal{C}_{loc}^{2,\eta}(\mathbb{R}^n)$, with $x_0 := \lim_{\alpha \rightarrow \infty} x_\alpha$, such that

$$\Delta U_\lambda = f(x_0) U_\lambda^{q-1}, \quad (3.4.26)$$

Since $\nabla U_\lambda(0) = 0$, it holds that

$$U_\lambda(x) = \lambda^{\frac{n-2}{2}} \left(1 + \frac{f(x_0) \lambda^2 |x|^2}{n(n-2)} \right)^{1-\frac{n}{2}} \quad (3.4.27)$$

for some $\lambda > 0$, where

$$\frac{\nu_\alpha}{\mu_\alpha} \rightarrow \lambda. \quad (3.4.28)$$

This yields (3.4.13), (3.4.14) and (3.4.15), thanks to (3.4.20), (3.4.27) and (3.4.24). \square

3.4.3 The sphere of dominance around a blow-up point

We denote

$$B_\alpha(x) = \mu_\alpha^{\frac{n-2}{2}} \left(\mu_\alpha^2 + \frac{f_\alpha(x_\alpha)}{n(n-2)} |x|^2 \right)^{1-\frac{n}{2}} \quad (3.4.29)$$

and

$$\theta_\alpha(x) = \sqrt{\mu_\alpha^2 + |x|^2}. \quad (3.4.30)$$

Our next goal is to extend the estimates from a ball of size μ_α to one of size ρ_α . We define the radius on which the estimates continue to hold as

$$r_\alpha = \sup \mathcal{R}_\alpha \quad (3.4.31)$$

where

$$\mathcal{R}_\alpha = \left\{ 0 < r \leq \rho_\alpha, \quad v_\alpha \leq (1 + \varepsilon)B_\alpha, \quad |\nabla(v_\alpha - B_\alpha)|_\xi \leq \varepsilon|\nabla B_\alpha|_\xi, \right. \\ \left. \text{and } B_0(r) \setminus B_0(2R_\alpha\mu_\alpha) \right\} \quad (3.4.32)$$

where

$$R_\alpha^2 = \frac{n(n-2)}{f_\alpha(x_\alpha)}. \quad (3.4.33)$$

The two following properties hold for r_α :

$$r_\alpha = O(\sqrt{\mu_\alpha}) \quad (3.4.34)$$

and

$$r_\alpha \gg \mu_\alpha. \quad (3.4.35)$$

By the previous lemma, we know that the $\mathcal{C}^{2,\eta}$ limit holds on balls of order ρ_α and by definition also of size r_α , which is to say that the two are comparable. As a result, (3.4.13) implies (3.4.34). In order to get the second estimate, it suffices to note that, by the definition of r_α and by (3.1.19),

$$\varepsilon \leq C\mu_\alpha^{\frac{n-2}{2}} r_\alpha^{2-n}, \quad (3.4.36)$$

which directly implies (3.4.35).

First order estimates of v_α on $B_0(8\delta_\alpha)$

Lemma 17. *Let $(\delta_\alpha)_\alpha$ $0 < \delta_\alpha \leq r_\alpha$ be a sequence of radii. Then for any $z_\alpha \in B_0(8\delta_\alpha)$ there holds :*

$$v_\alpha(z_\alpha) + |\nabla v_\alpha(z_\alpha)||z_\alpha| + |\nabla^2 v_\alpha(z_\alpha)||z_\alpha|^2 \leq CB_\alpha(z_\alpha). \quad (3.4.37)$$

Moreover, there exists a sequence of positive numbers $(\kappa_\alpha)_{\alpha \in \mathbb{N}}$ such that

$$(1 - \kappa_\alpha)B_\alpha(z_\alpha) \leq v_\alpha(z_\alpha). \quad (3.4.38)$$

If $\delta_\alpha \rightarrow 0$, then $\kappa_\alpha \rightarrow 0$.

Démonstration. For $x \in B_0(8)$:

$$\bar{v}_\alpha(x) = r_\alpha^{\frac{n-2}{2}} v_\alpha(r_\alpha x). \quad (3.4.39)$$

Then \bar{v}_α satisfies

$$\Delta_\xi \bar{v}_\alpha(x) + r_\alpha^2 \bar{h}_\alpha(x) = \bar{f}_\alpha(x) \bar{v}_\alpha^{q-1}(x) + r_\alpha^{2n} \frac{\bar{a}_\alpha(x)}{\bar{v}_\alpha^{q+1}(x)} - r_\alpha^n \frac{\bar{b}_\alpha(x)}{\bar{v}_\alpha(x)} \\ - r_\alpha^{\frac{n}{2}} \frac{\langle \nabla \bar{v}_\alpha(x), \bar{Y}_\alpha(x) \rangle}{\bar{v}_\alpha(x)} \left(\frac{r_\alpha^{\frac{n-2}{2}} \bar{d}_\alpha(x)}{\bar{v}_\alpha(x)} + \frac{r_\alpha^{\frac{3n-2}{n-2}} \bar{c}_\alpha(x)}{\bar{v}_\alpha^{q+1}(x)} \right) \\ - r_\alpha^{2n+2} \frac{\langle \nabla \bar{v}_\alpha(x), \bar{Y}_\alpha(x) \rangle^2}{\bar{v}_\alpha(x)} \frac{1}{\bar{v}_\alpha^{q+1}(x)} \quad (3.4.40)$$

where

$$\begin{aligned}\bar{u}_\alpha(x) &= \tilde{u}_\alpha(r_\alpha x), & \bar{h}_\alpha(x) &= \tilde{h}_\alpha(r_\alpha x), & \bar{f}_\alpha(x) &= \tilde{f}_\alpha(r_\alpha x), \\ \bar{c}_\alpha(x) &= \tilde{c}_\alpha(r_\alpha x), & \bar{d}_\alpha(x) &= \tilde{d}_\alpha(r_\alpha x), & \bar{Y}_\alpha(x) &= \tilde{Y}_\alpha(r_\alpha x).\end{aligned}\tag{3.4.41}$$

By the definition of r_α , we know that

$$\bar{v}_\alpha(x) \leq C \left(\frac{\mu_\alpha}{r_\alpha} \right)^{\frac{n-2}{2}}\tag{3.4.42}$$

in $B_0(1) \setminus B_0(\frac{1}{2})$. By the weak estimate we know that

$$\begin{aligned}\bar{v}_\alpha &\leq C \\ \left| \frac{\nabla \bar{v}_\alpha(x)}{\bar{v}_\alpha(x)} \right| &\leq C \\ \left| \frac{\nabla^2 \bar{v}_\alpha(x)}{\bar{v}_\alpha(x)} \right| &\leq C \\ r_\alpha^{2n} \bar{u}_\alpha &\leq C\end{aligned}\tag{3.4.43}$$

in $B_0(8) \setminus B_0(\frac{1}{2})$. We conclude the proof by Lemma 15.

Considering G_α the Green function of $\Delta_g + h_\alpha$ in M . For any sequence (z_α) of points in $B_0(8r_\alpha)$:

$$v_\alpha(z_\alpha) \geq \varphi_\alpha(z_\alpha) \int_{B_0(r_\alpha)} \varphi_\alpha(y) G_\alpha(\Phi_\alpha^{-1}(z_\alpha), \Phi_\alpha^{-1}(y))_\alpha v_\alpha^{q-1}(y) dy.\tag{3.4.44}$$

In particular,

$$\begin{aligned}\frac{v_\alpha(z_\alpha)}{B_\alpha(z_\alpha)} &\geq \varphi_\alpha(z_\alpha) \int_{B_0(\frac{6r_\alpha}{\mu_\alpha})} \varphi_\alpha(\mu_\alpha y) \tilde{f}_\alpha(\mu_\alpha y) \left(\mu_\alpha^{\frac{n-2}{2}} v_\alpha(\mu_\alpha y) \right)^{q-1} \\ &\quad \times G_\alpha(\Phi_\alpha^{-1}(z_\alpha), \Phi_\alpha^{-1}(\mu_\alpha y)) d_g(\Phi_\alpha^{-1}(z_\alpha), \Phi_\alpha^{-1}(\mu_\alpha y))^{n-2} \\ &\quad \times \left(\frac{\mu_\alpha^2 + \frac{f_\alpha(x_\alpha)}{n(n-2)} |z_\alpha|^2}{d_g(\Phi_\alpha^{-1}(z_\alpha), \Phi_\alpha^{-1}(\mu_\alpha y))^2} \right)^{\frac{n-2}{2}} dy.\end{aligned}\tag{3.4.45}$$

