Statistical inference for a partially observed interacting system of Hawkes processes

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Thèse de doctorat

Discipline : Mathématiques

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Presentée et soutenue le 9 Décembre 2019 devant le jury composé de :

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Remerciements

Je voudrais exprimer ma plus profonde gratitude à mes directeurs de thèse, Nicolas Fournier et Sylvain Delattre, pour avoir accepté d'encadrer mon doctorat, pour le temps qu'ils m'ont consacré pendant nos nombreuses discussions, pour des conseils qu'ils m'ont donnés, et pour tous ce qu'ils m'ont appris pendant ces années. Très sincèrement merci!

Je tiens à remercier chaleureusement mes deux rapporteurs, Émmanuelle Clément et Vincent Rivoirard , pour leurs commentaires précieux qui ont permis d'améliorer ce manuscrit. Ma reconnaissance va également à Ismael Castillo et Marc Hoffman pour avoir accepté de faire partie du jury de soutenance.

Je suis très reconnaissance envers la Fondation FSMP pour avoir participé au financement de ma thèse et ma master.

Au laboratoire j'ai pu bénéficier d'une bonne condition de travail et une ambiance sympathique. Un grand merci à tous mes amis qui sont anciens ou actuels membres du laboratoire: merci à Adeline, Florian, Sothea, Willem, et Yiyang qui ont organisé le groupe de travail des thésard, merci à Alexander, Alexandra, Flaminia, Liping, Malo, Michel, Pierre, Sandro, Sergi, Vivian, Willem, Zhuchao, avec qui j'ai partagé un même bureau, et merci à An, Armand, Carlo, Chenlin, David Eric, Francois, Guillaume, Henri, Isao, Léa, Lucas, Paul, Qiming, Wangru, Yating, Yi, Yoan pour de bons moments passés ensemble au cours de ces dernières années. Merci à Merci également à l'équipe administrative du laboratoire: Corinne, Fatima, Florence, Josette, Louise, Nathalie et Valérie , pour votre gentillesse et disponibilité.

J'ai aussi un grand merci pour tous mes professeurs de l'école à master. Merci spécialement à Julien Barral, Yueyun Hu et Zhan Shi, sans qui je n'aurais pas eu l'idée de venir en France. Merci à Jean Jacod et Camille Tardif, avec qui j'ai pu bénéficié beaucoup pendent les discussions.

Merci également à Bin, Binguang, Chao, Chaoen, Chuqi, Dan, Emily, Heshu, Hua, Huajie, Hui, Huihui, Jian, Jiaxin, Jingxuan, Kexin, Kun, Liqiong, Long, Loulou, Menglan, Nan, Ning, Peng, Qiaochu, Quan, Ran, Rangrang, Runqi, Ruotao, Saibo, Salawa, Shuai, Shuo, Sibo, Thuy, Vivienne, Wenjie, Wenqian, Xiang, Xingyu, Xiao, Xiaofeng, Xiaojie, Xiaoli, Xunwu, Xuwen, Yanni, Yao, Yaxin, Yi, Yichen, Yihan, Yijun, Yisheng, Yizhen, Yongxin, Yuan, Yuemeng, Zhiqiang, Zicheng.

Enfin, je remercie toute ma famille pour leur soutien et leurs encouragements constants, dans les moments de joie comme dans les moments de frustration.,

Abstract

We observe the actions of a K sub-sample of N individuals, during some time interval with length $t > 0$, for some large $K \leq N$. We model the relationships of individuals by i.i.d. Bernoulli(p) random variables, where $p \in (0, 1]$ is an unknown parameter. The rate of action of each individual depends on some unknown parameter $\mu > 0$ and on the sum of some function ϕ of the ages of the actions of the individuals which influence him. The function ϕ is unknown but we assume it rapidly decays. The aim of this thesis is to estimate the parameter p, which is the main characteristic of the interaction graph, in the asymptotic where the population size $N \to \infty$, the observed population size $K \to \infty$, and in large time $t \to \infty$. Let m_t be the average number of actions per individual up to time t, which depends on all the parameters of the model. In the subcritical case, where m_t increases linearly, we build an estimator of p with the rate of convergence $\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}} + \frac{N}{K\sqrt{m_t}}$. In the supercritical case, where m_t increases exponentially fast, we build an estimator of p with the rate of convergence $\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}}$.

In a second time, we study the asymptotic normality of those estimators. In the subcritical case, the work is very technical but rather general, and we are led to study three possible regimes, depending on the dominating term in $\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}} + \frac{N}{K\sqrt{m_t}} \to 0$.

In the supercritical case, we unfortunately suppose some additional conditions and consider only one of the two possible regimes.

Keywords. Multivariate Hawkes processes, Point processes, Statistical inference, Interaction graph, Stochastic interacting particles, Mean field limit, Central limit theorem.

Résumé

Nous observons les actions d'un sous-échantillon de K de N dindividus, pendant un intervalle de temps de longueur $t > 0$, pour certaines grandes $K \leq N$. Nous modélisons les relations des individus par i.i.d. Bernoulli (p) variables aléatoires, où $p \in (0, 1]$ est un paramètre inconnu. Le taux daction de chaque individu dépend dun paramètre inconnu $\mu > 0$ et sur la somme de quelque fonction ϕ des âges des actions des individus qui l'influencent. La fonction ϕ est inconnue mais nous supposons qu'elle se désintègre rapidement. Le but de cette thèse est d'estimer le paramètre p, qui est la principale caractéristique du graphe dinteraction, dans l'asymptotique où taille de la population $N \to \infty$, la taille de la population observée $K \to \infty$, et dans un temps long $t \to \infty$. Soit m_t le nombre moyen d'actions par individu jusqu'au temps t, qui dépend de tous les paramètres du modèle. Dans le cas sous-critique, où m_t augmente linéairement, nous construisons un estimateur de p avec le taux de convergence $\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}} + \frac{N}{K\sqrt{m_t}}$. Dans le cas supercritique, où m_t augmente rapidement de faon exponentielle, nous construisons un estimateur de p avec le taux de convergence $\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}}$.

Dans un second temps, nous étudions la normalité asymptotique de ces estimateurs. Dans le cas sous-critique, le travail est très technique mais assez génral, et nous sommes amenés à étudier trois régimes possibles, en fonction du terme dominant dans $\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}} + \frac{N}{K\sqrt{m_t}}$ à 0.

Dans le cas supercritique, nous supposons malheureusement quelques conditions supplémentaires et considérons seulement l'un des deux régimes possibles.

Mots-clés.

Processus Hawkes multivariés, Processus ponctuels, Inférence statistique, Graphe d'interaction, Particules à interaction stochastique, Limite de champ moyen, Théorème centrale limite.

Contents

Chapter 0

Introduction

0.1 Review of the thesis

We study mainly the statistical inference for a partially observed interacting system of Hawkes processes in chapter 1 and the central limit theorem for this partially observed interacting system of Hawkes processes in chapter 2.

0.2 Hawkes processes

In this section, we are going to give a short introduction of Hawkes process.

0.2.1 One dimensional Hawkes process

We consider $\mu > 0$ and $\phi : [0, \infty) \to [0, \infty)$. We always assume that the function ϕ is measurable and locally integrable. We consider $\Pi(dt, dz)$, a Poisson measure on $[0, \infty) \times [0, \infty)$ with intensity $dtdz$.

$$Z_t := \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \lambda_s\}} \Pi(ds, dz), \text{ where } \lambda_t := \mu + \int_0^{t-} \phi(t-s) dZ_s. \tag{0.1}$$

In this thesis, $\int_0^t$ means $\int_{[0,t]}$, and $\int_0^{t-}$ means $\int_{[0,t)}$. The solution $((Z_t)_{t\geq 0})$ is a counting processes. By [14, Proposition 1], the system (1) has a unique $(\mathcal{F}_t)_{t\geq 0}$-measurable càdlàg solution, where

$$\mathcal{F}_t = \sigma(\Pi(A) : A \in \mathcal{B}([0, t] \times [0, \infty))),$$

as soon as ϕ is locally integrable.

Remark 0.2.1. We usually say the function λ_t as rate function and call function ϕ kernel of the process Z_t. We denote by $\{t_i\}_{i\geq 1}$ the sequence of jump times of the counting process Z. Then we have another expression of the rate function:

$$\lambda_t = \mu + \sum_{t_i < t} \phi(t - t_i).$$

From the definition, we have the following martingale with respect to the filtration $\mathcal{F}_t$:

$$M_t := Z_t - \int_0^t \lambda_s ds = \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \lambda_s\}} \tilde{\Pi}(ds, dz),$$

where $\tilde{\Pi}(ds, dz) = \Pi(ds, dz) - dsdz$ is the compensated Poisson measure associated to $\Pi(ds, dz)$. Since Z_t counts the jump of M_t, we have the following equality for the quadratic covariation: $[M]_t = Z_t$. We refer to Jacod-Shiryaev [23, Chapter 1, Section 4e] for definitions and properties of pure jump martingales and of their quadratic variations.

Hawkes process is a simple point process, which has long memory, the clustering effect, the self-exciting property and is in general non-Markovian.

The property of one dimensional linear Hawkes processes have been well studied, see e.g. Chapter 12 of Daley and Vere-Jones in [13] for the introduction of the process, Brémaud and Massoulié in [8] for the analysis of the Bartlett spectrum of the process. In [31], Ogata gives some asymptotic behaviour of maximum likelihood for these processes.

Hawkes processes have a lot of illustrating representations. The most famous one is the following immigration-birth model given by Hawkes in [19]:

Immigration-Birth Representation

We count the number of individuals and denote it as Z_t. Each individual arrives either via immigration or by birth. The immigrations arrive according to a homogeneous Poisson process at rate μ. Then each individual produces children independently from each other. An individual who arrives at time t produces offspring according to an inhomogeneous Poisson process with intensity $\phi(t - s)$.

0.2.2 Two special kernels of one dimensional Hawkes process

Exponential kernel

The Hawkes process with exponential kernels has a lot of advantageous, especially the Markov property as follows:

Proposition 0.2.2. Consider the process (0.1) with exponential kernels $\phi(s) = \alpha e^{-\beta s}$ where $\alpha,\ \beta > 0$. Then the couple (Z_t, λ_t) is a Markov process and we have the following equation:

$$d\lambda_t = -\beta\lambda_t dt + \alpha dZ_t.$$

There is plenty of literature about this kind of Hawkes process, e.g. see [30], [16] and the application in Finance see [2].

In the non-exponential case, the Hawkes process usually cannot have the Markov property anymore. A famous example of a non-exponential kernel is the power-law one.

Power-law kernel

Consider the process (0.1) with power-law kernels $\phi(s) = \frac{\alpha\beta}{(1+\beta s)^\gamma}$ for $\alpha,\ \beta,\ \gamma > 0$. If we add $\gamma > \alpha$, it can ensure the stationarity of the process. The Hawkes with power-law kernel was proposed by Ogata in [32] for describing temporal clusters of seismic activity.

0.2.3 Nonlinear Hawkes Processes

A nonlinear Hawkes Process is a simple point process Z_t, such that:

$$Z_t := \int_0^t \int_0^\infty \mathbf{1}_{\{z \le \lambda_s\}} \Pi(ds, dz), \text{ where } \lambda_t := f\Big(\int_0^{t-} \phi(t - s) dZ_s\Big). \tag{0.2}$$

The Poisson measure $\Pi(ds, dz)$ and function ϕ are defined in (0.1) and $f : [0, \infty) \to [0, \infty)$. The study of nonlinear Hawkes Processes is much rarer than the linear case.

• the simulation see [10, P96-P116]

• the existence and uniqueness of a stationary nonlinear Hawkes process see Brémaud and Massoulié [7],

• a central limit theorem for nonlinear Hawkes processes see Zhu [47],

• a large deviations for Markovian nonlinear Hawkes processes see Zhu [49],

• some approximation of nonlinear Hawkes process see [42] and [43].

More studies of nonlinear Hawkes Processes see Zhu [48].

0.2.4 Multivariate Hawkes Processes

We consider $\phi_{ij} : [0,\infty) \to [0,\infty)$ for $i,j = 1,...,N$. μ_i for $i = 1,...,N$ are constants. We always assume that the function ϕ_{ij} are measurable and locally integrable. For $N \geq 1$, we consider an i.i.d. family $(\Pi^i(dt,dz))_{i=1,...,N}$ of Poisson measures on $[0,\infty)\times[0,\infty)$ with intensity $dtdz$. We consider the following system: for all $i \in \{1,...,N\}$, all $t \geq 0$,

$$Z_t^{i,N} := \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \lambda_s^{i,N}\}} \Pi^i(ds,dz), \text{ where } \lambda_t^{i,N} := \mu_i + \sum_{j=1}^N \int_0^{t-} \phi_{ij}(t-s) dZ_s^{j,N}. \tag{0.3}$$

The solution $((Z_t^{i,N})_{t\geq 0})_{i=1,...,N}$ is a family of counting processes. By [14, Proposition 1], the system (0.3) has a unique $(\mathcal{F}_t)_{t\geq 0}$-measurable càdlàg solution, where

$$\mathcal{F}_t = \sigma(\Pi^i(A) : A \in \mathcal{B}([0,t]\times[0,\infty)), i = 1,...,N),$$

as soon as ϕ is locally integrable. We usually assume that for any $i,j = 1,...,N$, $\int_0^\infty \phi_{ij} < \infty$. We introduce the $N \times N$ matrix $K_N(i,j) = \int_0^\infty \phi_{ij}(s)ds$ and let $\rho(K_N)$ is the spectral radius. Define the vectors $\boldsymbol{Z}_t^N = (Z_t^{1,N},...,Z_t^{N,N})$, $\boldsymbol{\mu} = (\mu_1,...,\mu_N)$. Then we will have the following proposition:

Proposition 0.2.3. ([1], Bacry, Delattre, Hoffmann and Muzy)
Assume $\rho(K_N) < 1$, then we have the following law of large numbers:

$$\sup_{0\leq u\leq 1} \|t^{-1}\boldsymbol{Z}_{ut}^N - u(I-K_N)^{-1}\boldsymbol{\mu}_N\| \to 0$$

as t going to ∞ convergences almost surely and in $L^2(P)$.

If we assume further that for any $i,j = 1,...,N$, $\int_0^\infty \sqrt{s}\phi_{ij}(s)ds < \infty$. Then, we have the following central limit theorem: as $t \to \infty$,

$$\Big(\sqrt{t}\Big(t^{-1}\boldsymbol{Z}_{ut}^N - u(I-K_N)^{-1}\boldsymbol{\mu}_N\Big)\Big)_{0\leq u\leq 1} \xrightarrow{d} \Big((I-K_N)^{-1}\Sigma^{\frac{1}{2}}\boldsymbol{B}_u^N\Big)_{0\leq u\leq 1}$$

where $\boldsymbol{B}_u^N$ is a N dimensional Brownian motion and Σ is the diagonal matrix with $\Sigma_{ii} = ((I-K_N)^{-1}\boldsymbol{\mu}_N)_i$ for $i = 1,...,N$.

And as the same of the case of one dimensional, there exists a unique stationary version of the multivariate Hawkes process satisfies (0.3). In [41], Torrisi gives the rate of convergence to the stationary version. Some studies of Bartlett spectrum of the multivariate Hawkes process can be found in Hawkes [20]. In [18], Hansen, Reynaud-Bouret and Rivoirard give some study of non-asymptotics estimates for multivariate Hawkes processes. The study of mean-field situations for Hawkes processes see e.g. [15], the non-linear case see e.g. [11].

0.2.5 Applications of Hawkes Processes

The Hawkes processes was first introduced as an immigration-birth model by Hawkes in [19]. Since then, there has been a huge literature of the application of the processes. In [32], Ogata

use the Hawkes process to give models for earthquake occurrences. In [6], Bray and Schoenberg review the Hawkes process among other model alternatives for earthquake forecasting. Pratiwi also gives a procedure for modeling earthquake based on these self-exciting point processes in [35] and another example about earthquake see [24].

We can see there are plenty of applications in genomics, for example see [17] by Gusto and [37] by Reynaud-Bouret. In [37], they use the hawkes process to model the process of the occurrences of a particular event along DNA sequence.

Hawkes processes also have a lot of applications in finance. In [21], Hewlett model the occurrence of buy and sell market orders on FX markets using a bivariate exponential Hawkes process. More examples in Finance see e.g. [2].

There is also some applications in neuroscience see e.g. [39] Sarma et al. And in [44], Truccolo uses autoregressive PPGLM models to treat spiking events from neurons as point events in these processes.

Reinhart also gives some applications of these self-exciting spatio-temporal point processes in [36]. Wu et a.l. also use Hawkes Processes to study Sporadic and Bursty Event in [45].

0.3 Statistical inference for Hawkes process

0.3.1 Motivation

Hawkes processes have been used to model interactions between multiple entities evolving through time. For an example in neurosciences, see Reynaud-Bouret et al. [38], where they use multivariate Hawkes processes to model the instantaneous firing rates of different neurons. In [12], Chevallier gives the mean-field of spiking neurons modeled via Hawkes processes. There are some more application examples in neroscience for example see Pakdaman et al. [33], [34]. In finance, Bauwens and Hautsch in [4] give an order book model. And in [28], Lu and Abergel give an order book model described by High-dimensional Hawkes processes with exponential kernels. They study the calibration problem and show a good agreement between the statistical properties of order book data and those of the model. Social networks interactions are considered in Blundell et al. [5], Simma-Jordan [40], Zhou et al. [46]. There are even some applications in criminology, see e.g. Mohler, Short, Brantingham, Schoenberg and Tita in [29]. Concerning the statistical inference for Hawkes processes, mainly the case of fixed finite dimension N has been studied, to our knowledge, in the asymptotic $t \to \infty$.

However, in the real world, we often need to consider the case when the number of individuals is large. For example, in the neurosciences, the number of the neurons are usually enormously large. So it is natural to consider the double asymptotic $t \to \infty$ *and* $N \to \infty$.

0.3.2 The system

We consider some unknown parameters $p \in (0,1]$, $\mu > 0$ and $\phi : [0,\infty) \to [0,\infty)$. We always assume that the function ϕ is measurable and locally integrable. For $N \geq 1$, we consider an i.i.d. family $(\Pi^i(dt,dz))_{i=1,...,N}$ of Poisson measures on $[0,\infty) \times [0,\infty)$ with intensity $dtdz$. And $(\theta_{ij})_{i,j=1,...,N}$ is a family of i.i.d. Bernoulli(p) random variables which is independent of the family $(\Pi^i(dt,dz))_{i=1,...,N}$. We consider the following system: for all $i \in \{1,...,N\}$, all $t \geq 0$,

$$Z_t^{i,N} := \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \lambda_s^{i,N}\}} \Pi^i(ds,dz), \text{ where } \lambda_t^{i,N} := \mu + \frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^{t-} \phi(t-s) dZ_s^{j,N}. \quad (0.4)$$

The solution $((Z_t^{i,N})_{t\geq 0})_{i=1,...,N}$ is a family of counting processes. By [14, Proposition 1], the system (0.4) has a unique $(\mathcal{F}_t)_{t\geq 0}$-measurable càdlàg solution, where

$$\mathcal{F}_t = \sigma(\Pi^i(A) : A \in \mathcal{B}([0,t]\times[0,\infty)), i = 1,...,N) \vee \sigma(\theta_{ij}, i, j = 1,...,N),$$

as soon as ϕ is locally integrable.

Remark 0.3.1. We usually say the function λ_t as rate function. And from the definition, we have the following martingale:

$$M_t^{i,N} := Z_t^{i,N} - \int_0^t \lambda_s^{i,N} ds = \int_0^t \int_0^\infty \mathbf{1}_{\{z\leq \lambda_s^{i,N}\}} \tilde{\Pi}^i(ds, dz),$$

where $\tilde{\Pi}^i(ds, dz) = \Pi^i(ds, dz) - dsdz$ is the compensated Poisson measure associated to $\Pi^i(ds, dz)$. Since the Poisson measures Π^i are independent, the martingales $M_t^{i,N}$ are orthogonal. More precisely, we have $[M^{i,N}, M^{j,N}]_t = 0$ if $i \neq j$ (because $Z_t^{i,N}$ is the number of jumps of $M_t^{i,N}$ and all jumps are size 1).

Let us provide an interpretation the process $((Z_t^{i,N})_{t\geq 0})_{i=1,...,N}$.

0.3.3 An illustrating example

We have N individuals. Each individual $j \in \{1,\dots,N\}$ is connected to the set of individuals $S_j = \{i \in \{1,\dots,N\} : \theta_{ij} = 1\}$. The only possible action of the individual i is to send a message to all the individuals of S_i. Here $Z_t^{i,N}$ stands for the number of messages sent by i during $[0,t]$.

The rate $\lambda_t^{i,N}$ at which i sends messages can be decomposed as the sum of two effects:

• he sends *new* messages at rate μ;

• he *forwards* the messages he received, after some delay (possibly infinite) depending on the age of the message, which induces a sending rate of the form $\frac{1}{N}\sum_{j=1}^N \theta_{ij} \int_0^{t-} \phi(t-s) dZ_s^{j,N}$.

If for example $\phi = \mathbf{1}_{[0,K]}$, then $N^{-1}\sum_{j=1}^N \theta_{ij} \int_0^{t-} \phi(t-s) dZ_s^{j,N}$ is precisely the number of messages that the i-th individual received between time $t-K$ and time t, divided by N.

0.3.4 Main Goals

In [14], Delattre and Fournier consider the case when one observes the whole sample of individuals $(Z_s^{i,N})_{i=1...N, 0\leq s\leq t}$ and they propose some estimator of the unknown parameter p.

However, in the real world, it is often impossible to observe the whole population. Our goal in the present thesis is to consider the case where one observes only a subsample of individuals.

In other words, we want to build some estimators of p when observing $(Z_s^{i,N})_{\{i=1,...,K,\ 0\leq s\leq t\}}$ with $1 \ll K \leq N$ and with t large. And then we establish a central limit theorem for this estimator, which allows to construct an asymptotic confidence interval of the parameter p.

Let $\Lambda = \int_0^\infty \phi(t)dt \in (0,\infty]$. In [14], we see that growth of $Z_t^{1,N}$ depends on the value of Λp. When $\Lambda p < 1$ (subcritical case), $Z_t^{1,N}$ increases (in average) linearly with time, while when $\Lambda p > 1$ (supercritical case), it increases exponentially. Thus the limit theorems will be different in the two cases. We will not consider the critical case when $\Lambda p = 1$.

0.3.5 The main result of the estimator

Assumptions

We will work under one of the two following conditions: either for some $q \geq 1$,

$$\mu \in (0,\infty), \quad \Lambda p \in (0,1) \quad \text{and} \quad \int_0^\infty s^q \phi(s) ds < \infty \tag{$H(q)$}$$

or

$$\mu \in (0,\infty), \quad \Lambda p \in (1,\infty) \quad \text{and} \quad \int_0^t |d\phi(s)| \text{ increases at most polynomially.} \tag{A}$$

In many applications, ϕ is smooth and decays fast. Hence what we have in mind is that in the subcritical case, $(H(q))$ is satisfied for all $q \geq 1$. In the supercritical case, (A) seems very reasonable.

The result in the subcritical case

For $N \geq 1$ and for $((Z_t^{i,N})_{t\geq 0})_{i=1,\dots,N}$ the solution of (0.4), we set $\bar{Z}_t^N := \frac{1}{N}\sum_{i=1}^N Z_t^{i,N}$ and $\bar{Z}_t^{N,K} := \frac{1}{K}\sum_{i=1}^K Z_t^{i,N}$. Next, we introduce

$$\varepsilon_t^{N,K} := t^{-1}(\bar{Z}_{2t}^{N,K} - \bar{Z}_t^{N,K}), \qquad \mathcal{V}_t^{N,K} := \frac{N}{K}\sum_{i=1}^K \Big[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \varepsilon_t^{N,K}\Big]^2 - \frac{N}{t}\varepsilon_t^{N,K}.$$

For $\Delta > 0$ such that $t/(2\Delta) \in \mathbb{N}^*$, we set

$$\mathcal{W}_{\Delta,t}^{N,K} := 2\mathcal{Z}_{2\Delta,t}^{N,K} - \mathcal{Z}_{\Delta,t}^{N,K}, \quad \mathcal{X}_{\Delta,t}^{N,K} := \mathcal{W}_{\Delta,t}^{N,K} - \frac{N-K}{K}\varepsilon_t^{N,K} \tag{0.5}$$

$$\text{where} \quad \mathcal{Z}_{\Delta,t}^{N,K} := \frac{N}{t}\sum_{a=t/\Delta}^{2t/\Delta} (\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \Delta\varepsilon_t^{N,K})^2. \tag{0.6}$$

Theorem 0.3.2. *We assume $(H(q))$ for some $q > 3$. There is a constant C depending only on q, p, μ, ϕ such that for all $\varepsilon \in (0,1)$, all $1 \leq K \leq N$, if setting $\Delta_t = t/(2\lfloor t^{1-4/(q+1)}\rfloor)$ for all $t \geq 1$,*

$$P\Big(\Big|\Psi\Big(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K}\Big) - (\mu,\Lambda,p)\Big| \geq \varepsilon\Big) \leq \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{K}} + \frac{N}{K\sqrt{t^{1-\frac{4}{1+q}}}} + \frac{N}{t\sqrt{K}}\Big) + CNe^{-C'K}$$

with $\Psi := \mathbf{1}_D\Phi : \mathbb{R}^3 \mapsto \mathbb{R}^3$, the function $\Phi := (\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)})$ being defined on $D := \{(u,v,w) \in \mathbb{R}^3 : w > u > 0$ and $v \geq 0\}$ by

$$\Phi^{(1)}(u,v,w) := u\sqrt{\frac{u}{w}}, \quad \Phi^{(2)}(u,v,w) := \frac{v + [u - \Phi^{(1)}(u,v,w)]^2}{u[u - \Phi^{(1)}(u,v,w)]},$$
$$\Phi^{(3)}(u,v,w) := \frac{1 - u^{-1}\Phi^{(1)}(u,v,w)}{\Phi^{(2)}(u,v,w)}.$$

We quote [14, Remark 2], which says that the mean number of actions per individual per unit of time increases linearly.

Remark 0.3.3. Assume $H(1)$. Then for all $\varepsilon > 0$,

$$\lim_{(N,t)\to(\infty,\infty)} P\Big(\Big|\frac{\bar{Z}_t^{N,K}}{t} - \frac{\mu}{1-\Lambda p}\Big| \geq \varepsilon\Big) = 0.$$

So roughly, if observing $((Z_s^{i,N})_{s\in[0,t]})_{i=1,\dots,K}$, we observe approximately Kt actions.

Remark 0.3.4. If the function ϕ decays fast, for example $\phi(s) = ae^{-bs}$ or $c\mathbf{1}_D$ where D is some compact set. In these situations, the function ϕ can satisfy the assumptions for arbitrary $q > 0$. Hence, we can say $\frac{N}{K\sqrt{t}}$ is almost equivalent to $\frac{N}{K\sqrt{t^{1-\frac{4}{1+q}}}}$.

Remark 0.3.5. We are going to consider two special cases:

- When $K \sim N$, we have

$$(\frac{1}{\sqrt{K}} + \frac{N}{K\sqrt{t^{1-\frac{4}{1+q}}}} + \frac{N}{t\sqrt{K}}) + CNe^{-C'K} \sim (\frac{1}{\sqrt{N}} + \frac{1}{\sqrt{t^{1-\frac{4}{1+q}}}} + \frac{\sqrt{N}}{t}) + CNe^{-C'N}.$$

Hence, in order to ensure the convergence, we just need $\frac{\sqrt{N}}{t} \to 0$.

- Assume $K \sim \gamma \log N$ and $\gamma C' > 1$, where C' is as in theorem 0.3.2, we have

$$(\frac{1}{\sqrt{K}} + \frac{N}{K\sqrt{t^{1-\frac{4}{1+q}}}} + \frac{N}{t\sqrt{K}}) + CNe^{-C'K} \sim (\frac{1}{\sqrt{\log N}} + \frac{N}{\log N\sqrt{t^{1-\frac{4}{1+q}}}} + \frac{N}{t\sqrt{\log N}}) + CN^{1-\gamma C'}.$$

Hence, in order to ensure the convergence, we just need $\frac{N}{\log N\sqrt{t^{1-\frac{4}{1+q}}}} + \frac{N}{t\sqrt{\log N}} \to 0$, which equivalent to $\frac{N}{\log N\sqrt{t^{1-\frac{4}{1+q}}}} \to 0$.

The result in the supercritical case

Here we define $\bar{Z}_t^{N,K}$ as previously and we set

$$\mathcal{U}_t^{N,K} := \Big[\frac{N}{K}\sum_{i=1}^{K}\Big(\frac{Z_t^{i,N} - \bar{Z}_t^{N,K}}{\bar{Z}_t^{N,K}}\Big)^2 - \frac{N}{\bar{Z}_t^{N,K}}\Big]\mathbf{1}_{\{\bar{Z}_t^{N,K}>0\}} \tag{0.7}$$

$$\text{and} \quad \mathcal{P}_t^{N,K} := \frac{1}{\mathcal{U}_t^{N,K}+1}\mathbf{1}_{\{\mathcal{U}_t^{N,K}\geq 0\}}. \tag{0.8}$$

Theorem 0.3.6. *We assume (A) and define α_0 by $p\int_0^\infty e^{-\alpha_0 t}\phi(t)dt = 1$ (recall that by (A), $\Lambda p = p\int_0^\infty \phi(t)dt > 1$). For all $\eta > 0$, there is a constant $C_\eta > 0$ (depending on p, μ, ϕ, η), such that for all $N \geq K \geq 1$, all $\varepsilon \in (0,1)$,*

$$P(|\mathcal{P}_t^{N,K} - p| \geq \varepsilon) \leq \frac{C_\eta e^{4\eta t}}{\varepsilon}\Big(\frac{N}{\sqrt{K}e^{\alpha_0 t}} + \frac{1}{\sqrt{K}}\Big).$$

Next, we quote [14, Remark 5].

Remark 0.3.7. Assume (A) and consider $\alpha_0 > 0$ such that $p\int_0^\infty e^{-\alpha_0 t}\phi(t)dt = 1$. Then for all $\eta > 0$,

$$\lim_{t\to\infty}\lim_{(N,K)\to(\infty,\infty)} P(\bar{Z}_t^{N,K} \in [e^{(\alpha_0-\eta)t}, e^{(\alpha_0+\eta)t}]) = 1.$$

So roughly, if observing $((Z_s^{i,N})_{s\in[0,t]})_{i=1,\dots,K}$, we observe around $Ke^{\alpha_0 t}$ actions.

0.3.6 On the choice of the estimators

In the whole aper, we denote by $\mathbb{E}_\theta$ the conditional expectation knowing $(\theta_{ij})_{i,j=1,\dots,N}$. Here we explain informally why the estimators should converge.

The subcritical case

We define $A_N(i,j) := N^{-1}\theta_{ij}$ and the matrix $(A_N(i,j))_{i,j\in\{1,\dots,N\}}$, as well as $Q_N := (I - \Lambda A_N)^{-1}$ on the event on which $I - \Lambda A_N$ is invertible.

Define $\widetilde{\varepsilon}_t^{N,K} := t^{-1}\bar{Z}_t^{N,K}$, $K \leq N$. We expect that, for t large enough, $Z_t^{i,N} \simeq \mathbb{E}_\theta[Z_t^{i,N}]$. And, by definition of $Z_t^{i,N}$, see (2.1), it is not hard to get

$$\mathbb{E}_\theta[Z_t^{i,N}] = \mu t + N^{-1}\sum_{j=1}^N \theta_{ij}\int_0^t \phi(t-s)\mathbb{E}_\theta[Z_s^{j,N}]ds.$$

Hence, assuming that $\gamma_N(i) = \lim_{t\to\infty} t^{-1}\mathbb{E}_\theta[Z_t^{i,N}]$ exists for each $i = 1, ..., N$ and observing that $\int_0^t \phi(t-s)sds \simeq \Lambda t$, we find that the vector $\boldsymbol{\gamma}_N = (\gamma_N(i))_{i=1,...,N}$ should satisfy $\boldsymbol{\gamma}_N = \mu\mathbf{1}_N + \Lambda A_N\boldsymbol{\gamma}_N$, where $\mathbf{1}_N$ is the vector defined by $\mathbf{1}_N(i) = 1$ for all $i = 1, \ldots, N$. Thus we deduce that $\boldsymbol{\gamma}_N = \mu(I - \Lambda A_N)^{-1}\mathbf{1}_N = \mu\boldsymbol{\ell}_N$, where we have set

$$\boldsymbol{\ell}_N := Q_N\mathbf{1}_N,\ \ell_N(i) := \sum_{j=1}^N Q_N(i,j),\ \bar{\ell}_N := \frac{1}{N}\sum_{i=1}^N \ell_N(i),\ \bar{\ell}_N^K := \frac{1}{K}\sum_{i=1}^K \ell_N(i)$$

So we expect that $Z_t^{i,N} \simeq \mathbb{E}_\theta[Z_t^{i.N}] \simeq \mu\ell_N(i)t$, whence $\widetilde{\varepsilon}_t^{N,K} = t^{-1}\bar{Z}_t^{N,K} \simeq \mu\bar{\ell}_N^K$.

We informally show that $\ell_N(i) \simeq 1 + \Lambda(1-\Lambda p)^{-1}L_N(i)$, where $L_N(i) := \sum_{j=1}^N A_N(i,j)$: when N is large, $\sum_{j=1}^N A_N^2(i,j) = N^{-2}\sum_{j=1}^N\sum_{k=1}^N \theta_{ik}\theta_{kj} \simeq pN^{-1}\sum_{k=1}^N \theta_{ik} = pL_N(i)$. And one gets convinced similarly that for any $n \in \mathbb{N}_*$, roughly, $\sum_{j=1}^N A_N^n(i,j) \simeq p^{n-1}L_N(i)$. So

$$\ell_N(i) = \sum_{n\geq 0}\Lambda^n\sum_{j=1}^N A_N^n(i,j) \simeq 1 + \sum_{n\geq 1}\Lambda^n p^{n-1}L_N(i) = 1 + \frac{\Lambda}{1-\Lambda p}L_N(i).$$

But $(NL_N(i))_{i=1,...,N}$ are i.i.d. Binomial(N,p) random variables, so that $\bar{\ell}_N^K \simeq 1 + \Lambda p(1-\Lambda p)^{-1} = (1-\Lambda p)^{-1}$. Finally, we have explained why $\widetilde{\varepsilon}_t^{N,K}$ should resemble $\mu(1-\Lambda p)^{-1}$.

Knowing $(\theta_{ij})_{i,j=1..N}$, the process $Z_t^{1,N}$ resembles a Poisson process, so that $\mathbb{V}\mathrm{ar}_\theta(Z_t^{1,N}) \simeq \mathbb{E}_\theta[Z_t^{1,N}]$, whence

$$\mathbb{V}\mathrm{ar}(Z_t^{1,N}) = \mathbb{V}\mathrm{ar}(\mathbb{E}_\theta[Z_t^{1,N}]) + \mathbb{E}[\mathbb{V}\mathrm{ar}_\theta(Z_t^{1,N})] \simeq \mathbb{V}\mathrm{ar}(\mathbb{E}_\theta[Z_t^{1,N}]) + \mathbb{E}[Z_t^{1,N}].$$

Writing an empirical version of this equality, we find

$$\frac{1}{K}\sum_{i=1}^K (Z_t^{i,N} - \bar{Z}_t^{N,K})^2 \simeq \frac{1}{K}\sum_{i=1}^K \Big(\mathbb{E}_\theta[Z_t^{i,N}] - \mathbb{E}_\theta[\bar{Z}_t^{N,K}]\Big)^2 + \bar{Z}_t^{N,K}.$$

And since $Z_t^{i,N} \simeq \mu\ell_N(i)t \simeq \mu[1 + (1-\Lambda p)^{-1}\Lambda L_N(i)]t$ as already seen a few lines above, we find

$$\frac{1}{K}\sum_{i=1}^K (Z_t^{i,N} - \bar{Z}_t^{N,K})^2 \simeq \frac{\mu^2t^2\Lambda^2}{K(1-\Lambda p)^2}\sum_{i=1}^K (L_N(i) - \bar{L}_N^K)^2 + \bar{Z}_t^{N,K}.$$

But $(NL_N(i))_{i=1,...,N}$ are i.i.d. Bernoulli(N,p) random variables, so that

$$\begin{aligned}
\widetilde{\mathcal{V}}_t^{N,K} :=& \frac{N}{K}\sum_{i=1}^K \Big[\frac{Z_t^{i,N}}{t} - \widetilde{\varepsilon}_t^{N,K}\Big]^2 - \frac{N}{t}\widetilde{\varepsilon}_t^{N,K} \\
=& \frac{N}{Kt^2}\Big[\sum_{i=1}^K (Z_t^{i,N} - \bar{Z}_t^{N,K})^2 - K\bar{Z}_t^{N,K}\Big] \\
\simeq& \frac{N\mu^2\Lambda^2}{K(1-\Lambda p)^2}\sum_{i=1}^K (L_N(i) - \bar{L}_N^K)^2 \simeq \frac{\mu^2\Lambda^2 p(1-p)}{(1-\Lambda p)^2}.
\end{aligned}$$

We finally build a third estimator. The temporal empirical variance

$$\frac{\Delta}{t}\sum_{k=1}^{t/\Delta}\Big[\bar{Z}^{N,K}_{k\Delta}-\bar{Z}^{N,K}_{(k-1)\Delta}-\frac{\Delta}{t}\bar{Z}^{N,K}_t\Big]^2$$

should resemble $\mathbb{V}\mathrm{ar}_\theta[\bar{Z}^{N,K}_\Delta]$ if $1\ll\Delta\ll t$. So we expect that:

$$\widetilde{\mathcal{W}}^{N,K}_{\Delta,t}:=\frac{N}{t}\sum_{k=1}^{t/\Delta}\Big[\bar{Z}^{N,K}_{k\Delta}-\bar{Z}^{N,K}_{(k-1)\Delta}-\Delta t^{-1}\bar{Z}^{N,K}_t\Big]^2\simeq\frac{N}{\Delta}\mathbb{V}\mathrm{ar}_\theta[\bar{Z}^{N,K}_\Delta].$$

To understand what $\mathbb{V}\mathrm{ar}_\theta[\bar{Z}^{N,K}_\Delta]$ looks like, we introduce the centered process $U^{i,N}_t:=Z^{i,N}_t-\mathbb{E}_\theta[Z^{i,N}_t]$ and the martingale $M^{i,N}_t:=Z^{i,N}_t-C^{i,N}_t$ where $C^{i,N}$ is the compensator of $Z^{i,N}$. An easy computation, see [14, Lemma 11], shows that, denoting by $\boldsymbol{U}^N_t$ and $\boldsymbol{M}^N_t$ the vectors $(U^{i,N}_t)_{i=1,...,N}$ and $(M^{i,N}_t)_{i=1,...,N}$,

$$\boldsymbol{U}^N_t=\boldsymbol{M}^N_t+A_N\int_0^t\phi(t-s)\boldsymbol{U}^N_s ds.$$

So for large times, we conclude that $\boldsymbol{U}^N_t\simeq\boldsymbol{M}^N_t+\Lambda A_N\boldsymbol{U}^N_t$, whence finally $\boldsymbol{U}^N_t\simeq Q\boldsymbol{M}^N_t$ and thus

$$\frac{1}{K}\sum_{i=1}^K U^{i,N}_t\simeq\frac{1}{K}\sum_{i=1}^K\sum_{j=1}^N Q(i,j)M^{j,N}_t=\frac{1}{K}\sum_{j=1}^N c^K_N(j)M^{j,N}_t,$$

where we have set $c^K_N(j)=\sum_{i=1}^K Q_N(i,j)$. But we obviously have $[M^{j,N},M^{i,N}]_t=\mathbf{1}_{\{i=j\}}Z^{j,N}_t$ (see [14, Remark 10]), so that

$$\mathbb{V}\mathrm{ar}_\theta[\bar{Z}^{N,K}_t]=\mathbb{V}\mathrm{ar}_\theta[\bar{U}^{N,K}_t]\simeq\frac{1}{K^2}\sum_{j=1}^N(c^K_N(j))^2Z^{j,N}_t.$$

Recalling that $Z^{j,N}_t\simeq\mu\ell_N(j)t$, we conclude that

$$\mathbb{V}\mathrm{ar}_\theta[\bar{Z}^{N,K}_t]\simeq K^{-2}\mu t\sum_{j=1}^N\Big(c^K_N(j)\Big)^2\ell_N(j),$$

whence

$$\widetilde{\mathcal{W}}^{N,K}_{\Delta,t}\simeq\frac{N}{\Delta}\mathbb{V}\mathrm{ar}_\theta[\bar{Z}^{N,K}_\Delta]\simeq\mu\frac{N}{K^2}\sum_{j=1}^N\Big(c^K_N(j)\Big)^2\ell_N(j).$$

To compute this last quantity, we start from $c^K_N(j)=\sum_{n\geq0}\sum_{i=1}^K\Lambda^nA^n_N(i,j)$. But we have $\sum_{i=1}^K A^2_N(i,j)=N^{-2}\sum_{i=1}^K\sum_{k=1}^N\theta_{ik}\theta_{kj}\simeq pKN^{-2}\sum_{k=1}^N\theta_{kj}=pKN^{-1}C_N(j)$. And one gets convinced similarly that for any $n\in\mathbb{N}_*$, roughly, $\sum_{i=1}^K A^n_N(i,j)\simeq KN^{-1}p^{n-1}C_N(j)$. So we conclude that $c^K_N(j)\simeq A^0_N(i,j)+\frac{K\Lambda}{N(1-\Lambda p)}C_N(j)$. Consequently, $c^K_N(j)\simeq1+\frac{K}{N}\frac{\Lambda p}{(1-\Lambda p)}$ for $j\in\{1,...,K\}$ and $c^K_N(j)\simeq\frac{K}{N}\frac{\Lambda p}{(1-\Lambda p)}$ for $j\in\{K+1,...,N\}$. We finally get, recalling that $\ell_N(j)\simeq(1-\Lambda p)^{-1}$,

$$\begin{aligned}\widetilde{\mathcal{W}}^{N,K}_{\Delta,t}\simeq&\mu\frac{N}{K^2}\sum_{j=1}^N\Big(c^K_N(j)\Big)^2\ell_N(j)\\\simeq&\mu\frac{N}{K^2}\Big(\frac{K}{1-\Lambda p}\Big[1+\frac{K\Lambda p}{N(1-\Lambda p)}\Big]^2+\frac{N-K}{1-\Lambda p}\Big[\frac{K\Lambda p}{N(1-\Lambda p)}\Big]^2\Big)\\\simeq&\frac{\mu}{(1-\Lambda p)^3}+\frac{(N-K)\mu}{K(1-\Lambda p)}.\end{aligned}$$

All in all, we should have $\widetilde{\mathcal{X}}^{N,K}_{\Delta,t} \simeq \frac{\mu}{(1-\Lambda p)^3}$.

It readily follows that $\Psi(\varepsilon^{N,K}_t, \mathcal{V}^{N,K}_t, \mathcal{X}^{N,K}_{\Delta,t})$ should resemble (μ, Λ, p).

The three estimators $\varepsilon^{N,K}_t, \mathcal{V}^{N,K}_t, \mathcal{X}^{N,K}_{\Delta,t}$ are very similar to $\widetilde{\varepsilon}^{N,K}_t$, $\widetilde{\mathcal{V}}^{N,K}_t$, $\widetilde{\mathcal{X}}^{N,K}_{\Delta,t}$ and should converge to the same limits. Let us explain why we have introduced $\varepsilon^{N,K}_t, \mathcal{V}^{N,K}_t, \mathcal{X}^{N,K}_{\Delta,t}$, of which the expressions are more complicated. The main idea is that, see [14, Lemma 16 (ii)], $\mathbb{E}[Z^{i,N}_t] = \mu\ell_N(i)t + \chi^N_i \pm t^{1-q}$ (under $(H(q))$), for some finite random variable χ^N_i. As a consequence, $t^{-1}\mathbb{E}[Z^{i,N}_{2t} - Z^{i,N}_t]$ converges to $\mu\ell_N(i)$ considerably much faster, if q is large, than $t^{-1}\mathbb{E}_\theta[Z^{i,N}_t]$ (for which the error is of order t^{-1}).

The supercritical case

We now turn to the supercritical case where $\Lambda p > 1$. We introduce the $N \times N$ matrix $A_N(i,j) = N^{-1}\theta_{ij}$.

We expect that $Z^{i,N}_t \simeq H_N \mathbb{E}_\theta[Z^{i,N}_t]$, when t is large, for some random $H_N > 0$ not depending on i. Since $\Lambda p > 1$, the process should increase like an exponential function, i.e. there should be $\alpha_N > 0$ such that for all $i = 1, \ldots, N$, $\mathbb{E}_\theta[Z^{i,N}_t] \simeq \gamma_N(i)e^{\alpha_N t}$ for t very large, where $\gamma_N(i)$ is some positive random constant. We recall that $\mathbb{E}_\theta[Z^{i,N}_t] = \mu t + N^{-1}\sum_{j=1}^N \theta_{ij}\int_0^t \phi(t-s)\mathbb{E}_\theta[Z^{j,N}_s]ds$. We insert $\mathbb{E}_\theta[Z^{i,N}_t] \simeq \gamma_N(i)e^{\alpha_N t}$ in this equation and let t go to infinite: we informally get $\boldsymbol{\gamma}_N = A_N\boldsymbol{\gamma}_N \int_0^\infty e^{-\alpha_N s}\phi(s)ds$. In other words, $\boldsymbol{\gamma}_N = (\gamma_N(i))_{i=1,\ldots,N}$ is an eigenvector of A_N for the eigenvalue $\rho_N := (\int_0^\infty e^{-\alpha_N s}\phi(s)ds)^{-1}$.

But A_N has nonnegative entries. Hence by the Perron-Frobenius theorem, it has a unique (up to normalization) eigenvector $\boldsymbol{V}_N$ with nonnegative entries (say, such that $\|\boldsymbol{V}_N\|_2 = \sqrt{N}$), and this vector corresponds to the maximum eigenvalue ρ_N of A_N. So there is a (random) constant κ_N such that $\boldsymbol{\gamma}_N \simeq \kappa_N \boldsymbol{V}_N$ and, furthermore, $(\int_0^\infty e^{-\alpha_N s}\phi(s)ds)^{-1} \simeq \rho_N$. All in all, we find that $Z^{i,N}_t \simeq \kappa_N H_N e^{\alpha_N t}\boldsymbol{V}_N(i)$. We define $\boldsymbol{V}^K_N = I_K\boldsymbol{V}_N$, where I_K is the $N \times N$-matrix defined by $I_K(i,j) = \mathbf{1}_{\{i=j\le K\}}$.

As in the subcritical case, the variance $K^{-1}\sum_{i=1}^K (Z^{i,N}_t - \bar{Z}^{N,K}_t)^2$ should look like

$$\frac{1}{K}\sum_{i=1}^K (\mathbb{E}_\theta[Z^{i,N}_t] - \mathbb{E}_\theta[\bar{Z}^{N,K}_t])^2 + \bar{Z}^{N,K}_t \simeq \frac{\kappa_N^2 H_N^2 e^{2\alpha_N t}}{K}\sum_{i=1}^K (V_N(i) - \bar{V}^K_N)^2 + \bar{Z}^{N,K}_t,$$

where as usual $\bar{V}^K_N := K^{-1}\sum_{i=1}^K V_N(i)$. We also get $\bar{Z}^{N,K}_t \simeq \kappa_N H_N \bar{V}^K_N e^{\alpha_N t}$. Finally,

$$\mathcal{U}^{N,K}_t = \frac{N}{K(\bar{Z}^{N,K}_t)^2}[\sum_{i=1}^K (Z^{i,N}_t - \bar{Z}^{N,K}_t)^2 - K\bar{Z}^{N,K}_t]\mathbf{1}_{\{\bar{Z}^{N,K}_t > 0\}} \simeq \frac{N}{K(\bar{V}^K_N)^2}\sum_{i=1}^K (V_N(i) - \bar{V}^K_N)^2.$$

Next, we consider the term $(\bar{V}^K_N)^{-2}\sum_{i=1}^K (V_N(i) - \bar{V}^K_N)^2$. By a rough estimation, $A^2_N(i,j) \simeq \frac{p^2}{N}$. Because $I_K A^2_N \boldsymbol{V}_N = \rho^2_N \boldsymbol{V}^K_N$, we have $\rho^2_N \boldsymbol{V}^K_N \simeq p^2 \bar{V}_N \mathbf{1}_K$, where $\mathbf{1}_K$ is the N dimensional vector of which the first K elements are 1 and others are 0. By the same reason, we have $\rho^2_N \boldsymbol{V}_N \simeq p^2 \bar{V}_N \mathbf{1}_N$. So $\boldsymbol{V}^K_N = I_K A_N \boldsymbol{V}_N/\rho_N \simeq k_N I_K A_N \mathbf{1}_N$, where $k_N = (p^2/\rho^3_N)\bar{V}_N$. In other words, the vector $(k_N)^{-1}\boldsymbol{V}^K_N$ is almost like the vector $\boldsymbol{L}^K_N = I_K A_N \mathbf{1}_N$. Finally, we expect that

$$\mathcal{U}^{N,K}_t \simeq \frac{N}{K}(\bar{V}^K_N)^{-2}\sum_{i=1}^K (V_N(i) - \bar{V}^K_N)^2 \simeq \frac{N}{K}(\bar{L}^K_N)^{-2}\sum_{i=1}^K (L_N(i) - \bar{L}^K_N)^2 \simeq p^{-2}p(1-p) = \frac{1}{p} - 1,$$

whence $\mathcal{P}^{N,K}_t \simeq p$.

0.3.7 Optimal rates in some toy models

The goal of this subsection is to verify, using some toy models, that the rates of convergence of our estimators, see Theorems 0.3.2 and 0.3.6, are not far from being optimal.

The first example

Consider $\alpha_0 \geq 0$ and two unknown parameters $\Gamma > 0$ and $p \in (0,1]$. Consider an i.i.d. family $(\theta_{ij})_{i,j=1\ldots N}$ of Bernoulli(p)-distributed random variables, where $N \geq 1$. We set $\lambda_t^{i,N} = N^{-1}\Gamma e^{\alpha_0 t}\sum_{j=1}^N \theta_{ij}$ and we introduce the processes $(Z_t^{1,N})_{t\geq 0}$, ..., $(Z_t^{N,N})_{t\geq 0}$ which are, conditionally on (θ_{ij}), independent inhomogeneous Poisson process with intensities $(\lambda_t^{1,N})_{t\geq 0}$, ..., $(\lambda_t^{N,N})_{t\geq 0}$. We only observe $(Z_s^{i,N})_{s\in[0,t],\, i=1,\ldots K}$, where $K \leq N$ and we want to estimate the parameter p in the asymptotic $(K,N,t) \to (\infty,\infty,\infty)$. This model is a simplified version of the one studied in our thesis. And roughly speaking, the mean number of jumps per individuals until time t resembles $m_t = \int_0^t e^{\alpha_0 s} ds$. When $\alpha_0 = 0$, this mimics the subcritical case, while when $\alpha_0 > 0$, this mimics the supercritical case. Remark that $(Z_t^{i,N})_{i=1,\ldots K}$ is a sufficient statistic, since α_0 is known.

We use the central limit theorem in order to perform a Gaussian approximation of $Z_t^{i,N}$. It is easy to show that:

$$\lambda_t^{i,N} = \Gamma e^{\alpha_0 t}\Big[\frac{1}{\sqrt{N}}\sqrt{p(1-p)}\frac{1}{\sqrt{Np(1-p)}}\sum_{j=1}^N(\theta_{ij}-p)+p\Big]$$

and $\frac{1}{\sqrt{Np(1-p)}}\sum_{j=1}^N(\theta_{ij}-p)$ converges in law to a Gaussian random variable $G_i \sim \mathcal{N}(0,1)$, where G_i is an i.i.d Gaussian family, as $N \to \infty$, for each i. Thus

$$\lambda_t^{i,N} \simeq \Gamma e^{\alpha_0 t}[\sqrt{N^{-1}p(1-p)}G_i + p].$$

Moreover, conditionally on $(\theta_{ij})_{i,j=1,\ldots,N}$, $Z_t^{i,N}$ is a Poisson random variable with mean $\int_0^t \lambda_s^{i,N} ds$. Thus, as t is large, we have $Z_t^{i,N} \simeq \int_0^t \lambda_s^{i,N} ds + \sqrt{\int_0^t \lambda_s^{i,N} ds} H_i$ where $(H_i)_{i=1,\ldots,N}$ is a family of $\mathcal{N}(0,1)$-distributed random variables, independent of $(G_i)_{i=1,\ldots,N}$. Since $(m_t)^{-1}N^{-1/2} \ll (m_t)^{-1}$, we obtain $(m_t)^{-1}Z_t^{i,N} \simeq \Gamma p + \Gamma\sqrt{N^{-1}p(1-p)}G_i + \sqrt{(m_t)^{-1}\Gamma p}H_i$, of which the law is nothing but $\mathcal{N}(\Gamma p, N^{-1}\Gamma^2 p(1-p) + (m_t)^{-1}\Gamma p)$.

By the above discussion, we construct the following toy model: one observes $(X_t^{i,N})_{i=1,\ldots K}$, where $(X_t^{i,N})_{i=1,\ldots N}$ are i.i.d and $\mathcal{N}(\Gamma p, N^{-1}\Gamma^2 p(1-p) + (m_t)^{-1}\Gamma p)$-distributed. Moreover we assume that Γp is known. So we can use the well-known statistic result: the empirical variance $S_t^{N,K} = K^{-1}\sum_{i=1}^K (X_t^{i,N} - \Gamma p)^2$ is the best estimator of $N^{-1}\Gamma^2 p(1-p) + (m_t)^{-1}\Gamma p$ (in any reasonnable sense). So $T_t^{N,K} = N(\Gamma p)^{-2}(S_t^{N,K} - (\Gamma p)/m_t)$ is the best estimator of $(\frac{1}{p}-1)$. As

$$\mathrm{Var}(S_t^{N,K}) = \frac{1}{K}\mathrm{Var}[(X_t^{1,N}-\Gamma p)^2] = \frac{2}{K}\Big(\frac{\Gamma^2 p(1-p)}{N} + \frac{\Gamma p}{m_t}\Big)^2,$$

we have

$$\mathrm{Var}(T_t^{N,K}) = \frac{2}{(\Gamma p)^4}\Big(\frac{\Gamma^2 p(1-p)}{\sqrt{K}} + \frac{N\Gamma p}{m_t\sqrt{K}}\Big)^2.$$

In other words, we cannot estimate $\Big(\frac{1}{p}-1\Big)$ with a precision better than $\Big(\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}}\Big)$, which implies that we cannot estimate p with a precision better than $\Big(\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}}\Big)$.

The second example

In the second part of this section, we are going to explain why there is a term $\frac{N}{K\sqrt{t^{1-\frac{4}{1+q}}}}$ in the subcritical case.

We consider discrete times $t = 1, ..., T$ and two unknown parameters $\mu > 0$ and $p \in (0, 1]$. Consider an i.i.d. family $(\theta_{ij})_{i,j=1...N}$ of Bernoulli(p)-distributed random variables, where $N \geq 1$. We set $Z_0^{i,N} = 0$ for all $i = 1, \dots, N$ and assume that, conditionally on $(\theta_{ij})_{i,j=1,...N}$ and $(Z_s^{j,N})_{s=0,...,t,j=1...,N}$, the random variables $(Z_{t+1}^{i,N} - Z_t^{i,N})$ (for $i = 1, \dots, N$) are independent and $\mathcal{P}(\lambda_t^{i,N})$-distributed, where $\lambda_t^{i,N} = \mu + \frac{1}{N}\sum_{j=1}^N \theta_{ij}(Z_t^{j,N} - Z_{t-1}^{j,N})$. This process $(Z_t^{i,N})_{i=1,...,N,t=0,...T}$ resembles the system of Hawkes processes studied in the present thesis.

By [1, theorem 2], we have when time t is large, the process $\boldsymbol{Z}_t^N$ is similar to a d-dimensional diffusion process $(I - A_N)^{-1}\Sigma^{\frac{1}{2}}\boldsymbol{B}_t + \mathbb{E}_\theta[\boldsymbol{Z}_t^N]$, where $\boldsymbol{B}_t$ is a N-dimensional Brownian Motion and Σ is the diagonal matrix such that $\Sigma_{ii} = ((I - A_N)^{-1}\mu)_i$. Hence $(Z_{t+1}^{i,N} - Z_t^{i,N}) - \mathbb{E}_\theta[Z_t^{i,N} - Z_{t-1}^{i,N}]$ (for $i = 1, \dots, N$ and $t = 1, ..., T$) are independent. Since $\mathbb{E}_\theta[\boldsymbol{Z}_t^N]$ is similar to $\frac{\mu t}{1-p}$ when both N and t are large. Hence $\lambda_t^{i,N} \simeq \mathbb{E}_\theta[\lambda_t^{i,N}] \simeq \frac{\mu}{1-p}$. Then by Gaussian approximation, we can roughly replace $(Z_t^{j,N} - Z_{t-1}^{j,N})_{j=1,...,N}$ in the expression of $(\lambda_t^{i,N})_{i=1,...,N}$ by $(\frac{\mu}{1-p} + Y_t^{j,N})_{j=1,...,N}$, for an i.i.d. array $(Y_t^{j,N})_{j=1,...,N,t=1,...,T}$ of $\mathcal{N}(0, \frac{\mu}{1-p})$-distributed random variables. Also, we replace the $\mathcal{P}(\lambda_t^{i,N})$ law by its Gaussian approximation.

We thus introduce the following model, with unknown parameters $\mu > 0$ and $p \in (0, 1)$. We start with three independent families of i.i.d. random variables, namely $(\theta_{ij})_{i,j=1,...,N}$ with law Bernoulli(p), and $(Y_t^{j,N})_{j=1,...,N,t=1,...,T}$ with law $\mathcal{N}(0, \frac{\mu}{1-p})$ and $(A_t^{j,N})_{j=1,...,N,t=1,...,T}$ with law $\mathcal{N}(0, 1)$. We then set, for each $t = 1, \dots, T$ and each $i = 1, \dots, N$,

$$a_t^{i,N} = \mu + \frac{1}{N}\sum_{j=1}^N \theta_{ij}\Big(\frac{\mu}{1-p} + Y_t^{j,N}\Big) \quad \text{and} \quad X_t^{i,N} = a_t^{i,N} + \sqrt{a_t^{i,N}}A_t^{i,N}.$$

We compute the covariances. First, for all $i = 1, \dots, N$ and all $t = 1, \dots, T$,

$$\begin{aligned}
\mathrm{Var}(X_t^{i,N}) &= \mathbb{E}[(a_t^{i,N} + \sqrt{a_t^{i,N}}A_t^{i,N} - \frac{\mu}{1-p})^2] \\
&= \mathbb{E}\Big[\Big(\frac{\mu}{N(1-p)}\sum_{k=1}^N(\theta_{ik} - p) + \frac{1}{N}\sum_{k=1}^N \theta_{ik}Y_t^{k,N} + \sqrt{a_t^{i,N}}A_t^{i,N}\Big)^2\Big] \\
&= \frac{p\mu^2}{N(1-p)} + \frac{p\mu^2}{N(1-p)^2} + \frac{\mu}{(1-p)}.
\end{aligned}$$

Next, for $i \neq j$ and all $t = 1, \dots, T$,

$$\begin{aligned}
\mathrm{Cov}(X_t^{i,N}, X_t^{j,N}) &= \mathbb{E}\Big[\Big(a_t^{i,N} + \sqrt{a_t^{i,N}}A_t^{i,N} - \frac{\mu}{(1-p)}\Big)\Big(a_t^{j,N} + \sqrt{a_t^{j,N}}A_t^{j,N} - \frac{\mu}{(1-p)}\Big)\Big] \\
&= \mathbb{E}\Big[\frac{1}{N^2}\sum_{k=1}^N \theta_{jk}\theta_{ik}(Y_t^{k,N})^2\Big] = \frac{p^2}{N}\frac{\mu^2}{(1-p)^2}.
\end{aligned}$$

For $s \neq t$ and $i = 1, \dots, N$,

$$\begin{aligned}
\mathrm{Cov}(X_t^{i,N}, X_s^{i,N}) =& \mathbb{E}\Big[\Big(a_t^{i,N} + \sqrt{a_t^{i,N}}A_t^{i,N} - \frac{\mu}{(1-p)}\Big)\Big(a_s^{i,N} + \sqrt{a_s^{i,N}}A_s^{i,N} - \frac{\mu}{(1-p)}\Big)\Big] \\
=& \Big(\frac{\mu}{1-p}\Big)^2 \mathrm{Var}\Big(\frac{1}{N}\sum_{j=1}^N \theta_{ij}\Big) = \frac{p\mu^2}{N(1-p)}.
\end{aligned}$$

Finally, for $s \neq t$ and $i \neq j$,

$$\mathrm{Cov}(X_t^{i,N}, X_s^{j,N}) = \mathbb{E}\Big[\Big(a_t^{i,N} + \sqrt{a_t^{i,N}} A_t^{i,N} - \mu - p\Big)\Big(a_s^{j,N} + \sqrt{a_s^{j,N}} A_t^{j,N} - \mu - p\Big)\Big] = 0.$$

Over all we have $\mathrm{Cov}(X_t^{i,N}, X_s^{j,N}) = C_{\mu,p,N}((i,t),(j,s))$, where

$$C_{\mu,p,N}((i,t),(j,s)) = \begin{cases} \frac{p\mu^2}{N(1-p)} + \frac{p\mu^2}{N(1-p)^2} + \frac{\mu}{(1-p)} & \text{if } i=j,\ t=s, \\ \frac{p^2}{N}\frac{\mu^2}{(1-p)^2} & \text{if } i\neq j,\ t=s, \\ \frac{p\mu^2}{N(1-p)} & \text{if } i=j,\ t\neq s, \\ 0 & \text{if } i\neq j,\ t\neq s. \end{cases}$$

From the covariance function above, we can ignore the covariance when $t \neq s$. So, we construct a new covariance function:

$$\widetilde{C}_{\mu,p,N}((i,t),(j,s)) = \begin{cases} \frac{p\mu^2}{N(1-p)} + \frac{p\mu^2}{N(1-p)^2} + \frac{\mu}{(1-p)} & \text{if } i=j,\ t=s, \\ \frac{p^2}{N}\frac{\mu^2}{(1-p)^2} & \text{if } i\neq j,\ t=s, \\ 0 & \text{if } i=j,\ t\neq s, \\ 0 & \text{if } i\neq j,\ t\neq s. \end{cases}$$

We thus consider the following toy model: for two unknown parameters $\mu > 0$ and $p \in (0,1)$, we observe $(U_s^{i,N})_{i=1,\dots,K,s=0,\dots,T}$, for some Gaussian array $(U_s^{i,N})_{i=1,\dots,N,s=0,\dots,T}$ with covariance matrix $\widetilde{C}_{\mu,p,N}$ defined above and we want to estimate p. If assuming that $\frac{\mu}{1-p}$ is known, it is well-known that the temporal empirical variance $S_T^{N,K} = \frac{1}{T}\sum_{t=1}^{T}(\bar{U}_t^{N,K} - \frac{\mu}{1-p})^2$, where $\bar{U}_t^{N,K} = \frac{1}{K}\sum_{i=1}^{K} U_t^{i,N}$, is the best estimator of $\frac{(2p-p^2)\mu^2}{NK(1-p)^2} + \frac{\mu}{K(1-p)} + \frac{p^2(K-1)}{NK}\frac{\mu^2}{(1-p)^2}$, (in all the usual senses). Consequently, $C_T^{N,K} = \frac{N}{K-1}(\frac{\mu}{1-p})^{-2}[KS_T^{N,K} - \frac{\mu}{1-p}]$ is the best estimator of p^2. And

$$\mathrm{Var}(C_T^{N,K}) = \frac{1}{T}\frac{N^2}{(K-1)^2}K^2\frac{1}{K^2}\Big[\rho + \frac{(K-1)\alpha}{N}\Big]^2 \simeq \frac{N^2}{TK^2}.$$

where $\rho = \frac{(2p-p^2)\mu^2}{N(1-p)^2} + \frac{\mu}{(1-p)}$ and $\alpha = \frac{p^2\mu^2}{(1-p)^2}$ Hence for this Gaussian toy model, it is not possible to estimate p^2 (and thus p) with a precision better than $\frac{N}{K}\frac{1}{\sqrt{T}}$.

Conclusion

Using the first example, it seems that it should not be possible to estimate p faster than $N/(\sqrt{K}e^{\alpha_0 t}) + 1/\sqrt{K}$. in the supercritical case. Using the two examples, it seems that it should not be possible to estimate p faster than $N/(t\sqrt{K}) + 1/\sqrt{K} + N/(K\sqrt{t})$ in the subcritical case.

0.3.8 Central limit theorem for the estimator

Recall that assumptions $(H(q))$ and (A) are defined at the beginning of section 0.3.5. In order to make the central limit theorem hold, we need stronger condition:

Assumptions

We will work under the following conditions: for some $q \geq 1$,

$$(H(q)) \qquad \text{and} \qquad \int_0^\infty (\phi(s))^2 ds < \infty \tag{$H'(q)$}$$

or

$$(A) \quad \text{and} \quad \phi(s) = e^{-bs} \quad \text{for some unknown } b > 0. \tag{A'}$$

Here b is a positive constant. Since $\Lambda = 1/b$, we thus assume that $p > b$.

