Robust observer design for nonlinear systems

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Synthèse d'observateur robuste pour les systèmes non linéaires

Robust observer design for nonlinear systems

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Résumé

L'estimation d'état d'un système non linéaire est essentielle pour la réussite des objectifs importants tels que : la surveillance, l'identification et le contrôle. Les observateurs sont des algorithmes qui estiment l'état actuel en utilisant, entre autres informations, les mesures effectuées par des capteurs. Le problème de synthèse d'observateur pour les systèmes non linéaires est un sujet de recherche majeur traité depuis plusieurs décennies dans le domaine de la théorie du contrôle. Récemment, les recherches ont également porté sur la synthèse des observateurs pour des modèles de plus en plus réels, qui peuvent prendre en compte des perturbations, des capteurs non linéaires et des sorties discrètes. Dans ce contexte, le but de cette thèse concerne la synthèse d'observateurs robustes pour certaines classes de systèmes non linéaires. Dans ce manuscrit, nous distinguons trois parties principales.

La première partie porte sur l'analyse des systèmes affines en état, affectés par le bruit, et l'estimation de l'état via le filtre de Kalman à grand gain. Les propriétés de convergence de cet observateur sont fortement influencées par deux variables : le paramètre de réglage du gain de l'observateur et l'entrée du système. Nous présentons un nouvel algorithme d'optimisation, basé sur une analyse de Lyapunov, qui adapte ces deux variables en fonction des perturbations affectant la dynamique et la sortie du système. La nouveauté de cette approche est qu'elle fournit une méthode systématique de réglage du gain et de sélection d'entrée simultanément ce qui améliore l'estimation de l'état en présence de telles perturbations et évite l'utilisation des méthodes basées de type essais-erreurs.

La deuxième partie concerne le problème de "redesign" d'observateurs pour des systèmes non linéaires sous une forme générale dont les sorties sont transformées par des fonctions non linéaires. En effet, l'observateur risque alors de ne pas estimer correctement l'état du système si elle ne prend pas en compte les non-linéarités des capteurs. Nous présentons une refonte d'observateur qui consiste en l'interconnexion de l'observateur originel avec un estimateur de sortie basé sur une inversion dynamique, et nous démontrons sa convergence asymptotique via des résultats du petit-gain. Nous illustrons notre méthode en utilisant deux classes de systèmes non linéaires courant dans la littérature : les systèmes affines en l'état avec injection de sortie, et les systèmes avec non-linéarité sous la forme canonique.

Enfin, la troisième partie étend notre approche présentée pour les systèmes continus aux systèmes dont les sorties sont non seulement transformées mais également discrétisées dans le temps. Cette propriété ajoutée introduit un défi important ; nous implémentons les techniques de "sample-and-hold" qui mènent à un gain de l'observateur basé sur des inégalités matricielles linéaires. La principale caractéristique des méthodes proposées est la possibilité d'adapter un grand nombre d'observateurs de la littérature à des scénarii plus réalistes. En effet, les capteurs classiques utilisés dans les applications d'ingénierie sont souvent non linéaires ou discrets, alors qu'une hypothèse récurrente dans la conception d'observateurs est la linéarité ou la continuité de la sortie.

Mots-clés

systèmes dynamiques non linéaires - systèmes affine en l'état - capteurs non linéaires - mesures discrètes - robustesse au bruit - observateurs non linéaires - observateurs à grand gain - observateurs interconnectés - filtre de Kalman - stabilité de Lyapunov - stabilité entrée-à-état - petit-gain - inégalités matricielles linéaires

Abstract

Estimating the state of a nonlinear system is an essential task for achieving important objectives such as: process monitoring, identification and control. Observers are algorithms that estimate the current state by using, among other information, sensor measurements. The problem of observer design for nonlinear systems has been a major research topic in control for many decades. Recently, there has been an increasing interest in the design of observers for more realistic models, which can include disturbances, sensor nonlinearities and discrete outputs. This thesis concerns the design of robust observers for selected classes of nonlinear systems and we can distinguish three main parts.

The first part studies state-affine systems affected by noise, and analyses the state estimation via the so-called high-gain Kalman filter. The convergence properties of this observer are strongly influenced by two variables: its tuning parameter and the properly excited system input. We present a new optimization algorithm, based on Lyapunov analyses, that adapts these variables in order to minimize the effect of both dynamic and output disturbances. The novelty of this approach is that it provides a systematic method of simultaneous tuning and input selection with the goal of improving state estimation in the face of disturbances, and that it avoids the use of trial-and-error based methods.

The second part studies the problem of observer redesign for general nonlinear systems whose outputs are transformed by nonlinear functions. Indeed, a given observer might not estimate the system state properly if it does not take into account sensor nonlinearities and, therefore, such an output mismatch needs to be addressed. We present an observer redesign that consists in the interconnection of the original observer with an output estimator based on a dynamic inversion, and we show its asymptotic convergence via small-gain arguments. We illustrate our method with two important classes of systems: state-affine systems up to output injection and systems with additive triangular nonlinearity.

Finally, the third part extends our redesign method to systems whose outputs are not only transformed but also discretized in time. This added assumption introduces important challenges; we now implement sample-and-hold techniques leading to an observer gain based on linear matrix inequalities. The main feature of our redesign methods is the possibility to adapt a large number of observers from the literature to more realistic scenarios. Indeed, classical sensors in engineering applications are often nonlinear or discrete, whereas a recurrent assumption in observer design is the linearity or continuity of the output.

Keywords

nonlinear dynamical systems - state-affine systems - nonlinear sensors - discrete measurements - noise robustness - nonlinear observers - high-gain observers - interconnected observers - Kalman filter - Lyapunov stability - input-to-state stability - small-gain - linear matrix inequalities

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Nomenclature

$|f|_\infty$ $\sup_{t\geq 0}|f(t)|$, where $f : \mathbb{R}^+ \to \mathbb{R}^n$ is a function

$|M|$ spectral norm of the real matrix M or $\lambda_{\max}(M'M)^{\frac{1}{2}}$

$\lambda_{\max}(M)$ maximum eigenvalue of the real symmetric matrix M

$\lambda_{\min}(M)$ minimum eigenvalue of the real symmetric matrix M

$\mathbb{Z}^+$ set of non-negative integers

$\mathbb{R}^+$ set of non-negative real numbers

$\mathbb{R}^n$ set of real numbers of n dimension

$\mathbb{R}^{n\times m}$ set of real matrices of $n \times m$ dimension

C^k class of functions whose derivatives of order up to k are continuous

$f(t^-)$ left limit of a function $f : [s,t) \to \mathbb{R}^n$ at t, or $\lim_{r\to t^-} f(r)$

$f^{(i)}$ ith derivative of the function $f : \mathbb{R}^+ \to \mathbb{R}^n$

$f^{-1}(0)$ the set of $t \in \mathbb{R}^+$ such that $f(t) = 0$ for a function $f : \mathbb{R}^+ \to \mathbb{R}^n$

I Identity matrix of appropriate dimension

$L^2_{loc,n}$ space of measurable and locally square integrable functions $f : \mathbb{R}^+ \to \mathbb{R}^n$

L^∞_n space of measurable and essentially bounded functions $f : \mathbb{R}^+ \to \mathbb{R}^n$

$L^\infty_{loc,n}$ space of measurable and locally essentially bounded functions $f : \mathbb{R}^+ \to \mathbb{R}^n$

M' transpose of the matrix M

$M \geq 0$ the matrix $M \in \mathbb{R}^{n\times n}$ satisfies $x'Mx \geq 0$, for all $x \in \mathbb{R}^n$

$M_{i,j}$ element in the ith row and jth column of the matrix M

a.e. almost everywhere (except for a Lebesgue zero measure subset)

DES Disturbance-to-Error Stability

EKF Extended Kalman Filter

GAS Global Asymptotic Stability

i-IOSS incremental Input/Output-to-State Stability

IOpS Input-to-Output practical Stability

IOS Input-to-Output Stability

IOSS Input/Output-to-State Stability

ISpS Input-to-State practical Stability

ISS Input-to-State Stability

LMI Linear Matrix Inequality

MPC Model Predictive Control

NCS Networked Control System

ODE Ordinary Differential Equation

OSS Output-to-State Stability

PDE Partial Differential Equation

pre-GAS pre-Global Asymptotic Stability

pre-ISS pre-Input-to-State Stability

UO Unboundedness Observability

Chapter 1

General introduction

1.1 Context

Every *physical system* is composed of different parts that follow a common mechanism. Therefore, partial information can sometimes be used in order to estimate what we ignore. This is the fundamental idea underlying the concept of *observer* for a system. Here, we consider models of physical systems that are dynamic or time dependant and that are governed by differential equations.

The model of a physical system is a rigorous description of the corresponding phenomena usually based on fundamental laws. Reliable modeling should be accurate; a sufficiently good representation of reality often requires a nonlinear structure and the consideration of unknown factors such as *disturbances*. The objectives of system modeling are diverse and include: phenomenon understanding, prediction, optimization and, crucially, control.

Modern control theory relies on the so-called state-space representation of a system, where we can distinguish the possibly multi-dimensional variables: *state* or system evolution, *input* or control, and *output* or measurements. In this context, the implementation of control laws can require full or partial state information from the output or sensor. Internal information is also needed for other important tasks, such as: identification, decision-making or monitoring. The sensor capabilities, however, are often limited by either physical, technological, economical or even safety constrains.

Observers are algorithms that estimate the current state by using the system structure and the available measurements. In this thesis, such algorithms are given by adjacent dynamical systems and the convergence of the state estimation is with respect to the Euclidean norm. Closely related notions to that of observer are the observability or detectability of a system, which distinguish those systems admitting an observer design.

The theory of observers has become an extremely rich field that lies at the interface between pure mathematical disciplines and practical applications. Although the design of observers for linear systems is considered a solved problem, its nonlinear counterpart has been under intensive research in the control community for several decades. As a consequence, even though currently there is no systematic design method, observers have been developed for specific classes of nonlinear systems and under different assumptions. An extremely desired feature of observers is their robustness, this is due to their practical nature. Indeed, their estimations should be satisfactory even in the presence of system disturbances. This makes the problem of robust observer design for nonlinear systems pivotal in the field of automatic control.

1.2 Motivations

It is not trivial how to adapt the design of a given observer to disturbances affecting the system. This problem is evident for uniformly observable systems and their high-gain observer, whose design depends on a tuning parameter that can amplify measurement noise; several optimal or adaptive strategies have been developed in the literature in order to tune this observer correctly. On the other hand, the system input plays a central role in the observability of nonlinear systems, and the design of inputs that render a nonlinear system observable is usually intricate. In the case of input design for state-affine or bilinear systems, researchers have developed algorithms to construct *sufficiently regular* inputs with prescribed properties. Surprisingly, no literature seems to aim for the simultaneous tuning and input selection in observers with the objective of improving state estimation in the presence of noise.

The focus in observer design is frequently on the nonlinearity of the system dynamics, however, modern sensors are constantly becoming more complex and output linearity might not match reality. This rises the question whether known observer designs can be adapted to an output nonlinearity. Similarly, we can also consider the case of discrete measurements, which cannot always be taken close enough in time; it is well-known that this can affect considerably the properties of an observer in the nonlinear case.

1.3 Contributions and thesis outline

The main contribution of this thesis is the design of three new observers for nonlinear systems, together with the proofs of their asymptotic convergence and of their robustness properties.

Chapter 2 provides a literature review on the subject of observability and observer design for nonlinear systems, especial attention is given to high-gain observers. The chapter also gathers central results from the input-to-state stability framework and introduces the notion of robust observer design. Some relevant and new preliminary results can be found at the end.

Chapter 3 proposes the first observer design, the systems under consideration are state-affine with dynamic and output disturbances. Although the observer structure is given by the known high-gain Kalman observer, our work develops an efficient optimization algorithm that improves the state estimation. That is, an offline algorithm for the simultaneous adaptation of the high-gain tuning and the system input to the disturbances affecting the system.

Chapter 4, which contains the main contribution of this thesis, introduces the second observer design as a general redesign method. More precisely, we first consider a general nonlinear system for which an observer design is available. We then redesign this observer in case the output is only measured after a nonlinear transformation. Our new observer is based on a dynamic inversion formula that avoids using the inverse of the transformation.

Chapter 5 provides the third observer design by generalizing the previous redesign method. The difference being that now the output is not only measured through a nonlinear transformation but it is also discretized in time. Nevertheless, this redesign method differs from the previous one and it imposes alternative constraints on the observer gain.

Finally, Chapter 6 summarizes the conclusions from the present thesis and indicates future research directions.

1.4 Publications

The work in this thesis gave rise to the following publications:

1. F. González de Cossío, M. Nadri, P. Dufour, *Observer design for nonlinear systems with output transformation.* IEEE Transactions on Automatic Control. Accepted as full paper.

2. F. González de Cossío, M. Nadri, P. Dufour, *Observer Design for nonlinear systems with sampled and transformed measurements.* In 2019 IEEE Conference on Decision and Control (CDC), Nice, France. Accepted.

3. F. González de Cossío, M. Nadri, P. Dufour, *Observer design for nonlinear systems with implicit output.* In 2018 IEEE Conference on Decision and Control (CDC), Miami Beach, FL, pp. 2170-2175.

4. F. González de Cossío, M. Nadri, P. Dufour, *Optimal observer design for disturbed state affine systems.* In 2018 Annual American Control Conference (ACC), Milwaukee, WI, pp. 2733-2738.

Chapter 2

Observer design for nonlinear systems

2.1 Introduction

In this thesis, we consider dynamical systems governed by ordinary differential equations (ODE) that are in the so-called state-space representation. That is,

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) \\ y(t) = h(x(t), u(t), t), \end{cases} \tag{2.1}$$

where $t \in \mathbb{R}^+$ represents time, $x(t) \in \mathbb{R}^n$ the state, $y(t) \in \mathbb{R}^p$ the output and $u(t) \in \mathbb{R}^m$ an input assumed continuous unless otherwise stated. The structure is described by the functions $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^p$, and the system is assumed forward-complete: for each initial condition $x(0) \in \mathbb{R}^n$, system (2.1) has a unique solution defined on all $\mathbb{R}^+$.

The observer design problem consists in defining an adjacent system to estimate the unknown state x of system (2.1) by using its output y and its input u. Indeed, an observer has the form

$$\begin{cases} \dot{z}(t) = \hat{f}(z(t), y(t), u(t), t) \\ \hat{x}(t) = \hat{h}(z(t), y(t), u(t), t), \end{cases}$$

where $z(t)$ evolves in some Euclidean space and where $\hat{x}(t) \in \mathbb{R}^n$ should estimate the state, that is,

$$|x(t) - \hat{x}(t)| \to 0, \quad \text{as } t \to \infty.$$

The observer is called global if such an estimation holds for all initial conditions $x(0)$, $z(0)$. We usually consider global observers and we denote the state estimation error by

$$e(t) = x(t) - \hat{x}(t), \quad \forall t \geq 0.$$

A type of convergence that is commonly desired is the exponential, often given by an inequality of the form

$$|e(t)| \leq c|e(0)| \exp(-rt),$$

for some positive constants c and r. In order to simplify the notation, we sometimes omit the explicit time-dependence if no confusion is possible. Moreover, we identify $n-$tuples $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with column matrices

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n\times 1}.$$

The internal information carried by the system state can be necessary for tasks such as: modeling, monitoring, diagnosis or, importantly, driving of the system through the classical control-loop. Therefore, observer design for nonlinear systems has been a research topic of utmost importance in the control community for several decades.

The main objective of this chapter is to introduce several popular observer designs from the literature with an emphasis on the so-called high-gain observers. We also discuss the closely related notions of observability of a nonlinear system and the corresponding representative system forms that facilitate the design of observers. Finally, we study the celebrated concept of input-to-state stability (ISS) and allied properties, which provide us with efficient tools for quantifying the robustness of observers with respect to system disturbances. Some selected results are gathered at the end, and will be used in our main result corresponding to Chapter 4.

2.2 Observer design for linear systems

The problem of observer design for linear systems was established and solved by: R.E. Kalman, 1960 [65] and D.G. Luenberger, 1964 [83]. However, the motivation behind the idea of using the output to estimate the state can be traced back to the so-called filtering in communication engineering and to the work of A. Kolmogorov, 1939 and N. Wiener, 1949 (more recent editions in [73] and [109] respectively).

Let us suppose that system (2.1) is linear, that is, of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t), \end{cases} \tag{2.2}$$

where A, B and C are constant real matrices of the appropriate dimensions. The celebrated *Luenberger observer* for system (2.2) is then given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - K(C\hat{x}(t) - y(t)), \tag{2.3}$$

where K is any matrix such that $A - KC$ is Hurwitz, that is, all its eigenvalues have strictly negative real part. We say that system (2.2) is: (i) *detectable* if such a matrix K exists, (ii) *observable* if the eigenvalues of $A - KC$ can be arbitrarily assigned by choosing K. The notion of observability is of interest because it allows an arbitrarily fast rate of exponential convergence to zero of the state estimation error $e = x - \hat{x}$.

Theorem 2.1 ([82]). *Consider system* (2.2) *and suppose that there exists a matrix K such that $A - KC$ is Hurwitz. Then, the state estimation error from* (2.3) *converges exponentially to zero.*

On the other hand, we can also estimate the state in the case of bounded and continuous time-varying matrices $A(t)$, $B(t)$ and $C(t)$ by using a high-gain version of the so-called Kalman filter. However, for a system of the form

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \end{cases} \tag{2.4}$$

it is known that a constant gain K as in (2.3) might not be enough to design an observer. Instead, we set $K(t) = S(t)^{-1}C(t)'$ for $S(t)$ given by the Riccati equation. The observer is described by

$$\begin{cases} \dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) - K(t)(C(t)\hat{x}(t) - y(t)) \\ \dot{S}(t) = -\theta S(t) - A(t)'S(t) - S(t)A(t) + C(t)'C(t), \end{cases} \tag{2.5}$$

where $S(0)$ is a symmetric and positive definite matrix and where the tuning parameter is such that $\theta > 2\sup_{t\geq 0}|A(t)|$. We require system (2.4) to satisfy the following property: there exist positive real numbers $a > 0$, $T > 0$ and $t_0 \geq T$ such that

$$\int_{t-T}^{t} \Phi(s,t-T)'C(s)'C(s)\Phi(s,t-T)ds \geq aI, \tag{2.6}$$

for all $t \geq t_0$ and where Φ is the transition matrix, namely, the solution to

$$\begin{cases} \frac{\partial \Phi}{\partial s}(s,t) = A(s)\Phi(s,t) \\ \Phi(t,t) = I, \end{cases}$$

for all s, $t \in \mathbb{R}^+$. Importantly, condition (2.6) is independent of the system input and the rate of convergence of the state estimation error can be tuned by θ.

Theorem 2.2 ([22])**.** *Consider system* (2.4)*, suppose that $A(t)$ and $C(t)$ satisfy condition* (2.6) *and that θ is large enough. Then, the state estimation error from* (2.5) *converges exponentially to zero.*

Remark 2.1. *Another possibility is that the matrix A in system* (2.4) *depends on the system input u. We can proceed as before in order to design an observer for the so-called state-affine system*

$$\begin{cases} \dot{x}(t) = A(u(t))x + B(u(t)) \\ y(t) = Cx(t) \end{cases}$$

with similar convergence properties [57]. However, we need to restrict the input set $\mathcal{U}$ by imposing the corresponding property (2.6) *for the input dependent Φ_u. System inputs satisfying this property are known as regularly persistent. Further details are given in Chapter 3.*

2.2.1 Noise and Kalman filtering

The original Kalman filter [65, 66] differs from (2.5) in various ways: the equation of the gain is instead written for the inverse matrix, there is no high-gain parameter θ and it involves additional symmetric and positive definite matrices which have a stochastic interpretation when the system is affected by noise. In fact, it is known that the Kalman filter is the least squares estimator in a *stochastic sense*.

The purpose of this section is to present a deterministic interpretation of the Kalman filter as given in [110]. We consider a disturbed linear system given by

$$\begin{cases} \dot{x}(t) = Ax(t) + Dd_1(t) \\ y(t) = Cx(t) + d_2(t), \end{cases}$$

where A, C and D are constant matrices, $d_1 \in L^2_{loc,d}$ represents dynamic uncertainties and $d_2 \in L^2_{loc,p}$ measurement noise. The goal is to filter the noise from the observations in order to estimate the state x. Given an initial condition $x(0) \in \mathbb{R}^n$, the system state and output have the expressions:

$$x(t) = \exp(At)x(0) + \int_0^t \exp(A(t-s))Dd_1(s)ds, \tag{2.7}$$

$$y(t) = C\exp(At)x(0) + \int_0^t C\exp(A(t-s))Dd_1(s)ds + d_2(t), \tag{2.8}$$

for all $t \in \mathbb{R}^+$. The least square filtering problem can be set as follows. Suppose that we observe a particular system output $\bar{y}$ on a time interval $[0,T]$ and that we collect all disturbances d_1, d_2 and initial conditions $x(0)$ such that (2.8) is satisfied on this interval (with $\bar{y}$). Then we aim to minimize over this collection the square sum:

$$|x(0)|^2_S + |d_1|^2_{2,T} + |d_2|^2_{2,T}, \tag{2.9}$$

where

$$|d_i|^2_{2,T} = \int_0^T |d_i(s)|^2 ds < \infty, \quad i = 1,2$$

and $|x(0)|^2_S = x(0)'Sx(0)$ for a symmetric and positive definite matrix S. If a minimum is achieved at some triplet $(d_1^*, d_2^*, x(0)^*)$, then we can use (2.7) to obtain the optimal estimate at time T as

$$\hat{x}(T) = \exp(AT)x(0)^* + \int_0^T \exp(A(T-s))Dd_1^*(s)ds.$$

Therefore, the state estimate is chosen by using the most likely triplet in the sense that uncertainty is minimized. Computing $\hat{x}$ then requires to apply an optimization algorithm at each given time which can be computationally expensive. The author in [110] shows that this optimization can be solved in a much more efficient way and that the equations of the Kalman filter arise naturally in this context.

Theorem 2.3 ([110]). *Suppose that $\bar{y} \in L^2_{loc,p}$ is observed and consider the system*

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + PC'(\bar{y} - C\hat{x}) \\ \dot{P} = AP + PA' - PC'CP + DD', \end{cases}$$

with the initial conditions $\hat{x}(0) = 0$ and $P(0) = S^{-1}$. Then, the least square filter with respect to (2.9) *is given by $\hat{x}$.*

The proof of this theorem consists first in showing the equality

$$\begin{aligned} |x(0)|^2_S + |d_1|^2_{2,T} + |d_2|^2_{2,T} =& |x(T) - \hat{x}(T)|^2_{P(T)^{-1}} \\ &+ |d_1 - D'P^{-1}(x - \hat{x})|^2_{2,T} + |\bar{y} - C\hat{x}|^2_{2,T}, \end{aligned}$$

where the triplet $(x(0), d_1, d_2)$ leads to the observed $\bar{y}$ on $[0, T]$. Thus,

$$|x(0)|^2_S + |d_1|^2_{2,T} + |d_2|^2_{2,T} \geq |\bar{y} - C\hat{x}|^2_{2,T},$$

where the right-hand side of this inequality is independent of the triplet. Finally, it is shown that there exists a unique triplet leading to $\bar{y}$ and such that

$$|x(T) - \hat{x}(T)|^2_{P(T)^{-1}} + |d_1 - D'P^{-1}(x - \hat{x})|^2_{2,T} = 0,$$

hence, minimizing (2.9).

2.3 Observer design for nonlinear systems

The Luenberger and the Kalman observer presented in the previous section provide satisfactory solutions to the observation problem of time-invariant or time-varying linear systems. However, the design of observers for nonlinear systems can be very challenging and is, in general, not systematic. A first approach to this problem can be made by considering the Extended Kalman Filter (EKF), which is based on linearizations along the system trajectories. Indeed, consider a nonlinear system of the form

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t)), \end{cases}$$

where f and h are of class C^1. Then, we can proceed as before by copying the dynamics of the system and adding a correcting term

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) - K(t)(h(\hat{x}(t)) - y(t)),$$

where the dynamic gain $K(t)$ is defined as for observer (2.5) but with:

$$A(t) = \frac{\partial f}{\partial x}(\hat{x}(t), u(t)), \quad C(t) = \frac{\partial h}{\partial x}(\hat{x}(t)).$$

Unfortunately, this technique has a major drawback since the obtained observer is only local and a good initial guess is crucial for the successful estimation of the system state [93]. Some other popular approaches to observer design for nonlinear systems include: the linearization of the error dynamics by nonlinear state transformations [75] or the less conservative technique from [69], exploiting the Lipschitz nonlinearities of the system by using Linear Matrix Inequalities (LMI's) [114, 2] or, instead, dominating this nonlinearities with the use of a high-gain parameter [46], using other types of nonlinearities such as monotonic [10] or those satisfying incremental quadratic constraints [13], and a large number of other different techniques and specific designs that constantly increases.

2.3.1 Observability of nonlinear systems

It is clear that for some systems the problem of observer design cannot be solved. For example, it can be shown that there exists a linear observer design for the linear system (2.2) precisely when the pair (A, C) is detectable. The slightly stronger condition of observability is equivalent to the full-rank of the matrix given by

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix},$$

which allows us to implement the tunable observer (2.3). As we have seen, the notion of observability for a time-varying linear system like (2.4) cannot be directly extended and, instead, a condition such as (2.6) is needed. In general, the observability of a nonlinear system can depend on the system input. Consider a forward-complete nonlinear system of the form

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x, u), \end{cases} \tag{2.10}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$, and where f and h are analytic functions satisfying $f(0, 0) = 0$ and $h(0, 0) = 0$. Let us denote by $x(t; x_0, u)$ the unique trajectory corresponding to the initial condition $x_0 = x(0)$ and to the input u.

Definition 2.1. *We say that an input u is universal for system* (2.10) *if for all initial conditions $x_0 \neq x_0'$ there exists $s \geq 0$ such that*

$$h(x(s; x_0, u), u(s)) \neq h(x(s; x_0', u), u(s)).$$

System (2.10) *is uniformly observable if every input $u \in L_m^\infty$ is universal.*

Remark 2.2. *The universal property of an input can be lost over time. This gives rise to the notion of persistency. That is, the input u is persistent if there exist $t_0 \geq T > 0$ such that for all $t \geq t_0$ and for all initial conditions $x_{t-T} \neq x_{t-T}'$ we have*

$$\int_{t-T}^{t} |h(x(s; x_{t-T}, u), u(s)) - h(x(s; x_{t-T}', u), u(s))|^2 ds > 0.$$

This notion is, of course, related to the regular persistence of an input as in Remark 2.1 and to allied properties from the next section such as local regularity.

A popular technique for observer design is to steer the system into a convenient form by a change of coordinates in order to facilitate the design of an observer, early work in this direction can be seen in [75]. Suppose that we have a system in the specific form

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x), \end{cases} \tag{2.11}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$, and for $f, g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ smooth functions. We define the transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$ coordinate-wise for $1 \leq i \leq n$ as

$$\phi_i(x) = L_f^{i-1}(h)(x), \quad \forall x \in \mathbb{R}^n, \tag{2.12}$$

where $L_f^i(h)$ denotes the *ith* Lie derivative of h along f. The Lie derivative can be defined recursively by

$$L_f^0(h) = h, \quad L_f(h)(x) = \frac{\partial h}{\partial x}(x) f(x), \quad L_f^i(h) = L_f(L_f^{i-1}(h)).$$

Theorem 2.4 ([47]). *If system* (2.11) *is uniformly observable, then there exists an open and dense subset $\mathcal{D}$ of $\mathbb{R}^n$ with the following properties: (i) Every $x_0 \in \mathcal{D}$ has a neighborhood $\mathcal{N}$ in $\mathbb{R}^n$ such that the restriction $\phi \restriction: \mathcal{N} \to \phi(\mathcal{N})$ is a diffeomorphism, (ii) the change of coordinates $z = \phi(x)$ for $x \in \mathcal{N}$ takes system* (2.11) *into the form:*

$$\begin{cases} \dot{z} = Az + \psi(z) + \omega(z)u \\ y = Cz, \end{cases}$$

where:

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}, \quad C' = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \psi(z) = \begin{bmatrix} 0 \\ \vdots \\ \psi_n(z) \end{bmatrix}, \quad \omega(z) = \begin{bmatrix} \omega_1(z_1) \\ \omega_2(z_1, z_2) \\ \vdots \\ \omega_n(z) \end{bmatrix}$$

and for some functions ψ_n, $\omega_1, \ldots, \omega_n$.

More generally, this result is valid for control-affine nonlinear systems with multi-dimensional inputs $u(t) \in \mathbb{R}^m$. The converse also holds: if the system can be steered into the specific form described in the previous theorem, then the system should be uniformly observable.

Theorem 2.4 is in fact a consequence of a more general result that requires the notion of *canonical flag*. Let us consider again the general nonlinear system described in (2.10). The study of the so-called observability normal forms is divided into two important dimension cases: $p > m$ and $m \geq p$. The former leads to the *phase variable representation* while the latter, studied next for $p = 1$, to the *uniform observability canonical form*.

In this case, the transformation ϕ defined in (2.12) depends on u and we can set

$$K_i(x,u) = \ker\left(\frac{\partial \phi_i}{\partial x}(x,u)\right),$$

that is, the kernel of the Jacobian matrix. The uniform canonical flag condition [48] applies to each $i = 1, \dots, n$ and requires that for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$:

$$\begin{aligned} &\dim(K_i(x,u)) = n - i, \\ &K_i(x,u) \text{ is independent of } u, \end{aligned} \tag{2.13}$$

where dim stands for the dimension of the kernel subspace of $\mathbb{R}^n$. The following theorem provides global conditions for steering system (2.10) into a convenient triangular structure. Let us define the function $\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\phi_0(x) = \phi(x, 0).$$

Theorem 2.5 ([48]). *Consider system* (2.10) *with* $p = 1$ *and suppose that it satisfies conditions* (2.13) *and that* ϕ_0 *is a global diffeomorphism. Then, the change of coordinates* $z = \phi_0(x)$ *takes system* (2.10) *into the form:*

$$\begin{cases} \dot{z} = \bar{f}(z,u) \\ y = \bar{h}(z_1, u), \end{cases}$$

for some functions $\bar{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and* $\bar{h} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ *such that:*

$$\begin{aligned} &\bar{f}_i(z_1, z_2, \dots, z_{i+1}, u), \quad \bar{f}_n(z,u) \\ &\frac{\partial \bar{f}_i}{\partial z_{i+1}} \neq 0, \quad \frac{\partial \bar{h}}{\partial z_1} \neq 0, \quad \forall z \in \mathbb{R}^n, \forall u \in \mathbb{R}^m. \end{aligned}$$

An extensive summary of results concerning system representations that characterize observability can be consulted in [48]. More recently, the authors in [19] also show that the class of systems for which there exist convenient forms (by immersion) can be extended to non-uniformly observable systems. Therefore, the problem of observer design for a large class of nonlinear systems reduces to specific system forms. The following section studies observer design for several representative systems.

2.3.2 High-gain observers

In nonlinear control, high-gain observers are a common state estimation tool due to their convenient structure. The pioneer work in [38] presents an observer adjustment of the Kalman filter, which is based on a fictitious noise that achieves a satisfactory linear control feedback. As with modern high-gain techniques, this adjustment implies a trade-off between noise rejection and margin recovery.

An important research direction concerning high-gain observers was initiated in the work of [42]. It deals with the peaking phenomenon in the absence of global growth conditions and with the possible destabilization of the closed-loop system; the authors present an appropriate control function that saturates precisely during the peaking. A contemporary but different research direction consists in dominating the Lipschitz nonlinearities of the system with the gain, this results in two main observer constructions: the Luenberger type [46] or the Kalman type [37].

Let us consider a nonlinear system in the representative form given in Theorem 2.5. As oppose to other systems considered in this section, this one allows a nonlinear output function. If we use the notation

$$\underline{x}_i = (x_1, \dots, x_i), \quad \forall x \in \mathbb{R}^n$$

then the system is described by

$$\begin{aligned} \dot{x}_1 &= f_1(\underline{x}_1, x_2, u) \\ \dot{x}_2 &= f_2(\underline{x}_2, x_3, u) \\ &\vdots \\ \dot{x}_{n-1} &= f_{n-1}(\underline{x}_{n-1}, x_n, u) \\ \dot{x}_n &= f_n(\underline{x}_n, u) \\ y &= h(x_1, u), \end{aligned} \tag{2.14}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}^m$, and where $f = (f_1, \dots, f_n)$ and h are analytic. Furthermore, we require each f_i to be Lipschitz continuous with respect to $\underline{x}_i$, uniformly in x_{i+1} and u, and we assume the existence of positive constants $\alpha < \beta$ such that

$$\alpha \leq \frac{\partial f_i}{\partial x_{i+1}} \leq \beta, \quad \alpha \leq \frac{\partial h}{\partial x_1} \leq \beta, \quad \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m. \tag{2.15}$$

The Luenberger-like observer from [48] takes the form

$$\dot{\hat{x}} = f(\hat{x}, u) + \Delta_\theta K(y - h(\hat{x}_1, u)), \tag{2.16}$$

where $\theta > 0$ and the constant gain matrix $K = [k_{n-1} \dots k_0]'$ are design parameters, and where

$$\Delta_\theta = \begin{bmatrix} \theta & & 0 \\ & \ddots & \\ 0 & & \theta^n \end{bmatrix}. \tag{2.17}$$

The design of K is intricate and it requires a preliminary result. Indeed, given any positive constants $\alpha < \beta$ and a collection of n continuous functions $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\alpha \leq g_i(t) \leq \beta, \quad \forall t \geq 0,$$

there exist real numbers k_0, $k_1, \dots, k_{n-1}$, a positive real number $\lambda > 0$ and a symmetric positive definite matrix S such that

$$SA(t) + A(t)'S \leq -\lambda I, \quad \forall t \geq 0$$

and where

$$A(t) = \begin{bmatrix} -k_{n-1}g_1(t) & g_2(t) & 0 & \dots & 0 & 0 \\ -k_{n-2}g_1(t) & 0 & g_3(t) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -k_1 g_1(t) & 0 & 0 & \dots & 0 & g_n(t) \\ -k_0 g_1(t) & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Crucially, the k_i, λ and S depend only on the given α and β and not on the choice of the collection of functions g_i. Since the proof of this result is inductive on n, it also provides an algorithm to find the aforementioned variables.

Finally, the design of K in (2.16) arises from the bounds in (2.15) and, in the proof of the observer convergence, the functions g_i are defined as:

$$g_1 = \frac{\partial h}{\partial x_1}(\delta_0, u), \quad g_{i+1} = \frac{\partial f_i}{\partial x_{i+1}}(\underline{x}_i, \delta_i, u),$$

where $\delta_i(t) \in \mathbb{R}$ comes from the application of the mean value theorem on a convenient interval. As one might expect, a Lyapunov function for the *re-scaled* state estimation error

$\epsilon_i = \theta^{n-i}(x_i - \hat{x}_i)$ is then given by $V(\epsilon) = \epsilon' S \epsilon$ and there is the following result. We remark that how large θ has to be depends on several factors, such as the Lipschitz constants of f_i and the state dimension n, which is a characteristic drawback of high-gain observers.

Theorem 2.6 ([48]). *Consider system* (2.14) *and suppose that f_i is Lipschitz continuous in $\bar{x}_i$, uniformly in x_{i+1} and u, and that f_i and h satisfy the inequalities in* (2.14)*. If we set the constant gain matrix K as described above and if $\theta > 0$ is large enough, then the state estimation error from* (2.16) *converges exponentially to zero.*

Similarly, we can also design an observer for systems in the form given by Theorem 2.4. That is, consider the nonlinear system given by

$$\begin{cases} \dot{x} = Ax + \zeta(x, u) \\ y = Cx, \end{cases} \tag{2.18}$$

where:

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}, \quad \zeta(x, u) = \begin{bmatrix} \zeta_1(x_1, u) \\ \zeta_2(x_1, x_2, u) \\ \vdots \\ \zeta_n(x, u) \end{bmatrix}, \tag{2.19}$$

for a nonlinear function $\zeta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Moreover, suppose that ζ is coordinate-wise Lipschitz continuous in the first entrance. That is, for each $i = 1, \dots, n$, there exists a positive constant c_i such that

$$|\zeta_i(\underline{x}_i, u) - \zeta_i(\underline{\hat{x}}_i, u)| \leq c_i |\underline{x}_i - \underline{\hat{x}}_i|, \tag{2.20}$$

where $\underline{x}_i = (x_1, \dots, x_i)$ and for all x, $\hat{x} \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. The following observer was introduced in [46],

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + \zeta(\hat{x}, u) + S_\infty^{-1} C'(y - C\hat{x}) \\ 0 = -\theta S_\infty - A' S_\infty - S_\infty A + C'C, \end{cases} \tag{2.21}$$

where $\theta > 0$ is large enough and where the observability of the pair (A, C) ensures the existence of a (constant) symmetric and positive definite matrix S_∞. This can be seen by noticing that S_∞ is the stationary solution of (2.5) for A and C constant matrices.

Theorem 2.7 ([46]). *Consider system* (2.18) *and suppose that the nonlinearity* ζ *satisfies* (2.20) *and that* θ *is large enough. Then, the state estimation error from* (2.21) *converges exponentially to zero.*

In more detail, let us denote by $S_{\infty,1}$ the solution of the second algebraic equation in (2.21) corresponding to $\theta = 1$; which should not be confused with the actual tuning of the observer. The authors in [46] then show that the state estimation error

$$e(t) = x(t) - \hat{x}(t)$$

satisfies the inequality

$$|e(t)| \leq k_\theta |e(0)| \exp\left(-\frac{\theta}{3}t\right), \quad \forall t \geq 0$$

as long as the parameter θ is large enough and for some $k_\theta > 0$. In fact, if we denote

$$c = \max_i c_i, \quad s = \max_{i,j} |(S_{\infty,1})_{i,j}|$$

then the tuning lower bound takes the form

$$\theta > \frac{6nc\sqrt{s}}{\lambda_{\min}(S_{\infty,1})},$$

where the right-hand side of the inequality is independent of θ. However, θ should dominate the product nc that depends on the system dimension and the Lipschitz constants of the functions ζ_i.

Remark 2.3. *The high-gain technique from [46] is generalized in [23]. The authors provide the more familiar observer form*

$$\dot{\hat{x}} = A\hat{x} + \zeta(\hat{x}, u) - \Delta_\theta K(C\hat{x} - y), \tag{2.22}$$

where K *is such that* $A - KC$ *is Hurwitz,* $\theta > 0$ *is large enough and* Δ_θ *is as in* (2.17).