□

Improved weak estimate of $\mathcal{L}_\xi Z_\alpha$ on $B_0(7\delta_\alpha)$

Lemma 18. *Let $(\delta_\alpha)_\alpha$ be a sequence of positive numbers such that $\delta_\alpha \gg \mu_\alpha$ and $\delta_\alpha \leq \sqrt{\mu_\alpha}$. We get for any $x \in B_0(7\delta_\alpha)$,*

$$\int_{B_0(6\delta_\alpha)} |x - y|^{2-n} |\mathcal{L}_\xi Z_\alpha(y)|_\xi^2 v_\alpha^{-q-1}(y) dy \leq C (B_\alpha(x) + O(1))\tag{3.4.46}$$

and as a consequence

$$\int_{B_0(6\delta_\alpha) \setminus B_0(\delta_\alpha)} |\mathcal{L}_\xi Z_\alpha|_\xi^2 dy \leq C \left(\mu_\alpha^{2n-2} \delta_\alpha^{2-3n} + \mu_\alpha^{\frac{3n}{2}-1} \delta_\alpha^{-2n} \right),\tag{3.4.47}$$

and there exists a sequence $s_\alpha \in (5\delta_\alpha, 6\delta_\alpha)$ such that

$$\int_{\partial B(s_\alpha)} |\mathcal{L}_\xi Z_\alpha|_\xi^2 d\sigma \leq C \left(\mu_\alpha^{2n-2} \delta_\alpha^{1-3n} + \mu_\alpha^{\frac{3n}{2}-1} \delta_\alpha^{-2n-1} \right).\tag{3.4.48}$$

Démonstration. We use the Green's representation theorem for $\Delta_\xi + \tilde{h}_\alpha$ in $B_0(7\delta_\alpha)$ in the 1st equation, and obtain

$$\int_{B_0(6\delta_\alpha)} |x-y|^{2-n} \frac{|\mathcal{L}_\xi Z_\alpha(y)|_\xi^2}{v_\alpha^{q+1}(y)} dy \leq C (B_\alpha(x) + H_1 + H_2 + H_3), \quad (3.4.49)$$

where

$$\begin{aligned} H_1 &= \int_{B_0(6\delta_\alpha)} \frac{\tilde{b}_\alpha(y)}{v_\alpha(y)} |x-y|^{2-n} dy, \\ H_2 &= \int_{B_0(6\delta_\alpha)} \langle \nabla v_\alpha(y), \tilde{V}(y) \rangle \left(\frac{\tilde{d}_\alpha(y)}{v_\alpha^2(y)} + \frac{\tilde{c}_\alpha(y)}{v_\alpha^{q+2}(y)} \right) |x-y|^{2-n} dy, \\ H_3 &= \int_{B_0(6\delta_\alpha)} \frac{\langle \nabla v_\alpha(y), \tilde{V}(y) \rangle^2}{v_\alpha^{q+3}(y)} |x-y|^{2-n} dy. \end{aligned} \quad (3.4.50)$$

Lemma 17 yields the following estimates :

$$H_1 \leq C \int_{B_0(6\delta_\alpha)} \mu_\alpha^{\frac{2-n}{n}} \theta_\alpha^{n-2}(y) |x-y|^{2-n} dy \leq C \mu_\alpha \left(\frac{\delta_\alpha^2}{\mu_\alpha} \right)^{\frac{n}{2}}, \quad (3.4.51)$$

$$\begin{aligned} H_2 &\leq C \int_{B_0(6\delta_\alpha)} \theta_\alpha^{-2}(y) \left(\mu_\alpha^{\frac{2-3n}{3}} \theta_\alpha^{3n-2}(y) + \mu_\alpha^{\frac{2-n}{2}} \theta_\alpha^{n-2}(y) \right) \\ &\quad \times |x-y|^{2-n} dx, \\ &\leq C \left(\frac{\delta_\alpha^2}{\mu_\alpha} \right)^{\frac{3n-2}{2}} + C \left(\frac{\delta_\alpha^2}{\mu_\alpha} \right)^{\frac{n-2}{2}} \end{aligned} \quad (3.4.52)$$

and

$$H_3 \leq C \int_{B_0(6\delta_\alpha)} \mu_\alpha^{\frac{2-3n}{2}} \theta_\alpha^{3n-2}(y) |x-y|^{2-n} dx \leq C \left(\frac{\delta_\alpha^2}{\mu_\alpha} \right)^{\frac{3n-2}{2}}. \quad (3.4.53)$$

As a consequence,

$$\int_{B_0(6\delta_\alpha)} |x-y|^{2-n} \frac{|\mathcal{L}_\xi Z_\alpha(y)|_\xi^2}{v_\alpha^{q+1}(y)} dy \leq C \left(\mu_\alpha^{\frac{n-2}{2}} \theta_\alpha^{2-n}(x) + 1 \right). \quad (3.4.54)$$

In particular, we get (3.4.47) and (3.4.48). \square

First order estimate of $\mathcal{L}_\xi Z_\alpha$ on $B_0(3\delta_\alpha)$

We use the previous improved weak estimate in order to get a first order estimate of $\mathcal{L}_\xi Z_\alpha$. For $x \neq 0$, let

$$\mathcal{G}_i(x)_j = -\frac{1}{4(n-1)\omega_{n-1}} |x|^{2-n} \left((3n-2)\delta_{ij} + (n-2) \frac{y_i y_j}{|x|^2} \right) \quad (3.4.55)$$

be the i -th fundamental solution of $\vec{\Delta}_\xi$ in \mathbb{R}^n . We define on \mathbb{R}^n the vector field

$$V_\alpha(x)_i = -\frac{n^2}{2(n-2)} \ln \left(1 + \frac{|x|^2}{\mu_\alpha^2} \right) \tilde{V}_\alpha(0)_i + \frac{n}{\mu_\alpha^2 + |x|^2} \langle x, \tilde{V}_\alpha(0) \rangle x_i \quad (3.4.56)$$

and a vector field R_α such that

$$\begin{aligned} \vec{\Delta}_\xi (V_\alpha + R_\alpha)(x) &= 2 \frac{n-1}{n-2} \left(-\frac{\langle \nabla^2 B_\alpha(x), \tilde{V}_\alpha(0) \rangle}{B_\alpha(x)} \right. \\ &\quad \left. + \frac{3n-2}{n-2} \frac{\langle \nabla B_\alpha(x), \tilde{V}_\alpha(0) \rangle \nabla B_\alpha(x)}{B_\alpha^2(x)} \right). \end{aligned} \quad (3.4.57)$$

Note, in particular, that

$$|\vec{\Delta}_\xi R_\alpha(x)| \leq C |\tilde{V}_\alpha(0)| \mu_\alpha^2 \theta_\alpha(x)^{-4}. \quad (3.4.58)$$

Thus,

$$|\mathcal{L}_\xi R_\alpha(x)| \leq \begin{cases} C |\tilde{V}_\alpha(0)| \mu_\alpha^2 \theta_\alpha^{-3}(x) & n = 5 \\ C |\tilde{V}_\alpha(0)| \mu_\alpha^2 \theta_\alpha^{-3}(x) \ln \left(1 + \frac{\theta_\alpha(x)}{\mu_\alpha} \right) & n = 4, \\ C |\tilde{V}_\alpha(0)| \mu_\alpha^2 \theta_\alpha^{-2}(x) & n = 3. \end{cases} \quad (3.4.59)$$

By direct calculation, we see that

$$\begin{aligned} (\mathcal{L}_\xi V_\alpha)_{ij}(x) = & -\frac{2n}{n-2} |\tilde{V}_\alpha(0)| \frac{|x|}{\mu_\alpha^2 + |x|^2} \left(\frac{x_j}{|x|} \frac{\tilde{V}_\alpha(0)_i}{|\tilde{V}_\alpha(0)|} + \frac{x_i}{|x|} \frac{\tilde{V}_\alpha(0)_j}{|\tilde{V}_\alpha(0)|} - \frac{2}{n} \left\langle \frac{x}{|x|}, \frac{\tilde{V}_\alpha(0)}{|\tilde{V}_\alpha(0)|} \right\rangle \right) \\ & - 4 \frac{|x|^3 |\tilde{V}_\alpha(0)|}{(\mu_\alpha^2 + |x|^2)^2} \left\langle \frac{x}{|x|}, \frac{\tilde{V}_\alpha(0)}{|\tilde{V}_\alpha(0)|} \right\rangle \\ & - 4n \frac{|x|^3 |\tilde{V}_\alpha(0)|}{(\mu_\alpha^2 + |x|^2)^2} \left\langle \frac{x}{|x|}, \frac{\tilde{V}_\alpha(0)}{|\tilde{V}_\alpha(0)|} \right\rangle \frac{x_i}{|x|} \frac{x_j}{|x|}. \end{aligned} \quad (3.4.60)$$