The result in subcritical case

Here we will assume $H'(q)$ for some $q \geq 1$. We then introduce the function $\Psi^{(3)}$ defined by

$$\Psi^{(3)}(u,v,w) = \frac{u^2(1-\sqrt{\frac{u}{w}})^2}{v+u^2(1-\sqrt{\frac{u}{w}})^2} \quad \text{if } u>0,\ v>0,\ w>0$$
$$\text{and} \quad \Psi^{(3)}(u,v,w) = 0 \text{ otherwise.}$$

We set

$$\hat{p}_{N,K,t} = \Psi^{(3)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{t,\Delta_t}^{N,K}),$$

with the choice

$$\Delta_t = (2\lfloor t^{1-4/(q+1)} \rfloor)^{-1} t \tag{0.9}$$

Theorem 0.3.8. *We assume that $p>0$ and that $H'(q)$ holds for some $q>3$. Define Δ_t by (2.2). We set $c_{p,\Lambda} := (1-\Lambda p)^2/(2\Lambda^2)$. We always work in the asymptotic $(N,K,t) \to (\infty,\infty,\infty)$ and in the regime $\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\lambda}K} \to 0$.*

(i) In the regime with dominating term $\frac{1}{\sqrt{K}}$, i.e. when $[\frac{1}{\sqrt{K}}]/[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}}] \to \infty$, it holds that

$$\sqrt{K}\Big(\hat{p}_{N,K,t} - p\Big) \xrightarrow{d} \mathcal{N}\Big(0, \frac{p^2(1-p)^2}{\mu^4}\Big).$$

(ii) In the regime with dominating term $\frac{N}{t\sqrt{K}}$, i.e. when $[\frac{N}{t\sqrt{K}}]/[\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}}] \to \infty$, we have

$$\frac{t\sqrt{K}}{N}\Big(\hat{p}_{N,K,t} - p\Big) \xrightarrow{d} \mathcal{N}\Big(0, \frac{2(1-\Lambda p)}{\mu^2\Lambda^4}\Big).$$

(iii) In the regime with dominating term $\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}$, i.e. when $[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]/[\frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}}] \to \infty$, imposing moreover that $\lim_{N,K\to\infty} \frac{K}{N} = \gamma \in [0,1]$,

$$\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\Big(\hat{p}_{N,K,t} - p\Big) \xrightarrow{d} \mathcal{N}\Big(0, \frac{3(1-p)^2}{2\mu^2\Lambda^2}\Big((1-\gamma)(1-\Lambda p)^3 + \gamma(1-\Lambda p)\Big)^2\Big).$$

We decided not to study the regimes where there are two or three dominating terms. We believe this is not very restrictive in practise. Furthermore, the study would be much more tedious, because it would be very difficult to study the correlations between the different terms.

Remark 0.3.9. This result allows us to construct an asymptotic confidence interval for p. We define

$$\hat{\mu}_{N,K,t} := \Psi^{(1)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K}), \quad \hat{\Lambda}_{N,K,t} := \Psi^{(2)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K})$$

where

$$\Psi^{(1)}(u,v,w) := u\sqrt{\frac{u}{w}}, \quad \Psi^{(2)}(u,v,w) := \frac{v+[u-\Psi^{(1)}(u,v,w)]^2}{u[u-\Psi^{(1)}(u,v,w)]}$$

if $u>0$, $v>0$, $w>u$ and $\Psi^{(1)}(u,v,w) = \Psi^{(2)}(u,v,w) = 0$ otherwise. By [26, Theorem 2.1], we have, in the regime $\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \to 0$,

$$\Big(\hat{\mu}_{N,K,t}, \hat{\Lambda}_{N,K,t}, \hat{p}_{N,K,t}\Big) \xrightarrow{P} (\mu,\Lambda,p).$$

Hence by Theorem 0.3.8, in the regime (i), (ii) or (iii), for $0<\alpha<1$,

$$\lim P\Big(|\hat{p}_{N,K,t} - p| \leq I_{N,K,t,\alpha}\Big) = 1-\alpha$$

where

$$I_{N,K,t,\alpha} = (\Phi)^{-1}(1-\frac{\alpha}{2})\Big(\frac{1}{\sqrt{K}}\frac{\hat{p}_{N,K,t}(1-\hat{p}_{N,K,t})}{(\hat{\mu}_{N,K,t})^2} + \frac{N}{t\sqrt{K}}\frac{\sqrt{2(1-\hat{\lambda}_{N,K,t}\hat{p}_{N,K,t})^2}}{\hat{\mu}_{N,K,t}(\hat{\Lambda}_{N,K,t})^2}$$
$$+ \frac{N}{K}\sqrt{\frac{\Delta_t}{t}}\sqrt{\frac{3(1-\hat{p}_{N,K,t})^2}{2\hat{\mu}^2_{N,K,t}\hat{\Lambda}^2_{N,K,t}}}\Big|(1-\frac{K}{N})(1-\hat{\Lambda}_{N,K,t}\hat{p}_{N,K,t})^3 + \frac{K}{N}(1-\hat{\Lambda}_{N,K,t}\hat{p}_{N,K,t})\Big|\Big)$$

and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{s^2}{2}}ds$.

Concerning the case $p = 0$, the following result shows that $\hat{p}_{N,K,t}$ is not always consistent.

Proposition 0.3.10. We assume that $p = 0$ and that $H'(q)$ holds for some $q > 3$. We set $c_{p,\Lambda} := (1-\Lambda p)^2/(2\Lambda^2)$. We always work in the asymptotic $(N,K,t) \to (\infty,\infty,\infty)$ and in the regime $\frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \to 0$.

(i) If $[\frac{N}{t\sqrt{K}}]/[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]^2 \to \infty$, we have

$$\hat{p}_{N,K,t} \xrightarrow{P} 0.$$

(ii) If $[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]^2/[\frac{N}{t\sqrt{K}}] \to \infty$, we have

$$\hat{p}_{N,K,t} \xrightarrow{d} X$$

where $P(X=1) = P(X=0) = \frac{1}{2}$.

The result in the supercritical case

Theorem 0.3.11. *We assume (A') and set $\alpha_0 = p-b$. In the regime where $(N,K,t) \to (\infty,\infty,\infty)$ with $\frac{N}{\sqrt{K}e^{\alpha_0 t}} + \frac{1}{\sqrt{K}} \to 0$ with dominating term $\frac{N}{\sqrt{K}e^{\alpha_0 t}}$ (i.e. with $[\frac{N}{\sqrt{K}e^{\alpha_0 t}}]/[\frac{1}{\sqrt{K}}] \to \infty$), it holds that,*

$$\frac{e^{\alpha_0 t}\sqrt{K}}{N}\Big(\mathcal{P}_t^{N,K} - p\Big) \xrightarrow{d} \mathcal{N}\Big(0, \frac{2(\alpha_0)^4 p^2}{\mu^2}\Big).$$

While our result in the subcritical case is rather general and satisfying, there are many restrictions in the supercritical case. First, we have not been able to deal with general functions ϕ. Second, we did not manage to prove a central limit theorem concerning a large Bernoulli random matrix (and its Perron-Frobenius eigenvalue and eigenvector) that would allow us to study the second regime where $[\frac{1}{\sqrt{K}}]/[\frac{N}{\sqrt{K}e^{\alpha_0 t}}] \to \infty$.

Chapter 1

Statistical inference for a partially observed interacting system of Hawkes processes

Abstract. We observe the actions of a K sub-sample of N individuals up to time t for some large $K \leq N$. We model the relationships of individuals by i.i.d. Bernoulli(p)-random variables, where $p \in (0, 1]$ is an unknown parameter. The rate of action of each individual depends on some unknown parameter $\mu > 0$ and on the sum of some function ϕ of the ages of the actions of the individuals which influence him. The function ϕ is unknown but we assume it rapidly decays. The aim of this paper is to estimate the parameter p asymptotically as $N \to \infty$, $K \to \infty$, and $t \to \infty$. Let m_t be the average number of actions per individual up to time t. In the subcritical case, where m_t increases linearly, we build an estimator of p with the rate of convergence $\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}} + \frac{N}{K\sqrt{m_t}}$. In the supercritical case, where m_t increases exponentially fast, we build an estimator of p with the rate of convergence $\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}}$.

1.1 Introduction

1.1.1 Motivation

The Hawkes processes were first introduced as an immigration-birth model by Hawkes in [19]. The properties of one-dimensional Hawkes processes have been well-studied, see e.g. Chapter 12 of Daley and Vere-Jones in [13] for the stability of the process, Brémaud and Massoulié in [8] for the analysis of the Bartlett spectrum of the process. We can also find some study of non-linear Hawkes processes from Zhu in [49], of their stability by Brémaud in [7]. Multivariate Hawkes processes were explored in Liniger [25]. Infinite dimensional Hawkes processes have been studied in [15].

Hawkes processes have a lot of applications. In [32], Ogata uses the Hawkes process to give models for earthquake occurrences. We can see there are plenty of applications in genomics, for example see [17] by Gusto-Schbath and [37] by Bouret-Schbath. In [37], they use the Hawkes process to model the process of the occurrences of a particular event along a DNA sequence. There are also some applications in neuroscience, see e.g. Bouret-Rivoirard-Malot [38]. In [38], they use multivariate Hawkes process to model the instantaneous firing rates of different neurons. There are applications in finance about market orders modelling, see e.g. Bauwens and Hautsch in [4]. There are even some applications in criminology, see e.g. Mohler, Short, Brantingham, Schoenberg and Tita in [29].

In the real world, we often need to consider the case when the number of individuals is large.

For example, in the neuroscience, the number of the neurons are usually enormously large. So it is very useful to consider the multivariate Hawkes process as the number of individuals goes to infinite. This problem seems to be rarely studied.

Next, we are going to give an example.

1.1.2 An illustrating example

We have N individuals. Each individual $j \in \{1, \dots, N\}$ is connected to the set of individuals $S_j = \{i \in \{1, \dots, N\} : \theta_{ij} = 1\}$. The only possible action of the individual i is to send a message to all the individuals of S_i. Here $Z_t^{i,N}$ stands for the number of messages sent by i during $[0, t]$. The counting process $(Z_s^{i,N})_{i=1\dots N, 0\le s\le t}$ is determined by its intensity process $(\lambda_s^{i,N})_{i=1\dots N, 0\le s\le t}$. It is informally defined by

$$P\Big(Z_t^{i,N} \textit{has a jump in } [t, t+dt] \Big| \mathcal{F}_t\Big) = \lambda_t^{i,N} dt, \ i = 1, ..., N$$

where $\mathcal{F}_t$ denotes the sigma-field generated by $(Z_s^{i,N})_{i=1\dots N, 0\le s\le t}$ and $(\theta_{ij})_{i,j=1,\dots,N}$.

The rate $\lambda_t^{i,N}$ at which i sends messages can be decomposed as the sum of two effects:

• he sends *new* messages at rate μ;

• he *forwards* the messages he received, after some delay (possibly infinite) depending on the age of the message, which induces a sending rate of the form $\frac{1}{N}\sum_{j=1}^N \theta_{ij} \int_0^{t-} \phi(t-s) dZ_s^{j,N}$.

If for example $\phi = \mathbf{1}_{[0,K]}$, then $N^{-1}\sum_{j=1}^N \theta_{ij} \int_0^{t-} \phi(t-s) dZ_s^{j,N}$ is precisely the number of messages that the i-th individual received between time $t-K$ and time t, divided by N.

1.1.3 Main Goals

We usually consider $(\theta_{ij})_{i,j=1,\dots,N}$ as a family of i.i.d. Bernoulli(p) random variables, where p is an unknown parameter. In [14], Delattre and Fournier consider the case where one observes the whole sample $(Z_s^{i,N})_{i=1\dots N, 0\le s\le t}$ and they propose some estimator of the unknown parameter p.

However, in the real world, it is often impossible to observe the whole population. Our goal in the present paper is to consider the case where one observes only a subsample of indivudals.

In other words, we want to build some estimators of p when observing $(Z_s^{i,N})_{\{i=1,\dots,K,\ 0\le s\le t\}}$ with $1 \ll K \le N$ and with t large. The paper [14] thus considers the special case where $K = N$.

Let $\Lambda = \int_0^\infty \phi(t)dt \in (0, \infty]$. In [14], we see that growth of $Z_t^{1,N}$ depends on the value of Λp. When $\Lambda p < 1$ (subcritical case), $Z_t^{1,N}$ increases (in average) linearly with time, while when $\Lambda p > 1$ (supercritical case), it increases exponentially. Thus the limit theorems will be different in the two cases. We will not consider the critical case when $\Lambda p = 1$.

1.2 Main results

1.2.1 Setting

We consider some unknown parameters $p \in (0, 1], \mu > 0$ and $\phi : [0, \infty) \to [0, \infty)$. We always assume that the function ϕ is measurable and locally integrable. For $N \ge 1$, we consider an i.i.d. family $(\Pi^i(dt, dz))_{i=1,\dots,N}$ of Poisson measures on $[0, \infty) \times [0, \infty)$ with intensity $dtdz$. And $(\theta_{ij})_{i,j=1,\dots,N}$ is a family of i.i.d. Bernoulli(p) random variables which is independent of the family

$(\Pi^i(dt,dz))_{i=1,...,N}$. We consider the following system: for all $i \in \{1,...,N\}$, all $t \geq 0$,

$$Z_t^{i,N} := \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \lambda_s^{i,N}\}} \Pi^i(ds,dz), \text{ where } \lambda_t^{i,N} := \mu + \frac{1}{N}\sum_{j=1}^N \theta_{ij} \int_0^{t-} \phi(t-s) dZ_s^{j,N}. \quad (1.1)$$

In this paper, $\int_0^t$ means $\int_{[0,t]}$, and $\int_0^{t-}$ means $\int_{[0,t)}$. The solution $((Z_t^{i,N})_{t\geq 0})_{i=1,...,N}$ is a family of counting processes. By [14, Proposition 1], the system (1) has a unique $(\mathcal{F}_t)_{t\geq 0}$-measurable càdlàg solution, where

$$\mathcal{F}_t = \sigma(\Pi^i(A) : A \in \mathcal{B}([0,t]\times[0,\infty)), i = 1,...,N) \vee \sigma(\theta_{ij}, i,j = 1,...,N),$$

as soon as ϕ is locally integrable.

1.2.2 Assumptions

Recall that $\Lambda = \int_0^\infty \phi(t)dt \in (0,\infty]$. We will work under one of the two following conditions: either for some $q \geq 1$,

$$\mu \in (0,\infty), \quad \Lambda p \in (0,1) \quad \text{and} \quad \int_0^\infty s^q \phi(s) ds < \infty \quad (H(q))$$

or

$$\mu \in (0,\infty), \quad \Lambda p \in (1,\infty] \quad \text{and} \quad \int_0^t |d\phi(s)| \text{ increases at most polynomially.} \quad (A)$$

In many applications, ϕ is smooth and decays fast. Hence what we have in mind is that in the subcritical case, $(H(q))$ is satisfied for all $q \geq 1$. In the supercritical case, (A) seems very reasonable.

Remark 1.2.1. There is a wide class of functions satisfy the assumptions $(H(q))$ or (A), especially the functions who decay fast. For example, any decreasing exponential function $\phi(s) = e^{-bs}$ satisfies $(H(q))$ is satisfied for all $q \geq 1$ if $\frac{\Lambda}{b} < 1$ and satisfies (A) when $\frac{\Lambda}{b} > 1$.

1.2.3 The result in the subcritical case

For $N \geq 1$ and for $((Z_t^{i,N})_{t\geq 0})_{i=1,...,N}$ the solution of (1.1), we set $\bar{Z}_t^N := N^{-1}\sum_{i=1}^N Z_t^{i,N}$ and $\bar{Z}_t^{N,K} := K^{-1}\sum_{i=1}^K Z_t^{i,N}$. Next, we introduce

$$\varepsilon_t^{N,K} := t^{-1}(\bar{Z}_{2t}^{N,K} - \bar{Z}_t^{N,K}), \qquad \mathcal{V}_t^{N,K} := \frac{N}{K}\sum_{i=1}^K \Big[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \varepsilon_t^{N,K}\Big]^2 - \frac{N}{t}\varepsilon_t^{N,K}.$$

For $\Delta > 0$ such that $t/(2\Delta) \in \mathbb{N}^*$, we set

$$\mathcal{W}_{\Delta,t}^{N,K} := 2\mathcal{Z}_{2\Delta,t}^{N,K} - \mathcal{Z}_{\Delta,t}^{N,K}, \quad \mathcal{X}_{\Delta,t}^{N,K} := \mathcal{W}_{\Delta,t}^{N,K} - \frac{N-K}{K}\varepsilon_t^{N,K} \quad (1.2)$$

$$\text{where} \quad \mathcal{Z}_{\Delta,t}^{N,K} := \frac{N}{t}\sum_{a=t/\Delta}^{2t/\Delta} (\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \Delta\varepsilon_t^{N,K})^2. \quad (1.3)$$

Theorem 1.2.2. *We assume $(H(q))$ for some $q > 3$. There is a constant C depending only on q, p, μ, ϕ such that for all $\varepsilon \in (0,1)$, all $1 \leq K \leq N$, if setting $\Delta_t = t/(2\lfloor t^{1-4/(q+1)}\rfloor)$ for all $t \geq 1$,*

$$P\Big(\Big|\Psi\Big(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K}\Big) - (\mu,\Lambda,p)\Big| \geq \varepsilon\Big) \leq \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{K}} + \frac{N}{K\sqrt{t^{1-\frac{4}{1+q}}}} + \frac{N}{t\sqrt{K}}\Big) + CNe^{-C'K}$$

with $\Psi := \mathbf{1}_D\Phi : \mathbb{R}^3 \mapsto \mathbb{R}^3$, *the function* $\Phi := (\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)})$ *being defined on* $D := \{(u,v,w) \in \mathbb{R}^3 : w > u > 0 \text{ and } v \geq 0\}$ *by*

$$\Phi^{(1)}(u,v,w) := u\sqrt{\frac{u}{w}}, \quad \Phi^{(2)}(u,v,w) := \frac{v + [u - \Phi^{(1)}(u,v,w)]^2}{u[u - \Phi^{(1)}(u,v,w)]},$$
$$\Phi^{(3)}(u,v,w) := \frac{1 - u^{-1}\Phi^{(1)}(u,v,w)}{\Phi^{(2)}(u,v,w)}.$$

We quote [14, Remark 2], which says that the mean number of actions per individual per unit of time increases linearly.

Remark 1.2.3. Assume $H(1)$. Then for all $\varepsilon > 0$,

$$\lim_{(N,t)\to(\infty,\infty)} P\Big(\Big|\frac{\bar{Z}_t^{N,K}}{t} - \frac{\mu}{1-\Lambda p}\Big| \geq \varepsilon\Big) = 0.$$

So roughly, if observing $((Z_s^{i,N})_{s\in[0,t]})_{i=1,\dots,K}$, we observe approximately Kt actions.

1.2.4 The result in the supercritical case

Here we define $\bar{Z}_t^{N,K}$ as previously and we set

$$\mathcal{U}_t^{N,K} := \Big[\frac{N}{K}\sum_{i=1}^{K}\Big(\frac{Z_t^{i,N} - \bar{Z}_t^{N,K}}{\bar{Z}_t^{N,K}}\Big)^2 - \frac{N}{\bar{Z}_t^{N,K}}\Big]\mathbf{1}_{\{\bar{Z}_t^{N,K}>0\}} \tag{1.4}$$

$$\text{and} \quad \mathcal{P}_t^{N,K} := \frac{1}{\mathcal{U}_t^{N,K}+1}\mathbf{1}_{\{\mathcal{U}_t^{N,K}\geq 0\}}. \tag{1.5}$$

Theorem 1.2.4. *We assume* (A) *and define* α_0 *by* $p\int_0^\infty e^{-\alpha_0 t}\phi(t)dt = 1$ *(recall that by* (A)*,* $\Lambda p = p\int_0^\infty \phi(t)dt > 1$*). For all* $\eta > 0$*, there is a constant* $C_\eta > 0$ *(depending on* p, μ, ϕ, η*), such that for all* $N \geq K \geq 1$*, all* $\varepsilon \in (0,1)$*,*

$$P(|\mathcal{P}_t^{N,K} - p| \geq \varepsilon) \leq \frac{C_\eta e^{4\eta t}}{\varepsilon}\Big(\frac{N}{\sqrt{K}e^{\alpha_0 t}} + \frac{1}{\sqrt{K}}\Big).$$

Next, we quote [14, Remark 5].

Remark 1.2.5. Assume (A) and consider $\alpha_0 > 0$ such that $p\int_0^\infty e^{-\alpha_0 t}\phi(t)dt = 1$. Then for all $\eta > 0$,

$$\lim_{t\to\infty}\lim_{(N,K)\to(\infty,\infty)} P(\bar{Z}_t^{N,K} \in [e^{(\alpha_0-\eta)t}, e^{(\alpha_0+\eta)t}]) = 1.$$

So roughly, if observing $((Z_s^{i,N})_{s\in[0,t]})_{i=1,\dots,K}$, we observe around $Ke^{\alpha_0 t}$ actions.

1.3 On the choice of the estimators

In the whole paper, we denote by $\mathbb{E}_\theta$ the conditional expectation knowing $(\theta_{ij})_{i,j=1,\dots,N}$. Here we explain informally why the estimators should converge.

1.3.1 The subcritical case

We define $A_N(i,j) := N^{-1}\theta_{ij}$ and the matrix $(A_N(i,j))_{i,j\in\{1,\dots,N\}}$, as well as $Q_N := (I - \Lambda A_N)^{-1}$ on the event on which $I - \Lambda A_N$ is invertible.

Define $\widetilde{\varepsilon}_t^{N,K} := t^{-1}\bar{Z}_t^{N,K}$, $K \le N$. We expect that, for t large enough, $Z_t^{i,N} \simeq \mathbb{E}_\theta[Z_t^{i,N}]$. And, by definition of $Z_t^{i,N}$, see (1.1), it is not hard to get

$$\mathbb{E}_\theta[Z_t^{i,N}] = \mu t + N^{-1}\sum_{j=1}^N \theta_{ij}\int_0^t \phi(t-s)\mathbb{E}_\theta[Z_s^{j,N}]ds.$$

Hence, assuming that $\gamma_N(i) = \lim_{t\to\infty} t^{-1}\mathbb{E}_\theta[Z_t^{i,N}]$ exists for each $i = 1, ..., N$ and observing that $\int_0^t \phi(t-s)sds \simeq \Lambda t$, we find that the vector $\boldsymbol{\gamma}_N = (\gamma_N(i))_{i=1,...,N}$ should satisfy $\boldsymbol{\gamma}_N = \mu\mathbf{1}_N + \Lambda A_N\boldsymbol{\gamma}_N$, where $\mathbf{1}_N$ is the vector defined by $\mathbf{1}_N(i) = 1$ for all $i = 1, \dots, N$. Thus we deduce that $\boldsymbol{\gamma}_N = \mu(I - \Lambda A_N)^{-1}\mathbf{1}_N = \mu\boldsymbol{\ell}_N$, where we have set

$$\boldsymbol{\ell}_N := Q_N\mathbf{1}_N,\ \ell_N(i) := \sum_{j=1}^N Q_N(i,j),\ \bar{\ell}_N := \frac{1}{N}\sum_{i=1}^N \ell_N(i),\ \bar{\ell}_N^K := \frac{1}{K}\sum_{i=1}^K \ell_N(i)$$

So we expect that $Z_t^{i,N} \simeq \mathbb{E}_\theta[Z_t^{i.N}] \simeq \mu\ell_N(i)t$, whence $\widetilde{\varepsilon}_t^{N,K} = t^{-1}\bar{Z}_t^{N,K} \simeq \mu\bar{\ell}_N^K$.

We informally show that $\ell_N(i) \simeq 1 + \Lambda(1-\Lambda p)^{-1}L_N(i)$, where $L_N(i) := \sum_{j=1}^N A_N(i,j)$: when N is large, $\sum_{j=1}^N A_N^2(i,j) = N^{-2}\sum_{j=1}^N\sum_{k=1}^N \theta_{ik}\theta_{kj} \simeq pN^{-1}\sum_{k=1}^N \theta_{ik} = pL_N(i)$. And one gets convinced similarly that for any $n \in \mathbb{N}_*$, roughly, $\sum_{j=1}^N A_N^n(i,j) \simeq p^{n-1}L_N(i)$. So

$$\ell_N(i) = \sum_{n\ge 0}\Lambda^n\sum_{j=1}^N A_N^n(i,j) \simeq 1 + \sum_{n\ge 1}\Lambda^n p^{n-1}L_N(i) = 1 + \frac{\Lambda}{1-\Lambda p}L_N(i).$$

But $(NL_N(i))_{i=1,...,N}$ are i.i.d. Binomial(N,p) random variables, so that $\bar{\ell}_N^K \simeq 1 + \Lambda p(1-\Lambda p)^{-1} = (1-\Lambda p)^{-1}$. Finally, we have explained why $\widetilde{\varepsilon}_t^{N,K}$ should resemble $\mu(1-\Lambda p)^{-1}$.

Knowing $(\theta_{ij})_{i,j=1..N}$, the process $Z_t^{1,N}$ resembles a Poisson process, so that $\mathbb{V}\mathrm{ar}_\theta(Z_t^{1,N}) \simeq \mathbb{E}_\theta[Z_t^{1,N}]$, whence

$$\mathbb{V}\mathrm{ar}(Z_t^{1,N}) = \mathbb{V}\mathrm{ar}(\mathbb{E}_\theta[Z_t^{1,N}]) + \mathbb{E}[\mathbb{V}\mathrm{ar}_\theta(Z_t^{1,N})] \simeq \mathbb{V}\mathrm{ar}(\mathbb{E}_\theta[Z_t^{1,N}]) + \mathbb{E}[Z_t^{1,N}].$$

Writing an empirical version of this equality, we find

$$\frac{1}{K}\sum_{i=1}^K (Z_t^{i,N} - \bar{Z}_t^{N,K})^2 \simeq \frac{1}{K}\sum_{i=1}^K \Big(\mathbb{E}_\theta[Z_t^{i,N}] - \mathbb{E}_\theta[\bar{Z}_t^{N,K}]\Big)^2 + \bar{Z}_t^{N,K}.$$

And since $Z_t^{i,N} \simeq \mu\ell_N(i)t \simeq \mu[1 + (1-\Lambda p)^{-1}\Lambda L_N(i)]t$ as already seen a few lines above, we find

$$\frac{1}{K}\sum_{i=1}^K (Z_t^{i,N} - \bar{Z}_t^{N,K})^2 \simeq \frac{\mu^2t^2\Lambda^2}{K(1-\Lambda p)^2}\sum_{i=1}^K (L_N(i) - \bar{L}_N^K)^2 + \bar{Z}_t^{N,K}.$$

But $(NL_N(i))_{i=1,...,N}$ are i.i.d. Bernoulli(N,p) random variables, so that

$$\begin{aligned}\widetilde{\mathcal{V}}_t^{N,K} :=& \frac{N}{K}\sum_{i=1}^K \Big[\frac{Z_t^{i,N}}{t} - \widetilde{\varepsilon}_t^{N,K}\Big]^2 - \frac{N}{t}\widetilde{\varepsilon}_t^{N,K}\\ =& \frac{N}{Kt^2}\Big[\sum_{i=1}^K (Z_t^{i,N} - \bar{Z}_t^{N,K})^2 - K\bar{Z}_t^{N,K}\Big]\\ \simeq& \frac{N\mu^2\Lambda^2}{K(1-\Lambda p)^2}\sum_{i=1}^K (L_N(i) - \bar{L}_N^K)^2 \simeq \frac{\mu^2\Lambda^2 p(1-p)}{(1-\Lambda p)^2}.\end{aligned}$$

We finally build a third estimator. The temporal empirical variance

$$\frac{\Delta}{t}\sum_{k=1}^{t/\Delta}\Big[\bar{Z}^{N,K}_{k\Delta}-\bar{Z}^{N,K}_{(k-1)\Delta}-\frac{\Delta}{t}\bar{Z}^{N,K}_t\Big]^2$$

should resemble $\mathbb{V}\mathrm{ar}_\theta[\bar{Z}^{N,K}_\Delta]$ if $1\ll\Delta\ll t$. So we expect that:

$$\widetilde{\mathcal{W}}^{N,K}_{\Delta,t}:=\frac{N}{t}\sum_{k=1}^{t/\Delta}\Big[\bar{Z}^{N,K}_{k\Delta}-\bar{Z}^{N,K}_{(k-1)\Delta}-\Delta t^{-1}\bar{Z}^{N,K}_t\Big]^2\simeq\frac{N}{\Delta}\mathbb{V}\mathrm{ar}_\theta[\bar{Z}^{N,K}_\Delta].$$

To understand what $\mathbb{V}\mathrm{ar}_\theta[\bar{Z}^{N,K}_\Delta]$ looks like, we introduce the centered process $U^{i,N}_t:=Z^{i,N}_t-\mathbb{E}_\theta[Z^{i,N}_t]$ and the martingale $M^{i,N}_t:=Z^{i,N}_t-C^{i,N}_t$ where $C^{i,N}$ is the compensator of $Z^{i,N}$. An easy computation, see [14, Lemma 11], shows that, denoting by $\boldsymbol{U}^N_t$ and $\boldsymbol{M}^N_t$ the vectors $(U^{i,N}_t)_{i=1,...,N}$ and $(M^{i,N}_t)_{i=1,...,N}$,

$$\boldsymbol{U}^N_t=\boldsymbol{M}^N_t+A_N\int_0^t\phi(t-s)\boldsymbol{U}^N_s ds.$$

So for large times, we conclude that $\boldsymbol{U}^N_t\simeq\boldsymbol{M}^N_t+\Lambda A_N\boldsymbol{U}^N_t$, whence finally $\boldsymbol{U}^N_t\simeq Q\boldsymbol{M}^N_t$ and thus

$$\frac{1}{K}\sum_{i=1}^K U^{i,N}_t\simeq\frac{1}{K}\sum_{i=1}^K\sum_{j=1}^N Q(i,j)M^{j,N}_t=\frac{1}{K}\sum_{j=1}^N c^K_N(j)M^{j,N}_t,$$

where we have set $c^K_N(j)=\sum_{i=1}^K Q_N(i,j)$. But we obviously have $[M^{j,N},M^{i,N}]_t=\mathbf{1}_{\{i=j\}}Z^{j,N}_t$ (see [14, Remark 10]), so that

$$\mathbb{V}\mathrm{ar}_\theta[\bar{Z}^{N,K}_t]=\mathbb{V}\mathrm{ar}_\theta[\bar{U}^{N,K}_t]\simeq\frac{1}{K^2}\sum_{j=1}^N(c^K_N(j))^2Z^{j,N}_t.$$

Recalling that $Z^{j,N}_t\simeq\mu\ell_N(j)t$, we conclude that $\mathbb{V}\mathrm{ar}_\theta[\bar{Z}^{N,K}_t]\simeq K^{-2}\mu t\sum_{j=1}^N\Big(c^K_N(j)\Big)^2\ell_N(j)$, whence

$$\widetilde{\mathcal{W}}^{N,K}_{\Delta,t}\simeq\frac{N}{\Delta}\mathbb{V}\mathrm{ar}_\theta[\bar{Z}^{N,K}_\Delta]\simeq\mu\frac{N}{K^2}\sum_{j=1}^N\Big(c^K_N(j)\Big)^2\ell_N(j).$$

To compute this last quantity, we start from $c^K_N(j)=\sum_{n\geq0}\sum_{i=1}^K\Lambda^nA^n_N(i,j)$. But we have $\sum_{i=1}^K A^2_N(i,j)=N^{-2}\sum_{i=1}^K\sum_{k=1}^N\theta_{ik}\theta_{kj}\simeq pKN^{-2}\sum_{k=1}^N\theta_{kj}=pKN^{-1}C_N(j)$. And one gets convinced similarly that for any $n\in\mathbb{N}_*$, roughly, $\sum_{i=1}^K A^n_N(i,j)\simeq KN^{-1}p^{n-1}C_N(j)$. So we conclude that $c^K_N(j)\simeq A^0_N(i,j)+\frac{K\Lambda}{N(1-\Lambda p)}C_N(j)$. Consequently, $c^K_N(j)\simeq1+\frac{K}{N}\frac{\Lambda p}{(1-\Lambda p)}$ for $j\in\{1,...,K\}$ and $c^K_N(j)\simeq\frac{K}{N}\frac{\Lambda p}{(1-\Lambda p)}$ for $j\in\{K+1,...,N\}$. We finally get, recalling that $\ell_N(j)\simeq(1-\Lambda p)^{-1}$,

$$\begin{aligned}\widetilde{\mathcal{W}}^{N,K}_{\Delta,t}\simeq&\mu\frac{N}{K^2}\sum_{j=1}^N\Big(c^K_N(j)\Big)^2\ell_N(j)\\\simeq&\mu\frac{N}{K^2}\Big(\frac{K}{1-\Lambda p}\Big[1+\frac{K\Lambda p}{N(1-\Lambda p)}\Big]^2+\frac{N-K}{1-\Lambda p}\Big[\frac{K\Lambda p}{N(1-\Lambda p)}\Big]^2\Big)\\\simeq&\frac{\mu}{(1-\Lambda p)^3}+\frac{(N-K)\mu}{K(1-\Lambda p)}.\end{aligned}$$

All in all, we should have $\widetilde{\mathcal{X}}^{N,K}_{\Delta,t}\simeq\frac{\mu}{(1-\Lambda p)^3}$.

It readily follows that $\Psi(\widetilde{\varepsilon}^{N,K}_t,\widetilde{\mathcal{V}}^{N,K}_t,\widetilde{\mathcal{X}}^{N,K}_{\Delta,t})$ should resemble (μ,Λ,p).

The three estimators $\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta,t}^{N,K}$ are very similar to $\widetilde{\varepsilon}_t^{N,K}$, $\widetilde{\mathcal{V}}_t^{N,K}$, $\widetilde{\mathcal{X}}_{\Delta,t}^{N,K}$ and should converge to the same limits. Let us explain why we have introduced $\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta,t}^{N,K}$, of which the expressions are more complicated. The main idea is that, see [14, Lemma 16 (ii)], $\mathbb{E}[Z_t^{i,N}] = \mu\ell_N(i)t + \chi_i^N \pm t^{1-q}$ (under $(H(q))$), for some finite random variable χ_i^N. As a consequence, $t^{-1}\mathbb{E}[Z_{2t}^{i,N} - Z_t^{i,N}]$ converges to $\mu\ell_N(i)$ considerably much faster, if q is large, than $t^{-1}\mathbb{E}_\theta[Z_t^{i,N}]$ (for which the error is of order t^{-1}).

1.3.2 The supercritical case

We now turn to the supercritical case where $\Lambda p > 1$. We introduce the $N \times N$ matrix $A_N(i,j) = N^{-1}\theta_{ij}$.

We expect that $Z_t^{i,N} \simeq H_N \mathbb{E}_\theta[Z_t^{i,N}]$, when t is large, for some random $H_N > 0$ not depending on i. Since $\Lambda p > 1$, the process should increase like an exponential function, i.e. there should be $\alpha_N > 0$ such that for all $i = 1, \dots, N$, $\mathbb{E}_\theta[Z_t^{i,N}] \simeq \gamma_N(i)e^{\alpha_N t}$ for t very large, where $\gamma_N(i)$ is some positive random constant. We recall that $\mathbb{E}_\theta[Z_t^{i,N}] = \mu t + N^{-1}\sum_{j=1}^N \theta_{ij}\int_0^t \phi(t-s)\mathbb{E}_\theta[Z_s^{j,N}]ds$. We insert $\mathbb{E}_\theta[Z_t^{i,N}] \simeq \gamma_N(i)e^{\alpha_N t}$ in this equation and let t go to infinite: we informally get $\boldsymbol{\gamma}_N = A_N\boldsymbol{\gamma}_N \int_0^\infty e^{-\alpha_N s}\phi(s)ds$. In other words, $\boldsymbol{\gamma}_N = (\gamma_N(i))_{i=1,\dots,N}$ is an eigenvector of A_N for the eigenvalue $\rho_N := (\int_0^\infty e^{-\alpha_N s}\phi(s)ds)^{-1}$.

But A_N has nonnegative entries. Hence by the Perron-Frobenius theorem, it has a unique (up to normalization) eigenvector $\boldsymbol{V}_N$ with nonnegative entries (say, such that $\|\boldsymbol{V}_N\|_2 = \sqrt{N}$), and this vector corresponds to the maximum eigenvalue ρ_N of A_N. So there is a (random) constant κ_N such that $\boldsymbol{\gamma}_N \simeq \kappa_N \boldsymbol{V}_N$. All in all, we find that $Z_t^{i,N} \simeq \kappa_N H_N e^{\alpha_N t}\boldsymbol{V}_N(i)$. We define $\boldsymbol{V}_N^K = I_K\boldsymbol{V}_N$, where I_K is the $N \times N$-matrix defined by $I_K(i,j) = \mathbf{1}_{\{i=j\le K\}}$.

As in the subcritical case, the variance $K^{-1}\sum_{i=1}^K (Z_t^{i,N} - \bar{Z}_t^{N,K})^2$ should look like

$$\frac{1}{K}\sum_{i=1}^K (\mathbb{E}_\theta[Z_t^{i,N}] - \mathbb{E}_\theta[\bar{Z}_t^{N,K}])^2 + \bar{Z}_t^{N,K} \simeq \frac{\kappa_N^2 H_N^2 e^{2\alpha_N t}}{K}\sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2 + \bar{Z}_t^{N,K},$$

where as usual $\bar{V}_N^K := K^{-1}\sum_{i=1}^K V_N(i)$. We also get $\bar{Z}_t^{N,K} \simeq \kappa_N H_N \bar{V}_N^K e^{\alpha_N t}$. Finally,

$$\mathcal{U}_t^{N,K} = \frac{N}{K(\bar{Z}_t^{N,K})^2}[\sum_{i=1}^K (Z_t^{i,N} - \bar{Z}_t^{N,K})^2 - K\bar{Z}_t^{N,K}]\mathbf{1}_{\{\bar{Z}_t^{N,K}>0\}} \simeq \frac{N}{K(\bar{V}_N^K)^2}\sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2.$$

Next, we consider the term $(\bar{V}_N^K)^{-2}\sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2$. By a rough estimation, $A_N^2(i,j) \simeq \frac{p^2}{N}$. Because $I_K A_N^2 \boldsymbol{V}_N = \rho_N^2 \boldsymbol{V}_N^K$, we have $\rho_N^2 \boldsymbol{V}_N^K \simeq p^2 \bar{V}_N \mathbf{1}_K$, where $\mathbf{1}_K$ is the N dimensional vector of which the first K elements are 1 and others are 0. By the same reason, we have $\rho_N^2 \boldsymbol{V}_N \simeq p^2 \bar{V}_N \mathbf{1}_N$. So $\boldsymbol{V}_N^K = I_K A_N \boldsymbol{V}_N/\rho_N \simeq k_N I_K A_N \mathbf{1}_N$, where $k_N = (p^2/\rho_N^3)\bar{V}_N$. In other words, the vector $(k_N)^{-1}\boldsymbol{V}_N^K$ is almost like the vector $\boldsymbol{L}_N^K = I_K A_N \mathbf{1}_N$. Finally, we expect that

$$\mathcal{U}_t^{N,K} \simeq \frac{N}{K}(\bar{V}_N^K)^{-2}\sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2 \simeq \frac{N}{K}(\bar{L}_N^K)^{-2}\sum_{i=1}^K (L_N(i) - \bar{L}_N^K)^2 \simeq p^{-2}p(1-p) = \frac{1}{p} - 1,$$

whence $\mathcal{P}_t^{N,K} \simeq p$.

1.4 Optimal rates in some toy models

The goal of this section is to verify, using some toy models, that the rates of convergence of our estimators, see Theorems 1.2.2 and 1.2.4, are not far from being optimal.

1.4.1 The first example

Consider $\alpha_0 \geq 0$ and two unknown parameters $\Gamma > 0$ and $p \in (0,1]$. Consider an i.i.d. family $(\theta_{ij})_{i,j=1\ldots N}$ of Bernoulli(p)-distributed random variables, where $N \geq 1$. We set $\lambda_t^{i,N} = N^{-1}\Gamma e^{\alpha_0 t}\sum_{j=1}^N \theta_{ij}$ and we introduce the processes $(Z_t^{1,N})_{t\geq 0}$, ..., $(Z_t^{N,N})_{t\geq 0}$ which are, conditionally on (θ_{ij}), independent inhomogeneous Poisson process with intensities $(\lambda_t^{1,N})_{t\geq 0}$, ..., $(\lambda_t^{N,N})_{t\geq 0}$. We only observe $(Z_s^{i,N})_{s\in[0,t],\, i=1,\ldots K}$, where $K \leq N$ and we want to estimate the parameter p in the asymptotic $(K,N,t) \to (\infty,\infty,\infty)$. This model is a simplified version of the one studied in our paper. And roughly speaking, the mean number of jumps per individuals until time t resembles $m_t = \int_0^t e^{\alpha_0 s} ds$. When $\alpha_0 = 0$, this mimics the subcritical case, while when $\alpha_0 > 0$, this mimics the supercritical case. Remark that $(Z_t^{i,N})_{i=1,\ldots K}$ is a sufficient statistic, since α_0 is known.

We use the central limit theorem in order to perform a Gaussian approximation of $Z_t^{i,N}$. It is easy to show that:

$$\lambda_t^{i,N} = \Gamma e^{\alpha_0 t}\Big[\frac{1}{\sqrt{N}}\sqrt{p(1-p)}\frac{1}{\sqrt{Np(1-p)}}\sum_{j=1}^N(\theta_{ij}-p)+p\Big]$$

and $\frac{1}{\sqrt{Np(1-p)}}\sum_{j=1}^N(\theta_{ij}-p)$ converges in law to a Gaussian random variable $G_i \sim \mathcal{N}(0,1)$, where G_i is an i.i.d Gaussian family, as $N \to \infty$, for each i. Thus

$$\lambda_t^{i,N} \simeq \Gamma e^{\alpha_0 t}[\sqrt{N^{-1}p(1-p)}G_i + p].$$

Moreover, conditionally on $(\theta_{ij})_{i,j=1,\ldots,N}$, $Z_t^{i,N}$ is a Poisson random variable with mean $\int_0^t \lambda_s^{i,N} ds$. Thus, as t is large, we have $Z_t^{i,N} \simeq \int_0^t \lambda_s^{i,N} ds + \sqrt{\int_0^t \lambda_s^{i,N} ds} H_i$ where $(H_i)_{i=1,\ldots,N}$ is a family of $\mathcal{N}(0,1)$-distributed random variables, independent of $(G_i)_{i=1,\ldots,N}$. Since $(m_t)^{-1}N^{-1/2} \ll (m_t)^{-1}$, we obtain $(m_t)^{-1}Z_t^{i,N} \simeq \Gamma p + \Gamma\sqrt{N^{-1}p(1-p)}G_i + \sqrt{(m_t)^{-1}\Gamma p}H_i$, of which the law is nothing but $\mathcal{N}(\Gamma p, N^{-1}\Gamma^2 p(1-p) + (m_t)^{-1}\Gamma p)$.

By the above discussion, we construct the following toy model: one observes $(X_t^{i,N})_{i=1,\ldots K}$, where $(X_t^{i,N})_{i=1,\ldots N}$ are i.i.d and $\mathcal{N}(\Gamma p, N^{-1}\Gamma^2 p(1-p) + (m_t)^{-1}\Gamma p)$-distributed. Moreover we assume that Γp is known. So we can use the well-known statistic result: the empirical variance $S_t^{N,K} = K^{-1}\sum_{i=1}^K (X_t^{i,N} - \Gamma p)^2$ is the best estimator of $N^{-1}\Gamma^2 p(1-p) + (m_t)^{-1}\Gamma p$ (in any reasonnable sense). So $T_t^{N,K} = N(\Gamma p)^{-2}(S_t^{N,K} - (\Gamma p)/m_t)$ is the best estimator of $(\frac{1}{p}-1)$. As

$$\mathrm{Var}(S_t^{N,K}) = \frac{1}{K}\mathrm{Var}[(X_t^{1,N}-\Gamma p)^2] = \frac{2}{K}\Big(\frac{\Gamma^2 p(1-p)}{N} + \frac{\Gamma p}{m_t}\Big)^2,$$

we have

$$\mathrm{Var}(T_t^{N,K}) = \frac{2}{(\Gamma p)^4}\Big(\frac{\Gamma^2 p(1-p)}{\sqrt{K}} + \frac{N\Gamma p}{m_t\sqrt{K}}\Big)^2.$$

In other words, we cannot estimate $\Big(\frac{1}{p}-1\Big)$ with a precision better than $\Big(\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}}\Big)$, which implies that we cannot estimate p with a precision better than $\Big(\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}}\Big)$.

1.4.2 The second example

In the second part of this section, we are going to explain why there is a term $\frac{N}{K\sqrt{t^{1-\frac{4}{1+q}}}}$ in the subcritical case.

We consider discrete times $t = 1, ..., T$ and two unknown parameters $\mu > 0$ and $p \in (0, 1]$. Consider an i.i.d. family $(\theta_{ij})_{i,j=1...N}$ of Bernoulli(p)-distributed random variables, where $N \geq 1$. We set $Z_0^{i,N} = 0$ for all $i = 1, \dots, N$ and assume that, conditionally on $(\theta_{ij})_{i,j=1,...N}$ and $(Z_s^{j,N})_{s=0,...,t,j=1...,N}$, the random variables $(Z_{t+1}^{i,N} - Z_t^{i,N})$ (for $i = 1, \dots, N$) are independent and $\mathcal{P}(\lambda_t^{i,N})$-distributed, where $\lambda_t^{i,N} = \mu + \frac{1}{N}\sum_{j=1}^N \theta_{ij}(Z_t^{j,N} - Z_{t-1}^{j,N})$. This process $(Z_t^{i,N})_{i=1,...,N,t=0,...T}$ resembles the system of Hawkes processes studied in the present paper.

By [1, theorem 2], we have when time t is large, the process $\boldsymbol{Z}_t^N$ is similar to a d-dimensional diffusion process $(I - A_N)^{-1}\Sigma^{\frac{1}{2}}\boldsymbol{B}_t + \mathbb{E}_\theta[\boldsymbol{Z}_t^N]$, where $\boldsymbol{B}_t$ is a N-dimensional Brownian Motion and Σ is the diagonal matrix such that $\Sigma_{ii} = ((I - A_N)^{-1}\mu)_i$. Hence $(Z_{t+1}^{i,N} - Z_t^{i,N}) - \mathbb{E}_\theta[Z_t^{i,N} - Z_{t-1}^{i,N}]$ (for $i = 1, \dots, N$ and $t = 1, ..., T$) are independent. Since $\mathbb{E}_\theta[\boldsymbol{Z}_t^N]$ is similar to $\frac{\mu t}{1-p}$ when both N and t are large. Hence $\lambda_t^{i,N} \simeq \mathbb{E}_\theta[\lambda_t^{i,N}] \simeq \frac{\mu}{1-p}$. Then by Gaussian approximation, we can roughly replace $(Z_t^{j,N} - Z_{t-1}^{j,N})_{j=1,...,N}$ in the expression of $(\lambda_t^{i,N})_{i=1,...,N}$ by $(\frac{\mu}{1-p} + Y_t^{j,N})_{j=1,...,N}$, for an i.i.d. array $(Y_t^{j,N})_{j=1,...,N,t=1,...,T}$ of $\mathcal{N}(0, \frac{\mu}{1-p})$-distributed random variables. Also, we replace the $\mathcal{P}(\lambda_t^{i,N})$ law by its Gaussian approximation.

We thus introduce the following model, with unknown parameters $\mu > 0$ and $p \in (0, 1)$. We start with three independent families of i.i.d. random variables, namely $(\theta_{ij})_{i,j=1,...,N}$ with law Bernoulli(p), and $(Y_t^{j,N})_{j=1,...,N,t=1,...,T}$ with law $\mathcal{N}(0, \frac{\mu}{1-p})$ and $(A_t^{j,N})_{j=1,...,N,t=1,...,T}$ with law $\mathcal{N}(0, 1)$. We then set, for each $t = 1, \dots, T$ and each $i = 1, \dots, N$,

$$a_t^{i,N} = \mu + \frac{1}{N}\sum_{j=1}^N \theta_{ij}\Big(\frac{\mu}{1-p} + Y_t^{j,N}\Big) \quad \text{and} \quad X_t^{i,N} = a_t^{i,N} + \sqrt{a_t^{i,N}}A_t^{i,N}.$$

We compute the covariances. First, for all $i = 1, \dots, N$ and all $t = 1, \dots, T$,

$$\begin{aligned}
\mathrm{Var}(X_t^{i,N}) &= \mathbb{E}[(a_t^{i,N} + \sqrt{a_t^{i,N}}A_t^{i,N} - \frac{\mu}{1-p})^2] \\
&= \mathbb{E}\Big[\Big(\frac{\mu}{N(1-p)}\sum_{k=1}^N(\theta_{ik} - p) + \frac{1}{N}\sum_{k=1}^N \theta_{ik}Y_t^{k,N} + \sqrt{a_t^{i,N}}A_t^{i,N}\Big)^2\Big] \\
&= \frac{p\mu^2}{N(1-p)} + \frac{p\mu^2}{N(1-p)^2} + \frac{\mu}{(1-p)}.
\end{aligned}$$

Next, for $i \neq j$ and all $t = 1, \dots, T$,

$$\begin{aligned}
\mathrm{Cov}(X_t^{i,N}, X_t^{j,N}) &= \mathbb{E}\Big[\Big(a_t^{i,N} + \sqrt{a_t^{i,N}}A_t^{i,N} - \frac{\mu}{(1-p)}\Big)\Big(a_t^{j,N} + \sqrt{a_t^{j,N}}A_t^{j,N} - \frac{\mu}{(1-p)}\Big)\Big] \\
&= \mathbb{E}\Big[\frac{1}{N^2}\sum_{k=1}^N \theta_{jk}\theta_{ik}(Y_t^{k,N})^2\Big] = \frac{p^2}{N}\frac{\mu^2}{(1-p)^2}.
\end{aligned}$$

For $s \neq t$ and $i = 1, \dots, N$,

$$\begin{aligned}
\mathrm{Cov}(X_t^{i,N}, X_s^{i,N}) =& \mathbb{E}\Big[\Big(a_t^{i,N} + \sqrt{a_t^{i,N}}A_t^{i,N} - \frac{\mu}{(1-p)}\Big)\Big(a_s^{i,N} + \sqrt{a_s^{i,N}}A_s^{i,N} - \frac{\mu}{(1-p)}\Big)\Big] \\
=& \Big(\frac{\mu}{1-p}\Big)^2 \mathrm{Var}\Big(\frac{1}{N}\sum_{j=1}^N \theta_{ij}\Big) = \frac{p\mu^2}{N(1-p)}.
\end{aligned}$$

Finally, for $s \neq t$ and $i \neq j$,

$$\mathrm{Cov}(X_t^{i,N}, X_s^{j,N}) = \mathbb{E}\Big[\Big(a_t^{i,N} + \sqrt{a_t^{i,N}}A_t^{i,N} - \mu - p\Big)\Big(a_s^{j,N} + \sqrt{a_s^{j,N}}A_t^{j,N} - \mu - p\Big)\Big] = 0.$$

Over all we have $\mathrm{Cov}(X_t^{i,N}, X_s^{j,N}) = C_{\mu,p,N}((i,t),(j,s))$, where

$$C_{\mu,p,N}((i,t),(j,s)) = \begin{cases} \frac{p\mu^2}{N(1-p)} + \frac{p\mu^2}{N(1-p)^2} + \frac{\mu}{(1-p)} & \text{if } i=j,\ t=s, \\ \frac{p^2}{N}\frac{\mu^2}{(1-p)^2} & \text{if } i\neq j,\ t=s, \\ \frac{p\mu^2}{N(1-p)} & \text{if } i=j,\ t\neq s, \\ 0 & \text{if } i\neq j,\ t\neq s. \end{cases}$$

From the covariance function above, we can ignore the covariance when $t \neq s$. So, we construct a new covariance function:

$$\widetilde{C}_{\mu,p,N}((i,t),(j,s)) = \begin{cases} \frac{p\mu^2}{N(1-p)} + \frac{p\mu^2}{N(1-p)^2} + \frac{\mu}{(1-p)} & \text{if } i=j,\ t=s, \\ \frac{p^2}{N}\frac{\mu^2}{(1-p)^2} & \text{if } i\neq j,\ t=s, \\ 0 & \text{if } i=j,\ t\neq s, \\ 0 & \text{if } i\neq j,\ t\neq s. \end{cases}$$

We thus consider the following toy model: for two unknown parameters $\mu > 0$ and $p \in (0,1)$, we observe $(U_s^{i,N})_{i=1,\dots,K,s=0,\dots,T}$, for some Gaussian array $(U_s^{i,N})_{i=1,\dots,N,s=0,\dots,T}$ with covariance matrix $\widetilde{C}_{\mu,p,N}$ defined above and we want to estimate p. If assuming that $\frac{\mu}{1-p}$ is known, it is well-known that the temporal empirical variance $S_T^{N,K} = \frac{1}{T}\sum_{t=1}^{T}(\bar{U}_t^{N,K} - \frac{\mu}{1-p})^2$, where $\bar{U}_t^{N,K} = \frac{1}{K}\sum_{i=1}^{K} U_t^{i,N}$, is the best estimator of $\frac{(2p-p^2)\mu^2}{NK(1-p)^2} + \frac{\mu}{K(1-p)} + \frac{p^2(K-1)}{NK}\frac{\mu^2}{(1-p)^2}$, (in all the usual senses). Consequently, $C_T^{N,K} = \frac{N}{K-1}(\frac{\mu}{1-p})^{-2}[KS_T^{N,K} - \frac{\mu}{1-p}]$ is the best estimator of p^2. And

$$\mathrm{Var}(C_T^{N,K}) = \frac{1}{T}\frac{N^2}{(K-1)^2}K^2\frac{1}{K^2}\Big[\rho + \frac{(K-1)\alpha}{N}\Big]^2 \simeq \frac{N^2}{TK^2}.$$

where $\rho = \frac{(2p-p^2)\mu^2}{N(1-p)^2} + \frac{\mu}{(1-p)}$ and $\alpha = \frac{p^2\mu^2}{(1-p)^2}$ Hence for this Gaussian toy model, it is not possible to estimate p^2 (and thus p) with a precision better than $\frac{N}{K}\frac{1}{\sqrt{T}}$.

1.4.3 Conclusion

Using the first example, it seems that it should not be possible to estimate p faster than $N/(\sqrt{K}e^{\alpha_0 t}) + 1/\sqrt{K}$. in the supercritical case. Using the two examples, it seems that it should not be possible to estimate p faster than $N/(t\sqrt{K}) + 1/\sqrt{K} + N/(K\sqrt{t})$ in the subcritical case.

1.5 Analysis of a random matrix in the subcritical case

1.5.1 Some notations

For $r \in [1,\infty)$ and $x \in \mathbb{R}^N$, we set $\|\boldsymbol{x}\|_r = (\sum_{i=1}^{N}|x_i|^r)^{\frac{1}{r}}$, and $\|\boldsymbol{x}\|_\infty = \max_{i=1\dots N}|x_i|$. For M a $N \times N$ matrix, we denote by $|||M|||_r$ is the operator norm associated to $\|\cdot\|_r$, that is $|||M|||_r = \sup_{\boldsymbol{x}\in R^n}\|M\boldsymbol{x}\|_r/\|\boldsymbol{x}\|_r$. We have the special cases

$$|||M|||_1 = \sup_{\{j=1,\dots,N\}}\sum_{i=1}^{N}|M_{ij}|, \quad |||M|||_\infty = \sup_{\{i=1,\dots,N\}}\sum_{j=1}^{N}|M_{ij}|.$$

We also have the inequality

$$|||M|||_r \leq |||M|||_1^{\frac{1}{r}}|||M|||_\infty^{1-\frac{1}{r}} \quad \text{for any} \quad r \in [1,\infty).$$

1.5.2 Some more notations

We define $A_N(i,j) := N^{-1}\theta_{ij}$ and the matrix $(A_N(i,j))_{i,j\in\{1,\dots,N\}}$, as well as $Q_N := (I - \Lambda A_N)^{-1}$ on the event on which $I - \Lambda A_N$ is invertible.

For $1 \le K \le N$, we introduce the N-dimensional vector $\mathbf{1}_K$ defined by $\mathbf{1}_K(i) = \mathbf{1}_{\{1\le i\le K\}}$ for $i = 1,\dots,N$, and the $N\times N$-matrix I_K defined by $I_K(i,j) = \mathbf{1}_{\{i=j\le K\}}$.

We assume here that $\Lambda p \in (0,1)$ and we set $a = \frac{1+\Lambda p}{2} \in (0,1)$. Next, we introduce the events

$$\Omega_N^1 := \Big\{\Lambda|||A_N|||_r \le a, \text{ for all } r\in[1,\infty]\Big\},$$
$$\mathcal{F}_N^{K,1} := \Big\{\Lambda|||I_K A_N|||_r \le \Big(\tfrac{K}{N}\Big)^{\frac{1}{r}} a, \text{for all } r\in[1,\infty)\Big\},$$
$$\mathcal{F}_N^{K,2} := \Big\{\Lambda|||A_N I_K|||_r \le \Big(\tfrac{K}{N}\Big)^{\frac{1}{r}} a, \text{ for all } r\in[1,\infty)\Big\},$$
$$\Omega_{N,K}^1 := \Omega_N^1\cap\mathcal{F}_N^{K,1},\quad \Omega_{N,K}^1 := \Omega_N^1\cap\mathcal{F}_N^{K,2},\quad \Omega_{N,K} = \Omega_{N,K}^1\cap\Omega_{N,K}^2.$$

We set $\boldsymbol{\ell}_N := Q_N\mathbf{1}_N$, $\ell_N(i) := \sum_{j=1}^N Q_N(i,j)$, $\bar\ell_N := \frac{1}{N}\sum_{i=1}^N \ell_N(i)$, $\bar\ell_N^K := \frac{1}{K}\sum_{i=1}^K \ell_N(i)$.

We also set $c_N^K(j) := \sum_{i=1}^K Q_N(i,j)$, $\bar c_N^K := \frac{1}{N}\sum_{j=1}^N c_N^K(j)$.

We let $\boldsymbol{L}_N := A_N\mathbf{1}_N$, $L_N(i) := \sum_{j=1}^N A_N(i,j)$, $\bar L_N := \frac{1}{N}\sum_{i=1}^N L_N(i)$, $\bar L_N^K := \frac{1}{K}\sum_{i=1}^K L_N(i)$ and $\boldsymbol{C}_N := A_N^*\mathbf{1}_N$, $C_N(j) := \sum_{i=1}^N A_N(i,j)$, $\bar C_N := \frac{1}{N}\sum_{j=1}^N C_N(j)$, $\bar C_N^K := \frac{1}{K}\sum_{j=1}^K C_N(i)$ and consider the event

$$\mathcal{A}_N := \{\|\boldsymbol{L}_N - p\mathbf{1}_N\|_2 + \|\boldsymbol{C}_N - p\mathbf{1}_N\|_2 \le N^{\frac14}\}.$$

where $\boldsymbol{L}_N$ is the vectors $(L_N(i))_{i=1,\dots,N}$. We also set $x_N(i) = \ell_N(i) - \bar\ell_N$, $\boldsymbol{x}_N = (x_N(i))_{i=1,\dots,N}$, $X_N(i) = L_N(i) - \bar L_N$ and $\boldsymbol{X}_N = (X_N(i))_{i=1,\dots,N}$. We finally put $X_N^K(i) = (L_N(i) - \bar L_N^K)\mathbf{1}_{\{i\le K\}}$ and $\boldsymbol{X}_N^K = (X_N^K(i))_{i=1,\dots,N} = \boldsymbol{L}_N^K - \bar L_N^K\mathbf{1}_K$, as well as $x_N^K(i) = (\ell_N(i) - \bar\ell_N^K)\mathbf{1}_{\{i\le K\}}$ and $\boldsymbol{x}_N^K = (x_N^K(i))_{i=1,\dots,N} = \boldsymbol{\ell}_N^K - \bar\ell_N^K\mathbf{1}_K$.

1.5.3 Review of some lemmas found in [14]

In this subsection we recall results from [14] showing that $\mathcal{A}_N$ and Ω_N^1 are big, and upper-bounds concerning x_n and $\boldsymbol{X}_N$.

Lemma 1.5.1. *We assume that $\Lambda p < 1$. Then $\Omega_{N,K}\subset\Omega_N^1\subset\{|||Q_N|||_r\le C$, for all $r\in[1,\infty]\}\subset\{\sup_{i=1\dots N}\ell_N(i)\le C\}$, where $C = (1-a)^{-1}$. For any $\alpha>0$, there exists a constant C_α such that*

$$P(\mathcal{A}_N)\ge 1 - C_\alpha N^{-\alpha}.$$

Proof. See [14, Notation 12 and Proposition 14, Step 1].

Lemma 1.5.2. *Assume that $\Lambda p<1$. Then,*

$$P(\Omega_N^1)\ge 1 - C\exp(-cN)$$

for some constants $C>0$ and $c>0$.

Proof. See [14, Lemma 13].

Lemma 1.5.3. *Assume that $\Lambda p<1$. Then*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1}\Big|\bar\ell_N - \frac{1}{1-\Lambda p}\Big|^2\Big]\le\frac{C}{N^2}.$$

Proof. See [14, Proposition 14].

Lemma 1.5.4. *Assume that $\Lambda p < 1$, set $b = \frac{2+\Lambda p}{3}$ and consider N_0 the smallest integer such that $a + \Lambda N_0^{-\frac{1}{4}} \leq b$. For all $N \geq N_0$,*

$$(i)\mathbf{1}_{\Omega_N^1 \cap \mathcal{A}_N} \|\boldsymbol{x}_N\|_2 \leq C\|\boldsymbol{X}_N\|_2, \quad (ii)\mathbb{E}[\|\boldsymbol{X}_N\|_2^4] \leq C, \quad (iii)\mathbb{E}[\|A_N \boldsymbol{X}_{\boldsymbol{N}}\|_2^2] \leq CN^{-1}.$$

Proof. See [14, Proof of Proposition 14, Steps 2 and 4].

Remark 1.5.5. In Lemma 1.5.4, the condition $\Lambda p < 1$ is not necessary for (ii) and (iii).

Lemma 1.5.6. *Assume that $\Lambda p < 1$ and set $k := \Lambda^{-1} \int_0^\infty s\phi(s)ds$, then for $n \geq 0$, $t \geq 0$,*

$$\int_0^t s\phi^{*n}(t-s)ds = \Lambda^n t - n\Lambda^n k + \varepsilon_n(t),$$

where $0 \leq \varepsilon_n(t) \leq C \min\{n^q \Lambda^n t^{1-q}, n\Lambda^n k\}$ and where $\phi^{*n}(s)$ is the n-times convolution of ϕ. We adopt the convention that $\phi^{*0} = \delta_0$, whence in particular $\int_0^t s\phi^{*0}(t-s)ds = t$.

Proof. See [14, Lemma 15].

1.5.4 Other preparation

In this subsection, we are going to prove that the set $\Omega_{N,K}$ is big, which will allow to work on the set $\Omega_{N,K}$ for all our study.

Lemma 1.5.7. *Assume that $\Lambda p < 1$. It holds that*

$$P(\Omega_{N,K}) \geq 1 - CNe^{-cK}$$

for some constants $C > 0$ and $c > 0$.

Proof. On $\Omega_{N,K}^1$, we have

$$N|||I_K A_N|||_1 = \sup_{j=1,\dots,N} \sum_{i=1}^K \theta_{ij} = \max\{X_1^{N,K}, ..., X_N^{N,K}\},$$

where $X_i^{N,K} = \sum_{j=1}^K \theta_{ij}$ for $i = 1, ..., N$ are i.i.d and Binomial(K, p)-distributed. So,

$$\begin{aligned} P\Big(\Lambda\frac{N}{K}|||I_K A_N|||_1 \geq a\Big) &= P\Big(\max\{X_1^{N,K}, ...X_N^{N,K}\} \geq \frac{Ka}{\Lambda}\Big) \leq NP\Big(X_1^{N,K} \geq \frac{Ka}{\Lambda}\Big) \\ &\leq NP\Big(|X_1^{N,K} - Kp| \geq K\Big(\frac{a}{\Lambda} - p\Big)\Big) \leq 2Ne^{-2K(\frac{a}{\Lambda}-p)^2}. \end{aligned}$$

The last equality follows from Hoeffding inequality. On the event $\Omega_N^1 \cap \{\Lambda\frac{N}{K}|||I_K A_N|||_1 \leq a\}$, we have

$$|||I_K A_N|||_r \leq |||I_K A_N|||_1^{\frac{1}{r}} \|I_K A_N|||_\infty^{1-\frac{1}{r}} \leq |||I_K A_N|||_1^{\frac{1}{r}} ||A_N|||_\infty^{1-\frac{1}{r}} \leq \Big(\frac{a}{\Lambda}\frac{K}{N}\Big)^{\frac{1}{r}}\Big(\frac{a}{\Lambda}\Big)^{1-\frac{1}{r}} = \frac{a}{\Lambda}\Big(\frac{K}{N}\Big)^{\frac{1}{r}}.$$

We conclude that $\Omega_{N,K}^1 = \Omega_N^1 \cap \{(\frac{N}{K})|||I_K A_N|||_1 \leq a\}$. And by Lemma 1.5.2, we deduce that $P(\Omega_{N,K}^1) \geq 1 - CNe^{-cK}$. By the same way, we prove that $P(\Omega_{N,K}^2) \geq 1 - CNe^{-cK}$. Finally by the definition of $\Omega_{N,K}$, we have $P(\Omega_{N,K}) \geq 1 - CNe^{-cK}$.

1.5.5 Matrix analysis for the first estimator

The aim of this subsection is to prove that $\bar{\ell}_N^K \simeq 1/(1-\Lambda p)$ and to study the rate of convergence.