Just as in the linear case, the natural extension of these results concerns time-varying continuous matrices. Moreover, let us suppose that the matrix in the system dynamics (2.18) depends continuously on the input so that we have

$$A(u) = \begin{bmatrix} 0 & a_1(u) & & 0 \\ & \ddots & \ddots & \\ & & \ddots & a_{n-1}(u) \\ 0 & & & 0 \end{bmatrix}, \quad C(u) = \begin{bmatrix} c_1(u) & 0 & \dots & 0 \end{bmatrix},$$

and the system

$$\begin{cases} \dot{x} = A(u)x + \zeta(x,u) \\ y = C(u)x. \end{cases} \tag{2.23}$$

As in Remark 2.1, it is useful to restrict the set $\mathcal{U}$ of admissible inputs. However, this time we are interested in small time intervals. The property we look for is the *local regularity* of a bounded input u: there exist positive real numbers $a > 0$ and $\theta_0 > 0$ such that for all $\theta \geq \theta_0$ we have

$$\int_{t-\frac{1}{\theta}}^{t} \Phi_u(s,t)'C(u)'C(u)\Phi_u(s,t)ds \geq a\theta\Delta_\theta^{-2}, \quad \forall t \geq \frac{1}{\theta} \tag{2.24}$$

where Δ_θ is as in (2.17) and, as before, Φ_u is the transition matrix:

$$\begin{cases} \frac{\partial \Phi_u}{\partial s}(s,t) = A(u(s))\Phi_u(s,t) \\ \Phi_u(t,t) = I, \end{cases}$$

for all s, $t \in \mathbb{R}^+$. The authors in [19] develop an exponential observer for system (2.23) given by

$$\begin{cases} \dot{\hat{x}} = A(u)\hat{x} + \zeta(\hat{x},u) - \Delta_\theta S^{-1}C(u)'(C(u)\hat{x} - y) \\ \dot{S} = -\theta(\gamma S + A(u)'S + SA(u) - C(u)'C(u)), \end{cases} \tag{2.25}$$

for $\theta > 0$ large enough and where $\gamma > 0$ guarantees the stability of the equation defining $S(t)$. Indeed, it can be shown that the local regularity of u implies that there exists $\gamma > 0$ and α_1, $\alpha_2 > 0$ such that for all $\theta \geq \theta_0$ we have

$$\alpha_1 I \leq S(t) \leq \alpha_2 I, \quad \forall t \geq \frac{1}{\theta}.$$

Theorem 2.8 ([19]). *Consider system* (2.23) *and suppose that the nonlinearity* $\zeta(x,u)$ *is globally Lipschitz in* x*, uniformly in* u*, that the bounded input* u *satisfies* (2.24) *and that* θ *is large enough. Then, the state estimation error from* (2.25) *converges exponentially to zero.*

The exponential rate of convergence of the state estimation error can then be arbitrarily set by choosing

$$\theta > \max\left\{\frac{2\alpha_2}{\gamma\alpha_1}c, \theta_0\right\},$$

where c is the Lipschitz constant of ζ. Their work extends previous results presented in [21].

Remark 2.4. *The authors in [39] show that it is possible to design a constant gain observer (Luenberger-like) for system* (2.23) *when the system input is locally regular. Furthermore, they define the weaker notion of strongly persistent input for which, in turn, they exhibit an observer design with dynamic gain.*

2.3.3 Observer design in the presence of noise

The nonlinearity ζ in the system dynamics of both (2.18) and (2.23) make us restrict our family of systems under consideration by imposing a specific form on the matrices A and C. Furthermore, the convergence to zero of the state estimation error e requires us to tune the observer with θ larger than a value depending on the Lipschitz constant of ζ. Two important problems arise from this fact: (i) strong peaking in the transient of e, which can destabilize the control loop [72], (ii) the measurement noise amplification as studied in this section.

The authors in [108] study the known problem of real-time signal differentiation by using a high-gain observer in the presence of measurement noise. Their work exhibits and quantifies the trade-off between the estimation of the signal derivative and the amplification of measurement noise. Let us consider a chain of integrators of the form

$$\begin{cases} \dot{x} = Ax + Bu^{(n)} \\ y = Cx + d, \end{cases} \tag{2.26}$$

for A and C in the canonical form (2.19) and where $B = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}'$, the signal $u : \mathbb{R}^+ \to \mathbb{R}$ is of class C^n and $d : \mathbb{R}^+ \to \mathbb{R}$ represents bounded measurement noise. Moreover, u and its derivatives are assumed to be bounded. Notice that the coordinates of x correspond to the derivatives of u, that is, a solution is given by

$$x = \begin{bmatrix} u \\ u^{(1)} \\ \vdots \\ u^{(n-1)} \end{bmatrix},$$

where $u^{(i)}$ stands for the $ith-$derivative of u with respect to time. Therefore, the output measurements are precisely the signal u affected by noise and the task is to estimate the $n-1$ derivatives of u. The linear high-gain observer for system (2.26) used in [108] is given by

$$\begin{cases} \dot{\hat{x}}_i = \hat{x}_{i+1} + \alpha_i \epsilon^{-i}(y - \hat{x}_1), \quad 1 \leq i \leq n-1 \\ \dot{\hat{x}}_n = \alpha_n \epsilon^{-n}(y - \hat{x}_1), \end{cases} \tag{2.27}$$

where the polynomial

$$s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$

is Hurwitz and where $\epsilon > 0$ is a tuning parameter. The following theorem states that the estimation error can be bounded coordinate-wise.

Theorem 2.9 ([108]). *Consider system* (2.26)*, observer* (2.27) *and* $1 \leq i \leq n$*. For an arbitrarily small quantity* $\delta > 0$ *there exists* $T > 0$ *large enough such that:*

$$|u^{(i-1)}(t) - \hat{x}_i(t)| \leq (p_{i+1}|u^{(n)}|_\infty) \cdot \epsilon^{n-i} + (q_{i+1}|d|_\infty) \cdot \epsilon^{-i} + \delta, \quad \forall t \geq T, \tag{2.28}$$

where the constants p_i*,* $q_i \geq 0$ *depend only on* $\alpha_1, \ldots \alpha_n$*.*

The right-hand side of inequality (2.28) is a strictly convex function of $\epsilon > 0$ that should be minimized. In the absence of measurement noise ($d = 0$), this is achieved by making $\epsilon \to 0$. Otherwise, a unique minimum is reached at

$$\epsilon_i^* = \left(\frac{i}{n-i}\right)^{1/n} \left(\frac{q_{i+1}|d|_\infty}{p_{i+1}|u^{(n)}|_\infty}\right)^{1/n},$$

which depends on the coordinate i under consideration. Nevertheless, their simulations show that ϵ_i^* does not vary much with i and that it is possible to choose ϵ^* as the average of these values. Importantly, these simulations also show that minimizing the error bound is not far from minimizing the state estimation error itself. Therefore, the authors provide a systematic way of tuning their linear high-gain observer in the presence of measurement noise.

Instead of system (2.26) we can consider

$$\begin{cases} \dot{x} = Ax + B\zeta(x) \\ y = Cx + d, \end{cases} \tag{2.29}$$

where A, B and C remain in the canonical form as before, and where $\zeta : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous. In the case of no disturbances ($d = 0$), using the observer in (2.22) implies there exist constants c_1, $c_2 > 0$ such that

$$|e(t)| \leq c_1 \theta^{n-1} |e(0)| \exp(-c_2 \theta t), \quad \forall t \geq 0.$$

In particular, this means that we can set a large exponential decay rate at the price of a *peaking* of order $n-1$. Furthermore, in practice, it can be hard to implement observer (2.22) for large tuning θ and for large state dimension n. This is due to the fact that θ^n appears in the observer gain. As a consequence, the authors in [11] present a novel high-gain observer for system (2.29) with a limited gain up to power two and that preserves the same observer properties. Their innovative technique consists in increasing the observer state dimension to $2n-2$ instead of n.

Although the observer developed in [11] seems to have high-frequency noise rejection properties, it still suffers from the so-called *peaking phenomenon*. A useful methodology to face this challenge and the noise amplification itself consists in a dynamic adaptation of the observer gain. This gives rise to an observer design methodology known as adaptive observers.

The authors in [25] implement a dynamic gain tuning θ in the classic observer design for a system of the form (2.29). The noise is not taken into consideration and θ can only increase. On the other hand, dynamic uncertainties and measurement noise is addressed in the work presented in [4]. The main idea is: (i) increase the tuning when the output error is large in order to achieve fast reconstruction, (ii) once it becomes small, decrease the tuning to balance the influence of both types of disturbances. Moreover, a switching delay timer is included to handle the peaking in the estimates. The authors in [20] study gain adaptation for a system of the form (2.23) by a mix approach: the observer switches between Kalman filtering and high-gain techniques. There exists a vast literature concerning adaptive observers which also includes papers such as [8, 98, 90] and many more.

2.4 Input-to-state stability and observers

In the previous section, we have seen the influence of different types of disturbances in the state estimation problem. However, we would like to have a general tool to determine if a given observer for a general nonlinear system has robustness properties with respect to measurement noise. The framework of Input-to-state stability (ISS) fits our needs since it quantifies the effect of external inputs: we can regard the error dynamics as a system whose input is precisely the measurement noise. This gives rise to the so-called ISS-observers or disturbance-to-error stable (DES) observers. In what follows, the operator sup is used to indicate either supremum or essential supremum, that is, up to a zero measure set.

2.4.1 Input-to-state stability

The concept of input-to-state stability was introduced by E. Sontag in his celebrated paper [101] as a test for the robustness of nonlinear systems to external perturbations, for a concise summary see [33]. As a motivational example, consider the linear system described by

$$\dot{x} = Ax + Bu, \tag{2.30}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ the input and A and B constant matrices. The solution corresponding to the initial condition $x(0)$ can be written as

$$x(t) = \exp(At)x(0) + \int_0^t \exp(A(t-s))Bu(s)ds, \quad \forall t \geq 0$$

and, if A is Hurwitz, we can dominate the system state with

$$|x(t)| \leq c_1 \exp(-c_2 t)|x(0)| + c_3 \sup_{s\in[0,t]} |u(s)|, \quad \forall t \geq 0. \tag{2.31}$$

The positive constants c_1, c_2 and c_3 are such that

$$|\exp(At)| \leq c_1 \exp(-c_2 t), \quad c_3 = c_1 c_2^{-1}|B|$$

and, thus, they are independent of $x(0)$ and u. The first term in (2.31) quantifies the effect of the initial condition for short times, while the second term accounts for the input impact. These ideas can be generalized for nonlinear systems by using *comparison functions*.

Definition 2.2. *A function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{K}$ if $\gamma(0) = 0$ and if it is strictly increasing and continuous. It is of class $\mathcal{K}_\infty$ if it is also unbounded. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{KL}$ if $\beta(r, t)$ is of class $\mathcal{K}$ for each $t \in \mathbb{R}^+$ and if $\beta(r, t)$ decreases to zero as $t \to \infty$ for each $r \in \mathbb{R}^+$. We use the notation $\gamma \in \mathcal{K}$ or $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$.*

For the following, we consider a nonlinear system of the form

$$\dot{x} = f(x, u), \tag{2.32}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ the input and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ a continuously differentiable function such that $f(0, 0) = 0$.

Definition 2.3. *System* (2.32) *is said to be input-to-state practically stable (ISpS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ and a constant $c \geq 0$ such that for all inputs $u \in L^\infty_m$ and all initial conditions $x(0) \in \mathbb{R}^n$, the solution x is defined on $\mathbb{R}^+$ and it holds that*

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(|u|_\infty) + c, \quad \forall t \geq 0. \tag{2.33}$$

The system is input-to-state stable (ISS) if the latter is satisfied with $c = 0$.

An alternative definition makes use of $\sup_{s\in[0,t]} \gamma(|u(s)|)$ instead of $\gamma(|u|_\infty)$, this is justified by the causality of the system. In such a case, the space of inputs can be defined by local boundedness ($L^\infty_{loc,m}$).

Remark 2.5. *We can recover a well-known stability property of system* (2.32) *if we fix $u = 0$. Indeed, the equilibrium point $x = 0$ is stable if for each $\epsilon > 0$ there exists $\delta > 0$ such that $|x(0)| < \delta \implies |x(t)| < \epsilon$, for all $t \in \mathbb{R}^+$. If additionally there exists $\delta > 0$ such that $|x(0)| < \delta \implies \lim_{t\to\infty} x(t) = 0$, we say the point is asymptotically stable. If for such a point the latter happens for all $x(0) \in \mathbb{R}^n$, then system* (2.32) *is called 0-globally asymptotically stable (0-GAS); which is equivalent to* (2.33) *with $u = 0$ and $c = 0$.*

The concept of ISS was initially motivated by the problem of disturbances in state feedback stabilization. In fact, it is of interest to analyze the influence of a time-varying disturbance d in the closed-loop system

$$\dot{x} = f(x, k(x) + d),$$

where k *stabilizes* the system for $d = 0$. The author in [101] shows that, for control-affine systems, k can be modified in order to achieve ISS when regarding d as the system input.

The elegant formulation of ISS has now become a standard tool in the literature in order to study robustness of nonlinear systems with respect to inputs in a wide range of situations.

Showing directly the ISS of a system can be challenging. Therefore, the following definition and theorem provide a Lyapunov characterization of ISS.

Definition 2.4. *A smooth function $V : \mathbb{R}^n \to \mathbb{R}^+$ is called an ISpS-Lyapunov function for system* (2.32) *if there exist functions $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3,\ \chi \in \mathcal{K}$, and a constant $c_L \geq 0$ such that for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$ we have*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),$$

and $|x| \geq \chi(|u|) + c_L$ implies

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\alpha_3(|x|).$$

The function is called ISS-Lyapunov if the latter holds with $c_L = 0$.

Theorem 2.10 ([103, 102])**.** *System* (2.32) *is ISpS if and only if it has an ISpS-Lyapunov function. The same equivalence holds for ISS.*

As an example, the linear system (2.30) is ISS precisely when A is Hurwitz. We can construct the ISS-Lyapunov function as $V(x) = x'Px$, where $A'P + PA < 0$.

2.4.2 Small-gain theorem

It is often the case in control analysis that a plant is connected with feedback laws or adjacent systems. The stability of the resulting closed-loop can be establish by using the so-called small-gain results. The classical small-gain theorem [113] arises in the context of input-output stability theory in the 1960s and guarantees the stability of the feedback loop if the loop gain is less than one. However, most of the early work concerning small-gain results considers normed-based linear gains. Further studies incorporate nonlinear function gains in the ISS framework [64] giving rise to a vast literature in the topic.

Consider a system described by

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u), \end{cases} \tag{2.34}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ an input and $y(t) \in \mathbb{R}^p$ the output. The functions f and h are assumed smooth.

Definition 2.5. *System* (2.34) *is input-to-output practically stable (IOpS) if there exist functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and a constant $c_d \geq 0$ such that for all $x(0) \in \mathbb{R}^n$ and all $u \in L_m^\infty$ we have*

$$|y(t)| \leq \beta(|x(0)|, t) + \gamma(|u|_\infty) + c_d, \quad \forall t \in [0, T),$$

where $[0, T)$ is the maximal interval of existence for x. If $c_d = 0$, then the system is called input-to-output stable (IOS).

In the case that $y = x$, the previous definition corresponds to input-to-state practical stability, and to ISS if in addition $c_d = 0$. The following definition is trivially satisfied in the case $y = x$.

Definition 2.6. *System* (2.34) *has the unboundedness observability (UO) property if there exist a function $\alpha \in \mathcal{K}$ and a constant $c_D \geq 0$ such that for each initial condition $x(0) \in \mathbb{R}^n$ and each input $u \in L_m^\infty$ we have*

$$|x(t)| \leq \alpha\left(|x(0)| + \sup_{s\in[0,t]} |(u(s), y(s))|\right) + c_D, \quad \forall t \in [0, T),$$

where $[0, T)$ is the maximal interval of existence for x.

Now let us consider an interconnected system given by

$$\begin{cases} \dot{x}_1 = f_1(x_1, y_2, u_1), & y_1 = h_1(x_1, y_2, u_1) \\ \dot{x}_2 = f_2(x_2, y_1, u_2), & y_2 = h_2(x_2, y_1, u_2) \end{cases} \tag{2.35}$$

and suppose that the first subsystem is IOpS when considering y_2 and u_1 as the input. This results in an inequality of the form

$$|y_1(t)| \leq \beta_1(|x_1(0)|, t) + \sup_{s\in[0,t]} \gamma_{11}(|y_2(s)|) + \gamma_{12}(|u_1|_\infty) + c_{d_1}.$$

Analogously, if the second subsystem is also IOpS we get

$$|y_2(t)| \leq \beta_2(|x_2(0)|, t) + \sup_{s\in[0,t]} \gamma_{21}(|y_1(s)|) + \gamma_{22}(|u_2|_\infty) + c_{d_2}.$$

Theorem 2.11 ([64])**.** *Suppose that each subsystem of* (2.35) *is IOpS as just described and* *that each subsystem has the UO property. If there exist functions ρ_1, $\rho_2 \in \mathcal{K}_\infty$ and a constant $r_0 \geq 0$ such that*

$$(I+\rho_2)\circ\gamma_{21}\circ(I+\rho_1)\circ\gamma_{11}(r)\leq r, \quad \forall r\geq r_0, \tag{2.36}$$

then the overall system (2.35) *is IOpS with input (u_1, u_2) and output (y_1, y_2) and it has the UO property.*

Remark 2.6. *The condition in* (2.36) *can be changed for*

$$(I+\rho_1)\circ\gamma_{11}\circ(I+\rho_2)\circ\gamma_{21}(r)\leq r, \quad \forall r\geq r_0.$$

If both subsystems have the UO property with $c_{D_1} = c_{D_2} = 0$, and if $c_{d_1} = c_{d_2} = r_0 = 0$, then the overall system (2.35) *is IOS. Notice that Theorem 2.11 can also be used to show the ISS of an interconnection of ISS subsystems. In such a case, the UO property is superfluous.*

The small-gain theorem has not only a trajectory based formulation but also a Lyapunov one. Let us consider an interconnected system described by

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, u_1) \\ \dot{x}_2 = f_2(x_1, x_2, u_2), \end{cases} \tag{2.37}$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$ and $f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ are locally Lipschitz. As before, we suppose that each subsystem is ISpS by using the Lyapunov functions V_1 and V_2. That is, V_i for $i \neq j$ satisfies for all $x_1 \in \mathbb{R}^{n_1}$, all $x_2 \in \mathbb{R}^{n_2}$ and all $u_i \in \mathbb{R}^{m_i}$:

$$\alpha_{i1}(|x_i|) \leq V_i(x_i) \leq \alpha_{i2}(|x_i|),$$

and $V_i(x_i) \geq \max\{\chi_i(V_j(x_j)), \gamma_i(|u_i|) + c_{Li}\}$ implies

$$\frac{\partial V_i}{\partial x_i}(x_i) f_i(x_1, x_2, u_i) \leq -\alpha_{i3}(V_i(x_i)),$$

where the functions α_{i1}, α_{i2}, $\alpha_{i3} \in \mathcal{K}_\infty$, χ_i, $\gamma_i \in \mathcal{K}$ and the constant $c_{Li} \geq 0$. We can state the following result by using the gains χ_1 and χ_2.

Theorem 2.12 ([63])**.** *Suppose that there exist Lyapunov functions V_1 and V_2 respectively for the first and second subsystem of* (2.37) *as described above. If there exists $r_0 \geq 0$ such that*

$$\chi_1 \circ \chi_2(r) < r, \quad \forall r > r_0, \tag{2.38}$$

then the overall system (2.37) *is ISpS with input* (u_1, u_2) *and state* (x_1, x_2)*.*

Remark 2.7. *Some remarks about the previous theorem. The condition in* (2.38) *can be changed for*

$$\chi_2 \circ \chi_1(r) < r, \quad \forall r > r_0.$$

If $c_{L1} = c_{L2} = r_0 = 0$*, then the overall system* (2.37) *is ISS and a zero input* $(u_1, u_2) = 0$ *implies global asymptotic stability.*

Remark 2.8. *The proof of Theorem 2.12 relies on the construction of an ISpS-Lyapunov function for the overall system, this function is defined on* $\mathbb{R}^{n_1+n_2}$ *as*

$$V(x_1, x_2) = \max\{\sigma(V_1(x_1)), V_2(x_2)\},$$

where $\sigma \in \mathcal{K}_\infty$ *is continuously differentiable on the positive real line, with a positive derivative there and, if* $\chi_1 \in \mathcal{K}_\infty$*, it satisfies*

$$\chi_2(r) < \sigma(r) < \chi_1^{-1}(r), \quad \forall r > 0.$$

Therefore, the function V *is only locally Lipschitz on* $\mathbb{R}^{n_1+n_2} - \{0\}$ *and differentiable almost everywhere. Nevertheless, this suffices to show the ISpS property.*

A natural generalization of the nonlinear small-gain results concerns interconnections of more than two ISS subsystems as in [32]. In such a case, the interconnection gains are collected in a matrix and bounds on its image establish the small-gain condition. The corresponding Lyapunov based approach is also studied in [35].

2.4.3 Observer robustness criterion

As we previously saw, the notion of ISS is a widely used tool for the study of the stability of nonlinear systems with inputs. On the other hand, the more general notion of IOS quantifies the effect of external perturbations on the system output. Another dual property is the so-called *output-to-state stability* (OSS) [104] which follows the same ideas and, in the linear case, corresponds to *detectability* of the system. The notion of *input/output to state stability* (IOSS), also studied in [104], combines the definitions of both ISS and OSS. The concept of IOSS, and its incremental version, are of particular interest to us given their connection with the so-called *ISS-observers*, observers that are robust with respect to measurement noise.

Here we consider a system described by

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x), \end{cases} \tag{2.39}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous, locally Lipschitz on x uniformly on bounded u, $f(0,0) = 0$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuously differentiable with $h(0) = 0$. We denote the trajectory and the output corresponding to the initial condition $x_0 = x(0)$ and to the input u by $x(t; x_0, u)$ and $y(t; x_0, u)$ respectively. The following condition is stronger than IOSS.

Definition 2.7. *System* (2.39) *is incrementally input/output-to-state stable or i-IOSS if there exist functions $\beta \in \mathcal{KL}$ and γ_1, $\gamma_2 \in \mathcal{K}$ such that for all inputs u, $\bar{u} \in L^\infty_{loc,m}$ and for all initial conditions x_0, $\bar{x}_0 \in \mathbb{R}^n$ it holds that*

$$\begin{aligned} |x(t; x_0, u) - x(t; \bar{x}_0, \bar{u})| \leq &\max\{\beta(|x_0 - \bar{x}_0|, t), \sup_{s\in[0,t]} \gamma_1(|u(s) - \bar{u}(s)|), \\ &\sup_{s\in[0,t]} \gamma_2(|y(s; x_0, u) - y(s; \bar{x}_0, \bar{u})|)\}, \quad \forall t \in [0, T), \end{aligned}$$

where $[0, T)$ is the common domain of definition.

A full-order observer for system (2.39) subject to measurement noise d_y and input noise d_u is specified by the dynamics

$$\dot{\hat{x}} = \hat{f}(\hat{x}, y + d_y, u + d_u),$$

and its robustness can be quantified as follows: there exist $\beta \in \mathcal{KL}$ and γ_1, $\gamma_2 \in \mathcal{K}$ such that for all u, $d_u \in L^\infty_{loc,m}$, all $d_y \in L^\infty_{loc,p}$ and all $x(0)$, $\hat{x}(0) \in \mathbb{R}^n$ we have

$$|x(t) - \hat{x}(t)| \leq \max\{\beta(|x(0) - \hat{x}(0)|, t), \sup_{s\in[0,t]} \gamma_1(|d_u(s)|), \sup_{s\in[0,t]} \gamma_2(|d_y(s)|)\},$$

for all $t \in \mathbb{R}^+$ in the maximal interval of existence for x (where $\hat{x}$ should exist). Such an *ISS-observer* makes the state estimation error converge to zero in the absence of noise and ensures a good estimation degradation in the case of disturbances. The following result sets a condition on the system for the existence of such an observer.

Proposition 2.1 ([104])**.** *System* (2.39) *is i-IOSS if it has an ISS-observer as described above.*

The authors in [100] first introduce a nonlinear observer design framework by using passivation of error dynamics. They then present a redesign of the observer injection term to achieve, under certain conditions, robustness with respect to measurement noise in an ISS sense as above. In the same context, the authors in [5] design an observer for a class of systems with Lipschitz nonlinearities and also consider its robustness with respect to both system uncertainties and measurement noise. An advantage of their observer is that it does not require state-space transformations. Indeed, consider a system of the form

$$\begin{cases} \dot{x} = Ax + f(x) + D_1 d \\ y = Cx + D_2 d, \end{cases} \tag{2.40}$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$, and where $d \in L^\infty_{loc,q}$ is a time-varying disturbance. The matrices A, C, D_1 and D_2 are constant and such that (A, C) is observable, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous: there exists a constant $c_f > 0$ such that

$$|f(x_1) - f(x_2)| \leq c_f |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^n.$$

The work in [5] proposes a Luenberger-like observer given by

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + L(y - C\hat{x}), \tag{2.41}$$

where $\hat{x}$ is the state estimate and L is a gain matrix to be designed.

Theorem 2.13 ([5])**.** *Consider system* (2.40) *and suppose that there exist $\alpha > 0$, a symmetric and positive definite matrix P and a matrix L such that*

$$\begin{bmatrix} (A - LC)'P + P(A - LC) + \alpha c_f^2 I & P \\ P & -\frac{\alpha}{2} I \end{bmatrix} < 0, \tag{2.42}$$

then (2.41) *is an ISS-observer.*

Remark 2.9. *The matrix inequality in* (2.42) *can be written as an LMI by using the change of variables $L = P^{-1}K$ for a matrix K. However, the observer design is not always possible: the existence of a matrix L such that $A - LC$ is Hurwitz is a necessary condition for* (2.42) *and a large Lipschitz constant c_f can make the LMI unfeasible.*

2.5 Some preliminary results

The purpose of this section is to develop the necessary tools for Chapter 4. It includes a useful reformulation of the small-gain theorem and some other results involving comparison functions. A comprehensive study of the function classes $\mathcal{K}$ and $\mathcal{KL}$ can be found in [70].

Remark 2.10. *The function classes $\mathcal{K}$ and $\mathcal{K}_\infty$ are closed under composition. Also, functions in $\mathcal{K}_\infty$ are invertible and their inverses remain in the class. A frequently used triangle-type inequality for $\gamma \in \mathcal{K}$ is*

$$\gamma(r_1 + r_2) \leq \gamma(2r_1) + \gamma(2r_2), \quad \forall r_1, r_2 \in \mathbb{R}^+.$$

Proposition 2.2 ([45])**.** *For any locally Lipschitz function $\phi : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$, there exist locally Lipschitz functions $\varphi : \mathbb{R}^{n_1} \to \mathbb{R}^+$ and $\alpha \in \mathcal{K}_\infty$ such that*

$$|\phi(z_1) - \phi(z_2)| \leq \varphi(z_1)\alpha(|z_1 - z_2|), \quad \forall z_1, z_2 \in \mathbb{R}^{n_1}.$$

We now reformulate the definition of ISpS by considering families of function pairs, this will simplify our exposition of Chapter 4. As can be seen in the proofs that follow, the classical results hold in this setting.

Definition 2.8. *Consider any family $\mathcal{G}$ formed by pairs of locally Lipschitz functions $(z, w) : I_{z,w} = [0, T_{z,w}) \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. We say that $\mathcal{G}$ is practically stable if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ and a constant $c \geq 0$ such that for all $(z, w) \in \mathcal{G}$ we have*

$$|z(t)| \leq \beta(|z(0)|, t) + \sup_{s \in [0,t]} \gamma(|w(s)|) + c, \quad \forall t \in I_{z,w}. \tag{2.43}$$

The functions β and γ and the constant c are respectively called decay rate, gain and constant of the practical stability.

In order to enlarge the class of admissible ISpS-Lyapunov functions, the next definition uses the fact that locally Lipschitz functions are differentiable almost everywhere (a.e.); this is known as Rademacher's theorem.

Definition 2.9. *A function $V : \mathbb{R}^+ \times \mathbb{R}^{n_1} \to \mathbb{R}^+$ is a Lyapunov function for the family $\mathcal{G}$ if: (i) for each $(z, w) \in \mathcal{G}$, $V^z = V(\cdot, z(\cdot))$ is everywhere continuous and locally Lipschitz outside*

$z^{-1}(0)$, *(ii) there exist* α_1, $\alpha_2 \in \mathcal{K}_\infty$, α_3, $\chi \in \mathcal{K}$ *and* $c_L \geq 0$ *such that for all* $(z, w) \in \mathcal{G}$:

$$\alpha_1(|z(t)|) \leq V^z(t) \leq \alpha_2(|z(t)|), \tag{2.44}$$

for all $t \in I_{z,w}$ *and*

$$\dot{V}^z(t) \leq -\alpha_3(|z(t)|), \tag{2.45}$$

if $|z(t)| \geq \chi(|w(t)|) + c_L$ *and for a.e.* $t \in I_{z,w} \setminus z^{-1}(0)$. *The functions* α_i, *the function* χ *and the constant* c_L *are respectively called Lyapunov-bounds, gain and constant.*

As usual, Lyapunov functions can be used for showing practical stability. The proof of the corresponding result goes along the lines of that in [102] and it is partly based on a well known comparison principle we next state.

Lemma 2.1 ([80])**.** *For any* $\hat{\alpha} \in \mathcal{K}$ *there exists* $\hat{\beta} \in \mathcal{KL}$ *with the following property. For every* $s \in (0, \infty]$ *and for every locally Lipschitz function* $\zeta : [0, s) \to \mathbb{R}^+$ *the inequality*

$$\begin{aligned}&\dot{\zeta}(t) \leq -\hat{\alpha}(\zeta(t)), \text{ for a.e. } t \in [0, s) \quad \text{implies}\\ &\zeta(t) \leq \hat{\beta}(\zeta(0), t), \text{ for every } t \in [0, s).\end{aligned}$$

Moreover, if we denote $\eta(r) = -\int_1^r \frac{dq}{\min\{q, \hat{\alpha}(q)\}}$ *then we can select*

$$\hat{\beta}(r, t) = \begin{cases} \eta^{-1}(\eta(r) + t), \text{ if } r > 0\\ 0, \text{ if } r = 0.\end{cases}$$

Theorem 2.14 (Lyapunov function)**.** *Consider any family* $\mathcal{G}$ *formed by pairs of locally Lipschitz functions. Then* $\mathcal{G}$ *is practically stable if it has a Lyapunov function.*

Proof. Fix for the moment $(z, w) \in \mathcal{G}$ and define the function $V^z = V(\cdot, z)$ and the subset

$$S = \{t \in [0, T_{z,w}) | V^z(t) \leq \Gamma(t)\},$$

where

$$\Gamma(t) = \alpha_2(\chi(\sup_{s \in [0,t]} |w(s)|) + c_L). \tag{2.46}$$

First, we prove by contradiction that for all $t_0 \in \mathbb{R}^+$ we have that

$$t_0 \in S \implies t \in S, \quad \forall t \in [t_0, T_{z,w}). \tag{2.47}$$

If this is not the case, then there is $t_0 \in S$ such that

$$t^* = \inf\{t \in [t_0, T_{z,w}) | V^z(t) \geq \Gamma(t) + \epsilon\} < \infty,$$

where $\epsilon > 0$ is small enough. Notice that the continuity of V^z and property (2.44) imply that $t^* > t_0$ and that $\alpha_2(|z(t^*)|) > \Gamma(t^*)$. In particular, $z(t^*) \neq 0$. It follows from the definition of Γ in (2.46) and from property (2.45) that

$$\dot{V}^z(t) \leq -\alpha_3(|z(t)|),$$

for a.e. $t \in B_{\epsilon_1}(t^*) \subseteq [t_0, T_{z,w}) \setminus z^{-1}(0)$ and where ϵ_1 is a small enough radius. This means that V^z does not increase on $B_{\epsilon_1}(t^*)$, which contradicts the choice of t^*. We conclude that our assumption in (2.47) must be true.

Let us now define

$$s^* = \inf S \leq \infty$$

and assume for now that s^* is finite. Notice that $[s^*, T_{z,w}) \subseteq S$, indeed, this is a consequence of $s^* \in S$ and of the claim shown in (2.47). Therefore, property (2.44) implies that

$$\begin{aligned} |z(t)| &\leq \gamma\left(\sup_{s \in [0,t]} |w(s)|\right) + c \\ &= \sup_{s \in [0,t]} \gamma(|w(s)|) + c, \end{aligned} \tag{2.48}$$

for all $t \in [s^*, T_{z,w})$ and where

$$\gamma = \alpha_1^{-1}(2\alpha_2(2\chi)) \in \mathcal{K}, \quad c = \alpha_1^{-1}(2\alpha_2(2c_L)) \geq 0.$$

On the other hand, it is clear that $S \cap [0, s^*)$ is empty and, as a consequence, also $z^{-1}(0) \cap [0, s^*)$. Then, properties (2.44) and (2.45) provide

$$\dot{V}^z(t) \leq -\hat{\alpha}(V^z(t)),$$

for a.e. $t \in [0, s^*)$ and where $\hat{\alpha} = \alpha_3(\alpha_2^{-1}) \in \mathcal{K}$. We can then apply Lemma 2.1 to get a function $\hat{\beta} \in \mathcal{KL}$ such that $V^z(t) \leq \hat{\beta}(V^z(0), t)$, for all $t \in [0, s^*)$. Hence, property (2.44) implies that

$$|z(t)| \leq \beta(|z(0)|, t), \tag{2.49}$$

for all $t \in [0, s^*)$ and where

$$\beta(r, t) = \alpha_1^{-1}(\hat{\beta}(\alpha_2(r), t)),$$

for all $r, t \in \mathbb{R}^+$. It is straight-forward to check that $\beta \in \mathcal{KL}$. We conclude the desired inequality in (2.43) by joining inequalities (2.48) and (2.49).

We proceed similarly if s^* is infinite. The only difference being that inequality (2.49) is satisfied on the whole interval $[0, T_{z,w})$. Finally, notice that β, γ and c are independent from the choice of (z, w), and that $c = 0$ precisely when $c_L = 0$. □

The stability of the interconnection of ISpS systems can be achieved by imposing a small-gain condition. The proof of the corresponding result in our framework follows the ideas of [63] and, hence, it requires the next lemma.

Lemma 2.2 ([63])**.** *Consider functions $\sigma_1 \in \mathcal{K}$ and $\sigma_2 \in \mathcal{K}_\infty$ and suppose that $\sigma_1(r) < \sigma_2(r)$, for all $r > 0$. Then, there exists a function $\sigma \in \mathcal{K}_\infty$ that is continuously differentiable on $r > 0$ and that satisfies:*

$$0 < \frac{\partial \sigma}{\partial r}(r), \quad \sigma_1(r) < \sigma(r) < \sigma_2(r), \quad \forall r > 0.$$

Remark 2.11. *The fact that the derivative of σ is strictly positive on $r > 0$ is crucial. This is not necessarily true for an arbitrary function $\mathcal{K}_\infty \cap C^1$ as shown in [70]. Notice that the function σ at zero is no more than continuous and that it appears in the definition of the overall Lyapunov function V in the proof below. However, this does not cause any further problems given our specific definition of Lyapunov function.*

In order to state the small-gain theorem in our setting, let us define the inverse set

$$\mathcal{G}^{-1} = \{(w, z) | (z, w) \in \mathcal{G}\}$$

and suppose that:

- there is a Lyapunov function for $\mathcal{G}$ with corresponding Lyapunov-bounds, gain and constant: α_{11}, α_{12} and α_{13}, χ_1 and c_{L1},

- there is a Lyapunov function for $\mathcal{G}^{-1}$ with corresponding Lyapunov-bounds, gain and constant: α_{21}, α_{22} and α_{23}, χ_2 and c_{L2}.

We can then define the mixed Lyapunov gains as:

$$\chi_{m1} = \alpha_{12}\left(2\chi_1(\alpha_{21}^{-1})\right), \quad \chi_{m2} = \alpha_{22}\left(2\chi_2(\alpha_{11}^{-1})\right). \tag{2.50}$$

Theorem 2.15 (small-gain). *Consider any family $\mathcal{G}$ formed by pairs of locally Lipschitz functions $(z, w) : I_{z,w} = [0, T_{z,w}) \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Suppose that there exist a Lyapunov function for $\mathcal{G}$ and a Lyapunov function for $\mathcal{G}^{-1}$ (defined above). Given the mixed Lyapunov gains in* (2.50)*, if χ_1 is of class $\mathcal{K}_\infty$ and if*

$$\chi_{m1}(\chi_{m2}(r)) < r, \quad \forall r > 0 \tag{2.51}$$

then $\mathcal{G} \times \{0\}$ is practically stable. That is, there exist a function $\beta \in \mathcal{KL}$ and a constant $c \geq 0$ such that for all $(z, w) \in \mathcal{G}$ we have

$$|(z(t), w(t))| \leq \beta(|(z(0), w(0))|, t) + c, \quad \forall t \in I_{z,w}.$$

Proof. The small-gain condition in (2.51) implies the existence of σ as in Lemma 2.2, that is:

$$0 < \frac{\partial \sigma}{\partial r}(r), \quad \chi_{m2}(r) < \sigma(r) < \chi_{m1}^{-1}(r), \quad \forall r > 0. \tag{2.52}$$

By Theorem 2.14, it suffices to find a Lyapunov function for the family $\mathcal{G} \times \{0\}$. Consider the Lyapunov functions V_1 and V_2, respectively for $\mathcal{G}$ and for $\mathcal{G}^{-1}$, and set

$$V(t, (z, w)) = \max\{\sigma(V_1(t, z)), V_2(t, w)\},$$

for all $t \in \mathbb{R}^+$ and $(z, w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Let us fix for the moment any $(z, w) \in \mathcal{G}$ and, as before, the notation V_1^z, V_2^w and $V^{z,w}$ indicates evaluation on the corresponding trajectories.

It is clear that $V^{z,w}$ is everywhere continuous. Moreover, V_1^z and V_2^w are zero precisely where their corresponding trajectories are zero. Hence, $V^{z,w}$ is also locally Lipschitz on

$I_{z,w} \setminus (z^{-1}(0) \cup w^{-1}(0))$. We consider the following partition of the interval $I_{z,w}$:

$$\begin{aligned} D &= \{t \in I_{z,w} | \sigma(V_1^z(t)) = V_2^w(t)\}, \\ D_z &= \{t \in I_{z,w} | \sigma(V_1^z(t)) > V_2^w(t)\}, \\ D_w &= \{t \in I_{z,w} | \sigma(V_1^z(t)) < V_2^w(t)\}, \end{aligned} \tag{2.53}$$

in order to write

$$V^{z,w}(t) = \begin{cases} \sigma(V_1^z(t)), \text{ if } t \in D_z \\ V_2^w(t), \text{ if } t \in D_w \cup D. \end{cases}$$

The open subsets D_z and D_w of $I_{z,w}$ are contained respectively in $I_{z,w} \setminus z^{-1}(0)$ and $I_{z,w} \setminus w^{-1}(0)$. In particular, this implies that $V^{z,w}$ is also locally Lipschitz on $D_z \cup D_w$ and, consequently, on the whole $I_{z,w} \setminus (z,w)^{-1}(0)$ as needed.