Note that

$$|\mathcal{L}_\xi V_\alpha(x)| \leq C |\tilde{V}_\alpha(0)| \theta_\alpha^{-1}(x). \quad (3.4.61)$$

Lemma 19. *Let $(\delta_\alpha)_\alpha$ be a sequence of positive numbers such that*

$$\frac{\mu_\alpha}{\delta_\alpha} \rightarrow 0 \quad \text{and} \quad \delta_\alpha \leq \min(r_\alpha, \sqrt{\mu_\alpha}). \quad (3.4.62)$$

For any $x \in B_0(3\delta_\alpha)$, we get the following estimate on $|\mathcal{L}_\xi(Z_\alpha - V_\alpha)(x)|$:

$$|\mathcal{L}_\xi Z_\alpha(x)| \leq \theta_\alpha^{-1}(x) + \mu_\alpha^{n-1} \delta_\alpha^{1-2n}. \quad (3.4.63)$$

Démonstration. Without making mention of the conformal change factor φ_α , We apply the Green representation theorem on the 2nd equation. Let $\mathcal{G}_{\alpha,i}$ be the i -th Green 1-form for $\vec{\Delta}_\xi$ with Neumann boundary conditions on $B_0(s_\alpha)$, $s_\alpha \leq 4\delta_\alpha$. Similarly, let

$$\mathcal{H}_{ij,\alpha}(x, y)_p = \partial_i \mathcal{G}_{\alpha,j}(x, y)_p + \partial_j \mathcal{G}_{\alpha,i}(x, y)_p - \frac{2}{n} \xi_{ij} \sum_{k=1}^n \partial_k \mathcal{G}_{\alpha,k}(x, y)_p. \quad (3.4.64)$$

There holds that

$$\begin{aligned} \mathcal{L}_\xi(Z_\alpha - V_\alpha - R_\alpha)_{ij}(x) = & \int_{B_0(s_\alpha)} \mathcal{H}_{ij,\alpha}(x, y)_p \vec{\Delta}_\xi(Z_\alpha - V_\alpha - R_\alpha)^p(y) dy \\ & + \int_{\partial B_0(s_\alpha)} \mathcal{H}_{ij,\alpha}(z_\alpha, y)_p \nu_p \mathcal{L}_\xi(Z_\alpha - V_\alpha - R_\alpha)^{pq}(y) d\sigma. \end{aligned} \quad (3.4.65)$$

Keeping in mind that R_α is negligible compared to V_α , we obtain the estimate

$$|\mathcal{L}_\xi(Z_\alpha - V_\alpha - R_\alpha)(x)|_\xi \leq C (I_1 + I_2 + I_3 + I_4 + J_1 + J_2), \quad (3.4.66)$$

where the bulk terms are

$$\begin{aligned}
I_1 &= \int_{B_0(6\delta_\alpha)} |x-y|^{1-n} |\mathcal{L}_\xi Z_\alpha(y)| dy \\
I_2 &= \left| 2^{\frac{n-1}{n-2}} \int_{B_0(6\delta_\alpha)} \frac{3n-2}{n-2} \left(\frac{\langle \nabla v_\alpha(y), \tilde{V}_\alpha(y) \rangle \nabla v_\alpha(y)}{v_\alpha^2(y)} - \frac{\langle \nabla B_\alpha(y), \tilde{V}_\alpha(y) \rangle \nabla B_\alpha(y)}{B_\alpha^2(y)} \right) |x-y|^{1-n} \right. \\
&\quad \left. - \left(\frac{\langle \nabla^2 v_\alpha(y), \tilde{V}_\alpha(y) \rangle}{v_\alpha(y)} - \frac{\langle \nabla^2 B_\alpha(y), \tilde{V}_\alpha(y) \rangle}{B_\alpha(y)} \right) |x-y|^{1-n} dy \right| \\
I_3 &= \left| \int_{B_0(6\delta_\alpha)} |x-y|^{1-n} \left(- \frac{\langle \nabla v_\alpha(y), \tilde{V}_\alpha(y) \rangle}{v_\alpha(y)} \nabla \ln \tilde{N}_\alpha(y) \right. \right. \\
&\quad \left. \left. + \operatorname{div} \tilde{V}_\alpha(y) \frac{\nabla v_\alpha(y)}{v_\alpha(y)} - \frac{\langle \nabla \tilde{V}_\alpha(y), \nabla v_\alpha(y) \rangle}{v_\alpha(y)} \right) dy \right| \\
I_4 &= \int_{B_0(6\delta_\alpha)} |x-y|^{1-n} dy
\end{aligned} \tag{3.4.67}$$

and the boundary terms

$$\begin{aligned}
J_1 &= \int_{\partial B_0} |x-y|^{1-n} |\mathcal{L}_\xi V_\alpha(x)| d\sigma, \\
J_2 &= \int_{\partial B_0} |x-y|^{1-n} |\mathcal{L}_\xi Z_\alpha(x)| d\sigma.
\end{aligned} \tag{3.4.68}$$

Then, by (3.4.61),

$$J_1 \leq C\delta_\alpha^{-1}, \tag{3.4.69}$$

and by (3.4.48),

$$J_2 \leq \mu_\alpha^{n-1} \delta_\alpha^{1-2n}. \tag{3.4.70}$$

Next, we see that

$$\begin{aligned}
I_2 &\leq C \int_{B_0(6\delta_\alpha)} |x-y|^{1-n} \theta_\alpha^{-2}(y) dy \\
&\leq C \theta_\alpha^{-1}(x) \int_{B_0\left(\frac{6\delta_\alpha}{\theta_\alpha(x)}\right)} \left| \frac{x}{\theta_\alpha(x)} - z \right|^{1-n} \frac{1}{\left(\frac{\mu_\alpha}{\theta_\alpha(x)}\right)^2 + |z|^2} dz
\end{aligned} \tag{3.4.71}$$

so that

$$|I_2| \leq \theta_\alpha^{-1}(x). \tag{3.4.72}$$

The term I_3 is in fact negligible when compared to I_2 and so

$$|I_3| \leq \theta_\alpha^{-1}(x) \tag{3.4.73}$$

also. It is also clear that

$$I_4 \leq C\delta_\alpha. \tag{3.4.74}$$

Coming back to (3.4.66) with all these estimates, we thus obtain that

$$|\mathcal{L}_\xi Z_\alpha(x)| \leq C (\theta_\alpha(x)^{-1} + \mu_\alpha^{n-1} \delta_\alpha^{1-2n}) + I_1. \tag{3.4.75}$$

It remains to estimate I_1 . We shall use an iterative argument to do it. Assume that

$$|\mathcal{L}_\xi Z_\alpha(x)| \leq C \left(\theta_\alpha(x)^{-\beta} + \mu_\alpha^{n-1} \delta_\alpha^{1-2n} \right) \tag{3.4.76}$$

for some $1 < \beta \leq n$. Note that, thanks to the weak estimate (3.4.11) on $\mathcal{L}_\xi Z_\alpha$, it holds for $\beta = n$. If (3.4.76) holds, we can write that

$$\begin{aligned}
I_1 &\leq C\mu_\alpha^{n-1}\delta_\alpha^{1-2n}\int_{B_0(6\delta_\alpha)}|x-y|^{1-n}dy \\
&\quad +C\int_{B_0(6\delta_\alpha)}|x-y|^{1-n}\theta_\alpha(y)^{-\beta}dy \\
&\leq C\mu_\alpha^{n-1}\delta_\alpha^{2-2n} \\
&\quad +\begin{cases} \theta_\alpha(x)^{1-\beta} & \text{if } \beta < n \\ \theta_\alpha(x)^{1-n}\ln\left(1+\frac{\theta_\alpha(x)}{\mu_\alpha}\right) & \text{if } \beta = n \end{cases}
\end{aligned} \tag{3.4.77}$$

Remember here that $\beta > 1$. Coming back to (3.4.75), we obtain that, if (3.4.76) holds for some $1 < \beta \leq n$, it necessarily also holds when β is replaced by $\beta - \frac{1}{2}$. Since, as already said, it holds for $\beta = n$, we obtain by induction that it holds for all $\beta = n - \frac{k}{2}$ as long as $n - \frac{k-1}{2} > 1$. Thus, it holds for $\beta = 1$. But this is exactly the estimate (3.4.63). \square

