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par

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sous la direction de

Jean-Michel CORON

Sujet de la thèse :

**Contrôle et stabilisation internes
de systèmes hyperboliques 1-D**

devant le jury composé de

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Chapter 1

Introduction

In France, the phrase “mathématiques appliquées” covers a large scope, part of which would be considered “pure mathematics” by most countries. As the compatriots of Descartes, we take pride in this linguistic peculiarity because it so eloquently highlights our taste for abstraction and theory. But it also illustrates the complex beauty of mathematics. Indeed, mathematics is a rigorous, sometimes poetic language describing objects of this (or other) world(s), but they are also a powerful tool to interact with them, and there is no clear boundary between the language and the tool, between theory and application. In the gray area in between, one finds, among others, control theory: wonderfully applicable and applied in its origins, but also profoundly mathematical in its evolution. This thesis focuses on some questions in control theory, namely the internal control and stabilization of some hyperbolic systems of first or second order.

1.1 A few general notions

1.1.1 Partial differential equations

Partial differential equations (PDEs) are equations that relate multivariable physical quantities to their variations in time and space. They are the most common tool to model dynamical phenomena such as heat diffusion, fluids, vibrations, electromagnetism, gravitation, chemical reactions. One of the first PDEs to be studied is the heat equation, which describes the evolution of temperature in a solid body Ω :

$$\begin{cases} u_t - c\Delta u = f, & \text{in } \Omega, \\ \partial_n u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

The first line describes the evolution of the temperature u inside the solid, with another heat source inside the solid described by the function f . The second line describes what happens at the boundary of the solid. These *boundary conditions* are of equal physical importance, as the nature and even the existence of solutions depend on what boundary conditions we set.

This example already shows the main ways of influencing the dynamics of a system modelled by a PDE: by acting inside, or at the boundary. For each one, there are a myriad of variations: the action can be localised in time and/or space, it can be reduced to changing the amplitude of prescribed force. Physically, these actions on a system, or *controls*, can take many forms: changing the entering flow of a water pipe, injecting heat in a solid, modulating the electric field in which a particle is moving...

There are many types of PDEs, and even though the most common fall into a finite number of categories, this multitude makes it impossible to formulate a general theory, even less a general method of resolution.

Fortunately, the aforementioned categories of PDEs have been studied for decades, some for centuries. Some deep questions have found answers, many remain open, making for a very exciting field of research. Roughly speaking, PDEs can be categorized according to the profound nature of the physical phenomena described by the equations: to name a few, kinetic equations (Boltzmann equation, Vlasov

equation) describe the evolution of a continuum of “particles” interacting with each other, dispersive equations (Schrödinger equation, KdV equation) describe wavelike phenomena where wavelength and propagation speed are linked, diffusion equations (heat equation) describe phenomena where a certain physical quantity spreads in space over time, and hyperbolic equations, on which the next section will focus.

1.1.2 Hyperbolic systems

In this thesis we will focus on a special class of PDEs, namely hyperbolic equations. Simply put, these equations describe phenomena where physical quantities travel along certain directions in space at finite speeds, like sound, electricity, or waves in general. Equivalently, this means the evolution of these quantities can be described by a differential equation along certain directions of time and space. More abstractly speaking, we can consider that the information about the system is traveling along these directions. This will be an important insight to understand the effect of a control on such systems. Now, if we know these directions, then we might be able to reconstruct the solution of such equations given their initial values in space. This is actually given as a qualitative definition by Boris L. Rozhddestvenskii ([131]): a hyperbolic system is such that, if initial data is given on a time-space surface that does not contain the directions along which the physical quantities remain constant, then there exists a unique solution close to this surface.

Let us illustrate this on a simple example. Consider the transport equation on the real line \mathbb{R} :

$$\alpha_t + \alpha_x = 0. \quad (1.2)$$

We are looking for lines in $\mathbb{R}_t^+ \times \mathbb{R}_x$ along which a solution of (1.2) would satisfy a simple differential equation. Here, it is easy to see that (1.2) implies that the gradient (in time and space) of α is orthogonal to the vector $(1, 1)$, which means that α is constant along any line $(t, x_0 + t)$, $t \in \mathbb{R}^+$, called characteristic lines. Now, every point of $\mathbb{R}_t^+ \times \mathbb{R}_x$ can be reached by such a line, so if the initial value of α is given, i.e. if α is given on the line $(0, x)$, $x \in \mathbb{R}$, then the solution to (1.2) is given by

$$\alpha(t, x) = \alpha(0, x - t), \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}. \quad (1.3)$$

Mathematically, a hyperbolic system has the general form

$$U_t + A(U, x)U_x + F(U, x) = 0 \quad (1.4)$$

where $A : \mathbb{R}^n \times \mathbb{R}^d \rightarrow M_n(\mathbb{R})$ is continuous, and $A(U, x)$ is diagonalizable with real eigenvalues for all $(U, x) \in \mathbb{R}^n \times \mathbb{R}^d$. In particular, if A is diagonal, and $F \equiv 0$, then we get a collection of transport equations similar to (1.2).

Note that the above characterization also covers equations of the second order, for example wave equations. For example, one of the ways to rewrite the wave equation

$$u_{tt} - u_{xx} = 0 \quad (1.5)$$

is to set

$$u^1 := u_t - u_x, \quad u^2 := u_t + u_x, \quad (1.6)$$

for which (1.5) rewrites

$$\begin{cases} u_t^1 + u_x^1 = 0, \\ u_t^2 - u_x^2 = 0. \end{cases} \quad (1.7)$$

As we have mentioned earlier, there are mainly two types of controls on PDEs: on the boundary, or inside the domain. In the case of hyperbolic equation, because of the finite travelling speed of the information, one should expect that the action on the system might not take effect immediately, depending on where one acts. Note that when acting on the boundary, one has to pay attention to the direction in which the information travels. The action or input should happen upstream of the “flow of information” in order to really influence the dynamics of the system.

1.1.3 Control theory: some definitions

Having modelled a system, and the way it can be acted upon, the next question is what we can make that system do. This has been narrowed down to three main fields of studies:

- Controllability: can the system be steered from point (state) A to point (state) B?
- Optimal control: can it be done in an optimal way (e.g. in minimal time, with a minimal cost...)? This places the problem in an optimization framework and has enriched the technical landscape of control theory.
- Stabilization: given an equilibrium point, can we design the control as a function of the state of the system (a closed-loop control) to make it stable if it is unstable, or improve its stability if it is already stable in some sense?

In this thesis, we focus on the first and third questions, and the links between them. Broadly speaking, controllability is about how well the information is transmitted and acted upon in a system. The better it is, the better our chances should be to stabilize that system by an input that is a function of the state.

1.1.3.1 Controllability

Let us now give a few definitions of control theory. In all these definitions, we consider a general control system in infinite dimension. We omit important subtleties regarding the existence of solutions, the nature of the solutions, and function spaces in general, as these are specific to each problem.

Definition 1.1.1 (Exact controllability). *A control system is **globally exactly controllable in time** $T > 0$ if for any initial and final states y_0, y_1 , there exists a control u such that the corresponding solution y of the system with initial value y_0 exists and satisfies $y(T) = y_1$.*

*It is **locally exactly controllable** around a trajectory if the above holds for y_0 (resp. y_1) close enough to the starting (resp. ending) point of that trajectory.*

*It is **globally small-time exactly controllable** if it is globally exactly controllable in any time $T > 0$.*

*It is **locally small-time exactly controllable** around an equilibrium point if for any $T > 0$ it is locally exactly controllable around that equilibrium point in time T .*

Sometimes, it is interesting to consider a less demanding goal than exact controllability. For systems for which 0 is an equilibrium point, knowing how to steer the state to 0 can be an important first step.

Definition 1.1.2 (Null controllability). *A control system is **globally null controllable in time** $T > 0$ if for any initial state y_0 there exists a control u such that the corresponding solution y of the system with initial value y_0 exists and satisfies $y(T) = 0$.*

*It is **locally null controllable** if the above holds only for small enough y_0 .*

*It is **small-time globally null controllable** if it is null controllable in any time $T > 0$.*

*It is **small-time locally null controllable** if for any $T > 0$, it is locally null controllable in time T .*

Sometimes, for intrinsic reasons (regularization phenomena for example), exact controllability is out of the question. However, for such systems one can still investigate the next best thing:

Definition 1.1.3 (Approximate controllability). *A control system is **approximately controllable in time** $T > 0$ if for any initial and final states y_0, y_1 , and any neighborhood of y_1 one can find a control such that the corresponding solution y with initial state y_0 exists and is such that $y(T)$ is in this neighborhood.*

A large range of methods have been developed to study the controllability of control systems. Let us first mention what is perhaps the simplest and most powerful characterization of controllability for finite-dimensional linear control systems:

Theorem 1.1.1 (J.LaSalle, 1960, Y.-C.Ho, R.Kalman, K.Narendra, 1963). *The control system*

$$\dot{X} = AX + Bu \quad (1.8)$$

where $X \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $A \in M_n(\mathbb{R}), B \in M_{n,m}(\mathbb{R})$ is controllable with $u \in L^\infty(0, T; \mathbb{R}^m)$ if and only if (A, B) satisfies the so-called Kalman rank condition:

$$\text{Span}\{A^j Bu, \quad j \in \{0, \dots, n-1\}, \quad u \in \mathbb{R}^m\} = \mathbb{R}^n. \quad (1.9)$$

But such a simple algebraic characterization is not always available, and in any case it does not give any indication on what control might work to steer the system from a given initial state to a given final state. Other methods address these issues: one can for example try to express the solutions of the control system as a function of the control, using for example the characteristics method as for system (1.2). The controllability problem can also be seen as the surjectivity of the end-point mapping, which maps a control to the final state of the corresponding trajectory starting at 0. Indeed, this mapping is onto if and only if any state can be reached from 0, which is equivalent to controllability for linear systems.

For linear systems in a Banach space setting, this can be tackled using the adjoint of that mapping (see [132, Theorem 4.15, page 97], and [34, Théorème II.19, pages 29-30] for unbounded operators):

Theorem 1.1.2. *Let H_1, H_2 be Hilbert spaces. Let $\mathcal{F} : H_1 \rightarrow H_2$ be a continuous linear operator. Then \mathcal{F} is onto if and only if there exists $C > 0$ such that*

$$\|\mathcal{F}^* x\|_{H_1} \geq C \|x\|_{H_2}, \quad \forall x \in H_2. \quad (1.10)$$

In that case, \mathcal{F} admits a continuous linear right inverse \mathcal{G} satisfying

$$\|\mathcal{G}\|_{\mathcal{L}(H_2, H_1)} \leq \frac{1}{C}. \quad (1.11)$$

In control theory, (1.10) is called *observability inequality*, after the notion of observability: a system with a prescribed output of the state is observable in time $T > 0$ if any two solutions with the same output on $[0, T]$ necessarily have the same initial state. As it turns out, controllability and observability are dual notions: the former asks what final states can be reached by acting in some way on the system, the latter asks if the initial state can be inferred by observing some output of the system. In particular, a linear control system is controllable if and only if its adjoint system is observable ([72]).

Many methods exist to prove observability inequalities: the Hilbert Uniqueness Method (HUM), Carleman inequalities, the moments method, spectral inequalities... One of the interesting features of this approach is that it does not require precise knowledge of the solutions, but simply to prove that they exist, and satisfy some inequalities.

1.1.3.2 Stability and stabilizability

Stability is a major notion in the study of physical systems. It is linked to the question of existence of solutions to an equation: often, when a solution exists only on a finite-time interval, it is because it blows up after some-time. But once we are sure that we can talk about the evolution of a system for all time, comes the question of its asymptotic behavior: we will not be there to see it, but can we still try to characterize how the evolution looks like after a very long time?

The question of stability is of particular practical interest: stability means that robustness with respect to the perturbations of the system, so that the system still has a “good trajectory” in the long run. For example, a self-driving car must have a stable steering mechanism in order to stay on course even if there are irregularities on the road. Instability can be even more spectacular: the collapse of the Tacoma Narrows bridge, caused by a resonance phenomenon, shows the importance of stability in practical applications.

Mathematically, stability is characterized by the fact that the system converges to an equilibrium. Let us consider an abstract evolution system, in some topological space E with norm $\|\cdot\|_E$:

$$\dot{X}(t) = f(t, X(t)). \quad (1.12)$$

We suppose, for the sake of simplicity, that its solutions are always defined, i.e. for all $X_0 \in E$, $s \in \mathbb{R}$, (1.12) has a solution $X(t), t \in [s, \infty)$ such that

$$X(s) = X_0. \quad (1.13)$$

Finally, suppose 0 is an equilibrium of that system, i.e.

$$f(t, 0) = 0, \quad \forall t \in \mathbb{R}. \quad (1.14)$$

Definition 1.1.4 (Asymptotic stability). *The equilibrium point 0 is said to be **locally asymptotically stable** if*

i) (uniform stability) For any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $X(t)$ is a solution of (1.12), then for every $s \leq \tau \in \mathbb{R}$,

$$\|X(s)\|_E \leq \delta \implies \|X(\tau)\|_E \leq \varepsilon. \quad (1.15)$$

ii) (Convergence) There exists $\delta > 0$ such that, if $X(t)$ is a solution of (1.12), then for all $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that for $s \in \mathbb{R}$

$$\|X(s)\|_E \leq \delta \implies \|X(s + T_\varepsilon)\|_E \leq \varepsilon. \quad (1.16)$$

*It is **globally asymptotically stable** if ii) holds for arbitrarily large $\delta > 0$.*

Stability can then be characterized by the speed of convergence: in particular, the most interesting kind of stability is the one we can actually witness without having to wait indefinitely:

Definition 1.1.5 (Finite-time stability). *The equilibrium point 0 is **locally finite-time stable** if it is uniformly stable, and if there exist $\delta, T > 0$ such that, if $X(t)$ is a solution of (1.12), then for every $s \in \mathbb{R}$,*

$$\|X(s)\|_E \leq \delta \implies X(s + T) = 0. \quad (1.17)$$

*It is **globally asymptotically stable** if it is uniformly stable and if there is $T > 0$ such that, for every $s \in \mathbb{R}$,*

$$X(s + T) = 0. \quad (1.18)$$

However, this notion is very strong, and it is more reasonable to explore exponential decay. Indeed this kind of convergence occurs naturally in heat diffusion, or in radioactive decay.

Definition 1.1.6 (Exponential stability). *The equilibrium point 0 is **locally exponentially stable** if there exist constants $\mu, C, \delta > 0$ such that, if $X(t)$ is a solution of (1.12), then for $s \leq \tau \in \mathbb{R}$,*

$$\|X(s)\|_E \leq \delta \implies \|X(\tau)\|_E \leq C e^{-\mu(\tau-s)} \|X(s)\|_E. \quad (1.19)$$

*The **exponential decay rate** of (1.12) is the supremum of the set of $\mu > 0$ such that (1.19) holds. The equilibrium point 0 is **globally exponentially stable** if the above holds for arbitrarily large $\delta > 0$.*

Even mathematically, this kind of decay is particularly interesting, since the exponential map is a central object in the study of evolution equations. Indeed, consider a simple linear system in finite dimension:

$$\dot{X} = AX, \quad A \in M_n(\mathbb{R}). \quad (1.20)$$

It is well known that any solution $X(t)$ to this system can be written:

$$X(t) = e^{At} X(0), \quad \forall t \in \mathbb{R}. \quad (1.21)$$

Now, if the eigenvalues of A have negative real parts, then it is clear that 0 is exponentially stable for (1.20). This leads to a local stability result for nonlinear autonomous systems, due to Lyapunov and known as *Lyapunov's first theorem*:

Theorem 1.1.3. *Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $f(0) = 0$. If every eigenvalue of $Df(0)$ has a negative real part, then 0 is locally exponentially stable for*

$$\dot{X} = f(X). \quad (1.22)$$

Spectral properties, though powerful in finite dimension, can fall short in infinite dimension, and some simple hyperbolic systems in particular have an exponentially stable linearized system without being even locally asymptotically stable. For infinite-dimensional systems, another set of techniques has been developed, resting on the physical intuition that stability is linked with the decay of some sort of energy. Mathematically, this translates into the so-called *Lyapunov functions*. Lyapunov's original works ([121]) treat the finite-dimensional case:

Definition 1.1.7. *Consider system (1.12) with state in \mathbb{R}^n , with f locally Lipschitz with respect to x , and such that for all $x \in \mathbb{R}^n$,*

$$t \in \mathbb{R} \mapsto f(t, x)$$

is measurable and L^1_{loc} . Let $U \subset \mathbb{R}^n$ be a neighbourhood of 0. A Lyapunov function for system (1.12) is a function $V \in C^1(\mathbb{R} \times U; \mathbb{R})$ such that

i) There exists $W \in C^0(U; \mathbb{R})$ such that

$$W(0) = 0, \quad W(x) > 0, \quad \forall x \in U \setminus \{0\}, \quad (1.23)$$

and such that

$$V(t, x) \geq W(x), \quad \forall (t, x) \in \mathbb{R} \times U. \quad (1.24)$$

ii) There exists $\eta \in C^0(U; \mathbb{R})$ such that

$$\eta(0) = 0, \quad \eta(x) \leq 0, \quad \forall x \in U \setminus \{0\}, \quad (1.25)$$

and such that

$$\nabla V(t, x) \cdot f(t, x) + \partial_t V(t, x) \leq \eta(x), \quad \forall (t, x) \in \mathbb{R} \times U. \quad (1.26)$$

A strict Lyapunov function is a Lyapunov function such that the inequality in (1.25) is strict.

These functions are a powerful tool to study the stability of a system, as this is reduced to finding a certain positive functional that will decrease along the trajectories of the system:

Theorem 1.1.4. *If system (1.12) satisfying the hypotheses of Definition 1.1.7 admits a strict Lyapunov function, then for a small enough initial condition at time 0, the corresponding solution of that system is defined on $[0, \infty)$, and 0 is locally asymptotically stable for system (1.12).*

Of course there is much more to the theory of Lyapunov functions: there are converse theorems that state that stable systems have a Lyapunov function, and stronger decay conditions on Lyapunov functions to prove exponential stability, for example. In the context of PDEs, for example, one of the possible extensions of the notion of Lyapunov functions is the following:

Definition 1.1.8. *A functional $V \in C^1(E, \mathbb{R})$ is a Lyapunov function for (1.12) if*

i) There exist constants $c, C > 0$ such that

$$c\|X\|_E \leq V(X) \leq C\|X\|_E, \quad \forall X \in E. \quad (1.27)$$

ii) If $X(t)$ is a solution of (1.12), then for all $s \leq t \in \mathbb{R}$

$$V(X(t)) \leq V(X(s)). \quad (1.28)$$

V is a strict Lyapunov function if, in the right-hand side of the above implication, the inequality is strict whenever $X(s) \neq 0$.

V is a local Lyapunov function if i) holds on a neighbourhood of 0, and if for all $s \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\|X(s)\| \leq \delta \implies V(X(t_2)) \leq V(X(t_1)), \quad \forall t_2 \geq t_1 \geq s. \quad (1.29)$$

Now let us go back to control theory: consider a general control system in some topological space E , with norm $\|\cdot\|_E$, with control in some other space U :

$$\dot{X} = f(X, u), \quad (1.30)$$

such that

$$f(0, 0) = 0. \quad (1.31)$$

A key feature in stabilization is that we will consider *closed-loop* controls. Indeed, in the previous section, we presented controllability with *open-loop* controls, where the choice of the control is planned beforehand and depends on the initial and final conditions. In stabilization problems, a form of automation is sought after, in the sense that the system reacts to its own state in a prescribed manner that does not depend on its initial state.

Technically speaking, the stabilization problem is that of finding controls of the form

$$u := h(x), \quad (1.32)$$

called *stationary feedback laws* ; sometimes even

$$u := h(t, x), \quad (1.33)$$

called *time-varying* feedback laws, such that the system

$$\dot{X} = f(X, h(t, X)) \quad (1.34)$$

is well-posed, and 0 is (locally/globally/asymptotically/exponentially/finite-time) stable.

In what follows, whenever there is no ambiguity about the equilibrium point under consideration, we will simply say that a system is (locally/globally/asymptotically/exponentially/finite-time) stable without specifying the equilibrium point.

Finally, let us specify some definitions used in this thesis in the context of autonomous linear partial differential equations.

Definition 1.1.9. *A control system is:*

- **(locally) exponentially stabilizable at decay rate $\lambda > 0$** if there exists a linear stationary feedback law such that the corresponding closed-loop system is well-posed and (locally) exponentially stable with decay rate λ .
- **(locally) rapidly stabilizable** if it is (locally) exponentially stabilizable at any positive decay rate.
- **(locally) finite-time stabilizable** if there exists a linear stationary feedback law such that the corresponding closed-loop system is (locally) finite-time stable.

1.2 From controllability to stabilization

In the previous section we have presented two central notions of control theory on which this thesis will focus. Although quite different in their definitions and in their practical aims, controllability and stabilizability are closely linked by many theoretical results. In a way, as controllability is a way of describing how well information is transmitted and acted upon in a system, one can imagine that if a system is controllable, i.e. information is well processed, then chances are that it can be stabilized too.

The main result we want to mention here can be found in [160], and concerns autonomous control systems in Hilbert spaces with bounded control operators:

Theorem 1.2.1. *If an autonomous linear system with a bounded control operator is null controllable, then it is exponentially stabilizable, at any decay rate.*

This result has since then been quantified via weakened observability inequalities in [146]. It holds in particular for finite-dimensional systems, and we will see that in this case, explicit feedbacks can be derived.

Conversely, some results highlight obstructions to stabilizability. Let us mention the famous result in finite dimension by Roger Brockett (see [36]):

Theorem 1.2.2 (Brockett condition). *A necessary condition for the control system*

$$\dot{X} = f(X, u), \quad X \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad (1.35)$$

to be locally asymptotically stabilizable at the equilibrium point $(x_e, 0)$ by a continuous, stationary feedback law vanishing at 0 is that the image of any neighborhood of $(x_e, 0)$ by f is a neighbourhood of 0.

In particular, there are some nonlinear systems that are locally controllable in small time but do not satisfy the Brockett condition. To overcome this obstacle, one can revert to time-varying feedback laws. Thus, in [47, 50], Jean-Michel Coron proves that nonlinear finite-dimensional controllable systems can be stabilized by means of time-varying feedbacks, when the control can be chosen continuous with respect to the initial and final conditions.

Of course this is not the whole story: some systems can be stabilized but are not controllable (see Example 1.4.1 hereafter), and some systems are approximately controllable but cannot be stabilized exponentially (see [160, Theorem 3.3]). And, last but not least, knowing that something can or cannot be done is but the first step. Doing it is the next one: let us now present a few methods that have been developed to stabilize controllable systems.

1.2.1 Finite-dimensional systems

For autonomous linear systems in finite dimension

$$\dot{X} = AX + Bu, \quad X \in \mathbb{R}^n, \quad A \in M_n(\mathbb{R}), \quad B \in M_{n,m}(\mathbb{R}), \quad u \in \mathbb{R}^m, \quad (1.36)$$

there is a powerful theorem which links controllability and stabilizability (see for example [53, Chapter 10, Section 10.1]):

Theorem 1.2.3. *Suppose that the control system (1.36) is controllable. Then, for any unitary polynomial $P \in \mathbb{R}[X]$, there exists a feedback law $K \in M_{m,n}(\mathbb{R})$ such that*

$$\chi(A + BK) = P, \quad (1.37)$$

where $\chi(A + BK)$ is the characteristic polynomial of $A + BK$. In particular, by Theorem 1.1.3, the control system (1.36) can be stabilized exponentially at any exponential decay rate.

After an appropriate reduction to the case of scalar controls, the proof of this result relies on the notion of canonical form (also known as Brunovski normal form). Indeed, all controllable systems with a scalar control can be rewritten, after an appropriate change of variables, in the form (see [53, Lemma 10.2]):

$$\dot{X} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & \cdots & \cdots & \cdots & -a_n \end{pmatrix} X + u(t) \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \quad (1.38)$$

This illustrates a general strategy of trying to bring all the dynamics of the system “within reach” of the control term. Indeed, one can see that the particular companion matrix form is a cascade structure where everything happens on the last line. In more algebraic terms, this form is the “natural form” to express and understand the Kalman rank condition (1.9).

This result for linear systems leads to a fundamental stabilizability theorem for nonlinear systems:

Theorem 1.2.4. *Consider the nonlinear control system*

$$\dot{X} = f(X, u), \quad X \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad f \in C^1(\mathbb{R}^{n+m}, \mathbb{R}^n), \quad (1.39)$$

where $f(0, 0) = 0$. Suppose that the linearized system

$$\dot{X} = \nabla f(0, 0) \cdot (X, u) \quad (1.40)$$

is controllable. Then system (1.39) is locally asymptotically stabilizable by means of continuous stationary feedback laws.

Another approach to the stabilization of controllable systems is to consider the following matrix, called controllability Gramian of the system (1.36):

$$C_T := \int_0^T e^{-tA} B B^* e^{-tA^*} dt. \quad (1.41)$$

It is a well-known result that (1.36) is controllable if and only if C_T is positive definite. Then, Dahlgard Lukes ([120]) and David Kleinman ([103]) prove that the feedback

$$K := -B^* C_T^{-1} \quad (1.42)$$

stabilizes (1.36) exponentially.

1.2.2 Infinite-dimensional systems

The power and simplicity of the canonical form make it tempting to try to generalize it to infinite-dimensional systems to study spectral assignability. Accordingly, in [135] David Russell builds a canonical form for a class of controllable linear first-order hyperbolic systems, in order to study what pole placement may be achieved using linear bounded feedback laws. In that case, the canonical form is a simple time-delay system and the transformation that maps the hyperbolic systems to their canonical form is a Volterra transformation of the second kind. We will see that this class of transformations plays an important role in Section 1.4. Another generalisation of the canonical form for hyperbolic systems can be found in [45], where the authors arrive at a canonical form, using the Laplace transform, to study spectral assignability by boundary input of a linear bounded feedback.

We will see in Section 1.4.1 that the canonical form shows some limitations in its infinite-dimensional generalization. On the other hand, the Gramian method has been successfully generalized to a whole class of infinite-dimensional control systems. In [140], to deal with infinite-dimensional systems with bounded control operators, Marshall Slemrod adds an exponential weight function inside the integral, for some $\omega > 0$:

$$C_T^\omega := \int_0^T e^{-2\omega t} e^{-tA} B B^* e^{-tA^*} dt, \quad (1.43)$$

and uses it to build a stabilizing feedback:

$$K_\omega := -B^* (C_T^\omega)^{-1} \quad (1.44)$$

which achieves an exponential stabilization at rate ω .

To handle unbounded control operators, Komornik chooses a different weight function in [108], by requiring that C_T^ω be the solution of an algebraic Riccati equation. This is explored further in [153], and yields a virtually explicit feedback of the form (1.44). However, this feedback requires the computation of the inverse of an operator, which can be quite computationally heavy, as indicated by Emmanuel Trélat, Gensheng Wang and Yashan Xu in [147]. In this paper an approach via Proper Orthogonal Reduction is proposed, where feedbacks are computed on appropriate finite-dimensional reductions, to approximate the infinite-dimensional feedback under suitable assumptions on the system.

Finally, in Section 1.4 we will present another method that has allowed us to build explicit (and computable) feedback laws to exponentially stabilize hyperbolic systems.

Aim of the thesis

In this thesis, we address some internal controllability and stabilization questions, and the link between these notions, for some hyperbolic systems in one space dimension. We first study the indirect controllability of systems of coupled semilinear wave equations, using inversion theorems of the Nash-Moser type, combined with the return method, in the second chapter of this thesis. In this chapter we also delve into the possibility of stabilizing these systems. The difficulties encountered in this endeavour suggest a different outlook on the stabilization of controllable systems. This is explored in the third, fourth and fifth chapters of this thesis. In the third and fourth, we prove the rapid and finite-time stabilization of a linear transport equation with a distributed scalar input, using a form of backstepping method. In the fifth, we adapt the same method to study the stabilization of a coupled hyperbolic system, namely, the linearized water tank system, modelled by the Saint-Venant equations.

1.3 Internal controllability of systems of semilinear coupled one-dimensional wave equations in 1-D with a single control

1.3.1 Controlling hyperbolic systems

As we have mentioned earlier, hyperbolic systems are characterized by the way information travels, at finite speed along characteristic manifolds. In the 1-D case, these manifolds are simply curves, and the propagation of information can be illustrated with simple diagrams.

Accordingly, the first results on the controllability of hyperbolic systems use these characteristics, as they are a convenient way to see how the input is processed inside the system: let us mention the fundamental result in [133, 134], further developed in [18], for the boundary control, observation, and stabilization of the linear wave equation, in particular the famous Geometric Control Condition (GCC). To quote Claude Bardos, Gilles Lebeau and Jeffrey Rauch in the latter reference: “To control, observe, or stabilize solutions of hyperbolic partial differential equations, it is necessary that we observe or control at least one point of each ray of geometric optics.”

This was later extended to nonlinear cases by [43], and local controllability results were proved in [78]. Let us mention also the survey [113] on the boundary controllability of quasilinear hyperbolic systems (in C^k spaces). The presence of the Laplacian operator in the wave operator makes it interesting to use duality methods: Enrique Zuazua obtained results for semilinear wave equations in [167, 166], with a constraint on the nonlinearity, first proving a linear result with the HUM method, then using a fixed-point theorem to get the semilinear result. Results for stronger nonlinearities were then obtained in [38, 37]. Carleman estimates have also been used to prove observability inequalities: in [164, 165] internal observability is established for semilinear wave equations with globally Lipschitz nonlinearities, and the observer supported in a neighbourhood of some portion of the boundary. In [79, 88] the method was improved to work for controllers with smaller supports and for stronger nonlinearities.

Knowing this, the question we address in Chapter 2 is that of so-called indirect controllability: can a system of coupled hyperbolic equations be controlled with less controls than equations? Physically speaking, can the interactions between the equations help us control them all, by acting on one of them only? This is a typical question of control theory, as it pushes the controllability to its limits, by asking *how much do we really need to act on in order to control everything? Why is it enough? Why can we not do it with less?* And its dual notion, partial observability, asks the question *how much do we really need to observe to know everything?* Several other questions then follow: how does it depend on the supports of the control (the observation) and/or the coupling? For hyperbolic equations there is often a minimal time, in the case of indirect control, on what does the minimal time depend? Intuitively speaking, one can consider that the information has to travel from the controlled zone to the coupling zone, then from the coupling zone to the whole domain.

In the linear case, some answers to these questions can be found in [6, 8, 3], where energy methods are used to obtain indirect controllability of various types of linear systems. In particular, in [8] geometric conditions on the supports of the coupling and the control appear (in particular they can have an empty intersection), and a control result is proved for a system of coupled wave equations, with a minimal time depending in particular on these supports:

Theorem 1.3.1 (F.Alabau-Boussouira, M.Léautaud, 2013). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary, a, b be smooth real-valued functions on Ω such that*

$$\begin{aligned} \Delta + a \text{ is a coercive operator on } H^2 \cap H_0^1(\Omega) \text{ for } \|\cdot\|_{L^2}, \\ b \geq 0 \text{ on } \Omega, \\ \{x \in \Omega, \quad b > 0\} \text{ satisfies the GCC for some time } T > 0. \end{aligned} \quad (1.45)$$

Then for p small enough in $\|\cdot\|_\infty$, positive, with a support satisfying the GCC in time T , the system

$$\begin{cases} u_{tt} - \Delta u + au + pv = bf, \\ v_{tt} - \Delta v + av + pu = 0, \\ u = v = 0 \text{ on } \partial\Omega. \end{cases} \quad (1.46)$$

is null controllable in time $T > 0$ with initial data in $H_0^1(\Omega) \times (H^2 \cap H_0^1(\Omega)) \times L^2(\Omega) \times H_0^1(\Omega)$.

Note that in the same paper analogous results are proved for a boundary control, and for the internal and boundary controllability of coupled Schrödinger equations and coupled heat equations.

A notable result in the quasilinear case can be found in [5, Theorem 3.1], for 1-D first order hyperbolic systems:

Theorem 1.3.2 (F.Alabau-Boussouira, J.-M.Coron, G.Olive, 2017). *Let $0 < a < b < L$ and $\Lambda_1, \Lambda_2, f_1, f_2 \in C^\infty(\mathbb{R}^2)$ be such that*

$$\Lambda_1(u, v) < \Lambda_2(u, v), \quad \forall (u, v) \in \mathbb{R}^2, \quad (1.47)$$

and

$$\frac{\partial f_2}{\partial u}(0, 0) \neq 0. \quad (1.48)$$

Then, the system

$$\begin{cases} u_t + \Lambda_1(u, v)u_x + f_1(u, v) = h, \\ v_t + \Lambda_2(u, v)v_x + f_2(u, v) = 0, \\ u(t, 0) = u(t, L), \quad v(t, 0) = v(t, L), \end{cases} \quad (1.49)$$

where h is the control, is locally controllable around 0 in time

$$T > (L - (b - a)) \max \left\{ \frac{1}{|\Lambda_1(0, 0)|}, \frac{1}{|\Lambda_2(0, 0)|} \right\} \quad (1.50)$$

for initial and final states in C^6 , with a $C^1([0, T] \times [0, L])$ control supported in $[\delta, T - \delta] \times [a + \delta, b - \delta]$ for every $0 < \delta < \min(T, (b - a)/2)/4$ such that

$$T - 4\delta > (L - (b - a - 8\delta)) \max \left\{ \frac{1}{\Lambda_1(0, 0)}, \frac{1}{\Lambda_2(0, 0)} \right\}. \quad (1.51)$$

Moreover, the control can be chosen to be a continuous $(C^6)^2 \rightarrow C^1$ function of the initial and final states.

The proof of this result is based on an altogether different approach, called the fictitious control method.

To prove the results of Chapter 2, we use the same approach to study the indirect controllability of coupled semilinear wave equations in 1-D in two different situations, one where a condition of the type (1.48) is satisfied, and one where it is not.

1.3.2 Main results

Let $T > 0$, and $0 < a < b < L$. We study the following class of systems:

$$\begin{cases} u_{tt} - \nu_1^2 u_{xx} = f_1(u, v) + h, & x \in (0, L), \\ v_{tt} - \nu_2^2 v_{xx} = f_2(u, v), & x \in (0, L), \\ u = 0 & \text{on } \{0, L\}, \\ v = 0 & \text{on } \{0, L\}, \end{cases} \quad (1.52)$$

where $h : [0, T] \times [0, L] \rightarrow \mathbb{R}$ is the control, with $\text{supp } h \subset [0, T] \times [a, b]$, and $f_1, f_2 \in C^\infty(\mathbb{R}^2)$, $f_1(0, 0) = f_2(0, 0) = 0$, $\nu_1, \nu_2 \neq 0$. We will also study the following particular system:

$$\begin{cases} u_{tt} - \nu_1^2 u_{xx} = h, & x \in (0, L), \\ v_{tt} - \nu_2^2 v_{xx} = u^3, & x \in (0, L), \\ u = 0 & \text{on } \{0, L\}, \\ v = 0 & \text{on } \{0, L\}. \end{cases} \quad (1.53)$$

These are systems of coupled semilinear wave equations, with different speeds, which we seek to control with a single control, which takes the form of a source term in the first equation with a support in $[0, L] \times [a, b]$. In both cases, we will study solutions with $C^k([0, T] \times [0, L])$ regularity in order to establish a controllability result with two controls. Thus, the initial and final conditions $((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f))$ have to satisfy some compatibility conditions, which will be given in Chapter 2 by (2.8).

In Chapter 2 we prove two controllability results: a local result for system (1.52), and a global result for system (1.53).

Theorem 1.3.3 (Christophe Zhang, 2017). *Let $R > 0$, and $0 \leq a < b \leq L$, $T > 0$ be such that*

$$T > 2(L - b) \max\left(\frac{1}{|\nu_1|}, \frac{1}{|\nu_2|}\right), \quad T > 2a \max\left(\frac{1}{|\nu_1|}, \frac{1}{|\nu_2|}\right). \quad (1.54)$$

If

$$\frac{\partial f_2}{\partial u}(0, 0) \neq 0, \quad (1.55)$$

then there exists $\eta > 0$ such that for initial and final conditions

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in (B_{C^{11}([0, L])} \times B_{C^{10}([0, L])}(0, \eta))^4$$

where $B_{C^k}(0, \eta)$ denotes the ball centered in 0 and with radius η in the usual C^k topology, satisfying compatibility conditions (2.8) at the order 11, there exists $h \in C^6([0, T] \times [0, L])$ such that

$$\text{supp } h \subset [0, T] \times [a, b], \quad (1.56)$$

and such that the corresponding solution $(u, v) \in C^6([0, T] \times [0, L])^2$ of (1.52) with initial values $((u_0, u_1), (v_0, v_1))$ satisfies

$$\begin{cases} u(T, \cdot) = u_0^f, & u_t(T, \cdot) = u_1^f, \\ v(T, \cdot) = v_0^f, & v_t(T, \cdot) = v_1^f \end{cases}$$

and

$$\|(u, v, h)\|_{(C^6)^3} \leq R. \quad (1.57)$$

The second theorem gives a controllability result for a system that does not satisfy (1.55). Note that, thanks to the homogeneity of the system, this result is global.

Theorem 1.3.4 (Christophe Zhang, 2017). *Let $0 \leq a < b \leq L$, $T > 0$ satisfying (1.54). There exists a constant $C > 0$ depending on T such that, for any given initial and final conditions*

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in (C^{11}([0, L]) \times C^{10}([0, L]))^4$$

satisfying compatibility conditions (2.8) at the order 11, there exists $h \in C^6([0, T] \times [0, L])$ such that

$$\text{supp } h \subset [0, T] \times [a, b], \quad (1.58)$$

and such that the corresponding solution $(u, v) \in C^6([0, T] \times [0, L])^2$ of (1.53) with initial values $((u_0, u_1), (v_0, v_1))$ satisfies

$$\begin{cases} u(T, \cdot) = u_0^f, & u_t(T, \cdot) = u_1^f, \\ v(T, \cdot) = v_0^f, & v_t(T, \cdot) = v_1^f \end{cases}$$

and

$$\|h\|_{C^6} \leq C \left(\|(u_0, u_1, u_0^f, u_1^f)\|_{(C^{11} \times C^{10})^2} + \|(v_0, v_1, v_0^f, v_1^f)\|_{(C^{11} \times C^{10})^2}^{\frac{1}{3}} \right). \quad (1.59)$$

1.3.3 The fictitious control method

To prove these results, we use a method called the ‘‘fictitious control method’’, first introduced in [47] and [83], and most recently developed in [5, 61, 76, 115, 77]. The idea is to first try to solve the problem with as many controls as equations (the fictitious controls), then to work from this solution to reduce the number of controls. In even broader terms, the strategy is to first find an ‘‘easy’’, but not exact solution, and then try to tweak it to find a real solution. Let us illustrate with a simple linear example, presented in detail in [53, Theorem 1.18]: consider the linear, time-varying, control problem

$$\dot{X} = A(t)X + B(t)u(t), \quad (1.60)$$

where $X \in \mathbb{R}^n$, $u(t) \in C^\infty([0, T]; \mathbb{R}^m)$, $A \in C^\infty(\mathbb{R}; M_n(\mathbb{R}))$, $B \in C^\infty(\mathbb{R}; M_{n,m}(\mathbb{R}))$. Define, by induction,

$$B_0(t) := B(t) \quad ; \quad B_i(t) := \dot{B}_{i-1}(t) - A(t)B_{i-1}(t), \quad \forall i \geq 1. \quad (1.61)$$

Under some conditions on the B_i , we have the following control result (see [40] and [139]):

Theorem 1.3.5 (A. Chang, 1965, L.Silverman, H.Meadows, 1965). *Let $T_0, T_1 > 0$. If there exists $\bar{t} \in [T_0, T_1]$ such that*

$$\text{Span}\{B_i(\bar{t})u, \quad u \in \mathbb{R}^m, \quad 0 \leq i \leq n-1\} = \mathbb{R}^n, \quad (1.62)$$

then (1.60) is controllable on the interval $[T_0, T_1]$.

Remark 1.3.1. *Note that if A, B are constant, then (1.62) becomes the usual Kalman rank condition (1.9) for the matrices (A, B) .*

This well-known result is usually proved using the controllability Gramian. In [53] however a new proof is given, which uses the fictitious control strategy.

Let $x_0, x_1 \in \mathbb{R}^n$. We first note that (1.62) is true in a neighbourhood of $[t_0, t_1] \subset [T_0, T_1]$ of \bar{t} . Then, let us consider the uncontrolled trajectory of (1.60) starting from x_0 (resp. ending at x_1), which we note a (resp. b). Then, let d be a cut-off function:

$$d \in C^\infty([T_0, T_1]), \quad d_{|[T_0, t_0+\varepsilon]} \equiv 0, \quad d_{|[t_1-\varepsilon, T_1]} \equiv 1, \quad (1.63)$$

for some $\varepsilon > 0$. Then,

$$\Gamma := (1-d)x_0 + dx_1 \quad (1.64)$$

has the right starting and ending points:

$$\Gamma(T_0) = x_0, \quad \Gamma(T_1) = x_1. \quad (1.65)$$

Now, if there existed $u \in C^\infty([T_0, T_1]; \mathbb{R}^m)$ such that

$$Bu = \dot{\Gamma} - A\Gamma =: q \quad (1.66)$$

then we would have found a solution to our problem. But we have little information on B itself, and it might well be that $q \notin \text{Im}(B)$, in other words, that the trajectory Γ uses too many controls. The idea at this point is rather to see $\dot{\Gamma} - A\Gamma$ as some sort of local error term that has to be corrected. Indeed if we can find $(r, u) \in C^\infty([T_0, T_1]; \mathbb{R}^n \times \mathbb{R}^m)$ such that

$$\dot{r} - Ar - Bu = q, \quad r(T_0) = r(T_1) = 0, \quad (1.67)$$

then, setting

$$X := \Gamma - r, \quad (1.68)$$

the controllability problem is solved with $(X, -u)$, as Γ already has the right starting and ending points, and r does not change them.

To solve (1.67), consider the following differential operator:

$$L(X, u) := \dot{X} - AX - Bu, \quad \forall (X, u) \in C^\infty([T_0, T_1]; \mathbb{R}^n \times \mathbb{R}^m), \quad (1.69)$$

so that we have

$$L(\Gamma, 0) = q. \quad (1.70)$$

Essentially, solving (1.67) is finding a right inverse M for the under-determined operator L , which would give the solution

$$(r, u) := M(q). \quad (1.71)$$

However we need to make sure that r satisfies the second part of (1.67). This is ensured by the fact that we can actually find a right inverse M that is also a linear differential operator, which in this case is built using (1.62). Then, noting that, by construction,

$$q(t) = 0, \quad \forall t \in [T_0, t_0] \cup [t_1, T_1]. \quad (1.72)$$

we get in particular that $r(T_0) = r(T_1) = 0$, so that by (1.68), the problem is solved.

Remark 1.3.2. *At this point, one could wonder why the right inverse M is not directly applied to 0, and not bother with an error term followed by a correction. But one has to keep in mind that we need a trajectory with the right starting and ending points, which is not guaranteed if we simply take $M(0)$.*

Let us now turn to an infinite-dimensional example, taken from [12]: consider the linear control problem

$$\begin{cases} \partial_t y - \Delta y = Ay + B\mathbf{1}_\omega h, & \text{on } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.73)$$

where $\Omega \in \mathbb{R}^d$, $y(t) \in L^2(\Omega)^n$, $A \in M_n(\mathbb{R})$, $m < n$, $h \in L^2((0, T) \times \Omega)^m$, $B \in M_{n,m}(\mathbb{R})$ and ω is an open subset of Ω .

Theorem 1.3.6 (F.Ammar-Khodja, A.Benabdallah, C.Dupaix, M.Gonzalez-Burgos, 2009). *System (1.73) is null-controllable in small-time for L^2 initial conditions if and only if (A, B) satisfies the Kalman rank condition.*

This extension of the Kalman rank condition, algebraic in nature, to a system of heat equations, can be understood thanks to the fictitious control method.

Let $y_0 \in L^2(\Omega)^n$. As before, the first step is to find a trajectory with the right initial (y_0) and final (0) conditions. This time, it is not as simple as ‘‘merging’’ two trajectories (for example, one cannot start from the end-point and go backwards as with (1.60)), but can still be done by studying the simpler control problem

$$\begin{cases} \partial_t y - \Delta y = Ay + \mathbf{1}_\omega h, & \text{on } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.74)$$

looking for controls $h \in L^2((0, T) \times \Omega)^m$ supported in $[\varepsilon, T - \varepsilon] \times \omega$. This system is null-controllable (using for example Carleman estimates), so we get a trajectory \hat{y} starting at y_0 and solution of (1.74) for some fictitious controls \hat{h} supported in $[\varepsilon, T - \varepsilon] \times \omega$.

Now let us use the fictitious controls to get the real control. Consider the following partial differential operator:

$$\mathcal{L}(y, h) = \partial_t y - \Delta y - Ay - B\mathbb{1}_\omega h, \quad \forall y \in H^1(0, T; H^2(\Omega)^n), \quad \forall h \in L^2((0, T) \times \Omega)^m, \quad (1.75)$$

so that we have

$$\mathcal{L}(\hat{y}, 0) = \hat{h}. \quad (1.76)$$

Finding a right inverse for this operator is equivalent to finding a left inverse for its adjoint \mathcal{L}^* :

$$\mathcal{L}^* f = \begin{pmatrix} -\partial_t f - \Delta f - A^* f \\ -B^* f \end{pmatrix}, \quad \forall f \in H^1(0, T; H^2(\Omega)^n). \quad (1.77)$$

Now let us apply the following operators to $\mathcal{L}^* f$:

$$\begin{cases} \mathcal{S} := (\mathcal{S}_1, \dots, \mathcal{S}_n), \\ \mathcal{S}_0(x_1, x_2) := -x_2, \\ \mathcal{S}_1(x_1, x_2) := \partial_t x_2 + \Delta x_2 - B^* x_1, \\ \dots \\ \mathcal{S}_k(x_1, x_2) := (\partial_t + \Delta)\mathcal{S}_{k-1}(x_1, x_2) - B^*(A^*)^{k-2}x_1 \end{cases} \quad (1.78)$$

so that

$$\mathcal{S} \circ \mathcal{L}^* f = \begin{pmatrix} B^* f \\ B^* A^* f \\ \vdots \\ B^*(A^*)^{n-1} f \end{pmatrix} = \begin{pmatrix} B^* \\ B^* A^* \\ \vdots \\ B^*(A^*)^{n-1} \end{pmatrix} f. \quad (1.79)$$

Then, thanks to the Kalman rank condition, there exists a matrix $M \in M_{n, nm}(\mathbb{R})$ such that

$$M \begin{pmatrix} B^* \\ B^* A^* \\ \vdots \\ B^*(A^*)^{n-1} \end{pmatrix} = I_n, \quad (1.80)$$

so that $M \circ \mathcal{S}$ is a linear left inverse for \mathcal{L}^* , hence $\mathcal{S}^* \circ M^*$ is a linear right inverse for \mathcal{L} .

Then, as in the previous example, thanks to the precautions on the support of \hat{h} , and the fact that we are dealing with linear differential operators,

$$(X, u) := (\hat{y}, 0) - \mathcal{S}^* \circ M^*(\hat{h}) \quad (1.81)$$

gives us a trajectory and a control that solve the null-controllability problem (1.73).

Remark 1.3.3. *The underlying general results that allowed us to find right inverses for the operators L and \mathcal{L} are a set of powerful theorems proved by Mikhail Gromov ([84, pages 150-156]). A generic underdetermined linear differential operator (in finite or infinite dimension) is **algebraically solvable**, i.e. has a right inverse that is also a linear differential operator. Algebraic solvability is crucial in the fictitious control method. Indeed, linear differential operators preserve the supports of the controls and trajectories we are working with, in finite or infinite dimension, which is crucial to find trajectories with the right initial and final conditions, and controls with the right support in infinite dimension.*

As we will see later on, there is also a nonlinear inversion theorem in [84], which will be very useful to obtain the results of Chapter 2.

1.3.4 The linear test and the fictitious control method

When dealing with nonlinear control systems, the go-to strategy is to look at the linearized system, and apply the so-called *linear test*: if it is controllable, then chances are, with a suitable inversion (fixed-point) theorem, the nonlinear system is locally controllable. This is always the case in finite dimension, around equilibrium points and around trajectories (see [53, Section 3.1]), thanks to the local inversion theorem (or, alternately, the implicit function theorem, or the Picard fixed-point theorem for contractions).

On the other hand, in infinite dimension the situation can be more complicated (as usual). Sometimes the general Picard fixed point theorem for Banach spaces still works, for example for the nonlinear 1-D KdV equation with a Neumann control on the right ([130]). The idea is to consider the nonlinearity like a perturbation of the linearized system around 0, in other words, a source term of the linear equation. Essentially, the solution of this controllability problem is given by the fixed point of a mapping that maps $y \in L^2(0, T; H^1(0, L))$ to the solution of

$$\begin{cases} z_t + z_x + z_{xxx} = -yy_x, \\ z(t, 0) = z(t, L) = z_x(t, L) = 0, \quad t \in (0, T), \\ z(0, x) = y_0, \quad x \in (0, L), \end{cases} \quad (1.82)$$

with some additional adjustments to get the wanted final value, which are possible thanks to the controllability of the linear system (without source term).

In other cases, one has to work a bit more to get the local controllability of the nonlinear system. Consider for example the following hyperbolic control system (see [53, Section 4.2.1]):

$$\begin{cases} y_t + a(y)y_x = 0, \\ y(t, 0) = u(t), \end{cases} \quad (1.83)$$

where $a \in C^2(\mathbb{R})$ and $a(0) > 0$. Its linearized system around $(0, 0)$

$$\begin{cases} y_t + a(0)y_x = 0, \\ y(t, 0) = u(t), \end{cases} \quad (1.84)$$

is controllable in times $T > L/a(0)$. Taking the same perturbative approach as before, if we try to consider the mapping that maps y to the solution of

$$\begin{cases} z_t + a(0)z_x = (a(0) - a(y))y_x, \\ z(t, 0) = 0. \\ z(0, x) = y_0, \end{cases} \quad (1.85)$$

a problem occurs with the spaces in which this mapping is defined. Indeed, it does not have the same nice regularity properties as the one defined by (1.82): we lose one derivative because the solution of (1.85) is not necessarily more regular than the source term. Instead, another fixed point theorem can be considered, with the mapping that maps $y \in C^1([0, T] \times [0, L])$ to the solution of

$$\begin{cases} z_t + a(y)z_x = 0, \\ z(0, x) = y_0, \quad z(T, x) = y_1. \end{cases} \quad (1.86)$$

This mapping is different in essence, as it links the coefficient of a linear differential operator, instead of the source term, to the solution of the corresponding equation. Moreover, this mapping requires the study of not one linear system (the linearized system), but a collection of linearized systems (all the linear systems “close to” the linearized system). Accordingly, it is dealt with thanks to a different, more powerful fixed-point theorem, namely Schauder’s theorem.

The problem encountered in (1.85), namely a loss of derivatives, occurs in many systems. In the above example it was handled with a particular mapping for which we can apply a “classical” fixed-point theorem. However, there is a more systematic approach, known as the Nash-Moser method (see [96, 97, 10] for a comprehensive overview and a general version on Hölder spaces). Let us illustrate it on an example due to Karine Beauchard ([28]), the 1-D bilinear Schrödinger equation:

$$\begin{cases} \psi_t(t, x) = i\psi_{xx}(t, x) + iu(t)x\psi(t, x), & (t, x) \in (0, T) \times (-1, 1), \\ \psi(t, -1) = \psi(t, 1) = 0, & t \in (0, T), \end{cases} \quad (1.87)$$

with states in the L^2 -sphere \mathbb{S} . For $\gamma \geq 0$ define the operator

$$S_\gamma := -\partial_{xx} - \gamma x, \quad D = H^2(-1, 1) \cap H_0^1(-1, 1), \quad (1.88)$$

and note $(\varphi_{k,\gamma}, \lambda_{k,\gamma})$ its eigenfunctions and eigenvalues. We will study the linearized system around the trajectory $(\psi_{1,\gamma}, \gamma)$:

$$\begin{cases} \psi_t(t, x) = i\psi_{xx}(t, x) + i\gamma x\psi(t, x) + iu(t)x\psi_{1,\gamma}(t, x), & (t, x) \in (0, T) \times (-1, 1), \\ \psi(t, -1) = \psi(t, 1) = 0, & t \in (0, T), \end{cases} \quad (1.89)$$

where $\psi_{1,\gamma}$ satisfies

$$\begin{cases} (\psi_{1,\gamma})_t = i(\psi_{1,\gamma})_{xx} + i\gamma x\psi_{1,\gamma}, & (t, x) \in (0, T) \times (-1, 1), \\ \psi_{1,\gamma}(t, -1) = \psi_{1,\gamma}(t, 1) = 0, & t \in (0, T), \\ \psi_{1,\gamma}(0, \cdot) = \varphi_{1,\gamma}(\cdot) \\ \|\psi_{1,\gamma}(t, \cdot)\|_{L^2(-1,1)} = 1, & t \in (0, T). \end{cases} \quad (1.90)$$

One can prove, using the moments method, that (1.89) is small-time controllable with state in

$$T_{\mathbb{S}}(\psi_{1,\gamma}(0, \cdot)) \cap H_{(0)}^3 := \{y \in H^3(-1, 1), \quad y(-1) = y''(-1) = y(1) = y''(1) = 0\}, \quad (1.91)$$

and $u \in L^2(0, T)$, for $\gamma > 0$ small enough. However, we cannot deduce a nonlinear controllability result in the same spaces. Indeed, consider the mapping

$$\mathcal{F} : u \mapsto y(T, \cdot) \in \mathbb{S} \quad (1.92)$$

where y is the solution of (1.87) with control u and initial value $\psi_{1,\gamma}(0, \cdot)$. Its differential around the constant function $\bar{u} \equiv \gamma$ is the endpoint mapping for (1.89), and the controllability of that linearized system corresponds to the existence of a right inverse

$$\mathcal{F}'(\bar{u})^{-1} : H_{(0)}^3 \cap T_{\mathbb{S}}(\psi_{1,\gamma}(T, \cdot)) \rightarrow L^2(0, T). \quad (1.93)$$

One could hope to use that right inverse in an iteration scheme to find a solution to the nonlinear controllability problem

$$\mathcal{F}(u) = \psi_1 \quad (1.94)$$

for some target state $\psi_1 \in H_{(0)}^3 \cap T_{\mathbb{S}}(\psi_{1,\gamma}(T, \cdot))$. The iteration scheme to solve (1.94) around \bar{u} , knowing that $\mathcal{F}'(\bar{u})$ has a right inverse, would be the sequence defined by

$$u_{n+1} = u_n - \mathcal{F}'(\bar{u})^{-1}P(\mathcal{F}(u_n) - \psi_1), \quad (1.95)$$

where P is the projection on the tangent space $T_{\mathbb{S}}(\psi_{1,\gamma}(T, \cdot))$. Normally, u_n would be expected to converge towards a solution. The problem is that although $\mathcal{F}'(\bar{u})^{-1}$ loses three derivatives, there is a priori no reason for the mapping \mathcal{F} to regain them: for an L^2 control, one cannot expect the solution of (1.87) to be more than H^2 . So at each iteration of (1.95), one derivative is lost. Nash’s idea (initially developed to imbed Riemannian manifolds in Euclidean spaces in [123]) was to apply a smoothing operator S_n at each step to compensate this loss of derivatives. To make sure the smoothing operator

does not hinder the convergence of the scheme, a Newton convergence scheme is used, which is much faster than (1.95):

$$u_{n+1} = u_n - S_n \mathcal{F}'(u_n)^{-1} (\mathcal{F}(u_n) - \psi_1). \quad (1.96)$$

Note that, as in the previous example, we now need to study all the linear systems close to the linearized system around \bar{u} . More precisely, we require that all $\mathcal{F}'(u)$ be invertible for u in a neighbourhood of \bar{u} . Moreover, because some margin is needed to have the right estimates and chose the right smoothing operators, the final nonlinear controllability result holds for states in

$$\begin{aligned} T_{\mathbb{S}}(\psi_{1,\gamma}(0, \cdot)) \cap H_{(0)}^7 &:= \{y \in H^7(-1, 1), \quad y^{(2k)}(-1) = y^{(2k)}(1) = 0, \quad k = 0, 1, 2, 3\}, \\ T_{\mathbb{S}}(\psi_{1,\gamma}(T, \cdot)) \cap H_{(0)}^7 & \end{aligned} \quad (1.97)$$

and with a positive minimal time. Note that in [31], Karine Beauchard improves her result from [28] to require less additional regularity, with the spaces

$$\begin{aligned} T_{\mathbb{S}}(\psi_{1,\gamma}(0, \cdot)) \cap H^{5+\varepsilon}(-1, 1) \cap H_{(0)}^5 &:= \{y \in H^5(-1, 1), \quad y^{(2k)}(-1) = y^{(2k)}(1) = 0, \quad k = 0, 1, 2\}, \\ T_{\mathbb{S}}(\psi_{1,\gamma}(T, \cdot)) \cap H^{5+\varepsilon}(-1, 1) \cap H_{(0)}^5 & \end{aligned} \quad (1.98)$$

Remark 1.3.4. *In that particular case, it turns out that the Nash-Moser method is not necessary to study the nonlinear system. Indeed in [30], Karine Beauchard and Camille Laurent prove that there is actually no loss of derivative, due to a hidden regularization effect. Thus the optimal spaces for the nonlinear controllability are the same as for the controllability of the linearized system, and hold for a generic potential function:*

Theorem 1.3.7 (K.Beauchard, C.Laurent, 2010). *Let $\mu \in H^3(-1, 1; \mathbb{R})$ be such that there exists a constant $C > 0$ such that*

$$\langle \mu \varphi_{1,0}, \varphi_{k,0} \rangle \geq \frac{C}{k^3}, \quad \forall k \geq 1, \quad (1.99)$$

Then, the 1-D bilinear Schrödinger equation

$$\begin{cases} \psi_t(t, x) = i\psi_{xx}(t, x) + iu(t)\mu(x)\psi(t, x), & (t, x) \in (0, T) \times (-1, 1), \\ \psi(t, -1) = \psi(t, 1) = 0, & t \in (0, T), \end{cases} \quad (1.100)$$

is small-time locally controllable around the trajectory $(\psi_{1,0}, 0)$ in $T_{\mathbb{S}}(\psi_{1,0}(0, \cdot)) \cap H_{(0)}^3$, with $u \in L^2$ a continuous function of the initial and final data.

In Chapter 2, we deal with the same a priori loss of derivatives, and the situation is analogous to the one in [5]. Indeed there is an a priori loss of derivatives, which will be detailed in Section 2.4.1 from the initial and final data to the control, and again the ideas of the Nash-Moser method are used to handle this issue.

However, our strategy, which we adapt from [5], differs from the examples above. Indeed, to prove Theorem 1.3.3, following the path of the previous example, we would set out to prove a controllability result for some related linear systems, using the fictitious control method as in the examples of Section 1.3.3. Then, we would try to find a way to use an inversion theorem to get a local controllability result for the nonlinear system. It turns out that it is not necessary to do things in two steps, but one can instead apply the fictitious control method directly to the nonlinear system. Indeed, as mentioned in Remark 1.3.3, there is a nonlinear inversion theorem (see [84, Section 2.3.2, Main Theorem]) which can be used to discard the fictitious controls. This inversion theorem, which is of the Nash-Moser type, relies on condition (1.55), which is basically a characterization of the controllability of linear systems close to the linearized system around 0 (if the dynamics of v does not depend on u there is no hope to control v through u). So, in a way, the study of the linearized system is absorbed into the fictitious control method. The first step is thus to solve the nonlinear problem with two controls, which is done using boundary control results in the book [113, Chapter 5]. We point out that this is

the only step in the proof that requires a 1-D domain, as the inversion theorem works in any space dimension, under the same condition (1.55).

On the other hand, (1.53) does not satisfy (1.55): the linearized system around 0 is not controllable. However, Theorem 1.3.4 shows that this condition is not necessary in the case of nonlinear systems, and we still obtain a controllability result.

This situation where the linearized system is not controllable is very common in control theory, for example with the nonviscous Burgers equation ([41, 42]), or the Euler equation ([48, 46, 81, 82]). To deal with this, Jean-Michel Coron introduced a method known as the return method. The idea is to look for another trajectory of the control system, a *return trajectory*, going from 0 back to 0 (hence the name), around which the linearized system is controllable, and then try once more to apply linear test methods. A more comprehensive outline, along with many references, can be found in [53, Chapter 6].

In our case, we will also build return trajectories, along which an adequate adaptation of (1.55) holds, or equivalently, the linearized system is controllable. This construction actually requires some legwork: we use the same method as Jean-Michel Coron, Sergio Guerrero, and Lionel Rosier in [60], where the controllability of a system of heat equations with cubic coupling is studied. The idea is to use the cascade structure of (1.53) to build a solution of the stationary problem

$$\begin{cases} -u_e'' = h_e, \\ -v_e'' = u_e^3, \\ u_e(0) = u_e(L) = v_e(0) = v_e(L) = 0, \end{cases} \quad (1.101)$$

by working upwards: first, we find a C^∞ function v_e whose second derivative is the cube of a C^∞ function, with a unique vanishing point, and a very smooth behaviour at the boundary. Then the solution of the stationary problem is given by

$$v_e := g, \quad u_e := \sqrt[3]{v_e''}, \quad h_e := u_e''. \quad (1.102)$$

To obtain a return trajectory, this solution is then perturbed in time and space. In contrast with [60], this must be done with extra care to control the support of \bar{u} . Indeed, (1.55) must be satisfied along the return trajectory, and this condition corresponds to \bar{u} being nonzero. Moreover, as the support of \bar{u} will be more or less the support of our final control, it has to satisfy a GCC, which will give us the minimal time for our control. As we will see, the method described above yields trajectories with very small supports, so that in order to satisfy a GCC, we will have to put many of these trajectories side by side. Although this method is 1-dimensional in essence, it can be easily extended to any space dimension by considering radial functions, as is done in [60]. So again, even in this particular case where we have to use a form of return method, the only point that specifically requires a 1-D spatial domain is the controllability problem with 2 controls.

Then, as we have pointed out before, we are not going to apply the usual linear test strategy around the return trajectory, but instead apply the nonlinear fictitious control method around these trajectories.

Finally, let us note that for now, it is likely that the spaces in Theorems 1.3.3 and 1.3.4 are not optimal, as the difference of regularity between the initial and final data and the control is 5, whereas the expected loss as shown in Section 2.4.1 is only 1. Moreover, it would be interesting to have a result in “natural” spaces for the wave equation, for example Sobolev spaces. To that effect let us point out that in the recently published [16], Pietro Baldi and Emanuele Haus prove a Nash-Moser-Hörmander inversion theorem for Sobolev spaces which could be applied to our problem. The Nash-Moser method has already been implemented in Sobolev spaces, as we have mentioned in the example of the Schrödinger equation (1.87), but Baldi and Haus’ theorem has the advantage of being sharp with respect to the loss of derivatives (see [16, Remark 2.5]). Indeed, when solving the linearized problem

$$\mathcal{F}'(u)h = g, \quad (1.103)$$

the right inverse of $\mathcal{F}'(u)$ is applied to g , so there is a loss of regularity from g to h . When solving the nonlinear problem

$$\mathcal{F}(u) = \mathcal{F}(0) + g, \quad (1.104)$$

the loss of regularity from g to u is the same. In contrast, previous inversion theorems required an arbitrarily small additional loss of regularity on the solution to the nonlinear problem.

1.3.5 Some comments about indirect stabilization

With Theorem 1.2.1 in mind, and knowing that system (1.52) is likely null-controllable in some Sobolev spaces, it is natural to look for exponentially stabilizing feedbacks for this system.

To stabilize physical systems such as waves, the most natural approach is to add a damping term. Consider for example the classical damped wave equation:

$$\begin{cases} u_{tt} - u_{xx} = -\lambda u_t, & \text{on } [0, L], \\ u(t, 0) = u(t, L) = 0, & \forall t \geq 0, \end{cases} \quad (1.105)$$

with initial conditions in $H_0^1 \times L^2$. Then one can derive an energy inequality for solutions in $(H^2 \cap H_0^1) \times H_0^1$:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^L |u_t|^2 + |u_x|^2 &= \int_0^L \Re(u_{xx} \overline{u_t}) - \lambda |u_t|^2 + \Re(u_{xt} \overline{u_x}) \\ &= -\lambda \int_0^L |u_t|^2. \end{aligned} \quad (1.106)$$

Hence, by density, V is a Lyapunov function for (1.105). Then we can show that 0 is asymptotically stable for system (1.105), using a weak version of the LaSalle invariance principle (see [143, Lemma 5.7.8, page 226], or [2, Proposition 1.3.6]):

Theorem 1.3.8. *Let V be a Lyapunov function for an autonomous system*

$$\dot{X} = f(X), \quad (1.107)$$

with f such that the system is well-posed in some Hilbert space, and $f(0) = 0$. Suppose that for any trajectory $X(t)$ of that system,

$$\frac{d}{dt} V(X(t)) = 0, \quad \forall t \geq 0 \implies X(t) \equiv 0. \quad (1.108)$$

Then, if the trajectories of that system are precompact, 0 is asymptotically stable for that system.

In the case of the damped wave equation, we can see from (1.106) that

$$\frac{1}{2} \frac{d}{dt} \int_0^L |u_t|^2 + |u_x|^2 = 0$$

if and only u is stationary. Stationary solutions of (1.105) are given by the equation

$$\begin{cases} -u_{xx} = 0, \\ u(0) = u(L) = 0, \end{cases} \quad (1.109)$$

for which the only solution is obviously 0. This proves implication (1.108) for system (1.105).

Now let $(u_0, u_1) \in (H^2 \cap H_0^1) \times H_0^1$, and let us note $u(t), t \geq 0$ the solution of (1.105) with initial conditions (u_0, u_1) . Then, $(u_1, (u_0)_{xx} - \lambda u_1) \in H_0^1 \times L^2$ and the solution of (1.105) with initial condition $(u_1, (u_0)_{xx} - \lambda u_1)$ is exactly $u_t(t), t \geq 0$.

The energy inequality (1.106) implies, by density, that (u_t, u_{tt}) is bounded in $H_1^0 \times L^2$. By Rellich's theorem, this means u_t is precompact in L^2 . Moreover, by (1.105) and boundedness of u_{tt} in L^2 , $u_{xx} - \lambda u_t$ is bounded in L^2 so in particular u_{xx} is bounded in L^2 . Again by Rellich's theorem, this

means u is precompact in H_0^1 . So we can apply the LaSalle invariance principle to prove that (1.105) is asymptotically stable (it is even exponentially stable) for the $H_0^1 \times L^2$ norm for initial conditions in $(H^2 \cap H_0^1) \times H_0^1$. Then, by density of $(H^2 \cap H_0^1) \times H_0^1$ in $H_0^1 \times L^2$, and by the energy inequality

$$\int_0^L |u_t(t_2)|^2 + |u_x(t_2)|^2 \leq \int_0^L |u_t(t_1)|^2 + |u_x(t_1)|^2, \quad \forall t_1 \leq t_2, \quad (1.110)$$

derived from (1.106), we get asymptotic stability for initial conditions in $H_0^1 \times L^2$.

For coupled wave equations, it is then natural to follow the same path: consider the simple system of coupled linear wave equations with a symmetric coupling

$$\begin{cases} u_{tt} - u_{xx} = Pv, \\ v_{tt} - v_{xx} = P^*u, \\ u(t, 0) = u(t, L) = 0, \quad \forall t \geq 0, \\ v(t, 0) = v(t, L) = 0, \quad \forall t \geq 0, \end{cases} \quad (1.111)$$

where $P \in \mathcal{L}(L^2)$, with initial conditions in $(H_0^1 \times L^2)^2$. Then, one can derive the following energy conservation law:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^L |u_t|^2 + |u_x|^2 + |v_t|^2 + |v_x|^2 &= \Re \left(\int_0^L (u_{xx} + Pv) \bar{u}_t + u_{tx} \bar{u}_x + (v_{xx} + P^*u) \bar{v}_t + v_{xt} \bar{v}_x \right) \\ &= \Re \left(\int_0^L Pv u_t + P^* u v_t \right) \\ &= \frac{d}{dt} \Re \langle u, Pv \rangle_{L^2} \end{aligned} \quad (1.112)$$

So the energy

$$E(u, v, u_t, v_t) = \frac{1}{2} \left(\int_0^L |u_t|^2 + |u_x|^2 + |v_t|^2 + |v_x|^2 \right) - \Re \langle u, Pv \rangle_{L^2} \quad (1.113)$$

is conserved for system (1.111). Note that this is some sort of total energy of the system, with a kinetic part (the squared terms) and an interaction part (the scalar product).

Now let us recall that we want to stabilize coupled wave equations with a feedback in the first equation only. Considering system (1.111), and the damping in (1.105), it seems natural to investigate what happens if we add a damping term in the first equation of (1.111). Intuitively, as we have just seen that (1.111) is conservative, adding a damping term should make the energy decay, and the system would become asymptotically stable. As it turns out, it takes some work to prove that intuition. To our knowledge, David Russell was the first to study the decay of solutions in [136] (and later [138]), in the more general framework of weak damping in general evolution equations in Hilbert spaces. In particular, [136, Proposition 1.1 and the subsequent remark] ([80] also gives a related result) establishes that compact perturbations of strongly continuous groups (not semigroups!) of operators cannot yield exponential stability. This negative result was later quantified by Fatiha Alabau, Piermarco Cannarsa and Vilmos Komornik in [1] (see also [2] for a synthesis), for partially damped coupled wave equations with a scalar weak coupling on a bounded domain $\Omega \subset \mathbb{R}^n, n \geq 1$:

$$\begin{cases} u_{tt} - A_1 u = \alpha v - Bu_t, \\ v_{tt} - A_2 v = \alpha u, \\ u(t, 0) = u(t, L) = 0, \quad \forall t \geq 0, \\ v(t, 0) = v(t, L) = 0, \quad \forall t \geq 0, \end{cases} \quad (1.114)$$

where A_1, A_2 are linear positive self-adjoint operators on $D(A_1^{\frac{1}{2}}) \times L^2 \times D(A_2^{\frac{1}{2}}) \times L^2$, and B is a bounded linear positive self-adjoint operator on L^2 . Indeed, the coupled terms, as they are of lower order, can be written as compact operators:

$$(u_1, u_2) \in H_0^1 \times L^2 \mapsto (0, \alpha u_1), \quad (1.115)$$

so that David Russell's result applies. In this context the authors show that it is still possible to obtain a decay rate: solutions with smooth initial conditions have a polynomial decay rate, under certain conditions. More precisely, if there exists an integer $j \geq 2$ such that

$$D(A_2^{\frac{j}{2}}) \subset D(A_1), \quad (1.116)$$

and if the energy

$$V(u, u_t, v, v_t) := \frac{1}{2} \left(\|u_t\|^2 + \|A_1^{\frac{1}{2}} u\|^2 + \|v_t\|^2 + \|A_2^{\frac{1}{2}} v\|^2 \right) + \alpha \langle u, v \rangle \quad (1.117)$$

satisfies the following estimate involving higher order terms, for some $n \geq 1$, $C > 0$, smooth enough initial conditions, and for all $T > 0$:

$$\int_0^T V(u(t), u_t(t), v(t), v_t(t)) dt \leq C \sum_{k=0}^j V \left(\left(\frac{d}{dt} \right)^{nk} u(0), \left(\frac{d}{dt} \right)^{nk+1} u(0), \left(\frac{d}{dt} \right)^{nk} v(0), \left(\frac{d}{dt} \right)^{nk+1} v(0) \right), \quad (1.118)$$

then it satisfies the following decay inequality for some other constant $C > 0$, and all $t > 0$:

$$V(u(t), u_t(t), v(t), v_t(t)) \leq \frac{C}{t^n} \sum_{k=0}^j V \left(\left(\frac{d}{dt} \right)^{nk} u(0), \left(\frac{d}{dt} \right)^{nk+1} u(0), \left(\frac{d}{dt} \right)^{nk} v(0), \left(\frac{d}{dt} \right)^{nk+1} v(0) \right). \quad (1.119)$$

Let us point out two important facts:

- The smoother the initial conditions, the stronger the polynomial decay can be.
- By density, (1.114) is asymptotically stable for initial solutions in $D(A_1^{\frac{1}{2}}) \times L^2 \times D(A_2^{\frac{1}{2}}) \times L^2$.

This result is extended to more general couplings in [7]: polynomial decay for smooth initial conditions holds for partially coercive operators, i.e. operators P such that there exists $\Pi_P \in \mathcal{L}(L^2)$ such that

$$\|\Pi_P\| = 1, \quad \langle Pv, v \rangle \geq c \|\Pi_P v\|^2, \quad \forall v \in L^2, \quad (1.120)$$

for some constant $c > 0$. In a way, this assumption corresponds to some downgraded information transmission for the coupling operator, but still sufficient for the damping in the first equation to stabilize both equations. Note that in the particular case of wave equations where the coupling and control operators are given by L^∞ functions, the authors give a geometric condition for the supports of these functions, called the piecewise multipliers geometric condition (PGMC). This condition relates to the aforementioned GCC, and implies polynomial stability for smooth initial conditions. In particular, the supports of the coupling and the control must intersect in dimension greater than 2, but in dimension 1 they can be of empty intersection.