The following verifies property (2.44) of $V^{z,w}$ by using:

$$\begin{aligned} \alpha_1 &= \frac{1}{2} \min\left\{\sigma\left(\alpha_{11}\left(\frac{\cdot}{2}\right)\right), \alpha_{21}\left(\frac{\cdot}{2}\right)\right\} \in \mathcal{K}_\infty, \\ \alpha_2 &= \max\{\sigma(\alpha_{12}), \alpha_{22}\} \in \mathcal{K}_\infty. \end{aligned}$$

Indeed, the corresponding properties of V_1^z and V_2^w imply:

$$\begin{aligned} \alpha_1(|(z(t), w(t))|) &\le \alpha_1(2|z(t)|) + \alpha_1(2|w(t)|) \\ &\le \max\{\sigma(V_1^z(t)), V_2^w(t)\} \\ &\le \max\{\sigma(\alpha_{12}(|z(t)|)), \alpha_{22}(|w(t)|)\} \\ &\le \alpha_2(|(z(t), w(t))|), \end{aligned}$$

for all $t \in \mathbb{R}^+$. We now continue by checking property (2.45) of $V^{z,w}$. For this purpose, we select the function

$$\alpha_3 = \frac{1}{2} \min\{\hat{\alpha}_1, \hat{\alpha}_2\} \in \mathcal{K},$$

where $\hat{\alpha}_1(0) = 0$ and

$$\begin{aligned} \hat{\alpha}_1 &= \frac{\partial \sigma}{\partial r}(\sigma^{-1}(\alpha_1)) \cdot \alpha_{13}(\alpha_{12}^{-1}(\sigma^{-1}(\alpha_1))), \\ \hat{\alpha}_2 &= \alpha_{23}(\alpha_{22}^{-1}(\alpha_1)) \end{aligned}$$

and we also select the non-negative constant

$$c_L = \alpha_1^{-1}(\sigma(\alpha_{12}(2c_{L1}))) + \alpha_1^{-1}(\alpha_{12}(2c_{L2})).$$

Fix a time $t_0 \in I_{z,w} \setminus (z,w)^{-1}(0)$ where $V^{z,w}$, V_1^z and V_2^w are differentiable and suppose that

$$|(z(t_0), w(t_0))| \geq c_L. \tag{2.54}$$

It then suffices to show the inequality

$$\dot{V^{z,w}}(t_0) \leq -\alpha_3(|(z(t_0), w(t_0))|). \tag{2.55}$$

Using (2.53), we have three possible cases:

1) Suppose that $t_0 \in D_z$, in particular, $z(t_0) \neq 0$. There exists $\epsilon_1 > 0$ such that

$$V^{z,w}(t) = \sigma(V_1^z(t)), \tag{2.56}$$

for all $t \in I_{z,w} \cap B_{\epsilon_1}(t_0)$. As a consequence,

$$\dot{V^{z,w}}(t_0) = \frac{\partial \sigma}{\partial r}(V_1^z(t_0)) \cdot \dot{V_1^z}(t_0). \tag{2.57}$$

Since

$$V_2^w(t_0) < \sigma(V_1^z(t_0)),$$

the inequality in (2.52) implies that

$$2\chi_1(\alpha_{21}^{-1}(V_2^w(t_0))) \leq \alpha_{12}^{-1}(V_1^z(t_0)).$$

From property (2.44) of V_1^z and V_2^w it follows that

$$\begin{aligned} \chi_1(|w(t_0)|) &\leq \chi_1(\alpha_{21}^{-1}(V_2^w(t_0))), \\ \alpha_{12}^{-1}(V_1^z(t_0)) &\leq |z(t_0)| \end{aligned}$$

and, therefore,

$$|z(t_0)| \geq 2\chi_1(|w(t_0)|). \tag{2.58}$$

On the other hand, (2.54) and (2.56) together with the properties of α_1 imply

$$\sigma(V_1^z(t_0)) \geq \sigma(\alpha_{12}(2c_{L1}))$$

and, hence,

$$|z(t_0)| \geq 2c_{L1}.$$

Summing this inequality with (2.58) leads to

$$|z(t_0)| \geq \chi_1(|w(t_0)|) + c_{L1}$$

and, using (2.56) and (2.57), property (2.45) of V_1^z implies:

$$\begin{aligned} V^{\dot{z},w}(t_0) &\leq -\frac{\partial\sigma}{\partial r}(\sigma^{-1}(V^{z,w}(t_0))) \cdot \alpha_{13}(\alpha_{12}^{-1}(\sigma^{-1}(V^{z,w}(t_0)))) \\ &\leq -\hat{\alpha}_1(|(z(t_0), w(t_0))|) \\ &< -\alpha_3(|(z(t_0), w(t_0))|). \end{aligned}$$

2) Suppose that $t_0 \in D_w$, in particular, $w(t_0) \neq 0$. The procedure in this case follows the same ideas. We have that:

$$V^{z,w}(t_0) = V_2^w(t_0), \quad V^{\dot{z},w}(t_0) = \dot{V_2^w}(t_0).$$

Inequality (2.52) implies

$$2\chi_2(\alpha_{11}^{-1}(V_1^z(t_0))) \leq \alpha_{22}^{-1}(V_2^w(t_0))$$

and, together with assumption (2.54), we have

$$|w(t_0)| \geq \chi_2(|z(t_0)|) + c_{L2}.$$

This in turn implies that

$$\begin{aligned} V^{\dot{z},w}(t_0) &\leq -\alpha_{23}(\alpha_{22}^{-1}(V^{z,w}(t_0))) \\ &\leq -\hat{\alpha}_2(|(z(t_0), w(t_0))|) \\ &< -\alpha_3(|(z(t_0), w(t_0))|). \end{aligned}$$

3) Now suppose that $t_0 \in D$ so that

$$\begin{aligned} V^{z,w}(t_0) &= \sigma(V_1^z(t_0)) \\ &= V_2^w(t_0). \end{aligned} \tag{2.59}$$

In particular, both $z(t_0)$ and $w(t_0)$ differ from zero. Just as in the previous two cases, we have:

$$\begin{aligned} \frac{\partial \sigma}{\partial r}(V_1^z(t_0)) \cdot \dot{V}_1^z(t_0) &< -\alpha_3(|(z(t_0), w(t_0))|), \\ \dot{V}_2^w(t_0) &< -\alpha_3(|(z(t_0), w(t_0))|). \end{aligned}$$

As a consequence, there exists $\epsilon_2 > 0$ such that

$$\begin{aligned} \frac{\sigma(V_1^z(t)) - \sigma(V_1^z(t_0))}{t - t_0} &< -\alpha_3(|(z(t_0), w(t_0))|), \\ \frac{V_2^w(t) - V_2^w(t_0)}{t - t_0} &< -\alpha_3(|(z(t_0), w(t_0))|), \end{aligned}$$

for all $t \in I_{z,w} \cap B_{\epsilon_2}(t_0)$, $t \neq t_0$. Then (2.59) leads to

$$\frac{V^{z,w}(t) - V^{z,w}(t_0)}{t - t_0} < -\alpha_3(|(z(t_0), w(t_0))|),$$

for all such t, which implies (2.55). Finally, notice that the functions α_1, α_2 and α_3 and the constant c_L are all independent from the choice of (z, w), and that $c = 0$ precisely when $c_{L1} = c_{L2} = 0$. $\square$

2.6 Conclusion

This chapter introduced the problem of observer design for dynamical systems. We presented the fundamental designs in the linear case known as the Luenberger observer and the Kalman filter, and we explored a deterministic interpretation of the latter. In the case of nonlinear systems, observability can depend on the input and this led to representative system structures for which it is possible to design observers more easily. In this context, we introduced the celebrated high-gain observers and we discussed the important issue of measurement noise amplification. The final part consisted on the theory of ISS and allied properties, on the corresponding small-gain theorems and on the quantification of observer robustness with respect to noise. We concluded by proving some useful results for Chapter 4.

Chapter 3

Optimal observer design for disturbed state-affine systems

3.1 Introduction

We analyzed in the previous chapter one of the standard methods of nonlinear observer design known as high-gain. A large enough tuning parameter is used in the observer structure to moderate the rate of convergence of the state estimation error. However, this method has a major drawback in practice where disturbances tend to affect the system: measurement noise sensitivity [76]. Therefore, a common strategy is to formulate an adaptive gain. Simply put, the gain is high when the state needs to be reconstructed and then it drops down to prevent noise amplification, see for example [4]. We remark that the systems involved in most of these studies are uniformly observable and must be represented in their canonical observability form, which is not always easy to obtain.

On the other hand, a common strategy for parameter identification using an observer relies on designing an input that optimizes estimations [40, 91]. In general, the selection of the input can have a considerable impact on the performance of an observer [16] and designing inputs with different types of regularity is not a trivial task. Nevertheless, in practice, this is usually done in a heuristic way. Some notable exceptions can be seen in [95, 96] where the authors consider state-affine systems.

The main part of this chapter, which is based on our work [51], considers the problem of disturbance robustness of an observer for state-affine systems. This class of systems is not necessarily in canonical form. Two types of bounded time-varying disturbances are distinguished: measurement noise affecting the output of the system and model uncertainties in the dynamics of the system. We investigate the robustness to noise of the high-gain Kalman observer and quantify a bound for the limiting estimation error. This bound depends on both the high-gain parameter and the system input, which should be properly excited. We present a novel approach of simultaneous off-line design of the high-gain parameter and the system input by minimizing this bound. Therefore, our method provides a new degree of freedom in the problem of noise attenuation.

The chapter is organized as follows. In the preliminaries, we state the basic results concerning observer design for state-affine systems and introduce the important concept of regular persistence of an input. We then present selected input design strategies from the literature that aim to optimize the observability for this class of systems. This is followed by our main contribution of this chapter and by the corresponding illustration of our method. Some brief concluding remarks are gathered at the end.

3.2 Preliminaries

3.2.1 Observer design for state-affine systems

State-affine systems are systems of the form

$$\begin{cases} \dot{x} = A(u)x + B(u) \\ y = Cx, \end{cases} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ the output, $u(t) \in \mathbb{R}^m$ a bounded and measurable input, $A : \mathbb{R}^m \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R}^m \to \mathbb{R}^n$ are continuous functions and C is a constant matrix. These systems are not uniformly observable: fixing an input u in (3.1) leads to a time-varying linear system and, thus, the observability properties should arise in the Gramian form of (2.6). Systems described by (3.1) are quite general and, in particular, include the so-called bilinear systems, for which the authors in [22, 28] present a high-gain version of the Kalman filter. Their results are then generalized to state-affine systems in [57]. Since B does not play a fundamental role in the observability properties of system (3.1), we set

$$B(u) = 0, \quad \forall u \in \mathbb{R}^m. \tag{3.2}$$

The observer from [57] takes the form

$$\begin{cases} \dot{\hat{x}} = A(u)\hat{x} - S^{-1}C'(C\hat{x} - y) \\ \dot{S} = -\theta S - A(u)'S - SA(u) + C'C, \end{cases} \tag{3.3}$$

where $\theta > 0$ and $S(0)$ is a symmetric and positive definite matrix. It is straight-forward to show that

$$\begin{aligned} S(t) = & \exp(-\theta t)\Phi_u(0,t)'S(0)\Phi_u(0,t) \\ & + \exp(-\theta t)\int_0^t \exp(\theta s)\Phi_u(s,t)'C'C\Phi_u(s,t)ds, \end{aligned} \tag{3.4}$$

where Φ_u is the transition matrix, namely, the solution on $\mathbb{R}^+ \times \mathbb{R}^+$ to

$$\begin{cases} \frac{\partial \Phi_u}{\partial s}(s,t) = A(u(s))\Phi_u(s,t) \\ \Phi_u(t,t) = I. \end{cases}$$

Therefore, we can see that the solution to the Riccati equation $S(t)$ remains symmetric and positive definite for all $t \in \mathbb{R}^+$. The following input constraint [16] keeps its minimum eigenvalue away from zero.

Definition 3.1. *The input u is regularly persistent for system* (3.1)-(3.2) *with respect to the triplet of positive real numbers (T, α, t_0) if $t_0 \geq T$ and if for all $t \geq t_0$ we have*

$$\int_{t-T}^{t} \Phi_u(s, t-T)'C'C\Phi_u(s, t-T)ds \geq \alpha I. \tag{3.5}$$

Theorem 3.1 ([57])**.** *Consider system* (3.1)-(3.2) *with a bounded and regularly persistent input u. Then, the state estimation from* (3.3) *converges exponentially to zero.*

In more detail, it can be shown that there exists a constant $k_{\theta,u} > 0$ such that the state estimation error $e = x - \hat{x}$ is bounded by

$$|e(t)|^2 \leq k_{\theta,u}|e(0)|^2 \exp(-\theta t), \quad \forall t \geq t_0.$$

As one might expect, the influence of the tuning parameter θ and of the input u appearing in (3.3) reflects on the decay rate and transient period of the state estimation error.

3.2.2 Optimal input design

A classical way to face the problem of dynamic and measurement disturbances in state estimation is to find a trade-off when tuning high-gain observers. However, the system input can also be implemented with the objective of improving the observer performance. The authors in [95] consider a discrete-time state-affine system with equations

$$\begin{cases} x_{k+1} = A(u_k)x_k + B(u_k) \\ y_k = C(u_k)x_k, \end{cases} \tag{3.6}$$

where $k \geq 0$ is an integer, $x_k \in \mathbb{R}^n$ the state, $y_k \in \mathbb{R}$ the output and $u_k \in \mathbb{R}$ the input to be designed. The functions $A(u_k)$, $B(u_k)$ and $C(u_k)$ are assumed to be bounded, and the transition matrix of the homogeneous part of the state equation is defined as

$$\Phi_u(k,l) = A(u_{k-1})A(u_{k-2})\dots A(u_l)$$

for $k \geq l$. If each of these matrices is invertible, then we can set $\Phi_u(l,k) = \Phi_u(k,l)^{-1}$.

Definition 3.2. *An input sequence (u_k) is regularly persistent for system (3.6) if $A(u_k)$ are invertible matrices and if there exist an integer $k_0 \geq 0$ and constants α_1, $\alpha_2 > 0$ such that*

$$\alpha_1 I \leq \Gamma_u(k,k_0) \leq \alpha_2 I, \quad \forall k \geq k_0,$$

where

$$\Gamma_u(k,k_0) = \sum_{i=k-k_0}^{k} \Phi_u(i,k)' C(u_i)' C(u_i) \Phi_u(i,k).$$

For such an input, it is possible to design an exponential observer which is a counterpart of the high-gain Kalman-filter for the time-continuous case [107]. Other desired properties for the input are: to be bounded, to have bounded consecutive variations and to remain close to a reference value u_{ref}. The following algorithm from [95] guarantees a prescribed degree of observability $\alpha > 0$ for system (3.6) under a cyclic input of period $N > n$:

$$
\begin{aligned}
&\min_u \sum_{k=0}^{N-1} (u_{ref} - u_k)^2, \\
&u_{min} < u_k < u_{max}, \quad 0 \le k \le N-1 \\
&|u_{k+1} - u_k| < v_{max}, \quad 0 \le k \le N-2 \\
&\Gamma(u_0, \dots, u_{N-1}) > \alpha I, \\
&\Gamma(u_1, \dots, u_{N-1}, u_0) > \alpha I, \\
&\vdots \\
&\Gamma(u_{N-1}, u_0, \dots, u_{N-2}) > \alpha I,
\end{aligned}
$$

where the conditions on Γ ensure a lower bound at any shifted time. Their subsequent work extensions are not based on a Gramian characterization of observability but, instead, they focus on the observability of the whole system [96]. The main idea being that input selection can be addressed by optimization when regarded as a control problem (observability-oriented optimal control formulation).

On the other hand, the authors in [40] design an explicit bang-bang controller for parameter identification based on a constrained Model Predictive Control (MPC) problem. Their goal is to design an input for an efficient estimation of the unknown time parameter $T > 0$ in the first order linear system

$$
\begin{cases}
\dot{x} = -\frac{1}{T}x + \frac{G}{T}u \\
y = x,
\end{cases}
$$

where $x(t) \in \mathbb{R}$ is the state, which coincides with the output $y(t)$, where $u(t) \in \mathbb{R}$ is the input to be designed and $G > 0$ the known static gain. The augmented model with state $x_a(t) = (x(t), \frac{1}{T})$ results in a state-affine system of the form

$$
\begin{cases}
\dot{x}_a = A_a(y,u)x_a \\
y = C_a x_a
\end{cases}, \quad A_a(y,u) = \begin{bmatrix} 0 & -y + Gu \\ 0 & 0 \end{bmatrix}, \quad C_a = \begin{bmatrix} 1 & 0 \end{bmatrix}
$$

for which a Kalman-like observer [16] can be designed under a persistent input to obtain the estimation $\hat{x}_a(t) = (\hat{x}(t), \frac{1}{\hat{T}(t)})$. Their technique consists in maximizing a norm of the sensitivities of x with respect to T, that is, of $x_s = \frac{\partial x}{\partial T}$ described by

$$
\dot{x}_s = \frac{1}{T^2}x - \frac{1}{T}x_s - \frac{G}{T^2}u,
$$

where $x_s(0) = 0$. The optimal input u^* is then given on a sequence of sampling times (t_k) by the optimization problem:

$$\max_{u_{inf} \leq u(t_k) \leq u_{sup}} \int_{t_k}^{t_k+N_p} x_s(t)^2 dt,$$

with u_{inf} and u_{sup} control bounds, N_p the prediction horizon and where $x_s(t)$ is computed for $t > t_k$ by using the Laplace transform at t_k and substituting T by its estimation $\hat{T}(t_k)$. The authors then proceed and give an optimal control law u^* that can be explicitly found offline and that results in an effective controller even for T varying on time periods.

Finally, the problem of constructing persistently exciting inputs for general nonlinear systems is addressed in [17], where the author proposes a relationship between the derivatives of output variables and the notion of persistent excitation. The general theory is then illustrated for the case of state-affine systems.

3.3 Robustness optimization based on Lyapunov techniques

The aim of this first part of our work is to *optimally* estimate the state vector of state-affine systems that are affected by noise. That is, we consider the system

$$\begin{cases} \dot{x}(t) = A(u(t))x(t) + d_1(t) \\ y(t) = Cx(t) + d_2(t), \end{cases} \tag{3.7}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}$ the output, $u(t) \in \mathbb{R}$ a continuous input, $d_1(t) \in \mathbb{R}^n$, $d_2(t) \in \mathbb{R}$ bounded and continuous disturbances, $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ a continuous matrix functional and $C \in \mathbb{R}^{1 \times n}$ a constant matrix. We use the previously presented observer

$$\begin{cases} \dot{\hat{x}} = A(u)\hat{x} + S^{-1}C'(y - C\hat{x}) \\ \dot{S} = -\theta S - A(u)'S - SA(u) + C'C, \end{cases} \tag{3.8}$$

where $S(0)$ is symmetric and positive definite and $\theta > 0$. Unfortunately, even for such choice of $S(0)$, the eigenvalues of $S(t)$ can still be arbitrarily close to zero for the so-called singular inputs; inputs where observability is lost. We add the next assumption to avoid this situation.

Assumption 3.1. *There exist real positive constants $c_1(u,\theta)$ and $c_2(u,\theta)$ such that, if t is large enough, then we have*

$$\begin{aligned} \lambda_{\max}\left(S(t)^2\right) &\leq c_1(u,\theta), \\ \lambda_{\min}\left(S(t)\right) &\geq c_2(u,\theta). \end{aligned} \tag{3.9}$$

The explicit dependencies of c_i on u and θ are usually omitted to simplify the notation.

Although the state estimation error might not converge to zero when disturbances are considered, it can still be bounded. Our goal is then to find a quantifiable bound for the estimation error

$$e = x - \hat{x} \tag{3.10}$$

in such a case. By minimizing the estimation error bound over an admissible space, we will provide a systematic way to find an optimal excitation u and an optimal observer tuning θ in terms of noise robustness.

3.3.1 Bound on the estimation error

The following Gronwall-type lemma is instrumental to the proof of the bound on the state estimation error.

Lemma 3.1 ([14])**.** *Consider the real numbers $t_0 < t_1$ and suppose that $f, g : [t_0, t_1] \to \mathbb{R}$ are continuous. If $h : [t_0, t_1] \to \mathbb{R}$ is of class C^1 and if*

$$\dot{h}(t) \leq f(t)h(t) + g(t), \quad \forall t \in [t_0, t_1],$$

then

$$h(t) \leq h(t_0) \exp\left(\int_{t_0}^{t} f(s)ds\right) + \int_{t_0}^{t} \exp\left(\int_{s}^{t} f(r)dr\right) g(s)ds, \quad \forall t \in [t_0, t_1].$$

For the sake of clarity, we denote the limiting disturbance magnitudes as

$$\begin{aligned} L_1 &= \limsup_{t\to\infty} |d_1(t)|, \\ L_2 &= \limsup_{t\to\infty} |d_2(t)|. \end{aligned} \tag{3.11}$$

Theorem 3.2. *Consider systems* (3.7) *and* (3.8) *for fixed* u *and* θ*, and suppose that Assumption 3.1 is fulfilled. Based on the definitions given in* (3.9), (3.10) *and* (3.11) *the following inequality is satisfied,*

$$\limsup_{t\to\infty} |e(t)| \leq \frac{2L_1 c_1 + 2L_2\sqrt{c_1\,\lambda_{\max}\left(C'C\right)}}{c_2(\theta\sqrt{c_1} + \lambda_{\min}\left(C'C\right))}.$$

Proof. Consider the Lyapunov function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ defined by

$$V(t, e(t)) = e(t)'S(t)e(t),$$

and its derivative along the error trajectories given by

$$\begin{aligned}
\dot{V} =& 2e'S\dot{e} + e'\dot{S}e \\
=& 2e'S(A(u)x + d_1 - A(u)\hat{x} \\
& - S^{-1}C'(Cx + d_2 - C\hat{x})) \\
& + e'(-\theta S - A(u)'S - SA(u) + C'C)e.
\end{aligned}$$

If we rearrange the equality shown above,

$$\begin{aligned}
\dot{V} =& 2e'SA(u)e - 2e'C'Ce - 2e'C'd_2 \\
& + 2e'Sd_1 - \theta e'Se - e'A(u)'Se \\
& - e'SA(u)e + e'C'Ce \\
=& - e'C'Ce - 2e'C'd_2 + 2e'Sd_1 - \theta e'Se.
\end{aligned}$$

This implies then that

$$\begin{aligned}
\dot{\sqrt{V}} =& \frac{1}{2\sqrt{V}}(-e'C'Ce - 2e'C'd_2 + 2e'Sd_1 \\
& - \theta e'Se).
\end{aligned}$$

On the other hand, by the Cauchy-Schwarz inequality we have

$$\begin{aligned} e'Sd_1 &\leq |Se||d_1|, \\ -e'C'd_2 &\leq |Ce||d_2|. \end{aligned}$$

Using (3.9) and denoting

$$\begin{aligned} \mu &= \frac{c_1}{c_2} > 0, \\ \mu_1 &= \frac{\lambda_{\max}\left(C'C\right)}{c_2} \geq 0 \end{aligned}$$

we obtain:

$$\begin{aligned} |Se|^2 &= e'S^2e \leq \mu V, \\ |Ce|^2 &= e'C'Ce \leq \mu_1 V. \end{aligned}$$

Similarly,

$$\frac{\lambda_{\min}\left(C'C\right)}{\sqrt{c_1}} V \leq e'C'Ce.$$

Putting all this together, we get that

$$\dot{\sqrt{V}} \leq \frac{\sqrt{V}}{2}(-\theta - \mu_2) + \sqrt{\mu}|d_1| + \sqrt{\mu_1}|d_2|,$$

where

$$\mu_2 = \frac{\lambda_{\min}\left(C'C\right)}{\sqrt{c_1}} \geq 0.$$

An inequality with this form is quite useful since Lemma 3.1 implies that for t large enough,

$$\begin{aligned} \sqrt{V(t, e(t))} \leq & \exp\left(\frac{t}{4}(-\theta - \mu_2)\right)\sqrt{V\left(\frac{t}{2}, e\left(\frac{t}{2}\right)\right)} \\ & + \sqrt{\mu}\int_{\frac{t}{2}}^{t} \exp\left((-\theta - \mu_2)\frac{(t-s)}{2}\right)|d_1(s)|ds \\ & + \sqrt{\mu_1}\int_{\frac{t}{2}}^{t} \exp\left((-\theta - \mu_2)\frac{(t-s)}{2}\right)|d_2(s)|ds. \end{aligned}$$

Notice that for any non-zero constant r,

$$\int_{\frac{t}{2}}^{t} \exp\left(r(t-s)\right) ds = -\frac{1}{r} + \frac{1}{r}\exp\left(r\frac{t}{2}\right)$$

and since V, d_1, d_2 are bounded, we can take the superior limit on both sides of the inequality to conclude that

$$\limsup_{t\to\infty} \sqrt{V(t,e(t))} \leq L_1 \frac{2\sqrt{\mu}}{\theta+\mu_2} + L_2 \frac{2\sqrt{\mu_1}}{\theta+\mu_2}.$$

Since also

$$|e|^2 \leq \frac{1}{c_2} V,$$

the terms can be expanded and arranged to conclude the result. □

3.3.2 Optimization of the bound

Theorem 3.2 provides a bound that can be minimized by playing simultaneously over positive tunings θ and admissible inputs u. To set this optimization problem correctly, we first need to specify a target function and its domain. We define the domain by considering inputs with the proper excitation and then redefine the bound in Theorem 3.2 as a function of u and θ by specifying an explicit choice of $c_1(u,\theta)$ and $c_2(u,\theta)$ in (3.9). This requires the concept of regular persistence of an input for the homogeneous part of system (3.7), see Definition 3.1.

Assumption 3.2. *The matrix functional $A(u(t))$ is bounded. This means that*

$$\sigma(u) = \sup_{t\geq 0} |A(u(t))| < \infty. \tag{3.12}$$

The explicit dependency of σ on u is usually omitted to simplify the notation.

The next lemma can be deduced from the work developed in [18]. The proof relies on the form of $S(t)$ given in (3.4) and on basic properties of the transition matrix.

Lemma 3.2. *Consider systems* (3.7) *and* (3.8)*, and suppose Assumption 3.2 holds. For all $\theta > 2\sigma$ and $t \geq 0$ we have*

$$S(t) \leq \beta_1 I,$$

where

$$\beta_1 = \frac{|C'C|}{\theta - 2\sigma} + |S(0)|.$$

Moreover, if u is regularly persistent for system (3.7) *with respect to* (T, α, t_0)*, then for all* $\theta > 0$ *and all* $t \geq t_0$ *we have*

$$\beta_2 I \leq S(t),$$

where

$$\beta_2 = \alpha \exp(-T(\theta + 2\sigma)).$$

Remark 3.1. *Lemma 3.2 implies that Assumption 3.1 is satisfied when u is regularly persistent. In fact, we can set the constants in* (3.9) *as:*

$$c_1 = \beta_1^2,$$

$$c_2 = \beta_2.$$

This selection of constants c_1 and c_2, together with Theorem 3.2, provides immediately the following corollary.

Corollary 3.1. *Consider systems* (3.7) *and* (3.8)*, and suppose that Assumption 3.2 is satisfied. If u is regularly persistent for system* (3.7) *with respect to* (T, α, t_0)*, then for any* $\theta > 2\sigma$ *we have that*

$$\limsup_{t\to\infty} |e(t)| \leq L_1 \left(\frac{2(\theta - 2\sigma)|S(0)| + 2|C'C|}{\alpha\theta(\theta - 2\sigma)\exp(-T(\theta + 2\sigma))} \right) + L_2 \left(\frac{2\sqrt{|C'C|}}{\alpha\theta \exp(-T(\theta + 2\sigma))} \right).$$

The bound in Corollary 3.1 has an advantage over the bound in Theorem 3.2. Namely, the influence of the tuning parameter θ and of the input u in the bound is now explicit. Our goal is to define a functional that assigns to each pair (u, θ) the bound given by Corollary 3.1. The optimal design of u and θ is then given by the minimization of such a functional.

Consider a set $\mathcal{U}$ of regularly persistent inputs for system (3.7) parametrized by a given bounded interval

$$\mathcal{P} = [p_{min}, p_{max}]$$

and suppose that

$$\sigma^* = \sup_{u\in\mathcal{U}} \sigma(u) < \infty,$$

where $\sigma(u)$ is defined as in (3.12). This constraint on $\mathcal{U}$ is usually set in order to respect physical limitations of the system. In a similar way, the tuning parameter θ can have magnitude requirements and should verify the lower bound in Corollary 3.1 given by $\theta > 2\sigma^*$. Consider then the bounded interval

$$\Theta = [\theta_{min}, \theta_{max}]$$

for some limit values $\theta_{max} > \theta_{min} > 2\sigma^*$. In order to define a functional

$$J : \Theta \times \mathcal{P} \to \mathbb{R}^+$$

by using the bound in Corollary 3.1, it suffices to assign to each $u \in \mathcal{U}$ a selection of the regularly persistent parameters T and α. We do this in a natural way: for each $u \in \mathcal{U}$ select and fix $T(u)$ such that the set

$$\mathcal{RP}(u) = \{\alpha \in (0, \infty) | \exists\, t_0 > 0 \text{ satisfying (3.5)}\}$$

is non-empty. Then simply define

$$\alpha(u) = \sup \mathcal{RP}(u). \tag{3.13}$$

Of course, the actual implementation of this design strategy of u and θ is not trivial and needs to be adapted to the specific system under consideration. This strategy depends on the properties of the resulting functional J. For example, suppose the parameterization of $\mathcal{U}$ and the assignment $T(u)$ are done in a continuous way. Then J results in a continuous functional with compact domain and a minimum is reached.

There exist several tools available in the software Matlab to perform the minimization. For example, if J is also strictly convex then the unique global minimum is easily obtained by the "fmincon" function, which is based on interior point algorithms [26]. Next we summarize the off-line design strategy for finding the optimal input u^* and the optimal tuning parameter θ^*

using such a functional J:

$$\begin{cases} \text{1. Choose a regularly persistent input space } \mathcal{U}, \\ \text{2. Parameterize } \mathcal{U} \text{ continuously with } \mathcal{P} \text{ interval,} \\ \text{3. Choose a tuning parameter space } \Theta, \\ \text{4. Set a continuous length assignment } T(u) \text{ and} \\ \text{compute } \alpha(u) \text{ as in (3.13),} \\ \text{5. The optimal design is obtained by:} \\ (\theta^*, p^*) = \underset{\theta\in\Theta, p\in\mathcal{P}}{\arg\min} J(\theta, p), \\ \text{where } J \text{ is given by the bound in Corollary 3.1 and} \\ \text{where } u^* \in \mathcal{U} \text{ corresponds to } p^*. \end{cases} \tag{3.14}$$

The next section shows an academic example of this off-line design strategy. The dimension of the considered state-affine system is two and the input space $\mathcal{U}$ is set to be a family of parametrized cosine functions.

3.4 Illustration

Let us consider system (3.7) in the specific case of $n = 2$ and set

$$A(u(t)) = \begin{bmatrix} 0 & u(t) \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{3.15}$$

Suppose that the physical limit of this system does not allow inputs with values outside the interval $[-1, 1]$. For any input u, we obtain in this case that

$$\sigma(u) = \sup_{t\geq 0} |u(t)|$$

and we need to define $\mathcal{U}$ such that $\sigma^* = 1$. On the other hand, the inputs have to be carefully selected since not every input of the system defined by (3.15) is regularly persistent [16]. Let us then choose the parametrized input space

$$\mathcal{U} = \{u = \cos(p_u \cdot t) | p_u \in [p_{min}, p_{max}]\}.$$

It is clear that in practice a realistic input frequency is lower and upper bounded. We use a rough approximation of the optimal frequency: $p_{min} = 0.1$ and $p_{max} = 14$, and we set the limit tunings: $\theta_{min} = 2.1 > 2\sigma^*$ and $\theta_{max} = 9$.

We proceed to show that the inputs $u \in \mathcal{U}$ are regularly persistent with respect to their period

$$T(u) = \frac{2\pi}{p_u}$$

and we give a formula for $\alpha(u)$. Since $A(u(t))A(u(s)) = 0$ for any $t, s \geq 0$, the transition matrix is simply given by

$$\begin{aligned}\Phi_u(s,t) &= \exp \int_t^s A(u(\tau))d\tau \\ &= \begin{bmatrix} 1 & \int_t^s u(\tau)d\tau \\ 0 & 1 \end{bmatrix},\end{aligned}$$

see for example [81]. It follows that the observability Gramian in the left hand side of inequality (3.5) is the symmetric matrix $G(u,t)$ described by

$$G(u,t) = \begin{bmatrix} T(u) & \int_{t-T(u)}^t \int_{t-T(u)}^s \cos(p_u \cdot \tau) d\tau ds \\ (\star) & \int_{t-T(u)}^t \left(\int_{t-T(u)}^s \cos(p_u \cdot \tau) d\tau \right)^2 ds \end{bmatrix},$$

where $(\star)$ is defined due to symmetry.

Then u is regularly persistent if we can find $\alpha > 0$ such that the matrix $G(u,t) - \alpha I$ has non-negative trace and non-negative determinant for all $t \in [T, T + p_u]$. After some simple computations, this is equivalent to stating

$$\begin{aligned} \frac{2\pi}{p_u} + \frac{\pi}{p_u^3} - 2\alpha \geq 0, \\ \alpha^2 - \left(\frac{2\pi}{p_u} + \frac{3\pi}{p_u^3} \right) \alpha + \frac{2\pi^2}{p_u^4} \geq 0. \end{aligned} \tag{3.16}$$

Finally, the largest admissible $\alpha(u)$ is the smallest root of the polynomial on the left hand side of the inequality in (3.16).

Let us now proceed with step five of the design strategy in (3.14). We suppose that the system is affected by a two-dimensional dynamic disturbance $d_1(t)$ represented, for simulation purposes, by pairs of uniform random numbers between 0 and $\sqrt{200}$. Similarly, the output measurements are corrupted by uniformly random noise between 0 and 10. Hence, $L_1 = 20$ and $L_2 = 10$. The plots of the disturbances are shown in Figure 3.1. We fix $S(0)$ as the identity matrix and the functional J is fully determined. The solver outputs the optimal design:

$$(\theta^*, p_u^*) = (3.3024, 11.1651). \tag{3.17}$$

Our goal now is to implement the optimal input and tuning given by (3.17), and to compare the performance of this optimal selection against other combinations of inputs and tunings that may arise from classical trail-and-error methods. The measure of performance used here is given by the mean of the norm of the real estimation error over a fix time length. Several simulations of the system were run by setting an initial error derived from $x(0) = (1, 1)$ and $\hat{x}(0) = (0, 0)$. Then the mean of the norm of the estimation error was computed over 50 time units for different combinations of tunings and inputs. The results can be seen in Table 3.1. It is essential to notice that these values do not correspond to the functional J but to the actual estimation error. Although the optimization domain did not include $\theta < 2.1$, some of these values were also simulated for comparison. Our method, summarized in (3.14), provided a close to optimal observer performance with a mean estimation error of 9.9. Moreover, notice the almost convex behavior of the table column-wise and row-wise.

θ/p_u	0.1	4	6	8	**11.1**	12	14
0.01	269.3	157.7	169.7	177.9	186.5	187.9	190.2
0.5	36.6	19.3	20.0	21.3	21.6	21.7	24.3
1	20.8	11.0	11.1	11.7	11.6	11.4	12.9
3.3	22.1	9.6	9.9	9.8	**9.9**	10.1	10.9
5	24.1	9.8	10.2	10.2	10.4	10.7	11.6
7	26.9	10.5	10.7	10.5	11.0	11.4	12.4
9	29.1	11.3	11.4	11.1	11.7	12.1	13.3

Table 3.1: Mean norm of the real estimation error. The value 9.9, corresponding to our strategy design, is in the lower 10.2%.

The system states and different estimations can be seen in Figure 3.2 for the first ten time units and for different inputs and tunings. Here we initialized the system at $x(0) = (10, 0)$ and $\hat{x}(0) = (5, 50)$. Small values of the tuning θ fail to approximate fast and correctly both states simultaneously (Figure 3.2, first and second row from the top). But if θ is too large then the effect of the noise gets amplified (Figure 3.2, bottom row, second column). The optimal combination θ^* and u^* is shown in the third row from the top of Figure 3.2.

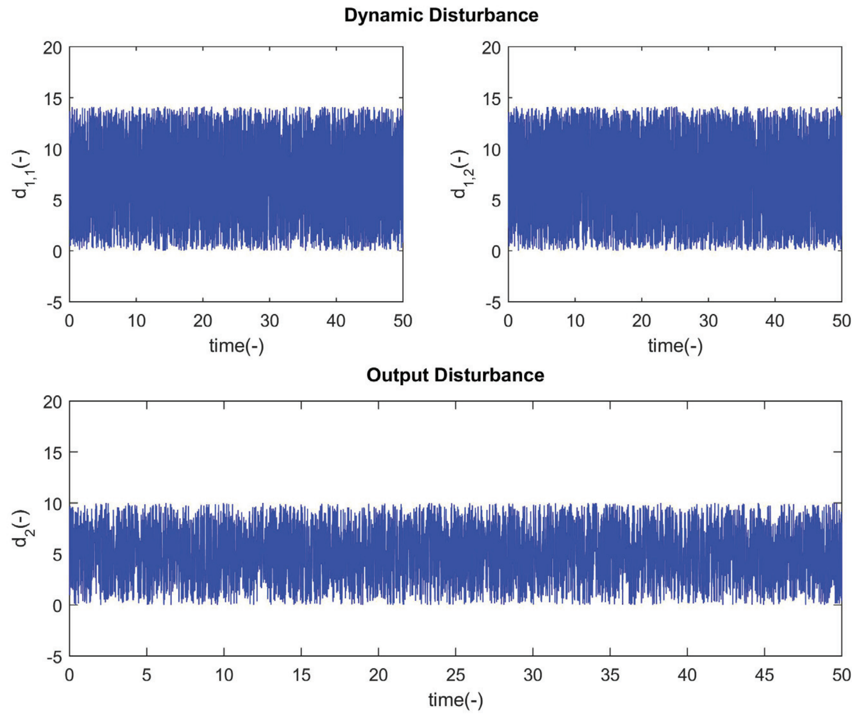

Figure 3.1: The disturbances that affect the system. The first row corresponds to the two entrances of the dynamic disturbance d_1. The second row corresponds to the output disturbance d_2.

3.5 Conclusion

In this chapter, we addressed the problem of robustness to measurement noise and model uncertainties of a high-gain observer for state-affine systems. For this, we first defined a cost functional based on an upper bound for the limiting estimation error that depends on the disturbances. We then formulated a feasible optimization algorithm for the off-line design of the input and the high-gain parameter, thus obtaining: (i) a new degree of freedom, given by the input, to improve the performance of the observer with respect to disturbances, (ii) a systematic way to directly tune the observer instead of using the classical trail-and-error method.

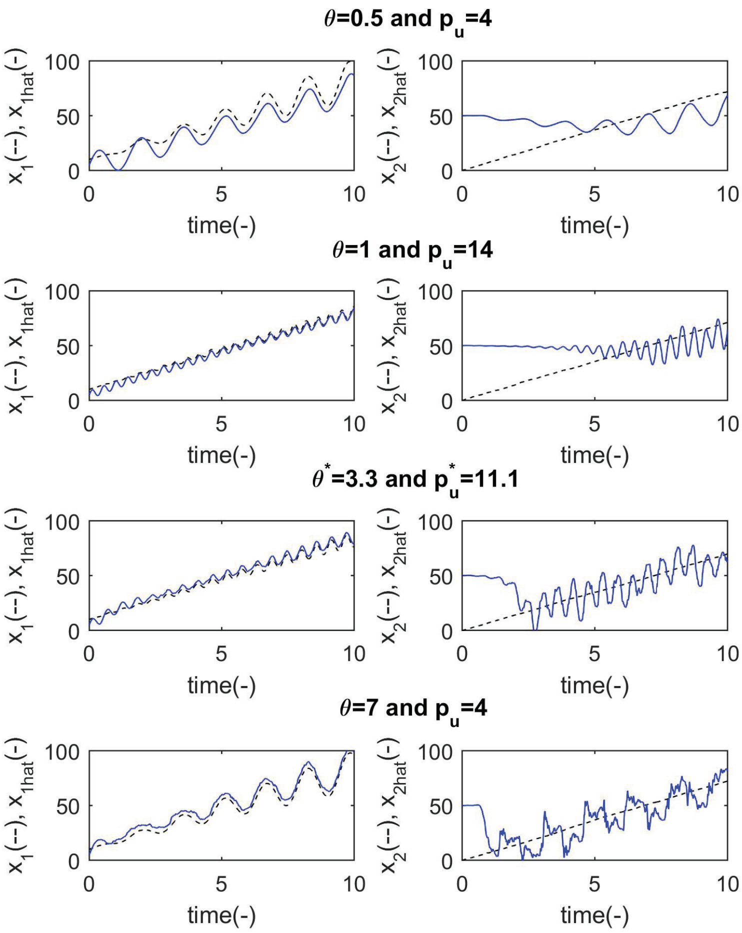

Figure 3.2: Influence of θ and p_u on the state estimation. In dotted line the states, in solid their estimations. The optimal design corresponds to the third row from the top.

Our approach, however, does not specify in general how to verify if a given input is regularly persistent or how to compute the corresponding parameters arising from this property. Therefore, further work needs to be done in this direction by combining our work with some of the results previously discussed in this chapter. Future studies can also focus on the development of an on-line design of the observer, this due to the speed of our optimization algorithm. Finally, this work opens a new perspective for optimal input design in parameter identification problems.