Remark 16. For $\delta_\alpha = r_\alpha$, we get that

$$|\mathcal{L}_\xi Z_\alpha| \leq \theta_\alpha^{-1} + \mu_\alpha^{n-1}\delta_\alpha^{1-2n} \tag{3.4.78}$$

implies

$$|\mathcal{L}_\xi Z_\alpha| \leq \left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-1}\theta_\alpha^{-n}. \tag{3.4.79}$$

Asymptotic profile on $B_0(2r_\alpha)$

Lemma 20. Up to a subsequence, it holds that

$$v_\alpha(0)r_\alpha^{n-2}v_\alpha(r_\alpha x) \rightarrow \frac{R_0^{n-2}}{|x|^{n-2}} + H(x) \quad \text{in } \mathcal{C}_{loc}^2(B_0(2)\setminus\{0\}), \tag{3.4.80}$$

where H is a non-negative superharmonic function in $B_0(2)$. We recall that, by (3.4.12),

$$v_\alpha(0) = \mu_\alpha^{1-\frac{n}{2}}. \tag{3.4.81}$$

Démonstration. **Step 1 :** Let

$$\check{v}_\alpha(x) = \mu_\alpha^{1-\frac{n}{2}}r_\alpha^{n-2}v_\alpha(r_\alpha x), \quad x \in B_0(2), \tag{3.4.82}$$

where μ_α is defined in (3.4.12). Then

$$\Delta_\xi \check{v}_\alpha = \check{F}_\alpha, \tag{3.4.83}$$

with

$$\begin{aligned}
\check{F}_\alpha &= -r_\alpha^2\check{h}_\alpha(x)\check{v}_\alpha(x) + \mu_\alpha^2r_\alpha^{-2}\check{f}_\alpha(x)\check{v}_\alpha^{q-1}(x) \\
&\quad + \mu_\alpha^{2-2n}r_\alpha^{4n-2}\frac{\check{\rho}_{1,\alpha}(x)+|\check{\Psi}_\alpha(x)+\check{\rho}_{2,\alpha}(x)\mathcal{L}_\xi Z_\alpha(x)|_\xi^2}{\check{v}_\alpha^{q+1}(x)} \\
&\quad - \mu_\alpha^{2-n}r_\alpha^{2n-2}\frac{\check{b}_\alpha(x)}{\check{v}_\alpha(x)} - \mu_\alpha^{\frac{2-n}{2}}r_\alpha^{n-2}\frac{\langle \nabla \check{v}_\alpha(x), \check{Y}_\alpha(x) \rangle^2}{\check{v}_\alpha^2(x)u_\alpha^{q+1}(\exp_{x_\alpha}(r_\alpha x))} \\
&\quad - \mu_\alpha^{1-\frac{n}{2}}r_\alpha^{n-1}\frac{\langle \nabla \check{v}_\alpha(x), \check{Y}_\alpha(x) \rangle}{\check{v}_\alpha(x)}\left(\frac{\check{d}_\alpha(x)}{v_\alpha(x)} + \frac{\check{c}_\alpha(x)}{v_\alpha(x)^{q+1}}\right)
\end{aligned} \tag{3.4.84}$$

This implies that

$$\begin{aligned} \Delta_{\check{g}_\alpha} \check{v}_\alpha &= \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \check{f}_\alpha \check{v}_\alpha^{q-1} + \left(\frac{r_\alpha^2}{\mu_\alpha}\right)^{2n} \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \frac{|\check{\rho}_{2,\alpha}|^2 |\mathcal{L}_\xi Z_\alpha(r_\alpha x)|^2}{\check{v}_\alpha^{q+1}} \\ &\quad - \left(\frac{r_\alpha^2}{\mu_\alpha}\right)^{2n-2} \frac{|\nabla \check{v}_\alpha, \check{Y}_\alpha|^2}{\check{v}_\alpha^{q+3}} + o(1); \end{aligned} \quad (3.4.85)$$

where

$$\check{f}_\alpha(x) = \tilde{f}_\alpha(r_\alpha x), \quad \check{\rho}_{2,\alpha} = \tilde{\rho}_{2,\alpha}(r_\alpha x), \quad \text{and} \quad \check{Y}_\alpha(x) = \tilde{Y}_\alpha(r_\alpha x). \quad (3.4.86)$$

By the definition (3.4.32), there holds for some positive C that

$$\left(\left(\frac{\mu_\alpha}{r_\alpha}\right)^2 + \frac{f_\alpha(x_\alpha)}{n(n-2)} |x|^2 \right)^{1-\frac{n}{2}} \leq \check{v}_\alpha(x) \leq \left(\frac{f_\alpha(x_\alpha) |x|^2}{n(n-2)} \right)^{1-\frac{n}{2}}. \quad (3.4.87)$$

Similarly,

$$|\nabla \check{v}_\alpha(x)| \leq \left(\frac{f_\alpha(x_\alpha) |x|^2}{n(n-2)} \right)^{-\frac{n}{2}}. \quad (3.4.88)$$

Moreover, for any $x \in B_0(2)$:

$$\frac{\check{a}_\alpha(x)}{\check{v}_\alpha^{q+1}(x)} \leq C \left(\left(\frac{\mu_\alpha}{r_\alpha}\right)^2 + \frac{f_\alpha(x_\alpha)}{n(n-2)} |x|^2 \right)^{\frac{n}{2}} \in L^\infty(B_0(2) \setminus \{0\}). \quad (3.4.89)$$

We recall that we've assumed $\check{Y}_\alpha \rightarrow 0$ in $\mathcal{C}^{0,\alpha}$. By standard elliptic theory, we see that

$$\check{v}_\alpha \rightarrow \check{v} \quad \text{in } \mathcal{C}_{loc}^1(B_0(2) \setminus \{0\}) \quad \text{as } \alpha \rightarrow \infty. \quad (3.4.90)$$

For $x \neq 0$,

$$\check{v}(x) = \frac{\lambda_0}{|x|^{n-2}} + H(x), \quad (3.4.91)$$

where H is a superharmonic function in $B_0(2)$ and $\lambda_0 = \left(\frac{n(n-2)}{f(x_0)}\right)^{1-\frac{n}{2}}$. Moreover, $H \geq 0$ in $B_0(2)$. If $r_\alpha < \rho_\alpha$, then $H(0) > 0$. Indeed, by the definition (3.4.32), there exists $y_\alpha \in B_0(r_\alpha)$ such that at least one of the following conditions hold :

1. $v_\alpha(y_\alpha) = (1 + \varepsilon) B_\alpha(y_\alpha)$,
2. $|\nabla v_\alpha(y_\alpha)|_\xi = (1 + \varepsilon) |\nabla B_\alpha(y_\alpha)|_\xi$,

Letting $\check{y}_\alpha = \frac{y_\alpha}{r_\alpha}$, we see that either $H(\check{y}_\alpha)$ or $\nabla H(\check{y}_\alpha)$ are non-zero, and since H is a non-negative superharmonic function, then $H(0) > 0$. Independently, we show that $H(0) \leq 0$. The Pohozaev identity writes as

$$\begin{aligned} &\int_{B_0(\delta r_\alpha)} (x^k \partial_k v_\alpha(x) + \frac{n-2}{2} v_\alpha(x)) \Delta_\xi v_\alpha(x) dx \\ &= \int_{\partial B_0(\delta r_\alpha)} \left(\frac{1}{2} \delta r_\alpha |\nabla v_\alpha(x)|_\xi^2 - \frac{n-2}{2} v_\alpha(x) \partial_\nu v_\alpha(x) - \delta r_\alpha (\partial_\nu v_\alpha(x))^2 \right) d\sigma. \end{aligned} \quad (3.4.92)$$

Thanks to (3.4.80), we can estimate the boundary terms as

$$\begin{aligned} &\int_{\partial B_0(\delta r_\alpha)} \left(\frac{1}{2} \delta r_\alpha |\nabla v_\alpha(x)|_\xi^2 - \frac{n-2}{2} v_\alpha(x) \partial_\nu v_\alpha(x) - \delta r_\alpha (\partial_\nu v_\alpha(x))^2 \right) d\sigma \\ &= \left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-2} \left(\int_{\partial B_0(\delta)} \left(\frac{1}{2} \delta |\nabla \Psi|^2 - \frac{n-2}{2} \Psi \partial_\nu \Psi - \delta (\partial_\nu \Psi)^2 \right) d\sigma + o(1) \right) \end{aligned} \quad (3.4.93)$$

where $\Psi(x) = R_0^{n-2} |x|^{2-n} + H(x)$. Simple computations lead then to

$$\begin{aligned} & \int_{\partial B_0(\delta r_\alpha)} \left(\frac{1}{2} \delta r_\alpha |\nabla v_\alpha(x)|_\xi^2 - \frac{n-2}{2} v_\alpha(x) \partial_\nu v_\alpha(x) - \delta r_\alpha (\partial_\nu v_\alpha(x))^2 \right) d\sigma \\ &= \left(\frac{\mu_\alpha}{r_\alpha} \right)^{n-2} \left(\frac{(n-2)^2}{2} \omega_{n-1} R_0^{n-2} H(0) + O(\delta) \right). \end{aligned} \quad (3.4.94)$$