Lemma 1.5.8. *Assume $\Lambda p < 1$. Then*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}|\bar{\ell}_N^K - 1 - \Lambda p \bar{\ell}_N|^2\Big] \le \frac{C}{NK}.$$

Proof. Recall that $\boldsymbol{\ell}_N = Q_N \mathbf{1}_{\boldsymbol{N}}$, whence $Q_N^{-1}\boldsymbol{\ell}_N = \mathbf{1}_N$. And since, $Q_N = (I - \Lambda A_N)^{-1}$, we have $Q_N^{-1}\boldsymbol{\ell}_N = (I - \Lambda A_N)\boldsymbol{\ell}_N = \mathbf{1}_N$ and thus $\boldsymbol{\ell}_N = \mathbf{1}_{\boldsymbol{N}} + \Lambda A_N \ell_N$. We conclude that

$$\bar{\ell}_N^K = \frac{1}{K}(\boldsymbol{\ell}_N, \mathbf{1}_K) = 1 + \frac{\Lambda}{K}\sum_{i=1}^K\sum_{j=1}^N A(i,j)\ell_N(j) = 1 + \frac{\Lambda}{K}\sum_{j=1}^N C_N^K(j)\ell_N(j),$$

where $C_N^K(j) := \sum_{i=1}^K A(i,j) = \frac{1}{N}\sum_{i=1}^K \theta_{ij}$. By some easy computing, we have

$$\mathbb{E}\Big[\Big(\sum_{j=1}^N \Big[C_N^K(j) - \frac{Kp}{N}\Big]^2\Big)^2\Big] \le C\frac{K^2}{N^2}, \tag{1.6}$$

whence

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\bar{\ell}_N^K - 1 - \Lambda p\bar{\ell}_N\Big|^2\Big]\\
=&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\frac{\Lambda}{K}\sum_{j=1}^N\Big(C_N^K(j) - \frac{Kp}{N}\Big)\ell_N(j)\Big|^2\Big]\\
=&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\frac{\Lambda}{K}\sum_{j=1}^N\Big(C_N^K(j) - \frac{Kp}{N}\Big)(\ell_N(j) - \bar{\ell}_N) + \bar{\ell}_N\frac{\Lambda}{K}\sum_{j=1}^N\Big(C_N^K(j) - \frac{Kp}{N}\Big)\Big|^2\Big]\\
\le&2\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\frac{\Lambda}{K}\sum_{j=1}^N\Big(C_N^K(j) - \frac{Kp}{N}\Big)(\ell_N(j) - \bar{\ell}_N)\Big|^2\Big]\\
&+ 2\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\bar{\ell}_N\frac{\Lambda}{K}\sum_{j=1}^N\Big(C_N^K(j) - \frac{Kp}{N}\Big)\Big|^2\Big].
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\bar{\ell}_N^K - 1 - \Lambda p\bar{\ell}_N\Big|^2\Big]\\
\le&2\Big(\frac{\Lambda}{K}\Big)^2\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{x}_N\|_2^4\Big]^{\frac{1}{2}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big(\sum_{j=1}^N\Big(C_N^K(j) - \frac{Kp}{N}\Big)^2\Big)^2\Big]^{\frac{1}{2}}\\
&+ 2\frac{\Lambda^2}{K^2}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\bar{\ell}_N\Big|^2\Big|\sum_{j=1}^N\Big(C_N^K(j) - \frac{Kp}{N}\Big)\Big|^2\Big]\\
\le&\frac{C}{NK}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{x}_N\|_2^4\Big]^{\frac{1}{2}} + 2\frac{\Lambda^2}{K^2}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\bar{\ell}_N\Big|^2\Big|\sum_{j=1}^N\Big(C_N^K(j) - \frac{Kp}{N}\Big)\Big|^2\Big].
\end{aligned}$$

By Lemma 1.5.4, we know that $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_\mathcal{N}}\|\boldsymbol{x}_N\|_2^4] \le C$. By Lemma 1.5.1, $\bar{\ell}_N$ and $\bar{\ell}_N^K$ are bounded on the set $\Omega_{N,K}$, whence, recalling (1.6), and since $\{C_N^K(j) - \frac{Kp}{N}\}_{j=1,\dots,N}$ are independent, we

conclude that

$$\begin{aligned}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\bar{\ell}_N\Big|^2\Big|\sum_{j=1}^N\Big(C_N^K(j)-\frac{Kp}{N}\Big)\Big|^2\Big]\leq& C\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\sum_{j=1}^N\Big(C_N^K(j)-\frac{Kp}{N}\Big)\Big|^2\Big]\\ \leq& C\mathbb{E}\Big[\sum_{j=1}^N\Big(C_N^K(j)-\frac{Kp}{N}\Big)^2\Big]\\ \leq&\frac{CK}{N}.\end{aligned}$$

Hence

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_\mathcal{N}}\Big|\bar{\ell}_N^K-1-\Lambda p\bar{\ell}_N\Big|^2\Big]\leq\frac{C}{NK}.$$

We finally apply Lemma 1.5.1 with e.g. $\alpha=2$ and get

$$\begin{aligned}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar{\ell}_N^K-1-\Lambda p\bar{\ell}_N\Big|^2\Big]&=\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_\mathcal{N}}\Big|\bar{\ell}_N^K-1-\Lambda p\bar{\ell}_N\Big|^2\Big]+\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_\mathcal{N}^c}\Big|\bar{\ell}_N^K-1-\Lambda p\bar{\ell}_N\Big|^2\Big]\\ &\leq\frac{c}{NK}+\frac{C}{N^2}\leq\frac{C}{NK}.\end{aligned}$$

The next lemma is the main result of the subsection.

Lemma 1.5.9. *If $\Lambda p<1$, we have*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar{\ell}_N^K-\frac{1}{1-\Lambda p}\Big|^2\Big]\leq\frac{C}{NK}.$$

Proof. Observing that $1/(1-\Lambda p)=1+\Lambda p/(1-\Lambda p)$, we write

$$\begin{aligned}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar{\ell}_N^K-\frac{1}{1-\Lambda p}\Big|^2\Big]\leq&2\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar{\ell}_N^K-1-\Lambda p\bar{\ell}_N\Big|^2\Big]+2\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\Lambda p\bar{\ell}_N-\frac{\Lambda p}{1-\Lambda p}\Big|^2\Big]\\ \leq&2\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar{\ell}_N^K-1-\Lambda p\bar{\ell}_N\Big|^2\Big]+2(\Lambda p)^2\mathbb{E}\Big[\mathbf{1}_{\Omega_N^1}\Big|\bar{\ell}_N-\frac{1}{1-\Lambda p}\Big|^2\Big].\end{aligned}$$

We complete the proof applying Lemmas 1.5.3 and 1.5.8.

1.5.6 Matrix analysis for the second estimator

The aim of this subsection is to prove that $\frac{N}{K}\|\boldsymbol{x}_N^K\|_2^2\simeq\Lambda^2p(1-p)/(1-\Lambda p)^2$ and to study the rate of convergence.

Lemma 1.5.10. *Assume that $p\in(0,1]$. It holds that*

$$\mathbb{E}[\|I_KA_N\boldsymbol{X}_N\|_2^2]\leq CKN^{-2}.$$

Proof. By Lemma 1.5.4, we already know that $\mathbb{E}[\|A_N\boldsymbol{X}_N\|_2^2]\leq\frac{C}{N}$, whence

$$\mathbb{E}\Big[\|I_KA_N\boldsymbol{X}_N\|_2^2\Big]=\sum_{i=1}^K\mathbb{E}\Big[\Big(\sum_{j=1}^N\frac{\theta_{ij}}{N}(L_N(j)-\bar{L}_N)\Big)^2\Big]=\frac{K}{N^2}\mathbb{E}\Big[\Big(\sum_{j=1}^N\theta_{1j}(L_N(j)-\bar{L}_N)\Big)^2\Big],$$

which equals $\frac{K}{N}\mathbb{E}[\|A_N\boldsymbol{X}_N\|_2^2]$ and thus is bounded by CKN^{-2}.

Lemma 1.5.11. *Assume that $\Lambda p<1$. It holds that*

$$\mathbb{E}[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{x}_N^K-\bar{\ell}_N\Lambda\boldsymbol{X}_N^K\|_2^2]\leq CN^{-1}.$$

Proof. By definition, $\boldsymbol{\ell}_N^K = I_K \boldsymbol{\ell}_N = \mathbf{1}_K + \Lambda I_K A_N \boldsymbol{\ell}_N$, so that

$$\bar{\ell}_N^K = \frac{1}{K}(\mathbf{1}_K, \boldsymbol{\ell}_N^K) = \frac{1}{K}(\mathbf{1}_K, I_K\boldsymbol{\ell}_N) = \frac{1}{K}(\mathbf{1}_K, \mathbf{1}_K + \Lambda I_K A_N \boldsymbol{\ell}_N) = 1 + \frac{\Lambda}{K}(I_K A_N \boldsymbol{\ell}_N, \mathbf{1}_K).$$

And, recalling that $\boldsymbol{x}_N^K = \boldsymbol{\ell}_N^K - \bar{\ell}_N^K \mathbf{1}_K$, we find

$$\begin{aligned}
\boldsymbol{x}_N^K &= \mathbf{1}_K + \Lambda I_K A_N \boldsymbol{\ell}_N - [1 + \frac{\Lambda}{K}(I_K A_N \boldsymbol{\ell}_N, \mathbf{1}_K)]\mathbf{1}_K \\
&= \Lambda I_K A_N \boldsymbol{\ell}_N - \frac{\Lambda}{K}(I_K A_N \boldsymbol{\ell}_N, \mathbf{1}_K)\mathbf{1}_K \\
&= \Lambda I_K A_N (\boldsymbol{\ell}_N - \bar{\ell}_N \mathbf{1}_N) - \frac{\Lambda}{K}(I_K A_N(\boldsymbol{\ell}_N - \bar{\ell}_N \mathbf{1}_N), \mathbf{1}_K)\mathbf{1}_K \\
&\quad + \bar{\ell}_N[\Lambda I_K A_N \mathbf{1}_N - \frac{\Lambda}{K}(I_K A_N \mathbf{1}_N, \mathbf{1}_K)\mathbf{1}_K] \\
&= \Lambda I_K A_N \boldsymbol{x}_N - \frac{\Lambda}{K}(I_K A_N \boldsymbol{x}_N, \mathbf{1}_K)\mathbf{1}_K + \bar{\ell}_N[\Lambda I_K A_N \mathbf{1}_N - \frac{\Lambda}{K}(I_K A_N \mathbf{1}_N, \mathbf{1}_K)\mathbf{1}_K] \\
&= \Lambda I_K A_N \boldsymbol{x}_N - \frac{\Lambda}{K}(I_K A_N \boldsymbol{x}_N, \mathbf{1}_K)\mathbf{1}_K + \Lambda \bar{\ell}_N \boldsymbol{X}_N^K.
\end{aligned}$$

We deduce that

$$\begin{aligned}
\boldsymbol{x}_N^K - \Lambda\bar{\ell}_N \boldsymbol{X}_N^K &= \Lambda I_K A_N \boldsymbol{x}_N - \frac{\Lambda}{K}(I_K A_N \boldsymbol{x}_N, \mathbf{1}_K)\mathbf{1}_K \\
&= \Lambda I_K A_N(\boldsymbol{x}_N - \Lambda\bar{\ell}_N \boldsymbol{X}_N) - \frac{\Lambda}{K}(I_K A_N \boldsymbol{x}_N, \mathbf{1}_K)\mathbf{1}_K + \bar{\ell}_N \Lambda^2 I_K A_N \boldsymbol{X}_N \\
&= \Lambda I_K A_N(\boldsymbol{x}_N - \Lambda\bar{\ell}_N \boldsymbol{X}_N) + \bar{\ell}_N \Lambda^2 I_K A_N \boldsymbol{X}_N - \frac{\Lambda}{K}\Big[\sum_{i=1}^K \sum_{j=1}^N A_N(i,j) x_N(j)\Big]\mathbf{1}_K \\
&= \Lambda I_K A_N(\boldsymbol{x}_N - \Lambda\bar{\ell}_N \boldsymbol{X}_N) + \bar{\ell}_N \Lambda^2 I_K A_N \boldsymbol{X}_N - \frac{\Lambda}{K}\sum_{j=1}^N \Big[C_N^K(j) - \frac{K}{N}p\Big] x_N(j)\mathbf{1}_K.
\end{aligned}$$

In the last step, we used that $\sum_{i=1}^N x(i) = 0$. As a conclusion,

$$\begin{aligned}
\|\boldsymbol{x}_N^K - \bar{\ell}_N \Lambda \boldsymbol{X}_N^K\|_2^2 \leq & 3(\Lambda \|I_K A_N(\boldsymbol{x}_N - \bar{\ell}_N \boldsymbol{X}_N)\|_2)^2 + 3(\Lambda^2 \bar{\ell}_N \|I_K A_N \boldsymbol{X}_N\|_2)^2 \\
& + 3\Lambda^2 K^{-1}\Big(\sum_{j=1}^N \Big[C_N^K(j) - \frac{K}{N}p\Big] x_N(j)\Big)^2.
\end{aligned}$$

By the Cauchy-Schwarz inequality, (1.6) and Lemma 1.5.4, we have

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big(\sum_{j=1}^N \Big[C_N^K(j) - \frac{K}{N}p\Big] x_N(j)\Big)^2\Big] \\
\leq &\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big(\sum_{j=1}^N \Big[C_N^K(j) - \frac{K}{N}p\Big]^2\Big)\Big]\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big(\sum_{j=1}^N x_N^2(j)\Big)\Big] \leq \frac{CK}{N}.
\end{aligned}$$

We also know from [14, Proposition 14, step 7, line 12] that $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{x}_N - \bar{\ell}_N \boldsymbol{X}_N\|_2^2] \leq \frac{C}{N}$. And also, by the definition, $\|A_N\|_2$ is bounded on $\mathcal{A}_\mathcal{N}$. So

$$\mathbb{E}\Big[|||I_K A_N|||_2^2 \|\boldsymbol{x}_N - \bar{\ell}_N \boldsymbol{X}_N\|_2^2\Big] \leq \frac{C}{N}.$$

Recalling Lemma 1.5.10 and that $\bar{\ell}_N$ is bounded on $\Omega_{N,K}$, the conclusion follows.

Lemma 1.5.12. *Assume that $\Lambda p < 1$. It holds that*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\|\boldsymbol{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\boldsymbol{X}_N^K\|_2^2\Big|\Big] \leq \frac{C\sqrt{K}}{N}.$$

Proof. We start from

$$\Big|\|\boldsymbol{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\boldsymbol{X}_N^K\|_2^2\Big| \le \|\boldsymbol{x}_N^K - (\Lambda\bar{\ell}_N)\boldsymbol{X}_N^K\|_2(\|\boldsymbol{x}_N^K\|_2 + (\Lambda\bar{\ell}_N)\|\boldsymbol{X}_N^K\|_2),$$

whence

$$\begin{aligned}&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\|\boldsymbol{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\boldsymbol{X}_N^K\|_2^2\Big|\Big]\\ &\le \mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{x}_N^K - (\Lambda\bar{\ell}_N)\boldsymbol{X}_N^K\|_2(\|\boldsymbol{x}_N^K\|_2 + (\Lambda\bar{\ell}_N)\|\boldsymbol{X}_N^K\|_2)\Big].\end{aligned}$$

By the Cauchy-Schwarz inequality.

$$\begin{aligned}&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{x}_N^K - (\Lambda\bar{\ell}_N)\boldsymbol{X}_N^K\|_2\Big(\|\boldsymbol{x}_N^K\|_2 + (\Lambda\bar{\ell}_N)\|\boldsymbol{X}_N^K\|_2\Big)\Big]\\ \le&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{x}_N^K - \bar{\ell}_N\Lambda\boldsymbol{X}_N^K\|_2^2\Big]^{\frac{1}{2}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big(\|\boldsymbol{x}_N^K\|_2 + (\Lambda\bar{\ell}_N)\|\boldsymbol{X}_N^K\|_2\Big)^2\Big]^{\frac{1}{2}}.\end{aligned}$$

Lemma 1.5.11 directly tells us that $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{x}_N^K - \bar{\ell}_N\Lambda\boldsymbol{X}_N^K\|_2^2] \le C/N$.

Next, it is easy to prove, using that $\|\boldsymbol{X}_N^K\|_2^2 = \sum_{i=1}^K (L_N(i) - \bar{L}_N^K)$, that $NL_N(1), \dots, NL_N(K)$ are i.i.d. and Binomial(N,p), that

$$\mathbb{E}\Big[\Big(\frac{N}{K}\|\boldsymbol{X}_N^K\|_2^2 - p(1-p)\Big)^2\Big] \le CK^{-1}, \tag{1.7}$$

whence, recalling that $\bar{\ell}_N$ is bounded on $\Omega_{N,K}$,

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}(\bar{\ell}_N)^2\frac{N}{K}\|\boldsymbol{X}_N^K\|_2^2\Big] \le C\mathbb{E}\Big[\Big(\frac{N}{K}\|\boldsymbol{X}_N^K\|_2^2 - p(1-p)\Big)^2\Big]^{\frac{1}{2}} + C \le C.$$

Then, by Lemma 1.5.11 again,

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{x}_N^K\|_2^2\Big] \le 2\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{x}_N^K - \bar{\ell}_N\Lambda\boldsymbol{X}_N^K\|_2^2\Big] + 2\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\bar{\ell}_N\Lambda\boldsymbol{X}_N^K\|_2^2\Big] \le C\frac{K}{N}.$$

The conclusion follows.

Lemma 1.5.13. *Assume that $\Lambda p < 1$. It holds that*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\frac{N}{K}(\bar{\ell}_N)^2\|\boldsymbol{X}_N^K\|_2^2 - \frac{p(1-p)}{(1-\Lambda p)^2}\Big|\Big] \le \frac{C}{\sqrt{K}}.$$

Proof. We define

$$d_N^K = \mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\frac{N}{K}(\bar{\ell}_N)^2\|\boldsymbol{X}_N^K\|_2^2 - \frac{p(1-p)}{(1-\Lambda p)^2}\Big|\Big].$$

Then $d_N^K \le a_N^K + b_N^K$, where

$$\begin{aligned}a_N^K &= \frac{N}{K}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|(\bar{\ell}_N)^2 - (1-\Lambda p)^{-2}\Big|\|\boldsymbol{X}_N^K\|_2^2\Big],\\ b_N^K &= (1-\Lambda p)^{-2}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\frac{N}{K}\|\boldsymbol{X}_N^K\|_2^2 - p(1-p)\Big|\Big].\end{aligned}$$

First, (1.7) directly implies that $b_N^K \le C/\sqrt{K}$. Next, (1.7) also implies that $\mathbb{E}[(\frac{N}{K})^2\|\boldsymbol{X}_N^K\|_2^4] \le C$, whence $a_N^K \le C/\sqrt{K}$ by Lemma 1.5.3. This completes the proof.

Here is the main lemma of this subsection.

Lemma 1.5.14. *Assume that $\Lambda p < 1$. It holds that*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\frac{N}{K}\|\boldsymbol{x}_N^K\|_2^2 - \frac{\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big|\Big] \le \frac{C}{\sqrt{K}}.$$

Proof. It directly follows from Lemmas 1.5.12 and 1.5.13 that

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\frac{N}{K}\|\boldsymbol{x}_N^K\|_2^2-\frac{\Lambda^2p(1-p)}{(1-\Lambda p)^2}\Big|\Big]\\
\leq&\Lambda^2\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\frac{N}{K}(\bar\ell_N)^2\|\boldsymbol{X}_N^K\|_2^2-\frac{p(1-p)}{(1-\Lambda p)^2}\Big|\Big]\\
&+\frac{N}{K}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\|\boldsymbol{x}_N^K\|_2^2-(\Lambda\bar\ell_N)^2\|\boldsymbol{X}_N^K\|_2^2\Big|\Big]\\
\leq&C\Big(\frac{1}{\sqrt{K}}+\frac{N}{K}\frac{\sqrt{K}}{N}\Big),
\end{aligned}$$

from which the conclusion.

1.5.7 Matrix analysis for the third estimator

We define $\mathcal{W}_{\infty,\infty}^{N,K}:=\frac{\mu N}{K^2}\sum_{j=1}^N(c_N^K(j))^2\ell_N(j)-\frac{N-K}{K}\bar\ell_N^K$, $\mathcal{X}_{\infty,\infty}^{N,K}:=\mathcal{W}_{\infty,\infty}^{N,K}-\frac{\mu(N-K)}{K}\bar\ell_N^K$. The aim of this subsection is to prove that $\mathcal{X}_{\infty,\infty}^{N,K}\simeq\mu/(1-\Lambda p)^3$ and to study the rate of convergence.

Lemma 1.5.15. *Assume that $\Lambda p<1$. It holds that*

$$\mathbb{E}[||\boldsymbol{F}_N^K||_2^2]\leq\frac{CK}{N},$$

where $\boldsymbol{F}_N^K:=\mathbf{1}_K^TA_N-\frac{1}{N}(\mathbf{1}_K^TA_N,\mathbf{1}_N^T)\mathbf{1}_N^T$ is a row vector.

Proof. Since the inequality $\sum_{i=1}^n(x_i-\bar x)^2\leq\sum_{i=1}^n(x_i-p)^2$, where $\bar x=\frac{1}{N}\sum_{i=1}^n x_i$, is correct for any real sequence $\{x_i\}_{i=1,\dots,n}$ and real number p. By definition,

$$\begin{aligned}
\mathbb{E}[||\boldsymbol{F}_N^K||_2^2]&=\mathbb{E}\Big[\sum_{j=1}^N\Big\{\frac{1}{N}\sum_{i=1}^K\theta_{ij}-\frac{1}{N^2}\sum_{i=1}^K\sum_{l=1}^N\theta_{il}\Big\}^2\Big]\\
&\leq\mathbb{E}\Big[\sum_{j=1}^N\Big\{\frac{1}{N}\sum_{i=1}^K\theta_{ij}-\frac{Kp}{N}\Big\}^2\Big]\leq\frac{1}{N}\mathbb{E}\Big[\Big\{\sum_{i=1}^K(\theta_{i1}-p)\Big\}^2\Big]\leq\frac{CK}{N}.
\end{aligned}$$

Lemma 1.5.16. *Assume that $\Lambda p<1$. It holds that*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\|\boldsymbol{t}_N^K\|_2^2\Big]\leq C\frac{K^2}{N^2}.$$

where $\boldsymbol{c}_N^K=\mathbf{1}_K^TQ_N$, $\bar c_N^K:=\frac{1}{N}\sum_{j=1}^N c_N^K(j)$, and

$$\boldsymbol{t}_N^K:=\boldsymbol{c}_N^K-\bar c_N^K\mathbf{1}_N^T-\mathbf{1}_K^T+\frac{K}{N}\mathbf{1}_N^T.$$

Proof. By definition, $\boldsymbol{c}_N^K:=\mathbf{1}_K^TQ_N$, $\bar c_N^K=\frac{1}{N}(\boldsymbol{c}_N^K,\mathbf{1}_N)$, $Q_N=(I-\Lambda A_N)^{-1}$, so that

$$\boldsymbol{c}_N^K=\mathbf{1}_K^T+\Lambda\boldsymbol{c}_N^KA_N,\qquad\bar c_N^K=\frac{1}{N}(\boldsymbol{c}_N^K,\mathbf{1}_N^T)=\frac{K}{N}+\frac{\Lambda}{N}(\boldsymbol{c}_N^KA_N,\mathbf{1}_N^T).$$

We deduce that

$$\begin{aligned}
\boldsymbol{t}_N^K&=\boldsymbol{c}_N^K-\bar c_N^K\mathbf{1}_N^T-\mathbf{1}_K^T+\frac{K}{N}\mathbf{1}_N^T\\
&=\Lambda\boldsymbol{c}_N^KA_N-\frac{\Lambda}{N}(\boldsymbol{c}_N^KA_N,\mathbf{1}_N^T)\mathbf{1}_N^T\\
&=\Lambda\boldsymbol{t}_N^KA_N-\frac{\Lambda}{N}(\boldsymbol{t}_N^KA_N,\mathbf{1}_N^T)\mathbf{1}_N^T+\Lambda\bar c_N^K\mathbf{1}_N^TA_N-\frac{\Lambda}{N}\bar c_N^K(\mathbf{1}_N^TA_N,1_N^T)\mathbf{1}_N^T\\
&\quad+\Lambda\mathbf{1}_K^TA_N-\frac{\Lambda}{N}(\mathbf{1}_K^TA_N,\mathbf{1}_N^T)\mathbf{1}_N^T-\Lambda\frac{K}{N}\mathbf{1}_N^TA_N+\frac{\Lambda}{N}\frac{K}{N}(\mathbf{1}_N^TA_N,\mathbf{1}_N^T)\\
&=\Lambda\boldsymbol{t}_N^KA_N-\frac{\Lambda}{N}(\boldsymbol{t}_N^KA_N,\mathbf{1}_N^T)\mathbf{1}_N^T+\Lambda\bar c_N^K\boldsymbol{X}_N^T-\Lambda\boldsymbol{F}_N^K-\Lambda\frac{K}{N}\boldsymbol{X}_N^T.
\end{aligned}\tag{1.8}$$

where $\boldsymbol{X}_N^T = \mathbf{1}_N^T A_N - \frac{1}{N}(\mathbf{1}_N^T A_N, \mathbf{1}_N^T)$. And it is clear that $\frac{N}{K}\bar{c}_N^K = \bar{\ell}_N^K$.

By Lemma 1.5.1, $\bar{\ell}_N$ and $\bar{\ell}_N^K$ are bounded on the set $\Omega_{N,K}$, whence, using Lemma 1.5.4,

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big(||\Lambda\bar{c}_N^K \boldsymbol{X}_N^T||_2^2 + ||\Lambda\frac{K}{N}\boldsymbol{X}_N^T||_2^2\Big)\Big] \leq C\frac{K^2}{N^2}.$$

Next, Lemma 1.5.15 tells us that

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\|\boldsymbol{F}_N^K\|_2^2\Big] \leq \mathbb{E}\Big[\|\boldsymbol{F}_N^K\|_2^2\Big] \leq \frac{CK^2}{N^2}.$$

Observing that $\sum_{i=1}^N t_N^K(i) = 0$, we see that

$$\begin{aligned}
||\frac{\Lambda}{N}(\boldsymbol{t}_N^K A_N, \mathbf{1}_N^T)\mathbf{1}_N^T||_2^2 =& \frac{\Lambda^2}{N}(\boldsymbol{t}_N^K A_N, \mathbf{1}_N^T)^2 \\
=& \frac{\Lambda^2}{N^3}\Big(\sum_{i=1}^N\sum_{j=1}^N(\theta_{ij}-p)t_N^K(i)\Big)^2 \\
=& \frac{\Lambda^2}{N}\Big(\sum_{i=1}^N(C_N(i)-p)t_N^K(i)\Big)^2,
\end{aligned}$$

so that

$$\begin{aligned}
\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\frac{\Lambda}{N}(\boldsymbol{t}_N^K A_N, \mathbf{1}_N^T)\mathbf{1}_N^T\|_2^2\Big] \leq& \frac{\Lambda^2}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{t}_N^K\|_2^2\|C_N - p\mathbf{1}_N\|_2^2\Big] \\
\leq& \frac{\Lambda^2}{N^{1/2}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{t}_N^K\|_2^2\Big]
\end{aligned}$$

by definition of $\mathcal{A}_N$. Since finally $||\Lambda\boldsymbol{t}_N^K A_N||_2 \leq |||\Lambda A_N|||_2||\Lambda\boldsymbol{t}_N^K||_2 \leq a||\Lambda\boldsymbol{t}_N^K||_2$ on $\Omega_{N,K}$ with $a = (1+\Lambda p)/2$, we conclude that

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{t}_N^K\|_2^2\Big] \leq C\frac{K^2}{N^2} + (a + \Lambda^2 N^{-1/2})\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{t}_N^K\|_2^2\Big].$$

Since $(a + \Lambda^2 N^{-1/2}) < (a+1)/2 < 1$ for all N large enough, we conclude that, for some constant $C > 0$, for all $N \geq 1$,

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{t}_N^K\|_2^2\Big] \leq C\frac{K^2}{N^2}.$$

Finally, observing that $\|\boldsymbol{t}_N^K\|_2^2$ is obviously bounded by CN on $\Omega_{N,K}$ and recalling that $\mathbb{P}(\mathcal{A}_N) \geq 1 - C/N^3$ by Lemma 1.5.1, we easily conclude that

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\|\boldsymbol{t}_N^K\|_2^2\Big] \leq C\frac{K^2+1}{N^2} \leq C\frac{K^2}{N^2}$$

as desired.

Lemma 1.5.17. *Assume that $\Lambda p < 1$. It holds that*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}||\boldsymbol{f}_N^K||_2^2\Big] \leq C\frac{K^2}{N^2}, \quad \mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\boldsymbol{f}_N^K, \mathbf{1}_K^T)\Big|\Big] \leq \frac{CK}{N}.$$

where $\boldsymbol{f}_N^K := \boldsymbol{t}_N^K I_K$.

Proof. The first inequality is obvious from Lemma 1.5.16 because $||\boldsymbol{f}_N^K|| \le ||\boldsymbol{t}_N^K||$. For the second inequality, by (1.8), we have

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\boldsymbol{f}_N^K, \mathbf{1}_K^T)\Big|\Big] = \mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\boldsymbol{t}_N^K I_K, \mathbf{1}_K^T)\Big|\Big] \\
&= \mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\Big(\Lambda \boldsymbol{t}_N^K A_N I_K - \frac{\Lambda}{N}(\boldsymbol{t}_N^K A_N, \mathbf{1}_N^T)\mathbf{1}_K^T + \Lambda \bar{\boldsymbol{c}}_N^K \boldsymbol{X}_N^T I_K - \Lambda \boldsymbol{F}_N^K I_K - \Lambda \frac{K}{N}\boldsymbol{X}_N^T I_K, \mathbf{1}_K^T\Big)\Big|\Big] \\
&\le \frac{CK}{N}\mathbb{E}\Big[\Big|(\boldsymbol{X}_N^T I_K, \mathbf{1}_K^T)\Big|\Big] + C\mathbb{E}\Big[\Big|(\boldsymbol{F}_N^K I_K, \mathbf{1}_K^T)\Big|\Big] \\
&\quad + \frac{CK}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\boldsymbol{t}_N^K A_N, \mathbf{1}_N^T)\Big|\Big] + \Lambda\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\boldsymbol{t}_N^K A_N I_K, \mathbf{1}_K^T)\Big|\Big].
\end{aligned}$$

We used that $(N/K)\bar{c}_N^K = \bar{\ell}_N^K$ is bounded on $\Omega_{N,K}$. First,

$$\begin{aligned}
\mathbb{E}\Big[\Big|(\boldsymbol{X}_N^T I_K, \mathbf{1}_K^T)\Big|^2\Big] &= \mathbb{E}\Big[\Big|\sum_{i=1}^K X_N(i)\Big|^2\Big] = \mathbb{E}\Big[\Big|\sum_{i=1}^K \Big(L_N(i) - p\Big) + K(p - \bar{L}_N)\Big|^2\Big] \\
&\le 2\mathbb{E}\Big[\Big|\sum_{i=1}^K \Big(L_N(i) - p\Big)\Big|^2\Big] + 2K^2\mathbb{E}\Big[(p - \bar{L}_N)^2\Big] \le \frac{CK}{N} \le C,
\end{aligned}$$

using only that $NL_N(1), \dots, NL_N(N)$ are i.i.d. and Binomial(N, p)-distributed. Next,

$$\begin{aligned}
(\boldsymbol{F}_N^K I_K, \mathbf{1}_K) &= \frac{1}{N}\Big(\sum_{j=1}^K\sum_{i=1}^K \theta_{ij} - \frac{K}{N}\sum_{i=1}^K\sum_{j=1}^N \theta_{ij}\Big) \\
&= \frac{1}{N}\Big[\frac{N-K}{N}\sum_{j=1}^K\sum_{i=1}^K(\theta_{ij} - p) - \frac{K}{N}\sum_{i=1}^K\sum_{j=k+1}^N(\theta_{ij} - p)\Big],
\end{aligned}$$

so that

$$\mathbb{E}[|(\boldsymbol{F}_N^K I_K, \mathbf{1}_K^T)|] \le \mathbb{E}\Big[\Big|(\boldsymbol{F}_N^K I_K, \mathbf{1}_K^T)\Big|^2\Big]^{\frac{1}{2}} \le \frac{C}{N}\Big[\frac{N-K}{N}K + \frac{K}{N}\sqrt{K(N-K)}\Big] \le \frac{CK}{N}.$$

Next, since $\sum_{i=1}^N t_N^K(i) = 0$,

$$\begin{aligned}
\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\boldsymbol{t}_N^K A_N, \mathbf{1}_N^T)\Big|\Big] &= \mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\frac{1}{N}\Big|\sum_{i,j=1}^N \theta_{ij} t_N^K(i)\Big|\Big] \\
&= \mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\frac{1}{N}\Big|\sum_{i,j=1}^N (\theta_{ij} - p) t_N^K(i)\Big|\Big] \\
&\le \frac{C}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\sum_{i=1}^N\Big(t_N^K(i)\Big)^2\Big]^{\frac{1}{2}}\mathbb{E}\Big[\sum_{i=1}^N\Big(\sum_{j=1}^N(\theta_{ij} - p)\Big)^2\Big]^{\frac{1}{2}} \le C
\end{aligned}$$

by Lemma 1.5.16. Finally,

$$\begin{aligned}
\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\boldsymbol{t}_N^K A_N I_K, \mathbf{1}_K^T)\Big|\Big] &= \frac{1}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\sum_{i,j=1}^K t_N^K(i)\theta_{ij}\Big|\Big] \\
&\le \frac{1}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\sum_{i,j=1}^K t_N^K(i)(\theta_{ij} - p)\Big|\Big] + \frac{Kp}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\sum_{i=1}^K t_N^K(i)\Big|\Big] \\
&\le \frac{1}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\sum_{i=1}^K\Big(t_N^K(i)\Big)^2\Big]^{\frac{1}{2}}\mathbb{E}\Big[\sum_{i=1}^K\Big(\sum_{j=1}^K(\theta_{ij} - p)\Big)^2\Big]^{\frac{1}{2}} + \frac{Kp}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\boldsymbol{f}_N^K, \mathbf{1}_K^T)\Big|\Big] \\
&\le C\frac{K}{N} + \frac{Kp}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\boldsymbol{f}_N^K, \mathbf{1}_K^T)\Big|\Big]
\end{aligned}$$

by Lemma 1.5.16. All this proves that

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\boldsymbol{f}_N^K,\mathbf{1}_K^T)\Big|\Big]\le C\frac{K}{N}+\frac{Kp}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\boldsymbol{f}_N^K,\mathbf{1}_K^T)\Big|\Big],$$

whence the conclusion since $Kp/N\le p<1$.

Lemma 1.5.18. *Assume that $\Lambda p<1$. It holds that*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\frac{N}{K^2}\Big(\bar{c}_N^K+\frac{N-K}{N}\Big)^2\sum_{j=1}^K\ell_N(j)+\frac{N}{K^2}\Big(\bar{c}_N^K-\frac{K}{N}\Big)^2\sum_{j=K+1}^N\ell_N(j)\\ -\frac{N-K}{K}\bar{\ell}_N^K-\frac{1}{(1-\Lambda p)^3}\Big|^2\Big]\le\frac{C}{NK}.$$

Proof. Recall that $\frac{N}{K}\bar{c}_N^K=\bar{\ell}_N^K$, whence

$$\begin{aligned}
&\Big(\bar{c}_N^K+\frac{N-K}{N}\Big)^2\sum_{j=1}^K\ell_N(j)+\Big(\bar{c}_N^K-\frac{K}{N}\Big)^2\sum_{j=K+1}^N\ell_N(j)\\
=&\Big(\bar{c}_N^K-\frac{K}{N}\Big)^2\sum_{j=1}^N\ell_N(j)+2\Big(\bar{c}_N^K-\frac{K}{N}\Big)\sum_{i=1}^K\ell_N(i)+\sum_{i=1}^K\ell_N(i)\\
=&\Big(\frac{K}{N}\Big)^2(\bar{\ell}_N^K-1)^2N\bar{\ell}_N+2\frac{K}{N}(\bar{\ell}_N^K-1)K\bar{\ell}_N^K+K\bar{\ell}_N^K\\
=&\frac{K^2}{N}(\bar{\ell}_N^K-1)^2\bar{\ell}_N+2\frac{K^2}{N}(\bar{\ell}_N^K-1)\bar{\ell}_N^K+K\bar{\ell}_N^K\\
=&2\frac{K^2}{N}\Big(-\bar{\ell}_N^K\bar{\ell}_N+(\bar{\ell}_N^K)^2\Big)+\frac{K^2}{N}\Big(\bar{\ell}_N-\bar{\ell}_N^K\Big)+\frac{K^2}{N}(\bar{\ell}_N^K)^2\bar{\ell}_N-\frac{K^2}{N}\bar{\ell}_N^K+K\bar{\ell}_N^K.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{N}{K^2}\Big(\bar{c}_N^K+\frac{N-K}{N}\Big)^2\sum_{j=1}^K\ell_N(j)+\frac{N}{K^2}\Big(\bar{c}_N^K-\frac{K}{N}\Big)^2\sum_{j=K+1}^N\ell_N(j)\\
&=2\Big(-\bar{\ell}_N^K\bar{\ell}_N+(\bar{\ell}_N^K)^2\Big)+\Big(\bar{\ell}_N-\bar{\ell}_N^K\Big)+\Big(\bar{\ell}_N^K\Big)^2\bar{\ell}_N+\frac{N-K}{K}\bar{\ell}_N^K.
\end{aligned}$$

On the event $\Omega_{N,K}$, we have $\bar{\ell}_N^K,\ \bar{\ell}_N$ are bounded. Hence,

$$\begin{aligned}
\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|-\bar{\ell}_N^K\bar{\ell}_N+(\bar{\ell}_N^K)^2\Big|^2\Big]&=\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar{\ell}_N^K\Big|^2\Big|\bar{\ell}_N^K-\bar{\ell}_N\Big|^2\Big]\\
&\le C\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar{\ell}_N^K-\frac{1}{1-\Lambda p}+\frac{1}{1-\Lambda p}-\bar{\ell}_N\Big|^2\Big]\\
&\le C\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar{\ell}_N^K-\frac{1}{1-\Lambda p}\Big|^2\Big]+C\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\frac{1}{1-\Lambda p}-\bar{\ell}_N\Big|^2\Big]\le\frac{C}{NK}
\end{aligned}$$

by Lemmas 1.5.3 and 1.5.9. Similarly,

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big(\Big|(\bar{\ell}_N^K)^2\bar{\ell}_N-\frac{1}{(1-\Lambda p)^3}\Big|^2+(\bar{\ell}_N-\bar{\ell}_N^K)^2\Big)\Big]\\
\le&C\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar{\ell}_N^K-\frac{1}{1-\Lambda p}\Big|^2\Big]+C\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\frac{1}{1-\Lambda p}-\bar{\ell}_N\Big|^2\Big]\le\frac{C}{NK}.
\end{aligned}$$

The conclusion follows.

Here is the main result of this subsection.

Lemma 1.5.19. *Assume that $\Lambda p < 1$. We have that*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{X}^{N,K}_{\infty,\infty} - \frac{\mu}{(1-\Lambda p)^3}\Big|\Big] \le \frac{C}{K}.$$

Proof. By definition,

$$\mathcal{X}^{N,K}_{\infty,\infty} - \frac{\mu}{(1-\Lambda p)^3} = \frac{\mu N}{K^2}\sum_{j=1}^{N}\Big(c^K_N(j)\Big)^2\ell_N(j) - \frac{\mu(N-K)}{K}\bar{\ell}^K_N - \frac{\mu}{(1-\Lambda p)^3} = \mu\sum_{\alpha=1}^{3} I^{\alpha}_{N,K},$$

where

$$\begin{aligned}
I^1_{N,K} =& \frac{N}{K^2}\sum_{j=1}^{K}\Big[c^K_N(j) - \bar{c}^K_N - \frac{N-K}{N}\Big]^2\ell_N(j) + \frac{N}{K^2}\sum_{j=K+1}^{N}\Big[c^K_N(j) - \bar{c}^K_N + \frac{K}{N}\Big]^2\ell_N(j),\\
I^2_{N,K} =& 2\frac{N}{K^2}\Big[\bar{c}^K_N + \frac{N-K}{N}\Big]\sum_{j=1}^{K}\ell_N(j)\Big[c^K_N(j) - \bar{c}^K_N - \frac{N-K}{N}\Big]\\
&+ 2\frac{N}{K^2}\Big[\bar{c}^K_N - \frac{K}{N}\Big]\sum_{j=K+1}^{N}\ell_N(j)\Big[c^K_N(j) - \bar{c}^K_N + \frac{K}{N}\Big],\\
I^3_{N,K} =& \frac{N}{K^2}\Big[\bar{c}^K_N + \frac{N-K}{N}\Big]^2\sum_{j=1}^{K}\ell_N(j) + \frac{N}{K^2}\Big[\bar{c}^K_N - \frac{K}{N}\Big]^2\sum_{j=K+1}^{N}\ell_N(j) - \frac{N-K}{K}\bar{\ell}^K_N - \frac{1}{(1-\Lambda p)^3}.
\end{aligned}$$

By Lemma 1.5.16, $\bar{\ell}_N$ and $\ell_N(j)$ are bounded on the set $\Omega_{N,K}$ for any $j = 1, ..., N$, whence

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}|I^1_{N,K}|\Big] \le C\frac{N}{K^2}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}||\boldsymbol{c}^K_N - \bar{c}^K_N\mathbf{1}^T_N - \mathbf{1}^T_K + \frac{K}{N}\mathbf{1}^T_N||^2_2\Big] = C\frac{N}{K^2}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}||\boldsymbol{t}^K_N||^2_2\Big] \le \frac{C}{N}.$$

Recall the result from Lemma 1.5.18: we have

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}|I^3_{N,K}|\Big] \le \frac{C}{\sqrt{NK}} \le \frac{C}{K}.$$

Next, we have $I^2_{N,K} = 2I^{2,1}_{N,K} + 2I^{2,2}_{N,K}$, where

$$\begin{aligned}
I^{2,1}_{N,K} =& \frac{N}{K^2}\sum_{j=1}^{K}\ell_N(j)\Big[c^K_N(j) - \bar{c}^K_N - \frac{N-K}{N}\Big],\\
I^{2,2}_{N,K} =& \frac{N}{K^2}\Big[\bar{c}^K_N - \frac{K}{N}\Big]\Big\{\sum_{j=1}^{K}\ell_N(j)\Big[c^K_N(j) - \bar{c}^K_N - \frac{N-K}{N}\Big] + \sum_{j=K+1}^{N}\ell_N(j)\Big[c^K_N(j) - \bar{c}^K_N + \frac{K}{N}\Big]\Big\}.
\end{aligned}$$

Since

$$\sum_{j=1}^{K}\Big[c^K_N(j) - \bar{c}^K_N - \frac{N-K}{N}\Big] + \sum_{j=K+1}^{N}\Big[c^K_N(j) - \bar{c}^K_N + \frac{K}{N}\Big] = 0,$$

we may write

$$\begin{aligned}
I^{2,2}_{N,K} =& \frac{N}{K^2}\Big[\bar{c}^K_N - \frac{K}{N}\Big]\Big\{\sum_{j=1}^{K}\Big(\ell_N(j) - \bar{\ell}_N\Big)\Big[c^K_N(j) - \bar{c}^K_N - \frac{N-K}{N}\Big]\\
&\qquad + \sum_{j=K+1}^{N}\Big(\ell_N(j) - \bar{\ell}_N\Big)\Big[c^K_N(j) - \bar{c}^K_N + \frac{K}{N}\Big]\Big\}\\
=& \frac{N}{K^2}\Big[\bar{c}^K_N - \frac{K}{N}\Big]\Big\{\sum_{j=1}^{K}x_N(j)\Big[c^K_N(j) - \bar{c}^K_N - \frac{N-K}{N}\Big] + \sum_{j=K+1}^{N}x_N(j)\Big[c^K_N(j) - \bar{c}^K_N + \frac{K}{N}\Big]\Big\}\\
=& \frac{N}{K^2}\Big[\bar{c}^K_N - \frac{K}{N}\Big](\boldsymbol{x}_N, \boldsymbol{t}_N).
\end{aligned}$$

Recalling that $\bar{c}_N^K = K\bar{\ell}_N^K/N$ and that $\bar{\ell}_N^K$ is bounded on $\Omega_{N,K}$, we conclude that $\mathbf{1}_{\Omega_{N,K}}|I_{N,K}^{2,2}| \leq C||\boldsymbol{x}_N||_2||\boldsymbol{t}_N||_2/K$. Using Lemmas 1.5.4 and 1.5.16, we readily conclude that $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}}|I_{N,K}^{2,2}|] \leq C/N$. Finally,

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\Omega_{N,K}}|I_{N,K}^{2,1}|] \leq& \frac{N}{K^2}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\sum_{j=1}^{K}(\ell_N(j)-\bar{\ell}_N^K)\Big[c_N^K(j)-\bar{c}_N^K-\frac{N-K}{N}\Big]\Big|\Big]\\ &+\frac{N}{K^2}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\sum_{j=1}^{K}\bar{\ell}_N^K\Big[c_N^K(j)-\bar{c}_N^K-\frac{N-K}{N}\Big]\Big|\Big]\\ \leq& \frac{N}{K^2}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\|\boldsymbol{x}_N\|_2^2\Big]^{\frac{1}{2}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\|\boldsymbol{f}_N^K\|_2^2\Big]^{\frac{1}{2}}+\frac{N}{K^2}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\boldsymbol{f}_N^K,\mathbf{1}_K^T)\Big|\Big] \leq \frac{C}{K}\end{aligned}$$

by Lemma 1.5.17. The proof is complete.

1.6 Some auxilliary processes

We first introduce a family of martingales: for $i = 1,\dots,N$, recalling (1.1),

$$M_t^{i,N} = \int_0^t\int_0^\infty \mathbf{1}_{\{z\leq\lambda_s^{i,N}\}}\widetilde{\pi}^i(ds,dz).$$

where $\widetilde{\pi}^i(ds,dz) = \pi^i(ds,dz) - dsdz$. We also introduce the family of centered processes $U_t^{i,N} = Z_t^{i,N} - \mathbb{E}_\theta[Z_t^{i,N}]$.

We denote by $\boldsymbol{Z}_t^N$ (resp. $\boldsymbol{U}_t^N$, $\boldsymbol{M}_t^N$) the N dimensional vector with coordinates $Z_t^{i,N}$ (resp. $U_t^{i,N}$, $M_t^{i,N}$) and set

$$\boldsymbol{Z}_t^{N,K} = I_K\boldsymbol{Z}_t^N, \quad \boldsymbol{U}_t^{N,K} = I_K\boldsymbol{U}_t^N,$$

as well as $\bar{Z}_t^{N,K} = K^{-1}\sum_{i=1}^K Z_t^{i,N}$ and $\bar{U}_t^{N,K} = K^{-1}\sum_{i=1}^K U_t^{i,N}$. By [14, Remark 10 and Lemma 11], we have the following equalities

$$\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}] = \mu\sum_{n\geq 0}\Big[\int_0^t s\phi^{*n}(t-s)ds\Big]I_K A_N^n\mathbf{1}_N, \tag{1.9}$$

$$\boldsymbol{U}_t^{N,K} = \sum_{n\geq 0}\int_0^t \phi^{*n}(t-s)I_K A_N^n\boldsymbol{M}_s^N ds, \tag{1.10}$$

$$[M^{i,N},M^{j,N}]_t = \mathbf{1}_{\{i=j\}}Z_t^{i,N}. \tag{1.11}$$

We recall that $\phi^{*0}(s)ds = \delta_0 ds$, whence in particular $\int_0^t s\phi^{*0}(t-s)ds = t$.

Lemma 1.6.1. *Assume $H(q)$ for some $q \geq 1$. There exists a constant C such that*

(i) for all r in $[1,\infty]$, all $t \geq 0$, a.s.,

$$\mathbf{1}_{\Omega_{N,K}}\|\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}]\|_r \leq CtK^{\frac{1}{r}},$$

(ii) for all r in $[1,\infty]$, all $t \geq s \geq 0$, a.s.,

$$\mathbf{1}_{\Omega_{N,K}}\|\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K} - \boldsymbol{Z}_s^{N,K} - \mu(t-s)\boldsymbol{\ell}_N^K]\|_r \leq C(\min\{1,s^{1-q}\})K^{\frac{1}{r}}.$$

Proof. (i) We start from (1.9). Recall that $\Lambda = \int_0^\infty \phi(s)ds$, whence

$$\int_0^\infty \phi^{*n}(s)ds \leq \Lambda^n, \qquad \int_0^t s\phi^{*n}(s)ds \leq t\int_0^\infty \phi^{*n}(s)ds \leq t\Lambda^n.$$

So on the event $\Omega_{N,K}$, on which we have $\Lambda|||I_K A_N|||_r \le (K/N)^{1/r}$ and $\Lambda|||A_N|||_r \le a < 1$, we have (observe that $||\mathbf{1}_K||_r = K^{1/r}$)

$$\begin{aligned}\|\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}]\|_r &\le \mu t K^{\frac{1}{r}} + \mu t \sum_{n\ge 1} \Lambda^n |||I_K A_N^n|||_r \|\mathbf{1}_N\|_r \\ &\le \mu t K^{\frac{1}{r}} + \mu t \sum_{n\ge 1} \Lambda^n |||I_K A_N|||_r |||A_N|||_r^{n-1} \|\mathbf{1}_N\|_r \le Ct K^{1/r}.\end{aligned}$$

(ii) By (1.9) and Lemma 1.5.6, we have

$$\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}] - \mathbb{E}_\theta[\boldsymbol{Z}_s^{N,K}] = \mu(t-s)\sum_{n\ge 0}\Lambda^n I_K A_N^n \mathbf{1}_N + \mu\Big(\sum_{n\ge 0}[\varepsilon_n(t)-\varepsilon_n(s)] I_K A_N^n \mathbf{1}_{\boldsymbol{N}}\Big)$$

with $0 \le \varepsilon_n(t) \le C\min\{n^q\Lambda^n t^{1-q}, n\Lambda^n k\}$. Since $\sum_{n\ge 0}\Lambda^n I_K A_N^n \mathbf{1}_{\boldsymbol{N}} = I_K Q_N \mathbf{1}_N = \boldsymbol{\ell}_N^K$ on the event $\Omega_{N,K}$,

$$\begin{aligned}&\|\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}] - \mathbb{E}_\theta[\boldsymbol{Z}_s^{N,K}] - \mu(t-s)\boldsymbol{\ell}_N^K\|_r \\ \le& C(\min\{1, s^{1-q}\})\|\mathbf{1}_N\|_r \sum_{n\ge 0} n^q \Lambda^n |||I_K A_N^n|||_r \\ \le& C(\min\{1, s^{1-q}\})N^{1/r}\Big(\sum_{n\ge 1} n^q\Lambda^n |||I_K A_N|||_r |||A_N|||_r^{n-1}\Big) \le C\min\{1, s^{1-q}\}K^{\frac{1}{r}}.\end{aligned}$$

We used the very same arguments as in point (i).

1.7 The first estimator in the subcritical case

Here we prove that $\varepsilon_t^{N,K} = t^{-1}(\bar{Z}_{2t}^{N,K} - \bar{Z}_t^{N,K}) \simeq \frac{\mu}{1-\Lambda p}$ and to study the rate of convergence.

Theorem 1.7.1. *Assume $(H(q))$ for some $q \ge 1$. There are some positive constants C, C' depending only on p, μ, ϕ and q such that for all $\varepsilon \in (0,1)$, all $N \ge K \ge 1$, all $t \ge 1$,*

$$P\Big(\Big|\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p}\Big| \ge \varepsilon\Big) \le CNe^{-C'K} + \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{NK}} + \frac{1}{\sqrt{Kt}} + \frac{1}{t^q}\Big).$$

Lemma 1.7.2. *Assume $(H(q))$ for some $q \ge 1$. There is a constant $C > 0$ such that a.s.,*

$$(i)\ \ \mathbf{1}_{\Omega_{N,K}}\Big|\mathbb{E}_\theta[\varepsilon_t^{N,K}] - \mu\bar{\ell}_N^K\Big| \le \frac{C}{t^q}, \quad (ii)\ \ \mathbf{1}_{\Omega_{N,K}}\mathbb{E}_\theta[|\bar{U}_t^{N,K}|^2] \le \frac{Ct}{K}.$$

Proof. By Lemma 1.6.1 (ii),

$$\Big|\mathbb{E}_\theta[\varepsilon_t^{N,K}] - \mu\bar{\ell}_N^K\Big| \le \frac{1}{K}\Big\|\mathbb{E}_\theta\Big[\frac{\boldsymbol{Z}_{2t}^{N,K} - \boldsymbol{Z}_t^{N,K}}{t}\Big] - \mu\boldsymbol{\ell}_N^K\Big\|_1 \le \frac{C}{t^q}.$$

which proves (i). Using (1.10),

$$\bar{U}_t^{N,K} = \frac{1}{K}\sum_{n\ge 0}\int_0^t \phi^{*n}(t-s)\sum_{i=1}^K\sum_{j=1}^N A_N^n(i,j)M_s^{j,N}ds.$$

Recalling (1.11), it is obvious that for $n \ge 1$,

$$\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K\sum_{j=1}^N A_N^n(i,j)M_s^{j,N}\Big)^2\Big] = \sum_{j=1}^N\Big(\sum_{i=1}^K A_N^n(i,j)\Big)^2\mathbb{E}_\theta[Z_s^{j,N}] \le |||I_K A_N|||_1^{2n}\sum_{j=1}^N\mathbb{E}_\theta[Z_s^{j,N}].$$

By Lemma 1.6.1-(i) with $r=1$, we have $\mathbf{1}_{\Omega_{N,K}}\mathbb{E}_\theta[\sum_{i=1}^N Z_t^{i,N}] \le CtN$ and $\mathbf{1}_{\Omega_{N,K}}\mathbb{E}_\theta[\sum_{i=1}^K Z_t^{i,N}] \le CtK$. Thus on $\Omega_{N,K}$,

$$\begin{aligned}
&\mathbb{E}_\theta[|\bar U_t^{N,K}|^2]^{\frac12}\\
&\le \frac1K\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K M_t^{i,N}\Big)^2\Big]^{\frac12} + \frac1K\sum_{n\ge1}\int_0^t \phi^{\star n}(t-s)\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K\sum_{j=1}^N A_N^n(i,j)M_s^{j,N}\Big)^2\Big]^{1/2}ds\\
&\le \frac1K\mathbb{E}_\theta\Big[\sum_{i=1}^K Z_t^{i,N}\Big]^{\frac12} + \frac{C}{K}\sum_{n\ge1}|||I_KA_N^n|||_1\int_0^t\mathbb{E}_\theta\Big[\sum_{i=1}^N Z_s^{i,N}\Big]^{\frac12}\phi^{*n}(t-s)ds\\
&\le \frac{C\sqrt t}{\sqrt K} + C\frac{(tN)^{\frac12}}{K}\sum_{n\ge1}\Lambda^n|||I_KA_N|||_1|||A_N|||_1^{n-1}.
\end{aligned}$$

We used that $\int_0^t\sqrt s\phi^{*n}(t-s)ds \le \sqrt t\int_0^t\phi^{*n}(t-s)ds \le \sqrt t\Lambda^n$. As a conclusion, still on $\Omega_{N,K}$, since $|||I_KA_N|||_1 \le CK/N$ and $\Lambda|||A_N|||_1 \le a<1$,

$$\mathbb{E}_\theta[|\bar U_t^{N,K}|^2]^{\frac12} \le C\sqrt t\Big(\frac{1}{\sqrt K}+\frac{1}{\sqrt N}\Big) \le \frac{C\sqrt t}{\sqrt K}$$

as desired.

Lemma 1.7.3. *Assume $(H(q))$ for some $q\ge1$. There is $C>0$ such that a.s.,*

$$\mathbf{1}_{\Omega_{N,K}}\mathbb{E}_\theta\Big[\Big|\varepsilon_t^{N,K}-\mu\bar\ell_N^K\Big|^2\Big] \le C\Big(\frac{1}{t^{2q}}+\frac{1}{tK}\Big).$$

Proof. It suffices to write

$$\begin{aligned}
\mathbb{E}_\theta\Big[\Big|\varepsilon_t^{N,K}-\mu\bar\ell_N^K\Big|^2\Big] &\le 2\mathbb{E}_\theta\Big[\Big|\varepsilon_t^{N,K}-\mathbb{E}_\theta[\varepsilon_t^{N,K}]\Big|^2\Big] + 2\Big|\mathbb{E}_\theta[\varepsilon_t^{N,K}]-\mu\bar\ell_N^K\Big|^2\\
&\le \frac{4}{t^2}\Big(\mathbb{E}_\theta[|\bar U_{2t}^{N,K}|^2]+\mathbb{E}_\theta[|\bar U_t^{N,K}|^2]\Big) + 2\Big|\mathbb{E}_\theta[\varepsilon_t^{N,K}]-\mu\bar\ell_N^K\Big|^2
\end{aligned}$$

and to use Lemma 1.7.2.

Finally, we can give the proof of Theorem 1.7.1.

Proof. By Lemmas 1.5.9 and 1.7.3, we have

$$\begin{aligned}
\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\varepsilon_t^{N,K}-\frac{\mu}{1-\Lambda p}\Big|\Big] &\le \mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\varepsilon_t^{N,K}-\mu\bar\ell_N^K\Big|^2\Big]^{\frac12} + \mu\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar\ell_N^K-\frac{1}{1-\Lambda p}\Big|^2\Big]^{\frac12}\\
&\le C\Big(\frac{1}{\sqrt{Kt}}+\frac{1}{t^q}+\frac{1}{\sqrt{NK}}\Big).
\end{aligned}$$

By Chebyshev's Inequality, we deduce

$$\begin{aligned}
P\Big(\Big|\varepsilon_t^{N,K}-\frac{\mu}{1-\Lambda p}\Big|\ge\varepsilon\Big) &\le P(\Omega_{N,K}^c) + P\Big(\Big\{\Big|\varepsilon_t^{N,K}-\frac{\mu}{1-\Lambda p}\Big|\ge\varepsilon\Big\}\cap\Omega_{N,K}\Big)\\
&\le P(\Omega_{N,K}^c) + \frac1\varepsilon\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\varepsilon_t^{N,K}-\frac{\mu}{1-\Lambda p}\Big|\Big]\\
&\le CNe^{-C'K} + \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{NK}}+\frac{1}{t^q}+\frac{1}{\sqrt{Kt}}\Big)
\end{aligned}$$

by Lemma 1.5.7.

1.8 The second estimator in the subcritical case

We now prove that $\mathcal{V}_t^{N,K} := \frac NK\sum_{i=1}^K\Big[\frac{Z_{2t}^{i,N}-Z_t^{i,N}}{t}-\varepsilon_t^{N,K}\Big]^2 - \frac Nt\varepsilon_t^{N,K} \simeq \frac{\mu^2\Lambda^2p(1-p)}{(1-\Lambda p)^2}$.

Theorem 1.8.1. *Assume $H(q)$ for some $q \geq 1$. There is $C > 0$ such that for all $t \geq 1$, a.s.,*

$$\mathbf{1}_{\Omega_{N,K}} \mathbb{E}_\theta\Big[\Big|\mathcal{V}_t^{N,K} - \mathcal{V}_\infty^{N,K}\Big|\Big] \leq C\Big(\frac{N}{t\sqrt{K}} + \frac{N}{t^q} + \frac{N}{K\sqrt{t}}\|\boldsymbol{x}_N^K\|_2\Big), \text{ where } \mathcal{V}_\infty^{N,K} := \frac{\mu^2 N}{K}\|\boldsymbol{x}_N^K\|_2^2.$$

We write $|\mathcal{V}_t^{N,K} - \mathcal{V}_\infty^{N,K}| \leq \Delta_t^{N,K,1} + \Delta_t^{N,K,2} + \Delta_t^{N,K,3}$, where

$$\Delta_t^{N,K,1} = \frac{N}{K}\Big|\sum_{i=1}^K \Big[(Z_{2t}^{i,N} - Z_t^{i,N})/t - \varepsilon_t^{N,K}\Big]^2 - \sum_{i=1}^K \Big[(Z_{2t}^{i,N} - Z_t^{i,N})/t - \mu\bar{\ell}_N^K\Big]^2\Big|,$$

$$\Delta_t^{N,K,2} = \frac{N}{K}\Big|\sum_{i=1}^K \Big[(Z_{2t}^{i,N} - Z_t^{i,N})/t - \mu\ell_N(i)\Big]^2 - (K/t)\varepsilon_t^{N,K}\Big|,$$

$$\Delta_t^{N,K,3} = 2\frac{N}{K}\Big|\sum_{i=1}^K \Big[Z_{2t}^{i,N} - Z_t^{i,N} - \mu\ell_N(i)\Big]\Big[\mu\ell_N(i) - \mu\bar{\ell}_N^K\Big]\Big|.$$

We also write $\Delta_t^{N,K,2} \leq \Delta_t^{N,K,21} + \Delta_t^{N,K,22} + \Delta_t^{N,K,23}$, where

$$\Delta_t^{N,K,21} = \frac{N}{K}\Big|\sum_{i=1}^K \Big[(Z_{2t}^{i,N} - Z_t^{i,N})/t - \mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]/t\Big]^2 - (K/t)\varepsilon_t^{N,K}\Big|,$$

$$\Delta_t^{N,K,22} = \frac{N}{K}\Big|\sum_{i=1}^K \Big\{\mathbb{E}_\theta[(Z_{2t}^{i,N} - Z_t^{i,N})/t] - \mu\ell_N(i)\Big\}^2\Big|,$$

$$\Delta_t^{N,K,23} = 2\frac{N}{K}\Big|\sum_{i=1}^K \Big[(Z_{2t}^{i,N} - Z_t^{i,N})/t - \mathbb{E}_\theta(Z_{2t}^{i,N} - Z_t^{i,N})/t\Big]\Big[\mathbb{E}_\theta(Z_{2t}^{i,N} - Z_t^{i,N})/t - \mu\ell_N(i)\Big]\Big|.$$

We next write $\Delta_t^{N,K,21} \leq \Delta_t^{N,K,211} + \Delta_t^{N,K,212} + \Delta_t^{N,K,213}$, where

$$\Delta_t^{N,K,211} = \frac{N}{K}\Big|\sum_{i=1}^K \Big\{(U_{2t}^{i,N} - U_t^{i,N})^2/t^2 - \mathbb{E}_\theta[(U_{2t}^{i,N} - U_t^{i,N})^2/t^2]\Big\}\Big|,$$

$$\Delta_t^{N,K,212} = \frac{N}{K}\Big|\sum_{i=1}^K \mathbb{E}_\theta[(U_{2t}^{i,N} - U_t^{i,N})^2/t^2] - \mathbb{E}_\theta[K\varepsilon_t^{N,K}/t]\Big|,$$

$$\Delta_t^{N,K,213} = \frac{N}{K}\Big|K\varepsilon_t^{N,K}/t - \mathbb{E}_\theta[K\varepsilon_t^{N,K}/t]\Big|,$$

At the last, we write $\Delta_t^{N,K,3} \leq \Delta_t^{N,K,31} + \Delta_t^{N,K,32}$, where

$$\Delta_t^{N,K,31} = 2\frac{N}{K}\Big|\sum_{i=1}^K \Big[(Z_{2t}^{i,N} - Z_t^{i,N})/t - \mathbb{E}_\theta[(Z_{2t}^{i,N} - Z_t^{i,N})/t]\Big]\Big[\mu\ell_N(i) - \mu\bar{\ell}_N^K\Big]\Big|,$$

$$\Delta_t^{N,K,32} = 2\Big|\sum_{i=1}^K \Big[\mathbb{E}_\theta[(Z_{2t}^{i,N} - Z_t^{i,N})/t] - \mu\ell_N(i)\Big]\Big[\mu\ell_N(i) - \mu\bar{\ell}_N^K\Big]\Big|.$$

Lemma 1.8.2. *Assume $H(q)$ for some $q \geq 1$. Then, on the set $\Omega_{N,K}$, for $t \geq 1$, a.s.,*

(i) $\mathbb{E}_\theta[\Delta_t^{N,K,1}] \leq C(Nt^{-2q} + NK^{-1}t^{-1})$,

(ii) $\mathbb{E}_\theta[\Delta_t^{N,K,22}] \leq CN/t^{2q}$,

(iii) $\mathbb{E}_\theta[\Delta_t^{N,K,23}] \leq CN/t^q$,

(iv) $\mathbb{E}_\theta[\Delta_t^{N,K,213}] \leq CNK^{-\frac{1}{2}}t^{-\frac{3}{2}}$,

(v) $\mathbb{E}_\theta[\Delta_t^{N,K,32}] \leq CN/t^q$.