Chapter 4

Observer design for nonlinear systems with output transformation

4.1 Introduction

As discussed in Chapter 2, the usual observers for linear systems are the Luenberger observer and Kalman observer which can be extended to nonlinear systems in specific forms. Therefore, one common strategy is to look for a coordinate transformation that steers the system into a convenient form [48, 16]. Early important contributions in this direction include: a linear form with output injection [75], a bilinear form with output injection and its Kalman-like observer [58, 22], and a triangular form for uniformly observable system and its Luenberger-like or high-gain observer [46]. Even though these observer designs mainly concern nonlinear systems, their outputs tend to be linear functions of the state. Similarly, a common goal of sensor manufacturers is achieving linearity. This is often complicated and a large number of classical sensors exhibit nonlinear behavior [105, 36]. For example, this is the case in: fuel cell power systems [9], image restoration [106], digital imaging [97], combustion control in automobiles [89] and engineering medicine [74].

A popular choice of an observer for general nonlinear systems with nonlinear output is the Extended Kalman Filter (EKF) in its deterministic version. Although simple and with good noise filtering properties, only local convergence is guaranteed. The observer proposed in [48] is not limited in this sense, however, the system needs to be in observability canonical form with a smooth nonlinear output. A different approach is based on the celebrated Kazantzis-

Kravaris observer [69, 7], which adapts Luenberger's ideas to the nonlinear framework; the design problem reduces to solving a system of partial differential equations (PDEs).

On the other hand, observer design for systems with monotonic nonlinearities is studied in [10], where the authors remove the Lipschitz condition and avoid high-gain by using the so-called circle criterion. In [43], they expand these results and also consider nonlinear outputs in the presence of model uncertainties. The work in [13] instead deals with a more general type of nonlinearities, those satisfying incremental quadratic constraints. Finally, the authors in [77] propose a simple observer design for systems in triangular form with a nonlinear output. This output function is not necessarily differentiable but it must satisfy an incremental sector condition.

However, these results tend to require systems in specific forms and sufficient conditions for the existence of the corresponding coordinate transformations are usually strong [48, 16]. Moreover, finding the right transformations can be difficult, especially in the multi-output case. Another limitation is that measurement noise is often ignored. As seen in Chapter 2, a natural framework for studying the robustness of an observer with respect to measurement noise is that of ISS. Indeed, we can consider the error dynamics as the state and the measurement noise as the input. In this context, ISS is also referred to as disturbance-to-error stability (DES) [99].

In the main part of this chapter, we consider the problem of robust observer design for nonlinear systems in the presence of a nonlinear output transformation. Our work has been the subject of [50, 53].

We first suppose that a DES observer has been designed for a given nonlinear system based on an output y. This design cannot be directly implemented if this output is not available and if we instead measure a nonlinear transformation $\psi(y)$. Therefore, we propose a new interconnected observer that estimates both: y and the system state. We suppose that ψ is a local diffeomorphism, however, ψ can be difficult to invert or its inverse might not be available in closed form. In order to study the robustness of the new observer, we consider model uncertainties on y and measurement noise on $\psi(y)$. We use small-gain arguments to show that the new observer is asymptotically convergent to a neighborhood of the origin that depends on the amplitude of the disturbances (or convergent to zero in their absence).

Our observer design is partially inspired by the Newton-Raphson method and [59], where the authors develop an explicit observer for systems defined on a manifold given by algebraic equations. Our work is also related to that of [54], where they require a Lyapunov function for the error dynamics of an observer that cannot be directly used; the output is only measured through a second linear system. We apply our general results to two families of systems: state affine systems up to output injection and systems with additive triangular nonlinearity. These families differ considerably since they represent non-uniformly and uniformly observable systems respectively. We finish by providing numerical examples for both cases.

The organization of this chapter is as follows. In the preliminaries, we start by motivating the study of systems with nonlinear output by briefly discussing the challenges in the sensor industry. We continue with a literature review concerning important results in observer design for such kind of systems, in particular, we discuss the work of [54]. We then present our main contribution, together with two case studies and the corresponding numerical examples. Finally, we write a summary of our work and selected perspectives in the last section.

4.2 Preliminaries

4.2.1 Motivations

Sensors or transducers are fundamental for controlling industrial processes and the final outcome strongly depends on their quality [36]. Popular sensors in control systems include: tachometers, resolvers, piezoelectric devices, flow meters, thermistors, etc. These instruments convert the sensed physical signal into the form of the device output, just as a piezoelectric accelerometer senses acceleration and converts it into an electric charge. It is not surprising that, as technology advances, the need of more precise sensors is constantly increasing. However, there are several undesirable properties of classical sensors such as noise, time delay, parameter drift and, importantly, nonlinearity. Fewer attention has been paid to the latter and usually rudimentary methods are applied in order to deal with nonlinearities, this includes look-up tables and calibration curves based on polynomial approximations [105]. Similar techniques are useful in the context of Wiener models, which consist of linear dynamics followed by a static nonlinearity; see for example [3] concerning glucose regulation and [55] that studies a pH neutralisation process.

Sensors are sometimes designed so that linearity between the system state and the output is achieved. This can be very challenging due to the nonlinear nature of most physical devices, for example in: image sensors [106], fiber optic displacement sensors, biosensors [74] and oxygen sensors, which are used for closed-loop combustion control in vehicles. Although the wide-band version of the latter does achieve linearity, the price increases considerably.

An important drawback of the use of nonlinear sensors is distortion. The early work of [112] concerns communication networks and suggests that nonlinear distortion can be tackled by using invertible nonlinear filtering. The authors in [61] consider the problem of recovering a band-limited signal $y(t)$ given by the measurements of a nonlinear sensor $y_\psi = \psi(y(t))$. For example, $y(t)$ can be a sum of cosine functions of different amplitudes and frequencies. It is clear that the signal $y(t)$ cannot be recover unless we set specific conditions on $\psi(y)$. The authors show that a sufficient condition for this is that $\psi(0) = 0$ and that $\frac{\partial \psi}{\partial y}(y)$ does not change sign.

4.2.2 Nonlinear output systems and observers

A turning process is a material removal process for creating rotational parts that consists of a turning machine and a cutting tool. Such a process can be modeled by a nonlinear system with an output resulting in a composition of a nonlinear function ψ and a linear function h, however, the nonlinearity is often compensated statistically. Instead, the authors in [94] develop an adaptive nonlinear control strategy in order to tackle the problem of turning force regulation by directly considering systems of the form

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y_\psi = \psi(h(x)), \end{cases}$$

where the measured output y_ψ is the main cutting force and where the value $h(x)$ represents the feed in millimeters per revolution.

On the other hand, the authors in [9] develop an observer design to estimate the partial hydrogen pressure in the anode channel of a fuel cell. Their subsequent work in [43] generalizes these results by considering systems of the form

$$\begin{cases} \dot{x} = Ax + G\gamma_1(Hx) + \rho(y_1, y_2, u) \\ y_1 = C_1 x, \quad y_2 = \gamma_2(C_2 x), \end{cases} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y_1(t) \in \mathbb{R}^r$ and $y_2(t) \in \mathbb{R}^{p_2}$ the outputs, and $u(t) \in \mathbb{R}^m$ an input. Here, the functions γ_i and ρ are assumed known. We remark that system (4.1) can model a single-link flexible joint robot with the linear outputs: motor position and velocity. However, the position sensor is known to be nonlinear and, thus, γ_2 plays an essential role in accurate modeling.

The functions $\gamma_i : \mathbb{R}^{p_i} \to \mathbb{R}^{p_i}$ are assumed continuously differentiable, $\rho : \mathbb{R}^{r+p_2} \times \mathbb{R}^m \to \mathbb{R}^n$ locally Lipschitz and system (4.1) forward-complete. Furthermore, suppose that there exists a constant $\epsilon > 0$ such that the following inequalities are satisfied:

$$\begin{aligned}
&\frac{\partial \gamma_1}{\partial v}(v) + \left(\frac{\partial \gamma_1}{\partial v}(v)\right)' \geq 0, \quad \forall v \in \mathbb{R}^{p_1}\\
&\frac{\partial \gamma_2}{\partial v}(v) + \left(\frac{\partial \gamma_2}{\partial v}(v)\right)' \geq \epsilon I, \quad \forall v \in \mathbb{R}^{p_2}.
\end{aligned} \tag{4.2}$$

One possible approach to observer design for system (4.1) would be to use the inverse of γ_2, which exists by (4.2), and to consider a linear output system as in [10]. However, as the authors state, it can be impossible to find the analytical expression of γ_2^{-1}. Instead, they directly incorporate the nonlinearity in the observer as described by

$$\begin{aligned}
\dot{\hat{x}} =& A\hat{x} + L_1(C_1\hat{x} - y_1) + L_2(\gamma_2(C_2\hat{x}) - y_2)\\
&+ G\gamma_1(H\hat{x} + K_1(C_1\hat{x} - y_1)) + \rho(y_1, y_2, u),
\end{aligned} \tag{4.3}$$

where L_1, L_2 and K_1 are matrices designed in the next theorem.

Theorem 4.1 ([43])**.** *Consider system* (4.1) *and suppose that the nonlinearities γ_1 and γ_2 satisfy the inequalities in* (4.2) *for some constant $\epsilon > 0$. If there exist a symmetric and positive definite matrix P, matrices L_1, L_2 and K_1, and constants $\zeta > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$ such that*

$$\begin{bmatrix} (A + L_1C_1 + \frac{1}{2}\epsilon L_2C_2)'P + P(A + L_1C_1 + \frac{1}{2}\epsilon L_2C_2) + \zeta I & B \\ B' & 0 \end{bmatrix} \leq 0, \tag{4.4}$$

where

$$B = P\begin{bmatrix} G & L_2 \end{bmatrix} + \begin{bmatrix} \lambda_1(H + K_1C_1) \\ \lambda_2C_2 \end{bmatrix}',$$

then the estimation error from observer (4.3) *converges exponentially to zero.*

Remark 4.1. *System* (4.1) *can be modified in different ways. For example, we can sum in* *the system dynamics a term* $Q\theta$ *with an unknown parameter* θ*, and use the nonlinear output* $y = \gamma_2(Cx+\theta)$*. This gives rise to an adaptive extension of observer* (4.3)*, which is especially useful in the context of fuel cell models [43]. Another modification consists on having a linear output, that is, a system of the form*

$$\begin{cases} \dot{x} = Ax + G\gamma(Hx) + \rho(y,u) \\ y = Cx, \end{cases}$$

where γ *satisfies an inequality as the first one in* (4.2)*. The corresponding observer* (4.3) *and condition* (4.4) *can be easily formulated in this case, in particular, the constant* ϵ *does not appear. Hence, in the original case, the output nonlinearity* γ_2 *is compensated in Theorem 4.1 by constraining the observer gains with a lower bound involving the derivative of* γ_2*.*

As discussed in Chapter 2, a popular approach for observer design for nonlinear systems consists in compensating the Lipschitz nonlinearities by using high-gain. Two popular research directions concern: (i) global results under global growth conditions [46], and (ii) the interactions between the peaking phenomena and the nonlinearities [42]. The proof of Theorem 4.1 follows a different approach which does not require global Lipschitz conditions and that avoids high-gain; this approach is instead based on the circle criterion.

Remark 4.2. *The circle criterion, in its simplest version, considers any time-invariant linear system of the form*

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx, \end{cases} \tag{4.5}$$

where $x(t) \in \mathbb{R}^n$*,* $y(t) \in \mathbb{R}$ *and* $u(t) \in \mathbb{R}$*, and the feedback interconnection*

$$u(t) = -\sigma(y(t), t), \tag{4.6}$$

for $\sigma : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ *a continuous nonlinear function that satisfies the sector condition*

$$0 \le \sigma(y,t)y \le y^2 M, \quad \forall y \in \mathbb{R}, \forall t \in \mathbb{R}^+,$$

for some constant $M \in \mathbb{R}$*. Furthermore, assume that the transfer function* G *of system* (4.5) *is minimal and that* $1 + MG$ *is strictly positive real. In this case, the circle criterion is known as the positivity theorem and it guarantees the global exponential stability at the origin of interconnection* (4.5)-(4.6)*, see for example [56].*

The previously presented results from [43] and other related work are all unified in [13, 115], where the authors characterize nonlinearities with incremental quadratic inequalities. More recently, systems with output nonlinearities and high-gain observers are studied in [77]. The authors consider nonlinear systems of the form

$$\begin{cases} \dot{x} = Ax + \zeta(x, u) \\ y_\psi = \psi(Cx), \end{cases} \tag{4.7}$$

where $x(t) \in \mathbb{R}^n$, $y_\psi(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}^m$ and where A, C and ζ have the canonical form (2.19). The function $\psi : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz such that $\psi(0) = 0$ and the function $\zeta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is coordinate-wise globally Lipschitz in x, uniformly in u. If $\zeta_i = 0$ for all $i \neq n$, then system (4.7) can model different mechanical systems in which the position x_1 is typically measure through a nonlinear sensor $y_\psi = \psi(x_1)$. This time, the nonlinear transformation ψ is constrained by the incremental sector condition: there exists a constant $\beta > 0$ such that

$$(y_1 - y_2)(\psi(y_1) - \psi(y_2)) \geq \beta(y_1 - y_2)^2, \quad \forall y_1, y_2 \in \mathbb{R}. \tag{4.8}$$

Remark 4.3. *The problem of observer design for system* (4.7) *can be solved with different approaches: (i) inverting the function ψ and using a standard high-gain observer, (ii) implementing the change of variables $z_1 = \psi(x_1)$ and $z_i = \dot{z}_{i-1}$, for $i \neq 1$, and using the standard high-gain observer, (iii) applying the early results from [48] which allow a nonlinear output, or (iv) using the just discussed results from [10, 43, 13]. However, these approaches require either a closed form for ψ^{-1}, smoothness assumptions on ψ and ζ, complicated LMI's to be solved, or other restrictive conditions.*

The authors in [77] propose a simple high-gain observer design, whose convergence is again inspired by the circle criterion, and that is given by

$$\dot{\hat{x}} = A\hat{x} + \zeta(\hat{x}, u) - \beta^{-1}\Delta_\theta K(\psi(C\hat{x}) - y_\psi), \tag{4.9}$$

where $\theta > 0$ is large enough and Δ_θ is the diagonal matrix with diagonal entries θ, $\theta^2 \ldots$,

θ^n. The matrix K is carefully selected as

$$K = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix},$$

where $\alpha_i > 0$ are chosen so that the following polynomial has only negative real roots,

$$s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n.$$

Theorem 4.2 ([77])**.** *Consider system* (4.7) *and suppose that ζ satisfies the Lipschitz condition* (2.20)*, that ψ satisfies the sector condition* (4.8) *and that $\theta > 0$ is large enough. Then, the estimation error from observer* (4.9) *converges exponentially to zero.*

Remark 4.4. *The authors of the previous theorem only assume that the functions ζ_i are locally Lipschitz. By imposing boundedness of the state and input, they can use saturation techniques to modify these functions and to achieve the globally Lipschitz conditions.*

Finally, we discuss a result that does not involve an output nonlinearity but that concerns observer redesign in face of an output modification. The authors in [54] develop a theoretical framework motivated by the problem of position, velocity and attitude estimation using integrated satellites and inertial measurements in navigation. They consider a cascade interconnection of a nonlinear system with a linear system:

$$\begin{cases} \dot{x} = f(x, u) \\ z = h(x, u), \end{cases} \tag{4.10}$$

$$\begin{cases} \dot{w} = Aw + B_u u + B_z z \\ y = Cw + D_u u + D_z z, \end{cases} \tag{4.11}$$

where $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^{n_l}$ and $y(t) \in \mathbb{R}^{p_l}$, where f and h and all signals are sufficiently smooth, and where (4.11) is composed by matrices of appropriate dimensions.

The authors assume that an observer design is readily available for system (4.10) when considering z as the output, that is,

$$\dot{\hat{x}} = f(\hat{x}, u) + g(\hat{x}, z, u), \tag{4.12}$$

where g is sufficiently smooth. This assumption can be formalized by considering the corresponding error dynamics in the following condition: there exist a continuously differentiable function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ and positive constants α_1, α_2, α_3 and α_4 such that for all $(t, e) \in \mathbb{R}^+ \times \mathbb{R}^n$:

$$\begin{aligned} &\alpha_1 |e|^2 \leq V(t, e) \leq \alpha_2 |e|^2, \\ &\frac{\partial V}{\partial t}(t, e) + \frac{V}{\partial e}(t, e)\mathcal{F}(t, e) \leq -\alpha_3 |e|^2, \\ &\left| \frac{\partial V}{\partial e}(t, e) \right| \leq \alpha_4 |e|, \end{aligned} \tag{4.13}$$

where $\mathcal{F}(t, e) = f(x(t), u(t)) - f(x(t) - e, u(t)) - g(x(t) - e, z(t), u(t))$. However, observer (4.12) cannot be directly implemented to estimate the state x since z is only available through system (4.11), that is, y is instead measured.

Let us also suppose that the functions g and

$$d(x, u, \dot{u}) = \frac{\partial h}{\partial u}(x, u)\dot{u} + \frac{\partial h}{\partial x}(x, u) f(x, u)$$

satisfy Lipschitz conditions, namely, there exist positive constants L_1 and L_2 such that:

$$\begin{aligned} &|g(\hat{x}, z(t), u(t)) - g(\hat{x}, \hat{z}, u(t))| \leq L_1 |z(t) - \hat{z}|, \\ &|d(x(t), u(t), \dot{u}(t)) - d(\hat{x}, u(t), \dot{u}(t))| \leq L_2 |x(t) - \hat{x}|, \end{aligned} \tag{4.14}$$

for all $t \in \mathbb{R}^+$, all $\hat{x} \in \mathbb{R}^n$ and all $\hat{z} \in \mathbb{R}^p$. The observer design proposed in [54] is given by the interconnection:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u) + g(\hat{x}, \hat{z}, u), \\ \hat{z} &= h(\hat{x}, u) + \xi, \\ \dot{\xi} &= -\frac{\partial h}{\partial x}(\hat{x}, u) g(\hat{x}, \hat{z}, u) + K_z(y - C\hat{w} - D_u u - D_z \hat{z}), \\ \dot{\hat{w}} &= A\hat{w} + B_u u + B_z \hat{z} + K_w(y - C\hat{w} - D_u u - D_z \hat{z}), \end{aligned} \tag{4.15}$$

where

$$K = [K_w', K_z']'$$

is a gain matrix to be determined. Their strategy consists in estimating $w_e = [w', z']'$ by $\hat{w}_e = [\hat{w}', \hat{z}']'$, this leads to consider the extended system

$$\begin{cases} \dot{w}_e = \mathcal{A} w_e + \mathcal{B}_u u + \mathcal{B}_d d(x, u, \dot{u}) \\ y = \mathcal{C} w_e + D_u u, \end{cases}$$

where:

$$\mathcal{A} = \begin{bmatrix} A & B_z \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_u = \begin{bmatrix} B_u \\ 0 \end{bmatrix}, \quad \mathcal{B}_d = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C & D_z \end{bmatrix}.$$

Theorem 4.3 ([54]). *Consider the interconnected system* (4.10)-(4.11) *and suppose that there exists a Lyapunov function as in* (4.13), *and that the Lipschitz conditions in* (4.14) *are satisfied. If there exist a symmetric and positive definite matrix P and a matrix K such that*

$$\begin{bmatrix} P\mathcal{A} + \mathcal{A}'P - X\mathcal{C} - \mathcal{C}'X' + I & P\mathcal{B}_d \\ \mathcal{B}_d'P & -\gamma^2 I \end{bmatrix} < 0,$$

where

$$X = PK, \quad \gamma = \frac{4\alpha_3}{4L_2^2 + \alpha_4^2 L_1^2},$$

then the estimation errors $e = x - \hat{x}$ and $w_e - \hat{w}_e$ from observer (4.15) *have globally exponentially stable dynamics.*

Remark 4.5. *The conclusions of Theorem 4.3 still hold if the LMI condition is substituted by $\mathcal{A} - K\mathcal{C}$ Hurwitz and $|H|_\infty < \gamma$, for the transfer matrix*

$$H(s) = (Is - \mathcal{A} + K\mathcal{C})^{-1}\mathcal{B}_d.$$

It is then shown that, under additional hypotheses on A, B_z, C and D_z and for γ large enough, there is always such a choice of K. These additional hypotheses include the detectability of the pair (A, C).

4.3 Problem statement

The aim of this second part of our work concerns observer redesign to adapt a given observer to a nonlinear transformation of the system output. Let us consider a nonlinear system of the form

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x) + d, \end{cases} \tag{4.16}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ the output, $u(t) \in \mathbb{R}^m$ a continuous input and $d(t) \in \mathbb{R}^p$ a disturbance. We suppose that f and h are of class C^2, that the system is forward-complete and we denote the input set by $u \in \mathcal{U}$.

We first assume that a "robust" observer has been designed for system (4.16) and that is given by

$$\begin{cases} \dot{\hat{x}} = \hat{f}(\hat{x}, g, y, u) \\ \dot{g} = G(g,u), \end{cases} \tag{4.17}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the state estimation, g a dynamic gain and $\hat{f}$ is locally Lipschitz.

We then consider the case where y is not directly available for measurements. Instead, we measure a nonlinear transformation $\psi(y)$ affected by noise. This situation arises frequently in engineering processes, where nonlinear sensor transformations are common. The new system takes the form

$$\begin{cases} \dot{x} = f(x,u) \\ y_\psi = \psi(y) + d_\psi, \end{cases} \tag{4.18}$$

where $y_\psi(t) \in \mathbb{R}^p$ is the measured output, $y(t) = h(x(t)) + d_y(t) \in \mathbb{R}^p$, the disturbances $d_y(t)$, $d_\psi(t) \in \mathbb{R}^p$ are bounded with bounded derivatives, and ψ is of class C^2. Here, d_y can represent model uncertainties while d_ψ measurement noise. As a consequence of changing the system output, observer (4.17) cannot be directly implemented and, thus, a redesign is needed.

For a general nonlinear system, there is no systematic way to adapt a given observer design to output transformations. We provide a novel method of observer redesign that faces this challenge.

Remark 4.6. *As an example, consider the uniformly observable systems. Many observers are easily designed for this class of systems, however, they are based on a linear output. If a* *nonlinear transformation of the output is instead measured, the convergence of these observers is no longer guaranteed. This example is studied in Section 4.6.2.*

Remark 4.7. *Other solutions to our problem include: (i) coordinate change to steer system* (4.18) *into a convenient form. This can be difficult and it is not systematic, especially for multi-output systems [48, 16]; (ii) using the EKF for system* (4.18) *without guarantee of its global convergence; (iii) inverting ψ and using observer* (4.17). *Unfortunately, the inverse of ψ might not be available in closed form or it can be difficult to compute.*

We cannot implement observer (4.17) since y is not directly known. However, we require this observer to be robust with respect to measurement noise. The corresponding error dynamics are given by

$$\dot{e} = \mathcal{F}(t, e, d), \tag{4.19}$$

where $e = x - \hat{x}$ and for

$$\begin{aligned} \mathcal{F}(t, e, d) =& f(x(t), u(t)) \\ & - \hat{f}(x(t) - e, g(t), h(x(t)) + d, u(t)), \end{aligned} \tag{4.20}$$

thus, what we need is the following assumption.

Assumption 4.1. *There exists a continuous function $\bar{V}(t, e) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, of class C^1 on $e \neq 0$, and functions $\bar{\alpha}_1$, $\bar{\alpha}_2 \in \mathcal{K}_\infty$ and $\bar{\alpha}_3$, $\bar{\chi} \in \mathcal{K}$ such that:*

$$\bar{\alpha}_1(|e|) \leq \bar{V}(t, e) \leq \bar{\alpha}_2(|e|),$$

for all $t \in \mathbb{R}^+$ and all $e \in \mathbb{R}^n$, and such that:

$$\frac{\partial \bar{V}}{\partial t}(t, e) + \frac{\partial \bar{V}}{\partial e}(t, e)\mathcal{F}(t, e, d) \leq -\bar{\alpha}_3(|e|),$$

whenever $|e| \geq \bar{\chi}(|d|)$ and for all $t \in \mathbb{R}^+$, $e \in \mathbb{R}^n - \{0\}$, $d \in \mathbb{R}^p$ and all $x(0) \in \mathbb{R}^n$.

Assumption 4.1 is equivalent to the ISS of system (4.19) as in [102] or, for time-varying Lyapunov functions, [41]. In this context, the ISS property guarantees the graceful degradation of observer (4.17) performance in the presence of measurement noise. Observers satisfying this property were first considered in [104] and they are known as disturbance-to-error stable (DES) observers [99]. There are methods to determine if certain observers are DES [5] or to redesign them if they are not [100].

Assumption 4.2. *The function $\psi : \mathbb{R}^p \to \mathbb{R}^p$ is of class C^2 and its Jacobian is invertible on all its domain. Moreover, there exists $\delta \in \mathcal{K}_\infty \cap C^2$ such that*

$$\delta(|\psi(y) - \psi(\hat{y})|) \geq |y - \hat{y}|, \quad \forall y, \hat{y} \in \mathbb{R}^p.$$

Assumption 4.2 implies in particular the injectivity of ψ. It is satisfied, for example, if $p = 1$ and if $|\frac{\partial \psi}{\partial y}|$ is bounded from below by a positive constant. The authors in [77] require this last condition to hold when the nonlinear output they study is continuously differentiable.

Remark 4.8. *Notice that Assumption 4.1 concerns the existence of a robust observer design for system* (4.16) *and, thus, depends on the properties of the functions f and h. However, Assumption 4.2 only constrains the output transformation ψ in system* (4.18) *and it is needed for redesigning the observer.*

4.4 New observer design

Let us consider the system with the transformed output given in (4.18). Its proposed observer design is presented as the following interconnection:

$$\begin{cases} \dot{\hat{x}} = \hat{f}(\hat{x}, g, \hat{y}, u) \\ \dot{g} = G(g, u) \\ \dot{\hat{y}} = \hat{h}(\hat{x}, \hat{y}, y_\psi, u), \end{cases} \tag{4.21}$$

where $\hat{f}$ and G are as in (4.17),

$$\begin{aligned} \hat{h}(\hat{x}, \hat{y}, y_\psi, u) =& \frac{\partial \psi}{\partial y}(\hat{y})^{-1} \frac{\partial \psi}{\partial y}(h(\hat{x})) \frac{\partial h}{\partial x}(\hat{x}) f(\hat{x}, u) \\ &+ \frac{\partial \psi}{\partial y}(\hat{y})^{-1} \varphi(\hat{x}, u) K(y_\psi - \psi(\hat{y})), \end{aligned}$$

and $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$ and $K : \mathbb{R}^p \to \mathbb{R}^p$ are locally Lipschitz functions defined next. We emphasize that the observer in (4.21) only requires knowledge of y_ψ and not directly of y. Figure 4.1 compares observers (4.17) and (4.21) and illustrates the relation of these observers with system (4.18).

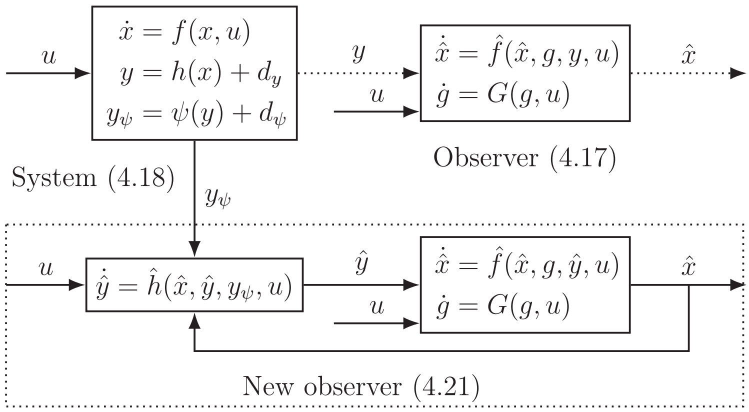

Figure 4.1: System (4.18) and observer (4.17) are represented on the top part of the diagram. The new observer (4.21) is represented on the bottom part as an interconnected system. Observer (4.17) requires the unavailable output y, while the new observer uses the measurements y_ψ instead.

In order to define the function φ, let us first consider the function ϕ given by

$$\phi(x, u, d_1, d_2, d_3) = \frac{\partial \psi}{\partial y}(h(x) + d_1)\left(\frac{\partial h}{\partial x}(x) f(x, u) + d_2\right) + d_3, \tag{4.22}$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and d_1, d_2, $d_3 \in \mathbb{R}^p$. According to Proposition 2.2, there exist locally Lipschitz functions $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$ and $\alpha \in \mathcal{K}_\infty$ such that,

$$|\phi(x, u, d_1, d_2, d_3) - \phi(\hat{x}, u, 0, 0, 0)| \leq \varphi(\hat{x}, u)\alpha(|(x - \hat{x}, d_1, d_2, d_3)|) \tag{4.23}$$

on all its domain and we can assume that $\varphi \geq 1$.

Remark 4.9. *The problem of finding α and φ as in* (4.23) *is simplified if ψ and each of the functions in expression* (4.22) *satisfy global Lipschitz conditions. We can then take α as the identity function and*

$$\varphi(\hat{x}, u) \geq c_\varphi(|f(\hat{x}, u)| + 1),$$

for some $c_\varphi \geq 1$ and for all $\hat{x} \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. There is a vast literature dealing with the computation of Lipschitz constants like those defining c_φ. Another convenient case is when system (4.18) *has bounded states, given the saturation techniques explained in Remark 4.13.*

We now continue with the definition of K. Consider a function $\rho \in \mathcal{K}_\infty \cap C^2$, a positive constant k and set $K : \mathbb{R}^p \to \mathbb{R}^p$ as

$$K(\xi) = \begin{cases} \left(k\frac{\rho(|\xi|)}{|\xi|}\right)\xi, \text{ if } \xi \neq 0 \\ 0, \text{ if } \xi = 0. \end{cases} \tag{4.24}$$

The choice of ρ guarantees that K is locally Lipschitz. Indeed, using that $\rho(0) = 0$ and L'Hôpital's rule, we can show that the function

$$\begin{cases} \frac{\rho(r)}{r}, \text{ if } r > 0 \\ \frac{\partial\rho}{\partial r}(0), \text{ if } r = 0 \end{cases}$$

is continuously differentiable on $\mathbb{R}^+$. The function ρ provides a degree of freedom for the design of the new observer (4.21).

4.5 Main results and proofs

Given initial conditions $\hat{x}(0) \in \mathbb{R}^n$ and $\hat{y}(0) \in \mathbb{R}^p$, there is a corresponding maximal interval of existence $[0, T)$ for the unique solution $(\hat{x}, \hat{y})$ of (4.21). We denote the estimation errors as

$$e(t) = x(t) - \hat{x}(t), \quad \xi(t) = y_\psi(t) - \psi(\hat{y}(t)),$$

for all $t \in [0, T)$. Notice that the solution is unbounded if T is finite, see for example [29]. We will see that this is not the case if ρ is properly chosen. The following results concern the definitions from Section 2.5.

Lemma 4.1. *Consider systems* (4.18) *and* (4.21)*, with* φ *and* K *as in* (4.23)-(4.24)*, and let Assumption 4.2 hold. If* $k > 1$*, then there exists a Lyapunov function for the family*

$$\mathcal{G} = \{(\xi, e)|x(0), \hat{x}(0) \in \mathbb{R}^n, \hat{y}(0) \in \mathbb{R}^p\}, \tag{4.25}$$

where $\xi = y_\psi - \psi(\hat{y})$ *and* $e = x - \hat{x}$ *are defined on* $[0, T)$*. Moreover, if the disturbances* d_y *and* d_ψ *are both zero then the corresponding Lyapunov-constant can be chosen to be zero.*

Proof. We propose the Lyapunov function simply as the norm. On the trajectory ξ, it takes the form

$$V_1^\xi(t) = |\xi(t)|,$$

for all $t \in [0, T)$. It is clear that property (2.44) is satisfied by defining both α_{11} and α_{12} as the identity function.

We next prove that property (2.45) is also satisfied on $[0, T) \setminus \xi^{-1}(0)$. From the definitions of ϕ in (4.22) and of the new observer in (4.21) we have that

$$\begin{aligned}\dot{V}_1^\xi &= \frac{\xi'}{|\xi|}\left(\frac{\partial\psi}{\partial y}(y)\dot{y} + \dot{d}_\psi - \frac{\partial\psi}{\partial y}(\hat{y})\dot{\hat{y}}\right)\\ &= \frac{\xi'}{|\xi|}(\phi(e+\hat{x}, u, d_y, \dot{d}_y, \dot{d}_\psi) - \phi(\hat{x}, u, 0, 0, 0))\\ &\quad - \frac{\xi'}{|\xi|}\varphi(\hat{x}, u)K(\xi),\end{aligned}$$

where ξ' denotes the transpose of ξ. It then follows from the construction of φ and K, respectively in (4.23) and (4.24), and by Remark 2.10 that

$$\begin{aligned}\dot{V}_1^\xi &\leq \varphi(\hat{x}, u)\alpha(|(e, d_y, \dot{d}_y, \dot{d}_\psi)|) - k\varphi(\hat{x}, u)\rho(|\xi|)\\ &\leq \varphi(\hat{x}, u)(\alpha(2|e|) + \alpha(c_d) - k\rho(|\xi|)),\end{aligned} \tag{4.26}$$

where

$$c_d = 2(|d_y|_\infty + |\dot{d}_y|_\infty + |\dot{d}_\psi|_\infty). \tag{4.27}$$

On the other hand, suppose that $\chi_1 \in \mathcal{K}$ is given by

$$\chi_1(r) = \rho^{-1}(2\alpha(2r)), \tag{4.28}$$

for all $r \in \mathbb{R}^+$, and set the non-negative constant

$$c_{L1} = \rho^{-1}(2\alpha(c_d)).$$

If $|\xi| \geq \chi_1(|e|) + c_{L1}$, then it follows that

$$\rho(|\xi|) \geq \alpha(2|e|) + \alpha(c_d).$$

Since $k > 1$ and $\varphi \geq 1$, we get that the negative term $\rho(|\xi|) - k\rho(|\xi|)$ dominates the last expression in (4.26). By using this and the inequalities in (4.26) we conclude that

$$\dot{V}_1^\xi \leq -(k-1)\rho(|\xi|)$$

and, as a consequence, we can define

$$\alpha_{13}(r) = (k-1)\rho(r),$$

for all $r \in \mathbb{R}^+$.

Finally, notice that if both disturbances d_y and d_ψ are the zero function then c_d in (4.27) is zero and the same holds for the Lyapunov-constant c_{L1}. □

Lemma 4.2. *Consider systems* (4.18) *and* (4.21)*, with* φ *and* K *as in* (4.23)-(4.24)*, and let Assumptions 4.1 and 4.2 hold. There exists a Lyapunov function for the family*

$$\mathcal{G}^{-1} = \{(e,\xi) | x(0), \hat{x}(0) \in \mathbb{R}^n, \hat{y}(0) \in \mathbb{R}^p\}, \tag{4.29}$$

where $e = x - \hat{x}$ *and* $\xi = y_\psi - \psi(\hat{y})$ *are defined on* $[0,T)$*. Moreover, if the disturbances* d_y *and* d_ψ *are both zero then the corresponding Lyapunov-constant can be chosen to be zero.*

Proof. Consider the Lyapunov function $\bar{V}$ from Assumption 4.1 and the corresponding functions $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\alpha}_3$ and $\bar{\chi}$. Set $V_2 = \bar{V}$, which on the trajectory takes the form $V_2^e = V_2(\cdot, e)$, and define $\alpha_{21} = \bar{\alpha}_1$ and $\alpha_{22} = \bar{\alpha}_2$. We next prove that property (2.45) holds on $[0,T) \setminus e^{-1}(0)$. From the definition of $\mathcal{F}$ in (4.20) we have that

$$\dot{e} = \mathcal{F}(\cdot, e, \hat{y} - h(x))$$

and, therefore, we can select $\alpha_{23} = \bar{\alpha}_3$ to get

$$\begin{aligned} \dot{V_2^e} &= \frac{\partial \bar{V}}{\partial t}(\cdot, e) + \frac{\partial \bar{V}}{\partial z}(\cdot, e)\dot{e} \\ &\le -\alpha_{23}(|e|), \end{aligned} \tag{4.30}$$

whenever $|e| \ge \bar{\chi}(|\hat{y} - h(x)|)$. We also define the class $\mathcal{K}$ function

$$\chi_2(r) = \bar{\chi}(2\delta(2r)),$$

for all $r \in \mathbb{R}^+$, and the non-negative constant

$$c_{L2} = \bar{\chi}(2|d_y|_\infty) + \bar{\chi}(2\delta(2|d_\psi|_\infty)).$$

Assumption 4.2 then implies that

$$\begin{aligned}\chi_2(|\xi|) + c_{L2} &\geq \bar{\chi}(2\delta(|\psi(y) - \psi(\hat{y})|)) + \bar{\chi}(2|d_y|_\infty) \\ &\geq \bar{\chi}(2|y - \hat{y}|) + \bar{\chi}(2|d_y|_\infty) \\ &\geq \bar{\chi}(|y - \hat{y}| + |d_y|_\infty) \\ &\geq \bar{\chi}(|\hat{y} - h(x)|)\end{aligned}$$

which, together with (4.30), provides the needed property.

Finally, notice that c_{L2} is zero precisely when the disturbances d_y and d_ψ are zero. □

We are now ready to state our main result. It establishes a condition on the gain ρ in order to guarantee the asymptotic convergence to a neighborhood of zero (or to zero itself) of the state estimation error given by the new observer in (4.21).

Theorem 4.4 (Observer gain design)**.** *Consider systems* (4.18) *and* (4.21)*, with* φ *and* K *as in* (4.23)-(4.24)*, and let Assumptions 4.1 and 4.2 hold. For any* $k > 1$ *and any* $\rho \in \mathcal{K}_\infty \cap C^2$ *satisfying*

$$\rho(r) > 2\alpha(2\bar{\alpha}_1^{-1}(\bar{\alpha}_2(2\bar{\chi}(2\delta(4r))))), \quad \forall r > 0 \tag{4.31}$$

there exist a class $\mathcal{KL}$ *function* β *and a constant* $c \geq 0$ *such that for all* $x(0)$, $\hat{x}(0) \in \mathbb{R}^n$ *and all* $\hat{y}(0) \in \mathbb{R}^p$ *the estimation errors* $e = x - \hat{x}$ *and* $\xi = y_\psi - \psi(\hat{y})$ *are defined on* $\mathbb{R}^+$ *and*

$$|(\xi(t), e(t))| \leq \beta(|(\xi(0), e(0))|, t) + c, \quad \forall t \geq 0. \tag{4.32}$$

Moreover, if the disturbances d_y *and* d_ψ *are both zero then* c *is zero as well.*

Proof. Using Lemmas 4.1 and 4.2, we deduce that there exist Lyapunov functions for the family $\mathcal{G}$ given in (4.25) and for the family $\mathcal{G}^{-1}$ given in (4.29). Since the first two Lyapunov-bounds in the proof of Lemma 4.1 can be taken as the identity function, the small-gain condition in (2.51) is given by:

$$\chi_{m1} = 2\chi_1(\alpha_{21}^{-1}), \quad \chi_{m2} = \alpha_{22}(2\chi_2). \tag{4.33}$$

It is then straight-forward to verify the equivalence of this condition with the inequality in (4.31). Indeed, by using the expression of χ_1 in (4.28) and the functions defined in the proof of Lemma 4.2 we have that

$$2\chi_1(\alpha_{21}^{-1}(\alpha_{22}(2\chi_2(r)))) < r, \quad \forall r > 0$$

precisely when ρ satisfies (4.31). Hence, Theorem 2.15 concludes that $\mathcal{G} \times \{0\}$ is practically stable. That is, there exist a function $\beta \in \mathcal{KL}$ and a constant $c \geq 0$ such that (4.32) is satisfied on $[0, T)$. As a consequence, this interval is necessarily the whole $\mathbb{R}^+$.