As we have noted above, the obstruction to exponential stability comes from the fact that the coupling operator is compact because it involves zero-order terms. In contrast, coupled velocities yield quite different results:

$$\begin{cases} u_{tt} - A_1 u = \alpha v_t - \rho(x, u_t), \\ v_{tt} - A_2 v = \alpha u_t, \\ u(t, 0) = u(t, L) = 0, \quad \forall t \geq 0, \\ v(t, 0) = v(t, L) = 0, \quad \forall t \geq 0. \end{cases} \quad (1.121)$$

Indeed, in [9] the authors prove that the whole system is as stable as the first, damped, equation. In particular, if the first equation is exponentially stable, then system (1.121) is exponentially stable.

Let us finally mention the work of Alain Haraux and Mohamed Ali Jendoubi in [89], who give a unified approach with carefully chosen Lyapunov functions, of a more involved form than the usual total energy (1.117), and recover the aforementioned results of polynomial stability (Section 7), as well as a general result for so-called strong couplings, under some assumptions on the coupling operator (Section 5).

Now, let us point out some specificities in the case of nonlinear systems. Consider once again the system of coupled wave equations with a cubic coupling (1.53), and consider the equilibrium:

$$(\gamma u_e, \gamma^3 v_e, \gamma h_e)$$

where $\gamma > 0$ and (u_e, v_e, h_e) are defined by (1.101). The system (1.53) is locally controllable around that equilibrium. We can linearize (1.53) around that equilibrium:

$$\begin{cases} u_{tt} - u_{xx} = h, \\ v_{tt} - v_{xx} = 3\gamma^2 u_e^2 u, \\ u(t, 0) = u(t, L) = 0, \quad \forall t \geq 0, \\ v(t, 0) = v(t, L) = 0, \quad \forall t \geq 0. \end{cases} \quad (1.122)$$

Given the computations we made for system (1.111) and for (1.105), let us consider the following feedback law:

$$h_\gamma(u, u_t, v, v_t) = -u_t + 3\gamma^2 u_e^2 v, \quad (1.123)$$

which yields the following closed-loop system:

$$\begin{cases} u_{tt} - u_{xx} = 3\gamma^2 u_e^2 v - u_t, \\ v_{tt} - v_{xx} = 3\gamma^2 u_e^2 u, \\ u(t, 0) = u(t, L) = 0, \quad \forall t \geq 0, \\ v(t, 0) = v(t, L) = 0, \quad \forall t \geq 0. \end{cases} \quad (1.124)$$

Then, the result of Fatiha Alabau and Matthieu Léautaud in [7] gives us the asymptotic stability of the linearized system (1.124) as long as $\gamma > 0$. However this not enough to get stability results for the nonlinear system.

Instead, we can add some terms to the Lyapunov function and the feedback. The nonlinear system can be rewritten with $(\tilde{u}, \tilde{v}, \tilde{h}) := (u - \gamma u_e, v - \gamma^3 v_e, h - \gamma h_e)$:

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = \tilde{h}, \quad x \in [0, L], \\ \tilde{v}_{tt} - \tilde{v}_{xx} = 3\gamma^2 u_e^2 \tilde{u} + 3\gamma u_e \tilde{u}^2 + \tilde{u}^3, \quad x \in [0, L], \\ \tilde{u}(t, 0) = \tilde{u}(t, L) = 0, \quad \forall t \geq 0, \\ \tilde{v}(t, 0) = \tilde{v}(t, L) = 0, \quad \forall t \geq 0. \end{cases} \quad (1.125)$$

The new “total energy” has additional terms due to the nonlinear coupling:

$$V^\gamma(u, u_t, v, v_t) = \frac{1}{2} \int_0^L |u_t|^2 + |\tilde{u}_x|^2 + |v_t|^2 + |\tilde{v}_x|^2 - \Re \left(\int_0^L \tilde{v} (\tilde{u}^3 + 3\tilde{u}^2 \gamma u_e + 3\gamma^2 u_e^2 \tilde{u}) \right), \quad (1.126)$$

and with additional terms in the feedback

$$\tilde{h}(u, u_t, v, v_t) = -u_t + 3\tilde{v}(\tilde{u} + u_\gamma)^2 \quad (1.127)$$

V^γ is a Lyapunov function. However, it is not strict, which brings the following alternative: prove that the trajectories are precompact ([11] gives a powerful method to do that for single nonlinear equations); or try to obtain decay estimates following the method in [2], or the approach of [89].

An important feature of (1.53) is that the linearized system around 0 has no coupling term. Thus it seems difficult to tackle the stabilization directly. But the above study, if it yields some form of stability around the non-zero equilibria $(\gamma u_e, \gamma^3 v_e, \gamma h_e)$, could be used to bring the system close to 0, using the so-called *phantom tracking method*. One way of doing this would be to consider γ as a function of time, with the same expression of \tilde{h} , and add a penalizing term in the energy:

$$H(\gamma, u, u_t, v, v_t) := V^\gamma(u, u_t, v, v_t) + (\gamma - V^\gamma)^2. \quad (1.128)$$

Differentiating in time, we get:

$$\frac{d}{dt}H(\gamma, u, u_t, v, v_t) = -(1 - 2(\gamma - V^\gamma)) \int_0^L |u_t|^2 + \dot{\gamma} \rho(\gamma, u, u_t, v, v_t) \quad (1.129)$$

where

$$\begin{aligned} \rho(\gamma, u, u_t, v, v_t) = & 2(\gamma - V^\gamma) \\ & + (1 - 2(\gamma - V^\gamma)) \left(\int_0^L h_e u + \gamma (u'_e)^2 + 3\gamma^2 (v''_e v + v_e u^3 + u_e v^3) \right. \\ & \left. - 9\gamma^3 u^2 u_e v_e + 3\gamma^4 u u_e^2 v_e + 3\gamma^5 ((v'_e)^2 - 2u_e^3 v_e) \right). \end{aligned} \quad (1.130)$$

The closed-loop system with the additional variable γ now writes

$$\begin{cases} u_{tt} - u_{xx} = -u_t + \gamma h_e + 3u^2(v - \gamma^3 v_e), & x \in [0, L], \\ v_{tt} - v_{xx} = u^3, & x \in [0, L], \\ \dot{\gamma} = -\rho(\gamma, u, u_t, v, v_t), \\ \tilde{u}(t, 0) = \tilde{u}(t, L) = 0, & \forall t \geq 0, \\ \tilde{v}(t, 0) = \tilde{v}(t, L) = 0, & \forall t \geq 0. \end{cases} \quad (1.131)$$

If $\gamma(0)$ is small enough then we have the following energy inequality:

$$\frac{d}{dt}H(\gamma, u, u_t, v, v_t) = -(1 - 2(\gamma - V^\gamma)) \int_0^L |u_t|^2 - \rho(\gamma, u, u_t, v, v_t)^2 \leq 0. \quad (1.132)$$

Now the precompactness problem reappears for these augmented trajectories. Note however that the energy inequality implies that H is bounded on trajectories of (1.131), so in particular γ is also bounded, hence it is precompact as it is a scalar variable. So the main problem remains the u, u_t, v, v_t components.

Let us point out, as a final note, that this approach is not likely to yield exponential stability, as it rests on the weak LaSalle principle. Moreover, it all stems from a particular way of choosing the feedback, which, although natural, is not the only possibility for feedback design.

Accordingly, the next section will focus on another method of feedback design that achieves exponential stabilization of controllable system.

1.4 Stabilization of hyperbolic systems with a distributed scalar input

1.4.1 Main results

In this part, we turn to first-order linear 1-D hyperbolic systems with a distributed scalar input and proportional boundary conditions, of the general form:

$$\begin{cases} Y_t + A(x)Y_x + B(x)Y = u(t)\Phi(x), & \forall x \in [0, L] \\ Y_+(t) = GY_-(t), & \forall t \geq 0, \end{cases} \quad (1.133)$$

where $A, B \in C^1([0, L], M_n(\mathbb{R}))$, $G \in M_n(\mathbb{R})$. We suppose that A is already in diagonal form, with non-zero eigenvalues λ_i . Then Y_+ (resp. Y_-) is the vector of ingoing boundary conditions $Y_i(t, \sigma)$ with $\sigma = 0$ if $\lambda_i > 0$ and $\sigma = L$ otherwise (resp. the vector of outgoing boundary conditions $Y_i(t, \sigma)$ with $\sigma = 0$ if $\lambda_i < 0$ and $\sigma = L$ otherwise). Finally, $u(t)\Phi(x)$ is an internal control force, with a fixed spatial profile Φ , and our control is the amplitude of that force, u .

Let us point out some results related to the stabilization of these systems. As we have mentioned in Section 1.2, the study of pole placement is closely linked with the stabilizability of a system.

It is natural to first consider linear, bounded feedbacks, for which David Russell proves in [137]:

Theorem 1.4.1 (D.L.Russell, 1978). *Consider the system (1.133) with $n = 2$ and*

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and note $(e_k, \lambda_k)_{k \in \mathbb{Z}}$ the eigenfunctions and eigenvalues of the operator $-A\partial_x - B(x)$ with boundary conditions

$$\alpha_0 Y_1(t, 0) + \beta_0 Y_2(t, 0) = 0, \quad \alpha_1 Y_1(t, L) + \beta_1 Y_2(t, L) = 0, \quad |\alpha_i|^2 + |\beta_i|^2 \neq 0, \quad i = 0, 1. \quad (1.134)$$

If the resulting system is controllable, then for any complex sequence $(\rho_k)_{k \in \mathbb{Z}}$ satisfying

$$\sum_{k \in \mathbb{Z}} \left| \frac{\rho_k - \lambda_k}{\phi_k} \right|^2 < \infty, \quad (1.135)$$

where the ϕ_k are the coefficients of the expansion of Φ in $(e_k)_{k \in \mathbb{Z}}$, there exists a linear bounded feedback law $u(t) = \langle U, F \rangle$ such that the closed-loop operator $-A\partial_x - B + \langle \cdot, F \rangle \Phi$ has eigenvalues $(\rho_k)_{k \in \mathbb{Z}}$.

The quite limiting sufficient condition (1.135) was then proved to be a necessary and sufficient condition in a more general setting which comprises our hyperbolic systems (see [144]):

Theorem 1.4.2 (S.H.Sun, 1981). *Consider the general control system*

$$\frac{dY}{dt} = AY + Bu, \quad Y \in H, u \in \mathbb{R}, \quad (1.136)$$

where H is a Hilbert space and A is a linear operator, potentially unbounded. Suppose that

1. A has discrete spectrum $(\lambda_k)_{k \geq 1}$, $\lambda_k \neq \lambda_j$ for $j \neq k$, and the associated normalized eigenfunctions e_k form a Riesz basis.
2. $\inf_{k \neq j} |\lambda_k - \lambda_j| > 0$.
3. $\sup_{k \geq 1} \sum_{j \geq 1, j \neq k} \frac{1}{|\lambda_k - \lambda_j|^2} < \infty$.

Let $(\rho_k)_{k \geq 1}$ be a sequence of complex numbers. Then, there exists a bounded feedback law $u(t) = \langle Y, k \rangle$ such that the operator $A + \langle \cdot, k \rangle B$ has eigenvalues $(\rho_k)_{k \geq 1}$, if and only if

1. $B^* e_k \neq 0, \quad \forall k \geq 1$.
2. $\sum_{k \in \mathbb{Z}} \left| \frac{\rho_k - \lambda_k}{B^* e_k} \right|^2 < \infty$.

Finally, let us mention an extension of this study to so-called admissible unbounded feedback laws, in [129]. Building on Sun's result, and without using canonical forms, Richard Rebarber gives a sufficient condition for pole placement by a certain class of unbounded feedback laws, with:

- generalized spectral spacing conditions (conditions 1 and 2 in the above theorem) using cardinal functions,

- a relaxation of condition (1.135), requiring boundedness instead of summable squares,
- an additional condition depending on the spacing of the initial spectrum.

Note that these results do not yield exponential stabilization for systems of the form (1.133) unless we consider unbounded feedback laws. In this latter case, [129] gives a formula for a feedback law that achieves the desired pole placement. However, this formula requires to know a cardinal function for which the poles coincide with the initial spectrum, which might be difficult in practice.

Systems of the form (1.133) appear naturally in physical problems. For example, as is mentioned in [135], a linear wave equation which can be rewritten as a 2×2 first order hyperbolic system, the problem of a vibrating damped string, or the plucking of a string, can be modelled by systems of the form (1.133). In a different field altogether, chemical tubular reactors, in particular plug flow reactors (see [125, 127]), are modeled by hyperbolic systems of the form (1.133). The control is the temperature of the reactor, and instead of proportional boundary conditions there is a given input at the boundary. Numerous other examples of hyperbolic control systems with boundary control (channels, electric circuits, other types of chemical reactors...) can be found in [24, Chapter 1].

In Chapter 5, we will focus on yet another example, the so-called *water-tank system*. Introduced by François Dubois, Nicolas Petit and Pierre Rouchon in [75], it models a 1-D tank containing an inviscid, incompressible, irrotational fluid¹, in the approximation that its acceleration is small compared with the gravitational constant, and that the height of the liquid is small compared with the length of the tank². In this setting, the motion of the fluid can be modelled by the Saint-Venant equations ([19, 20, 21]) on the interval $[0, L]$:

$$\begin{cases} \partial_t H + \partial_x(HV) = 0, \\ \partial_t V + V\partial_x V + g\partial_x H = -U(t), \\ \frac{ds}{dt}(t) = U(t), \\ \frac{dD}{dt}(t) = s(t), \\ V(t, 0) = V(t, L) = 0, \quad \forall t \geq 0, \end{cases} \quad (1.137)$$

where g is the gravitational constant, H is the height of the water, and V its averaged velocity, D is the horizontal displacement of the tank, s its horizontal speed. The penultimate equation corresponds to the fact that the sides of the tank are not permeable, and this implies that one never spills water. Mathematically, one can obtain mass conservation by integrating the first equation:

$$\begin{aligned} 0 &= \int_0^L (\partial_t H + \partial_x(HV)) \\ &= \frac{d}{dt} \int_0^L H(t, x) dx + H(t, L)V(t, L) - H(t, 0)V(t, 0) \\ &= \frac{d}{dt} \int_0^L H(t, x) dx, \quad \forall t \geq 0. \end{aligned} \quad (1.138)$$

It is clear that this system has natural equilibrium points, corresponding to a motionless tank and fluid:

$$(H_e, 0, 0, D), \quad H_e > 0, \quad D \in \mathbb{R}.$$

In [52], Jean-Michel Coron proves the following theorem, using the return method and quasi-static deformations (very slow movements of the whole system):

¹Water, or beer, for example

²So I am not really stabilizing coffee mugs, as I like to tell my friends. I can only hope none of them reads my thesis up to this footnote.

Theorem 1.4.3 (Coron, 2002). *There exists $T > 0$ such that the water tank system is locally controllable around the equilibrium point $(H_e, 0, 0, 0)$ for initial and final states*

$$(H^0, V^0, s^0, D^0), (H^1, V^1, s^1, D^1) \in C^1([0, L]) \times C^1([0, L]) \times \mathbb{R} \times \mathbb{R}$$

satisfying

$$\begin{aligned} \int_0^L H_0(x) dx &= \int_0^L H_1(x) dx = LH_e, \\ H_x^0(0) &= H_x^0(L), \quad H_x^1(0) = H_x^1(L), \end{aligned} \quad (1.139)$$

and with $|s^1 - s^0| + |D^1 - s^0 T - D^0|$ small enough.

From this theorem, it follows that any equilibrium point $(H_e, 0, 0, D^0)$ can be steered to any other equilibrium point $(H_e, 0, 0, D^1)$.

This raises the following open problem: is the water-tank stabilizable?

In Chapter 5, we give a partial answer. First, we study the linearized water-tank system around another class of equilibria corresponding to constant acceleration: $(H^\gamma := 1 - \gamma x, \gamma)$, which are no longer uniform. Moreover, as the equilibrium corresponds to a moving tank, we do not include speed and position in the state for the linearized control system:

$$\begin{cases} \partial_t \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} 0 & H^\gamma \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} 0 & -\gamma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = -u(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ v(t, 0) = v(t, L) = 0, \quad \forall t \geq 0, \end{cases} \quad (1.140)$$

which is of the form (1.133). Again, we can integrate the first equation to derive conservation of mass:

$$\frac{d}{dt} \int_0^L h(t, x) dx = 0, \quad \forall t \geq 0, \quad (1.141)$$

and as initial perturbations of the system do not change the mass, this yields the following mass condition:

$$\int_0^L h(t, x) dx = 0, \quad \forall t \geq 0. \quad (1.142)$$

and we prove that it is controllable for states in Sobolev spaces that satisfy the boundary conditions of (1.140) and (1.142), for $\gamma > 0$ small enough, but not if $\gamma = 0$. As it so happens, mass condition (1.142) translates into the fact that the controller $(0 \ 1)^T$ fails to act on the kernel $\text{Span}\{(1 \ 0)^T\}$ of the operator

$$\begin{pmatrix} 0 & H^\gamma \\ 1 & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & -\gamma \\ 0 & 0 \end{pmatrix}.$$

To circumvent this obstacle, we consider a controller with an added component:

$$\begin{pmatrix} \nu \\ 1 \end{pmatrix}$$

with $\nu \neq 0$ carefully chosen so that we can apply the backstepping method. With this new controller, we have a fictitious control system:

$$\begin{cases} \partial_t \begin{pmatrix} \mathbf{h} \\ \mathbf{v} \end{pmatrix} + \begin{pmatrix} 0 & H^\gamma \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} \mathbf{h} \\ \mathbf{v} \end{pmatrix} + \begin{pmatrix} 0 & -\gamma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{h} \\ \mathbf{v} \end{pmatrix} = -u(t) \begin{pmatrix} \nu \\ 1 \end{pmatrix}, \\ \mathbf{v}(t, 0) = \mathbf{v}(t, L) = 0, \quad \forall t \geq 0, \end{cases} \quad (1.143)$$

which we can now separate in two components, one along the vector $(1 \ 0)^T$, and the one with conserved mass:

$$\begin{pmatrix} \mathbf{h} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} h_0 \\ 0 \end{pmatrix} + \begin{pmatrix} h \\ v \end{pmatrix}. \quad (1.144)$$

Then, (1.143) rewrites:

$$\begin{cases} \dot{h}_0 = -u(t)\nu, \\ \partial_t \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} 0 & H^\gamma \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} 0 & -\gamma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = -u(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ v(t, 0) = v(t, L) = 0, \quad \forall t \geq 0, \end{cases} \quad (1.145)$$

Then, under the same assumptions, we prove the following theorem:

Theorem 1.4.4 (J.-M. Coron, A. Hayat, S. Xiang, C. Zhang, 2019). *For any $\mu > 0$, there exists $\gamma_0 > 0$ such that, for any $\gamma \in (0, \gamma_0)$, there exists $\nu \neq 0$ such that there exists an unbounded feedback law $u(t) := \langle [h_0, (h, v)], F \rangle$ such that the associated closed-loop system (1.145) is exponentially stable in H^1 norm, with decay rate μ .*

In particular, this gives us a way to stabilize system (1.140) by adding a form of integrator.

Remark 1.4.1. *As we will see, one of the requirements on μ and γ is that the target system be controllable. This will give a quantitative relationship between these two parameters:*

$$\gamma \leq ce^{-2\mu L} \quad (1.146)$$

for some $c > 0$. So, in particular, the smaller the acceleration, the greater decay rate one can achieve with a well-chosen feedback.

The proof of this result draws from the method used in [59] to stabilize a linearized Schrödinger equation, which is a mix of the backstepping method for PDEs and the pole-shifting properties of finite-dimensional systems. As in classical PDE backstepping, which we will present below, this allows us to build explicit feedbacks.

However, although the basic outline of the proof is inspired from [59], the technical developments prove quite different due to the hyperbolic nature of the system. In fact, Chapter 3 is devoted to the elaboration of this method, first developed on a simpler hyperbolic system, namely a periodic linear transport equation:

$$\begin{cases} \alpha_t + \alpha_x + \mu\alpha = u(t)\varphi(x), & x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), & \forall t \geq 0, \end{cases} \quad (1.147)$$

The simplicity of this example makes for lighter and more practical computations, while at the same time providing a nice illustration of deeper technical issues. Moreover, the choice of this toy model is motivated by the fact that the uncontrolled linearized system (1.140) when $\gamma = 0$ is equivalent, after an appropriate variable change and a transformation to obtain one equation instead of two, to a periodic transport equation.

The results for this simplified system are stronger than for the water tank, and are obtained in the framework of periodic Sobolev spaces :

$$H_{per}^m = \left\{ f \in H^m, \quad f^{(i)}(0) = f^{(i)}(L), \forall i \in \{0, \dots, m-1\} \right\}. \quad (1.148)$$

Theorem 1.4.5 (Christophe Zhang, 2018). *Let $m \geq 1$. If (1.147) is exactly controllable in H_{per}^m , then for any $\lambda > 0$, there exists an unbounded feedback law $u(t) := \langle \alpha(t), F^\lambda \rangle$ such that the closed-loop operator associated to (1.147) generates an exponentially stable C^0 semigroup on H_{per}^m , with decay rate λ .*

The feedbacks F^λ are explicit, and even allow for finite-time stabilization, which we will present in Chapter 4:

Theorem 1.4.6 (Christophe Zhang, 2018). *Let $m \geq 1$. If (1.147) is exactly controllable in H_{per}^m , then there exists an unbounded feedback law $u(t) := \langle \alpha(t), F \rangle$ such that the closed-loop operator associated to (1.147) generates a C^0 semigroup that goes to 0 in time L .*

1.4.2 A classic example

Simply put, the idea of backstepping for PDEs is to try to transform the considered system into a simpler, stable system, the target system. The possibility of doing this depends on the feedback we choose for our system, so in a way, the feedback law becomes a parameter of such a transformation. The stabilization problem thus becomes that of finding if there exists a value for this parameter (a feedback) such that the transformation is invertible and maps the trajectories of the resulting closed-loop system into those of the target system. To our knowledge, this method was first presented to stabilize an unstable heat equation with a boundary control in [33], and results on boundary stabilization of classes of parabolic systems were then developed in [32, 17]. Let us give a rapid account of the feedback design strategy in these works. Consider the following 1-D heat equation, with internal antidamping $\lambda > 0$:

$$\begin{cases} u_t - u_{xx} = \lambda u, \\ u(0) = 0, \quad u(1) = U(t). \end{cases} \quad (1.149)$$

The goal is to find a transformation T_λ :

$$w(t, x) := u(t, x) - \int_0^x k(x, y)u(t, y)dy \quad (1.150)$$

that maps the solutions of (1.149) to those of the exponentially stable target system:

$$\begin{cases} w_t - w_{xx} = 0, \\ w(0) = 0, \quad w(1) = 0, \end{cases} \quad (1.151)$$

as it is well-known that the solutions of (1.151) decay exponentially with decay rate 1/4:

$$\|w(t)\|_{L^2} \leq e^{-\frac{t}{4}} \|w(0)\|_{L^2}. \quad (1.152)$$

Then, from (1.150) and the right boundary condition of (1.151), we will get the natural control design:

$$U(t) = \int_0^1 k(1, y)u(t, y)dy. \quad (1.153)$$

Replacing w by (1.150) in (1.151), and performing several integrations by parts using the boundary conditions of (1.149), we find that k should satisfy the following the following kernel equations on $\mathcal{T} := \{0 \leq y \leq x \leq 1\}$:

$$\begin{cases} k_{xx} - k_{yy} = \lambda k, \\ k(x, 0) = 0, \\ k(x, x) = -\lambda \frac{x}{2} \end{cases} \quad (1.154)$$

which is a form of wave equation on a triangular domain. To solve this PDE, let us first perform a variable change:

$$\xi = x + y, \quad \eta = x - y, \quad G(\xi, \eta) = k(x, y), \quad (1.155)$$

which yields a new equation on a new domain \mathcal{T}' :

$$\begin{cases} 4G_{\xi\eta}(\xi, \eta) = \lambda G(\xi, \eta), \\ G(\xi, \xi) = 0, \\ G(\xi, 0) = -\lambda \frac{\xi}{4}. \end{cases} \quad (1.156)$$

Now, by successive integrations, (1.156) can be rewritten as an integral equation:

$$\begin{cases} G(\xi, \eta) = -\frac{\lambda}{4}(\xi - \eta) + \frac{\lambda}{4} \int_\eta^\xi \int_0^\eta G(s, \sigma) d\sigma ds, \\ G(\xi, \xi) = 0, \\ G(\xi, 0) = -\lambda \frac{\xi}{4}. \end{cases} \quad (1.157)$$

This equation can be solved using a classic fixed-point iterative method, where successive approximations are made with the following iterative scheme:

$$G_{n+1}(\xi, \eta) = -\frac{\lambda}{4}(\xi - \eta) + \frac{\lambda}{4} \int_{\eta}^{\xi} \int_0^{\eta} G_n(s, \sigma) d\sigma ds, \quad (1.158)$$

which leads to the following explicit expression for k , in the original variables x, y :

$$k(x, y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2 - y^2)}\right)}{\sqrt{\lambda(x^2 - y^2)}}, \quad (1.159)$$

where I_1 is the first Bessel function. This makes (1.153) an explicit full-state feedback.

Now that we know that $k \in L^2$ we can compute the inverse T_{λ}^{-1} :

$$u(t, x) = w(t, x) + \int_0^x l(x, y) w(t, y) dy, \quad (1.160)$$

where, with almost the same computations as before, l is given by:

$$l(x, y, \lambda) = k(x, y, -\lambda). \quad (1.161)$$

Thus, T_{λ} is indeed invertible, and maps the solutions of (1.149) to those of (1.151), which proves the stability of (1.149) with feedback (1.153). More precisely, we have estimates on the decay of our system, using (1.152):

$$\begin{aligned} \|y(t)\|_{L^2} &\leq \| (T^{\lambda})^{-1} \| \| T^{\lambda} \| e^{-\frac{t}{4}} \|y_0\|_{L^2} \\ &\leq C_1 \lambda^2 e^{C_2 \sqrt{\lambda}} e^{-\frac{t}{4}} \|y_0\|_{L^2}, \end{aligned} \quad (1.162)$$

for some constants $C_1, C_2 > 0$. Thus, by looking for an invertible transformation under a certain form, degrees of freedom are resolved and the problem of stabilization becomes one of solving the PDE satisfied by the kernel of the Volterra transformation.

Since this seminal work, this backstepping strategy has been applied to many different PDE systems, obtaining exponential or rapid stabilization for the wave equation ([109] and [141]), for the Korteweg-de Vries equation ([39], more recently [155, 156]). A general presentation for an application to first-order hyperbolic systems can be found in [24, chapter 7].

1.4.3 ODE backstepping

In its origins, however, backstepping is something quite different. Initially, the word ‘‘backstepping’’ comes from a method in finite dimension (see for example [53, 110, 143] for a comprehensive overview of this technique). Given a system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad f \in C^1(\mathbb{R}^{n+m}, \mathbb{R}^n) \quad (1.163)$$

for which a Lyapunov function V and a stabilizing C^1 feedback law $u = \alpha(x)$ are already known, one can find a stabilizing feedback law for (1.163) with an added integrator:

$$\begin{aligned} \dot{x} &= f(x, \xi) \\ \dot{\xi} &= u, \end{aligned} \quad (1.164)$$

where the state is (x, ξ) and the control is u . The idea is that we are adding extra steps in the stabilization problem, and the question is whether knowledge on the initial stabilization problem can be used even though there are extra steps. And indeed, using a Lyapunov function approach, one can design a feedback law that builds on α and accounts for these extra steps. Indeed, consider

$$V_b(x, \xi) := V(x) + \frac{1}{2}(\xi - \alpha(x))^2. \quad (1.165)$$

Then, along a solution of (1.164),

$$\frac{d}{dt}V_b(x(t), \xi(t)) = \nabla V(x) \cdot f(x, \xi) + (\xi - \alpha(x))(u - \alpha'(x)f(x, \alpha(x))). \quad (1.166)$$

As $f \in C^1$, there exists a continuous function G such that

$$f(x, \xi_1) - f(x, \xi_2) = G(x, \xi_1, \xi_2)(\xi_1 - \xi_2), \forall (x, \xi_1, \xi_2) \in \mathbb{R}^{n+m+m}. \quad (1.167)$$

Thus,

$$\begin{aligned} \frac{d}{dt}V_b(x(t), \xi(t)) &= \nabla V(x) \cdot f(x, \alpha(x)) + \\ &(\xi - \alpha(x))(u - \alpha'(x)f(x, \alpha(x)) + \nabla V(x) \cdot G(x, \xi, \alpha(x))). \end{aligned} \quad (1.168)$$

so that by setting, for example,

$$u(x, \xi) := \alpha'(x)f(x, \alpha(x)) - \nabla V(x) \cdot G(x, \xi, \alpha(x)) - (\xi - \alpha(x)), \quad (1.169)$$

we get

$$\frac{d}{dt}V_b(x(t), \xi(t)) \leq 0, \quad (1.170)$$

which proves the stability of system (1.164) with the feedback law (1.169). The essence of this strategy is that we use the control u to get $\xi(t)$ to behave as much like the feedback $\alpha(x(t))$ as possible, by adding a penalizing term $(\xi - \alpha(x))$ in V_b : we try to reproduce the action of α through the integrator. In a way, we are taking α one step back, through the integrator, that is, backstepping it. This can be extended to chains of integrators.

Another way of approaching the problem of an added integrator is to consider that we are performing a change of variables on our system, by setting

$$\begin{aligned} z_1 &:= x, \\ z_2 &:= \xi - \alpha(x). \end{aligned} \quad (1.171)$$

Note that by its “lower-triangular structure”, this change of variables is invertible provided α is a smooth enough function of x . Then, (1.164) becomes

$$\begin{aligned} \dot{z}_1 &= f(z_1, z_2 + \alpha(z_1)) \\ \dot{z}_2 &= u - \alpha'(z_1)f(z_1, z_2 + \alpha(z_1)), \end{aligned} \quad (1.172)$$

which we can rewrite

$$\begin{aligned} \dot{z}_1 &= f(z_1, \alpha(z_1)) + G(z_1, z_2 + \alpha(z_1), \alpha(z_1))z_2 \\ \dot{z}_2 &= u - \alpha'(z_1)f(z_1, z_2 + \alpha(z_1)). \end{aligned} \quad (1.173)$$

In this setting, we can use the feedback to “symmetrize” the coupling term $G(z_1, z_2 + \alpha(z_1), \alpha(z_1))z_2$ and compensate $\alpha'(z_1)f(z_1, z_2 + \alpha(z_1))$, so that, defining

$$\begin{aligned} u &:= \alpha'(z_1)f(z_1, z_2 + \alpha(z_1)) - \nabla V(z_1) \cdot G(z_1, z_2 + \alpha(z_1), \alpha(z_1)) \\ V^z(z_1, z_2) &:= V(z_1) + \frac{1}{2}z_2^2. \end{aligned} \quad (1.174)$$

we get

$$\frac{d}{dt}V^z(t) = \nabla V(z_1) \cdot f(z_1, \alpha(z_1)) \leq 0. \quad (1.175)$$

More generally, systems of nonlinear ODEs with an underlying integrator chain structure and with one scalar control at the bottom (or similar systems of PDEs, see [56] or [118] for instance) can be transformed using an invertible, nonlinear change of variables. This nonlinear change of variables is constructed recursively, variable by variable. The example (1.164) shows how the first step goes.

Then, at step N , we go down one equation, and we see the current equation (in the new variable set z_1, \dots, z_N)

$$\dot{z}_N = \tilde{f}_N(z_1, \dots, z_N, x_{N+1}) \quad (1.176)$$

as a control system for which a stabilizing feedback law $\alpha_N(z_1, \dots, z_N)$ is known, and where the variable x_{N+1} of the following equation plays the role of a control input. Then, we change that variable in order to consider the error term

$$z_{N+1} := x_{N+1} - \alpha_N(z_1, \dots, z_N),$$

as in (1.171), and we are now trying to use x_{N+2} to stabilize z_{N+1} to 0, so that x_{N+1} behaves like the feedback law we know for z_N . We are, once again, backstepping the feedback law α_N through to the next equation.

Example 1.4.1. *Let us note that this method can be used to stabilize uncontrollable systems. Indeed consider the following nonlinear system:*

$$\begin{aligned} \dot{x}_1 &= -x_1^2 x_2 \\ \dot{x}_2 &= u. \end{aligned} \quad (1.177)$$

It is clear that trajectories starting with $x_1 = 0$ stay at $x_1 = 0$ regardless of the control input. Thus (1.177) is not controllable. However, the backstepping method can still be applied. Indeed, $\alpha(x_1) := x_1$ clearly defines a stabilizing feedback for the first equation of (1.177): indeed it is well-known that 0 is asymptotically stable for

$$\dot{x}_1 = -x_1^2 \alpha(x_1) = x_1^3. \quad (1.178)$$

Then, consider the following functional:

$$V(x_1, x_2) := \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - x_1)^2. \quad (1.179)$$

V is clearly a positive definite quadratic form, moreover, differentiating in time along a trajectory of (1.177), we get

$$\begin{aligned} \frac{d}{dt}V(x_1(t), x_2(t)) &= x_1 \dot{x}_1 + (x_2 - x_1)(\dot{x}_2 - \dot{x}_1) \\ &= -x_1^3 x_2 + (x_2 - x_1)(u + x_1^2 x_2) \\ &= -x_1^4 + (x_2 - x_1)(u + x_1^2 x_2 - x_1^3), \end{aligned} \quad (1.180)$$

so that setting the feedback law

$$u := -x_1^2 x_2 + x_1^3 + x_1 - x_2 = (x_1^2 + 1)(x_1 - x_2) \quad (1.181)$$

ensures that

$$\frac{d}{dt}V(x_1(t), x_2(t)) \leq 0, \quad (1.182)$$

and thus stabilizes (1.177).

1.4.4 PDE backstepping

As mentioned earlier it is possible to directly apply the ODE backstepping method, with a finite number of steps, to some systems of PDEs. It seems also natural to try and apply the above method on a semi-discretization of the system under consideration, as the boundary control ensures that the resulting ODE system has some sort of integrator chain structure. This is done in [17] and [32], where a backstepping change of variable with an increasing number of steps is applied to an increasingly fine discretization of the heat equation of the form (1.149). However, with the traditional backstepping change of variables, where natural Lyapunov functions are considered, the norm of the feedback law obtained increases to infinity with the number of points in the discretization (or, equivalently,

the number of backstepping steps). The reason for this is that the traditional change of variables transforms the semi-discretized system into a tridiagonal system that is no longer parabolic, which in particular replaces all its poles. This total pole placement is however not necessary, as there are only a finite number of unstable eigenvalues for system (1.149). Thus, choosing a parabolic target system for a less demanding pole placement, the authors manage to design a feedback that remains finite when $N \rightarrow \infty$, leading to the example presented in Section 1.4.2. Remarkably, the change of variables converges to a Volterra transformation of the second kind, and this is where the idea of using Volterra transformations of the second kind originated.

This development shows the central difference between the finite-dimensional method, where one uses, equivalently, natural Lyapunov functions, or lower triangular changes of variables, and the infinite-dimensional method, where one has to pay attention to the pole placement involved, which may call for a different target system. However, the new method for PDEs has inherited the name “backstepping” because the spirit remains the same: a recursive change of variables is used to synthesize the action of the control by climbing down the integrator chain (or going from the passive boundary to the active boundary for PDEs); or, equivalently, to reap all the destabilizing terms and bring them to the last equation (or to the boundary, for PDEs), where the control input can deal with them. As a result, the transformations in both cases have a triangular structure. In the case of a Volterra transformation of the second kind, one can see that it is “spatially causal”: indeed in equation (1.150) one can see that the value of $w(x)$ depends only on the values of v on $[0, x]$.

1.4.5 The evolution of the method

1.4.5.1 From Volterra to Fredholm transformations

Volterra transformations of the second kind are one of the ingredients that made the new backstepping method so powerful on PDEs. Interestingly, the use of a Volterra transformation of the second kind also appears in the study of spectral assignability, which is the infinite-dimensional analog of pole-shifting.

All in all, Volterra transformations of the second kind seem to work well with control systems. But on the other hand, as mentioned previously these transformations have a specific triangular structure. For some stabilization problems, this might be too constraining, and one could consider looking for a larger class of invertible transformations, along with suitable target systems, to use the backstepping method. And indeed, more recently, general kernel operators, also known as Fredholm transformations, have been considered:

$$f(t, x) \mapsto \int_0^L k(x, y) f(t, y) dy. \quad (1.183)$$

Even though they require more work, as one has to prove their invertibility from scratch, they have proven to be more suitable regarding the position of the control (see [66, 65] for example), and have also allowed to find stabilizing boundary feedbacks for hyperbolic systems (see [62] for integro-differential systems, and in [63] for general balance laws). A striking feature is that the Fredholm transformations in [66, 65] are actually still compact perturbations of the identity, although it is not clear whether they are actually Volterra transformations of the second kind. Let us also note that in [63] the Fredholm operator is sought in the form

$$f(t, x) \mapsto f(t, x) - \int_0^L K(x, y) f(t, y) dy$$

where $K(x, y)$ is a lower triangular matrix with zero diagonal for all $(x, y) \in [0, L]^2$.

In [142] the authors achieve rapid boundary stabilization of the Euler-Bernoulli beam system. In [87] the authors stabilize a spatially non-causal reaction-diffusion equation exponentially. Remarkably, the system is spatially non-causal, and accordingly the authors apply the backstepping method with a mixed Fredholm-Volterra transformation instead of a Volterra transformation of the second kind.

1.4.5.2 From boundary control to distributed scalar inputs

Another significant evolution is the extension of the backstepping method to systems with a distributed scalar input, which no longer bear any resemblance to finite-dimensional cascade systems:

$$\frac{dY}{dt} + AY = u(t)\Phi(x), \quad (1.184)$$

where u is the control. Notwithstanding their differing nature, the backstepping method can still be applied on these systems. For example, by adding some additional assumptions on the controller $\Phi(x)$, the authors of [150] and [154] for parabolic systems, and [159] for first-order hyperbolic systems similar to the ones in Section 1.4.1, are able to apply a Volterra transformation of the second kind to their system, which moves some terms to the boundary but still leaves the input inside the domain. Then, they apply a second invertible differential transformation to their simpler target system to move the input to the boundary, which allows them to design an explicit feedback law.

As in the case of boundary control, on some systems the backstepping method seems to work better with Fredholm transformations. For example, in [59], the authors use a Fourier approach to find a suitable backstepping transformation, in the form of a Fredholm transformation, to achieve rapid stabilization for the bilinear Schrödinger equation.

In this case, and also in the results presented in Chapters 3 and 5, we will see that the Fredholm transformations that are found by the backstepping method are not compact perturbations of the identity anymore.

1.4.5.3 Target systems

As we have mentioned, the choice of a target system is an important ingredient in PDE backstepping, primarily because it ensures the convergence of the backstepping change of variables on the discretization of the system, and essentially because it encodes what destabilizing terms we want to remove, or what stabilizing terms we want to add. For example, in the works on the heat and wave equation ([109, 141, 17, 32]), antidamping terms are removed, and stability is enhanced by adding internal damping:

$$u_t - \Delta u = \lambda_1 u \quad (1.185)$$

becomes

$$u_t - \Delta u = -\lambda_2 u \quad (1.186)$$

where $\lambda_1, \lambda_2 > 0$.

In some cases, however, backstepping can achieve more. In [68] and in [14, 13] the authors derive a Volterra transformation of the second kind that moves the internal coupling terms to the boundary. Remarkably, in [68], this allows for a complete cancellation of the boundary input, and yields finite-time stabilization of the linearized system, with a minimal time due to the hyperbolic nature of the system.

Finally, in Chapter 5 we encounter another kind of target system. Indeed, we have found that adding boundary damping rather than internal damping seems to work better to find a backstepping transformation.

1.4.5.4 Nonlinear systems

Another advantage of obtaining explicit feedback laws is that they can be used to stabilize nonlinear systems. This approach can take different forms, depending on the system under consideration. For example, in [39] the authors study a nonlinear KdV equation with a boundary control. Using the backstepping method, they first compute explicit feedback laws for the linearized system for any given exponential decay rate. Then, they build a solution u to the nonlinear equation step by step, on the intervals $[nT, (n+1)T]$. On each interval, they study the image of u by the backstepping transformation, and prove that it decays exponentially. Finally, by patching all the intervals together, they prove that the image of u by the backstepping transformation decays exponentially on \mathbb{R}^+ . Thus, by invertibility of the backstepping transformation, u decays exponentially as well.

Another approach is given in [68]. The authors plug the linear feedback, which stabilizes the linearized system in finite time, into the nonlinear system, and prove the exponential stability of the nonlinear closed-loop system using Lyapunov functions.

Regarding the systems studied in Chapters 3 and 5, as the feedbacks are explicit as well, one can hope that the results for linear systems can be extended to nonlinear systems. However a new difficulty arises: the feedback laws are not bounded for the state space norm. As a consequence the system obtained by plugging the linear feedback law into the nonlinear system is not that straightforward to study.

1.4.5.5 Null-controllability and finite-time stabilization

As mentioned earlier, in [68] the structure of the system makes it possible to chose a target system that converges to 0 in finite time. The same is achieved in [159], where the second transformation actually maps the system to a hyperbolic system with zero input at the boundary.

Even when it seems difficult to aim for target systems with finite-time convergence, backstepping can help achieve finite-time stabilization. Indeed, a strategy has been developed in [67, 155, 156], using the explicit feedback laws obtained by the backstepping method. The general strategy is to divide the interval $[0, T]$ in smaller intervals $[t_n, t_{n+1}]$, the length of which tends to 0, and on which one gets exponential stabilization with decay rates λ_n , with $\lambda_n \rightarrow \infty$, by applying feedbacks k_{λ_n} . Then, for well-chosen t_n, λ_n , the trajectory thus obtained reaches 0 in time T , with a piecewise H^1 , explicit, closed-loop control. For example, in [155] the author derives an explicit feedback law to stabilize a linearized KdV equation exponentially, with a Dirichlet control on the left boundary. This yields the following decay estimates for a given exponential decay rate λ , for the state y and the feedback $u := k_\lambda(y)$:

$$\begin{aligned} \|y(t)\|_{L^2} &\leq e^{4(1+L)^2\sqrt{\lambda}-\lambda t} \|y(0)\|_{L^2}, \\ |u(t)| &\leq e^{6(1+L)^2\sqrt{\lambda}-\lambda t} \|y(0)\|_{L^2} \end{aligned} \quad (1.187)$$

so that the above defined piecewise feedback law yields the following estimates on $[t_n, t_{n+1}]$:

$$\begin{aligned} \|y(t)\|_{L^2} &\leq e^{-\sum_{k=0}^n \lambda_n(t_{n+1}-t_k) + 4(1+L)^2 \sum_{k=0}^n \sqrt{\lambda_n}} \|y(0)\|_{L^2}, \\ |u(t)| &\leq e^{-\sum_{k=0}^n \lambda_n(t_{n+1}-t_k) + 6(1+L)^2 \sum_{k=0}^n \sqrt{\lambda_n}} \|y(0)\|_{L^2}. \end{aligned} \quad (1.188)$$

Choosing for example

$$t_n := T - \frac{1}{n^2}, \quad \lambda := 2n^8, \quad (1.189)$$

then ensures that the state reaches 0 in finite time.

Although this provides an explicit control to steer the system to 0, the norm of the feedback laws k_{λ_n} tends to infinity, and the closed-loop flow Φ does not satisfy the uniform stability condition:

$$\forall \varepsilon > 0, \exists \eta > 0, \quad \|y_0\|_{L^2} \leq \eta \implies \|\Phi(t, t'; y_0)\|_{L^2} \leq \varepsilon, \quad \forall t' \geq t. \quad (1.190)$$

However, the previous construction of the control can be used, with some adequate modifications (see [67] and [156]) to design a time-varying, periodic feedback, with some regularity in the state variable, which stabilizes the system in finite time.

Theorem 1.4.6 translates an altogether different situation: indeed in this case, the norm of the feedback law F_λ achieving exponential decay rate λ is bounded when $\lambda \rightarrow \infty$, and

$$F_\lambda \xrightarrow{N \rightarrow \infty} F_\infty, \quad (1.191)$$

where F_∞ is a linear form, and the convergence is in the sense of the coefficients of the linear forms in some basis of the state space.

To test whether F_∞ is actually a feedback law achieving finite-time stabilization, we then use the semi-group formalism.

1.4.6 A finite-dimensional example

The results stated in Section 1.4.1 are obtained by a uniform pole-shifting. As we have seen in the example of the canonical form for hyperbolic systems, it seems that Volterra transformations of the second kind on our systems are unlikely to yield more than a compact perturbation of the spectrum, which is too weak to achieve the exponential stabilization we want.

On the other hand, we have seen that controllability implies a powerful pole-shifting property in finite dimension. Moreover, in [66, 59] a controllability assumption is made, and is crucial in proving the invertibility of the Fredholm transformation. This marks another evolution of the backstepping method, which, in the finite-dimensional case, did not rely on the controllability of the system (see the example with system (1.177)). In the wake of this evolution, the method we use to prove Theorems 1.4.5 and 1.4.4 draws on the strategy of proof in [59], and combines the use of controllability with the spirit of backstepping transformations and target systems.

Let us now give a finite-dimensional example to illustrate the role controllability can play in the backstepping method for PDEs. Consider the finite-dimensional control system

$$\dot{x} = Ax + Bu(t), \quad x \in \mathbb{C}^n, A \in M_n(\mathbb{C}), B \in M_{n,1}(\mathbb{C}). \quad (1.192)$$

Now suppose that (A, B) is controllable, and let us try to invertibly transform system (1.192) into another controllable system, namely

$$\dot{x} = \tilde{A}x, \quad (1.193)$$

which is exponentially stable if \tilde{A} is well chosen.

Suppose that $x(t)$ is a solution of system (1.192) with $u(t) = Kx(t)$. Such a transformation T would map (1.192) into

$$(T\dot{x}) = T\dot{x} = T(A + BK)x.$$

In order for Tx to be a solution of (1.193), we need

$$T(A + BK)x = \tilde{A}Tx.$$

To find such a T , let us suppose without loss of generality that (A, B) is in canonical form, using the fact that (A, B) is controllable. Now, as (\tilde{A}, B) is also controllable, it can be put in canonical form with an invertible matrix T :

$$T^{-1}\tilde{A}T = c(\tilde{A}) \quad (1.194)$$

Now, as in the proof of the pole-shifting theorem (Theorem 1.2.3), there exists a unique K such that

$$A + BK = c(\tilde{A}) \quad (1.195)$$

which yields

$$T(A + BK) = \tilde{A}T. \quad (1.196)$$

Now notice that as we assumed that (A, B) was in canonical form, this implies that

$$TB = B. \quad (1.197)$$

Injecting the above equation into (1.196), we get the following equations:

$$\begin{aligned} TA + BK &= \tilde{A}T, \\ TB &= B, \end{aligned} \quad (1.198)$$

for which we just proved the following theorem:

Theorem 1.4.7. *If (A, B) and (\tilde{A}, B) are controllable, then there exists a unique pair (T, K) satisfying conditions (1.198).*

What this proof shows is that controllability can be very useful when one wants to transform systems into other systems. In the finite-dimensional case, using the canonical form is the most efficient way of writing it. However, in order to gain some insight on the infinite-dimensional case, there is a different proof, relying on the spectral properties of A and \tilde{A} , which can be found in [59]. The idea is that the controllability of A allows to build a basis for the space state, in which T can then be constructed. Indeed, suppose A is diagonalizable with eigenvectors $(e_n, \lambda_n)_{1 \leq n \leq N}$, and suppose that \tilde{A} and A have no mutual eigenvalues. Then, let us project (1.198) on e_n :

$$\lambda_n T e_n + (K e_n) B = \tilde{A} T e_n, \quad (1.199)$$

from which we get the following relationship

$$T e_n = (K e_n)(\tilde{A} - \lambda_n I)^{-1} B, \quad \forall n \in \{1, \dots, N\}. \quad (1.200)$$

Then, using the Kalman rank condition on the pair (\tilde{A}, B) , one can prove that the $f_n := ((\tilde{A} - \lambda_n I)^{-1} B)$ form a basis of \mathbb{R}^N .

Knowing this, write

$$\begin{aligned} B &= \sum_{n=1}^N b_n e_n, \\ B &= \sum_{n=1}^N \tilde{b}_n f_n, \end{aligned} \quad (1.201)$$

and $T B$ is written naturally in this basis:

$$T B = \sum_{n=1}^N (K e_n) b_n f_n, \quad (1.202)$$

so that the second equation of (1.198) becomes

$$\sum_{n=1}^N (K e_n) b_n f_n = \sum_{n=1}^N \tilde{b}_n f_n. \quad (1.203)$$

Using the Kalman rank condition on (A, B) , one can prove that $b_n \neq 0$ so that the $(K e_n)$ are uniquely determined. The only thing that remains to prove is the invertibility of T , as the $(K e_n)$ could be 0. In the end the invertibility is proven thanks to the Hautus test on the pair (\tilde{A}, B) , and the uniqueness is given by the $T B = B$ condition.

1.4.7 Making educated guesses

The strategy described above can be translated into a heuristic for PDEs. Consider now the same systems (1.192) and (1.193), but where A, \tilde{A}, B are differential operators and the state space is some Hilbert space (for example, L^2).

Suppose that A and \tilde{A} admit Riesz bases of eigenfunctions. Translating (1.199) in terms of differential operators, we get an ODE that $T e_n$ would satisfy if a backstepping transformation T existed. We can then solve this ODE, which is analog to writing (1.200). Then, using the controllability of the target system as in the finite-dimensional case, we can try and prove that the $(T e_n)$ form a Riesz basis of the state space, under some conditions on the $(K e_n)$. If that is the case, then *for every suitable feedback law, we have built an invertible transformation*.

Note however that at this point, we do not know if one of these is an actual backstepping transformation: for a suitable feedback law, our approach is to *make an "educated guess" at what the corresponding backstepping transformation could be*.

Now, to continue our search for a backstepping transformation, we recall that (1.199) is not the genuine backstepping operator equality: indeed we have injected the $T B = B$ condition in the first

equation of (1.196), which was actually very convenient, as it separated the variables T, K in finite dimension, and removed the non-local terms of the kernel equations in infinite dimension ([59], or Chapters 3 and 5).

So, for our educated guess to be consistent, we need some form of $TB = B$ condition to be satisfied by (T, K) . In finite dimension, this is relatively easy, as it relies on decompositions in two different bases. In infinite dimension, this can prove trickier. In [59] the controller function is in the domain of definition of the invertible transformations derived from the kernel equations. However, in Chapters 3 and 5, this is no longer the case, and we resort to a weaker form of the $TB = B$ condition. To our knowledge, this feature is novel in the development of the backstepping method.

To understand this weaker form, let us recall that we are trying to check whether our guess is accurate, that is, checking rigorously if, for a given feedback, the corresponding transformation we have built really is a backstepping transformation. More precisely, we want to check that this transformation satisfies the operator equality (1.198) on some domain depending on the feedback law. Accordingly, a careful study of this operator equality (see Chapter 5, Section 5.5.3) yields a condition on the feedback law which is actually the weaker form of the $TB = B$ condition:

$$\langle TB^{(N)}, \tilde{\phi}_m \rangle \xrightarrow{N \rightarrow \infty} \langle \mathcal{I}, \tilde{\phi}_m \rangle, \quad \forall m \in \mathbb{Z}, \quad (1.204)$$

for a specific approximation $B^{(N)}$ of B . Finally we process this condition by using pointwise convergence theorems for Fourier-type series, inspired by the Dirichlet convergence theorem for Fourier series (see for example [101]):

$$\sum_{n=-N}^N \langle f, e_n \rangle e_n(x) \xrightarrow{N \rightarrow \infty} \frac{f(x^+) + f(x^-)}{2}, \quad f \in C_{pm}^1, \quad \forall x \in [0, L]. \quad (1.205)$$

This kind of technical development is new in the landscape of backstepping, and we believe it to be linked to the growth of the eigenvalues of the hyperbolic operators we have considered. Indeed, in [59], the eigenvalues have quadratic growth and the technical developments are quite different. In particular, the Riesz basis property of the (Te_n) is profoundly different in nature, which leads to a backstepping transformation T defined on the whole state space.

1.4.8 Backstepping in higher dimension

As far as we know, PDE backstepping has only been applied on 1-D systems, except for some parabolic systems on a parallelepiped, under some assumptions on the diffusion and reaction coefficients, in [100, 122], or extension to higher dimension of feedback laws elaborated in 1-D ([119]). Indeed the triangular (or spatially causal) structure of Volterra transformations of the second kind makes them difficult to define on higher-dimensional domains, and geometrical constraints on the system seem necessary as they help reduce the problem to a collection of 1-D problems.

In contrast, our method relies on general kernel operators, and one could consider extending them to higher-dimensional domains, as long as the differential operators involved have nice spectral properties. This would involve more complicated kernel equations, and the analog of the equations (1.199) would be PDEs instead of ODEs, for which Riesz basis properties could be more challenging to prove.

1.5 Conclusion and prospects

In this thesis, we have studied various problems of internal control for some hyperbolic systems in 1-D.

We have studied indirect controllability for systems of quasilinear and semilinear coupled wave equations, with an internal control. We have given a natural sufficient condition for local controllability around trajectories of the system, together with a natural relation between the control time and the support of the control. To obtain this result we used the fictitious control method, together with a

local inversion theorem by Mikhail Gromov, of the Nash-Moser type, to deal with the inherent loss of derivatives in such systems. This method allowed for control supports that are subsets of an interval, as long as they satisfy a form of Geometrical Control Condition. This flexibility made it possible to construct non-trivial trajectories around which the system is locally controllable, as is illustrated in the particular case of semilinear wave equations with a cubic coupling.

We have studied the stabilization of controllable hyperbolic systems by a distributed scalar feedback. We built on the most recent developments of the backstepping method for PDEs to obtain explicit feedbacks. Although based on a recent application to the bilinear Schrödinger equation, the method we develop proved to be quite different in its technical developments. We believe this to be linked with the growth of the eigenvalues of the differential operators involved. Using this method, we have obtained explicit stationary feedbacks that stabilize controllable linear periodic transport equations, for arbitrarily large exponential decay rates. In this framework it turns out that a new stationary feedback law can be derived from these feedback laws, which achieves finite-time stabilization. Finally, we have applied our method on the linearized water tank system around non-uniform steady states. This last situation presented an additional challenge, due to the conservation of mass. In fact, the system under consideration is only controllable up to a missing direction which corresponds to variation of mass. We have overcome this obstacle by adding an integrator to the feedback loop.

Let us now list a few questions and prospects.

Regarding internal controllability, the regularities of the control and the trajectories in our result are probably not optimal, due to technical specificities of the inversion theorem we have used. It would also be interesting to investigate a higher-dimensional version of this result, keeping in mind that part of our proof is specific to 1-D systems. On another note, our proof is constructive, in a way, but the Nash-Moser scheme contained in the inversion theorem suggests that the control could be approximated numerically. Finally, the question of boundary indirect controllability or observability was studied in the linear case, and recent works seem to indicate that it might be more challenging than internal indirect controllability, but it remains open for nonlinear systems.

Regarding the stabilization of controllable hyperbolic systems, several prospects should be pointed out. Concerning the water tank itself, we have provided a feedback law to stabilize a linearized system. Therefore the stabilization of the nonlinear system, around non-uniform or uniform steady-states, is still in question. Moreover, the feedback loop we have provided is relatively explicit, so that a numerical illustration seems within reach.

Some general questions on the backstepping method should also be raised. As we have presented in this introduction, there are many variants of the backstepping method. How does one know which one to choose for a given PDE? Could the variant we used on hyperbolic systems, and the one used on the bilinear Schrödinger equation, be understood in a broader framework, which could, in particular, explain the influence of the growth of the eigenvalues? Is the assumption of exact controllability absolutely necessary, knowing that the backstepping method can work with approximate controllability in some cases? Finally, up to now the backstepping method has been elaborated on 1-D systems, and recent extensions to higher space dimensions are restricted to systems with a tensor product structure. Could a backstepping method be designed for more general higher-dimensional systems?

Part I

Internal controllability of coupled wave equations

Chapter 2

Internal Controllability of Systems of Semilinear Coupled One-Dimensional Wave Equations with One Control

This chapter is taken from the following article (also referred to as [161]):

Christophe Zhang. Internal controllability of systems of semilinear coupled one-dimensional wave equations with one control. *SIAM Journal on Control and Optimization*, Society for Industrial and Applied Mathematics, 2018, 56 (4), pp.3092 - 3127.

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Abstract We study systems of two coupled wave equations in one space dimension, with one control, spatially supported on an arbitrarily small interval. We obtain the controllability of such systems under certain conditions on the coupling. To do this we apply the “fictitious control method” in two cases: general systems with a controllable linearised system, and a particular case where the linearised system is not controllable, namely a cubic coupling.

In the latter case, our proof requires to find nontrivial trajectories of the control system that go from 0 to 0 and having a controllable linearized system. We build these trajectories by adapting (in 1

space dimension) a construction developed by Jean-Michel Coron, Sergio Guerrero and Lionel Rosier for the study of coupled parabolic systems.

2.1 Main results and outline of proof

2.1.1 Control systems

Let $T > 0$, and $0 < a < b < L$. We study the following class of systems:

$$\begin{cases} u_{tt} - \nu_1^2 u_{xx} = f_1(u, v) + h, & x \in (0, L), \\ v_{tt} - \nu_2^2 v_{xx} = f_2(u, v), & x \in (0, L), \\ u = 0 & \text{on } \{0, L\}, \\ v = 0 & \text{on } \{0, L\}, \end{cases} \quad (2.1)$$

where $h : [0, T] \times [0, L] \rightarrow \mathbb{R}$ is the control, with $\text{supp } h \subset [0, T] \times [a, b]$, and $f_1, f_2 \in C^\infty(\mathbb{R}^2)$, $f_1(0, 0) = f_2(0, 0) = 0$, $\nu_1, \nu_2 \neq 0$. In what follows we shall note, for any $\nu \neq 0$,

$$\square_\nu := \partial_{tt} - \nu^2 \partial_{xx}$$

We will also study the following particular system:

$$\begin{cases} \square_{\nu_1} u = h, & x \in (0, L), \\ \square_{\nu_2} v = u^3, & x \in (0, L), \\ u = 0 & \text{on } \{0, L\}, \\ v = 0 & \text{on } \{0, L\}. \end{cases} \quad (2.2)$$

These are systems of coupled semilinear wave equations, with different speeds, which we seek to control with a single control, which takes the form of a source term in the first equation with a support in $[0, L] \times [a, b]$. In both cases, we will study solutions with $C^k((0, T] \times [0, L])$ regularity in order to establish a controllability result with two controls. Thus, the initial and final conditions $((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f))$ have to satisfy some compatibility conditions. For example, the conditions of order 1 and 2 read as:

$$\forall \beta \in \{0, L\}, \begin{cases} u_0(\beta) = u_1(\beta) = (u_0^f)(\beta) = (u_1^f)(\beta) = 0, \\ u_0''(\beta) = u_1''(\beta) = (u_0^f)''(\beta) = (u_1^f)''(\beta) = 0, \\ v_0(\beta) = v_1(\beta) = (v_0^f)(\beta) = (v_1^f)(\beta) = 0, \\ v_0''(\beta) = v_1''(\beta) = (v_0^f)''(\beta) = (v_1^f)''(\beta) = 0. \end{cases} \quad (2.3)$$

To write the compatibility conditions of order $k \geq 3$, the idea is to first write the time derivatives of u and v as a function of their lower order derivatives.

There exists a multivariate polynomial $Q_{n,i}^{f_i}$ such that

$$\left(\frac{d}{dt}\right)^n (f_i(u, v)) = Q_{n,i}^{f_i}(J_t^n(u, v)), \quad i = 1, 2, \quad (2.4)$$

where $J_t^n(u, v)$ denotes the n -jet of time derivatives of u and v , that is

$$(u, v, u_t, v_t, \dots, \partial_t^n u, \partial_t^n v).$$

Now, define by recurrence the following family of operators:

$$\begin{cases} \mathcal{D}_1^{(i)} = \partial_t, \\ \mathcal{D}_2^{(i)} = \partial_{xx} + f_i(\cdot, \cdot), \\ \mathcal{D}_n^{(i)} = \partial_{xx} \circ \mathcal{D}_{n-2}^{(i)} + Q_{n-2,i}^{f_i}(J_t^{n-2}(\cdot, \cdot)), \text{ for } 3 \leq n \leq k. \end{cases} \quad (2.5)$$

Then, near the corners $\Gamma := \{(0, 0), (0, L), (T, 0), (T, L)\}$, using the equations of system (2.1) and keeping in mind that the control h is supported away from the corners, we have

$$\begin{cases} \partial_t^n u = \mathcal{D}_n^{(1)}(u, v) \\ \partial_t^n v = \mathcal{D}_n^{(2)}(u, v) \end{cases} \quad (2.6)$$

Now, thanks to the boundary conditions,

$$\partial_t^n u(c) = \partial_t^n v(c) = 0, \quad \forall c \in \Gamma, \forall n \leq k.$$

Moreover, it is clear thanks to the recurrence in (2.5) that there exist multivariate polynomials $P_{n,i}^{f_i}$ such that:

$$\mathcal{D}_n^{(i)}(u, v) = P_{n,i}^{f_i} (J_x^n(u, v), J_x^{n-1}(u_t, v_t), J_t^n(u, v)), \quad \forall n \leq k, i = 1, 2, \quad (2.7)$$

where $J_x^n(u, v)$ denotes the n -jet of space derivatives of u and v . Now, (2.6) can be written in the corners using only $u_0, u_1, u_0^f, u_1^f, v_0, v_1, v_0^f, v_1^f$, which gives the following compatibility conditions of order k :

$$\begin{cases} P_{n,i}^{f_i} (J_x^n(u_0, v_0)(0), J_x^{n-1}(u_1, v_1)(0), (0, \dots, 0)) = 0, \\ P_{n,i}^{f_i} (J_x^n(u_0, v_0)(L), J_x^{n-1}(u_1, v_1)(L), (0, \dots, 0)) = 0, \\ P_{n,i}^{f_i} (J_x^n(u_0^f, v_0^f)(0), J_x^{n-1}(u_1^f, v_1^f)(0), (0, \dots, 0)) = 0, \\ P_{n,i}^{f_i} (J_x^n(u_0^f, v_0^f)(L), J_x^{n-1}(u_1^f, v_1^f)(L), (0, \dots, 0)) = 0, \end{cases} \quad \forall n \leq k, i = 1, 2. \quad (2.8)$$

The existence and uniqueness of solutions to these systems can be derived from TaTsiens Li's general results on quasilinear wave equations (see [112] or [113, chapter 5, section 5.2]).

In this chapter we prove two controllability results: a local result for system (2.1), and a global result for system (2.2).

Theorem 2.1.1. *Let $R > 0$, and $0 \leq a < b \leq L$, $T > 0$ such that*

$$T > 2(L - b) \max\left(\frac{1}{|\nu_1|}, \frac{1}{|\nu_2|}\right), \quad T > 2a \max\left(\frac{1}{|\nu_1|}, \frac{1}{|\nu_2|}\right). \quad (2.9)$$

If

$$\frac{\partial f_2}{\partial u}(0, 0) \neq 0, \quad (2.10)$$

then there exists $\eta > 0$ such that for initial and final conditions

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in (B_{C^{11}}([0, L]) \times B_{C^{10}}([0, L])(0, \eta))^4$$

where $B_{C^k}(0, \eta)$ denotes the ball centered in 0 and with radius η in the usual C^k topology, satisfying (2.8) at the order 11, there exists $h \in C^6([0, T] \times [0, L])$ such that

$$\text{supp } h \subset [0, T] \times [a, b], \quad (2.11)$$

and such that the corresponding solution $(u, v) \in C^6([0, T] \times [0, L])^2$ of (2.1) with initial values $((u_0, u_1), (v_0, v_1))$ satisfies

$$\begin{cases} u(T, \cdot) = u_0^f, & u_t(T, \cdot) = u_1^f, \\ v(T, \cdot) = v_0^f, & v_t(T, \cdot) = v_1^f \end{cases}$$

and

$$\|(u, v, h)\|_{(C^6)^3} \leq R. \quad (2.12)$$

Condition (2.10) is necessary and sufficient for the controllability of linear systems (if the dynamics of v does not depend on u there is no hope to control v through u). In contrast, the following theorem shows that it is not necessary in the case of nonlinear systems: system (2.2) does not satisfy (2.10), but we still obtain a controllability result. Moreover, thanks to the system's homogeneity, the result is global.

Theorem 2.1.2. *Let $0 \leq a < b \leq L$, $T > 0$ satisfying (2.9). There exists a constant $C > 0$ depending on T such that, for any given initial and final conditions*

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in (C^{11}([0, L]) \times C^{10}([0, L]))^4$$

satisfying (2.8) at the order 11, there exists $h \in C^6([0, T] \times [0, L])$ such that

$$\text{supp } h \subset [0, T] \times [a, b], \quad (2.13)$$

and such that the corresponding solution $(u, v) \in C^6([0, T] \times [0, L])^2$ of (2.2) with initial values $((u_0, u_1), (v_0, v_1))$ satisfies

$$\begin{cases} u(T, \cdot) = u_0^f, & u_t(T, \cdot) = u_1^f, \\ v(T, \cdot) = v_0^f, & v_t(T, \cdot) = v_1^f \end{cases}$$

and

$$\|h\|_{C^6} \leq C \left(\|(u_0, u_1, u_0^f, u_1^f)\|_{(C^{11} \times C^{10})^2} + \|(v_0, v_1, v_0^f, v_1^f)\|_{(C^{11} \times C^{10})^2}^{\frac{1}{3}} \right). \quad (2.14)$$

2.1.2 Related results

Control of hyperbolic equations

Fundamental results for the controllability and stabilisation of the linear wave equation can be found in [18]. For quasilinear wave equations, boundary controllability results for scalar systems with C^2 regularity can be found in [113, chapter 5], and can be adapted to coupled systems with the same number of controls and equations, and for C^k regularity. For the semilinear wave equation, local controllability results have been obtained using the implicit function theorem (see [78] and the generalisation by [43]). To get global boundary and internal controllability for the semilinear wave equation, under some growth constraints on the nonlinearity, Enrique Zuazua used HUM (Hilbert Uniqueness Method) and introduced a suitable fixed-point method in [167] and [166]. These results have since then been improved successively by [38] and [37], where authors study the one-sided and internal controllability of a semilinear wave equation with an iterated logarithm nonlinearity. Another powerful method to prove controllability results is the Carleman estimates method. It was first used for the semilinear wave equation in [164] and [165], where a new Carleman estimate was established to prove internal observability. The estimate worked for globally Lipschitz nonlinearities, with the observer supported in a neighbourhood of some portion of the boundary. More recently, Carleman estimates were used in [79] to obtain internal controllability of the semilinear wave equation in any space dimension. The control is supported in a neighbourhood of a portion of the boundary (earlier works required the controller to be supported in the neighbourhood of the whole boundary), and the nonlinearity is superlinear. The method of Carleman estimates was also used in [88] for mechanical systems of several coupled linear hyperbolic equations (a multilayer Rao-Nakra plate). This yields internal controllability results, with the same number of controls and equations, and controllers supported on an arbitrarily small neighbourhood of some portion of the boundary.

Systems with less controls than equations

Linear case. Regarding controllability with a reduced number of controls, results for boundary and internal control of coupled linear symmetric wave systems have been proved by Fatiha Alabau-Boussouira ([8] and [3]) in any space dimension, using energy methods, with more or less strong

assumptions on the coupling operators, and in particular in the case where the control domain and the coupling domain do not intersect. This was then used in [4] to prove the existence of insensitizing controls for a single wave equation, as this is linked to the controllability of linear cascade systems in one space dimension, with the same speed in both equations. Other methods have been used to deal with a reduced number of controllers, albeit on different types of systems: on the related question of partial observability on a sphere, on top of some results proven by Lions in [116] and [117], [107] shows a way to deal with a reduced number of controllers using the Fourier expansion of the solutions. They prove that for a generic choice of coupling parameters, and provided the initial conditions of the unobserved components are zero, and the initial conditions of the observed components are orthogonal to a finite-dimensional space (possibly trivial, for example in the one-dimensional case), then partial observability holds.

Nonlinear case. The link between cascade controllability and desensitizing controls has also been explored for semilinear equations in [145], where the author proves the controllability of cascade systems of the form:

$$\begin{cases} \square u + f(u) = h + \xi, \\ \square v + f'(u)v = 0, \\ u = 0, v(t, 0) = \frac{\partial u}{\partial n} \chi_{\Gamma_0} \text{ on } \partial\Omega, \end{cases} \quad (2.15)$$

where Γ_0 is a portion of the boundary, and where f is subject to a growth constraint to have global well-posedness. To prove the controllability of such systems, the author first establishes the controllability of a linear problem, using a form of HUM combined with Carleman estimates. Then, using the Schauder fixed-point theorem, he establishes the controllability of the nonlinear problem.

In other cases, as for system (2.2), the linearised system around 0 is not controllable. A classical tool to handle this problem in finite dimension is the use of iterated Lie brackets, see for example [99, chapter 2], [124, chapter 3], and [53, chapter 3]. However, this tool does not work for many partial differential equations (see for example [53, chapter 5]). In particular it does not work for our control system (2.2). In that case, a method to handle this situation is the return method. It consists in looking for trajectories going from 0 to 0 and such that the linearised system around them is controllable (return trajectories). This method has been introduced in [47] for the stabilization of driftless control systems and in [51] and in [49] for the controllability of the Euler equations of incompressible fluids. It is also used in [60] for parabolic systems with cubic coupling. Following this method, in [60] the authors build return trajectories, using the structure of the coupling. Then, using Carleman estimates, they prove the controllability of a family of related parabolic linear systems close to the return trajectory, from which they deduce null-controllability using Kakutani's fixed-point theorem.

In yet other cases, a phenomenon of loss of derivatives can occur: this can be handled with an inversion theorem of the Nash-Moser type, with a stronger condition on the linearised system. A well-known case is the local controllability of the 1-D Schrödinger equation, which was proven in [28] and [29] using a Nash-Moser implicit function theorem. More recently, the controllability of a system of coupled quasilinear first order hyperbolic systems with one control was proved in [5], using the “fictitious control method” and a Nash-Moser type inversion theorem proven by Gromov, which we will explain in the following section. More precisely the result concerns systems of the form:

$$\begin{cases} u_t + \Lambda_1(u, v)u_x + f_1(u, v) = h, \\ v_t + \Lambda_2(u, v)v_x + f_2(u, v) = 0, \end{cases}$$

with

$$\frac{\partial f_2}{\partial u}(0, 0) \neq 0. \quad (2.16)$$

The work presented in this chapter draws from all these situations: we study semilinear systems, as in [145], but of a more general form than (2.15). The idea would then be to prove a controllability

result for some sort of linearised system, then use a fixed point theorem (or an inversion theorem) to conclude. However, because of a phenomenon of loss of derivatives, we rather follow the same path as [5] to get Theorem 2.1.1. This approach, in addition to dealing with the loss of derivatives in our system, has the advantage of not being as computation heavy as Carleman estimates, and allows for general control domains, whereas the use of Carleman estimates usually requires the control domain to be the neighborhood of some portion of the boundary. For Theorem 2.1.2 however, condition (2.10) (analogous to (2.16)) is not satisfied. This corresponds to the fact that the linearized system is not controllable, and so we build return trajectories as in [60].

Finally, a remark on the control time is in order: for hyperbolic systems, the control time is usually linked to the speeds of propagation and the size of the domain, as this represents how fast the deformation produced by the control reaches every point of the domain. Now, in the case of a reduced number of controls, one can expect that the indirect action of the control should mean additional control time, or that the control time should not depend only on the geometry of the domains and the propagation speeds, but on some other parameters. For example, in the results of [8], the authors point out that the control time they obtain depends on all the parameters of the system, not only the geometry of the control and coupling domains. Likewise, in [3] and [4] the control times depend on observability times not only for a single equation, but also for the coupling operator, and in [145], the control time depends on the choice of some function used to establish Carleman estimates. On the other hand, in [5] as well as in our theorems, the control time is the same as for scalar equations. Indeed, as we will see in what follows, applying the fictitious control method does not change the control time when removing one control, and the control time depends only on the size of the support of the control. Physically speaking, we use the coupling to transmit information from one equation to another (this corresponds to conditions (2.10) and (2.16)) everywhere in the domain, so that the action of the control on the first equation can be transmitted without delay.

2.1.3 The fictitious control method

The fictitious control method was introduced in [47] and [83], and successfully used in [64], [5] and [61]. The idea is to first prove a controllability result with two controls (the fictitious controls), then reduce the number of controls, using some sort of fixed-point theorem, namely Theorem 2.2.1.