On the other hand, the LHS writes as

$$\int_{B_0(\delta r_\alpha)} \left(x^k \partial_k v_\alpha(x) + \frac{n-2}{2} v_\alpha(x) \right) \Delta_\xi v_\alpha(x) dx = J_1 + J_2 + J_3 + J_4, \quad (3.4.95)$$

where

$$\begin{aligned} J_1 &= - \int_{B_0(\delta r_\alpha)} \left(x^k \partial_k v_\alpha(x) + \frac{n-2}{2} v_\alpha(x) \right) \\ &\quad \times \left(\tilde{h}_\alpha(x) v_\alpha(x) + \frac{\tilde{b}_\alpha(x)}{v_\alpha(x)} + \langle \nabla v_\alpha(x), \tilde{Y}_\alpha(x) \rangle \left(\frac{\tilde{d}_\alpha(x)}{v_\alpha^2(x)} + \frac{\tilde{c}_\alpha(x)}{v_\alpha^{q+2}(x)} \right) \right) dx \\ J_2 &= \int_{B_0(\delta r_\alpha)} \left(x^k \partial_k v_\alpha(x) + \frac{n-2}{2} v_\alpha(x) \right) \tilde{f}_\alpha(x) v_\alpha^{q-1}(x) dx \\ J_3 &= \int_{B_0(\delta r_\alpha)} \left(x^k \partial_k v_\alpha(x) + \frac{n-2}{2} v_\alpha(x) \right) \frac{\tilde{\rho}_{1,\alpha}(x) + |\tilde{\Psi}_\alpha(x) + \tilde{\rho}_{2,\alpha}(x) \mathcal{L}_\xi Z_\alpha(x)|^2}{v_\alpha^{q+1}(x)} dx \\ J_4 &= - \int_{B_0(\delta r_\alpha)} \left(x^k \partial_k v_\alpha(x) + \frac{n-2}{2} v_\alpha(x) \right) \frac{\langle \nabla v_\alpha(x), \tilde{Y}_\alpha(x) \rangle^2}{v_\alpha^{q+3}(x)} dx \end{aligned} \quad (3.4.96)$$

We find estimates for each quantity in turn. In the case of J_1 , we notice that

$$\left| \int_{B_0(\delta r_\alpha)} \tilde{h}_\alpha(x) B_\alpha^2(x) dx \right| \leq C \begin{cases} \mu_\alpha^2 & \text{if } n = 5 \\ \mu_\alpha^2 \ln \left(\frac{r_\alpha}{\mu_\alpha} \right) & \text{if } n = 4 \\ \delta r_\alpha \mu_\alpha & \text{if } n = 3 \end{cases} \quad (3.4.97)$$

Then we have that

$$\left| \int_{B_0(\delta r_\alpha)} \tilde{b}_\alpha(x) dx \right| \leq C (\delta r_\alpha)^n, \quad (3.4.98)$$

and

$$\begin{aligned} \left| \int_{B_0(\delta r_\alpha)} \frac{\langle \nabla B_\alpha(x), \tilde{Y}_\alpha(x) \rangle}{B_\alpha^q(x)} \left(\tilde{d}_\alpha(x) + \frac{\tilde{c}_\alpha(x)}{v_\alpha^q(x)} \right) dx \right| &\leq C \int_{B_0(\delta r_\alpha)} \theta_\alpha^{-1}(x) dx \\ &\leq C (\delta r_\alpha)^{n-1}. \end{aligned} \quad (3.4.99)$$

For J_3 , we obtain

$$\begin{aligned} \left| \int_{B_0(\delta r_\alpha)} \frac{|\mathcal{L}_\xi Z_\alpha(x)|^2}{B_\alpha^q(x)} dx \right| &\leq \int_{B_0(\delta r_\alpha)} \left(\frac{\mu_\alpha}{r_\alpha} \right)^{2n-2} \mu_\alpha^{-n} \theta_\alpha^n(x) dx \\ &\leq C \left(\frac{\mu_\alpha}{r_\alpha} \right)^{n-2} r_\alpha \delta^{2n}, \end{aligned} \quad (3.4.100)$$

while for J_4 , we get

$$\begin{aligned} \left| \int_{B_0(\delta r_\alpha)} \frac{\langle \nabla B_\alpha(x), \tilde{Y}_\alpha(x) \rangle^2}{B_\alpha^{q+3}(x)} dx \right| &\leq \int_{B_0(\delta r_\alpha)} \theta_\alpha^{2n-2}(x) \mu_\alpha^{-n} dx \\ &\leq C \left(\frac{\mu_\alpha}{r_\alpha} \right)^n \left(\frac{r_\alpha}{\mu_\alpha} \right)^{2n} r_\alpha^{-2} \delta^{3n-2}. \end{aligned} \quad (3.4.101)$$

For J_2 , lengthy, yet straightforward computations as those seen in [Vâl19] lead to

$$J_2 = o\left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-2}. \quad (3.4.102)$$

We conclude that

$$H(0) = o\left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-2} (1 + O(\delta)), \quad \forall \alpha, \quad \forall \delta > 0, \quad (3.4.103)$$

and thus $H(0) = 0$.

3.4.4 Stability theorem proof

Consider the sets \mathcal{S}_α and let

$$16\delta_\alpha := \min_{1 \leq i < j \leq N_\alpha} |x_{i,\alpha} - x_{j,\alpha}|. \quad (3.4.104)$$

We first prove that $\delta_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$. Assuming that the contrary holds, we can apply the results of Lemma 16 with $x_\alpha = x_{1,\alpha}$ and $\rho_\alpha = \delta$ for some $\delta > 0$ fixed. This contradicts (3.4.32). We reorder the elements of the sets \mathcal{S}_α in order of distance, so that

$$16\delta_\alpha = |x_{1,\alpha} - x_{2,\alpha}|. \quad (3.4.105)$$

For $R > 1$, let $1 \leq M_{R,\alpha}$ be such that

$$\begin{aligned} |x_{1,\alpha} - x_{i_\alpha,\alpha}| &\leq R\delta_\alpha \quad \text{for } i_\alpha \in \{1, \dots, M_{R,\alpha}\}, \\ |x_{1,\alpha} - x_{i_\alpha,\alpha}| &> R\delta_\alpha \quad \text{for } i_\alpha \in \{M_{R,\alpha} + 1, \dots, N_\alpha\}. \end{aligned} \quad (3.4.106)$$

For $x \in B_0(8\delta_\alpha)$, we define the rescaled quantities

$$\check{v}_\alpha(x) := \delta_\alpha^{\frac{n-2}{2}} \varphi_\alpha(\delta_\alpha x) u_\alpha \circ \Phi_\alpha^{-1}(\delta_\alpha x) \quad (3.4.107)$$

and

$$\check{Z}_\alpha(x) = \delta_\alpha^{n-1} \varphi_\alpha^{-q+2}(\delta_\alpha x) (\Phi_\alpha)_* W_\alpha(\delta_\alpha x). \quad (3.4.108)$$

In the exponential chart, the elements of \mathcal{S}_α become

$$\check{x}_{i,\alpha} := \delta_\alpha^{-1} \exp_{x_{1,\alpha}}^{-1}(x_{i,\alpha}), \quad (3.4.109)$$

where $1 \leq i \leq N_i$. Note that $B_{x_{i,\alpha}}(8\delta_\alpha)$ and $B_{x_{j,\alpha}}(8\delta_\alpha)$ are disjoint. We define two types of concentration points : the first

$$\sup_{B_{\check{x}_{i,\alpha}}(8)} \left(\check{v}_\alpha(x)^q + \left| \frac{\nabla \check{v}_\alpha(x)}{\check{v}_\alpha(x)} \right|^n + \left| \frac{\nabla^2 \check{v}_\alpha(x)}{\check{v}_\alpha(x)} \right|^{\frac{n}{2}} + |\mathcal{L}_\xi \check{Z}_\alpha(x)| \right) = O(1) \quad (3.4.110)$$

and the second

$$\sup_{B_{\check{x}_{i,\alpha}}(8)} \left(\check{v}_\alpha(x)^q + \left| \frac{\nabla \check{v}_\alpha(x)}{\check{v}_\alpha(x)} \right|^n + \left| \frac{\nabla^2 \check{v}_\alpha(x)}{\check{v}_\alpha(x)} \right|^{\frac{n}{2}} + |\mathcal{L}_\xi \check{Z}_\alpha(x)| \right) \rightarrow \infty. \quad (3.4.111)$$