Proof. (i) Recalling the definition $\varepsilon_t^{N,K} = t^{-1}(\bar{Z}_{2t}^{N,K} - \bar{Z}_t^{N,K})$,

$$\Delta_t^{N,K,1} = \frac{N}{K}\Big|\sum_{i=1}^{K}\Big[\mu\bar{\ell}_N^K - \varepsilon_t^{N,K}\Big]\Big[2(Z_{2t}^{i,N} - Z_t^{i,N})/t - \mu\bar{\ell}_N^K - \varepsilon_t^{N,K}\Big]\Big| = N\Big(\varepsilon_t^{N,K} - \mu\bar{\ell}_N^K\Big)^2,$$

whence by Lemma 1.7.3,

$$\mathbb{E}_\theta[\Delta_t^{N,1,K}] = N\mathbb{E}_\theta\Big[\Big(\varepsilon_t^{N,K} - \mu\bar{\ell}_N^K\Big)^2\Big] \le C\Big(Nt^{-2q} + NK^{-1}t^{-1}\Big).$$

(ii) We use Lemma 1.6.1-(ii) with $r = 2$:

$$\mathbb{E}_\theta[\Delta_t^{N,K,22}] = \frac{N}{K}\sum_{i=1}^{K}\Big\{\mathbb{E}_\theta[(Z_{2t}^{i,N} - Z_t^{i,N})/t] - \mu\ell_N(i)\Big\}^2 \le CN/t^{2q}.$$

(iii) By Lemma 1.6.1-(i) with $r = \infty$ and 1.6.1-(ii),

$$\mathbb{E}_\theta[\Delta_t^{N,K,23}] \le \frac{4N}{K}\Big\|\mathbb{E}_\theta\Big[\Big(\boldsymbol{Z}_{2t}^{N,K} - \boldsymbol{Z}_t^{N,K}\Big)/t\Big] - \mu\ell_N^K\Big\|_\infty\Big\|\mathbb{E}_\theta\Big[\boldsymbol{Z}_{2t}^{N,K} + \boldsymbol{Z}_t^{N,K}\Big]\Big\|_1 t^{-1} \le \frac{CN}{t^q}.$$

(iv) Since

$$\Delta_t^{N,K,213} = Nt^{-2}\Big|\bar{U}_{2t}^{N,K} - \bar{U}_t^{N,K}\Big| \le Nt^{-2}\Big(|\bar{U}_{2t}^{N,K}| + |\bar{U}_t^{N,K}|\Big)$$

and thanks to Lemma 1.7.2-(ii), we deduce that

$$\mathbb{E}_\theta[\Delta_t^{N,K,213}] \le CNK^{-\frac{1}{2}}t^{-\frac{3}{2}}.$$

(v) Since $\max_{j=1,\dots,N}[\ell_N(j)]$ is bounded on the set $\Omega_{N,K}$, by Lemma 1.6.1-(ii) with $r = 1$,

$$\mathbb{E}_\theta[\Delta_t^{N,K,32}] \le C\mathbb{E}_\theta\Big[\frac{N}{Kt}\|\boldsymbol{Z}_{2t}^{N,K} - \boldsymbol{Z}_t^{N,K} - \mu t\boldsymbol{\ell}_N^K\|_1\Big] \le \frac{CN}{t^q}.$$

The proof is complete.

Lemma 1.8.3. *Assume $H(q)$ for some $q \ge 1$. We have, for all $t \ge 1$, on the set $\Omega_{N,K}$, a.s.*

$$\mathbb{E}_\theta[\Delta_t^{N,K,212}] \le C/t.$$

Proof. We write $\mathbb{E}_\theta[\Delta_t^{N,K,212}] \le t^{-2}\frac{N}{K}\sum_{i=1}^{K} a_i$, where $a_i = |\mathbb{E}_\theta[(U_{2t}^{i,N} - U_t^{i,N})^2 - (Z_{2t}^{i,N} - Z_t^{i,N})]|$, and then

$$a_i = b_i + d_i \quad \text{where} \quad a_i = \mathbb{E}_\theta[(R_t^{i,N})^2] \quad \text{and} \quad b_i = 2\mathbb{E}_\theta[(M_{2t}^{i,N} - M_t^{i,N})R_t^{i,N}],$$

where, recalling (1.10), we have $U_{2t}^{i,N} - U_t^{i,N} = M_{2t}^{i,N} - M_t^{i,N} + R_t^{i,N}$, with

$$R_t^{i,N} = \sum_{n\ge1}\int_0^{2t}\beta_n(t,2t,s)\sum_{j=1}^{N}A_N^n(i,j)M_s^{j,N}ds \quad \text{with} \quad \beta_n(t,2t,s) = \phi^{\star n}(2t-s) - \phi^{\star n}(t-s).$$

This uses that $\mathbb{E}_\theta[(M_{2t}^{i,N} - M_t^{i,N})^2] = \mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]$ by (1.11). By the proof of [14, Lemma 21, lines 10 and 15], we have $b_i \le CtN^{-1}$ and $d_i \le CtN^{-1}$, whence the conclusion.

Before considering the term $\Delta_t^{N,K,31}$, we review [14, Lemma 22] (observing that $\Omega_{N,K} \subset \Omega_N^1$).

Lemma 1.8.4. *Assume $H(q)$ for some $q \ge 1$. Then for all $t \ge 1$ and $k,l,a,b \in \{1,\dots,N\}$, all $r,s,u,v \in [0,t]$, on the set $\Omega_{N,K}$ a.s,*

(i) $|\mathrm{Cov}_\theta(Z_r^{k,N}, Z_s^{l,N})| = |\mathrm{Cov}_\theta(U_r^{k,N}, U_s^{l,N})| \le Ct(N^{-1} + \mathbf{1}_{\{k=l\}})$,

(ii) $|\mathrm{Cov}_\theta(Z_r^{k,N}, M_s^{l,N})| = |\mathrm{Cov}_\theta(U_r^{k,N}, M_s^{l,N})| \le Ct(N^{-1} + \mathbf{1}_{\{k=l\}})$,

(iii) $|\mathrm{Cov}_\theta(Z_r^{k,N}, \int_0^s M_{\tau-} dM_\tau^{l,N})| = |\mathrm{Cov}_\theta(U_r^{k,N}, \int_0^s M_{\tau-} dM_\tau^{l,N})| \le Ct^{\frac{3}{2}}(N^{-1} + \mathbf{1}_{\{k=l\}})$,

(iv) $|\mathbb{E}_\theta[M_r^{k,N} M_s^{k,N} M_u^{l,N}]| \le \frac{Ct}{N}$, *if* $\#\{k,l\} = 2$,

(v) $|\mathrm{Cov}_\theta(M_r^{k,N} M_s^{l,N}, M_u^{a,N} M_v^{b,N})| = 0$, *if* $\#\{k,l,a,b\} = 4$,

(vi) $|\mathrm{Cov}_\theta(M_r^{k,N} M_s^{l,N}, M_u^{a,N} M_v^{b,N})| \le Ct/N^2$, *if* $\#\{k,a,b\} = 3$,

(vii) $|\mathrm{Cov}_\theta(M_r^{k,N} M_s^{l,N}, M_u^{a,N} M_v^{a,N})| \le CN^{-1}t^{\frac{3}{2}}$, *if* $\#\{k,a\} = 2$,

(viii) $|\mathrm{Cov}_\theta(M_r^{k,N} M_s^{l,N}, M_u^{a,N} M_v^{b,N})| \le Ct^2$.

Lemma 1.8.5. *Assume $H(q)$ for some $q \ge 1$. Then for $t \ge 1$ on $\Omega_{N,K}$ a.s.,*

$$\mathbb{E}_\theta[(\Delta_t^{N,K,31})^2] \le \frac{CN^2}{tK^2} \sum_{i=1}^K \Big(\ell_N(i) - \bar{\ell}_N^K\Big)^2.$$

Proof. By definition of $\Delta_t^{N,K,31}$,

$$\mathbb{E}_\theta[(\Delta_t^{N,K,31})^2] = \frac{4\mu^2 N^2}{t^2 K^2} \sum_{i,j=1}^K (\ell_N(i) - \bar{\ell}_N^K)(\ell_N(j) - \bar{\ell}_N^K)\mathrm{Cov}_\theta(U_{2t}^{i,N} - U_t^{i,N}, U_{2t}^{j,N} - U_t^{j,N}).$$

By Lemma 1.8.4 (i), we have $\mathrm{Cov}_\theta[U_{2t}^{i,N} - U_t^{i,N}, U_{2t}^{j,N} - U_t^{j,N}] \le Ct(\mathbf{1}_{\{i=j\}} + \frac{1}{N})$. We deduce that

$$\begin{aligned}\mathbb{E}_\theta[(\Delta_t^{N,K,31})^2] &\le \frac{C\mu^2 N^2}{t^2 K^2} t \sum_{i,j=1}^K \Big(\mathbf{1}_{\{i=j\}} + \frac{1}{N}\Big)\Big\{[\ell_N(i) - \bar{\ell}_N^K]^2 + [\ell_N(j) - \bar{\ell}_N^K]^2\Big\} \\ &\le \frac{C}{t}\frac{N^2}{K^2} \sum_{i=1}^K \Big(\ell_N(i) - \bar{\ell}_N^K\Big)^2.\end{aligned}$$

We ,finally used that $K/N \le 1$.

Next, we deal with the term $\Delta_t^{N,K,211}$.

Lemma 1.8.6. *Assume $H(q)$ for some $q \ge 1$. Then for all $t \ge 1$, a.s. on the set $\Omega_{N,K}$, we have*

$$\mathbb{E}_\theta[(\Delta_t^{N,K,211})^2] \le \frac{CN^2}{Kt^2}.$$

Proof. First, $\mathbb{E}_\theta[(\Delta_t^{N,K,211})^2] = \frac{N^2}{K^2 t^4} \sum_{i,j=1}^K a_{ij}$, where $a_{ij} = \mathrm{Cov}_\theta[(U_{2t}^{i,N} - U_t^{i,N})^2, (U_{2t}^{j,N} - U_t^{j,N})^2]$. Let $\Gamma_{k,l,a,b}(t) = \sup_{r,s,u,v\in[0,2t]} |\mathrm{Cov}_\theta(M_r^{k,N} M_s^{l,N}, M_u^{a,N} M_v^{b,N})|$. By the proof of [14, Lemma 24 lines 9 to 12], we have

$$a_{ij} \le C \sum_{k,l,a,b=1}^N (\mathbf{1}_{\{i=k\}} + N^{-1})(\mathbf{1}_{\{i=l\}} + N^{-1})(\mathbf{1}_{\{j=a\}} + N^{-1})(\mathbf{1}_{\{j=b\}} + N^{-1})\Gamma_{k,l,a,b}(t).$$

Hence,

$$\sum_{i,j=1}^K a_{ij} \le C[R_1^K + R_2^K + R_3^K + R_4^K + R_5^K + R_6^K],$$

where

$$
\begin{aligned}
R_1^K &= \frac{1}{N^4}\sum_{i,j=1}^{K}\sum_{k,l,a,b=1}^{N}\Gamma_{k,l,a,b}(t) = \frac{K^2}{N^4}\sum_{k,l,a,b=1}^{N}\Gamma_{k,l,a,b}(t),\\
R_2^K &= \frac{1}{N^3}\sum_{i,j=1}^{K}\sum_{k,l,a,b=1}^{N}\mathbf{1}_{\{i=k\}}\Gamma_{k,l,a,b=1}(t) = \frac{K}{N^3}\sum_{i=1}^{K}\sum_{l,a,b=1}^{N}\Gamma_{i,l,a,b}(t),\\
R_3^K &= \frac{1}{N^2}\sum_{i,j=1}^{K}\sum_{k,l,a,b=1}^{N}\mathbf{1}_{\{i=k\}}\mathbf{1}_{\{j=a\}}\Gamma_{k,l,a,b}(t) = \frac{1}{N^2}\sum_{k,a=1}^{K}\sum_{b,l=1}^{N}\Gamma_{k,l,a,b}(t),\\
R_4^K &= \frac{1}{N^2}\sum_{i,j=1}^{K}\sum_{k,l,a,b=1}^{N}\mathbf{1}_{\{i=k\}}\mathbf{1}_{\{i=l\}}\Gamma_{k,l,a,b}(t) = \frac{K}{N^2}\sum_{k=1}^{K}\sum_{a,b=1}^{N}\Gamma_{k,k,a,b}(t),\\
R_5^K &= \frac{1}{N}\sum_{i,j=1}^{K}\sum_{k,l,a,b=1}^{N}\mathbf{1}_{\{i=k\}}\mathbf{1}_{\{i=l\}}\mathbf{1}_{\{j=a\}}\Gamma_{k,l,a,b}(t) = \frac{1}{N}\sum_{k,a=1}^{K}\sum_{b=1}^{N}\Gamma_{k,k,a,b}(t),\\
R_6^K &= \sum_{i,j=1}^{K}\sum_{k,l,a,b=1}^{N}\mathbf{1}_{\{i=k\}}\mathbf{1}_{\{i=l\}}\mathbf{1}_{\{j=a\}}\mathbf{1}_{\{j=b\}}\Gamma_{k,l,a,b}(t) = \sum_{k,a=1}^{K}\Gamma_{k,k,a,a}(t).
\end{aligned}
$$

By Lemma 1.8.4-(v)-(viii), we see that $\Gamma_{k,l,a,b}(t) \le Ct^2\mathbf{1}_{\{\#\{k,l,a,b\}<4\}}$, so that

$$
R_1^K \le Ct^2\frac{K^2}{N}, \quad R_2^K \le Ct^2\frac{K^2}{N}, \quad \text{and} \quad R_3^K \le Ct^2K.
$$

Also, from Lemma 1.8.4-(vi)-(viii), we have $\Gamma_{k,k,a,b}(t) \le C(\mathbf{1}_{\{\#\{k,a,b\}=3\}}N^{-2}t + \mathbf{1}_{\{\#\{k,a,b\}<3\}}t^2)$, whence

$$
R_4^K \le C\Big(\frac{K^2}{N^2}t + \frac{K^2}{N}t^2\Big) \le C\frac{K^2}{N}t^2 \quad \text{and} \quad R_5^K \le C\Big(Kt^2 + \frac{K^2}{N^2}t\Big) \le CKt^2.
$$

Finally, from Lemma 1.8.4-(vii)-(viii), $\Gamma_{k,k,a,a}(t) \le C(\mathbf{1}_{\{\#\{k,a\}=2\}}N^{-1}t^{\frac{3}{2}} + \mathbf{1}_{\{\#\{k,a\}=1\}}t^2)$, so that

$$
R_6^K \le C\Big(\frac{K^2}{N}t^{\frac{3}{2}} + Kt^2\Big) \le CKt^2.
$$

All in all, we deduce that $\sum_{i,j}^{K} a_{ij} \le CKt^2$.

Then we can give prove of Theorem 1.8.1.

Proof. Recalling that

$$
\begin{aligned}
|\mathcal{V}_t^{N,K} - \mathcal{V}_\infty^{N,K}| =& \Delta_t^{N,K,1} + \Delta_t^{N,K,211} + \Delta_t^{N,K,212} + \Delta_t^{N,K,213} + \Delta_t^{N,K,22}\\
&+ \Delta_t^{N,K,23} + \Delta_t^{N,K,31} + \Delta_t^{N,K,32},
\end{aligned}
$$

Lemmas 1.8.2, 1.8.3, 1.8.5 and 1.8.6 allow us to conclude that

$$
\begin{aligned}
\mathbf{1}_{\Omega_{N,K}}\mathbb{E}_\theta[|\mathcal{V}_t^{N,K} - \mathcal{V}_\infty^{N,K}|] &\le C\Big(\frac{N}{t\sqrt{K}} + \frac{N}{K^{\frac{1}{2}}t^{\frac{3}{2}}} + \frac{N}{t^q} + \frac{N}{t^{2q}} + \frac{N}{tK} + \frac{N}{K\sqrt{t}}\Big[\sum_{i=1}^{K}(\ell_N(i) - \bar{\ell}_N^K)^2\Big]^{\frac{1}{2}}\Big)\\
&\le C\Big(\frac{N}{t\sqrt{K}} + \frac{N}{t^q} + \frac{N}{K\sqrt{t}}\|\boldsymbol{x}_N^K\|_2\Big)
\end{aligned}
$$

as desired.

Corollary 1.8.7. *Assume $H(q)$ for some $q > 3$. There exists some constants $C > 0$ and $C' > 0$ depending only on p, μ, ϕ, q such that for all $\varepsilon \in (0,1)$, such that, for $t \ge 1$,*

$$
P\Big(\Big|\mathcal{V}_t^{N,K} - \frac{\mu^2\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big| \ge \varepsilon\Big) \le CNe^{-C'K} + \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}}\Big).
$$

Proof. By Theorem 1.8.1 and Lemma 1.5.14 (since $\mathcal{V}_\infty^{N,K} = \mu^2 \frac{N}{K} ||\boldsymbol{x}_K^N||^2 = \mu^2 \frac{N}{K} \sum_{i=1}^K (\ell_N(i) - \bar{\ell}_N^K)^2$), we have

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}} \Big| \mathcal{V}_t^{N,K} - \frac{\mu^2\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big|\Big] \\
&\le \mathbb{E}[\mathbf{1}_{\Omega_{N,K}} |\mathcal{V}_t^{N,K} - \mathcal{V}_\infty^{N,K}|] + \mu^2 \mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}} \Big| \frac{N}{K} \sum_{i=1}^K \Big(\ell_N(i) - \bar{\ell}_N^K\Big)^2 - \frac{\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big|\Big] \\
&\le C\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}} \Big(\frac{N}{t\sqrt{K}} + \frac{N}{t^q} + \frac{N}{K\sqrt{t}} \|\boldsymbol{x}_N^K\|_2\Big)\Big] + \frac{C}{\sqrt{K}} \\
&\le C\Big(\frac{1}{\sqrt{K}} + \frac{\sqrt{N}}{\sqrt{tK}} + \frac{N}{t^q} + \frac{N}{t\sqrt{K}}\Big).
\end{aligned}$$

By the classical inequality $\frac{N}{t\sqrt{K}} + \frac{1}{\sqrt{K}} \ge 2\frac{\sqrt{N}}{\sqrt{Kt}}$, we end with

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}} \Big| \mathcal{V}_t^{N,K} - \frac{\mu^2\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big|\Big] \le C\Big(\frac{N}{t^q} + \frac{N}{t\sqrt{K}} + \frac{1}{\sqrt{K}}\Big).$$

Using Lemma 1.5.7 and Chebyshev's inequality, we conclude that

$$P\Big(\Big|\mathcal{V}_t^{N,K} - \frac{\mu^2\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big| \ge \varepsilon\Big) \le CNe^{-CK} + \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{K}} + \frac{N}{t^q} + \frac{N}{t\sqrt{K}}\Big).$$

Next, we get rid of the term $\frac{N}{t^q}$. We assume without loss of generality that $C \ge 1$. When $t \le \sqrt{K}$, then $\frac{N}{t\sqrt{K}} \ge 1$, so that

$$P\Big(\Big|\mathcal{V}_t^{N,K} - \mu^2\Lambda^2 p(1-p)/(1-\Lambda p)^2\Big| \ge \varepsilon\Big) \le 1 \le CNe^{-CK} + \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}}\Big).$$

When now $t \ge \sqrt{K}$, then $\frac{N}{t\sqrt{K}} \ge \frac{N}{t} \ge \frac{N}{t^q}$. So

$$P\Big(\Big|\mathcal{V}_t^{N,K} - \mu^2\Lambda^2 p(1-p)/(1-\Lambda p)^2\Big| \ge \varepsilon\Big) \le CNe^{-CK} + \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}}\Big).$$

This completes the proof.

1.9 The third estimator in the subcritical case

Recall that by definition,

$$\mathcal{W}_{\Delta,t}^{N,K} = 2\mathcal{Z}_{2\Delta,t}^{N,K} - \mathcal{Z}_{\Delta,t}^{N,K}, \quad \mathcal{Z}_{\Delta,t}^{N,K} = \frac{N}{t} \sum_{i=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}_{i\Delta}^{N,K} - \bar{Z}_{(i-1)\Delta}^{N,K} - \Delta\varepsilon_t^{N,K}\Big)^2,$$

$$\mathcal{X}_{\Delta,t}^{N,K} = \mathcal{W}_{\Delta,t}^{N,K} - \frac{N-K}{K}\varepsilon_t^{N,K}.$$

The goal of this section is to check that $\mathcal{X}_{\Delta,t}^{N,K} \simeq \frac{\mu}{(1-\Lambda p)^3}$, and more precisely to prove the following estimate.

Theorem 1.9.1. *Assume $H(q)$ for some $q \ge 3$. Then a.s., for all $t \ge 4$ and all $\Delta \in [1, t/4]$ such that $t/(2\Delta)$ is a positive integer,*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}} \Big| \mathcal{X}_{\Delta,t}^{N,K} - \frac{\mu}{(1-\Lambda p)^3}\Big|\Big] \le C\Big(\frac{N}{K}\sqrt{\frac{\Delta}{t}} + \frac{N^2}{K\Delta^{\frac{1}{2}(q+1)}} + \frac{Nt}{K\Delta^{\frac{q}{2}+1}} + \frac{N}{K\sqrt{Kt}}\Big).$$

In the whole section, we assume that $t \geq 4$ and that $\Delta \in [1, t/4]$ is such that $t/(2\Delta)$ is a positive integer. First, we recall that $\mathcal{W}^{N,K}_{\infty,\infty} := \frac{\mu N}{K^2}\sum_{j=1}^N (c^K_N(j))^2 \ell_N(j)$ and write

$$\begin{aligned}|\mathcal{X}^{N,K}_{\Delta,t} - \mathcal{X}^{N,K}_{\infty,\infty}| \leq& |\mathcal{W}^{N,K}_{\Delta,t} - \mathcal{W}^{N,K}_{\infty,\infty}| + \frac{N-K}{K}\Big|\varepsilon^{N,K}_t - \bar{\ell}^K_N\Big| \\ \leq& D^{N,K,1}_{\Delta,t} + 2D^{N,K,1}_{2\Delta,t} + D^{N,K,2}_{\Delta,t} + 2D^{N,K,2}_{2\Delta,t} + D^{N,K,3}_{\Delta,t} + 2D^{N,K,3}_{2\Delta,t} + D^{N,K,4}_{\Delta,t} + \frac{N}{K}\Big|\varepsilon^{N,K}_t - \bar{\ell}^K_N\Big|,\end{aligned}$$

where

$$\begin{aligned}D^{N,K,1}_{\Delta,t} =& \frac{N}{t}\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \Delta\varepsilon^{N,K}_t\Big)^2 - \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \Delta\mu\bar{\ell}^K_N\Big)^2\Big|, \\ D^{N,K,2}_{\Delta,t} =& \frac{N}{t}\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \Delta\mu\bar{\ell}^K_K\Big)^2 \\ & - \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta}]\Big)^2\Big|, \\ D^{N,K,3}_{\Delta,t} =& \frac{N}{t}\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta}]\Big)^2 \\ & - \mathbb{E}_\theta\Big[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta}]\Big)^2\Big]\Big|,\end{aligned}$$

and finally

$$\begin{aligned}D^{N,K,4}_{\Delta,t} =& \Big|\frac{2N}{t}\mathbb{E}_\theta\Big[\sum_{a=\frac{t}{2\Delta}+1}^{\frac{t}{\Delta}} \Big(\bar{Z}^{N,K}_{2a\Delta} - \bar{Z}^{N,K}_{2(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^{N,K}_{2a\Delta} - \bar{Z}^{N,K}_{2(a-1)\Delta}]\Big)^2\Big] \\ & - \frac{N}{t}\mathbb{E}_\theta\Big[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta}]\Big)^2\Big] - \mathcal{W}^{N,K}_{\infty,\infty}\Big|.\end{aligned}$$

For the first term $D^{N,K,1}_{\Delta,t}$, we have the following lemma.

Lemma 1.9.2. *Assume $H(q)$ for some $q \geq 1$. Then a.s. on the set $\Omega_{N,K}$*

$$\mathbb{E}_\theta[D^{N,K,1}_{\Delta,t}] \leq C\Delta\Big(\frac{N}{t^{2q}} + \frac{N}{Kt}\Big).$$

Proof. Recalling that $\varepsilon^{N,K}_t := t^{-1}(\bar{Z}^{N,K}_{2t} - \bar{Z}^{N,K}_t)$, we have

$$\begin{aligned}D^{N,K,1}_{\Delta,t} =& \frac{N}{t}\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} [\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \Delta\varepsilon^{N,K}_t]^2 - \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} [\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \Delta\mu\bar{\ell}^K_N]^2\Big| \\ =& N\Delta(\mu\bar{\ell}^K_N - \varepsilon^{N,K}_t)^2,\end{aligned}$$

Lemma 1.7.3 completes the proof.

Next, we consider the term $D^{N,K,2}_{\Delta,t}$.

Lemma 1.9.3. *Assume $H(q)$ for some $q \geq 1$. Then a.s. on the set $\Omega_{N,K}$,*

$$\mathbb{E}_\theta[D^{N,K,2}_{\Delta,t}] \leq CNt^{1-q}$$

Proof. First, we have

$$D^{N,K,2}_{\Delta,t} = \frac{2N}{t}\Big| \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\Delta\mu\bar\ell^K_N - \mathbb{E}_\theta[\bar Z^{N,K}_{a\Delta} - \bar Z^{N,K}_{(a-1)\Delta}]\Big) \Big(2(\bar Z^{N,K}_{a\Delta} - \bar Z^{N,K}_{(a-1)\Delta}) - \mathbb{E}_\theta\Big[\bar Z^{N,K}_{a\Delta} - \bar Z^{N,K}_{(a-1)\Delta}\Big] - \Delta\mu\bar\ell^K_N\Big)\Big|,$$

whence

$$\mathbb{E}_\theta[D^{N,K,2}_{\Delta,t}] \leq \frac{2N}{t} \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big|\Delta\mu\bar\ell^K_N - \mathbb{E}_\theta[\bar Z^{N,K}_{a\Delta} - \bar Z^{N,K}_{(a-1)\Delta}]\Big|\Big(\mathbb{E}_\theta\Big[\bar Z^{N,K}_{a\Delta} - \bar Z^{N,K}_{(a-1)\Delta}\Big] + \Delta\mu\bar\ell^K_N\Big).$$

By Lemma 1.6.1-(i)-(ii) with $r = 1$, since $(a-1)\Delta \geq t$, we conclude that on $\Omega_{N,K}$, a.s.,

$$\Big|\Delta\mu\bar\ell^K_N - \mathbb{E}_\theta[\bar Z^{N,K}_{a\Delta} - \bar Z^{N,K}_{(a-1)\Delta}]\Big| \leq Ct^{1-q} \quad \text{and} \quad \mathbb{E}_\theta\Big[\bar Z^{N,K}_{a\Delta} - \bar Z^{N,K}_{(a-1)\Delta}\Big] \leq C\Delta\bar\ell^K_N + C \leq C\Delta$$

since $\bar\ell^K_N$ is bounded on $\Omega_{N,K}$. The conclusion follows.

Next we consider the term $D^{N,K,4}_{\Delta,t}$.

Lemma 1.9.4. *Assume $H(q)$ for some $q \geq 1$. On $\Omega_{N,K}$, there is a $\sigma((\theta_{ij})_{i,j=1\ldots N})$-measurable finite random variable $\mathcal{Y}^{N,K}$ such that for all $1 \leq \Delta \leq \frac{x}{2}$, a.s. on $\Omega_{N,K}$,*

$$\mathrm{Var}_\theta(\bar U^{N,K}_{x+\Delta} - \bar U^{N,K}_x) = \frac{\Delta}{N}\mathcal{W}^{N,K}_{\infty,\infty} - \mathcal{Y}^{N,K} + r_{N,K}(x,\Delta),$$

where, for some constant C, $|r_{N,K}(x,\Delta)| \leq Cx\Delta^{-q}K^{-1}$.

Proof. Recalling (1.10), we write

$$\bar U^{N,K}_{x+\Delta} - \bar U^{N,K}_x = \sum_{n\geq 0}\int_0^{x+\Delta} \beta_n(x, x+\Delta, s)\frac{1}{K}\sum_{i=1}^K\sum_{j=1}^N A^n_N(i,j)M^{j,N}_s ds,$$

where $\beta_n(x, x+\Delta, s) = \phi^{\star n}(x+\Delta-s) - \phi^{\star n}(x-s)$. Set $V^{N,K}_{x,\Delta} = \mathrm{Var}_\theta(\bar U^{N,K}_{x+\Delta} - \bar U^{N,K}_x)$. Recall that $\mathbb{E}[M^{i,N}_s M^{j,N}_t] = \mathbf{1}_{\{i=j\}}\mathbb{E}_\theta[Z^{i,N}_{s\wedge t}]$, see (1.11). We thus have

$$V^{N,K}_{x,\Delta} = \sum_{m,n\geq 0}\int_0^{x+\Delta}\int_0^{x+\Delta} \beta_m(x,x+\Delta,r)\beta_n(x,x+\Delta,s)\frac{1}{K^2}\sum_{i,k=1}^K\sum_{j=1}^N A^m_N(i,j)A^n_N(k,j)\mathbb{E}_\theta[Z^{j,N}_{s\wedge r}]drds.$$

In view of [14, Lemma 28, Step 2], we have $\mathbb{E}_\theta[Z^{j,N}_s] = \mu\ell_N(j)s - X^N_j + R^N_j(s)$, where

$$X^N_j = \mu\kappa\sum_{n\geq 0} n\Lambda^n \sum_{l=1}^N A^n_N(j,l) \quad \text{and} \quad R^N_j(s) = \mu\sum_{n\geq 0}\varepsilon_n(s)\sum_{l=1}^N A^n_N(j,l).$$

Recall that κ and $\varepsilon_n(s)$ were defined in Lemma 1.5.6. Also, there is a constant C such that, for all $j = 1, ..., N$, we have $0 \leq X^N_j \leq C$ and $|R^N_j(s)| \leq C(s^{1-q} \wedge 1)$. Then we can write that

$V^{N,K}_{x,\Delta} = I - M + Q$, where

$$I = \sum_{n,m\geq 0}\int_0^{x+\Delta}\int_0^{x+\Delta}\beta_n(x,x+\Delta,s)\beta_m(x,x+\Delta,r)\frac{1}{K^2}\sum_{i,k=1}^{K}\sum_{j=1}^{N}A^m_N(i,j)A^n_N(k,j)\mu\ell_N(j)(r\wedge s)drds$$
$$M = \sum_{n,m\geq 0}\int_0^{x+\Delta}\int_0^{x+\Delta}\beta_n(x,x+\Delta,s)\beta_m(x,x+\Delta,r)\frac{1}{K^2}\sum_{i,k=1}^{K}\sum_{j=1}^{N}A^m_N(i,j)A^n_N(k,j)X^N_j drds$$
$$Q = \sum_{n,m\geq 0}\int_0^{x+\Delta}\int_0^{x+\Delta}\beta_n(x,x+\Delta,s)\beta_m(x,x+\Delta,r)\frac{1}{K^2}\sum_{i,k=1}^{K}\sum_{j=1}^{N}A^m_N(i,j)A^n_N(k,j)R^N_j(r\wedge s)drds$$

First, we consider M. Using that $|\int_0^{x+\Delta}\beta_n(x,x+\Delta,r)dr| \leq Cn^q\Lambda^n x^{-q}$, see [14, Lemma 15 (ii)] and that X^N_j is bounded by some constant not depending on t, we conclude that on $\Omega_{N,K}$,

$$\begin{aligned}
|M| \leq& C\sum_{m,n\geq 0} m^q n^q \Lambda^{m+n}x^{-2q}K^{-2}\sum_{i,k=1}^{K}\sum_{j=1}^{N}A^m_N(i,j)A^n_N(k,j)\\
\leq& Cx^{-2q}NK^{-2}\sum_{m,n\geq 1} m^q n^q\Lambda^{m+n}|||I_K A^n_N|||_1|||I_K A^m_N|||_1\\
\leq& Cx^{-2q}NK^{-2}\sum_{m,n\geq 1} m^q n^q\Lambda^{m+n}|||I_K A_N|||_1^2|||A_N|||_1^{m+n-2}\\
\leq& Cx^{-2q}N^{-1} \leq Cx\Delta^{-q}K^{-1}.
\end{aligned}$$

Next, we consider Q. We write

$$\begin{aligned}
|Q| \leq& C\sum_{m,n\geq 1}\int_0^{x+\Delta}\int_0^{x+\Delta}\Big|\beta_m(x,x+\Delta,r)\Big|\Big|\beta_n(x,x+\Delta,s)\Big|\\
&\qquad\qquad\frac{N}{K^2}|||I_K A_N|||_1^2|||A_N|||_1^{m+n-2}[(r\wedge s)^{1-q}\wedge 1]drds\\
&+ 2C\sum_{m\geq 0}\int_0^{x+\Delta}\int_0^{x+\Delta}\Big|\beta_0(x,x+\Delta,s)\Big|\Big|\beta_m(x,x+\Delta,r)\Big|\frac{1}{K}|||I_K A^m_N|||_1[(r\wedge s)^{1-q}\wedge 1]drds\\
\leq& Q_1 + Q_2 + 2Q_3 + 2Q_4.
\end{aligned}$$

where, using that $x-\Delta \geq \frac{x}{2}$ and that $(r\wedge s)^{1-q} \leq x^{1-q}$ if $r\wedge s \geq x-\Delta$,

$$\begin{aligned}
Q_1 =& \frac{C}{x^{q-1}}\sum_{m,n\geq 1}\int_{x-\Delta}^{x+\Delta}\int_{x-\Delta}^{x+\Delta}\Big|\beta_m(x,x+\Delta,r)\Big|\Big|\beta_n(x,x+\Delta,s)\Big|\frac{N}{K^2}|||I_K A_N|||_1^2|||A_N|||_1^{m+n-2}drds,\\
Q_2 =& C\sum_{m,n\geq 1}\int_0^{x-\Delta}\int_0^{x+\Delta}\Big|\beta_m(x,x+\Delta,r)\Big|\Big|\beta_n(x,x+\Delta,s)\Big|\frac{N}{K^2}|||I_K A_N|||_1^2|||A_N|||_1^{m+n-2}drds,\\
Q_3 =& \frac{C}{x^{q-1}}\sum_{m\geq 0}\int_0^{x+\Delta}\int_{x-\Delta}^{x+\Delta}\Big|\beta_0(x,x+\Delta,s)\Big|\Big|\beta_m(x,x+\Delta,r)\Big|\frac{1}{K}|||I_K A^m_N|||_1 drds,\\
Q_4 =& C\sum_{m\geq 0}\int_0^{x+\Delta}\int_0^{x-\Delta}\Big|\beta_0(x,x+\Delta,s)\Big|\Big|\beta_m(x,x+\Delta,r)\Big|\frac{1}{K}|||I_K A^m_N|||_1 drds.
\end{aligned}$$

In view of [14, Lemma 15-(ii)], we have the inequalities $\int_0^{x+\Delta}|\beta_n(x,x+\Delta,s)|ds \leq 2\Lambda^n$ and

$\int_0^{x-\Delta} |\beta_m(x, x+\Delta, r)|dr \le Cm^q\Lambda^m\Delta^{-q}$. Hence, on $\Omega_{N,K}$,

$$Q_1 \le Cx^{1-q} \sum_{m,n\ge 1} \Lambda^{m+n} NK^{-2} |||I_K A_N|||_1^2 |||A_N|||_1^{m+n-2} \le CN^{-1}x^{1-q} \le Cx\Delta^{-q}K^{-1},$$
$$Q_2 \le C\Delta^{-q} \sum_{m,n\ge 1} m^q\Lambda^{m+n} NK^{-2} |||I_K A_N|||_1^2 |||A_N|||_1^{m+n-2} \le C\Delta^{-q}N^{-1} \le Cx\Delta^{-q}K^{-1}.$$

Since furthermore $|\beta_0(x, x+\Delta, s)| = |\delta_{\{s=x+\Delta\}} - \delta_{\{s=x\}}| \le \delta_{\{s=x+\Delta\}} + \delta_{\{s=x\}}$, we have

$$Q_3 \le Cx^{1-q} \sum_{m\ge 0} \Lambda^m K^{-1} |||I_K A_N^m|||_1 \le Cx^{1-q}K^{-1} \le Cx\Delta^{-q}K^{-1},$$
$$Q_4 \le C\Delta^{-q} \sum_{m\ge 0} m^q\Lambda^m K^{-1} |||I_K A_N^m|||_1 \le C\Delta^{-q}K^{-1} \le Cx\Delta^{-q}K^{-1}.$$

All in all, on $\Omega_{N,K}$, we have $Q \le Cx\Delta^{-q}K^{-1}$.

Finally we consider I. We recall from [14, Lemma 15 (iii)] that there are $0 \le \kappa_{m,n} \le (m+n)\kappa$ and a function $\varepsilon_{m,n} : (0,\infty)^2 \mapsto \mathbb{R}$ satisfying $|\varepsilon_{m,n}(t, t+\Delta)| \le C(m+n)^q\Lambda^{m+n}t\Delta^{-q}$ such that

$$\begin{aligned}\gamma_{m,n}(x, x+\Delta) =& \int_0^{x+\Delta}\int_0^{x+\Delta} (s\wedge u)\beta_m(x, x+\Delta, s)\beta_n(x, x+\Delta, u)duds \\ =& \Delta\Lambda^{m+n} - \kappa_{m,n}\Lambda^{m+n} + \varepsilon_{m,n}(x, x+\Delta).\end{aligned}$$

Then we can write I as:

$$I = \mu \sum_{m,n\ge 0} \gamma_{m,n}(x, x+\Delta)\frac{1}{K^2}\sum_{i,k=1}^K\sum_{j=1}^N A_N^m(i,j)A_N^n(k,j)\ell_N(j) = I_1 - I_2 + I_3,$$

where

$$\begin{aligned}I_1 =& \mu\Delta \sum_{m,n\ge 0} \Lambda^{m+n}\frac{1}{K^2}\sum_{i,k=1}^K\sum_{j=1}^N A_N^m(i,j)A_N^n(k,j)\ell_N(j), \\ I_2 =& \mu \sum_{m,n\ge 0} \kappa_{m,n}\Lambda^{m+n}\frac{1}{K^2}\sum_{i,k=1}^K\sum_{j=1}^N A_N^m(i,j)A_N^n(k,j)\ell_N(j), \\ I_3 =& \mu \sum_{m,n\ge 0} \varepsilon_{m,n}(x, x+\Delta)\frac{1}{K^2}\sum_{i,k=1}^K\sum_{j=1}^N A_N^m(i,j)A_N^n(k,j)\ell_N(j).\end{aligned}$$

Recalling that $\mathcal{W}_{\infty,\infty}^{N,K} := \frac{\mu N}{K^2}\sum_{j=1}^N (c_N^K(j))^2\ell_N(j)$ by definition and that $\sum_{m\ge 0}\Lambda^m A_N^m(i,j) = Q_N(i,j)$,

$$\begin{aligned}I_1 &= \mu\Delta \sum_{m,n\ge 0} \Lambda^{m+n}\frac{1}{K^2}\sum_{i,k}^K\sum_{j=1}^N A_N^m(i,j)A_N^n(k,j)\ell_N(j) \\ &= \mu\Delta\frac{1}{K^2}\sum_{j=1}^N \left(c_N^K(j)\right)^2\ell_N(j) = \Delta\frac{1}{N}\mathcal{W}_{\infty,\infty}^{N,K}.\end{aligned}$$

Next, we set $\mathcal{Y}^{N,K} = I_2$. It is obvious that $\mathcal{Y}^{N,K}$ is a $(\theta_{ij})_{i,j=1\dots N}$ measurablefunction and well-defined on $\Omega_{N,K}$. Finally, using that $\varepsilon_{m,n}(x, x+\Delta) \le C(m+n)^q\Lambda^{m+n}x\Delta^{-q}$ and that ℓ_N is

bounded on $\Omega_{N,K}$ (we have to treat separately the case $n=0$ or $m=0$),

$$\begin{aligned}
I_3 \leq & C\frac{x}{K^2\Delta^q}\sum_{m\geq 0} m^q\Lambda^m \sum_{i,k=1}^{K} A_N^m(k,i) + C\frac{xN}{\Delta^q K^2}\sum_{m,n\geq 1}(n+m)^q\Lambda^{m+n}|||I_K A_N^n|||_1|||I_K A_N^m|||_1 \\
\leq & C\frac{x}{K\Delta^q}\sum_{m\geq 0} m^q\Lambda^m|||I_K A_N^m|||_1 + C\frac{xN}{\Delta^q K^2}\sum_{m,n\geq 1}(n+m)^q\Lambda^{m+n}|||I_K A_N|||_1^2|||A_N|||_1^{m+n-2} \\
\leq & C\frac{x}{N\Delta^q},
\end{aligned}$$

still on $\Omega_{N,K}$. All in all, we have verified that $V_{x,\Delta}^{N,K} = I - M + Q$, with

$$|M| + |Q| + |I - \Delta N^{-1}\mathcal{W}_{\infty,\infty}^{N,K} + \mathcal{Y}^{N,K}| \leq Cx\Delta^{-q}K^{-1},$$

which completes the proof.

Next, we consider the term $D_{\Delta,t}^{N,K,4}$.

Lemma 1.9.5. *Assume $H(q)$ for some $q \geq 1$. Then a.s. on $\Omega_{N,K}$, for $1 \leq \Delta \leq \frac{t}{4}$, we have:*

$$\mathbb{E}_\theta[D_{\Delta,t}^{N,K,4}] \leq C\frac{Nt}{K\Delta^{1+q}}.$$

Proof. Recalling that $U_t^{i,N} = Z_t^{i,N} - \mathbb{E}_\theta[Z_t^{i,N}]$, we see that

$$D_{\Delta,t}^{N,K,4} = \Big|\frac{2N}{t}\sum_{a=t/(2\Delta)+1}^{t/\Delta}\mathrm{Var}(\bar{U}_{2a\Delta}^{N,K} - \bar{U}_{2(a-1)\Delta}^{N,K}) - \frac{N}{t}\sum_{a=t/\Delta+1}^{2t/\Delta}\mathrm{Var}(\bar{U}_{a\Delta}^{N,K} - \bar{U}_{(a-1)\Delta}^{N,K}) - \mathcal{W}_{\infty,\infty}^{N,K}\Big|.$$

By Lemma 1.9.4, we have

$$\mathrm{Var}_\theta(\bar{U}_{x+\Delta}^{N,K} - \bar{U}_x^{N,K}) = \frac{\Delta}{N}\mathcal{W}_{\infty,\infty}^{N,K} - \mathcal{Y}^{N,K} + r_{N,K}(x,\Delta).$$

Since $a \in \{t/(2\Delta)+1, ..., t/\Delta\}$, $x = 2(a-1)\Delta \geq t$ satisfies $2\Delta \leq \frac{x}{2}$ and for $a \in \{t/\Delta+1, ..., 2t/\Delta\}$, $x = (a-1)\Delta \geq t$ satisfies $\Delta \leq x/2$. Then we conclude that

$$\begin{aligned}
D_{\Delta,t}^{N,K,4} = & \Big|\frac{2N}{t}\sum_{a=t/(2\Delta)+1}^{t/\Delta}\Big[\frac{2\Delta}{N}\mathcal{W}_{\infty,\infty}^{N,K} - \mathcal{Y}^{N,K} + r_{N,K}(2(a-1)\Delta, 2\Delta)\Big] \\
& - \frac{N}{t}\sum_{a=\frac{t}{\Delta}+1}^{2t/\Delta}\Big[\frac{\Delta}{N}\mathcal{W}_{\infty,\infty}^{N,K} - \mathcal{Y}^{N,K} + r_{N,K}((a-1)\Delta, \Delta)\Big] - \mathcal{W}_{\infty,\infty}^{N,K}\Big| \\
= & \Big|\frac{2N}{t}\sum_{a=t/(2\Delta)+1}^{t/\Delta} r_{N,K}(2(a-1)\Delta, 2\Delta) - \frac{N}{t}\sum_{a=t/\Delta+1}^{2t/\Delta} r_{N,K}((a-1)\Delta, \Delta)\Big|.
\end{aligned}$$

But $|r_{N,K}(x,\Delta)| \leq Cx\Delta^{-q}K^{-1}$, whence finally

$$D_{\Delta,t}^{N,K,4} \leq C\frac{N}{t}\frac{t}{\Delta}\Big(\frac{t}{\Delta^q K}\Big) = \frac{CNt}{K\Delta^{1+q}}$$

as desired.

To treat the last term $D_{\Delta,t}^{N,K,3}$, we need this following Lemma.

Lemma 1.9.6. *Assume $H(q)$ for some $q \geq 1$. On the set $\Omega_{N,K}$, for all $t, x, \Delta \geq 1$, we have*

$$\mathrm{Var}\Big[\Big(\bar{U}_{x+\Delta}^{N,K} - \bar{U}_x^{N,K}\Big)^2\Big] \leq C\Big(\frac{\Delta^2}{K^2} + \frac{t^2}{K^2\Delta^{4q}}\Big) \quad \text{if} \quad \frac{t}{2} \leq x - \Delta \leq x + \Delta \leq 2t \tag{1.12}$$

and

$$\mathrm{Cov}_\theta\Big((\bar U^{N,K}_{x+\Delta}-\bar U^{N,K}_x)^2,(\bar U^{N,K}_{y+\Delta}-\bar U^{N,K}_y)^2\Big)\le C\Big(\frac{\sqrt{t}}{K\Delta^{q-1}}+\frac{t^2}{K^2\Delta^{4q}}+\frac{\sqrt{t}}{K^2\Delta^{q-\frac{3}{2}}}\Big) \tag{1.13}$$
$$\text{if}\quad \frac{t}{2}\le y-\Delta\le y+\Delta\le x-2\Delta\le x+\Delta\le 2t.$$

Proof. Step 1: recalling (1.10), for $z\in[x,x+\Delta]$, we write

$$U^{i,N}_z-U^{i,N}_x=\sum_{n\ge0}\int_0^z\beta_n(x,z,r)\sum_{j=1}^N A^n_N(i,j)M^{j,N}_r dr=\Gamma^{i,N}_{x,z}+X^{i,N}_{x,z},$$

where $\beta_n(x,z,r)=\phi^{\star n}(z-r)-\phi^{\star n}(x-r)$ and where

$$\Gamma^{i,N}_{x,z}=\sum_{n\ge0}\int_{x-\Delta}^z\beta_n(x,z,r)\sum_{j=1}^N A^n_N(i,j)(M^{j,N}_r-M^{j,N}_{x-\Delta})dr,$$
$$X^{i,N}_{x,z}=\sum_{n\ge0}\Big(\int_{x-\Delta}^z\beta_n(x,z,r)dr\Big)\sum_{j=1}^N A^n_N(i,j)M^{j,N}_{x-\Delta}+\sum_{n\ge0}\int_0^{x-\Delta}\beta_n(x,z,r)\sum_{j=1}^N A^n_N(i,j)M^{j,N}_r dr.$$

We set $\bar\Gamma^{N,K}_{x,z}=K^{-1}\sum_{i=1}^K\Gamma^{i,N}_{x,z}$ and $\bar X^{N,K}_{x,z}=K^{-1}\sum_{i=1}^K X^{i,N}_{x,z}$. We write

$$\bar X^{N,K}_{x,z}=\sum_{n\ge0}\Big(\int_{x-\Delta}^z\beta_n(x,z,r)dr\Big)O^{N,K,n}_{x-\Delta}+\sum_{n\ge0}\int_0^{x-\Delta}\beta_n(x,z,r)O^{N,K,n}_r dr.$$

where

$$O^{N,K,n}_r=\frac{1}{K}\sum_{i=1}^K\sum_{j=1}^N A^n_N(i,j)M^{j,N}_r.$$

By (1.11), we have $[M^{i,N},M^{j,N}]_t=\mathbf{1}_{\{i=j\}}Z^{i,N}_t$. Hence, for $n\ge1$,

$$[O^{N,K,n},O^{N,K,n}]_r=\frac{1}{K^2}\sum_{j=1}^N\Big(\sum_{i=1}^K A^n_N(i,j)\Big)^2Z^{j,N}_r\le\frac{N}{K^2}|||I_KA_N|||_1^2|||A_N|||_1^{2n-2}\bar Z^N_r.$$

And when $n=0$, we have

$$[O^{N,K,0},O^{N,K,0}]_r=\frac{1}{K^2}\sum_{j=1}^N\Big(\sum_{i=1}^K A^0_N(i,j)\Big)^2Z^{j,N}_r=\frac{1}{K}\bar Z^{N,K}_r.$$

By Lemma 1.7.2, we have, on $\Omega_{N,K}$,

$$\mathbb{E}_\theta[(\bar Z^{N,K}_t)^2]\le2\mathbb{E}_\theta[\bar Z^{N,K}_t]^2+2\mathbb{E}_\theta[(\bar U^{N,K}_t)^2]\le Ct^2.$$

Hence, by the Doob's inequality, when $n\ge1$:

$$\mathbb{E}_\theta\Big[\sup_{[0,2t]}\Big(O^{N,K,n}_r\Big)^4\Big]\le\frac{CN^2}{K^4}|||I_KA_N|||_1^4|||A_N|||_1^{4n-4}\mathbb{E}_\theta\Big[\Big(\bar Z^N_{2t}\Big)^2\Big]\le\frac{C}{N^2}|||A_N|||_1^{4n-4}t^2. \tag{1.14}$$

By the same way,

$$\mathbb{E}_\theta\Big[\sup_{[x-\Delta,x+\Delta]}\Big(O^{N,K,n}_r-O^{N,K,n}_{x-\Delta}\Big)^4\Big]\le\frac{C}{N^2}|||A_N|||_1^{4n-4}\Delta^2, \tag{1.15}$$

and in the case $n = 0$, by Doob's inequality,

$$\mathbb{E}_\theta\Big[\sup_{[x-\Delta,x+\Delta]}\Big(O_r^{N,K,0} - O_{x-\Delta}^{N,K,0}\Big)^4\Big] \le CK^{-2}\Delta^2. \tag{1.16}$$

Step 2: We recall the result of [14, Lemma 15]:

$$\Big|\int_{x-\Delta}^{z}\beta_n(x,z,r)dr\Big| + \int_0^{x-\Delta}\Big|\beta_n(x,z,r)\Big|dr \le Cn^q\Lambda^n\Delta^{-q}.$$

So we conclude that

$$|\bar{X}_{x,z}^N| \le C\sum_{n\ge0} n^q\Lambda^n\Delta^{-q}\sup_{[0,2t]}|O_r^{N,K,n}| = C\sum_{n\ge1} n^q\Lambda^n\Delta^{-q}\sup_{[0,2t]}|O_r^{N,K,n}|.$$

Recalling (1.14), on the set $\Omega_{N,K}$, by using the Minkowski inequality we conclude that

$$\mathbb{E}[(\bar{X}_{x,z}^N)^4]^{\frac14} \le C\sum_{n\ge1} n^q\Lambda^n|||A_N|||_1^{n-1}\Delta^{-q}N^{-\frac12}\sqrt{t} \le C\Delta^{-q}N^{-\frac12}\sqrt{t}.$$

Step 3: We rewrite

$$\bar{\Gamma}_{x,z}^{N,K} = \sum_{n\ge0}\int_{x-\Delta}^{z}\beta_n(x,z,r)[O_r^{N,K,n} - O_{x-\Delta}^{N,K,n}]dr.$$

Since $\int_{x-\Delta}^{x}|\beta_n(x,z,r)|dr \le 2\Lambda^n$ by [14, Lemma 15], using (1.15)-(1.16) and the Minkowski inequality,

$$\mathbb{E}[(\bar{\Gamma}_{x,z}^{N,K})^4]^{\frac14} \le C\Big\{\Delta^{\frac12}K^{-\frac12} + \sum_{n\ge1}\Lambda^n\frac{1}{\sqrt{N}}|||A_N|||_1^{n-1}\Delta^{\frac12}\Big\} \le C\Delta^{\frac12}(K^{-\frac12}+N^{-\frac12}) \le C\Delta^{\frac12}K^{-\frac12}.$$

Step 4: Since, see Step 1,

$$\Big(\bar{U}_{x+\Delta}^{N,K} - \bar{U}_x^{N,K}\Big)^4 = \Big(\bar{\Gamma}_{x,x+\Delta}^{N,K} + \bar{X}_{x,x+\Delta}^{N,K}\Big)^4 \le 8\Big[(\bar{\Gamma}_{x,x+\Delta}^{N,K})^4 + (\bar{X}_{x,x+\Delta}^{N,K})^4\Big],$$

we deduce from Steps 2 and 3 that (1.12) holds true.

Step 5: The aim of this step is to show that, for x, y, Δ as in the statement, it holds true that

$$\mathrm{Cov}_\theta\Big((\bar{U}_{x+\Delta}^{N,K} - \bar{U}_x^{N,K})^2, (\bar{U}_{y+\Delta}^{N,K} - \bar{U}_y^{N,K})^2\Big) \le |\mathrm{Cov}_\theta[(\bar{\Gamma}_{x,x+\Delta}^{N,K})^2, (\bar{\Gamma}_{y,y+\Delta}^{N,K})^2]| + \frac{C}{K^2}\Big(\frac{t^2}{\Delta^{4q}} + \frac{\sqrt{t}}{\Delta^{q-\frac32}}\Big).$$

We write

$$(\bar{U}_{x+\Delta}^{N,K} - \bar{U}_x^{N,K})^2 = (\bar{\Gamma}_{x,x+\Delta}^{N,K})^2 + (\bar{X}_{x,x+\Delta}^{N,K})^2 + 2\bar{\Gamma}_{x,x+\Delta}^{N,K}\bar{X}_{x,x+\Delta}^{N,K},$$

and the same formula for y. Then we use the bilinearity of the covariance. We have the term $\mathrm{Cov}_\theta[(\bar{\Gamma}_{x,x+\Delta}^{N,K})^2, (\bar{\Gamma}_{y,y+\Delta}^{N,K})^2]$, and it remains to verify that

$$\begin{aligned} R := \mathbb{E}_\theta\Big[&(\bar{\Gamma}_{x,x+\Delta}^{N,K})^2(\bar{X}_{y,y+\Delta}^{N,K})^2 + 2(\bar{\Gamma}_{x,x+\Delta}^{N,K})^2|\bar{\Gamma}_{y,y+\Delta}^{N,K}\bar{X}_{y,y+\Delta}^{N,K}| + (\bar{X}_{x,x+\Delta}^{N,K})^2(\bar{\Gamma}_{y,y+\Delta}^{N,K})^2 \\ &+ (\bar{X}_{x,x+\Delta}^{N,K})^2(\bar{X}_{y,y+\Delta}^{N,K})^2 + 2(\bar{X}_{x,x+\Delta}^{N,K})^2|\bar{\Gamma}_{y,y+\Delta}^{N,K}\bar{X}_{y,y+\Delta}^{N,K}| + 2|\bar{\Gamma}_{x,x+\Delta}^{N,K}\bar{X}_{x,x+\Delta}^{N,K}|(\bar{\Gamma}_{y,y+\Delta}^{N,K})^2 \\ &+ 2|\bar{X}_{x,x+\Delta}^{N,K}\bar{\Gamma}_{x,x+\Delta}^{N,K}|(\bar{X}_{y,y+\Delta}^{N,K})^2 + 4|\bar{\Gamma}_{x,x+\Delta}^{N,K}\bar{X}_{x,x+\Delta}^{N,K}\bar{\Gamma}_{y,y+\Delta}^{N,K}\bar{X}_{y,y+\Delta}^{N,K}|\Big] \end{aligned}$$

is bounded by $\frac{C}{K^2}(\frac{t^2}{\Delta^{4q}} + \frac{\sqrt{t}}{\Delta^{q-\frac32}})$.

By Steps 2 and 3, we know that $\mathbb{E}[(\bar{\Gamma}^{N,K}_{x,z})^4] \le C\Delta^2 K^{-2}$ and $\mathbb{E}[(\bar{X}^{N,K}_{x,z})^4] \le Ct^2\Delta^{-4q}N^{-2}$, and the sames inequalities hold true with y instead of x. Using furthermore the Hölder inequality, one may verify that, setting $a = C\Delta^2K^{-2}$ and $b = Ct^2\Delta^{-4q}N^{-2}$, we have

$$R \le \sqrt{ab} + 2a^{3/4}b^{1/4} + \sqrt{ab} + b + 2a^{1/4}b^{3/4} + 2a^{3/4}b^{1/4} + 2a^{1/4}b^{3/4} + 4\sqrt{ab},$$

which is easily bounded by $C(b + b^{1/4}a^{3/4})$, from which the conclusion follows.

Step 6: Here we want to verify that

$$\mathcal{I} := |\mathrm{Cov}_\theta[(\bar{\Gamma}^{N,K}_{x,x+\Delta})^2, (\bar{\Gamma}^{N,K}_{y,y+\Delta})^2]| \le \frac{C\sqrt{t}}{K\Delta^{q-1}}.$$

We recall from [14, Lemma 30, Step 6] that for any r, s in $[x-\Delta, x+\Delta]$, any u, v in $[y-\Delta, y+\Delta]$, any j, l, δ, ε in $\{1, \ ...N\}$,

$$\Big|\mathrm{Cov}_\theta\Big[(M^{j,N}_r - M^{j,N}_{x-\Delta})(M^{l,N}_s - M^{l,N}_{x-\Delta}), (M^{\delta,N}_u - M^{\delta,N}_{y-\Delta})(M^{\varepsilon,N}_v - M^{\varepsilon,N}_{y-\Delta})\Big| \le C\mathbf{1}_{\{j=l\}}\sqrt{t}\Delta^{1-q}.$$

We start from

$$\bar{\Gamma}^{N,K}_{x,x+\Delta} = \sum_{n\ge 0}\int_{x-\Delta}^{x+\Delta}\beta_n(x, x+\Delta, r)\frac{1}{K}\sum_{i=1}^{K}\sum_{j=1}^{N}A_N(i,j)(M^{j,N}_r - M^{j,N}_{x-\Delta})dr.$$

So

$$\begin{aligned}\mathcal{I} = &\sum_{m,n,a,b\ge 0}\int_{x-\Delta}^{x+\Delta}\int_{x-\Delta}^{x+\Delta}\int_{y-\Delta}^{y+\Delta}\int_{y-\Delta}^{y+\Delta}\beta_m(x, x+\Delta, r)\beta_n(x, x+\Delta, s)\\ &\beta_a(y, y+\Delta, u)\beta_b(y, y+\Delta, v)\frac{1}{K^4}\sum_{i,k,\alpha,\gamma=1}^{K}\sum_{j,l,\delta,\varepsilon=1}^{N}A^m_N(i,j)A^n_N(k,l)A^a_N(\alpha,\delta)A^b_N(\gamma,\varepsilon)\\ &\mathrm{Cov}_\theta\Big[(M^{j,N}_r - M^{j,N}_{x-\Delta})(M^{l,N}_s - M^{l,N}_{x-\Delta}), (M^{\delta,N}_u - M^{\delta,N}_{y-\Delta})(M^{\varepsilon,N}_v - M^{\varepsilon,N}_{y-\Delta})\Big]dvdudsdr\\ \le &C\sqrt{t}\Delta^{1-q}\sum_{m,n,a,b\ge 0}\Lambda^{m+n+a+b}\frac{1}{K^4}\sum_{i,k,\alpha,\gamma=1}^{K}\sum_{j,\delta,\varepsilon=1}^{N}A^m_N(i,j)A^n_N(k,j)A^a_N(\alpha,\delta)A^b_N(\gamma,\varepsilon).\end{aligned}$$

We used again the result [14, Lemma 15]: $\int_{x-\Delta}^{x+\Delta}|\beta_m(x, x+\Delta, r)|dr \le 2\Lambda^m$. And we observe one more time that $A^0_N(i,j) = \mathbf{1}_{\{i=j\}}$ and, when $m \ge 1$, $\sum_{i=1}^{K}A^m_N(i,j) \le |||I_K A_N|||_1|||A_N|||_1^{m-1}$. We

now treat separately the cases where m,n,a,b vanish and find, on $\Omega_{N,K}$,

$$\begin{aligned}
\mathcal{I} \leq & \frac{C\sqrt{t}}{K^4\Delta^{q-1}} \sum_{m,n,a,b\geq 1} \sum_{j,\delta,\varepsilon=1}^{N} \Lambda^{m+n+a+b} |||I_K A_N|||_1^4 |||A_N|||_1^{m+n+a+b-4} \\
& + \frac{4C\sqrt{t}}{K^4\Delta^{q-1}} \sum_{n,a,b\geq 1} \sum_{j,\delta,\varepsilon=1}^{N} \sum_{i=1}^{K} \mathbf{1}_{\{i=j\}} \Lambda^{n+a+b} |||I_K A_N|||_1^3 |||A_N|||_1^{n+a+b-3} \\
& + \frac{2C\sqrt{t}}{K^4\Delta^{q-1}} \sum_{a,b\geq 1} \sum_{j,\delta,\varepsilon=1}^{N} \Big\{ \sum_{i,k=1}^{K} \mathbf{1}_{\{i=k=j\}} + \sum_{\alpha,\gamma=1}^{K} \mathbf{1}_{\{\alpha=\delta,\varepsilon=\gamma\}} + \sum_{i,\alpha=1}^{K} \mathbf{1}_{\{i=j,\alpha=\delta\}} \Big\} \\
& \qquad \times \Lambda^{a+b} |||I_K A_N|||_1^2 |||A_N|||_1^{a+b-2} \\
& + \frac{2C\sqrt{t}}{K^4\Delta^{q-1}} \sum_{a\geq 1} \sum_{j,\delta,\varepsilon=1}^{N} \Big\{ 2 \sum_{i,k,\alpha=1}^{K} \mathbf{1}_{\{i=k=j,\alpha=\delta\}} + 2 \sum_{i,\alpha,\gamma=1}^{K} \mathbf{1}_{\{i=j,\alpha=\delta,\varepsilon=\gamma\}} \Big\} \Lambda^{a} |||I_K A_N|||_1 |||A_N|||_1^{a-1} \\
& + \frac{2C\sqrt{t}}{K^4\Delta^{q-1}} \sum_{i,k,\alpha,\gamma=1}^{K} \sum_{j,\delta,\varepsilon=1}^{N} \mathbf{1}_{\{i=k=j,\alpha=\delta,\varepsilon=\gamma\}} \\
\leq & \frac{2C\sqrt{t}}{K^4\Delta^{q-1}} \Big(\frac{K^4}{N^4} N^3 + \frac{K^3}{N^3} N^2 K + \frac{K^2}{N^2}(KN^2 + K^2 N + K^2 N) + \frac{K}{N}(K^2 N + K^3) + K^3 \Big) \\
\leq & \frac{C\sqrt{t}\Delta^{1-q}}{K}.
\end{aligned}$$

Step 7: We conclude from Steps 5 and 6 that on the set $\Omega_{N,K}$,

$$\Big| \mathrm{Cov}_\theta \Big[(\bar{U}^{N,K}_{x+\Delta} - \bar{U}^{N,K}_x)^2, (\bar{U}^{N,K}_{y+\Delta} - \bar{U}^{N,K}_y)^2 \Big] \Big| \leq C \Big[\frac{\sqrt{t}}{K\Delta^{q-1}} + \frac{t^2}{K^2\Delta^{4q}} + \frac{\sqrt{t}}{K^2\Delta^{q-3/2}} \Big],$$

which proves (1.13).

We can now study $D^{N,K,3}_{\Delta,t}$.

Lemma 1.9.7. *Assume $H(q)$ for some $q \geq 1$. On the set $\Omega_{N,K}$, for all $1 \leq \Delta \leq \frac{t}{2}$,*

$$\mathbb{E}_\theta[(D^{N,K,3}_{\Delta,t})^2] \leq C \Big(\frac{N^2}{K^2}\frac{\Delta}{t} + \frac{N^2}{K^2}\frac{t}{\Delta^{4q+1}} + \frac{N^2}{K}\frac{\sqrt{t}}{\Delta^{q+1}} + \frac{N^2}{K^2}\frac{t^2}{\Delta^{4q+2}} + \frac{N^2}{K^2}\frac{\sqrt{t}}{\Delta^{q+\frac{1}{2}}} \Big).$$

Proof. Recall that by definition

$$\begin{aligned}
D^{N,K,3}_{\Delta,t} = & \frac{N}{t} \Big| \sum_{a=t/\Delta+1}^{2t/\Delta} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta}] \Big)^2 \\
& - \mathbb{E}_\theta \Big[\sum_{a=t/\Delta+1}^{2t/\Delta} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta}] \Big)^2 \Big] \Big|.
\end{aligned}$$

Since now $\bar{U}^{N,K}_r = \bar{Z}^{N,K}_r - \mathbb{E}_\theta[\bar{Z}^{N,K}_r]$,

$$\mathbb{E}_\theta[(D^{N,K,3}_{\Delta,t})^2] = \frac{N^2}{t^2} \mathrm{Var}_\theta \Big(\sum_{a=t/\Delta+1}^{2t/\Delta} (\bar{U}^{N,K}_{a\Delta} - \bar{U}^{N,K}_{(a-1)\Delta})^2 \Big) = \frac{N^2}{t^2} \sum_{a,b=t/\Delta+1}^{2t/\Delta} K_{a,b},$$

where $K_{a,b} = \mathrm{Cov}_\theta[(\bar{U}^{N,K}_{a\Delta} - \bar{U}^{N,K}_{(a-1)\Delta})^2, (\bar{U}^{N,K}_{b\Delta} - \bar{U}^{N,K}_{(b-1)\Delta})^2]$. By Lemma 1.9.6, for $|a-b| \leq 2$,

$$|K_{a,b}| \leq \Big\{ \mathrm{Var}_\theta[(\bar{U}^{N,K}_{a\Delta} - \bar{U}^{N,K}_{(a-1)\Delta})^2] \mathrm{Var}[(\bar{U}^{N,K}_{b\Delta} - \bar{U}^{N,K}_{(b-1)\Delta})^2] \Big\}^{\frac{1}{2}} \leq C \Big(\frac{\Delta^2}{K^2} + \frac{t^2}{K^2\Delta^{4q}} \Big).$$

If now $|a-b|\geq 3$, we set $x=(a-1)\Delta$, $y=(b-1)\Delta$ in (1.13) and get

$$|K_{a,b}|\leq C\Big(\frac{\sqrt{t}}{K\Delta^{q-1}}+\frac{t^2}{K^2\Delta^{4q}}+\frac{\sqrt{t}}{K^2\Delta^{q-\frac{3}{2}}}\Big).$$

Finally we conclude that

$$\begin{aligned}\mathbb{E}_\theta[(D^{N,K,3}_{\Delta,t})^2]\leq& C\frac{N^2}{t^2}\frac{t}{\Delta}\Big(\frac{\Delta^2}{K^2}+\frac{t^2}{K^2\Delta^{4q}}\Big)+C\frac{N^2}{t^2}\frac{t^2}{\Delta^2}\Big(\frac{\sqrt{t}}{K\Delta^{q-1}}+\frac{t^2}{K^2\Delta^{4q}}+\frac{\sqrt{t}}{K^2\Delta^{q-\frac{3}{2}}}\Big)\\ \leq& C\Big(\frac{N^2}{K^2}\frac{\Delta}{t}+\frac{N^2}{K^2}\frac{t}{\Delta^{4q+1}}+\frac{N^2}{K}\frac{\sqrt{t}}{\Delta^{q+1}}+\frac{N^2}{K^2}\frac{t^2}{\Delta^{4q+2}}+\frac{N^2}{K^2}\frac{\sqrt{t}}{\Delta^{q+\frac{1}{2}}}\Big).\end{aligned}$$

which completes the proof.