In this chapter, we apply it to second order hyperbolic systems, which present the same problem of loss of derivatives as the systems in [5]. This loss of derivatives is handled by using Gromov's notion of algebraic solvability, which allows for differential operators to be inverted in a special way under some condition (infinitesimal inversion) on their derivative. This yields local results around the equilibrium, but we will also work around other trajectories than the stationary trajectory at the equilibrium, in the spirit of the return method, paying close attention to the regularities involved. Indeed, condition (2.10) from Theorem 2.1.1 is identical to condition (2.16), and is crucial to solve the system algebraically (see Proposition 2.2.2). If, as in Theorem 2.1.2, it is not satisfied, then, following the spirit of the return method, one can build trajectories of the system along which such a condition is verified, at least on some appropriate spatial domain.

We can thus sum up our strategy of proof in three steps:

1. Find smooth trajectories around which Theorem 2.2.1 can be used (when necessary).
2. Prove a local controllability result with two controls (fictitious controls) around the return trajectory, using classical boundary control results.
3. Use Theorem 2.2.1 to reduce the number of controls to one.

Remark 2.1.1. *In this method, the controllability of the linearised system is not used directly to obtain controllability of the nonlinear system using a fixed point theorem. Rather, the corresponding condition (2.10) gives us some sort of indication that information is “well transmitted” from the first equation to the second equation, so that what happens with one control in each equation, can be translated into a single control in the first equation.*

this chapter is organised as follows: in section 2, we illustrate Gromov’s ideas on a linear example, and then prove Theorem 2.1.1, which is a case where we do not need to find return trajectories. This will allow us to present how Gromov’s ideas can be applied to a system of nonlinear wave equations. In section 3 we prove Theorem 2.1.2. In this case we need to find return trajectories, and the application of Theorem 2.2.1 around those trajectories will require a more detailed knowledge of the supports of the return trajectory. Finally section 4 is devoted to possible improvements and further questions on this topic.

2.2 First case: the linearised system is controllable

As mentioned in section 2.1.3, we build on the method presented in [5]. One of the main ingredients of this method is the theory of differential operators, and the notion of algebraic solvability, which we briefly present in the subsection below. The use of algebraic solvability in the study of control systems first appears in [47], where it was used to prove the stabilisability of finite dimensional systems without drift with time-varying feedbacks. It was first used in the context of partial differential equations in [64] for the control of the Navier-Stokes equation.

But first let us give an informal explanation of our method in the case of a linear system: first we have to rewrite the control problem using differential operators. We note \mathcal{D} the operator associated with the equation of our control problem. Then, the control problem, given initial and final conditions, consists in finding (u, v) with those initial and final conditions, and a control h such that

$$\mathcal{D}(u, v, h) = 0.$$

This corresponds to an inversion problem, but with a twist: one has to find an inverse image with the right initial and final conditions. Now, using the solutions to forward- and backward-evolving Cauchy problems corresponding to the initial and final conditions, one can build functions (u, v) with the right initial and final conditions. The nonlinear version of this is done at the beginning of subsection 2.2.2. In general, one can do this so that for some $\eta > 0$,

$$(h_1, h_2) := \mathcal{D}(u, v, 0) = 0, \forall t \notin [\eta, T - \eta].$$

Now suppose \mathcal{D} is invertible. We can make the following computation, the nonlinear version of which is made in subsection 2.2.2:

$$\mathcal{D}((u, v, 0) + \mathcal{D}^{-1}(-h_1, -h_2)) = (h_1, h_2) - (h_1, h_2) = 0.$$

This seems to yield a solution to the control problem, however we still need to check that the “corrective term” does not change the initial and final conditions. This is where Gromov’s notion of algebraic solvability comes into play: the right property for \mathcal{D} is not to be invertible, but to be algebraically solvable. That is, that the inverse can also be written as a differential operator:

$$\mathcal{D}^{-1}(-h_1, -h_2) = \sum_r a_r \partial_r(-h_1) + \sum_r b_r \partial_r(-h_2)$$

for some functions a_r, b_r . With this additional property, one can see that, because $-h_1, -h_2$ vanish for $t \notin [\eta, T - \eta]$,

$$\mathcal{D}^{-1}(-h_1, -h_2) = 0, \forall t \notin [\eta, T - \eta].$$

Hence, $(u, v, 0) + \mathcal{D}^{-1}(-h_1, -h_2)$ still has the right initial and final conditions.

2.2.1 Differential relations and Gromov’s theorem

In this section we sum up some basic notions regarding differential operators, and Gromov’s local inversion theorem for differential operators. More details can be found in [84].

In what follows, \mathcal{Q} is the closure of a non-empty open bounded smooth subset of \mathbb{R}^2 , and $p, q, r \in \mathbb{N}^*$. We note $n_{r,p} := 2 + p \text{ card}\{(\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid \alpha_1 + \alpha_2 \leq r\}$. Recall the definition of the r -jet of a function $z \in C^r(\mathcal{Q})^p$:

$$J^r z(t, x) = \left((t, x), z(t, x), \dots, \frac{\partial^{|\alpha|} z}{\partial t^{\alpha_1} \partial x^{\alpha_2}}, \dots, \frac{\partial^r z}{\partial t^{\alpha_1} \partial x^{\alpha_2}} \right) \in \mathbb{R}^{n_{r,p}}, \forall (t, x) \in \mathcal{Q}.$$

Definition 2.2.1. A map $\mathcal{D} : C^r(\mathcal{Q})^p \rightarrow C^0(\mathcal{Q})^q$ is a C^∞ nonlinear differential operator of order r if there exists $F \in C^\infty(\mathbb{R}^{n_{r,p}}, \mathbb{R}^q)$ such that

$$\mathcal{D}(z) = F(J^r z), \forall z \in C^r(\mathcal{Q})^p.$$

This clearly implies that \mathcal{D} is C^∞ (with the usual C^r, C^0 topologies), and we denote by

$$\mathcal{L}_z : C^r(\mathcal{Q})^p \rightarrow C^0(\mathcal{Q})^q$$

its Fréchet differential at $z \in C^r(\mathcal{Q})^p$.

We now define some sort of manifold, over which we can invert these operators:

Definition 2.2.2. A subset \mathcal{A} of $C^d(\mathcal{Q})^p$ is a differential relation of order $d \in \mathbb{N}$ if there exists $\mathcal{R} \subset \mathbb{R}^{n_{d,p}}$ such that

$$\mathcal{A} = \{z \in C^d(\mathcal{Q})^p \mid \forall (t, x) \in \mathcal{Q}, J^d z(t, x) \in \mathcal{R}\}.$$

It is said to be open if \mathcal{R} is an open subset of $\mathbb{R}^{n_{d,p}}$. For $k \in \mathbb{N}$, we note

$$\mathcal{A}^k := \mathcal{A} \cap C^k(\mathcal{Q})^p$$

For classical local inversion theorems, one needs the differential at one point to be invertible. Here the requirement is somewhat stronger: we need the differential at any point to be invertible, with the extra property that the inverse of each differential is also a linear differential operator.

Definition 2.2.3. Let $\mathcal{A} \subset C^d(\mathcal{Q})^p$ be a differential relation of order d , and let \mathcal{D} be a differential operator of order r . We say that \mathcal{D} admits an infinitesimal inversion of order $s \in \mathbb{N}$ over \mathcal{A} if there exists a family of linear differential operators of order s

$$z \in \mathcal{A}, \mathcal{M}_z : C^s(\mathcal{Q})^q \rightarrow C^0(\mathcal{Q})^p,$$

such that:

1. For every $g \in C^s(\mathcal{Q})^q$, $z \mapsto \mathcal{M}_z(g)$ is a differential operator of order d (possibly nonlinear) and it is a C^∞ -differential operator in (z, g) .
2. (Algebraic solvability) For every $z \in \mathcal{A}^{d+r}$,

$$\mathcal{L}_z \circ \mathcal{M}_z = \text{Id}_{C^{r+s}(\mathcal{Q})}.$$

We can now state Gromov's inversion theorem (see [84, Section 2.3.2, main theorem]):

Theorem 2.2.1 (Gromov). Let $\mathcal{A} \subset C^d(\mathcal{Q})^p$ be a non-empty open differential relation of order d , and let \mathcal{D} be a differential operator of order r . Assume that \mathcal{D} admits an infinitesimal inversion of order s over \mathcal{A} . Let

$$\sigma_0 > \max(d, 2r + s), \tag{2.17}$$

$$\nu \in (0, \infty). \tag{2.18}$$

Then, there exists a family of sets $\mathcal{B}_z \subset C^{\sigma_0+s}(\mathcal{Q})^q$ and a family of operators $\mathcal{D}_z^{-1} : \mathcal{B}_z \rightarrow \mathcal{A}$ where $z \in \mathcal{A}^{\sigma_0+r+s}$, such that:

1. (Neighbourhood property) For every $z \in \mathcal{A}^{\sigma_0+r+s}$, $0 \in \mathcal{B}_z$ and

$$\mathcal{B} := \bigcup_{z \in \mathcal{A}^{\sigma_0+r+s}} \{z\} \times \mathcal{B}_z$$

is an open subset of $C^{\sigma_0+r+s}(\mathcal{Q})^p \times C^{\sigma_0+s}(\mathcal{Q})^q$.

2. (Inversion property)

$$\mathcal{D}(\mathcal{D}_z^{-1}(g)) = \mathcal{D}(z) + g, \quad \forall (z, g) \in \mathcal{B}. \quad (2.19)$$

3. (Normalisation property)

$$\mathcal{D}_z^{-1}(0) = z, \quad \forall z \in \mathcal{A}^{\sigma_0+r+s}. \quad (2.20)$$

4. (Regularity and continuity) Let $\sigma_0 \leq \sigma_1 \leq \eta_1$, then for all $z \in \mathcal{A}^{\eta_1+r+s}$ and $g \in \mathcal{B}_z^{\sigma_1+s} := \mathcal{B}_z \cap C^{\sigma_1+s}$,

$$\mathcal{D}_z^{-1}(g) \in \mathcal{A}^k, \quad \forall k < \sigma_1. \quad (2.21)$$

Moreover,

$$(z, g) \mapsto \mathcal{D}_z^{-1}(g) \in C^0(\mathcal{A}^{\sigma_0+r+s} \times \mathcal{B}_z^{\sigma_1+s}, \mathcal{A}^k), \quad \forall k < \sigma_1. \quad (2.22)$$

Finally, if $\eta_1 > \sigma_1$, then (2.21) and (2.22) hold for $k = \sigma_1$.

5. (Locality) For every $(t, x) \in \mathcal{Q}$, and for every $(z_1, g_1), (z_2, g_2) \in \mathcal{B}$, if we have

$$(z_1, g_1)(\tilde{t}, \tilde{x}) = (z_2, g_2)(\tilde{t}, \tilde{x}), \quad \forall (\tilde{t}, \tilde{x}) \in B((t, x), \nu) \cap \mathcal{Q},$$

then,

$$\mathcal{D}_{z_1}^{-1}(g_1)(t, x) = \mathcal{D}_{z_2}^{-1}(g_2)(t, x).$$

Remark 2.2.1. The neighbourhood property allows to relate the domains of inversion for each local inversion to each other: local inverses at two “neighbouring” points will be defined on domains that have “neighbouring” sizes. In particular that means the domains of inversions are bound to overlap. The locality property tells us that when this happens (albeit locally), the images of the local inverses agree locally. In the linear case, this corresponds to the fact that when a function vanishes on an open set, its image by any linear differential operator also vanishes on this open set (see the beginning of the section).

2.2.2 From two controls to one: algebraic solvability

As in the linear case, we first build a trajectory (u, v) with the right initial and final conditions, but with $\mathcal{D}(u, v, 0)$ potentially non-zero on some restricted domain. In terms of control theory, this amounts to solving the control problem with *two* controls (the fictitious controls), with restricted supports. In fact, for systems of the form

$$\begin{cases} \square_{\nu_1} u = f_1(u, v) + h_1, & x \in [0, L], \\ \square_{\nu_2} v = f_2(u, v) + h_2, & x \in [0, L], \\ u = 0 & \text{on } \{0, L\}, \\ v = 0 & \text{on } \{0, L\}, \end{cases} \quad (2.23)$$

where $f_1(0, 0) = f_2(0, 0) = 0$, we have the following local controllability result, which is a consequence of boundary control results presented in [113, chapter , sections 5.2 and 5.3]:

Proposition 2.2.1. Let $k \geq 2$, $0 \leq a < b \leq L$, $T > 0$ such that (2.9) holds. For every $0 < \delta < \min(T/2, (b-a)/2)$ satisfying

$$\begin{aligned} T - 2\delta &> 2(L - b + 2\delta) \max\left(\frac{1}{|\nu_1|}, \frac{1}{|\nu_2|}\right), \\ T - 2\delta &> 2(a + 2\delta) \max\left(\frac{1}{|\nu_1|}, \frac{1}{|\nu_2|}\right), \end{aligned} \quad (2.24)$$

there exists $\eta > 0$ such that, for initial and final conditions

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in (B_{C^k([0, T] \times [0, L])}(0, \eta) \times B_{C^{k-1}([0, T] \times [0, L])}(0, \eta))^4$$

satisfying (2.8) at the order k , there exist controls $h_1, h_2 \in C^{k-2}([0, T] \times [0, L])$ and constants $C_1, C_2 > 0$ depending on T, δ, k satisfying

$$\text{supp } h_i \subset [\delta, T - \delta] \times [a + \delta, b - \delta], \quad i = 1, 2, \quad (2.25)$$

$$\|h_i\|_{C^{k-2}} \leq C_1 \|((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f))\|_{(C^k \times C^{k-1})^4}, \quad i = 1, 2, \quad (2.26)$$

such that the corresponding solution of (2.23) with initial values $((u_0, u_1), (v_0, v_1))$ satisfies

$$\begin{cases} u(T, \cdot) = u_0^f, & u_t(T, \cdot) = u_1^f, \\ v(T, \cdot) = v_0^f, & v_t(T, \cdot) = v_1^f. \end{cases}$$

$$\|(u, v)\|_{(C^k)^2} \leq C_2 \|((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f))\|_{(C^k \times C^{k-1})^4}. \quad (2.27)$$

This result is a particular case of Proposition 2.3.2 which we will prove in the following section, when dealing with the cubic coupling.

For now, let $R > 0$, $0 \leq a < b \leq L$, and let $T > 0$ be such that (2.9) holds. Let $0 < \delta < \min(T/2, (b-a)/2)/2$ such that (2.24) holds for 2δ (note that it also holds for δ). Define

$$\mathcal{Q}_\delta := [\delta, T - \delta] \times [a + \delta, b - \delta],$$

$$\mathcal{Q}_{2\delta} := [2\delta, T - 2\delta] \times [a + 2\delta, b - 2\delta],$$

and let $\mathcal{Q} \subset [0, T] \times [a, b]$ be a smooth closed set such that

$$\mathcal{Q}_\delta \subset \overset{\circ}{\mathcal{Q}}.$$

Define the following nonempty open differential relation of order 2:

$$\mathcal{A} = \left\{ (u, v, h) \in (C^2(\mathcal{Q}))^3 \mid \forall (t, x) \in \mathcal{Q}, \frac{\partial f_2}{\partial u}(u(t, x), v(t, x)) \neq 0 \right\}.$$

We define the following nonlinear differential operator $\mathcal{D} : C^2(\mathcal{Q})^3 \rightarrow C^0(\mathcal{Q})^2$ of order $r = 2$:

$$\mathcal{D}((u, v, h)) = (\square_{\nu_1} u - f_1(u, v) - h, \square_{\nu_2} v - f_2(u, v)), \quad \forall (u, v, h) \in C^2(\mathcal{Q})^3,$$

and its differential at $(u, v, h) \in C^2([0, T] \times [0, L])^3$:

$$\mathcal{L}_{(u, v, h)}(\tilde{u}, \tilde{v}, \tilde{h}) = \left(\square_{\nu_1} \tilde{u} - Df_1(u, v) \cdot (\tilde{u}, \tilde{v}) - \tilde{h}, \square_{\nu_2} \tilde{v} - Df_2(u, v) \cdot (\tilde{u}, \tilde{v}) \right), \quad \forall (\tilde{u}, \tilde{v}, \tilde{h}) \in C^2([0, T] \times [0, L])^3.$$

We now have the following result, thanks to the definition of \mathcal{A} :

Proposition 2.2.2. \mathcal{D} admits an infinitesimal inversion of order 2 over \mathcal{A} .

Proof. Let $h_1, h_2 \in C^4(\mathcal{Q})$, $(u, v, h) \in \mathcal{A}$. Using the fact that $\frac{\partial f_2}{\partial u}(u, v)$ never vanishes, if we set:

$$\begin{aligned} \tilde{v} &= 0, \\ \tilde{u} &= -\frac{h_2}{\frac{\partial f_2}{\partial u}(u, v)}, \\ \tilde{h} &= \square_{\nu_1} \tilde{u} - \frac{\partial f_1}{\partial u}(u, v) \tilde{u} - h_1, \end{aligned}$$

then we have

$$\mathcal{L}_{(u, v, h)}(\tilde{u}, \tilde{v}, \tilde{h}) = (h_1, h_2).$$

Moreover, the above formulae clearly show that $(u, v, h) \mapsto \mathcal{L}_{(u, v, h)}(\tilde{u}, \tilde{v}, \tilde{h})$ is a (nonlinear, C^∞ with the usual topology of $C^2(\mathcal{Q})$) differential operator of order 2 on $C^2(\mathcal{Q})$, and $(u, v, h, \tilde{u}, \tilde{v}, \tilde{h}) \mapsto \mathcal{L}_{(u, v, h)}(\tilde{u}, \tilde{v}, \tilde{h})$ is also C^∞ . \square

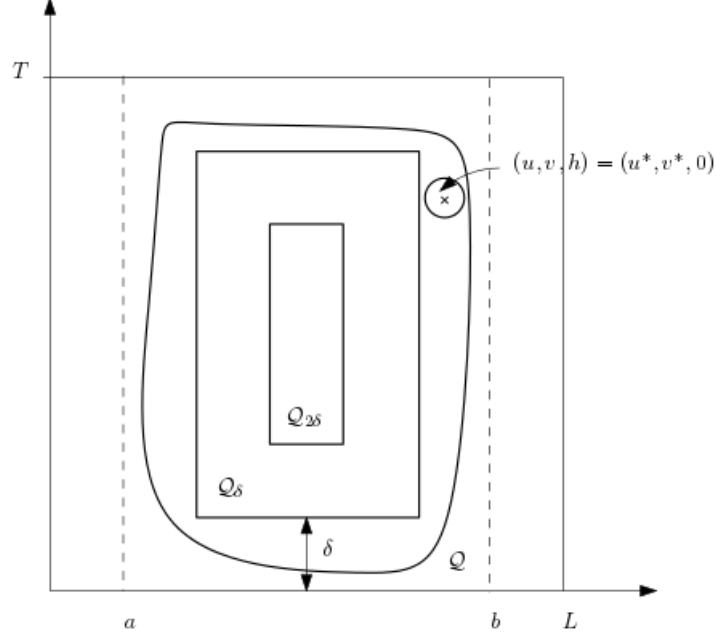


Figure 2.1: *Matching trajectories with two controls and with a single control on the appropriate domain.*

We can now apply Theorem 2.2.1 with $d = 2$, $s = 2$, $r = 2$, $\sigma_0 = 7$, $\nu = \delta/2$. This yields a collection of open sets, which all contain 0,

$$\mathcal{B}_z \subset (C^9(\mathcal{Q}))^2, \quad z \in \mathcal{A}^{11},$$

the open subset of $(C^{11}(\mathcal{Q}))^3 \times (C^9(\mathcal{Q}))^2$

$$\mathcal{B} = \bigcup_{z \in \mathcal{A}^{11}} \{z\} \times \mathcal{B}_z,$$

and the collection of operators

$$\mathcal{D}_z^{-1} : \mathcal{B}_z \rightarrow \mathcal{A}, \quad z \in \mathcal{A}^{11}.$$

Now, thanks to condition (2.10),

$$(0, 0, 0) \in \mathcal{A},$$

$$\mathcal{D}(0, 0, 0) = (0, 0),$$

and

$$((0, 0, 0), (0, 0)) \in \mathcal{B},$$

so that, thanks to the neighbourhood property of Theorem 2.2.1, there exists $\varepsilon > 0$ such that

$$(B_{C^{11}(\mathcal{Q})}((0, 0, 0), \varepsilon))^3 \times (B_{C^9(\mathcal{Q})}((0, 0), \varepsilon))^2 \subset \mathcal{B}. \quad (2.28)$$

By the continuity property of Theorem 2.2.1 with $\eta_1 = \sigma_1 = \sigma_0 = 7$, there exists $\eta > 0$ such that for $\|((u, v, h), (h_1, h_2))\|_{(C^{11})^3 \times (C^9)^2} \leq \eta$,

$$\|\mathcal{D}_{(u,v,h)}^{-1}(h_1, h_2)\|_{(C^6)^3} \leq R.$$

Proposition 2.2.1 with $k = 11$ yields $\eta' > 0$ such that for any initial and final conditions

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in (B_{C^{11}([0,T] \times [0,L])}(0, \eta') \times B_{C^{10}([0,T] \times [0,L])}(0, \eta'))^4,$$

there exist two controls $h_1, h_2 \in C^9([0, T] \times [0, L])$, supported in $\mathcal{Q}_{2\delta}$ (condition (2.25)), that steer system (2.23) from the given initial conditions to the given final conditions, with the corresponding trajectory (u^*, v^*) satisfying (2.27). Together with (2.28), this implies that there exists $\eta' \geq \eta'' > 0$ such that for initial and final conditions

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in (B_{C^{11}([0, T] \times [0, L])}(0, \eta'')) \times B_{C^{10}([0, T] \times [0, L])}(0, \eta''))^4,$$

the corresponding trajectory of system (2.23) satisfies

$$\mathcal{D}(u_{|\mathcal{Q}}^*, v_{|\mathcal{Q}}^*, 0) = (h_{1|\mathcal{Q}}, h_{2|\mathcal{Q}}), \quad (2.29)$$

$$\left((u_{|\mathcal{Q}}^*, v_{|\mathcal{Q}}^*, 0), (-h_{1|\mathcal{Q}}, -h_{2|\mathcal{Q}}) \right) \in \mathcal{B}. \quad (2.30)$$

$$\|((u^*, v^*, 0), (h_1, h_2))\|_{(C^{11})^3 \times (C^9)^2} \leq \min(R, \eta). \quad (2.31)$$

Let us now set, keeping in mind the regularity property of Theorem 2.2.1 with $\eta_1 = \sigma_1 = \sigma_0 = 7$,

$$(u, v, h) = \mathcal{D}_{(u_{|\mathcal{Q}}^*, v_{|\mathcal{Q}}^*, 0)}^{-1}(-h_{1|\mathcal{Q}}, -h_{2|\mathcal{Q}}) \in \mathcal{A}^6.$$

Then, by the inversion property of Theorem 2.2.1, and (2.29),

$$\mathcal{D}(u, v, h) = \mathcal{D}(u_{|\mathcal{Q}}^*, v_{|\mathcal{Q}}^*, 0) - (h_{1|\mathcal{Q}}, h_{2|\mathcal{Q}}) = (0, 0).$$

Now, let us show that $(u, v, h) = (u^*, v^*, 0)$ on $\mathring{\mathcal{Q}} \setminus \mathcal{Q}_{\delta'}$. This will allow us to extend (u, v, h) on $([0, T] \times [0, L]) \setminus \mathcal{Q}$.

Let $(t, x) \in \mathring{\mathcal{Q}} \setminus \mathcal{Q}_{\delta}$. As the h_i are supported in $\mathcal{Q}_{2\delta}$,

$$((u^*, v^*, 0), (-h_1, -h_2)) = ((u^*, v^*, 0), (0, 0)) \quad \text{on } B\left((t, x), \frac{\delta}{2}\right) \cap \mathcal{Q}. \quad (2.32)$$

Thus, using the locality property of Theorem 2.2.1,

$$\mathcal{D}_{(u_{|\mathcal{Q}}^*, v_{|\mathcal{Q}}^*, 0)}^{-1}(-h_{1|\mathcal{Q}}, -h_{2|\mathcal{Q}})(t, x) = \mathcal{D}_{(u_{|\mathcal{Q}}^*, v_{|\mathcal{Q}}^*, 0)}^{-1}(0, 0)(t, x), \quad (2.33)$$

that is, using the normalisation property:

$$(u, v, h)(t, x) = (u^*, v^*, 0)(t, x). \quad (2.34)$$

We can now extend (u, v, h) by setting

$$(u, v, h)(t, x) = (u^*, v^*, 0)(t, x), \quad \forall (t, x) \in [0, T] \times [0, L] \setminus \mathcal{Q}. \quad (2.35)$$

Then,

$$\text{supp } h \subset [0, T] \times [a, b],$$

and (u, v) satisfies the same initial, boundary and final conditions as (u^*, v^*) :

$$\begin{cases} (u, v)(0, \cdot) = (u_0, v_0), & (u_t, v_t)(0, \cdot) = (u_1, v_1) \\ (u, v)(T, \cdot) = (u_0^f, v_0^f), & (u_t, v_t)(T, \cdot) = (u_1^f, v_1^f) \end{cases} \quad (2.36)$$

$$\begin{cases} u(\cdot, 0) = u(\cdot, L) = 0 \\ v(\cdot, 0) = v(\cdot, L) = 0 \end{cases} \quad (2.37)$$

and

$$\begin{cases} \square_{\nu_1} u = f_1(u, v) + h, \\ \square_{\nu_2} v = f_2(u, v), \end{cases} \quad (2.38)$$

Finally, we get (2.12) from (2.31) and the continuity property of Theorem 2.2.1. This proves Theorem 2.1.1.

Remark 2.2.2. *Theorem 2.1.1 actually holds for coupled quasilinear equations:*

$$\begin{cases} \partial_{tt}u - \partial_x(K_1(u, \partial_x u)) = f_1(u, v) + h, & x \in [0, L], \\ \partial_{tt}v - \partial_x(K_2(v, \partial_x v)) = f_2(u, v), & x \in [0, L], \\ u = 0 & \text{on } \{0, L\}, \\ v = 0 & \text{on } \{0, L\}, \end{cases} \quad (2.39)$$

where $f_1(0, 0) = f_2(0, 0) = 0$, $K_1, K_2 \in C^\infty(\mathbb{R}^2)$, and $K_1(0, 0) = K_2(0, 0) = 0$. One can check that when one modifies the recurrence relation in (2.5) to match the new equations, the operators can still be written using only $J_x^n(u, v)$, $J_x^{n-1}(u_t, v_t)$ and $J_t^n(u, v)$, and thus the compatibility conditions will have the same form as (2.8).

Indeed, in this case we can still use Li's results for the perturbed quasilinear system, as we consider the "perturbations" around 0. This will yield a "universal" time condition, because the propagation speeds are close to $\min(\sqrt{\partial_2 K_1(0, 0)}, \sqrt{\partial_2 K_2(0, 0)})$ for the perturbed system. On the other hand if we work around a nonzero trajectory (return method), the perturbed quasilinear system could present quite smaller propagation speeds. The final time condition would then depend on the return trajectories that are found.

Theorem 2.2.2. *Let $R > 0$, $0 \leq a < b \leq L$, $T > 0$ such that*

$$\begin{aligned} T &> 2(L - b) \max \left(\left(\sqrt{\partial_2 K_1(0, 0)} \right)^{-1}, \left(\sqrt{\partial_2 K_2(0, 0)} \right)^{-1} \right), \\ T &> 2a \max \left(\left(\sqrt{\partial_2 K_1(0, 0)} \right)^{-1}, \left(\sqrt{\partial_2 K_2(0, 0)} \right)^{-1} \right). \end{aligned} \quad (2.40)$$

If

$$\frac{\partial f_2}{\partial u}(0, 0) \neq 0, \quad (2.41)$$

then there exists $\eta > 0$ such that for initial and final conditions

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in B_{(C^{11}([0, L]) \times C^{10}([0, L]))^4}(0, \eta)$$

compatible at the order 11, there exists $h \in C^6([0, T] \times [0, L])$ such that

$$\text{supp } h \subset [0, T] \times [a, b], \quad (2.42)$$

and such that the corresponding solution $(u, v) \in C^6([0, T] \times [0, L])$ of (2.39) with initial values $((u_0, u_1), (v_0, v_1))$ satisfies

$$\begin{cases} u(T, \cdot) = u_0^f, & u_t(T, \cdot) = u_1^f, \\ v(T, \cdot) = v_0^f, & v_t(T, \cdot) = v_1^f \end{cases}$$

and inequality (2.12) holds.

2.3 Second case: an example with an uncontrollable linearized system

We now turn to system (2.2). As mentioned before, it does not satisfy condition (2.10), in other words, the linearised system around 0 is not controllable:

$$\begin{cases} \square_{\nu_1} u = h, \\ \square_{\nu_2} v = 0, \\ u|_{\partial\Omega} = 0, \\ v|_{\partial\Omega} = 0, \end{cases} \quad (2.43)$$

the control h gives us no influence on the dynamics of v .

Thus, the computations from the beginning of subsection 2.2.2 do not hold: we cannot work around the stationary trajectory 0, thus we need to find another trajectory around which to work. More precisely, keeping in mind Proposition 2.2.2, we look for a return trajectory $(\bar{u}, \bar{v}, \bar{h})$ going from 0 to 0 such that for some smooth closed set $\mathcal{Q} \subset [0, T] \times [a, b]$, we have

$$\forall (t, x) \in \mathcal{Q}, \frac{\partial f_2}{\partial u}(\bar{u}(t, x), \bar{v}(t, x)) = 3\bar{u}^2(t, x) \neq 0. \quad (2.44)$$

Additionally, \mathcal{Q} will have to satisfy some properties so that a result with two controls can be proved.

To find such a trajectory, we follow the same idea as in [60], where return trajectories are built for coupled heat equations with a cubic coupling. The additional derivative in time simply adds terms and makes for heavier computations. However, condition (2.44) will account for additional work.

We will then prove and use a more general controllability result with two controls. After that, the application of Gromov's theorem is rather straightforward.

2.3.1 A preliminary construction: elementary trajectories

In this subsection, we describe a construction of a smooth trajectory of system (2.2) that goes from 0 to 0. For now we consider condition (2.44) but without any special requirements for \mathcal{Q} .

In what follows, we suppose, without loss of generality (by scaling the space variable) that $\nu_2 = 1$.

To build trajectories that start at 0 and return there, the idea is to use the cascade structure of the equation: first we find a $C^\infty([-1, 1] \times [0, 1])$ function \bar{v} such that $\square\bar{v}$ is the third power of a $C^\infty([-1, 1] \times [0, 1])$ function \bar{u} . By setting the right conditions at the start and end times, this gives us a return trajectory. The corresponding control will then be $\square_{\nu_1}\bar{u}$.

Let us recall that $x \mapsto \sqrt[3]{x}$ is C^∞ on \mathbb{R}^* . So, by composition, the cubic root of a C^∞ function f is C^∞ at all the points where f is non-zero. At the points where f vanishes, by Taylor's formula, a fairly simple sufficient condition for $\sqrt[3]{f}$ to be C^∞ at those points is for f to vanish, along with its first and second derivatives, while its third derivative is non-zero.

Now, to find functions whose image by the wave operator is a third power of a C^∞ function, we consider the solutions to the corresponding stationary problem, namely functions whose Laplacian is the third power of a C^∞ function. The solution of this problem corresponds to the following proposition, proven (with 1/2 instead of 3/4) in [60]:

Proposition 2.3.1 (Coron, Guerrero, Rosier). *There exist $\delta', \delta'', g \in C^\infty([0, 1])$, $G \in C^\infty([0, 1])$ such that*

$$\begin{cases} g'' = G, \\ g(z) = 1 - z^2 \text{ on } [0, \delta''], \\ g(z) = e^{-\frac{1}{1-z^2}} \text{ on } [1 - \delta', 1), \\ G(z) \left(z - \frac{3}{4}\right) > 0 \text{ for } z \in (0, 1) \setminus \left\{\frac{3}{4}\right\}, \\ G(z) = \left(z - \frac{3}{4}\right)^3 \text{ on } \left[\frac{3}{4} - \frac{\delta''}{2}, \frac{3}{4} + \frac{\delta''}{2}\right], \end{cases} \quad (2.45)$$

In a sense, this proposition gives us the simplest example of functions the second derivative of which is the third power of a smooth function: $G = g''$ vanishes exponentially in 1, and has only one vanishing point on $[0, 1)$, around which it has a cubic behaviour. The idea of the construction is then to perturb this function of space and make it evolve in time, so slightly as to preserve the properties 2.45 of the stationary problem. Let $0 \leq a < b \leq L$, and $T > 0$ such that (2.9) holds.

Let $0 < \delta < \min(T/2, (b - a)/2)$ such that (2.46) holds. Set λ_0 to be a function such that

$$\begin{aligned} \lambda_0(t) &= e^{-\sqrt{\frac{1}{t(T-t)}}} \quad \forall t \in \left(0, \frac{\delta}{2}\right] \cup \left[T - \frac{\delta}{2}, T\right), \\ \lambda_0(0) &= \lambda_0(T) = 0, \\ \lambda_0(t) &> 0, \quad \forall t \in (0, T), \\ \lambda_0([\delta, T - \delta]) &= \{1\}, \end{aligned} \quad (2.46)$$

and write $\lambda := \varepsilon\lambda_0$ for some ε to be determined.

Remark 2.3.1. In [60], the authors take

$$\lambda(t) = \varepsilon t^2(1-t)^2. \quad (2.47)$$

In our case however, we will see that we need to fit a rectangle of the form $[\delta, T - \delta] \times [x_0 - \xi, x_0 + \xi]$ inside the support of \bar{u} , see Figure 2.3. With a polynomial as in (2.47), the smaller $\delta > 0$ gets, the smaller ξ has to be. This in itself would not be an obstruction to prove our controllability result, but using definition (2.46) has the advantage to fix the width of the rectangle for all δ satisfying (2.24).

Set

$$\begin{aligned} f_0(t) &= e^{-\frac{1}{\varepsilon(t-T)}}, \quad \forall t \in (0, T), \\ f_0(0) &= f_0(T) = 0, \end{aligned} \quad (2.48)$$

Finally, let g_0 be the solution to the stationary problem (see Proposition 2.3.1). Let $x_0 \in (0, L)$, and choose $\varepsilon \leq \min(x_0, L - x_0)$. We now look for \bar{v} in the form

$$\bar{v}(t, x) = \sum_{i=0}^3 f_i(t) g_i \left(\frac{|x - x_0|}{\lambda(t)} \right). \quad (2.49)$$

Note that the fact that f_0 vanishes faster than λ at 0 and T compensates the singularity that occurs in the term $|x - x_0|/\lambda(t)$ of the first term of the sum. We will see that the f_i have a similar property, thus ensuring that functions of the form above are indeed C^∞ . We also require that the g_i satisfy

$$\text{supp } g_i \subset \left[\frac{3}{4} - \frac{\delta''}{2}, \frac{3}{4} + \frac{\delta''}{2} \right], \quad \forall i \in \{1, 2, 3\}, \quad (2.50)$$

where δ'' is as defined in Proposition 2.3.1, so that

$$\text{supp } (\bar{u}, \bar{v}, \bar{h}) \subset [0, T] \times [x_0 - \varepsilon, x_0 + \varepsilon]. \quad (2.51)$$

Let us then set, in order to simplify the notations for our computations:

$$z := \frac{|x - x_0|}{\lambda(t)},$$

$$V(t, x) := \square \bar{v} = \bar{v}_{tt} - \bar{v}_{xx},$$

which we note, in the new set of variables,

$$\mathcal{V}(t, z) := V(t, \lambda(t)z).$$

We are now looking for functions f_i and g_i such that $V^{\frac{1}{3}}$ is of class C^∞ . In order to achieve this, we will work with the new set of variables (t, z) , and study \mathcal{V} . We now need to have precise knowledge of the behaviour of \mathcal{V} when it vanishes.

More precisely, the aim is to write \mathcal{V} near $\frac{3}{4}$ as:

$$\lambda^2 \mathcal{V} = \left(z - \frac{3}{4} \right)^3 \varphi(t, z), \quad \text{with } \varphi \in C^\infty \left([0, T] \times \left[\frac{3}{4} - \frac{\delta''}{2}, \frac{3}{4} + \frac{\delta''}{2} \right] \right), \quad \varphi < 0 \text{ for } t \neq 0, T.$$

Note that φ has to be negative because of the minus sign in the wave operator. Hence, we look for \mathcal{V} satisfying

$$\begin{aligned} \mathcal{V}_z \left(\cdot, \frac{3}{4} \right) &= 0, \\ \mathcal{V}_{zz} \left(\cdot, \frac{3}{4} \right) &= 0, \\ \mathcal{V}_{zzz} &\leq -C f_0 \text{ on } [0, T] \times \left[\frac{3}{4} - \frac{\delta''}{2}, \frac{3}{4} + \frac{\delta''}{2} \right], \text{ where } C > 0. \end{aligned}$$

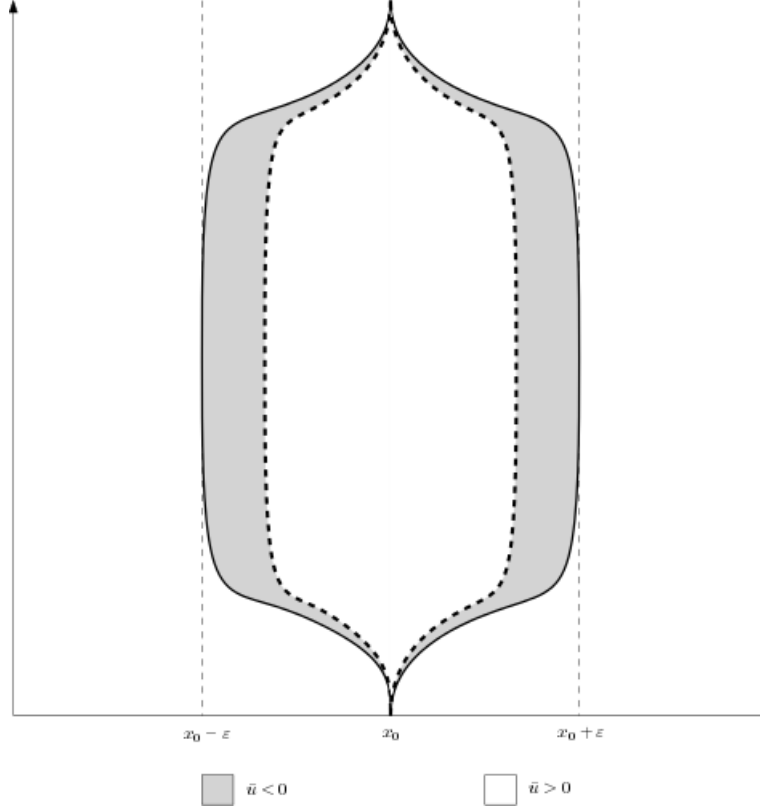


Figure 2.2: The support of the trajectory $(\bar{u}, \bar{v}, \bar{h})$. The dashed line represents the vanishing points of $\square \bar{v}$ (or, equivalently, \bar{u}).

Additionally, since we have the following condition on G :

$$G(z)(z - \frac{3}{4}) > 0 \text{ for } z \in (0, 1) \setminus \{\frac{3}{4}\},$$

we will make sure to have

$$\mathcal{V}(t, z) \left(z - \frac{3}{4} \right) < 0, \quad \forall (t, z) \in (0, T) \times \left((0, 1) \setminus \left\{ \frac{3}{4} \right\} \right).$$

Let us now compute \mathcal{V} and its first, second and third derivatives:

$$\begin{aligned} \bar{v}_{tt} &= \sum_{i=0}^3 \ddot{f}_i g_i - 2\dot{f}_i z \frac{\dot{\lambda}}{\lambda} g'_i - f_i \left(z \frac{\lambda \ddot{\lambda} - 2\dot{\lambda}^2}{\lambda^2} g'_i - z^2 \left(\frac{\dot{\lambda}}{\lambda} \right)^2 g''_i \right), \\ \bar{v}_{xx} &= \lambda^{-2} f_0 G + \sum_{i=1}^3 f_i \lambda^{-2} g''_i. \end{aligned}$$

$$\begin{aligned}
\lambda^2 \mathcal{V} &= -(1 - z^2 \dot{\lambda}^2) f_0 G + \lambda^2 \ddot{f}_0 g_0 - 2 \dot{f}_0 z \dot{\lambda} \lambda g'_0 - z (\lambda \ddot{\lambda} - 2 \dot{\lambda}^2) f_0 g'_0 + \sum_{i=1}^3 (\lambda^2 \ddot{f}_i g_i - 2 \dot{f}_i z \dot{\lambda} \lambda g'_i \\
&\quad - f_i (z (\lambda \ddot{\lambda} - 2 \dot{\lambda}^2) g'_i - z^2 \dot{\lambda}^2 g''_i) - f_i g''_i) \\
&= -(1 - z^2 \dot{\lambda}^2) f_0 G + \lambda^2 \ddot{f}_0 g_0 - 2 \dot{f}_0 z \dot{\lambda} \lambda g'_0 - z (\lambda \ddot{\lambda} - 2 \dot{\lambda}^2) f_0 g'_0 + \sum_{i=1}^3 (\lambda^2 \ddot{f}_i g_i - 2 \dot{f}_i z \dot{\lambda} \lambda g'_i \\
&\quad - f_i [z (\lambda \ddot{\lambda} - 2 \dot{\lambda}^2) g'_i + (1 - z^2 \dot{\lambda}^2) g''_i]).
\end{aligned}$$

Now, for ε small enough (note that this depends on the value of δ),

$$1 - (\varepsilon z \dot{\lambda}_0(t))^2 > \frac{1}{2}, \quad \forall (t, z) \in [0, T] \times [0, 1] \quad (2.52)$$

and, using the notation λ ,

$$\left\| \frac{1}{1 - \left(\frac{3}{4} \dot{\lambda}\right)^2} \right\|_{C^2([0, T])} \leq 10. \quad (2.53)$$

Now, if we impose

$$\begin{cases} g_i^{(j)} \left(\frac{3}{4}\right) = 0 \quad \forall i \in \{1, 2, 3\}, j \in \{0, 1, 2\} \quad (i, j) \neq (1, 2) \\ g_1^{(2)} \left(\frac{3}{4}\right) = 1 \end{cases} \quad (2.54)$$

and if we define f_1 by

$$f_1 := \frac{1}{1 - \left(\frac{3}{4} \dot{\lambda}\right)^2} \left(\lambda^2 g_0 \left(\frac{3}{4}\right) \ddot{f}_0 - 2 \frac{3}{4} \dot{\lambda} \lambda g'_0 \left(\frac{3}{4}\right) \dot{f}_0 - \frac{3}{4} (\lambda \ddot{\lambda} - 2 \dot{\lambda}^2) g'_0 \left(\frac{3}{4}\right) f_0 \right), \quad (2.55)$$

we get:

$$\lambda^2 \mathcal{V} \left(\cdot, \frac{3}{4} \right) = 0.$$

We now compute the first derivative of \mathcal{V} :

$$\begin{aligned}
\lambda^2 \mathcal{V}_z &= -(1 - z^2 \dot{\lambda}^2) f_0 G' + (2z \dot{\lambda}^2 - z (\lambda \ddot{\lambda} - 2 \dot{\lambda}^2)) f_0 G - 2z \dot{\lambda} \lambda \dot{f}_0 G + \lambda^2 \ddot{f}_0 g'_0 - 2 \dot{f}_0 \dot{\lambda} \lambda g'_0 \\
&\quad - (\lambda \ddot{\lambda} - 2 \dot{\lambda}^2) f_0 g'_0 + \sum_{i=1}^3 \lambda^2 \ddot{f}_i g'_i - 2 \dot{f}_i z \dot{\lambda} \lambda g''_i - 2 \dot{f}_i \dot{\lambda} \lambda g'_i - f_i [(\lambda \ddot{\lambda} - 2 \dot{\lambda}^2) g'_i - 2z \dot{\lambda}^2 g''_i] \\
&\quad - f_i [z (\lambda \ddot{\lambda} - 2 \dot{\lambda}^2) g''_i + (1 - z^2 \dot{\lambda}^2) g_i^{(3)}] \\
&= -(1 - z^2 \dot{\lambda}^2) f_0 G' + (4z \dot{\lambda}^2 + z \lambda \ddot{\lambda}) f_0 G - 2z \dot{\lambda} \lambda \dot{f}_0 G + \lambda^2 \ddot{f}_0 g'_0 - 2 \dot{f}_0 \dot{\lambda} \lambda g'_0 - (\lambda \ddot{\lambda} - 2 \dot{\lambda}^2) f_0 g'_0 \\
&\quad + \sum_{i=1}^3 \lambda^2 \ddot{f}_i g'_i - 2 \dot{f}_i \dot{\lambda} \lambda (z g''_i + g'_i) - f_i [(\lambda \ddot{\lambda} - 2 \dot{\lambda}^2) g'_i + z (\lambda \ddot{\lambda} - 4 \dot{\lambda}^2) g''_i + (1 - z^2 \dot{\lambda}^2) g_i^{(3)}].
\end{aligned}$$

Again, we impose

$$g_i^{(3)} \left(\frac{3}{4}\right) = \begin{cases} 0 & \text{if } i \in \{1, 3\}, \\ 1 & \text{if } i = 2, \end{cases} \quad (2.56)$$

and we set

$$\begin{aligned}
f_2 &:= \frac{1}{1 - \left(\frac{3}{4} \dot{\lambda}\right)^2} \left[\lambda^2 g'_0 \left(\frac{3}{4}\right) \ddot{f}_0 - \dot{\lambda} \lambda g'_0 \left(\frac{3}{4}\right) \dot{f}_0 - 2 \dot{\lambda} \lambda g_0 \left(\frac{3}{4}\right) \dot{f}_0 \right. \\
&\quad \left. - (\lambda \ddot{\lambda} - 2 \dot{\lambda}^2) g'_0 \left(\frac{3}{4}\right) f_0 - 2 \frac{3}{4} \dot{\lambda} \lambda f_1 - \frac{3}{4} (\lambda \ddot{\lambda} - 4 \dot{\lambda}^2) f_1 \right] \quad (2.57)
\end{aligned}$$

so that

$$\lambda^2 \mathcal{V}_z \left(\cdot, \frac{3}{4} \right) = 0.$$

Finally,

$$\begin{aligned} \lambda^2 \mathcal{V}_{zz} &= -(1 - z^2 \dot{\lambda}^2) f_0 G'' + (6z \dot{\lambda}^2 + z \lambda \ddot{\lambda}) f_0 G' - 2z \dot{\lambda} \lambda \dot{f}_0 G' + 6 \dot{\lambda}^2 f_0 G - 4 \dot{\lambda} \lambda \dot{f}_0 G + \lambda^2 \ddot{f}_0 G \\ &\quad + \sum_{i=1}^3 \lambda^2 \ddot{f}_i g_i'' - 2 \dot{f}_i \dot{\lambda} \lambda (2g_i'' + z g_i^{(3)}) - f_i \left[(2\lambda \ddot{\lambda} - 6 \dot{\lambda}^2) g_i'' + z(\lambda \ddot{\lambda} - 6 \dot{\lambda}^2) g_i^{(3)} + (1 - z^2 \dot{\lambda}^2) g_i^{(4)} \right]. \end{aligned}$$

Again we impose

$$g_i^{(4)} \left(\frac{3}{4} \right) = \begin{cases} 0 & \text{if } i \in \{1, 2\}, \\ 1 & \text{if } i = 3, \end{cases} \quad (2.58)$$

then, by setting:

$$f_3 = \frac{1}{1 - (\frac{3}{4} \dot{\lambda})^2} \left[-(2\lambda \dot{\lambda} + 2 \dot{\lambda}^2) f_1 - 4\lambda \dot{\lambda} \dot{f}_1 + \lambda^2 \ddot{f}_1 - \frac{3}{4} (\lambda \ddot{\lambda} - 6 \dot{\lambda}^2) f_2 - 2 \frac{3}{4} \dot{\lambda} \lambda \dot{f}_2 \right] \quad (2.59)$$

we get:

$$\lambda^2 \mathcal{V}_{zz} \left(\cdot, \frac{3}{4} \right) = 0.$$

Now all that remains is to estimate the third derivative: on $[0, T] \times \left[\frac{3}{4} - \frac{\delta''}{2}, \frac{3}{4} + \frac{\delta''}{2} \right]$, by definition of G , we have

$$\lambda^2 \mathcal{V}_{zzz} = -6K_{\frac{3}{4}} (1 - z^2 \dot{\lambda}^2) f_0 + \mathcal{R}_0 + \mathcal{R}, \quad (2.60)$$

with:

$$\mathcal{R}_0 := z(8 \dot{\lambda}^2 + \ddot{\lambda} \lambda) f_0 G'' - 2z \dot{\lambda} \lambda \dot{f}_0 G'' + (12 \dot{\lambda}^2 + \lambda \ddot{\lambda}) f_0 G' - 6 \dot{\lambda} \lambda \dot{f}_0 G' + \lambda^2 \ddot{f}_0 G', \quad (2.61)$$

and

$$\mathcal{R} := \sum_{i=1}^3 \lambda^2 \ddot{f}_i g_i^{(3)} - 2 \dot{f}_i \dot{\lambda} \lambda (3g_i^{(3)} + z g_i^{(4)}) - f_i \left[(3 \ddot{\lambda} \lambda - 12 \dot{\lambda}^2) g_i^{(3)} + (z \lambda \ddot{\lambda} - 8z \dot{\lambda}^2) g_i^{(4)} + (1 - z^2 \dot{\lambda}^2) g_i^{(5)} \right]. \quad (2.62)$$

Let us note that (2.48), combined with the properties of exponential functions, yields

$$\left(\frac{d}{dt} \right)^n f_0 = F_n(t) f_0(t), \quad \forall n \in \mathbb{N}, \quad (2.63)$$

where the F_n are rational fractions, the poles of which are 0 and T . Now, one can see in (2.61), (2.55), (2.57) and (2.59) that the divergent behaviour of these fractions near 0 and T is always compensated by the exponential behaviour of λ and its derivatives. Furthermore, differentiating the f_i does not change this fact. Hence, keeping (2.53) in mind:

$$\begin{aligned} \mathcal{R}_0 &= \varepsilon^2 \mathcal{O}(f; t, z), \\ f_1^{(n)} &= \varepsilon^2 \mathcal{O}(f_0; t), \quad \forall n \in \mathbb{N}, \\ f_2^{(n)} &= \varepsilon^2 \mathcal{O}(f_0; t), \quad \forall n \in \mathbb{N}, \\ f_3^{(n)} &= \varepsilon^4 \mathcal{O}(f_0; t), \quad \forall n \in \mathbb{N}, \end{aligned} \quad (2.64)$$

where the notation $\mathcal{O}(f; t)$ (resp. $\mathcal{O}(f; t, z)$) means f times a bounded function of time on $[0, T]$ (resp. time and space). Hence, near $\frac{3}{4}A$, we have

$$\mathcal{R}_0 + \mathcal{R} = \varepsilon^2 \mathcal{O}(f_0; t, z), \quad (2.65)$$

the dominant term being $\ddot{f}_1(1-z^2\dot{\lambda}^2)g_i^{(5)}$. Consequently, using (2.60) and (2.65), for a small enough ε , there exists a constant $C > 0$ such that:

$$\lambda^2 \mathcal{V}_{zzz} \leq -Cf_0 \text{ on } \left[\frac{3}{4} - \frac{\delta''}{2}, \frac{3}{4} + \frac{\delta''}{2} \right]. \quad (2.66)$$

Thus, on $[0, T] \times \left[\frac{3}{4} - \frac{\delta''}{2}, \frac{3}{4} + \frac{\delta''}{2} \right]$, we can write, thanks to the Taylor-Laplace formula:

$$\lambda^2 \mathcal{V} = \left(z - \frac{3}{4} \right)^3 \varphi(t, z), \text{ with } \varphi \in C^\infty \left([0, T] \times \left[\frac{3}{4} - \frac{\delta''}{2}, \frac{3}{4} + \frac{\delta''}{2} \right] \right), \varphi < 0 \text{ for } t \neq 0, T.$$

Additionally, by definition of f_0 , φ/λ^2 vanishes exponentially for $t = 0, T$, and (2.50) ensures that φ vanishes exponentially for $z = 1$, so that

$$\left(\frac{\varphi}{\lambda^2} \right)^{\frac{1}{3}} \in C^\infty \left([0, T] \times \left[\frac{3}{4} - \frac{\delta''}{2}, \frac{3}{4} + \frac{\delta''}{2} \right] \right).$$

We now have

$$\mathcal{V}^{\frac{1}{3}} \in C^\infty \left([0, T] \times \left[\frac{3}{4} - \frac{\delta''}{2}, \frac{3}{4} + \frac{\delta''}{2} \right] \right) \quad (2.67)$$

Moreover, on $[0, T] \times \left(\left[0, \frac{3}{4} - \frac{\delta''}{2} \right] \cup \left[\frac{3}{4} - \frac{\delta''}{2}, 1 \right] \right)$, thanks to the constraint on the supports of the g_i , we have:

$$\lambda^2 \mathcal{V} = -(1-z^2\dot{\lambda}^2)f_0G + \underbrace{\lambda^2 \ddot{f}_0 g_0 - 2\dot{f}_0 z \dot{\lambda} \lambda g'_0 - z(\lambda \ddot{\lambda} - 2\dot{\lambda}^2) f_0 g'_0}_{\varepsilon^2 \mathcal{O}(f_0; t, x)}. \quad (2.68)$$

As, thanks to Proposition 2.3.1, we have

$$|G| > 2 \text{ on } \left[0, \frac{3}{4} - \frac{\delta''}{2} \right] \cup \left[\frac{3}{4} + \frac{\delta''}{2}, 1 - \delta' \right],$$

for small enough ε , we have:

$$|\lambda^2 \mathcal{V}| > 0 \text{ on }]0, T[\times \left(\left[0, \frac{3}{4} - \frac{\delta''}{2} \right] \cup \left[\frac{3}{4} + \frac{\delta''}{2}, 1 - \delta' \right] \right).$$

Now, let us recall that, on $[1 - \delta', 1)$,

$$\begin{aligned} g_0(z) &= e^{-\frac{1}{1-z^2}}, \\ g'_0(z) &= \frac{-2z}{(1-z^2)^2} e^{-\frac{1}{1-z^2}}, \\ G(z) = g''_0(z) &= \frac{6z^4 - 2}{(1-z^2)^4} e^{-\frac{1}{1-z^2}}, \end{aligned}$$

So that g_0/G and g'_0/G are bounded near 1, allowing us to write

$$\lambda^2 \mathcal{V} = -f_0 G + \varepsilon^2 \mathcal{O}(f_0; t) \mathcal{O}_{1-}(G; z). \quad (2.69)$$

The notation $\mathcal{O}_{1-}(G; z)$ meaning G times a bounded function of space on $[1 - \delta', 1]$. So for small enough ε , there exists a function a with positive values on $]0, T[$, such that

$$\lambda^2 \mathcal{V}(t, z) \leq -a(t)G(z) < 0, \quad \forall (t, z) \in (-0, T) \times [1 - \delta', 1).$$

Finally, for all $t \in [0, T]$, $\mathcal{V}(t, \cdot)$ vanishes exponentially at $z = 1$, and for all $z \in [1 - \delta', 1]$, $\mathcal{V}(\cdot, z)$ vanishes exponentially for $t = 0, T$. Hence,

$$\mathcal{V}^{\frac{1}{3}} \in C^\infty \left([0, T] \times \left(\left[0, \frac{3}{4} - \frac{\delta''}{2} \right] \cup \left[\frac{3}{4} + \frac{\delta''}{2}, 1 \right] \right) \right). \quad (2.70)$$

This, together with (2.67), proves that

$$\mathcal{V}^{\frac{1}{3}} \in C^\infty ([0, T] \times [0, 1]). \quad (2.71)$$

Now, as $x \mapsto |x|$ is C^∞ on $\mathbb{R} \setminus \{0\}$, by composition we deduce from (2.71) that

$$V^{\frac{1}{3}} \in C^\infty ([0, T] \times ((0, L) \setminus \{x_0\})).$$

To deal with the missing point x_0 , let us recall that for all $t \in]-1, 1[$, for all $x \in [0, L]$ such that $|x - x_0| \leq \delta'' \lambda(t)$ (i.e. $z \leq \delta''$),

$$\begin{aligned} \lambda^2 \mathcal{V}(t, z) &= -f_0 G + \lambda^2 \ddot{f}_0 g_0 - 2\dot{f}_0 z \dot{\lambda} \lambda g'_0 - z \left(\lambda \ddot{\lambda} - 2\dot{\lambda}^2 \right) f_0 g'_0 + z^2 \dot{\lambda}^2 f_0 G \\ &= 2f_0 + \ddot{f}_0 (\lambda^2 - |x - x_0|^2) + 4\dot{f}_0 \frac{\dot{\lambda}}{\lambda} |x - x_0|^2 + 2 \left(\frac{\ddot{\lambda}}{\lambda} - 2 \left(\frac{\dot{\lambda}}{\lambda} \right)^2 \right) f_0 |x - x_0|^2 - 2 \left(\frac{\dot{\lambda}}{\lambda} \right)^2 f_0 |x - x_0|^2 \\ &= 2f_0 + \lambda^2 \ddot{f}_0 + \psi(t) |x - x_0|^2, \end{aligned} \quad (2.72)$$

where $\psi \in C^\infty([0, T])$, and ψ vanishes exponentially for $t = 0, T$, along with all its derivatives. We now see that the terms in $|x - x_0|$ of V are actually in $|x - x_0|^2$, which compensates the singularity at 0 of the map $x \mapsto |x|$. Thus, from the smoothness of $\mathcal{V}^{\frac{1}{3}}$ we get, by composition, $V^{\frac{1}{3}} \in C^\infty([0, T] \times [0, L])$. Thus we have proved that, by choosing g_i that verify (2.50), (2.54), (2.56) and (2.58), we get

$$V^{\frac{1}{3}} \in C^\infty([0, T] \times [0, L]).$$

Finally, we set

$$\begin{aligned} \bar{v}(x, t) &:= \sum_{i=0}^3 f_i(t) g_i \left(\frac{|x - x_0|}{\lambda(t)} \right), \\ \bar{u} &:= (\square \bar{v})^{\frac{1}{3}}, \\ \bar{h} &:= \square \bar{u}. \end{aligned}$$

where λ is defined by (2.46), the g_i are some functions satisfying (2.50), (2.54), (2.56), and (2.58), and the f_i are defined by (2.48), (2.55), (2.57), and (2.59).

Let us check that we have indeed built a return trajectory: for $i \in \{0, \dots, 3\}$, the f_i vanish at -1 and 1 , along with all their derivatives. Hence,

$$\begin{aligned} \bar{u}(-1, \cdot) &= \bar{v}(-1, \cdot) = \bar{u}_t(-1, \cdot) = \bar{v}_t(-1, \cdot) = 0, \\ \bar{u}(1, \cdot) &= \bar{v}(1, \cdot) = \bar{u}_t(1, \cdot) = \bar{v}_t(1, \cdot) = 0. \end{aligned}$$

□

Remark 2.3.2. *Most of the work in the construction above comes from the vanishing points $(t, (3/4)\lambda(t))$ “in the middle of the domain”. So one could wonder, would it not be simpler to try and build a function that only vanishes, along with all its derivatives, at the points $(t, \lambda(t))$?*

Let us remind that our strategy to build the return trajectory is to start from a solution to the stationary problem, and then make it evolve through time so as to stay “not too far away from it”. But the reason we have vanishing points “in the middle of the domain” has to do with that same stationary problem. More precisely, the stationary problem consists in finding functions that vanish, along with their derivatives, on the boundary of the domain. In our case this condition corresponds to

$$g(z) = e^{-\frac{1}{1-z^2}} \text{ on } [1 - \delta', 1]. \quad (2.73)$$

We further require that the Laplacians of these functions be third powers of C^∞ functions. In our case this condition becomes

$$G(z) \left(z - \frac{3}{4} \right) > 0,$$

$$G(z) = \left(z - \frac{3}{4} \right)^3 \text{ on } \left[\frac{3}{4} - \frac{\delta''}{2}, \frac{3}{4} + \frac{\delta''}{2} \right].$$

Now, we could instead demand that G be non-negative (or non-positive). But then, by convexity arguments (or Hopf's maximum principle), we would get

$$g'(1) < 0,$$

Which contradicts condition (2.73). But that condition is very helpful in proving the smoothness of $\mathcal{V}^{\frac{1}{3}}$ near the boundary. Giving it up would mean setting more conditions on the g_i functions near the boundary, so we would have to give up condition (2.50), and then set additional conditions on the g_i to make sure V is well defined (as $\lambda(0) = \lambda(T) = 0$), preserve the sign of \mathcal{V} or more generally its smoothness, in particular near the boundary... Which would probably be more trouble than what we had to do at the vanishing points $(t, (3/4)\lambda(t))$.

2.3.2 Covering sets and return trajectories

As mentioned at the beginning of this section, we want to work on a smooth subset of $[0, T] \times [a, b]$ where $u \neq 0$. However, to do so we need more than the elementary trajectory described above: rather, we use the elementary trajectory as a building block for our final return trajectory. Indeed, let $0 < \delta < \min((b-a)/4, T/2)$ such that (2.24) is satisfied. The preliminary construction gives us a real number $\varepsilon > 0$ (after the right rescaling of the space variable) and, for any $x_0 \in [a + \delta + \varepsilon, b - \delta - \varepsilon]$, a trajectory $(\bar{u}, \bar{v}, \bar{h})$ such that $\bar{u} \neq 0$ on $\Lambda_{\varepsilon, x_0} := \{(t, x) \mid |x - x_0| < (3/4)\varepsilon\lambda_0(t)\}$, which contains any rectangle of the form $[\delta, T - \delta] \times [x_0 - \xi, x_0 + \xi]$ with $\xi < (3/4)\varepsilon$. Moreover, each of these rectangles can be fit into the interior of a smooth closed subset of $\Lambda_{\varepsilon, x_0}$.

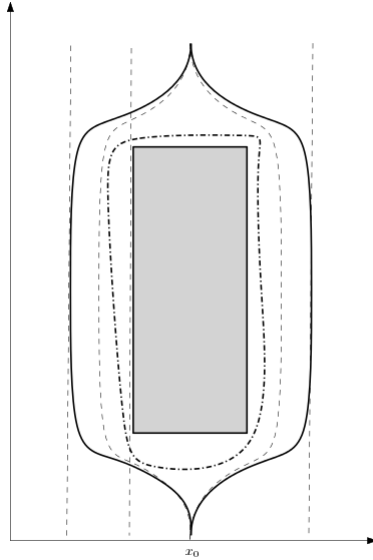


Figure 2.3: The support of the preliminary construction with a rectangle fit inside the line of vanishing points of \bar{u} .

Now there are cases (if $[a, b]$ is too long and ε - and consequently, ξ - too small), where none of the rectangles $[\delta, T - \delta] \times [x_0 - \xi, x_0 + \xi]$ satisfies the Geometric Control Condition (GCC). Thus we

cannot apply Proposition 2.2.1 with controls supported in some $[\delta, T - \delta] \times [x_0 - \xi, x_0 + \xi]$, as time condition (2.94) does not hold in these cases. So we need to build a return trajectory $(\bar{u}, \bar{v}, \bar{h})$ such that $\bar{u} \neq 0$ on a smooth closed set \mathcal{Q} containing a set \mathcal{Q}_δ that satisfies the GCC.

Now there is a simple type of set that would fit our needs for \mathcal{Q}_δ : in Section 2 we worked in $[\delta, T - \delta] \times [a + \delta, b + \delta]$, but we do not need the whole rectangle in general for the GCC to be satisfied. We can in fact work with a number of much smaller rectangles, as long as they are close enough to each other:

Definition 2.3.1. *Let $0 < \delta < \min((b - a)/4, T/2)$, such that (2.24) is satisfied. A δ -covering set of $[0, T] \times [a, b]$ for system (2.1) is a union of rectangles of the form $[\delta, T - \delta] \times [a_i, b_i]$, $1 \leq i \leq N$ for some $N \geq 1$, such that*

$$\begin{aligned} a_1 &= a + \delta, \\ b_N &= b - \delta, \\ 0 &< (a_{i+1} - b_i) \max\left(\frac{1}{|\nu_1|}, \frac{1}{|\nu_2|}\right) < T - 2\delta, \quad 1 \leq i \leq N - 1. \end{aligned} \tag{2.74}$$

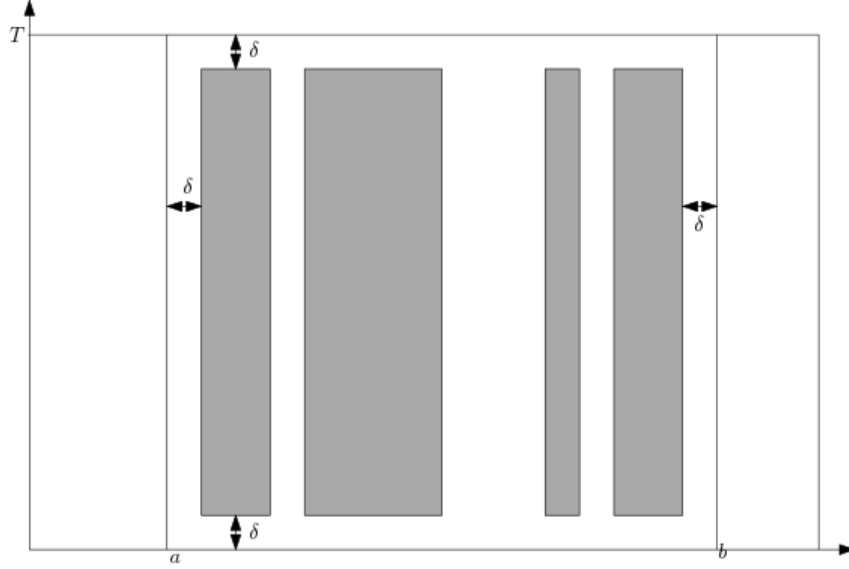


Figure 2.4: An example of a δ -covering set.

Now the idea is to add the elementary trajectories obtained by the preliminary construction on disjoint supports centered in $x_i \in [a + \delta + \varepsilon, b - \delta - \varepsilon]$, that are close enough, and with a small enough ε so that the rectangles $[\delta, T - \delta] \times [x_0 - \varepsilon/2, x_0 + \varepsilon/2]$ form a δ -covering set. Take $\varepsilon_0 \leq (b - a - 2\delta)/2$ small enough for the preliminary construction to work, and such that $\varepsilon_0 \max(1/|\nu_1|, 1/|\nu_2|) < T - 2\delta$. We then define the following sequence: take $N \in \mathbb{N}$ large enough so that

$$\varepsilon := \frac{b - a - 2\delta}{2N - 1} \leq \varepsilon_0$$

and define, for $1 \leq i \leq N$,

$$x_i := a + \delta + \left(2i - \frac{3}{2}\right) \varepsilon,$$

and $(\bar{u}_i, \bar{v}_i, \bar{h}_i)$ the trajectory obtained by the preliminary construction corresponding to the chosen ε , centered in x_i . Let \mathcal{Q}_i be a smooth closed subset of $\Lambda_{\varepsilon, x_i}$ containing $[\delta, T - \delta] \times [x_i - \varepsilon/2, x_i + \varepsilon/2]$ in its interior. Then,

$$\mathcal{Q}_\delta := \bigcup_{i=1}^N [\delta, T - \delta] \times [x_i - \varepsilon/2, x_i + \varepsilon/2]$$

is a δ -covering set,

$$\mathcal{Q} := \left(\bigcup_i \mathcal{Q}_i \right)$$

is a smooth closed set such that $\mathcal{Q}_\delta \subset \overset{\circ}{\mathcal{Q}}$, and we can define

$$(\bar{u}, \bar{v}, \bar{h}) := \sum_{i=1}^N (\bar{u}_i, \bar{v}_i, \bar{h}_i), \quad (2.75)$$

which is supported in $[0, T] \times [a, b]$, and satisfies (2.44).

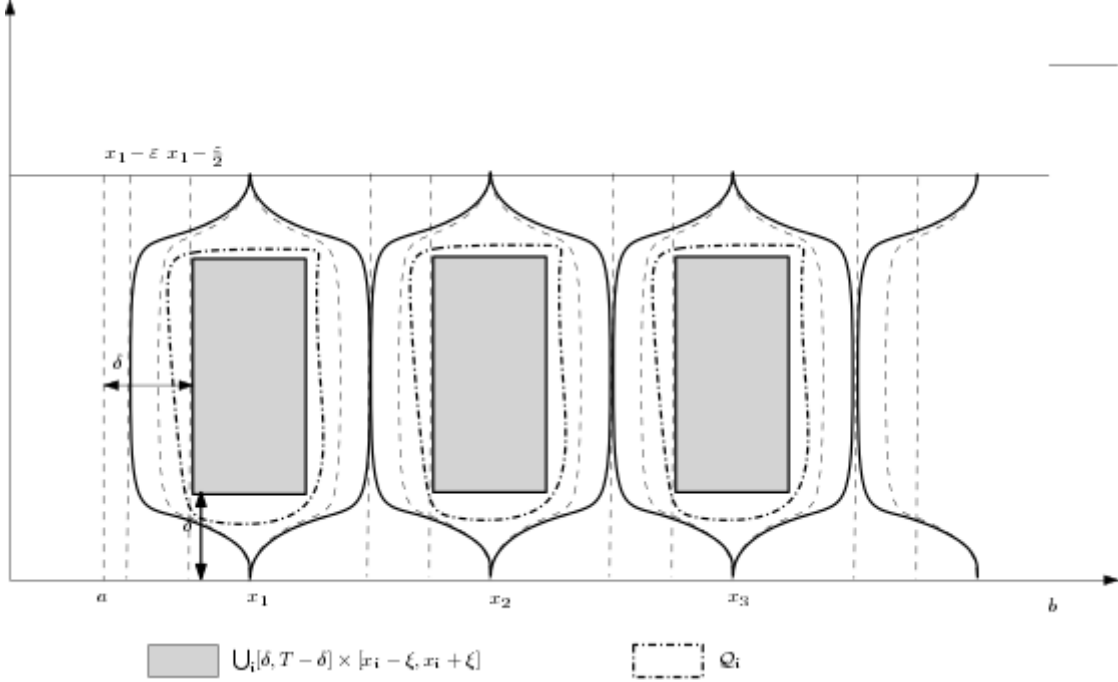


Figure 2.5: Putting elementary trajectories side by side. The rectangles form a covering set.