Finally, if both disturbances d_y and d_ψ are zero then in Lemmas 4.1 and 4.2 both Lyapunov-constants c_{L1} and c_{L2} are zero as well. The details of the proofs in Section 2.5 imply then $c = 0$. □

Remark 4.10. *In practice, the design of the new observer* (4.21) *starts by proposing an observer as* (4.17) *and by finding a Lyapunov function $\bar{V}$ for its error dynamics, together with the functions $\bar{\alpha}_1$, $\bar{\alpha}_2$ and $\bar{\chi}$ from Assumption 4.1. We then require δ from Assumption 4.2 and the functions φ and α from* (4.23) *(see Remark 4.9). Finally, the lower bound in* (4.31) *itself can be used to construct such a ρ and K is then given by* (4.24).

Remark 4.11. *From the proofs of Lemmas 4.1 and 4.2, we have that the new observer recovers a type of DES property: there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $x(0)$, $\hat{x}(0) \in \mathbb{R}^n$ and all $\hat{y}(0) \in \mathbb{R}^p$, and for all bounded, Lipschitz and differentiable d_y and d_ψ we have*

$$|(\xi(t), e(t))| \leq \beta(|(\xi(0), e(0))|, t) + \gamma(|(d_y, d_\psi, \dot{d}_y, \dot{d}_\psi)|_\infty), \quad \forall t \geq 0.$$

Finally, notice that we can find the explicit decay rate β and constant c in (4.32) by following the next steps:

1. Find the Lyapunov-bounds and gain from Assumption 4.1. Use them to construct ρ satisfying the inequality in (4.31).
2. Get the Lyapunov-bounds, gains and constants from Lemma 4.1 and Lemma 4.2.
3. Consider the mixed gains in (4.33) and choose an in-between function σ as explained in Section 2.5.
4. Compute the bounds, gain and constant of the Lyapunov function for $\mathcal{G} \times \{0\}$ as in the proof of Theorem 2.15 in Section 2.5.

5. Compute the corresponding decay rate β and constant c as shown in the proofs of Lemma 2.1 and Theorem 2.14 in Section 2.5.

The following section concerns two classical and widely used families of nonlinear systems and it helps to illustrate the design of the new observer.

4.6 Particular cases

The aim of this section is to study Theorem 4.4 when system (4.16) belongs to the family of: (i) state affine systems, (ii) systems with additive triangular nonlinearity. We add the following assumption on ψ throughout this section.

Assumption 4.3. *The function ψ and its derivative are both Lipschitz continuous.*

This assumption is not a critical design requirement but it simplifies the computations of φ and α in (4.23) as explained in Remark 4.9. Moreover, it can be easily met in the case of bounded states as shown in Remark 4.13 at the end of this section.

4.6.1 State-affine systems up to output injection

Consider $A : \mathbb{R}^m \to \mathbb{R}^{n\times n}$ a continuous matrix functional, $\eta : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^n$ a nonlinear and continuous function and $C \in \mathbb{R}^{p\times n}$ a constant matrix. Our goal is to estimate the state of the system given by:

$$\begin{cases} \dot{x} = A(u)x + \eta(Cx, u) \\ y_\psi = \psi(y) + d_\psi, \end{cases} \tag{4.34}$$

where $y = Cx + d_y$ and where $\psi : \mathbb{R}^p \to \mathbb{R}^p$ satisfies Assumption 4.2. We need the next condition on η.

Assumption 4.4. *The function η is Lipschitz continuous with respect to its first entrance. That is, there exists a positive constant c_η such that*

$$|\eta(y, u) - \eta(\hat{y}, u)| \leq c_\eta |y - \hat{y}|,$$

for all y, $\hat{y} \in \mathbb{R}^p$ and all $u \in \mathbb{R}^m$.

The following observer [22, 57], which cannot be directly implemented, plays the role of observer (4.17) and it is described by:

$$\begin{cases} \dot{\hat{x}} = A(u)\hat{x} + \eta(y,u) + S^{-1}C'(y - C\hat{x}) \\ \dot{S} = -\theta S - A(u)'S - SA(u) + C'C, \end{cases} \tag{4.35}$$

where $\theta > 0$ is a tuning parameter and $S(0) \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix. It is known that $S(t)$ maintains these properties for all $t \in \mathbb{R}^+$.

Notice that the equation defining S in (4.35) coincides with the one in [18]. As in Definition 3.1, we require the input u to be regularly persistent for the homogeneous part of system (4.34) given by

$$\begin{cases} \dot{x} = A(u)x \\ y = Cx, \end{cases}$$

and with respect to some $a > 0$, $t_0 > 0$ and $T \geq t_0$.

Assumption 4.5. *The input set $\mathcal{U}$ consists of continuous, bounded and regularly persistent inputs $u : \mathbb{R}^+ \to \mathbb{R}^m$.*

Remark 4.12. *Assumptions 4.4 and 4.5 concern the convergence properties of the initial observer* (4.35) *and, as shown below, substitute Assumption 4.1 in the case of state-affine systems up to output injection.*

Let us consider $0 < a \leq \lambda_{\min}(S(0))$ to simplify and denote

$$b_1 = \sup_{|u|\leq c_u} |A(u)|,$$

where $|u|_\infty \leq c_u$. If $\theta \geq 3b_1 > 0$, we can deduce the following bounds as in [18],

$$s_1 I_n \leq S(t) \leq s_2 I_n, \tag{4.36}$$

for all $t \in \mathbb{R}^+$ and where

$$s_1 = a\exp(-t_0(\theta + 2b_1)), \quad s_2 = b_1^{-1}|C|^2 + |S(0)|$$

are positive constants.

We next show that observer (4.35) satisfies Assumption 4.1. We define the function $\bar{V}$:

$\mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ as

$$\bar{V}(t,e) = e'S(t)e,$$

for all $t \in \mathbb{R}^+$ and $e \in \mathbb{R}^n$. According to (4.36), we can choose the class $\mathcal{K}_\infty$ functions:

$$\bar{\alpha}_1(r) = s_1 r^2, \quad \bar{\alpha}_2(r) = s_2 r^2, \tag{4.37}$$

for all $r \in \mathbb{R}^+$. Moreover, the following holds for all $t \in \mathbb{R}^+$, $e \in \mathbb{R}^n$ and $d \in \mathbb{R}^p$. On one hand, the definition of S in (4.35) provides

$$\begin{aligned} \frac{\partial \bar{V}}{\partial t}(t,e) &= -\theta e'S(t)e - e'A(u(t))'S(t)e \\ &\quad - e'S(t)A(u(t))e + e'C'Ce \\ &= -\theta e'S(t)e - 2e'S(t)A(u(t))e + e'C'Ce, \end{aligned} \tag{4.38}$$

where we used that $e'A(u(t))'Se$, as a real number, coincides with its transpose. On the other hand, $\mathcal{F}$ from (4.20) takes the form

$$\begin{aligned} \mathcal{F}(t,e,d) &= A(u(t))e - S(t)^{-1}C'(Ce+d) \\ &\quad + \eta(Cx(t),u(t)) - \eta(Cx(t)+d,u(t)). \end{aligned}$$

It then follows that

$$\begin{aligned} \frac{\partial \bar{V}}{\partial e}(t,e)\mathcal{F}(t,e,d) =& 2e'S(t)\mathcal{F}(t,e,d) \\ =& 2e'S(t)A(u(t))e - 2e'C'(Ce+d) \\ &+ 2e'S(t)(\eta(Cx(t),u(t)) - \eta(Cx(t)+d,u(t))). \end{aligned} \tag{4.39}$$

Putting (4.38) and (4.39) together we obtain that

$$\begin{aligned} \frac{\partial \bar{V}}{\partial t}(t,e) + \frac{\partial \bar{V}}{\partial e}(t,e)\mathcal{F}(t,e,d) =& -\theta e'S(t)e + e'C'Ce - 2e'C'(Ce+d) \\ &+ 2e'S(t)(\eta(Cx(t),u(t)) - \eta(Cx(t)+d,u(t))), \end{aligned}$$

and using Assumption 4.4 and the bounds in (4.36), we get

$$\begin{aligned}\frac{\partial \bar{V}}{\partial t}(t,e)+\frac{\partial \bar{V}}{\partial e}(t,e)\mathcal{F}(t,e,d) &\leq -\theta e' S(t)e + 2|C||e||d| + 2c_\eta |S(t)||e||d| \\ &\leq -\theta s_1 |e|^2 + 2|C||e||d| + 2c_\eta s_2 |e||d|.\end{aligned}$$

We now define the class $\mathcal{K}_\infty$ functions

$$\bar{\chi}(r) = s_1^{-1} r, \quad \bar{\alpha}_3(r) = s_1(\theta - b_2) r^2, \tag{4.40}$$

where

$$\theta > b_2 = 2(|C| + c_\eta s_2).$$

Hence, the inequality $|e| \geq \bar{\chi}(|d|)$ implies that

$$\begin{aligned}\frac{\partial \bar{V}}{\partial t}(t,e)+\frac{\partial \bar{V}}{\partial e}(t,e)\mathcal{F}(t,e,d) &\leq s_1(-\theta + 2|C| + 2c_\eta s_2)|e|^2 \\ &= -\bar{\alpha}_3(|e|)\end{aligned}$$

and we are required to tune $\theta > \max\{3b_1, b_2\}$, which is possible since b_1 and b_2 are independent of θ.

We can now design the corresponding new observer (4.21). Notice that in this case, the function f is globally Lipschitz in x, uniformly in u such that $|u| \leq c_{\mathcal{U}}$. Therefore, we are in the situation of Remark 4.9 and we can choose α as the identity function and

$$\varphi(\hat{x}, u) \geq c_\varphi(|A(u)\hat{x} + \eta(C\hat{x}, u)| + 1), \tag{4.41}$$

for some $c_\varphi \geq 1$ and for all $\hat{x} \in \mathbb{R}^n$ and $|u| \leq c_u$. By using the functions in (4.37) and (4.40) and by using the small-gain condition in (4.31), we get

$$\rho(r) = 17(s_1^{-3} s_2)^{\frac{1}{2}} \delta(4r), \tag{4.42}$$

for all $r \in \mathbb{R}^+$ and where δ is given in Assumption 4.2. As a consequence, K in (4.24) is given by:

$$K(\xi) = \begin{cases} 17k(s_1^{-3} s_2)^{\frac{1}{2}} \delta(4|\xi|)|\xi|^{-1} \cdot \xi, \text{ if } \xi \neq 0 \\ 0, \text{ if } \xi = 0, \end{cases} \tag{4.43}$$

for $k > 1$. The observer in (4.21) takes the form:

$$\begin{cases} \dot{\hat{x}} = A(u)\hat{x} + \eta(\hat{y}, u) + S^{-1}C'(\hat{y} - C\hat{x}) \\ \dot{\hat{y}} = \frac{\partial\psi}{\partial y}(\hat{y})^{-1}\frac{\partial\psi}{\partial y}(C\hat{x})C(A(u)\hat{x} + \eta(C\hat{x}, u)) \\ \qquad + \frac{\partial\psi}{\partial y}(\hat{y})^{-1}\varphi(\hat{x}, u)K(y_\psi - \psi(\hat{y})), \end{cases} \tag{4.44}$$

where S is as in (4.35). We summarize our results in the following corollary.

Corollary 4.1 (state-affine case). *Consider systems* (4.34) *and* (4.44) *and let Assumptions 4.2, 4.3, 4.4 and 4.5 hold. For any tuning parameter θ such that*

$$\theta > \max\{3b_1, 2|C| + 2c_\eta(b_1^{-1}|C|^2 + |S(0)|)\},$$

where $b_1 = \sup_{|u|\leq c_u}|A(u)|$, there exist a function $\beta \in \mathcal{KL}$ and a constant $c \geq 0$ such that for all $x(0)$, $\hat{x}(0) \in \mathbb{R}^n$ and all $\hat{y}(0) \in \mathbb{R}^p$, the estimation errors $e = x - \hat{x}$ and $\xi = y_\psi - \psi(\hat{y})$ are defined on $\mathbb{R}^+$ and

$$|(\xi(t), e(t))| \leq \beta(|(\xi(0), e(0))|, t) + c, \quad \forall t \geq 0. \tag{4.45}$$

Moreover, if the disturbances d_y and d_ψ are both zero then c is zero as well.

Proof. It follows from our previous development and from Theorem 4.4. Indeed, we showed that Assumption 4.1 is satisfied by using Assumptions 4.4 and 4.5. $\square$

Finally, we compute the explicit decay rate β and the constant c in (4.45) as explained at the end of Section 4.4. The function ρ is already given in (4.42), this is Step 1). For Steps 2) and 3), we suppose that δ is linear with slope $c_\delta > 0$ so that:

$$\delta(r) = c_\delta r, \quad \rho(r) = c_\rho r, \quad c_\rho = 68(s_1^{-3}s_2)^{\frac{1}{2}}c_\delta,$$

for all $r \in \mathbb{R}^+$. The mixed Lyapunov gains then take the form:

$$\chi_{m1}(r) = c_{\chi_{m1}}r^{\frac{1}{2}}, \qquad c_{\chi_{m1}} = 2s_1^{-\frac{1}{2}}c_\rho^{-1},$$
$$\chi_{m2}(r) = c_{\chi_{m2}}r^2, \qquad c_{\chi_{m2}} = 64s_1^{-2}s_2c_\delta^2,$$

for all $r \in \mathbb{R}^+$. As described in Section 2.5, we can select the function between the gains as

$$\sigma(r) = c_\sigma r^2, \quad c_\sigma = \frac{c_{\chi_{m1}}^{-2} + c_{\chi_{m2}}}{2} > 0,$$

for all $r \in \mathbb{R}^+$. Simple computations in Steps 4) and 5) show that the decay rate is given by

$$\beta(r,t) = r\sqrt{c_1} \exp\left(-\frac{c_2}{c_1}t\right),$$

for all $r, t \in \mathbb{R}^+$ and with the positive constants:

$$\begin{aligned} c_1 &= \frac{8\max\{s_2, c_\sigma\}}{\min\{s_1, c_\sigma\}}, \\ c_2 &= \frac{1}{4}\min\{2(k-1)c_\rho, (\theta - b_2)s_1 s_2^{-1}\}. \end{aligned}$$

Furthermore, the constant of the practical stability is

$$\begin{aligned} c = 16 &\left(\frac{c_1}{s_1 \min\{s_1, c_\sigma\}}\right)^{\frac{1}{2}} (|d_y|_\infty + 2c_\delta |d_\psi|_\infty)^{\frac{1}{2}} \\ &+ 64 c_\rho^{-1} \left(\frac{c_1 c_\sigma}{\min\{s_1, c_\sigma\}}\right)^{\frac{1}{2}} (|d_y|_\infty + |\dot{d}_y|_\infty + |\dot{d}_\psi|_\infty). \end{aligned}$$

4.6.2 Systems with additive triangular nonlinearity

Let us now consider the canonical matrices $A \in \mathbb{R}^n$ and $C \in \mathbb{R}^{1\times n}$ and a nonlinear and continuous triangular function $\zeta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ in the form of (2.19). Our goal is to estimate the state of the system:

$$\begin{cases} \dot{x} = Ax + \zeta(x,u) \\ y_\psi = \psi(y) + d_\psi, \end{cases} \tag{4.46}$$

where $y = Cx + d_y$ and where $\psi : \mathbb{R} \to \mathbb{R}$ satisfies Assumption 4.2. As in Chapter 2, we add the following assumption on ζ.

Assumption 4.6. *The function ζ is coordinate-wise Lipschitz continuous in the first entrance. That is, for each $i = 1, \ldots, n$, there exists a positive constant $c_{\zeta i}$ such that*

$$|\zeta_i(\underline{x}_i, u) - \zeta_i(\hat{\underline{x}}_i, u)| \le c_{\zeta i}|\underline{x}_i - \hat{\underline{x}}_i|,$$

where $\underline{x}_i$ is defined below (2.20) *and for all x, $\hat{x} \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.*

As opposed to the state-affine case, the nonlinearity ζ in system (4.46) depends on the full-state x which complicates its estimation. The observer in (2.21) plays the role of observer (4.17) and we recall its form

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + \zeta(\hat{x}, u) + S_\infty^{-1} C'(y - C\hat{x}) \\ 0 = -\theta S_\infty - A' S_\infty - S_\infty A + C'C, \end{cases} \tag{4.47}$$

where $\theta > 0$ is a tuning parameter. Similar to the previous case, the matrix S_∞ is symmetric and positive definite. Let us denote the eigenvalue extrema of S_∞ by

$$s_1 = \lambda_{\min}(S_\infty), \quad s_2 = \lambda_{\max}(S_\infty)$$

and define $S_{\infty,1}$ as the solution of the second equation in (4.47) corresponding to the unitary tuning, that is,

$$S_{\infty,1} = \theta^{-1} \Delta_\theta S_\infty \Delta_\theta, \tag{4.48}$$

where Δ_θ is as in (2.17). Finally, we name the following maxima:

$$c_\zeta = \max_{i=1,\dots,n} c_{\zeta i}, \quad s_m = \max_{i,j=1,\dots,n} |(S_{\infty,1})_{i,j}|. \tag{4.49}$$

We next show that observer (4.47) satisfies Assumption 4.1 by using similar techniques to those of [46]. In this case, the input set $\mathcal{U}$ can be taken simply as the set of continuous functions $u : \mathbb{R}^+ \to \mathbb{R}^m$. We define the function $\bar{V} : \mathbb{R}^n \to \mathbb{R}^+$ as the norm induced by S_∞, that is,

$$\begin{aligned} \bar{V}(e) &= (e' S_\infty e)^{\frac{1}{2}} \\ &= |e|_{S_\infty}, \end{aligned}$$

for all $e \in \mathbb{R}^n$. Clearly, we can select the class $\mathcal{K}_\infty$ functions given by

$$\bar{\alpha}_1(r) = s_1^{\frac{1}{2}} r, \quad \bar{\alpha}_2(r) = s_2^{\frac{1}{2}} r,$$

for all $r \in \mathbb{R}^+$. Moreover, the following holds for all $t \in \mathbb{R}^+$, $e \in \mathbb{R}^n - \{0\}$ and $d \in \mathbb{R}$. The function $\mathcal{F}$ from (4.20) takes the form

$$\begin{aligned}\mathcal{F}(t,e,d) = Ae - S_\infty^{-1}C'(Ce+d)\\ + \zeta(x(t),u(t)) - \zeta(x(t)-e,u(t)).\end{aligned}$$

It then follows that

$$\begin{aligned}\frac{\partial \bar{V}}{\partial e}(e)\mathcal{F}(t,e,d) =& \frac{1}{2}|e|_{S_\infty}^{-1}(2e'S_\infty\mathcal{F}(t,e,d))\\ \leq& -\frac{\theta}{2}|e|_{S_\infty} - |e|_{S_\infty}^{-1}e'C'd\\ &+ |e|_{S_\infty}^{-1}e'S_\infty\cdot(\zeta(x(t),u(t)) - \zeta(x(t)-e,u(t)))\\ \leq& -\frac{\theta}{2}|e|_{S_\infty} + s_1^{-\frac{1}{2}}|C||d|\\ &+ |\zeta(x(t),u(t)) - \zeta(x(t)-e,u(t))|_{S_\infty}.\end{aligned} \tag{4.50}$$

On the other hand, computing the induced norm and using (4.48) we have

$$|\zeta(x(t),u(t)) - \zeta(x(t)-e,u(t))|_{S_\infty}^2 = \sum_{i,j=1}^{n}\frac{\theta}{\theta^{i+j}}(S_{\infty,1})_{i,j}\bar{\zeta}(i,t,e)\bar{\zeta}(j,t,e), \tag{4.51}$$

where $\bar{\zeta}(i,t,e) = \zeta_i(\underline{x}_i(t),u(t)) - \zeta_i(\underline{x}_i(t)-\underline{e}_i,u(t))$. Therefore, Assumption 4.6 implies:

$$\begin{aligned}|\zeta(x(t),u(t)) - \zeta(x(t)-e,u(t))|_{S_\infty}^2 \leq& \sum_{i,j=1}^{n}\frac{\theta s_m c_\zeta^2}{\theta^{i+j}}|\underline{e}_i||\underline{e}_j|\\ \leq& n^2 s_m c_\zeta^2\,\lambda_{\min}(S_{\infty,1})^{-1}|e|_{S_\infty}^2,\end{aligned}$$

where we used the inequality $|\theta^{-i}\underline{e}_i| \leq |\Delta_\theta^{-1}e|$. Putting (4.50) and (4.51) together,

$$\begin{aligned}\frac{\partial \bar{V}}{\partial e}(e)\mathcal{F}(t,e,d) \leq& -\frac{\theta}{2}|e|_{S_\infty} + s_1^{-\frac{1}{2}}|C||d|\\ &+ n s_m^{\frac{1}{2}} c_\zeta\,\lambda_{\min}(S_{\infty,1})^{-\frac{1}{2}}|e|_{S_\infty}\\ \leq& -\bar{\alpha}_3(|e|),\end{aligned}$$

whenever $|e| \geq \bar{\chi}(|d|)$ and where:

$$\bar{\chi}(r) = s_1^{-1}r, \quad \bar{\alpha}_3(r) = \frac{s_1^{\frac{1}{2}}}{2}(\theta - b)r,$$

for all $r \in \mathbb{R}^+$ and for

$$\theta > b = 2\left(n s_m^{\frac{1}{2}} c_\zeta \, \lambda_{\min}\left(S_{\infty,1}\right)^{-\frac{1}{2}} + 1\right).$$

Notice that such θ exists since $S_{\infty,1}$ corresponds to the unitary tuning.

In order to construct the new observer (4.21) and given that we are in the case of Remark 4.9, we can choose α as the identity function and:

$$\begin{aligned} \varphi(\hat{x}, u) &\geq c_\varphi(|A\hat{x} + \zeta(\hat{x}, u)| + 1), \\ \rho(r) &= 17(s_1^{-3} s_2)^{\frac{1}{2}} \delta(4r) \end{aligned}$$

for some $c_\varphi \geq 1$ and all $\hat{x} \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $r \in \mathbb{R}^+$. The new observer (4.21) then takes the form:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + \zeta(\hat{x}, u) + S_\infty^{-1} C'(\hat{y} - C\hat{x}) \\ \dot{\hat{y}} = \frac{\partial \psi}{\partial y}(\hat{y})^{-1} \frac{\partial \psi}{\partial y}(C\hat{x}) C(A\hat{x} + \zeta(\hat{x}, u)) \\ \qquad + \frac{\partial \psi}{\partial y}(\hat{y})^{-1} \varphi(\hat{x}, u) K(y_\psi - \psi(\hat{y})), \end{cases} \tag{4.52}$$

where S is as in (4.47) and K as in (4.24) with $k > 1$.

Corollary 4.2 (triangular case)**.** *Consider systems* (4.46) *and* (4.52) *and let Assumptions 4.2, 4.3 and 4.6 hold. For any tuning*

$$\theta > 2\left(n s_m^{\frac{1}{2}} c_\zeta \, \lambda_{\min}\left(S_{\infty,1}\right)^{-\frac{1}{2}} + 1\right),$$

where c_ζ and s_m are as in (4.49)*, there exist a function $\beta \in \mathcal{KL}$ and a constant $c \geq 0$ such that for all $x(0)$, $\hat{x}(0) \in \mathbb{R}^n$ and all $\hat{y}(0) \in \mathbb{R}$ the estimation errors $e = x - \hat{x}$ and $\xi = y_\psi - \psi(\hat{y})$ are defined on $\mathbb{R}^+$ and*

$$|(\xi(t), e(t))| \leq \beta(|(\xi(0), e(0))|, t) + c, \quad \forall t \geq 0.$$

Moreover, if the disturbances d_y and d_ψ are both zero then c is zero as well.

Proof. It follows from our previous development and from Theorem 4.4. Indeed, we showed that Assumption 4.1 is satisfied by using the specific forms of A and C and by using Assumption 4.6. $\square$

Remark 4.13. *If the states of system* (4.46) *are uniformly bounded and if ζ is of class C^1 then Assumption 4.6 can be met by saturating ζ. Indeed, denote by X the state space and let*

$\mu > 0$ be such that $|x| \leq \mu$, for all $x \in X$. Set each $\zeta_i^s : \mathbb{R}^i \times \mathbb{R}^m \to \mathbb{R}$ as

$$\zeta_i^s(x_1, \ldots, x_i, u) = \zeta_i(\mu \operatorname{sat}(\mu^{-1} x_1), \ldots, \mu \operatorname{sat}(\mu^{-1} x_i), u),$$

where $\operatorname{sat} : \mathbb{R} \to \mathbb{R}$ *is given by* $\operatorname{sat}(t) = \min\{1, |t|\} \operatorname{sign}(t)$. *Replacing ζ by ζ^s defines an equivalent system and the latter function satisfies Assumption 4.6. We can proceed similarly for ψ and Assumption 4.3 but using a non-constant and smooth saturation.*

4.7 Numerical results

In this section, we study specific cases of systems (4.34) and (4.46) in order to provide numerical examples of Corollaries 4.1 and 4.2.

4.7.1 Example for state-affine systems

Let us suppose that system (4.34) is given by:

$$A(u) = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}, \quad \eta(y, u) = \begin{bmatrix} \sin(y) \\ u^2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

for all $y, u \in \mathbb{R}$. The input and the nonlinear function are defined as:

$$u(t) = \cos(t), \quad \psi(y) = \sin(y) + 2y.$$

The function ψ is a diffeomorphism whose inverse has no closed form, and it is straightforward to check that Assumptions 4.2, 4.3 and 4.4 are satisfied. Moreover, we can select δ as the identity function and it can be shown that u is regularly persistent, see for example [51]. The lower bound of θ in Corollary 4.1 has a value of 6 when $S(0) = I_2$. A simple computation shows that, according to (4.41) and (4.43), we can select

$$\varphi(\hat{x}, u) K(\xi) = \lambda(|\hat{x}_2| + 1) \cdot \xi, \tag{4.53}$$

for all $\hat{x} \in \mathbb{R}^2$ and $u, \xi \in \mathbb{R}$ and where $\lambda > 0$ is to be tuned.

We initialize the system and the observers at: $x(0) = (10, 0)$, $\hat{x}(0) = (4, 9)$ and $\hat{y}(0) = C\hat{x}(0)$

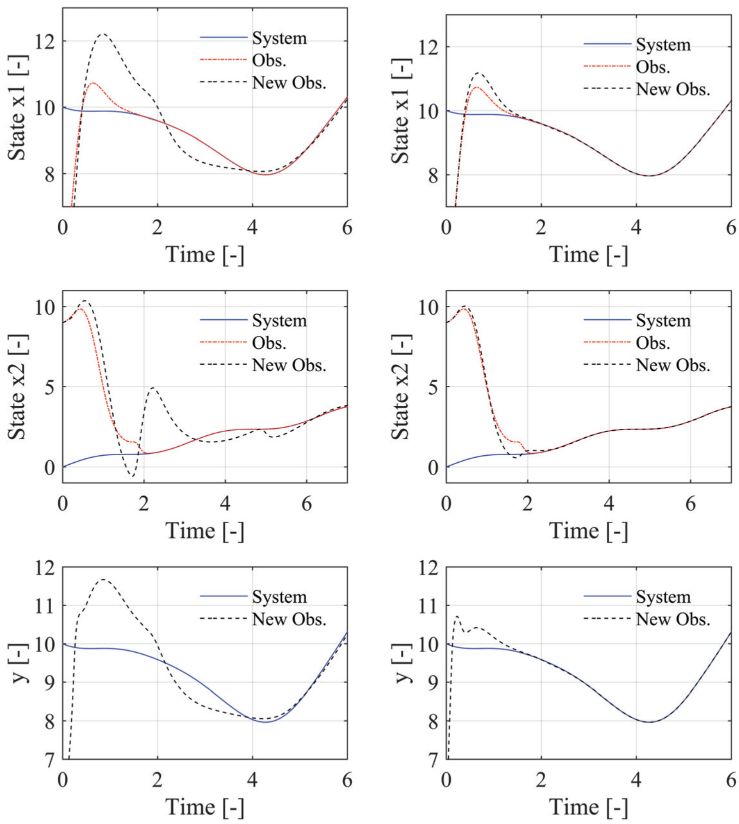

Figure 4.2: Comparison of the new observer with the observer that uses y in the case: $d_y = 0$, $d_\psi = 0$. Estimation of the system states of (4.34) (blue solid) by observer (4.35) (red pointed-dashed) and by the new observer (4.44) (black dashed). The columns correspond respectively to the tunings $\lambda = 0.5$ and $\lambda = 2$ in (4.53).

and we set the tuning parameter $\theta = 7$. The results are shown in Figures 4.2 and 4.3. We simulate two disturbance cases: (i) d_y and d_ψ are both zero (Figure 4.2), (ii) d_y and d_ψ are uniformly distributed numbers respectively between ± 0.8 and ± 1.3 (Figure 4.3).

Discussions on Figure 4.2: In the top two rows, we can see that observer (4.35), which uses y, correctly estimates the states of the system. As also depicted in the top two rows, the new observer (4.44) achieves at most the performance of observer (4.35). Moreover, the performance is closer for the larger choice of λ (right column). Finally, the quality of the state estimation given by the new observer (4.44) depends of the estimation of y by $\hat{y}$, as can be seen in the third row.

Discussions on Figure 4.3: In the top two rows, we can see that the state estimation from observer (4.35) converges to a neighborhood of the system state, as expected from a DES

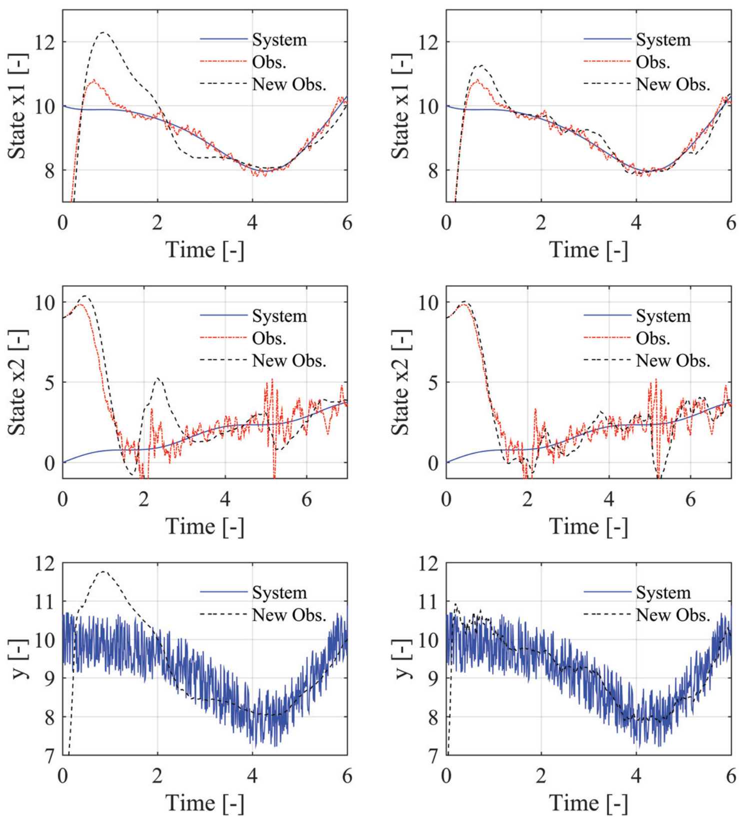

Figure 4.3: Comparison of the new observer with the observer that uses y in the case: $d_y \neq 0$, $d_\psi \neq 0$. Estimation of the system states of (4.34) (blue solid) by observer (4.35) (red pointed-dashed) and by the new observer (4.44) (black dashed). The columns correspond respectively to the tunings $\lambda = 0.5$ and $\lambda = 2$ in (4.53).

observer. The performance of observer (4.35) and the new observer (4.44) are again similar for the larger λ. Moreover, the new observer (4.44) outperforms observer (4.35) in terms of noise robustness. This is not surprising since noise amplification is a common problem for high-gain observers when their tuning parameter θ is large [4]. The noise, in contrast, is averaged out by the definition of $\hat{y}$.

These figures show that, for λ large enough, the new observer in (4.44) recovers the performance of observer (4.35). Furthermore, the design of the new observer (4.44) seems to be more robust against measurement noise.

4.7.2 Example for systems with triangular nonlinearity

Here we consider the example studied by the authors in [77]. The system is in the form of (4.46) and it is described by:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$\zeta(x,u) = \begin{bmatrix} 0 \\ -x_1 - 2x_2 + ax_1^2x_2 + u \end{bmatrix},$$

for all $x \in \mathbb{R}^2$ and $u \in \mathbb{R}$ and for some $a > 0$. They also define the input and the nonlinear output as:

$$u(t) = b\sin(2t), \quad \psi(y) = \frac{1}{3}y^3 - \frac{1}{2}y^2 + y,$$

for some $b > 0$. As opposed to [77], we consider disturbances. Notice that ψ is a diffeomorphism whose derivative is bounded from below by 0.75 and, therefore, Assumption 4.2 is met. However, ψ does not satisfy Assumption 4.3 and ζ satisfies only locally Assumption 4.6.

As in [71] or [77], for $a = 0.25$ and $b = 0.2$, every state starting at

$$X = \{x \in \mathbb{R}^2 | 1.5x_1^2 + x_1x_2 + 0.5x_2^2 \leq \sqrt{2}\}$$

remains in that set for all positive times. Therefore, in Remark 4.13 we can replace ψ and ζ by their saturated versions ψ^s and ζ^s. These new functions satisfy Assumptions 4.2, 4.3 and 4.6. Similar to the first case, we set

$$\varphi(\hat{x},u)K(\xi) = \lambda(|\hat{x}_1| + |\hat{x}_2| + |\hat{x}_1^2\hat{x}_2| + 1)\cdot\xi,$$

for all $\hat{x} \in \mathbb{R}^2$ and $u, \xi \in \mathbb{R}$ and where $\lambda > 0$. On the other hand, the observer from [77] is given by

$$\dot{\hat{z}} = A\hat{z} + \begin{bmatrix} 0 \\ \zeta_0(\hat{z},u) \end{bmatrix} + \begin{bmatrix} (8/3)\epsilon^{-1} \\ (4/3)\epsilon^{-2} \end{bmatrix} (y_\psi - \psi(C\hat{z})), \tag{4.54}$$

where $\zeta_0(\hat{z},u) = -\hat{z}_1 - 2\hat{z}_2 + a\,\mathrm{sat}(\hat{z}_1^2\hat{z}_2) + u$, $y_\psi = \psi(Cx + d_y) + d_\psi$ and where $\epsilon > 0$ is to be tuned.

In order to make a fair comparison of observers (4.52) and (4.54): first we choose two far

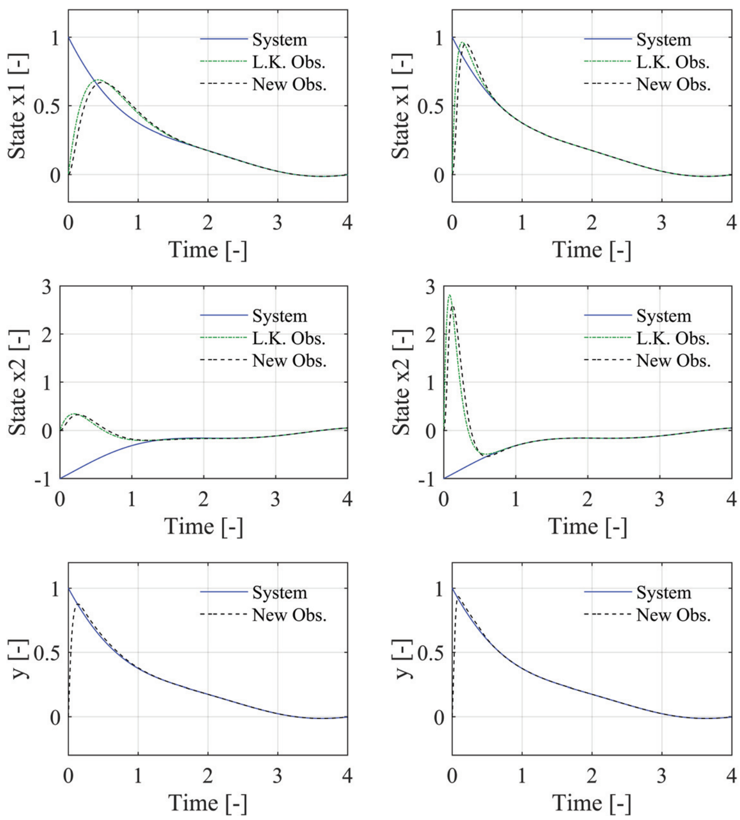

Figure 4.4: Comparison of the new observer with the observer of J. Lei and H.K. Khalil [77] in the case: $d_y = 0$, $d_\psi = 0$. Estimation of the system states in (4.46) (blue solid) by observer (4.54) (green pointed-dashed) and by the new observer (4.52) (black dashed). The left column corresponds to the tunings $\theta = 2$ and $\epsilon = 0.5$ and the right column to $\theta = 9$ and $\epsilon = 0.11$.

enough values of ϵ, then we try to improve or to match this performance by tuning θ and λ.

We initialize the system and the observers at: $x(0) = (1, -1)$, $\hat{x}(0) = \hat{z}(0) = (0, 0)$ and $\hat{y} = C\hat{x}(0)$ and we fix the value $\lambda = 20$. The results can be seen in Figures 4.4 and 4.5. We simulate two cases: (i) d_y and d_ψ are both zero (Figure 4.4), (ii) d_y and d_ψ are uniformly distributed numbers respectively between ± 0.3 and ± 0.4 (Figure 4.5).

Discussions on Figure 4.4: In the top two rows, we see that both the new observer (4.52) and observer (4.54) render quite similar estimations that converge to the system states. Furthermore, higher tuning of θ and ϵ^{-1} leads to faster state reconstruction at the price of higher peaking (right column). The third row shows that tuning θ does not have a very strong effect in the estimation of y given by $\hat{y}$.

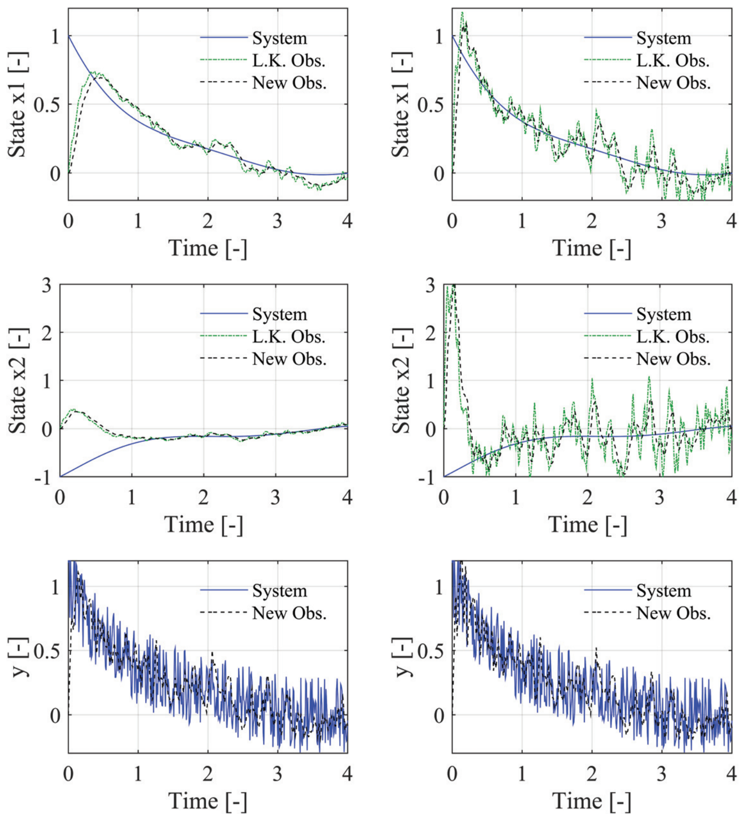

Figure 4.5: Comparison of the new observer with the observer of J. Lei and H.K. Khalil [77] in the case: $d_y \neq 0$, $d_\psi \neq 0$. Estimation of the system states in (4.46) (blue solid) by observer (4.54) (green pointed-dashed) and by the new observer (4.52) (black dashed). The left column corresponds to the tunings $\theta = 2$ and $\epsilon = 0.5$ and the right column to $\theta = 9$ and $\epsilon = 0.11$.