A cluster with only the first type of points, *i.e.* where all bubbles are of a comparable size. Assume $\check{x}_{i,\alpha}$ corresponds to the first type. Since for all $j \leq M_{R,\alpha}$,

$$|\check{x}_{i,\alpha} - \check{x}_{j,\alpha}|^{\frac{n-2}{2}} \check{v}_\alpha(\check{x}_{i,\alpha}) \geq 1, \quad (3.4.112)$$

then

$$\check{v}(\check{x}_{i,\alpha}) \geq 2C(R). \quad (3.4.113)$$

Since \check{v}_α is uniformly bounded in \mathcal{C}^2 , there exists $r_i > 0$ such that

$$\inf_{B_{\check{x}_{i,\alpha}}(r_i)} \check{v}_\alpha \geq C(R). \quad (3.4.114)$$

By following the arguments of Lemmas 13 and 16, there exists a $\mathcal{C}^2(B_0(R))$ limit,

$$\check{v} = \lim_{\alpha \rightarrow \infty} \check{v}_\alpha \quad (3.4.115)$$

such that

$$\Delta_\xi \check{v} = f(0)\check{v}^{q-1}; \quad (3.4.116)$$

since \check{v} has at least two maxima, this leads to a contradiction.

A cluster with both type of points, i.e. where there exists at least one pair of bubbles such that one is much greater than the other. Around the second type of concentration point, we consider two cases : either

$$\sup_{B_{\check{x}_j,\alpha}(8)} \check{v}_\alpha(x) \leq M \quad \text{and} \quad \sup_{B_{\check{x}_j,\alpha}(8)} \left| \frac{\nabla \check{v}_\alpha(x)}{\check{v}_\alpha(x)} \right|^n + \left| \frac{\nabla^2 \check{v}_\alpha(x)}{\check{v}_\alpha(x)} \right|^{\frac{n}{2}} + |\mathcal{L}_\xi \check{Z}_\alpha(x)| \rightarrow \infty \quad (3.4.117)$$

or

$$\sup_{B_{\check{x}_j,\alpha}(8)} \check{v}_\alpha(x) \rightarrow \infty. \quad (3.4.118)$$

By similar arguments to those of Lemma 16. From Lemma 20, we know that

$$|\check{v}_\alpha - \check{B}_\alpha| = o(\delta_\alpha^{\frac{n-2}{2}}). \quad (3.4.119)$$

where

$$\check{B}_\alpha(x) = \check{\mu}_\alpha^{\frac{n-2}{2}} \left(\check{\mu}_\alpha^{\frac{n-2}{2}} - \frac{\check{f}(\check{x}_{i,\alpha})}{n(n-2)} |x|^2 \right) \quad (3.4.120)$$

with

$$\check{\mu}_\alpha = \frac{\mu_\alpha}{\delta_\alpha} = \check{u}_\alpha(\check{x}_{i,\alpha})^{-q+2}. \quad (3.4.121)$$

Up to a subsequence,

$$\check{u}_\alpha(\check{x}_{j,\alpha}) \check{u}_\alpha(x) \rightarrow \frac{\lambda_j}{|x - \check{x}_j|^{n-2}} + H_j(x) \quad (3.4.122)$$

in $B_{\check{x}_i}(\frac{1}{2} \setminus \{\check{x}_j\})$, with $\lambda_j > 0$, where H_j is superharmonic in $B_{\check{x}_j,\alpha}(\frac{1}{2})$ with $H(\check{x}_i) = 0$. This means that $\check{u}_\alpha \rightarrow 0$ in $\mathcal{C}^0(B_{\check{x}_i,\alpha}(\frac{1}{2}) \setminus B_{\check{x}_i,\alpha}(\frac{1}{4}))$. By the Harnack type result, Lemma 15, we get a contradiction.

A cluster with only the second type of points. Let $\check{G}_\alpha(x, \cdot)$ be the Green function of the operator $\Delta_\xi + \delta_\alpha^2 \check{h}_\alpha$ in $B_x(3R)$. It converges to the Green function of Δ_ξ in $\mathcal{C}_{loc}^1(B_x(3R) \setminus \{x\})$. Since $\Delta_\xi + h_0$ is coercive, for any $y \in B_x(2R)$, and since $Y_\alpha \rightarrow 0$,

$$\begin{aligned} \check{u}_\alpha(x) &\geq \int_{B_0(\frac{1}{2})} \check{G}_\alpha(x, y) \check{f}_\alpha(y) \check{v}_\alpha^{q-1}(y) dy \\ &\quad + \int_{B_{\check{x}_2,\alpha}(\frac{1}{2})} \check{G}_\alpha(x, y) \check{f}_\alpha(y) \check{v}_\alpha^{q-1}(y) dy. \end{aligned} \quad (3.4.123)$$

This yields

$$\check{v}_\alpha(x) \geq (1 + o(1))(\check{B}_{1,\alpha}(x) + \check{B}_{2,\alpha}(x)) - \frac{C}{R^{n-2}} \left(\mu_{1,\alpha}^{\frac{n-2}{2}} + \mu_{2,\alpha}^{\frac{n-2}{2}} \right) \quad (3.4.124)$$

For $|x| \leq \frac{1}{4}$, $x \neq 0$, we approximate the RHS with $\check{B}_{1,\alpha}$ to get

$$\left(\frac{\mu_{1,\alpha}}{\mu_{2,\alpha}}\right)^{\frac{n-2}{2}} |x|^{n-2} (|x - \check{x}_2|^{2-n} - CR^{2-n} + o(1)) \leq o(1) + \frac{C}{R^{n-2}} |x|^{2-n} \quad (3.4.125)$$

We divide the previous equation by $|x|$ and take $x \rightarrow 0$ to get, for R large,

$$\limsup_{\alpha \rightarrow \infty} \left(\frac{\mu_{1,\alpha}}{\mu_{2,\alpha}}\right)^{\frac{n-2}{2}} \leq C \frac{16^{n-2}}{R^{n-2} - C16^{n-2}}. \quad (3.4.126)$$

By switching the roles of $\check{x}_{1,\alpha}$ and $\check{x}_{2,\alpha}$, we obtain

$$\limsup_{\alpha \rightarrow \infty} \left(\frac{\mu_{2,\alpha}}{\mu_{1,\alpha}}\right)^{\frac{n-2}{2}} \leq C \frac{16^{n-2}}{R^{n-2} - C16^{n-2}}. \quad (3.4.127)$$

This is a contradiction. \square

3.5 Annex

3.5.1 Standard elliptic theory for the Lamé operator

If X is a 1-form in M , the Lamé operator is written in coordinate form as :

$$\overrightarrow{\Delta}_g X_i = \nabla^j \nabla_j X_i + \nabla^j \nabla_i X^j - \frac{2}{n} \nabla_i (\operatorname{div}_g X). \quad (3.5.1)$$

The operator $\overrightarrow{\Delta}_g$ is uniformly elliptic on M . It satisfies the strong ellipticity condition (also known as the Legendre-Hadamard condition) : for any $x \in M$ and any $\eta \in T_x^* M$:

$$(L(x, \xi)\eta)_i \eta^i = |\xi|_g^2 |\eta|_g^2 + \left(1 - \frac{2}{n}\right) |\langle \xi, \eta \rangle|_g^2 \geq |\xi|_g^2 |\eta|_g^2. \quad (3.5.2)$$

The Lamé operator is self-adjoint on $H^1(M)$ on any closed manifold M , since by integration by parts one gets, for any 1 forms X and Y ,

$$\int_M \langle \overrightarrow{\Delta}_g X, Y \rangle_g dv_g = \frac{1}{2} \int_M \langle \mathcal{L}_g X, \mathcal{L}_g Y \rangle_g dv_g. \quad (3.5.3)$$

This implies that for any 1-form X on M ,

$$\overrightarrow{\Delta}_g X = 0 \iff \mathcal{L}_g X = 0. \quad (3.5.4)$$

The standard elliptic theory for (self-adjoint) strongly elliptic operators acting on vector bundles on a compact manifold apply (see Theorem 5.20 in Giaquinta-Martinazzi) :

Proposition 3. *For any $p > 1$, there exists constants $C_1 = C_1(g, p)$ and $C_2 = C_2(g, p)$ such that for any 1-form X in M :*

$$\|X\|_{W^{2,p}(M)} \leq C_1 \|\overrightarrow{\Delta}_g X\|_{L^p(M)} + C_2 \|X\|_{L^1(M)}. \quad (3.5.5)$$

In addition, X satisfies

$$\int_M \langle X, K \rangle_g dv_g = 0 \quad (3.5.6)$$

for all conformal Killing 1-forms K , then we can choose $C_2 = 0$.