2.3.3 Local controllability with two controls and Gromov inversion

We now have our return trajectory $(\bar{u}, \bar{v}, \bar{h})$. Now let $R > 0$, and notice that for all $\kappa > 0$, $(\kappa\bar{u}, \kappa^3\bar{v}, \kappa\bar{h})$ is also a return trajectory, with the same support. Thus, we can now suppose without loss of generality, that

$$\|(\bar{u}, \bar{v}, \bar{h})\|_{(C^{11})^3} \leq \frac{R}{2}. \quad (2.76)$$

Let $u, v \in C^k([0, T] \times [0, L])$, $h_1, h_2 \in C^{k-2}([0, T] \times [0, L])$. Let us consider the trajectory $(\bar{u} + u, \bar{v} + v)$, controlled by $(\bar{h} + h_1, h_2)$, we get the following control system for u and v :

$$\begin{cases} \square_{\nu_1} u = h_1, \\ \square_{\nu_2} v = u^3 + 3\bar{u}u^2 + 3\bar{u}^2u + h_2, \\ u(\cdot, 0) = 0, \\ u(\cdot, L) = 0, \\ v(\cdot, 0) = 0, \\ v(\cdot, L) = 0. \end{cases} \quad (2.77)$$

This is a coupled semilinear system with a source term, and falls in the category of systems (2.23). The aim of this section is to prove the following proposition:

Proposition 2.3.2. *Let $k \geq 2$, $0 \leq a < b \leq L$, $T > 0$ such that*

$$T > 2(L - b) \max\left(\frac{1}{\nu_1}, \frac{1}{\nu_2}\right), \quad T > 2a \max\left(\frac{1}{\nu_1}, \frac{1}{\nu_2}\right).$$

For every $0 < \delta < \min(T/2, (b - a)/2)$ satisfying (2.24), for every δ -covering set \mathcal{Q}_δ of $[0, T] \times [a, b]$, there exists $\eta > 0$ and constants $C_1, C_2 > 0$ depending on T, δ, k such that, for initial and final values

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in (B_{C^k([0, T] \times [0, L])}(0, \eta) \times B_{C^{k-1}([0, T] \times [0, L])}(0, \eta))^4$$

satisfying (2.8) at the order k , there exist controls $h_1, h_2 \in C^{k-1}([0, T] \times [0, L])$ satisfying

$$\text{supp } h_i \subset \mathcal{Q}_\delta, \quad i = 1, 2, \quad (2.78)$$

$$\|h_i\|_{C^{k-2}} \leq C_1 \|((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f))\|_{(C^k \times C^{k-1})^4}, \quad i = 1, 2, \quad (2.79)$$

such that the corresponding solution of (2.23) with initial values $((u_0, u_1), (v_0, v_1))$ satisfies

$$\begin{cases} u(T, \cdot) = u_0^f, & u_t(T, \cdot) = u_1^f, \\ v(T, \cdot) = v_0^f, & v_t(T, \cdot) = v_1^f. \end{cases} \quad (2.80)$$

$$\|(u, v)\|_{(C^k)^2} \leq C_2 \|((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f))\|_{(C^k \times C^{k-1})^4}. \quad (2.81)$$

Remark 2.3.3. *It is clear, by Definition 2.3.1, that for any $0 < \delta < \min((b - a)/4, T/2)$ such that (2.24) is satisfied, $[\delta, T - \delta] \times [a + \delta, b - \delta]$ is a δ -covering set of $[0, T] \times [a, b]$. Thus Proposition 2.3.2 implies Proposition 2.2.1.*

To prove this proposition, we use the following propositions, which are particular cases of more general quasilinear results proved in [112] (see also [113, chapter 5, sections 5.3 and 5.4]):

Proposition 2.3.3 (two-sided control). *Let $k \geq 2$, $L > 0$, $T > 0$, $F \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$, $\nu_1, \nu_2 > 0$. If*

$$T > L \max\left(\frac{1}{\nu_1}, \frac{1}{\nu_2}\right),$$

then there exists $\eta > 0$ and a constant $C > 0$ depending on T, k , such that for any initial and final values

$$(U_0, U_1, U_0^f, U_1^f) \in B_{(C^k([0, L])^2 \times C^{k-1}([0, L])^2)^2}(0, \eta)$$

there exist controls H_1 and $H_2 \in C^k([0, T], \mathbb{R}^2)$ satisfying compatibility conditions

$$\begin{cases} P_{n,i}^{f_i}(J_x^n(U_0)(0), J_x^{n-1}(U_1)(0), (0, \dots, 0)) = \partial_t^n H_{1i}(0), \\ P_{n,i}^{f_i}(J_x^n(U_0)(L), J_x^{n-1}(U_1)(L), (0, \dots, 0)) = \partial_t^n H_{2i}(0), \\ P_{n,i}^{f_i}(J_x^n(U_0^f)(0), J_x^{n-1}(U_1^f)(0), (0, \dots, 0)) = \partial_t^n H_{1i}(T), \\ P_{n,i}^{f_i}(J_x^n(U_0^f)(L), J_x^{n-1}(U_1^f)(L), (0, \dots, 0)) = \partial_t^n H_{2i}(T), \end{cases} \quad \forall n \leq k, i = 1, 2. \quad (2.82)$$

such that the solution to the vector system

$$\begin{cases} \partial_{tt} U - \begin{pmatrix} \nu_1^2 & 0 \\ 0 & \nu_2^2 \end{pmatrix} \partial_{xx} U = F(U), \quad x \in (0, L) \\ U(t, 0) = H_1, \\ U(t, L) = H_2, \\ U(0) = U_0, \\ U_t(0) = U_1, \end{cases} \quad (2.83)$$

satisfies

$$\begin{cases} U(T) = U_0^f, \\ U_t(T) = U_1^f, \end{cases}$$

$$\|U\|_{C^k} \leq C\|(U_0, U_1), (U_0^f, U_1^f)\|_{(C^k \times C^{k-1})^2}. \quad (2.84)$$

Proposition 2.3.4 (one-sided control). *Let $k \geq 2$, $L > 0$, $T > 0$, $F \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$, $\nu_1, \nu_2 > 0$. If*

$$T > 2L \max\left(\frac{1}{\nu_1}, \frac{1}{\nu_2}\right),$$

then there exists $\eta > 0$ and a constant $C > 0$ depending on T, k , such that for any initial and final values

$$(U_0, U_1, U_0^f, U_1^f) \in B_{(C^k([0,L])^2 \times C^{k-1}([0,L])^2)^2}(0, \eta)$$

there exists a control $H \in C^k([0, T], \mathbb{R}^2)$ satisfying compatibility conditions

$$\begin{cases} P_{n,i}^{f_i}(J_x^n(U_0)(0), J_x^{n-1}(U_1)(0), (0, \dots, 0)) = \partial_t^n H_i(0) \quad (\text{resp. } 0), \\ P_{n,i}^{f_i}(J_x^n(U_0)(L), J_x^{n-1}(U_1)(L), (0, \dots, 0)) = 0 \quad (\text{resp. } \partial_t^n H_i(0)), \\ P_{n,i}^{f_i}(J_x^n(U_0^f)(0), J_x^{n-1}(U_1^f)(0), (0, \dots, 0)) = \partial_t^n H_i(T) \quad (\text{resp. } 0), \\ P_{n,i}^{f_i}(J_x^n(U_0^f)(L), J_x^{n-1}(U_1^f)(L), (0, \dots, 0)) = 0 \quad (\text{resp. } \partial_t^n H_i(T)), \end{cases} \quad \forall n \leq k, i = 1, 2. \quad (2.85)$$

such that the solution to the vector system

$$\begin{cases} \partial_{tt}U - \begin{pmatrix} \nu_1^2 & 0 \\ 0 & \nu_2^2 \end{pmatrix} \partial_{xx}U = F(U), \quad x \in (0, L), \\ U(t, L) = 0 \quad (\text{resp. } U(t, L) = H(t)), \\ U(t, 0) = H, \quad (\text{resp. } U(t, 0) = 0), \\ U(0) = U_0, \\ U_t(0) = U_1, \end{cases} \quad (2.86)$$

satisfies

$$\begin{cases} U(T) = U_0^f, \\ U_t(T) = U_1^f, \end{cases}$$

$$\|U\|_{C^k} \leq C\|(U_0, U_1), (U_0^f, U_1^f)\|_{(C^k \times C^{k-1})^2}. \quad (2.87)$$

Proof of Proposition 2.3.2. Let us note

$$\mathcal{Q}_\delta = \bigcup_{1 \leq i \leq N} [\delta, T - \delta] \times [a_i, b_i],$$

for some $N \geq 1$. for every $1 \leq i \leq N - 1$, let $0 < \delta_i < \min((b_{i+1} - a_{i+1})/2, (b_i - a_i)/2)$ such that

$$T - 2\delta_i > (a_{i+1} - b_i + 4\delta_i) \max\left(\frac{1}{\nu_1}, \frac{1}{\nu_2}\right). \quad (2.88)$$

Thanks to Propositions 2.3.3 and 2.3.4, Definition 2.3.1 and conditions (2.24) and (2.88), there exists $\eta > 0$ such that for initial and final values

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in (B_{C^k([0,L])}(0, \eta) \times B_{C^{k-1}([0,L])}(0, \eta))^4$$

satisfying (2.8),

- There exist boundary controls $u_1^{(i)}, u_2^{(i)} \in C^k([0, T - 2\delta])$ at $b_i - \delta_i$ and $a_{i+1} + \delta_i$ that steer (u, v) on $[b_i - \delta_i, a_{i+1} + \delta_i]$ from $(y_0, y_1)|_{[b_i - \delta_i, a_{i+1} + \delta_i]}$ to $(z_0, z_1)|_{[b_i - \delta_i, a_{i+1} + \delta_i]}$.
- There exist two boundary controls $u_1, u_2 \in C^k([0, T - 2\delta])$ at $a + 2\delta$ and $b - 2\delta$ that steer (u, v) on $[0, a + 2\delta]$ from $(y_0, y_1)|_{[0, a + 2\delta]}$ to $(z_0, z_1)|_{[0, a + 2\delta]}$, and from $(y_0, y_1)|_{[b - 2\delta, L]}$ to $(z_0, z_1)|_{[b - 2\delta, L]}$ while satisfying the boundary conditions of the system at 0 and L .

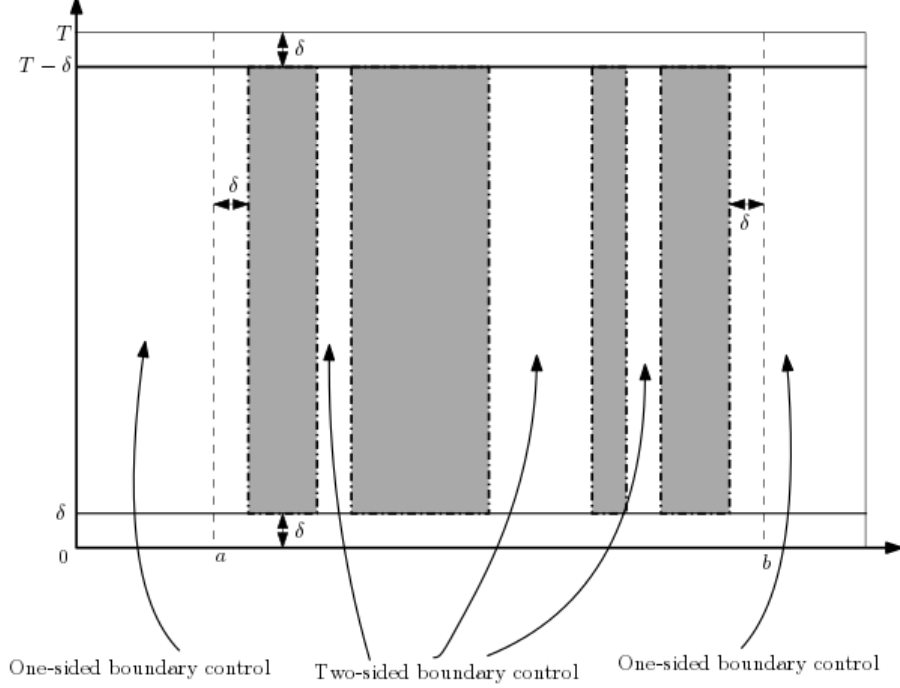


Figure 2.6: Using boundary control results outside of the covering set.

We note $\delta_0 = \delta_N := \delta$, and (u^*, v^*) the corresponding trajectory on $[0, a + 2\delta] \cup [b - 2\delta, L] \cup \bigcup_{1 \leq i \leq N-1} [b_i - \delta_i, a_{i+1} + \delta_i]$. Then, (2.87) and (2.84) imply

$$\|(u^*, v^*)\|_{C^k} \leq C \|(u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)\|_{(C^k \times C^{k-1})^4} \quad (2.89)$$

for some constant $C > 0$.

On the other hand, for $\eta > 0$ small enough, for initial and final conditions

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in (B_{C^k}([0, T] \times [0, L])(0, \eta) \times B_{C^{k-1}}([0, T] \times [0, L])(0, \eta))^4,$$

for $1 \leq i \leq N$ the forward evolving solutions $(u_f^{(i)}, v_f^{(i)})$ of the vector equations

$$\begin{cases} \square_{\nu_1} u = f_1(\bar{u} + u, \bar{v} + v) - f_1(\bar{u}, \bar{v}), \\ \square_{\nu_2} v = f_2(\bar{u} + u, \bar{v} + v) - f_2(\bar{u}, \bar{v}), \\ (u, v)(t, a_i) = (u, v)(t, b_i) = (0, 0), \\ (u, v)(0, \cdot) = (u_0|_{[a_i, b_i]}, v_0|_{[a_i, b_i]}), \\ (u, v)_t(0, \cdot) = (u_1|_{[a_i, b_i]}, v_1|_{[a_i, b_i]}). \end{cases},$$

are defined on $[0, T - 2\delta] \times [a_i, b_i]$. Let us also note $(u_b^{(i)}, v_b^{(i)})$ the backward evolving solutions of the

vector equations on $[0, T - 2\delta] \times [a_i, b_i]$

$$\left\{ \begin{array}{l} \square_{\nu_1} u = f_1(\bar{u} + u, \bar{v} + v) - f_1(\bar{u}, \bar{v}), \\ \square_{\nu_2} v = f_2(\bar{u} + u, \bar{v} + v) - f_2(\bar{u}, \bar{v}), \\ (u, v)(t, a_i) = (u, v)(t, b_i) = (0, 0), \\ (u, v)(T - 2\delta, \cdot) = (u_0^f|_{[a_i, b_i]}, v_0^f|_{[a_i, b_i]}), \\ (u, v)_t(T - 2\delta, \cdot) = (u_1^f|_{[a_i, b_i]}, v_1^f|_{[a_i, b_i]}). \end{array} \right.$$

Then we define (\tilde{u}, \tilde{v}) by

$$(\tilde{u}, \tilde{v}) = \left(u_f^{(i)}, v_f^{(i)} \right) \phi + \left(u_b^{(i)}, v_b^{(i)} \right) (1 - \phi), \text{ on } [a_i, b_i], \forall i \leq N,$$

where ϕ is a time cut-off function such that

$$\phi(0) = 1, \quad \phi(T - 2\delta) = 0.$$

Note that, by well-posedness of the Cauchy problems, there exists $C' > 0$ such that the norm of (\tilde{u}, \tilde{v}) satisfies

$$\|(\tilde{u}, \tilde{v})\|_{(C^k)^2} \leq C' \|(u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)\|_{(C^k \times C^{k-1})^4}. \quad (2.90)$$

Finally, let us define (u^{**}, v^{**}) by smoothly extending (u^*, v^*) on $\bigcup_{1 \leq i \leq N} [a_i + \delta_{i-1}, b_i - \delta_i]$ with

$$\|(u^{**}, v^{**})\|_{(C^k)^2} \leq C'' \|(u^*, v^*)\|_{C^k}, \quad (2.91)$$

where C'' is a constant depending on the a_i, b_i . Then, we define (u, v) by

$$(u, v) = \xi(u^{**}, v^{**}) + (1 - \xi)(\tilde{u}, \tilde{v}),$$

where ξ is a space cut-off function satisfying

$$\xi = 1 \text{ on } [0, a + \delta] \cup [b - \delta, L] \cup \bigcup_{1 \leq i \leq N-1} [b_i, a_{i+1}],$$

$$\xi = 0 \text{ on } \bigcup_{1 \leq i \leq N} [a_i + \delta_{i-1}, b_i - \delta_i].$$

Then, by construction, we have

$$\left\{ \begin{array}{ll} u(0, \cdot) = u_0, & v(0, \cdot) = v_0, \\ u_t(0, \cdot) = u_1, & v_t(0, \cdot) = v_1, \\ u(T - 2\delta, \cdot) = u_0^f, & v(T - 2\delta, \cdot) = v_0^f, \\ u_t(T - 2\delta, \cdot) = u_1^f, & v_t(T - 2\delta, \cdot) = v_1^f, \end{array} \right.$$

and

$$\begin{aligned} \text{supp}(\square_{\nu_1} u - f_1(\bar{u} + u, \bar{v} + v) - f_1(\bar{u}, \bar{v})) &\subset \mathcal{Q}_\delta, \\ \text{supp}(\square_{\nu_2} v - f_2(\bar{u} + u, \bar{v} + v) - f_2(\bar{u}, \bar{v})) &\subset \mathcal{Q}_\delta, \end{aligned}$$

Finally, (2.89), (2.90) and (2.91) imply that there exists a constant $C_2 > 0$ such that (2.81) holds, and, by continuity of the f_i , noting

$$h_i := \square_{\nu_i} u - f_i(\bar{u} + u, \bar{v} + v) - f_i(\bar{u}, \bar{v}), \quad i = 1, 2,$$

there exists a constant $C_1 > 0$ such that (2.79) holds. \square

Now, we define

$$\mathcal{A} = \left\{ (u, v, h) \in (C^2(\mathcal{Q}))^3 \mid \forall (t, x) \in \mathcal{Q}, u(t, x) \neq 0 \right\},$$

which is clearly nonempty, and

$$\forall (u, v, h) \in C^2(\mathcal{Q})^3, \mathcal{D}(u, v, h) = (\square_{\nu_1} u - h, \square_{\nu_2} v - u^3).$$

Then, we have the following proposition, similar to Proposition 2.2.2:

Proposition 2.3.5. \mathcal{D} admits an infinitesimal inversion of order 2 over \mathcal{A} .

Moreover, thanks to (2.75) and (2.44),

Proposition 2.3.6.

$$(\bar{u}, \bar{v}, \bar{h})|_{\mathcal{Q}} \in \mathcal{A}.$$

Now, we can use Theorem 2.2.1: there exists $\eta > 0$ such that for initial and final conditions

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in (B_{C^{11}([0, L])}(0, \eta) \times B_{C^{10}([0, L])}(0, \eta))^4$$

the corresponding trajectories of system (2.77) with two controls u^*, v^*, h_1, h_2 are small enough in $(C^{11})^2 \times (C^9)^2$ norm so that \mathcal{D} can be inverted locally around $(\bar{u} + u^*, \bar{v} + v^*, \bar{h})$, and so that, by the continuity property, $(u, v, h) := \mathcal{D}_{(\bar{u} + u^*, \bar{v} + v^*, \bar{h})}^{-1}(\theta_1, \theta_2)$ satisfies

$$\|(u - \bar{u}, v - \bar{v}, h - \bar{h})\|_{(C^6)^3} \leq \frac{R}{2}. \quad (2.92)$$

Together with (2.76), this yields

$$\|(u, v, h)\|_{(C^6)^3} \leq R. \quad (2.93)$$

This proves the following local controllability result:

Theorem 2.3.1. Let $R > 0$, and $0 \leq a < b \leq L$, $T > 0$ such that

$$T > 2(L - b) \max\left(\frac{1}{|\nu_1|}, \frac{1}{|\nu_2|}\right), \quad T > 2a \max\left(\frac{1}{|\nu_1|}, \frac{1}{|\nu_2|}\right). \quad (2.94)$$

There exists $\eta > 0$ such that for given initial and final conditions

$$((u_0, u_1), (v_0, v_1), (u_0^f, u_1^f), (v_0^f, v_1^f)) \in (B_{C^{11}([0, L])}(0, \eta) \times B_{C^{10}([0, L])}(0, \eta))^4$$

satisfying (2.8), there exists $h \in C^6([0, T] \times [0, L])$ satisfying

$$\text{supp } h \subset [0, T] \times [a, b].$$

such that the corresponding solution $(u, v) \in C^6([0, T] \times [0, L])$ of (2.2) with initial values $((u_0, u_1), (v_0, v_1))$ satisfies

$$\begin{cases} u(T, \cdot) = u_0^f, & u_t(T, \cdot) = u_1^f, \\ v(T, \cdot) = v_0^f, & v_t(T, \cdot) = v_1^f \end{cases}$$

and (2.93) holds.

Now let $(u_0, u_1, v_0, v_1, u_0^f, u_1^f, v_0^f, v_1^f) \in (C^{11}([0, L]) \times C^{10}([0, L]))^4$ such that (2.8) is satisfied. Let us note

$$M := \|(u_0, u_1, u_0^f, u_1^f)\|_{(C^{11} \times C^{10})^2} + \|(v_0, v_1, v_0^f, v_1^f)\|_{(C^{11} \times C^{10})^2}^{\frac{1}{3}},$$

and $\alpha := \frac{\eta}{2M}$. Then,

$$\begin{aligned} \|\alpha u_0\|_{C^{11}} &\leq \eta, & \|\alpha u_1\|_{C^{10}} &\leq \eta, & \|\alpha u_0^f\|_{C^{11}} &\leq \eta, & \|\alpha u_1^f\|_{C^{10}} &\leq \eta, \\ \|\alpha^3 v_0\|_{C^{11}} &\leq \eta, & \|\alpha^3 v_1\|_{C^{10}} &\leq \eta, & \|\alpha^3 v_0^f\|_{C^{11}} &\leq \eta, & \|\alpha^3 v_1^f\|_{C^{10}} &\leq \eta, \end{aligned}$$

and these functions satisfy (2.8). We can now apply Theorem 2.3.1, and for any support and time $T > 0$ compatible with that support, we get (u, v, h) with initial and final conditions $(\alpha u_0, \alpha u_1, \alpha u_0^f, \alpha u_1^f, \alpha^3 v_0, \alpha^3 v_1, \alpha^3 v_0^f, \alpha^3 v_1^f)$ such that

$$\begin{cases} \square_{\nu_1} u = h, \\ \square_{\nu_2} v = u^3, \\ u|_{\partial\Omega} = 0, \\ v|_{\partial\Omega} = 0. \end{cases}$$

Then we also have

$$\begin{cases} \square_{\nu_1} \alpha^{-1} u = \alpha^{-1} h, \\ \square_{\nu_2} \alpha^{-3} v = (\alpha^{-1} u)^3, \\ \alpha^{-1} u(0) = \alpha^{-1} u(L) = 0, \\ \alpha^{-3} v(0) = \alpha^{-3} v(L) = 0, \end{cases}$$

Thus, $\alpha^{-1} h$ steers (u_0, u_1, v_0, v_1) to $(u_0^f, u_1^f, v_0^f, v_1^f)$ in T . Finally, to get estimate (2.14), recall (2.93)

$$\|h\|_{C^6} \leq R,$$

hence, in terms of the original control system,

$$\begin{aligned} \|\alpha^{-1} h\|_{C^6} &\leq \alpha^{-1} R \\ &\leq \frac{2R}{\eta} \left(\|(u_0, u_1, u_0^f, u_1^f)\|_{(C^{11} \times C^{10})^2} + \|(v_0, v_1, v_0^f, v_1^f)\|_{(C^{11} \times C^{10})^2}^{\frac{1}{3}} \right). \end{aligned}$$

This proves Theorem 2.1.2.

2.3.4 A general criterion for internal controllability

Let us now give a general definition, which gives the main criterion our return trajectories must fulfill to apply our method:

Definition 2.3.2. *A suitable return trajectory for time $T > 0$ is a trajectory $(\bar{u}, \bar{v}, \bar{h}) \in C^{11}([0, T] \times [0, L])^3$ of system (2.1), such that*

$$\begin{aligned} \bar{u}(0, \cdot) &= 0, \quad \bar{v}(0, \cdot) = 0, \\ \bar{u}_t(0, \cdot) &= 0, \quad \bar{v}_t(0, \cdot) = 0, \\ \bar{u}(T, \cdot) &= 0, \quad \bar{v}(T, \cdot) = 0, \\ \bar{u}_t(T, \cdot) &= 0, \quad \bar{v}_t(T, \cdot) = 0, \end{aligned}$$

$$\text{supp } (\bar{u}, \bar{v}, \bar{h}) \subset [0, T] \times [a, b],$$

$$\mathcal{D}(\bar{u}, \bar{v}, \bar{h}) = (0, 0),$$

and such that there exists $0 < \delta < \min(T/2, (b-a)/2)$ satisfying (2.24), a δ -covering set \mathcal{Q}_δ , a smooth closed set \mathcal{Q} such that $\mathcal{Q}_\delta \subset \overset{\circ}{\mathcal{Q}}$ such that

$$\forall (t, x) \in \mathcal{Q}, \quad \bar{u}(t, x) \neq 0.$$

We can now give a general statement to sum up our work on system (2.2):

Proposition 2.3.7. *Let $0 \leq a < b \leq L$, and $T > 0$ such that (2.9) holds. Suppose condition (2.10) does not hold. If one can find a suitable return trajectory, then system (2.1) is locally controllable in time T for $(C^{11} \times C^{10})^4$ initial and final conditions, with C^6 trajectories, and with a C^6 control supported in $[0, T] \times [a, b]$.*

2.4 Further questions

2.4.1 Regularity

Our method requires somewhat specific regularities: C^{11} (C^{10} for the time-derivative) for the initial and final data. As is often the case when using a Nash-Moser scheme, these regularities are probably not optimal. However, if we require for example C^k regularity for the control, $k \geq 2$, the initial and final data have to be at least one notch smoother. Indeed, note

$$w := u_t - \nu_1 u_x,$$

and consider to the following computation, where one requires the control and the trajectories to be C^k :

$$\begin{aligned} \frac{d}{dt} w(t, x - \nu_1 t) &= w_t - \nu_1 w_x \\ &= w_t + \nu_1 w_x - 2\nu_1 w_x \\ &= h(t, x - \nu_1 t) - 2\nu_1 (u_{tx} - \nu_1 u_{xx}) \\ &= h(t, x - \nu_1 t) - 2\nu_1 \frac{d}{dt} u_x(t, x - \nu_1 t), \end{aligned}$$

hence, for a fixed $t > 0$ and for characteristics going from $\{0\} \times [0, L]$ to $\{t\} \times [0, L]$,

$$\int_0^t h(s, x - \nu_1 s) ds = u_t(t, x - \nu_1 t) - u_1(x) - \nu_1 u_x(t, x - \nu_1 t) + \nu_1 u'_0(x) - 2\nu_1 (u_x(t, x - \nu_1 t) - u'_0(x)). \quad (2.95)$$

Now the left-hand side of (2.95) is a C^k function of x , so we need to upgrade the regularity of u_0 to C^{k+1} , and that of u_1 , $x \mapsto u_x(t, x)$ and $x \mapsto u_t(t, x)$ to C^k . This shows a partial derivative loss (in the space dimension) between the trajectory and the control, and, taking $t = T$, a derivative loss between the initial and final data for u , and the control.

For linear cascade systems with smooth coefficients, the same procedure can be repeated on the second equation to establish similar losses of derivatives, showing that because u has increased spatial regularity, the initial and final data for v have to be $C^{k+2} \times C^{k+1}$. Note that with two controls, this would not be the case, as each control “absorbs” the loss of derivatives in each equation.

Furthermore, it would be interesting to consider other iteration schemes such as the one presented in [53], section 4.2.1, where one considers the following linear system:

$$\begin{cases} \square u = f_1(0, 0) + g_1^v(a, b)v + g_1^u(a, 0)u + h \\ \square v = f_2(0, 0) + g_2^v(0, b)v + g_2^u(a, b)u, \end{cases} \quad (2.96)$$

where, for $i \in \{1, 2\}$,

$$g_i^u(u, v) = \begin{cases} \frac{f_i(u, v) - f_i(0, v)}{u} & \text{for } u \neq 0 \\ \partial_u f_i(0, v) & \text{for } u = 0. \end{cases}$$

$$g_i^v(u, v) = \begin{cases} \frac{f_i(u, v) - f_i(u, 0)}{v} & \text{for } v \neq 0 \\ \partial_v f_i(u, 0) & \text{for } v = 0. \end{cases}$$

Then, by superposition one can restrict to the study of

$$\begin{cases} \square u = h \\ \square v = g_2^u(a, b)u. \end{cases} \quad (2.97)$$

But ultimately, this only shifts the problem of the C^k regularity gap between data and control, although we now have a linear system instead of a semilinear one.

On the other hand, there are some situations where the Nash-Moser theorem eventually proves unnecessary. For instance, as mentioned in the introduction, controllability results for the 1-D Schrödinger equation were first proved using a Nash-Moser implicit function theorem ([28], [29]), because of an a priori loss of derivatives. Karine Beauchard and Camille Laurent later discovered that, because of a regularizing effect, there was actually no loss of derivatives, and proved more general results on the bilinear control of the linear and nonlinear 1-D Schrödinger equation in [30], using a classical inversion theorem on optimal function spaces. Similarly, in [85] and [86], it was proved that the problem of isometric embedding could be solved using a classical iteration scheme, instead of a Nash-Moser one. It would be interesting to know if a similar do-over is possible for our result, keeping in mind that the argument we gave above excludes the possibility of a regularizing effect that would cancel all loss of derivatives. We could hope for a result with a C^2 control and $(C^4 \times C^3)^2$ initial and final conditions, for example.

This would also be interesting in terms of numerical analysis, to compute approximate controls and trajectories for the considered system. Indeed, the proof of Gromov's inversion theorem relies on a Nash-Moser type iterative scheme, which could be computed. However, such an iterative scheme would be very heavy to implement, because of the regularization at each step. A classical inversion theorem, relying on a Newton scheme, would be lighter to implement.

Finally, it would be interesting to investigate a H^k version of this result, using other versions of the Nash-Moser implicit function theorem.

2.4.2 Other systems with an uncontrollable linearised system

Our scheme of proof also allows to prove a controllability result for systems of the form

$$\begin{cases} \square_{\nu_1} u = G(u, v) + h, & G \in C^\infty(\mathbb{R}^2), \\ \square_{\nu_2} v = u^3, \\ u|_{\partial\Omega} = 0, \\ v|_{\partial\Omega} = 0. \end{cases} \quad (2.98)$$

Indeed, this simply adds a term in the definition of \bar{h} when we build our return trajectory. However, \bar{h} is no longer supported in $[0, T] \times [a, b]$. The other steps remain unchanged, as the additional G term does not prevent the differential operator \mathcal{D} from being algebraically solvable. So we get a local internal controllability result with the same time conditions, but no condition on the support of the control. Finally, if G is homogeneous of degree 1, we can use the scaling argument to deduce a global result.

In addition to adding a coupling term to the first equation, we can also change the power of the coupling term in the second equation. There are two cases:

1. *Even powers* As such, our method cannot work for even powers: indeed, u^{2k} has nonnegative values. In particular, by the same convexity argument as in Remark 2.3.2, solutions to the stationary problem cannot vanish smoothly in 1. So the perturbative approach would allow us to build smooth return trajectories only if u (and thus h) is spatially supported in all of $[0, L]$. Another way of answering this question would be to switch to complex values, as is done in the appendix of [60] for the quadratic case.
2. *Odd powers* Thanks to Proposition 2.3.7, we know that the part that requires the most work is the construction of return trajectories: say the power of the coupling is $2k + 1$, $k \in \mathbb{N}^*$, in order to control all the derivatives of v up to $2k + 1$, we would have to look for v in the form

$$\sum_{i=0}^{2k+1} f_i(t) g_i \left(\frac{|x - \frac{b+a}{2}|}{\lambda} \right).$$

This would call for ever longer computations, and for now there is no indication that there might or might not be new difficulties with these additional terms.

2.4.3 Boundary controllability

In this chapter we have explored a method to prove internal controllability with one control. However, to our knowledge there is no result for boundary controllability with one control for semilinear systems such as (2.1). Although boundary controllability is relatively easy to establish for simple equations, or when there are the same number of controls and equations, we cannot use results on the inversion of differential operators to reduce the number of controls.

Part II

Stabilization of 1-D linear hyperbolic systems with a scalar input

Chapter 3

Internal rapid stabilization of a 1-D linear transport equation with a scalar feedback

This chapter is taken from the following preprint (also referred to as [162]):

Christophe Zhang. Internal rapid stabilization of a 1-D linear transport equation with a scalar feedback. 2018. (hal-01905098)

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Abstract We use the backstepping method to study the stabilization of a 1-D linear transport equation on the interval $(0, L)$, by controlling the scalar amplitude of a piecewise regular function of the space variable in the source term. We prove that if the system is controllable in a periodic Sobolev space of order greater than 1, then the system can be stabilized exponentially in that space and, for any given decay rate, we give an explicit feedback law that achieves that decay rate.

3.1 Introduction

We study the linear 1-D hyperbolic equation

$$\begin{cases} y_t + y_x + a(x)y = u(t)\tilde{\varphi}(x), & x \in [0, L], \\ y(t, 0) = y(t, L), & \forall t \geq 0, \end{cases} \quad (3.1)$$

where a is continuous, real-valued, $\tilde{\varphi}$ is a given real-valued function that will have to satisfy certain conditions, and at time t , $y(t, \cdot)$ is the state and $u(t)$ is the control. As the system can be transformed into

$$\begin{cases} \alpha_t + \alpha_x + \mu\alpha = u(t)\varphi(x), & x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), & \forall t \geq 0, \end{cases} \quad (3.2)$$

through the state transformation

$$\alpha(t, x) := e^{\int_0^x a(s)ds - \mu x} y(x, t),$$

where $\mu = \int_0^L a(s)ds$, and with

$$\varphi(x) := e^{\int_0^x a(s)ds - \mu x} \tilde{\varphi}(x),$$

we will focus on systems of the form (3.2) in this article.

These systems are an example of linear hyperbolic systems with a distributed scalar input. Such systems appear naturally in physical problems. For example, as is mentioned in [135], a linear wave equation which can be rewritten as a 2×2 first order hyperbolic system, the problem of a vibrating damped string, or the plucking of a string, can be modelled thus. In a different field altogether, chemical tubular reactors, in particular plug flow reactors (see [125, 127]), are modeled by hyperbolic systems with a distributed scalar input (the temperature of the reactor jacket), albeit with a boundary input instead of proportional boundary conditions. Let us cite also the water tank system, introduced by François Dubois, Nicolas Petit and Pierre Rouchon in [75]. It models a 1-D tank containing an inviscid, incompressible, irrotational fluid, in the approximation that its acceleration is small compared with the gravitational constant, and that the height of the liquid is small compared with the length of the tank. In this setting, the motion of the fluid can be modelled by the Saint-Venant equations on the interval $[0, L]$ with impermeable boundary conditions (which correspond to proportional boundary conditions after a variable change), and the control is the force applied to the tank itself, which takes the form of a distributed scalar input.

3.1.1 Notations and definitions

We note ℓ^2 the space of summable square series $\ell^2(\mathbb{Z})$. To simplify the notations, we will note L^2 the space $L^2(0, L)$ of complex-valued L^2 functions, with its hermitian product

$$\langle f, g \rangle = \int_0^L f(x)\overline{g(x)}dx, \quad \forall f, g \in L^2, \quad (3.3)$$

and the associated norm $\|\cdot\|$. We also use the following notation

$$e_n(x) = \frac{1}{\sqrt{L}} e^{\frac{2i\pi}{L}nx}, \quad \forall n \in \mathbb{Z}, \quad (3.4)$$

the usual Hilbert basis for L^2 . For a function $f \in L^2$, we will note $(f_n) \in \ell^2$ its coefficients in this basis:

$$f = \sum_{n \in \mathbb{Z}} f_n e_n.$$

Note that with this notation, we have

$$\bar{f} = \sum_{n \in \mathbb{Z}} \overline{f_{-n}} e_n,$$

so that, in particular, if f is real-valued:

$$f_{-n} = \overline{f_n}, \quad \forall n \in \mathbb{Z}.$$

Functions of L^2 can also be seen as L -periodic functions on \mathbb{R} , by the usual L -periodic continuation: in this article, for any $f \in L^2$ we will also note f its L -periodic continuation on \mathbb{R} .

We will use the following definition of the convolution product on L -periodic functions:

$$f \star g = \sum_{n \in \mathbb{Z}} f_n g_n e_n = \int_0^L f(s) g(\cdot - s) ds \in L^2, \quad \forall f, g \in L^2, \quad (3.5)$$

where $g(x - s)$ should be understood as the value taken in $x - s$ by the L -periodic continuation of g .

Let us now note \mathcal{E} the space of finite linear combinations of the $(e_n)_{n \in \mathbb{Z}}$. Then, any sequence $(f_n)_{n \in \mathbb{Z}}$ defines an element f of \mathcal{E}' :

$$\langle e_n, f \rangle = \overline{f_n}.$$

On this space of linear forms, derivation can be defined by duality:

$$f' = \left(\frac{2i\pi n}{L} f_n \right)_{n \in \mathbb{Z}}, \quad \forall f \in \mathcal{E}'.$$

We also define the following spaces:

Definition 3.1.1. Let $m \in \mathbb{N}$. We note H^m the usual Sobolev spaces on the interval $(0, L)$, equipped with the Hermitian product

$$\langle f, g \rangle_m = \int_0^L \partial^m f \overline{\partial^m g} + f \overline{g}, \quad \forall f, g \in H^m,$$

and the associated norm $\|\cdot\|_m$.

For $m \geq 1$ we also define $H_{(pw)}^m$ the space of piecewise H^m functions, that is, $f \in H_{(pw)}^m$ if there exists a finite number d of points $(\sigma_j)_{1 \leq j \leq d} \in [0, L]$ such that, noting $\sigma_0 := 0$ and $\sigma_{d+1} := L$, f is H^m on every $[\sigma_j, \sigma_{j+1}]$ for $0 \leq j \leq d$. This space can be equipped with the norm

$$\|f\|_{m,pw} := \sum_{j=0}^d \|f|_{[\sigma_j, \sigma_{j+1}]}\|_{H^m(\sigma_j, \sigma_{j+1})}.$$

For $s > 0$, we also define the periodic Sobolev space H_{per}^s as the subspace of L^2 functions $f = \sum_{n \in \mathbb{Z}} f_n e_n$ such that

$$\sum_{n \in \mathbb{Z}} \left(1 + \left| \frac{2i\pi n}{L} \right|^{2s} \right) |f_n|^2 < \infty.$$

H^s is a Hilbert space, equipped with the Hermitian product

$$\langle f, g \rangle_s = \sum_{n \in \mathbb{Z}} \left(1 + \left| \frac{2i\pi n}{L} \right|^{2s} \right) f_n \overline{g_n}, \quad \forall f, g \in H^s,$$

and the associated norm $\|\cdot\|_s$, as well as the Hilbert basis

$$(e_n^s) := \left(\frac{e_n}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2s}}} \right).$$

Note that for $m \in \mathbb{N}$, H_{per}^m is a closed subspace of H^m , with the same scalar product and norm, thanks to the Parseval identity. Moreover,

$$H_{per}^m = \left\{ f \in H^m, \quad f^{(i)}(0) = f^{(i)}(L), \forall i \in \{0, \dots, m-1\} \right\}.$$

3.1.2 Main result

To stabilize (3.2), we will be considering linear feedbacks of the form

$$\langle \alpha(t), F \rangle = \sum_{n \in \mathbb{Z}} \overline{F_n} \alpha_n(t) = \int_0^L \overline{F}(s) \alpha(s) ds$$

where $F \in \mathcal{E}'$ and $(F_n) \in \mathbb{C}^{\mathbb{Z}}$ are its Fourier coefficients, and F is real-valued, that is,

$$F_{-n} = \overline{F_n}, \quad \forall n \in \mathbb{Z}.$$

In fact, the integral notation will appear as purely formal, as the (F_n) will have a prescribed growth, so that $F \notin L^2$. The associated closed-loop system now writes

$$\begin{cases} \alpha_t + \alpha_x + \mu \alpha = \langle \alpha(t), F \rangle \varphi(x), & x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), & \forall t \geq 0. \end{cases} \quad (3.6)$$

This is a linear transport equation, which we seek to stabilize with an internal, scalar feedback, given by a real-valued feedback law. This article aims at proving the following class of stabilization results:

Theorem 3.1.1 (Rapid stabilization in Sobolev norms). *Let $m \geq 1$. Let $\varphi \in H_{(pw)}^m \cap H_{per}^{m-1}$ such that*

$$\frac{c}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2m}}} \leq |\varphi_n| \leq \frac{C}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2m}}}, \quad \forall n \in \mathbb{Z}, \quad (3.7)$$

where $c, C > 0$ are the optimal constants for these inequalities. Then, for every $\lambda > 0$, for all $\alpha_0 \in H_{per}^m$ the closed-loop system (3.6) with the stationary feedback law $F \in \mathcal{E}'$ given by

$$\langle e_n, F \rangle := -\frac{1 - e^{\lambda L}}{1 + e^{-\lambda L}} \frac{2}{L \varphi_n}, \quad \forall n \in \mathbb{Z},$$

has a solution $\alpha(t)$ which satisfies the estimate

$$\|\alpha(t)\|_m \leq \left(\frac{C}{c} \right)^2 e^{(\mu + \lambda)L} e^{-\lambda t} \|\alpha_0\|_m, \quad \forall t \geq 0. \quad (3.8)$$

Note that the estimate (3.8) is constructive, as it only depends on c, C, μ and λ . Though it is not necessarily sharp for a given controller φ and the corresponding feedback law F , it is the “least worse” a priori estimate one can get, in a sense that we will elaborate further on. The growth restriction (3.7) on the Fourier coefficients of φ can be written, more intuitively, and for some other constants $c', C' > 0$,

$$\frac{c'}{1 + \left| \frac{2i\pi n}{L} \right|^m} \leq |\varphi_n| \leq \frac{C'}{1 + \left| \frac{2i\pi n}{L} \right|^m}, \quad \forall n \in \mathbb{Z},$$

and corresponds to the necessary and sufficient condition for the controllability of system (3.2) in H_{per}^m , in time $T \geq L$. This is obtained using the moments method, and we refer to [137, Equation (2.19) and pages 199-200] for more details. The controllability of system (3.2), in turn, will allow us to use a form of backstepping method to stabilize it.

On the other hand, the additional regularity $\varphi \in H_{(pw)}^m$ gives us the following equality, first using the fact that $\varphi \in H_{per}^{m-1}$, then by integration by parts on each interval $[\sigma_j, \sigma_{j+1}]$, using the fact that $\partial^{m-1} \varphi \in H_{(pw)}^1$:

$$\begin{aligned} \varphi_n &= \frac{(-1)^{m-1}}{\left(\frac{2i\pi n}{L} \right)^{m-1}} \langle \varphi, \partial^{m-1} e_n \rangle \\ &= \frac{1}{\left(\frac{2i\pi n}{L} \right)^{m-1}} \langle \partial^{m-1} \varphi, e_n \rangle \\ &= -\frac{\tau_n^\varphi}{\left(\frac{2i\pi n}{L} \right)^m} + \frac{1}{\left(\frac{2i\pi n}{L} \right)^m} \sum_{j=0}^d \langle \chi_{[\sigma_j, \sigma_{j+1}]} \partial^m \varphi, e_n \rangle, \quad \forall n \in \mathbb{Z}^*, \end{aligned} \quad (3.9)$$

where

$$\tau_n^\varphi := \frac{1}{\sqrt{L}} \left(\partial^{m-1} \varphi(L) - \partial^{m-1} \varphi(0) + \sum_{j=1}^d e^{-\frac{2i\pi}{L} n \sigma_j} (\partial^{m-1} \varphi(\sigma_j^-) - \partial^{m-1} \varphi(\sigma_j^+)) \right), \quad \forall n \in \mathbb{Z}.$$

Note that, thanks to condition (3.7), there exists $C_1, C_2 > 0$ such that

$$C_1 \leq |\tau_n^\varphi| \leq C_2, \quad n \in \mathbb{Z},$$

so that these numbers are the eigenvalues of a diagonal isomorphism of any periodic Sobolev space into itself, which we note τ^φ . Moreover, it is clear from the definition of its coefficients that τ^φ is a sum of translations. Also, note that $\tau_n^\varphi \neq 0$, and thus, $\varphi \notin H_{per}^m$. Finally, note that

$$\left(\sum_{j=0}^d \langle \chi_{[\sigma_j, \sigma_{j+1}]} \partial^m \varphi, e_n \rangle \right) \in \ell^2. \quad (3.10)$$

3.1.3 Related results

To investigate the stabilization of infinite-dimensional systems, there are four main types of approaches.

The first type of approach relies on abstract methods, such as the Gramian approach and the Riccati equations (see for example [153, 151, 108]). This method is a generalization of the well-known Gramian method in finite dimension (see [120, 103]), these feedback laws involve the solution to an algebraic Riccati equation, and the inversion of an infinite-dimensional Gramian operator, which makes them difficult to compute in practice.

The second approach relies on Lyapunov functions. Many results on the boundary stabilization of first-order hyperbolic systems, linear and nonlinear, have been obtained using this approach: see for example the book [24], and the recent results in [91, 92]. However, this approach can be limited, as it is sometimes impossible to obtain an arbitrary decay rate using Lyapunov functions (see [53, Remark 12.9, page 318] for a finite dimensional example).

The third approach is related to pole-shifting results in finite dimension. Indeed, it is well-known that if a linear finite-dimensional system is controllable, than its poles can be arbitrarily reassigned (shifted) with an appropriate linear feedback law (see [53]). There have been some generalizations of this powerful property to infinite-dimensional systems, notably hyperbolic systems. Let us cite [137], in which the author uses a sort of canonical form to prove a pole-shifting result for a class of hyperbolic systems with a distributed scalar control. In this paper, the feedback laws under consideration are bounded and pole-shifting property is not as strong as in finite dimension. This is actually inevitable, as was proved in [144], in a very general setting: bounded feedback laws can only achieve weak pole-shifting, which is not sufficient for exponential stabilization. However, if one allows for unbounded feedback laws, it is possible to obtain stronger pole-shifting, and in particular exponential stabilization in some cases. This is extensively studied in [129]. In this paper, the author gives a formula for a feedback law that achieves the desired pole placement. However, this formula requires to know a cardinal function for which the poles coincide with the initial spectrum, which might be difficult in practice.

The fourth approach, which we will be using in this article, is the backstepping method. This name originally refers to a way of designing feedbacks for finite-dimensional stabilizable systems with an added chain of integrators (see [53, 143, 110], and [56] or [118] for some applications to partial differential equations). Another way of applying this approach to partial differential equations was then developed in [17] and [32]: when applied to the discretization of the heat equation, the backstepping approach yielded a change of coordinates which was equivalent to a Volterra transform of the second kind. Backstepping then took yet another successful form, consisting in mapping the system to stable target system, using a Volterra transformation of the second kind (see [111] for a comprehensive

introduction to the method):

$$f(t, x) \mapsto f(t, x) - \int_0^x k(x, y)f(t, y)dy.$$

This was used to prove a host of results on the boundary stabilization of partial differential equations: let us cite for example [109] and [141] for the wave equation, [155, 156] for the Korteweg-de Vries equation, [24, chapter 7] for an application to first-order hyperbolic systems, and also [68], which combines the backstepping method with Lyapunov functions to prove finite-time stabilization in H^2 for a quasilinear 2×2 hyperbolic system.

The backstepping method has the advantage of providing explicit feedback laws, which makes it a powerful tool to prove other related results, such as null-controllability or small-time stabilization (stabilization in an arbitrarily small time). This is done in [67], where the authors give an explicit control to bring a heat equation to 0, then a time-varying, periodic feedback to stabilize the equation in small time. In [156], the author obtains the same kind of results for the Korteweg-de Vries equation.

In some cases, the method was used to obtain stabilization with an internal feedback. This was done in [150] and [154] for parabolic systems, and [159] for first-order hyperbolic systems. The strategy in these works is to first apply a Volterra transformation as usual, which still leaves an unstable source term in the target, and then apply a second invertible transformation to reach a stable target system. Let us note that in the latter reference, the authors study a linear transport equation and get finite-time stabilization. However, their controller takes a different form than ours, and several hypotheses are made on the space component of the controller so that a Volterra transform can be successfully applied to the system. This is in contrast with the method in this article, where the assumption we make on the controller corresponds to the exact null-controllability of the system.

In this paper, we use another application of the backstepping method, which uses another type of linear transformations, namely, Fredholm transformations:

$$f(t, x) \mapsto \int_0^L k(x, y)f(t, y)dy.$$

These are more general than Volterra transformations, but they require more work: indeed, Volterra transformations are always invertible, but the invertibility of a Fredholm transform is harder to check. Even though it is sometimes more involved and technical, the use of a Fredholm transformation proves more effective for certain types of control: for example, in [66] for the Korteweg-de Vries equation and [65] for a Kuramoto-Sivashinsky, the position of the control makes it more appropriate to use a Fredholm transformation. Other boundary stabilization results using a Fredholm transformation can be found in [62] for integro-differential hyperbolic systems, and in [63] for general hyperbolic balance laws.

Fredholm transformations have also been used in [59], where the authors prove the rapid stabilization of the Schrödinger equation with an internal feedback. Their method of proof relies on the assumption that the system is controllable, and the technical developments are quite different from the work in previous references. This is a new development in the evolution of the backstepping method. Indeed, the original form of the backstepping method, and the backstepping method with Volterra transformations of the second kind, could be applied to uncontrollable systems. Hence, a controllability assumption makes for potentially powerful additional information, for example when one considers the more general Fredholm transformations instead of Volterra transformations of the second kind. It is interesting to note that this is a common feature with the pole-shifting approach, although in this setting it leads to an explicit feedback law given by its Fourier coefficients.

3.1.4 The backstepping method revisited: a finite-dimensional example

Let us now give a finite-dimensional example to illustrate the role controllability can play in the backstepping method for PDEs. Consider the finite-dimensional control system

$$\dot{x} = Ax + Bu(t), \quad x \in \mathbb{C}^n, A \in M_n(\mathbb{C}), B \in M_{n,1}(\mathbb{C}). \quad (3.11)$$

Assume that (A, B) is controllable. Suppose that $x(t)$ is a solution of system (3.11) with $u(t) = Kx(t)$. Now, in the spirit of PDE backstepping, let us try to invertibly transform the resulting closed-loop system into another controllable system, namely

$$\dot{x} = \tilde{A}x, \quad (3.12)$$

which can be exponentially stable if \tilde{A} is well chosen.

Such a transformation T would map the closed loop system to

$$(\dot{T}x) = T\dot{x} = T(A + BK)x.$$

In order for Tx to be a solution of (3.12), we would need

$$T(A + BK) = \tilde{A}T. \quad (3.13)$$

One can see quite clearly that this matrix equation is not well-posed, in that if it has a solution, it has an infinity of solutions. Moreover, the variables T and K are not separated because of the TBK term, and as a result the equation is nonlinear. Hence, we can add the following constraint to equation (3.13), to separate the variables, make the equation linear in (T, K) , and get a uniqueness property:

$$TB = B. \quad (3.14)$$

Injecting the above equation into (3.13), we get the following equations:

$$\begin{aligned} TA + BK &= \tilde{A}T, \\ TB &= B, \end{aligned} \quad (3.15)$$

Now for this set of equations, one can prove the following theorem, using the Brunovski normal form (or canonical form):

Theorem 3.1.2. *If (A, B) and (\tilde{A}, B) are controllable, then there exists a unique pair (T, K) satisfying conditions (3.15).*

This shows that controllability can be very useful when one wants to transform systems into other systems. In the finite-dimensional case, using the canonical form is the most efficient way of writing it. However, in order to gain some insight on the infinite-dimensional case, there is a different proof, relying on the spectral properties of A and \tilde{A} , which can be found in [59]. The idea is that the controllability of A allows to build a basis for the space state, in which T can then be constructed. Indeed, suppose A is diagonalizable with eigenvectors $(e_n, \lambda_n)_{1 \leq n \leq N}$, and suppose that \tilde{A} and A have no mutual eigenvalues. Then, let us project (3.15) on e_n :

$$\lambda_n T e_n + (K e_n) B = \tilde{A} T e_n, \quad (3.16)$$

from which we get the following relationship

$$T e_n = (K e_n) (\tilde{A} - \lambda_n I)^{-1} B, \quad \forall n \in \{1, \dots, N\}. \quad (3.17)$$

Then, using the Kalman rank condition on the pair (\tilde{A}, B) , one can prove that the $f_n := ((\tilde{A} - \lambda_n I)^{-1} B)$ form a basis of \mathbb{R}^N .

Knowing this, write

$$\begin{aligned} B &= \sum_{n=1}^N b_n e_n, \\ B &= \sum_{n=1}^N \tilde{b}_n f_n, \end{aligned} \quad (3.18)$$

and TB is written naturally in this basis:

$$TB = \sum_{n=1}^N (Ke_n)b_n f_n, \quad (3.19)$$

so that the second equation of (3.15) becomes

$$\sum_{n=1}^N (Ke_n)b_n f_n = \sum_{n=1}^N \tilde{b}_n f_n. \quad (3.20)$$

Using the Kalman rank condition on (A, B) , one can prove that $b_n \neq 0$ so that the (Ke_n) are uniquely determined. The only thing that remains to prove is the invertibility of T , as the (Ke_n) could be 0. In the end the invertibility is proven thanks to the Hautus test on the pair (\tilde{A}, B) , and the uniqueness is given by the $TB = B$ condition.

3.1.5 Structure of the article

The structure of this article follows the outline of the proof given above: in Section 2, we look for candidates for the backstepping transformation in the form of Fredholm transformations. Formal calculations (and a formal $TB = B$ condition) lead to a PDE analogous to (3.16) which we solve, which is analogous to the derivation of (3.17). Using the properties of Riesz bases and the controllability assumption, we prove that such candidates are indeed invertible, under some conditions on the feedback coefficients (F_n) . For consistency, we then determine the feedback law (F_n) such that the corresponding transformation indeed satisfies a weak form of the $TB = B$ condition. Then, in Section 3, we check that the corresponding transformation indeed satisfies an operator equality analogous to (3.15), making it a valid backstepping transformation. We check the well-posedness of the closed-loop system for the feedback law obtained in Section 2, which allows us to prove the stability result. Finally, Section 4 gives a few remarks on the result, as well as further questions on this stabilization problem.

3.2 Definition and properties of the transformation

Let $\lambda' > 0$ be such that $\lambda' - \mu > 0$, and $m \geq 1$. Let $\varphi \in H^m \cap H_{per}^{m-1}$ be a real-valued function satisfying (3.7). We consider the following target system:

$$\begin{cases} z_t + z_x + \lambda' z = 0, & x \in (0, L), \\ z(t, 0) = z(t, L), & t \geq 0. \end{cases} \quad (3.21)$$

Then it is well-known that, taking $\alpha_0 \in L^2$, the solution to (3.21) with initial condition α_0 writes

$$z(t, x) = e^{-\lambda' t} \alpha_0(x - t), \quad \forall (t, x) \in \mathbb{R}^+ \times (0, L).$$

Hence,

Proposition 3.2.1. *For all $s \geq 0$, the system (3.21) is exponentially stable for $\|\cdot\|_s$, for initial conditions in H_{per}^s .*

3.2.1 Kernel equations

As mentioned in the introduction, we want to build backstepping transformations T as a kernel operator of the Fredholm type:

$$f(t, x) \mapsto \int_0^L k(x, y) f(t, y) dy.$$

To have an idea of what this kernel looks like, we can do the following formal computation for some Fredholm operator T : first the boundary conditions

$$\left(\int_0^L k(0, y) \alpha(y) dy \right) = \left(\int_0^L k(L, y) \alpha(y) dy \right),$$

then the equation of the target system, for $x \in [0, L]$:

$$\begin{aligned} 0 &= \left(\int_0^L k(x, y) \alpha(y) dy \right)_t + \left(\int_0^L k(x, y) \alpha(y) dy \right)_x + \lambda' \left(\int_0^L k(x, y) \alpha(y) dy \right) \\ &= \left(\int_0^L k(x, y) \alpha_t(y) dy \right) + \left(\int_0^L k_x(x, y) \alpha(y) dy \right) + \lambda' \left(\int_0^L k(x, y) \alpha(y) dy \right) \\ &= \left(\int_0^L k(x, y) (-\alpha_x(y) - \mu \alpha(y) + \langle \alpha, F \rangle \varphi(y)) dy \right) + \left(\int_0^L (k_x(x, y) + \lambda' k(x, y)) \alpha(y) dy \right) \\ &= \left(\int_0^L k_y(x, y) \alpha(y) dy \right) - (k(x, L) \alpha(L) - k(x, 0) \alpha(0)) + \left(\int_0^L k(x, y) \langle \alpha, F \rangle \varphi(y) dy \right) + \\ &\quad \left(\int_0^L (k_x(x, y) + (\lambda' - \mu) k(x, y)) \alpha(y) dy \right) \\ &= \left(\int_0^L k(x, y) \left(\int_0^L \bar{F}(s) \alpha(s) ds \right) \varphi(y) dy \right) - (k(x, L) \alpha(L) - k(x, 0) \alpha(0)) \\ &\quad + \left(\int_0^L (k_y(x, y) + k_x(x, y) + (\lambda' - \mu) k(x, y)) \alpha(y) dy \right) \\ &= \left(\int_0^L \bar{F}(s) \left(\int_0^L k(x, y) \varphi(y) dy \right) \alpha(s) ds \right) - (k(x, L) \alpha(L) - k(x, 0) \alpha(0)) \\ &\quad + \left(\int_0^L (k_y(x, y) + k_x(x, y) + (\lambda' - \mu) k(x, y)) \alpha(y) dy \right). \end{aligned}$$

Now, suppose we have the formal $TB = B$ condition

$$\int_0^L k(x, y) \varphi(y) dy = \varphi(x), \quad \forall x \in [0, L].$$

Then, we get, noting $\lambda := \lambda' - \mu > 0$,

$$\left(\int_0^L (k_y(x, y) + k_x(x, y) + \lambda k(x, y) + \varphi(x) \bar{F}(y)) \alpha(y) dy \right) - (k(x, L) \alpha(L) - k(x, 0) \alpha(0)) = 0.$$

Hence the kernel equation:

$$\begin{cases} k_x + k_y + \lambda k = -\varphi(x) \bar{F}(y), \\ k(0, y) = k(L, y), \\ k(x, 0) = k(x, L), \end{cases} \quad (3.22)$$

together with the $TB = B$ condition

$$\langle k(x, \cdot), \varphi(\cdot) \rangle = \varphi(x), \quad \forall x \in [0, L]. \quad (3.23)$$

3.2.2 Construction of Riesz bases for Sobolev spaces

To study the solution to the kernel equation, we project it along the variable y . Let us write heuristically

$$k(x, y) = \sum_{n \in \mathbb{Z}} k_n(x) e_n(y),$$

so that

$$\int_0^L k(x, y) \alpha(y) dy = \sum_{n \in \mathbb{Z}} \alpha_n k_{-n}(x).$$

Projecting the kernel equations (3.22), we get

$$k'_n + \lambda_n k_n = -\overline{F_{-n}} \varphi, \quad (3.24)$$

where

$$\lambda_n = \lambda + \frac{2i\pi}{L} n. \quad (3.25)$$

Note that

$$\frac{2i\pi p}{L} \frac{1}{\lambda_{n+p}} + \lambda_n \frac{1}{\lambda_{n+p}} = 1, \quad \forall n, p \in \mathbb{Z}. \quad (3.26)$$

Now consider the L^2 function given by

$$\Lambda_n^\lambda(x) = \frac{\sqrt{L}}{1 - e^{-\lambda L}} e^{-\lambda_n x}, \quad \forall n \in \mathbb{Z}, \quad \forall x \in [0, L]. \quad (3.27)$$

Then, for all $m \geq 0$, $\Lambda_n^\lambda \in H^m$, and we have

$$\langle \Lambda_n^\lambda, e_p \rangle = \frac{1}{\sqrt{L}} \int_0^L \frac{\sqrt{L}}{1 - e^{-\lambda L}} e^{-\lambda_n x} e^{-\frac{2i\pi p}{L} x} dx = \frac{1}{1 - e^{-\lambda L}} \int_0^L e^{-\lambda_{n+p} x} dx = \frac{1}{\lambda_{n+p}}, \quad \forall n, p \in \mathbb{Z},$$

so that, using (3.26),

$$(\Lambda_n^\lambda)' + \lambda_n \Lambda_n^\lambda = \sum_{p \in \mathbb{Z}} e_p \text{ in } \mathcal{E}'.$$

Remark 3.2.1. In \mathcal{E}' , $\sum_{p \in \mathbb{Z}} e_p$ is the equivalent of the Dirac comb, or the “Dirac distribution” on the space of functions on $[0, L]$. So, in a sense, Λ_n^λ is the elementary solution of (3.24).

Let us now define, in analogy with the elementary solution method,

$$k_{n,\lambda} = -\overline{F_{-n}} \Lambda_n^\lambda \star \varphi \in H_{per}^m, \quad \forall n \in \mathbb{Z}. \quad (3.28)$$

The regularity comes from the definition of the convolution product, (3.7) and (3.25), and one can check, using (3.26), that $k_{n,\lambda}$ is a solution of (3.24).

The next step to build an invertible transformation is to find conditions under which $(k_{n,\lambda})$ is some sort of basis. More precisely we use the notion of Riesz basis (see [44, Chapter 4])

Definition 3.2.1. A Riesz basis in a Hilbert space H is the image of an orthonormal basis of H by an isomorphism.

Proposition 3.2.2. Let H be a Hilbert space. A family of vectors $(f_k)_{k \in \mathbb{N}} \in H$ is a Riesz basis if and only if it is complete (i.e., $\overline{\text{Span}(f_k)} = H$) and there exists constants $C_1, C_2 > 0$ such that, for any scalar sequence (a_k) with finite support,

$$C_1 \sum |a_k|^2 \leq \left\| \sum a_k f_k \right\|_H^2 \leq C_2 \sum |a_k|^2. \quad (3.29)$$

Let us now introduce the following growth condition:

Definition 3.2.2. Let $s \geq 0$, $(u_n) \in \mathbb{C}^{\mathbb{Z}}$ (or $u \in \mathcal{E}'$). We say that (u_n) (or u) has s -growth if

$$c\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2s}} \leq |u_n| \leq C\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2s}}, \quad \forall n \in \mathbb{Z}, \quad (3.30)$$

for some $c, C > 0$. The optimal constants for these inequalities are called growth constants.

Remark 3.2.2. The inequalities (3.30) can also be written, more intuitively, and for some other positive constants,

$$c(1 + |n|^s) \leq |u_n| \leq C(1 + |n|^s), \quad \forall n \in \mathbb{Z}. \quad (3.31)$$

We can now establish the following Riesz basis properties for the $(k_{n,\lambda})$:

Proposition 3.2.3. Let $s \geq 0$. If (F_n) has s -growth, then the family of functions

$$(k_{n,\lambda}^s) := \left(\frac{k_{n,\lambda}}{\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2s}}} \right)$$

is a Riesz basis for H_{per}^m .

Proof. We use the characterization of Riesz bases given in Proposition 3.2.2. First, let us prove the completeness of $(k_{n,\lambda}^s)$. Let $f \in H_{per}^m$ be such that

$$\langle f, k_{n,\lambda}^s \rangle_m = 0, \quad \forall n \in \mathbb{Z}.$$

Then for all $n \in \mathbb{Z}$ we get

$$0 = \langle \Lambda_n^\lambda \star \varphi, f \rangle_m = \sum_{p \in \mathbb{Z}} \frac{\overline{f_p} \varphi_p}{\lambda_{n+p}} \left(1 + \left| \frac{2i\pi p}{L} \right|^{2m} \right) = \left\langle \Lambda_n^\lambda, \sum_{p \in \mathbb{Z}} \left(1 + \left| \frac{2i\pi p}{L} \right|^{2m} \right) f_p \overline{\varphi_p} e_p \right\rangle,$$

as, thanks to (3.7), and using the fact that $f \in H_{per}^m$,

$$\sum_{p \in \mathbb{Z}} \left(1 + \left| \frac{2i\pi p}{L} \right|^{2m} \right) f_p \overline{\varphi_p} e_p \in L^2.$$

Now, (Λ_n^λ) is a complete family of L^2 , as it is a Riesz basis, so that

$$f_p \varphi_p = 0, \quad \forall p \in \mathbb{Z}.$$

Recalling condition (3.7), this yields

$$f_p = 0, \quad \forall p \in \mathbb{Z},$$

which proves the completeness of $(k_{n,\lambda}^s)$.

Now let $I \subset \mathbb{Z}$ be a finite set, and $(a_n) \in \mathbb{C}^I$. Then,

$$\begin{aligned} \left\| \sum_{n \in I} a_n k_{n,\lambda}^s \right\|_m^2 &= \left\| \sum_{n \in I} -a_n \frac{\overline{F_{-n}}}{\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2s}}} \Lambda_n^\lambda \star \varphi \right\|_m^2 \\ &= \left\| \sum_{n \in I} a_n \frac{\overline{F_{-n}}}{\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2s}}} \sum_{p \in \mathbb{Z}} \frac{\varphi_p}{\lambda_{n+p}} e_p \right\|_m^2 \\ &= \left\| \sum_{p \in \mathbb{Z}} \varphi_p \sum_{n \in I} \frac{a_n \overline{F_{-n}}}{\lambda_{n+p} \sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2s}}} e_p \right\|_m^2 \\ &= \sum_{p \in \mathbb{Z}} \left(1 + \left| \frac{2i\pi p}{L} \right|^{2m} \right) |\varphi_p|^2 \left| \sum_{n \in I} \frac{a_n \overline{F_{-n}}}{\lambda_{n+p} \sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2s}}} \right|^2. \end{aligned}$$

Now, using condition (3.7), we have

$$c^2 \sum_{p \in \mathbb{Z}} \left| \sum_{n \in I} \frac{a_n \overline{F_{-n}}}{\lambda_{n+p} \sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2s}}} \right|^2 \leq \left\| \sum_{n \in I} a_n k_{n,\lambda}^s \right\|_m^2 \leq C^2 \sum_{p \in \mathbb{Z}} \left| \sum_{n \in I} \frac{a_n \overline{F_{-n}}}{\lambda_{n+p} \sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2s}}} \right|^2,$$

where $c, C > 0$ are the decay constants in condition (3.7).

This last inequality can be rewritten

$$c^2 \left\| \sum_{n \in I} \frac{a_n \overline{F_{-n}}}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2s}}} \Lambda_n^\lambda \right\|^2 \leq \left\| \sum_{n \in I} a_n k_{n,\lambda}^s \right\|_m^2 \leq C^2 \left\| \sum_{n \in I} \frac{a_n \overline{F_{-n}}}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2s}}} \Lambda_n^\lambda \right\|^2,$$

as

$$\Lambda_n^\lambda = \sum_{p \in \mathbb{Z}} \frac{1}{\lambda_{n+p}} e_p.$$

We now use the fact that (Λ_n^λ) is a Riesz basis of L^2 : indeed, it is the image of the Hilbert basis (e_n) by the isomorphism

$$\Lambda^\lambda : f \in L^2 \mapsto \Lambda_0^\lambda f.$$

The norms of Λ^λ and its inverse are rather straightforward to compute using piecewise constant functions, we have

$$\begin{aligned} \|\Lambda^\lambda\| &= \frac{\sqrt{L}}{1 - e^{-\lambda L}}, \\ \|(\Lambda^\lambda)^{-1}\| &= \frac{1 - e^{-\lambda L}}{\sqrt{L}} e^{\lambda L}, \end{aligned}$$

so that

$$\frac{1}{\|(\Lambda^\lambda)^{-1}\|^2} \sum_{n \in I} \left| \frac{a_n \overline{F_{-n}}}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2s}}} \right|^2 \leq \left\| \sum_{n \in I} \frac{a_n \overline{F_{-n}}}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2s}}} \Lambda_n^\lambda \right\|^2 \leq \|\Lambda^\lambda\|^2 \sum_{n \in I} \left| \frac{a_n \overline{F_{-n}}}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2s}}} \right|^2,$$

and we finally get, using the fact that (F_n) has s -growth,

$$c^2 C_1^2 \frac{1}{\|(\Lambda^\lambda)^{-1}\|^2} \sum_{n \in I} |a_n|^2 \leq \left\| \sum_{n \in I} a_n k_{n,\lambda}^s \right\|_m^2 \leq C^2 C_2^2 \|\Lambda^\lambda\|^2 \sum_{n \in I} |a_n|^2.$$

where $C_1, C_2 > 0$ are the growth constants of (F_n) , so that the constants in the inequalities above are optimal. Hence, using again point 2. of Proposition 3.2.2, $(k_{n,\lambda}^s)$ is a Riesz basis of H_{per}^m . \square

We now have candidates for the backstepping transformation, under some conditions on F :

Corollary 3.2.1. *Let $m \in \mathbb{N}^*$, and F such that (F_n) has m -growth, with growth constants $C_1, C_2 > 0$. Define*

$$T^\lambda \alpha := \sum_{n \in \mathbb{Z}} \sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2m}} \alpha_n k_{-n,\lambda}^m = \sum_{n \in \mathbb{Z}} \alpha_n k_{-n,\lambda} \in H_{per}^m, \quad \forall \alpha \in H_{per}^m, \quad (3.32)$$

where $\alpha = \sum_{n \in \mathbb{Z}} \alpha_n e_n$. Then, $T^\lambda : H_{per}^m \rightarrow H_{per}^m$ is an isomorphism. Moreover,

$$\begin{aligned} \|T^\lambda\| &\leq \frac{CC_2 \sqrt{L}}{1 - e^{-\lambda L}}, \\ \|(T^\lambda)^{-1}\| &\leq \frac{1 - e^{-\lambda L}}{cC_1 \sqrt{L}} e^{\lambda L}. \end{aligned} \quad (3.33)$$

Proof. The invertibility of T^λ is clear thanks to the Riesz basis property of $(k_{-n,\lambda}^m)$, and (3.33) comes from the fact that, as mentioned at the end of the proof of Proposition 3.2.3, all the constants in the inequalities are optimal. \square

3.2.3 Definition of the feedback law

In order to further determine the feedback law, and define our final candidate for the backstepping transformation, the idea is now to return to the $TB = B$ condition (3.23), as we have used it in the formal computations of section 3.2.1, in the equation (3.23). However, in this case, $\varphi \notin H_{per}^m$, and so it is not clear whether $T^\lambda \varphi$ is well-defined.

We can nonetheless obtain a $TB = B$ condition in some weak sense: indeed, let us set

$$\varphi^{(N)} := \sum_{n=-N}^N \varphi_n e_n \in H_{per}^m, \quad \forall N \in \mathbb{N}.$$

Then,

$$\varphi^{(N)} \xrightarrow[N \rightarrow \infty]{H^{m-1}} \varphi$$

and

$$\begin{aligned} T^\lambda \varphi^{(N)} &= \sum_{n=-N}^N -\varphi_n \overline{F_n} \Lambda_{-n}^\lambda \star \varphi \\ &= \sum_{n=-N}^N \sum_{p \in \mathbb{Z}} \frac{-\varphi_n \overline{F_n} \varphi_p}{\lambda_{-n+p}} e_p \\ &= \sum_{p \in \mathbb{Z}} \varphi_p \left(\sum_{n=-N}^N \frac{-\varphi_n \overline{F_n}}{\lambda_{-n+p}} \right) e_p. \end{aligned}$$

Now, notice that one can apply the Dirichlet convergence theorem for Fourier series (see for example [101]) to $\Lambda_p^\lambda, p \in \mathbb{Z}$ at 0:

$$\sum_{n=-N}^N \frac{1}{\lambda_{-n+p}} = \sum_{n=-N}^N \frac{1}{\lambda_{n+p}} \xrightarrow[N \rightarrow \infty]{} \sqrt{L} \frac{\Lambda_p^\lambda(0) + \Lambda_p^\lambda(L)}{2} = \frac{L}{1 - e^{-\lambda L}} \frac{1 + e^{-\lambda L}}{2}.$$

Let us note

$$K(\lambda) := \frac{2}{L} \frac{1 - e^{-\lambda L}}{1 + e^{-\lambda L}},$$

and set

$$F_n := -\frac{K(\lambda)}{\varphi_n}, \quad \forall n \in \mathbb{Z}. \quad (3.34)$$

This defines a feedback law $F \in \mathcal{E}'$ which is real-valued, as φ is real-valued, and which has m -growth thanks to condition (3.7), so that T^λ is a valid backstepping transformation. Moreover,

$$\langle T^\lambda \varphi^{(N)}, e_p \rangle = \varphi_p K(\lambda) \sum_{n=-N}^N \frac{1}{\lambda_{-n+p}} \xrightarrow[N \rightarrow \infty]{} \varphi_p, \quad \forall p \in \mathbb{Z}, \quad (3.35)$$

which corresponds to the $TB = B$ condition in some weak sense.

With this feedback law, the backstepping transformation now writes

$$T^\lambda \alpha = \sum_{n \in \mathbb{Z}} \alpha_n k_{-n, \lambda}, \quad \forall \alpha \in H_{per}^m, \quad (3.36)$$

and

$$\begin{aligned} \|T^\lambda\| &= \frac{CK(\lambda)\sqrt{L}}{c(1 - e^{-\lambda L})}, \\ \|(T^\lambda)^{-1}\| &= \frac{C(1 - e^{-\lambda L})}{cK(\lambda)\sqrt{L}} e^{\lambda L}. \end{aligned} \quad (3.37)$$

3.2.4 Regularity of the feedback law

Finally, in order to study the well-posedness of the closed-loop system corresponding to (3.34), we need some information on the regularity of F .

Let us first begin by a general lemma for linear forms with coefficients that have m -growth:

Lemma 3.2.1. *Let $m \geq 0$, and $G \in \mathcal{E}'$ with m -growth.*

Then, for all $s > 1/2$, G is defined on H_{per}^{m+s} , is continuous for $\|\cdot\|_{m+s}$, but not for $\|\cdot\|_{m+\sigma}$, for $-m \leq \sigma < 1/2$.

In particular, the feedback law $F \in \mathcal{E}'$ defined by (3.34) defines a linear form on H_{per}^{m+1} which is continuous for $\|\cdot\|_{m+1}$ but not for $\|\cdot\|_m$.

Proof. Let $s > 1/2$, and let $\alpha \in H_{per}^{m+s}$. Using the growth conditions (3.30), we can do the following computations for $\alpha \in H_{per}^{m+s}$:

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |G_n| |\alpha_n| &\leq C \sum_{n \in \mathbb{Z}} \sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2m}} |\alpha_n| \\ &\leq C' \sum_{n \in \mathbb{Z}} \frac{1}{1 + |n|^s} \sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2m+2s}} |\alpha_n| \\ &\leq C' \left(\sqrt{\sum_{n \in \mathbb{Z}} \frac{1}{(1 + |n|^s)^2}} \right) \|\alpha\|_{m+s} \end{aligned}$$

where $C, C' > 0$ are constants that do not depend on α , and where the last inequality is obtained using the Cauchy-Schwarz inequality. Thus G is defined on H_{per}^{m+s} by

$$\langle \alpha, G \rangle := \sum_{n \in \mathbb{Z}} G_n \alpha_n, \quad \forall \alpha \in H_{per}^{m+s},$$

and G is continuous on H_{per}^{m+s} .