Discussions on Figure 4.5: The top two rows show that both observers (4.52) and (4.54) provide similar estimations, this time converging to neighborhoods of the states. Increasing the value of ϵ^{-1} in observer (4.54) amplifies the noise, this is slightly less visible for the larger choice of θ and the new observer (4.52) (right column).

These figures suggest that both observers have similar performances if they are properly tuned. It is clear that observer (4.54) has a simpler design. Nevertheless, our methodology provides a new observer design for a much more general family of systems.

4.8 Conclusion

The method presented in this chapter concerns observer redesign for nonlinear systems in the presence of output transformations. The new observer consists in an interconnection of the original observer dynamics with an estimator of the unavailable output. By using the small-gain theorem, we showed that the new observer converges asymptotically to a neighborhood of zero that depends on the amplitude of the disturbances and their derivatives. We then studied the new observer design for two major families of systems that differ in their observability properties. The simulations showed that the new observer recovers the performance of the initial observer and that its performance is comparable to that of other related observers. Unlike previous studies on observer design for systems with nonlinear outputs: (i) the generality of our approach considers systems without a specific form at the cost of a more intricate design, (ii) we quantified the influence of different types of system disturbances in our design and concluded its robustness.

Future work could focus on relaxing the conditions on the initial observer by considering weaker versions of the DES property, such as quasi-DES developed in [99]. Another promising direction consists on using the results from [15] concerning the augmentation of an immersion into a diffeomorphism. Indeed, this could be used to extend our observer redesign to the case of a system with bounded state and an output transformation $\psi : \mathbb{R}^{p_1} \to \mathbb{R}^{p_2}$ with $p_1 < p_2$. However, our new observer would require a modification as in [12] in order to guarantee that $\hat{y}$ remains in a convenient region. Another interesting work direction concerns observer design for nonlinear systems with implicit output. That is, we can suppose that the output y is measured and related to the state through an implicit equation of the form $\psi(y, x) = 0$; such a formulation arises, for example, in the field of dynamic vision [85].

Chapter 5

Sampled-data observer design for nonlinear systems

5.1 Introduction

Recently, there has been an increasing interest in state estimation for nonlinear systems whose output is only available at discrete times. This and similar output constraints arise frequently in engineering applications within: networked systems [60], sampled-data systems and quantized systems [44]. Indeed, a large number of modern control systems consist of subparts which are interconnected by limited digital communication networks. In this context, continuous and precise data is not an accurate representation of reality. Examples of network control systems include: mobile sensor networks, remote surgery, automated highway systems, etc. On the other hand, nonlinear quantization occurs when the output values are limited to a countable set. This is the case for sensors such as optical incremental encoders, which are known to destabilize the control loop.

The problem of observer design for systems with discrete output measurements can be traced back to the introduction of the continuous-discrete Kalman filter in a stochastic framework [62], initially motivated by the problem of orbit determination of space vehicles. In the context of high-gain observers, the authors in [37] adapt the well-known observer design for uniformly observable systems [46] to the case of a discrete output. It is later shown in [87] that such a hybrid design is also possible with a constant gain. The authors in [86] propose an impulsive observer for state-affine systems with sampled output by modifying the Kalman-like design from [57]. In this setting, a new interesting trade-off arises between the high-gain parameter and the measurement step size.

On the other hand, the authors in [92] develop a Luenberger-like observer for Lipschitz nonlinear systems with a discrete output by implementing a sample-and-hold technique. Their method, based on Lyapunov analyses, is inspired by sampled-data control techniques and LMI's. Instead of holding the last measurement, the authors in [68] suggest to use a given observer design for a general nonlinear system based on a continuous-time output and to interconnect it with an output predictor in case the measurements are instead discrete. This observer redesign preserves the robustness properties of the original observer for small enough sampling periods. In order to tackle sampling period constraints, the authors in [6] develop a self-triggered observer design for systems with triangular nonlinearities and output measured at variable sampled times. That is, the sampling time is used as an additional tuning parameter of the observer. Finally, robust observer design for nonlinear networked control systems (NCS's) can be found in [84]. The authors derive sufficient conditions on the network in order to preserve the stability of a given observer under communication constraints.

In this chapter, we extend our previous approach to deal simultaneously with both output nonlinearities and output non-uniform discretization in time. Our work has been the subject of a communication [52].

One way to solve this problem consists in applying the results of the previous chapter together with the observer redesign from [68], which tackles the case of a discretized output. Nevertheless, this would result in an observer with three interconnections when it is expected that only two may suffice. Therefore, we instead propose to directly extend our observer design from Chapter 4 by implementing the output $\psi(y(t_k))$ with sample-and-hold. The resulting interconnected observer consists of: (i) a subsystem with continuous dynamics based on the structure of the original observer, (ii) a subsystem with switched dynamics arising from the sample-and-hold techniques. Its convergence is shown by defining a Lyapunov-Krasovskii functional, based on [92, 88], and by using the small-gain theorem for switched systems [111]. This approach leads to two LMI's that depend, among other parameters, on the maximum time between two consecutive samplings.

The organization of this chapter is as follows. We first review the classical results involving nonlinear systems with a discrete output, this includes a general observer redesign method that addresses the transition from continuous to discrete measurements. We then proceed with the introduction of a Lyapunov functional that arises in the context of control and sampled-data systems, and we present in detail the known small-gain theorem for interconnected switched systems. The subsequent section includes our main contribution, which is illustrated by a case study and a numerical example. Finally, we gather some concluding remarks concerning our work in the last section.

5.2 Preliminaries

5.2.1 Observer design for systems with sampled output

Let us consider again the representative form for uniformly observable systems but this time with a sampled output, that is,

$$\begin{cases} \dot{x}(t) = Ax(t) + \zeta(x(t), u(t)) \\ y(t_k) = Cx(t_k), \end{cases} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$, $y(t_k) \in \mathbb{R}$ and $u(t) \in \mathbb{R}^m$, for A, C and ζ in the canonical form (2.19) and where (t_k) is a strictly increasing sequence in $\mathbb{R}^+$. We suppose that $t_0 = 0$ and $t_{k+1} = t_k + \delta$ for a constant $\delta > 0$ that can be chosen. Let us further assume that the lower triangular function ζ is Lipschitz continuous in x, uniformly in u, so that there is a constant $L > 0$ such that

$$|\zeta(x, u) - \zeta(\hat{x}, u)| \leq L|x - \hat{x}|, \quad \forall x, \hat{x} \in \mathbb{R}^n, \forall u \in \mathbb{R}^m. \tag{5.2}$$

To simplify the notation, let us denote the left limit of a function $f : [s, t) \to \mathbb{R}^n$ at t as

$$f(t^-) = \lim_{r \to t^-} f(r).$$

We have seen in Chapter 2 that system (5.1) admits a Luenberger-like observer in case of a continuous output $y(t)$. The authors in [87] propose a constant gain observer for the discrete output case given by:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + \zeta(\hat{x}(t), u(t)), \quad t \in [t_k, t_{k+1}) \\ \hat{x}(t_{k+1}) = \hat{x}(t_{k+1}^-) - \rho\delta S_\infty^{-1} C'(C\hat{x}(t_{k+1}^-) - y(t_{k+1})) \\ S_\infty = \exp(-\theta\delta)\exp(-\theta\delta A')S_\infty \exp(-\theta\delta A) + \rho\delta C'C, \end{cases} \tag{5.3}$$

where θ and ρ are positive tuning parameters and where S_∞ is symmetric and positive definite. Crucially, this gain matrix is also constant.

Theorem 5.1 ([87])**.** *Consider system* (5.1) *and suppose that* ζ *satisfies* (5.2)*. There exist explicit functions* $\alpha_1 : \mathbb{R}^+ \to \mathbb{R}^+$ *and* $\alpha_2 : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for every positive numbers* r*,* δ *and* ρ*, and for every* θ *satisfying*

$$\max\left\{1, 2L\sqrt{n\alpha_2(r)\alpha_1(r)^{-1}}\right\} < \theta \leq \delta^{-1}\min\{r, \alpha_1(r)\},$$

the state estimation from observer (5.3) *converges exponentially to zero.*

Remark 5.1. *It can be shown that the matrix* S_∞ *arises as the limit of* $S(t_k)$ *when* $k \to \infty$*, that is, of the solution to the continuous-discrete Ricatti equation*

$$\begin{cases} \dot{S}(t) = -\theta S(t) - A'S(t) - S(t)A, \quad t \in [t_k, t_{k+1}) \\ S(t_{k+1}) = S(t_{k+1}^-) + \rho\delta C'C, \end{cases}$$

where $S(0)$ *is symmetric and positive definite. Hence, the proof of Theorem 5.1 relies on bounds of the form*

$$\rho\alpha_1(r)I \leq \theta^{-1}\Delta_\theta S(t)\Delta_\theta \leq \rho\alpha_2(r)I.$$

A different study concerns continuous-discrete state-affine systems of the form

$$\begin{cases} \dot{x} = A(u)x + B(u) \\ y(t_k) = Cx(t_k), \end{cases} \tag{5.4}$$

where $x(t) \in \mathbb{R}^n$, $y(t_k) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$, $A : \mathbb{R}^m \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R}^m \to \mathbb{R}^n$ are continuous functions, and C a constant matrix. As opposed to the previous case, the system output here is allowed to be multidimensional. Let us assume that the sequence (t_k) is as before and uniformly spaced by a constant $\delta > 0$. The observer proposed in [86] has the form

$$\begin{cases} \dot{\hat{x}}(t) = A(u(t))\hat{x}(t) + B(u(t)), \quad t \in [t_k, t_{k+1}) \\ \hat{x}(t_{k+1}) = \hat{x}(t_{k+1}^-) - \rho\delta_k S^{-1}(t_{k+1})C'(C\hat{x}(t_{k+1}^-) - y(t_{k+1})) \\ \dot{S}(t) = -\theta S(t) - A(u(t))'S(t) - S(t)A(u(t)), \quad t \in [t_k, t_{k+1}) \\ S(t_{k+1}) = S(t_{k+1}^-) + \delta C'C, \end{cases} \tag{5.5}$$

where θ and ρ are positive tuning parameters and where $S(0)$ is a symmetric and positive definite matrix. Notice that, as expected, the matrix gain is dynamic in this case.

Theorem 5.2 ([86]). *Consider system* (5.4) *and suppose that u is a bounded and regularly persistent input as in Definition 3.1. If $\rho \geq 1$ and if θ and δ^{-1} are large enough, then the state estimation error from observer* (5.5) *converges exponentially to zero.*

The previous results can be seen as extensions of their continuous output analogues presented in Chapters 2 and 3. An observer redesign method for adapting a general observer to a discrete output can be found in [68]. There, the authors first consider a forward-complete nonlinear system of the form

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) + d, \end{cases} \tag{5.6}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$ and $d(t) \in \mathbb{R}$, and for functions f and h, respectively of class C^1 and C^2, such that $f(0) = 0$ and $h(0) = 0$. They assume that a *robust* observer is readily available for system (5.6) and that it is given by

$$\begin{cases} \dot{z} = \hat{f}(z, y) \\ \hat{x} = \hat{h}(z), \end{cases} \tag{5.7}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the state estimation and $z(t) \in \mathbb{R}^k$, and where $\hat{f}$ and $\hat{h}$ are functions of class C^1 satisfying $\hat{f}(0,0) = 0$ and $\hat{h}(0) = 0$.

The robustness of observer (5.7) with respect to measurement noise can be defined in an ISS style as in Chapter 2 but with some important differences. For this, let us introduce the function classes $\mathcal{K}^+$ and $\mathcal{N}$ which respectively denote the positive and continuous functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$, and the non-decreasing continuous functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho(0) = 0$.

Definition 5.1. *System* (5.7) *is called a robust observer for system* (5.6) *with respect to* *measurement errors if the following holds:*

(i) there exist $\sigma \in \mathcal{KL}$, γ, $p \in \mathcal{N}$, $\mu \in \mathcal{K}^+$ *and* $a \in \mathcal{K}_\infty$ *such that for all* $x(0) \in \mathbb{R}^n$, *all* $z(0) \in \mathbb{R}^k$ *and all* $d \in L^\infty_{loc,1}$,

$$|x(t) - \hat{x}(t)| \leq \sigma(|x(0)| + |z(0)|, t) + \sup_{s\in[0,t]} \gamma(|d(s)|), \quad \forall t \geq 0,$$
$$|z(t)| \leq \mu(t) \left(a(|x(0)| + |z(0)|) + \sup_{s\in[0,t]} p(|d(s)|) \right), \quad \forall t \geq 0.$$

(ii) for all $x(0) \in \mathbb{R}^n$ *there exists* $z(0) \in \mathbb{R}^k$ *such that,*

$$d = 0 \implies x(t) = \hat{x}(t), \quad \forall t \geq 0.$$

The goal is then to estimate the state of system (5.6) in the case of a discrete output, that is, of a system of the form

$$\begin{cases} \dot{x} = f(x) \\ y(t_k) = h(x(t_k)) + d(t_k), \end{cases} \tag{5.8}$$

where (t_k) is an increasing sequence in $\mathbb{R}^+$ such that $t_0 = 0$ and $t_{k+1} - t_k \leq r$ for a constant $r > 0$. The corresponding observer redesign proposed in [68] is given by

$$\begin{cases} \dot{z}(t) = \hat{f}(z(t), w(t)), \quad t \in [t_k, t_{k+1}), \\ \hat{x}(t) = \hat{h}(z(t)), \quad t \in [t_k, t_{k+1}), \\ \dot{w}(t) = L_f h(\hat{x}(t)), \quad t \in [t_k, t_{k+1}), \\ w(t_{k+1}) = y(t_{k+1}), \end{cases} \tag{5.9}$$

where we recall that the Lie derivative of h along f is defined as $L_f h(x) = \frac{\partial h}{\partial x}(x) f(x)$, for all $x \in \mathbb{R}^n$. Therefore, observer (5.9) uses the structure of observer (5.7) and, to deal with the output discretization, instead of $y(t)$ it employs $w(t)$ as an output predictor.

We say that observer (5.9) is a robust sampled-data observer for system (5.8) with respect to measurement errors if similar conditions to the ones from Definition 5.1 hold. Indeed, it suffices to consider a function τ in the sampling sequence:

$$t_{k+1} = t_k + r \exp(-\tau(t_k)), \quad \forall k \geq 0 \tag{5.10}$$

and to do the following changes: part (i) also holds for all non-negative $\tau \in L^{\infty}_{loc,1}$ and $|w(0)|$ is added to $|x(0)| + |z(0)|$, and part (ii) requires $z(0)$ and $w(0)$ to work for all such τ.

Theorem 5.3 ([68])**.** *Suppose that system* (5.6) *is forward-complete and that it has a robust observer* (5.7) *with respect to measurement errors. Furthermore, suppose there exist a constant $c \geq 0$ satisfying*

$$rc < 1, \tag{5.11}$$

and $\bar{\sigma} \in \mathcal{KL}$ such that for all $x(0) \in \mathbb{R}^n$, all $z(0) \in \mathbb{R}^k$ and all $d \in L^{\infty}_{loc,1}$,

$$|L_f h(\hat{x}(t)) - L_f h(x(t))| \leq \bar{\sigma}(|x(0)| + |z(0)|, t) + c \sup_{s \in [0,t]} |d(s)|, \quad \forall t \geq 0,$$

where $\hat{x}$ is given by (5.7)*. Then,* (5.9) *is a robust sampled-data observer for system* (5.8) *with respect to measurement errors.*

The previous theorem can be applied, for example, to the case of triangular Lipschitz systems which are given by

$$\begin{cases} \dot{x} = Ax + \zeta(x) \\ y = Cx, \end{cases}$$

where A, C and ζ are in canonical form as in (2.19), and for ζ_i Lipschitz continuous with constant $L \geq 0$ and such that $\zeta_i(0) = 0$. Recall from Remark 2.3 the observer given by

$$\dot{\hat{x}} = A\hat{x} + \zeta(\hat{x}) - \Delta_\theta K(C\hat{x} - y),$$

where K is such that $A - KC$ is Hurwitz, $\theta > 0$ and Δ_θ is as in (2.17). Thus, there exists a symmetric and positive definite matrix P and a constant $\mu > 0$ such that

$$P(A - KC) + (A - KC)'P + 2\mu I \leq 0$$

and it can be shown that, for θ large enough, condition (5.11) constrains the sampling sequence with the inequality

$$2r(L + \theta)\frac{|P||K|}{\mu}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} < 1,$$

where $r > 0$ is as in (5.10). Notice that the latter illustrates the intuition behind Theorem 5.3, that is, it is possible to adapt an observer design for a continuous output to the case of relatively small sampling periods.

5.2.2 Stability of sampled-data systems

Sampled-data systems also arise as processes coupled with state-feedback controllers based on sample-and-hold blocks as in [88]. Consider a linear time-invariant system

$$\dot{x} = Ax + Bu,$$

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ the input, and for A and B matrices of the appropriate dimensions. Let us suppose that (t_k) is a strictly increasing sampling sequence starting at $t_0 = 0$, that there is a constant $\delta > 0$ such that

$$t_{k+1} - t_k \leq \delta, \quad \forall k \geq 0 \tag{5.12}$$

and let us denote the time since the last sampling as

$$\rho(t) = t - t_k, \quad t \in [t_k, t_{k+1}).$$

The piece-wise constant control $u(t) = Kx(t_k)$, for $t \in [t_k, t_{k+1})$, is given by a gain matrix K and the closed-loop can be represented as an impulsive system with an enlarged state

$$\xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}, \quad \forall t \in [t_k, t_{k+1}),$$

and whose dynamics are given by

$$\begin{cases} \dot{\xi}(t) = F\xi(t), \quad t \in [t_k, t_{k+1}), \\ \xi(t_{k+1}) = \begin{bmatrix} \xi_1(t_{k+1}^-) \\ \xi_1(t_{k+1}^-) \end{bmatrix}, \end{cases} \tag{5.13}$$

where

$$F = \begin{bmatrix} A & BK \\ 0 & 0 \end{bmatrix}.$$

Notice that, by continuity of x, we have in fact $\xi_1(t_{k+1}^-) = x(t_{k+1})$ and that at each sampling time t_{k+1} the control is restarted to $\xi_2(t_{k+1}) = x(t_{k+1})$.

The authors in [88] propose a Lyapunov function $V : \mathbb{R}^{2n} \times [0, \delta] \to \mathbb{R}^+$ for system (5.13) that consists of three parts:

$$V(\xi, \rho) = V_1(\xi_1) + V_2(\xi, \rho) + V_3(\xi, \rho),$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^n$ and where

$$\begin{aligned}
V_1(\xi_1) &= \xi_1' P \xi_1, \\
V_2(\xi, \rho) &= \xi' \left(\int_{-\rho}^{0} (s + \delta)(F \exp(Fs))' \bar{R} F \exp(Fs) ds \right) \xi \\
V_3(\xi, \rho) &= (\delta - \rho)(\xi_1 - \xi_2)' Q (\xi_1 - \xi_2),
\end{aligned}$$

for symmetric and positive definite matrices P, R and Q to be designed, and for

$$\bar{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}.$$

In particular, we have that

$$V_2(\xi(t), \rho(t)) = \int_{t-\rho(t)}^{t} (\delta - t + s) \dot{x}(s)' R \dot{x}(s) ds, \quad \forall t \geq 0$$

which is closely related to Lyapunov functions appearing in delay differential equations and network control systems. For the following theorem, whose proof relies on V, we use the notation:

$$\begin{aligned}
M_1 &= \begin{bmatrix} P \\ 0 \end{bmatrix} \bar{F} + \bar{F}' \begin{bmatrix} P & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} Q \begin{bmatrix} I & -I \end{bmatrix} \\
&\quad - N \begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} N' + \delta \bar{F}' R \bar{F}, \qquad (5.14) \\
M_2 &= \begin{bmatrix} I \\ -I \end{bmatrix} Q \bar{F} + \bar{F}' Q' \begin{bmatrix} I & -I \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} A & BK \end{bmatrix}.
\end{aligned}$$

Theorem 5.4 ([88]). *Consider system* (5.13) *and suppose there exist a matrix N and sym-*

metric and positive definite matrices P, R and Q such that

$$
\begin{aligned}
&M_1 + \delta M_2 < 0, \\
&\begin{bmatrix} M_1 & \delta N \\ \delta N' & -\delta R \end{bmatrix} < 0,
\end{aligned}
\tag{5.15}
$$

where M_1 and M_2 are as in (5.14)*. Then, there exist positive constants c and λ such that for all $\xi_1(0) = \xi_2(0) \in \mathbb{R}^n$ we have*

$$|\xi(t)| \leq c|\xi(0)| \exp(-\lambda t), \quad \forall t \geq 0.$$

Notice that the Lyapunov function along the trajectory $V(\xi, \rho)$ is not standard in the sense that it is discontinuous at each sampling time t_k. Nevertheless, it satisfies desirable properties whenever (5.15) is satisfied: there exist positive constants c_1, c_2 and c_3 such that

$$
\begin{aligned}
&c_1|x(t)|^2 \leq V(\xi(t), \rho(t)) \leq c_2|\xi(t)|^2, \quad \forall t \geq 0, \\
&\dot{V}(\xi(t), \rho(t)) \leq -c_3 V(\xi(t), \rho(t)), \quad \forall t \geq 0.
\end{aligned}
$$

Thus, $V(\xi, \rho)$ is lower bounded by a function of the norm of x and not of ξ, which turns out to suffice. Crucially, it also holds

$$V(\xi(t_k), 0) \leq \lim_{t \to t_k^-} V(\xi(t), \rho(t)), \quad \forall k \geq 0.$$

In our work corresponding to this chapter, we will use a Lyapunov function that is partly inspired by V and the studies from [92].

Remark 5.2. *Letting δ tend to zero in* (5.12) *leads to a continuous control and the matrix conditions in* (5.15) *can be shown equivalent to*

$$(A + BK)'P + P(A + BK) < 0$$

for a symmetric and positive definite matrix P. This is consistent with the stability of the closed-loop system $\dot{x} = (A + BK)x$.

5.2.3 Small-gain theorem for interconnected switched systems

A switched system consists of: (i) a family of dynamical systems indexed by an arbitrary set $\mathcal{P}$ and described by

$$\dot{x} = f_p(x,u), \quad p \in \mathcal{P},$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and for f_p locally Lipschitz such that $f_p(0,0) = 0$, (ii) a switching signal $\sigma : \mathbb{R}^+ \to \mathcal{P}$ which is a piece-wise constant and right-continuous function. The switched system then has the form

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t)), \quad t \geq 0.$$

It is assumed that $u \in L^\infty_{loc,m}$ and that the set of switching time instants has no accumulation points.

Input-to-state stability has been studied for switched nonlinear systems since the early work of [78] and continues to be a subject under intense research. The authors in [111] provide a Lyapunov based small-gain theorem for interconnected switched systems, and a trajectory approach can be found in [34]. In what follows, we discuss the Lyapunov based result. Consider systems given by

$$\begin{cases} \dot{x}_1 = f_{1,\sigma_1}(x_1, x_2) \\ \dot{x}_2 = f_{2,\sigma_2}(x_2, x_1), \end{cases} \tag{5.16}$$

where $x_i \in \mathbb{R}^{n_i}$ and for f_{i,σ_i} as above. Although it is possible to allow both switched systems to contain ISS and non-ISS subsystems, here we focus in the case that all subsystems are ISS.

Suppose that for each different $i, j \in \{1,2\}$ there exists a family of class C^1 functions $V_{i,p_i} : \mathbb{R}^{n_i} \to \mathbb{R}^+$, $p_i \in \mathcal{P}_i$, satisfying the following conditions:

1. $\exists\ \alpha_{i,1}, \alpha_{i,2} \in \mathcal{K}_\infty$ such that for all $x_i \in \mathbb{R}^{n_i}$ and all $p_i \in \mathcal{P}_i$,

$$\alpha_{i,1}(|x_i|) \leq V_{i,p_i}(x_i) \leq \alpha_{i,2}(|x_i|),$$

2. $\exists\ \phi_i \in \mathcal{K}_\infty$, $c_i > 0$ such that for all $x_i \in \mathbb{R}^i$, all $x_j \in \mathbb{R}^{n_j}$ and all $p_i \in \mathcal{P}_i$,

$$|x_i| \geq \phi_i(|x_j|) \implies \frac{\partial V_{i,p_i}}{\partial x_i}(x_i) f_{i,p_i}(x_i, x_j) \leq -c_i V_{i,p_i}(x_i),$$

3. $\exists\ \mu_i \geq 1$ such that for all $x_i \in \mathbb{R}^{n_i}$ and all $p_i,\ q_i \in \mathcal{P}_i$,

$$V_{i,p_i}(x_i) \leq \mu_i V_{i,q_i}(x_i),$$

4. $\exists\ \tau_{a,i} > 0,\ N_{0,i} \in \mathbb{Z}^+$ such that for all $t_2 \geq t_1 \geq 0$,

$$N_i(t_2, t_1) \leq N_{0,i} + \frac{t_2 - t_1}{\tau_{a,i}},$$
$$\frac{\ln(\mu_i)}{\tau_{a,i}} < c_i,$$

   where $N_i(t_2, t_1)$ is the number of switchings of σ_i in $(t_1, t_2]$.

Notice that conditions 1)-3) concern the dynamics of each switched system independently, while the small-gain condition is a joint requirement. On the other hand, 4) is a condition on the signals that limits their switching speed. We conclude this section with the statement of the theorem.

Theorem 5.5 (small-gain [111]). *Consider system* (5.16)*, suppose that* 1) − 4) *hold and define the mixed gains*

$$\chi_{m_i}(r) = \alpha_{i,2}(\phi_i(\alpha_{j,1}^{-1}(r))) \exp(N_{0,i} \ln(\mu_i)), \quad \forall r \geq 0$$

for $i, j \in \{1, 2\}$ such that $i \neq j$. If the small-gain condition

$$\chi_{m_1}(\chi_{m_2}(r)) < r, \quad \forall r > 0$$

is satisfied, then system (5.16) *is globally asymptotically stable. That is, there exists $\beta \in \mathcal{KL}$ such that for all $x_1(0) \in \mathbb{R}^{n_1}$ and all $x_2(0) \in \mathbb{R}^{n_2}$,*

$$|(x_1(t), x_2(t))| \leq \beta(|(x_1(0), x_2(0))|, t), \quad \forall t \geq 0.$$

5.3 Problem statement

The aim of this third and final part of our work is to extend the results from the previous chapter concerning observer redesign. In this case, we suppose that the system output is transformed by a nonlinear function and available only at discrete times. We consider a nonlinear system of the form

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ y(t) = h(x(t)), \end{cases} \tag{5.17}$$

where $x(t) \in \mathbb{R}^n$ is the state and $y(t) \in \mathbb{R}^p$ the output. We assume that f and h are of class C^2 and that the system is forward-complete. We suppose that an observer for system (5.17) has been designed and that it is described by

$$\dot{\hat{x}}(t) = \hat{f}(\hat{x}(t), y(t)), \tag{5.18}$$

where $\hat{x}$ is the state estimation. Similar to the problem studied in Chapter 4, y is considered not measured and, instead, the available output is a sampled and nonlinear transformation of y. This leads to a different and more complex nonlinear observer design problem.

Our goal is to design an observer for the system given by

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ y_\psi(t_k) = \psi(y(t_k)), \end{cases} \tag{5.19}$$

where (t_k) is an increasing sequence of sampling times such that $t_0 = 0$ and such that there exist positive constants $\underline{\delta}$ and $\bar{\delta}$ with

$$\underline{\delta} \leq t_{k+1} - t_k \leq \bar{\delta}, \quad \forall k \geq 0. \tag{5.20}$$

As in Chapter 4, our first assumption is that observer (5.18) is robust with respect to measurement noise in an ISS sense, that is, disturbance-to-error stable (DES). More precisely, if the output of (5.17) is affected by noise $y = h(x) + d_y$, then the corresponding error dynamics $e = x - \hat{x}$ are given by

$$\dot{e}(t) = \mathcal{F}(t, e(t), d_y(t)),$$

where $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is defined as

$$\mathcal{F}(t, e, d) = f(x(t)) - \hat{f}(x(t) - e, h(x(t)) + d), \tag{5.21}$$

and we have the following assumption.

Assumption 5.1. *There exists a class C^1 function $\bar{V} : \mathbb{R}^n \to \mathbb{R}^+$ such that the following conditions hold:*

1. $\exists\ \bar{\alpha}_1,\ \bar{\alpha}_2 \in \mathcal{K}_\infty$ *such that for all* $e \in \mathbb{R}^n$,

$$\bar{\alpha}_1(|e|) \leq \bar{V}(e) \leq \bar{\alpha}_2(|e|),$$

2. $\exists\ \bar{\chi} \in \mathcal{K}_\infty$ *and a constant* $\bar{c} > 0$ *such that for all* $t \in \mathbb{R}^+$, $e \in \mathbb{R}^n$, $d \in \mathbb{R}^p$ *and* $x(0) \in \mathbb{R}^n$,

$$|e| \geq \bar{\chi}(|d|) \implies \frac{\partial \bar{V}}{\partial e}(e)\mathcal{F}(t,e,d) \leq -\bar{c}\bar{V}(e).$$

Assumption 5.2. *The function* $\psi : \mathbb{R}^p \to \mathbb{R}^p$ *from* (5.19) *is of class* C^2 *and its Jacobian is invertible on all its domain. Moreover, there exists a constant* $c_\psi > 0$ *such that*

$$|y - \hat{y}| \leq c_\psi |\psi(y) - \psi(\hat{y})|, \quad \forall y, \hat{y} \in \mathbb{R}^p.$$

Assumption 5.3. *The function* $\varphi : \mathbb{R}^n \to \mathbb{R}^p$ *given by*

$$\varphi(x) = \frac{\partial \psi}{\partial y}(h(x))\frac{\partial h}{\partial x}(x)f(x)$$

is Lipschitz continuous. That is, there exists a constant $c_L > 0$ *such that*

$$|\varphi(x) - \varphi(\hat{x})| \leq c_L|x - \hat{x}|, \quad \forall x, \hat{x} \in \mathbb{R}^n.$$

The system state is frequently bounded in physical estimation problems. Saturation techniques can then be used in order to satisfy Assumption 5.3, see for example [54].

5.4 New observer design

Inspired by our work in Chapter 4, the proposed sample-and-hold observer for the continuous-discrete system (5.19) is defined by the interconnection:

$$\begin{cases} \dot{\hat{x}} = \hat{f}(\hat{x}, \hat{y}) \\ \dot{\hat{y}} = \frac{\partial \psi}{\partial y}(\hat{y})^{-1}\left(\frac{\partial \psi}{\partial y}(h(\hat{x}))\frac{\partial h}{\partial x}(\hat{x})f(\hat{x}) + K\xi_k\right) \\ \xi_k(t) = y_\psi(t_k) - \psi(\hat{y}(t_k)), \quad t \in [t_k, t_{k+1}), \end{cases} \tag{5.22}$$

where $\hat{f}$ is as in Assumption 5.1 and K is a matrix to be designed. Notice that observer (5.22) consists of continuous dynamics interconnected with a switched system and that it only requires $y_\psi(t_k) = \psi(y(t_k))$ as an output measure. Our main result of this chapter is the following.

Theorem 5.6. *Consider systems* (5.19) *and* (5.22)*, and let Assumptions 5.1, 5.2 and 5.3 hold. Suppose that the following conditions are satisfied.*

1. *there exist a constant $\alpha > 0$ and matrices: $P = P' > 0$, $\tilde{K}$, N_1, N_2 and N_3 in $\mathbb{R}^{p \times p}$ such that the LMI conditions* (5.24) *and* (5.25) *are satisfied with*

$$\lambda_1 = -\alpha\bar{\delta} + 1, \quad \lambda_2 = \exp(-\alpha\bar{\delta}), \quad \lambda_3 = \bar{\delta}\exp(-0.5\alpha\bar{\delta}),$$

2. *there exists a constant $c > c_L^2/\alpha$ such that*

$$\bar{\alpha}_2(\bar{\chi}(c_\psi c^{1/2}\,\lambda_{\min}(P)^{-1/2}\bar{\alpha}_1^{-1}(r))) < r, \quad \forall r > 0. \tag{5.23}$$

Then, choosing $K = P^{-1}\tilde{K}$, there exists $\beta \in \mathcal{KL}$ such that for all $x(0)$, $\hat{x}(0) \in \mathbb{R}^n$ and all $\hat{y}(0) \in \mathbb{R}^p$ the estimation errors $e = x - \hat{x}$ and $\xi = y_\psi - \psi(\hat{y})$ are defined on $\mathbb{R}^+$ and

$$|(e(t), \xi(t))| \leq \beta(|(e(0), \xi(0))|, t), \quad \forall t \geq 0.$$

The previous theorem sets conditions ensuring, in particular, the asymptotic convergence to zero of e. Its proof, which consists of two lemmas, is shown in Section 5.5 and it is based on the small-gain theorem for switched systems [111]. Therefore, we search for ISS Lyapunov functions V_1 and V_2 by first considering $e = x - \hat{x}$ as the state and $\xi = y_\psi - \psi(\hat{y})$ as the input and then by doing the opposite.

Remark 5.3. *The observer in* (5.22) *can be seen as an interconnection consisting of a regular and a switched subsystem, however, applying the general results from [111] requires careful analysis of their proofs. In particular: (i) the switching signal*

$$\sigma(t) = k, \quad \forall t \in [t_k, t_{k+1})$$

satisfies the so-called average dwell-time constraint, (ii) we use a single Lyapunov function V_2 but discontinuous at each time t_{k+1}; the uniform ratio bound condition is replaced by (5.29)*,*

$$\begin{bmatrix} (\alpha-\lambda_1)P+\lambda_2(-N_1-N_1') & \lambda_1 P-(1+\bar{\delta})\tilde{K}+\lambda_2(N_1-N_2') & (1+\bar{\delta})P+\lambda_2(-N_3') & 0 \\ \lambda_1 P-(1+\bar{\delta})\tilde{K}'+\lambda_2(N_1'-N_2) & -\lambda_1 P+\bar{\delta}(\tilde{K}+\tilde{K}')+\lambda_2(N_2+N_2') & -\bar{\delta}P+\lambda_2 N_3' & -\bar{\delta}\tilde{K}' \\ (1+\bar{\delta})P+\lambda_2(-N_3) & -\bar{\delta}P+\lambda_2 N_3 & -I & \bar{\delta}P \\ 0 & -\bar{\delta}\tilde{K} & \bar{\delta}P & -\bar{\delta}P \end{bmatrix} < 0, \tag{5.24}$$

$$\begin{bmatrix} (\alpha-\lambda_1)P+\lambda_2(-N_1-N_1') & \lambda_1 P-\tilde{K}+\lambda_2(N_1-N_2') & P+\lambda_2(-N_3') & 0 & \lambda_3 N_1 \\ \lambda_1 P-\tilde{K}'+\lambda_2(N_1'-N_2) & -\lambda_1 P+\lambda_2(N_2+N_2') & \lambda_2 N_3' & -\bar{\delta}\tilde{K}' & \lambda_3 N_2 \\ P+\lambda_2(-N_3) & \lambda_2 N_3 & -I & \bar{\delta}P & \lambda_3 N_3 \\ 0 & -\bar{\delta}\tilde{K} & \bar{\delta}P & -\bar{\delta}P & 0 \\ \lambda_3 N_1' & \lambda_3 N_2' & \lambda_3 N_3' & 0 & -\bar{\delta}P \end{bmatrix} < 0. \tag{5.25}$$

and (iii) as in the proof of the original small-gain result [63], we can see that the condition: there exists $\alpha_{2,2} \in \mathcal{K}_\infty$ such that

$$V_2(t) \leq \alpha_{2,2}(|\xi(t)|), \quad \forall t \geq 0$$

is not crucial as long as the modified implications (5.26) *and* (5.28) *hold, together with the corresponding small-gain condition.*

5.5 Proofs of main results

In this section, we present the two Lyapunov function lemmas which are instrumental for the proof of Theorem 5.6.

Lemma 5.1. *Consider systems* (5.19) *and* (5.22)*, and let Assumptions 5.1 and 5.2 hold. There exists a Lyapunov function V_1 from $\xi = y_\psi - \psi(\hat{y})$ to $e = x - \hat{x}$. That is, a class C^1 function $V_1 : \mathbb{R}^n \to \mathbb{R}^+$ such that the following conditions hold:*

1. *$\exists$ α_{11}, $\alpha_{12} \in \mathcal{K}_\infty$ such that for all $x(0) \in \mathbb{R}^n$, all $\hat{x}(0) \in \mathbb{R}^n$, all $\hat{y}(0) \in \mathbb{R}^p$ and all $t \in \mathbb{R}^+$,*
$$\alpha_{11}(|e(t)|) \leq V_1(e(t)) \leq \alpha_{12}(|e(t)|),$$

2. *$\exists$ $\chi_1 \in \mathcal{K}_\infty$ and a constant $c_1 > 0$ such that for all $x(0) \in \mathbb{R}^n$, all $\hat{x}(0) \in \mathbb{R}^n$, all $\hat{y}(0) \in \mathbb{R}^p$ and all $t \in \mathbb{R}^+$,*
$$|e(t)| \geq \chi_1(|\xi(t)|) \implies \dot{V}_1(e(t)) \leq -c_1 V_1(e(t)). \tag{5.26}$$

Proof. Consider the Lyapunov function $\bar{V}$ from Assumption 5.1 and the corresponding functions $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\alpha}_3$ and $\bar{\chi}$. Set $V_1 = \bar{V}$ and define $\alpha_{21} = \bar{\alpha}_1$ and $\alpha_{22} = \bar{\alpha}_2$. From the definition of $\mathcal{F}$ in (5.21) we have that

$$\dot{e}(t) = \mathcal{F}(t, e(t), \hat{y}(t) - y(t))$$

and, therefore, we can select $c_1 = \bar{c}$ to get

$$\begin{aligned} \dot{V}_1(e(t)) &= \frac{\partial \bar{V}}{\partial z}(e(t))\dot{e}(t) \\ &\leq -\bar{c}\bar{V}(e(t)) \\ &= -c_1 V_1(e(t)), \end{aligned} \tag{5.27}$$

whenever $|e(t)| \geq \bar{\chi}(|\hat{y}(t) - y(t)|)$. We also define the class $\mathcal{K}$ function

$$\chi_1(r) = \bar{\chi}(c_\psi r), \quad \forall r \geq 0,$$

where c_ψ is from Assumption 5.2 and hence

$$\begin{aligned} \chi_1(|\xi(t)|) &= \bar{\chi}(c_\psi |\psi(y(t)) - \psi(\hat{y}(t))|) \\ &\geq \bar{\chi}(|y(t) - \hat{y}(t)|) \end{aligned}$$

which, together with (5.27), provides the needed property. □

The following lemma allows us to transform a nonlinear matrix inequality into an LMI at the cost of increasing the dimension.

Lemma 5.2. *[Schur complement [1]] Consider a symmetric real matrix*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{bmatrix}.$$

Then, $S < 0$ if and only if $S_{22} < 0$ and $S_{11} - S_{12}S_{22}^{-1}S_{12}' < 0$.