We now turn to the case of \mathbb{R}^n . For any $1 \leq i \leq n$, we define the 1-form $\mathbb{R}^n \setminus \{0\}$ by :

$$\mathcal{G}_i(y)_j = -\frac{1}{4(n-1)\omega_{n-1}}|y|^{2-n} \left((3n-2)\delta_{ij} + (n-2)\frac{y_i y_j}{|y|^2} \right) \quad (3.5.7)$$

for any $y \neq 0$. Note that the matrices $(\mathcal{G}_i(y)_j)_{ij}$ thus defined are symmetric : for any $y \neq 0$,

$$\mathcal{G}_i(y)_j = \mathcal{G}_j(y)_i. \quad (3.5.8)$$

Let X be a field of 1-form in \mathbb{R}^n . For any $R > 0$ and for any $x \in B_0(R)$ there holds :

$$\begin{aligned} X_i(x) &= \int_{B_0(R)} \mathcal{G}_i(x-y)_j \overrightarrow{\Delta}_\xi X(y)^j dx \\ &+ \int_{\partial B_0(R)} \mathcal{L}_\xi X(y)^{kl} \nu_k(y) \mathcal{G}_i(x-y)_l d\sigma \\ &- \int_{\partial B_0(R)} \mathcal{L}_\xi (\mathcal{G}_i(x-\cdot))_{kl}(y) \nu(y)^k X(y)^l d\sigma. \end{aligned} \quad (3.5.9)$$

If Y is a smooth 1-form in $L^1(\mathbb{R}^n)$, then

$$W_i(x) = \int_{\mathbb{R}^n} \mathcal{G}_i(x-y)_j Y^j(y) dy = (\mathcal{G} * Y)_i(x) \quad (3.5.10)$$

satisfies

$$\overrightarrow{\Delta}_\xi W_i(x) = Y_i(x). \quad (3.5.11)$$

The system (3.1.15) is invariant up to adding a conformal Killing 1-form in M to W_α . Let

$$K_R = \{X \in H^1(M)(B_0(R)), \mathcal{L}_\xi X = 0\} \quad (3.5.12)$$

is the subspace of 1-forms associated to the kernel to the Neumann problem for Δ_ξ in $B_0(R)$. The H^1 orthogonal space is defined as the space of 1-forms $Y \in H^1(B_0(R))$ such that for any $X \in K_R$:

$$\int_{B_0(R)} \langle Y, X \rangle_\xi dx = 0. \quad (3.5.13)$$

For any 1-form $X \in B_0(B_0(R))$, we define the orthogonal projection on K_R by

$$\pi_R(X) = \sum_{j=1}^m \left(\int_{B_0(R)} \langle K_j, X \rangle dx \right) K_j. \quad (3.5.14)$$

The existence of Green 1-forms satisfying Neumann boundary conditions :

Proposition 4. *For any $1 \leq i \leq n$ and any $R > 0$, there exists a unique $\mathcal{G}_{i,R}$ defined in $B_0(R) \times B_0(R) \setminus D$, where $D = \{(x,x), x \in B_0(R)\}$ there holds :*

$$\begin{aligned} (X - \pi_R(X))_i(x) &= \int_{B_0(R)} \mathcal{G}_{i,R}(x,y)_j \overrightarrow{\Delta}_\xi X(y)^j dx \\ &+ \int_{\partial B_0(R)} \mathcal{L}_\xi X(y)^{kl} \nu_k(y) \mathcal{G}_{i,R}(x,y)_k d\sigma. \end{aligned} \quad (3.5.15)$$

Moreover, $\mathcal{G}_{i,R}$ is continuously differentiable in $B_0(R) \times B_0(R) \setminus D$. Furthermore, if K denotes any compact set in $B_0(R)$, there holds for any $x, y \in M$

$$|x-y| |\nabla \mathcal{G}_{i,R}(x,y)| + |\mathcal{G}_{i,R}(x,y)| \leq C(\delta) |x-y|^{2-n}, \quad (3.5.16)$$

where

$$\delta = \frac{1}{R} d(K, \partial B_0(R)) > 0. \quad (3.5.17)$$

3.5.2 Limiting equation

The following lemma has been proved in [Väl19].

Lemma 21. *Let u be a bounded subharmonic function defined on \mathbb{R}^n . If there exists $0 < \varepsilon \leq u$ which bounds u from below and $\alpha > 0$ such that $u^{-\alpha}$ is a subharmonic function, then u is a constant.*

- [ACI08] Paul T. Allen, Adam Clausen, and James Isenberg. Near-constant mean curvature solutions of the Einstein constraint equations with non-negative Yamabe metrics. *Classical Quantum Gravity*, 25(7):075009, 15, 2008.
- [Aub82] Thierry Aubin. *Nonlinear analysis on manifolds. Monge-Ampère equations*, volume 252 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982.
- [BCS05] Robert Beig, Piotr T. Chruściel, and Richard Schoen. KIDs are non-generic. *Ann. Henri Poincaré*, 6(1):155–194, 2005.
- [BI04] Robert Bartnik and Jim Isenberg. The constraint equations. In *The Einstein equations and the large scale behavior of gravitational fields*, pages 1–38. Birkhäuser, Basel, 2004.
- [CB09] Yvonne Choquet-Bruhat. *General relativity and the Einstein equations*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2009.
- [CB15] Yvonne Choquet-Bruhat. *Introduction to general relativity, black holes, and cosmology*. Oxford University Press, Oxford, 2015. With a foreword by Thibault Damour.
- [CBG69] Yvonne Choquet-Bruhat and Robert Geroch. Global aspects of the Cauchy problem in general relativity. *Comm. Math. Phys.*, 14:329–335, 1969.
- [CBIP06] Yvonne Choquet-Bruhat, James Isenberg, and Daniel Pollack. The Einstein-scalar field constraints on asymptotically Euclidean manifolds. *Chinese Ann. Math. Ser. B*, 27(1):31–52, 2006.
- [CBIP07] Yvonne Choquet-Bruhat, James Isenberg, and Daniel Pollack. The constraint equations for the Einstein-scalar field system on compact manifolds. *Classical Quantum Gravity*, 24(4):809–828, 2007.
- [CBY80] Yvonne Choquet-Bruhat and James W. York, Jr. The Cauchy problem. In *General relativity and gravitation, Vol. 1*, pages 99–172. Plenum, New York-London, 1980.
- [CCI11] Piotr T. Chruściel, Justin Corvino, and James Isenberg. Construction of N -body initial data sets in general relativity. *Comm. Math. Phys.*, 304(3):637–647, 2011.

- [CD03] Piotr T. Chruściel and Erwann Delay. On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications. *Mém. Soc. Math. Fr. (N.S.)*, (94):vi+103, 2003.
- [CGS89] Luis A. Caffarelli, Basilis Gidas, and Joel Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.*, 42(3):271–297, 1989.
- [CIP05] Piotr T. Chruściel, James Isenberg, and Daniel Pollack. Initial data engineering. *Comm. Math. Phys.*, 257(1):29–42, 2005.
- [CS06] Justin Corvino and Richard M. Schoen. On the asymptotics for the vacuum Einstein constraint equations. *J. Differential Geom.*, 73(2):185–217, 2006.
- [CS14] Alessandro Carlotto and Richard Schoen. Localizing solutions of the einstein constraint equations. *arXiv:1407.4766*, 2014.
- [DGH12] Mattias Dahl, Romain Gicquaud, and Emmanuel Humbert. A limit equation associated to the solvability of the vacuum Einstein constraint equations by using the conformal method. *Duke Math. J.*, 161(14):2669–2697, 2012.
- [DH09] Olivier Druet and Emmanuel Hebey. Stability and instability for Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds. *Math. Z.*, 263(1):33–67, 2009.
- [DHKM17] James Dilts, Michael Holst, Tamara Kozareva, and David Maxwell. Numerical Bifurcation Analysis of the Conformal Method. 2017.
- [DHR04] Olivier Druet, Emmanuel Hebey, and Frédéric Robert. *Blow-up theory for elliptic PDEs in Riemannian geometry*, volume 45 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2004.
- [DP15] Olivier Druet and Bruno Premoselli. Stability of the Einstein-Lichnerowicz constraint system. *Math. Ann.*, 362(3-4):839–886, 2015.
- [Dru10] Olivier Druet. *La notion de stabilité pour des équations aux dérivées partielles elliptiques*, volume 19 of *Ensaio Matemáticos [Mathematical Surveys]*. Sociedade Brasileira de Matemática, Rio de Janeiro, 2010.
- [Eva10] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.
- [FB52] Y. Fourès-Bruhat. Théorème d’existence pour certains systèmes d’équations aux dérivées partielles non linéaires. *Acta Math.*, 88:141–225, 1952.
- [GHL04] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, third edition, 2004.
- [Heb14] Emmanuel Hebey. *Compactness and stability for nonlinear elliptic equations*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2014.
- [HMM18] Mike Holst, David Maxwell, and Rafe Mazzeo. Conformal fields and the structure of the space of solutions of the einstein constraint equations. *arXiv:1711.01042*, 2018.