On the other hand, let $-m \leq \sigma < 1/2$, and consider, for $N \geq 1$,

$$\gamma^{(N)} := \sum_{|n| \geq N} \frac{1}{G_n (1 + |n|^{1+s})} e_n \in H_{per}^{m+s}.$$

We have

$$\|\gamma^{(N)}\|_{m+\sigma}^2 = \sum_{|n| \geq N} \frac{\left(1 + \left| \frac{2i\pi n}{L} \right|^{2m+2\sigma}\right)}{|G_n|^2} \frac{1}{(1 + |n|^{1+s})^2} \leq C \sum_{|n| \geq N} \frac{1}{1 + |n|^{2+2s-2\sigma}}$$

for some constant $C > 0$. Then,

$$\begin{aligned} |\langle \gamma^{(N)}, G \rangle| &= \sum_{|n| \geq N} \frac{1}{1 + |n|^{1+s}} \\ &\geq c \sum_{|n| \geq N} |n|^{1+s-2\sigma} \frac{1}{1 + |n|^{2+2s-2\sigma}} \\ &\geq cN^{1+s-2\sigma} \sum_{|n| \geq N} \frac{1}{1 + |n|^{2+2s-2\sigma}} \\ &\geq c'N^{1+s-2\sigma} \sqrt{\sum_{|n| \geq N} \frac{1}{1 + |n|^{2+2s-2\sigma}}} \|\gamma^{(N)}\|_{m+\sigma} \end{aligned}$$

for some constants $c, c' > 0$. Now, we know that there exists constants $c'', C'' > 0$ such that

$$\frac{c''}{N^{1+2s-2\sigma}} \leq \sum_{|n| \geq N} \frac{1}{1 + |n|^{2+2s-2\sigma}} \leq \frac{C''}{N^{1+2s-2\sigma}},$$

So that

$$N^{1+s-2\sigma} \sqrt{\sum_{|n| \geq N} \frac{1}{1 + |n|^{2+2s-2\sigma}}} \geq c'' N^{\frac{1}{2}-\sigma} \xrightarrow{N \rightarrow \infty} \infty.$$

This proves that G is not continuous for $\|\cdot\|^{m+\sigma}$. \square

Let us now give a more precise description of the domain of definition and regularity of F . Recalling the identity (3.9), we can derive the following identity for F_n from (3.34):

$$F_n = (-1)^m \frac{K(\lambda)}{\tau_{-n}^\varphi} \left(\frac{2i\pi n}{L} \right)^m + (-1)^m \frac{K(\lambda)}{\tau_{-n}^\varphi} \left(\frac{2i\pi n}{L} \right)^m \frac{\overline{\sum_{j=0}^d \langle \chi_{[\sigma_j, \sigma_{j+1}]} \partial^m \varphi, e_n \rangle}}{\tau_{-n}^\varphi - \sum_{j=0}^d \langle \chi_{[\sigma_j, \sigma_{j+1}]} \partial^m \varphi, e_n \rangle}, \quad \forall n \in \mathbb{Z}^*, \quad (3.38)$$

so that

$$\left(\frac{1}{\left(\frac{2i\pi n}{L} \right)^m} \left(F_n - (-1)^m \frac{K(\lambda)}{\tau_{-n}^\varphi} \left(\frac{2i\pi n}{L} \right)^m \right) \right)_{n \in \mathbb{Z}^*} \in \ell^2. \quad (3.39)$$

Let us then note

$$h_n := (-1)^m \frac{K(\lambda)}{\tau_{-n}^\varphi} \left(\frac{2i\pi n}{L} \right)^m, \quad \forall n \in \mathbb{Z},$$

and h the associated linear form in \mathcal{E}' .

Proposition 3.2.4. *The linear form h defines the following linear form on $\tau^\varphi(H_{(pw)}^{m+1})$, continuous for $\|\cdot\|_{m+1, pw}$:*

$$\langle \alpha, h \rangle = \sqrt{L} \frac{K(\lambda)}{2} (\partial^m ((\tau^\varphi)^{-1} \alpha)(0) + \partial^m ((\tau^\varphi)^{-1} \alpha)(L)), \quad \forall \alpha \in \tau^\varphi(H_{(pw)}^{m+1}). \quad (3.40)$$

Moreover, $\tilde{F} := F - h$ is continuous for $\|\cdot\|_m$, so that F is defined on $\tau^\varphi(H_{(pw)}^{m+1}) \cap H_{per}^m$, and is continuous for $\|\cdot\|_{m+1, pw}$, but not $\|\cdot\|_m$.

Proof. It is clear, by definition of H_{per}^m , and using (3.39), that for $\alpha \in H_{per}^m$, the expression:

$$\langle \alpha, F - h \rangle = \sum_{n \in \mathbb{Z}} \alpha_n (\overline{F_n} - \overline{h_n}) = \frac{K(\lambda) \alpha_0}{\varphi_0} + \sum_{n \neq 0} \left(\frac{2i\pi n}{L} \right)^m \alpha_n \frac{1}{\left(\frac{2i\pi n}{L} \right)^m} (\overline{F_n} - \overline{h_n}) \quad (3.41)$$

defines a continuous linear form on H_{per}^m .

On the other hand, let $\alpha \in \tau^\varphi(H_{(pw)}^{m+2})$, then

$$\sum_{n=-N}^N \alpha_n \overline{h_n} = \sqrt{L} K(\lambda) \sum_{n=-N}^N \left(\frac{2i\pi n}{L} \right)^m \frac{\alpha_n}{\tau_n^\varphi} \frac{1}{\sqrt{L}}$$

we can use the Dirichlet convergence theorem (see [101]) on $\partial^m ((\tau^\varphi)^{-1} \alpha) \in H_{(pw)}^2$ at 0, so that

$$\begin{aligned} \sum_{n=-N}^N \alpha_n \overline{h_n} &= \sqrt{L} K(\lambda) \sum_{n=-N}^N \left(\frac{2i\pi n}{L} \right)^m \frac{\alpha_n}{\tau_n^\varphi} \frac{1}{\sqrt{L}} \\ &\xrightarrow{N \rightarrow \infty} \sqrt{L} \frac{K(\lambda)}{2} (\partial^m ((\tau^\varphi)^{-1} \alpha)(0) + \partial^m ((\tau^\varphi)^{-1} \alpha)(L)). \end{aligned}$$

Now, we know that $H_{(pw)}^{m+2}$ is dense in $H_{(pw)}^{m+1}$ for the $H_{(pw)}^{m+1}$ norm. As τ^φ is a sum of translations, it is continuous for $\|\cdot\|_{m+1,pw}$, so that $\tau^\varphi(H_{(pw)}^{m+2})$ is dense in $\tau^\varphi(H_{(pw)}^{m+1})$ for $\|\cdot\|_{m+1,pw}$.

Moreover, using the Sobolev inequality for H^1 and L^∞ (see for example [35, Chapter 8, Theorem 8.8]), we get the continuity of h for $\|\cdot\|_{m+1,pw}$, so that we can extend it from $\tau^\varphi(H_{(pw)}^{m+2})$ to $\tau^\varphi(H_{(pw)}^{m+1})$ by density. We also get that h is not continuous for $\|\cdot\|_m$, as $\alpha \in H^m \mapsto \partial^m \alpha(0)$ and $\alpha \in H^m \mapsto \partial^m \alpha(L)$ are not continuous for $\|\cdot\|_m$.

Thus, $F = \tilde{F} + h$ is defined on $\tau^\varphi(H_{(pw)}^{m+1}) \cap H_{per}^m$, is continuous for $\|\cdot\|_{m+1}$ but not for $\|\cdot\|_m$. \square

3.3 Well-posedness and stability of the closed-loop system

Let $m \geq 1$, $\varphi \in H_{(pw)}^m \cap H_{per}^{m-1}$ satisfying growth condition (3.7). Let the feedback law F be defined by (3.34).

3.3.1 Operator equality

Now that we have completely defined the feedback F and the transformation T^λ , let us check that we have indeed built a backstepping transformation. As in the finite dimensional example of subsection 3.1.4, this corresponds to the formal operator equality

$$T(A + BK) = (A - \lambda I)T.$$

Let us define the following domain:

$$D_m := \left\{ \alpha \in \tau^\varphi(H_{(pw)}^{m+1}) \cap H_{per}^m, \quad -\alpha_x - \mu\alpha + \langle \alpha, F \rangle \varphi \in H_{per}^m \right\}. \quad (3.42)$$

Notice that, as $\varphi \in H_{(pw)}^m$, the condition $\alpha \in H_{(pw)}^{m+1} \supset \tau^\varphi(H_{(pw)}^{m+1})$ is necessary for $-\alpha_x - \mu\alpha + \langle \alpha, F \rangle \varphi$ to be in H_{per}^m . Let us first check the following property:

Proposition 3.3.1. *For $m \geq 1$, D_m is dense in H_{per}^m for $\|\cdot\|_m$.*

Proof. It is clear that $H_{per}^{m+1} \subset \tau^\varphi(H_{(pw)}^{m+1})$, so that

$$\mathcal{K}_m := \left\{ \alpha \in H_{per}^{m+1}, \quad \langle \alpha, F \rangle = 0 \right\} \subset D_m.$$

Now, by Lemma 3.2.1, as F has m -growth, \mathcal{K}_m is dense in H_{per}^{m+1} for $\|\cdot\|_m$, as the kernel of the linear form F which is not continuous for $\|\cdot\|_m$. As H_{per}^{m+1} is dense in H_{per}^m , then D_m is dense in H_{per}^m for $\|\cdot\|_m$. \square

Now, on this dense domain, let us establish the operator equality:

Proposition 3.3.2.

$$T^\lambda(-\partial_x - \mu I + \langle \cdot, F \rangle \varphi)\alpha = (-\partial_x - \lambda I)T^\lambda\alpha \quad \text{in } H_{per}^m, \quad \forall \alpha \in D_m. \quad (3.43)$$

Proof. First let us rewrite (3.43) in terms of λ :

$$T^\lambda(-\partial_x + \langle \cdot, F \rangle \varphi)\alpha = (-\partial_x - \lambda I)T^\lambda\alpha \quad \text{in } H_{per}^m, \quad \forall \alpha \in D_m(F).$$

Let $\alpha \in D_m$. By definition of the domain D_m , the left-hand side of (3.43) is a function of $H_{per}^m \subset \mathcal{E}'$, and by construction of T^λ , the right-hand side of (3.43) is a function of $H_{per}^{m-1} \subset \mathcal{E}'$. To prove that these functions are equal, it is thus sufficient to prove their equality in \mathcal{E}' . Let us then write each term of the equality against e_n for $n \in \mathbb{Z}$. One has

$$\begin{aligned}
\langle (-\partial_x - \lambda I)T^\lambda \alpha, e_n \rangle &= \left\langle T^\lambda \alpha, \frac{2i\pi n}{L} e_n \right\rangle - \lambda \langle T^\lambda \alpha, e_n \rangle \\
&= -\lambda_n \langle T^\lambda \alpha, e_n \rangle.
\end{aligned}$$

Let us now prove that

$$\langle T^\lambda(-\alpha_x + \langle \alpha, F \rangle \varphi), e_n \rangle = -\lambda_n \langle T^\lambda \alpha, e_n \rangle, \quad \forall n \in \mathbb{Z}. \quad (3.44)$$

Now, as we only have $\alpha_x \in H_{per}^{m-1}$, $T^\lambda \alpha_x$ is not defined *a priori*. In order to allow for more computations, let us define

$$\begin{aligned}
\alpha^{(N)} &:= \sum_{n=-N}^N \alpha_n e_n, \quad \forall N \in \mathbb{N}, \\
\varphi^{(N)} &:= \sum_{n=-N}^N \varphi_n e_n,
\end{aligned}$$

so that we have, by property of the partial Fourier sum of a H_{per}^m function,

$$-\alpha_x^{(N)} + \langle \alpha, F \rangle \varphi^{(N)} \xrightarrow[N \rightarrow \infty]{H^m} -\alpha_x + \langle \alpha, F \rangle \varphi,$$

so that in particular,

$$\langle T^\lambda(-\alpha_x^{(N)} + \langle \alpha, F \rangle \varphi^{(N)}), e_n \rangle \xrightarrow[N \rightarrow \infty]{} \langle T^\lambda(-\alpha_x + \langle \alpha, F \rangle \varphi), e_n \rangle \quad (3.45)$$

Let $N \in \mathbb{N}$. We can now write

$$\begin{aligned}
\langle T^\lambda(-\alpha_x^{(N)} + \langle \alpha, F \rangle \varphi^{(N)}), e_n \rangle &= -\langle T^\lambda \alpha_x^{(N)}, e_n \rangle + \langle \alpha, F \rangle \langle T^\lambda \varphi^{(N)}, e_n \rangle \\
&= -\left\langle \sum_{p=-N}^N \frac{2i\pi p}{L} \alpha_p k_{-p, \lambda}, e_n \right\rangle + \langle \alpha, F \rangle \langle T^\lambda \varphi^{(N)}, e_n \rangle.
\end{aligned}$$

Now, using (3.24), we get

$$\frac{2i\pi p}{L} k_{-p, \lambda} = (k_{-p, \lambda})_x + \lambda k_{-p, \lambda} + \overline{F_p} \varphi,$$

so that

$$-T^\lambda \alpha_x^{(N)} = \sum_{p=-N}^N \alpha_p ((k_{-p, \lambda})_x + \lambda k_{-p, \lambda} + \overline{F_p} \varphi).$$

Hence

$$-\langle T^\lambda \alpha_x^{(N)}, e_n \rangle = -\left\langle (T^\lambda \alpha^{(N)})_x, e_n \right\rangle - \lambda \langle T^\lambda \alpha^{(N)}, e_n \rangle - \langle \alpha^{(N)}, F \rangle \varphi_n,$$

and finally,

$$\begin{aligned}
\langle T^\lambda(-\alpha_x^{(N)} + \langle \alpha, F \rangle \varphi^{(N)}), e_n \rangle &= -\lambda_n \langle T^\lambda \alpha^{(N)}, e_n \rangle + \left(\langle \alpha - \alpha^{(N)}, F \rangle \right) \varphi_n \\
&\quad + \langle \alpha, F \rangle \left(\langle T^\lambda \varphi^{(N)} - \varphi, e_n \rangle \right). \quad (3.46)
\end{aligned}$$

To deal with the third term of the right-hand side of this equality, recall that we have chosen a feedback law so that the $TB = B$ condition (3.23) holds. Thus,

$$\langle T^\lambda \varphi^{(N)} - \varphi, e_n \rangle \xrightarrow[N \rightarrow \infty]{} 0. \quad (3.47)$$

To deal with the second term, recall that F is the sum of a regular part \tilde{F} and a singular part h :

$$\langle \alpha - \alpha^{(N)}, F \rangle = \langle \alpha - \alpha^{(N)}, \tilde{F} \rangle + \langle \alpha - \alpha^{(N)}, h \rangle.$$

Now, by definition of $\alpha^{(N)}$ and continuity of \tilde{F} for $\|\cdot\|_m$,

$$\langle \alpha - \alpha^{(N)}, \tilde{F} \rangle \xrightarrow{N \rightarrow \infty} 0. \quad (3.48)$$

On the other hand, for all $N \in \mathbb{N}$,

$$\begin{aligned} \langle \alpha^{(N)}, h \rangle &= K(\lambda) \sum_{n=-N}^N \frac{\alpha_n}{\tau_n^\varphi} \left(\frac{2i\pi n}{L} \right)^m \\ &= \frac{K(\lambda)}{2} \sum_{n=-N}^N \left(\frac{\alpha_n}{\tau_n^\varphi} + (-1)^m \frac{\alpha_{-n}}{\tau_{-n}^\varphi} \right) \left(\frac{2i\pi n}{L} \right)^m. \\ &= \sqrt{L} \frac{K(\lambda)}{2} \partial^{m-1} \tilde{\tau}^\varphi \alpha_x^{(N)}(0), \end{aligned} \quad (3.49)$$

where

$$\tilde{\tau}^\varphi f = \sum_{n \in \mathbb{Z}} \left(\frac{f_n}{\tau_n^\varphi} + (-1)^{m-1} \frac{f_{-n}}{\tau_{-n}^\varphi} \right) e_n, \quad \forall f \in L^2.$$

Now, notice that, by definition of τ^φ and D_m ,

$$\tilde{\tau}^\varphi (-\alpha_x - \mu\alpha + \langle \alpha, F \rangle \varphi) \in H_{per}^m. \quad (3.50)$$

Moreover, using (3.9), we have for $n \in \mathbb{Z}^*$:

$$\begin{aligned} \frac{\varphi_n}{\tau_n^\varphi} + (-1)^{m-1} \frac{\varphi_{-n}}{\tau_{-n}^\varphi} &= \frac{\varphi_n}{\tau_n^\varphi} + (-1)^{m-1} \frac{\overline{\varphi_n}}{\tau_n^\varphi} \\ &= \frac{-1 - (-1)^{m-1} (-1)^m}{\left(\frac{2i\pi}{L} n \right)^m} + \frac{r_n}{\left(\frac{2i\pi}{L} n \right)^m} \\ &= \frac{r_n}{\left(\frac{2i\pi}{L} n \right)^m}, \end{aligned}$$

where $r_n \in \ell^2$. Hence, $\tilde{\tau}^\varphi \varphi \in H_{per}^m$. This, together with (3.50), yields

$$\tilde{\tau}^\varphi \alpha_x \in H_{per}^m.$$

This implies that

$$\tilde{\tau}^\varphi \alpha_x^{(N)} \xrightarrow{N \rightarrow \infty} \tilde{\tau}^\varphi \alpha_x,$$

as $\tilde{\tau}^\varphi \alpha_x^{(N)}$ is the partial sum of $\tilde{\tau}^\varphi \alpha_x$.

Hence, by continuity of $\alpha \mapsto \partial^{m-1} \alpha(0)$ for $\|\cdot\|_m$, (3.49) implies that

$$\langle \alpha - \alpha^{(N)}, h \rangle \xrightarrow{N \rightarrow \infty} 0. \quad (3.51)$$

Finally, (3.46), (3.47), (3.48), (3.51), and the continuity of T^λ yield

$$\langle T^\lambda (-\alpha_x^{(N)} + \langle \alpha, F \rangle \varphi^{(N)}), e_n \rangle \xrightarrow{N \rightarrow \infty} -\lambda_n \langle T^\lambda \alpha, e_n \rangle.$$

This, put together with (3.45), gives (3.44) by uniqueness of the limit, which in turn proves (3.43). \square

Remark 3.3.1. When $\varphi \in H^m$, τ^φ is simply $(1/\sqrt{L})(\partial^{m-1}\varphi(L) - \partial^{m-1}\varphi(0))Id$, F is defined on $H^{m+1} \cap H_{per}^m$, and $\tilde{\tau}^\varphi\alpha$ is simply, up to a constant factor, the symmetrisation $\alpha + (-1)^{m-1}\alpha(L - \cdot)$, which is H_{per}^m if $\alpha \in H^m \cap H_{per}^{m-1}$.

3.3.2 Well-posedness of the closed-loop system

The operator equality we have established in the previous section means that T^λ transforms, if they exist, solutions of the closed-loop system with a well-chosen feedback into solutions of the target system. Let us now check that the closed-loop system in question is indeed well-posed in some sense.

Proposition 3.3.3. *The operator $A+BK := -\partial_x - \mu\alpha + \langle F, \cdot \rangle\varphi$ defined on D_m is a dense restriction of the infinitesimal generator of a C^0 -semigroup on H_{per}^m .*

Proof. We know from Lemma 3.3.1 that $A+BK$ is densely defined on $D_m \subset H_{per}^m$.

Now, define the following semigroup on H_{per}^m :

$$S_{\lambda'}(t)\alpha := e^{-\lambda't}\alpha(\cdot - t), \quad \forall \alpha \in H_{per}^m, \quad t \geq 0, \quad (3.52)$$

which corresponds to the target system (3.21). Its infinitesimal is given by

$$D^{\lambda'} := H_p^{m+1}er, \quad (3.53)$$

$$-\partial_x - \lambda'I.$$

Now, define a second semigroup on H_{per}^m :

$$S(t)\alpha := (T^\lambda)^{-1}S_{\lambda'}(t)T^\lambda\alpha, \quad \forall \alpha \in H_{per}^m, \quad t \geq 0. \quad (3.54)$$

The infinitesimal generator of $S(t)$ is given, when it exists, by the limit of

$$\frac{S(t)\alpha - \alpha}{t} = (T^\lambda)^{-1} \frac{S_{\lambda'}(t)T^\lambda\alpha - T^\lambda\alpha}{t}, \quad (3.55)$$

so, by (3.53), the domain of the infinitesimal generator of $S(t)$ is $(T^\lambda)^{-1}(H_{per}^{m+1})$, and the infinitesimal generator itself is given by

$$\frac{S(t)\alpha - \alpha}{t} \xrightarrow[t \rightarrow 0^+]{H^m} (T^\lambda)^{-1}(-\partial_x - \lambda'I)T^\lambda\alpha. \quad (3.56)$$

In particular, by (3.43),

$$(T^\lambda)^{-1}(-\partial_x - \lambda'I)T^\lambda\alpha = (-\partial_x - \mu I + \langle \cdot, F \rangle\varphi)\alpha = (A+BK)\alpha, \quad (3.57)$$

which proves the proposition. □

3.3.3 Stability of the closed-loop system

We can now prove Theorem 3.1.1.

Let $S(t)$ the semigroup defined by (3.54), $\alpha \in H_{per}^m$.

By definition of $S(t)$, and using (3.37), we then get, for $t \geq 0$,

$$\begin{aligned} \|S(t)\alpha\|_m &\leq \|(T^\lambda)^{-1}\| \|S_{\lambda'}(t)T^\lambda\alpha\|_m \\ &\leq \|(T^\lambda)^{-1}\| e^{-\lambda't} \|T^\lambda\alpha\|_m \\ &\leq \|(T^\lambda)^{-1}\| \|T^\lambda\| e^{-\lambda't} \|\alpha\|_m \\ &\leq \left(\frac{C}{c}\right)^2 e^{\lambda L} e^{-\lambda't} \|\alpha\|_m, \end{aligned}$$

which proves the exponential stability of the semigroup $S(t)$.

Now consider the particular case where $C = c > 0$, and $\mu = 0$ to simplify notations:

$$\varphi_0 := C, \quad \varphi_n := \frac{C}{\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2m}}}, \quad \forall n \in \mathbb{Z}^*, \quad (3.58)$$

so that

$$\|\alpha \star \varphi\|_m = C\|\alpha\|, \quad \forall \alpha \in L^2, \quad \|\alpha \star F\| = \frac{1}{C}\|\alpha\|_m, \quad \forall \alpha \in H_{per}^m. \quad (3.59)$$

Now let $\varepsilon > 0$. Keeping in mind that $(\chi_{[0,1/n]})_{n>0}$ and $(\chi_{[L-1/n,L]})_{n>0}$ are maximizing sequences for Λ^λ and $(\Lambda^\lambda)^{-1}$ respectively, . Using (3.32), (3.34), (3.52), we get

$$\begin{aligned} S\left(L - \frac{1}{n}\right)(\chi_{[0,1/n]} \star \varphi) &= (T^\lambda)^{-1} S_{\lambda'}\left(L - \frac{1}{n}\right) T^\lambda(\chi_{[0,1/n]} \star \varphi) \\ &= (T^\lambda)^{-1} S_{\lambda'}\left(L - \frac{1}{n}\right) (\varphi \star (\Lambda^\lambda(\chi_{[0,1/n]}))) \\ &= \frac{e^{-\lambda(L-1/n)}\sqrt{L}}{1 - e^{-\lambda L}} (T^\lambda)^{-1} \varphi \star (\chi_{[L-1/n,L]} e^{-\lambda(\cdot-L+1/n)}) \\ &= e^{-\lambda(L-1/n)} e^{\lambda(L-1/n)} (\chi_{[L-1/n,L]} \star \varphi), \quad \forall n > 0, \end{aligned} \quad (3.60)$$

so that

$$\left\| S\left(L - \frac{1}{n}\right)(\chi_{[0,1/n]} \star \varphi) \right\|_m = e^{-\lambda(L-1/n)} e^{\lambda(L-1/n)} \|\chi_{[0,1/n]} \star \varphi\|_m, \quad \forall n > 0. \quad (3.61)$$

Then, there exists $n > 0$ such that

$$\left\| S\left(L - \frac{1}{n}\right)(\chi_{[0,1/n]} \star \varphi) \right\|_m > e^{-\lambda(L-1/n)} (e^{\lambda L} - \varepsilon) \|\chi_{[0,1/n]} \star \varphi\|_m. \quad (3.62)$$

This shows that estimate (3.8) can be critical in some cases.

3.3.4 Application

Let $m = 1, \lambda > 0$, and let us suppose, to simplify the computations, that $a \equiv 0$. Define

$$\varphi(x) = L - x, \quad \forall x \in (0, L), \quad (3.63)$$

so that $\varphi \in H^1$ but is not periodic, and satisfies (3.7), with

$$\begin{aligned} \varphi_n &= -\frac{iL^{\frac{3}{2}}}{2\pi n}, \quad \forall n \in \mathbb{Z}^*, \\ \varphi_0 &= \frac{L^{\frac{3}{2}}}{2}. \end{aligned}$$

Then,

$$\langle \alpha, F \rangle = -\frac{2K(\lambda)}{L^{\frac{3}{2}}} \alpha_0 - K(\lambda) \frac{\alpha_x(0) + \alpha_x(L)}{2}, \quad \forall \alpha \in H^2 \cap H_{per}^1, \quad (3.64)$$

and

$$D_1 = \left\{ \alpha \in H^2 \cap H_{per}^1, \quad \frac{2K(\lambda)}{L^{\frac{3}{2}}} \alpha_0 + \left(\frac{1}{L} - K(\lambda)\right) \alpha_x(0) - \frac{1}{L} \alpha_x(L) = 0 \right\},$$

so that

$$\begin{cases} \alpha_t + \alpha_x = \left(-\frac{2K(\lambda)}{L^{\frac{3}{2}}} \alpha_0 - K(\lambda) \frac{\alpha_x(0) + \alpha_x(L)}{2}\right) \varphi(x), & x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), & \forall t \geq 0, \end{cases} \quad (3.65)$$

has a unique solution for initial conditions in D_1 .

The backstepping transformation can be written as:

$$T^\lambda \alpha = \frac{\sqrt{L}}{1 - e^{-\lambda L}} \left(e^{-\lambda x} \left(-\frac{K(\lambda)}{\sqrt{L}} \alpha_x - \frac{2K(\lambda)}{L^2} \alpha_0 \right) \right) \star \varphi, \quad \forall \alpha \in H_{per}^1. \quad (3.66)$$

Let $\alpha(t) \in D_1$ be the solution of the closed loop system (3.65) with initial condition $\alpha^0 \in D_1$, and let us note $z(t) := T^\lambda \alpha(t)$, then

$$\begin{aligned} z_t &= \frac{\sqrt{L}}{1 - e^{-\lambda L}} \left(e^{-\lambda x} \left(-\frac{K(\lambda)}{\sqrt{L}} \alpha_{xt} - \frac{2K(\lambda)}{L^2} \alpha'_0 \right) \right) \star \varphi. \\ &= \frac{\sqrt{L}}{1 - e^{-\lambda L}} \left(e^{-\lambda x} \left(-\frac{K(\lambda)}{\sqrt{L}} (-\alpha_{xx} + \langle \alpha, F \rangle \varphi_x) - \frac{2K(\lambda)}{L^2} \alpha'_0 \right) \right) \star \varphi. \\ &= \frac{\sqrt{L}}{1 - e^{-\lambda L}} \left(e^{-\lambda x} \left(-\frac{K(\lambda)}{\sqrt{L}} (-\alpha_{xx} - \langle \alpha, F \rangle) - \frac{2K(\lambda)}{L^2} \alpha'_0 \right) \right) \star \varphi. \\ z_x &= \frac{\sqrt{L}}{1 - e^{-\lambda L}} \left(-e^{-\lambda x} \frac{K(\lambda)}{\sqrt{L}} \alpha_{xx} \right) \star \varphi - \lambda z \\ z_t + z_x + \lambda z &= \frac{\sqrt{L}}{1 - e^{-\lambda L}} \left(e^{-\lambda x} \left(\frac{K(\lambda)}{\sqrt{L}} \langle \alpha, F \rangle - \frac{2K(\lambda)}{L^2} \alpha'_0 \right) \right) \star \varphi. \end{aligned}$$

By projecting the closed loop system on e_0 , we get

$$\alpha'_0 = \langle \alpha, F \rangle \varphi_0 = \langle \alpha, F \rangle \frac{L^{\frac{3}{2}}}{2}$$

so that

$$z_t + z_x + \lambda z = 0.$$

In particular,

$$\frac{d}{dt} \|z\|_1^2 = -2\lambda \|z\|_1^2. \quad (3.67)$$

Let us now set

$$V(\alpha) := \|z\|_1^2, \quad \forall \alpha \in H_{per}^1.$$

Now, notice that

$$\begin{aligned} \|T^\lambda \alpha\|_1^2 &= \frac{L}{(1 - e^{-\lambda L})^2} \sum_{n \in \mathbb{Z}} \left(1 + \left| \frac{2i\pi n}{L} \right|^2 \right) |\varphi_n|^2 \left| \left\langle e^{-\lambda x} \left(-\frac{K(\lambda)}{\sqrt{L}} \alpha_x - \frac{2K(\lambda)}{L^2} \alpha_0 \right), e_n \right\rangle \right|^2 \\ &\geq C \left\| e^{-\lambda x} \left(-\frac{K(\lambda)}{\sqrt{L}} \alpha_x - \frac{2K(\lambda)}{L^2} \alpha_0 \right) \right\|^2 \\ &\geq C e^{2\lambda L} \left\| -\frac{K(\lambda)}{\sqrt{L}} \alpha_x - \frac{2K(\lambda)}{L^2} \alpha_0 \right\|^2 \\ &\geq C' K(\lambda)^2 e^{2\lambda L} \|\alpha\|_1^2. \end{aligned}$$

Together with (3.67), this shows that V is a Lyapunov function, and (3.65) is exponentially stable.

3.4 Further remarks and questions

3.4.1 Controllability and the $TB = B$ condition

In the introduction we have mentioned that the growth constraint on the Fourier coefficients of φ actually corresponds to the exact null controllability condition in some Sobolev space for the control system (3.2). As we have mentioned in the finite dimensional example, the controllability condition is essential to solve the operator equation: in our case, formal computations lead to a family of functions that turns out to be a Riesz basis precisely thanks to that rate of growth. Moreover, that rate of growth is essential for the compatibility of the $TB = B$ condition and the invertibility of the backstepping transformation. Indeed, as the transformation is constructed formally using a formal $TB = B$ condition, that same $TB = B$ condition fixes the value of the coefficients of F_n , giving them the right rate of growth for T^λ to be an isomorphism.

In that spirit, it would be interesting to investigate if a backstepping approach is still valid if the conditions on φ are weakened. For example, if we suppose approximate controllability instead of exact controllability, i.e.

$$\varphi_n \neq 0, \quad \forall n \in \mathbb{Z},$$

then F can still be defined using a weak $TB = B$ condition. However, it seems delicate to prove, in the same direct way as we have done, that T^λ is an isomorphism, as we only get the completeness of the corresponding $(k_{n,\lambda})$, but not the Riesz basis property.

Finally, it should be noted that, while in [59] the $TB = B$ condition is well-defined, in our case, it only holds in a rather weak sense. This is probably because of a lack of regularization, indeed in [59] the backstepping transformation has nice properties, as it can be decomposed in Fredholm form, i.e. as the sum of an isomorphism and a compact operator. Accordingly, the Riesz basis in that case is quadratically close to the orthonormal basis given by the eigenvectors of the Laplacian operator. That is not the case for our backstepping transformation, as it is closely linked to the operator Λ^λ , which does not have any nice spectral properties.

Nonetheless, it appears that thanks to some information on the regularity of F , a weak sense is sufficient and allows us to prove the operator equality by convergence.

3.4.2 Regularity of the feedback law

As we have pointed out in Section 3.2.4, if φ is such that system (3.2) is controllable in H_{per}^m , then the feedback law F defined by (3.34) is continuous for $\|\cdot\|_{m+1}$ but not for $\|\cdot\|_m$. This was actually to be expected, as we have mentioned in the introduction that Shun Hua Sun proved that bounded feedback laws can only achieve “compact” perturbations of the spectrum, which is not enough to get exponential stabilization. More precisely, it would be possible to get exponential stabilization only with very singular controllers. With a distributed control such as ours, it is necessary to consider unbounded feedback laws.

Moreover, the application in Section 3.3.4 shows that even though the feedback is not continuous, and is given by its Fourier coefficients, in practice it can be expressed quite simply for some controllers.

3.4.3 Null-controllability and finite-time stabilization

As we have mentioned in the introduction, one of the advantages of the backstepping method is that it can provide an explicit expression for feedbacks, thus allowing the construction of explicit controls for null controllability, as well as time-varying feedbacks that stabilize the system in finite time $T > 0$.

The general strategy (as is done in [67], [155]) is to divide the interval $[0, T]$ in smaller intervals $[t_n, t_{n+1}]$, the length of which tends to 0, and on which one applies feedbacks to get exponential stabilization with decay rates λ_n , with $\lambda_n \rightarrow \infty$. Then, for well-chosen t_n, λ_n , the trajectory thus obtained reaches 0 in time T . Though this provides an explicit control to steer the system to 0, the norm of the operators applied successively to obtain the control tends to infinity. As such, it does not provide a reasonably regular feedback. However, the previous construction of the control can be used, with some adequate modifications (see [67] and [156]) to design a time-varying, periodic feedback, with some regularity in the state variable, which stabilizes the system in finite time.

Let us first note that, due to the hyperbolic nature of the system, there is a minimal control time, and thus small-time stabilization cannot be expected. Moreover, even for $T > L$, the estimates we have established on the backstepping transforms prevent us from applying the strategy we have described above: indeed, for any sequences $(t_n) \rightarrow T$, $\lambda_n \rightarrow \infty$, we have

$$\|\alpha(t)\|_m \leq \prod_{k=0}^n \left(\frac{C}{c}\right)^{2n} e^{n\mu L} \exp\left(\sum_{k=0}^n -\lambda_k(t_{k+1} - t_k - L)\right) \|\alpha_0\|_m, \quad \forall t \in [t_n, t_{n+1}],$$

where c, C are the decay constants in (3.7). Moreover, as $t_{k+1} - t_k \rightarrow 0$, we have

$$\exp\left(\sum_{k=0}^n -\lambda_k(t_{k+1} - t_k - L)\right) \xrightarrow{n \rightarrow \infty} \infty.$$

Another approach could be to draw from [63] and apply a second transformation to design a more efficient feedback law. It would also be interesting to adapt the strategy in [159], inspired from [150], to our setting.

3.4.4 Nonlinear systems

Finally, another prospect, having obtained explicit feedbacks that stabilize the linear system, is to investigate the stabilization of nonlinear transport equations. This has been done in [65], where the authors show that the feedback law obtained for the linear Korteweg-de Vries equation also stabilize the nonlinear equation. However, as in [59], the feedback law we have obtained is not continuous in the norm for which the system is stabilize. This would require some nonlinear modifications to the feedback law in order to stabilize the nonlinear system.

Chapter 4

Finite-time internal stabilization of a linear 1-D transport equation

This chapter is taken from the following article (also referred to as [163]):

Christophe Zhang, Finite-time internal stabilization of a linear 1-D transport equation, *Systems & Control Letters*, Volume 133, 2019, 104529.

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Abstract We consider a 1-D linear transport equation on the interval $(0, L)$, with an internal scalar control. We prove that if the system is controllable in a periodic Sobolev space of order greater than 1, then the system can be stabilized in finite time, and we give an explicit feedback law.

4.1 Introduction

We study the linear 1-D hyperbolic equation

$$\begin{cases} y_t + y_x + a(x)y = u(t)\tilde{\varphi}(x), & x \in [0, L], \\ y(t, 0) = y(t, L), & \forall t \geq 0, \end{cases} \quad (4.1)$$

where a is continuous, real-valued, $\tilde{\varphi}$ is a given real-valued function of space, and at time t , $y(t, \cdot)$ is the state and $u(t)$ is the control. As in [162], the system can be transformed into

$$\begin{cases} \alpha_t + \alpha_x + \mu\alpha = u(t)\varphi(x), & x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), & \forall t \geq 0, \end{cases} \quad (4.2)$$

through the state transformation

$$\alpha(t, x) := e^{\int_0^x a(s)ds - \mu x} y(t, x),$$

where $\mu = \int_0^L a(s)ds$, and with

$$\varphi(x) := e^{\int_0^x a(s)ds - \mu x} \tilde{\varphi}(x),$$

so that we focus on systems of the form (4.2) in this article. Hyperbolic systems with an internal control of this form model a variety of physical systems: let us cite the water tank system (introduced in [75] and further studied in [52, 126]), which is modelled by Saint-Venant equations with boundary conditions analog to our periodic boundary conditions, and the plug-flow reactor system, where the control is the temperature of the reactor, and there is a given input at the boundary (see [125, 127]).

4.1.1 Notations and definitions

We note ℓ^2 the space of summable square series $\ell^2(\mathbb{Z}, \mathbb{C})$. To simplify the notations, we will note L^2 the space $L^2(0, L)$ of complex-valued L^2 functions on the interval $(0, L)$, with its hermitian product

$$\langle f, g \rangle = \int_0^L f(x)\overline{g(x)}dx, \quad \forall f, g \in L^2, \quad (4.3)$$

and the associated norm $\|\cdot\|$. Functions of L^2 can also be seen as L -periodic functions on \mathbb{R} , by the usual L -periodic extension: in this article, for any $f \in L^2$ we will also note f its L -periodic extension on \mathbb{R} .

We also use the following notation

$$e_n(x) = \frac{1}{\sqrt{L}} e^{\frac{2i\pi}{L}nx}, \quad \forall n \in \mathbb{Z}, \quad (4.4)$$

the usual Hilbert basis for L^2 . For a function $f \in L^2$, we will note $(f_n)_{n \in \mathbb{Z}} \in \ell^2$ its coefficients in this basis:

$$f = \sum_{n \in \mathbb{Z}} f_n e_n.$$

Note that with this notation, we have

$$\bar{f} = \sum_{n \in \mathbb{Z}} \overline{f_{-n}} e_n,$$

so that, in particular, if f is real-valued:

$$f_{-n} = \overline{f_n}, \quad \forall n \in \mathbb{Z}.$$

We will use the following definition of the convolution product on L -periodic functions:

$$\begin{aligned} f \star g &= \sum_{n \in \mathbb{Z}} f_n g_n e_n \\ &= \int_0^L f(s)g(\cdot - s)ds \in L^2, \quad \forall f, g \in L^2, \end{aligned} \quad (4.5)$$

where $g(x-s)$ should be understood as the value taken in $x-s$ by the L -periodic extension of g .

Let us now note \mathcal{E} the space of finite linear combinations of the $(e_n)_{n \in \mathbb{Z}}$. Then, any sequence $(f_n)_{n \in \mathbb{Z}}$ defines an element f of \mathcal{E}' :

$$\langle e_n, f \rangle = \overline{f_n}, \quad \forall n \in \mathbb{Z}. \quad (4.6)$$

On this space of linear forms, we can extend our previous definition of convolution:

$$\langle e_n, f \star g \rangle = \overline{f_n g_n} \quad (4.7)$$

derivation can be defined by duality from (4.6):

$$f' = \left(\frac{2i\pi n}{L} f_n \right)_{n \in \mathbb{Z}}, \quad \forall f \in \mathcal{E}'. \quad (4.8)$$

We also define the following spaces:

Definition 4.1.1. *Let $m \in \mathbb{N}$. We note H^m the usual Sobolev spaces on the interval $(0, L)$, equipped with the Hermitian product*

$$\langle f, g \rangle_m = \int_0^L f \bar{g} + \partial^m f \overline{\partial^m g}, \quad \forall f, g \in H^m,$$

and the associated norm $\|\cdot\|_m$.

For $m \geq 1$ we also define $H_{(pw)}^m$ the space of piecewise H^m functions, that is, $f \in H_{(pw)}^m$ if there exists a finite number d of points $(\sigma_j)_{1 \leq j \leq d} \in [0, L]$ such that, noting $\sigma_0 := 0$ and $\sigma_{d+1} := L$, f is H^m on every $[\sigma_j, \sigma_{j+1}]$ for $0 \leq j \leq d$. This space can be equipped with the norm

$$\|f\|_{m,pw} := \sum_{j=0}^d \|f|_{[\sigma_j, \sigma_{j+1}]}\|_{H^m(\sigma_j, \sigma_{j+1})}. \quad (4.9)$$

For $s > 0$, we also define the periodic Sobolev space H_{per}^s as the subspace of L^2 functions $f = \sum_{n \in \mathbb{Z}} f_n e_n$ such that

$$\sum_{n \in \mathbb{Z}} \left(1 + \left| \frac{2i\pi n}{L} \right|^{2s} \right) |f_n|^2 < \infty.$$

Note that (4.9) does not depend on the choice of a suitable partition (σ_j) , so $\|\cdot\|_{m,pw}$ is well-defined. Also, H_{per}^s is a Hilbert space, equipped with the Hermitian product

$$\langle f, g \rangle_s = \sum_{n \in \mathbb{Z}} \left(1 + \left| \frac{2i\pi n}{L} \right|^{2s} \right) f_n \bar{g}_n, \quad \forall f, g \in H_{per}^s,$$

and the associated norm $\|\cdot\|_s$, as well as the Hilbert basis

$$(e_n^s) := \left(\frac{e_n}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2s}}} \right).$$

Finally, for $m \in \mathbb{N}$, H_{per}^m is a closed subspace of H^m , with the same scalar product and norm, thanks to the Parseval identity. Moreover,

$$H_{per}^m = \{f \in H^m, \quad f^{(i)}(0) = f^{(i)}(L), \quad \forall i \in \{0, \dots, m-1\}\}. \quad (4.10)$$

4.1.2 Main result

To stabilize (4.2), we will be considering linear feedbacks, that is, formally,

$$\langle \alpha(t), F \rangle = \sum_{n \in \mathbb{Z}} \overline{F_n} \alpha_n(t) = \int_0^L \overline{F}(s) \alpha(s) ds$$

where $F \in \mathcal{E}'$ and $(F_n)_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ are its Fourier coefficients, and F is “real-valued”:

$$F_{-n} = \overline{F_n}, \quad \forall n \in \mathbb{Z}.$$

In fact, the integral notation will appear as purely formal, as the $(F_n)_{n \in \mathbb{Z}}$ will have a prescribed growth, so that $F \notin L^2$. The associated closed-loop system now writes

$$\begin{cases} \alpha_t + \alpha_x + \mu \alpha = \langle \alpha(t), F \rangle \varphi(x), & x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), & \forall t \geq 0. \end{cases} \quad (4.11)$$

This is a linear transport equation, which we seek to stabilize with an internal, scalar feedback, given by a real-valued feedback law. In [162], we proved the following theorem for system (4.2) when it is controllable:

Theorem 4.1.1 (Rapid stabilization in Sobolev norms). *Let $m \geq 1$. Let $\varphi \in H_{(pw)}^m \cap H_{per}^{m-1}$ such that*

$$\frac{c}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2m}}} \leq |\varphi_n| \leq \frac{C}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2m}}}, \quad (4.12)$$

$\forall n \in \mathbb{Z},$

where $c, C > 0$. Then, for every $\lambda \geq 0$ there exists a stationary feedback law F^λ such that for all initial data $\alpha^0 \in H_{per}^m$ the closed-loop system (4.11) has a solution $\alpha(t) \in H_{per}^m$ which satisfies

$$\|\alpha(t)\|_m \leq \left(\frac{C}{c} \right)^2 e^{(\mu+\lambda)L} e^{-\lambda t} \|\alpha^0\|_m, \quad \forall t \geq 0. \quad (4.13)$$

The growth condition (4.12) is equivalent to the exact controllability of (4.2) (see [137, Equation (2.19) and pages 199-200] where the author uses the moments method). In particular, one can see that if c (resp. C) is the largest (resp. smallest) constant such that (4.12) holds, then the constant in (4.13) can be critical in some particular cases (see [162]).

Now, for $\lambda > 0$, the corresponding feedback law obtained in [162, Section 2.3] using the backstepping method is the linear form $F^{\lambda-\mu} \in \mathcal{E}'$ defined by

$$F_n^{\lambda-\mu} := -\frac{K(\lambda-\mu)}{\varphi_n}, \quad \forall n \in \mathbb{Z}, \quad (4.14)$$

where

$$K(\lambda-\mu) := \frac{2}{L} \frac{1 - e^{-(\lambda-\mu)L}}{1 + e^{-(\lambda-\mu)L}} \xrightarrow{\lambda \rightarrow \infty} \frac{2}{L}, \quad (4.15)$$

so that

$$F^\lambda \xrightarrow[\lambda \rightarrow \infty]{\mathcal{E}'} F^\infty \quad (4.16)$$

where

$$F_n^\infty := -\frac{2}{\varphi_n L}, \quad \forall n \in \mathbb{Z}. \quad (4.17)$$

Moreover, when $\lambda \rightarrow \infty$, the stability estimate in Theorem 4.1.1 becomes, for $t > L$,

$$\|\alpha(t)\|_m = 0.$$

This would suggest that taking the limit feedback F^∞ could result in finite-time stabilization of (4.2). This is indeed the case, and in this article we will prove the following theorem:

Theorem 4.1.2 (Finite-time stabilization in Sobolev norms). *Let $m \geq 1$. Let $\varphi \in H_{(pw)}^m \cap H_{per}^{m-1}$, satisfying (4.12) for some $c, C > 0$. Then, if the feedback law is defined by (4.17), for any initial data $\alpha^0 \in H_{per}^m$ the corresponding closed-loop system (4.11) has a solution $\alpha(t)$ which satisfies*

$$\|\alpha(t)\|_m = 0, \quad \forall t \geq L.$$

4.1.3 Related results

To investigate the stabilization of infinite-dimensional systems, there are three main types of approaches: the Gramian method (see for example [153, 151, 108]), Lyapunov functions (see for example [55], the book [24], and the recent results in [91, 92], which study the boundary stabilization of hyperbolic systems), and the backstepping method. The latter is derived from a method in finite dimension, also called backstepping, used to stabilize stabilizable systems with an added chain of integrators (see [110, 53, 143] for an overview of the finite-dimensional case, and [56] or [118] for applications to partial differential equations). Another way of applying this approach to partial differential equations was then pioneered and developed in [17] and [32]. This new form of backstepping consisted in mapping the system to a stable target system, using a Volterra transformation of the second kind (see [111] for a comprehensive introduction to the method):

$$f(t, x) \mapsto f(t, x) - \int_0^x k(x, y) f(t, y) dy.$$

This was used to prove many results on the boundary stabilization of partial differential equations (see for example [109, 141, 155, 156, 68], and also [24, chapter 7]).

In some cases, the method was used to obtain stabilization with an internal feedback (see [150, 154, 159]). We point out that in the latter reference, a system resembling (4.1) is studied, which leads to finite-time stabilization. However, several hypotheses are made on the space component of the controller so that a Volterra transformation of the second kind can be successfully applied to the system, whereas in this article and in [162], we simply assume the exact controllability of the system.

Another recent development of the backstepping method is the use of Fredholm transformations:

$$f(t, x) \mapsto \int_0^L k(x, y) f(t, y) dy,$$

to map the control system to a stable target system (see for example [66, 65, 62, 63] for boundary stabilization problems, [59] for an internal stabilization problem). These are more general than Volterra transformation of the second kind, but one has to check that the transformation under consideration is actually invertible, whereas Volterra transformations of the second kind are always invertible if the kernel k has enough regularity.

Because backstepping provides explicit feedback laws, it has helped prove null-controllability or small-time stabilization (stabilization in an arbitrarily small time) results for some systems: see [67] for the heat equation, and [156] for the Korteweg-de Vries equation. In this article, we use the explicit feedback laws obtained by the backstepping method in [162] to design an explicit stationary feedback law that achieves finite-time stabilization.

4.1.4 Structure of the article

In Section 2, we derive an expression for the exponentially stable semigroup corresponding to the explicit feedback laws obtained in [162] for exponential stabilization. Then, in Section 3, we study the semigroup obtained when $\lambda \rightarrow \infty$. In particular, we derive its infinitesimal generator and prove that it corresponds to a closed-loop system which goes to 0 in finite time, which yields a feedback law achieving stabilization in finite time. Finally, Section 4 is devoted to some comments on the result, and on further questions.

4.2 The exponentially stable semigroup

We recall some specifics of Theorem 4.1.1, which can be found in more detail in [162].

4.2.1 Backstepping transformation

To prove Theorem 4.1.1, the backstepping method was used. This method consists in mapping our system into a stable target system, here

$$\begin{cases} z_t + z_x + \lambda' z = 0, & x \in (0, L), \\ z(t, 0) = z(t, L), & t \geq 0, \end{cases} \quad (4.18)$$

with $\lambda' > 0$. To find an invertible transformation that does this, the idea is to write it as a Fredholm operator:

$$T : \alpha(t, x) \mapsto \int_0^L k(x, y) \alpha(t, y) dy$$

so that the mapping condition becomes a partial differential equation in k (the kernel equation). This equation contains non-local terms, which are resolved by adding a natural constraint to the kernel equation:

$$\int_0^L k(x, y) \varphi(y) dy = \varphi(x), \quad \forall x \in [0, L], \quad (4.19)$$

which turns the non-local terms (left hand side) into local terms (right hand side). This constraint is sometimes called the $TB = B$ condition (see [162, 59]).

From this kernel equation, conditions on F for the invertibility of T can be derived. Then, using a weak version of (4.19) condition, a suitable feedback is computed, so that a candidate for the backstepping transformation can be derived:

$$T^\lambda \alpha = \sum_{n \in \mathbb{Z}} \alpha_n \overline{F_n^\lambda} \Lambda_n^\lambda \star \varphi, \quad \forall \alpha \in H_{per}^m, \quad (4.20)$$

where $\lambda := \lambda' - \mu$, $\overline{F_n^\lambda}$ is defined by (4.14), and

$$\begin{aligned} \Lambda_n^\lambda(x) &:= \frac{\sqrt{L}}{1 - e^{-\lambda L}} e^{-\lambda_n x} \\ &= \Lambda(x) e_{-n}(x), \quad \forall n \in \mathbb{Z}, \quad \forall x \in [0, L], \end{aligned} \quad (4.21)$$

where

$$\lambda_n = \lambda + \frac{2i\pi n}{L}, \quad \forall n \in \mathbb{Z}, \quad (4.22)$$

and where Λ is the L -periodic function defined by

$$\Lambda(x) = \frac{L}{1 - e^{-\lambda L}} e^{-\lambda x}, \quad \forall x \in [0, L].$$

4.2.2 Well-posedness of the closed-loop system

Now that a candidate for the backstepping transformation has been determined, it must be proved that it is indeed a backstepping transformation, and that the closed-loop with the feedback defined above is well-posed. We first define the domains

$$\begin{aligned} D_m^\lambda &:= \{ \alpha \in \tau^\varphi(H_{(pw)}^{m+1}) \cap H_{per}^m, \\ &\quad -\alpha_x - \mu \alpha + \langle \alpha, F^\lambda \rangle \varphi \in H_{per}^m \} \end{aligned} \quad (4.23)$$

where τ^φ is the diagonal operator defined by the eigenvalues

$$\tau_n^\varphi := \frac{1}{\sqrt{L}} \left(\sum_{j=1}^d e^{-\frac{2i\pi}{L} n \sigma_j} (\partial^{m-1} \varphi(\sigma_j^-) - \partial^{m-1} \varphi(\sigma_j^+)) + \partial^{m-1} \varphi(L) - \partial^{m-1} \varphi(0) \right), \quad \forall n \in \mathbb{Z}.$$

In [162], we investigate the regularity of the feedback law, using the controllability condition (4.12). This helps to prove that the corresponding closed-loop operator

$$A + BK := -\partial_x - \mu I + \langle \cdot, F^\lambda \rangle \varphi$$

is densely defined and closed. Finally, to check that the mapping property between systems (4.11) and (4.18) is verified, one proves the operator equality

$$T^\lambda (-\partial_x + \langle \cdot, F^\lambda \rangle \varphi) \alpha = (-\partial_x - \lambda I) T^\lambda \alpha \quad \text{in } H_{per}^m, \quad \forall \alpha \in D_m^\lambda. \quad (4.24)$$

This operator equality implies that the unbounded operator $A + BK$ is a dense restriction of the infinitesimal generator of an exponentially stable semigroup $S^\lambda(t)$.

Now, the basic idea is that for a given initial condition α^0 , for each feedback F^λ one has a trajectory of the closed-loop system

$$\alpha_t^\lambda + \alpha_x^\lambda + \mu \alpha^\lambda = \langle \alpha^\lambda(t), F^\lambda \rangle \varphi(x) \quad (4.25)$$

and one hopes that the α^λ converge in some sense towards a trajectory α^∞ , which should satisfy the closed-loop equation

$$\alpha_t^\infty + \alpha_x^\infty + \mu \alpha^\infty = \langle \alpha^\infty(t), F^\infty \rangle \varphi(x). \quad (4.26)$$

However, one can write equation (4.25) only for $\alpha^0 \in D_m^\lambda$, and to use this equation to study the convergence of the α^λ , one would need $\alpha \in \bigcap_{\lambda>0} D_m^\lambda$. This is too restrictive since we would like a

statement for $\alpha^0 \in H_{per}^m$. Thus, rather than consider the equations (4.25), we will work in the more general framework of semigroups.

4.2.3 Expression of the semigroup

In [162], an expression of the semigroup S^λ is given using the transformation T^λ . Here, to study what happens when $\lambda \rightarrow \infty$, we need to expand that expression.

First, we derive from (4.20) and (4.21) the following expression for the backstepping transformation:

$$T^\lambda \alpha = \varphi \star \left(\Lambda(\alpha \star \tilde{F}^\lambda) \right), \quad \forall \alpha \in H_{per}^m, \quad (4.27)$$

where $\tilde{F}^\lambda \in \mathcal{E}'$ is defined by:

$$\langle e_n, \tilde{F}^\lambda \rangle = F_n^\lambda, \quad n \in \mathbb{Z}.$$

Now, define the following operators:

$$\begin{aligned} C_\varphi f &= \varphi \star f \in H_{per}^m, \quad \forall f \in L^2, \\ C_{\tilde{F}^\lambda} f &= \tilde{F}^\lambda \star f \in L^2, \quad \forall f \in H_{per}^m, \\ M_\Lambda f &= \Lambda f, \quad \forall f \in L^2. \end{aligned}$$

Then, by definition of F^λ , it follows that

$$\begin{aligned} C_\varphi \circ C_{\tilde{F}^\lambda} &= -K(\lambda) Id_{H_{per}^m}, \\ C_{\tilde{F}^\lambda} \circ C_\varphi &= -K(\lambda) Id_{L^2}, \\ M_\Lambda \circ M_{\frac{1}{\lambda}} &= Id_{L^2}, \end{aligned} \quad (4.28)$$

where K is defined by (4.15). Moreover, with these notations, we have

$$T^\lambda = C_\varphi \circ M_\Lambda \circ C_{\tilde{F}^\lambda} \quad (4.29)$$

hence

$$(T^\lambda)^{-1} = \frac{1}{K(\lambda)^2} C_\varphi \circ M_{\frac{1}{\Lambda}} \circ C_{\tilde{F}^\lambda}, \quad (4.30)$$

i.e.

$$(T^\lambda)^{-1}\alpha = \frac{1}{K(\lambda)^2} \varphi \star \left(\frac{1}{\Lambda} (\alpha \star \tilde{F}^\lambda) \right), \quad \forall \alpha \in H_{per}^m. \quad (4.31)$$

Now, recall that for all initial data $z_0 \in H_{per}^m$, the solution of system (4.18) can be written

$$z(t, x) = e^{-\lambda't} z_0(x - t), \quad \forall (t, x) \in \mathbb{R}^+ \times (0, L). \quad (4.32)$$

Thus, by the expression of the semigroup (see [162, Subsection 3.2]), for all initial data $\alpha^0 \in D_m^\lambda$, the solution of system (4.11) can be written:

$$\begin{aligned} \alpha(t, x) &= (T^\lambda)^{-1} e^{-\lambda't} (T^\lambda \alpha^0)(x - t), \\ &\forall (t, x) \in \mathbb{R}^+ \times (0, L). \end{aligned} \quad (4.33)$$

Now, notice that convolution and translation commute, so for $(t, x) \in [0, L] \times (0, L)$, we get, using (4.27) and (4.31),

$$\begin{aligned} \alpha(t, x) &= \frac{1}{K(\lambda)^2} \varphi \star \left(\frac{1}{\Lambda} \left(\tilde{F}^\lambda \star \right. \right. \\ &\quad \left. \left. \left(e^{-\lambda't} \left(\varphi \star \left(\Lambda(\alpha^0 \star \tilde{F}^\lambda) \right) (\cdot - t) \right) \right) \right) \right) \\ &= -\frac{e^{-\lambda't}}{K(\lambda)} \varphi \star \left(\frac{1}{\Lambda} \left(\left(\Lambda(\alpha^0 \star \tilde{F}^\lambda) \right) (\cdot - t) \right) \right) \\ &= -\frac{e^{-\lambda't}}{K(\lambda)} \varphi \star \left(\chi_{[0,t]} e^{\lambda(t-L)} \alpha^0 \star \tilde{F}^\lambda(\cdot - t + L) \right. \\ &\quad \left. + \chi_{[t,L]} e^{\lambda t} \alpha^0 \star \tilde{F}^\lambda(\cdot - t) \right) \\ &= -\frac{e^{-\mu t}}{K(\lambda)} \varphi \star \left(\chi_{[0,t]} e^{-\lambda L} \alpha^0 \star \tilde{F}^\lambda(\cdot - t + L) \right. \\ &\quad \left. + \chi_{[t,L]} \alpha^0 \star \tilde{F}^\lambda(\cdot - t) \right). \end{aligned}$$

This expression is derived for $\alpha^0 \in D_m^\lambda$, but it is actually well-defined on all of H_{per}^m , as $\alpha^0 \star \tilde{F}^\lambda \in L^2$ when $\alpha^0 \in H_{per}^m$. This gives us an expression for $S^\lambda(t)$ on all of H_{per}^m :

$$\begin{aligned} S^\lambda(t)\alpha^0 &= \frac{-e^{-\mu t}}{K(\lambda)} \varphi \star \left(\chi_{[t,L]} \alpha^0 \star \tilde{F}^\lambda(\cdot - t) \right. \\ &\quad \left. + \chi_{[0,t]} e^{-\lambda L} \alpha^0 \star \tilde{F}^\lambda(\cdot - t + L) \right), \\ &\forall t \in [0, L], \quad \forall \alpha^0 \in H_{per}^m, \end{aligned} \quad (4.34)$$

which defines $S^\lambda(t)$ for $t \geq 0$ by the semigroup property.

4.3 The limit semigroup

Now, notice that, from (4.15) and (4.17),

$$F^\lambda = \frac{LK(\lambda)}{2} F^\infty,$$

so that we can write

$$S^\lambda(t)\alpha = -\frac{Le^{-\mu t}}{2}\varphi \star \left(\chi_{[0,t]}e^{-\lambda L}\alpha \star \tilde{F}^\infty(\cdot - t + L) \right. \\ \left. + \chi_{[t,L]}\alpha \star \tilde{F}^\infty(\cdot - t) \right), \quad \forall t \in [0, L], \quad \forall \alpha \in H_{per}^m.$$

Then it is clear that

$$\chi_{[0,t]}e^{-\lambda L}\alpha \star \tilde{F}^\infty(\cdot - t + L) \xrightarrow{\lambda \rightarrow \infty} 0, \quad \forall t \in [0, L], \quad \forall \alpha \in H_{per}^m,$$

so that, after convolution with φ ,

$$S^\lambda(t)\alpha \xrightarrow{\lambda \rightarrow \infty} S^\infty(t)\alpha, \quad \forall t \in [0, L], \quad \forall \alpha \in H_{per}^m,$$

where

$$S^\infty(t)\alpha := -\frac{Le^{-\mu t}}{2}\varphi \star \left(\chi_{[t,L]}\alpha \star \tilde{F}^\infty(\cdot - t) \right), \\ \forall t \geq 0, \quad \forall \alpha \in H_{per}^m,$$

with the convention that $\chi_{[t,L]} \equiv 0$ when $t \geq L$. Hence we have defined a new semigroup $S^\infty(t)$ on H_{per}^m , which we now study in order to establish Theorem 4.1.2.

4.3.1 A useful semigroup

Consider the semigroup given by

$$S_0(t)\alpha = e^{-\mu t}\chi_{[t,L]}\alpha(\cdot - t), \quad \forall t \geq 0, \quad \forall \alpha \in L^2.$$

This is actually a contraction semigroup, the infinitesimal generator of which is given by

$$D(A_0) = \{ \alpha \in H^1, \quad \alpha(0) = 0 \} \\ A_0 = -\partial_x - \mu I \tag{4.35}$$

where the derivative is to be understood as the usual derivative of a Sobolev function, not as the derivative in \mathcal{E}' . Note that this semigroup is associated to the following transport equation:

$$\begin{cases} y_t + y_x + \mu y = 0, & x \in [0, L], \\ y(t, 0) = 0, & \forall t \geq 0, \end{cases} \tag{4.36}$$

and that in particular

$$S_0(t)\alpha = 0, \quad \forall t \geq L, \quad \forall \alpha \in L^2. \tag{4.37}$$

4.3.2 Infinitesimal generator

Now let us compute the infinitesimal generator of S^∞ . First, notice that

$$S^\infty(t)\alpha = -\frac{L}{2}\varphi \star S_0(t) \left(\alpha \star \tilde{F}^\infty \right). \tag{4.38}$$

Now, let us define the following domain, in the same spirit as in section 4.2.1:

$$D_m^\infty := \{ \alpha \in \tau^\varphi(H_{(pw)}^{m+1}) \cap H_{per}^m, \\ -\alpha_x - \mu\alpha + \langle \alpha, F^\infty \rangle \varphi \in H_{per}^m \}. \tag{4.39}$$

This domain is dense in H_{per}^m , as it contains the following dense subspace (see [162, Proposition 3.1]):

$$\{ \alpha \in H_{per}^{m+1}, \quad \langle \alpha, F \rangle = 0 \}.$$

Let us now prove that on this domain, S^∞ has an infinitesimal generator. For $\alpha \in D_m^\infty$, we have

$$r := \tilde{F}^\infty \star (-\alpha_x - \mu\alpha + \langle \alpha, F^\infty \rangle \varphi) \in L^2. \quad (4.40)$$

Thus, taking the Fourier coefficients, we get:

$$\alpha_n \overline{F_n^\infty} = -\frac{r_n + \mu\alpha_n \overline{F_n^\infty}}{\left(\frac{2i\pi n}{L}\right)} + i\frac{\langle \alpha, F^\infty \rangle}{\pi n}, \quad \forall n \neq 0.$$

Now, note that

$$\sum_{n \in \mathbb{Z}^*} i\frac{\langle \alpha, F^\infty \rangle}{\pi n} e_n(x) = \frac{2}{L} \langle \alpha, F^\infty \rangle \left(\frac{x}{\sqrt{L}} - \frac{\sqrt{L}}{2} \right),$$

so that

$$\alpha \star \tilde{F}^\infty = \tilde{r} + \frac{2}{L} \langle \alpha, F^\infty \rangle \left(\frac{x}{\sqrt{L}} - \frac{\sqrt{L}}{2} \right), \quad (4.41)$$

where

$$\tilde{r} = \frac{\alpha_0 F_0^\infty}{\sqrt{L}} - \sum_{n \in \mathbb{Z}^*} \frac{r_n + \mu\alpha_n \overline{F_n^\infty}}{\left(\frac{2i\pi n}{L}\right)} e_n \in H_{per}^1. \quad (4.42)$$

Hence, $\alpha \star \tilde{F}^\infty \in H^1$, and, from (4.41) and (4.42) we get

$$\begin{aligned} (\alpha \star \tilde{F}^\infty)_x &= - \left(r - \frac{r_0}{\sqrt{L}} + \mu\alpha \star \tilde{F}^\infty - \mu \frac{\alpha_0 F_0^\infty}{\sqrt{L}} \right) \\ &\quad + \frac{2}{L\sqrt{L}} \langle \alpha, F^\infty \rangle. \end{aligned}$$

Now, by (4.40), (4.17) and by definition of the convolution product,

$$r_0 = -\mu F_0^\infty \alpha_0 - \frac{2\langle \alpha, F^\infty \rangle}{L},$$

so that, again by (4.40),

$$\begin{aligned} (\alpha \star \tilde{F}^\infty)_x &= -r + \mu\alpha \star \tilde{F}^\infty \\ &= -\tilde{F}^\infty \star (-\alpha_x + \langle \alpha, F^\infty \rangle \varphi) \quad \text{in } L^2. \end{aligned} \quad (4.43)$$

On the other hand, we know, by the Dirichlet convergence theorem (see [101]) applied to $\alpha \star \tilde{F}^\infty \in H^1$ at point 0, that

$$\frac{\alpha \star \tilde{F}^\infty(0) + \alpha \star \tilde{F}^\infty(L)}{2} = \sum_{n \in \mathbb{Z}} \frac{\alpha_n \overline{F_n^\infty}}{\sqrt{L}} = \frac{\langle \alpha, F^\infty \rangle}{\sqrt{L}}.$$

On the other hand, by (4.41),

$$\begin{aligned} (\alpha \star \tilde{F}^\infty - \tilde{r})(0) &= -\frac{\langle \alpha, F^\infty \rangle}{\sqrt{L}} \\ &= -(\alpha \star \tilde{F}^\infty - \tilde{r})(L), \end{aligned} \quad (4.44)$$

thus, as \tilde{r} is periodic,

$$\tilde{r}(0) = \frac{\alpha \star \tilde{F}^\infty(0) + \alpha \star \tilde{F}^\infty(L)}{2} = \frac{\langle \alpha, F^\infty \rangle}{\sqrt{L}}. \quad (4.45)$$

From (4.44) and (4.45), we get

$$\alpha \star \tilde{F}^\infty(0) = \tilde{r}(0) - \frac{\langle \alpha, F^\infty \rangle}{\sqrt{L}} = 0,$$

so that $\alpha \star \tilde{F}^\infty \in D(A_0)$.

We can now compute the infinitesimal generator of S^∞ : let $\alpha \in D_m^\infty$. Then, thanks to the above, $\alpha \star \tilde{F}^\infty \in D(A_0)$, which means in particular that

$$\frac{S_0(t)(\alpha \star \tilde{F}^\infty) - (\alpha \star \tilde{F}^\infty)}{t} \xrightarrow[t \rightarrow 0^+]{L^2} -(\alpha \star \tilde{F}^\infty)_x - \mu(\alpha \star \tilde{F}^\infty).$$

This, together with (4.38) and (4.12), implies that

$$\begin{aligned} \frac{S^\infty(t)\alpha - \alpha}{t} &= -\frac{L}{2}\varphi \star \left(\frac{S_0(t)(\alpha \star \tilde{F}^\infty) - (\alpha \star \tilde{F}^\infty)}{t} \right) \\ &\xrightarrow[t \rightarrow 0^+]{H^m} \frac{L}{2}\varphi \star \left((\alpha \star \tilde{F}^\infty)_x + \mu(\alpha \star \tilde{F}^\infty) \right). \end{aligned}$$

By (4.43), we have

$$\begin{aligned} \varphi \star \left((\alpha \star \tilde{F}^\infty)_x + \mu(\alpha \star \tilde{F}^\infty) \right) &= \\ &= \frac{2}{L}(-\alpha_x - \mu\alpha + \langle \alpha, F^\infty \rangle \varphi) \end{aligned}$$

so that, finally,

$$\frac{S^\infty(t)\alpha - \alpha}{t} \xrightarrow[t \rightarrow 0^+]{H^m} -\alpha_x - \mu\alpha + \langle \alpha, F^\infty \rangle \varphi.$$

This, together with (4.39), means that the infinitesimal generator of $S^\infty(t)$ can be given by the domain D_m^∞ and the unbounded operator $-\partial_x - \mu I + \langle \cdot, F^\infty \rangle \varphi$. Hence, $S^\infty(t)$ corresponds to the closed loop system

$$\begin{cases} \alpha_t + \alpha_x + \mu\alpha = \langle \alpha(t), F^\infty \rangle \varphi(x), & x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), & \forall t \geq 0, \end{cases} \quad (4.46)$$

which is well-posed. Moreover, by (4.37) and (4.38),

$$S^\infty(t)\alpha^0 = 0, \quad \forall t \geq L, \forall \alpha^0 \in H_{per}^m, \quad (4.47)$$

which proves Theorem 4.1.2.

4.4 An explicit example

Consider the control system

$$\begin{cases} \alpha_t + \alpha_x = u(t)(L - x), & x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), & \forall t \geq 0. \end{cases} \quad (4.48)$$

In this case, $\varphi(x) = L - x$, so $\varphi \in H^1$ and the Fourier coefficients of the controller are

$$\begin{aligned} \varphi_n &= -\frac{iL^{\frac{3}{2}}}{2\pi n}, \quad \forall n \in \mathbb{Z}^*, \\ \varphi_0 &= \frac{L^{\frac{3}{2}}}{2}. \end{aligned} \quad (4.49)$$

so that (4.12) is clearly satisfied for $m = 1$. Now, from (4.49) and (4.17) we get

$$\begin{aligned} F_n^\infty &= \frac{2}{L} \frac{2i\pi n}{L\sqrt{L}}, \quad \forall n \in \mathbb{Z}^*, \\ F_0^\infty &= -\frac{4}{L^{\frac{5}{2}}}. \end{aligned} \quad (4.50)$$

Now, using the Dirichlet convergence theorem, we have for $\alpha \in H^2 \cap H_{per}^1$,

$$\begin{aligned} \sum_{n=-N}^N \overline{F_n^\infty} \alpha_n &= -\frac{2}{L} \sum_{n=-N}^N \frac{2i\pi n}{L} \alpha_n \frac{1}{\sqrt{L}} - \frac{4}{L^{\frac{5}{2}}} \alpha_0 \\ &\xrightarrow{N \rightarrow \infty} -\frac{\alpha_x(0) + \alpha_x(L)}{L} - \frac{4}{L^{\frac{5}{2}}} \alpha_0, \end{aligned} \quad (4.51)$$

so that

$$\begin{aligned} \langle \alpha, F^\infty \rangle &= -\frac{\alpha_x(0) + \alpha_x(L)}{L} - \frac{4}{L^{\frac{5}{2}}} \alpha_0, \\ &\quad \forall \alpha \in H^2 \cap H_{per}^1. \end{aligned} \quad (4.52)$$

One can see from the above expression that even though our method defines F^∞ by its Fourier coefficients, with some controllers the feedback law can be expressed quite simply.

Now, let us consider solutions of the closed-loop system (4.48) with $u(t) = \langle \alpha(t), F^\infty \rangle$, with initial conditions in the domain

$$\begin{aligned} D_1^\infty &= \left\{ \alpha \in H^2 \cap H_{per}^1, \right. \\ &\quad \left. -\alpha_x + \langle \alpha, F^\infty \rangle(L-x) \in H_{per}^1 \right\}, \end{aligned} \quad (4.53)$$

which can be rewritten as

$$\begin{aligned} D_1^\infty &= \left\{ \alpha \in H^2 \cap H_{per}^1, \right. \\ &\quad \left. \alpha_x(0) = -\frac{2}{L\sqrt{L}} \alpha_0 \in H_{per}^1 \right\}. \end{aligned} \quad (4.54)$$

Indeed, $-\alpha_x + \langle \alpha, F^\infty \rangle(L-x) \in H^1$, so the above condition simply corresponds to its being periodic in addition.

Let $\alpha^0 \in D_1^\infty$, and note $\alpha(t)$ the corresponding solution of (4.48). We can make the following computations for $t \geq 0$, using (4.50) for the first, the periodicity of α , and differentiating the first

equation of (4.48) in space for the second, and (4.54) for the third:

$$\begin{aligned}
\alpha(t) \star \tilde{F}^\infty &= -\frac{2}{L\sqrt{L}}\alpha_x - \frac{4}{L^3}\alpha_0, \\
(\alpha \star \tilde{F}^\infty)_t &= -\frac{2}{L\sqrt{L}}\alpha_{xt} - \frac{4}{L^3}(\alpha_0)_t \\
&= -\frac{2}{L\sqrt{L}}\alpha_{tx} - \frac{4}{L^3\sqrt{L}}\int_0^L \alpha_t \\
&= \frac{2}{L\sqrt{L}}((\alpha_x - \langle \alpha, F^\infty \rangle(L-x))_x) \\
&\quad - \frac{4}{L^3\sqrt{L}}\int_0^L \alpha_t \\
&= \frac{2}{L\sqrt{L}}(\alpha_{xx} + \langle \alpha, F^\infty \rangle) \\
&\quad - \frac{4}{L^3\sqrt{L}}\int_0^L \langle \alpha, F^\infty \rangle(L-x) - \alpha_x \\
&= \frac{2}{L\sqrt{L}}(\alpha_{xx} + \langle \alpha, F^\infty \rangle) \\
&\quad - \frac{4}{L^3\sqrt{L}}\langle \alpha, F^\infty \rangle \int_0^L (L-x)dx \\
&= \frac{2}{L\sqrt{L}}\alpha_{xx} \\
&= -(\alpha \star \tilde{F}^\infty)_x, \\
\alpha(t) \star \tilde{F}^\infty(0) &= -\frac{2}{L\sqrt{L}}\alpha_x(0) - \frac{4}{L^3}\alpha_0 \\
&= 0.
\end{aligned} \tag{4.55}$$

So in particular we can see quite clearly how $\alpha \star \tilde{F}^\infty$ satisfies the equation (4.36) with $\mu = 0$. In particular,

$$\alpha(t) \star \tilde{F}^\infty = 0, \quad \forall t \geq L, \tag{4.56}$$

which implies, using the first equation of (4.55), that $\alpha_x(t)$ is a constant function of space, i.e. $\alpha(t)$ is an affine function of space. However, it is also periodic, so we get

$$\alpha(t) = 0, \quad \forall t \geq L. \tag{4.57}$$

4.5 Comments and further questions

4.5.1 Backstepping and finite-time stabilization

As we have mentioned in the introduction, one of the advantages of the backstepping method is that it can provide explicit feedback laws for exponential stabilization. This allows the construction of explicit controls for null controllability ([155, 67]) as well as time-varying feedbacks that stabilize the system in finite time $T > 0$ ([156, 67]).