Lemma 1.9.8. *Under the assumption $H(q)$ for some $q\geq 3$ and the the set $\Omega_{N,K}$, we have:*

$$\mathbb{E}_\theta\Big[\Big|\mathcal{W}^{N,K}_{\Delta,t}-\mathcal{W}^{N,K}_{\infty,\infty}\Big|\Big]\leq C\Big(\frac{N}{K}\sqrt{\frac{\Delta}{t}}+\frac{N^2}{K\Delta^{\frac{1}{2}(q+1)}}+\frac{Nt}{K\Delta^{\frac{q}{2}+1}}\Big).$$

Proof. We summarize all the above Lemmas and conclude that, on $\Omega_{N,K}$,

$$\begin{aligned}&\mathbb{E}_\theta\Big[\Big|\mathcal{W}^{N,K}_{\Delta,t}-\mathcal{W}^{N,K}_{\infty,\infty}\Big|\Big]\\ \leq&\mathbb{E}_\theta\Big[D^{N,K,1}_{\Delta,t}+2D^{N,K,1}_{2\Delta,t}+D^{N,K,2}_{\Delta,t}+2D^{N,K,2}_{2\Delta,t}+D^{N,K,3}_{\Delta,t}+2D^{N,K,3}_{2\Delta,t}+D^{N,K,4}_{\Delta,t}\Big]\\ \leq&C\Big(\frac{N}{K}\frac{\Delta}{t}+\frac{N\Delta}{t^{2q}}+\frac{N}{t^{q-1}}+\frac{Nt}{K\Delta^{1+q}}\Big)\\ &+C\sqrt{\frac{N^2}{K^2}\frac{\Delta}{t}+\frac{N^2}{K^2}\frac{t}{\Delta^{4q+1}}+\frac{N^2}{K}\frac{\sqrt{t}}{\Delta^{q+1}}+\frac{N^2}{K^2}\frac{t^2}{\Delta^{4q+2}}+\frac{N^2}{K^2}\frac{\sqrt{t}}{\Delta^{q+\frac{1}{2}}}}.\end{aligned}$$

Since $1\leq\Delta\leq t$ and $q\geq 3$, we conclude, after some tedious but direct computations, that

$$\mathbb{E}_\theta\Big[\Big|\mathcal{W}^{N,K}_{\Delta,t}-\mathcal{W}^{N,K}_{\infty,\infty}\Big|\Big]\leq C\Big(\frac{N}{K}\sqrt{\frac{\Delta}{t}}+\frac{N^2}{K\Delta^{\frac{1}{2}(q+1)}}+\frac{Nt}{K\Delta^{\frac{q}{2}+1}}\Big).$$

The most difficult terms are

$$\sqrt{\frac{N^2t^{1/2}}{K\Delta^{q+1}}}=\sqrt{\frac{N^2}{K\Delta^{(q+1)/2}}}\sqrt{\frac{t^{1/2}}{\Delta^{(q+1)/2}}}\leq\frac{N^2}{K\Delta^{(q+1)/2}}+\frac{t^{1/2}}{\Delta^{(q+1)/2}}\leq\frac{N^2}{K\Delta^{(q+1)/2}}+\frac{Nt}{K\Delta^{q/2+1}}$$

and

$$\sqrt{\frac{N^2}{K^2}\frac{\sqrt{t}}{\Delta^{q+\frac{1}{2}}}}\leq\frac{N}{K}\Big(\sqrt{\frac{\Delta}{t}}+\frac{t}{\Delta^{q+1}}\Big)\leq\frac{N}{K}\Big(\sqrt{\frac{\Delta}{t}}+\frac{t}{\Delta^{q/2+1}}\Big).$$

The proof is complete.

Next we prove the main result of this section. *Proof.* [Proof of Theorem 1.9.1] We start from

$$\begin{aligned}&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{X}^{N,K}_{\Delta,t}-\frac{\mu}{(1-\Lambda p)^3}\Big|\Big]\\ \leq&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{W}^{N,K}_{\Delta,t}-\mathcal{W}^{N,K}_{\infty,\infty}\Big|\Big]+\frac{N}{K}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\varepsilon^{N,K}_t-\mu\bar{\ell}^K_N\Big|\Big]+\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{X}^{N,K}_{\infty,\infty}-\frac{\mu}{(1-\Lambda p)^3}\Big|\Big]\\ \leq&C\Big(\frac{N}{K}\sqrt{\frac{\Delta}{t}}+\frac{N^2}{K\Delta^{\frac{1}{2}(q+1)}}+\frac{Nt}{K\Delta^{\frac{q}{2}+1}}+\frac{1}{\sqrt{K}}\Big)+C\frac{N}{Kt^q}+C\frac{N}{K\sqrt{Kt}}+\frac{C}{K}\end{aligned}$$

by Lemmas 1.9.8, 1.7.3 and 1.5.19. Since $t\geq\Delta\geq 1$, we have $\frac{N}{Kt^q}\leq\frac{Nt}{K\Delta^{\frac{q}{2}+1}}$ and we conclude that

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{X}^{N,K}_{\Delta,t}-\frac{\mu}{(1-\Lambda p)^3}\Big|\Big]\leq C\Big(\frac{N}{K}\sqrt{\frac{\Delta}{t}}+\frac{N^2}{K\Delta^{\frac{1}{2}(q+1)}}+\frac{Nt}{K\Delta^{\frac{q}{2}+1}}+\frac{1}{K}+\frac{N}{K\sqrt{Kt}}\Big),$$

which was our goal.

Next, we write down the probability estimate.

Corollary 1.9.9. *Assume $H(q)$ for some $q \geq 1$. We have*

$$P\Big(\Big|\mathcal{X}^{N,K}_{\Delta,t} - \frac{\mu}{(1-\Lambda p)^3}\Big| \geq \varepsilon\Big) \\ \leq \frac{C}{\varepsilon}\Big(\frac{N}{K}\sqrt{\frac{\Delta}{t}} + \frac{N^2}{K\Delta^{\frac{1}{2}(q+1)}} + \frac{Nt}{K\Delta^{\frac{q}{2}+1}} + \frac{1}{K} + \frac{N}{K\sqrt{Kt}}\Big) + CNe^{-C'K}.$$

Under $H(q)$ for some $q > 3$ and with the choice $\Delta_t \sim t^{\frac{4}{(q+1)}}$, this gives

$$P\Big(\Big|\mathcal{X}^{N,K}_{\Delta_t,t} - \frac{\mu}{(1-\Lambda p)^3}\Big| \geq \varepsilon\Big) \leq \frac{C}{\varepsilon}\Big(\frac{1}{K} + \frac{N}{K\sqrt{t^{1-\frac{4}{1+q}}}} + \frac{N^2}{Kt^2}\Big) + CNe^{-C'K}.$$

Proof. The first assertion immediately follows from Theorem 1.9.1 and Lemma 1.5.7. The second assertion is not difficult.

1.10 The final result in the subcritical case.

We summarize the rates we obtained for the three estimators: by Theorem 1.7.1 and Corollaries 1.8.7 and 1.9.9, we have, under $H(q)$ for some $q > 3$, for all $\epsilon \in (0,1)$, all $t \geq 1$, all $N \geq K \geq 1$,

$$P\Big(\Big|\varepsilon^{N,K}_t - \frac{\mu}{1-\Lambda p}\Big| \geq \varepsilon\Big) \leq CNe^{-C'K} + \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{NK}} + \frac{1}{\sqrt{Kt}} + \frac{1}{t^q}\Big),$$
$$P\Big(\Big|\mathcal{V}^{N,K}_t - \frac{\mu^2\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big| \geq \varepsilon\Big) \leq CNe^{-C'K} + \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}}\Big),$$
$$P\Big(\Big|\mathcal{X}^{N,K}_{\Delta_t,t} - \frac{\mu}{(1-\Lambda p)^3}\Big| \geq \varepsilon\Big) \leq \frac{C}{\varepsilon}\Big(\frac{1}{K} + \frac{N}{K\sqrt{t^{1-\frac{4}{1+q}}}} + \frac{N^2}{Kt^2}\Big) + CNe^{-C'K}.$$

Proof. [Proof of Theorem 1.2.2] One easily verifies that Ψ is C^∞ in the domain D, that

$$(u,v,w) = \Big(\frac{\mu}{1-\Lambda p}, \frac{\mu^2\Lambda^2 p(1-p)}{(1-\Lambda p)^2}, \frac{\mu}{(1-\Lambda p)^3}\Big) \in D$$

and that $\Psi(u,v,w) = (\mu, \Lambda, p)$. Hence there is a constant c such that for any $N \geq 1$, $t \geq 1$, any $\varepsilon \in (0, 1/c)$,

$$P\Big(\Big|\Psi(\varepsilon^{N,K}_t, \mathcal{V}^{N,K}_t, \mathcal{X}^{N,K}_{\Delta_t,t}) - (\mu,\Lambda,p)\Big| \geq \varepsilon\Big) \\ \leq P\Big(\Big|\varepsilon^{N,K}_t - u\Big| + \Big|\mathcal{V}^{N,K}_t - v\Big| + \Big|\mathcal{X}^{N,K}_{\Delta_t,t} - w\Big| \geq c\varepsilon\Big) \\ \leq \frac{C}{\varepsilon}\Big(\frac{1}{\sqrt{K}} + \frac{N}{K\sqrt{t^{1-\frac{4}{1+q}}}} + \frac{N}{t\sqrt{K}}\Big) + CNe^{-C'K},$$

which completes the proof.

1.11 Analysis of a random matrix for the supercritical case

We define the matrix A_N by $A_N(i,j) := N^{-1}\theta_{ij}$, $i,j \in \{1,...,N\}$. We assume here that $p \in (0,1]$ and we introduce the events:

$$\Omega^2_N := \Big\{\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N A_N(i,j) > \frac{p}{2} \quad \text{and} \quad |NA^2_N(i,j) - p^2| < \frac{p^2}{2N^{3/8}} \quad \text{for all } i,j = 1,...,N\Big\},$$

$$\Omega^{K,2}_N := \Big\{\frac{1}{K}\sum_{i=1}^K\sum_{j=1}^N A_N(i,j) > \frac{p}{2}\Big\} \cap \Omega^2_N.$$

Lemma 1.11.1. *One has*

$$P(\Omega_N^{K,2}) \geq 1 - Ce^{-cN^{\frac{1}{4}}}.$$

Proof. By [14, lemma 33], we already have $P(\Omega_N^2) \geq 1 - Ce^{-cN^{\frac{1}{4}}}$. We recall the Hoeffding inequality for the Binomial(n,q) random variables. For all $x \geq 0$ and X is a Binomial(n,q) distributed, we have:

$$P\Big(|X - nq| \geq x\Big) \leq 2\exp(-2x^2/n).$$

Since $N\sum_{i=1}^K \sum_{j=1}^N A_N(i,j) = \sum_{i=1}^K \sum_{j=1}^N \theta_{ij}$ is Binomial(NK,p) distributed,

$$P\Big(K^{-1}\sum_{i=1}^K\sum_{j=1}^N A_N(i,j) \leq \frac{p}{2}\Big) \leq P\Big(\Big|N\sum_{i=1}^K\sum_{j=1}^N A_N(i,j) - NKp\Big| \geq \frac{NKp}{2}\Big) \leq 2\exp\Big(-\frac{NKp^2}{2}\Big).$$

So we have

$$P(\Omega_N^{K,2}) \geq 1 - 2\exp\Big(-\frac{NKp^2}{2}\Big) - Ce^{-cN^{\frac{1}{4}}} \geq 1 - Ce^{-cN^{\frac{1}{4}}}.$$

Next we apply the Perron-Frobenius theorem and recall some lemma in [1].

Lemma 1.11.2. *On the event $\Omega_N^{K,2}$, the spectral radius ρ_N of A_N is a simple eigenvalue of A_N and $\rho_N \in [p(1-\frac{1}{2N^{\frac{3}{8}}}), p(1+\frac{1}{2N^{\frac{3}{8}}})]$. There is a row eigenvector $\mathbf{V}_N \in \mathbb{R}_+^N$ of A_N for the eigenvalue ρ_N such that $||\mathbf{V}_N||_2 = \sqrt{N}$. We also have $\mathbf{V}_N(i) > 0$ for all $i = 1, ...N$.*

Proof. See [14, lemma 34].

We set $\mathbf{V}_N^K := I_K\mathbf{V}_N$ and let $(e_1, \ldots, e_N)$ the canonical basis of $\mathbb{R}^N$. Recall that $\mathbf{1}_N = \sum_{i=1}^N e_i$.

Lemma 1.11.3. *There exists $N_0 \geq 1$ (depending only on p) such that for all $N \geq N_0$, on the set $\Omega_N^{K,2}$, these properties hold true for all $i,j,k,l = 1, ..., N$:*

(i) for all $n \geq 2$, $A_N^n(i,j) \leq (\frac{3}{2})A_N^n(k,l)$,

(ii) $V_N(i) \in [\frac{1}{2}, 2]$,

(iii) for all $n \geq 0$, $||A_N^n\mathbf{1}_N||_2 \in [\sqrt{N}\frac{\rho_N^n}{2}, 2\sqrt{N}\rho_N^n]$,

(iv) for all $n \geq 2$, $A_N^n(i,j) \in [\rho_N^n/(3N), 3\rho_N^n/N]$,

(v) for all $n \geq 0$, all $r \in [1,\infty]$, $\Big\|A_N^n\boldsymbol{e}_j/||A_N^n\boldsymbol{e}_j||_r - \mathbf{V}_N/||\mathbf{V}_N||_r\Big\|_r \leq 12(2N^{-\frac{3}{8}})^{\lfloor\frac{n}{2}\rfloor}$,

(vi) for all $n \geq 1$, $||A_N^n e_j||_2 \leq 3\rho_N^n/(p\sqrt{N})$ and for all $n \geq 0$, $||A_N^n\mathbf{1}_N||_\infty \leq 3\rho_N^n/p$.

(vii) for all $n \geq 0$, all $r \in [1,\infty]$, $\Big\|I_KA_N^n\mathbf{1}_N/\|I_KA_N^n\mathbf{1}_N\|_r - \mathbf{V}_N^K/\|\mathbf{V}_N^K\|_r\Big\|_r \leq 3(2N^{-\frac{3}{8}})^{\lfloor\frac{n}{2}\rfloor+1}$,

(viii) for all $n \geq 0$, all $r \in [1,\infty]$, $\Big\|I_KA_N^n\boldsymbol{e}_j/||I_KA_N^n\boldsymbol{e}_j||_r - \mathbf{V}_N^K/||\mathbf{V}_N^K||_r\Big\|_r \leq 12(2N^{-\frac{3}{8}})^{\lfloor\frac{n}{2}\rfloor}$,

(ix) for all $n \geq 0$ $\|I_KA_N^n\mathbf{1}_N\|_2 \in \Big[\sqrt{K}\rho_N^n/8, 8\sqrt{K}\rho_N^n\Big]$.

Proof. The proof of (i)-(vi) see [14, Lemma 35]. For the point (vii), we set for $\boldsymbol{x}, \boldsymbol{y} \in (0,\infty)^N$

$$d_K(\boldsymbol{x},\boldsymbol{y}) = \log\Big[\frac{\max_{i=1,\ldots,K}(\frac{x_i}{y_i})}{\min_{i=1,\ldots,K}(\frac{x_i}{y_i})}\Big].$$

Clearly one has $d_K(I_K A_N^n \mathbf{1}_N, I_K \mathbf{V}_N) \leq d_N(A_N^n \mathbf{1}_N, \mathbf{V}_N)$. Moreover from [14, Step 3 of the proof of Lemma 35] one has $d_N(A_N^n \mathbf{1}_N, \mathbf{V}_N) \leq (2N^{-3/8})^{\lfloor n/2 \rfloor + 1}$. Therefore we can apply [14, Lemma 39] and we obtain that

$$||I_K A_N^n \mathbf{1}_N / ||I_K A_N^n \mathbf{1}_N||_r - \mathbf{V}_N^K / ||\mathbf{V}_N^K||_r||_r \leq 3 d_K(I_K A_N^n \mathbf{1}_N, I_K \mathbf{V}_N) \leq 3(2N^{-\frac{3}{8}})^{\lfloor \frac{n}{2} \rfloor + 1}.$$

Let us prove $(viii)$. The case $n \in \{0, 1\}$ is straightforward. In [14, Lemma 35 step 4], we already have for all $n \geq 2$, $d_N(A_N^n e_j, V_N) \leq 4(2N^{-3/8})^{\lfloor n/2 \rfloor}$. Therefore

$$||I_K A_N^n \boldsymbol{e}_j / ||I_K A_N^n \boldsymbol{e}_J||_r - \mathbf{V}_N^K / ||\mathbf{V}_N^K||_r||_r \leq 3 d_K(I_K A_N^n \boldsymbol{e}_j, I_K \mathbf{V}_N) \leq 4(2N^{-\frac{3}{8}})^{\lfloor \frac{n}{2} \rfloor}.$$

which finishes the proof of $(viii)$.

We now verify (ix). We write $A_N^n \mathbf{1}_N = ||A_N^n \mathbf{1}_N||_2 (\|\boldsymbol{V}_N\|_2^{-1} \boldsymbol{V}_N + Z_{N,n})$, where $Z_{N,n} = ||A_N^n \mathbf{1}_N||_2^{-1} A_N^n \mathbf{1}_N - \|\boldsymbol{V}_N\|_2^{-1} \boldsymbol{V}_N$. By (vii), we already have $||Z_{N,n}||_2 \leq 3(2N^{-3/8})^{\lfloor n/2 \rfloor + 1}$. Multiplying each side by I_K, we obtain that $I_K A_N^n \mathbf{1}_N = ||A_N^n \mathbf{1}_N||_2 (\|\boldsymbol{V}_N\|_2^{-1} \boldsymbol{V}_N^K + I_K Z_{N,n})$

Thus

$$\Big| \frac{||I_K A_N^n \mathbf{1}_N||_2}{||A_N^n \mathbf{1}_N||_2} - \frac{\|\boldsymbol{V}_N^K\|_2}{\|\boldsymbol{V}_N\|_2} \Big| \leq \|I_K Z_{N,n}\|_2 \leq \|Z_{N,n}\|_2 \leq 3(2N^{-\frac{3}{8}})^{\lfloor \frac{n}{2} \rfloor + 1}.$$

So for all $n \geq 0$, we have

$$||I_K A_N^n \mathbf{1}_N||_2 \in \Big[\Big(\frac{\|\boldsymbol{V}_N^K\|_2}{\|\boldsymbol{V}_N\|_2} - CN^{-\frac{3}{8}} \Big) ||A_N^n \mathbf{1}_N||_2, \Big(\frac{\|\boldsymbol{V}_N^K\|_2}{\|\boldsymbol{V}_N\|_2} + CN^{-\frac{3}{8}} \Big) ||A_N^n \mathbf{1}_N||_2 \Big].$$

Finally, recalling (ii) and (iii), we deduce (ix).

Lemma 1.11.4. *We have*

$$\mathbb{E}\Big[\|\boldsymbol{\mathcal{L}}_N^K - (\bar{L}_N^K)^5 \boldsymbol{L}_N^K\|_2^2 \Big] \leq \frac{C}{N}$$

where $\boldsymbol{\mathcal{L}}_N := A_N^6 \mathbf{1}_N$, $\boldsymbol{\mathcal{L}}_N^K = I_K \boldsymbol{\mathcal{L}}_N$ *and* $\mathcal{L}_N(i) = \sum_{j=1}^N A_N^6(i,j)$, $\bar{L}_N^K = \frac{1}{K} \sum_{i=1}^K L_N(i)$.

Proof. We write

$$\begin{aligned}
\|\boldsymbol{\mathcal{L}}_N^K - (\bar{L}_N^K)^5 \boldsymbol{L}_N^K\|_2 &= \|I_K A_N^6 \mathbf{1}_N - (\bar{L}_N^K)^5 I_K A_N \mathbf{1}_N\|_2 \\
&\leq \sum_{k=1}^5 \|(\bar{L}_N^K)^{5-k} I_K A_N^{k+1} \mathbf{1}_N - (\bar{L}_N^K)^{6-k} I_K A_N^k \mathbf{1}_N\|_2 \qquad (1.17) \\
&\leq \sum_{k=1}^5 \|I_K A_N^{k+1} \mathbf{1}_N - (\bar{L}_N^K) I_K A_N^k \mathbf{1}_N\|_2 \qquad (1.18)
\end{aligned}$$

First we study the term corresponding to $k = 1$. We have

$$\mathbb{E}\Big[\|I_K A_N^2 \mathbf{1}_N - \bar{L}_N^K I_K A_N \mathbf{1}_N\|_2^2 \Big] \leq 2\mathbb{E}\Big[\|I_K A_N \boldsymbol{L}_N - \bar{L}_N I_K A_N \mathbf{1}_N\|_2^2 + \|(\bar{L}_N - \bar{L}_N^K) I_K A_N \mathbf{1}_N\|_2^2 \Big]$$

By Lemma 1.5.10 we have $\mathbb{E}[\|I_K A_N (\boldsymbol{L}_N - \bar{L}_N \mathbf{1}_N)\|_2^2] \leq \frac{CK}{N^2}$. Besides we have

$$\begin{aligned}
&\mathbb{E}\Big[\|(\bar{L}_N - \bar{L}_N^K) I_K A_N \mathbf{1}_N\|_2^2 \Big] \\
&\leq 2\mathbb{E}\Big[\|(\bar{L}_N - p) I_K A_N \mathbf{1}_N\|_2^2 \Big] + 2\mathbb{E}\Big[\|(p - \bar{L}_N^K) I_K A_N \mathbf{1}_N\|_2^2 \Big] \\
&\leq 2\mathbb{E}[(\bar{L}_N - p)^4]^{\frac{1}{2}} \mathbb{E}\Big[\|I_K A_N \mathbf{1}_N\|_2^4 \Big]^{\frac{1}{2}} + 2\mathbb{E}[(p - \bar{L}_N^K)^4]^{\frac{1}{2}} \mathbb{E}\Big[\|I_K A_N \mathbf{1}_N\|_2^4 \Big]^{\frac{1}{2}}
\end{aligned}$$

using the Cauchy-Schwarz inequality

$$\leq C\Big(\frac{1}{N^2} K + \frac{1}{NK} K\Big) \leq \frac{C}{N}$$

since $\|I_K A_N \mathbf{1}_N\|_2 \le \sqrt{K}$, $\mathbb{E}[(\bar{L}_N - p)^4]^{\frac{1}{2}} \le \frac{C}{N^2}$ and $\mathbb{E}[(\bar{L}_N^K - p)^4]^{\frac{1}{2}} \le \frac{C}{NK}$ ($NL_N(1), \dots, NL_N(K)$ are i.i.d. and Binomial(N,p)). So

$$\mathbb{E}\Big[\|I_K A_N^2 \mathbf{1}_N - \bar{L}_N^K I_K A_N \mathbf{1}_N\|_2^2\Big] \le \frac{C}{N}.$$

Next, we consider the other terms, for any $k \ge 2$. We have

$$\begin{aligned}
&\mathbb{E}\Big[\|I_K A_N^{k+1}\mathbf{1}_N - (\bar{L}_N^K) I_K A_N^k \mathbf{1}_N\|_2^2\Big] \\
&\le \mathbb{E}\Big[|||I_K A_N|||_2^2 \, |||A_N|||_2^{2k-4} \, \|A_N^2 \mathbf{1}_N - (\bar{L}_N^K) A_N \mathbf{1}_N\|_2^2\Big] \\
&\le \Big(\frac{K}{N}\Big)^2 \mathbb{E}\Big[\|A_N^2 \mathbf{1}_N - (\bar{L}_N^K) A_N \mathbf{1}_N\|_2^2\Big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{since } |||I_K A_N|||_2 \le K/N \\
&\le 2\Big(\frac{K}{N}\Big)^2 \Big\{\mathbb{E}\Big[\|A_N^2 \mathbf{1}_N - \bar{L}_N A_N \mathbf{1}_N\|_2^2\Big] + \mathbb{E}\Big[|\bar{L}_N - \bar{L}_N^K|^2 \|A_N \mathbf{1}_N\|_2^2\Big]\Big\} \\
&\le 2\Big(\frac{K}{N}\Big)^2 \Big\{\mathbb{E}\Big[\|A_N \boldsymbol{X}_N\|_2^2\Big] \\
&\qquad + 2\mathbb{E}\Big[|\bar{L}_N - p|^2 \|A_N \mathbf{1}_N\|_2^2\Big] + 2\mathbb{E}\Big[|p - \bar{L}_N^K|^2 \|A_N \mathbf{1}_N\|_2^2\Big]\Big\} \\
&\le C\Big(\frac{K}{N}\Big)^2 \Big[\frac{1}{N} + \frac{1}{N^2} N + \frac{1}{NK} N\Big] \le \frac{C}{N}.
\end{aligned}$$

Recalling (1.17), we conclude that

$$\mathbb{E}[\|\boldsymbol{\mathcal{L}}_N^K - (\bar{L}_N^K)^5 \boldsymbol{L}_N^K\|_2^2] \le \frac{C}{N}$$

which completes the proof.

Lemma 1.11.5. *We have*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}} \Big| H_N^K - \Big(\frac{1}{p} - 1\Big)\Big|\Big] \le \frac{C}{\sqrt{K}}, \text{ where } \quad H_N^K := \frac{N}{K} \sum_{i=1}^K \Big(\frac{L_N(i) - \bar{L}_N^K}{\bar{L}_N^K}\Big)^2.$$

Proof. Since $\bar{L}_N^K \ge p/2$ on $\Omega_N^{K,2}$, we have

$$\begin{aligned}
\Big|H_N^K - \Big(\frac{1}{p} - 1\Big)\Big| &\le \Big|\frac{N}{K} \frac{\|\boldsymbol{L}_N^K - \bar{L}_N^K \mathbf{1}_K\|_2^2}{(\bar{L}_N^K)^2} - \frac{p(1-p)}{(\bar{L}_N^K)^2}\Big| + p(1-p)\Big|\frac{1}{(\bar{L}_N^K)^2} - \frac{1}{p^2}\Big| \\
&\le C\Big|\frac{N}{K}\|\boldsymbol{X}_N^K\|_2^2 - p(1-p)\Big| + C|\bar{L}_N^K - p|.
\end{aligned}$$

Using (1.7) and the fact that $\mathbb{E}[(\bar{L}_N^K - p)^2] \le \frac{C}{NK}$, we obtain

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}} \Big| H_N^K - (\frac{1}{p} - 1)\Big|\Big] \le C\mathbb{E}\Big[\Big|\frac{N}{K}\|\boldsymbol{X}_N^K\|_2^2 - p(1-p)\Big| + |\bar{L}_N^K - p|\Big] \le \frac{C}{\sqrt{K}}.$$

Proposition 1.11.6. We set $\bar{V}_N^K = \frac{1}{K}\sum_{i=1}^K V_N(i)$ and

$$\mathcal{U}_\infty^{N,K} := \frac{N}{K}(\bar{V}_N^K)^{-2} \sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2 \quad \text{on } \Omega_N^{K,2}. \tag{1.19}$$

There exists $N_0 \ge 1$ and $C > 0$ (depending only on p) such that for all $N \ge N_0$,

$$\frac{N}{K}\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}} ||\mathbf{V}_N^K - \bar{V}_N^K \mathbf{1}_K||_2^2\Big] \le C, \quad \mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}} \Big|\mathcal{U}_\infty^{N,K} - \Big(\frac{1}{p} - 1\Big)\Big|\Big] \le \frac{C}{\sqrt{K}}.$$

Proof. We start from

$$\Big|\mathcal{U}_\infty^{N,K} - \Big(\frac{1}{p} - 1\Big)\Big| \le \Big|\mathcal{U}_\infty^{N,K} - \mathcal{H}_N^K\Big| + \Big|\mathcal{H}_N^K - H_N^K\Big| + \Big|H_N^K - \Big(\frac{1}{p} - 1\Big)\Big|$$

where $\mathcal{H}_N^K = \frac{N}{K}\sum_{i=1}^K \Big(\frac{\mathcal{L}_N(i) - \bar{\mathcal{L}}_N^K}{\bar{\mathcal{L}}_N^K}\Big)^2$ and $\bar{\mathcal{L}}_N^K = \frac{1}{K}\sum_{i=1}^K \mathcal{L}_N(i)$.

Step 1: First we check that $\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}}\Big|H_N^K - \mathcal{H}_N^K\Big|\Big] \le C/\sqrt{K}$. We notice that

$$H_N^K = \frac{N}{K}||(\bar{L}_N^K)^5 \boldsymbol{L}_N^K - (\bar{L}_N^K)^6 \mathbf{1}_K||_2^2/(\bar{L}_N^K)^{12}.$$

Thus

$$\begin{aligned}|H_N^K - \mathcal{H}_N^K| \le \frac{N}{K}\Big|&\|(\bar{L}_N^K)^5(\boldsymbol{L}_N^K - \bar{L}_N^K\mathbf{1}_K)\|_2^2\Big(1/(\bar{L}_N^K)^{12} - 1/(\bar{\mathcal{L}}_N^K)^2\Big)\\ &+(1/\bar{\mathcal{L}}_N^K)^2\Big(\|(\bar{L}_N^K)^5(\boldsymbol{L}_N^K - \bar{L}_N^K\mathbf{1}_K)\|_2^2 - \|\boldsymbol{\mathcal{L}}_N^K - \bar{\mathcal{L}}_N^K\mathbf{1}_K\|_2^2\Big)\Big|.\end{aligned}$$

On the set $\Omega_N^{K,2}$, by Lemma 1.11.3 (iv), we have that $(\bar{L}_N^K)^6 \ge \frac{p^6}{64}$ and $\bar{\mathcal{L}}_N^K \ge \frac{(\rho_N)^6}{3} \ge \frac{p^6}{192}$, and the function $\frac{1}{x^2}$ is globally Lipschitz and bounded on the interval $[\frac{p^6}{192}, \infty)$. So

$$\begin{aligned}\mathbf{1}_{\Omega_N^{K,2}}|H_N^K - \mathcal{H}_N^K| &\le \frac{N}{K}\Big|\|(\bar{L}_N^K)^5(\boldsymbol{L}_N^K - \bar{L}_N^K\mathbf{1}_K)\|_2^2\Big(1/(\bar{L}_N^K)^{12} - 1/(\bar{\mathcal{L}}_N^K)^2\Big)\\ &\quad + (1/\bar{\mathcal{L}}_N^K)^2\Big(\|(\bar{L}_N^K)^5(\boldsymbol{L}_N^K - \bar{L}_N^K\mathbf{1}_K)\|_2^2 - \|\boldsymbol{\mathcal{L}}_N^K - \bar{\mathcal{L}}_N^K\mathbf{1}_K\|_2^2\Big)\Big|\\ &\le C\frac{N}{K}\Big(\|(\bar{L}_N^K)^5(\boldsymbol{L}_N^K - \bar{L}_N^K\mathbf{1}_K)\|_2^2\Big|(\bar{L}_N^K)^6 - \bar{\mathcal{L}}_N^K\Big|\\ &\quad + \Big|\|(\bar{L}_N^K)^5(\boldsymbol{L}_N^K - \bar{L}_N^K\mathbf{1}_K)\|_2^2 - \|\boldsymbol{\mathcal{L}}_N^K - \bar{\mathcal{L}}_N^K\mathbf{1}_K\|_2^2\Big|\Big).\end{aligned}$$

Next, we use the inequality $|a^2 - b^2| \le (a-b)^2 + 2a|a-b|$ for $a,\ b \ge 0$. So

$$\begin{aligned}\mathbf{1}_{\Omega_N^{K,2}}|H_N^K - \mathcal{H}_N^K| &\le C\frac{N}{K}\Big(\|(\bar{L}_N^K)^5(\boldsymbol{L}_N^K - \bar{L}_N^K\mathbf{1}_K)\|_2^2\Big|(\bar{L}_N^K)^6 - \bar{\mathcal{L}}_N^K\Big|\\ &\quad + \Big|\|(\bar{L}_N^K)^5(\boldsymbol{L}_N^K - \bar{L}_N^K\mathbf{1}_K)\|_2^2 - \|\boldsymbol{\mathcal{L}}_N^K - \bar{\mathcal{L}}_N^K\mathbf{1}_K\|_2^2\Big|\Big)\\ &\le C\frac{N}{K}\Big\{\|(\bar{L}_N^K)^5)(\boldsymbol{L}_N^K - \bar{L}_N^K\mathbf{1}_K)\|_2^2\frac{1}{\sqrt{K}}J_N^K + (I_N^K)^2\\ &\quad + \|(\bar{L}_N^K)^5(\boldsymbol{L}_N^K - \bar{L}_N^K\mathbf{1}_K)\|_2^2 I_N^K\Big\},\end{aligned}$$

where

$$J_N^K = \|[\bar{\mathcal{L}}_N^K - (\bar{L}_N^K)^6]\mathbf{1}_K\|_2 = \sqrt{K}\Big|\bar{\mathcal{L}}_N^K - (\bar{L}_N^K)^6\Big|,\ I_N^K = \|(\boldsymbol{\mathcal{L}}_N^K - \bar{\mathcal{L}}_N^K\mathbf{1}_K) - (\bar{L}_N^K)^5(\boldsymbol{L}_N^K - \bar{L}_N^K\mathbf{1}_K)\|_2.$$

Because

$$\Big((\boldsymbol{\mathcal{L}}_N^K - \bar{\mathcal{L}}_N^K\mathbf{1}_K) - (\bar{L}_N^K)^5(\boldsymbol{L}_N^K - \bar{L}_N^K\mathbf{1}_K), \mathbf{1}_K\Big) = 0.$$

it implies

$$(J_N^K)^2 + (I_N^K)^2 = \|\boldsymbol{\mathcal{L}}_N^K - (\bar{L}_N^K)^5\boldsymbol{L}_N^K\|_2^2.$$

And by Lemma 1.11.4, we conclude

$$\frac{N}{K}\mathbb{E}\Big[\Big\{(I_N^K)^2 + (J_N^K)^2\Big\}\Big] = \frac{N}{K}\mathbb{E}\Big[\|\boldsymbol{\mathcal{L}}_N^K - (\bar{L}_N^K)^5\boldsymbol{L}_N^K\|_2^2\Big] \le \frac{C}{K}.$$

By (1.7), we conclude that $\mathbb{E}\Big[\Big(\frac{N}{K}\Big)^2\|X_N^K\|_2^4\Big] \le C$. Finally,

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}}\Big|H_N^K - \mathcal{H}_N^K\Big|\Big]\\
&\le C\frac{N}{K}\mathbb{E}\Big[\Big\{\|\boldsymbol{X}_N^K\|_2^2\frac{1}{\sqrt{K}}J_N^K + (I_N^K)^2 + \|\boldsymbol{X}_N^K\|_2^2 I_N^K\Big\}\Big]\\
&\le C\frac{N}{K}\mathbb{E}\Big[\Big\{\|\boldsymbol{X}_N^K\|_2^2 J_N^K + (I_N^K)^2 + \|\boldsymbol{X}_N^K\|_2^2 I_N^K\Big\}\Big]\\
&\le C\frac{N}{K}\mathbb{E}\Big[\Big\{(I_N^K)^2 + (J_N^K)^2\Big\}\Big] + C\mathbb{E}\Big[\Big(\frac{N}{K}\Big)^2\|\boldsymbol{X}_N^K\|_2^4\Big]^{\frac{1}{2}}\mathbb{E}\Big[\Big\{(I_N^K)^2 + (J_N^K)^2\Big\}\Big]^{\frac{1}{2}}\\
&\le \frac{C}{\sqrt{K}}.
\end{aligned}$$

Step 2: By (ii) and (iv) in Lemma 1.11.3, we have the following inequality under the set $\Omega_N^{K,2}$:

$$\begin{aligned}
\Big|\mathcal{U}_\infty^{N,K} - \mathcal{H}_{N,K}\Big| &= \frac{N}{K}\Big|\sum_{i=1}^K\Big[(V_N(i)/\bar{V}_N^K)^2 - (\mathcal{L}_N(i)/\bar{\mathcal{L}}_N^K)^2\Big]\Big|\\
&\le C\frac{N}{K}\sum_{i=1}^K\Big|V_N(i)/\bar{V}_N^K - \mathcal{L}_N(i)/\bar{\mathcal{L}}_N^K\Big|
\end{aligned}$$

Then we use the lemma 1.11.3 (v): on the set $\Omega_N^{K,2}$ we have

$$\begin{aligned}
\frac{N}{K}\sum_{i=1}^K\Big|V_N(i)/\bar{V}_N^K - \mathcal{L}_N(i)/\bar{\mathcal{L}}_N^K\Big| &= N\Big\|\|I_K A_N^6\mathbf{1}_N\|_1^{-1}I_K A_N^6\mathbf{1}_N - \|\mathbf{V}_N^K\|_1^{-1}\mathbf{V}_N^K\Big\|_1\\
&\le CN(N^{-\frac{3}{8}})^{3+1} \le \frac{C}{\sqrt{N}}
\end{aligned}$$

So we have the following inequality:

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}}\Big|\mathcal{U}_\infty^{N,K} - \mathcal{H}_{N,K}\Big|\Big] \le \frac{C}{\sqrt{N}}.$$

Step 3: From the two previous steps and lemma 1.11.5, it follows that

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}}\Big|\mathcal{U}_\infty^{N,K} - \Big(\frac{1}{p} - 1\Big)\Big|\Big] \le \frac{C}{\sqrt{K}}.$$

Moreover, by lemma 1.11.3 (ii), $\bar{V}_N^K$ is bounded by 2 on the set $\Omega_N^{K,2}$, thus

$$\frac{N}{K}\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}}\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2^2\Big] = \mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}}(\bar{V}_N^K)^2|\mathcal{U}_\infty^{N,K}|\Big] \le C.$$

1.12 The estimator in the supercritical case

Recall the definition in (1.4), the aim of this section is to prove $\mathcal{P}_t^{N,K} \simeq p$. Recall (1.9) and (1.10). We start from

$$\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}] = \mu\sum_{n\ge 0}\Big[\int_0^t s\phi^{*n}(t-s)ds\Big]I_K A_N^n\mathbf{1}_N = v_t^{N,K}\boldsymbol{V}_N^K + \boldsymbol{I}_t^{N,K}, \tag{1.20}$$

$$\boldsymbol{U}_t^{N,K} = \boldsymbol{Z}_t^{N,K} - \mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}] = \sum_{n\ge 0}\Big[\int_0^t \phi^{*n}(t-s)\Big]I_K A_N^n\boldsymbol{M}_s^N ds = \boldsymbol{M}_t^{N,K} + \boldsymbol{J}_t^{N,K} \tag{1.21}$$

where

$$v_t^{N,K} = \mu \sum_{n\geq 0} \frac{\|I_K A_N^n \mathbf{1}_N\|_2}{\|\mathbf{V}_N^K\|_2} \int_0^t s\phi^{*n}(t-s)ds, \tag{1.22}$$

$$\boldsymbol{I}_t^{N,K} = \mu \sum_{n\geq 0} \Big[\int_0^t s\phi^{*n}(t-s)ds \Big] \Big[I_K A_N^n \mathbf{1}_N - \frac{\|I_K A_N^n \mathbf{1}_N\|_2}{\|\boldsymbol{V}_N^K\|_2} \boldsymbol{V}_N^K \Big] \tag{1.23}$$

and

$$\boldsymbol{J}_t^{N,K} = \sum_{n\geq 1} \Big[\int_0^t \phi^{*n}(t-s) \Big] I_K A_N^n \boldsymbol{M}_s^N ds. \tag{1.24}$$

Lemma 1.12.1. *Assume (A). For all $\eta > 0$, there exists $N_\eta \geq 1$ and $C_\eta < \infty$ such that for all $N \geq N_\eta$, $t \geq 0$, on the set $\Omega_N^{K,2}$, we have*

$$\|\boldsymbol{I}_t^{N,K}\|_2 \leq C_\eta t \sqrt{K} N^{-\frac{3}{8}}.$$

Proof. In view of (1.23), Lemma 1.11.3 (vii) yields

$$\begin{aligned}
\|\boldsymbol{I}_t^{N,K}\|_2 &\leq \mu \sum_{n\geq 0} \Big[\int_0^t s\phi^{*n}(t-s)ds \Big] \Big\| I_K A_N^n \mathbf{1}_N - \frac{\|I_K A_N^n \mathbf{1}_N\|_2}{\|\boldsymbol{V}_N^K\|_2} \boldsymbol{V}_N^K \Big\|_2 \\
&\leq C_\eta t \sqrt{K} \sum_{n\geq 0} \Big[\int_0^t \phi^{*n}(t-s)ds \Big] (N^{-\frac{3}{8}})^{\lfloor \frac{n}{2} \rfloor + 1} \\
&\leq C_\eta t \sqrt{K} N^{-\frac{3}{8}} \sum_{n\geq 0} \Lambda^n (N^{-\frac{3}{8}})^{\lfloor \frac{n}{2} \rfloor} \\
&\leq C_\eta t \sqrt{K} N^{-\frac{3}{8}}.
\end{aligned}$$

Lemma 1.12.2. *Assume (A). For all $\eta > 0$, there exists $N_\eta \geq 1$ and $C_\eta < \infty$ such that for all $N \geq N_\eta$, $t \geq 0$, on the set $\Omega_N^{K,2}$, we have*

$$\mathbb{E}_\theta \Big[\|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K} \mathbf{1}_K\|_2^2 \Big]^{\frac{1}{2}} \leq C_\eta \sqrt{\frac{K}{N}} \Big[e^{\frac{1}{2}(\alpha_0+\eta)t} + \frac{\|\boldsymbol{V}_N^K - \bar{V}_N^K \mathbf{1}_K\|_2}{\|\boldsymbol{V}_N^K\|_2} e^{(\alpha_0+\eta)t} \Big]$$

where $\bar{J}_t^{N,K} = \frac{1}{K}(\boldsymbol{J}_t^{N,K}, \mathbf{1}_K)$.

Proof. In view of (1.24), by Minkowski inequality we have

$$\begin{aligned}
&\mathbb{E}_\theta \Big[\|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K} \mathbf{1}_K\|_2^2 \Big]^{\frac{1}{2}} \\
&\leq \sum_{n\geq 1} \int_0^t \phi^{*n}(t-s) \mathbb{E}_\theta \Big[\|I_K A_N^n \boldsymbol{M}_s^N - \overline{I_K A_N^n \boldsymbol{M}_s^N} \mathbf{1}_K\|_2^2 \Big]^{\frac{1}{2}}.
\end{aligned}$$

where $\overline{I_K A_N^n \boldsymbol{M}_s^N} := \frac{1}{K} \sum_{j=1}^N \sum_{i=1}^K A_N^n(i,j) M_s^{j,N}$.
In [14, Lemma 44 (i)], it is shown that $\max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N})^2] \leq C_\eta e^{2(\alpha_0+\eta)t}$ on Ω_N^2. Using (1.11), we conclude that on Ω_N^2:

$$\begin{aligned}
\mathbb{E}_\theta \Big[\|I_K A_N^n \boldsymbol{M}_s^N - \overline{I_K A_N^n \boldsymbol{M}_s^N} \mathbf{1}_K\|_2^2 \Big] &= \sum_{i=1}^K \sum_{j=1}^N \Big(A_N^n(i,j) - \frac{1}{K} \sum_{k=1}^K A_N^n(k,j) \Big)^2 \mathbb{E}_\theta[Z_s^{j,N}] \\
&\leq C_\eta e^{(\alpha_0+\eta)s} \sum_{j=1}^N \|I_K A_N^n \boldsymbol{e}_j - \overline{I_K A_N^n \boldsymbol{e}_j} \mathbf{1}_K\|_2^2.
\end{aligned}$$

Using (*viii*) in Lemma 1.11.3 and and the inequality $\big|||\boldsymbol{x}-\bar{x}\mathbf{1}_N||_2-||\boldsymbol{y}-\bar{y}\mathbf{1}_N||_2\big|\leq||\boldsymbol{x}-\boldsymbol{y}||_2$ for all $x,\ y\in\mathbb{R}^N$, we deduce that on $\Omega_N^{K,2}$:

$$\begin{aligned}
&\|I_KA_N^n\boldsymbol{e}_j-\overline{I_KA_N^n\boldsymbol{e}_j}\mathbf{1}_K\|_2\\
&\leq\Big\|I_KA_N^n\boldsymbol{e}_j-\frac{1}{\|\boldsymbol{V}_N^K\|_2}\|I_KA_N^n\boldsymbol{e}_j\|_2\boldsymbol{V}_N^K\Big\|_2+\frac{\|I_KA_N^n\boldsymbol{e}_j\|_2}{\|\boldsymbol{V}_N^K\|_2}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2\\
&=\|I_KA_N^n\boldsymbol{e}_j\|_2\Big(\Big\|\frac{I_KA_N^n\boldsymbol{e}_j}{\|I_KA_N^n\boldsymbol{e}_j\|_2}-\frac{\boldsymbol{V}_N^K}{\|\boldsymbol{V}_N^K\|_2}\Big\|_2+\frac{\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2}{\|\boldsymbol{V}_N^K\|_2}\Big)\\
&\leq C\|I_KA_N^n\boldsymbol{e}_j\|_2\Big(N^{-\frac{3}{8}\lfloor\frac{n}{2}\rfloor}+\frac{\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2}{\|\boldsymbol{V}_N^K\|_2}\Big).
\end{aligned}$$

From Lemma 1.11.3 (*iv*) it follows that on the event $\Omega_N^{K,2}$ for all $n\geq2$, $\|I_KA_N^n\boldsymbol{e}_j\|_2\leq\frac{3\sqrt{K}}{N}\rho_N^n$. So on the event $\Omega_N^{K,2}$,

$$\begin{aligned}
&\mathbb{E}_\theta[\|\boldsymbol{J}_t^{N,K}-\bar{J}_t^{N,K}\mathbf{1}_K\|_2^2]^{\frac12}\\
&\leq C_\eta\sqrt{\frac{K}{N}}\sum_{n\geq1}\rho_N^n\Big[(2N^{-\frac38})^{\lfloor\frac{n}{2}\rfloor}+\frac{\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2}{\|\boldsymbol{V}_N^K\|_2}\Big]\int_0^t\phi^{*n}(t-s)e^{\frac{(\alpha_0+\eta)s}{2}}ds.
\end{aligned}$$

Using [14, lemma 43 (iii) and (iv)], we deduce that on the event $\Omega_N^{K,2}$

$$\mathbb{E}_\theta[\|\boldsymbol{J}_t^{N,K}-\bar{J}_t^{N,K}\mathbf{1}_K\|_2^2]^{\frac12}\leq C_\eta\sqrt{\frac{K}{N}}\Big[e^{\frac12(\alpha_0+\eta)t}+\frac{\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2}{\|\boldsymbol{V}_N^K\|_2}e^{(\alpha_0+\eta)t}\Big].$$

Lemma 1.12.3. *There exists $N_0\geq1$ such that for all $N\geq N_0$, for all $t\geq0$, on the event $\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac14v_t^{N,K}>0\}$, we have the following inequality:*

$$\mathcal{D}_t^{N,K}\leq16\mathcal{D}_t^{N,K,1}+128\frac{N}{K}\|\mathbf{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2\,\mathcal{D}_t^{N,K,2}+\Big|\mathcal{U}_\infty^{N,K}-\Big(\frac1p-1\Big)\Big|$$

where

$$\mathcal{D}_t^{N,K}=\Big|\mathcal{U}_t^{N,K}-\Big(\frac1p-1\Big)\Big|,\tag{1.25}$$

$$\mathcal{D}_t^{N,K,1}=\frac{1}{(v_t^{N,K})^2}\Big|\frac{N}{K}\|\boldsymbol{Z}_t^{N,K}-\bar{Z}_t^{N,K}\mathbf{1}_K\|_2^2-N\bar{Z}_t^{N,K}-\frac{N}{K}(v_t^{N,K})^2\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2\Big|,\tag{1.26}$$

$$\mathcal{D}_t^{N,K,2}=\Big|\frac{\bar{Z}_t^{N,K}}{v_t^{N,K}}-\bar{V}_N^K\Big|,\tag{1.27}$$

Proof. Recall definitions (1.4) and (1.19). On the event $\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac14v_t^{N,K}>0\}$, we have

$$\begin{aligned}
|\mathcal{U}_t^{N,K}-\mathcal{U}_\infty^{N,K}|\leq\frac{1}{(\bar{Z}_t^{N,K})^2}\Big|\frac{N}{K}\|\boldsymbol{Z}_t^{N,K}-\bar{Z}_t^{N,K}\mathbf{1}_K\|_2^2-N\bar{Z}_t^{N,K}-(v_t^{N,K})^2\frac{N}{K}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2\Big|\\
+\frac{N}{K}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2\Big|\Big(\frac{v_t^{N,K}}{\bar{Z}_t^{N,K}}\Big)^2-\frac{1}{(\bar{V}_N^K)^2}\Big|.
\end{aligned}$$

By [14, lemma 35 (ii)], we have $\bar{V}_N^K\geq\frac12$ on $\Omega_N^{K,2}$. Since $|\frac{1}{x^2}-\frac{1}{y^2}|=|\frac{(x-y)(x+y)}{x^2y^2}|\leq128|x-y|$, for $x,y\geq\frac14$, on the event $\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac14v_t^{N,K}>0\}$, we have

$$\Big|\Big(\frac{v_t^{N,K}}{\bar{Z}_t^{N,K}}\Big)^2-\frac{1}{(\bar{V}_N^K)^2}\Big|\leq128\mathcal{D}_t^{N,K,2}.$$

Finally on the event $\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \geq \frac{1}{4} v_t^{N,K} > 0\}$, we obtain

$$\begin{aligned}\mathcal{D}_t^{N,K} &\leq \Big|\mathcal{U}_t^{N,K} - \mathcal{U}_\infty^{N,K}\Big| + \Big|\mathcal{U}_\infty^{N,K} - \Big(\frac{1}{p} - 1\Big)\Big| \\ &\leq 16\mathcal{D}_t^{N,K,1} + 128\frac{N}{K}\|\boldsymbol{V}_N^K - \bar{V}_N^K \mathbf{1}_K\|_2^2 \mathcal{D}_t^{N,K,2} + \Big|\mathcal{U}_\infty^{N,K} - \Big(\frac{1}{p} - 1\Big)\Big|.\end{aligned}$$

Before the analysis of the term $\mathcal{D}_t^{N,K,2}$, we still need the following fact:

Lemma 1.12.4. *Assume (A). For any $\eta > 0$, we can find $N_\eta \geq 1$, $t_\eta > 0$ and $0 < c_\eta < C_\eta < \infty$, such that for all $N \geq N_\eta$, $t \geq t_\eta$ on the set $\Omega_N^{K,2}$*

$$c_\eta e^{(\alpha_0 - \eta)t} \leq v_t^{N,K} \leq C_\eta e^{(\alpha_0 + \eta)}$$

where $v_t^{N,K}$ is defined in (1.22).

Proof. We work on the set $\Omega_N^{K,2}$. Recall Lemma 1.11.3 (ii) and (ix). We can conclude that $\frac{1}{2}\sqrt{K} \leq \|\mathbf{V}_N^K\|_2 \leq 2\sqrt{K}$ and $\|I_K A_N^n \mathbf{1}_N\|_2 \in [\sqrt{K}\rho_N^n/8, 8\sqrt{K}\rho_N^n]$. So there exists $0 < c < C < \infty$ such that

$$c\frac{\|A_N^n \mathbf{1}_N\|_2}{\|\mathbf{V}_N\|_2} \leq \frac{\|I_K A_N^n \mathbf{1}_N\|_2}{\|\mathbf{V}_N^K\|_2} \leq C\frac{\|A_N^n \mathbf{1}_N\|_2}{\|\mathbf{V}_N\|_2}.$$

Therefore we have $c v_t^{N,N} \leq v_t^{N,K} \leq C v_t^{N,N}$. Moreover, in view of [14, (i) and (ii) Lemma 43], we already have $c_\eta e^{(\alpha_0-\eta)t} \leq v_t^{N,N} \leq C_\eta e^{(\alpha_0+\eta)}$. The proof is finished.

Lemma 1.12.5. *Assume (A). For all $\eta > 0$, there exists $N_\eta \geq 1$, $t_\eta \geq 0$ and $C_\eta < \infty$ such that for all $N \geq N_\eta$, all $t \geq t_\eta$, on the event $\Omega_N^{K,2}$,*

$$(i) \quad \mathbb{E}_\theta[\mathcal{D}_t^{N,K,2}] \leq C_\eta e^{2\eta t}\Big(\frac{1}{\sqrt{K}} + e^{-\alpha_0 t}\Big).$$

$$(ii) \quad P_\theta\Big(\bar{Z}_t^{N,K} \leq \frac{1}{4} v_t^{N,K}\Big) \leq C_\eta e^{2\eta t}\Big(\frac{1}{\sqrt{K}} + e^{-\alpha_0 t}\Big).$$

Proof. Recalling (1.20) and (1.21), we can write

$$\mathcal{D}_t^{N,K,2} = \Big|\frac{\bar{Z}_t^{N,K}}{v_t^{N,K}} - \bar{V}_N^K\Big| \leq (v_t^{N,K})^{-1}\Big(|\bar{I}_t^{N,K}| + |\bar{U}_t^{N,K}|\Big).$$

We fix $\eta > 0$ and work with N large enough and on $\Omega_N^{K,2}$. By the lemma 1.12.1, we have: $|\bar{I}_t^{N,K}| \leq \frac{1}{\sqrt{K}}\|\boldsymbol{I}_t^{N,K}\|_2 \leq C_\eta t N^{-\frac{3}{8}}$.
From [14, proof of Lemma 44, step 3], we have $\mathbb{E}_\theta[(J_t^{i,N})^2] \leq C_\eta N^{-1} e^{2(\alpha_0+\eta)t}$. Thus

$$\mathbb{E}_\theta[(\bar{J}_t^{N,K})^2] \leq K^{-1}\sum_{i=1}^K \mathbb{E}_\theta[(J_t^{i,N})^2] \leq C_\eta \frac{1}{N} e^{2(\alpha_0+\eta)t}.$$

In view of [14, Lemma 44 (i)], we already have $\max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N})^2] \leq C_\eta e^{2(\alpha_0+\eta)t}$. Then by (1.11) we deduce that

$$\mathbb{E}[(\bar{M}_t^{N,K})^2] = \frac{1}{K^2}\sum_{i=1}^K \mathbb{E}_\theta[Z_t^{i,N}] \leq C_\eta \frac{1}{K} e^{(\alpha_0+\eta)t}.$$

Over all, we deduce that $\mathbb{E}[|\bar{U}_t^{N,K}|] \leq \frac{C}{\sqrt{K}} e^{(\alpha_0+\eta)t}$. According to Lemma 1.12.4, there exists $t_\eta \geq 0$ such that for all $t \geq t_\eta$, $v_t^{N,K} \geq c_\eta e^{(\alpha_0-\eta)t}$ and we finally obtain (i):

$$\mathbb{E}_\theta[\mathcal{D}_t^{N,K,2}] = \mathbb{E}_\theta\Big[(v_t^{N,K})^{-1}\Big(|\bar{I}_t^{N,K}| + |\bar{U}_t^{N,K}|\Big)\Big] \leq C_\eta e^{2\eta t}\Big(\frac{1}{\sqrt{K}} + e^{-\alpha_0 t}\Big).$$

Now we prove (ii). Because of $\bar{V}_N^K \geq \frac{1}{2}$ we have $\{\bar{Z}_t^{N,K} \leq \frac{v_t^N}{4}\} \subset \{\mathcal{D}_t^{N,K,2} = \left|\frac{\bar{Z}_t^{N,K}}{v_t^N} - \bar{V}_N^K\right| \geq \frac{1}{4}\}$. Hence

$$P_\theta\Big(\bar{Z}_t^{N,K} \leq \frac{1}{4} v_t^{N,K}\Big) \leq 4\mathbb{E}_\theta[\mathcal{D}_t^{N,K,2}] \leq C_\eta e^{2\eta t}\Big(\frac{1}{\sqrt{K}} + e^{-\alpha_0 t}\Big).$$

Lemma 1.12.6. *Assume (A). For all $\eta > 0$, there exists $N_\eta \geq 1$ and $C_\eta < \infty$ such that for all $N \geq N_\eta$, all $t \geq 0$, on $\Omega_N^{K,2}$:*

(i) $\mathbb{E}_\theta[(\boldsymbol{M}_t^{N,K} - \bar{M}_t^{N,K}\mathbf{1}_K, \mathbf{V}_N^K - \bar{V}_N^K\mathbf{1}_K)^2] \leq C_\eta \|\mathbf{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2^2\, e^{(\alpha_0+\eta)t}$.

(ii) $\mathbb{E}_\theta[|X_t^{N,K}|] \leq C_\eta \frac{N}{\sqrt{K}} e^{(\alpha_0+\eta)t}$, *where* $X_t^{N,K} := \frac{N}{K}(\|\boldsymbol{M}_t^{N,K} - \bar{M}_t^{N,K}\mathbf{1}_K\|_2^2 - K\bar{Z}_t^{N,K})$.

(iii) $\mathbb{E}_\theta[\|\boldsymbol{M}_t^{N,K} - \bar{M}_t^{N,K}\mathbf{1}_K\|_2^2] \leq CNe^{(\alpha_0+\eta)t}$.

Proof. We fix $\eta > 0$ and work with N large enough and on $\Omega_N^{K,2}$. We already from [14, Lemma 44 (i)] that $\max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N})^2] \leq C_\eta e^{2(\alpha_0+\eta)t}$. Thus

$$\begin{aligned}\mathbb{E}_\theta\Big[\Big(\boldsymbol{M}_t^{N,K} - \bar{M}_t^{N,K}\mathbf{1}_K, \mathbf{V}_N^K - \bar{V}_N^K\mathbf{1}_K\Big)^2\Big] &= \sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2 \mathbb{E}_\theta[Z_t^{i,N}] \\ &\leq C_\eta \|\mathbf{V}_N^K - \bar{V}_N^K\|_2^2 e^{(\alpha_0+\eta)t}\end{aligned}$$

which completes the proof of (i).

By Itô's formula, we have

$$\|\boldsymbol{M}_t^{N,K}\|_2^2 = \sum_{i=1}^K (M_t^{i,N})^2 = 2\sum_{i=1}^K \int_0^t M_{s-}^{i,N} dM_s^{i,N} + \sum_{i=1}^K Z_t^{i,N},$$

hence

$$\begin{aligned}X_t^{N,K} &= \frac{N}{K}\Big(\|\boldsymbol{M}_t^{N,K}\|_2^2 - K(\bar{M}_t^{N,K})^2 - K\bar{Z}_t^{N,K}\Big) \\ &= \frac{N}{K}\Big(2\sum_{i=1}^K \int_0^t M_{s-}^{i,N} dM_s^{i,N} - K(\bar{M}_t^{N,K})^2\Big).\end{aligned}$$

It follows that

$$\mathbb{E}_\theta[|X_t^{N,K}|] \leq \frac{N}{K}\Big(2\mathbb{E}_\theta\Big[\Big|\sum_{i=1}^K \int_0^t M_{s-}^{i,N} dM_s^{i,N}\Big|\Big] + \mathbb{E}_\theta[\bar{Z}_t^{N,K}]\Big).$$

Besides, using Cauchy-Schwarz inequality

$$\begin{aligned}\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K \int_0^t M_{s-}^{i,N} dM_s^{i,N}\Big)^2\Big] &= \sum_{i=1}^K \mathbb{E}_\theta\Big[\int_0^t (M_{s-}^{i,N})^2 dZ_s^{i,N}\Big] \\ &\leq \sum_{i=1}^K \mathbb{E}_\theta\Big[\sup_{[0,t]}(M_s^{i,N})^4\Big]^{\frac{1}{2}} \mathbb{E}_\theta\Big[(Z_t^{i,N})^2\Big]^{\frac{1}{2}} \\ &\leq C\sum_{i=1}^K \mathbb{E}_\theta\Big[(Z_t^{i,N})^2\Big]\end{aligned}$$

since $\mathbb{E}_\theta[\sup_{[0,t]}(M_s^{i,N})^4] \leq C\mathbb{E}_\theta[(Z_t^{i,N})^2]$ by Doob's inequality. So

$$\mathbb{E}_\theta[|X_t^{N,K}|] \leq \frac{N}{K}\Big(2\mathbb{E}_\theta\Big[\Big|\sum_{i=1}^K \int_0^t M_{s-}^{i,N} dM_s^{i,N}\Big|\Big] + \mathbb{E}_\theta[\bar{Z}_t^{N,K}]\Big) \leq C_\eta \frac{N}{\sqrt{K}} e^{(\alpha_0+\eta)t}$$

This completes the proof of (ii). Finally, we have

$$\frac{N}{K}\mathbb{E}_\theta\Big[\|\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\|_2^2\Big]\le\mathbb{E}_\theta[|X_t^{N,K}|]+N\mathbb{E}_\theta[\bar{Z}_t^{N,K}]\le C_\eta N e^{(\alpha_0+\eta)t}.$$

This completes the proof of (iii).

Next we consider the term $\mathcal{D}_t^{N,K,1}$.

Lemma 1.12.7. *Assume (A). For all $\eta>0$, there are $N_\eta\ge 1$, $t_\eta\ge 0$ and $C_\eta<\infty$ such that for all $N\ge N_\eta$, all $t\ge t_\eta$, we have:*

$$\mathbb{E}[\mathbf{1}_{\Omega_N^{K,2}}\mathcal{D}_t^{N,K,1}]\le C_\eta e^{4\eta t}\Big(\frac{1}{\sqrt{K}}+\Big(\frac{\sqrt{N}}{e^{\alpha_0 t}}\Big)^{\frac{3}{2}}+\frac{N}{\sqrt{K}}e^{-\alpha_0 t}\Big).$$

Proof. Recalling (1.20) and (1.21), we start from $\boldsymbol{Z}_t^{N,K}=\boldsymbol{M}_t^{N,K}+\boldsymbol{J}_t^{N,K}+v_t^{N,K}\boldsymbol{V}_N^K+\boldsymbol{I}_t^{N,K}$. In view of (1.26), we have:

$$\begin{aligned}
\mathcal{D}_t^{N,K,1}&=\frac{1}{(v_t^{N,K})^2}\Big|\frac{N}{K}\|\boldsymbol{I}_t^{N,K}-\bar{I}_t^{N,K}\mathbf{1}_K+\boldsymbol{J}_t^{N,K}-\bar{J}_t^{N,K}\mathbf{1}_K\|_2^2+\frac{N}{K}\|\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\|_2^2\\
&\quad-NZ_t^{N,K}+2\frac{N}{K}\Big(\boldsymbol{I}_t^{N,K}-\bar{I}_t^{N,K}\mathbf{1}_K+\boldsymbol{J}_t^{N,K}-\bar{J}_t^{N,K}\mathbf{1}_K,v_t^{N,K}(V_N^K-\bar{V}_N^K\mathbf{1}_K)\\
&\quad+\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\Big)+2\frac{N}{K}v_t^{N,K}\Big(\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K,\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\Big)\Big|\\
&\le\frac{1}{(v_t^{N,K})^2}\Big[2\frac{N}{K}\|\boldsymbol{I}_t^{N,K}-\bar{I}_t^{N,K}\mathbf{1}_K\|_2^2+2\frac{N}{K}\|\boldsymbol{J}_t^{N,K}-\bar{J}_t^{N,K}\mathbf{1}_K\|_2^2+|X_t^{N,K}|\\
&\quad+2\frac{N}{K}\Big(\|\boldsymbol{I}_t^{N,K}-\bar{I}_t^{N,K}\mathbf{1}_K\|_2+\|\boldsymbol{J}_t^{N,K}-\bar{J}_t^{N,K}\mathbf{1}_K\|_2\Big)\Big(v_t^{N,K}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2\\
&\quad+\|\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\|_2\Big)+2\frac{N}{K}\Big|v_t^{N,K}\Big(\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K,\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\Big)\Big|\Big].
\end{aligned}$$

We fix $\eta>0$ and work with N and t large enough and on $\Omega_N^{K,2}$. Using Lemmas 1.12.1, 1.12.2, 1.12.4, 1.12.6 together with the fact that $c\sqrt{K}\le\|\boldsymbol{V}_N^K\|_2\le C\sqrt{K}$ on $\Omega_N^{K,2}$ (by Lemma 1.11.3 (ii)), we deduce the following bound on the set $\Omega_N^{K,2}$:

$$\begin{aligned}
&\mathbb{E}_\theta\big[\mathcal{D}_t^{N,K,1}\big]\le C_\eta e^{-2(\alpha_0-\eta)t}\Big\{N^{\frac{1}{4}}e^{2\eta t}+e^{(\alpha_0+\eta)t}+e^{2(\alpha_0+\eta)t}\frac{1}{\|\boldsymbol{V}_N^K\|_2^2}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2\\
&+\frac{N}{\sqrt{K}}e^{(\alpha_0+\eta)t}+\frac{2N}{K}\Big[t\sqrt{K}N^{-\frac{3}{8}}+\sqrt{\frac{K}{N}}e^{\frac{\alpha_0+\eta}{2}t}+\sqrt{\frac{K}{N}}e^{(\alpha_0+\eta)}\frac{1}{\|\boldsymbol{V}_N^K\|_2}\|\boldsymbol{V}_N^K-\bar{V}_N^K\|_2\Big]\Big[\sqrt{K}e^{\frac{\alpha_0+\eta}{2}t}\\
&+e^{(\alpha_0+\eta)t}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2\Big]+e^{\frac{1}{2}(\alpha_0+\eta)t}\frac{N}{K}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2\Big\}
\end{aligned}$$

By proposition 1.11.6, we finally obtain:

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{\Omega_N^{K,2}}\mathcal{D}_t^{N,K,1}]&\le C_\eta e^{-2(\alpha_0-\eta)t}\Big|N^{\frac{5}{8}}te^{\frac{\alpha_0+\eta}{2}}+\frac{N}{\sqrt{K}}e^{(\alpha_0+\eta)t}+N^{\frac{1}{8}}e^{(\alpha_0+\eta)t}\\
&\quad+e^{\frac{3(\alpha_0+\eta)}{2}t}+e^{\frac{3}{2}(\alpha_0+\eta)t}+e^{2(\alpha_0+\eta)t}\frac{1}{\sqrt{N}}+N^{\frac{1}{4}}e^{2\eta t}\Big|\\
&\le C_\eta e^{4\eta t}\Big|N^{\frac{5}{8}}e^{-\frac{3}{2}\alpha_0 t}+\frac{N}{\sqrt{K}}e^{-\alpha_0 t}+e^{-\frac{1}{2}\alpha_0 t}+\frac{1}{\sqrt{N}}\Big|.
\end{aligned}$$

Since $\frac{N}{\sqrt{K}}e^{-\alpha_0 t}+\frac{1}{\sqrt{N}}\ge e^{-\frac{\alpha_0}{2}t}$, $N^{\frac{5}{8}}e^{-\frac{3}{2}\alpha_0 t}\le(\sqrt{N}e^{-\alpha_0 t})^{\frac{3}{2}}$, one gets

$$\mathbb{E}[\mathbf{1}_{\Omega_N^{K,2}}\mathcal{D}_t^{N,K,1}]\le C_\eta e^{4\eta t}\Big(\frac{1}{\sqrt{N}}+\Big(\frac{\sqrt{N}}{e^{\alpha_0 t}}\Big)^{\frac{3}{2}}+\frac{N}{\sqrt{K}}e^{-\alpha_0 t}\Big).$$

1.13 Proof of the main theorem in the supercritical case.

In this section we prove Theorem 1.2.4 and Remark 1.2.5.