- [HNT09] Michael Holst, Gabriel Nagy, and Gantumur Tsogtgerel. Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions. *Comm. Math. Phys.*, 288(2):547–613, 2009.
- [HPP08] Emmanuel Hebey, Frank Pacard, and Daniel Pollack. A variational analysis of Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds. *Comm. Math. Phys.*, 278(1):117–132, 2008.
- [Hua09] Lan-Hsuan Huang. On the center of mass of isolated systems with general asymptotics. *Classical Quantum Gravity*, 26(1):015012, 25, 2009.
- [Hua10] Lan-Hsuan Huang. Solutions of special asymptotics to the Einstein constraint equations. *Classical Quantum Gravity*, 27(24):245002, 10, 2010.
- [IM96] James Isenberg and Vincent Moncrief. A set of nonconstant mean curvature solutions of the Einstein constraint equations on closed manifolds. *Classical Quantum Gravity*, 13(7):1819–1847, 1996.
- [IÓM04] James Isenberg and Niall Ó Murchadha. Non-CMC conformal data sets which do not produce solutions of the Einstein constraint equations. *Classical Quantum Gravity*, 21(3):S233–S241, 2004. A spacetime safari: essays in honour of Vincent Moncrief.
- [Ise95] James Isenberg. Constant mean curvature solutions of the Einstein constraint equations on closed manifolds. *Classical Quantum Gravity*, 12(9):2249–2274, 1995.
- [Lic44] André Lichnerowicz. L'intégration des équations de la gravitation relativiste et le problème des n corps. *J. Math. Pures Appl. (9)*, 23:37–63, 1944.
- [Max09] David Maxwell. A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature. *Math. Res. Lett.*, 16(4):627–645, 2009.
- [Max11] David Maxwell. A model problem for conformal parameterizations of the Einstein constraint equations. *Comm. Math. Phys.*, 302(3):697–736, 2011.
- [Max14a] David Maxwell. The conformal method and the conformal thin-sandwich method are the same. *Classical Quantum Gravity*, 31(14):145006, 34, 2014.
- [Max14b] David Maxwell. Initial data in general relativity described by expansion, conformal deformation and drift. *arXiv:1407.1467*, 2014.
- [Max15] David Maxwell. Conformal parameterizations of slices of flat Kasner spacetimes. *Ann. Henri Poincaré*, 16(12):2919–2954, 2015.
- [MW13] Li Ma and Juncheng Wei. Stability and multiple solutions to Einstein-scalar field Lichnerowicz equation on manifolds. *J. Math. Pures Appl. (9)*, 99(2):174–186, 2013.
- [NX12] Quoc Anh Ngô and Xingwang Xu. Existence results for the Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds. *Adv. Math.*, 230(4-6):2378–2415, 2012.
- [NX14] Quoc Anh Ngô and Xingwang Xu. Existence results for the Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds in the positive case. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 9(3):451–485, 2014.

- [ÓMY74] Niall Ó Murchadha and James W. York, Jr. Initial-value problem of general relativity. I. General formulation and physical interpretation. *Phys. Rev. D* (3), 10:428–436, 1974.
- [Pau13] Frédéric Paulin. *Groupes et géométrie*. 2013.
- [Pet16] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, Cham, third edition, 2016.
- [Poh65] S. I. Pohožaev. On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Dokl. Akad. Nauk SSSR*, 165:36–39, 1965.
- [Pre14] Bruno Premoselli. The Einstein-scalar field constraint system in the positive case. *Comm. Math. Phys.*, 326(2):543–557, 2014.
- [Pre15] Bruno Premoselli. Effective multiplicity for the Einstein-scalar field Lichnerowicz equation. *Calc. Var. Partial Differential Equations*, 53(1-2):29–64, 2015.
- [Pre16] Bruno Premoselli. Stability and instability of the Einstein-Lichnerowicz constraint system. *Int. Math. Res. Not. IMRN*, (7):1951–2025, 2016.
- [PW16] Bruno Premoselli and Juncheng Wei. Non-compactness and infinite number of conformal initial data sets in high dimensions. *J. Funct. Anal.*, 270(2):718–747, 2016.
- [PY03] Harald P. Pfeiffer and James W. York, Jr. Extrinsic curvature and the Einstein constraints. *Phys. Rev. D* (3), 67(4):044022, 8, 2003.
- [Rat09] Jesse Ratzkin. Pohozaev-type identities. 2009.
- [Sch89] Richard M. Schoen. Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In *Topics in calculus of variations (Montecatini Terme, 1987)*, volume 1365 of *Lecture Notes in Math.*, pages 120–154. Springer, Berlin, 1989.
- [Str84] Michael Struwe. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.*, 187(4):511–517, 1984.
- [Vâl19] Caterina Vâlcu. The Constraint Equations in the Presence of a Scalar Field: The Case of the Conformal Method with Volumetric Drift. *Comm. Math. Phys.*, 2019.
- [Wal84] Robert M. Wald. *General relativity*. University of Chicago Press, Chicago, IL, 1984.
- [Yor73] James W. York, Jr. Conformally invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial-value problem of general relativity. *J. Mathematical Phys.*, 14:456–464, 1973.
- [Yor99] James W. York, Jr. Conformal “thin-sandwich” data for the initial-value problem of general relativity. *Phys. Rev. Lett.*, 82(7):1350–1353, 1999.

Autour des équations de contrainte en relativité générale

Résumé. Le but à long terme de mon travail de recherche est de trouver une alternative viable à la méthode conforme, qui nous permettrait de mieux comprendre la structure géométrique de l'espace des solutions des équations de contrainte. L'avantage du modèle de Maxwell (the drift model) par rapport aux modèles plus classiques est la présence des paramètres supplémentaires. Le prix à payer, par contre, sera la complexité analytique du système correspondant. Ma thèse a été structurée en deux parties :

- a. *Existence sous la condition de petitesse des données initiales.* Nous avons montré que le système de Maxwell est raisonnable dans le sens où nous pouvons le résoudre, malgré sa forte nonlinéarité, sous des conditions de petitesse sur ses coefficients, en dimension $3 \leq n \leq 5$. Par conséquent, l'ensemble des solutions est non-vide.
- b. *Stabilité.* Nous montrons la stabilité des solutions du système : ce résultat est obtenu en dimension $3 \leq n \leq 5$, dans le cas où la métrique est localement conformément plate, et le drift est petit.

Mots-clés : Relativité générale, Analyse asymptotique, Équations au dérivées partielles, Physique mathématique, Équations de contrainte

On the Constraint Equations in General Relativity

Abstract. The long-term goal of my work is to find a viable alternative to the conformal method, which would allow us to better understand the geometry of the space of solutions of the constraint equations. The advantage of Maxwell's model (the drift model) is the presence of additional parameters. Its downside, however, is that it proves to be much more difficult from an analytic standpoint. My thesis is structured in two parts:

- a. *Existence under suitable smallness conditions.* We show that Maxwell's system is sufficiently reasonable: it can be solved even given the presence of focusing non linearities. We prove this under smallness conditions of its coefficients, and in dimensions $3 \leq n \leq 5$. An immediate consequence is that the set of solutions is non-empty.
- b. *Stability.* We verify that the solutions of the system are stable: this result holds in dimensions $3 \leq n \leq 5$, when the metric is conformally flat and the drift is small.

Keywords: General relativity, Asymptotic analysis, Partial differential equations, Mathematical physics, Constraint equations

Image de couverture : Matthew Cusick, Cat's Wave (2015)



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