The general strategy in these articles is to divide the interval $[0, T]$ in smaller intervals $[t_n, t_{n+1}]$ on which the feedback corresponding to some $\lambda_n > 0$ is applied. The idea is then to chose the t_n so that the length of the intervals $[t_n, t_{n+1}]$ tends to 0 fast enough to compensate the growth of the norm of the feedback law as $\lambda_n \rightarrow \infty$. Building from this, the authors design a time-varying feedback law that stabilizes the system in finite-time.

Here, the feedback is stationary, and we do not need to define it piecewise: indeed, the norm of the feedback law F^λ is bounded when $\lambda \rightarrow \infty$. This comes from the fact that we used a special type of convergence to define the feedback law, using a weak version of (4.19). Indeed, in [162], we set

$$\varphi^{(N)} := \sum_{n=-N}^N \varphi_n e_n \in H_{per}^m, \quad \forall N \in \mathbb{N}.$$

Then,

$$\begin{aligned} T^\lambda \varphi^{(N)} &= \sum_{n=-N}^N -\varphi_n \overline{F_n^\lambda} \Lambda_{-n}^\lambda \star \varphi \\ &= \sum_{n=-N}^N \sum_{p \in \mathbb{Z}} \frac{-\varphi_n \overline{F_n^\lambda} \varphi_p}{\lambda_{-n+p}} e_p \\ &= \sum_{p \in \mathbb{Z}} \varphi_p \left(\sum_{n=-N}^N \frac{-\varphi_n \overline{F_n^\lambda}}{\lambda_{-n+p}} \right) e_p. \end{aligned}$$

and F^λ is defined by

$$\frac{1}{-\varphi_n \overline{F_n^\lambda}} = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{\lambda_{-n+p}}$$

in order to have

$$\langle T^\lambda \varphi^{(N)}, e_n \rangle \xrightarrow{N \rightarrow \infty} \varphi_n, \quad n \in \mathbb{Z},$$

which is the weak version of (4.19).

Now, if the convergence of the right-hand side had been absolute, the limit would have gone to 0 when $\lambda \rightarrow \infty$. However, here the sum converges in a special way due to the Dirichlet convergence theorem (see for example [101]), which is why it remains positive (and thus F^λ remains bounded) when $\lambda \rightarrow \infty$.

Hence, a weaker $TB = B$ condition seems to allow for better behavior of the feedback law when $\lambda \rightarrow \infty$.

4.5.2 Regularity of the feedback law

A remarkable point of this application of the backstepping method, both for rapid and finite-time stabilization, is that the feedback law is not regular on the state space: indeed, it is continuous for $\|\cdot\|_{m+1}$ but not for $\|\cdot\|_m$.

On the other hand, it seems that a continuous feedback law would have a more restricted action on the eigenvalues of the system. Indeed, in [137] it is proved that if the sequence of complex numbers $(\rho_n)_{n \in \mathbb{Z}}$ satisfies

$$\left(\left| \frac{\rho_n - \frac{2i\pi n}{L}}{\varphi_n} \right| \right) \in \ell^2, \quad (4.58)$$

then there exists a *bounded* feedback law such that the resulting closed-loop system has eigenvalues $(\rho_n)_{n \in \mathbb{Z}}$. It is clear that (4.58) does not allow for a uniform pole-shifting as we have done in [162]. But even though (4.58) is not a necessary condition, subsequent works such as [135, 45, 129] turn to unbounded feedback laws, as they are proved to allow for more eigenvalue displacement, and in particular uniform pole-shifting. *A fortiori*, the stronger notion of finite-time stabilization, in which case the operator associated to the closed-loop system has an empty spectrum (see for example [135, Theorem 3 and comments]), probably requires an unbounded feedback law.

Chapter 5

Exponential stabilization of the linearized water tank system

This chapter is taken from a work in progress with Jean-Michel Coron, Amaury Hayat and Shenguan Xiang.

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5.1 Introduction

5.1.1 Equations of the problem

We consider a water tank modelled by the Saint-Venant equations with no friction and no slope,

$$\begin{cases} \partial_t H + \partial_x(HV) = 0, \\ \partial_t V + V\partial_x V + g\partial_x H = -U(t). \end{cases} \quad (5.1)$$

where H is the height of the water, V its averaged velocity and U is control input. Without loss of generality we can suppose that $g = 1$. The water is localized inside the water tank, which implies the following Dirichlet boundary conditions:

$$V(t, 0) = V(t, L) = 0. \quad (5.2)$$

Moreover, integrating the first equation of (5.1) we obtain the conservation of the mass of the water, we have that

$$\int_0^L H(t, x) dx \text{ does not change with respect to time.} \quad (5.3)$$

First derived in 1871 by Barré de Saint-Venant [19, 21, 20], the Saint-Venant equations are among the most famous equations in fluid dynamics and represent flow under shallow water approximation. Despite their apparent simplicity, they capture a large number of physical behaviors, which made them a ground tool for practical application in particular in the regulation of canals for agriculture management and in the regulation of navigable rivers.

The stabilization of the Saint-Venant equations by boundary controls is a well-studied problem. The first result goes back to 1999 with [57] where the stability of the homogeneous linearized Saint-Venant was shown, using proportional boundary conditions. This was extended in [58] to the nonlinear homogeneous Saint-Venant equations. Later, in 2008, using a semigroup approach and the method of the characteristics, the stabilization of the nonlinear homogeneous equations was achieved for sufficiently small friction and slope [73, 128]. The same type of result was shown in [54] using a Lyapunov approach while [27] dealt with the inhomogeneous Saint-Venant equations in the particular case where the steady-states are uniform. In 2017, the stabilization was achieved for arbitrary large friction but in the absence of slope [25], and very recently for any section profil and any source term [95, 94]. Other results exists using different boundary conditions for instance PI controllers [23, Chapter 8], [74, 26, 157, 158, 148, 93] or full-state feedbacks resulting of a backstepping approach [68] (see [71, 70] for its application on variant systems based on the Saint-Venant equations). The stabilization of the Saint-Venant equations by internal control, however, has seldom been studied while being very interesting mathematically and corresponding to physical situations, for instance a water tank subject to an acceleration.

5.1.2 Main result

The water tank problem is interesting in that it has been studied for a long time and is rich enough to have led to several interesting results. Among the control results in this setting, one can cite [75, 126, 52] where the authors show, among others, that the linearized homogeneous Saint-Venant equations with null velocity at the boundaries and subject to a scalar control force are not locally approximately controllable around their uniform steady-states. This implies that they are not stabilizable either.

Now, consider the non-uniform steady-states corresponding to a (small) acceleration $U(t) = \gamma$ with $\gamma > 0$ fixed: $H^* = H^\gamma$, $V^* = 0$ with $H^\gamma(0) = 1$ and

$$H^\gamma(x) = 1 - \gamma x. \quad (5.4)$$

The linearized equations around this steady-state expressed with the variables $h = H - H^\gamma$ and $v = V - V^\gamma$ denoting the perturbations and the internal control $u(t) = -(U(t) - \gamma)$

$$\partial_t \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} 0 & H^\gamma \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} 0 & -\gamma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = -u(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (5.5)$$

with the boundary conditions:

$$v(t, 0) = v(t, L) = 0. \quad (5.6)$$

While, condition (5.3) becomes

$$\int_0^L h(t, x) dx = 0. \quad (5.7)$$

As $H^\gamma \neq 0$, the transport matrix is diagonalizable and the system is thus strictly hyperbolic. We now recall the definition of exponential stability:

Definition 5.1.1. *The system (5.5)–(5.6) is exponentially stable with decay rate μ if there exists a constant $C > 0$ such that for any $(h_0, v_0) \in H^1((0, L); \mathbb{R}^2)$ satisfying the compatibility conditions $v_0(0) = v_0(L) = 0$ corresponding to (5.6), the system (5.5)–(5.6) has a unique solution $(h, v) \in C^0([0, +\infty), H^1((0, L); \mathbb{R}^2))$ and*

$$\|h(t, \cdot), v(t, \cdot)\|_{H^1((0, L); \mathbb{R}^2)} \leq C e^{-\mu t} \|h_0, v_0\|_{H^1((0, L); \mathbb{R}^2)}, \quad \forall t \in [0, +\infty). \quad (5.8)$$

In this article we give a way of stabilizing system (5.5) exponentially for a small enough γ .

To state our main result let us introduce some notations. We know from [137] (see also Section 5.2.3) that the family of eigenvectors associated to the problem (5.5)–(5.6) form a Riesz basis of $(L^2)^2$, let us note them $(h_n^\gamma, v_n^\gamma)_{n \in \mathbb{Z}}$. Then, we denote by \mathcal{D}_γ the space of finite linear combinations of the $(h_n^\gamma, v_n^\gamma)_{n \in \mathbb{Z}}$, then, any sequence $(F_n)_{n \in \mathbb{Z}}$ defines an element F of \mathcal{D}'_γ :

$$\langle (h_n^\gamma, v_n^\gamma)^T, F \rangle = \overline{F_n}. \quad (5.9)$$

This gives us a general framework to talk about linear feedback laws. The actual domain of definition of our feedback laws, and their regularity, will be closely studied later on in Subsection 5.6.2.

Theorem 5.1.1. *For any $\mu > 0$, there exists $\gamma_0 > 0$ such that, for any $\gamma \in (0, \gamma_0)$, there exists $\nu \neq 0$ such that the control u of the feedback form*

$$u(t) := \left\langle \begin{pmatrix} h_0^\gamma \\ v_0^\gamma \end{pmatrix}, F_1^\gamma \right\rangle \int_0^t e^{\nu L/L^\gamma \langle (h_0^\gamma, v_0^\gamma)^T, F_1^\gamma \rangle (t-\tau)} \frac{\nu L}{L^\gamma} \left\langle \begin{pmatrix} h \\ v \end{pmatrix} (\tau, \cdot), F_1^\gamma \right\rangle d\tau + \left\langle \begin{pmatrix} h \\ v \end{pmatrix} (t, \cdot), F_1^\gamma \right\rangle, \quad (5.10)$$

where $L_\gamma := \frac{2}{\gamma} (1 - \sqrt{1 - \gamma L})$ and $F_1^\gamma \in \mathcal{D}'_\gamma$ is given by

$$\begin{aligned} \langle (h_n, v_n)^T, F_1^\gamma \rangle &= -\frac{\tanh(\mu L)}{H^\gamma(0)} \frac{(h_n)^2(0)}{\int_0^L \frac{L}{L_\gamma \sqrt{1 - \gamma x}} \exp\left(-\int_0^x \frac{3\gamma}{4(1 - \gamma x)} ds\right) v_n(x) dx}, \quad \forall n \in \mathbb{Z}^*, \\ \langle (h_0, v_0)^T, F_1^\gamma \rangle &= -2 \frac{\tanh(\mu L)}{H^\gamma(0)} \frac{(h_0)^2(0)}{\nu}, \end{aligned} \quad (5.11)$$

stabilizes the system (5.5)–(5.6) exponentially in $(H^1)^2$ norm, for initial conditions in $(H^1)^2$ satisfying the boundary conditions (5.6), with decay rate $\frac{\mu}{2}$.

Remark 5.1.1. *This result works for any small $\gamma > 0$, therefore one could wonder whether it could be extended to $\gamma = 0$. However, when $\gamma = 0$ no such F exists as the system (5.5)–(5.6) is not controllable (see [52]), thus this result is sharp in this sense.*

This result will be shown using a strategy inspired from the backstepping approach. Backstepping originally referred to a way of designing more effective feedback laws for finite dimensional systems in a recursive way (see [104], [99], [149] for instance). Later, this method has been modified and adapted to partial differential equations (see [17] and [32]). The key idea is to use an invertible transformation mapping the original system to a target system for which the stability is easy to prove. The stabilization problem becomes then a problem of existence of an isomorphism between two systems. As the class of transformation could be very large, this is potentially a very complicated problem, it is usually simplified by restricting to the Volterra transformation of second kind which have the advantage of being convenient to use and naturally invertible in most cases. These transformations were extensively used in the last decades, for instance for the heat equation [33, 32, 17], for first order hyperbolic linear then quasilinear systems [152, 68], and for many particular cases (see [155, 156] for the KdV equations, [69] for coupled PDE-ODE systems, or [111] for an overview), the goal of each new study being to show that such a transformation exists. However, restricting to only a special type of invertible transformation necessarily restricts the cases where this method can be applied. Moreover, Volterra transformations of the second kind are usually used to move a complexity in the dynamics to the boundaries, to be dealt with an appropriate control. Therefore it could be ill-adapted to an internal control stabilization problem, where the boundary conditions are fixed and cannot be changed, although some results exist by applying a second invertible transform (see [150] or [154]).

In this article we opt for a more general approach, following the strategy of [59] by searching for general kernel operators, namely Fredholm transforms. This requires more work as a Fredholm transform is not always invertible, but we have an additional information: the system is controllable. This will be a key ingredient to find an invertible transformation and derive an explicit feedback law to stabilize the system exponentially.

5.1.3 The backstepping method: a finite-dimensional example

Our strategy of proof combines the use of controllability to stabilize systems with the backstepping method. It was developed to stabilize the Schrödinger equation in [59], and adapted to hyperbolic systems in [162]. We illustrate it with a finite-dimensional example.

Consider the finite-dimensional control system

$$\dot{x} = Ax + Bu(t), \quad x \in \mathbb{C}^n, A \in \mathcal{M}_n(\mathbb{C}), B \in \mathcal{M}_{n,1}(\mathbb{C}). \quad (5.12)$$

Suppose that (5.12) is controllable. Then, it is well known (see for example [53]) that for every polynomial $P \in \mathbb{C}[X]$ there exists a feedback $K \in \mathcal{M}_{1,n}(\mathbb{R})$ such that P is the characteristic polynomial of $A + BK$.

This pole-shifting property for controllable systems can be formulated in another way, by trying to invertibly transform system (5.12) into another system with shifted poles, namely

$$\dot{x} = (A - \lambda I)x, \quad (5.13)$$

which is asymptotically stable for a large enough λ .

More generally we can try to invertibly transform system (5.12) into

$$\dot{x} = \tilde{A}x, \quad (5.14)$$

which is exponentially stable if \tilde{A} is well chosen.

Suppose that $x(t)$ is a solution of system (5.12) with $u(t) = Kx(t)$ for some control function v . Such a transformation T would map (5.12) into

$$(\dot{T}x) = T\dot{x} = T(A + BK)x.$$

In order for Tx to be a solution of (5.13), we need

$$T(A + BK)x = \tilde{A}Tx.$$

To find such a T , let us suppose without loss of generality that (A, B) is in canonical form, using the fact that (A, B) is controllable. Now, suppose that (\tilde{A}, B) is also controllable, so that it can be put in canonical form with an invertible matrix T :

$$T^{-1}\tilde{A}T = c(\tilde{A}). \quad (5.15)$$

Now, it is well-known that there exists a unique K such that

$$A + BK = c(\tilde{A}), \quad (5.16)$$

which yields

$$T(A + BK) = \tilde{A}T. \quad (5.17)$$

Now notice that as we assumed that (A, B) was in canonical form, this implies that

$$TB = B. \quad (5.18)$$

Injecting the above equation into (5.17), we get the following equations:

$$\begin{aligned} TA + BK &= \tilde{A}T, \\ TB &= B, \end{aligned} \quad (5.19)$$

for which we just proved the following theorem:

Theorem 5.1.2. *If (A, B) and (\tilde{A}, B) are controllable, then there exists a unique pair (T, K) satisfying conditions (5.19).*

The controllability of (5.12) and (5.14) is crucial here, as it allows us to use the control canonical form. However, another proof can be found in [59], which is more adaptable to the context of PDEs: the idea is to suppose that A and \tilde{A} are diagonalizable. Then, the controllability of (\tilde{A}, B) allows to build a basis for the space state using the eigenvectors of A , in which T can then be constructed. The $TB = B$ condition along with the controllability of the first system help define the coefficients of the feedback K , and finally the controllability of the second system ensures the invertibility of T with K thus defined.

This other approach to pole-shifting, which links controllability to stabilization, can be used in infinite dimension. In our case, the controllability of (5.59)–(5.38) will have the same importance: it will also allow us to build some sort of basis for the state space, and find a general form for the backstepping transformation, depending on F .

5.2 Properties of the system and presentation of the method

5.2.1 Transforming the system

Let us consider the system (5.5)–(5.6). Using the change of variable

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{H^\gamma}} & 1 \\ -\sqrt{\frac{1}{H^\gamma}} & 1 \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix}, \quad (5.20)$$

we get the system in Riemann coordinates:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_x + \delta_0(x) \begin{pmatrix} 1 & \frac{1}{3} \\ -\frac{1}{3} & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = u(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (5.21)$$

where $\lambda_1 = \lambda_2 = \sqrt{H^\gamma}$, $\delta_0(x) = -\frac{3}{4} \frac{\gamma}{\sqrt{1-\gamma x}}$, and with the boundary conditions:

$$\begin{aligned} \xi_1(t, 0) &= -\xi_2(t, 0), \\ \xi_2(t, L) &= -\xi_1(t, L). \end{aligned} \quad (5.22)$$

Thus also implies the following condition:

$$\int_0^L \sqrt{H^\gamma(x)} (\xi_1 - \xi_2)(x) dx = 0. \quad (5.23)$$

We would like to simplify the matrix in front of the transport term. To this aim, let us introduce a change of variable in space: $y = \frac{2}{\gamma} (1 - \sqrt{1 - \gamma x})$ and define

$$L_\gamma = \frac{2}{\gamma} (1 - \sqrt{1 - \gamma L}), \quad (\gamma \text{ is supposed sufficiently small}). \quad (5.24)$$

By a slight abuse of notation we used again ξ , now defined on $y \in [0, L_\gamma]$, to denote the solutions to this last system, so that these equations become:

$$\partial_t \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_y \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \delta_1(y) \begin{pmatrix} 1 & \frac{1}{3} \\ -\frac{1}{3} & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = u(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (5.25)$$

where $\delta_1(y) = -\frac{3}{4} \frac{\gamma}{(1 - \gamma y/2)}$, and with the boundary conditions:

$$\begin{aligned} \xi_1(t, 0) &= -\xi_2(t, 0), \\ \xi_2(t, L_\gamma) &= -\xi_1(t, L_\gamma). \end{aligned} \quad (5.26)$$

As well as the conservation law,

$$\int_0^{L_\gamma} (1 - \frac{\gamma}{2} y)^2 (\xi_1(y) - \xi_2(y)) dy = 0. \quad (5.27)$$

This could be expressed in a more compact form using the following notations:

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, J_0 = \begin{pmatrix} 1 & \frac{1}{3} \\ -\frac{1}{3} & -1 \end{pmatrix}. \quad (5.28)$$

We also define

$$J = \begin{pmatrix} 0 & \frac{1}{3} \\ -\frac{1}{3} & 0 \end{pmatrix}, \quad (5.29)$$

which will be used later on. Looking at (5.25) and (5.28), the transport matrix Λ has now a simple form, as expected, but the length of the domain depends now on γ . We arrange this by using a scaling simultaneously on time and space and we define

$$w(t, z) := \xi(L_\gamma t/L, y(z)), \quad \text{with } y(z) = \frac{L_\gamma}{L} z. \quad (5.30)$$

For convenience we renote $x := z \in [0, L]$ so that x still denotes the space variable, then $w(x)$ satisfies

$$\begin{aligned} \partial_t w + \Lambda \partial_x w + \delta J_0 w &= u(L_\gamma t/L) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ w_1(t, 0) &= -w_2(t, 0), \\ w_2(t, L) &= -w_1(t, L), \end{aligned} \tag{5.31}$$

with

$$\delta(x) = (L_\gamma/L) \delta_1(L_\gamma x/L), \text{ for all } x \in [0, L], \tag{5.32}$$

§so that

$$\delta(x) = -\frac{3}{4}\gamma \left(1 + \frac{1}{2}\gamma(L+x) + O(\gamma^2)\right). \tag{5.33}$$

And the condition

$$\int_0^L \left(1 - \frac{1 - \sqrt{1 - \gamma L}}{L} x\right)^2 (w_1(x) - w_2(x)) dx = 0, \tag{5.34}$$

which, from now on, will be called the “missing direction” as this cannot be changed, whatever the control, and restricts necessarily the admissible perturbation or the reachable states.

And, finally, we use a diagonal change of coordinates

$$\zeta(t, x) := \exp\left(\int_0^x \delta(s) ds\right) w(t, x), \tag{5.35}$$

with

$$\exp\left(\int_0^x \delta(s) ds\right) = \left(1 - \frac{\gamma L_\gamma}{2L} x\right)^{3/2} = 1 - \frac{3}{4}\gamma x + O(\gamma^2). \tag{5.36}$$

This last operation is used to remove the diagonal coefficients of the source term (see [111, Chapter 9], [98] for more details on the interest of this change of coordinates). The system then becomes

$$\partial_t \zeta + \Lambda \partial_x \zeta + \delta J \zeta = u(L_\gamma t/L) \exp\left(\int_0^x \delta(s) ds\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{5.37}$$

with boundary conditions

$$\begin{aligned} \zeta_1(t, 0) &= -\zeta_2(t, 0), \\ \zeta_2(t, L) &= -\zeta_1(t, L). \end{aligned} \tag{5.38}$$

Hence a condition

$$\int_0^L \left(1 - \frac{1 - \sqrt{1 - \gamma L}}{L} x\right)^{1/2} (\zeta_1(x) - \zeta_2(x)) dx = 0. \tag{5.39}$$

This will be our system in the following, together with the boundary conditions (5.38).

5.2.2 Spaces and notations

In this subsection, we define several notations which will be used throughout the article. Some of them will be introduced later on in the article but are gathered here as a glossary for the reader's convenience. To simplify the computations and the statements we denote

$$\begin{aligned} (L^2)^2 &= L^2((0, L); \mathbb{C}^2), \\ (H^s)^2 &= H^s((0, L); \mathbb{C}^2), \text{ for any } s \geq 0. \end{aligned} \tag{5.40}$$

Similarly for any $s \in \mathbb{N}$ we denote $C^s = C^s([0, L], \mathbb{R}^2)$ and we note C_{pw}^s the space of piecewise C^s functions, i.e. functions f such that there exists a subdivision $\{\sigma_i\}, i \in \{1, \dots, n-1\}$ for some $n \geq 1$, such that

$$f|_{[\sigma_i, \sigma_{i+1}]} \in C^1([\sigma_i, \sigma_{i+1}]), \quad \forall i \in \{1, \dots, n-1\}. \tag{5.41}$$

For any family we denote for simplicity $(a_n)_{n \in \mathbb{Z}} = (a_n)$, the index being specified when the family is not considered over the whole \mathbb{Z} . The scalar product correspond to the L^2 norm is defined by

$$\langle f, g \rangle := \frac{1}{2L} \int_0^L f_1(x) \overline{g_1(x)} + f_2(x) \overline{g_2(x)} dx. \quad (5.42)$$

We now present the following families of functions, whose existence will be justified later on :

- $(f_n)_{n \in \mathbb{Z}}$ denote the eigenfunctions of the operator given by (5.46)–(5.47) and associated to the original system (5.37)–(5.38) and forming an orthonormal Riesz basis.
- $(\tilde{f}_n, \tilde{\phi}_n)_{n \in \mathbb{Z}}$ denote the eigenfunctions forming a Riesz basis and the associated biorthonormal family of the operator given by (5.61) and associated to the target system (5.60).
- (ψ_n, χ_n) denote the eigenfunctions forming a Riesz basis and the associated biorthonormal family of the operator associated to the system (5.31).
- $(\tilde{\psi}_n, \tilde{\chi}_n)$ denote the eigenfunctions forming a Riesz basis and the associated biorthonormal family of the operator associated to the system (5.210).

Let us now note \mathcal{E} the space of finite linear combinations of the $(f_n)_{n \in \mathbb{Z}}$. Then, any sequence $(F_n)_{n \in \mathbb{Z}}$ defines an element F of \mathcal{E}' :

$$\langle f_n, F \rangle = \overline{F_n}. \quad (5.43)$$

\mathcal{E} and \mathcal{E}' are linked to the spaces \mathcal{D}_γ and \mathcal{D}'_γ by the changes of variables performed in the previous section.

Finally we define the spaces

$$X^s := \{f \in (L^2)^2, \quad (\tau^{\mathcal{I}})^{-1}(\Lambda \partial_x f + \delta(x) J f) \in (H^{s-1})^2\}, \quad s \geq 1, \quad (5.44)$$

where $\tau^{\mathcal{I}}$ is an isomorphism of H^s defined by (5.289), and we endow them with the norms:

$$\|f\|_{X^s} := \|(\tau^{\mathcal{I}})^{-1}(\Lambda \partial_x f + \delta(x) J f)\|_{(H^{s-1})^2} + \|f\|_{L^2}, \quad s \geq 1. \quad (5.45)$$

5.2.3 A system of eigenvectors for the open-loop system (i.e. without feedback):

Considering (5.37), let us define the following operator:

$$\mathcal{A} := \Lambda \partial_x + \delta(x) J, \quad (5.46)$$

defined on the domain

$$D(\mathcal{A}) := \{(f_1, f_2) \in (H^1)^2, \quad f_1(0) + f_2(0) = 0, \quad f_1(L) + f_2(L) = 0\}. \quad (5.47)$$

Its adjoint is clearly defined by:

$$D(\mathcal{A}^*) := \{(f_1, f_2) \in (H^1)^2, \quad f_1(0) + f_2(0) = 0, \quad f_1(L) + f_2(L) = 0\}, \quad (5.48)$$

so that

$$\mathcal{A}^* = -\mathcal{A}, \quad \text{if } \gamma \in \mathbb{R}. \quad (5.49)$$

We know from [137] that \mathcal{A} has a family of eigenfunctions, which we note (f_n) , that form a Riesz basis of $(L^2)^2$. From (5.49) we know that the (f_n) form an orthonormal basis, and the corresponding eigenvalues μ_n are all imaginary. Moreover, they satisfy the following asymptotic behaviour, given the boundary conditions we have set:

$$\mu_n = \frac{i\pi n}{L} + \mathcal{O}\left(\frac{1}{n}\right), \quad \forall n \in \mathbb{Z}. \quad (5.50)$$

Moreover, given the definition of \mathcal{A} , we can easily derive a few additional properties (see Appendix 5.A for the proof):

Proposition 5.2.1. *The (f_n, μ_n) satisfy the following:*

$$(i) \quad \mu_{-n} = \overline{\mu_n} = -\mu_n, \quad \forall n \in \mathbb{Z}. \quad (5.51)$$

In particular, $\mu_0 = 0$.

$$(ii) \quad f_{-n} = \overline{f_n} = (-f_{n,2}(\cdot), -f_{n,1}(\cdot)), \quad \forall n \in \mathbb{Z}. \quad (5.52)$$

In particular, $f_{n,1}(0), f_{n,1}(L) \in \mathbb{R}$, and

$$f_{0,1}(x) + f_{0,2}(x) = 0, \quad \forall x \in [0, L]. \quad (5.53)$$

Finally let us introduce the spaces:

$$D(\mathcal{A}^s) := \{\alpha \in (L^2)^2, \sum_{n \in \mathbb{Z}^*} (1 + |\mu_n|^{2s}) |\langle \alpha, f_n \rangle|^2 < \infty\}, \quad s \geq 0, \quad (5.54)$$

which corresponds to H^s , so that we have

$$D(\mathcal{A}^1) = D(\mathcal{A}). \quad (5.55)$$

5.3 Dealing with mass conservation

5.3.1 A new system

Let us now consider again our system (5.37):

$$\partial_t \zeta + \Lambda \partial_x \zeta + \delta J \zeta = u(L_\gamma t/L) \exp\left(\int_0^x \delta(s) ds\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with boundary conditions (5.38). In order to identify the control in the following we define

$$\mathcal{I} := \exp\left(\int_0^x \delta(s) ds\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (5.56)$$

As we will see later on this control term has a drawback : from (5.53), we get

$$\langle \mathcal{I}, f_0 \rangle = 0, \quad (5.57)$$

which means that the control cannot act on this direction and therefore the system is not fully controllable. Physically this comes from the fact that the control does not add or spill any water, thus the mass is conserved.

To overcome this difficulty we introduce the following virtual control

$$\mathcal{I}_\nu := \mathcal{I} + \nu f_0, \quad (5.58)$$

together with the following virtual system

$$\begin{cases} \partial_t \mathbf{Z} + \Lambda \partial_x \mathbf{Z} + \delta(s) J \mathbf{Z} = \langle \mathbf{Z}(t, \cdot), F \rangle \mathcal{I}_\nu, \\ \mathbf{Z}_1(t, 0) = -\mathbf{Z}_2(t, 0), \quad \forall t \geq 0, \\ \mathbf{Z}_1(t, L) = -\mathbf{Z}_2(t, L), \quad \forall t \geq 0, \end{cases} \quad (5.59)$$

where F is a linear feedback to be determined.

5.3.2 Our target system

In past applications of the backstepping method, the most frequently used target system is simply the homogeneous system corresponding to the damped operator $\mathcal{A} - \mu Id$, for some $\mu > 0$. This choice can be easily understood: by adding a damping large enough the solution is likely to be decaying with a decay rate large enough. As it appears, in our case it is more practical to consider a target system where the dissipation occurs instead at the boundary:

$$\begin{cases} \partial_t z + \Lambda \partial_x z + \delta(x) J z = 0, \\ z_1(t, 0) = -e^{-2\mu L} z_2(t, 0), \\ z_2(t, L) = -z_1(t, L). \end{cases} \quad (5.60)$$

5.3.2.1 A system of eigenvectors for the target system

As this target system has boundary conditions different from the original system, let us define a new operator:

$$D(\tilde{\mathcal{A}}) := \{(f_1, f_2) \in (H^1)^2, \quad \tilde{\mathcal{A}} := \Lambda \partial_x + \delta(x) J, \quad f_1(0) + e^{-2\mu L} f_2(0) = 0, \quad f_1(L) + f_2(L) = 0\}. \quad (5.61)$$

The eigenvectors $(\tilde{f}_n, \tilde{\mu}_n)$ of $\tilde{\mathcal{A}}$ form a Riesz basis of $(L^2)^2$, and, using again the results in [137], we have the following asymptotic development for $\tilde{\mu}_n$:

$$\tilde{\mu}_n = \mu + \frac{i\pi n}{L} + O\left(\frac{1}{n}\right). \quad (5.62)$$

Moreover, the (\tilde{f}_n) admit a biorthogonal family which we note $(\tilde{\phi}_n)$. Note that it is a well known fact that the $(\tilde{\phi}_n, \tilde{\mu}_n)$ are the eigenvectors of the adjoint operator:

$$D(\tilde{\mathcal{A}}^*) := \{(f_1, f_2) \in (H^1)^2, \quad \tilde{\mathcal{A}}^* := -\Lambda \partial_x - \delta(x) J, \quad f_1(0) + e^{2\mu L} f_2(0) = 0, \quad f_1(L) + f_2(L) = 0\}. \quad (5.63)$$

Again, this allows us to define the following spaces:

$$D(\tilde{\mathcal{A}}^s) := \{\alpha \in (L^2)^2, \quad \sum (1 + |\tilde{\mu}_n|^{2s}) |\langle \alpha, \tilde{\phi}_n \rangle|^2 < \infty\}, \quad s \geq 0, \quad (5.64)$$

and

$$D((\tilde{\mathcal{A}}^*)^s) := \{\alpha \in (L^2)^2, \quad \sum (1 + |\tilde{\mu}_n|^{2s}) |\langle \alpha, \tilde{f}_n \rangle|^2 < \infty\}, \quad s \geq 0. \quad (5.65)$$

Now notice that when $\gamma = 0$, from the expression of δ_1 and (5.32), $\delta(x) = 0$ and the operator becomes:

$$D(\tilde{\mathcal{A}}^{(0)}) := \{(f_1, f_2) \in (H^1)^2, \quad \tilde{\mathcal{A}}^{(0)} := \Lambda \partial_x, \quad f_1(0) + e^{-2\mu L} f_2(0) = 0, \quad f_1(L) + f_2(L) = 0\}, \quad (5.66)$$

for which the eigenvectors and eigenvalues are

$$\tilde{f}_n^{(0)} = \begin{pmatrix} e^{(\mu + \frac{i\pi n}{L})x} \\ -e^{(\mu + \frac{i\pi n}{L})(2L-x)} \end{pmatrix}, \quad \tilde{\mu}_n^{(0)} = \mu + \frac{i\pi n}{L}, \quad \forall n \in \mathbb{Z}, \quad (5.67)$$

and the corresponding biorthogonal family is given by:

$$\tilde{\phi}_n^{(0)} = \begin{pmatrix} e^{(-\mu + \frac{i\pi n}{L})x} \\ -e^{(-\mu + \frac{i\pi n}{L})(2L-x)} \end{pmatrix}, \quad \forall n \in \mathbb{Z}. \quad (5.68)$$

These eigenfunctions are not normal in L^2 space. Thanks to [137], we know that $\{\tilde{f}_n^{(0)}/\|\tilde{f}_n^{(0)}\|_{L^2}\}$ form a Riesz basis of L^2 space. Moreover, since

$$\|\tilde{f}_n^{(0)}\|_{L^2}^2 = \frac{e^{4\mu L} - 1}{2\mu}, \quad (5.69)$$

we know that

$$\{\tilde{f}_n^{(0)}\} \text{ form a Riesz basis of } L^2 \text{ space.} \quad (5.70)$$

Now note that, by integration by parts,

$$\begin{aligned} \overline{\tilde{\mu}_n^{(0)}} \langle (1, 1), \tilde{\phi}_n^{(0)} \rangle &= \langle (1, 1), \Lambda \partial_x \tilde{\phi}_n^{(0)} \rangle \\ &= \overline{\left(\tilde{\phi}_n^{(0)}\right)_1(L)} - \overline{\left(\tilde{\phi}_n^{(0)}\right)_1(0)} - \overline{\left(\tilde{\phi}_n^{(0)}\right)_2(L)} + \overline{\left(\tilde{\phi}_n^{(0)}\right)_2(0)} \\ &= 2(-1)^n e^{-\mu L} - 1 - e^{-2\mu L}, \end{aligned} \quad (5.71)$$

which is clearly bounded away from 0 for μ large enough. Hence $(\mathcal{A}^{(0)}, (1, 1))$ is controllable.

5.3.2.2 Exponential stability of the target system

In this section we show the following proposition

Proposition 5.3.1. *For any $\lambda \in (0, \mu)$, there exists $\gamma_0 > 0$ such that if $\gamma \in (0, \gamma_0)$, the target system (5.60) is exponentially stable with decay rate λ (for the H^p norm, for any $p \in \mathbb{N}$).*

Proof. First, from [114] note that the system is well-posed in H^p . More precisely, let $T > 0$, there exists a constant $C(T) > 0$ such that for any $z_0 \in H^p$ the system (5.60) with initial condition z_0 has a unique solution $z \in C^0([0, T], H^p)$ and

$$\|z(t, \cdot)\|_{H^p} \leq C(T) \|z_0\|_{H^p}. \quad (5.72)$$

We now define the following Lyapunov function candidate V

$$V(Z) = \sum_{n=0}^p \|\Theta(x)(\tilde{\mathcal{A}}^n Z)\|_{L^2(0, L)}^2, \quad \forall Z \in H^p(0, L), \quad (5.73)$$

where $\Theta = \text{diag}(\sqrt{\theta_1}, \sqrt{\theta_2})$ with θ_1 and θ_2 two positive C^1 functions to be selected later on. Obviously V is equivalent to the square of the H^p norm in the sense that there exists positive constants C_1 and C_2 such that for any $Z \in H^p(0, L)$,

$$C_1 \|Z\|_{H^p(0, L)}^2 \leq V(Z) \leq C_2 \|Z\|_{H^p(0, L)}^2. \quad (5.74)$$

Now observe that for a solution z to the system (5.60), one has

$$V(z) = \sum_{n=0}^p \langle \Theta(x) \partial_t^n z(t, \cdot), \Theta(x) \partial_t^n z(t, \cdot) \rangle. \quad (5.75)$$

Let $n \in \{0, \dots, p\}$, from (5.60) $\partial_t^n z$ is also a solution to (5.60), and therefore, differentiating $V(z)$ along time, one has

$$\dot{V}(z) = -2\Re \left(\sum_{n=0}^p \langle \Lambda \partial_x (\partial_t^n z)(t, \cdot), \Theta^2 \partial_t^n z(t, \cdot) \rangle + \langle \delta J \partial_t^n z(t, \cdot), \Theta^2 \partial_t^n z(t, \cdot) \rangle \right). \quad (5.76)$$

Thus, integrating by parts and using the boundary conditions of (5.60),

$$\begin{aligned} \dot{V}(z) &= -2\lambda V \\ &\quad - \sum_{n=0}^p [(\theta_1(L) - \theta_2(L))(\partial_t^n z_1)^2(t, L) + (\theta_2(0) - \theta_1(0)e^{-4\mu L})(\partial_t^n z_2)^2(t, 0)] \\ &\quad - \sum_{n=0}^p \Re \left(\langle (-\Lambda(\Theta^2)' - 2\lambda\Theta^2 + 2\Theta^2\delta J)\partial_t^n z, \partial_t^n z \rangle \right). \end{aligned} \quad (5.77)$$

Our goal is now to choose Θ such that the two last sums are nonnegative. Recognizing a quadratic form in the integrals, it suffices to ensure that

$$\begin{aligned} \theta_1(L) &\geq \theta_2(L), \\ \theta_2(0) - \theta_1(0)e^{-4\mu L} &\geq 0 \\ (-\Lambda(\Theta^2)' - 2\lambda\Theta^2 + 2\Theta^2\delta J) &\text{ is definite semi-positive.} \end{aligned} \quad (5.78)$$

Denoting $\Xi_1 = \theta_1 \exp(2\lambda(x - L))$ and $\Xi_2 = \theta_2 \exp(-2\lambda(x - L))$, (5.78) is equivalent to

$$\begin{aligned} \Xi_1(L) &\geq \Xi_2(L), \\ \Xi_2(0) - \Xi_1(0)e^{-4(\mu-\lambda)L} &\geq 0, \\ -\Xi_1'\Xi_2' &\geq \left(\frac{\delta}{3}\right)^2 (\Xi_1 \exp(-2\lambda(x - L)) - \Xi_2 \exp(2\lambda(x - L)))^2. \end{aligned} \quad (5.79)$$

Following [22, Proposition 1], the existence of Ξ_1, Ξ_2 positive and of class C^1 satisfying these conditions is equivalent to the existence of η positive and of class C^1 satisfying

$$\begin{aligned} \eta(L) &\leq 1, \\ \eta(0) &= e^{-2(\mu-\lambda)L}, \\ \eta' &= \left| \frac{\delta}{3} \right| |\exp(-2\lambda(x - L)) - \eta^2 \exp(2\lambda(x - L))|, \end{aligned} \quad (5.80)$$

which, using the first condition in the third one, is in fact equivalent to

$$\begin{aligned} \eta(L) &\leq 1, \\ \eta(0) &= e^{-2(\mu-\lambda)L}, \\ \eta' &= \left| \frac{\delta}{3} \right| (\exp(-2\lambda(x - L)) - \eta^2 \exp(2\lambda(x - L))). \end{aligned} \quad (5.81)$$

We will now show the existence of such η by exhibiting a super-solution to the two last equations of (5.81) satisfying also the first condition. Let us introduce ξ being the C^1 solution of

$$\begin{aligned} \xi' &= \left\| \frac{\delta}{3} \right\|_{L^\infty(0,L)} \left(e^{2\lambda(L-x)} \right), \\ \xi(0) &= e^{-2(\mu-\lambda)L}. \end{aligned} \quad (5.82)$$

This system can be easily solved and

$$\xi(x) = e^{-2(\mu-\lambda)L} + \frac{\left\| \frac{\delta}{3} \right\|_{L^\infty(0,L)}}{6\lambda} \left(e^{2\lambda L} - e^{2\lambda(L-x)} \right). \quad (5.83)$$

Let us now set

$$\gamma_0 = \min \left(\frac{7}{16L}, 6\lambda \frac{1 - e^{-2(\mu-\lambda)L}}{(e^{2\lambda L} - e^{2\lambda(L-x)})} \right), \quad (5.84)$$

and assume that $\gamma \in (0, \gamma_0)$, then one has from the definition of δ given in (5.32),

$$\xi(L) \leq 1. \quad (5.85)$$

Thus, looking at (5.82) and the two last equations of (5.81) and by comparison (see for instance [90]), η exists on $[0, L]$ and in addition $\eta \leq \xi$ on $[0, L]$. Thus, choosing such η , we have

$$\dot{V}(z) \leq -2\lambda V, \quad (5.86)$$

thus, using (5.74),

$$\|z(t, \cdot)\|_{H^p} \leq \sqrt{\frac{C_2}{C_1}} \|z_0\|_{H^p} e^{-\lambda t}, \text{ on } [0, T]. \quad (5.87)$$

But, as $T > 0$ was chosen arbitrary and C_1, C_2 and λ are independent of T this is also true on $[0, +\infty)$. This ends the proof of Proposition 5.3.1 \square

Remark 5.3.1. *As it can be seen in condition (5.84), when γ is small enough we can actually achieve a decay rate as close as we want from μ , as was expected looking at the eigenvalues of $\tilde{\mathcal{A}}$ given by (5.62).*

Finally, following [22] again, if the maximal solution of the two last equations of (5.81) does not exist on $[0, L]$ or does not satisfies the first condition of (5.81), then there does not exist any Lyapunov function with a decay rate larger of equal to λ of the form

$$V(Z) = \sum_{n=0}^p \langle \tilde{\mathcal{A}}^n Z, Q(x)(\tilde{\mathcal{A}}^n Z) \rangle, \quad \forall Z \in H^p(0, L), \quad (5.88)$$

where $Q \in C^1([0, L], M_2^+(\mathbb{R}))$, with $M_2^+(\mathbb{R})$ the space of positive definite matrix on \mathbb{R}^2 . Note that the form (5.88) includes all the Lyapunov functions of the form (5.73). This implies that we cannot get a uniform bound on γ and in particular that for any $\gamma > 0$ there exists $\mu_\gamma > 0$ such that there does not exist any Lyapunov function of the form (5.88) with a decay rate larger of equal to $\mu_\gamma > 0$. Indeed let $\gamma > 0$, and assume by contradiction for any $\mu > 0$ there exists a Lyapunov function of the form (5.88) with decay rate larger or equal to μ , then there exists a function η on $[0, L]$ satisfying (5.81). Besides,

$$\eta' \geq \inf_{[0, L]} \left| \frac{\delta}{3} \right| \left(e^{\mu(L-x)} - e^{-\mu(L-x)} \right), \quad (5.89)$$

hence, integrating and as $\eta(0) \geq 0$,

$$\eta(L) \geq \frac{1}{\mu} \inf_{[0, L]} \left| \frac{\delta}{3} \right| (\text{ch}(\mu L) - 1), \quad (5.90)$$

and using the first condition of (5.81),

$$\frac{1}{\mu} \inf_{[0, L]} \left| \frac{\delta}{3} \right| (\text{ch}(\mu L) - 1) \leq 1. \quad (5.91)$$

As $\inf_{[0, L]} |\delta| > 0$, there exists $\mu_1 > 0$ such that there is contradiction.

5.3.3 Outline of the proof

Now, notice that, thanks to the fact that $\Lambda \partial_x f_0 + \delta(s) J f_0 = 0$, projecting the system (5.59) on the $(f_n)_{n \in \mathbb{Z}}$ and denoting

$$\mathbf{Z} = \zeta_0 f_0 + \zeta, \quad \langle \zeta, f_0 \rangle = 0, \quad (5.92)$$

one has

$$\dot{\zeta}_0 = \nu (\langle \zeta, F \rangle + \zeta_0 \langle f_0, F \rangle), \quad (5.93)$$

and

$$\begin{cases} \partial_t \zeta + \Lambda \partial_x \zeta + \delta J \zeta = (\langle \zeta(t, \cdot), F \rangle + \zeta_0 \langle f_0, F \rangle) \mathcal{I}, \\ \zeta_1(t, 0) = -\zeta_2(t, 0), \\ \zeta_2(t, L) = -\zeta_1(t, L). \end{cases} \quad (5.94)$$

We will show later (see Remark 5.4.7) that (5.39) is equivalent to $\langle \zeta, f_0 \rangle = 0$. Thus ζ is exactly the solution of (5.37), (5.38), (5.39) with control $u(L_\gamma t/L) := (\langle \zeta(t, \cdot), F \rangle + \zeta_0 \langle f_0, F \rangle)$. This control is indeed an implementable feedback law as it depends only on the state and past states of the system, with ζ_0 acting as a kind of integrator on the system. Physically speaking, we can think of it as “virtual mass” that is added or removed from the real, physical system described by ζ .

Then, the key result to prove Theorem 5.1.1 is the following:

Proposition 5.3.2. *For any $\mu > 0$, there exists $\gamma_0 > 0$ such that, for any $\gamma \in (0, \gamma_0)$, there exists $\nu \neq 0$ such that the control u of the feedback form*

$$u(t) = \langle \mathbf{Z}, F \rangle, \quad (5.95)$$

where $F \in \mathcal{E}'$ is given by

$$\begin{aligned} \langle f_0, F \rangle &= -2 \tanh(\mu L) \frac{(f_{0,1}(0))^2}{\nu}, \\ \langle f_n, F \rangle &= -2 \tanh(\mu L) \frac{(f_{n,1}(0))^2}{\langle \mathcal{I}, f_n \rangle}, \quad \forall n \in \mathbb{Z}^*, \end{aligned} \quad (5.96)$$

stabilizes (5.59) exponentially, with decay rate $\frac{3}{4}\mu$.

Indeed, to design a feedback ensuring the exponential stability of (5.37), (5.38), (5.39), it suffices to design a linear feedback ensuring the exponential stability of the virtual system (5.59). Then, for any initial condition of system (5.37), (5.38),

$$\zeta^0 \in D(\mathcal{A}), \quad \langle \zeta^0, f_0 \rangle = 0, \quad (5.97)$$

one can implement that feedback on system (5.59) with initial condition

$$\zeta_0(0) = 0, \quad \zeta(0) = \zeta^0.$$

This will yield, in particular, exponential stabilization of the ζ part.

Then, as the transformations (5.20) and (5.35) and the scaling introduced in (5.25) and (5.30) define a diffeomorphism, this implies the exponential stability of the initial system (5.5)–(5.7), albeit with a different decay rate (because of the time scaling in (5.30)). Finally, the inverse transformations of (5.20) and (5.35) allow us to recover (5.11) in the original variables from the definition of F given by (5.271) (this is given in more detail in Appendix 5.C), which completes the proof of Theorem 5.1.1.

The following sections are thus devoted to the proof of Proposition 5.3.2, by applying the backstepping method described in Section 5.1.3 to virtual system (5.59) and target system (5.60).

Section 5.4 handles the controllability of both systems, proving that there exists $\nu \neq 0$ such (5.59) (Lemma 5.4.5), and (5.60) (Lemma 5.4.7) are both controllable.

Section 5.5 builds candidates for a suitable backstepping transformation for our problem, and gives a sufficient condition to find such a backstepping transformation.

Section 5.6 then builds a suitable backstepping transformation, along with the associated feedback law, and checks that this feedback law is indeed exponentially stabilizing in some sense.

5.4 Controllability

The goal of this section is to achieve some controllability results for the original system and the target system. Thanks to the moment theory, those results can be obtained by several estimates, such as

Lemma 5.4.1, Lemma 5.4.5, Lemma 5.4.6, (5.179), (5.200), (5.229), which are exactly some key points for our stabilization problem.

When $\gamma = 0$ the target system (5.60) becomes quite simple (see (5.66)) and obviously controllable. Hence, it is rather easy to prove its controllability with $\gamma > 0$ small. On the other hand, as already mentioned, the initial system (5.37)–(5.38) is not controllable when $\gamma = 0$. Therefore, we mainly focus in this section on the controllability of system (5.37)–(5.38) which will be the object of Section 5.4.1 and Corollary 5.4.1. Then, as the operator \mathcal{A} given by (5.46) and the operator $\tilde{\mathcal{A}}$ given by (5.61) share many common properties, almost all the calculations and estimates in Section 5.4.1.1–5.4.1.6 also hold for $\tilde{\mathcal{A}}$, which will lead to the controllability of System (5.60). This will be the object of Section 5.4.2 and Theorem 5.4.5.

5.4.1 Controllability of System (5.37)–(5.38)

As system (5.37), (5.38) is obtained from (5.31) by an isomorphism, it suffices to prove the controllability of System (5.31)

5.4.1.1 Asymptotic calculation: Perturbed operators and normalized eigenfunctions

Let us define the operator

$$T := \Lambda \partial_x, \quad (5.98)$$

$$T_\gamma := \mathcal{S}_0 = \Lambda \partial_x + \delta(x) J_0, \quad (5.99)$$

$$D(\mathcal{S}_0) := \{(w_1, w_2) \in (H^1)^2 : w_1(0) = -w_2(0), w_1(L) = -w_2(L)\}, \quad (5.100)$$

associated to the system (5.31).

We want to find an asymptotic formulation of eigenvalues and eigenfunctions of this operator. Then the adjoint operator is given by

$$\mathcal{S}_0^* := -\Lambda \partial_x + \bar{\delta}(x) J_0^*, \quad (5.101)$$

$$D(\mathcal{S}_0^*) = D(\mathcal{S}_0). \quad (5.102)$$

Hence, \mathcal{S}_0 is neither self-adjoint nor anti-adjoint. In fact, it is even not a normal operator:

$$\mathcal{S}_0 \mathcal{S}_0^* - \mathcal{S}_0^* \mathcal{S}_0 = 2\delta_x(x). \quad (5.103)$$

Note that this operator has the same eigenvalues $\mu_n(\gamma)$ as the operator \mathcal{A} . Concerning the relation between \mathcal{S}_0 and \mathcal{S}_0^* . If $(\psi_n(\gamma), \mu_n(\gamma))$ are eigenfunctions of \mathcal{S}_0 , then $(\chi_n(\gamma), \bar{\mu}_n(\gamma))$ are eigenfunctions of \mathcal{S}_0^* . Moreover

$$\langle \psi_n(\gamma), \chi_m(\gamma) \rangle = 0, \text{ if } n \neq m. \quad (5.104)$$

The eigenvalues of T are $\mu_n^{(0)} := i\pi n/L$. They are thus simple and isolated. The normalized eigenfunctions are $\psi_n^{(0)} = (e^{i\pi n x/L}, -e^{-i\pi n x/L})$. Similarly, the eigenvalues of T_γ are $\mu_n(\gamma)$. In fact, due to the fact that an eigenfunction multiplied by a scalar number is still an eigenfunction, it is convenient to consider normalized eigenfunctions $\psi_n(\gamma)$, as what is done in Kato's noval book [102, page 92]:

$$\langle \psi_n(\gamma), \chi_n^{(0)} \rangle = 1, \text{ (resp. } \langle \psi_n^{(0)}, \chi_n(\gamma) \rangle = 1). \quad (5.105)$$

This normalization formula is standard and is convenient to perform symbol calculation. For this reason we cannot assume $\langle \psi_n(\gamma), \chi_n(\gamma) \rangle = 1$ at the same time.

5.4.1.2 L^2 -normalized eigenfunctions and Riesz basis

We are also interested in the L^2 -normalized eigenfunctions:

$$\hat{\psi}_n(\gamma) := \psi_n(\gamma) / \|\psi_n(\gamma)\|_{L^2}, \hat{\chi}_n(\gamma) := \chi_n(\gamma) / \|\chi_n(\gamma)\|_{L^2}. \quad (5.106)$$

The following theorem by Russell tells us that those eigenfunctions form a Riesz basis when γ is sufficiently small.

Theorem 5.4.1 (Russell [137]). *There exists $r_R > 0$ such that, $\forall \gamma \in (-r_R, r_R)$,*

- (1) *both $\{\hat{\psi}_n(\gamma)\}_n$ and $\{\hat{\chi}_n(\gamma)\}_n$ form a Riesz basis of L^2*
- (2) *T_γ has simple isolated eigenvalues $\mu_n(\gamma)$. Moreover,*

$$\mu_n(\gamma) = \frac{i\pi n}{L} + \gamma \mathcal{O}\left(\frac{1}{n}\right), \quad \forall n \in \mathbb{Z}^*. \quad (5.107)$$

5.4.1.3 The moment method

The moments method consists in decomposing the state and the control term in a Riesz basis of eigenfunctions, which yields an infinity of independent ODEs, and studying the moments problem given by these ODEs. The main issues, as indicated above, are that the eigenfunctions form a Riesz basis, and that the projection of the control term on each direction is away from 0 (hence observable). We also refer to the book [15] for a good introduction of this method.

This suggests that we study

$$a_n := \langle \psi_n(\gamma), (1, 1) \rangle \text{ and } b_n := \langle \chi_n(\gamma), (1, 1) \rangle, \quad \forall n \in \mathbb{Z}, \quad (5.108)$$

Indeed, at least formally, we are able to decompose (1,1) by

$$(1, 1) = \sum_{n \in \mathbb{Z}} d_n \psi_n(\gamma), \quad (5.109)$$

thus, using (5.104)

$$d_n \langle \psi_n(\gamma), \chi_n(\gamma) \rangle = \langle (1, 1), \chi_n(\gamma) \rangle = \bar{b}_n. \quad (5.110)$$

For that moment let us assume that the following lemma holds (it will be proved in Sections 5.4.1.6–5.4.1.7):

Lemma 5.4.1. *There exists $\gamma_0, c, C > 0$ such that for any $\gamma \in (0, \gamma_0)$ we have*

- (i) *$(\psi_n(\gamma))_{n \in \mathbb{Z}}$ is a Riesz basis of L^2 ;*
- (ii) *$|\langle \psi_n(\gamma), \chi_n(\gamma) \rangle| \in (1/2, 2)$;*
- (iii) *$|\mu_n(\gamma) - \mu_n^{(0)}| < \frac{1}{4L}$;*
- (iv) *b_n is away from zero in some sense. More precisely, we have*

$$b_0 = 0, \quad (5.111)$$

$$\gamma \frac{c}{n} < |b_n| < \frac{C}{n}, \quad \forall n \in \mathbb{Z}^*. \quad (5.112)$$

Thanks to Lemma 5.4.1 and classical moment theory, we can conclude that the system is not yet controllable but there is only one dimension missing corresponding to the moment b_0 . From (5.109), the missing direction corresponds therefore to $\text{Span}\{\psi_0\}$. In fact we will show later on that this missing direction corresponds exactly to the condition (5.34) which is the condition of mass conservation in the original system (5.7) (see Remark 5.4.7). This also means that any state that keeps a constant mass is reachable, more precisely

Theorem 5.4.2. *If $T \geq 2L$, then system (5.31) is $D(\mathcal{H}_{(0)}^s)$ controllable with $D(\mathcal{H}_{(0)}^{s-1})$ controls, where*

$$D(\mathcal{H}_{(0)}^s) = \left\{ f := \sum_{n \in \mathbb{Z}^*} f_n \psi_n \in H^s \mid \sum_{n \in \mathbb{Z}^*} (1 + n^{2s}) f_n^2 < +\infty \right\}. \quad (5.113)$$

Remark 5.4.1. *We see again, with another argument than [52], that the system is not controllable when $\gamma = 0$, since $a_n = b_n = 0$ when n is even.*

Remark 5.4.2. *This control time is not sharp as it depends on the localization of the eigenvalues, which we do not know for perturbed operators.*

5.4.1.4 Asymptotic calculation: holomorphic extension

We will now use Kato's method [102] of asymptotic calculation with the help of complex analysis to obtain an explicit formulation, and remainder estimates, for the eigenvalues, and an insight on the eigenfunctions' asymptotic behavior.

Let us consider eigenfunctions on the space $C([0, 1])$. From now on, $\|\cdot\|$ denotes the L^∞ norm. In the preceding formulas, γ was defined for sufficiently small real numbers. Now we extend those formulas to $\gamma \in \mathbb{C}$ with $|\gamma|$ small: at least formally this extension is true. In fact, this complex extension enables us to use holomorphic techniques concerning asymptotic calculation. Once we will get estimates that we require, we will use them with a real γ , as several properties are better in this case. Moreover, the operators $\{T_\gamma\}$ are of *type (A)* (see Kato Chapter 7 Section 2), hence the extended formulas are holomorphic for $|\gamma|$ small.

Remark 5.4.3. *We choose the L^∞ norm for asymptotic information on boundary points, because it will be useful in the following. Getting an estimation on the L^2 norm, though, would be much simpler.*

Let us define,

$$A(\gamma) := T_\gamma - T = \delta(x)J_0, \quad (5.114)$$

$$A^{(1)}(\gamma) = T_\gamma - T - \gamma T^{(1)}, \quad T^{(1)} = -\frac{3}{4}J_0, \quad (5.115)$$

where T_γ is still given by (5.99). Direct calculation yields the existence of $d \in (0, 1/(8L))$ such that, if $|\gamma| < d$, then

$$|\delta(x)| < |\gamma|, \quad |\delta(x) + \frac{3}{4}\gamma| < L|\gamma|^2, \quad (5.116)$$

thus

$$\|A(\gamma)\| < 2|\gamma|, \quad \|A^{(1)}(\gamma)\| < 2L|\gamma|^2. \quad (5.117)$$

Theorem 5.4.3. *The resolvent $R(\xi) := (T - \xi Id)^{-1}$ is defined on $D_0 := \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} \mu_n^{(0)}$. Besides, T has compact resolvent, i.e. $R(\xi)$ is compact for all $\xi \in D_0$.*

The proof is straightforward: let $\xi \in D_0$, $v = R(\xi)u$, if $u = \sum_{n \in \mathbb{Z}} a_n \psi_n^{(0)}$ and $v = \sum_{n \in \mathbb{Z}} b_n \psi_n^{(0)}$, then

$$b_n = \frac{a_n}{\mu_n^{(0)} - \xi}. \quad (5.118)$$

By slightly changing the notations and basically following the same calculations as in Kato [102], we get the following result.

Theorem 5.4.4. *For any $n \in \mathbb{Z}$ there exists $d_n > 0$ (convergence radii) such that*

- (i) (Kato, [102, page 377 Theorem 2.4]) T_γ has compact resolvent;
- (ii) (Kato, [102, page 382, Example 2.14]) $\mu_n(\gamma)$ and $\psi_n(\gamma)$ are holomorphic in B_{d_n} ;

In fact $d_n \geq d$: let us define

$$\mathbb{D} := \bigcup_n B(i\pi n/L; 1/L), \quad \text{with } B(a; r) := \{x \in \mathbb{C}; |x - a| < r\}, \quad (5.119)$$

$$\Gamma_n := \{x \in \mathbb{C}; |x - i\pi n/L| = 1/L\}, \quad (5.120)$$

and the perturbation of the resolvent

$$R(\xi, \gamma) := (T_\gamma - \xi Id)^{-1}. \quad (5.121)$$

Simple symbol calculation leads to the second Neumann series of $R(\xi, \gamma)$:

$$R(\xi, \gamma) = R(\xi)(1 + A(\gamma)R(\xi))^{-1}. \quad (5.122)$$

It suffices to let $\|A(\gamma)R(\xi)\| < 1$.

For $\xi \in \mathbb{C} \setminus \mathbb{D}$, we estimate the norm of $R(\xi)$: if $u = \sum_{n \in \mathbb{Z}} a_n \psi_n^{(0)} \in (\mathcal{L}_{(0)})^2$, then

$$\|R(\xi)u\|_{L^\infty} \leq \sum_{n \in \mathbb{Z}} \left| \frac{a_n}{\mu_n^{(0)} - \xi} \right| \leq \left(\sum_{n \in \mathbb{Z}} a_n^2 \right)^{\frac{1}{2}} \left(\sum_{n \in \mathbb{Z}} \frac{1}{|\mu_n^{(0)} - \xi|^2} \right)^{\frac{1}{2}} < 2L\|u\|_{L^2} \leq 2L\|u\|_{L^\infty}, \quad (5.123)$$

which gives $\|R(\xi)\| < 2L$.

On the other hand, for $|\gamma| < d$, we have

$$\|A(\gamma)\| < 2|\gamma| < 1/(4L). \quad (5.124)$$

Therefore

$$\|A(\gamma)R(\xi)\| < 1 \text{ for } \xi \in \mathbb{C} \setminus \mathbb{D} \text{ and } |\gamma| < d. \quad (5.125)$$

Hence $R(\xi, \gamma)$ is holomorphic. From now on, we will always let ξ and γ be in this domain to guarantee convergences of calculations. In the following subsection our aim will be to derive asymptotic estimates on $\mu_n(\gamma)$ and $\psi_n(\gamma)$, first by direct estimation, then using majoring series.

5.4.1.5 Asymptotic calculation: Direct estimation

In order to estimate $\mu_n(\gamma)$ and $\psi_n(\gamma)$, we need to decompose $R(\xi)$. More precisely, for each $\mu_n^{(0)}$, we have the following Laurent series:

$$R(\xi) = -(\xi - \mu_n^{(0)})^{-1}P_n + \sum_{m=0}^{+\infty} (\xi - \mu_n^{(0)})^m S_n^{m+1}, \quad (5.126)$$

with

$$P_n := -\frac{1}{2\pi i} \int_{\Gamma_n} R(\xi) d\xi, \quad (5.127)$$

$$S_n := \frac{1}{2\pi i} \int_{\Gamma_n} (\xi - \mu_n^{(0)})^{-1} R(\xi) d\xi. \quad (5.128)$$

Moreover, thanks to the explicit formula (5.118), we get

$$P_n u = \langle u, \psi_n^{(0)} \rangle \psi_n^{(0)}, \quad (5.129)$$

$$S_n u = \sum_{k \neq n} \frac{\langle u, \psi_k^{(0)} \rangle}{\mu_k^{(0)} - \mu_n^{(0)}} \psi_k^{(0)}. \quad (5.130)$$

The following lemma gives the expansion of eigenvalues:

Lemma 5.4.2 ([102], Chapter 2 Section 3.1). *One has*

$$|\mu_n(\gamma) - \mu_n^{(0)} - \sum_{p=1}^m \gamma^p \mu_n^{(p)}| \leq \frac{|\gamma|^{m+1}}{d^m(d - |\gamma|)}, \quad \forall |\gamma| < d. \quad (5.131)$$

Therefore,

$$|\mu_n(\gamma) - \mu_n^{(0)}| \leq \frac{3|\gamma|}{2d} \leq \frac{1}{2L}, \quad \forall |\gamma| < \min\left\{\frac{d}{3L}, \frac{d}{3}\right\}. \quad (5.132)$$

As for eigenfunctions, the explicit asymptotic formulation of $\psi_n(\gamma)$ reads

$$\psi_n(\gamma) = \psi_n^{(0)} - \gamma S_n T^{(1)} \psi_n^{(0)} + \dots = \psi_n^{(0)} + \gamma \psi_n^{(1)} + \gamma^2 \psi_n^{(2)} \dots \quad (5.133)$$

Thanks to (5.130), $\psi_n^{(k)}$ can be calculated, *e.g.* the first order term is given by

$$\psi_n^{(1)} = -S_n T^{(1)} \psi_n^{(0)} = \frac{3}{4} S_n J_0 \psi_n^{(0)} = \frac{3L}{4} \sum_{k \neq n} \frac{\langle J_0 \psi_n^{(0)}, \psi_k^{(0)} \rangle}{i\pi(k-n)} \psi_k^{(0)}. \quad (5.134)$$

The remainder of the zeroth order reads as

$$\psi_n(\gamma) - \psi_n^{(0)} = -S_n \left(1 + (A(\gamma) - \mu_n(\gamma) + \mu_n^{(0)}) S_n \right)^{-1} A(\gamma) \psi_n^{(0)}. \quad (5.135)$$

Let us define $s_n := \|S_n\|$. Suppose that $u = \sum_k a_k \psi_k^{(0)} \in L^\infty$, then

$$\|S_n u\|_{L^\infty} \leq \sum_{k \neq n} \left| \frac{a_k}{\mu_k^{(0)} - \mu_n^{(0)}} \right| \leq \frac{L}{\pi} \left(\sum_k |a_k|^2 \right)^{1/2} \left(\sum_{k \neq n} \frac{1}{(n-k)^2} \right)^{1/2} \leq L \|u\|_{L^2} \leq L \|u\|_{L^\infty}. \quad (5.136)$$

On the other hand, we choose u as $\psi_{n+1}^{(0)}$, which gives

$$S_n \psi_{n+1}^{(0)} = \frac{L \psi_{n+1}^{(0)}}{i\pi}. \quad (5.137)$$

Therefore

$$L/\pi \leq s_n \leq L, \quad \forall n \in \mathbb{Z}. \quad (5.138)$$

Combining (5.135), (5.138), (5.124) and (5.132), we get

$$\|\psi_n(\gamma) - \psi_n^{(0)}\|_{L^\infty} \leq 4L|\gamma|, \quad \forall |\gamma| < d^{(1)} := \min\left\{\frac{d}{3L}, \frac{d}{3}\right\}. \quad (5.139)$$

Let us continue to estimate the remainder of the first order.

$$\begin{aligned} & \psi_n(\gamma) - \psi_n^{(0)} - \gamma \psi_n^{(1)} \\ &= -S_n \left(1 + (A(\gamma) - \mu_n(\gamma) + \mu_n^{(0)}) S_n \right)^{-1} A(\gamma) \psi_n^{(0)} - \gamma \frac{3}{4} S_n J_0 \psi_n^{(0)} \\ &= \gamma \frac{3}{4} \left(S_n \left(1 + (A(\gamma) - \mu_n(\gamma) + \mu_n^{(0)}) S_n \right)^{-1} - S_n \right) J_0 \psi_n^{(0)} \\ & \quad - S_n \left(1 + (A(\gamma) - \mu_n(\gamma) + \mu_n^{(0)}) S_n \right)^{-1} A^{(1)}(\gamma) \psi_n^{(0)} \\ &= -\gamma \frac{3}{4} S_n (A(\gamma) - \mu_n(\gamma) + \mu_n^{(0)}) S_n \left(1 + (A(\gamma) - \mu_n(\gamma) + \mu_n^{(0)}) S_n \right)^{-1} J_0 \psi_n^{(0)} \\ & \quad - S_n \left(1 + (A(\gamma) - \mu_n(\gamma) + \mu_n^{(0)}) S_n \right)^{-1} A^{(1)}(\gamma) \psi_n^{(0)}. \end{aligned} \quad (5.140)$$

Combing (5.135), (5.138), (5.124) and (5.132), we get the existence of $d^{(2)} > 0$ and $C^{(2)}$ which are independent of $n \in \mathbb{Z}$ such that

$$\|\psi_n(\gamma) - \psi_n^{(0)} - \gamma \psi_n^{(1)}\|_{L^\infty} \leq C^{(2)} |\gamma|^2, \quad \forall |\gamma| < d^{(2)}. \quad (5.141)$$

Remark 5.4.4. We observe that, by using this direct method, the best estimates that we can get are Lemma 5.4.2 and

$$\|\psi_n(\gamma) - \psi_n^{(0)} - \gamma \psi_n^{(1)} - \gamma^2 \psi_n^{(2)} - \dots - \gamma^k \psi_n^{(k)}\| \leq C^{(k)} |\gamma|^{k+1}, \quad \forall |\gamma| < d^{(k)}. \quad (5.142)$$

Clearly there is no asymptotic behavior with respect to n .

5.4.1.6 Asymptotic calculation: majorizing series for better estimation

The so called majorizing series provides a more explicit and more systematic way of estimating remainder terms. This method is heavily used for high order remainder terms estimates, since it is rather difficult to perform calculation as (5.140) then. One can see [102][Chapter 2 Section 3.2, page 382 Example 2.14] for a perturbation of the Laplace operator. To give a comprehensive view of the method, we start by considering a reduced case: the eigenvalues and eigenfunctions of $T + \gamma T^{(1)}$. Let us define

$$p_n := \|\frac{3}{4}J_0P_n\|, q_n := \|\frac{3}{4}J_0S_n\|, r_n := [(p_n s_n)^{1/2} + q_n^{1/2}]^{-2}. \quad (5.143)$$

Lemma 5.4.3. *There exists c and C independent of n such that*

$$c < p_n, q_n, r_n, \|P_n\| < C, \forall n \in \mathbb{Z}. \quad (5.144)$$

Proof. Suppose that $u = \sum_k a_k \psi_k^{(0)} \in L^\infty$, then

$$\|P_n u\|_{L^\infty} = |a_n| \|\psi_n^{(0)}\|_{L^\infty} \leq \|u\|_{L^2} \leq \|u\|_{L^\infty}, \quad (5.145)$$

which completes the right hand side inequality of (5.144).

On the other hand, for the left hand side of (5.144), let us choose u as $\psi_n^{(0)}$ or $\psi_{n+1}^{(0)}$, which gives

$$\begin{aligned} S_n \psi_{n+1}^{(0)} &= \frac{L \psi_{n+1}^{(0)}}{i\pi}, \quad J_0 S_n \psi_{n+1}^{(0)} = J_0 \frac{L \psi_{n+1}^{(0)}}{i\pi}, \\ P_n \psi_n^{(0)} &= \psi_n^{(0)}, \quad J_0 P_n \psi_n^{(0)} = J_0 \psi_n^{(0)}. \end{aligned} \quad (5.146)$$

□

We obtain the asymptotic expression of $\psi_n(\gamma)$ similarly as in [102, page 382 Example 2.14] (which is based on majorizing series and [102, Chapter 2 Section 3 Examples 3.2-3.7]):

Lemma 5.4.4. *For any $|\gamma| < \min\{r_n\}$ we have*

$$\|\psi_n(\gamma) - \psi_n^{(0)} - \gamma \psi_n^{(1)}\| \leq |\gamma|^2 \frac{s_n}{(p_n q_n s_n)^{1/2}} ((p_n s_n)^{1/2} + q_n^{1/2})^2 (p_n s_n + q_n + (p_n q_n s_n)^{1/2}). \quad (5.147)$$

By inserting (5.144) into (5.147), we get

$$\|\psi_n(\gamma) - \psi_n^{(0)} - \gamma \psi_n^{(1)}\| \leq C |\gamma|^2, \forall |\gamma| < \min\{r_n\}, \forall n \in \mathbb{Z}. \quad (5.148)$$

However, as we have seen that p_n, q_n, s_n are bounded by constant instead of c/n , when considering not only $T + \gamma T^{(1)}$ but the whole T_γ , the best asymptotic estimate that we can get from majorizing series is (still)

$$\|\psi_n(\gamma) - \psi_n^{(0)} - \gamma \psi_n^{(1)} - \gamma^2 \psi_n^{(2)} - \dots - \gamma^k \psi_n^{(k)}\| \leq C^{(k)} \gamma^{k+1}, \quad (5.149)$$

with $C^{(k)}$ independent of n .

Remark 5.4.5 (Comparison with Laplacian). *The asymptotic calculation for the Laplacian operator contains some factor $1/n$ which comes from (5.130). More precisely, it comes from the localization of the eigenvalues and the fact that for the Laplacian operator case $\mu_n \sim n^2$. Indeed, for the same reason, we can expect some asymptotic behavior of eigenvalues' gap for fractional Laplacian Δ^s when $s > 1$.*

5.4.1.7 The controllability thanks to boundary conditions

From now on we will assume that $\gamma \in \mathbb{R}$ to simplify the calculation. With the two last subsections, we have an estimate on $\|\psi_n(\gamma) - \psi_n^{(0)} - \gamma\psi_n^{(1)} - \gamma^2\psi_n^{(2)} - \dots - \gamma^k\psi_n^{(k)}\|$ that it not totally satisfactory and that does not depend on n . However, notice that, from (5.110), what we need to conclude the study of the controllability is the value of

$$(\psi_n(\gamma) - \psi_n^{(0)} - \gamma\psi_n^{(1)} - \gamma^2\psi_n^{(2)} - \dots - \gamma^k\psi_n^{(k)}, (1, 1)). \quad (5.150)$$

It sounds thus natural to investigate (5.150) directly.