1.13.1 Proof of Theorem 1.2.4

By Lemma 1.12.3, on the event $\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \geq \frac{1}{4} v_t^{N,K} > 0\}$,
we already have the following inequality:

$$\mathcal{D}_t^{N,K} \leq 16\mathcal{D}_t^{N,K,1} + 128\frac{N}{K}\|\boldsymbol{V}_N^K - \bar{V}_N^K \mathbf{1}_N\|_2^2 \mathcal{D}_t^{N,K,2} + \Big|\mathcal{U}_\infty^{N,K} - \Big(\frac{1}{p} - 1\Big)\Big|.$$

Thus

$$\mathbf{1}_{\Omega_N^{K,2}} \mathbb{E}_\theta\Big[\mathbf{1}_{\{\bar{Z}_t^{N,K} \geq v_t^{N,K}/4 > 0\}} \Big|\mathcal{U}_t^{N,K} - \Big(\frac{1}{p} - 1\Big)\Big|\Big] \leq \mathbf{1}_{\Omega_N^{K,2}} \Big|\mathcal{U}_\infty^{N,K} - \Big(\frac{1}{p} - 1\Big)\Big| + C_\eta 16 \mathbb{E}_\theta[\mathcal{D}_t^{N,K,1}]$$
$$+ 128\frac{N}{K}\|\boldsymbol{V}_N^K - \bar{V}_N^K \mathbf{1}_N\|_2^2 \; \mathbb{E}_\theta[\mathcal{D}_t^{N,K,2}].$$

From Proposition 1.11.6 and Lemmas 1.12.5, 1.12.7 it follows that

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}} \mathbf{1}_{\{\bar{Z}_t^{N,K} \geq v_t^{N,K}/4 > 0\}} \Big|\mathcal{U}_t^{N,K} - \Big(\frac{1}{p} - 1\Big)\Big|\Big] \leq C_\eta e^{4\eta t}\Big(\frac{1}{\sqrt{K}} + \Big(\frac{\sqrt{N}}{e^{\alpha_0 t}}\Big)^{\frac{3}{2}} + \frac{N}{\sqrt{K}} e^{-\alpha_0 t}\Big).$$

Moreover, by Lemmas 1.11.1 and 1.12.5 we have:

$$P(\Omega_N^{K,2}) \geq 1 - Ce^{-cN^{\frac{1}{4}}}, \quad P_\theta\Big(\bar{Z}_t^{N,K} \leq \frac{1}{4} v_t^{N,K}\Big) \leq C_\eta e^{2\eta t}\Big(\frac{1}{\sqrt{K}} + e^{-\alpha_0 t}\Big).$$

Hence, by the Chebyshev's inequality, we obtain:

$$\begin{aligned} P(|\mathcal{P}_t^{N,K} - p| \geq \varepsilon) &\leq (C_\eta/\varepsilon) e^{4\eta t}\Big(\frac{1}{\sqrt{K}} + \Big(\frac{\sqrt{N}}{e^{\alpha_0 t}}\Big)^{\frac{3}{2}} + \frac{N}{\sqrt{K}} e^{-\alpha_0 t}\Big) \\ &\quad + Ce^{-cN^{\frac{1}{4}}} + C_\eta e^{2\eta t}\Big(\frac{1}{\sqrt{K}} + e^{-\alpha_0 t}\Big) \\ &\leq (C_\eta'/\varepsilon) e^{4\eta t}\Big(\frac{1}{\sqrt{K}} + \Big(\frac{\sqrt{N}}{e^{\alpha_0 t}}\Big)^{\frac{3}{2}} + \frac{N}{\sqrt{K}} e^{-\alpha_0 t}\Big). \end{aligned}$$

Finally, using that $(\frac{\sqrt{N}}{e^{\alpha_0 t}})^{\frac{3}{2}} \leq \frac{N}{\sqrt{K}} e^{-\alpha_0 t}$, we get:

$$P(|\mathcal{P}_t^{N,K} - p| \geq \varepsilon) \leq \frac{C_\eta e^{4\eta t}}{\varepsilon}\Big(\frac{N}{\sqrt{K} e^{\alpha_0 t}} + \frac{1}{\sqrt{K}}\Big).$$

The proof is complete.

1.13.2 Proof of Remark 1.2.5

By the Lemma 1.12.5, for $N \geq N_\eta$, we have that:

$$\mathbf{1}_{\Omega_N^{K,2}} \mathbb{E}_\theta\Big[\Big|\frac{\bar{Z}_t^{N,K}}{v_t^{N,K}} - \bar{V}_N^K\Big|\Big] = \mathbf{1}_{\Omega_N^{K,2}} \mathbb{E}_\theta[\mathcal{D}_t^{N,K,2}] \leq C_\eta e^{2\eta t}\Big(\frac{1}{\sqrt{K}} + e^{-\alpha_0 t}\Big).$$

From lemma 1.11.3 (ii), we have for all $V_N(i) \in [\frac{1}{2}, 2]$. So $\bar{V}_N^K = (\frac{1}{K}\sum_{i=1}^K V_N(i)) \in [\frac{1}{2}, 2]$ on the set $\Omega_N^{K,2}$. From lemma 1.11.1, we have $P(\Omega_N^{K,2}) \geq 1 - Ce^{-cN^{\frac{1}{4}}}$. From Lemma 1.12.4, for $t \geq t_\eta$ we get $v_t^{N,K} \in [a_\eta e^{(\alpha_0+\eta)}, b_\eta e^{(\alpha_0-\eta)t}]$ for some $a_\eta < b_\eta$. So we deduce that for $N \geq N_\eta$, $t \geq t_\eta$,

$$P\Big(\bar{Z}_t^{N,K} \in [\frac{a_\eta}{2} e^{(\alpha_0-\eta)t}, 2b_\eta e^{(\alpha_0+\eta)t}]\Big) \geq 1 - Ce^{-cN^{\frac{1}{4}}} - C_\eta e^{2\eta t}\Big(\frac{1}{\sqrt{K}} + e^{-\alpha_0 t}\Big).$$

This implies that for any $\eta > 0$,

$$\lim_{t\to\infty} \lim_{(N,K)\to(\infty,\infty)} P(\bar{Z}_t^{N,K} \in [e^{(\alpha_0-\eta)t}, e^{(\alpha_0+\eta)t}]) = 1.$$

Chapter 2

Central limit theorem for a partially observed interacting system of Hawkes processes

Abstract We observe the actions of a K sub-sample of N individuals up to time t for some large $K \leq N$. We model the relationships of individuals by i.i.d. Bernoulli(p)-random variables, where $p \in (0,1]$ is an unknown parameter. The rate of action of each individual depends on some unknown parameter $\mu > 0$ and on the sum of some function ϕ of the ages of the actions of the individuals which influence him. The parameters μ and ϕ are considered as nuisance parameters. The aim of this paper is to obtain a central limit theorem for the estimator of p that we introduced in [26], both in the subcritical and supercritical cases.

2.1 Introduction

2.1.1 Setting

We consider some unknown parameters $p \in (0,1], \mu > 0$ and $\phi : [0,\infty) \to [0,\infty)$. We always assume that the function ϕ is measurable and locally integrable. For $N \geq 1$, we consider an i.i.d. family $(\Pi^i(dt,dz))_{i=1,\dots,N}$ of Poisson measures on $[0,\infty) \times [0,\infty)$ with intensity $dtdz$, together with $(\theta_{ij})_{i,j=1,\dots,N}$, a family of i.i.d. Bernoulli(p) random variables independent of the family $(\Pi^i(dt,dz))_{i=1,\dots,N}$. We consider the following system: for all $i \in \{1,...,N\}$, all $t \geq 0$,

$$Z_t^{i,N} = \int_0^t \int_0^\infty \mathbf{1}_{\{z \leq \lambda_s^{i,N}\}} \Pi^i(ds,dz), \text{ where } \lambda_t^{i,N} = \mu + \frac{1}{N}\sum_{j=1}^N \theta_{ij} \int_0^{t-} \phi(t-s) dZ_s^{j,N}. \tag{2.1}$$

In this paper, $\int_0^t$ means $\int_{[0,t]}$, and $\int_0^{t-}$ means $\int_{[0,t)}$. The solution $((Z_t^{i,N})_{t\geq 0})_{i=1,\dots,N}$ is a family of counting processes. By [14, Proposition 1], the system (1.1) has a unique $(\mathcal{F}_t)_{t\geq 0}$-measurable càdlàg solution, where

$$\mathcal{F}_t = \sigma(\Pi^i(A) : A \in \mathcal{B}([0,t] \times [0,\infty)), i = 1,...,N) \vee \sigma(\theta_{ij}, i,j = 1,...,N),$$

as soon as ϕ is locally integrable.

2.1.2 An illustrating example

Let us provide an interpretation of the process $((Z_t^{i,N})_{t\geq 0})_{i=1,\dots,N}$. It describes the activity of N individuals along the time. Each individual $j \in \{1,\dots,N\}$ influences the set of individuals

$S_j = \{i \in \{1, \dots, N\} : \theta_{ij} = 1\}$. The only possible action of the individual j is to send a message to all the individuals of S_j. Here $Z_t^{i,N}$ stands for the number of messages sent by i during $[0, t]$.

The rate $\lambda_t^{i,N}$ at which i sends messages can be decomposed as the sum of two effects:

• he sends *new* messages at rate μ;

• he *forwards* the messages he received, after some delay (possibly infinite) depending on the age of the message, which induces a sending rate of the form $\frac{1}{N}\sum_{j=1}^N \theta_{ij} \int_0^{t-} \phi(t-s) dZ_s^{j,N}$.

If for example $\phi = \mathbf{1}_{[0,K]}$, then $N^{-1}\sum_{j=1}^N \theta_{ij} \int_0^{t-} \phi(t-s) dZ_s^{j,N}$ is precisely the number of messages that the i-th individual received between time $t-K$ and time t, divided by N.

2.1.3 Motivations and main goals

In the real world, the number of individuals is often large. So it is necessary to construct consistent estimators in the asymptotic where N and t tend simultaneously to infinity. In our context, we only observe the activity of some (or all) individuals, we do not know at all the graph corresponding to the relationships between individuals. Our goal is to estimate p, which can be seen as the main characteristic of the graph of interactions, since it represents the proportion of open edges. In [14], Delattre and Fournier consider the case when one observes the whole sample $(Z_s^{i,N})_{i=1\dots N, 0\le s\le t}$ and they propose some estimator of the unknown parameter p. In [26], we build some estimator of p when observing $(Z_s^{i,N})_{\{i=1,\dots,K,\ 0\le s\le t\}}$ with $1 \ll K \le N$ and with t large. In this work, we establish a central limit theorem for this estimator, which allows to construct an asymptotic confidence interval of the parameter p.

2.1.4 Assumptions

We will work under the following conditions: for some $q \ge 1$,

$$\mu \in (0,\infty), \quad \Lambda \in (0,\infty), \quad \Lambda p \in [0,1), \quad \int_0^\infty s^q \phi(s) ds < \infty \quad \text{and} \quad \int_0^\infty (\phi(s))^2 ds < \infty \quad (H'(q))$$

or

$$\mu \in (0,\infty), \quad \Lambda p \in (1,\infty) \quad \text{and} \quad \phi(s) = e^{-bs} \quad \text{for some unknown } b > 0. \qquad (A')$$

Here b is a positive constant. Since $\Lambda = 1/b$, we thus assume that $p > b$.

2.1.5 The result in subcritical case

Here we will assume $H(q)$ for some $q \ge 1$. We first recall the estimator we built in Chapter 1. For $N \ge 1$ and for $((Z_t^{i,N})_{t\ge 0})_{i=1,\dots,N}$ the solution of system (2.1), we set $\bar{Z}_t^N = N^{-1}\sum_{i=1}^N Z_t^{i,N}$, and $\bar{Z}_t^{N,K} = K^{-1}\sum_{i=1}^K Z_t^{i,N}$. Next, we introduce:

$$\varepsilon_t^{N,K} = \frac{1}{t}(\bar{Z}_{2t}^{N,K} - \bar{Z}_t^{N,K}), \qquad \mathcal{V}_t^{N,K} = \frac{N}{K}\sum_{i=1}^K \Big[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \varepsilon_t^{N,K}\Big]^2 - \frac{N}{t}\varepsilon_t^{N,K}.$$

And for $\Delta > 0$ such that $t/(2\Delta) \in \mathbb{N}^*$.

$$\mathcal{X}_{t,\Delta}^{N,K} := \mathcal{W}_{\Delta,t}^{N,K} - \frac{N-K}{K}\varepsilon_t^{N,K},$$

where

$$\mathcal{W}_{\Delta,t}^{N,K} = 2\mathcal{Z}_{2\Delta,t}^{N,K} - \mathcal{Z}_{\Delta,t}^{N,K}, \qquad \mathcal{Z}_{\Delta,t}^{N,K} = \frac{N}{t}\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} (\bar{Z}_{a\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K} - \Delta\varepsilon_t^{N,K})^2.$$

We then introduce the function $\Psi^{(3)}$ defined by

$$\Psi^{(3)}(u,v,w) = \frac{u^2(1-\sqrt{\frac{u}{w}})^2}{v+u^2(1-\sqrt{\frac{u}{w}})^2} \quad \text{if } u>0,\ v>0,\ w>0 \quad \text{and} \quad \Psi^{(3)}(u,v,w)=0 \text{ otherwise.}$$

We set

$$\hat{p}_{N,K,t} = \Psi^{(3)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{t,\Delta_t}^{N,K}),$$

with the choice

$$\Delta_t = (2\lfloor t^{1-4/(q+1)} \rfloor)^{-1} t \tag{2.2}$$

It was shown in [26, Theorem 2.1] that under $(H(q))$ for some $q>3$, for some constants $C,c>0$ (depending only on q,p,μ,ϕ), for all $\varepsilon\in(0,1)$, all $1\le K\le N$, all $t\ge 1$,

$$P(|\hat{p}_{N,K,t}-p|>\varepsilon) \le \frac{C}{\epsilon}\Big(\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}}\Big) + CNe^{-cK}.$$

We also showed using a toy model in [26, Section 13] that this rate of convergence is likely to be optimal. Finally, to have an idea of the orders of magnitude, we recall that roughly, in the subcritical case (where $\Lambda p\in(0,1)$), each individual has around 1 jump per unit of time, in the sense that, see [14, Remark 2], under $H(1)$, $t^{-1}\bar{Z}_t^N$ goes in probability to the constant $(1-\Lambda p)^{-1}\mu>0$ as $N\to\infty$ and $t\to\infty$. Hence, when observing a sample of K individuals during $[0,t]$, one observes around Kt jumps. Here is the main result of the present paper in the subcritical case.

Theorem 2.1.1. *We assume that $p>0$ and that $H(q)$ holds for some $q>3$. Define Δ_t by (2.2). We set $c_{p,\Lambda} := (1-\Lambda p)^2/(2\Lambda^2)$. We always work in the asymptotic $(N,K,t)\to(\infty,\infty,\infty)$ and in the regime $\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\lambda}K}\to 0$.*

(i) In the regime with dominating term $\frac{1}{\sqrt{K}}$, i.e. when $[\frac{1}{\sqrt{K}}]/[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}}]\to\infty$, it holds that

$$\sqrt{K}\Big(\hat{p}_{N,K,t}-p\Big) \xrightarrow{d} \mathcal{N}\Big(0, \frac{p^2(1-p)^2}{\mu^4}\Big).$$

(ii) In the regime with dominating term $\frac{N}{t\sqrt{K}}$, i.e. when $[\frac{N}{t\sqrt{K}}]/[\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]\to\infty$, we have

$$\frac{t\sqrt{K}}{N}\Big(\hat{p}_{N,K,t}-p\Big) \xrightarrow{d} \mathcal{N}\Big(0, \frac{2(1-\Lambda p)}{\mu^2\Lambda^4}\Big).$$

(iii) In the regime with dominating term $\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}$, i.e. when $[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]/[\frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}}]\to\infty$, imposing moreover that $\lim_{N,K\to\infty}\frac{K}{N}=\gamma\in[0,1]$,

$$\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\Big(\hat{p}_{N,K,t}-p\Big) \xrightarrow{d} \mathcal{N}\Big(0, \frac{3(1-p)^2}{2\mu^2\Lambda^2}\Big((1-\gamma)(1-\Lambda p)^3 + \gamma(1-\Lambda p)\Big)^2\Big).$$

We decided not to study the regimes where there are two or three dominating terms. We believe this is not very restrictive in practise. Furthermore, the study would be much more tedious, because it would be very difficult to study the correlations between the different terms.

Remark 2.1.2. This result allows us to construct an asymptotic confidence interval for p. We define

$$\hat{\mu}_{N,K,t} := \Psi^{(1)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K}), \quad \hat{\Lambda}_{N,K,t} := \Psi^{(2)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{\Delta_t,t}^{N,K})$$

where

$$\Psi^{(1)}(u,v,w) := u\sqrt{\frac{u}{w}}, \quad \Psi^{(2)}(u,v,w) := \frac{v+[u-\Psi^{(1)}(u,v,w)]^2}{u[u-\Psi^{(1)}(u,v,w)]}$$

if $u > 0$, $v > 0$, $w > u$ and $\Psi^{(1)}(u,v,w) = \Psi^{(2)}(u,v,w) = 0$ otherwise. By [26, Theorem 2.1], we have, in the regime $\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \to 0$,

$$\Big(\hat{\mu}_{N,K,t}, \hat{\Lambda}_{N,K,t}, \hat{p}_{N,K,t}\Big) \xrightarrow{P} (\mu, \Lambda, p).$$

Hence by Theorem 2.1.1, in the regime (i), (ii) or (iii), for $0 < \alpha < 1$,

$$\lim P\Big(|\hat{p}_{N,K,t} - p| \le I_{N,K,t,\alpha}\Big) = 1 - \alpha$$

where

$$I_{N,K,t,\alpha} = (\Phi)^{-1}(1 - \frac{\alpha}{2})\Big(\frac{1}{\sqrt{K}}\frac{\hat{p}_{N,K,t}(1-\hat{p}_{N,K,t})}{(\hat{\mu}_{N,K,t})^2} + \frac{N}{t\sqrt{K}}\frac{\sqrt{2(1-\hat{\lambda}_{N,K,t}\hat{p}_{N,K,t})^2}}{\hat{\mu}_{N,K,t}(\hat{\Lambda}_{N,K,t})^2}$$
$$+ \frac{N}{K}\sqrt{\frac{\Delta_t}{t}}\sqrt{\frac{3(1-\hat{p}_{N,K,t})^2}{2\hat{\mu}^2_{N,K,t}\hat{\Lambda}^2_{N,K,t}}}\Big|(1-\frac{K}{N})(1-\hat{\Lambda}_{N,K,t}\hat{p}_{N,K,t})^3 + \frac{K}{N}(1-\hat{\Lambda}_{N,K,t}\hat{p}_{N,K,t})\Big|$$

and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{s^2}{2}}ds$.

Concerning the case $p = 0$, the following result shows that $\hat{p}_{N,K,t}$ is not always consistent.

Proposition 2.1.3. We assume that $p = 0$ and that $H(q)$ holds for some $q > 3$. We set $c_{p,\Lambda} := (1-\Lambda p)^2/(2\Lambda^2)$. We always work in the asymptotic $(N,K,t) \to (\infty,\infty,\infty)$ and in the regime $\frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \to 0$.

(i) If $[\frac{N}{t\sqrt{K}}]/[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]^2 \to \infty$, we have

$$\hat{p}_{N,K,t} \xrightarrow{P} 0.$$

(ii) If $[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]^2/[\frac{N}{t\sqrt{K}}] \to \infty$, we have

$$\hat{p}_{N,K,t} \xrightarrow{d} X$$

where $P(X=1) = P(X=0) = \frac{1}{2}$.

2.1.6 The result in the supercritical case

Here we will assume A and first recall the estimator we built in [26], $\bar{Z}_t^{N,K}$ being defined as previously. We set

$$\mathcal{U}_t^{N,K} := \Big[\frac{N}{K}\sum_{i=1}^{K}\Big(\frac{Z_t^{i,N} - \bar{Z}_t^{N,K}}{\bar{Z}_t^{N,K}}\Big)^2 - \frac{N}{\bar{Z}_t^{N,K}}\Big]\mathbf{1}_{\{\bar{Z}_t^{N,K}>0\}} \tag{2.3}$$

$$\text{and} \quad \mathcal{P}_t^{N,K} := \frac{1}{\mathcal{U}_t^{N,K}+1}\mathbf{1}_{\{\mathcal{U}_t^{N,K}\ge 0\}}. \tag{2.4}$$

It was shown in [26, Theorem 2.3] that we assume A (actually for a much more general class of functions ϕ), for all $\eta > 0$, for some constant $C_\eta > 0$ (depending only on η, p, μ, b), for all $\varepsilon \in (0,1)$, all $\eta > 0$, for all $1 \le K \le N$, all $t \ge 1$,

$$P(|\mathcal{P}_t^{N,K} - p| > \varepsilon) \le \frac{C_\eta e^{\eta t}}{\epsilon}\Big(\frac{N}{\sqrt{K}e^{\alpha_0 t}} + \frac{1}{\sqrt{K}}\Big),$$

where $\alpha_0 = p - b > 0$ (it is determined by the equation $p\int_0^\infty e^{-\alpha_0 t}\phi(t)dt = 1$). We also showed using a toy model in [26, Section 13] that this rate of convergence is likely to be optimal. Finally, to have an idea of the orders of magnitude, we recall that assumption A, see [14, Remark 5], for any $\eta > 0$, $\lim_{(N,t)\to(\infty,\infty)} P(\bar{Z}_t^N \in [e^{(\alpha_0-\eta)t}, e^{(\alpha_0+\eta)t}]) = 1$. Hence, when observing a sample of K individuals during $[0,t]$, one observes around $Ke^{\alpha_0 t}$ jumps. Here is the main result of the present paper in the supercritical case.

Theorem 2.1.4. *We assume (A) and set $\alpha_0 = p - b$. In the regime where $(N,K,t) \to (\infty,\infty,\infty)$ with $\frac{N}{\sqrt{K}e^{\alpha_0 t}} + \frac{1}{\sqrt{K}} \to 0$ with dominating term $\frac{N}{\sqrt{K}e^{\alpha_0 t}}$ (i.e. with $[\frac{N}{\sqrt{K}e^{\alpha_0 t}}]/[\frac{1}{\sqrt{K}}] \to \infty$), it holds that,*

$$\frac{e^{\alpha_0 t}\sqrt{K}}{N}\Big(\mathcal{P}_t^{N,K} - p\Big) \xrightarrow{d} \mathcal{N}\Big(0, \frac{2(\alpha_0)^4 p^2}{\mu^2}\Big).$$

While our result in the subcritical case is rather general and satisfying, there are many restrictions in the supercritical case. First, we have not been able to deal with general functions ϕ. Second, we did not manage to prove a central limit theorem concerning a large Bernoulli random matrix (and its Perron-Frobenius eigenvalue and eigenvector) that would allow us to study the second regime where $[\frac{1}{\sqrt{K}}]/[\frac{N}{\sqrt{K}e^{\alpha_0 t}}] \to \infty$.

2.1.7 Reference and fields of application

Hawkes processes were first introduced as an birth-immigration model by Hawkes in [19]. The properties of one dimensional Hawkes processes have been well studied, see e.g. Chapter 12 of Daley and Vere-Jones in [13] for the stability of the process, Brémaud and Massoulié in [8] for the analysis of the Bartlett spectrum of the process. Some limit theorems of some large systems modeled by interacting Hawkes processes also have also been estiablished by Delattre, Fournier and Hoffmann, [15]. In [1], Bacry, Delattre, Hoffmann and Muzy prove a law of large numbers and a functional central limit theorem for finite dimensional Hawkes processes observed over a time interval $[0,T]$, as $T \to \infty$. Zhu proves some large deviation principles for Markovian nonlinear Hawkes processes in the subcritical case in [49] and central limit theorem of stationary and ergodic nonlinear Hawkes process in [47].

Hawkes processes have a lot of applications:

- earthquake seismology, see e.g. Ogata [32],
- finance about market orders modelling, see e.g. Bauwens and Hautsch [4] or Lu and Abergel [28],
- neuroscience, see e.g. Brémaud-Massoulié [7],
- criminology, see e.g. Mohler, Short, Brantingham, Schoenberg and Tita [29],
- genomics, see e.g. Gusto and Schbath [17].
- social networks interactions, see Blundell et al. [5] and Zhou et al. [46].

For more examples see the references [15].

2.1.8 Plan of the paper

Sections 2.2 to 2.6 are devoted to the study of the subcritical case. After some preliminaries stated in Section 2.2, we study some random matrix in Section 2.3, establish some limit theorems for the first and second estimator in Section 2.4, and for the third one in Section 2.5. We conclude the study of the subcritical case in Section 2.6.

Concerning the supercritical case, we study the random matrix in Section 2.7, the stochastic processes in Section 2.8, and conclude the proof in Section 2.9.

An appendix containing some technical results lies at the end of the paper.

2.1.9 Important notation

In the whole paper, we denote by $\mathbb{E}_\theta$ the conditionnal expectation knowing $(\theta_{ij})_{i,j=1,...N}$.

2.2 Preliminaries for the subcritical case

2.2.1 Some notations

For $r \in [1,\infty)$ and $\boldsymbol{x} \in \mathbb{R}^N$, we set $\|\boldsymbol{x}\|_r = (\sum_{i=1}^N |x_i|^r)^{\frac{1}{r}}$, and $\|\boldsymbol{x}\|_\infty = \max_{i=1...N} |x_i|$. For M a $N \times N$ matrix, we denote by $|||M|||_r$ is the operator norm associated to $\|\cdot\|_r$, that is $|||M|||_r = \sup_{\boldsymbol{x}\in R^n} \|M\boldsymbol{x}\|_r/\|\boldsymbol{x}\|_r$. We have the special cases

$$|||M|||_1 = \sup_{j=1,\dots,N} \sum_{i=1}^N |M_{ij}|, \quad |||M|||_\infty = \sup_{i=1,\dots,N} \sum_{j=1}^N |M_{ij}|.$$

We also have the inequality

$$|||M|||_r \le |||M|||_1^{\frac{1}{r}} |||M|||_\infty^{1-\frac{1}{r}} \quad \text{for any} \quad r \in [1,\infty).$$

We define $A_N(i,j) := N^{-1}\theta_{ij}$ for $i,j = 1,\dots,N$, as well as $Q_N := (I - \Lambda A_N)^{-1}$ on the event on which $I - \Lambda A_N$ is invertible.

For $1 \le K \le N$, we introduce the N-dimensional vector $\mathbf{1}_K$ defined by $\mathbf{1}_K(i) = \mathbf{1}_{\{1\le i\le K\}}$ for $i = 1,\dots,N$, and the $N \times N$-matrix I_K defined by $I_K(i,j) = \mathbf{1}_{\{i=j\le K\}}$.

We assume here that $\Lambda p \in (0,1)$ and we set $a = \frac{1+\Lambda p}{2} \in (0,1)$. Next, we introduce the events

$$\begin{aligned}
\Omega_N^1 &:= \Big\{\Lambda|||A_N|||_r \le a, \text{ for all } r \in [1,\infty]\Big\},\\
\mathcal{F}_N^{K,1} &:= \Big\{\Lambda|||I_K A_N|||_r \le \Big(\tfrac{K}{N}\Big)^{\frac{1}{r}} a, \text{for all } r \in [1,\infty)\Big\},\\
\mathcal{F}_N^{K,2} &:= \Big\{\Lambda|||A_N I_K|||_r \le \Big(\tfrac{K}{N}\Big)^{\frac{1}{r}} a, \text{ for all } r \in [1,\infty)\Big\},\\
\Omega_{N,K}^1 := \Omega_N^1 \cap \mathcal{F}_N^{K,1}, &\quad \Omega_{N,K}^2 := \Omega_N^1 \cap \mathcal{F}_N^{K,2}, \quad \Omega_{N,K} = \Omega_{N,K}^1 \cap \Omega_{N,K}^2.
\end{aligned}$$

Recall that $c_{p,\Lambda} = (1-\Lambda p)^2/(2\Lambda^2)$.

Lemma 2.2.1. *Assume that $\Lambda p < 1$. It holds that*

$$P(\Omega_{N,K}) \ge 1 - CNe^{-c_{p,\Lambda}K}$$

for some constants $C > 0$.

Proof. On $\Omega_{N,K}^1$, we have

$$N|||I_K A_N|||_1 = \sup_{j=1,\dots,N} \sum_{i=1}^K \theta_{ij} = \max\{X_1^{N,K},\dots,X_N^{N,K}\},$$

where $X_i^{N,K} = \sum_{j=1}^K \theta_{ij}$ for $i = 1,\dots,N$ are i.i.d and Binomial(K,p)-distributed. So,

$$\begin{aligned}
P\Big(\Lambda\frac{N}{K}|||I_K A_N|||_1 \ge a\Big) &= P\Big(\max\{X_1^{N,K},\dots X_N^{N,K}\} \ge \frac{Ka}{\Lambda}\Big) \le NP\Big(X_1^{N,K} \ge \frac{Ka}{\Lambda}\Big)\\
&\le NP\Big(|X_1^{N,K} - Kp| \ge K\Big(\frac{a}{\Lambda} - p\Big)\Big) \le 2Ne^{-2K(\frac{a}{\Lambda}-p)^2} = 2Ne^{-c_{p,\Lambda}K}.
\end{aligned}$$

The last equality follows from Hoeffding inequality. On the event $\Omega_N^1 \cap \{\Lambda \frac{N}{K} |||I_K A_N|||_1 \le a\}$, we have

$$|||I_K A_N|||_r \le |||I_K A_N|||_1^{\frac{1}{r}} \|I_K A_N|||_\infty^{1-\frac{1}{r}} \le |||I_K A_N|||_1^{\frac{1}{r}} ||A_N|||_\infty^{1-\frac{1}{r}} \le \Big(\frac{a}{\Lambda}\frac{K}{N}\Big)^{\frac{1}{r}} \Big(\frac{a}{\Lambda}\Big)^{1-\frac{1}{r}} = \frac{a}{\Lambda}\Big(\frac{K}{N}\Big)^{\frac{1}{r}}.$$

We conclude that $\Omega_{N,K}^1 = \Omega_N^1 \cap \{\Lambda(\frac{N}{K}) |||I_K A_N|||_1 \le a\}$. And from the proof of [14, Lemma 13], we find that that $P(\Omega_N^1) \ge 1 - CNe^{-c_{p,\Lambda} N}$. Hence

$$P(\Omega_{N,K}^1) \ge P(\Omega_N^1) + P\Big(\Lambda \frac{N}{K} |||I_K A_N|||_1 \le a\Big) - 1 \ge 1 - CNe^{-c_{p,\Lambda} K}.$$

By the same way, we prove that $P(\Omega_{N,K}^2) \ge 1 - CNe^{-c_{p,\Lambda} K}$. Finally by the definition of $\Omega_{N,K}$, we have $P(\Omega_{N,K}) \ge P(\Omega_{N,K}^1) + P(\Omega_{N,K}^2) - 1 \ge 1 - CNe^{-c_{p,\Lambda} K}$.

Next, we set $\boldsymbol{\ell}_N := Q_N \mathbf{1}_N$, i.e. $\ell_N(i) := \sum_{j=1}^N Q_N(i,j)$, as well as $\bar{\ell}_N := \frac{1}{N}\sum_{i=1}^N \ell_N(i)$, $\bar{\ell}_N^K := \frac{1}{K}\sum_{i=1}^K \ell_N(i)$. We also set $c_N^K(j) := \sum_{i=1}^K Q_N(i,j)$, $\bar{c}_N^K := \frac{1}{N}\sum_{j=1}^N c_N^K(j)$.

We let $\boldsymbol{L}_N := A_N \mathbf{1}_N$, i.e. $L_N(i) := \sum_{j=1}^N A_N(i,j)$. We also let $\bar{L}_N := \frac{1}{N}\sum_{i=1}^N L_N(i)$, $\bar{L}_N^K := \frac{1}{K}\sum_{i=1}^K L_N(i)$ and $\boldsymbol{C}_N := A_N^* \mathbf{1}_N$, i.e. $C_N(j) := \sum_{i=1}^N A_N(i,j)$, $\bar{C}_N := \frac{1}{N}\sum_{j=1}^N C_N(j)$, $\bar{C}_N^K := \frac{1}{K}\sum_{j=1}^K C_N(j)$ and consider the event

$$\mathcal{A}_N := \{\|\boldsymbol{L}_N - p\mathbf{1}_N\|_2 + \|\boldsymbol{C}_N - p\mathbf{1}_N\|_2 \le N^{\frac{1}{4}}\}. \tag{2.5}$$

We also set $x_N(i) = \ell_N(i) - \bar{\ell}_N$, $\boldsymbol{x}_N = (x_N(i))_{i=1,\dots,N}$, $X_N(i) = L_N(i) - \bar{L}_N$ and $\boldsymbol{X}_N = (X_N(i))_{i=1,\dots,N}$. We finally put $X_N^K(i) = (L_N(i) - \bar{L}_N^K)\mathbf{1}_{\{i\le K\}}$ and $\boldsymbol{X}_N^K = (X_N^K(i))_{i=1,\dots,N} = \boldsymbol{L}_N^K - \bar{L}_N^K \mathbf{1}_K$, as well as $x_N^K(i) = (\ell_N(i) - \bar{\ell}_N^K)\mathbf{1}_{\{i\le K\}}$ and $\boldsymbol{x}_N^K = (x_N^K(i))_{i=1,\dots,N} = \boldsymbol{\ell}_N^K - \bar{\ell}_N^K \mathbf{1}_K$. Next, we are going to review some important results in [14].

Lemma 2.2.2. *We assume that $\Lambda p < 1$. Then $\Omega_{N,K} \subset \Omega_N^1 \subset \{|||Q_N|||_r \le C$, for all $r \in [1,\infty]\} \subset \{\sup_{i=1\dots N} \ell_N(i) \le C\}$, where $C = (1-a)^{-1}$. For any $\alpha > 0$, there exists a constant C_α such that*

$$P(\mathcal{A}_N) \ge 1 - C_\alpha N^{-\alpha}.$$

Proof. See [14, Notation 12 and Proposition 14, Step 1].

2.2.2 Some auxilliary processes

We first introduce a family of martingales: for $i = 1, \dots, N$, recalling (2.1),

$$M_t^{i,N} = \int_0^t \int_0^\infty \mathbf{1}_{\{z \le \lambda_s^{i,N}\}} \widetilde{\pi}^i(ds, dz),$$

where $\widetilde{\pi}^i(ds,dz) = \pi^i(ds,dz) - dsdz$. We also introduce the family of centered processes $U_t^{i,N} = Z_t^{i,N} - \mathbb{E}_\theta[Z_t^{i,N}]$.

We denote by $\boldsymbol{Z}_t^N$ (resp. $\boldsymbol{U}_t^N$, $\boldsymbol{M}_t^N$) the N dimensional vector with coordinates $Z_t^{i,N}$ (resp. $U_t^{i,N}$, $M_t^{i,N}$) and set

$$\boldsymbol{Z}_t^{N,K} = I_K \boldsymbol{Z}_t^N, \quad \boldsymbol{U}_t^{N,K} = I_K \boldsymbol{U}_t^N,$$

as well as $\bar{Z}_t^{N,K} = K^{-1}\sum_{i=1}^K Z_t^{i,N}$ and $\bar{U}_t^{N,K} = K^{-1}\sum_{i=1}^K U_t^{i,N}$. By [14, Remark 10 and Lemma 11], we have the following equalities:

$$\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}] = \mu \sum_{n\ge 0} \Big[\int_0^t s\phi^{*n}(t-s)ds\Big] I_K A_N^n \mathbf{1}_N, \tag{2.6}$$

$$\boldsymbol{U}_t^{N,K} = \sum_{n\ge 0} \int_0^t \phi^{*n}(t-s) I_K A_N^n \boldsymbol{M}_s^N ds, \tag{2.7}$$

$$[M^{i,N}, M^{j,N}]_t = \mathbf{1}_{\{i=j\}} Z_t^{i,N}. \tag{2.8}$$

We use the convention that $\phi^{*0}(s)ds = \delta_0 ds$, whence in particular $\int_0^t s\phi^{*0}(t-s)ds = t$.

Lemma 2.2.3. *Assume $H(q)$ for some $q \geq 1$. There exists a constant C such that*

(i) for all r in $[1,\infty]$, all $t \geq 0$, a.s.,

$$\mathbf{1}_{\Omega_{N,K}} \|\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}]\|_r \leq Ct K^{\frac{1}{r}}.$$

(ii) For any $r \in [1,\infty]$, for all $t \geq s \geq 0$,

$$\mathbf{1}_{\Omega_{N,K}} \|\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K} - \boldsymbol{Z}_s^{N,K} - \mu(t-s)\boldsymbol{\ell}_N^K]\|_r \leq C(\min\{1, s^{1-q}\})K^{\frac{1}{r}}.$$

(iii) For all $t \geq s+1 \geq 1$, on $\Omega_{N,K}$, we have a.s.,

$$\mathbb{E}_\theta[(\bar{U}_t^{N,K} - \bar{U}_s^{N,K})^4] \leq \frac{C(t-s)^2}{K^2} \quad \text{and} \quad \mathbb{E}_\theta[(\bar{Z}_t^{N,K} - \bar{Z}_s^{N,K})^4] \leq C(t-s)^4.$$

Proof. See [26, Lemma 5.1] for the proofs of (i) and (ii). For (iii), we deduce from (2.7) that

$$\bar{U}_t^{N,K} = K^{-1} \sum_{n\geq 0} \int_0^t \phi^{\star n}(t-s) \sum_{i=1}^K \sum_{j=1}^N A_N^n(i,j) M_s^{j,N} ds.$$

We set $\phi(s) = 0$ for $s \leq 0$. Separating the cases $n = 0$ and $n \geq 1$, using the Minkowski inequality, we see that on $\Omega_{N,K}$, we have

$$\begin{aligned}
&\mathbb{E}_\theta[(\bar{U}_t^{N,K} - \bar{U}_s^{N,K})^4]^{\frac{1}{4}} \\
\leq &\mathbb{E}_\theta[(\bar{M}_t^{N,K} - \bar{M}_s^{N,K})^4]^{\frac{1}{4}} \\
&+ \frac{1}{K} \sum_{n\geq 1} \int_0^\infty \Big(\phi^{\star n}(t-u) - \phi^{\star n}(s-u)\Big) \mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K \sum_{j=1}^N A_N^n(i,j) M_u^{j,N}\Big)^4\Big]^{\frac{1}{4}} du.
\end{aligned}$$

By [14, Lemma 16 (iii)], we already know that, on $\Omega_{N,K}$, $\max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^2] \leq C(t-s)^2$. For the first term ($n = 0$), we use (2.8) and Burkholder's inequality:

$$\begin{aligned}
\mathbb{E}_\theta[(\bar{M}_t^{N,K} - \bar{M}_s^{N,K})^4] =& \frac{1}{K^4} \mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K (M_t^{i,N} - M_s^{i,N})\Big)^4\Big] \\
\leq& \frac{C}{K^4} \mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K (Z_t^{i,N} - Z_s^{i,N})\Big)^2\Big] \\
\leq& \frac{C(t-s)^2}{K^2}
\end{aligned}$$

For the second term ($n \geq 1$), we use again (2.8) and by Burkholder's inequality and we get

$$\begin{aligned}
\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K \sum_{j=1}^N A_N^n(i,j) M_u^{j,N}\Big)^4\Big] \leq& C\mathbb{E}_\theta\Big[\Big([\sum_{i=1}^K \sum_{j=1}^N A_N^n(i,j) M^{j,N}, \sum_{i=1}^K \sum_{j=1}^N A_N^n(i,j) M^{j,N}]_u\Big)^2\Big] \\
\leq& C\mathbb{E}_\theta\Big[\Big(\sum_{j=1}^N \Big(\sum_{i=1}^K A_N^n(i,j)\Big)^2 Z_u^{j,N}\Big)^2\Big] \\
\leq& C\mathbb{E}_\theta\Big[\Big(\sum_{j=1}^N |||I_K A_N^n|||_1^2 Z_u^{j,N}\Big)^2\Big] \\
\leq& C\mathbb{E}_\theta\Big[\Big(\sum_{j=1}^N |||I_K A_N|||_1^2 |||A_N|||_1^{2(n-1)} Z_u^{j,N}\Big)^2\Big] \\
\leq& CN^2u^2 |||I_K A_N|||_1^4 |||A_N|||_1^{4(n-1)}.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\frac{1}{K}\sum_{n\geq 1}\int_0^\infty \Big(\phi^{\star n}(t-u)-\phi^{\star n}(s-u)\Big)\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K\sum_{j=1}^N A_N^n(i,j)M_u^{j,N}\Big)^4\Big]^{\frac{1}{4}}du\\
\leq&\frac{C}{K}\sum_{n\geq 1}|||I_KA_N|||_1|||A_N|||_1^{n-1}\int_0^t\sqrt{Nu}\Big(\phi^{\star n}(t-u)-\phi^{\star n}(s-u)\Big)du\\
\leq&\frac{C(t-s)^{1/2}}{N^{1/2}}\sum_{n\geq 0}\Lambda^n|||A_N|||_1^n\leq\frac{C(t-s)^{1/2}}{N^{1/2}}.
\end{aligned}$$

We used first that for all $n\geq 1$, it holds that

$$\begin{aligned}
\int_0^t\sqrt{u}(\phi^{\star n}(t-u)-\phi^{\star n}(s-u))du=&\int_0^t\sqrt{t-u}\phi^{\star n}(u)du-\int_0^s\sqrt{s-u}\phi^{\star n}(u)du\\
\leq&\int_0^s[\sqrt{t-u}-\sqrt{s-u}]\phi^{\star n}(u)du+\int_s^t\sqrt{t-u}\phi^{\star n}(u)du\\
\leq&2\sqrt{t-s}\int_0^\infty\phi^{\star n}(u)du\leq 2\Lambda^n\sqrt{t-s}.
\end{aligned}$$

We next used that on $\Omega_{N,K}$, we have $\Lambda|||A_N|||_1\leq a<1$ and $\Lambda|||I_KA_N|||_1\leq aK/N$. This completes the first part of (iii).

For the second part, by [26, Lemma 5.1 (ii)], we have $\mathbb{E}_\theta[\bar{Z}_t^{N,K}-\bar{Z}_s^{N,K}]\leq C(t-s)$ on $\Omega_{N,K}$, whence

$$\mathbb{E}_\theta[(\bar{Z}_t^{N,K}-\bar{Z}_s^{N,K})^4]\leq 4\Big\{\mathbb{E}_\theta[\bar{Z}_t^{N,K}-\bar{Z}_s^{N,K}]^4+\mathbb{E}_\theta[(\bar{U}_t^{N,K}-\bar{U}_s^{N,K})^4]\Big\}\leq C(t-s)^4$$

as desired.

2.3 Some limit theorems for the random matrix in the subcritical case

2.3.1 First estimator

As we will see, the first estimator $\varepsilon_t^{N,K}$ is closely linked to $\bar{\ell}_N^K$. For this last quantity, we will only use the following easy inequality, of which the proof can be found in [26, Lemma 3.9].

Lemma 2.3.1. *If $\Lambda p<1$, there is $C>0$ such that for all $1\leq K\leq N$,*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar{\ell}_N^K-\frac{1}{1-\Lambda p}\Big|^2\Big]\leq\frac{C}{NK}.$$

2.3.2 Second estimator

The second estimator $\mathbf{V}_t^{N,K}$ is related to $\mathbf{V}_\infty^{N,K}=\frac{N}{K}\|\boldsymbol{x}_N^K\|_2^2$, which we now study.

Theorem 2.3.2. *Assume $\Lambda p<1$. Then, in distribution, as $(N,K)\to(\infty,\infty)$, in the regime $Ne^{-c_{p,\Lambda}K}\to 0$,*

$$\mathbf{1}_{\Omega_{N,K}}\sqrt{K}\Big(\mathbf{V}_\infty^{N,K}-\frac{\Lambda^2p(1-p)}{(1-\Lambda p)^2}\Big)\longrightarrow\mathcal{N}\Big(0,\Big(\Lambda^2\frac{p(1-p)}{(1-\Lambda p)^2}\Big)^2\Big).$$

The proof relies on four lemmas.

Lemma 2.3.3. *Assume that $\Lambda p < 1$. There is $C > 0$ such that for all $1 \le K \le N$,*

$$\mathbb{E}[||(I_K A_N)^T \mathbf{X}_N^K)||_2^2] \le \frac{CK^2}{N^3}.$$

Proof. Recall that $\mathbf{X}_N^K = \mathbf{L}_N^K - \bar{L}_N^K \mathbf{1}_K$. By symmetry, we have

$$\begin{aligned}\mathbb{E}[\|(I_K A_N)^T \mathbf{X}_N^K\|_2^2] =& \frac{K}{N^2}\mathbb{E}\Big[\Big(\sum_{j=1}^K \theta_{j1}(L_N(j) - \bar{L}_N^K)\Big)^2\Big] \\ \le& \frac{2K}{N^2}\Big\{\mathbb{E}\Big[\Big(\sum_{j=1}^K \theta_{j1}(L_N(j) - p)\Big)^2\Big] + \mathbb{E}\Big[\Big(\sum_{j=1}^K \theta_{j1}(p - \bar{L}_N^K)\Big)^2\Big]\Big\}.\end{aligned}$$

First, since $\theta_{j1} \le 1$, we obviously have

$$\frac{K}{N^2}\mathbb{E}\Big[\Big(\sum_{j=1}^K \theta_{j1}(p - \bar{L}_N^K)\Big)^2\Big] \le \frac{K^3}{N^2}\mathbb{E}[(p - \bar{L}_N^K)^2] \le \frac{CK^2}{N^3}.$$

Next,

$$\begin{aligned}&\frac{K}{N^2}\mathbb{E}\Big[\Big(\sum_{j=1}^K \theta_{j1}(L_N(j) - p)\Big)^2\Big] \\ \le& \frac{2K}{N^4}\Big\{\mathbb{E}\Big[\Big(\sum_{j=1}^K\sum_{i=2}^N \theta_{j1}(\theta_{ji} - p)\Big)^2\Big] + \mathbb{E}\Big[\Big(\sum_{j=1}^K \theta_{j1}(\theta_{j1} - p)\Big)^2\Big]\Big\} \\ \le& \frac{4K}{N^4}\Big\{\mathbb{E}\Big[\Big(\sum_{j=1}^K\sum_{i=2}^N (\theta_{j1} - p)(\theta_{ji} - p)\Big)^2\Big] + p^2\mathbb{E}\Big[\Big(\sum_{j=1}^K\sum_{i=2}^N (\theta_{ji} - p)\Big)^2\Big] + \mathbb{E}\Big[\Big(\sum_{j=1}^K \theta_{j1}(\theta_{j1} - p)\Big)^2\Big]\Big\}.\end{aligned}$$

This is controled by CK^2/N^3 as desired, because $\mathbb{E}[(\sum_{j=1}^K\sum_{i=2}^N(\theta_{j1} - p)(\theta_{ji} - p))^2] \le CKN$ (since the family $\{(\theta_{ji} - p), i = 2, \dots, N, j = 1, \dots, K\}$ is independant and centered), because $\mathbb{E}[(\sum_{j=1}^K\sum_{i=2}^N(\theta_{ji} - p))^2] \le CNK$ (for similar reasons), and $\mathbb{E}[(\sum_{j=1}^K \theta_{j1}(\theta_{j1} - p))^2] \le CK^2$.

Lemma 2.3.4. *Assume that $0 < p \le 1$. There is $C > 0$ such that for all $1 \le K \le N$,*

$$\mathbb{E}\Big[\Big|\Big(I_K A_N \mathbf{X}_N, \mathbf{X}_N^K\Big)\Big|\Big] \le \frac{CK}{N^2}.$$

Proof. By definition, we have

$$\begin{aligned}(I_K A_N \mathbf{X}_N, \mathbf{X}_N^K) =& \frac{1}{N}\sum_{i,j=1}^K (\theta_{ij} - p) X_N(j) X_N^K(i) \\ =& \frac{1}{N}\Big[\sum_{i,j=1}^K (\theta_{ij} - p)(L_N(j) - p) X_N^K(i) + (p - \bar{L}_N)\sum_{i,j=1}^K (\theta_{ij} - p) X_N^K(i)\Big] \\ =& \frac{1}{N}\Big[\sum_{i,j=1}^K (\theta_{ij} - p)(L_N(j) - p)(L_N(i) - p) + (p - \bar{L}_N^K)\sum_{i,j=1}^K (\theta_{ij} - p)(L_N(j) - p) \\ &+ (p - \bar{L}_N)\sum_{i,j=1}^K (\theta_{ij} - p)(L_N(i) - p) + (p - \bar{L}_N)(p - \bar{L}_N^K)\sum_{i,j=1}^K (\theta_{ij} - p)\Big].\end{aligned}$$

We start with the first term:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{i,j=1}^{K}(\theta_{ij}-p)(L_N(j)-p)(L_N(i)-p)\Big)^2\Big]\\
=&\frac{1}{N^4}\mathbb{E}\Big[\Big(\sum_{i,j=1}^{K}\sum_{m,n=1}^{N}(\theta_{ij}-p)(\theta_{jm}-p)(\theta_{in}-p)\Big)^2\Big]\\
=&\frac{1}{N^4}\mathbb{E}\Big[\sum_{i,j,i',j'=1}^{K}\sum_{m,n,m',n'=1}^{N}(\theta_{ij}-p)(\theta_{jm}-p)(\theta_{in}-p)(\theta_{i'j'}-p)(\theta_{j'm'}-p)(\theta_{i'n'}-p)\Big]\leq\frac{CK^2}{N^2}
\end{aligned}$$

since the family $\{(\theta_{ij}-p), i,j=1,\dots,N\}$ is i.i.d., centered, and bounded. For the second term, we write, using the Cauchy-Schwarz inequality,

$$\mathbb{E}\Big[\Big|(p-\bar{L}_N^K)\sum_{i,j=1}^{K}(\theta_{ij}-p)(L_N(j)-p)\Big|\Big]\leq\frac{1}{N}\mathbb{E}[(p-\bar{L}_N^K)^2]^{\frac{1}{2}}\mathbb{E}\Big[\Big(\sum_{i,j=1}^{K}\sum_{k=1}^{N}(\theta_{ij}-p)(\theta_{jk}-p)\Big)^2\Big]^{\frac{1}{2}}.$$

This is dominated by $\frac{\sqrt{K}}{N}$, because on the first hand, we have the equality $\mathbb{E}[(p-\bar{L}_N^K)^2]=\frac{1}{N^2K^2}\mathbb{E}[(\sum_{i=1}^{K}\sum_{j=1}^{N}(\theta_{ij}-p))^2]=\frac{\mathbb{E}[(\theta_{11}-p)^2]}{NK}\leq\frac{C}{NK}$, and on the other hand,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{i,j=1}^{K}\sum_{k=1}^{N}(\theta_{ij}-p)(\theta_{jk}-p)\Big)^2\Big]\\
=&\mathbb{E}\Big[\sum_{i,j,i',j'=1}^{K}\sum_{k,k'=1}^{N}(\theta_{ij}-p)(\theta_{i'j'}-p)(\theta_{jk}-p)(\theta_{j'k'}-p)\Big]\leq CNK^2.
\end{aligned}$$

For the third term, using Cauchy-Schwarz inequality, we can write (by the previous discussion, we have $\mathbb{E}[(p-\bar{L}_N)^2]=\mathbb{E}[(p-\bar{L}_N^N)^2]\leq\frac{C}{N^2}$),

$$\begin{aligned}
&\mathbb{E}\Big[\Big|(p-\bar{L}_N)\sum_{i,j=1}^{K}(\theta_{ij}-p)(L_N(i)-p)\Big|\Big]\\
\leq&\frac{1}{N}\mathbb{E}[(p-\bar{L}_N)^2]^{\frac{1}{2}}\mathbb{E}\Big[\Big(\sum_{i,j=1}^{K}\sum_{k=1}^{N}(\theta_{ij}-p)(\theta_{ik}-p)\Big)^2\Big]^{\frac{1}{2}}\\
\leq&\frac{1}{N}\sqrt{\frac{C}{N^2}}\mathbb{E}\Big[\sum_{i,j,i',j'=1}^{K}\sum_{k,k'=1}^{N}(\theta_{ij}-p)(\theta_{ik}-p)(\theta_{i'j'}-p)(\theta_{i'k'}-p)\Big]^{\frac{1}{2}}\\
\leq&\frac{1}{N}\sqrt{\frac{C}{N^2}}\sqrt{K^2N+K^4}=C\frac{K}{N^{3/2}}+C\frac{K^2}{N^2}.
\end{aligned}$$

Finally, we study the last term. We observe that $\mathbb{E}[(\sum_{i,j=1}^{K}(\theta_{ij}-p))^2]=\mathbb{E}[\sum_{i,j=1}^{K}(\theta_{ij}-p)^2]=CK^2$ and $\mathbb{E}[(p-\bar{L}_N^K)^4]=\frac{1}{N^4K^4}\mathbb{E}[(\sum_{i=1}^{K}\sum_{j=1}^{N}(\theta_{ij}-p))^4]\leq\frac{C}{N^2K^2}$. Hence

$$\begin{aligned}
&\mathbb{E}\Big[\Big|(p-\bar{L}_N)(p-\bar{L}_N^K)\sum_{i,j=1}^{K}(\theta_{ij}-p)\Big|\Big]\\
\leq&\mathbb{E}[(p-\bar{L}_N)^4]^{\frac{1}{4}}\mathbb{E}[(p-\bar{L}_N^K)^4]^{\frac{1}{4}}\mathbb{E}\Big[\Big(\sum_{i,j=1}^{K}(\theta_{ij}-p)\Big)^2\Big]^{\frac{1}{2}}\\
\leq&C\Big(\frac{1}{N^4}\Big)^{1/4}\Big(\frac{1}{N^2K^2}\Big)^{1/4}\sqrt{K^2}=\frac{\sqrt{K}}{N\sqrt{N}}.
\end{aligned}$$

This completes the proof.

Recalling the definition of $\mathcal{A}_N$ in (2.5) first, we have the following lemma.

Lemma 2.3.5. *Assume $\Lambda p < 1$. There is $C > 0$ such that for all $1 \le K \le N$,*

$$\frac{N}{K}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\Big(\|\boldsymbol{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\mathbf{X}_N^K\|_2^2\Big) - \|\boldsymbol{x}_N^K - \bar{\ell}_N\Lambda\mathbf{X}_N^K\|_2^2\Big|\Big] \le \frac{C}{N}.$$

Proof. We start from

$$\begin{aligned}&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\Big(\|\boldsymbol{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\mathbf{X}_N^K\|_2^2\Big) - \|\boldsymbol{x}_N^K - \bar{\ell}_N\Lambda\mathbf{X}_N^K\|_2^2\Big|\Big]\\ &= 2\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\Lambda\bar{\ell}_N(\mathbf{X}_N^K, \Lambda\bar{\ell}_N\mathbf{X}_N^K - \boldsymbol{x}_N^K)\Big|\Big]\end{aligned}$$

By [26, Lemma 3.11], we already know that

$$\boldsymbol{x}_N^K - \Lambda\bar{\ell}_N\boldsymbol{X}_N^K = \Lambda I_K A_N(\boldsymbol{x}_N - \Lambda\bar{\ell}_N\boldsymbol{X}_N) - \frac{\Lambda}{K}(I_K A_N\boldsymbol{x}_N, \mathbf{1}_K)\mathbf{1}_K + \bar{\ell}_N\Lambda^2 I_K A_N\boldsymbol{X}_N.$$

Since by definition $(\mathbf{1}_{\boldsymbol{K}}, \mathbf{X}_N^K) = 0$, we conclude that

$$(\boldsymbol{x}_N^K - \Lambda\bar{\ell}_N\mathbf{X}_N^K, \mathbf{X}_N^K) = \Lambda(I_K A_N(\boldsymbol{x}_N - \Lambda\bar{\ell}_N\mathbf{X}_N), \mathbf{X}_N^K) + \Lambda^2\bar{\ell}_N(I_K A_N\mathbf{X}_N, \mathbf{X}_N^K) = e_{N,K,1} + e_{N,K,2},$$

the last equality standing for a definition. By Lemma 2.3.3, and [26, Lemma 4.11], we have:

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}e_{N,K,1}] =& \mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Lambda\Big((\boldsymbol{x}_N - \Lambda\bar{\ell}_N\mathbf{X}_N), (I_K A_N)^T\mathbf{X}_N^K\Big)\Big]\\ \le& \Lambda\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{x}_N - \Lambda\bar{\ell}_N\mathbf{X}_N\|_2^2\Big]^{\frac{1}{2}}\mathbb{E}\Big[\|(I_K A_N)^T\mathbf{X}_N^K\|_2^2\Big]^{\frac{1}{2}} \le \frac{CK}{N^2}\end{aligned}$$

Next, by Lemma 2.2.2, we know that $\bar{\ell}_N$ is bounded in $\Omega_{N,K}$, and by Lemma 2.3.4, we conclude that

$$\mathbb{E}[\mathbf{1}_{\Omega_{N,K}}e_{2,N,K}] \le \frac{CK}{N^2}$$

as desired.

Lemma 2.3.6. *Assume $\Lambda p < 1$. Then, in distribution, as $(N, K) \to (\infty, \infty)$, in regime $Ne^{-c_{p,\Lambda}K} \to 0$,*

$$\mathbf{1}_{\Omega_{N,K}}\sqrt{K}\Big[\frac{N}{K}(\bar{\ell}_N\|\mathbf{X}_N^K\|_2)^2 - \frac{p(1-p)}{(1-\Lambda p)^2}\Big] \xrightarrow{d} \mathcal{N}\Big(0, \Big[\frac{p(1-p)}{(1-\Lambda p)^2}\Big]^2\Big).$$

Proof. Recall [14, Proposition 14]: we already have $\mathbb{E}[\mathbf{1}_{\Omega_N^1}|\bar{\ell}_N - \frac{1}{1-\Lambda p}|^2] \le \frac{C}{N^2}$. By Lemma 2.2.2, we also know that $\bar{\ell}_N$ is bounded by some constant C on $\Omega_{N,K}$. Also, it is easily checked, see e.g. [26, Equation (7)], that $\mathbb{E}[\frac{N^2}{K^2}\|\boldsymbol{X}_N^K\|_2^4] \le C$. All in all,

$$\begin{aligned}&\sqrt{K}\frac{N}{K}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|(\bar{\ell}_N)^2 - \Big(\frac{1}{1-\Lambda p}\Big)^2\Big|\|\mathbf{X}_N^K\|_2^2\Big]\\ \le&\sqrt{K}\mathbb{E}\Big[\frac{N^2}{K^2}\|\boldsymbol{X}_N^K\|_2^4\Big]^{\frac{1}{2}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar{\ell}_N - \frac{1}{1-\Lambda p}\Big|^2\Big]^{\frac{1}{2}} \le \frac{C\sqrt{K}}{N} \le \frac{C}{\sqrt{N}},\end{aligned}$$

whence it suffices to prove that $\mathbf{1}_{\Omega_{N,K}}\sqrt{K}[\frac{N}{K}(\|\mathbf{X}_N^K\|_2)^2 - p(1-p)] \xrightarrow{d} \mathcal{N}(0, p^2(1-p)^2)$. We recall that $\mathbf{1}_{\Omega_{N,K}}$ tends to 1 in probability. Hence it suffices to verify that $\sqrt{K}[\frac{N}{K}(\|\mathbf{X}_N^K\|_2)^2 - p(1-p)] \xrightarrow{d} \mathcal{N}(0, p^2(1-p)^2)$. But

$$\|\mathbf{X}_N^K\|_2^2 = \sum_{i=1}^K(L_N(i) - \bar{L}_N^K)^2 = \sum_{i=1}^K(L_N(i) - p)^2 - K(p - \bar{L}_N^K)^2.$$

As seen in the proof of Lemma 2.3.4, we have $\mathbb{E}[(p-\bar{L}_N^K)^2] \le \frac{C}{NK}$, so that $\sqrt{K}\frac{N}{K}\mathbb{E}[K(p-\bar{L}_N^K)^2] \le \frac{C}{\sqrt{K}}$. Hence, our goal is to verify that

$$\xi_{N,K} = \sqrt{K}\Big[\frac{N}{K}(\sum_{i=1}^{K}(L_N(i)-p)^2 - p(1-p)\Big] \xrightarrow{d} \mathcal{N}(0, p^2(1-p)^2).$$

Recalling that $L_N(i) = N^{-1}\sum_{j=1}^{N}\theta_{ij}$, we can check that

$$\xi_{N,K} = \frac{1}{N\sqrt{K}}\sum_{i=1}^{K}\sum_{j=1}^{N}[(\theta_{ij}-p)^2 - p(1-p)] + \frac{1}{N\sqrt{K}}\sum_{i=1}^{K}\sum_{j=1}^{N}\sum_{j'=1,j'\neq j}^{N}(\theta_{ij}-p)(\theta_{ij'}-p).$$

The first term tends to 0 in probability, because by the central limit theorem, we have convergence in distribution of $\frac{1}{\sqrt{NK}}\sum_{i=1}^{K}\sum_{j=1}^{N}[(\theta_{ij}-p)^2 - p(1-p)]$. And, using the central limit theorem again, we find that

$$\frac{1}{N\sqrt{K}}\sum_{i=1}^{K}\sum_{j=1}^{N}\sum_{j'=1,j'\neq j}^{N}(\theta_{ij}-p)(\theta_{ij'}-p) \xrightarrow{d} \mathcal{N}(0, p^2(1-p)^2).$$

The proof is complete.

Finally, we give the

Proof. [Proof of Theorem 2.3.2] Recalling that $\mathbf{V}_\infty^{N,K} = \frac{N}{K}\|\boldsymbol{x}_N^K\|_2^2$, we write

$$\begin{aligned}\sqrt{K}\Big(\mathbf{V}_\infty^{N,K} - \frac{\Lambda^2 p(1-p)}{(1-\Lambda p)^2}\Big) =& \frac{N}{\sqrt{K}}\Big(\|\boldsymbol{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\mathbf{X}_N^K\|_2^2\Big)\\ &+ \frac{N(\Lambda\bar{\ell}_N)^2}{\sqrt{K}}\|\mathbf{X}_N^K\|_2^2 - \frac{\Lambda^2 p(1-p)\sqrt{K}}{(1-\Lambda p)^2}.\end{aligned}$$

By Lemma 2.3.6, it suffices to check that $\zeta_{N,K} = \mathbf{1}_{\Omega_{N,K}}\frac{N}{\sqrt{K}}\Big(\|\boldsymbol{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\mathbf{X}_N^K\|_2^2\Big) \to 0$ in probability. Since moreover $\mathbf{1}_{\mathcal{A}_N} \to 1$ a.s. by Lemma 2.2.2, we only have to verify that $\mathbf{1}_{\mathcal{A}_N}\zeta_{N,K} \to 0$ in probability. We write

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\mathcal{A}_N}\zeta_{N,K}] \le& \frac{N}{\sqrt{K}}\mathbb{E}[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\|\boldsymbol{x}_N^K - \Lambda\bar{\ell}_N\mathbf{X}_N^K\|_2^2]\\ &+ \frac{N}{\sqrt{K}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\Big|\Big(\|\boldsymbol{x}_N^K\|_2^2 - (\Lambda\bar{\ell}_N)^2\|\mathbf{X}_N^K\|_2^2\Big) - \|\boldsymbol{x}_N^K - \bar{\ell}_N\Lambda\mathbf{X}_N^K\|_2^2\Big|\Big].\end{aligned}$$

By [26, Lemma 4.11], the first term is bounded by $C/\sqrt{K}$. By Lemma 2.3.5, the second term is bounded by $C\sqrt{K}/N$.