Lemma 5.3. *Consider systems* (5.19) *and* (5.22), *and let Assumptions 5.2 and 5.3 hold. Suppose there exist a constant $\alpha > 0$ and matrices: $P = P' > 0$, $\tilde{K}$, N_1, N_2 and N_3 in $\mathbb{R}^{p \times p}$ such that the LMI conditions* (5.24) *and* (5.25), *with*

$$\lambda_1 = -\alpha\bar{\delta} + 1, \quad \lambda_2 = \exp(-\alpha\bar{\delta}), \quad \lambda_3 = \bar{\delta}\exp(-0.5\alpha\bar{\delta}),$$

are satisfied. If we set $K = P^{-1}\tilde{K}$, then there exists a Lyapunov-Krasovskii functional from $e = x - \hat{x}$ to $\xi = y_\psi - \psi(\hat{y})$. That is, a function $V_2 : \mathbb{R}^+ \to \mathbb{R}^+$, depending on ξ and differentiable on $t \neq t_k$, such that the following conditions hold:

1. *$\exists\ \alpha_{21} \in \mathcal{K}_\infty$ such that for all $x(0) \in \mathbb{R}^n$, all $\hat{x}(0) \in \mathbb{R}^n$, all $\hat{y}(0) \in \mathbb{R}^p$ and $t \in \mathbb{R}^+$,*

$$\alpha_{21}(|\xi(t)|) \leq V_2(t),$$

2. *$\exists\ \chi_2 \in \mathcal{K}_\infty$ and a constant $c_2 > 0$ such that for all $x(0) \in \mathbb{R}^n$, all $\hat{x}(0) \in \mathbb{R}^n$, all $\hat{y}(0) \in \mathbb{R}^p$ and all $t \neq t_k$,*

$$V_2(t) \geq \chi_2(|e(t)|) \implies \dot{V_2}(t) \leq -c_2 V_2(t), \tag{5.28}$$

3.
$$V_2(t_{k+1}) \leq \lim_{t \to t_{k+1}^-} V_2(t), \quad \forall k \geq 0. \tag{5.29}$$

Proof. We have that

$$\dot{\xi}(t) = \phi(t) - K\xi_k(t),$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^p$ is given by Assumption 5.3 and

$$\phi(t) = \varphi(x(t)) - \varphi(\hat{x}(t)),$$

hence,

$$|\phi(t)| \leq c_L |e(t)|, \quad \forall t \geq 0. \tag{5.30}$$

We use a slight modification of the Lyapunov-Krasovskii functional [88, 92]:

$$\begin{aligned} V_2(t) =& \xi(t)' P \xi(t) + \int_{t-\tau(t)}^{t} (\bar{\delta} - t + s) \exp(-\alpha(t-s)) \dot{\xi}(s)' P \dot{\xi}(s) ds \\ &+ (\bar{\delta} - \tau(t))(\xi(t) - \xi_k(t))' P (\xi(t) - \xi_k(t)), \end{aligned} \tag{5.31}$$

where $\alpha > 0$ and $P \in \mathbb{R}^{p \times p}$ symmetric and positive definite matrix are to be chosen and where $\tau(t) = t - t_k$ for $t \in [t_k, t_{k+1})$. Notice that the last two terms in (5.31) are non-negative and therefore condition 3) is satisfied. Since $V_2(t_k) = \xi(t_k)' P \xi(t_k)$, we also have

$$\lambda_{\min}(P) |\xi(t_k)|^2 \leq V_2(t_k) \leq \lambda_{\max}(P) |\xi(t_k)|^2, \quad \forall k \geq 0.$$

Moreover, for the augmented state

$$\bar{\xi}(t) = \begin{bmatrix} \xi(t)' & \xi_k(t)' & \phi(t)' \end{bmatrix}'$$

we can show there exists a constant $c_{22} > 0$ such that

$$\lambda_{\min}(P)|\xi(t)|^2 \leq V_2(t) \leq c_{22} \max_{s\in[-\tau(t),0]} |\bar{\xi}(t+s)|^2, \quad \forall t \geq 0.$$

The augmented state $\bar{\xi}$ is used to express the time derivative of V_2. Indeed, by the Leibniz integral rule we have that:

$$\begin{aligned}
\dot{V}_2(t) =& 2\xi(t)'P\dot{\xi}(t) + \bar{\delta}\dot{\xi}(t)'P\dot{\xi}(t) \\
&- \alpha \int_{t-\tau(t)}^{t} (\bar{\delta} - t + s)\exp(-\alpha(t-s))\dot{\xi}(s)'P\dot{\xi}(s)ds \\
&- \int_{t-\tau(t)}^{t} \exp(-\alpha(t-s))\dot{\xi}(s)'P\dot{\xi}(s)ds \\
&+ 2(\bar{\delta} - \tau(t))(\xi(t) - \xi_k(t))'P\dot{\xi}(t) \\
&- (\xi(t) - \xi_k(t))'P(\xi(t) - \xi_k(t))
\end{aligned}$$

and, hence,

$$\begin{aligned}
\dot{V}_2(t) + \alpha V_2(t) \leq& \bar{\xi}(t)'Q_1\bar{\xi}(t) + \bar{\delta}\bar{\xi}(t)'Q_2\bar{\xi}(t) \\
&- \exp(-\alpha\bar{\delta}) \int_{t-\tau(t)}^{t} \dot{\xi}(s)'P\dot{\xi}(s)ds \\
&+ (-\alpha\bar{\delta} + 1)\bar{\xi}(t)'Q_3\bar{\xi}(t) \\
&+ (\bar{\delta} - \tau(t))\bar{\xi}(t)'Q_4\bar{\xi}(t) \\
&+ \alpha\bar{\xi}(t)'\bar{P}\bar{\xi}(t),
\end{aligned} \tag{5.32}$$

where:

$$Q_1 = \begin{bmatrix} 0 & -PK & P \\ -K'P & 0 & 0 \\ P & 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 \\ -K' \\ I \end{bmatrix} P \begin{bmatrix} 0 & -K & I \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} -P & P & 0 \\ P & -P & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_4 = \begin{bmatrix} 0 & -PK & P \\ -K'P & PK + K'P & -P \\ P & -P & 0 \end{bmatrix},$$

$$\bar{P} = \begin{bmatrix} P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By using

$$0 \le \int_{t-\tau(t)}^{t} \begin{bmatrix} \dot{\xi}(s) \\ \bar{\xi}(t) \end{bmatrix}' \begin{bmatrix} P & -N' \\ -N & NP^{-1}N' \end{bmatrix} \begin{bmatrix} \dot{\xi}(s) \\ \bar{\xi}(t) \end{bmatrix} ds,$$

for any matrix $N = [N_1', N_2', N_3']' \in \mathbb{R}^{3p \times p}$, we can dominate the integral appearing in (5.32):

$$\begin{aligned} -\int_{t-\tau(t)}^{t} \dot{\xi}(s)'P\dot{\xi}(s)ds \le & \bar{\xi}(t)'Q_6\bar{\xi}(t) \\ & + \tau(t)\bar{\xi}(t)'NP^{-1}N'\bar{\xi}(t), \end{aligned} \tag{5.33}$$

where

$$Q_6 = \begin{bmatrix} -N_1 - N_1' & N_1 - N_2' & -N_3' \\ N_1' - N_2 & N_2 + N_2' & N_3' \\ -N_3 & N_3 & 0 \end{bmatrix}.$$

Take any constant $c > \frac{c_L^2}{\alpha}$ and suppose that

$$V_2(t) \ge c|e(t)|^2, \tag{5.34}$$

that is, choose the gain

$$\chi_2(r) = cr^2, \quad \forall r \ge 0.$$

Let us denote

$$Q_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix}$$

so that by (5.30) and (5.34) we get

$$0 \leq \bar{\xi}(t)' Q_5 \bar{\xi}(t) + \frac{c_L^2}{c} V_2(t). \tag{5.35}$$

Finally, we define the matrices

$$\begin{aligned}
R_1 &= Q_1 + \bar{\delta} Q_2 + (-\alpha\bar{\delta} + 1) Q_3 + Q_5 + \exp(-\alpha\bar{\delta}) Q_6 + \alpha \bar{P}, \\
R_2 &= \exp(-\alpha\bar{\delta}) N P^{-1} N', \\
R_3 &= Q_4,
\end{aligned}$$

in order to write from (5.32), (5.33) and (5.35) that

$$\begin{aligned}
\dot{V}_2(t) + \left(\alpha - \frac{c_L^2}{c}\right) V_2(t) \leq & \bar{\xi}(t)'(R_1 + \tau(t) R_2 \\
& + (\bar{\delta} - \tau(t)) R_3) \bar{\xi}(t).
\end{aligned}$$

As in [88], the time dependence in the matrix appearing in the right hand side above can be removed since the conditions:

$$R_1 + \bar{\delta} R_3 < 0 \quad \text{and} \quad R_1 + \bar{\delta} R_2 < 0, \tag{5.36}$$

imply there exists a constant $c_R > 0$ such that

$$\dot{V}_2(t) + \left(\alpha - \frac{c_L^2}{c}\right) V_2(t) \leq -c_R |\bar{\xi}(t)|^2 < 0$$

and, therefore, we can choose

$$c_2 = \alpha - \frac{c_L^2}{c} > 0$$

to conclude

$$\dot{V}_2(t) \leq -c_2 V_2(t).$$

The conditions in (5.36) can be expressed as the LMI's in (5.24) and (5.25) by using Lemma

5.2 twice. First with:

$$
\begin{aligned}
S_{11} &= R_1 - \bar{\delta}Q_2 + \bar{\delta}Q_4,\\
S_{12} &= \begin{bmatrix} 0 & -\bar{\delta}PK & \bar{\delta}P \end{bmatrix}',\\
S_{22} &= -\bar{\delta}P,
\end{aligned}
$$

and then with:

$$
\begin{aligned}
S_{11} &= R_1 - \bar{\delta}Q_2,\\
S_{12} &= \begin{bmatrix} 0 & \bar{\delta}\lambda_3 N_1 \\ -\bar{\delta}K'P & \bar{\delta}\lambda_3 N_2 \\ \bar{\delta}P & \bar{\delta}\lambda_3 N_3 \end{bmatrix},\\
S_{22} &= \begin{bmatrix} -\bar{\delta}P & 0 \\ 0 & -\bar{\delta}P \end{bmatrix},
\end{aligned}
$$

and by setting $\tilde{K} = PK$. □

Proof of Theorem 5.6: Using Lemmas 5.1 and 5.3 define:

$$
\begin{aligned}
\chi_{m_1}(r) &= \alpha_{12}(\chi_1(\alpha_{21}^{-1}(r))), \quad \forall r \geq 0,\\
\chi_{m_2}(r) &= \chi_2(\alpha_{11}^{-1}(r)), \quad \forall r \geq 0.
\end{aligned}
$$

Then, the following implications hold for all $t \in \mathbb{R}^+$:

$$
\begin{aligned}
V_1(e(t)) \geq \chi_{m_1}(V_2(t)) &\implies |e(t)| \geq \chi_1(|\xi(t)|),\\
V_2(t) \geq \chi_{m_2}(V_1(e(t))) &\implies V_2(t) \geq \chi_2(|e(t)|).
\end{aligned}
$$

As in the original theorem [63], the small-gain condition arises from the composition of the ISS gains when comparing directly V_1 with V_2 as above. That is, the gains are constrained by

$$
\chi_{m_1}(\chi_{m_2}(r)) < r, \quad \forall r > 0,
$$

which is equivalent to the condition in (5.23). We can conclude the global asymptotic convergence of the estimation error of the new observer (5.22) by considering Remark 5.3 and by applying Theorem 5.5. □

5.6 Illustration

5.6.1 Uniformly observable systems

Consider the nonlinear system in triangular form given by

$$\begin{cases} \dot{x}(t) = Ax(t) + \zeta(x(t), u(t)) \\ y(t) = Cx(t), \end{cases} \tag{5.37}$$

where the matrices A, C and the function ζ are as in (2.19). As in Chapter 4, our theory can be easily extended to the case when $f(x, u)$ depends on a bounded and continuous input $u(t) \in \mathbb{R}^m$. We aim to estimate the state of system (5.37) by using a sampled and transformed version of the output, that is, we consider:

$$\begin{cases} \dot{x}(t) = Ax(t) + \zeta(x(t), u(t)) \\ y_\psi(t_k) = \psi(y(t_k)), \end{cases} \tag{5.38}$$

where the sequence (t_k) satisfies (5.20) and where $\psi : \mathbb{R} \to \mathbb{R}$ is a given function satisfying Assumption 5.2. We introduce the usual assumption on the nonlinearity of the dynamics.

Assumption 5.4. *For each $i = 1, \ldots, n$, there exists a positive constant $c_{\zeta i}$ such that*

$$|\zeta_i(\underline{x}_i, u) - \zeta_i(\underline{\hat{x}}_i, u)| \le c_{\zeta i} |\underline{x}_i - \underline{\hat{x}}_i|,$$

for all x, $\hat{x} \in \mathbb{R}^n$, all $u \in \mathbb{R}^m$ and where $\underline{x}_i = (x_1, \ldots, x_i) \in \mathbb{R}^i$.

We consider the observer for system (5.37) introduced in [46],

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + \zeta(\hat{x}, u) + S_\infty^{-1} C'(y - C\hat{x}) \\ 0 = -\theta S_\infty - A' S_\infty - S_\infty A + C'C, \end{cases} \tag{5.39}$$

where $\theta > 0$ and the matrix S_∞ is symmetric and positive definite. The same computations as in Chapter 4 then show that Assumption 5.1 is satisfied with the linear functions:

$$\bar{\alpha}_1(r) = s_1^{\frac{1}{2}} r, \quad \bar{\alpha}_2(r) = s_2^{\frac{1}{2}} r, \quad \bar{\chi}(r) = s_1^{-1} r,$$

for all $r \in \mathbb{R}^+$, and with

$$\theta > 2\left(n s_m^{\frac{1}{2}} c_\zeta \, \lambda_{\min}\left(S_{\infty,1}\right)^{-\frac{1}{2}} + 1\right),$$

where s_1, s_2, s_m, c_ζ and $S_{\infty,1}$ are as in Section 4.6.2. The rest of the hypotheses from Theorem 5.6 are illustrated in the numerical example of the following section.

5.6.2 Numerical example

We study the system dynamics as in [77] but we consider a discrete output with a different nonlinearity. For this, suppose that system (5.38) is specified by:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$\zeta(x, u) = \begin{bmatrix} 0 \\ -x_1 - 2x_2 + a x_1^2 x_2 + u \end{bmatrix}$$

$$u(t) = b \sin(2t), \quad \psi(y) = \sin(y) + 2y,$$

for $a = 0.25$, $b = 0.2$ and assume that the sampling is uniform with $\bar{\delta} = 0.005$. Notice that ψ is a diffeomorphism with an inverse that has no closed form and that Assumption 5.2 is met with $c_\psi = 1$. Although ζ is not Lipschitz continuous, Assumptions 5.3 and 5.4 are satisfied if we use saturation techniques as in Section 4.7.2. We have that the LMI's (5.24) and (5.25) are feasible for $\alpha = 100$ with solutions $P = 18$ and $K = 110$, moreover, condition (5.23) reduces to

$$\frac{c_L^2}{0.05P} < \alpha,$$

for $c_L = 7.5$ and where $\theta = 0.4$. We consider a larger $\bar{\delta} = 0.1$ for the simulation results in Fig. 5.1. The new observer corresponding to (5.22) correctly estimates the states from (5.38) only using $\psi(y(t_k))$.

5.7 Perspectives: the hybrid approach

The proof of Theorem 5.6 relies on the small-gain theorem for switched systems [111] presented in Section 5.2.3. Here, we discuss a possible framework for an alternative and more elegant approach to our previous results.

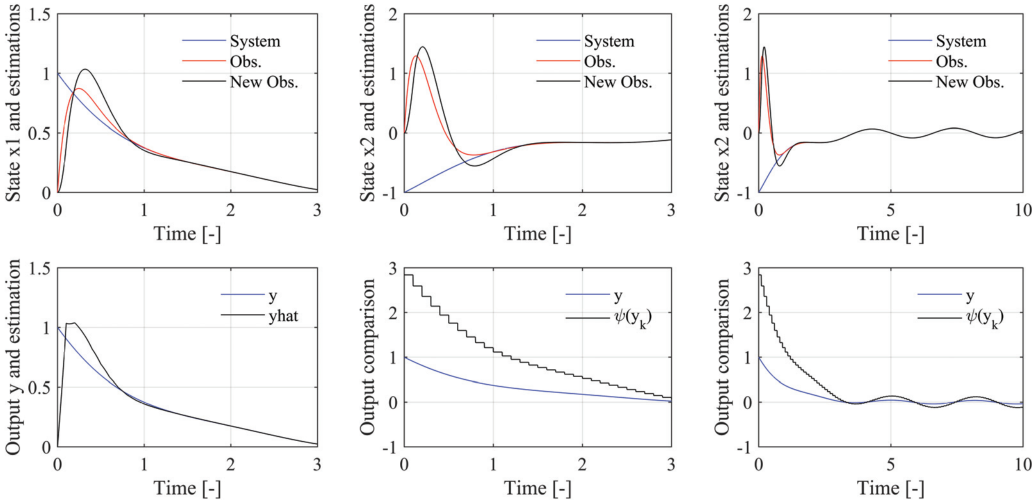

Figure 5.1: First row: estimation of the system states of (5.37) (blue) by observer (5.39) (red, lower peaking) and by the new observer (5.22) (black, higher peaking). Second row: output and its estimation from (5.22), and the comparison of the unavailable output and the actual measurements $\psi(y(t_k))$. The initial conditions are $x(0) = (1, -1)$, $\hat{x}(0) = (0, 0)$, $\hat{y}(0) = C\hat{x}(0)$ and the sampling time distance $\bar{\delta} = 0.1$.

The strategy in [111] consists in modeling the interconnection of switched systems in the so-called hybrid framework. Indeed, hybrid systems have both continuous and discrete features and can represent seemingly different systems including: impulsive, switched and sampled-data. The hybrid framework [49] has now become standard and, partly due to its generality, increasingly popular in recent years. A hybrid system is a dynamical system denoted by

$$\begin{cases} \dot{z} \in F(z, w), & (z, w) \in C \\ z^+ \in G(z, w), & (z, w) \in \mathrm{D}, \end{cases} \tag{5.40}$$

where C and D are closed subsets of $\mathbb{R}^n \times \mathbb{R}^m$, and where F and G are set-valued maps from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$. A solution pair to system (5.40) consists of a *hybrid arc* $z : \mathrm{dom}(z) \to \mathbb{R}^n$ and a *hybrid input* $w : \mathrm{dom}(w) \to \mathbb{R}^m$, these notions stand for the following conditions. The shared domain of definition should be a *hybrid time domain* $E = \mathrm{dom}(z) = \mathrm{dom}(w) \subseteq \mathbb{R}^+ \times \mathbb{Z}^+$, meaning: for each $(T, J) \in E$, $E' = E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a *compact hybrid domain*, that is, $E' = \cup_{j=0}^{J'}([t_j, t_{j+1}], j)$ for some finite non-decreasing sequence of times $0 = t_0, t_1, \ldots, t_{J'+1}$. Furthermore, each $w(\cdot, j)$ should be Lebesgue measurable and locally essentially bounded, and each $z(\cdot, j)$ locally absolutely continuous. Finally, z and w should satisfy: (i) $(z(t, j), w(t, j)) \in C$ and $\dot{z}(t, j) \in F(z(t, j), w(t, j))$, for all $j \in \mathbb{Z}^+$ and almost all

$t \in \mathbb{R}^+$ such that $(t, j) \in \mathrm{dom}(z)$, (ii) $(z(t, j), w(t, j)) \in D$ and $z(t, j+1) \in G(z(t, j), w(t, j))$, for all $(t, j) \in \mathrm{dom}(z)$ such that $(t, j+1) \in \mathrm{dom}(z)$. Therefore, $z(t, j)$ represents the system state after t time units and j jumps. Results involving existence and uniqueness of solutions for hybrid systems require careful analyses [49], we assume local existence throughout the rest of this section.

Not only switched but also sampled-data systems can be modeled in the hybrid framework and, as detailed in the following remark, the latter could be a better fit for our setup.

Remark 5.4. *The estimation error dynamics of* (5.19) *and* (5.22) *can be written as*

$$\begin{cases} \dot{e}(t) = \mathcal{F}_1(t, e(t), \bar{\xi}(t)), & t \geq 0 \\ \dot{\xi}(t) = \mathcal{F}_2(t, e(t), \xi(t_k)), & t \in [t_k, t_{k+1}) \end{cases}$$

where $e = x - \hat{x}$, $\bar{\xi} = y - \hat{y}$ *and* $\xi = \psi(y) - \psi(\hat{y})$, *and where:*

$$\begin{aligned} \mathcal{F}_1(t, e, \bar{\xi}) =& f(x(t)) - \hat{f}(x(t) - e, y(t) - \bar{\xi}), \\ \mathcal{F}_2(t, e, \xi_k) =& \frac{\partial \psi}{\partial y}(h(x(t))) \frac{\partial h}{\partial x}(x(t)) f(x(t)) \\ &- \frac{\partial \psi}{\partial y}(h(x(t) - e)) \frac{\partial h}{\partial x}(x(t) - e) f(x(t) - e) - K\xi_k, \end{aligned}$$

for a properly designed constant matrix K*.*

Motivated by the previous remark, a first step towards the reformulation of our problem is to reconsider the general sampled-data system from Section 5.2.2 described by

$$\dot{x}(t) = Ax(t) + Bu(t_k), \quad t \in [t_k, t_{k+1}), \tag{5.41}$$

where $x(t) \in \mathbb{R}^{n'}$ and $u(t) \in \mathbb{R}^{m'}$, A and B are constant matrices, and (t_k) is an increasing sequence of sampling times. Let us further assume that there exist positive real numbers T_1 and T_2 such that

$$T_1 \leq t_{k+1} - t_k \leq T_2, \quad \forall k \geq 0$$

and that the control law is specified by $u = Kx$, for a constant gain K. Suppose that $n' = m'$, then system (5.41) has the associated hybrid system (5.40) with $n = 2n' + 2$ and:

$$z = \begin{bmatrix} x \\ u \\ \tau \\ T \end{bmatrix} \in \mathbb{R}^n, \quad F(z) = \begin{bmatrix} Ax + Bu \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad G(z) = \begin{bmatrix} x \\ Kx \\ 0 \\ [\underline{\delta}, \bar{\delta}] \end{bmatrix},$$

here τ is a timer that measures the time since the last jump and T represents the inter-sampling time; the flow and jump sets are defined as

$$C = \{z \in \mathbb{R}^n | \tau \leq T\}, \quad D = \{z \in \mathbb{R}^n | \tau \geq T\}.$$

The notion of Lyapunov function can be extended naturally to hybrid systems. In this case, the authors in [24] introduce the function $V : \mathbb{R}^n \to \mathbb{R}^+$ given by

$$V(z) = \begin{bmatrix} x' & u' \end{bmatrix} P(\tau, T) \begin{bmatrix} x \\ u \end{bmatrix},$$

where $P : [0, T_2] \times [T_1, T_2] \to \mathbb{R}^{(n-2)\times(n-2)}$ is a matrix functional such that each $P(\tau, T)$ is symmetric and positive definite. Similarly, ISS can be defined in the hybrid context and its Lyapunov characterization can be found in [27].

Definition 5.2. *The hybrid system* (5.40) *is pre-input-to-state stable (pre-ISS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that all solutions satisfy*

$$|z(t,j)| \leq \max\{\beta(|z(0,0)|, t+j), \gamma(|w|_{(t,j)})\}, \quad \forall (t,j) \in \mathrm{dom}(z). \tag{5.42}$$

The previous norm is defined as

$$|w|_{(t,j)} = \max\left\{ \operatorname*{ess\,sup}_{(s,k)\in N_1} |w(s,k)|, \sup_{(s,k)\in N_2} |w(s,k)| \right\},$$

where N_1 (N_2) is the subset of $\mathrm{dom}(w)$ *consisting of pairs (s,k) such that $(s, k+1) \notin (\in)\, \mathrm{dom}(w)$ and $s + k \leq t + j$, and where* ess sup *stands for the essential supremum. We say that system* (5.40) *is pre-globally asymptotically stable (pre-GAS) if* (5.42) *holds with $w = 0$.*

Here, the prefix "pre" indicates that completeness of solutions to (5.40) is not included in the previous definition.

Remark 5.5. *The idea is then to find a more adequate small-gain theorem to show the stability of the system in the previous remark. To our knowledge, possible alternatives are given by: (i) the very general small-gain results from [67], (ii) writing* (5.41) *as a fast-varying delay system with $\tau(t) = t - t_k$, $u(t) = Kx(t - \tau(t))$ and using the corresponding small-gain theorem [31], and (iii) modeling sampled-data systems in the hybrid framework and using the small-gain theorem [30, 79]. We next discuss the results from the latter work.*

An interconnected hybrid system without external inputs can be written as

$$\begin{cases} \dot{z}_1 \in F_1(z_1, z_2), \quad \dot{z}_2 \in F_2(z_1, z_2), \quad (z_1, z_2) \in C \\ z_1^+ \in G_1(z_1, z_2), \quad z_2^+ \in G_2(z_1, z_2), \quad (z_1, z_2) \in D, \end{cases} \tag{5.43}$$

where C and D are closed subsets of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and where F_i and G_i are set-valued maps from $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ to $\mathbb{R}^{n_i}$. We make use of the Clarke derivative defined for a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and a vector $v \in \mathbb{R}^n$ as

$$f^\circ(x; v) = \limsup_{h \to 0^+, y \to x} \frac{f(y + hv) - f(y)}{h},$$

which coincides with the directional derivative whenever $f \in C^1$. Let us suppose that for different $i, j \in \{1, 2\}$ there exist locally Lipschitz functions $V_i : \mathbb{R}^{n_i} \to \mathbb{R}^+$ satisfying the following conditions:

1. $\exists\ \alpha_{i,1}, \alpha_{i,2} \in \mathcal{K}_\infty$ such that for all $z_i \in \mathbb{R}^{n_i}$,

$$\alpha_{i,1}(|z_i|) \le V_i(z_i) \le \alpha_{i,2}(|z_i|),$$

2. $\exists\ \chi_i \in \mathcal{K}_\infty$ and $\alpha_{i,3} \in \mathcal{K}$ such that for all $(z_1, z_2) \in C$ and all $w_i \in F_i(z_1, z_2)$,

$$V_i(z_i) \ge \chi_i(V_j(z_j)) \implies V_i^\circ(z_i; w_i) \le -\alpha_{i,3}(V_i(z_i)),$$

3. $\exists\ \lambda_i \in \mathcal{K}$ such that for all $r > 0$, all $(z_1, z_2) \in D$ and all $w_i \in G_i(z_1, z_2)$,

$$\lambda_i(r) < r, \quad V_i(w_i) \le \max\{\lambda_i(V_i(z_i)), \chi_i(V_j(z_j))\}.$$

Theorem 5.7 ([79]). *Consider the overall system* (5.43) *and suppose that 1)-3) hold. If the small-gain condition*

$$\chi_1(\chi_2(r)) < r, \quad \forall r > 0$$

is satisfied, then the system is pre-GAS.

Applications of the small-gain theorem for hybrid systems arise in different contexts including: the natural decomposition of a hybrid system, networked control systems, emulation with event-triggered sampling, and quantized-feedback control as shown in [79].

To summarize: our main results corresponding to Chapter 5 consist in the construction of two Lyapunov functions together with a small-gain argument, which we based on [111]. Although the switched system setting does not correspond directly to ours, the steps in the proof of their main result can still be followed to conclude the asymptotic convergence of our proposed observer. On the other hand, this section suggests that a possibly more suitable framework is that of hybrid systems. This is due to Theorem 5.7 together with the fact that hybrid systems can model adequately the estimation error dynamics implied by our observer. However, it is not clear yet how to construct the Lyapunov functions for the hybrid subsystems.

5.8 Conclusion

In this chapter, we proposed an observer redesign method that adapts a given observer to a discretized and nonlinearly transformed version of the system output. The main idea behind our method consists in interconnecting the given observer with a dynamic inversion of the nonlinear transformation based on sample-and-hold techniques. We formulated two LMI's and a small-gain type constraint that guarantee the asymptotic convergence to zero of the proposed observer. A simulation example illustrates the latter and also the effectiveness of our approach. Finally, we presented some brief ideas on an alternative approach to our results by using the hybrid framework. Future work can address the case of sampled and implicit outputs and the corresponding applications that arise in many fields such as machine vision.

Chapter 6

Conclusions and perspectives

6.1 Summary

The work presented in this thesis concerns observer design for nonlinear systems with robustness guarantees in the presence of disturbances. Three novel observer designs were introduced: a Kalman-like observer with optimal tuning and input selection concerning state-affine systems, an observer redesign for general nonlinear systems in the presence of nonlinear sensors, and an extension of this redesign in the case of sampled measurements.

Chapter 2 provided a general overview of observability and of observer design. We first discussed cornerstone results involving representative system forms, which then led to specific Luenberger-like and Kalman-like observer designs. We emphasized the advantages and disadvantages of high-gain observers, with a special focus on measurement noise sensitivity. It was then convenient to deepen in the input-to-state stability framework, first by studying the central results: Lyapunov and small-gain theorems, and then by introducing a related notion of observer robustness.

The problem of tuning high-gain observers has been widely studied in the literature, and the same holds for regularly persistent input design in the case of state-affine systems. Nevertheless, in Chapter 3, we developed a novel strategy for simultaneous tuning and input selection of a Kalman-like observer in the face of dynamic and output disturbances. This approach is based on an estimation error bound functional, given by Lyapunov analyses, and an algorithm to minimize it. The simulations were successful in the sense that the actual state estimation improved with the optimal design.

Observer design for state-affine systems and for many other classes of systems is usually based on linearity of the system output, however, nonlinear sensors arise frequently in practice. Chapter 4, which constitutes the main part of this thesis, presented a general and robust observer redesign approach to adapt a given observer for a nonlinear system to a nonlinear transformation of the system output. The main idea of the proposed approach is to consider an additional subsystem to estimate the output and to feed it back to the given observer, which is assumed measurement robust in the ISS sense. Thus, the new observer results in an interconnected system whose asymptotic convergence was shown by means of an adapted version of the Lyapunov based small-gain theorem. Our redesign was made explicit in detailed applications for two important classes of systems which differ considerably in their observability properties.

In many engineering applications, sensors provide measurements in discrete time and several observers have been designed in the literature under this constraint. In Chapter 5, we extended our observer redesign in order to handle sampled and transformed outputs. This extension consists in a sample-and-hold technique together with the dynamic inversion equations developed in the previous chapter. As a consequence, the new observer is a continuous-discrete system whose gain arises as the solution of some LMI's, and whose asymptotic convergence was shown via the small-gain theorem for switched systems. We studied the new observer design for uniformly observable systems and illustrated the feasibility of the LMI's. Finally, we discussed the hybrid framework as an alternative approach for future work.

6.2 Perspectives

The work from Chapter 3 can be extended in different research directions. For example, the error minimization algorithm can be coupled with the design of regularly persistent inputs. Or, if the main objective is to control the system, then a different functional could represent a trade-off between persistency of excitation and regulation constraints. Furthermore, it is interesting to consider our bounding strategy for state-affine systems up to triangular nonlinearity. With respect to the observer redesign presented in Chapter 4 or 5, new studies should target the main assumptions. The classes of systems which admit a robust observer in the ISS sense can be significantly enlarged when considering weaker stability notions. Our results could also be extended for an output transformation $\psi : \mathbb{R}^{p_1} \to \mathbb{R}^{p_2}$ with $p_1 \neq p_2$, or for a time-varying $\psi(y, t)$. In Chapter 5, further efforts can be done in order to improve the feasibility of the resulting LMI and to allow for larger sampling periods; new results might be possible within the hybrid system framework. As previously mentioned, these studies are a first step towards the case of sampled and implicit outputs arising in fields such as machine vision.

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Appendix A

Résumé étendu en français

A.1 Introduction générale

Contexte

Chaque *système physique* est composé de différentes parties qui suivent un mécanisme commun. Par conséquent, des informations partielles peuvent parfois être utilisées afin d'estimer ce que nous ignorons. C'est l'idée fondamentale à la base du concept d'*observateur* pour un système. Ici, nous considérons des modèles des systèmes physiques dynamiques (dépendants du temps) et régis par des équations différentielles.

Le modèle d'un système physique est une description rigoureuse des phénomènes correspondants généralement basée sur des lois fondamentales. Une modélisation fiable doit être précise. Une représentation suffisamment bonne de la réalité nécessite souvent une structure non linéaire et la prise en compte de facteurs inconnus tels que les *perturbations*. Les objectifs de la modélisation d'un système sont divers et comprennent : la compréhension des phénomènes, la prévision, l'optimisation et, surtout, le contrôle.

La théorie du contrôle moderne est basée sur la représentation dite d'état d'un système, dans laquelle nous pouvons distinguer les variables multidimensionnelles : *état* ou évolution du système, *entrée* ou contrôle, et *sortie* ou mesure. Dans ce contexte, la mise en œuvre de lois de commande peut nécessiter des informations d'état complètes ou partielles provenant des sortie ou des capteurs. Des informations internes sont également nécessaires pour d'autres tâches importantes, telles que : l'identification, la prise de décision ou la surveillance. Cependant, les capacités des capteurs sont souvent limitées par des contraintes physiques, technologiques, économiques ou même de sécurité.

Les observateurs sont des algorithmes qui estiment l'état actuel à l'aide de la structure du système et des mesures disponibles. Dans cette thèse, de tels algorithmes sont donnés par des systèmes dynamiques adjacents et la convergence de l'estimation d'état est dans la norme Euclidienne. Des notions étroitement liées à l'observateur sont celles d'observabilité ou détectabilité d'un système, qui distinguent les systèmes admettant un modèle d'observateur.

La théorie des observateurs est devenue un domaine extrêmement riche qui se situe à l'interface entre les disciplines des mathématiques pures et les applications pratiques. Bien que la conception d'observateurs pour les systèmes linéaires soit considérée comme un problème résolu, son homologue non linéaire fait l'objet de recherches intensives dans la communauté du contrôle depuis plusieurs décennies. En conséquence, bien qu'il n'existe actuellement aucune méthode de conception systématique, des observateurs ont été développés pour des classes spécifiques de systèmes non linéaires et sous différentes hypothèses. Une caractéristique extrêmement recherchée des observateurs est leur robustesse, en raison de leur caractère pratique. En effet, leurs estimations devraient être satisfaisantes même en présence de perturbations du système. Cela rend le problème de la conception d'observateur robuste pour les systèmes non linéaires crucial dans le domaine de la commande automatique.

Motivations

Il n'est pas anodin d'adapter la conception d'un observateur aux perturbations affectant le système. Ce problème est évident pour les systèmes uniformément observables et leur observateur à grand gain, dont la conception dépend d'un paramètre de réglage pouvant amplifier le bruit de mesure. Plusieurs stratégies d'ajustement optimal ou adaptatif ont été développées afin de régler cet observateur correctement. D'autre part, les entrées du système jouent un rôle central dans l'observabilité des systèmes non linéaires, et la conception des entrées permettant de rendre un système non linéaire observable est généralement complexe. Dans le cas de la conception des entrées pour les systèmes affine ou bilinéaire, les chercheurs ont mis au point des algorithmes pour construire des entrées *suffisamment régulières* avec les propriétés prescrites. De manière surprenante, aucune littérature ne semble avoir pour objectif le réglage simultané et la sélection des entrées chez les observateurs dans le but d'améliorer l'estimation de l'état en présence de bruit.

La conception des observateurs met souvent l'accent sur la non-linéarité de la dynamique du système. Cependant, les capteurs modernes deviennent de plus en plus complexes et la linéarité de la sortie peut ne pas correspondre à la réalité. Cela soulève la question de savoir si les conceptions d'observateur connues peuvent être adaptées à une non-linéarité de sortie. De même, nous pouvons également considérer le cas des mesures discrètes, qui ne peuvent pas toujours être prises assez près dans le temps ; il est bien connu que cela peut affecter considérablement les propriétés d'un observateur dans le cas non linéaire.

Organisation et contributions de la thèse

La contribution principale de cette thèse est la conception de trois nouveaux observateurs pour les systèmes non linéaires, ainsi que les preuves de leur convergence asymptotique et de leurs propriétés de robustesse.

Le chapitre 2 de la thèse propose une revue de la littérature sur le thème de l'observabilité et de la conception d'observateur pour les systèmes non linéaires, une attention particulière étant accordée aux observateurs à grand gain. Le chapitre rassemble également les résultats centraux du cadre de stabilité des entrées à l'état et introduit la notion de conception par observateur robuste. Quelques résultats préliminaires pertinents et nouveaux peuvent être trouvés à la fin.

Le chapitre 3 de la thèse (partie 1 ici) propose le premier concept d'observateur, où les systèmes considérés sont affines en l'état avec des perturbations dynamiques et de sortie. Bien que la structure d'observateur soit donnée par l'observateur de Kalman à grand gain connu, notre travail développe un algorithme d'optimisation efficace. Plus en détail, un algorithme pour l'adaptation simultanée du réglage et de l'entrée aux perturbations affectant le système.

Le chapitre 4 (partie 2 ici), qui contient la principale contribution de cette thèse, présente le deuxième concept d'observateur en tant que méthode de refonte générale. Plus précisément, nous considérons d'abord un système non linéaire général pour lequel une synthèse d'observateur est disponible. Nous repensons ensuite cet observateur au cas où la sortie ne serait mesurée qu'après une transformation non linéaire. Notre nouvel observateur est basé sur une formule d'inversion dynamique qui évite d'utiliser l'inverse de la transformation.

Le chapitre 5 (partie 3 ici) présente une extension de l'approche précédente dans le cas où la sortie est non seulement mesurée par une transformation non linéaire, mais elle est également discrétisée dans le temps. Néanmoins, notre méthode diffère de la précédente et elle impose des contraintes alternatives sur le gain de l'observateur.

A.2 Chapitre 3 : design d'observateur optimal pour des systèmes affines en l'état perturbés

Dans le chapitre 2 de cette thèse, nous avons analysé l'une des méthodes standard de synthèse d'observateurs non linéaires connues sous le nom du grand gain. Un paramètre de réglage suffisamment grand est utilisé dans la structure de l'observateur pour régler le taux de convergence de l'erreur d'estimation. Cependant, cette méthode présente un inconvénient majeur dans la pratique où les perturbations ont tendance à affecter le système : la sensibilité de l'estimation au bruit [76]. Par conséquent, une stratégie pour palier à ce problème consiste à formuler un gain adaptatif de manière que le gain soit élevé lorsque l'état doit être reconstruit, puis il diminue pour éviter l'amplification du bruit, voir par exemple [4]. La plupart de ces études considèrent les systèmes uniformément observables et peuvent se mettre sous une forme d'observabilité canonique, ce qui n'est pas toujours facile à obtenir.

D'un autre côté, une approche connue pour l'identification des paramètres à l'aide d'un observateur repose sur la conception d'une entrée, qui optimise les estimations (entrée riche) [40, 91]. En général, la sélection de l'entrée peut avoir un impact considérable sur les performances d'un observateur [16] et la conception d'entrées régulièrement persistantes n'est pas une tâche triviale. Néanmoins, en pratique, cela se fait généralement de manière heuristique. Quelques exceptions notables peuvent être vues dans [95, 96] où les auteurs considèrent les systèmes affines en l'état.

Dans cette partie qui est basée sur notre travail [51], on analyse le problème de la robustesse aux perturbations d'un observateur pour les systèmes affines en l'état. Cette classe de systèmes n'est pas nécessairement sous forme canonique. On distingue deux types de perturbations bornées : le bruit de mesure affectant la sortie du système et les incertitudes du modèle sur la dynamique du système. Nous étudions la robustesse d'un observateur de type Kalman et quantifions une borne limite pour l'erreur d'estimation. Cette borne dépend à la fois du paramètre du gain et de l'entrée du système qui doit être régulièrement persistante. Nous présentons une nouvelle approche de conception hors ligne permettant le réglage du gain de l'observateur et de l'entrée du système en minimisant cette borne. Par conséquent, notre méthode offre un nouveau degré de liberté dans le problème de l'atténuation du bruit.