Let us calculate the boundary value of $\psi_n^{(1)}$, more precisely, a combination of boundary values:

$$l_n^{(1)} := \psi_{n,1}^{(1)}(1) - \psi_{n,1}^{(1)}(0) - \psi_{n,2}^{(1)}(1) + \psi_{n,2}^{(1)}(0). \quad (5.151)$$

We observe that $l_n^{(0)} := \psi_{n,1}^{(0)}(1) - \psi_{n,1}^{(0)}(0) - \psi_{n,2}^{(0)}(1) + \psi_{n,2}^{(0)}(0) = 2((-1)^n - 1)$. Direct calculation shows that

$$l_n^{(1)} = \frac{3L}{2i\pi} \sum_{k \neq n} \frac{\langle J_0 \psi_n^{(0)}, \psi_k^{(0)} \rangle}{(k-n)} ((-1)^k - 1). \quad (5.152)$$

On the other hand we know that

$$\langle J_0 \psi_n^{(0)}, \psi_k^{(0)} \rangle = \frac{(-1)^{n+k} - 1}{i\pi} \left(\frac{1}{n-k} + \frac{1}{3} \frac{1}{n+k} \right). \quad (5.153)$$

Notice that if n is odd we will have that $((-1)^k - 1)((-1)^{n+k} - 1) = 0$, therefore $l_n^{(1)} = 0$. We thus only need to consider the case when $n = 2m$ is even. Then

$$l_n^{(1)} = -\frac{3L}{2\pi^2} \sum_{k \neq 2m} ((-1)^k - 1)^2 \frac{1}{k-2m} \left(\frac{1}{2m-k} + \frac{1}{3} \frac{1}{2m+k} \right) \quad (5.154)$$

$$= -\frac{6L}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{2k+1-2m} \left(\frac{1}{2m-2k-1} + \frac{1}{3} \frac{1}{2m+2k+1} \right) \quad (5.155)$$

$$= \frac{6L}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{2k+1-2m} \left(\frac{1}{2k+1-2m} - \frac{1}{3} \frac{1}{2m+2k+1} \right). \quad (5.156)$$

We know that

$$\sum_{k \in \mathbb{Z}} \left(\frac{1}{2k+1-2m} \right)^2 = \sum_{k \in \mathbb{Z}} \left(\frac{1}{2k+1} \right)^2 = \frac{\pi^2}{4} = :C_0 > 1. \quad (5.157)$$

If $m = 0$, then $l_n^{(1)} > 0$. If $m \neq 0$, without loss of generality let us suppose that $m > 0$, then

$$\left| \frac{1}{3} \sum_k \frac{1}{2k+1-2m} \frac{1}{2m+2k+1} \right| \quad (5.158)$$

$$= \left| \frac{1}{3} \left(\sum_{k \geq m} + \sum_{k \leq -m-1} + \sum_{k=-m}^{m-1} \right) \frac{1}{2k+1-2m} \frac{1}{2m+2k+1} \right| \quad (5.159)$$

$$\leq \frac{2}{3} C_0 + \frac{2m}{4m-1} \quad (5.160)$$

$$< C_0 - \frac{1}{3}. \quad (5.161)$$

Therefore,

$$|l_n^{(1)}| > \frac{2L}{\pi^2} \text{ if } n \text{ is even, } |l_n^{(1)}| = 0 \text{ if } n \text{ is odd.} \quad (5.162)$$

Remark 5.4.6. *The way that we get this uniform bound relies on the the matrix J_0 , more precisely, the diagonal matrix Λ . Indeed, this is the key point to get the controllability: in (5.99) if we replace $\delta(x)J_0$ by $\delta(x)\Lambda$, then the operator becomes easier as the coupled term disappear; thanks to the expansion of $\delta(x)$, (5.33), if we further replace $\delta(x)\Lambda$ by $-\frac{3}{4}\gamma\Lambda$, then the operator becomes*

$$T_{\gamma,s} := \Lambda \left(\partial_x - \frac{3}{4}\gamma \right), \quad (5.163)$$

$$D(T_{\gamma,s}) := \{(w_1, w_2) \in (H^1)^2 : w_1(0) = -w_2(0), w_1(L) = -w_2(L)\}, \quad (5.164)$$

for which the controlability is rather easy to obtain. Maybe with the help of some perturbation arguments we can prove the controllability of the operator $\Lambda\partial_x + \Lambda\delta(x)$, of course we need to deal with some loss of derivative issues as the normal fixed point argument can not be applied here. However, in our case we need to consider $J_0 = \Lambda + J$, where J is of the same order as Λ , hence cannot be ignored.

We observe from (5.144) and (5.134) that $\|\psi_n^{(1)}\|_{L^\infty}$ are $O(1)$ rather than $O((1/n)^\alpha)$. However, in (5.150), we still have a γ^k in front of $\|\psi_n^{(k)}\|_{L^\infty}$, which, when γ is small, gives good estimate.

In fact, thanks to the above calculation, (5.99), (5.148), we get the following:

$$\delta(x) = -\frac{3}{4}\gamma + R_0(\gamma), \quad (5.165)$$

$$\psi_{n,1}(\gamma) = e^{i\pi nx/L} + \gamma\psi_{n,1}^{(1)} + R_{n,1}(\gamma), \quad (5.166)$$

$$\psi_{n,2}(\gamma) = -e^{-i\pi nx/L} + \gamma\psi_{n,2}^{(1)} + R_{n,2}(\gamma). \quad (5.167)$$

Therefore, the existence of $r^{(2)} > 0$ and $C_2 > 0$ such that

$$\|\psi_n^{(1)}\|_{L^\infty} \leq L, \quad (5.168)$$

$$\|R_0(\gamma)\|_{L^\infty}, \|R_{n,1}(\gamma)\|_{L^\infty}, \|R_{n,2}(\gamma)\|_{L^\infty} \leq C_2|\gamma|^2. \quad (5.169)$$

Moreover, there exists $r^{(3)} \in (0, r^{(2)})$ such that, if $|\gamma| < r^{(3)}$, then we have

$$4/5 < \|\psi_{n,1}(\gamma)\|_{L^\infty}, \|\psi_{n,2}(\gamma)\|_{L^\infty} < 6/5, \quad (5.170)$$

$$4/5 < \|\psi_{n,1}(\gamma)\|_{L^2}, \|\psi_{n,2}(\gamma)\|_{L^2} < 6/5. \quad (5.171)$$

The same estimates hold for $\chi_n(\gamma)$, hence

$$\langle \psi_n(\gamma), \chi_n(\gamma) \rangle = 1 + \langle \psi_n(\gamma), \chi_n(\gamma) - \chi_n^{(0)} \rangle \in (3/4, 5/4). \quad (5.172)$$

Furthermore, for $n \in \mathbb{Z}^*$ we observe from the definition of $\psi_n(\gamma)$ and (5.165)–(5.169) that

$$\mu_n(\gamma) \int_0^L \psi_{n,1}(\gamma) + \psi_{n,2}(\gamma) \quad (5.173)$$

$$= 2\langle \mu_n(\gamma)\psi_n(\gamma), (1, 1) \rangle \quad (5.174)$$

$$= 2\langle \Lambda\partial_x\psi_n(\gamma) + \delta(x)J_0\psi_n(\gamma), (1, 1) \rangle \quad (5.175)$$

$$= (\psi_{n,1}(\gamma)(1) - \psi_{n,1}(\gamma)(0) - \psi_{n,2}(\gamma)(1) + \psi_{n,2}(\gamma)(0)) + \frac{2}{3} \int_0^L \delta(x)(\psi_{n,1} - \psi_{n,2})$$

$$= \left(2((-1)^n - 1) + \gamma l_n^{(1)} + O(\gamma^2) \right) - \frac{1}{2}\gamma \left(e^{i\pi nx/L} + e^{-i\pi nx/L} \right) + O(\gamma^2)$$

$$= 2((-1)^n - 1) + \gamma l_n^{(1)} + O(\gamma^2). \quad (5.176)$$

We remark here that $O(\gamma^2)$ means uniformly bounded by $C\gamma^2$ with some $C > 0$ independent of $n \in \mathbb{Z}^*$ and γ small. The same for other similar notations, as $O(\gamma)$. By inserting (5.162) into (5.176), we get

$$\mu_n(\gamma) \int_0^1 \psi_{n,1}(\gamma) + \psi_{n,2}(\gamma) = \begin{cases} 0 + \gamma l_n^{(1)} + R_n(\gamma), & \text{when } n \text{ is even,} \\ -4 + 0 + R_n(\gamma), & \text{when } n \text{ is odd,} \end{cases} \quad (5.177)$$

with $|R_n(\gamma)| \leq C_3|\gamma|^2$. Therefore, there exists $r^{(4)} \in (0, r^{(3)})$ such that, if $|\gamma| < r^{(4)}$, then we have

$$|\gamma| \frac{L}{\pi^2} < |\mu_n(\gamma) \langle \psi_n(\gamma), (1, 1) \rangle| < 5, \quad \forall n \in \mathbb{Z}^*. \quad (5.178)$$

We are able to perform the same calculation for $\chi_n(\gamma)$ which are eigenfunctions of \mathcal{S}_0^* , thus

$$|\gamma| \frac{L}{\pi^2} < |\bar{\mu}_n(\gamma) \langle \chi_n(\gamma), (1, 1) \rangle| < 5, \quad \forall n \in \mathbb{Z}^*. \quad (5.179)$$

Remark 5.4.7 (χ_0 and the “missing direction”). *As for the case when $n = 0$, thanks to the diffeomorphism (5.35) and the equation (5.53), we know that*

$$\psi_{0,1}(x) + \psi_{0,2}(x) = 0, \quad \forall x \in [0, L], \quad (5.180)$$

$$\partial_x \psi_{0,1} + \frac{2}{3} \delta \psi_{0,1} = 0, \quad (5.181)$$

thus ψ_0 can be calculated explicitly,

$$\psi_{0,1} = -\psi_{0,2} = \left(1 - \frac{\gamma L \gamma}{2L} x\right)^{-1} = \left(1 - \frac{1 - \sqrt{1 - \gamma L}}{L} x\right)^{-1}, \quad (5.182)$$

which of course satisfies

$$\langle \psi_0(\gamma), (1, 1) \rangle = 0. \quad (5.183)$$

The same reason and similar calculations for χ_0 lead to

$$\chi_{0,1}(x) + \chi_{0,2}(x) = 0, \quad (5.184)$$

$$\partial_x \chi_{0,1}(x) - \frac{4}{3} \delta(x) \chi_{0,1}(x) = 0, \quad (5.185)$$

thus

$$\chi_{0,1}(x) = \left(1 - \frac{1 - \sqrt{1 - \gamma L}}{L} x\right)^2, \quad (5.186)$$

$$\langle \chi_0(\gamma), (1, 1) \rangle = 0. \quad (5.187)$$

We observe from (5.187) that we can not cover direction ψ_0 with our control, due to the moment theory. More precisely, the quantity

$$\langle w, \chi_0 \rangle = \int_0^L \left(1 - \frac{1 - \sqrt{1 - \gamma L}}{L} x\right)^2 (w_1(x) - w_2(x)) dx, \quad (5.188)$$

which is precisely the term appearing in (5.34), remains constant in time for any control $u(t)$. Hence the controller $(1, 1)^T$ is compatible with the conservation of mass (5.34). The equivalence between (5.34) and (5.188) is not a coincidence: from (5.34) we know that there is at least one direction for which we can not control; on the other hand, (5.187) gives us the only uncontrollable direction of the system. Algebraically, these two directions have to be the same.

Thanks to Theorem 5.4.1 and (5.171), for $|\gamma| < r_R$, we know that

$$(\psi_n(\gamma))_{n \in \mathbb{Z}^*} \text{ form a Riesz basis of } \mathcal{L}_{(0)}^2, \quad (5.189)$$

where

$$\mathcal{L}_{(0)}^2 := \{f \in L^2 \mid \langle f, \chi_0 \rangle = 0\}. \quad (5.190)$$

Combining (5.108), (5.110), (5.50), (5.172), (5.179), and (5.189), this complete the proof of Lemma 5.4.1.

5.4.1.8 The controllability of the transformed system (5.37)–(5.38)

Let us look now at the operator \mathcal{A} given by (5.46)–(5.47) and associated to system (5.37)–(5.38). As stated previously, the eigenvalues of the operators (5.99) and of (5.46) are the same; and the eigenfunctions of (5.46) can be generated by applying the diffeomorphism (5.35) on the eigenfunctions of (5.99). Indeed, with $(f_n)_{n \in \mathbb{Z}^*}$ the family of eigenfunctions of \mathcal{A} that form a Riesz basis of $(L_{(0)}^2)^2$, and depends of course of γ , and from (ii) in Lemma 5.4.1, there exists uniformly bounded positive constants $(a_n)_{n \in \mathbb{Z}^*}$ such that

$$f_n(\gamma) := a_n \exp\left(\int_0^x \delta(s) ds\right) \psi_n(\gamma). \quad (5.191)$$

Remark 5.4.8. *The generated eigenfunctions $f_n(\gamma)$ no longer satisfy the normalized condition: $\langle f_n(\gamma), \phi_n^{(0)} \rangle = 1$.*

As the change of coordinates (5.35) used to remove the diagonal coefficients of the source term is an isomorphism in H^s , Theorem 5.4.2 directly leads to the controllability of this system and thus of (5.37)–(5.38):

Corollary 5.4.1. *If $T \geq 2L$, then system (5.37)–(5.38) is $D(\mathcal{A}_{(0)}^s)$ controllable with $D(\mathcal{A}_{(0)}^{s-1})$ controls, where*

$$D(\mathcal{A}_{(0)}^s) = \{f \in D(\mathcal{A}^s) \mid \langle f, f_0 \rangle = 0\}. \quad (5.192)$$

In particular, by using the same technique we are able to prove the following estimate (compare to (5.179))

$$m_0 |\mu_n^{-1}| \leq |\langle \mathcal{I}, a_n^{-1} f_n \rangle| \leq M_0 |\mu_n^{-1}|, \quad \forall n \in \mathbb{Z}^*, \quad (5.193)$$

for some constants $m_0, M_0 > 0$, which implies, from the uniform boundedness of (a_n) ,

$$m |\mu_n^{-1}| \leq |\langle \mathcal{I}, f_n \rangle| \leq M |\mu_n^{-1}|, \quad \forall n \in \mathbb{Z}^*, \quad (5.194)$$

for some constants $m, M > 0$. We can be even more precise, using the fact that $\mathcal{I} \in (H^1)^2$: let us write, for $n \in \mathbb{Z}^*$,

$$\begin{aligned} -\mu_n \langle \mathcal{I}, f_n \rangle &= \langle \mathcal{I}, \mathcal{A} f_n \rangle \\ &= \int_0^L \mathcal{I}_1 \partial_x f_{n,1} - \mathcal{I}_2 \partial_x f_{n,2} - \langle \delta(x) J \mathcal{I}, f_n \rangle \\ &= [\mathcal{I}_1 f_{n,1}]_0^L - [\mathcal{I}_2 f_{n,2}]_0^L - \int_0^L \partial_x \mathcal{I}_1 f_{n,1} - \partial_x \mathcal{I}_2 f_{n,2} - \langle \delta(x) J \mathcal{I}, f_n \rangle \\ &= 2 \left(e^{\int_0^L \delta} f_{n,1}(L) - f_{n,1}(0) \right) - \langle \Lambda \partial_x \mathcal{I} + \delta(x) J \mathcal{I}, f_n \rangle \\ &= 2 \left(e^{\int_0^L \delta} f_{n,1}(L) - f_{n,1}(0) \right) - \langle \delta(x) J_0 \mathcal{I}, f_n \rangle. \end{aligned} \quad (5.195)$$

Now, as $\Lambda \partial_x \mathcal{I} + \delta(x) J \mathcal{I} \in L^2$, this gives us an asymptotic expansion for the coefficients of \mathcal{I} :

$$\langle \mathcal{I}, f_n \rangle = \frac{2 \left(f_{n,1}(0) - e^{\int_0^L \delta} f_{n,1}(L) \right)}{\mu_n} + \frac{1}{\mu_n} \langle \delta(x) J_0 \mathcal{I}, f_n \rangle, \quad \forall n \in \mathbb{Z}^*. \quad (5.196)$$

Thanks to the expansion of f_n and $\delta(x)$, we know that, for any $n \in \mathbb{Z}^*$

$$f_{n,1}(0) - e^{\int_0^L \delta} f_{n,1}(L) = (1 - (-1)^n) + \gamma \left(\frac{3L}{4} (-1)^n + f_{n,1}^{(1)}(0) - f_{n,1}^{(1)}(L) \right) + O(\gamma^2), \quad (5.197)$$

which, combined with the fact that

$$\langle \delta(x) J_0 \mathcal{I}, f_n \rangle = O(\gamma^2), \quad (5.198)$$

leads to

$$-\mu_n \langle \mathcal{I}, f_n \rangle = (1 - (-1)^n) + \gamma \left(\frac{3L}{4} (-1)^n + f_{n,1}^{(1)}(0) - f_{n,1}^{(1)}(L) \right) + O(\gamma^2). \quad (5.199)$$

Similar calculations as in Subsection 5.4.1.7, lead to the existence of $m, M > 0$ such that

$$m \leq |\mu_n \langle \mathcal{I}, f_n \rangle| \leq M, \quad \forall n \in \mathbb{Z}^*. \quad (5.200)$$

We now study the system with our virtual control \mathcal{I}_ν : thanks to the fact that

$$\langle f_0, f_n \rangle = \delta_{0,n}, \quad (5.201)$$

we get the following lemma which leads to the controllability of the system with control \mathcal{I}_ν .

Lemma 5.4.5. *Let $0 < |\gamma| < \gamma_0$. Let $\nu \neq 0$. There exist $m, M > 0$ such that*

$$m \leq |\langle \mathcal{I}_\nu, f_0 \rangle| \leq M, \quad (5.202)$$

$$m \leq |\mu_n \langle \mathcal{I}_\nu, f_n \rangle| \leq M, \quad \forall n \in \mathbb{Z}^*. \quad (5.203)$$

And thus, as all the moment are bounded away from 0, we recover the controllability of the system. This explains the definition of \mathcal{I}_ν given by (5.58).

5.4.1.9 Asymptotic calculation: is $O(1)$ sharp for the transformed operator \mathcal{A} ?

We have seen from the above calculation that $\gamma \psi_n^{(1)}$ and $R_{1,n}(\gamma)$ are $O(\gamma)$ and $O(\gamma^2)$ respectively. Thanks to (5.162), at least for L^∞ norm the $O(1)$ type estimates are sharp: there is no decay with respect to n . As we have indicated in Remark 5.4.6, the diagonal matrix plays an important role in the estimation. Hence it is natural to ask whether the normalized eigenfunctions of \mathcal{A} are better than those of \mathcal{S}_0 , as the diagonal coefficients of J are 0. The normalized eigenfunctions are given by

$$\mathcal{A} \bar{\psi}_n(\gamma) = \mu_n(\gamma) \bar{\psi}_n(\gamma), \quad \langle \bar{\psi}_n(\gamma), \bar{\chi}_n^{(0)} \rangle = 1. \quad (5.204)$$

In this situation, everything we have defined for the calculation of eigenfunctions from (5.99) to (5.150) remain the same if we replace J_0 by J , except for the part of controllability: as $(1, 1)$ is replaced by $\exp\left(\int_0^x \delta(s) ds\right)(1, 1)$.

Therefore, from (5.134) and using (5.29) we know that

$$\bar{\psi}_n^{(1)} = \frac{3L}{4} \sum_{k \neq n} \frac{\langle J \psi_n^{(0)}, \psi_k^{(0)} \rangle}{i\pi(k-n)} \psi_k^{(0)} = -\frac{L}{2\pi^2} \sum_{k \neq n} \frac{((-1)^{k-n} - 1)}{n^2 - k^2} \psi_k^{(0)}. \quad (5.205)$$

It is easy to find that

$$\|\bar{\psi}_n^{(1)}\|_{L^2} \longrightarrow_{|n| \rightarrow +\infty} 0. \quad (5.206)$$

Moreover, with the help of more precise estimates we are able to prove that

$$\|\bar{\psi}_n^{(1)}\|_{L^\infty} \longrightarrow_{|n| \rightarrow +\infty} 0. \quad (5.207)$$

The interpolation implies that the same result holds for the L^p norm. Maybe for the transformed operator we are able to conclude some better asymptotic estimates rather than $O(1)$. On the other way around, as we know that the controllability is strongly related to those estimates, see (5.108). The system is controllable only if b_n is away from 0 in some sense. If for the transformed operator $\Lambda \partial_x + \delta(x)J$ we are able to get some better decay estimates, it is rather interesting to investigate its controllability with the control term as $u(t)(1, 1)$, since this case is “critical” in some sense (with respect to Remark 5.4.6; or, at least, more complicated.)

5.4.2 Controllability of the target system

As previously, it suffices to investigate the properties of the operator

$$\tilde{T}_\gamma := \tilde{\mathcal{S}}_0 = \Lambda \partial_x + \delta(x)J, \quad (5.208)$$

$$D(\tilde{\mathcal{S}}_0) := \{(w_1, w_2) \in (H^1)^2 : w_1(0) = -e^{2\mu L}w_2(0), w_1(L) = -w_2(L)\}, \quad (5.209)$$

and the controllability of

$$\begin{aligned} \partial_t w + \Lambda \partial_x w + \delta J w &= u(t)\mathcal{I}_\nu, \\ w_1(t, 0) &= -e^{2\mu L}w_2(t, 0), w_2(t, L) = -w_1(t, L). \end{aligned} \quad (5.210)$$

As in Section 5.4.1 we define adjoint operators $\tilde{\mathcal{S}}_0^*$, eigenvalues $\tilde{\mu}_n$, and normalized eigenfunctions $(\tilde{f}_n, \tilde{\phi}_n)$. As indicated at the beginning of Section 5.4 we are able to perform the same calculation for Operator $\tilde{\mathcal{S}}_0$ as in Section 5.4.1.1–5.4.1.6, we omit the explicit calculation for readers' convenience. More precisely, we have

Lemma 5.4.6. *Let $\mu \neq 0$. There exists $\bar{r}_\mu, c_\mu, C_\mu > 0$ such that for any $\gamma \in B_{\bar{r}_\mu}$ and any $n \in \mathbb{Z}$ we have*

$$|\tilde{\mu}_n - \tilde{\mu}_n^{(0)}| < 1/(4L), \quad (5.211)$$

$$\{\tilde{\phi}_n / \|\tilde{\phi}_n\|_{L^2}\}_n \text{ is a Riesz basis of } L^2, \quad (5.212)$$

$$\{\tilde{f}_n / \|\tilde{f}_n\|_{L^2}\}_n \text{ is a Riesz basis of } L^2, \quad (5.213)$$

$$\|\tilde{\phi}_n - \tilde{\phi}_n^{(0)}\|_{L^\infty}, \|\tilde{f}_n - \tilde{f}_n^{(0)}\|_{L^\infty} \leq C_\mu |\gamma|, \quad (5.214)$$

$$\|\tilde{\phi}_n - \tilde{\phi}_n^{(0)} - \gamma \tilde{\phi}_n^{(1)}\|_{L^\infty} \leq C_\mu |\gamma|^2, \quad (5.215)$$

$$\|\tilde{f}_n - \tilde{f}_n^{(0)} - \gamma \tilde{f}_n^{(1)}\|_{L^\infty} \leq C_\mu |\gamma|^2, \quad (5.216)$$

$$\|\tilde{\phi}_n^{(1)}\|_{L^\infty}, \|\tilde{f}_n^{(1)}\|_{L^\infty} \leq C_\mu, \quad (5.217)$$

$$c_\mu \leq \|\tilde{\phi}_n\|_{L^2}, \|\tilde{f}_n\|_{L^2}, \|\tilde{\phi}_n\|_{L^\infty}, \|\tilde{f}_n\|_{L^\infty} \leq C_\mu. \quad (5.218)$$

Remark 5.4.9. *Due to the fact that $\|\tilde{f}_n^{(0)}\|_{L^\infty}$, (5.69) and the Riesz basis (5.70) strongly depend on μ , the norm of \tilde{S}_n depends on the value of μ (see (5.136) for instance). This is one of the key points which result in the existence of c_μ, C_μ and (especially) \bar{r}_μ .*

Thanks to (5.71) and Lemma 5.4.6, for $\forall n \in \mathbb{Z}$, we have

$$\tilde{\mu}_n \langle \tilde{\phi}_n, (1, 1) \rangle = (\tilde{\phi}_{n,1}(\gamma)(L) - \tilde{\phi}_{n,1}(\gamma)(0) - \tilde{\phi}_{n,2}(\gamma)(L) + \tilde{\psi}_{n,2}(\gamma)(0)) + O(\gamma) \quad (5.219)$$

$$= (\tilde{\phi}_{n,1}(L) - \tilde{\phi}_{n,1}(0) - \tilde{\phi}_{n,2}(L) + \tilde{\phi}_{n,2}(0)) + O(\gamma) \quad (5.220)$$

$$= 2(-1)^n e^{-\mu L} - 1 - e^{-2\mu L} + O(\gamma). \quad (5.221)$$

Moreover, similar calculation as (5.195) leads to

$$\tilde{\mu}_n \langle \tilde{\phi}_n, \mathcal{I} \rangle = [\tilde{\phi}_{n,1} \mathcal{I}_1]_0^L - [\tilde{\phi}_{n,2} \mathcal{I}_2]_0^L + O(\gamma) \quad (5.222)$$

$$= (\tilde{\phi}_{n,1}(L) - \tilde{\phi}_{n,1}(0) - \tilde{\phi}_{n,2}(L) + \tilde{\phi}_{n,2}(0)) + O(\gamma) \quad (5.223)$$

$$= 2(-1)^n e^{-\mu L} - 1 - e^{-2\mu L} + O(\gamma). \quad (5.224)$$

Then, we further get

$$\tilde{\mu}_n \langle \tilde{\phi}_n, f_0 \rangle = \langle \Lambda \partial_x \tilde{\phi}_n + \delta(x)J \tilde{\phi}_n, f_0 \rangle \quad (5.225)$$

$$= \langle \Lambda \partial_x \tilde{\phi}_n, f_0 \rangle + O(\gamma) \quad (5.226)$$

$$= [\tilde{\phi}_{n,1} f_{0,1} - \tilde{\phi}_{n,2} f_{0,2}]_0^L + O(\gamma) \quad (5.227)$$

$$= (e^{-2\mu L} - 1) + O(\gamma). \quad (5.228)$$

Therefore, combining (5.224) and (5.228) we get

$$\tilde{\mu}_n \langle \tilde{\phi}_n, \mathcal{I}_\nu \rangle = \tilde{\mu}_n \langle \tilde{\phi}_n, \mathcal{I} + \nu f_0 \rangle = 2(-1)^n e^{-\mu L} - 1 - e^{-2\mu L} + \nu(e^{-2\mu L} - 1) + O(\gamma). \quad (5.229)$$

Remark 5.4.10. *Similar estimates hold for \tilde{f}_n .*

Remark 5.4.11. *One can compare estimate (5.229) to (5.176) where we calculated until γ^2 order, while in (5.221) only γ^1 order is needed. That is because the zeroth order in (5.176) becomes 0 when n is even, hence we need to use the first order terms as dominate term in this case. However, as we have seen in (5.71) the zeroth order of (5.221) is already away from 0, even for $n = 0$, thus we no longer need to estimate the second order.*

Hence we get the following lemma:

Lemma 5.4.7. *Let $\mu > 3/L$. Let $0 < |\nu| < 1$. There exists $\tilde{\gamma}_\mu$ such that for any $\gamma \in (-\tilde{\gamma}_\mu, \tilde{\gamma}_\mu)$ and for any $n \in \mathbb{Z}$ we have*

- (i) $\{\tilde{\phi}_n\}_n$ (resp. $\{\tilde{f}_n\}_n$) is a Riesz basis of L^2 ;
- (ii) $|\langle \tilde{f}_n, \tilde{\phi}_n \rangle| \in (1/2, 2)$;
- (iii) $|\tilde{\mu}_n - \tilde{\mu}_n^{(0)}| < 1/(4L)$;
- (iv) $0 < 1/2 < |\tilde{\mu}_n \langle \tilde{f}_n, \mathcal{I}_\nu \rangle|, |\tilde{\mu}_n \langle \tilde{\phi}_n, \mathcal{I}_\nu \rangle| < 3/2$.

Since Lemma 5.4.7 and the classical moment theory lead to the controllability of System (5.210), while System (5.60) is obtained from System (5.210) by an isomorphism, System (5.60) is also controllable.

Theorem 5.4.5. *If $T \geq 2L$, then system (5.60) is $D(\tilde{\mathcal{A}}^s)$ controllable with $D(\tilde{\mathcal{A}}^{s-1})$ controls.*

5.5 A heuristic construction

From now on, let $\mu > 0$ be the desired decay rate, and $\gamma, \nu > 0$ such that (5.59) and (5.60) are both controllable.

5.5.1 Kernel equations

Suppose there exists a solution u to (5.59), and a solution z to (5.60). We are looking for an invertible transformation that maps u to z , under the form of a Fredholm transformation:

$$\begin{pmatrix} z_1(t, x) \\ z_2(t, x) \end{pmatrix} = T(u(t, \cdot))(x) := \int_0^L K(x, y) \cdot \begin{pmatrix} u_1(t, y) \\ u_2(t, y) \end{pmatrix} dy \quad (5.230)$$

where $K(x, y)$ is a 2×2 matrix. Using (5.60), (5.59) and (5.38), we would like the kernel to satisfy the following equation:

$$\Lambda \partial_x K + \partial_y K \Lambda + \delta(x) JK - K \delta(y) J = - \exp \left(\int_0^x \delta(s) ds \right) \begin{pmatrix} F_1 & F_2 \\ F_1 & F_2 \end{pmatrix} (y) \quad (5.231)$$

where F_1, F_2 are the coefficients of the feedback, together with the (formal) so called $TB = B$ condition:

$$\int_0^L K \mathcal{I}_\nu dy = \mathcal{I}_\nu, \quad (5.232)$$

and the following boundary conditions:

$$\begin{aligned}
k_{11}(0, y) &= -e^{-2\mu L} k_{21}(0, y), \\
k_{12}(0, y) &= -e^{-2\mu L} k_{22}(0, y), \\
k_{11}(L, y) &= -k_{21}(L, y), \\
k_{12}(L, y) &= -k_{22}(L, y),
\end{aligned} \tag{5.233}$$

and

$$\begin{aligned}
k_{11}(x, L) &= -k_{12}(x, L), \\
k_{21}(x, L) &= -k_{22}(x, L), \\
k_{11}(x, 0) &= -e^{2\mu L} k_{12}(x, 0), \\
k_{21}(x, 0) &= -e^{2\mu L} k_{22}(x, 0).
\end{aligned} \tag{5.234}$$

To study solutions to (5.231), (5.232), (5.233), and (5.234), let us derive equations for the family of functions

$$g_n := (Tf_n)_{n \in \mathbb{Z}^*}.$$

From (5.231), integrating against the f_n and using the boundary conditions (5.233) and (5.234) we get for any $n \in \mathbb{Z}$,

$$\begin{aligned}
-\int_0^L \exp\left(\int_0^x \delta(s) ds\right) \begin{pmatrix} F_1 & F_2 \\ F_1 & F_2 \end{pmatrix} (y) f_n(y) dy &= \int_0^L (\Lambda \partial_x K(x, y) f_n(y) + \partial_y K(x, y) \Lambda f_n(y) \\
&\quad + \delta(x) J K(x, y) f_n(y) - K(x, y) \delta(y) J f_n(y)) dy \\
&= \Lambda \partial_x g_n + \delta(x) J g_n - T \mathcal{A} f_n \\
&= \Lambda \partial_x g_n - \mu_n g_n + \delta(x) J g_n,
\end{aligned}$$

hence the following equation for (g_n) :

$$\begin{cases} \Lambda \partial_x g_n - \mu_n g_n + \delta(x) J g_n = -\langle f_n, F \rangle \mathcal{I}_\nu, \\ g_{n,1}(0) + e^{-2\mu L} g_{n,2}(0) = 0, \quad g_{n,1}(L) + g_{n,2}(L) = 0. \end{cases} \tag{5.235}$$

Now, we study solutions to (5.235) for any given F .

By property of biorthogonal families, we have the following decomposition:

$$g_n = \sum_{p \in \mathbb{Z}} \langle g_n, \tilde{\phi}_p \rangle \tilde{f}_p \text{ in } L^2, \quad \forall n \in \mathbb{Z}. \tag{5.236}$$

Taking the scalar product of equation (5.235) with the $\tilde{\phi}_p$, we get

$$\langle \tilde{\mathcal{A}} g_n, \tilde{\phi}_p \rangle - \mu_n \langle g_n, \tilde{\phi}_p \rangle = -\langle f_n, F \rangle \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle, \quad \forall n \in \mathbb{Z}, \forall p \in \mathbb{Z}. \tag{5.237}$$

As \mathcal{A} is anti-hermitian, (5.237) becomes

$$\langle g_n, \tilde{\mathcal{A}}^* \tilde{\phi}_p \rangle - \mu_n \langle g_n, \tilde{\phi}_p \rangle = -\langle f_n, F \rangle \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle, \quad \forall n \in \mathbb{Z}, \forall p \in \mathbb{Z},$$

hence finally

$$\langle g_n, \tilde{\phi}_p \rangle = -\frac{\langle f_n, F \rangle \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle}{\tilde{\mu}_p - \mu_n}, \quad \forall n \in \mathbb{Z}, \forall p \in \mathbb{Z}. \tag{5.238}$$

5.5.2 Riesz basis property

Obviously, such a family (g_n) defines an operator T . Now we study the invertibility of this operator T thus defined. The first obvious observation from (5.238) is that T is injective if and only if

$$\langle f_n, F \rangle \neq 0, \quad \forall n \in \mathbb{Z}. \tag{5.239}$$

Next, considering (5.238), we turn to the family of functions given by

$$k_n := \sum_{p \in \mathbb{Z}} \frac{1}{\widetilde{\mu}_p - \mu_n} \widetilde{f}_p, \quad \forall n \in \mathbb{Z}. \quad (5.240)$$

Then, we have the following property for the (k_n) :

Lemma 5.5.1. (k_n) is a Riesz basis of $L^2(0, L)^2$.

Proof. First, let us perform the computations of (5.237) with the (f_n) :

$$\langle \mathcal{A}f_n, \widetilde{\phi}_p \rangle - \mu_n \langle f_n, \widetilde{\phi}_p \rangle = 0, \quad \forall n \in \mathbb{Z}. \quad (5.241)$$

This time, due to the dissipative boundary conditions in 0 for the $\widetilde{\phi}_p$, boundary terms appear in the integration by parts (which amounts to taking the adjoint $\widetilde{\mathcal{A}}^*$):

$$\begin{aligned} \mu_n \langle f_n, \widetilde{\phi}_p \rangle &= \langle \mathcal{A}f_n, \widetilde{\phi}_p \rangle \\ &= \langle f_n, \widetilde{\mathcal{A}}^* \widetilde{\phi}_p \rangle + f_{n,2}(0) \overline{\widetilde{\phi}_{p,2}(0)} - f_{n,1}(0) \overline{\widetilde{\phi}_{p,1}(0)} \\ &= \widetilde{\mu}_p \langle f_n, \widetilde{\phi}_p \rangle - f_{n,1}(0) \overline{\widetilde{\phi}_{p,1}(0)} (1 - e^{-2\mu L}), \end{aligned}$$

where the last equality is obtained using boundary conditions given by (5.38) and (5.60). From this we get the following decomposition for f_n :

$$f_n = \sum_{p \in \mathbb{Z}} \frac{f_{n,1}(0) \overline{\widetilde{\phi}_{p,1}(0)} (1 - e^{-2\mu L})}{\widetilde{\mu}_p - \mu_n} \widetilde{f}_p = f_{n,1}(0) \widetilde{\tau}_{\mathcal{A}} k_n, \quad \forall n \in \mathbb{Z}. \quad (5.242)$$

where $\widetilde{\tau}_{\mathcal{A}}$ is the operator defined by

$$\widetilde{\tau}_{\mathcal{A}} \widetilde{f}_p := \overline{\widetilde{\phi}_{p,1}(0)} (1 - e^{-2\mu L}) \widetilde{f}_p, \quad \forall p \in \mathbb{Z}. \quad (5.243)$$

Let us now give a crucial Lemma:

Lemma 5.5.2. *There exists $\gamma^* > 0$ such that for any $\gamma \in (0, \gamma^*)$, there exist $m, M > 0$ such that*

$$\begin{aligned} m &\leq \left| \overline{\widetilde{\phi}_{p,1}(0)} \right| \leq M, \quad \forall p \in \mathbb{Z}, \\ m &\leq |f_{p,1}(0)| \leq M, \quad \forall p \in \mathbb{Z}. \end{aligned} \quad (5.244)$$

Proof. When $\gamma = 0$, $f_{p,1}(0) = 1$. Using (5.191) and estimate (5.139), there exists $\gamma^* > 0$ such that this remains true for any $\gamma \in (0, \gamma^*)$. Similarly one can show looking at (5.60)–(5.61) that $\overline{\widetilde{\phi}_{p,1}(0)} \neq 0$ and independent of p , thus, using estimate (5.211), the result holds. \square

We can assume that we have chosen $\gamma < \gamma^*$ without losing the controllability of (5.59) and (5.60). Then, it is clear thanks to Lemma 5.5.2 that $\widetilde{\tau}_{\mathcal{A}}$ is an isomorphism of $(L^2)^2$. Moreover, thanks to the same lemma and the definition of a Riesz basis, the family of functions defined by

$$\left(\frac{f_n}{f_{n,1}(0)} \right)$$

forms a Riesz basis of $(L^2)^2$, so that the family

$$(k_n) = \left(\widetilde{\tau}_{\mathcal{A}}^{-1} \frac{f_n}{f_{n,1}(0)} \right)$$

forms a Riesz basis of $(L^2)^2$. \square

We now recall a result about Riesz basis (see for instance [44, Chapter 4])

Proposition 5.5.1. *A family of vector $(f_k)_{k \in \mathbb{Z}}$ of a Hilbert space H is a Riesz basis if and only if it is complete (i.e.) and there exists positive constants c and C such that, for any scalar sequence (a_k) with finite support,*

$$c \sum |a_k|^2 \leq \left\| \sum a_k f_k \right\|_H^2 \leq C \sum |a_k|^2. \quad (5.245)$$

We can now prove the following Riesz basis property for the (g_n) :

Proposition 5.5.2. *The family*

$$\left(\frac{1}{\langle f_n, F \rangle} g_n \right)_{n \in \mathbb{Z}}$$

is a Riesz basis of $D(\tilde{\mathcal{A}})$.

Proof. We first prove that the above family is a Riesz sequence: let I be a finite subset of \mathbb{Z} , and $(a_n) \in \mathbb{C}^I$. Then,

$$\begin{aligned} \left\| \sum_{n \in I} \frac{a_n}{\langle f_n, F \rangle} g_n \right\|_{D(\tilde{\mathcal{A}})}^2 &= \left\| \sum_{n \in I} \frac{a_n}{\langle f_n, F \rangle} \sum_{p \in \mathbb{Z}} \frac{\langle f_n, F \rangle \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle}{\tilde{\mu}_p - \mu_n} \tilde{f}_p \right\|_{D(\tilde{\mathcal{A}})}^2 \\ &= \left\| \sum_{p \in \mathbb{Z}} \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle \left(\sum_{n \in I} \frac{a_n}{\tilde{\mu}_p - \mu_n} \right) \tilde{f}_p \right\|_{D(\tilde{\mathcal{A}})}^2 \\ &= \left\| \sum_{p \in \mathbb{Z}} \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle \left(\sum_{n \in I} \frac{a_n}{\tilde{\mu}_p - \mu_n} \right) \tilde{f}_p \right\|^2 \\ &\quad + \left\| \sum_{p \in \mathbb{Z}} \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle \left(\sum_{n \in I} \frac{a_n}{\tilde{\mu}_p - \mu_n} \right) \tilde{\mu}_p \tilde{f}_p \right\|^2. \end{aligned} \quad (5.246)$$

as (\tilde{f}_p) is Riesz basis for L^2 and from Proposition 5.5.1, there exist $C_1, C_2 > 0$ such that

$$\begin{aligned} C_1 \sum_{p \in \mathbb{Z}} (1 + |\tilde{\mu}_p|^2) \left| \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle \right|^2 \left| \sum_{n \in I} \frac{a_n}{\tilde{\mu}_p - \mu_n} \right|^2 \\ \leq \left\| \sum_{n \in I} \frac{a_n}{\langle f_n, F \rangle} g_n \right\|_{D(\tilde{\mathcal{A}})}^2 \leq C_2 \sum_{p \in \mathbb{Z}} (1 + |\tilde{\mu}_p|^2) \left| \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle \right|^2 \left| \sum_{n \in I} \frac{a_n}{\tilde{\mu}_p - \mu_n} \right|^2. \end{aligned} \quad (5.247)$$

Now, similar estimates as what is done to obtain (5.200) (though even simpler in this case, as only first order of γ is required), also thanks to the controllability of our system, we know that there exists constants $m, M > 0$ such that

$$m \leq (1 + |\tilde{\mu}_p|^2) \left| \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle \right|^2 \leq M. \quad (5.248)$$

Moreover,

$$\sum_{p \in \mathbb{Z}} \left| \sum_{n \in I} \frac{a_n}{\tilde{\mu}_p - \mu_n} \right|^2 = \left\| \sum_{n \in I} a_n k_n \right\|^2, \quad (5.249)$$

so that, from Proposition 5.5.1 and as (k_n) is a Riesz basis, there exist constants $C_1, C_2 > 0$ such that

$$C_1 \sum_{n \in I} |a_n|^2 \leq \sum_{p \in \mathbb{Z}} \left| \sum_{n \in I} \frac{a_n}{\tilde{\mu}_p - \mu_n} \right|^2 \leq C_2 \sum_{n \in I} |a_n|^2, \quad (5.250)$$

and we get, putting (5.247) and (5.250) together, and for some constants $C_1, C_2 > 0$,

$$C_1 \sum_{n \in I} |a_n|^2 \leq \left\| \sum_{n \in I} a_n \frac{g_n}{\langle f_n, F \rangle} \right\|_{D(\tilde{\mathcal{A}})}^2 \leq C_2 \sum_{n \in I} |a_n|^2. \quad (5.251)$$

Now let us prove the the above family is complete. Let $\alpha \in D(\tilde{\mathcal{A}})$ be such that

$$\left\langle \alpha, \frac{g_n}{\langle F, f_n \rangle} \right\rangle_{D(\tilde{\mathcal{A}})} = 0, \quad \forall n \in \mathbb{Z}. \quad (5.252)$$

Then, (5.252) and the definition of (k_n) given by (5.240) implies that

$$\begin{aligned} 0 &= \sum_{p \in \mathbb{Z}} (1 + |\mu_p|^2) \langle \alpha, \tilde{f}_p \rangle \frac{\langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle}{\tilde{\mu}_p - \mu_n} \\ &= \left\langle \sum_{p \in \mathbb{Z}} (1 + |\mu_p|^2) \langle \alpha, \tilde{f}_p \rangle \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle \tilde{\phi}_p, k_n \right\rangle, \quad \forall n \in \mathbb{Z}. \end{aligned} \quad (5.253)$$

Thanks to (5.239), this implies

$$\left\langle \sum_{p \in \mathbb{Z}} (1 + |\mu_p|^2) \langle \alpha, \tilde{f}_p \rangle \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle \tilde{\phi}_p, k_n \right\rangle = 0, \quad \forall n \in \mathbb{Z}, \quad (5.254)$$

which implies, thanks to the completeness of the Riesz basis (k_n) , that

$$\sum_{p \in \mathbb{Z}} (1 + |\mu_p|^2) \langle \alpha, \tilde{f}_p \rangle \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle \tilde{\phi}_p = 0 \quad (5.255)$$

hence, thanks to the controllability of the system,

$$\alpha = 0.$$

This proves the completeness of the family, and, together with (5.251), the proposition. \square

Corollary 5.5.1. *If there exist constants $c, C > 0$ such that*

$$c(1 + |n|) \leq |\langle F, f_n \rangle| \leq C(1 + |n|), \quad \forall n \in \mathbb{Z}, \quad (5.256)$$

then the expression, for $\alpha \in D(\mathcal{A})$,

$$T\alpha := \sum_{n \in \mathbb{Z}} \langle \alpha, f_n \rangle g_n = - \sum_{n \in \mathbb{Z}} \langle \alpha, f_n \rangle \langle f_n, F \rangle \sum_{p \in \mathbb{Z}} \frac{\langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle}{\tilde{\mu}_p - \mu_n} \tilde{f}_p \quad (5.257)$$

defines an isomorphism of $D(\mathcal{A}) \rightarrow D(\tilde{\mathcal{A}})$.

5.5.3 Finding a suitable candidate: the operator equality

Now, for any $F \in \mathcal{E}'$ satisfying (5.256), we have used the heuristic kernel equation (5.231), (5.233), (5.234) to build an isomorphism. The next step is to find a feedback law F such that the corresponding isomorphism T is indeed a backstepping transformation, i.e. satisfies the following operator inequality:

$$T(-\mathcal{A} + \mathcal{I}_\nu F) = -\tilde{\mathcal{A}}T \quad (5.258)$$

on the domain

$$D_F = \{\alpha \in D(\mathcal{A}), \quad -\mathcal{A}\alpha + \langle \alpha, F \rangle \mathcal{I}_\nu \in D(\mathcal{A})\}. \quad (5.259)$$

Let $F \in \mathcal{E}'$, satisfying (5.256). To prove (5.258), it suffices to prove

$$\langle T(-\mathcal{A}\alpha + \langle \alpha, F \rangle \mathcal{I}_\nu), \tilde{\phi}_m \rangle = \langle -\tilde{\mathcal{A}}T\alpha, \tilde{\phi}_m \rangle, \quad \forall \alpha \in D_F, \quad \forall m \in \mathbb{Z}. \quad (5.260)$$

Let $m \in \mathbb{Z}$, we consider the left-hand side of (5.260), which can be rewritten

$$\langle -\mathcal{A}\alpha + \langle \alpha, F \rangle \mathcal{I}_\nu, T^* \tilde{\phi}_m \rangle_{D(\mathcal{A}) \times D(\mathcal{A})'}. \quad (5.261)$$

Now, to evaluate the linear form $T^* \tilde{\phi}_m$ on the function $\mathcal{A}\alpha + \langle \alpha, F \rangle \mathcal{I}_\nu$, we approximate the function by its truncated expansion in the basis $(f_n)_{n \in \mathbb{Z}}$:

$$\begin{aligned} \langle T(-\mathcal{A}\alpha^{(N)} + \langle \alpha, F \rangle \mathcal{I}_\nu^{(N)}), \tilde{\phi}_m \rangle &= \langle -\mathcal{A}\alpha^{(N)} + \langle \alpha, F \rangle \mathcal{I}_\nu^{(N)}, T^* \tilde{\phi}_m \rangle_{D(\mathcal{A}) \times D(\mathcal{A})'} \\ &\xrightarrow{N \rightarrow \infty} \langle -\mathcal{A}\alpha + \langle \alpha, F \rangle \mathcal{I}_\nu, T^* \tilde{\phi}_m \rangle_{D(\mathcal{A}) \times D(\mathcal{A})'} \end{aligned} \quad (5.262)$$

where, for $N > 0$,

$$\begin{aligned} \alpha^{(N)} &:= \sum_{n=-N}^N \langle \alpha, f_n \rangle f_n, \\ \mathcal{I}_\nu^{(N)} &:= \sum_{n=-N}^N \langle \mathcal{I}_\nu, f_n \rangle f_n. \end{aligned} \quad (5.263)$$

We can then make the following computations for $\alpha \in D_F$ and $N > 0$, using (5.235) and (5.257):

$$\begin{aligned} T(-\mathcal{A}\alpha^{(N)} + \langle \alpha, F \rangle \mathcal{I}_\nu^{(N)}) &= \langle \alpha, F \rangle T\mathcal{I}_\nu^{(N)} + \sum_{n=-N}^N -\mu_n \langle \alpha, f_n \rangle g_n \\ &= \langle \alpha, F \rangle T\mathcal{I}_\nu^{(N)} + \sum_{n=-N}^N \langle \alpha, f_n \rangle (-\tilde{\mathcal{A}}g_n - \langle f_n, F \rangle \mathcal{I}_\nu) \\ &= -\tilde{\mathcal{A}}T\alpha^{(N)} - \langle \alpha^{(N)}, F \rangle \mathcal{I}_\nu + \langle \alpha, F \rangle T\mathcal{I}_\nu^{(N)}. \end{aligned} \quad (5.264)$$

Now, as

$$\langle \tilde{\mathcal{A}}T\alpha^{(N)}, \tilde{\phi}_m \rangle \xrightarrow{N \rightarrow \infty} \langle \tilde{\mathcal{A}}T\alpha, \tilde{\phi}_m \rangle, \quad \forall m \in \mathbb{Z}, \quad \forall \alpha \in D_F, \quad (5.265)$$

to get (5.260) from (5.264), it suffices to have

$$\begin{aligned} \langle \alpha^{(N)}, F \rangle &\xrightarrow{N \rightarrow \infty} \langle \alpha, F \rangle, \quad \forall \alpha \in D_F, \\ \langle T\mathcal{I}_\nu^{(N)}, \tilde{\phi}_m \rangle &\xrightarrow{N \rightarrow \infty} \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle, \quad \forall m \in \mathbb{Z}. \end{aligned} \quad (5.266)$$

Recall that the formal $TB = B$ condition (5.232) had been added to the kernel equation (5.231) to take away the nonlocal term of the equation and thus make it easier to study, and allowing us to build isomorphisms. Consistently, when checking whether such isomorphisms are actually backstepping transformations, the $TB = B$ condition reappears in a weak form, precisely given by the second limit of (5.266). Notice that this is the “real” $TB = B$ condition, as, according to Corollary 5.5.1, $T\mathcal{I}_\nu$ is not defined and thus (5.232) has no real mathematical meaning.

5.6 Backstepping transformation and feedback law

To find an actual backstepping transformation, we want to construct a feedback F that satisfies (5.266). We will first study the second limit of (5.266), which will determine the value of F . Then, we will check that F thus defined satisfies the first limit of (5.266), so that the corresponding isomorphism T is indeed a backstepping transformation.

5.6.1 Construction of the feedback law

Keeping the notations of (5.263), first note that

$$\begin{aligned} T\mathcal{I}_\nu^{(N)} &= - \sum_{n=-N}^N \langle \mathcal{I}_\nu, f_n \rangle \langle f_n, F \rangle \sum_{p \in \mathbb{Z}} \frac{\langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle}{\tilde{\mu}_p - \mu_n} \tilde{f}_p \\ &= - \sum_{p \in \mathbb{Z}} \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle \left(\sum_{n=-N}^N \frac{\langle \mathcal{I}_\nu, f_n \rangle \langle f_n, F \rangle}{\tilde{\mu}_p - \mu_n} \right) \tilde{f}_p. \end{aligned} \quad (5.267)$$

Hence, for $m \in \mathbb{Z}$, using (5.243), and recalling that $\langle \tilde{f}_p, \tilde{\phi}_m \rangle = 0$, for any $p \neq m$ from the definition of a biorthogonal family,

$$\begin{aligned} \langle T\mathcal{I}_\nu^{(N)}, \tilde{\phi}_m \rangle &= - \left\langle \sum_{p \in \mathbb{Z}} \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle \left(\sum_{n=-N}^N \frac{\langle \mathcal{I}_\nu, f_n \rangle \langle f_n, F \rangle}{\tilde{\mu}_p - \mu_n} \right) \tilde{f}_p, \tilde{\phi}_m \right\rangle \\ &= - \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle \left\langle \sum_{p \in \mathbb{Z}} \left(\sum_{n=-N}^N \frac{\langle \mathcal{I}_\nu, f_n \rangle \langle f_n, F \rangle}{\tilde{\mu}_p - \mu_n} \right) \tilde{f}_p, \tilde{\phi}_m \right\rangle \\ &= - \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle \left\langle \sum_{n=-N}^N \langle \mathcal{I}_\nu, f_n \rangle \langle f_n, F \rangle \sum_{p \in \mathbb{Z}} \frac{1}{\tilde{\mu}_p - \mu_n} \tilde{f}_p, \tilde{\phi}_m \right\rangle \\ &= - \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle \left\langle \sum_{n=-N}^N \langle \mathcal{I}_\nu, f_n \rangle \langle f_n, F \rangle \tilde{\tau}_A^{-1} \frac{f_n}{f_{n,1}(0)}, \tilde{\phi}_m \right\rangle \\ &= - \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle \left\langle \tilde{\tau}_A^{-1} \sum_{n=-N}^N \langle \mathcal{I}_\nu, f_n \rangle \langle f_n, F \rangle \frac{f_n}{f_{n,1}(0)}, \tilde{\phi}_m \right\rangle. \end{aligned} \quad (5.268)$$

Moreover, by definition (5.243) of $\tilde{\tau}_A, (f_n), (\tilde{\phi}_n)$, we have:

$$(\tilde{\tau}_A^{-1})^* \tilde{\phi}_m = \frac{1}{\tilde{\phi}_{m,1}(0)(1 - e^{-2\mu L})} \tilde{\phi}_m, \quad (5.269)$$

hence

$$\begin{aligned} \langle T\mathcal{I}_\nu^{(N)}, \tilde{\phi}_m \rangle &= - \frac{\langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle}{\tilde{\phi}_{m,1}(0)(1 - e^{-2\mu L})} \left\langle \sum_{n=-N}^N \langle \mathcal{I}_\nu, f_n \rangle \langle f_n, F \rangle \frac{f_n}{f_{n,1}(0)}, \tilde{\phi}_m \right\rangle \\ &= - \frac{\langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle}{\tilde{\phi}_{m,1}(0)(1 - e^{-2\mu L})} \sum_{n=-N}^N \langle \mathcal{I}_\nu, f_n \rangle \frac{\langle f_n, F \rangle}{f_{n,1}(0)} \langle f_n, \tilde{\phi}_m \rangle. \end{aligned} \quad (5.270)$$

Now let us set

$$\langle f_n, F \rangle := -2 \tanh(\mu L) \frac{(f_{n,1}(0))^2}{\langle \mathcal{I}_\nu, f_n \rangle}, \quad \forall n \in \mathbb{Z}, \quad (5.271)$$

so that

$$\langle T\mathcal{I}_\nu^{(N)}, \tilde{\phi}_m \rangle = \frac{2 \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle}{\tilde{\phi}_{m,1}(0)(1 + e^{-2\mu L})} \sum_{n=-N}^N f_{n,1}(0) \langle f_n, \tilde{\phi}_m \rangle. \quad (5.272)$$

The sum in (5.272) is analog to the Dirichlet sum which appears in [162] and that we recall here: for $f \in C_{pw}^1([0, L])$, then

$$\sum_{n=-N}^N \langle f, e_p \rangle \xrightarrow{N \rightarrow +\infty} \frac{f(0) + f(L)}{2} \quad (5.273)$$

where $(e_p)_{p \in \mathbb{Z}} = (e^{\frac{2i\pi px}{L}})_{p \in \mathbb{Z}}$. Note that this can be extended to functions $f \in (C^1)^2$ such that $f_1(0) = -f_2(0)$ and $f_1(L) = -f_2(L)$ and the basis $(E_p)_{p \in \mathbb{Z}} = \left(\left(e^{\frac{i\pi px}{L}}, -e^{-\frac{i\pi px}{L}} \right) \right)_{p \in \mathbb{Z}}$. Indeed, using the gluing (5.362), from each of these functions one can define $\underline{f} \in C_{pw}^1([0, 2L])$ and $(\underline{E}_p)_{p \in \mathbb{Z}} = (e^{\frac{2i\pi px}{2L}})_{p \in \mathbb{Z}}$, thus

$$\sum_{n=-N}^N \langle f, E_p \rangle \xrightarrow{N \rightarrow +\infty} \frac{f(0) + f(2L)}{2} = \frac{f_1(0) - f_2(0)}{2}. \quad (5.274)$$

As it turns out, the sum in (5.272) has the same remarkable property of converging towards the mean of the left and right values of $\tilde{\phi}_m$. This comes from a powerful equiconvergence theorem proved in [105] for a Schrödinger operator (see [106] for its generalization on operators of order $n \geq 2$):

Theorem 5.6.1. *Let $(u_k) \subset L^2(0, L)$ be a complete orthonormal system of eigenfunctions associated to the eigenvalues λ_p of the Schrödinger operator $-\partial_{xx} + q$, where q is a locally integrable function. Let us denote for any $f \in L^2(0, L)$, $\mu > 0$, and $x \in (0, L)$,*

$$\sigma_\mu(f, x) = \sum_{|\operatorname{Re}(\sqrt{\lambda_p})| < \mu} \langle f, u_p \rangle u_p(x). \quad (5.275)$$

Similarly, let $(\hat{u}_k) \subset L^2(0, L)$ be a complete orthonormal system of eigenfunctions associated to the eigenvalues $\hat{\lambda}_p$ of the Schrödinger operator $-\partial_{xx} + \hat{q}$, where \hat{q} is a locally integrable function and denote,

$$\hat{\sigma}_\mu(f, x) = \sum_{|\operatorname{Re}(\sqrt{\hat{\lambda}_p})| < \mu} \langle f, \hat{u}_p \rangle \hat{u}_p(x). \quad (5.276)$$

Then, given any compact interval $K \subset (0, L)$, for any $f \in L^2(0, L)$, the following holds

$$\lim_{\mu \rightarrow +\infty} \sup_{x \in K} |\sigma_\mu(f, x) - \hat{\sigma}_\mu(f, x)| = 0. \quad (5.277)$$

This theorem can be adapted to our operator:

Proposition 5.6.1. *Let us denote*

$$\sigma_\mu(f, x) = \sum_{|\operatorname{Im}(\mu_p)| < \mu} \langle f, f_p \rangle f_p(x) \quad (5.278)$$

and

$$p_\mu(f, x) = \sum_{|\operatorname{Im}(\mu_p^{(0)})| < \mu} \langle f, E_p \rangle E_p(x) \quad (5.279)$$

where $(E_p)_{p \in \mathbb{Z}} = \left(\left(e^{\frac{i\pi px}{L}}, -e^{-\frac{i\pi px}{L}} \right) \right)_{p \in \mathbb{Z}}$. One has for any compact $K \subset [0, L]$

$$\lim_{\mu \rightarrow +\infty} \sup_{x \in K} |\sigma_\mu(f, x) - p_\mu(f, x)| = 0. \quad (5.280)$$

This result is a generalization of the work of Komornik in [105]. A way to adapt the proof of [105, Theorem 2] is given in Appendix 5.B.

Now let us recall that for $m \in \mathbb{Z}$, $\tilde{\phi}_m$ is the solution of a linear ODE given by the operator $\tilde{\mathcal{A}}^*$. Thus $\tilde{\phi}_m \in C_{pw}^1(0, 2L)$ (recall the definition of \underline{f} given in (5.362)) and, applying Proposition 5.6.1 and the Dirichlet convergence theorem given by (5.274), we get:

$$\sum_{n=-N}^N f_{n,1}(0) \langle f_n, \tilde{\phi}_m \rangle = \overline{\sum_{n=-N}^N f_{n,1}(0) \langle \tilde{\phi}_m, f_n \rangle} \xrightarrow{N \rightarrow \infty} \frac{\tilde{\phi}_{m,1}(0) - \tilde{\phi}_{m,2}(0)}{2} \quad (5.281)$$

Now, using the boundary conditions (5.63) for $\tilde{\phi}_m$, we have

$$\frac{\overline{\tilde{\phi}_{m,1}(0) - \tilde{\phi}_{m,2}(0)}}{2} = \frac{\overline{\tilde{\phi}_{m,1}(0)(1 + e^{-2\mu L})}}{2} \quad (5.282)$$

so that finally, if F is defined by (5.271), we get from (5.272):

$$\langle T\mathcal{I}_\nu^{(N)}, \tilde{\phi}_m \rangle \xrightarrow{N \rightarrow \infty} \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle, \quad \forall m \in \mathbb{Z}, \quad (5.283)$$

which is the second limit of (5.266).

5.6.2 Regularity of the feedback law

Let us now study the regularity of the feedback law defined by (5.271). First note that, thanks to (5.194) that was deduced from the moments condition that gave us the controllability of the system, and thanks to Lemma 5.5.2,

$$c(1 + |\mu_n|) \leq |\langle f_n, F \rangle| \leq C(1 + |\mu_n|), \quad n \in \mathbb{Z}, \quad (5.284)$$

for some constants $c, C > 0$, which is exactly to (5.256) for some other constants, thanks to (5.50). Then, we have the following regularity result, analogous to [162, Lemma 2.1]:

Lemma 5.6.1. *$F \in \mathcal{E}'$ defined by (5.271) defines a linear form on $D(\mathcal{A}^2)$ which is continuous for $\|\cdot\|_{D(\mathcal{A}^2)}$ but not for $\|\cdot\|_{D(\mathcal{A})}$.*

Corollary 5.6.1. *If F is defined by (5.271), then the domain D_F given by (5.259) defines a dense domain of $D(\mathcal{A})$.*

Proof. It is clear from (5.259) that:

$$\mathcal{K}_F := \{\alpha \in D(\mathcal{A}^2), \quad \langle \alpha, F \rangle = 0\} \subset D_F. \quad (5.285)$$

Now, \mathcal{K}_F is the kernel of a non-continuous linear form, so it is dense in $D(\mathcal{A}^2)$ for $\|\cdot\|_{D(\mathcal{A})}$, which is in turn dense in $D(\mathcal{A})$ for $\|\cdot\|_{D(\mathcal{A})}$, hence the density of D_F in $D(\mathcal{A})$. \square

Now, to obtain the second limit of (5.266), we need a more precise knowledge of the regularity of F .

Recall that (5.196) had given more information on the growth of the coefficients of \mathcal{I}_ν . Now, taking the inverse, and using (5.194) we get the following property for the coefficients of the feedback law F :

$$\left(\frac{1}{\mu_n} \left(\langle f_n, F \rangle + \frac{\tanh(\mu L)}{\tau_n^{\mathcal{I}}} f_{n,1}(0) \mu_n \right) \right) \in \ell^2(\mathbb{Z}), \quad (5.286)$$

where

$$\tau_n^{\mathcal{I}} := \left(e^{\int_0^L \delta f_{n,1}(L)} / f_{n,1}(0) - 1 \right), \quad \forall n \in \mathbb{Z}. \quad (5.287)$$

Note that we have, thanks to Lemma 5.5.2 and (5.200),

$$c \leq |\tau_n^{\mathcal{I}}| \leq C, \quad \forall n \in \mathbb{Z}, \quad (5.288)$$

for some constants $c, C > 0$, so that the operator defined by

$$\tau^{\mathcal{I}} \alpha := \sum_{n \in \mathbb{Z}} \tau_n^{\mathcal{I}} \langle \alpha, f_n \rangle f_n, \quad \forall \alpha \in (L^2)^2 \quad (5.289)$$

is an isomorphism of $D(\mathcal{A}^s), \forall s \geq 0$.

Remark 5.6.1. *Because it is a diagonal operator, $\tau^{\mathcal{I}}$ commutes with \mathcal{A} .*

We recall now the definition of the spaces X^s for $s \geq 0$

$$X^s := \{f \in (L^2_{(0)})^2, \quad (\tau^{\mathcal{I}})^{-1}(\Lambda \partial_x f + \delta(x)Jf) \in (H^{s-1})^2\}. \quad (5.290)$$

Let us then note:

$$h_n := -\frac{\tanh(\mu L)}{\tau_n^{\mathcal{I}}} f_{n,1}(0)\mu_n, \quad n \in \mathbb{Z}, \quad (5.291)$$

and $h \in \mathcal{E}'$ the linear form defined by:

$$\langle f_n, h \rangle = h_n, \quad \forall n \in \mathbb{Z}. \quad (5.292)$$

Then, we have the following proposition:

Proposition 5.6.2. *The linear form h defines the following linear form on $X^2 \cap D(\mathcal{A})$, continuous for $\|\cdot\|_{X^2}$:*

$$\langle \alpha, h \rangle = -\tanh(\mu L) \frac{(\mathcal{A}(\tau^{\mathcal{I}})^{-1}\alpha)_1(0) - (\mathcal{A}(\tau^{\mathcal{I}})^{-1}\alpha)_2(0)}{2} \quad (5.293)$$

Moreover, $\tilde{F} := F - h$ is continuous for $\|\cdot\|_{D(\mathcal{A})}$, so that F is actually defined on $X^2 \cap D(\mathcal{A})$, and is continuous for $\|\cdot\|_{X^2}$, but not $\|\cdot\|_{D(\mathcal{A})}$.

Remark 5.6.2. *This proposition means that h is the "singular part" of F , i.e. the part that limits the regularity of F , the rest $F - h$ being continuous for $\|\cdot\|_{D(\mathcal{A})}$. In the following we will therefore study specifically this singular part.*

Proof of Proposition 5.6.2. The continuity of \tilde{F} follows directly from (5.286). On the other hand, let $\alpha \in X^3 \cap D(\mathcal{A})$. Then, $\mathcal{A}(\tau^{\mathcal{I}})^{-1}\alpha \in (H^2)^2$, and thus satisfies the conditions of the Dirichlet convergence theorem, so that using again Proposition 5.6.1, we get:

$$\begin{aligned} \langle \alpha, h \rangle &= -\sum_{n \in \mathbb{Z}} \langle \alpha, f_n \rangle \frac{\tanh(\mu L)}{\tau_n^{\mathcal{I}}} f_{n,1}(0)\mu_n \\ &= -\tanh(\mu L) \lim_{N \rightarrow \infty} p_N(\mathcal{A}(\tau^{\mathcal{I}})^{-1}\alpha, 0) \\ &= -\tanh(\mu L) \frac{(\mathcal{A}(\tau^{\mathcal{I}})^{-1}\alpha)_1(0) - (\mathcal{A}(\tau^{\mathcal{I}})^{-1}\alpha)_2(0)}{2}. \end{aligned} \quad (5.294)$$

It is clear from this last expression that h is continuous for $\|\cdot\|_{X^2}$, by the continuous injection of $(H^1)^2 \rightarrow (L^\infty)^2$. Now let us show that $X^3 \cap D(\mathcal{A})$ is dense in $X^2 \cap D(\mathcal{A})$ for the X^2 norm. Let $\alpha \in X^2 \cap D(\mathcal{A})$, and $\varepsilon > 0$. Then, by density of $(H^2)^2$ in $(H^1)^2$ there exists $a_\varepsilon \in (H^2)^2$ such that

$$\|a_\varepsilon - \mathcal{A}(\tau^{\mathcal{I}})^{-1}\alpha\|_{(H^1)^2} \leq \varepsilon. \quad (5.295)$$

Now,

$$\langle \mathcal{A}(\tau^{\mathcal{I}})^{-1}\alpha, f_0 \rangle = 0 \quad (5.296)$$

so that

$$\langle a_\varepsilon, f_0 \rangle \leq \varepsilon, \quad (5.297)$$

and, setting

$$\tilde{a}_\varepsilon := a_\varepsilon - \langle a_\varepsilon, f_0 \rangle f_0, \quad (5.298)$$

we get, thanks to (5.297) and the smoothness of f_0 as the solution of a linear ODE,

$$\begin{aligned} \tilde{a}_\varepsilon &\in (H^2)^2, \\ \|\tilde{a}_\varepsilon - \mathcal{A}(\tau^{\mathcal{I}})^{-1}\alpha\|_{(H^1)^2} &\leq C\varepsilon, \end{aligned} \quad (5.299)$$

For some constant $C > 0$ depending only on $\|f_0\|_{(H^1)^2}$. Let us now set:

$$\alpha_\varepsilon := \langle \alpha, f_0 \rangle f_0 + \sum_{n \in \mathbb{Z}^*} \tau_n^{\mathcal{I}} \frac{\langle \tilde{u}_\varepsilon, f_n \rangle}{\mu_n} f_n \in D(\mathcal{A}). \quad (5.300)$$

Then,

$$\mathcal{A}(\tau^{\mathcal{I}})^{-1} \alpha_\varepsilon = a_\varepsilon, \quad (5.301)$$

so that $\alpha_\varepsilon \in X^3$. Moreover, by (5.295) and (5.288),

$$\|\mathcal{A}\alpha_\varepsilon - \mathcal{A}\alpha\| \leq C\varepsilon, \quad (5.302)$$

for some constant $C > 0$. As $|\mu_n| \geq \delta > 0$, $\forall |n| \geq 1$, and by definition of α_ε , $\langle \alpha_\varepsilon - \alpha, f_0 \rangle = 0$, we get

$$\|\alpha_\varepsilon - \alpha\| \leq C'\varepsilon, \quad (5.303)$$

and (5.295), (5.301), (5.302), and (5.303) yield

$$\|\alpha_\varepsilon - \alpha\|_{X^2} \leq C''\varepsilon, \quad (5.304)$$

This proves the density of $X^3 \cap D(\mathcal{A})$ in $X^2 \cap D(\mathcal{A})$. We can then continuously extend (5.294) to $X^2 \cap D(\mathcal{A})$, which proves the rest of the proposition. \square

Now that we have some knowledge on F defined by (5.271), we get some more knowledge on the corresponding domain D_F :

Lemma 5.6.2.

$$D_F \subset X^2. \quad (5.305)$$

Proof. Recall that

$$D_F = \{\alpha \in D(\mathcal{A}), \quad -\mathcal{A}\alpha + \langle \alpha, F \rangle \mathcal{I}_\nu \in D(\mathcal{A})\}. \quad (5.306)$$

In particular,

$$(\tau^{\mathcal{I}})^{-1} (-\mathcal{A}\alpha + \langle \alpha, F \rangle \mathcal{I}_\nu) \in D(\mathcal{A}). \quad (5.307)$$

Moreover, by (5.289) and (5.196),

$$(\tau^{\mathcal{I}})^{-1} \mathcal{I}_\nu - \sum_{n \in \mathbb{Z}} \frac{f_{n,1}(0)}{\mu_n} f_n \in D(\mathcal{A}). \quad (5.308)$$

Now, let

$$\varphi(x) := \begin{pmatrix} \frac{1}{2} - \frac{x}{2L} \\ \frac{1}{2} - \frac{x}{2L} \end{pmatrix} \in (H^1)^2, \quad (5.309)$$

it is clear by integration by parts that

$$\varphi - \sum_{n \in \mathbb{Z}} \frac{f_{n,1}(0)}{\mu_n} f_n \in D(\mathcal{A}), \quad (5.310)$$

so that

$$(\tau^{\mathcal{I}})^{-1} \mathcal{I}_\nu \in (H^1)^2 \quad (5.311)$$

and, finally, by (5.307) and (5.311),

$$\mathcal{A}(\tau^{\mathcal{I}})^{-1} \alpha \in (H^1)^2. \quad (5.312)$$

\square

Now we can tackle the first limit of (5.266). Let $\alpha \in D_F$. As

$$\alpha^{(N)} \xrightarrow[N \rightarrow \infty]{D(\mathcal{A})} \alpha, \quad (5.313)$$

, by Proposition 5.6.2, we only need to study the ‘‘singular part’’ h of the feedback F . By (5.263) and (5.52), we have:

$$\begin{aligned} \langle \alpha^{(N)}, h \rangle &= - \sum_{n=-N}^N \langle \alpha, f_n \rangle \frac{\tanh(\mu L)}{\tau_n^{\mathcal{I}}} f_{n,1}(0) \mu_n \\ &= - \frac{\tanh(\mu L)}{2} \sum_{n=-N}^N \left(\frac{\langle \alpha, f_n \rangle}{\tau_n^{\mathcal{I}}} \mu_n + \frac{\langle \alpha, f_{-n} \rangle}{\tau_{-n}^{\mathcal{I}}} \mu_{-n} \right) f_{n,1}(0) \\ &= - \frac{\tanh(\mu L)}{2} \sum_{n=-N}^N \langle \mathcal{A}((\tau^{\mathcal{I}})^{-1} \alpha - \sigma((\tau^{\mathcal{I}})^{-1} \alpha)), f_n \rangle f_{n,1}(0), \end{aligned} \quad (5.314)$$

where

$$\sigma(f)(x) := \sum_{n \in \mathbb{Z}} \langle f, f_{-n} \rangle f_n, \quad \forall f \in (L^2)^2. \quad (5.315)$$

Note that we used that for $f \in D(\mathcal{A})$, $\mathcal{A}\sigma(f) = -\sigma(\mathcal{A}f)$. Now, as $\alpha \in X^2 \cap D(\mathcal{A})$, we have

$$\mathcal{A}(\tau^{\mathcal{I}})^{-1} \alpha = \Lambda \partial_x (\tau^{\mathcal{I}})^{-1} \alpha + \delta(x) J (\tau^{\mathcal{I}})^{-1} \alpha \in (H^1)^2 \quad (5.316)$$

so

$$\Lambda \partial_x (\tau^{\mathcal{I}})^{-1} \alpha \in (L^2)^2, \quad (5.317)$$

hence

$$(\tau^{\mathcal{I}})^{-1} \alpha \in (H^1)^2. \quad (5.318)$$

Repeating the same argument, we get

$$(\tau^{\mathcal{I}})^{-1} \alpha \in (H^2)^2 \cap D(\mathcal{A}). \quad (5.319)$$

We now use the following lemma:

Lemma 5.6.3. *Let $f \in (H^2)^2 \cap D(\mathcal{A})$. Then,*

$$f - \sigma(f) \in D(\mathcal{A}^2). \quad (5.320)$$

Proof. Using the regularity of f , we write, by double integration by parts:

$$\begin{aligned} \langle f, f_n \rangle &= \frac{1}{\mu_n} \langle \mathcal{A}f, f_n \rangle \\ &= \frac{(\mathcal{A}f_1(L) + \mathcal{A}f_2(L)) f_{n,1}(L) - (\mathcal{A}f_1(0) + \mathcal{A}f_2(0)) f_{n,1}(0)}{(\mu_n)^2} \\ &\quad + \frac{1}{(\mu_n)^2} \langle \mathcal{A}^2 f, f_n \rangle, \end{aligned} \quad (5.321)$$

Hence, using (5.51) and (5.52).

$$\langle f, f_n \rangle - \langle f, f_{-n} \rangle = \frac{1}{(\mu_n)^2} \langle \mathcal{A}^2 f, f_n - f_{-n} \rangle. \quad (5.322)$$

As $\mathcal{A}^2 f \in (L^2)^2$, this proves the Lemma. \square

From Lemma 5.6.3, we get

$$\mathcal{A}((\tau^{\mathcal{I}})^{-1}\alpha - \sigma((\tau^{\mathcal{I}})^{-1}\alpha)) \in D(\mathcal{A}). \quad (5.323)$$

Hence, the last sum of (5.314) converges absolutely. As $\alpha \in X^2 \cap D(\mathcal{A})$, by Proposition 5.6.2 and unicity of the limit, we get

$$\langle \alpha^{(N)}, F \rangle \xrightarrow{N \rightarrow \infty} \langle \alpha, F \rangle, \quad (5.324)$$

which is the first limit of (5.266).