2.3.3 Third estimator

The third estimator $\boldsymbol{X}_{t,\Delta_t}^{N,K}$ is closely related to

$$\boldsymbol{X}_{\infty,\infty}^{N,K} := \mathcal{W}_{\infty,\infty}^{N,K} - \frac{(N-K)\mu}{K}\bar{\ell}_N^K, \quad \textit{where} \quad \mathcal{W}_{\infty,\infty}^{N,K} := \mu\frac{N}{K^2}\sum_{j=1}^{N}\Big(\sum_{i=1}^{K}Q_N(i,j)\Big)^2\ell_N(j).$$

We thus study the convergence of $\boldsymbol{X}_{\infty,\infty}^{N,K}$. The following easy estimate will be sufficient for our task.

Lemma 2.3.7. *When (N,K) tends to (∞,∞), with $K \le N$, we have*

$$\mathbf{1}_{\Omega_{N,K}}\sqrt{K}\Big(\boldsymbol{X}_{\infty,\infty}^{N,K} - \frac{\mu}{(1-\Lambda p)^3}\Big) \longrightarrow 0$$

in probability.

Proof. By [26, lemma 4.19], we have:

$$\sqrt{K}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\boldsymbol{X}^{N,K}_{\infty,\infty}-\frac{\mu}{(1-\Lambda p)^3}\Big|\Big]\leq\frac{C}{\sqrt{K}}.$$

which includes the result of the statement.

We will also need the following estimate, asserting that, setting $A^{N,K}_{\infty,\infty}:=\sum_{i=1}^N\Big(c^K_N(i)\Big)^2\ell_N(i)$, it holds true that $\frac{(A^{N,K}_{\infty,\infty})^2}{2K^2}$ is close to $\frac{1}{2}(\frac{N-K}{N(1-\Lambda p)}+\frac{K}{N(1-\Lambda p)^3})^2$.

Lemma 2.3.8. *When (N,K) tends to (∞,∞), with $K\leq N$ and in the regime $\lim_{N,K\to\infty}\frac{K}{N}=\gamma\in[0,1]$, we have*

$$\lim\frac{A^{N,K}_{\infty,\infty}}{K}=\frac{1-\gamma}{(1-\Lambda p)}+\frac{\gamma}{(1-\Lambda p)^3}$$

in probability.

Proof. We have $\mathcal{W}^{N,K}_{\infty,\infty}=(\mu N/K^2)A^{N,K}_{\infty,\infty}$, so that

$$\frac{A^{N,K}_{\infty,\infty}}{K}=\frac{K}{\mu N}\boldsymbol{X}^{N,K}_{\infty,\infty}+\frac{N-K}{N}\bar{\ell}^K_N.$$

Since $\bar{\ell}^K_N\to\frac{1}{1-\Lambda p}$ by [26, Lemma 4.9] and since $\boldsymbol{X}^{N,K}_{\infty,\infty}\to\frac{\mu}{(1-\Lambda p)^3}$ by Lemma 2.3.7, the conclusion follows.

2.4 The limit theorems for the first and the second estimators

Since the dominating error term cannot come from the first estimator, we only need to recall the following result, which is an immediate consequence of [26, Lemma 6.3].

Lemma 2.4.1. *Assume $H(q)$ for some $q\geq1$, in the regime $\frac{K}{t^{2q}}\to0$, we have:*

$$\lim_{t,N,K\to\infty}\mathbf{1}_{\Omega_{N,K}}\sqrt{K}\mathbb{E}_\theta\Big[\Big|\varepsilon^{N,K}_t-\mu\bar{\ell}^K_N\Big|\Big]=0$$

almost surely.

Recall the definition of $\mathcal{V}^{N,K}_t$, see Section 2.1.5. The main result of this section is the following limit theorem.

Theorem 2.4.2. *Assume $H(q)$ for some $q>1$. When $(N,K,t)\to(\infty,\infty,\infty)$ in the regime where $\frac{t\sqrt{K}}{N}(\frac{N}{t^q}+\sqrt{\frac{N}{Kt}})+Ne^{-c_{p,\Lambda}K}\to0$ we have*

$$\mathbf{1}_{\Omega_{N,K}}\frac{t\sqrt{K}}{N}(\mathcal{V}^{N,K}_t-\mathcal{V}^{N,K}_\infty)\xrightarrow{d}\mathcal{N}\Big(0,\frac{2\mu^2}{(1-\Lambda p)^2}\Big).$$

We split $\mathcal{V}^{N,K}_t-\mathcal{V}^{N,K}_\infty=\Delta^{N,K,1}_t+\Delta^{N,K,2}_t+\Delta^{N,K,3}_t$, where

$$\Delta^{N,K,1}_t=\frac{N}{K}\Big\{\sum_{i=1}^K\Big[\frac{Z^{i,N}_{2t}-Z^{i,N}_t}{t}-\varepsilon^{N,K}_t\Big]^2-\sum_{i=1}^K\Big[\frac{Z^{i,N}_{2t}-Z^{i,N}_t}{t}-\mu\bar{\ell}^K_N\Big]^2\Big\},$$

$$\Delta^{N,K,2}_t=\frac{N}{K}\Big\{\sum_{i=1}^K\Big[\frac{Z^{i,N}_{2t}-Z^{i,N}_t}{t}-\mu\ell_N(i)\Big]^2-\frac{K}{t}\varepsilon^{N,K}_t\Big\},$$

$$\Delta^{N,K,3}_t=2\frac{N}{K}\sum_{i=1}^K\Big[\frac{Z^{i,N}_{2t}-Z^{i,N}_t}{t}-\mu\ell_N(i)\Big]\Big[\mu\ell_N(i)-\mu\bar{\ell}^K_N\Big].$$

We also write $\Delta_t^{N,K,2} = \Delta_t^{N,K,21} + \Delta_t^{N,K,22} + \Delta_t^{N,K,23}$, where

$$\Delta_t^{N,K,21} = \frac{N}{K}\Big\{\sum_{i=1}^K \Big[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \frac{\mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]}{t}\Big]^2 - \frac{K}{t}\varepsilon_t^{N,K}\Big\},$$

$$\Delta_t^{N,K,22} = \frac{N}{K}\sum_{i=1}^K \Big\{\frac{\mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]}{t} - \mu\ell_N(i)\Big\}^2,$$

$$\Delta_t^{N,K,23} = 2\frac{N}{K}\sum_{i=1}^K \Big[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \frac{\mathbb{E}_\theta(Z_{2t}^{i,N} - Z_t^{i,N})}{t}\Big]\Big[\frac{\mathbb{E}_\theta(Z_{2t}^{i,N} - Z_t^{i,N})}{t} - \mu\ell_N(i)\Big].$$

We next write $\Delta_t^{N,K,21} = \Delta_t^{N,K,211} + \Delta_t^{N,K,212} + \Delta_t^{N,K,213}$, where

$$\Delta_t^{N,K,211} = \frac{N}{K}\sum_{i=1}^K \Big\{\frac{(U_{2t}^{i,N} - U_t^{i,N})^2}{t^2} - \frac{\mathbb{E}_\theta[(U_{2t}^{i,N} - U_t^{i,N})^2]}{t^2}\Big\},$$

$$\Delta_t^{N,K,212} = \frac{N}{K}\Big\{\sum_{i=1}^K \frac{\mathbb{E}_\theta[(U_{2t}^{i,N} - U_t^{i,N})^2]}{t^2} - \mathbb{E}_\theta[\varepsilon_t^{N,K}]\frac{K}{t}\Big\},$$

$$\Delta_t^{N,K,213} = \frac{N}{K}\Big\{\mathbb{E}_\theta[\varepsilon_t^{N,K}]\frac{K}{t} - \varepsilon_t^{N,K}\frac{K}{t}\Big\}.$$

At last, we write $\Delta_t^{N,K,3} = \Delta_t^{N,K,31} + \Delta_t^{N,K,32}$, where

$$\Delta_t^{N,K,31} = 2\frac{N}{K}\sum_{i=1}^K \Big[\frac{Z_{2t}^{i,N} - Z_t^{i,N}}{t} - \frac{\mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]}{t}\Big]\Big[\mu\ell_N(i) - \mu\bar{\ell}_N^K\Big],$$

$$\Delta_t^{N,K,32} = 2\sum_{i=1}^K \Big[\frac{\mathbb{E}_\theta[Z_{2t}^{i,N} - Z_t^{i,N}]}{t} - \mu\ell_N(i)\Big]\Big[\mu\ell_N(i) - \mu\bar{\ell}_N^K\Big].$$

We next summarizes some estimates of [26].

Lemma 2.4.3. *Assume $H(q)$ for some $q \geq 1$. Then, on the set $\Omega_{N,K}$, for all $t \geq 1$, a.s.,*

(i) $\mathbb{E}_\theta[|\Delta_t^{N,K,1}|] \leq C(Nt^{-2q} + NK^{-1}t^{-1})$,

(ii) $\mathbb{E}_\theta[|\Delta_t^{N,K,22}|] \leq CNt^{-2q}$,

(iii) $\mathbb{E}_\theta[|\Delta_t^{N,K,23}|] \leq CNt^{-q}$,

(iv) $\mathbb{E}_\theta[|\Delta_t^{N,K,213}|] \leq CNK^{-\frac{1}{2}}t^{-\frac{3}{2}}$,

(v) $\mathbb{E}_\theta[|\Delta_t^{N,K,32}|] \leq CNt^{-q}$.

(vi) $\mathbb{E}_\theta[|\Delta_t^{N,K,212}|] \leq Ct^{-1}$.

(vii) $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}|\Delta_t^{N,K,31}|] \leq CN^{1/2}K^{-1/2}t^{-1/2}$.

Proof. Points (i)-(v) can be found in [26, Lemma 7.2]. For point (vi), see [26, Lemma 7.3]. Finally, for (vii), we first use [26, Lemma 7.5] which tells us that

$$\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}\mathbb{E}_\theta[|\Delta_t^{N,K,31}|] \leq C\frac{N}{K\sqrt{t}}\Big[\sum_{i=1}^K(\ell_N(i) - \bar{\ell}_N^K)^2\Big]^{1/2} = C\frac{N}{K\sqrt{t}}||\boldsymbol{x}_N^K||_2.$$

To conclude, we use [26, Lemmas 4.14, 7.5] which implies that $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}\cap\mathcal{A}_N}||\boldsymbol{x}_N^K||_2] \leq CK^{1/2}N^{-1/2}$.

One immediately deduce the following result (recall that $\Delta_t^{N,K,3} = \Delta_t^{N,K,31} + \Delta_t^{N,K,32}$).

Corollary 2.4.4. *Assume $H(q)$ for some $q > 1$. When $(N, K, t) \to (\infty, \infty, \infty)$ in the regime $\frac{t\sqrt{K}}{N}(\frac{N}{t^q} + \sqrt{\frac{N}{Kt}}) \to 0$,*

$$\lim \frac{t\sqrt{K}}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\Delta_t^{N,K,1} + \Delta_t^{N,K,212} + \Delta_t^{N,K,213} + \Delta_t^{N,K,22} + \Delta_t^{N,K,23} + \Delta_t^{N,K,3}\Big|\Big] = 0.$$

Next, we study the limit behaviour of the intensity $\lambda_t^{i,N}$, recall (2.1).

Lemma 2.4.5. *Assume $H(q)$ for some $q \geq 1$. Then on the event $\Omega_{N,K}$, we have*

$$\sup_{t\in\mathbb{R}_+}\max_{i=1,\dots,N}\mathbb{E}_\theta[\lambda_t^{i,N}] \leq \frac{\mu}{1-a}$$

Proof. We directly find, observing that $\mathbb{E}_\theta[Z_t^{i,N}] = \int_0^t \mathbb{E}_\theta[\lambda_s^{i,N}]ds$, that

$$\max_{i=1,\dots,N}\mathbb{E}_\theta[\lambda_t^{i,N}] = \mu + \max_{i=1,\dots,N}\Big\{\sum_{j=1}^N A_N(i,j)\int_0^t \phi(t-s)\mathbb{E}_\theta[\lambda_s^{j,N}]ds\Big\}.$$

We define $a_N(t) = \sup_{0\leq s\leq t}\max_{i=1,\dots,N}\mathbb{E}_\theta[\lambda_s^{i,N}]$. By definition, on $\Omega_{N,K}$ we have the bound $\Lambda\max_{i=1,\dots,N}\{\sum_{j=1}^N A_N(i,j)\} \leq a < 1$, whence, since $\Lambda = \int_0^\infty \phi(s)ds$,

$$a_N(t) \leq \mu + a_N(t)a,$$

which completes the result.

Lemma 2.4.6. *Assume $H(q)$ for some $q \geq 1$. On $\Omega_{N,K}$, for all $t \geq 1$, we a.s. have*

$$\frac{1}{K}\sum_{i=1}^K \mathbb{E}_\theta\Big[\Big(\lambda_t^{i,N} - \mu\ell_N(i)\Big)^2\Big]^{\frac{1}{2}} \leq \frac{C}{t^q} + \frac{C}{\sqrt{N}}.$$

Proof. By definition, we have $\boldsymbol{\ell}_N = Q_N\mathbf{1}_N = (I - \Lambda A_N)^{-1}\mathbf{1}_N$, so that $\boldsymbol{\ell}_N = \mathbf{1}_N + \Lambda A_N\boldsymbol{\ell}_N$. So, writing $\Lambda = \int_0^t \phi(t-s)ds + \int_t^\infty \phi(s)ds$, we find

$$\lambda_t^{i,N} - \mu\ell_N(i) = \frac{1}{N}\sum_{j=1}^N \theta_{ij}\Big(\int_0^t \phi(t-s)dZ_s^j - \mu\ell_N(j)\int_0^t \phi(t-s)ds\Big) - \frac{\mu}{N}\sum_{j=1}^N \theta_{ij}\ell_N(j)\int_t^\infty \phi(s)ds.$$

This implies, on $\Omega_{N,K}$, that

$$\begin{aligned}\mathbb{E}_\theta\Big[\Big(\lambda_t^{i,N} - \mu\ell_N(i)\Big)^2\Big]^{\frac{1}{2}} \leq& \mathbb{E}_\theta\Big[\Big(\frac{1}{N}\sum_{j=1}^N \theta_{ij}\Big(\int_0^t \phi(t-s)dZ_s^j - \mu\ell_N(j)\int_0^t \phi(t-s)ds\Big)\Big)^2\Big]^{\frac{1}{2}}\\ &+ \mu\mathbb{E}_\theta\Big[\Big(\frac{1}{N}\sum_{j=1}^N \theta_{ij}\ell_N(j)\int_t^\infty \phi(s)ds\Big)^2\Big]^{\frac{1}{2}}.\end{aligned}$$

Recalling that $M_t^{i,N} = Z_t^{i,N} - \int_0^t \lambda_s^{i,N}ds$ and using that $\ell_N(j)$ is uniformly bounded on $\Omega_{N,K}$,

$$\begin{aligned}\mathbb{E}_\theta\Big[\Big(\lambda_t^{i,N} - \mu\ell_N(i)\Big)^2\Big]^{\frac{1}{2}} \leq& \mathbb{E}_\theta\Big[\Big(\frac{1}{N}\sum_{j=1}^N \theta_{ij}\int_0^t \phi(t-s)dM_s^j\Big)^2\Big]^{\frac{1}{2}}\\ &+ \mathbb{E}_\theta\Big[\Big(\frac{1}{N}\sum_{j=1}^N \theta_{ij}\int_0^t \phi(t-s)|\lambda_s^{j,N} - \mu\ell(j)_N|ds\Big)^2\Big]^{\frac{1}{2}} + C\int_t^\infty \phi(s)ds.\end{aligned}$$

By Lemma 2.4.5 and assumption $H(q)$, using (2.8),

$$\begin{aligned}\mathbb{E}_\theta\Big[\Big(\frac{1}{N}\sum_{j=1}^N\theta_{ij}\int_0^t\phi(t-s)dM_s^j\Big)^2\Big]^{\frac12}=&\frac{1}{N}\mathbb{E}_\theta\Big[\sum_{j=1}^N\int_0^t\Big(\theta_{ij}\phi(t-s)\Big)^2dZ_s^j\Big]^{\frac12}\\=&\frac{1}{N}\mathbb{E}_\theta\Big[\sum_{j=1}^N\int_0^t\Big(\theta_{ij}\phi(t-s)\Big)^2\lambda_s^jds\Big]^{\frac12}\\\le&\frac{1}{\sqrt{N}}\Big[\int_0^t\Big(\phi(t-s)\Big)^2\max_{j=1,\dots,N}\mathbb{E}_\theta[\lambda_s^j]ds\Big]^{\frac12}\\\le&\frac{C}{\sqrt{N}}\Big(\int_0^t(\phi(t-s))^2ds\Big)^{\frac12}\le\frac{C}{\sqrt{N}}.\end{aligned}$$

Defining $F_t^{K,N}:=\frac{1}{K}\sum_{i=1}^K\mathbb{E}_\theta[(\lambda_t^{i,N}-\mu\ell_N(i))^2]^{\frac12}$, we thus have, by Minkowski's inequality,

$$\begin{aligned}F_t^{K,N}\le&\frac{1}{KN}\sum_{j=1}^N\sum_{i=1}^K\theta_{ij}\int_0^t\phi(t-s)\mathbb{E}_\theta\Big[\Big|\lambda_s^{j,N}-\mu\ell_N(j)\Big|^2\Big]^{\frac12}ds+C\int_t^\infty\phi(s)ds+\frac{C}{\sqrt{N}}\\\le&\int_0^t\frac{N}{K}|||I_KA_N|||_1\phi(t-s)F_s^{N,N}ds+C\int_t^\infty\phi(s)ds+\frac{C}{\sqrt{N}}\\\le&\int_0^t\frac{a}{\Lambda}\phi(t-s)F_s^{N,N}ds+C\int_t^\infty\phi(s)ds+\frac{C}{\sqrt{N}},\end{aligned}$$

because $N|||I_KA_N|||_1=\max_{j=1,\dots,N}\sum_{i=1}^K\theta_{ij}$ and $\frac{N}{K}|||I_KA_N|||_1\le a/\Lambda$ on $\Omega_{N,K}$.

Defining $g_N(t):=C\int_t^\infty\phi(s)ds+\frac{C}{\sqrt{N}}$, we conclude that on $\Omega_{N,K}$, for all $K=1,\dots,N$,

$$F_t^{K,N}\le\int_0^t\frac{a}{\Lambda}\phi(t-s)F_s^{N,N}ds+g_N(t).\tag{2.9}$$

Since $\int_0^\infty(1+s^q)\phi(s)ds<\infty$ from $H(q)$, we have $g_N(t)\le C(\frac{1}{t^q}\wedge1)+CN^{-1/2}$. Moreover, by Lemma 2.4.5, $F_t^{N,N}\le C$, so that $\int_0^t(\frac{a}{\Lambda})^n\phi^{*n}(t-s)F_s^{N,N}ds\le Ca^n\to0$ as $n\to\infty$. Hence, iterating (2.9) (using it once with some fixed $K\in\{1,\dots,N\}$ and then always with $K=N$), one concludes that on $\Omega_{N,K}$,

$$\begin{aligned}F_t^{N,N}\le&\sum_{n\ge1}\int_0^t\Big(\frac{a}{\Lambda}\Big)^n\phi^{*n}(t-s)g_N(s)ds+g_N(t)\\\le&\sum_{n\ge1}\int_0^{\frac{t}{2}}\Big(\frac{a}{\Lambda}\Big)^n\phi^{*n}(t-s)g_N(s)ds+\sum_{n\ge1}\int_{\frac{t}{2}}^t\Big(\frac{a}{\Lambda}\Big)^n\phi^{*n}(t-s)g_N(s)ds+g_N(t)\\\le&C\sum_{n\ge1}\int_{\frac{t}{2}}^t\Big(\frac{a}{\Lambda}\Big)^n\phi^{*n}(s)ds+g_N\Big(\frac{t}{2}\Big)\sum_{n\ge1}\int_0^\infty\Big(\frac{a}{\Lambda}\Big)^n\phi^{*n}(s)ds+g_N(t),\end{aligned}$$

because g_N is non-increasing and bounded. Recalling that $\int_0^\infty\phi^{*n}(s)ds=\Lambda^n$ and, see [14, Proof of Lemma 15-(ii)], that

$$\int_r^\infty\phi^{\star n}(u)du\le Cn^q\Lambda^nr^{-q},$$

we conclude that (since $a\in(0,1)$)

$$F_t^{N,N}\le C\Big(\frac{t}{2}\Big)^{-q}\sum_{n\ge1}n^qa^n+g_N\Big(\frac{t}{2}\Big)\frac{a}{1-a}+g_N(t)\le\frac{C}{t^q}+\frac{C}{\sqrt{N}}.$$

This completes the proof.

Lemma 2.4.7. *For all $t \geq s+1 \geq 1$, on $\Omega_{N,K}$, we have a.s.,*

$$\max_{i=1,\dots,N} \mathbb{E}_\theta[(U_t^{i,N} - U_s^{i,N})^4] \leq (t-s)^2 \quad \textit{and} \quad \max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^4] \leq C(t-s)^4.$$

Proof. Recalling (2.7), we may write

$$U_t^{i,N} = \sum_{n\geq 0} \int_0^t \phi^{\star n}(t-s) \sum_{j=1}^N A_N^n(i,j) M_s^{j,N} ds.$$

Hence, by the Minkowski inequality, we see that on $\Omega_{N,K}$, we have

$$\begin{aligned}\mathbb{E}_\theta[(U_t^{i,N} - U_s^{i,N})^4]^{\frac{1}{4}} \leq & \mathbb{E}_\theta[(M_t^{i,N} - M_s^{i,N})^4]^{\frac{1}{4}} \\ & + \sum_{n\geq 1} \int_0^t \Big(\phi^{\star n}(t-u) - \phi^{\star n}(s-u)\Big) \mathbb{E}_\theta\Big[\Big(\sum_{j=1}^N A_N^n(i,j) M_u^{j,N}\Big)^4\Big]^{\frac{1}{4}} du.\end{aligned}$$

By [14, Lemma 16 (iii)], we already know that, on $\Omega_{N,K}$, $\max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^2] \leq C(t-s)^2$. For the first term ($n=0$), we use (2.8) and Burkholder's inequality:

$$\mathbb{E}_\theta[(M_t^{i,N} - M_s^{i,N})^4] \leq C\mathbb{E}_\theta\Big[(Z_t^{i,N} - Z_s^{i,N})^2\Big] \leq C(t-s)^2.$$

For the second term ($n \geq 1$), we use again (2.8) and Burkholder's inequality and we get

$$\begin{aligned}\mathbb{E}_\theta\Big[\Big(\sum_{j=1}^N A_N^n(i,j) M_u^{j,N}\Big)^4\Big] \leq & C\mathbb{E}_\theta\Big[\Big([\sum_{j=1}^N A_N^n(i,j) M^{j,N}, \sum_{j=1}^N A_N^n(i,j) M^{j,N}]_u\Big)^2\Big] \\ \leq & C\mathbb{E}_\theta\Big[\Big(\sum_{j=1}^N (A_N^n(i,j))^2 Z_u^{j,N}\Big)^2\Big] \\ = & C \sum_{j,j'=1}^N (A_N^n(i,j))^2 (A_N^n(i,j'))^2 \mathbb{E}_\theta[Z_u^{j,N} Z_u^{j',N}] \\ \leq & C\Big(\sum_{j=1}^N (A_N^n(i,j))^2\Big)^2 u^2 \\ \leq & C\Big(\sum_{j=1}^N A_N^n(i,j)\Big)^4 u^2 \leq C|||A_N|||_1^{4n} u^2.\end{aligned}$$

It implies that

$$\begin{aligned}& \sum_{n\geq 1} \int_0^\infty \Big(\phi^{\star n}(t-u) - \phi^{\star n}(s-u)\Big) \mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K \sum_{j=1}^N A_N^n(i,j) M_u^{j,N}\Big)^4\Big]^{\frac{1}{4}} du \\ \leq & C \sum_{n\geq 1} |||A_N|||_1^n \int_0^t \sqrt{u}\Big(\phi^{\star n}(t-u) - \phi^{\star n}(s-u)\Big) du \\ \leq & C(t-s)^{1/2} \sum_{n\geq 1} \Lambda^n |||A_N|||_1^n \leq (t-s)^{1/2}\end{aligned}$$

since we showed that $\int_0^t \sqrt{u}(\phi^{\star n}(t-u) - \phi^{\star n}(s-u))du \leq 2\Lambda^n \sqrt{t-s}$ in the proof of Lemma 2.2.3 and since, as usual, on $\Omega_{N,K}$, we have $\Lambda|||A_N|||_1 < a < 1$. This completes the first part of this Lemma.

For the second part, we recall from Lemma 2.2.3 (ii) with $K = N$ and $r = \infty$ that we have $\max_{i=1,\dots,N} \mathbb{E}_\theta[Z_t^{i,N} - Z_s^{i,N}] \le C(t-s)$ on $\Omega_{N,N} \supset \Omega_{N,K}$, whence

$$\max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_t^{i,N} - Z_s^{i,N})^4] \le 8\Big\{ \max_{i=1,\dots,N} \mathbb{E}_\theta[Z_t^{i,N} - Z_s^{i,N}]^4 + \max_{i=1,\dots,N} \mathbb{E}_\theta[(U_t^{i,N} - U_s^{i,N})^4] \Big\} \le C(t-s)^4$$

as desired.

Lemma 2.4.8. *Assume $H(q)$ for some $q \ge 1$. As $(t, N, K) \to (\infty, \infty, \infty)$, in the regime $Ne^{-c_{p,\Lambda} K} \to 0$,*

$$\mathbf{1}_{\Omega_{N,K}} \frac{t\sqrt{K}}{N} \Delta_t^{N,K,211} = \mathbf{1}_{\Omega_{N,K}} \frac{1}{t\sqrt{K}} \sum_{i=1}^K \Big\{ (U_{2t}^{i,N} - U_t^{i,N})^2 - \mathbb{E}_\theta[(U_{2t}^{i,N} - U_t^{i,N})^2] \Big\} \xrightarrow{d} \mathcal{N}\Big(0, \frac{2\mu^2}{(1-\Lambda p)^2}\Big).$$

Proof. We work on the set $\Omega_{N,K}$. Recalling (2.7), we have

$$U_t^{i,N} = \sum_{n \ge 0} \sum_{j=1}^N \int_0^t \phi^{*n}(t-s) A_N^n(i,j) M_s^{j,N} ds$$

and we write

$$(U_{2t}^{i,N} - U_t^{i,N})^2 = (M_{2t}^{i,N} - M_t^{i,N})^2 + 2T_t^{i,N}(M_{2t}^{i,N} - M_t^{i,N}) + (T_t^{i,N})^2,$$

where

$$T_t^{i,N} = \sum_{n \ge 1} \sum_{j=1}^N \int_0^{2t} \phi^{*n}(2t-s) A_N^n(i,j) M_s^{j,N} ds - \sum_{n \ge 1} \sum_{j=1}^N \int_0^t \phi^{*n}(t-s) A_N^n(i,j) M_s^{j,N} ds.$$

We treat these terms one by one and set $\phi(s) = 0$ for $s \le 0$ as usual.

Step 1. Here we verify that

$$\lim \mathbf{1}_{\Omega_{N,K}} \frac{1}{t\sqrt{K}} \mathbb{E}_\theta\Big[\sum_{i=1}^K \Big| (T_t^{i,N})^2 - \mathbb{E}_\theta[(T_t^{i,N})^2] \Big| \Big] = 0.$$

We will check that for all $i = 1, \dots, K$, we have $\mathbb{E}_\theta[(T_t^{i,N})^2] \le Ct/N$, which of course suffices.

Setting $\beta_n(s,t,r) = \phi^{\star n}(t-r) - \phi^{\star n}(s-r)$, we may write

$$T_t^{i,N} = \sum_{n \ge 1} \int_0^{2t} \beta_n(t, 2t, u) \sum_{j=1}^N A_N^n(i,j) M_u^{j,N} du. \tag{2.10}$$

Hence

$$\mathbb{E}_\theta[(T_t^{i,N})^2] = \sum_{m,n \ge 1} \int_0^{2t} \int_0^{2t} \beta_m(t,2t,u) \beta_n(t,2t,v) \sum_{j,k=1}^N A_N^m(i,j) A_N^n(i,k) \mathbb{E}_\theta[M_u^{j,N} M_v^{k,N}] dv du.$$

It is obvious that $\int_0^{2t} \beta_n(t,2t,u) \le 2\Lambda^n$ for any $n \ge 0$. Using (2.8) and that $M^{j,N}$ and $M^{k,N}$ are martingales, we see that $\mathbb{E}_\theta[M_u^{j,N} M_v^{k,N}] = \mathbf{1}_{\{j=k\}} \mathbb{E}_\theta[Z_{u \wedge v}^{j,N}] \le C(u \wedge v)$ (on $\Omega_{N,K}$, due to Lemma

2.2.3-(i) with $r = \infty$), whence

$$\begin{aligned}\mathbb{E}_\theta[(T_t^{i,N})^2] &= \sum_{m,n\geq 1}\int_0^{2t}\int_0^{2t}\beta_m(t,2t,u)\beta_n(t,2t,v)\sum_{j,k=1}^N A_N^m(i,j)A_N^n(i,k)\mathbb{E}_\theta[M_u^{j,N}M_v^{k,N}]dvdu\\ &\leq Ct\sum_{m,n\geq 1}\int_0^{2t}\int_0^{2t}\beta_m(t,2t,u)\beta_n(t,2t,v)dvdu\sum_{j=1}^N A_N^m(i,j)A_N^n(i,j)\\ &\leq Ct\sum_{m,n\geq 1}\Lambda^{m+n}\sum_{j=1}^N A_N^m(i,j)A_N^n(i,j)\\ &\leq Ct\sum_{j=1}^N(Q_N(i,j)-\mathbf{1}_{\{i=j\}})^2 \leq \frac{Ct}{N}\end{aligned}$$

The reason of the last step comes from the fact that by [14, Equation (8)], on Ω_N^1, we have $\mathbf{1}_{\{i=j\}} \leq Q_N(i,j) \leq \mathbf{1}_{\{i=j\}} + \Lambda C N^{-1}$.

Step 2. Here we verify that

$$\lim \mathbf{1}_{\Omega_{N,K}}\frac{1}{t\sqrt{K}}\mathbb{E}_\theta\Big[\Big|\sum_{i=1}^K\Big(T_t^{i,N}(M_{2t}^{i,N}-M_t^{i,N})-\mathbb{E}_\theta[T_t^{i,N}(M_{2t}^{i,N}-M_t^{i,N})]\Big)\Big|\Big]=0.$$

Actually, this will follows from the estimate (on $\Omega_{N,K}$)

$$x := \mathbb{V}ar_\theta\Big[\sum_{i=1}^K(T_t^{i,N}(M_{2t}^{i,N}-M_t^{i,N}))\Big] \leq C\frac{Kt^2}{N}$$

that we now verify. We start from

$$\begin{aligned}x =&\mathbb{E}_\theta\Big[\sum_{i,j=1}^K\Big(T_t^{i,N}(M_{2t}^{i,N}-M_t^{i,N}))-\mathbb{E}_\theta[T_t^{i,N}(M_{2t}^{i,N}-M_t^{i,N}))]\Big)\\ &\Big(T_t^{j,N}(M_{2t}^{j,N}-M_t^{j,N}))-\mathbb{E}_\theta[T_t^{j,N}(M_{2t}^{j,N}-M_t^{j,N}))]\Big)\Big].\end{aligned}$$

Recalling (2.10) and setting $\alpha_N(u,t,i,j) = \sum_{n\geq 1}\beta_n(t,2t,u)A_N^n(i,j)$,

$$\begin{aligned}x \leq \sum_{i,j=1}^K\int_0^{2t}\int_0^{2t}\sum_{k,m=1}^N &|\alpha_N(s,t,i,k)\alpha_N(u,t,j,m)|\\ &|\mathbb{C}ov_\theta[(M_{2t}^{i,N}-M_t^{i,N})M_s^{k,N},(M_{2t}^{j,N}-M_t^{j,N})M_u^{m,N}]|dsdu.\end{aligned}$$

But

$$\int_0^{2t}|\alpha_N(s,t,i,k)|ds \leq \sum_{n\geq 1}A_N^n(i,k)\int_0^{2t}|\beta_n(t,2t,s)|ds \leq 2\sum_{n\geq 1}A_N^n(i,k)\Lambda^n \leq 2(Q_N(i,k)-\mathbf{1}_{\{i=k\}})$$

which is bounded by C/N, as seen a few lines above.

And by [14, Lemma 22], we already know that, for s and u in $[0,2t]$, still on $\Omega_{N,K}$,

$$|\mathbb{C}ov_\theta[(M_{2t}^{i,N}-M_t^{i,N})M_s^{k,N},(M_{2t}^{j,N}-M_t^{j,N})M_u^{m,N}]| \leq C(\mathbf{1}_{\#\{k,i,j,m\}=3}N^{-2}t+\mathbf{1}_{\#\{k,i,j,m\}\leq 2}t^2).$$

Hence we conclude that

$$x \leq \frac{C}{N^2}\sum_{i,j=1}^K\sum_{k,m=1}^N(\mathbf{1}_{\#\{k,i,j,m\}=3}N^{-2}t+\mathbf{1}_{\#\{k,i,j,m\}\leq 2}t^2) \leq \frac{C}{N^2}\Big(N^2K\times N^{-2}t+NK\times t^2\Big),$$

which is bounded by CKt^2/N as desired.

Step 3. It only remains to show that

$$\mathbf{1}_{\Omega_{N,K}}\frac{1}{t\sqrt{K}}\Big[\sum_{i=1}^{K}(M_{2t}^{i,N}-M_t^{i,N})^2-\sum_{i=1}^{K}\mathbb{E}_\theta[(M_{2t}^{i,N}-M_t^{i,N})^2]\Big]$$

converges to some Gaussian random variable with variance $2\mu^2/(1-\Lambda p)^2$. By Ito's formula, we have

$$(M_{2t}^{i,N}-M_t^{i,N})^2=2\int_t^{2t}(M_{s-}^{i,N}-M_t^{i,N})dM_s^{i,N}+Z_{2t}^{i,N}-Z_t^{i,N}.$$

By [26, Lemma 6.2-(ii)], we know that $\mathbf{1}_{\Omega_{N,K}}\mathbb{E}_\theta[|\bar{U}_t^{N,K}|^2]\leq\frac{Ct}{K}$. This directly implies that $\mathbf{1}_{\Omega_{N,K}}\frac{1}{t\sqrt{K}}\sum_{i=1}^{K}\{(Z_{2t}^{i,N}-Z_t^{i,N})-\mathbb{E}_\theta[Z_{2t}^{i,N}-Z_t^{i,N}]\}=\mathbf{1}_{\Omega_{N,K}}\frac{\sqrt{K}}{t}[\bar{U}_{2t}^{N,K}-\bar{U}_t^{N,K}|\to 0$.

We introduce $N_u^{t,i,N}:=\int_t^{t+\sqrt{u}t}(M_{s-}^{i,N}-M_t^{i,N})dM_s^{i,N}$. We observe that for $t\geq 0$ fixed, $(N_u^{t,i,N})_{u\in[0,1]}$ is a martingale in the filtration $\mathcal{F}^N_{t+\sqrt{u}t}$. We will prove that, as $(t,N,K)\to(\infty,\infty,\infty)$,

$$\Big(\frac{1}{t\sqrt{K}}\sum_{i=1}^{K}N_u^{t,i,N}\Big)_{u\in[0,1]}\overset{(d)}{\to}\Big(\frac{\mu}{\sqrt{2}(1-\Lambda p)}B_u\Big)_{u\in[0,1]},\tag{2.11}$$

where $(B_u)_{u\in[0,1]}$ is a Brownian motion. Since we have $\frac{1}{t\sqrt{K}}\sum_{i=1}^{K}\int_t^{2t}(M_{s-}^{i,N}-M_t^{i,N})dM_s^{i,N}=\frac{1}{t\sqrt{K}}\sum_{i=1}^{K}N_1^{t,i,N}$, this will complete the proof. To prove (2.11), by Jacod-Shiryaev [23, Theorem VIII-3-8], it suffices to verify that, as $(t,N,K)\to(\infty,\infty,\infty)$,

(a) $[\frac{1}{t\sqrt{K}}\sum_{i=1}^{K}N_u^{t,i,N}]_u\to\frac{\mu^2}{2(1-\Lambda p)^2}u$ in probability, for all $u\in[0,1]$ fixed,

(b) $\sup_{u\in[0,1]}\frac{1}{t\sqrt{K}}\sum_{i=1}^{K}|N_u^{t,i,N}-N_{u-}^{t,i,N}|\to 0$ in probability.

Point (b) is not difficult: using that the Poisson measures are independant in (2.1) and that the jumps of $M^{i,N}$ are always equal to 1, one has, using (2.8),

$$\begin{aligned}\frac{1}{t\sqrt{K}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\sup_{u\in[0,1]}\sum_{i=1}^{K}|N_u^{t,i,N}-N_{u-}^{t,i,N}|\Big]\leq&\frac{C}{t\sqrt{K}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\sup_{u\in[0,1]}\max_{i=1,\dots,K}\Big|M_{t+t\sqrt{u}}^{i,N}-M_t^{i,N}\Big|\Big]\\\leq&\frac{C}{t\sqrt{K}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\sup_{u\in[0,1]}\Big|\sum_{i=1}^{K}(M_{t+t\sqrt{u}}^{i,N}-M_t^{i,N})^2\Big|^{\frac12}\Big]\\\leq&\frac{C}{t\sqrt{K}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\sup_{u\in[0,1]}\sum_{i=1}^{K}(M_{t+t\sqrt{u}}^{i,N}-M_t^{i,N})^2\Big]^{\frac12}\\\leq&\frac{C}{t\sqrt{K}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\sum_{i=1}^{K}(Z_{2t}^{i,N}-Z_t^{i,N})\Big|\Big]^{\frac12}\leq\frac{C}{\sqrt{t}}.\end{aligned}$$

Concerning point (a), we fix u and write,

$$\Big[\frac{1}{t\sqrt{K}}\sum_{i=1}^{K}N_\cdot^{t,i,N}\Big]_u=\frac{1}{t^2K}\sum_{i=1}^{K}\int_t^{t+\sqrt{u}t}(M_{s-}^{i,N}-M_t^{i,N})^2dZ_s^{i,N}=I^1_{t,N,K,u}+I^2_{t,N,K,u}+I^3_{t,N,K,u},$$

where, recalling that $Z_t^{i,N} = M_t^{i,N} + \int_0^t \lambda_s^{i,N} ds$,

$$I^1_{t,N,K,u} = \frac{1}{t^2K}\sum_{i=1}^K \int_t^{t+\sqrt{u}t} (M_{s-}^{i,N} - M_t^{i,N})^2 dM_s^{i,N},$$
$$I^2_{t,N,K,u} = \frac{1}{t^2K}\sum_{i=1}^K \int_t^{t+\sqrt{u}t} (M_s^{i,N} - M_t^{i,N})^2 (\lambda_s^{i,N} - \mu\ell_N(i)) ds,$$
$$I^3_{t,N,K,u} = \frac{1}{t^2K}\sum_{i=1}^K \mu\ell_N(i) \int_t^{t+\sqrt{u}t} (M_s^{i,N} - M_t^{i,N})^2 ds.$$

Step 3.1. Here we verify that $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}} I^1_{t,N,K,u}] \to 0$. By (2.8), we have

$$\begin{aligned}
\mathbb{E}_\theta[(I^1_{t,N,K,u})^2] &= \frac{1}{K^2t^4}\sum_{i=1}^K \mathbb{E}_\theta\Big[\int_t^{t+\sqrt{u}t} (M_{s-}^{i,N} - M_t^{i,N})^4 dZ_s^{i,N}\Big] \\
&= \frac{1}{K^2t^4}\sum_{i=1}^K \mathbb{E}_\theta\Big[\int_t^{t+\sqrt{u}t} (M_s^{i,N} - M_t^{i,N})^4 \lambda_s^{i,N} ds\Big] \\
&\le \frac{1}{K^2t^4}\sum_{i=1}^K \int_t^{t+\sqrt{u}t} \Big\{\mathbb{E}_\theta[(M_s^{i,N} - M_t^{i,N})^4 |\lambda_s^{i,N} - \mu\ell_N(i)|] + \mu\mathbb{E}_\theta[(M_s^{i,N} - M_t^{i,N})^4]|\ell_N(i)|\Big\} ds.
\end{aligned}$$

Hence, using the Cauchy-Schwarz and Burkholder inequalities,

$$\begin{aligned}
&\mathbb{E}_\theta[(I^1_{t,N,K,u})^2] \\
&\le \frac{1}{K^2t^4}\sum_{i=1}^K \int_t^{t+\sqrt{u}t} \Big\{\mathbb{E}_\theta[(M_s^{i,N} - M_t^{i,N})^8]^{\frac12}\mathbb{E}_\theta[|\lambda_s^{i,N} - \mu\ell_N(i)|^2]^{\frac12} + \mu\mathbb{E}_\theta[(M_s^{i,N} - M_t^{i,N})^4]|\ell_N(i)|\Big\} ds \\
&\le \frac{C}{K^2t^4}\sum_{i=1}^K \int_t^{t+\sqrt{u}t} \Big\{\mathbb{E}_\theta[(Z_s^{i,N} - Z_t^{i,N})^4]^{\frac12}\mathbb{E}_\theta[|\lambda_s^{i,N} - \mu\ell_N(i)|^2]^{\frac12} + \mu\mathbb{E}_\theta[(M_s^{i,N} - M_t^{i,N})^4]|\ell_N(i)|\Big\} ds.
\end{aligned}$$

By Lemma 2.4.7, we know that on $\Omega_{N,K}$, for all $s \ge t$, we have $\max_{i=1,\dots,N} \mathbb{E}_\theta[(M_s^{i,N} - M_t^{i,N})^4] \le C(t-s)^2$, as well as $\max_{i=1,\dots,N} \mathbb{E}_\theta[(Z_s^{i,N} - Z_t^{i,N})^4] \le C(t-s)^4$. Hence, recalling that ℓ_N is bounded on $\Omega_{N,K}$,

$$\mathbb{E}_\theta[(I^1_{t,N,K,u})^2] \le \frac{C}{K^2t^2}\sum_{i=1}^K \int_t^{t+\sqrt{u}t} \Big(1 + \mathbb{E}_\theta[|\lambda_s^{i,N} - \mu\ell_N(i)|^2]^{\frac12}\Big) ds \le \frac{C}{Kt}\Big(\frac{1}{t^q} + \frac{1}{\sqrt{N}}\Big),$$

which ends the step. We used Lemma 2.4.6 for the last inequality.

Step 3.2. Similarly, one verifies that, on $\Omega_{N,K}$,

$$\begin{aligned}
\mathbb{E}_\theta[|I^2_{t,N,K,u}|] &\le \frac{1}{Kt^2}\sum_{i=1}^K \int_t^{t+\sqrt{u}t} \mathbb{E}_\theta[(M_s^{i,N} - M_t^{i,N})^4]^{\frac12}\mathbb{E}_\theta[|\lambda_s^{i,N} - \mu\ell_N(i)|^2]^{\frac12} \\
&\le \frac{C}{Kt}\sum_{i=1}^K \int_t^{2t} \mathbb{E}_\theta[|\lambda_s^{i,N} - \mu\ell_N(i)|^2]^{\frac12} ds \le \frac{C}{t^q} + \frac{C}{\sqrt{N}}.
\end{aligned}$$

Step 3.3. Finally, we have to prove that $I^3_{t,N,K,u} \to \mu^2 u/[2(1-\Lambda p)^2]$ in probability as $(t,N,K) \to (\infty,\infty,\infty)$. Using the Itô formula and (2.8), we write

$$\begin{aligned}
&(M_s^{i,N} - M_t^{i,N})^2 \\
&= 2\int_t^s (M_{r-}^{i,N} - M_t^{i,N}) dM_r^{i,N} + Z_s^{i,N} - Z_t^{i,N} \\
&= 2\int_t^s (M_{r-}^{i,N} - M_t^{i,N}) dM_r^{i,N} + U_s^{i,N} - U_t^{i,N} + \mathbb{E}_\theta[Z_s^{i,N} - Z_t^{i,N} - \mu(s-t)\ell_N(i)] + \mu(s-t)\ell_N(i).
\end{aligned}$$

and $I^3_{t,N,K,u} = I^{3,1}_{t,N,K,u} + I^{3,2}_{t,N,K,u} + I^{3,3}_{t,N,K,u} + I^{3,4}_{t,N,K,u}$, where

$$
\begin{aligned}
I^{3,1}_{t,N,K,u} =& \frac{2}{t^2K}\sum_{i=1}^K \mu\ell_N(i)\int_t^{t+\sqrt{u}t}\int_t^s (M^{i,N}_{r-} - M^{i,N}_t)dM^{i,N}_r ds,\\
I^{3,2}_{t,N,K,u} =& \frac{1}{t^2K}\sum_{i=1}^K \mu\ell_N(i)\int_t^{t+\sqrt{u}t} (U^{i,N}_s - U^{i,N}_t)ds,\\
I^{3,3}_{t,N,K,u} =& \frac{1}{t^2K}\sum_{i=1}^K \mu\ell_N(i)\int_t^{t+\sqrt{u}t} \mathbb{E}_\theta[Z^{i,N}_s - Z^{i,N}_t - \mu(s-t)\ell_N(i)]ds,\\
I^{3,4}_{t,N,K,u} =& \frac{1}{t^2K}\sum_{i=1}^K \mu^2(\ell_N(i))^2 \times \frac{ut^2}{2} = \frac{\mu^2 u}{2K}\sum_{i=1}^K(\ell_N(i))^2.
\end{aligned}
$$

First,

$$
2I^{3,4}_{t,N,K,u} = \mu^2 u(\bar\ell^K_N)^2 + \frac{\mu^2 u}{K}\sum_{i=1}^K(\ell_N(i) - \bar\ell^K_N)^2 = \mu^2 u(\bar\ell^K_N)^2 + \frac{\mu^2 u}{K}||\boldsymbol{x}^K_N||_2^2 = \mu^2 u(\bar\ell^K_N)^2 + \frac{\mu^2 u}{N}\mathcal{V}^{N,K}_\infty
$$

and we immediately deduce from Lemmas 2.3.1 and 2.3.2 that $I^{3,4}_{t,N,K,u}$ tends to $\mu^2 u/[2(1-\Lambda p)^2]$.

For the second term, we recall (2.7) and we write for $s \geq t$,

$$
U^{i,N}_s - U^{i,N}_t = \sum_{n\geq 0}\int_0^s (\phi^{\star n}(s-u) - \phi^{\star n}(t-u))\sum_{j=1}^N A^n_N(i,j)M^{j,N}_u du,
$$

so that, by Minkowski's inequality and separating as usual the terms $n=0$ and $n\geq 1$,

$$
\begin{aligned}
\mathbb{E}_\theta[|I^{3,2}_{t,N,K,u}|^2]^{\frac12} \leq& \frac{C}{t^2K}\int_t^{t+\sqrt{u}t} \mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K \ell_N(i)(U^{i,N}_s - U^{i,N}_t)\Big)^2\Big]^{\frac12} ds\\
\leq& \frac{C}{t^2K}\int_t^{t+\sqrt{u}t} \Big\{\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K \ell_N(i)(M^{i,N}_s - M^{i,N}_t)\Big)^2\Big]^{\frac12}\\
&+ \sum_{n\geq 1}\int_0^s (\phi^{\star n}(s-r) - \phi^{\star n}(t-r))\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K\sum_{j=1}^N \ell_N(i)A^n_N(i,j)M^{j,N}_r\Big)^2\Big]^{\frac12} dr\Big\} ds.
\end{aligned}
$$

By (2.8), we see that on $\Omega_{N,K}$, for all $t \leq s \leq 2t$, we have

$$
\begin{aligned}
\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K \ell_N(i)(M^{i,N}_s - M^{i,N}_t)\Big)^2\Big]^{\frac12} =& \mathbb{E}_\theta\Big[\sum_{i=1}^K(\ell_N(i))^2(Z^{i,N}_s - Z^{i,N}_t)\Big]^{\frac12}\\
=& \Big\{\sum_{i=1}^K(\ell_N(i))^2\mathbb{E}_\theta\Big[Z^{i,N}_s - Z^{i,N}_t\Big]\Big\}^{\frac12}\\
\leq& C\sqrt{Kt}
\end{aligned}
$$

by Lemma 2.2.3-(i) with $r=\infty$ and since $\ell_N(i)$ is bounded on $\Omega_{N,K}$. Next, for $n\geq 1$,

$$\begin{aligned}\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K\sum_{j=1}^N \ell_N(i)A_N^n(i,j)M_r^{j,N}\Big)^2\Big] =& \sum_{j=1}^N\Big(\sum_{i=1}^K \ell_N(i)A_N^n(i,j)\Big)^2\mathbb{E}_\theta[Z_r^{j,N}]\\ \leq& C\sum_{j=1}^N\Big(\sum_{i=1}^K A_N^n(i,j)\Big)^2\mathbb{E}_\theta[Z_r^{j,N}]\\ \leq& C\sum_{j=1}^N |||I_KA_N^n|||_1^2\mathbb{E}_\theta[Z_r^{j,N}]\\ \leq& \frac{CK^2}{N}|||A_N|||_1^{2n-2}r\end{aligned}$$

because $|||I_KA_N|||_1\leq CK/N$ on $\Omega_{N,K}$ and by and Lemma 2.2.3-(i) again. So, for all $u\in[0,1]$ (recall that $\int_0^\infty \phi^{\star n}(u)du=\Lambda^n$),

$$\begin{aligned}\mathbb{E}_\theta[|I_{t,N,K,u}^{3,2}|^2]^{\frac12} \leq& \frac{C}{t^2K}\int_t^{t+\sqrt{u}t}\Big\{\sqrt{Kt}+\sum_{n\geq1}\int_0^s(\phi^{\star n}(s-r)-\phi^{\star n}(t-r))\frac{K}{\sqrt{N}}|||A_N|||_1^{n-1}\sqrt{r}dr\Big\}ds\\ \leq& \frac{C}{t^2K}\int_t^{t+\sqrt{u}t}\Big\{\sqrt{Kt}+\sum_{n\geq1}\sqrt{s}\frac{K}{\sqrt{N}}\Lambda^n|||A_N|||_1^{n-1}\Big\}ds\leq\frac{C}{\sqrt{Kt}}.\end{aligned}$$

We finally used that $\Lambda|||A_N|||_1\leq a<1$ on $\Omega_{N,K}$.

By Minkowski's inequality and Lemma 2.2.3-(ii) with $r=1$, we find, on $\Omega_{N,K}$,

$$\begin{aligned}\mathbb{E}_\theta[|I_{t,N,K,u}^{3,3}|] \leq& \frac{1}{Kt^2}\sum_{i=1}^K\int_t^{t+\sqrt{u}t}|\ell_N(i)|\Big|\mathbb{E}_\theta[Z_s^{i,N}-Z_t^{i,N}-\mu(s-t)\ell_N(i)]\Big|ds\\ \leq& \frac{C}{Kt^2}\sum_{i=1}^K\int_t^{t+\sqrt{u}t}\Big|\mathbb{E}_\theta[Z_s^{i,N}-Z_t^{i,N}-\mu(s-t)\ell_N(i)]\Big|ds\leq\frac{C}{t^q}.\end{aligned}$$

Finally, we set $\mathbb{N}_u^{t,i,N}:=M_{t+ut}^{i,N}-M_t^{i,N}$. Then $\mathbb{N}_u^{t,i,N}$ is a martingale for the filtration $\mathcal{F}_{t+ut}^N$ with parameter $0\leq u\leq 1$ and we have, by (2.8),

$$[\mathbb{N}_\cdot^{t,i,N},\mathbb{N}_\cdot^{t,j,N}]_u=\mathbf{1}_{\{i=j\}}(Z_{t+ut}^{i,N}-Z_t^{i,N}).$$

Then we have, on $\Omega_{N,K}$, using the change variables $s=t+at$,

$$\begin{aligned}\mathbb{E}_\theta[(I_{t,N,K,u}^{3,1})^2]=&\frac{2}{K^2t^2}\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K\mu\ell_N(i)\int_0^{\sqrt{u}}\int_t^{t+at}(M_{r-}^{i,N}-M_t^{i,N})dM_r^{i,N}da\Big)^2\Big]\\ =&\frac{1}{K^2t^2}\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K\mu\ell_N(i)\int_0^{\sqrt{u}}\int_0^a\mathbb{N}_{b-}^{t,i,N}d\mathbb{N}_b^{t,i,N}da\Big)^2\Big]\\ =&\frac{\mu^2}{K^2t^2}\sum_{i=1}^K\sum_{i'=1}^K\ell_N(i)\ell_N(i')\int_0^{\sqrt{u}}\int_0^{\sqrt{u}}\mathbb{E}_\theta\Big[\int_0^a\mathbb{N}_{b-}^{t.i,N}d\mathbb{N}_b^{t,i,N}\int_0^{a'}\mathbb{N}_{b'-}^{t,i',N}d\mathbb{N}_{b'}^{t,i',N}\Big]dada'\\ \leq&\frac{C}{K^2t^2}\sum_{i=1}^K\int_0^1\int_0^1\mathbb{E}_\theta\Big[\Big(\int_0^{a\wedge a'}\mathbb{N}_{b-}^{t.i,N}d\mathbb{N}_b^{t,i,N}\Big)^2\Big]dada'\\ \leq&\frac{C}{K^2t^2}\sum_{i=1}^K\mathbb{E}_\theta\Big[\int_0^1(\mathbb{N}_{b-}^{t.i,N})^2dZ_{t+bt}^{i,N}\Big]\\ \leq&\frac{C}{K^2t^2}\sum_{i=1}^K\mathbb{E}_\theta\Big[\Big(\sup_{0\leq b\leq1}(\mathbb{N}_b^{t.i,N})^2\Big)Z_{2t}^{i,N}\Big].\end{aligned}$$

Hence, using the Cauchy-Schwarz and Burkholder inequalities,

$$\mathbb{E}_\theta[(I^{3,1}_{t,N,K,u})^2] \leq \frac{C}{K^2t^2}\sum_{i=1}^K \mathbb{E}_\theta\Big[\sup_{0\leq b\leq 1}(\mathbb{N}_b^{t.i,N})^4\Big]^{\frac{1}{2}}\mathbb{E}_\theta[(Z^{i,N}_{2t})^2]^{\frac{1}{2}} \leq \frac{C}{K^2t^2}\sum_{i=1}^K \mathbb{E}_\theta[(Z^{i,N}_{2t})^2] \leq \frac{C}{K}$$

by Lemma 2.4.7.

Finally, we can give the

Proof. [Proof of Theorem 2.4.2] Recall that we work when $(N,K,t)\to(\infty,\infty,\infty)$ in the regime where $\frac{t\sqrt{K}}{N}(\frac{N}{t^q}+\sqrt{\frac{N}{Kt}})+Ne^{-c_{p,\Lambda}K}\to 0$. In the beginning of the section, we have written

$$\mathbf{V}_t^{N,K}-\mathbf{V}_\infty^{N,K}=\Delta_t^{N,K,1}+\Delta_t^{N,K,211}+\Delta_t^{N,K,212}+\Delta_t^{N,K,213}+\Delta_t^{N,K,22}+\Delta_t^{N,K,23}+\Delta_t^{N,K,3}.$$

We have seen in Corollary 2.4.4 that the terms $1, 212, 213, 22, 23, 3$, when multiplied by $t\sqrt{K}/N$, all tend to 0, while Lemma 2.4.8 tells us that

$$\mathbf{1}_{\Omega_{N,K}}\frac{t\sqrt{K}}{N}\Delta_t^{N,K,211}\xrightarrow{d}\mathcal{N}\Big(0,\frac{2\mu^2}{(1-\Lambda p)^2}\Big),$$

which completes the proof.

2.5 Some limit theorems for the third estimator

The aim of this section is to prove the following theorem.

Theorem 2.5.1. *Assume $H(q)$ for some $q>3$, $K\leq N$ and $\lim_{N,K\to\infty}\frac{K}{N}=\gamma\leq 1$, $\Delta_t = t/(2\lfloor t^{1-4/(q+1)}\rfloor)\sim t^{4/(q+1)}/2$ (for t large). If $(N,K,t)\to(\infty,\infty,\infty)$ and $\frac{1}{\sqrt{K}}+\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}+\frac{N}{t\sqrt{K}}+Ne^{-c_{p,\Lambda}K}\to 0$,*

$$\lim_{N,K,t\to+\infty}\mathbf{1}_{\Omega_{N,K}}\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\Big(\mathcal{X}^{N,K}_{\Delta_t,t}-\boldsymbol{X}^{N,K}_{\infty,\infty}\Big)\longrightarrow\mathcal{N}\Big(0,\frac{3}{2}\Big(\frac{1-\gamma}{(1-\Lambda p)}+\frac{\gamma}{(1-\Lambda p)^3}\Big)^2\Big)$$

Recall the definition in section 2.1.5 and we define:

$$\begin{aligned}
&\mathcal{X}^{N,K}_{\Delta,t}-\mathcal{X}^{N,K}_{\infty,\infty}\\
=&(\mathcal{W}^{N,K}_{\Delta,t}-\mathcal{W}^{N,K}_{\infty,\infty})+\frac{N-K}{K}\Big(\varepsilon_t^{N,K}-\mu\bar{\ell}_N^K\Big)\\
=&D^{N,K,1}_{\Delta,t}+2D^{N,K,1}_{2\Delta,t}+D^{N,K,2}_{\Delta,t}+2D^{N,K,2}_{2\Delta,t}+D^{N,K,3}_{\Delta,t}+2D^{N,K,3}_{2\Delta,t}+D^{N,K,4}_{\Delta,t}+\frac{N-K}{K}\Big(\varepsilon_t^{N,K}-\mu\bar{\ell}_N^K\Big),
\end{aligned}$$

where

$$D^{N,K,1}_{\Delta,t} = \frac{N}{t}\Big\{ \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \Delta\varepsilon^{N,K}_t\Big)^2 - \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \Delta\mu\bar{\ell}^K_N\Big)^2\Big\},$$

$$\begin{aligned} D^{N,K,2}_{\Delta,t} =& \frac{N}{t}\Big\{ \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \Delta\mu\bar{\ell}^K_K\Big)^2 \\ &- \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta}]\Big)^2\Big\}, \end{aligned}$$

$$\begin{aligned} D^{N,K,3}_{\Delta,t} =& \frac{N}{t}\Big\{ \sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta}]\Big)^2 \\ &- \mathbb{E}_\theta\Big[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta}]\Big)^2\Big]\Big\}, \end{aligned}$$

and finally

$$\begin{aligned} D^{N,K,4}_{\Delta,t} =& \Big\{ \frac{2N}{t}\mathbb{E}_\theta\Big[\sum_{a=\frac{t}{2\Delta}+1}^{\frac{t}{\Delta}} \Big(\bar{Z}^{N,K}_{2a\Delta} - \bar{Z}^{N,K}_{2(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^{N,K}_{2a\Delta} - \bar{Z}^{N,K}_{2(a-1)\Delta}]\Big)^2\Big] \\ &- \frac{N}{t}\mathbb{E}_\theta\Big[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}} \Big(\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta} - \mathbb{E}_\theta[\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-1)\Delta}]\Big)^2\Big] - \mathcal{W}^{N,K}_{\infty,\infty}\Big\}. \end{aligned}$$

2.5.1 Some small terms of the estimator

First, we are going to prove the terms $D^{N,K,1}_{\Delta,t}$, $D^{N,K,2}_{\Delta,t}$, $D^{N,K,4}_{\Delta,t}$ are small.

Lemma 2.5.2. *Assume $H(q)$ for some $q \geq 1$. Then for all $t \geq 4$ and all $\Delta \in [1, t/4]$ such that $t/(2\Delta)$ is a positive integer,*

(i) $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}}|D^{N,K,1}_{\Delta,t}|] \leq C\Delta\Big(\frac{N}{t^{2q}} + \frac{N}{Kt}\Big)$,

(ii) $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}}|D^{N,K,2}_{\Delta,t}|] \leq C\frac{N}{t^{q-1}}$,

(iii) $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}}|D^{N,K,4}_{\Delta,t}|] \leq C\frac{Nt}{K\Delta^{1+q}}$,

(iv) $\frac{N}{K}\mathbb{E}[\mathbf{1}_{\Omega_{N,K}}|\varepsilon^{N,K}_t - \mu\bar{\ell}^K_N|] \leq \frac{CN}{Kt^q} + \frac{CN}{K\sqrt{tK}}$.

Proof. The results follow easily from [26, Lemmas 6.3, 8.2, 8.3 and 8.5].

We then deduce the following corollary.