Dans la suite, basé sur les observateurs pour les systèmes affines en l'état donnés dans la littérature et sur le concept important de la persistance régulière d'une entrée (voir une résumé au chapitre 3), nous présentons notre principale contribution de cette partie. Une illustration peut être trouver à la fin du chapitre 3.

Optimisation de la robustesse basée sur les techniques de Lyapunov

L'objectif de cette première partie de notre travail est d'estimer *de manière optimale* le vecteur d'état des systèmes affines en l'état affectés par le bruit. Autrement dit, nous considérons le système

$$\begin{cases} \dot{x}(t) = A(u(t))x(t) + d_1(t) \\ y(t) = Cx(t) + d_2(t), \end{cases} \tag{A.1}$$

où $x(t) \in \mathbb{R}^n$ est l'état, $y(t) \in \mathbb{R}$ la sortie, $u(t) \in \mathbb{R}$ une entrée, $d_1(t) \in \mathbb{R}^n$, $d_2(t) \in \mathbb{R}$ perturbations bornées et continues, $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ une matrice temps variant et $C \in \mathbb{R}^{1\times n}$ une matrice constante. Nous utilisons l'observateur présenté dans le chapitre 3

$$\begin{cases} \dot{\hat{x}} = A(u)\hat{x} + S^{-1}C'(y - C\hat{x}) \\ \dot{S} = -\theta S - A(u)'S - SA(u) + C'C, \end{cases} \tag{A.2}$$

où $S(0)$ est une matrice symétrique définie positive et $\theta > 0$. Malheureusement, même pour un tel choix de $S(0)$, les valeurs propres de $S(t)$ peuvent encore être arbitrairement proches de zéro pour les entrées dites singulières ; entrées où l'observabilité est perdue. Nous ajoutons l'hypothèse suivante pour éviter cette situation.

Hypothèse A.1. *Il existe des constantes réelles positives $c_1(u, \theta)$ et $c_2(u, \theta)$ telles que, si t est suffisamment grand, alors nous avons*

$$\begin{aligned} \lambda_{\max}\left(S(t)^2\right) &\leq c_1(u, \theta), \\ \lambda_{\min}\left(S(t)\right) &\geq c_2(u, \theta). \end{aligned} \tag{A.3}$$

Les dépendances explicites de c_i en u et θ sont généralement omises pour simplifier la notation.

En utilisant le lemme de Gronwall : lemme 3.1, nous pouvons montrer qu'une borne supérieure de l'erreur d'estimation peut être explicitée. Ce résultat est donné dans le théorème suivant. Nous utilisons la notation :

$$e = x - \hat{x}, \tag{A.4}$$

$$\begin{aligned} L_1 &= \limsup_{t\to\infty} |d_1(t)|, \\ L_2 &= \limsup_{t\to\infty} |d_2(t)|. \end{aligned} \tag{A.5}$$

Théorème A.1. *Considérons les systèmes* (A.1) *et* (A.2) *pour u et θ fixés, et supposons que l'hypothèse A.1 est satisfaite. En utilisant les définitions données dans* (A.3), (A.4) *et* (A.5), *l'inégalité suivante est satisfaite,*

$$\limsup_{t\to\infty} |e(t)| \leq \frac{2L_1 c_1 + 2L_2\sqrt{c_1\,\lambda_{\max}\left(C'C\right)}}{c_2(\theta\sqrt{c_1} + \lambda_{\min}\left(C'C\right))}.$$

Bien que l'erreur d'estimation d'état puisse ne pas converger vers zéro lorsque des perturbations sont prises en compte, elle peut néanmoins être bornée. Notre objectif est alors dans ce cas de trouver une borne quantifiable et optimale pour l'erreur d'estimation e. En minimisant l'erreur d'estimation restante dans un espace admissible, nous fournissons alors un moyen systématique de trouver une excitation optimale u et un réglage optimal du gain de l'observateur θ pour garantir une robustesse au bruit.

Optimisation de la borne d'erreur d'estimation

Le théorème A.1 fournit une borne qui peut être minimisée en jouant simultanément sur le paramètre de réglage θ et l'entrée admissible u. Pour aborder correctement ce problème d'optimisation, nous devons d'abord spécifier une fonction coût et son domaine. Nous définissons le domaine en considérant les entrées avec l'excitation appropriée, puis redéfinissons la borne donnée dans ce théorème A.1 en fonction de u et θ en spécifiant un choix explicite de $c_1(u, \theta)$ et $c_2(u, \theta)$ dans (A.3). Cela nécessite d'utiliser le concept de persistance régulière d'une entrée pour la partie homogène du système (A.1), voir Définition 3.1.

Hypothèse A.2. *La matrice $A(u(t))$ est bornée. Cela signifie que,*

$$\sigma(u) = \sup_{t \geq 0} |A(u(t))| < \infty. \tag{A.6}$$

La dépendance explicite de σ en u est généralement omise pour simplifier la notation.

Le lemme suivant peut être déduit du travail développé dans [18]. La preuve repose sur la forme de $S(t)$ donnée dans (3.4) et sur les propriétés de base de la matrice de transition.

Lemme A.1. *Considérons les systèmes* (A.1) *et* (A.2)*, et supposons que l'hypothèse A.2 est vraie. Pour tous $\theta > 2\sigma$ et $t \geq 0$, nous avons*

$$S(t) \leq \beta_1 I,$$

où

$$\beta_1 = \frac{|C'C|}{\theta - 2\sigma} + |S(0)|.$$

De plus, si u est régulièrement persistant pour le système (A.1) *par rapport à le triplet (T, α, t_0), alors pour tous les $\theta > 0$ et tous les $t \geq t_0$ nous avons*

$$\beta_2 I \leq S(t),$$

où

$$\beta_2 = \alpha \exp(-T(\theta + 2\sigma)).$$

Remarque A.1. *Lemme A.1 implique que l'hypothèse A.1 est satisfaite lorsque u est régulièrement persistant. En fait, nous pouvons définir les constantes dans* (A.3) *comme :*

$$c_1 = \beta_1^2,$$

$$c_2 = \beta_2.$$

Cette sélection de constantes c_1 et c_2, avec le théorème A.1, permet la formulation du corollaire suivant.

Corollaire A.1. *Considérons les systèmes* (A.1) *et* (A.2)*, et supposons que l'hypothèse A.2 est satisfaite. Si u est régulièrement persistant pour le système* (A.1) *par rapport à (T, α, t_0),* *alors pour tout $\theta > 2\sigma$ nous avons*

$$\limsup_{t\to\infty} |e(t)| \leq L_1 \left(\frac{2(\theta - 2\sigma)|S(0)| + 2|C'C|}{\alpha\theta(\theta - 2\sigma)\exp(-T(\theta + 2\sigma))} \right) + L_2 \left(\frac{2\sqrt{|C'C|}}{\alpha\theta \exp(-T(\theta + 2\sigma))} \right).$$

Considérons un ensemble $\mathcal{U}$ d'entrées régulièrement persistantes pour le système (A.1) paramétrées sur un intervalle borné donné par

$$\mathcal{P} = [p_{min}, p_{max}]$$

et supposons que

$$\sigma^* = \sup_{u\in\mathcal{U}} \sigma(u) < \infty,$$

où $\sigma(u)$ est défini comme dans (A.6). Cette contrainte sur $\mathcal{U}$ est généralement définie afin de respecter les limitations physiques du système. De la même manière, le paramètre de réglage θ peut avoir des exigences d'amplitude et doit vérifier la borne inférieure dans le corollaire A.1 donnée par $\theta > 2\sigma^*$. Considérons alors l'intervalle borné

$$\Theta = [\theta_{min}, \theta_{max}]$$

pour certaines valeurs limites $\theta_{max} > \theta_{min} > 2\sigma^*$. Afin de définir un

$$J : \Theta \times \mathcal{P} \to \mathbb{R}^+$$

fonctionnel en utilisant la borne du corollaire A.1, il suffit d'assigner à chaque $u \in \mathcal{U}$ régulièrement persistant une sélection des paramètres T et α. Nous le faisons de manière naturelle : pour chaque $u \in \mathcal{U}$ sélectionnez et fixez $T(u)$ de telle sorte que l'ensemble

$$\mathcal{RP}(u) = \{\alpha \in (0, \infty) | \exists t_0 > 0 \text{ satisfaisant (3.5)}\}$$

n'est pas vide. Ensuite, on défini

$$\alpha(u) = \sup \mathcal{RP}(u). \tag{A.7}$$

Bien sûr, la mise en œuvre réelle de cette stratégie de conception de u et θ n'est pas anodine et doit être adaptée au système spécifique considéré. Cette stratégie dépend des propriétés de la fonctionnelle résultante J. Par exemple, supposons que le paramétrage de $\mathcal{U}$ et l'affectation $T(u)$ se fassent de manière continue. Ensuite, J se traduit par une fonctionnelle continue avec un domaine définie sur un compact et sa borne minimum est atteinte.

Il existe plusieurs outils disponibles dans le logiciel Matlab pour effectuer cette minimisation. Par exemple, si J est également strictement convexe, le minimum global unique est facilement obtenu par la fonction "fmincon", qui est basée sur des algorithmes de points intérieurs [26].

Nous résumons maintenant la stratégie de conception hors ligne de l'entrée optimale u^* et du paramètre de réglage optimal θ^* pour une telle J de la manière suivante :

$$\begin{cases} \text{1. Choisir un espace d'entrées régulièrement persistantes } \mathcal{U}, \\ \text{2. Paramétrer } \mathcal{U} \text{ en continu avec } \mathcal{P} \text{ intervalle,} \\ \text{3. Choisir un espace de paramètre de réglage } \Theta, \\ \text{4. Définir une longueur continue d'affectation } T(u) \text{ et} \\ \text{calculer } \alpha(u) \text{ comme dans (A.7),} \\ \text{5. La conception optimale est obtenue par :} \\ (\theta^*, p^*) = \underset{\theta\in\Theta, p\in\mathcal{P}}{\arg\min}\, J(\theta, p), \\ \text{où } J \text{ est donné par la borne dans le corollaire A.1 et} \\ \text{où } u^* \in \mathcal{U} \text{ correspond à } p^*. \end{cases}$$

La fin du chapitre 3 montre un exemple académique de cette approche de conception hors ligne.

A.3 Chapitre 4 : design d'observateur pour des systèmes non linéaires à transformation de sortie

Les observateurs habituellement utilisés pour les systèmes linéaires sont l'observateur de Luenberger et l'observateur de Kalman qui peuvent être étendus aux systèmes non linéaires sous des formes spécifiques. Par conséquent, une stratégie courante consiste à rechercher un changement de coordonnées qui permet d'écrire le système vers une forme spécifique [48, 16]. Les premières contributions importantes dans ce sens comprennent : une forme linéaire avec injection de sortie [75], une forme bilinéaire avec injection de sortie et son observateur de type Kalman [58, 22], et une forme triangulaire pour un système uniformément observable et son observateur de type Luenberger ou à grand gain [46]. Même si ces synthèses d'observateurs concernent principalement des systèmes non linéaires, leurs sorties sont souvent des fonctions linéaires de l'état. De même, un objectif commun des fabricants de capteurs est d'atteindre la linéarité. Ceci est souvent compliqué et un grand nombre de capteurs classiques présente un comportement non linéaire [105, 36]. Par exemple, c'est le cas dans : les systèmes d'alimentation à piles à combustible [9], la restauration d'images [106], l'imagerie numérique [97], le contrôle de la combustion dans les automobiles [89] et l'ingénierie médicale [74].

Un choix populaire d'observateur pour les systèmes non linéaires généraux à sortie non linéaire est le filtre de Kalman étendu dans sa version déterministe. Bien que simple et avec de bonnes propriétés de filtrage du bruit, seule la convergence locale est garantie. L'observateur proposé dans [48] n'est pas limité en ce sens, cependant, le système doit être sous forme canonique d'observabilité avec une sortie non linéaire régulière. Une approche différente est basée sur le célèbre observateur de Kazantzis-Kravaris [69, 7], qui adapte les idées de Luenberger au cadre non linéaire ; le problème de conception se réduit à résoudre un système d'équations aux dérivées partielles.

D'autre part, la conception d'observateurs pour les systèmes avec des non-linéarités monotones est étudiée dans [10], où les auteurs suppriment la condition de Lipschitz et évitent les grands gains en utilisant ce que l'on appelle le critère du cercle. Dans [43], ils développent ces résultats et considèrent également les sorties non linéaires en présence d'incertitudes du modèle. Le travail dans [13] traite plutôt d'un type plus général de non-linéarités, celles qui satisfont aux contraintes quadratiques incrémentielles. Enfin, les auteurs de [77] proposent une conception d'observateur simple pour des systèmes sous forme triangulaire avec une sortie non linéaire. Cette fonction de sortie n'est pas nécessairement différenciable mais elle doit satisfaire une condition de secteur.

Cependant, ces résultats ont tendance à nécessiter des systèmes sous des formes spécifiques et des conditions suffisantes pour l'existence des transformations de coordonnées correspondantes sont généralement fortes [48, 16]. De plus, trouver les bonnes transformations peut être difficile, en particulier dans le cas de plusieurs sorties. Une autre limitation est que le bruit de mesure est souvent négligé. Un cadre naturel pour étudier la robustesse d'un observateur par rapport au bruit de mesure est celui de la stabilité entrée-état (ISS). En effet, on peut considérer la dynamique d'erreur comme l'état et le bruit de mesure comme l'entrée.

Dans ce contexte, l'ISS est également appelé stabilité de la perturbation à l'erreur (DES) [99].

Dans cette deuxième partie, qui est basée sur notre travail [50, 53], nous considérons le problème de la conception robuste d'observateurs pour les systèmes non linéaires en présence d'une transformation non linéaire de sortie. En effet, l'objectif de ce chapitre concerne la refonte de l'observateur pour adapter un observateur donné à une transformation non linéaire de la sortie du système.

Ce travail est inspiré de problèmes pratiques ; les capteurs sont parfois conçus pour que la linéarité entre l'état du système et la sortie soit atteinte, et cela peut être très difficile en raison de la nature non linéaire de la plupart des appareils physiques. Par exemple, considérer les capteurs d'image [106], les capteurs de déplacement à fibre optique, les biocapteurs [74] et les capteurs d'oxygène, qui sont utilisés pour la combustion en boucle fermée contrôle dans les véhicules. Bien que la version large bande de ce dernier atteigne la linéarité, le prix augmente considérablement.

Formulation du problème

Considérons un système non linéaire de la forme

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) + d, \end{cases} \tag{A.8}$$

où $x(t) \in \mathbb{R}^n$ est l'état, $y(t) \in \mathbb{R}^p$ la sortie, $u(t) \in \mathbb{R}^m$ une entrée continue et $d(t) \in \mathbb{R}^p$ est une perturbation. Nous supposons que f et h sont de classe C^2, que le système est complet et nous désignons par $u \in \mathcal{U}$ l'ensemble d'entrées.

Nous supposons d'abord qu'un observateur "robuste" dans le sens ISS a été conçu pour le système (A.8) et est donné par

$$\begin{cases} \dot{\hat{x}} = \hat{f}(\hat{x}, g, y, u) \\ \dot{g} = G(g, u), \end{cases} \tag{A.9}$$

où $\hat{x}(t) \in \mathbb{R}^n$ est l'estimation d'état, g un gain dynamique et $\hat{f}$ est localement de type Lipschitz.

Nous considérons ensuite le cas où y n'est pas directement disponible pour les mesures. Au lieu de cela, nous mesurons une transformation non linéaire $\psi(y)$ affectée par le bruit. Cette situation se produit fréquemment dans les processus d'ingénierie, où les transformations de capteurs non linéaires sont courantes. Le nouveau système prend la forme

$$\begin{cases} \dot{x} = f(x, u) \\ y_\psi = \psi(y) + d_\psi, \end{cases} \tag{A.10}$$

où $y_\psi(t) \in \mathbb{R}^p$ est la sortie mesurée, $y(t) = h(x(t)) + d_y(t) \in \mathbb{R}^p$, les perturbations $d_y(t)$, $d_\psi(t) \in \mathbb{R}^p$ sont bornées par des dérivés bornées et ψ est de classe C^2. Ici, d_y peut représenter les incertitudes du modèle tandis que d_ψ mesure le bruit. En conséquence de la modification de la sortie du système, l'observateur (A.9) ne peut pas être directement implémenté et, par conséquent, une refonte est nécessaire.

Pour un système non linéaire général, il n'y a aucun moyen systématique d'adapter l'observateur donné aux transformations de sortie. Nous proposons une nouvelle méthode de "redesign" des observateurs qui fait face à ce défi.

Nous ne pouvons pas implémenter l'observateur (A.9) car y n'est pas directement connu. Cependant, nous exigeons que cet observateur soit robuste vis-à-vis du bruit de mesure. La dynamique d'erreur correspondante est donnée par

$$\dot{e} = \mathcal{F}(t, e, d), \tag{A.11}$$

où $e = x - \hat{x}$ et pour

$$\mathcal{F}(t, e, d) = f(x(t), u(t)) - \hat{f}(x(t) - e, g(t), h(x(t)) + d, u(t)),$$

ainsi, nous avons besoin de l'hypothèse suivante.

Hypothèse A.3. *Il existe une fonction continue $\bar{V}(t, e) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, de classe C^1 sur $e \neq 0$, et des fonctions $\bar{\alpha}_1$, $\bar{\alpha}_2 \in \mathcal{K}_\infty$ et $\bar{\alpha}_3$, $\bar{\chi} \in \mathcal{K}$ telles que :*

$$\bar{\alpha}_1(|e|) \leq \bar{V}(t, e) \leq \bar{\alpha}_2(|e|),$$

pour tous les $t \in \mathbb{R}^+$ et tous les $e \in \mathbb{R}^n$, et tels que :

$$\frac{\partial \bar{V}}{\partial t}(t, e) + \frac{\partial \bar{V}}{\partial e}(t, e)\mathcal{F}(t, e, d) \leq -\bar{\alpha}_3(|e|),$$

chaque fois que $|e| \geq \bar{\chi}(|d|)$ *et pour tous* $t \in \mathbb{R}^+$, $e \in \mathbb{R}^n - \{0\}$, $d \in \mathbb{R}^p$ *et tous* $x(0) \in \mathbb{R}^n$.

L'hypothèse A.3 est équivalente à l'ISS du système (A.11) comme dans [102] ou, pour les fonctions Lyapunov variant dans le temps [41]. Dans ce contexte, la propriété ISS garantit la dégradation contrôlée des performances de l'observateur (A.9) en présence de bruit de mesure. Les observateurs satisfaisant cette propriété ont d'abord été considérés dans [104] et ils sont connus comme observateurs stables de perturbation à l'erreur (DES) [99]. Il existe des méthodes pour déterminer si certains observateurs sont DES [5] ou pour les repenser s'ils ne le sont pas [100].

Hypothèse A.4. *La fonction* $\psi : \mathbb{R}^p \to \mathbb{R}^p$ *est de classe* C^2 *et son Jacobien est inversible sur tout son domaine. De plus, il existe* $\delta \in \mathcal{K}_\infty \cap C^2$ *tel que*

$$\delta(|\psi(y) - \psi(\hat{y})|) \geq |y - \hat{y}|, \quad \forall y, \hat{y} \in \mathbb{R}^p.$$

L'hypothèse A.4 implique notamment l'injectivité de ψ. Il est satisfait, par exemple, si $p = 1$ et si $|\frac{\partial \psi}{\partial y}|$ est borné par le bas par une constante positive. Les auteurs de [77] exigent que cette dernière condition soit vérifiée lorsque la sortie non linéaire qu'ils étudient est continuellement différenciable.

Remarque A.2. *Notez que l'hypothèse A.3 concerne l'existence d'une conception d'observateur robuste pour le système* (A.8) *et, par conséquent, dépend des propriétés des fonctions* f *et* h. *Cependant, l'hypothèse A.4 ne contraint que la transformation de sortie* ψ *dans le système* (A.10) *et elle est nécessaire pour repenser l'observateur.*

Synthèse de l'observateur

Considérons le système avec la sortie transformée donnée dans (A.10). La conception proposée pour l'observateur est présentée comme l'interconnexion suivante :

$$\begin{cases} \dot{\hat{x}} = \hat{f}(\hat{x}, g, \hat{y}, u) \\ \dot{g} = G(g, u) \\ \dot{\hat{y}} = \hat{h}(\hat{x}, \hat{y}, y_\psi, u), \end{cases} \tag{A.12}$$

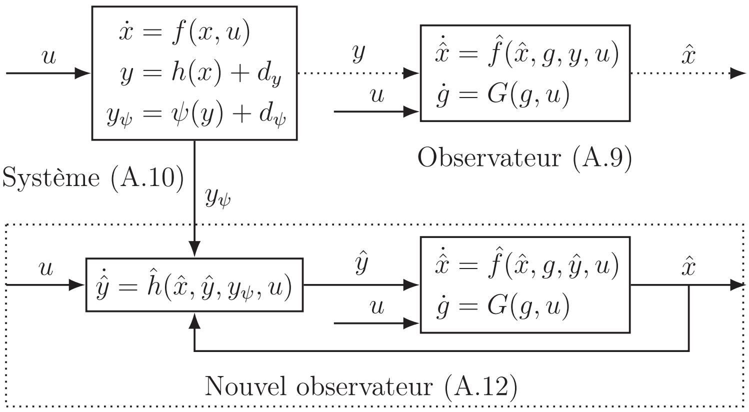

FIGURE A.1 : Le système (A.10) et l'observateur (A.9) sont représentés dans la partie supérieure du diagramme. Le nouvel observateur (A.12) est représenté dans la partie inférieure comme un système inter-connecté. L'observateur (A.9) nécessite la sortie non disponible y, tandis que le nouvel observateur utilise à la place les mesures y_ψ.

où $\hat{f}$ et G sont comme dans (A.9),

$$\hat{h}(\hat{x}, \hat{y}, y_\psi, u) = \frac{\partial \psi}{\partial y}(\hat{y})^{-1} \frac{\partial \psi}{\partial y}(h(\hat{x})) \frac{\partial h}{\partial x}(\hat{x}) f(\hat{x}, u) + \frac{\partial \psi}{\partial y}(\hat{y})^{-1} \varphi(\hat{x}, u) K(y_\psi - \psi(\hat{y})),$$

et $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$ et $K : \mathbb{R}^p \to \mathbb{R}^p$ sont localement des fonctions Lipschitz définies par la suite. Nous soulignons que l'observateur dans (A.12) ne requiert que la connaissance de y_ψ et non directement de y. La figure A.1 compare les observateurs (A.9) et (A.12) et illustre la relation de ces observateurs avec le système (A.10).

Afin de définir la fonction φ, considérons d'abord la fonction ϕ donnée par

$$\phi(x, u, d_1, d_2, d_3) = \frac{\partial \psi}{\partial y}(h(x) + d_1) \left(\frac{\partial h}{\partial x}(x) f(x, u) + d_2 \right) + d_3, \tag{A.13}$$

pour tous les $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ et d_1, d_2, $d_3 \in \mathbb{R}^p$. Selon la proposition 2.2, il existe localement des fonctions Lipschitz $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$ et $\alpha \in \mathcal{K}_\infty$ telles que,

$$|\phi(x, u, d_1, d_2, d_3) - \phi(\hat{x}, u, 0, 0, 0)| \leq \varphi(\hat{x}, u) \alpha(|(x - \hat{x}, d_1, d_2, d_3)|) \tag{A.14}$$

sur tout son domaine et on peut supposer que $\varphi \geq 1$.

Remarque A.3. *Le problème de trouver α et φ comme dans* (A.14) *est simplifié si ψ et chacune des fonctions dans l'expression* (A.13) *satisfont les conditions Lipschitz globales. On* *peut alors prendre* α *comme fonction d'identité et*

$$\varphi(\hat{x}, u) \geq c_\varphi(|f(\hat{x}, u)| + 1),$$

pour certain $c_\varphi \geq 1$ *et pour tous les* $\hat{x} \in \mathbb{R}^n$ *et tous les* $u \in \mathbb{R}^m$*. Il existe une vaste littérature traitant du calcul des constantes de Lipschitz comme celles définissant* c_φ*. Un autre cas pratique est lorsque le système* (A.10) *a des états bornés, compte tenu des techniques de saturation (voir chapitre 4).*

Nous continuons maintenant avec la définition de K. Considérons une fonction $\rho \in \mathcal{K}_\infty \cap C^2$, une constante positive k et définissons $K : \mathbb{R}^p \to \mathbb{R}^p$ comme

$$K(\xi) = \begin{cases} \left(k\frac{\rho(|\xi|)}{|\xi|}\right)\xi, \text{ si } \xi \neq 0 \\ 0, \text{ si } \xi = 0. \end{cases} \tag{A.15}$$

Le choix de ρ garantit que K est localement Lipschitz. En effet, en utilisant $\rho(0) = 0$ et la règle de L'Hôpital, nous pouvons montrer que la fonction

$$\begin{cases} \frac{\rho(r)}{r}, \text{ si } r > 0 \\ \frac{\partial \rho}{\partial r}(0), \text{ si } r = 0 \end{cases}$$

est différenciable continûment sur $\mathbb{R}^+$. La fonction ρ offre un degré de liberté pour la synthèse d'observateur (A.12).

Résultat principal

Étant données les conditions initiales $\hat{x}(0) \in \mathbb{R}^n$ et $\hat{y}(0) \in \mathbb{R}^p$, il existe un intervalle maximal d'existence correspondant $[0, T)$ pour la solution unique $(\hat{x}, \hat{y})$ de (A.12). Les erreurs d'estimation sont données par

$$e(t) = x(t) - \hat{x}(t), \quad \xi(t) = y_\psi(t) - \psi(\hat{y}(t)),$$

pour tout $t \in [0, T)$. Notez que la solution est illimitée si T est fini, voir par exemple [29]. Nous verrons que ce n'est pas le cas si ρ est correctement choisi. Les deux lemmes suivants concernent les définitions de la section 2.5.

Lemme A.2. *Considérons les systèmes* (A.10) *et* (A.12)*, avec φ et K comme dans* (A.14)-(A.15)*, et soit l'hypothèse A.4 vérifiée. Si $k > 1$, alors il existe une fonction Lyapunov pour la famille*

$$\mathcal{G} = \{(\xi, e) | x(0), \hat{x}(0) \in \mathbb{R}^n, \hat{y}(0) \in \mathbb{R}^p\},$$

où $\xi = y_\psi - \psi(\hat{y})$ et $e = x - \hat{x}$ sont définis sur $[0, T)$. De plus, si les perturbations d_y et d_ψ sont toutes les deux nulles, la constante de Lyapunov correspondante peut être choisie nulle.

Lemme A.3. *Considérons les systèmes* (A.10) *et* (A.12)*, avec φ et K comme dans* (A.14)-(A.15)*, et soient les hypothèse A.3 et A.4 vérifiées. Il existe une fonction Lyapunov pour la famille*

$$\mathcal{G}^{-1} = \{(e, \xi) | x(0), \hat{x}(0) \in \mathbb{R}^n, \hat{y}(0) \in \mathbb{R}^p\},$$

où $e = x - \hat{x}$ et $\xi = y_\psi - \psi(\hat{y})$ sont définis sur $[0, T)$. De plus, si les perturbations d_y et d_ψ sont toutes les deux nulles, la constante de Lyapunov correspondante peut être choisie nulle.

Cela étant établit, nous avons, maintenant, les éléments pour énoncer notre principal résultat. Il établit une condition sur le gain ρ afin de garantir la convergence asymptotique vers un voisinage de zéro (ou vers zéro lui-même) de l'erreur d'estimation d'état donnée par le nouvel observateur dans (A.12).

Théorème A.2 (Design du gain de l'observateur)**.** *Considérons les systèmes* (A.10) *et* (A.12)*, avec φ et K comme dans* (A.14)-(A.15)*, et soient les hypothèses A.3 et A.4 vérifiées. Pour tout $k > 1$ et tout $\rho \in \mathcal{K}_\infty \cap C^2$ satisfaisant*

$$\rho(r) > 2\alpha(2\bar{\alpha}_1^{-1}(\bar{\alpha}_2(2\bar{\chi}(2\delta(4r))))), \quad \forall r > 0 \tag{A.16}$$

il existe une classe $\mathcal{KL}$ fonction β et une constante $c \geq 0$ telles que pour tout $x(0)$, $\hat{x}(0) \in \mathbb{R}^n$ et tout $\hat{y}(0) \in \mathbb{R}^p$ les erreurs d'estimation $e = x - \hat{x}$ et $\xi = y_\psi - \psi(\hat{y})$ sont définies sur $\mathbb{R}^+$ et

$$|(\xi(t), e(t))| \leq \beta(|(\xi(0), e(0))|, t) + c, \quad \forall t \geq 0. \tag{A.17}$$

De plus, si les perturbations d_y et d_ψ sont toutes deux nulles alors c est également nulle.

Remarque A.4. *Dans la pratique, la conception du nouvel observateur* (A.12) *commence par proposer un observateur comme* (A.9) *et par trouver une fonction de Lyapunov $\bar{V}$ pour sa dynamique d'erreur, avec les fonctions $\bar{\alpha}_1$, $\bar{\alpha}_2$ et $\bar{\chi}$ de l'hypothèse A.3. Nous avons alors* *besoin de δ dans l'hypothèse A.4 et des fonctions φ et α à* (A.14) *(voir Remarque A.3). Enfin, la borne inférieure dans* (A.16) *peut être utilisée pour construire un tel ρ et K est alors donné par* (A.15)*.*

Remarque A.5. *A partir des preuves (voir chapitre 4), le nouvel observateur récupère un type de propriété DES : il existe $\beta \in \mathcal{KL}$ et $\gamma \in \mathcal{K}$ de telle sorte que pour tous $x(0)$, $\hat{x}(0) \in \mathbb{R}^n$ et tout $\hat{y}(0) \in \mathbb{R}^p$, et pour toutes d_y et d_ψ délimitées, Lipschitz et différenciables nous avons*

$$|(\xi(t), e(t))| \leq \beta(|(\xi(0), e(0))|, t) + \gamma(|(d_y, d_\psi, \dot{d}_y, \dot{d}_\psi)|_\infty), \quad \forall t \geq 0.$$

Enfin, notez que nous pouvons trouver le taux de décroissance explicite β et la constante c dans (A.17) en suivant les étapes suivantes :

1. Trouver les bornes et le gain de Lyapunov à partir de l'hypothèse A.3. Utilisez-les pour construire ρ satisfaisant l'inégalité dans (A.16).
2. Obtenir les limites de Lyapunov, les gains et les constantes des Lemmes A.2 et A.3.
3. Choisir une fonction intermédiaire σ comme expliqué dans la section 2.5.
4. Calculer les bornes, le gain et la constante de la fonction Lyapunov pour $\mathcal{G} \times \{0\}$ comme dans la preuve du théorème 2.15 dans la section 2.5.
5. Calculer le taux de décroissance correspondant β et la constante c comme indiqué dans les preuves de lemme 2.1 et le théorème 2.14 dans la section 2.5.

Notre résultats ont été illustrés sur deux cas importantes de systèmes non linéaires (voir chapitre 4).

A.4 Chapitre 5 : extension au cas de sortie discrète et transformée

Dans le chapitre 5 de ce manuscrit, l'approche précédente du théorème A.2 a été étendue pour traiter simultanément des non-linéarités de sortie et de la discrétisation non uniforme de sortie dans le temps. Notre travail a fait l'objet d'une communication [52].

Une façon de résoudre ce problème consiste à appliquer les résultats de la partie précédente avec le "redesign" de l'observateur de [68], qui aborde le cas d'une sortie discrétisée. Néanmoins, cela se traduirait par un observateur avec trois interconnexions alors que l'on s'attend à ce que seulement deux puissent suffire. Par conséquent, nous proposons plutôt d'étendre directement notre conception d'observateur à partir du chapitre 4 en implémentant la sortie $\psi(y(t_k))$ avec "sample-and-hold". L'observateur interconnecté résultant se compose de : (i) un sous-système avec une dynamique continue basée sur la structure de l'observateur d'origine, (ii) un sous-système avec une dynamique commutée découlant des techniques de "sample-and-hold". Sa convergence est montrée en définissant une fonctionnelle Lyapunov-Krasovskii, basée sur [92, 88], et en utilisant le théorème du petit-gain pour les systèmes commutés [111]. Cette approche conduit à deux LMIs qui dépendent, entre autres paramètres, du temps maximum entre deux échantillonnages consécutifs.

Notre objectif est de concevoir un observateur pour le système donné par

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ y_\psi(t_k) = \psi(y(t_k)), \end{cases} \tag{A.18}$$

où $x(t) \in \mathbb{R}^n$, $y(t) = h(x(t)) \in \mathbb{R}^p$ et $y_\psi(t) \in \mathbb{R}^p$. Nous supposons que f, h et ψ sont de classe C^2 et que le système est complet. De plus, (t_k) est une séquence croissante de temps d'échantillonnage telle que $t_0 = 0$ et telle qu'il existe des constantes positives $\underline{\delta}$ et $\bar{\delta}$ avec

$$\underline{\delta} \leq t_{k+1} - t_k \leq \bar{\delta}, \quad \forall k \geq 0.$$

Comme dans la partie 2, notre première hypothèse est la existence d'un observateur

$$\dot{\hat{x}}(t) = \hat{f}(\hat{x}(t), y(t)),$$

où $\hat{x}(t) \in \mathbb{R}^n$ est l'estimation d'état. Cet observateur doit être robuste en ce qui concerne le bruit de mesure dans le sens ISS, c'est-à-dire stable à la perturbation par rapport à l'erreur (DES). Plus précisément, si la sortie est affectée par le bruit $y = h(x) + d_y$, alors la dynamique d'erreur correspondante $e = x - \hat{x}$ est donnée par

$$\dot{e}(t) = \mathcal{F}(t, e(t), d_y(t)),$$

où $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ est défini comme

$$\mathcal{F}(t, e, d) = f(x(t)) - \hat{f}(x(t) - e, h(x(t)) + d),$$

et nous avons l'hypothèse suivante.

Hypothèse A.5. *Il existe une fonction de classe C^1, dénoté par $\bar{V} : \mathbb{R}^n \to \mathbb{R}^+$, telle que les conditions suivantes sont remplies :*

1. *$\exists\ \bar{\alpha}_1,\ \bar{\alpha}_2 \in \mathcal{K}_\infty$ de tels sortes que pour tous $e \in \mathbb{R}^n$,*

$$\bar{\alpha}_1(|e|) \leq \bar{V}(e) \leq \bar{\alpha}_2(|e|),$$

2. *$\exists\ \bar{\chi} \in \mathcal{K}_\infty$ et une constante $\bar{c} > 0$ tels que pour tous $t \in \mathbb{R}^+$, $e \in \mathbb{R}^n$, $d \in \mathbb{R}^p$ et $x(0) \in \mathbb{R}^n$,*

$$|e| \geq \bar{\chi}(|d|) \implies \frac{\partial \bar{V}}{\partial e}(e)\mathcal{F}(t, e, d) \leq -\bar{c}\bar{V}(e).$$

Hypothèse A.6. *La fonction $\psi : \mathbb{R}^p \to \mathbb{R}^p$ dans* (A.18) *est de classe C^2 et son Jacobien est inversible sur tout son domaine. De plus, il existe une constante $c_\psi > 0$ telle que*

$$|y - \hat{y}| \leq c_\psi |\psi(y) - \psi(\hat{y})|, \quad \forall y, \hat{y} \in \mathbb{R}^p.$$

Hypothèse A.7. *La fonction $\varphi : \mathbb{R}^n \to \mathbb{R}^p$ donnée par*

$$\varphi(x) = \frac{\partial \psi}{\partial y}(h(x))\frac{\partial h}{\partial x}(x)f(x)$$

est Lipschitz continue. Autrement dit, il existe une constante $c_L > 0$ telle que

$$|\varphi(x) - \varphi(\hat{x})| \leq c_L|x - \hat{x}|, \quad \forall x, \hat{x} \in \mathbb{R}^n.$$

L'état du système est souvent borné par des problèmes physiques d'estimation. Des techniques de saturation peuvent alors être utilisées afin de satisfaire l'hypothèse A.7, voir par exemple [54].

Inspiré par nos travaux du chapitre 4, l'observateur proposé pour le système continu-discret (A.18) est défini par le système donné par l'interconnexion :

$$\begin{cases} \dot{\hat{x}} = \hat{f}(\hat{x}, \hat{y}) \\ \dot{\hat{y}} = \frac{\partial \psi}{\partial y}(\hat{y})^{-1} \left(\frac{\partial \psi}{\partial y}(h(\hat{x}))\frac{\partial h}{\partial x}(\hat{x})f(\hat{x}) + K\xi_k \right) \\ \xi_k(t) = y_\psi(t_k) - \psi(\hat{y}(t_k)), \quad t \in [t_k, t_{k+1}), \end{cases} \tag{A.19}$$

où $\hat{f}$ est comme dans l'hypothèse A.5 et K est une matrice à concevoir. Notez que l'observateur (A.19) consiste en une dynamique continue interconnectée avec un système commuté et qu'il ne nécessite que $y_\psi(t_k) = \psi(y(t_k))$ comme mesure de sortie. Notre résultat principal de cette partie est le suivant.

Théorème A.3. *Considérons les systèmes* (A.18) *et* (A.19)*, et supposons les hypothèses A.5, A.6 et A.7 vérifiées. Supposons aussi que les conditions suivantes soient remplies.*

1. *il existe une constante $\alpha > 0$ et des matrices : $P = P' > 0$, $\tilde{K}$, N_1, N_2 et N_3 dans $\mathbb{R}^{p\times p}$ de telles sortes que les conditions LMI* (5.24) *et* (5.25) *soient satisfaites avec*

$$\lambda_1 = -\alpha\bar{\delta} + 1, \quad \lambda_2 = \exp(-\alpha\bar{\delta}), \quad \lambda_3 = \bar{\delta}\exp(-0.5\alpha\bar{\delta}),$$

2. *il existe une constante $c > c_L^2/\alpha$ telle que*

$$\bar{\alpha}_2(\bar{\chi}(c_\psi c^{1/2}\,\lambda_{\min}(P)^{-1/2}\bar{\alpha}_1^{-1}(r))) < r, \quad \forall r > 0.$$

Ensuite, en choisissant $K = P^{-1}\tilde{K}$, il existe $\beta \in \mathcal{KL}$ de telle sorte que pour tout $x(0)$, $\hat{x}(0) \in \mathbb{R}^n$ et tout $\hat{y}(0) \in \mathbb{R}^p$ les erreurs d'estimation $e = x - \hat{x}$ et $\xi = y_\psi - \psi(\hat{y})$ sont définies sur $\mathbb{R}^+$ et

$$|(e(t), \xi(t))| \leq \beta(|(e(0), \xi(0))|, t), \quad \forall t \geq 0.$$

Le théorème précédent pose des conditions assurant notamment la convergence asymptotique à zéro de e. Sa preuve, qui consiste en deux lemmes, est présentée dans la section 5.5 et est basée sur le théorème du petit-gain pour les systèmes commutés [111]. Par conséquent, nous recherchons les fonctions ISS Lyapunov V_1 et V_2 en considérant d'abord $e = x - \hat{x}$ comme état et $\xi = y_\psi - \psi(\hat{y})$ comme entrée puis en faisant le contraire.

Remarque A.6. *L'observateur dans* (A.19) *peut être vu comme une interconnexion composée d'un sous-système ordinaire et d'un sous-système commuté, cependant, l'application des résultats généraux de [111] nécessite une analyse minutieuse de leurs preuves.*

Notre résultats ont été illustrés sur un cas important de système non linéaires (voir chapitre 5).