Thus, by the results of the two previous subsections and (5.266), we have the following proposition:

Proposition 5.6.3. *Let F be defined by (5.271), and T accordingly defined by (5.257). Then, T satisfies (5.258) on the domain D_F .*

5.6.3 Well-posedness and stability of the closed-loop system

Now that we have constructed a pair (T, F) that satisfies (5.258), let us check that the closed-loop system (5.59) corresponding to the feedback F is well-posed in some sense. The idea is to use the dynamics of the target system (5.60), as (5.258) essentially means that T exchanges the dynamics of the target system and the closed-loop system. We have seen in Section 5.3.2 that (5.60) is well-posed, more specifically, looking at Section 5.3.2.2, $\tilde{\mathcal{A}}$ with domain $D(\tilde{\mathcal{A}}^2) \subset D(\tilde{\mathcal{A}})$ generates a contraction semigroup on $D(\tilde{\mathcal{A}})$ for the norm defined by the Lyapunov function (5.73) with $p = 1$. We note that semigroup $\tilde{S}(t), t \geq 0$.

Accordingly, before we study the dynamics generated by $-\mathcal{A} + \mathcal{I}_\nu F$, let us give a more precise characterization of its domain $D_F \subset D(\mathcal{A})$. We start with the following lemma:

Lemma 5.6.4. *The operator $-\mathcal{A} + \mathcal{I}_\nu F$ admits a Riesz basis of eigenvectors in $D(\mathcal{A})$, given by*

$$h_p := T^{-1} \frac{\tilde{f}_p}{\tilde{\mu}_p}, \quad \forall p \in \mathbb{Z}, \quad (5.325)$$

with corresponding eigenvalues $(-\tilde{\mu}_p)_{p \in \mathbb{Z}}$.

Proof. It is clear that the normalized family $(\tilde{f}_p/\tilde{\mu}_p)_{p \in \mathbb{Z}}$ is a Riesz basis of $D(\tilde{\mathcal{A}})$, so that, applying the isomorphism T^{-1} , $(h_p)_{p \in \mathbb{Z}}$ is a Riesz basis of $D(\mathcal{A})$. Let us now show that

$$h_p \in D_F, \quad \forall p \in \mathbb{Z}. \quad (5.326)$$

Let us denote the following decomposition along the orthonormal basis $(f_n)_{n \in \mathbb{Z}}$:

$$h_p = \sum_{n \in \mathbb{Z}} a_{n,p} f_n, \quad (\mu_n a_{n,p})_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}), \quad \forall p \in \mathbb{Z}. \quad (5.327)$$

From the definitions of T and h_p ,

$$\tilde{f}_p = \tilde{\mu}_p T h_p = \sum_{n \in \mathbb{Z}} \tilde{\mu}_p a_{n,p} g_n, \quad \forall p \in \mathbb{Z}, \quad (5.328)$$

which implies, from (5.236) and (5.238),

$$\tilde{f}_p = - \sum_{n \in \mathbb{Z}} \tilde{\mu}_p a_{n,p} \langle f_n, F \rangle \sum_{k \in \mathbb{Z}} \frac{\langle \mathcal{I}_\nu, \tilde{\phi}_k \rangle}{\tilde{\mu}_k - \mu_n} \tilde{f}_k, \quad \forall p \in \mathbb{Z}. \quad (5.329)$$

Now, as in Subsection 5.6.1, consider the truncatures along the Riesz basis $(g_n/\langle f_n, F \rangle)_{n \in \mathbb{Z}}$,

$$\tilde{f}_p^{(N)} := \sum_{n=-N}^N \tilde{\mu}_p a_{n,p} \langle f_n, F \rangle \frac{g_n}{\langle f_n, F \rangle}, \quad \forall N \in \mathbb{N}^*, \quad \forall p \in \mathbb{Z}, \quad (5.330)$$

so that

$$\begin{aligned}
\langle \tilde{f}_p^{(N)}, \tilde{\phi}_m \rangle &= -\tilde{\mu}_p \left\langle \sum_{n=-N}^N a_{n,p} \langle f_n, F \rangle \sum_{k \in \mathbb{Z}} \frac{\langle \mathcal{I}_\nu, \tilde{\phi}_k \rangle}{\tilde{\mu}_k - \mu_n} \tilde{f}_k, \tilde{\phi}_m \right\rangle \\
&= -\tilde{\mu}_p \left\langle \sum_{k \in \mathbb{Z}} \langle \mathcal{I}_\nu, \tilde{\phi}_k \rangle \left(\sum_{n=-N}^N \frac{a_{n,p} \langle f_n, F \rangle}{\tilde{\mu}_k - \mu_n} \right) \tilde{f}_k, \tilde{\phi}_m \right\rangle \\
&= -\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle \left\langle \sum_{n=-N}^N \frac{a_{n,p} \langle f_n, F \rangle}{\tilde{\mu}_m - \mu_n} \tilde{f}_m, \tilde{\phi}_m \right\rangle \\
&= -\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle \left\langle \sum_{k \in \mathbb{Z}} \left(\sum_{n=-N}^N \frac{a_{n,p} \langle f_n, F \rangle}{\tilde{\mu}_k - \mu_n} \right) \tilde{f}_k, \tilde{\phi}_m \right\rangle \\
&= -\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle \left\langle \sum_{n=-N}^N a_{n,p} \langle f_n, F \rangle \sum_{k \in \mathbb{Z}} \frac{\tilde{f}_k}{\tilde{\mu}_k - \mu_n}, \tilde{\phi}_m \right\rangle \\
&= -\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle \left\langle \sum_{n=-N}^N a_{n,p} \langle f_n, F \rangle k_n, \tilde{\phi}_m \right\rangle \\
&= -\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle \left\langle \sum_{n=-N}^N a_{n,p} \langle f_n, F \rangle \tilde{\tau}_A^{-1} \frac{f_n}{f_{n,1}(0)}, \tilde{\phi}_m \right\rangle, \\
&\quad \forall p \in \mathbb{Z}, \forall m \in \mathbb{Z}, \forall N \in \mathbb{N}^*,
\end{aligned} \tag{5.331}$$

where we used the biorthogonality of the families $(\tilde{f}_p)_{p \in \mathbb{Z}}$ and $(\tilde{\phi}_p)_{p \in \mathbb{Z}}$, and the relation (5.242). Using again the aforementioned biorthogonality, and the convergences

$$\begin{aligned}
\tilde{f}_p^{(N)} &\xrightarrow[N \rightarrow \infty]{D(\tilde{A})} \tilde{f}_p, \\
\sum_{n=-N}^N a_{n,p} \langle f_n, F \rangle \tilde{\tau}_A^{-1} \frac{f_n}{f_{n,1}(0)} &\xrightarrow[N \rightarrow \infty]{L^2} \sum_{n \in \mathbb{Z}} a_{n,p} \langle f_n, F \rangle \tilde{\tau}_A^{-1} \frac{f_n}{f_{n,1}(0)},
\end{aligned} \tag{5.332}$$

given by (5.194), (5.271) and (5.327), we get

$$\left\langle \frac{\tilde{f}_p}{\langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle}, \tilde{\phi}_m \right\rangle = \left\langle \frac{\tilde{f}_p}{\langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle}, \tilde{\phi}_m \right\rangle = - \left\langle \tilde{\mu}_p \sum_{n \in \mathbb{Z}} a_{n,p} \langle f_n, F \rangle \tilde{\tau}_A^{-1} \frac{f_n}{f_{n,1}(0)}, \tilde{\phi}_m \right\rangle. \tag{5.333}$$

Note that the first equality holds as $\langle \tilde{f}_p / \langle \mathcal{I}_\nu, \tilde{\phi}_m \rangle, \tilde{\phi}_m \rangle = 0$ whenever $p \neq m$. Therefore, by property of the Riesz basis $(\tilde{\phi}_m)_{m \in \mathbb{Z}}$, and by continuity and invertibility of $\tilde{\tau}_A$,

$$\tilde{\tau}_A \frac{\tilde{f}_p}{\langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle} = -\tilde{\mu}_p \sum_{n \in \mathbb{Z}} a_{n,p} \langle f_n, F \rangle \frac{f_n}{f_{n,1}(0)}. \tag{5.334}$$

Using the expression of $\tilde{\tau}_A$ given in (5.243), we finally get

$$\overline{\tilde{\phi}_{p,1}(0)} (1 - e^{-2\mu L}) \frac{\tilde{f}_p}{\langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle} = -\tilde{\mu}_p \sum_{n \in \mathbb{Z}} a_{n,p} \langle f_n, F \rangle \frac{f_n}{f_{n,1}(0)}, \tag{5.335}$$

so that, by property of the orthonormal basis $(f_n)_{n \in \mathbb{Z}}$,

$$a_{n,p} = - \frac{\overline{\tilde{\phi}_{p,1}(0)} (1 - e^{-2\mu L})}{\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle} \frac{f_{n,1}(0)}{\langle f_n, F \rangle} \langle \tilde{f}_p, f_n \rangle. \tag{5.336}$$

Let us now compute $\langle h_p, F \rangle$ with truncatures of the h_p :

$$\begin{aligned} h_p^{(N)} &:= \sum_{n=-N}^N a_{n,p} f_n. \\ \langle h_p^{(N)}, F \rangle &= \sum_{n=-N}^N a_{n,p} \langle f_n, F \rangle \\ &= -\frac{\overline{\tilde{\phi}_{p,1}(0)}(1 - e^{-2\mu L})}{\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle} \sum_{n=-N}^N f_{n,1}(0) \langle \tilde{f}_p, f_n \rangle. \end{aligned} \quad (5.337)$$

Now, using Proposition 5.6.1, similar to (5.281), we have,

$$\sum_{n=-N}^N f_{n,1}(0) \langle \tilde{f}_p, f_n \rangle \xrightarrow{N \rightarrow \infty} \frac{\tilde{f}_{p,1}(0) - \tilde{f}_{p,2}(0)}{2}, \quad (5.338)$$

so that

$$\langle h_p, F \rangle = -\frac{\overline{\tilde{\phi}_{p,1}(0)}(1 - e^{-2\mu L})}{\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle} \frac{\tilde{f}_{p,1}(0) - \tilde{f}_{p,2}(0)}{2}. \quad (5.339)$$

Using (5.336) together with (5.339), we get

$$\begin{aligned} \langle -\mathcal{A}h_p + \mathcal{I}_\nu \langle h_p, F \rangle, f_n \rangle &= \frac{\overline{\tilde{\phi}_{p,1}(0)}(1 - e^{-2\mu L})}{\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle} \frac{f_{n,1}(0)}{\langle f_n, F \rangle} \langle \tilde{f}_p, f_n \rangle \mu_n \\ &\quad - \langle \mathcal{I}_\nu, f_n \rangle \frac{\overline{\tilde{\phi}_{p,1}(0)}(1 - e^{-2\mu L})}{\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle} \frac{\tilde{f}_{p,1}(0) - \tilde{f}_{p,2}(0)}{2} \\ &= \frac{\overline{\tilde{\phi}_{p,1}(0)}(1 - e^{-2\mu L})}{\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle} \left(-\frac{\langle \mathcal{I}_\nu, f_n \rangle}{2 \tanh(\mu L) f_{n,1}(0)} \langle \tilde{f}_p, f_n \rangle \mu_n \right. \\ &\quad \left. - \langle \mathcal{I}_\nu, f_n \rangle \frac{\tilde{f}_{p,1}(0) - \tilde{f}_{p,2}(0)}{2} \right). \end{aligned} \quad (5.340)$$

Note that, proceeding as for (5.241), we have

$$\begin{aligned} \tilde{\mu}_p \langle \tilde{f}_p, f_n \rangle &= -\overline{\mu_n} \langle \tilde{f}_p, f_n \rangle + (\tilde{f}_{p,2} \overline{f_{n,2}(0)} - \tilde{f}_{p,1}(0) \overline{f_{n,1}(0)}) \\ &= \mu_n \langle \tilde{f}_p, f_n \rangle - \tilde{f}_{p,1}(0) f_{n,1}(0) (1 - e^{2\mu L}) \end{aligned} \quad (5.341)$$

where we used the boundary conditions given by (5.47) and (5.61) together with (5.51). Thus

$$\langle \tilde{f}_p, f_n \rangle = \frac{-\tilde{f}_{p,1}(0) f_{n,1}(0) (1 - e^{2\mu L})}{\tilde{\mu}_p - \mu_n}. \quad (5.342)$$

Therefore, using again the boundary conditions given by (5.47) and (5.61) and also (5.271), we have

$$\begin{aligned}
\langle -\mathcal{A}h_p + \langle h_p, F \rangle \mathcal{I}_\nu, f_n \rangle &= \frac{\overline{\tilde{\phi}_{p,1}(0)(1 - e^{2\mu L})}}{\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle} \left(-\frac{\langle \mathcal{I}_\nu, f_n \rangle}{2 \tanh(\mu L)} - \frac{\tilde{f}_{p,1}(0)(1 - e^{2\mu L})}{\tilde{\mu}_p - \mu_n} \mu_n \right. \\
&\quad \left. - \langle \mathcal{I}_\nu, f_n \rangle \frac{\tilde{f}_{p,1}(0) - \tilde{f}_{p,2}(0)}{2} \right). \\
&= -\frac{\overline{\tilde{\phi}_{p,1}(0)(1 - e^{2\mu L})}}{\tilde{\mu}_p \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle} \langle \mathcal{I}_\nu, f_n \rangle \\
&\quad \left(\tilde{f}_{p,1}(0) \frac{1 + e^{2\mu L}}{2} \frac{\mu_n}{\tilde{\mu}_p - \mu_n} + \tilde{f}_{p,1}(0) \frac{1 + e^{2\mu L}}{2} \right) \\
&= -\frac{\tilde{f}_{p,1}(0) \overline{\tilde{\phi}_{p,1}(0)(1 - e^{4\mu L})}}{2 \langle \mathcal{I}_\nu, \tilde{\phi}_p \rangle} \frac{\langle \mathcal{I}_\nu, f_n \rangle}{\tilde{\mu}_p - \mu_n}.
\end{aligned} \tag{5.343}$$

This shows, thanks to (5.194) and (5.50), that

$$(-\mathcal{A} + \mathcal{I}_\nu F)h_p \in D(\mathcal{A}). \tag{5.344}$$

Then, we can apply (5.258) to the $(h_p)_{p \in \mathbb{Z}}$, thanks to Proposition 5.6.3:

$$\begin{aligned}
T(-\mathcal{A} + \mathcal{I}_\nu F)h_p &= -\tilde{\mathcal{A}}Th_p \\
&= -\tilde{\mathcal{A}} \frac{\tilde{f}_p}{\tilde{\mu}_p} \\
&= -\tilde{\mu}_p \frac{\tilde{f}_p}{\tilde{\mu}_p}, \quad \forall p \in \mathbb{Z},
\end{aligned} \tag{5.345}$$

so that

$$(-\mathcal{A} + \mathcal{I}_\nu F)h_p = -\tilde{\mu}_p h_p, \quad \forall p \in \mathbb{Z}. \tag{5.346}$$

□

We can now prove the following proposition which gives a precise characterization of the elements of D_F :

Proposition 5.6.4. *The domain D_F satisfies the following equality:*

$$D_F = T^{-1}D(\tilde{\mathcal{A}}^2). \tag{5.347}$$

Proof. Given Lemma 5.6.4, one clearly has the following characterization of D_F :

$$D_F = \left\{ f \in D(\mathcal{A}) : f = \sum_{n \in \mathbb{Z}} f_p h_p, (\tilde{\mu}_p f_p)_{p \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \right\}. \tag{5.348}$$

Then, let $\alpha \in D(\tilde{\mathcal{A}}^2)$, with the following decomposition:

$$\alpha := \sum_{p \in \mathbb{Z}} \alpha_p \frac{\tilde{f}_p}{\tilde{\mu}_p}, \quad (\tilde{\mu}_p \alpha_p)_{p \in \mathbb{Z}} \in \ell^2(\mathbb{Z}). \tag{5.349}$$

Then,

$$T^{-1}\alpha := \sum_{p \in \mathbb{Z}} \alpha_p T^{-1} \frac{\tilde{f}_p}{\tilde{\mu}_p} = \sum_{p \in \mathbb{Z}} \alpha_p h_p. \tag{5.350}$$

Now, by construction of T , $T^{-1}\alpha \in D(\mathcal{A})$, and as $(\tilde{\mu}_p \alpha_p)_{p \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$, it follows from (5.348) that

$$T^{-1}\alpha \in D_F,$$

hence

$$T^{-1}D(\tilde{\mathcal{A}}^2) \subset D_F. \quad (5.351)$$

The converse inclusion is a consequence of the operator equality (5.258). Indeed, let $\alpha \in D_F$, then

$$T\alpha \in D(\tilde{\mathcal{A}}), \quad \tilde{\mathcal{A}}T\alpha = -T(-\mathcal{A} + \mathcal{I}_\nu F)\alpha \in D(\tilde{\mathcal{A}}),$$

so that

$$T\alpha \in D(\tilde{\mathcal{A}}^2),$$

hence,

$$D_F \subset T^{-1}D(\tilde{\mathcal{A}}^2). \quad (5.352)$$

□

Then, we have the following result:

Proposition 5.6.5. *The mapping*

$$\begin{aligned} S : \mathbb{R}^+ &\rightarrow \mathcal{L}(D(\mathcal{A})) \\ t &\mapsto T^{-1}\tilde{S}(t)T \end{aligned} \quad (5.353)$$

defines an exponentially stable C^0 -semigroup on $D(\mathcal{A})$, and its infinitesimal generator is the unbounded operator $(-\mathcal{A} + \mathcal{I}_\nu F, D_F)$.

Proof. By continuity and invertibility of T , (5.353) clearly defines a C^0 -semigroup, and the domain D_S of its infinitesimal generator is clearly $T^{-1}(D(\tilde{\mathcal{A}}^2))$. Now, Proposition 5.6.4 implies that

$$D_S = D_F.$$

Let $\alpha \in D_S$. Then, $T\alpha \in D(\tilde{\mathcal{A}}^2)$ so that, by definition of S , the definition of the infinitesimal generator $\tilde{\mathcal{A}}$ of $\tilde{S}(t)$, and (5.258),

$$\frac{\tilde{S}(t)T\alpha - T\alpha}{t} \xrightarrow[t \rightarrow 0^+]{D(\tilde{\mathcal{A}})} -\tilde{\mathcal{A}}T\alpha = T(-\mathcal{A} + \mathcal{I}_\nu F)\alpha. \quad (5.354)$$

Hence, applying the isomorphism T^{-1} to both sides of (5.354), we get

$$\frac{S(t)\alpha - \alpha}{t} \xrightarrow[t \rightarrow 0^+]{D(\mathcal{A})} (-\mathcal{A} + \mathcal{I}_\nu F)\alpha, \quad (5.355)$$

which proves the second part of the proposition.

Let $\alpha \in D(\mathcal{A})$. Then, using the equivalence of $\|\cdot\|_{D(\tilde{\mathcal{A}})}$ and the $(H^1)^2$ norm, and estimate (5.87) with $p = 1$, we can write

$$\begin{aligned} \|S(t)\alpha\|_{D(\mathcal{A})} &= \|T^{-1}\tilde{S}(t)T\alpha\|_{D(\mathcal{A})} \\ &\leq \|T^{-1}\| \|\tilde{S}(t)T\alpha\|_{D(\tilde{\mathcal{A}})} \\ &\leq C \|T^{-1}\| \|e^{-\frac{3\mu}{4}s} T\alpha\|_{D(\tilde{\mathcal{A}})} \\ &\leq C \|T^{-1}\| \|T\| \|e^{-\frac{3\mu}{4}s} \alpha\|_{D(\mathcal{A})}, \quad \forall t \geq 0. \end{aligned} \quad (5.356)$$

Hence, $S(t), t \geq 0$ is an exponentially stable semi-group. □

This ends the proof of Proposition 5.3.2.

Appendix 5.A Proof of Proposition 5.2.1

Proof. Note that if we conjugate the eigenvalue equation, using the fact that the μ_n are all imaginary, we get, for $n \in \mathbb{Z}$:

$$\begin{aligned} \partial_x \overline{f_{n,1}} + \frac{\delta}{3} \overline{f_{n,2}} &= -\mu_n \overline{f_{n,1}}, \\ -\partial_x \overline{f_{n,2}} - \frac{\delta}{3} \overline{f_{n,1}} &= -\mu_n \overline{f_{n,2}}, \end{aligned} \quad (5.357)$$

which proves (5.51) and the first equality of (5.52). Moreover (5.357) can be written

$$\mathcal{A} \begin{pmatrix} -\overline{f_{n,2}} \\ -\overline{f_{n,1}} \end{pmatrix} = \mu_n \begin{pmatrix} -\overline{f_{n,2}} \\ -\overline{f_{n,1}} \end{pmatrix}. \quad (5.358)$$

Now, as

$$\left\| \begin{pmatrix} -\overline{f_{n,2}} \\ -\overline{f_{n,1}} \end{pmatrix} \right\| = \|f_n\| \quad (5.359)$$

and, according to (5.358), these two functions of $L^2(0, L)^2$ are solutions to the same eigenvalue problem, this means we have

$$\begin{aligned} -\overline{f_{n,2}} &= f_{n,1} \\ \text{or} \\ \overline{f_{n,2}} &= f_{n,1}. \end{aligned} \quad (5.360)$$

Now let us recall that, by (5.168) and (5.169), the f_n are L^∞ -close to the E_n , which satisfy the first relation of (5.360). So for a small enough γ , we have

$$\overline{f_{n,1}} = -f_{n,2}. \quad (5.361)$$

□

Appendix 5.B Proof of Proposition 5.6.1

Proof of Proposition 5.6.1. As [105, Theorem 2] deals with scalar second-order equation, we first define a map R on $L^2(0, L)^2$, gluing the two components to form a function of $L^2(0, L)$ and apply the first order operator twice to recover a second order scalar equation. For $f \in H^1((0, L); \mathbb{R}^2)$ satisfying $f^1(L) = -f^2(L)$ we define $R(f) := \underline{f}$ on $(0, 2L)$ by

$$\underline{f} = \mathbb{1}_{[0, L]} f_1 - \mathbb{1}_{[L, 2L]} f_2(2L - \cdot) \quad (5.362)$$

This is a natural mapping, given the boundary condition $f_1(L) = -f_2(L)$, and defines an isomorphism between $H^1(0, 2L)$ and $\{f \in H^1((0, L); \mathbb{R}^2) | f_1(L) = -f_2(L)\}$. We extend this definition to L^2 functions by density, and the resulting map R is, up to a constant, an isometry from $L^2(0, L)^2 \rightarrow L^2(0, 2L)$ for their usual scalar products.

Now, notice that the (f_p) are also the eigenfunctions of the operator $-\mathcal{A}^2$, for which the following expression can be derived:

$$\begin{aligned} -(\Lambda \partial_x + \delta(x)J)^2 &= -\partial_x^2 - \delta(x)^2 J^2 - \Lambda \partial_x (\delta(x)J) - \delta(x)J \Lambda \partial_x \\ &= -\partial_x^2 - \delta(x)^2 J^2 - \delta'(x)\Lambda J - \delta(x)\Lambda J \partial_x - \delta(x)J \Lambda \partial_x \\ &= -\partial_x^2 - \delta(x)^2 J^2 - \delta'(x)\Lambda J, \end{aligned} \quad (5.363)$$

the last equality being obtained thanks to the relation

$$\begin{aligned} \Lambda J + J \Lambda &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1/3 \\ -1/3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1/3 \\ -1/3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1/3 \\ 1/3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1/3 \\ -1/3 & 0 \end{pmatrix} \\ &= 0. \end{aligned} \quad (5.364)$$

Hence,

$$-\mathcal{A}^2 = -\partial_x^2 + \frac{1}{3} \begin{pmatrix} \delta(x)^2/3 & -\delta'(x) \\ -\delta'(x) & \delta(x)^2/3 \end{pmatrix}, \quad (5.365)$$

and

$$-\mathcal{A}^2 R^{-1} \underline{f}_k = -\mu_k^2 R^{-1} \underline{f}_k$$

i.e.

$$-R\mathcal{A}^2 R^{-1} \underline{f}_k = -\mu_k^2 \underline{f}_k. \quad (5.366)$$

Now, it is clear from (5.362) that

$$R\Lambda\partial_x\alpha = \partial_x(R\alpha), \quad \forall \alpha \in H^1, \quad \alpha_1(L) = -\alpha_2(L). \quad (5.367)$$

Also, for $a \in L^2(0, L)$, we define

$$\underline{a} := R \begin{pmatrix} a \\ -a \end{pmatrix}. \quad (5.368)$$

and we have

$$R(af) = \underline{a}f, \quad \forall f \in L^2(0, L)^2. \quad (5.369)$$

Finally,

$$R \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} f \right) = \mathbb{1}_{[0, L]} f_2 - \mathbb{1}_{[L, 2L]} f_1(2L - \cdot) = -\underline{f}(2L - \cdot). \quad (5.370)$$

From (5.367), (5.369), and (5.370), (5.366) becomes

$$-\partial_x^2 \underline{f}_k + \frac{\delta^2}{9} \underline{f}_k + \frac{(\delta')}{3} \underline{f}_k(2L - \cdot) = -\mu_k^2 \underline{f}_k \quad (5.371)$$

so that, using (5.370),

$$\underline{f}_k(2L - x) = \overline{f}_k(x), \quad \forall x \in (0, 2L), \quad (5.372)$$

and we finally get

$$-\partial_x^2 \overline{f}_k + \frac{\delta^2}{9} \overline{f}_k + \frac{(\delta')}{3} \overline{f}_k = -\mu_k^2 \overline{f}_k \quad (5.373)$$

Thus $(\underline{f}_p)_{p \in \mathbb{Z}}$ is a family of eigenvectors of the operator \mathcal{L} defined by

$$\mathcal{L}u = -\partial_x^2 u + \frac{1}{9} \delta^2 u + \frac{1}{3} (\delta)' \overline{u}, \quad (5.374)$$

with eigenvalues $-\lambda_p^2$ which, from (5.51), are real and nonnegative. Besides, $(\underline{f}_p)_{p \in \mathbb{Z}}$ is still an orthonormal basis of $L^2(0, 2L)$. Observe that $(\underline{E}_p)_{p \in \mathbb{Z}} = (e^{i\pi p x/L})_{p \in \mathbb{Z}}$ and is an orthonormal basis of $L^2(0, 2L)$ and a family of eigenvectors of the operator $\mathcal{L}_0 = -\partial_x^2$ with associated eigenvalues $(\pi^2 p^2 / L^2)_{p \in \mathbb{Z}}$. Note that \mathcal{L} can be written as

$$\mathcal{L}u = -\partial_x^2 u - qu - q_1 u(2L - \cdot) \quad (5.375)$$

where q and q_1 are both L^1 (and in fact C^∞) functions on $(0, 2L)$. We now extend periodically the functions \underline{f}_p , \underline{E}_p , q and q_1 on $(-L, 3L)$ as follows :

$$\begin{aligned} f_p(\cdot) &= f_p(\cdot + 2L), \quad \text{on } (-L, 0), \\ f_p(\cdot) &= f_p(\cdot - 2L), \quad \text{on } (2L, 3L), \end{aligned} \quad (5.376)$$

and similarly for \underline{E}_p , q and q_1 . As $f_p(2L) = f_p(0)$ from (5.362) and (5.47), the functions f_p thus constructed are continuous on $[-2L, 2L]$. Besides,

$$\|q\|_{L^1(-L, 3L)} = 2\|q\|_{L^1(0, 2L)}, \quad \|q_1\|_{L^1(-L, 3L)} = 2\|q_1\|_{L^1(0, 2L)}. \quad (5.377)$$

Then, let a compact interval $K \subset [0, L)$ and consider the restriction of \underline{f}_p to K , one can easily see from (5.362) that its gluing map corresponds to the restriction of \underline{f}_p on $\{x \in [0, 2L) | x \in K \text{ or } 2L - x \in K\}$ which is a compact set symmetrical with respect to L and with two connected components. This means that, in order to end the proof of Proposition 5.6.1, it suffices to show the same type of result as [105, Theorem 2] applied to $(\underline{f}_p)_{p \in \mathbb{Z}}$ and $(\underline{E}_p)_{p \in \mathbb{Z}}$, but on compacts of $(-L, 3L)$, symmetrical with respect to L with two connected components only. Observe that [105, Theorem 2] is only given for compact interval but is also true for compacts with finite number of connected components. Now, the two only differences between our case with [105, Theorem 2] are that \mathcal{L} has a non-local term which is the third term $q_1 u(2L - \cdot)$ in the right-hand side of (5.375), and that the \underline{f}_p are continuous but their derivatives are not always continuous and have discontinuities on $\mathcal{D} := \{x = 0, x = 2L\}$. Observe, however, that in the proof of [105, Theorem 2], the fact that \underline{f}_p are eigenvectors of \mathcal{L} is only used through the Titchmarsh formula [105], and note that a generalized Titchmarsh formula still holds for this operator and we have, for $p \in \mathbb{Z}$, and $x \in (-L, 3L)$, $t \in (0, \min(|3L - x|, |x + L|))$

$$\begin{aligned}
\underline{f}_p(x+t) + \underline{f}_p(x-t) &= 2\underline{f}_p(x) \cos(\sqrt{-\mu_p^2}t) \\
&+ \int_{x-t}^{x+t} q(\xi) \underline{f}_p(\xi) \frac{\sin(\sqrt{-\mu_p^2}(t - |x - \xi|))}{\sqrt{-\mu_p^2}} d\xi \\
&+ \int_{x-t}^{x+t} q_1(\xi) \underline{f}_p(2L - \xi) \frac{\sin(\sqrt{-\mu_p^2}(t - |x - \xi|))}{\sqrt{-\mu_p^2}} d\xi \\
&+ \sum_{x_1 \in (x-t, x+t) \cap \mathcal{D}} \frac{\sin(\sqrt{-\mu_p^2}(t - |x - x_1|))}{\sqrt{-\mu_p^2}} (\underline{f}_p'(x_1^+) - \underline{f}_p'(x_1^-)).
\end{aligned} \tag{5.378}$$

As expected, the two differences with [105, Theorem 2] are now translated in the appearance of the third and fourth terms that do not appear in the case of [105, Theorem 2]. Now observe that in the proof of [105, Theorem 2], the second term of the right-hand side is each time bounded using the L^∞ norm of \underline{f}_p and $Q = \|q\|_{L^1(0,L)}$. Thus, to adapt the proof of [105, Theorem 2], all we have to do is to provide the same type of bounds at each step for the third and fourth terms. As for any compact $K \subset (-L, 3L)$ symmetrical with respect to L , $\|\underline{f}_p\|_{L^\infty(K)} = \|\underline{f}_p(2L - \cdot)\|_{L^\infty(K)}$, the same bounds hold in our case for the third term replacing Q by $\|q\|_{L^1(-L,3L)} + \|q_1\|_{L^1(-L,3L)}$.¹ To deal with the fourth term, we need to study the jump discontinuities $\underline{f}_p'(x_1^+) - \underline{f}_p'(x_1^-)$. From the definition of \underline{f}_p , one has

$$\begin{aligned}
\partial_x \underline{f}_p - \frac{\delta}{3} \underline{f}_p(2L - x) + \mu_p \underline{f}_p &= 0 \text{ for } x \in (0, L) \cup (2L, 3L), \\
\partial_x \underline{f}_p + \frac{\delta}{3} \underline{f}_p(2L - x) + \mu_p \underline{f}_p &= 0 \text{ for } x \in (-L, 0) \cup (L, 2L).
\end{aligned} \tag{5.379}$$

Thus for $x_1 \in \mathcal{D}$, one has

$$\underline{f}_p'(x_1^+) - \underline{f}_p'(x_1^-) = 2\delta(x_1) \underline{f}_p(x_1) \tag{5.380}$$

which implies, for $x_1 \in K$, that

$$\left| \underline{f}_p'(x_1^+) - \underline{f}_p'(x_1^-) \right| \leq C \|\underline{f}_p\|_{L^\infty(K)}, \tag{5.381}$$

where C is a constant independent of p . With this in mind, and noting that for any compact K , $K \cap \mathcal{D}$ has a finite cardinal $\mathcal{N}_K \leq 2$ depending only on K , we can bound the fourth term of (5.378) as in [105, Theorem 2]. More precisely we have :

¹For the sake of rigor let us note that when $K = [a, b]$ is symmetrical with respect to L , $K_R = [a - R, b + R]$ is also symmetrical with respect to L .

1. For Lemma 1 of [105], with $R > 0$ such that $x + 2R \in K$, integrating the fourth term on $(0, R)$ gives

$$\begin{aligned} & \left| \int_0^R \sum_{x_1 \in (x-t, x+t) \cap \mathcal{D}} \frac{\sin(\sqrt{-\mu_p^2}(t - |x - x_1|))}{\sqrt{-\mu_p^2}} (\underline{f}_p'(x_1^+) - \underline{f}_p'(x_1^-)) dt \right| \\ & \leq AC \|\underline{f}_p\|_{L^\infty(K)} \int_0^R \sum_{x_1 \in (x-t, x+t) \cap \mathcal{D}} |t - |x - x_1|| dt \\ & \leq AC \|\underline{f}_p\|_{L^\infty(K)} \mathcal{N}_K R^2, \end{aligned} \quad (5.382)$$

where we used in the last line that $x_1 \in (x - t, x + t)$ and where $A = 1$ but corresponds to the constant A in [105]. Thus, this bound is similar to the bound obtained in [105] for the second term of (5.378) replacing Q with $Q = C\mathcal{N}_K$. The proof of the first part a) of Theorem 1 in [105] follows directly.

2. For the part b) of Theorem 1 in [105], one has, with $x \in K$ and K_R is the compact extension of K given by $\{x \in \mathcal{B}_y(R) | y \in K\}$

$$\begin{aligned} & \left| \int_0^R \cos(\mu t) \sum_{x_1 \in (x-t, x+t) \cap \mathcal{D}} \frac{\sin(\sqrt{-\mu_p^2}(t - |x - x_1|))}{\sqrt{-\mu_p^2}} (f_p'(x_1^+) - f_p'(x_1^-)) dt \right| \\ & \leq C \|\underline{f}_p\|_{L^\infty(K_R)} \int_0^R \sum_{x_1 \in (x-t, x+t) \cap \mathcal{D}} \left| \frac{\sin(\sqrt{-\mu_p^2}(t - |x - x_1|))}{\sqrt{-\mu_p^2}} \right| dt \end{aligned} \quad (5.383)$$

Now, for $t \in [0, R]$ and $x_1 \in (x - t, x + t)$, we have $(t - |x - x_1|) \in [0, R]$, which implies that

$$\left| \frac{\sin(\sqrt{-\mu_p^2}(t - |x - x_1|))}{\sqrt{-\mu_p^2}} \right| \leq 2 \frac{R + 1}{1 + |\sqrt{-\mu_p^2}|}. \quad (5.384)$$

Thus, (5.383) becomes

$$\begin{aligned} & \left| \int_0^R \cos(\mu t) \sum_{x_1 \in (x-t, x+t) \cap \mathcal{D}} \frac{\sin(\sqrt{-\mu_p^2}(t - |x - x_1|))}{\sqrt{-\mu_p^2}} (f_p'(x_1^+) - f_p'(x_1^-)) dt \right| \\ & \leq C \|\underline{f}_p\|_{L^\infty(K_R)} 2 \frac{R + 1}{1 + |\sqrt{-\mu_p^2}|} \mathcal{N}_K R \end{aligned} \quad (5.385)$$

and we have once again a bound similar to the bound obtained in [105] for the second term of (5.378) with $Q = C\mathcal{N}_K$.

3. Finally, one can do the same with the proof of Theorem 2 by noting that Lemma 3 and 4 still holds identically and that

$$\begin{aligned} & \int_0^R \frac{\sin(\mu t)}{\pi t} \sum_{x_1 \in (x-t, x+t) \cap \mathcal{D}} \frac{\sin(\sqrt{-\mu_p^2}(t - |x - x_1|))}{\sqrt{-\mu_p^2}} (f_p'(x_1^+) - f_p'(x_1^-)) dt \\ & = \sum_{\{|x - x_1| \leq R\} \cap \mathcal{D}} \int_{|x - x_1|}^R \frac{\sin(\mu t)}{\pi t} \frac{\sin(\sqrt{-\mu_p^2}(t - |x - x_1|))}{\sqrt{-\mu_p^2}} (f_p'(x_1^+) - f_p'(x_1^-)) dt. \end{aligned} \quad (5.386)$$

Thus, we can overall apply the results of [105] by replacing Q with $Q = 2(\|q\|_{L^1(0,2L)} + \|q_1\|_{L^1(0,2L)}) + C\mathcal{N}_K$. Thus, we still have from [105, Theorem 2]

$$\lim_{\mu \rightarrow +\infty} \sup_{x \in K} \left| \sum_{|Im(\mu_p)| < \mu} \langle f, \underline{f_p} \rangle \underline{f_p}(x) - \sum_{|Im(\mu_p^{(0)})| < \mu} \langle f, \underline{E_p} \rangle \underline{E_p}(x) \right| = 0. \quad (5.387)$$

Choosing now the compact $K = [-L/2, 5L/2] \in (-L, 3L)$ and symmetrical with respect to L , one can see that for any $x \in [0, L]$, $x \in K$ and $2L - x \in K$. Thus

$$\lim_{\mu \rightarrow +\infty} \sup_{x \in K} |\sigma_\mu(f, x) - p_\mu(f, x)| = 0. \quad (5.388)$$

This ends the proof of Lemma 5.6.1 □

Appendix 5.C Expression of the feedback coefficients before and after variable changes

In this appendix, we justify the form of (5.11) from (5.94) and (5.271). Let $(h_n, v_n)_{n \in \mathbb{Z}}$ be the Riesz basis of $(L^2)^2$ of eigenvectors associated to the problem (5.5)–(5.6). Let us now define

$$S = \begin{pmatrix} \sqrt{\frac{1}{H\gamma}} & 1 \\ -\sqrt{\frac{1}{H\gamma}} & 1 \end{pmatrix} \quad (5.389)$$

and

$$r : z \rightarrow L_\gamma z / L - \frac{\gamma L_\gamma^2}{4L^2} z^2, \quad (5.390)$$

which is a bijection from $[0, L]$ to $[0, L]$. As the transformations (5.20) and (5.35) and the scaling introduced in (5.25) and (5.30) define a diffeomorphism, the family $(f_n)_{n \in \mathbb{Z}}$ given by

$$f_n = \exp\left(\int_0^z \delta(x) dx\right) S(r(\cdot)) \begin{pmatrix} h_n \\ v_n \end{pmatrix} (r(\cdot)), \quad (5.391)$$

is also a Riesz basis of $(L^2)^2$ but this time of eigenvectors of \mathcal{A} given by (5.46). As \mathcal{A} is anti-adjoint this basis is orthogonal, and we can form an orthonormal basis that we denote again by (f_n) . This basis is therefore suitable to apply the results of Section 5.4–5.6. Thus the exponential stability holds provided that $F \in \mathcal{L}(D(\mathcal{A}); \mathbb{R})$ satisfies (5.271), which we are now going to show using (5.11) and (5.94). Let be a feedback $F \in \mathcal{E}'$, one has from (5.94) and (5.20)–(5.37)

$$u(L_\gamma t / L) = (\langle \zeta(t, \cdot), F \rangle + \zeta_0 \langle f_0, F \rangle), \quad (5.392)$$

and

$$\langle \zeta(t, \cdot), F \rangle = \left\langle \exp\left(\int_0^z \delta(x) dx\right) S(r(\cdot)) \begin{pmatrix} h \\ v \end{pmatrix} (L_\gamma t / L, r(\cdot)), F \right\rangle. \quad (5.393)$$

Thus there exists $F_1 \in \mathcal{D}'_\gamma$ such that

$$\langle \zeta(L / L_\gamma t, \cdot), F \rangle = \left\langle \begin{pmatrix} h \\ v \end{pmatrix} (t, \cdot), F_1 \right\rangle, \quad (5.394)$$

and therefore from (5.391)

$$\langle f_n, F \rangle = \left\langle \begin{pmatrix} h_n \\ v_n \end{pmatrix}, F_1 \right\rangle. \quad (5.395)$$

Now if F_1 satisfies (5.11), one has using (5.389) and (5.391)

$$\langle f_n, F \rangle = -2 \tanh(\mu L) \frac{(f_{n,1}(0))^2}{\int_0^L f_{n,1}(x) + f_{n,2}(x) dx}, \quad \forall n \in \mathbb{Z}, \quad (5.396)$$

which is exactly (5.271), noting that $f_{n,1}$ is real from (5.52). Finally, as we restrict ourselves to solutions of the system (5.59), (5.38) with $\zeta_0(0) = 0$ (see (5.97)), and from (5.394), the control under the form (5.10) corresponds exactly to (5.392).

Finally, to get the final exponential decay rate for (5.5)–(5.6) with feedback F_1^γ , let us recall that we operated a scaling in time in (5.30), so the decay rate is

$$\frac{3}{4} \mu \frac{L}{L^\gamma} \xrightarrow{\gamma \rightarrow 0} \frac{3}{4} \mu. \quad (5.397)$$

In particular, for $\gamma > 0$ small enough,

$$\frac{3}{4} \mu \frac{L}{L^\gamma} \geq \frac{\mu}{2}, \quad (5.398)$$

which gives us the decay rate of Theorem 5.1.1.

Remark 5.C.1. *Note that the conditions (5.11) and (5.271) remain the same when the basis under consideration is renormalized.*

Bibliography

- [1] Fatiha Alabau, Piermarco Cannarsa, and Vilmos Komornik. Indirect internal stabilization of weakly coupled evolution equations. J. Evol. Equ., 2(2):127–150, 2002.
- [2] Fatiha Alabau-Boussouira. On some recent advances on stabilization for hyperbolic equations. In Control of partial differential equations, volume 2048 of Lecture Notes in Math., pages 1–100. Springer, Heidelberg, 2012.
- [3] Fatiha Alabau-Boussouira. A hierarchic multi-level energy method for the control of bidiagonal and mixed n -coupled cascade systems of PDE's by a reduced number of controls. Adv. Differential Equations, 18(11-12):1005–1072, 2013.
- [4] Fatiha Alabau-Boussouira. Insensitizing exact controls for the scalar wave equation and exact controllability of 2-coupled cascade systems of PDE's by a single control. Math. Control Signals Systems, 26(1):1–46, 2014.
- [5] Fatiha Alabau-Boussouira, Jean-Michel Coron, and Guillaume Olive. Internal Controllability of First Order Quasi-linear Hyperbolic Systems with a Reduced Number of Controls. SIAM J. Control Optim., 55(1):300–323, 2017.
- [6] Fatiha Alabau-Boussouira and Matthieu Léautaud. Indirect controllability of locally coupled systems under geometric conditions. C. R. Math. Acad. Sci. Paris, 349(7-8):395–400, 2011.
- [7] Fatiha Alabau-Boussouira and Matthieu Léautaud. Indirect stabilization of locally coupled wave-type systems. ESAIM Control Optim. Calc. Var., 18(2):548–582, 2012.
- [8] Fatiha Alabau-Boussouira and Matthieu Léautaud. Indirect controllability of locally coupled wave-type systems and applications. J. Math. Pures Appl. (9), 99(5):544–576, 2013.
- [9] Fatiha Alabau-Boussouira, Zhiqiang Wang, and Lixin Yu. A one-step optimal energy decay formula for indirectly nonlinearly damped hyperbolic systems coupled by velocities. ESAIM Control Optim. Calc. Var., 23(2):721–749, 2017.
- [10] Serge Alinhac and Patrick Gérard. Opérateurs pseudo-différentiels et théorème de Nash-Moser. Savoirs Actuels. [Current Scholarship]. InterEditions, Paris; Éditions du Centre National de la Recherche Scientifique (CNRS), Meudon, 1991.
- [11] Faouzia Aloui, Imen Ben Hassen, and Alain Haraux. Compactness of trajectories to some nonlinear second order evolution equations and applications. J. Math. Pures Appl. (9), 100(3):295–326, 2013.
- [12] Farid Ammar-Khodja, Assia Benabdallah, Cédric Dupaix, and Manuel González-Burgos. A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems. J. Evol. Equ., 9(2):267–291, 2009.
- [13] Jean Auriol, Federico Bribiesca-Argomedo, David Bou Saba, Michael Di Loreto, and Florent Di Meglio. Delay-robust stabilization of a hyperbolic PDE-ODE system. Automatica J. IFAC, 95:494–502, 2018.

- [14] Jean Auriol and Florent Di Meglio. Two-sided boundary stabilization of heterodirectional linear coupled hyperbolic PDEs. IEEE Trans. Automat. Control, 63(8):2421–2436, 2018.
- [15] Sergei A. Avdonin and Sergei A. Ivanov. Families of exponentials. Cambridge University Press, Cambridge, 1995. The method of moments in controllability problems for distributed parameter systems, Translated from the Russian and revised by the authors.
- [16] Pietro Baldi and Emanuele Haus. A Nash-Moser-Hörmander implicit function theorem with applications to control and Cauchy problems for PDEs. J. Funct. Anal., 273(12):3875–3900, 2017.
- [17] Andras Balogh and Miroslav Krstić. Infinite Dimensional Backstepping-Style Feedback Transformations for a Heat Equation with an Arbitrary Level of Instability. European Journal of Control, 8(2):165–175, 2002.
- [18] Claude Bardos, Gilles Lebeau, and Jeffrey Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J. Control Optim., 30(5):1024–1065, 1992.
- [19] Adhémar Barré de Saint-Venant. Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et à l’introduction des marées dans leur lit. Comptes Rendus de l’Académie des Sciences, 53:147–154, 1871.
- [20] Adhémar Barré de Saint-Venant. Mémoire sur la perte de force vive d’un fluide aux endroits où sa section d’écoulement augmente brusquement ou rapidement. Mémoires de l’Académie des Sciences de l’Institut de France, 44:193–243, 1888.
- [21] Adhémar Barré de Saint-Venant. Mémoire sur la prise en considération de la force centrifuge dans le calcul du mouvement des eaux courantes et sur la distinction des torrents et des rivières. Mémoires de l’Académie des Sciences de l’Institut de France, 44:245–273, 1888.
- [22] Georges Bastin and Jean-Michel Coron. On boundary feedback stabilization of non-uniform linear 2×2 hyperbolic systems over a bounded interval. Systems Control Lett., 60(11):900–906, 2011.
- [23] Georges Bastin and Jean-Michel Coron. Stability and Boundary Stabilisation of 1-D Hyperbolic Systems. Number 88 in Progress in Nonlinear Differential Equations and Their Applications. Springer International, 2016.
- [24] Georges Bastin and Jean-Michel Coron. Stability and boundary stabilization of 1-D hyperbolic systems, volume 88 of Progress in Nonlinear Differential Equations and their Applications. Birkhäuser/Springer, [Cham], 2016. Subseries in Control.
- [25] Georges Bastin and Jean-Michel Coron. A quadratic Lyapunov function for hyperbolic density-velocity systems with nonuniform steady states. Systems & Control Letters, 104:66–71, 2017.
- [26] Georges Bastin and Jean-Michel Coron. Exponential stability of PI control for Saint-Venant equations with a friction term. working paper or preprint, December 2018.
- [27] Georges Bastin, Jean-Michel Coron, and Brigitte d’Andréa Novel. On Lyapunov stability of linearised Saint-Venant equations for a sloping channel. Networks and Heterogeneous Media, 4(2):177–187, 2009.
- [28] Karine Beauchard. Local controllability of a 1-D Schrödinger equation. J. Math. Pures Appl. (9), 84(7):851–956, 2005.
- [29] Karine Beauchard and Jean-Michel Coron. Controllability of a quantum particle in a moving potential well. J. Funct. Anal., 232(2):328–389, 2006.

- [30] Karine Beauchard and Camille Laurent. Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control. J. Math. Pures Appl. (9), 94(5):520–554, 2010.
- [31] Beauchard, Karine. Controllability of a quantum particle in a 1D variable domain. ESAIM: COCV, 14(1):105–147, 2008.
- [32] Dejan M. Bosković, Andras Balogh, and Miroslav Krstić. Backstepping in infinite dimension for a class of parabolic distributed parameter systems. Math. Control Signals Systems, 16(1):44–75, 2003.
- [33] Dejan M. Bošković, Miroslav Krstić, and Weijiu Liu. Boundary control of an unstable heat equation via measurement of domain-averaged temperature. IEEE Trans. Automat. Control, 46(12):2022–2028, 2001.
- [34] Haim Brezis. Analyse fonctionnelle. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master’s Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].
- [35] Haim Brezis. Functional analysis, Sobolev spaces and partial differential equations. Universitext. Springer, New York, 2011.
- [36] R. W. Brockett. Asymptotic stability and feedback stabilization. In Differential Geometric Control Theory, pages 181–191. Birkhauser, 1983.
- [37] Piermarco Cannarsa, Vilmos Komornik, and Paola Loreti. Controllability of semilinear wave equations with infinitely iterated logarithms. Control Cybernet., 28(3):449–461, 1999. Recent advances in control of PDEs.
- [38] Piermarco Cannarsa, Vilmos Komornik, and Paola Loreti. One-sided and internal controllability of semilinear wave equations with infinitely iterated logarithms. Discrete Contin. Dyn. Syst., 8(3):745–756, 2002.
- [39] Eduardo Cerpa and Jean-Michel Coron. Rapid stabilization for a Korteweg-de Vries equation from the left Dirichlet boundary condition. IEEE Trans. Automat. Control, 58(7):1688–1695, 2013.
- [40] Alicia Chang. An algebraic characterization of controllability. IEEE Transactions on Automatic Control, 10(1):112–113, January 1965.
- [41] Marianne Chapouly. Global controllability of nonviscous Burgers type equations. C. R. Math. Acad. Sci. Paris, 344(4):241–246, 2007.
- [42] Marianne Chapouly. Global controllability of nonviscous and viscous Burgers-type equations. SIAM J. Control Optim., 48(3):1567–1599, 2009.
- [43] William C. Chewning. Controllability of the nonlinear wave equation in several space variables. SIAM J. Control Optimization, 14(1):19–25, 1976.
- [44] Ole Christensen. An introduction to frames and Riesz bases. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, [Cham], second edition, 2016.
- [45] B. M. N. Clarke and D. Williamson. Control canonical forms and eigenvalue assignment by feedback for a class of linear hyperbolic systems. SIAM J. Control Optim., 19(6):711–729, 1981.
- [46] J.-M. Coron. On the controllability of 2-D incompressible perfect fluids. J. Math. Pures Appl. (9), 75(2):155–188, 1996.
- [47] Jean-Michel Coron. Global asymptotic stabilization for controllable systems without drift. Math. Control Signals Systems, 5(3):295–312, 1992.

- [48] Jean-Michel Coron. Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles bidimensionnels. C. R. Acad. Sci. Paris Sér. I Math., 317(3):271–276, 1993.
- [49] Jean-Michel Coron. Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles bidimensionnels. C.R. Acad. Sci. Paris, 317:271–276, 1993.
- [50] Jean-Michel Coron. On the stabilization in finite time of locally controllable systems by means of continuous time-varying feedback law. SIAM J. Control Optim., 33(3):804–833, 1995.
- [51] Jean-Michel Coron. On the controllability of 2-D incompressible perfect fluids. J. Math. Pures Appl. (9), 75(2):155–188, 1996.
- [52] Jean-Michel Coron. Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations. ESAIM Control Optim. Calc. Var., 8:513–554, 2002. A tribute to J. L. Lions.
- [53] Jean-Michel Coron. Control and nonlinearity, volume 136 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2007.
- [54] Jean-Michel Coron, Georges Bastin, and Brigitte d'Andréa Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. SIAM Journal on Control and Optimization, 47(3):1460–1498, 2008.
- [55] Jean-Michel Coron, Georges Bastin, and Brigitte d'Andréa Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. SIAM J. Control Optim., 47(3):1460–1498, 2008.
- [56] Jean-Michel Coron and Brigitte d'Andréa Novel. Stabilization of a rotating body beam without damping. IEEE Trans. Automat. Control, 43(5):608–618, 1998.
- [57] Jean-Michel Coron, Brigitte d'Andréa Novel, and Georges Bastin. A Lyapunov approach to control irrigation canals modeled by Saint-Venant equations. In CD-Rom Proceedings, Paper F1008-5, ECC99, Karlsruhe, Germany, pages 3178–3183, 1999.
- [58] Jean-Michel Coron, Brigitte d'Andréa Novel, and Georges Bastin. A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. IEEE Transactions on Automatic Control, 52(1):2–11, 2007.
- [59] Jean-Michel Coron, Ludovick Gagnon, and Morgan Morancey. Rapid stabilization of a linearized bilinear 1-D Schrödinger equation. J. Math. Pures Appl. (9), 115:24–73, 2018.
- [60] Jean-Michel Coron, Sergio Guerrero, and Lionel Rosier. Null controllability of a parabolic system with a cubic coupling term. SIAM J. Control Optim., 48(8):5629–5653, 2010.
- [61] Jean-Michel Coron and Jean-Philippe Guilleron. Control of three heat equations coupled with two cubic nonlinearities. working paper or preprint, November 2016.
- [62] Jean-Michel Coron, Long Hu, and Guillaume Olive. Stabilization and controllability of first-order integro-differential hyperbolic equations. J. Funct. Anal., 271(12):3554–3587, 2016.
- [63] Jean-Michel Coron, Long Hu, and Guillaume Olive. Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation. Automatica J. IFAC, 84:95–100, 2017.
- [64] Jean-Michel Coron and Pierre Lissy. Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components. Invent. Math., 198(3):833–880, 2014.
- [65] Jean-Michel Coron and Qi Lü. Local rapid stabilization for a Korteweg-de Vries equation with a Neumann boundary control on the right. J. Math. Pures Appl. (9), 102(6):1080–1120, 2014.

- [66] Jean-Michel Coron and Qi Lü. Fredholm transform and local rapid stabilization for a Kuramoto-Sivashinsky equation. J. Differential Equations, 259(8):3683–3729, 2015.
- [67] Jean-Michel Coron and Hoai-Minh Nguyen. Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. Arch. Ration. Mech. Anal., 225(3):993–1023, 2017.
- [68] Jean-Michel Coron, Rafael Vazquez, Miroslav Krstić, and Georges Bastin. Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping. SIAM J. Control Optim., 51(3):2005–2035, 2013.
- [69] Florent Di Meglio, Federico Bribiesca Argomedo, Long Hu, and Miroslav Krstic. Stabilization of coupled linear heterodirectional hyperbolic PDE-ODE systems. Automatica J. IFAC, 87:281–289, 2018.
- [70] Ababacar Diagne, Mamadou Diagne, Shuxia Tang, and Miroslav Krstic. Backstepping stabilization of the linearized *saint-venant-exner* model. Automatica J. IFAC, 76:345–354, 2017.
- [71] Ababacar Diagne, Shuxia Tang, Mamadou Diagne, and Miroslav Krstic. State feedback stabilization of the linearized bilayer *saint-venant* model. IFAC-PapersOnLine, 49(8):130–135, 2016.
- [72] Szymon Dolecki and David L. Russell. A general theory of observation and control. SIAM J. Control Optimization, 15(2):185–220, 1977.
- [73] Valérie Dos Santos and Christophe Prieur. Boundary control of open channels with numerical and experimental validations. IEEE transactions on Control systems technology, 16(6):1252–1264, 2008.
- [74] Valérie Dos Santos, Georges Bastin, Jean-Michel Coron, and B. d’Andréa Novel. Boundary control with integral action for hyperbolic systems of conservation laws: stability and experiments. Automatica J. IFAC, 44(5):1310–1318, 2008.
- [75] François Dubois, Nicolas Petit, and Pierre Rouchon. Motion planning and nonlinear simulations for a tank containing a fluid. In 1999 European Control Conference (ECC), pages 3232–3237. IEEE, 1999.
- [76] Michel Duprez and Pierre Lissy. Indirect controllability of some linear parabolic systems of m equations with $m - 1$ controls involving coupling terms of zero or first order. J. Math. Pures Appl. (9), 106(5):905–934, 2016.
- [77] Michel Duprez and Pierre Lissy. Positive and negative results on the internal controllability of parabolic equations coupled by zero- and first-order terms. J. Evol. Equ., 18(2):659–680, 2018.
- [78] H. O. Fattorini. Local controllability of a nonlinear wave equation. Math. Systems Theory, 9(1):30–45, 1975.
- [79] Xiaoyu Fu, Jiongmin Yong, and Xu Zhang. Exact controllability for multidimensional semilinear hyperbolic equations. SIAM J. Control Optim., 46(5):1578–1614, 2007.
- [80] J. S. Gibson. A note on stabilization of infinite-dimensional linear oscillators by compact linear feedback. SIAM J. Control Optim., 18(3):311–316, 1980.
- [81] Olivier Glass. Contrôlabilité exacte frontière de l’équation d’Euler des fluides parfaits incompressibles en dimension 3. C. R. Acad. Sci. Paris Sér. I Math., 325(9):987–992, 1997.
- [82] Olivier Glass. Exact boundary controllability of 3-D Euler equation. ESAIM Control Optim. Calc. Var., 5:1–44, 2000.
- [83] Manuel González-Burgos and Rosario Pérez-García. Controllability of some coupled parabolic systems by one control force. Comptes Rendus Mathématique, 340(2):125–130, 2005.

- [84] Mikhael Gromov. Partial differential relations, volume 9 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1986.
- [85] Matthias Günther. On the perturbation problem associated to isometric embeddings of Riemannian manifolds. Ann. Global Anal. Geom., 7(1):69–77, 1989.
- [86] Matthias Günther. Isometric embeddings of Riemannian manifolds. In Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), pages 1137–1143. Math. Soc. Japan, Tokyo, 1991.
- [87] C. Guo, C. Xie, and C. Zhou. Stabilization of a spatially non-causal reaction-diffusion equation by boundary control. Internat. J. Robust Nonlinear Control, 24(1):1–17, 2014.
- [88] Scott W. Hansen and Oleg Imanuvilov. Exact controllability of a multilayer Rao-Nakra plate with clamped boundary conditions. ESAIM Control Optim. Calc. Var., 17(4):1101–1132, 2011.
- [89] Alain Haraux and Mohamed Ali Jendoubi. A Liapunov function approach to the stabilization of second order coupled systems. North-West. Eur. J. Math., 2:121–144, i, 2016.
- [90] Philip Hartman. Ordinary differential equations, volume 38 of Classics in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Corrected reprint of the second (1982) edition [Birkhäuser, Boston, MA; MR0658490 (83e:34002)], With a foreword by Peter Bates.
- [91] Amaury Hayat. Exponential stability of general 1-D quasilinear systems with source terms for the C^1 norm under boundary conditions. preprint, October 2017.
- [92] Amaury Hayat. On boundary stability of inhomogeneous 2×2 1-D hyperbolic systems for the C^1 norm. preprint, July 2018.
- [93] Amaury Hayat. PI controllers for the general Saint-Venant equations. working paper or preprint, January 2019.
- [94] Amaury Hayat and Peipei Shang. Exponential stability of density-velocity systems with boundary conditions and source term for the h^2 norm. working paper or preprint, 2019.
- [95] Amaury Hayat and Peipei Shang. A quadratic Lyapunov function for Saint-Venant equations with arbitrary friction and space-varying slope. Automatica J. IFAC, 100:52–60, 2019.
- [96] Lars Hörmander. On the Nash-Moser implicit function theorem. Ann. Acad. Sci. Fenn. Ser. A I Math., 10:255–259, 1985.
- [97] Lars Hörmander. The Nash-Moser theorem and paradifferential operators. In Analysis, et cetera, pages 429–449. Academic Press, Boston, MA, 1990.
- [98] Long Hu and Guillaume Olive. Minimal time for the exact controllability of one-dimensional first-order linear hyperbolic systems by one-sided boundary controls. working paper or preprint, January 2019.
- [99] Alberto Isidori. Nonlinear control systems. Communications and Control Engineering Series. Springer-Verlag, Berlin, third edition, 1995.
- [100] Lukas Jadachowski, Thomas Meurer, and Andreas Kugi. Backstepping observers for linear PDEs on higher-dimensional spatial domains. Automatica J. IFAC, 51:85–97, 2015.
- [101] Jean-Pierre Kahane and Pierre-Gilles Lemarié-Rieusset. Séries de Fourier et ondelettes. Cassini, 1998.

- [102] Tosio Kato. Perturbation theory for linear operators. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
- [103] David Kleinman. An easy way to stabilize a linear constant system. IEEE Transactions on Automatic Control, 15(6):692–692, December 1970.
- [104] Daniel E Koditschek. Adaptive techniques for mechanical systems. Proc. 5th. Yale University Conference, page pp. 259–265, 1987.
- [105] V. Komornik. An equiconvergence theorem for the Schrödinger operator. Acta Math. Hungar., 44(1-2):101–114, 1984.
- [106] V. Komornik. On the equiconvergence of eigenfunction expansions associated with ordinary linear differential operators. Acta Math. Hungar., 47(1-2):261–280, 1986.
- [107] V. Komornik and P. Loreti. Partial observability of coupled linear systems. Acta Math. Hungar., 86(1-2):49–74, 2000.
- [108] Vilmos Komornik. Rapid boundary stabilization of linear distributed systems. SIAM J. Control Optim., 35(5):1591–1613, 1997.
- [109] Miroslav Krstić, Bao-Zhu Guo, Andras Balogh, and Andrey Smyshlyaev. Output-feedback stabilization of an unstable wave equation. Automatica J. IFAC, 44(1):63–74, 2008.
- [110] Miroslav Krstic, Petar V. Kokotovic, and Ioannis Kanellakopoulos. Nonlinear and Adaptive Control Design. John Wiley & Sons, Inc., New York, NY, USA, 1st edition, 1995.
- [111] Miroslav Krstic and Andrey Smyshlyaev. Boundary control of PDEs, volume 16 of Advances in Design and Control. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008. A course on backstepping designs.
- [112] Ta-Tsien Li and Bo-Peng Rao. Exact boundary controllability for quasi-linear hyperbolic systems. SIAM J. Control Optim., 41(6):1748–1755, 2003.
- [113] Tatsien Li. Controllability and observability for quasilinear hyperbolic systems, volume 3 of AIMS Series on Applied Mathematics. American Institute of Mathematical Sciences (AIMS), Springfield, MO; Higher Education Press, Beijing, 2010.
- [114] Tatsien Li. Controllability and observability for quasilinear hyperbolic systems, volume 3 of AIMS Series on Applied Mathematics. American Institute of Mathematical Sciences (AIMS), Springfield, MO; Higher Education Press, Beijing, 2010.
- [115] Thibault Liard and Pierre Lissy. A Kalman rank condition for the indirect controllability of coupled systems of linear operator groups. Math. Control Signals Systems, 29(2):Art. 9, 35, 2017.
- [116] J.-L. Lions. Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1, volume 8 of Recherches en Mathématiques Appliquées [Research in Applied Mathematics]. Masson, Paris, 1988. Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.
- [117] J.-L. Lions. Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 2, volume 9 of Recherches en Mathématiques Appliquées [Research in Applied Mathematics]. Masson, Paris, 1988. Perturbations. [Perturbations].
- [118] Wei Jiu Liu and Miroslav Krstić. Backstepping boundary control of Burgers’ equation with actuator dynamics. Systems Control Lett., 41(4):291–303, 2000.
- [119] Wei Jiu Liu and Miroslav Krstić. Boundary feedback stabilization of homogeneous equilibria in unstable fluid mixtures. Internat. J. Control, 80(6):982–989, 2007.

- [120] Dahlard L. Lukes. Stabilizability and optimal control. Funkcial. Ekvac., 11:39–50, 1968.
- [121] A. M. Lyapunov. The general problem of the stability of motion. Taylor & Francis, Ltd., London, 1992. Translated from Edouard Davaux’s French translation (1907) of the 1892 Russian original and edited by A. T. Fuller, With an introduction and preface by Fuller, a biography of Lyapunov by V. I. Smirnov, and a bibliography of Lyapunov’s works compiled by J. F. Barrett, Lyapunov centenary issue, Reprint of Internat. J. Control 55 (1992), no. 3 [MR1154209 (93e:01035)], With a foreword by Ian Stewart.
- [122] Thomas Meurer. Control of higher-dimensional PDEs. Communications and Control Engineering Series. Springer, Heidelberg, 2013. Flatness and backstepping designs.
- [123] John Nash. The imbedding problem for Riemannian manifolds. Ann. of Math. (2), 63:20–63, 1956.
- [124] Henk Nijmeijer and Arjan van der Schaft. Nonlinear dynamical control systems. Springer-Verlag, New York, 1990.
- [125] Yuri Orlov and Denis Dochain. Discontinuous feedback stabilization of minimum-phase semi-linear infinite-dimensional systems with application to chemical tubular reactor. IEEE Trans. Automat. Control, 47(8):1293–1304, 2002.
- [126] Nicolas Petit and Pierre Rouchon. Dynamics and solutions to some control problems for water-tank systems. IEEE Trans. Automat. Control, 47(4):594–609, 2002.
- [127] J.L. Pitarch, Mohsen Rakhshan, Mohammad Mardani, Mokhtar Sadeghi, and C Prada. Distributed nonlinear control of a plug-flow reactor under saturation. IFAC-PapersOnLine, 49:87–92, 12 2016.
- [128] Christophe Prieur, Joseph Winkin, and Georges Bastin. Robust boundary control of systems of conservation laws. Mathematics of Control, Signals, and Systems, 20(2):173–197, 2008.
- [129] Richard Rebarber. Spectral assignability for distributed parameter systems with unbounded scalar control. SIAM J. Control Optim., 27(1):148–169, 1989.
- [130] Lionel Rosier. Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. ESAIM Control Optim. Calc. Var., 2:33–55, 1997.
- [131] Boris L. Rozhdestvenskii. Hyperbolic partial differential equation. http://www.encyclopediaofmath.org/index.php?title=Hyperbolic_partial_differential_equation&oldid=16785.
- [132] Walter Rudin. Functional analysis. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973. McGraw-Hill Series in Higher Mathematics.
- [133] David L. Russell. Boundary value control of the higher-dimensional wave equation. SIAM J. Control, 9:29–42, 1971.
- [134] David L. Russell. Boundary value control theory of the higher-dimensional wave equation. II. SIAM J. Control, 9:401–419, 1971.
- [135] David L. Russell. Control theory of hyperbolic equations related to certain questions in harmonic analysis and spectral theory. J. Math. Anal. Appl., 40:336–368, 1972.
- [136] David L. Russell. Decay rates for weakly damped systems in Hilbert space obtained with control-theoretic methods. J. Differential Equations, 19(2):344–370, 1975.
- [137] David L. Russell. Canonical forms and spectral determination for a class of hyperbolic distributed parameter control systems. J. Math. Anal. Appl., 62(1):186–225, 1978.

- [138] David L. Russell. A general framework for the study of indirect damping mechanisms in elastic systems. J. Math. Anal. Appl., 173(2):339–358, 1993.
- [139] L. M. Silverman and H. E. Meadows. Controllability and time-variable unilateral networks. IEEE Trans. Circuit Theory, CT-12:308–314, 1965.
- [140] Marshall Slemrod. A note on complete controllability and stabilizability for linear control systems in Hilbert space. SIAM J. Control, 12:500–508, 1974.
- [141] Andrey Smyshlyaev, Eduardo Cerpa, and Miroslav Krstić. Boundary stabilization of a 1-D wave equation with in-domain antidamping. SIAM J. Control Optim., 48(6):4014–4031, 2010.
- [142] Andrey Smyshlyaev, Bao-Zhu Guo, and Miroslav Krstic. Arbitrary decay rate for Euler-Bernoulli beam by backstepping boundary feedback. IEEE Trans. Automat. Control, 54(5):1134–1140, 2009.
- [143] Eduardo D. Sontag. Mathematical control theory, volume 6 of Texts in Applied Mathematics. Springer-Verlag, New York, second edition, 1998. Deterministic finite-dimensional systems.
- [144] Shun Hua Sun. On spectrum distribution of completely controllable linear systems. SIAM J. Control Optim., 19(6):730–743, 1981. Translated from the Chinese by L. F. Ho.
- [145] Louis Tebou. Some results on the controllability of coupled semilinear wave equations: the desensitizing control case. SIAM J. Control Optim., 49(3):1221–1238, 2011.
- [146] Emmanuel Trélat, Gengsheng Wang, and Yashan Xu. Characterization by observability inequalities of controllability and stabilization properties. working paper or preprint, November 2018.
- [147] Emmanuel Trélat, Gengsheng Wang, and Yashan Xu. Stabilization of infinite-dimensional linear control systems by POD reduced-order Riccati feedback. working paper or preprint, June 2019.
- [148] Ngoc-Tu Trinh, Vincent Andrieu, and Cheng-Zhong Xu. Output regulation for a cascaded network of 2×2 hyperbolic systems with PI controller. Automatica J. IFAC, 91:270–278, 2018.
- [149] John Tsinias. Sufficient Lyapunov-like conditions for stabilization. Math. Control Signals Systems, 2(4):343–357, 1989.
- [150] Daisuke Tsubakino, Miroslav Krstić, and Shinji Hara. Backstepping control for parabolic pdes with in-domain actuation. 2012 American Control Conference (ACC), pages 2226–2231, 2012.
- [151] Jose Manuel Urquiza. Rapid exponential feedback stabilization with unbounded control operators. SIAM J. Control Optim., 43(6):2233–2244, 2005.
- [152] Rafael Vazquez, Jean-Michel Coron, Miroslav Krstic, and Georges Bastin. Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping. In 2011 50th IEEE Conference on Decision and Control and European Control Conference, pages 1329–1334. IEEE, 2011.
- [153] Ambroise Vest. Rapid stabilization in a semigroup framework. SIAM J. Control Optim., 51(5):4169–4188, 2013.
- [154] Frank Woittennek, Siqian Wang, and Torsten Knüppel. Backstepping design for parabolic systems with in-domain actuation and Robin boundary conditions. IFAC Proceedings Volumes, 47(3):5175–5180, 2014. 19th IFAC World Congress.
- [155] S. Xiang. Null controllability of a linearized Korteweg-de Vries equation by backstepping approach. SIAM J. Control Optim., 57(2):1493–1515, 2019.

- [156] Shengquan Xiang. Small-time local stabilization for a Korteweg–de Vries equation. Systems Control Lett., 111:64–69, 2018.
- [157] Cheng-Zhong Xu and Gauthier Sallet. Proportional and integral regulation of irrigation canal systems governed by the st venant equation. IFAC Proceedings Volumes, 32(2):2274–2279, 1999.
- [158] Cheng-Zhong Xu and Gauthier Sallet. Multivariable boundary PI control and regulation of a fluid flow system. Math. Control Relat. Fields, 4(4):501–520, 2014.
- [159] Xin Yu, Chao Xu, Huacheng Jiang, Arthi Ganesan, and Guojie Zheng. Backstepping synthesis for feedback control of first-order hyperbolic PDEs with spatial-temporal actuation. Abstr. Appl. Anal., pages Art. ID 643640, 13, 2014.
- [160] Jerzy Zabczyk. Mathematical control theory. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. An introduction, Reprint of the 1995 edition.
- [161] Christophe Zhang. Internal controllability of systems of semilinear coupled one-dimensional wave equations with one control. SIAM J. Control Optim., 56(4):3092–3127, 2018.
- [162] Christophe Zhang. Internal rapid stabilization of a 1-D linear transport equation with a scalar feedback. preprint, October 2018.
- [163] Christophe Zhang. Finite-time internal stabilization of a linear 1- d transport equation. Systems & Control Letters, 133:104529, 2019.
- [164] Xu Zhang. Explicit observability estimate for the wave equation with potential and its application. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 456(1997):1101–1115, 2000.
- [165] Xu Zhang. Explicit observability inequalities for the wave equation with lower order terms by means of Carleman inequalities. SIAM J. Control Optim., 39(3):812–834, 2000.
- [166] E. Zuazua. Exact boundary controllability for the semilinear wave equation. In Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. X (Paris, 1987–1988), volume 220 of Pitman Res. Notes Math. Ser., pages 357–391. Longman Sci. Tech., Harlow, 1991.
- [167] E. Zuazua. Exact controllability for semilinear wave equations in one space dimension. Ann. Inst. H. Poincaré Anal. Non Linéaire, 10(1):109–129, 1993.