Corollary 2.5.3. *Assume $H(q)$ for some $q > 3$. If we choose $\Delta_t = t/(2\lfloor t^{1-4/(q+1)}\rfloor) \sim t^{4/(q+1)}/2$ (for t large) then, in the regime $\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \to 0$, we have the convergence in probability*

$$\lim_{N,K,t\to+\infty} \mathbf{1}_{\Omega_{N,K}} \frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\Big\{|D^{N,K,1}_{\Delta_t,t}| + |D^{N,K,2}_{\Delta_t,t}| + |D^{N,K,4}_{\Delta_t,t}| + \frac{N}{K}|\varepsilon^{N,K}_t - \mu\bar{\ell}^K_N|\Big\} = 0.$$

Next, we consider the term $D^{N,K,3}_{\Delta,t}$. For $0 \le v \le 1$, we define:

$$\mathbb{X}^{N,K}_{\Delta,t,v} := \sum_{a=[\frac{vt}{\Delta}]+1}^{[\frac{2vt}{\Delta}]} \Big\{ (\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta})^2 - \mathbb{E}_\theta[(\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta})^2] \Big\}, \tag{2.12}$$

where

$$\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta} := \frac{1}{K} \sum_{j=1}^{N} c^K_N(j)(M^{j,N}_{a\Delta} - M^{j,N}_{(a-1)\Delta}). \tag{2.13}$$

Lemma 2.5.4. *Assume $H(q)$ for some $q \ge 2$, then we have*

$$\frac{K}{N}\sqrt{\frac{t}{\Delta}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|D^{N,K,3}_{\Delta,t} - \frac{N}{t}\mathbb{X}^{N,K}_{\Delta,t,1}\Big|\Big] \le \frac{CK}{N\Delta} + \frac{CKt^{\frac{3}{4}}}{\Delta^{1+\frac{q}{2}}\sqrt{N}} + \frac{C\sqrt{K}}{\sqrt{N\Delta}} + \frac{Ct^{\frac{3}{4}}}{\Delta^{1+\frac{q}{2}}}.$$

Before the proof, we need some preparations. For $a \in \{t/(2\Delta)+1, ..., 2t/\Delta\}$, we write that $U^{i,N}_{a\Delta} - U^{i,N}_{(a-1)\Delta} = \Gamma^{i,N}_{(a-1)\Delta,a\Delta} + X^{i,N}_{(a-1)\Delta,a\Delta}$, where

$$\begin{aligned}\Gamma^{i,N}_{(a-1)\Delta,a\Delta} :=& \sum_{n\ge1}\Big\{\int_0^{a\Delta}\phi^{*n}(a\Delta-s)\sum_{j=1}^{N}A^n_N(i,j)[M^{j,N}_s - M^{j,N}_{a\Delta}]ds \\ &- \int_0^{(a-1)\Delta}\phi^{*n}((a-1)\Delta-s)\sum_{j=1}^{N}A^n_N(i,j)[M^{j,N}_s - M^{j,N}_{(a-1)\Delta}]ds\Big\},\end{aligned} \tag{2.14}$$

$$\begin{aligned}X^{i,N}_{(a-1)\Delta,a\Delta} :=& \sum_{n\ge0}\sum_{j=1}^{N}\Big\{\int_0^{a\Delta}\phi^{*n}(a\Delta-s)dsA^n_N(i,j)M^{j,N}_{a\Delta} \\ &- \int_0^{(a-1)\Delta}\phi^{*n}((a-1)\Delta-s)dsA^n_N(i,j)M^{j,N}_{(a-1)\Delta}\Big\}.\end{aligned} \tag{2.15}$$

And we define the mean values as

$$\bar\Gamma^{N,K}_{(a-1)\Delta,a\Delta} := \frac{1}{K}\sum_{i=1}^{K}\Gamma^{i,N}_{(a-1)\Delta,a\Delta}, \quad \bar X^{N,K}_{(a-1)\Delta,a\Delta} := \frac{1}{K}\sum_{i=1}^{K}X^{i,N}_{(a-1)\Delta,a\Delta}.$$

First, we consider the term $\bar\Gamma^{N,K}_{(a-1)\Delta,a\Delta}$. Then we can write

$$\bar\Gamma^{N,K}_{(a-1)\Delta,a\Delta} = C^{N,K}_{a\Delta} + B^{N,K}_{a\Delta} - C^{N,K}_{(a-1)\Delta} - B^{N,K}_{(a-1)\Delta},$$

where

$$C^{N,K}_{a\Delta} := \frac{1}{K}\sum_{i=1}^{K}\sum_{j=1}^{N}\sum_{n\ge1}\int_0^{\Delta}\phi^{*n}(s)A^n_N(i,j)(M^{j,N}_{(a\Delta-s)} - M^{j,N}_{a\Delta})ds, \tag{2.16}$$

$$B^{N,K}_{a\Delta} := \frac{1}{K}\sum_{i=1}^{K}\sum_{j=1}^{N}\sum_{n\ge1}\int_{\Delta}^{a\Delta}\phi^{*n}(s)A^n_N(i,j)(M^{j,N}_{(a\Delta-s)} - M^{j,N}_{a\Delta})ds. \tag{2.17}$$

Lemma 2.5.5. *Assume $H(q)$ for some $q \ge 1$. On the set $\Omega_{N,K}$, for $a \in \{t/(2\Delta)+1, ..., 2t/\Delta\}$, we a.s. have*

$$\mathbb{E}_\theta[(B^{N,K}_{a\Delta})^2] \le \frac{C}{N}\Delta^{1-2q}.$$

Proof. We work on the set $\Omega_{N,K}$. Recall (2.8). By [14, Lemma 16.(iii)] and Cauchy-Schwarz inequality we have:

$$\begin{aligned}\mathbb{E}_\theta[(M^{j,N}_{(a\Delta-s)} - M^{j,N}_{a\Delta})(M^{j',N}_{(a\Delta-s')} - M^{j',N}_{a\Delta})] &\le \mathbf{1}_{j=j'}\mathbb{E}_\theta[Z^{j,N}_{a\Delta} - Z^{j,N}_{(a\Delta-s)}]^{\frac12}\mathbb{E}_\theta[Z^{j,N}_{a\Delta} - Z^{j,N}_{(a\Delta-s')}]^{\frac12} \\ &\le \mathbf{1}_{j=j'}\sqrt{ss'}.\end{aligned}$$

From Lemma 2.10.1, we already have $\int_r^\infty \sqrt{u}\phi^{\star n}(u)du \le C\Lambda^n n^q r^{\frac{1}{2}-q}$. Hence,

$$\begin{aligned}
&\mathbb{E}_\theta[(B^{N,K}_{a\Delta})^2] = \frac{1}{K^2}\sum_{i,i'=1}^{K}\sum_{j,j'=1}^{N}\sum_{n,m\ge 1}\int_\Delta^{a\Delta}\int_\Delta^{a\Delta}\phi^{*n}(s)\phi^{*m}(s')A^n_N(i,j)A^m_N(i',j')\\
&\mathbb{E}_\theta[(M^{j,N}_{(a\Delta-s)}-M^{j,N}_{a\Delta})(M^{j',N}_{(a\Delta-s')}-M^{j',N}_{a\Delta})]dsds'\\
&\le \frac{CN}{K^2}\Big(\sum_{n,m\ge 1}\Lambda^{n+m}|||I_KA_N|||_1^2|||A_N|||_1^{n+m-2}\int_\Delta^\infty\int_\Delta^\infty\sqrt{ss'}\phi^{*n}(s)\phi^{*m}(s')dsds'\Big)\\
&\le \frac{C}{N}\Big(\sum_{n\ge 1}n^q\Lambda^n|||A_N|||_1^{n-1}\Big)^2(\Delta)^{1-2q} \le \frac{C}{N}\Delta^{1-2q}
\end{aligned}$$

Lemma 2.5.6. *Assume $H(q)$ for some $q \ge 1$. On the set $\Omega_{N,K}$, for $a \in \{t/(2\Delta)+1, ..., 2t/\Delta\}$, we a.s. have*

$$\mathbb{E}_\theta[(C^{N,K}_{a\Delta})^4] \le \frac{C}{N^2}.$$

Proof. We write

$$C^{N,K}_{a\Delta} = \sum_{n\ge 1}\int_0^\Delta \phi^{*n}(s)O^{N,K,n}_{s,s,a\Delta}$$

where for $r \ge 0$ and $0 \le s \le a\Delta$,

$$O^{N,K,n}_{r,s,a\Delta} = \frac{1}{K}\sum_{i=1}^{K}\sum_{j=1}^{N}A^n_N(i,j)(M^{j,N}_{(a\Delta-s)}-M^{j,N}_{a\Delta-s+r}).$$

When, we fix s, $\{O^{N,K,n}_{r,s,a\Delta}\}_{r\ge 0}$ is a family of martingale for the filtration $(\mathcal{F}_{a\Delta-s+r})_{r\ge 0}$. By (2.8), we have $[M^{i,N}, M^{j,N}]_t = \mathbf{1}_{\{i=j\}}Z^{i,N}_t$. Hence, for $n \ge 1$, on $\Omega_{N,K}$,

$$\begin{aligned}
[O^{N,K,n}_{.,s,a\Delta}, O^{N,K,n}_{.,s,a\Delta}]_r =& \frac{1}{K^2}\sum_{j=1}^{N}\Big(\sum_{i=1}^{K}A^n_N(i,j)\Big)^2(Z^{j,N}_{a\Delta-s+r}-Z^{j,N}_{a\Delta-s})\\
\le& \frac{N}{K^2}|||I_KA^n_N|||_1^2(\bar{Z}^N_{a\Delta-s+r}-\bar{Z}^N_{a\Delta-s})\\
\le& \frac{N}{K^2}|||I_KA_N|||_1^2|||A_N|||_1^{2n-2}(\bar{Z}^N_{a\Delta-s+r}-\bar{Z}^N_{a\Delta-s})\\
\le& \frac{1}{N}|||A_N|||_1^{2n-2}(\bar{Z}^N_{a\Delta-s+r}-\bar{Z}^N_{a\Delta-s})
\end{aligned}$$

Since $|||I_KA_N|||_1^2 \le \frac{K^2}{N^2}$ on $\Omega_{N,K}$, by Burkholder-Davis-Gundy inequality, we have on $\Omega_{N,K}$,

$$\mathbb{E}_\theta[(O^{N,K,n}_{r,s,a\Delta})^4] \le 4\mathbb{E}_\theta\Big[\Big([O^{N,K,n}_{,s,a\Delta}, O^{N,K,n}_{,s.a\Delta}]_r\Big)^2\Big] \le \frac{C|||A_N|||_1^{4n-4}}{N^2}\mathbb{E}_\theta[(\bar{Z}^N_{a\Delta-s+r}-\bar{Z}^N_{a\Delta-s})^2].$$

From [14, lemma 16 (iii)], we already have $\sup_{i=1,...,N}\mathbb{E}_\theta[(Z^{i,N}_t - Z^{i,N}_s)^2] \le C(t-s)^2$. Recalling the second part of Lemma 2.10.1, by Minkowski inequality, we deduce:

$$\begin{aligned}
\mathbb{E}_\theta[(C^{N,K}_{a\Delta})^4]^{\frac{1}{4}} \le& \sum_{n\ge 1}\int_0^\Delta \phi^{*n}(s)\mathbb{E}_\theta[(O^{N,K,n}_{s,s,a\Delta})^4]^{\frac{1}{4}}ds\\
\le& \frac{1}{\sqrt{N}}\sum_{n\ge 0}|||A_N|||_1^n\int_0^\Delta\sqrt{s}\phi^{*(n+1)}(s)ds\\
\le& \frac{1}{\sqrt{N}}\sum_{n\ge 0}\sqrt{n+1}\,\Lambda^{n+1}|||A_N|||_1^n ds \le \frac{C}{\sqrt{N}}.
\end{aligned}$$

This completes the proof.

Lemma 2.5.7. *Assume $H(q)$ for some $q \geq 1$ and $a, b \in \{\frac{t}{\Delta}+1, ..., \frac{2t}{\Delta}\}$. Then a.s. on the set $\Omega_{N,K}$*

$$\mathbb{C}ov_\theta[(C^{N,K}_{a\Delta} - C^{N,K}_{(a-1)\Delta})^2, (C^{N,K}_{b\Delta} - C^{N,K}_{(b-1)\Delta})^2] \leq \frac{C\sqrt{t}}{N\Delta^{q-1}}, \ \textbf{\textit{if}}\ |a-b| \geq 4.$$

Proof. Because

$$\begin{aligned}
&\mathbb{C}ov_\theta[(C^{N,K}_{a\Delta} - C^{N,K}_{(a-1)\Delta})^2, (C^{N,K}_{b\Delta} - C^{N,K}_{(b-1)\Delta})^2] \\
=&\frac{1}{K^4}\sum_{i,k,i',k'=1}^{K}\sum_{j,l,j',l'=1}^{N}\sum_{m,n,m',n'\geq 1}\int_0^\Delta\int_0^\Delta\int_0^\Delta\int_0^\Delta \phi^{*n}(s)\phi^{*m}(t)\phi^{*n'}(s')\phi^{*m'}(s') \\
&\times A^n_N(i,j)A^m_N(k,l)A^{n'}_N(i',j')A^{m'}_N(k',l') \\
&\times \mathbb{C}ov_\theta[(M^{j,N}_{(a\Delta-s)} - M^{j,N}_{a\Delta} - M^{j,N}_{((a-1)\Delta-s)} + M^{j,N}_{(a-1)\Delta}) \\
&\qquad \times (M^{j',N}_{(a\Delta-s')} - M^{j',N}_{a\Delta} - M^{j',N}_{((a-1)\Delta-s')} + M^{j',N}_{(a-1)\Delta}), \\
&\quad (M^{l,N}_{(b\Delta-r)} - M^{l,N}_{b\Delta} - M^{l,N}_{((b-1)\Delta-r)} + M^{l,N}_{(b-1)\Delta}) \\
&\qquad \times (M^{l',N}_{(b\Delta-r')} - M^{l',N}_{b\Delta} + (M^{l',N}_{((b-1)\Delta-r')} - M^{l',N}_{(b-1)\Delta})]dsdrds'dr'.
\end{aligned}$$

We define $\zeta^{j,N}_{a\Delta,s} := M^{j,N}_{(a\Delta-s)} - M^{j,N}_{a\Delta}$ for $0 \leq s \leq \Delta$. Then we can rewrite that:

$$\begin{aligned}
&\mathbb{C}ov_\theta[(M^{j,N}_{(a\Delta-s)} - M^{j,N}_{a\Delta} - M^{j,N}_{((a-1)\Delta-s)} + M^{j,N}_{(a-1)\Delta}) \\
&\qquad \times (M^{j',N}_{(a\Delta-s')} - M^{j',N}_{a\Delta} - M^{j',N}_{((a-1)\Delta-s')} + M^{j',N}_{(a-1)\Delta}), \\
&\quad (M^{l,N}_{(b\Delta-r)} - M^{l,N}_{b\Delta} - M^{l,N}_{((b-1)\Delta-r)} + M^{l,N}_{(b-1)\Delta}) \\
&\qquad \times (M^{l',N}_{(b\Delta-r')} - M^{l',N}_{b\Delta} + (M^{l',N}_{((b-1)\Delta-r')} - M^{l',N}_{(b-1)\Delta})] \\
=&\mathbb{C}ov_\theta[(\zeta^{j,N}_{a\Delta,s} - \zeta^{j,N}_{(a-1)\Delta,s})(\zeta^{j',N}_{a\Delta,s'} - \zeta^{j',N}_{(a-1)\Delta,s'}), (\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})(\zeta^{l',N}_{b\Delta,r'} - \zeta^{l',N}_{(b-1)\Delta,r'})].
\end{aligned}$$

Because $0 \leq s, s', r, r' \leq \Delta$, we have:

$$\mathbb{E}_\theta[\zeta^{j,N}_{(a-1)\Delta,s}\zeta^{j',N}_{a\Delta,s'}] = \mathbb{E}_\theta[\zeta^{j,N}_{a\Delta,s}\zeta^{j',N}_{(a-1)\Delta,s'}] = \mathbb{E}_\theta[\zeta^{l,N}_{(b-1)\Delta,r}\zeta^{l',N}_{b\Delta,r'}] = \mathbb{E}_\theta[\zeta^{l,N}_{b\Delta,r}\zeta^{l',N}_{(b-1)\Delta,r'}] = 0$$

Without loss and generality, we assume $a - b \geq 4$ and $s \leq s'$, first we notice that

$$\begin{aligned}
&\mathbb{C}ov_\theta[\zeta^{j,N}_{a\Delta,s}\zeta^{j',N}_{(a-1)\Delta,s'}, (\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})(\zeta^{l',N}_{b\Delta,r'} - \zeta^{l',N}_{(b-1)\Delta,r'})] \\
=&\mathbb{E}_\theta\Big[\zeta^{j,N}_{a\Delta,s}|\mathcal{F}_{(a-1)\Delta}\Big]\zeta^{j',N}_{(a-1)\Delta,s'} \\
\times&\Big((\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})(\zeta^{l',N}_{b\Delta,r'} - \zeta^{l',N}_{(b-1)\Delta,r'}) - \mathbb{E}_\theta\Big[(\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})(\zeta^{l',N}_{b\Delta,r'} - \zeta^{l',N}_{(b-1)\Delta,r'})\Big]\Big) = 0.
\end{aligned}$$

And by the same reason we conclude that

$$\mathbb{C}ov_\theta[\zeta^{j,N}_{(a-1)\Delta,s}\zeta^{j',N}_{a\Delta,s'}, (\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})(\zeta^{l',N}_{b\Delta,r'} - \zeta^{l',N}_{(b-1)\Delta,r'})] = 0.$$

When $j \neq j'$, the covariance vanishes because

$$\mathbb{E}_\theta[(\zeta^{j,N}_{a\Delta,s} - \zeta^{j,N}_{(a-1)\Delta,s})(\zeta^{j',N}_{a\Delta,s'} - \zeta^{j',N}_{(a-1)\Delta,s'})|\mathcal{F}_{b\Delta}] = 0.$$

Next we assume that $j = j'$, we have

$$\begin{aligned}
K :=&\mathbb{C}ov_\theta[(\zeta^{j,N}_{a\Delta,s} - \zeta^{j,N}_{(a-1)\Delta,s})(\zeta^{j',N}_{a\Delta,s'} - \zeta^{j',N}_{(a-1)\Delta,s'}), (\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})(\zeta^{l',N}_{b\Delta,r'} - \zeta^{l',N}_{(b-1)\Delta,r'})] \\
=&\mathbb{C}ov_\theta[(\zeta^{j,N}_{a\Delta,s}\zeta^{j',N}_{a\Delta,s'} + \zeta^{j,N}_{(a-1)\Delta,s}\zeta^{j',N}_{(a-1)\Delta,s'}), (\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})(\zeta^{l',N}_{b\Delta,r'} - \zeta^{l',N}_{(b-1)\Delta,r'})].
\end{aligned}$$

Since $\mathbb{E}_\theta[\zeta^{j,N}_{a\Delta,s}\zeta^{j,N}_{a\Delta,s'}|\mathcal{F}_{a\Delta-s'}] = \mathbb{E}_\theta[(M^{j,N}_{a\Delta})^2 - (M^{j,N}_{a\Delta-s})^2|\mathcal{F}_{a\Delta-s'}]$, writung as usual $(M^{j,N}_{a\Delta})^2 - (M^{j,N}_{a\Delta-s})^2 = 2\int_{a\Delta-s}^{a\Delta} M^{j,N}_{\tau-}dM^{j,N}_\tau + Z^{j,N}_{a\Delta} - Z^{j,N}_{a\Delta-s}$, we find that

$$\begin{aligned}K =&\mathbb{C}ov_\theta[Z^{j,N}_{a\Delta} - Z^{j,N}_{a\Delta-s} + Z^{j,N}_{(a-1)\Delta} - Z^{j,N}_{(a-1)\Delta-s}, (\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})(\zeta^{l',N}_{b\Delta,r'} - \zeta^{l',N}_{(b-1)\Delta,r'})]\\ =&\mathbb{C}ov_\theta[U^{j,N}_{a\Delta} - U^{j,N}_{a\Delta-s} + U^{j,N}_{(a-1)\Delta} - U^{j,N}_{(a-1)\Delta-s}, (\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})(\zeta^{l',N}_{b\Delta,r'} - \zeta^{l',N}_{(b-1)\Delta,r'})].\end{aligned}$$

Recalling that $\beta_n(x,z,r) = \phi^{\star n}(z-r) - \phi^{\star n}(x-r)$, we can write

$$U^{j,N}_{a\Delta} - U^{j,N}_{a\Delta-s} = \sum_{n\geq 0}\int_0^{a\Delta}\beta_n(a\Delta-s, a\Delta, r)\sum_{j=1}^N A^n_N(i,j)M^{j,N}_r dr = R^{j,N}_{a\Delta,a\Delta-s} + T^{j,N}_{a\Delta,a\Delta-s},$$

where

$$\begin{aligned}R^{j,N}_{a\Delta,a\Delta-s} =& \sum_{n\geq 0}\int_{(a-1)\Delta-s}^{a\Delta}\beta_n(x,z,r)\sum_{j=1}^N A^n_N(i,j)(M^{j,N}_r - M^{j,N}_{(a-1)\Delta-s})dr,\\ T^{j,N}_{a\Delta,a\Delta-s} =& \sum_{n\geq 0}\Big(\int_{(a-1)\Delta-s}^{a\Delta}\beta_n(x,z,r)dr\Big)\sum_{j=1}^N A^n_N(i,j)M^{j,N}_{(a-1)\Delta-s}\\ &+\sum_{n\geq 0}\int_0^{(a-1)\Delta-s}\beta_n(x,z,r)\sum_{j=1}^N A^n_N(i,j)M^{j,N}_r dr.\end{aligned}$$

The conditional expectation of $R^{j,N}_{a\Delta,a\Delta-s}$ knowing $\mathcal{F}_{b\Delta}$ vanishes. Hence

$$K = \mathbb{C}ov_\theta[T^{j,N}_{a\Delta,a\Delta-s} + T^{j,N}_{(a-1)\Delta,(a-1)\Delta-s}, (\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})(\zeta^{l',N}_{b\Delta,r'} - \zeta^{l',N}_{(b-1)\Delta,r'})].$$

Recall [14, proof of Lemma 30, Step 1] (and notice that $T^{j,N}_{a\Delta,a\Delta-s}$ is exactly the $X^{j,N}_{a\Delta-s,a\Delta}$ in [14]), we have $\sup_{i=1,\dots,N}\mathbb{E}_\theta[(T^{j,N}_{a\Delta,a\Delta-s})^4] \leq Ct^2\Delta^{-4q}$. Since $r\leq\Delta$,

$$\begin{aligned}\mathbb{E}_\theta[(\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})^4]^{\frac14} &\leq \mathbb{E}_\theta[(M^{l,N}_{(b\Delta-r)} - M^{l,N}_{b\Delta})^4]^{\frac14} + \mathbb{E}_\theta[(M^{l,N}_{((b-1)\Delta-r)} - M^{l,N}_{(b-1)\Delta})^4]^{\frac14}\\ &\leq C\sqrt{\Delta},\end{aligned}$$

whence

$$\begin{aligned}|K| \leq&\{\mathbb{E}_\theta[(T^{j,N}_{a\Delta,a\Delta-s})^2]^{\frac12} + \mathbb{E}_\theta[(T^{j,N}_{(a-1)\Delta,(a-1)\Delta-s})^2]^{\frac12}\}\\ &\times\mathbb{E}_\theta[(\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})^4]^{\frac14}\mathbb{E}_\theta[(\zeta^{l',N}_{b\Delta,r'} - \zeta^{l',N}_{(b-1)\Delta,r'})^4]^{\frac14}\\ \leq& Ct^{1/2}\Delta^{-q}\Delta.\end{aligned}$$

Moreover, by symmetry, we conclude that, when $|a-b|\geq 4$,

$$\begin{aligned}&\mathbb{C}ov_\theta[(\zeta^{j,N}_{a\Delta,s} - \zeta^{j,N}_{(a-1)\Delta,s})(\zeta^{j',N}_{a\Delta,s'} - \zeta^{j',N}_{(a-1)\Delta,s'}), (\zeta^{l,N}_{b\Delta,r} - \zeta^{l,N}_{(b-1)\Delta,r})(\zeta^{l',N}_{b\Delta,r'} - \zeta^{l',N}_{(b-1)\Delta,r'})]\\ \leq& C\Big(\mathbf{1}_{l=l'} + \mathbf{1}_{j=j'}\Big)\sqrt{t}\Delta^{1-q}.\end{aligned}$$

Recall the definition of $\Omega_{N,K}$, we have $|||I_K A^n_N|||_1 \leq |||I_K A_N|||_1|||A_N|||_1^{n-1} \leq \frac{CK}{N}|||A_N|||_1^{n-1}$,

which gives us

$$\begin{aligned}
&\mathbb{C}ov_\theta[(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})^2,(C^{N,K}_{b\Delta}-C^{N,K}_{(b-1)\Delta})^2]\\
\leq&\frac{C\sqrt{t}\Delta^{1-q}}{K^4}\sum_{i,k,i',k'=1}^{K}\sum_{j,l,j',l'=1}^{N}\sum_{m,n,m',n'\geq 1}\\
&\qquad\Lambda^{n+m+n'+m'}A^n_N(i,j)A^m_N(k,l)A^{n'}_N(i',j')A^{m'}_N(k',l')\Big(\mathbf{1}_{l=l'}+\mathbf{1}_{j=j'}\Big)\\
\leq&\frac{C\sqrt{t}\Delta^{1-q}}{K^4}N^3\sum_{m,n,m',n'\geq 1}\Lambda^{n+m+n'+m'}|||I_KA^n_N|||_1|||I_KA^m_N|||_1|||I_KA^{n'}_N|||_1|||I_KA^{m'}_N|||_1\\
\leq&\sum_{n,m,n',m'\geq 1}\frac{1}{K^4}N^3\Big(\frac{K}{N}\Big)^4\Lambda^4(\Lambda|||A_N|||_1)^{n+m+n'+m'-4}\frac{C\sqrt{t}}{\Delta^{q-1}}\leq\frac{C\sqrt{t}}{N\Delta^{q-1}}.
\end{aligned}$$

The proof is finished.

Lemma 2.5.8. *Assume $H(q)$ for some $q\geq 1$. Then a.s. on the set $\Omega_{N,K}$*

$$\frac{K}{N}\sqrt{\frac{t}{\Delta}}\frac{N}{t}\mathbb{E}_\theta\Big[\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\Big\{(\bar\Gamma^{N,K}_{(a-1)\Delta,a\Delta})^2-\mathbb{E}_\theta[(\bar\Gamma^{N,K}_{(a-1)\Delta,a\Delta})^2]\Big\}\Big|\Big]\leq\frac{CK\sqrt{t}}{N\Delta^{(q+1)}}+\frac{CK}{N\Delta}+\frac{CKt^{\frac{3}{4}}}{\Delta^{(1+\frac{q}{2})}\sqrt{N}}.$$

Proof. We start from

$$(\bar\Gamma^{N,K}_{(a-1)\Delta,a\Delta})^2=(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})^2+2(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})+(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})^2.$$

By Lemma 2.5.5, we have

$$\begin{aligned}
&\mathbb{E}_\theta\Big[\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})^2-\mathbb{E}_\theta\Big[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})^2\Big]\Big|\Big]\\
\leq&2\mathbb{E}_\theta\Big[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})^2\Big]\leq\frac{Ct}{N\Delta^{2q}}.
\end{aligned}$$

And by lemma 2.5.6 and lemma 2.5.7

$$\begin{aligned}
&\mathbb{E}_\theta\Big[\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})^2-\mathbb{E}_\theta[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})^2]\Big|^2\Big]\\
=&\mathbb{V}ar_\theta\Big[\sum_{a=\frac{vt}{\Delta}+1}^{\frac{2vt}{\Delta}}(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})^2\Big]\\
\leq&\sum_{\substack{t/\Delta+1\leq a,b\leq 2t/\Delta\\|a-b|\leq 3}}\mathbb{E}_\theta\Big[(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})^4\Big]^{\frac{1}{2}}\mathbb{E}_\theta\Big[(C^{N,K}_{b\Delta}-C^{N,K}_{(b-1)\Delta})^4\Big]^{\frac{1}{2}}\\
&+\sum_{\substack{t/\Delta+1\leq a,b\leq 2t/\Delta\\|a-b|\geq 4}}\mathbb{C}ov_\theta\Big[(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})^2,(C^{N,K}_{b\Delta}-C^{N,K}_{(b-1)\Delta})^2\Big]\\
\leq&C\Big[\frac{t}{\Delta}\frac{1}{N^2}+\frac{t^{\frac{5}{2}}}{\Delta^{q+1}N}\Big].
\end{aligned}$$

Moreover we have, by Lemmas 2.5.5 and 2.5.6,

$$\begin{aligned}
&\mathbb{E}_\theta\Big[\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})-\mathbb{E}_\theta[(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})]\Big|\Big]\\
\le&4\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\Big\{\mathbb{E}_\theta\Big[\Big|B^{N,K}_{a\Delta}C^{N,K}_{a\Delta}\Big|\Big]+\mathbb{E}_\theta\Big[\Big|B^{N,K}_{(a-1)\Delta}C^{N,K}_{a\Delta}\Big|\Big]+\mathbb{E}_\theta\Big[\Big|B^{N,K}_{a\Delta}C^{N,K}_{(a-1)\Delta}\Big|\Big]\\
&\qquad+\mathbb{E}_\theta\Big[\Big|B^{N,K}_{(a-1)\Delta}C^{N,K}_{(a-1)\Delta}\Big|\Big]\Big\}\\
\le&4\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\Big\{\mathbb{E}_\theta\Big[\Big|B^{N,K}_{a\Delta}\Big|^2\Big]^{\frac12}+\Big|B^{N,K}_{(a-1)\Delta}\Big|^2\Big]^{\frac12}\Big\}\Big\{\mathbb{E}_\theta\Big[\Big|C^{N,K}_{a\Delta}\Big|^2\Big]^{\frac12}+\mathbb{E}_\theta\Big[\Big|C^{N,K}_{(a-1)\Delta}\Big|^2\Big]^{\frac12}\Big\}\\
\le&\frac{Ct}{\Delta^{q+\frac12}}\frac{1}{N}
\end{aligned}$$

Overall, we have:

$$\begin{aligned}
&\frac{K}{N}\sqrt{\frac{t}{\Delta}}\frac{N}{t}\mathbb{E}_\theta\Big[\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\Big\{(\bar\Gamma^{N,K}_{(a-1)\Delta,a\Delta})^2-\mathbb{E}_\theta[(\bar\Gamma^{N,K}_{(a-1)\Delta,a\Delta})^2]\Big\}\Big|\Big]\\
\le&\frac{K}{\sqrt{t\Delta}}\Big\{\mathbb{E}_\theta\Big[\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})^2-\mathbb{E}_\theta\Big[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})^2\Big]\Big|\Big]\\
&+\mathbb{E}_\theta\Big[\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})^2-\mathbb{E}_\theta\Big[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})^2\Big]\Big|\Big]\\
&+2\mathbb{E}_\theta\Big[\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})\\
&\qquad-\mathbb{E}_\theta\Big[(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})\Big]\Big|\Big]\Big\}\\
\le&\frac{K}{\sqrt{t\Delta}}\Big\{\frac{1}{N}\frac{Ct}{\Delta^{2q}}+\Big[\frac{t}{\Delta}\frac{1}{N^2}+\frac{t^{\frac52}}{\Delta^{q+1}N}\Big]^{\frac12}+\frac{1}{N}\frac{Ct}{\Delta^{q+\frac12}}\Big\}\le C\Big\{\frac{K}{N\Delta}+\frac{Kt^{\frac34}}{\Delta^{(1+\frac{q}{2})}\sqrt{N}}+\frac{K\sqrt{t}}{N\Delta^{(q+1)}}\Big\}
\end{aligned}$$

The proof is finished.

Recall that $c^K_N(j):=\sum_{i=1}^K Q_N(i,j)$. Next, we will prove that $\bar X^{N,K}_{(a-1)\Delta,a\Delta}$ is close to $\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}$,

Lemma 2.5.9. *Assume $H(q)$ for some $q\ge 2$. Then a.s. on the set $\Omega_{N,K}$, one has*

$$\begin{aligned}
\mathbb{E}_\theta[(\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}-\bar X^{N,K}_{(a-1)\Delta,a\Delta})^2]&\le\frac{C}{N}\Big[\frac{1}{(a\Delta)^{2q-1}}+\frac{1}{((a-1)\Delta)^{2q-1}}\Big],\\
\frac{K}{\sqrt{t\Delta}}\mathbb{E}_\theta\Big[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}-\bar X^{N,K}_{(a-1)\Delta,a\Delta}\Big|^2\Big]&\le\frac{CK}{N\sqrt{t}\Delta^{2q-\frac12}}.
\end{aligned}$$

Proof. By Lemma 2.10.2, on the set $\Omega_{N,K}$, we have:

$$
\begin{aligned}
&\mathbf{1}_{\Omega_{N,K}}\mathbb{E}_\theta[(\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}-\bar{X}^{N,K}_{(a-1)\Delta,a\Delta})^2]\\
\leq&\frac{1}{K^2}\mathbb{E}_\theta\Big[\Big|\sum_{j=1}^N\Big\{\sum_{n\geq 0}\sum_{i=1}^K\Big(\int_0^{a\Delta}\phi^{*n}(a\Delta-s)ds-\Lambda^n\Big)A^n_N(i,j)\Big\}M^{j,N}_{a\Delta}\Big|^2\Big]\\
&+\frac{1}{K^2}\mathbb{E}_\theta\Big[\Big|\sum_{j=1}^N\Big\{\sum_{n\geq 0}\sum_{i=1}^K\Big(\int_0^{(a-1)\Delta}\phi^{*n}((a-1)\Delta-s)ds-\Lambda^n\Big)A^n_N(i,j)\Big\}M^{j,N}_{(a-1)\Delta}\Big|^2\Big]\\
=&\frac{1}{K^2}\sum_{j=1}^N\Big\{\sum_{n\geq 0}\sum_{i=1}^K\Big(\int_0^{a\Delta}\phi^{*n}(a\Delta-s)ds-\Lambda^n\Big)A^n_N(i,j)\Big\}^2\mathbb{E}_\theta[Z^{j,N}_{a\Delta}]\\
&+\frac{1}{K^2}\sum_{j=1}^N\Big\{\sum_{n\geq 0}\sum_{i=1}^K\Big(\int_0^{(a-1)\Delta}\phi^{*n}((a-1)\Delta-s)ds-\Lambda^n\Big)A^n_N(i,j)\Big\}^2\mathbb{E}_\theta[Z^{j,N}_{(a-1)\Delta}]\\
\leq&\frac{1}{K^2}\sum_{j=1}^N\Big\{\sum_{n\geq 1}n\int_{(a\Delta)/n}^\infty\phi(s)ds\Lambda^{n-1}|||I_KA^n_N|||_1\Big)\Big\}^2\mathbb{E}_\theta[Z^{j,N}_{a\Delta}]\\
&+\frac{1}{K^2}\sum_{j=1}^N\Big\{\sum_{n\geq 1}n\int_{(a-1)\Delta/n}^\infty\phi(s)ds\Lambda^{n-1}|||I_KA^n_N|||_1\Big\}^2\mathbb{E}_\theta[Z^{j,N}_{(a-1)\Delta}]\\
&\leq\frac{1}{N^2}\sum_{j=1}^N\Big\{\sum_{n\geq 1}n\int_{(a\Delta)/n}^\infty\phi(s)ds\Lambda^{n-1}|||A_N|||_1^{n-1}\Big)\Big\}^2\mathbb{E}_\theta[Z^{j,N}_{a\Delta}]\\
&+\frac{1}{N^2}\sum_{j=1}^N\Big\{\sum_{n\geq 1}n\int_{(a-1)\Delta/n}^\infty\phi(s)ds\Lambda^{n-1}|||A_N|||_1^{n-1}\Big\}^2\mathbb{E}_\theta[Z^{j,N}_{(a-1)\Delta}]\\
\leq&\frac{1}{N^2}\sum_{j=1}^N\Big\{\sum_{n\geq 1}n^{1+q}(a\Delta)^{-q}\int_0^\infty s^q\phi(s)ds\Lambda^{n-1}|||A_N|||_1^{n-1}\Big)\Big\}^2\mathbb{E}_\theta[Z^{j,N}_{a\Delta}]\\
&+\frac{1}{N^2}\sum_{j=1}^N\Big\{\sum_{n\geq 1}n^{1+q}[(a-1)\Delta]^{-q}\int_0^\infty s^q\phi(s)ds\Lambda^{n-1}|||A_N|||_1^{n-1}\Big\}^2\mathbb{E}_\theta[Z^{j,N}_{(a-1)\Delta}]\\
&\leq\frac{1}{N^2(a\Delta)^{2q}}\mathbb{E}_\theta\Big[\sum_{j=1}^N Z^{j,N}_{a\Delta}\Big]+\frac{1}{N^2((a-1)\Delta)^{2q}}\mathbb{E}_\theta\Big[\sum_{j=1}^N Z^{j,N}_{(a-1)\Delta}\Big]\\
&\leq\frac{C}{N}\Big[\frac{1}{(a\Delta)^{2q-1}}+\frac{1}{((a-1)\Delta)^{2q-1}}\Big].
\end{aligned}
$$

For $q\geq 2$, we always have $\sum_{a=1}^\infty a^{1-2q}<+\infty$, which concludes the result.

Lemma 2.5.10. *Assume $H(q)$ for some $q\geq 2$. Then a.s. on the set $\Omega_{N,K}$,*

(i) $\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}\Big|^2\Big]\leq\frac{C\Delta}{K}$,

(ii) $\frac{K}{\sqrt{\Delta t}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\sum_{a=\frac{t}{\Delta}}^{\frac{2t}{\Delta}}\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}\Big|\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}-\bar{\mathcal{X}}^{N,K}_{(a-1)\Delta,a\Delta}\Big|\Big]\leq\frac{C\sqrt{K}}{\Delta^{q-\frac{1}{2}}\sqrt{Nt}}$.

Proof. By [26, Lemma 4.19] and the definition of $\mathcal{X}^{N,K}_{\infty,\infty}$, we have on $\Omega_{N,K}$,

$$
\Big|\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\sum_{j=1}^N\Big(c^K_N(j)\Big)^2\ell_N(j)\Big]\Big|\leq\frac{CK^2}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big\{\Big|\frac{\mu(N-K)}{K}\bar{\ell}^K_N\Big|+\Big|\frac{\mu}{(1-\Lambda p)^3}\Big|+\frac{1}{K}\Big\}\Big]\leq CK.
$$

Recalling [14, Lemma 16 (ii)], we already know that

$$
\sup_{j=1,\dots,N}\Big|\mathbb{E}_\theta[Z^{j,N}_{a\Delta}-Z^{j,N}_{(a-1)\Delta}-\Delta\ell_N(j)]\Big|\leq C\Delta^{1-q}.
$$

From [26, Lemma 4.16], we easily conclude that

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\|\boldsymbol{c}_N^K\|_2^2\Big] \le 2\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\|\boldsymbol{t}_N^K\|_2^2 + \|\bar{c}_N^K\mathbf{1}_N^T - \mathbf{1}_K^T + \frac{K}{N}\mathbf{1}_N^T\|^2\Big] \le \frac{CK^2}{N^2}(1+N) + CK \le CK.$$

Recalling the definition of $\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}$ and (2.8), we have

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}\Big|^2\Big]\\
=&\frac{1}{K^2}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big\{\sum_{j=1}^N\Big(c_N^K(j)\Big)^2\mathbb{E}_\theta[Z^{j,N}_{a\Delta} - Z^{j,N}_{(a-1)\Delta} - \Delta\ell_N(j)] + \Delta\sum_{j=1}^N\Big(c_N^K(j)\Big)^2\ell_N(j)\Big\}\Big]\\
\le&\frac{C\Delta}{K}
\end{aligned}$$

which completes the proof of (i). From Lemma 2.5.9, we conclude that

$$\begin{aligned}
&\frac{K}{\sqrt{\Delta t}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}\Big|\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta} - \bar{X}^{N,K}_{(a-1)\Delta,a\Delta}\Big|\Big]\\
\le&\frac{K}{\sqrt{\Delta t}}\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}\Big|^2\Big]^{\frac12}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta} - \bar{X}^{N,K}_{(a-1)\Delta,a\Delta}\Big|^2\Big]^{\frac12}\\
\le&\frac{C\sqrt{K}}{\Delta^{q-\frac12}\sqrt{Nt}}\sum_{a=\frac{t}{\Delta}}^{\frac{2t}{\Delta}}a^{\frac12-q} \le \frac{C\sqrt{K}}{\Delta^{q-\frac12}\sqrt{Nt}}.
\end{aligned}$$

For the last step, we used that since $q \ge 2$, we always have $\sum_{a=1}^\infty a^{\frac12-q} < +\infty$.

Lemma 2.5.11. *Assume $H(q)$ for some $q \ge 1$. Then*

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\frac{K}{N}\sqrt{\frac{t}{\Delta}}\frac{N}{t}\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\bar{\Gamma}^{N,K}_{(a-1)\Delta,a\Delta}\bar{X}^{N,K}_{(a-1)\Delta,a\Delta} - \mathbb{E}_\theta[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\bar{\Gamma}^{N,K}_{(a-1)\Delta,a\Delta}\bar{X}^{N,K}_{(a-1)\Delta,a\Delta}]\Big|\Big]\\
\le&\frac{CK}{N\Delta^q\sqrt{t}} + \frac{C\sqrt{tK}}{\Delta^{q+\frac12}\sqrt{N}} + \frac{C\sqrt{K}}{\sqrt{N\Delta}} + \frac{Ct^{\frac34}}{\Delta^{1+\frac{q}{2}}}.
\end{aligned}$$

Proof. Recalling (2.16) and (2.17), we write

$$\begin{aligned}
&\bar{\Gamma}^{N,K}_{(a-1)\Delta,a\Delta}\bar{X}^{N,K}_{(a-1)\Delta,a\Delta} = \bar{\Gamma}^{N,K}_{(a-1)\Delta,a\Delta}(\bar{X}^{N,K}_{(a-1)\Delta,a\Delta} - \mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}) + \mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}\bar{\Gamma}^{N,K}_{(a-1)\Delta,a\Delta}\\
=&\bar{\Gamma}^{N,K}_{(a-1)\Delta,a\Delta}(\bar{X}^{N,K}_{(a-1)\Delta,a\Delta} - \mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}) + \mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}(C^{N,K}_{a\Delta} + B^{N,K}_{a\Delta} - C^{N,K}_{(a-1)\Delta} - B^{N,K}_{(a-1)\Delta}).
\end{aligned}$$

From Lemmas 2.5.9, 2.5.5 and 2.5.6, we get

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\bar{\Gamma}^{N,K}_{(a-1)\Delta,a\Delta}\Big(\bar{X}^{N,K}_{(a-1)\Delta,a\Delta} - \mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}\Big)\Big|\Big]^2\\
\le&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta} - \bar{X}^{N,K}_{(a-1)\Delta,a\Delta}\Big|^2\Big]\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}(\bar{\Gamma}^{N,K}_{(a-1)\Delta,a\Delta})^2\Big]\\
=&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta} - \bar{X}^{N,K}_{(a-1)\Delta,a\Delta}\Big|^2\Big]\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big(C^{N,K}_{a\Delta} + B^{N,K}_{a\Delta} - C^{N,K}_{(a-1)\Delta} - B^{N,K}_{(a-1)\Delta}\Big)^2\Big]\\
\le&4\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta} - \bar{X}^{N,K}_{(a-1)\Delta,a\Delta}\Big|^2\Big]\\
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big\{\Big(C^{N,K}_{a\Delta}\Big)^2 + \Big(B^{N,K}_{a\Delta}\Big)^2 + \Big(C^{N,K}_{(a-1)\Delta}\Big)^2 + \Big(B^{N,K}_{(a-1)\Delta}\Big)^2\Big\}\Big]\\
&\le \frac{C}{N}\Big[(a\Delta)^{1-2q} + \Big((a-1)\Delta\Big)^{1-2q}\Big]\Big(\frac{1}{N} + \frac{1}{N}\Delta^{1-2q}\Big) \le \Big[(a\Delta)^{1-2q} + \Big((a-1)\Delta\Big)^{1-2q}\Big]\frac{C}{N^2}.
\end{aligned}$$

And by Lemmas 2.5.5 and 2.5.10, we have:

$$\begin{aligned}&\mathbf{1}_{\Omega_{N,K}}\mathbb{E}_\theta\Big[\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})\Big|\Big]^2\\ \leq&\mathbf{1}_{\Omega_{N,K}}\mathbb{E}_\theta\Big[\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}\Big|^2\Big]\mathbb{E}_\theta\Big[\Big(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta}\Big)^2\Big]\leq\frac{C}{NK\Delta^{2q-2}}.\end{aligned}$$

Next, we consider the term $\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})$. We can write

$$\begin{aligned}\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})=\frac{1}{K^2}\sum_{i=1}^K\sum_{j,j'=1}^N\sum_{n\geq1}\int_0^\Delta\phi^{*n}(s)A^n_N(i,j)c^K_N(j')\\ (M^{j,N}_{a\Delta-s}-M^{j,N}_{a\Delta}-M^{j,N}_{(a-1)\Delta-s}+M^{j,N}_{(a-1)\Delta})(M^{j',N}_{a\Delta}-M^{j',N}_{(a-1)\Delta}).\end{aligned}$$

We set for $1\leq j,j',l,l'\leq N$ and $a,\ b\in\{t/(2\Delta)+1,...,2t/\Delta\}$

$$\begin{aligned}\Upsilon_{a,b}(j,j',l,l'):=&\mathbb{C}ov_\theta[(M^{j,N}_{a\Delta-s}-M^{j,N}_{a\Delta}-M^{j,N}_{(a-1)\Delta-s}+M^{j,N}_{(a-1)\Delta})(M^{j',N}_{a\Delta}-M^{j',N}_{(a-1)\Delta}),\\ &(M^{l,N}_{b\Delta-s}-M^{l,N}_{b\Delta}-M^{l,N}_{(b-1)\Delta-s}+M^{l,N}_{(b-1)\Delta})(M^{l',N}_{b\Delta}-M^{l',N}_{(b-1)\Delta})]\end{aligned}$$

It is obvious that without any condition on (a,b), we have that on $\Omega_{N,K}$

$$\begin{aligned}&|\Upsilon_{a,b}(j,j',l,l')|\\ \leq&\Big\{\mathbb{E}_\theta\Big[\Big(M^{j,N}_{a\Delta}-M^{j,N}_{(a-1)\Delta}\Big)^4\Big]^{\frac14}+\mathbb{E}_\theta\Big[\Big(M^{j,N}_{a\Delta-s}-M^{j,N}_{(a-1)\Delta-s}\Big)^4\Big]^{\frac14}\Big\}\mathbb{E}_\theta\Big[\Big(M^{j',N}_{a\Delta}-M^{j',N}_{(a-1)\Delta}\Big)^4\Big]^{\frac14}\\ &\Big\{\mathbb{E}_\theta\Big[\Big(M^{l,N}_{b\Delta}-M^{l,N}_{(b-1)\Delta}\Big)^4\Big]^{\frac14}+\mathbb{E}_\theta\Big[\Big(M^{l,N}_{b\Delta-s}-M^{l,N}_{(b-1)\Delta-s}\Big)^4\Big]^{\frac14}\Big\}\mathbb{E}_\theta\Big[\Big(M^{l',N}_{b\Delta}-M^{l',N}_{(b-1)\Delta}\Big)^4\Big]^{\frac14}\\ \leq&C\Delta^2\end{aligned}$$

and $\mathbf{1}_{\#\{j,j',l,l'\}=4}|\Upsilon_{a,b}(j,j',l,l')|=0$.

Next, we consider the case when $a-b\geq4$. Recalling that $\zeta^{j,N}_{a\Delta,s}:=M^{j,N}_{(a\Delta-s)}-M^{j,N}_{a\Delta}$ for $0\leq s\leq\Delta$,

$$\begin{aligned}\Upsilon_{a,b}(j,j',l,l')=&\mathbb{C}ov_\theta[(\zeta^{j,N}_{a\Delta,s}-\zeta^{j,N}_{(a-1)\Delta,s})\zeta^{j',N}_{a\Delta,\Delta},(\zeta^{l,N}_{b\Delta,r}-\zeta^{l,N}_{(b-1)\Delta,r})\zeta^{l',N}_{b\Delta,\Delta}]\\ =&\mathbb{C}ov_\theta[\zeta^{j,N}_{a\Delta,s}\zeta^{j',N}_{a\Delta,\Delta},(\zeta^{l,N}_{b\Delta,r}-\zeta^{l,N}_{(b-1)\Delta,r})\zeta^{l',N}_{b\Delta,\Delta}].\end{aligned}$$

Using the same strategy as the proof in Lemma 2.5.7, we have

$$\begin{aligned}&|\mathbb{C}ov_\theta[\zeta^{j,N}_{a\Delta,s}\zeta^{j',N}_{a\Delta,\Delta},(\zeta^{l,N}_{b\Delta,r}-\zeta^{l,N}_{(b-1)\Delta,r})\zeta^{l',N}_{b\Delta,\Delta}]|\\ =&|\mathbb{C}ov_\theta[T^{j,N}_{a\Delta,(a-1)\Delta},(\zeta^{l,N}_{b\Delta,r}-\zeta^{l,N}_{(b-1)\Delta,r})\zeta^{l',N}_{b\Delta,\Delta}]|\\ \leq&\{\mathbb{E}_\theta[(T^{j,N}_{a\Delta,(a-1)\Delta})^2]^{\frac12}\}\mathbb{E}_\theta[(\zeta^{l,N}_{b\Delta,r}-\zeta^{l,N}_{(b-1)\Delta,r})^4]^{\frac14}\mathbb{E}_\theta[(\zeta^{l',N}_{b\Delta,\Delta})^4]^{\frac14}\\ \leq&Ct^{1/2}\Delta^{-q}\Delta.\end{aligned}$$

Hence, by symmetry, for $|a-b|\geq4$, we have $|\Upsilon_{a,b}(j,j',l,l')|\leq C\Big(\mathbf{1}_{l=l'}+\mathbf{1}_{j=j'}\Big)\sqrt{t}\Delta^{1-q}$. Hence,

still for $|a-b|\geq 4$,

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathbb{C}ov_\theta\Big[\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta}),\mathcal{Y}^{N,K}_{(b-1)\Delta,b\Delta}(C^{N,K}_{b\Delta}-C^{N,K}_{(b-1)\Delta})\Big]\Big|\Big]\\
=&\frac{1}{K^4}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\sum_{i,i'=1}^{K}\sum_{l,l',j,j'=1}^{N}\sum_{n,n'\geq 1}\int_0^\Delta\int_0^\Delta\\
&\qquad\phi^{*n}(s)\phi^{*n'}(s')A^n_N(i,j)A^{n'}_N(i',l)c^K_N(j')c^K_N(l')\Upsilon_{a,b}(j,j',l,l')dsds'\Big|\Big]\\
\leq&\frac{t^{1/2}}{K^4\Delta^{q-1}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\sum_{j=1}^{N}\Big(c^K_N(j)-1\Big)c^K_N(j)\Big|\sum_{l=1}^{N}c^K_N(l)\Big|\sum_{n\geq 1}N\Lambda^n|||I_KA_N|||_1|||A_N|||_1^{n-1}\Big]\\
\leq&\frac{Ct^{1/2}}{K^2\Delta^{q-1}}.
\end{aligned}$$

The last step follows from Lemma 2.5.10 in which we have proved that $\mathbb{E}[\mathbf{1}_{\Omega_{N,K}}\|\boldsymbol{c}^K_N\|_2^2]\leq CK$ and from the fact that on the event $\Omega_{N,K}$, it holds that $|||I_KA_N|||_1\leq\frac{K}{N}$, $|\sum_{l=1}^N c^K_N(l)|=K|\bar{\ell}^K_N|\leq CK$.

Next, when $|a-b|\leq 4$,

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathbb{C}ov_\theta\Big[\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta}),\mathcal{Y}^{N,K}_{(b-1)\Delta,b\Delta}(C^{N,K}_{b\Delta}-C^{N,K}_{(b-1)\Delta})\Big]\Big|\Big]\\
\leq&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\mathbb{E}_\theta\Big[\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}\Big|^2\Big]\mathbb{E}_\theta\Big[\Big(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta}\Big)^2\Big]\mathbb{E}_\theta\Big[\Big|\mathcal{Y}^{N,K}_{(b-1)\Delta,b\Delta}\Big|^2\Big]\mathbb{E}_\theta\Big[\Big(C^{N,K}_{b\Delta}-C^{N,K}_{(b-1)\Delta}\Big)^2\Big]\Big]\\
\leq&\frac{C\Delta}{NK}.
\end{aligned}$$

Finally,

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\mathbb{V}ar_\theta\Big[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})\Big]\Big]\leq\frac{Ct}{NK}+\frac{Ct^{5/2}}{K^2\Delta^{q+1}}.$$

Overall we conclude that

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\frac{K}{N}\sqrt{\frac{t}{\Delta}}\frac{N}{t}\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\bar{\Gamma}^{N,K}_{(a-1)\Delta,a\Delta}\bar{X}^{N,K}_{(a-1)\Delta,a\Delta}-\mathbb{E}_\theta[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\bar{\Gamma}^{N,K}_{(a-1)\Delta,a\Delta}\bar{X}^{N,K}_{(a-1)\Delta,a\Delta}]\Big|\Big]\\
\leq&\frac{K}{\sqrt{\Delta t}}\Big\{\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\Big(\Big|\bar{\Gamma}^{N,K}_{(a-1)\Delta,a\Delta}(\bar{X}^{N,K}_{(a-1)\Delta,a\Delta}-\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta})\Big|\\
&+\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}(B^{N,K}_{a\Delta}-B^{N,K}_{(a-1)\Delta})\Big|\Big)\Big]+\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\mathbb{V}ar_\theta\Big[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}(C^{N,K}_{a\Delta}-C^{N,K}_{(a-1)\Delta})\Big]\Big]^{\frac{1}{2}}\Big\}\\
\leq&\frac{CK}{N\Delta^q\sqrt{t}}+\frac{C\sqrt{tK}}{\Delta^{q+\frac{1}{2}}\sqrt{N}}+\frac{C\sqrt{K}}{\sqrt{N\Delta}}+\frac{Ct^{\frac{3}{4}}}{\Delta^{1+\frac{q}{2}}}.
\end{aligned}$$

The proof is finished.

Finally, we can give the proof of Lemma 2.5.4.

Proof. Recalling (2.13) and (2.12), as well as Lemmas 2.5.8, 2.5.11, 2.5.10-(ii) and 2.5.9,

$$\begin{aligned}
&\frac{K}{N}\sqrt{\frac{t}{\Delta}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|D^{N,K,3}_{\Delta,t}-\frac{N}{t}\mathbb{X}^{N,K}_{\Delta,t,1}\Big|\Big]\\
\leq&\frac{2K}{\sqrt{t\Delta}}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big(\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\Big\{(\bar\Gamma^{N,K}_{(a-1)\Delta,a\Delta})^2-\mathbb{E}_\theta[(\bar\Gamma^{N,K}_{(a-1)\Delta,a\Delta})^2]\Big\}\Big|\\
&\quad+\Big|\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\bar\Gamma^{N,K}_{(a-1)\Delta,a\Delta}\bar X^{N,K}_{(a-1)\Delta,a\Delta}-\mathbb{E}_\theta[\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\bar\Gamma^{N,K}_{(a-1)\Delta,a\Delta}\bar X^{N,K}_{(a-1)\Delta,a\Delta}]\Big|\\
&\quad+\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}\Big|\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}-\bar X^{N,K}_{(a-1)\Delta,a\Delta}\Big|+\sum_{a=\frac{t}{\Delta}+1}^{\frac{2t}{\Delta}}\Big|\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta}-\bar X^{N,K}_{(a-1)\Delta,a\Delta}\Big|^2\Big)\Big]\\
\leq&\frac{CK}{N\Delta}+\frac{CKt^{\frac34}}{\Delta^{(1+\frac{q}{2})}\sqrt{N}}+\frac{CK\sqrt{t}}{N\Delta^{(q+1)}}+\frac{CK}{N\Delta^q\sqrt{t}}+\frac{C\sqrt{tK}}{\Delta^{q+\frac12}\sqrt{N}}\\
&\quad+\frac{C\sqrt{K}}{\sqrt{N\Delta}}+\frac{Ct^{\frac34}}{\Delta^{1+\frac{q}{2}}}+\frac{C\sqrt{K}}{\Delta^{q-\frac12}\sqrt{Nt}}+\frac{CK}{N\sqrt{t}\Delta^{2q+\frac12}}\\
\leq&\frac{CKt^{\frac34}}{\Delta^{1+\frac{q}{2}}\sqrt{N}}+\frac{C\sqrt{K}}{\sqrt{N\Delta}}+\frac{Ct^{\frac34}}{\Delta^{1+\frac{q}{2}}},
\end{aligned}$$

which completes the proof.

2.5.2 The convergence of $\mathbb{X}^{N,K}_{\Delta_t,t,v}$

The aim of this subsection is to prove the following Lemma.

Lemma 2.5.12. *Assume $H(q)$ for some $q>3$. For $t\geq 1$, set $\Delta_t = t/(2\lfloor t^{1-4/(q+1)}\rfloor)\sim t^{4/(q+1)}/2$ (as $t\to\infty$). In the limit $(N,K,t)\to(\infty,\infty,\infty)$ and in the regime where $\frac{K}{N}\to\gamma\leq 1$ and where $\frac{1}{\sqrt{K}}+\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}+\frac{N}{t\sqrt{K}}+Ne^{-c_{p,\Lambda}K}\to 0$, it holds that*

$$\left(\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\frac{N}{t}\mathbb{X}^{N,K}_{\Delta_t,t,v}\right)_{v\geq 0}\xrightarrow{d}\frac{1}{\sqrt{2}}\Big(\frac{1-\gamma}{(1-\Lambda p)}+\frac{\gamma}{(1-\Lambda p)^3}\Big)(B_{2v}-B_v)_{v\geq 0}$$

for the Skorohod topology, where B is a standard Brownian motion.

We start by applying the Ito formula to write

$$(K\mathcal{Y}^{N,K}_{(a-1)\Delta,a\Delta})^2=Q_{a,N,K}+\sum_{j=1}^N\Big(c^K_N(j)\Big)^2\Big(Z^{j,N}_{a\Delta}-Z^{j,N}_{(a-1)\Delta}\Big)\tag{2.18}$$

where $Q_{a,N,K}=2\int_{(a-1)\Delta}^{a\Delta}\sum_{j=1}^N c^K_N(j)(M^{j,N}_{s-}-M^{j,N}_{(a-1)\Delta})\sum_{j=1}^N c^K_N(j)dM^{j,N}_s$. First, we verify that:

Lemma 2.5.13. *Assume $H(q)$ for some $q\geq 1$, $0\leq v\leq 1$,*

$$\mathbb{E}_\theta\Big[\Big|\frac{1}{K\sqrt{\Delta t}}\sum_{a=[\frac{vt}{\Delta}]+1}^{[\frac{2vt}{\Delta}]}\sum_{j=1}^N\Big(c^K_N(j)\Big)^2\Big(Z^{j,N}_{a\Delta}-Z^{j,N}_{(a-1)\Delta}-\mathbb{E}_\theta[Z^{j,N}_{a\Delta}-Z^{j,N}_{(a-1)\Delta}]\Big)\Big|\Big]\leq\frac{C}{\sqrt{\Delta t}}.$$

Proof.

$$\frac{1}{K\sqrt{\Delta t}}\sum_{a=[\frac{vt}{\Delta}]+1}^{[\frac{2vt}{\Delta}]}\sum_{j=1}^{N}\Big(c_N^K(j)\Big)^2\Big(Z_{a\Delta}^{j,N}-Z_{(a-1)\Delta}^{j,N}-\mathbb{E}_\theta[Z_{a\Delta}^{j,N}-Z_{(a-1)\Delta}^{j,N}]\Big)$$
$$=\frac{1}{K\sqrt{\Delta t}}\Big\{\sum_{j=1}^{N}\Big(c_N^K(j)\Big)^2\Big(Z_{2vt}^{j,N}-Z_{vt}^{j,N}-\mu vt\ell_N(j)\Big)$$
$$+\sum_{j=1}^{N}\mathbb{E}_\theta\Big[\Big(c_N^K(j)\Big)^2\Big(\mu vt\ell_N(j)-Z_{2vt}^{j,N}+Z_{vt}^{j,N}\Big)\Big]\Big\}.$$

Recalling [14, (8)], we already know that $\mathbf{1}_{\{i=j\}}\le Q_N(i,j)\le \mathbf{1}_{\{i=j\}}+\Lambda CN^{-1}$ for all $i,j=1,\dots,N$ on $\Omega_{N,K}\subset\Omega_N^1$. So $|c_N^K(j)|=|\sum_{i=1}^K Q_N(i,j)|$ is bounded by some constant C for $j=1,...,K$ and smaller than $\frac{CK}{N}$ for $K+1\le j\le N$. By [14, Lemma 16-(ii)], we also know that

$$\max_{j=1,\dots,N}\mathbb{E}_\theta\Big[\Big|\Big(Z_{2vt}^{j,N}-Z_{vt}^{j,N}-\mu vt\ell_N(j)\Big)\Big|\Big]\le C.$$

Hence

$$\mathbb{E}_\theta\Big[\frac{1}{K\sqrt{\Delta t}}\sum_{j=1}^{N}\Big|\Big(c_N^K(j)\Big)^2\Big(Z_{2vt}^{j,N}-Z_{vt}^{j,N}-\mu vt\ell_N(j)\Big)\Big|\Big]\le\frac{C}{\sqrt{\Delta t}}.$$

So

$$\mathbb{E}_\theta\Big[\Big|\frac{1}{K\sqrt{\Delta t}}\sum_{a=[\frac{vt}{\Delta}]+1}^{[\frac{2vt}{\Delta}]}\sum_{j=1}^{N}\Big(c_N^K(j)\Big)^2\Big(Z_{a\Delta}^{j,N}-Z_{(a-1)\Delta}^{j,N}-\mathbb{E}_\theta[Z_{a\Delta}^{j,N}-Z_{(a-1)\Delta}^{j,N}]\Big)\Big|\Big]\le\frac{C}{\sqrt{\Delta t}},$$

which ends the proof.

We next define

$$\mathcal{L}_{N,K}^{t,\Delta}(u):=\frac{1}{K\sqrt{\Delta t}}\sum_{a=1}^{[\frac{t}{\Delta}u]}Q_{a,N,K},\qquad\text{for } 0\le u\le 2.$$

We notice $\mathbb{E}[Q_{a,N,K}|\mathcal{F}_{a-1}]=0$. So $\mathcal{L}_{N,K}^{t,\Delta}(u)$ is a martingale for the filtration $\mathcal{F}_{[\frac{t}{\Delta}u]}$. Recalling the equality (2.18) and definition (2.12), as well as Lemma 2.5.13, we conclude the following estimate.

Corollary 2.5.14. *Assume $H(q)$ for some $q\ge 1$, then for $0\le v\le 1$,*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\mathcal{L}_{N,K}^{t,\Delta}(2v)-\mathcal{L}_{N,K}^{t,\Delta}(v)-\frac{K}{\sqrt{\Delta t}}\mathbb{X}_{\Delta,t,v}^{N,K}\Big|\Big]\le\frac{C}{\sqrt{\Delta t}}.$$

Next we will prove the convergence of $\mathcal{L}_{N,K}^{t,\Delta}(u)$ to a Brownian motion.

Lemma 2.5.15. *Assume $H(q)$ for some $q\ge 1$. Then a.s. on the set $\Omega_{N,K}$, for all $\Delta\ge 1$:*

$$\mathbb{E}_\theta[(Q_{a,N,K})^4]\le C(K\Delta)^4.$$

Proof. For $0\le u\le 1$, we set

$$q_{a,N,K}(u):=\int_{(a-1)\Delta}^{[(a-1)+u]\Delta}\sum_{j=1}^{N}c_N^K(j)(M_s^{j,N}-M_{(a-1)\Delta}^{j,N})d\Big(\sum_{j=1}^{N}c_N^K(j)M_s^{j,N}\Big).$$

It is obvious $q_{a,N,K}(1)=Q_{a,N,K}$. And

$$[q_{a,N,K}(.),q_{a,N,K}(.)]_u=\int_{(a-1)\Delta}^{(a-1+u)\Delta}\Big(\sum_{j=1}^{N}c_N^K(j)(M_s^{j,N}-M_{(a-1)\Delta}^{j,N})\Big)^2\sum_{j=1}^{N}\Big(c_N^K(j)\Big)^2dZ_s^{j,N}.$$

From [14, (8)], we already have on the event $\Omega_{N,K}$, $\mathbf{1}_{\{i=j\}} \leq Q_N(i,j) \leq \mathbf{1}_{\{i=j\}} + \frac{C}{N}$ for all $i,j = 1,...,N$. Recalling that $c_N^K(i) = \sum_{j=1}^K Q_N(j,i)$,

$$1 \leq c_N^K(i) \leq 1 + \frac{CK}{N} \text{ when } 1 \leq i \leq K \text{ and } 0 \leq c_N^K(i) \leq \frac{CK}{N} \text{ when } (K+1) \leq i \leq N.$$

Then we conclude that

$$\sum_{j=1}^N \Big(c_N^K(j)\Big)^2 dZ_s^{j,N} \leq C\Big(K d\bar{Z}_s^{N,K} + \frac{K^2}{N} d\bar{Z}_s^{N,N}\Big).$$

On $\Omega_{N,K}$, we have

$$\begin{aligned}
&\mathbb{E}_\theta[(q_{a,N,K}(u))^4] \leq 4\mathbb{E}_\theta[([q_{a,N,K}(.), q_{a,N,K}(.)]_u)^2] \\
=&4\mathbb{E}_\theta\Big[\Big(\int_{(a-1)\Delta}^{(a-1+u)\Delta} \Big(\sum_{j=1}^N c_N^K(j)(M_s^{j,N} - M_{(a-1)\Delta}^{j,N})\Big)^2 d\sum_{j=1}^N \Big(c_N^K(j)\Big)^2 Z_s^{j,N}\Big)^2\Big] \\
\leq&4\mathbb{E}_\theta\Big[\sup_{0\leq s\leq u\Delta}\Big(\sum_{j=1}^N c_N^K(j)(M_{(a-1)\Delta+s}^{j,N} - M_{(a-1)\Delta}^{j,N})\Big)^4 \Big(\sum_{j=1}^N \Big(c_N^K(j)\Big)^2 (Z_{(a-1+u)\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N})\Big)^2\Big] \\
\leq&8\mathbb{E}_\theta\Big[\sup_{0\leq s\leq u\Delta}\Big(\sum_{j=1}^N c_N^K(j)(M_{(a-1)\Delta+s}^{j,N} - M_{(a-1)\Delta}^{j,N})\Big)^8 + \Big(\sum_{j=1}^N \Big(c_N^K(j)\Big)^2 (Z_{(a-1+u)\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N})\Big)^4\Big] \\
\leq&C\mathbb{E}_\theta\Big[\Big(\sum_{j=1}^N \Big(c_N^K(j)\Big)^2 (Z_{(a-1+u)\Delta}^{j,N} - Z_{(a-1)\Delta}^{j,N})\Big)^4\Big] \\
\leq&C\mathbb{E}_\theta\Big[\Big(K(\bar{Z}_{(a-1+u)\Delta}^{N,K} - \bar{Z}_{(a-1)\Delta}^{N,K}) + \frac{K^2}{N}(\bar{Z}_{(a-1+u)\Delta}^{N,N} - \bar{Z}_{(a-1)\Delta}^{N,N})\Big)^4\Big] \leq C(Ku\Delta)^4.
\end{aligned}$$

We used the Burkholder-Davies-Gundy inequality in the fourth step as well as Lemma 2.2.3-(iii) in the last one.

Next, we are going to prove that the jumps of $\mathcal{L}_{N,K}^{t,\Delta}(u)$ are not large.

Lemma 2.5.16. *Assume $H(q)$ for some $q \geq 1$.*

$$\mathbf{1}_{\Omega_{N,K}}\mathbb{E}_\theta\Big[\sup_{0\leq u\leq 2}\Big|\mathcal{L}_{N,K}^{t,\Delta}(u) - \mathcal{L}_{N,K}^{t,\Delta}(u-)\Big|\Big] \leq C\Big(\frac{\Delta}{t}\Big)^{\frac{1}{4}}.$$

Proof. First, we notice $\mathcal{L}_{N,K}^{t,\Delta}(u)$ is a pure jump process. So

$$\mathcal{L}_{N,K}^{t,\Delta}(u) - \mathcal{L}_{N,K}^{t,\Delta}(u-) = \frac{1}{K\sqrt{\Delta t}}\Big([\frac{t}{\Delta}u] - [\frac{t}{\Delta}u-]\Big)Q_{[\frac{t}{\Delta}u],N,K}.$$

Then by Lemma 2.5.15, we have

$$\begin{aligned}
\mathbb{E}_\theta\Big[\sup_{0\leq u\leq 2}\Big|\mathcal{L}_{N,K}^{t,\Delta}(u) - \mathcal{L}_{N,K}^{t,\Delta}(u-)\Big|\Big] =& \frac{1}{K\sqrt{\Delta t}}\mathbb{E}_\theta\Big[\sup_{\{i=1...[\frac{2t}{\Delta}]\}}|Q_{[i\Delta],N,K}|\Big] \\
\leq& \frac{1}{K\sqrt{\Delta t}}\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^{[\frac{2t}{\Delta}]}|Q_{[i\Delta],N,K}|^4\Big)^{\frac{1}{4}}\Big] \\
\leq& \frac{1}{K\sqrt{\Delta t}}\mathbb{E}_\theta\Big[\sum_{i=1}^{[\frac{2t}{\Delta}]}|Q_{[i\Delta],N,K}|^4\Big]^{\frac{1}{4}} \\
\leq& C\Big(\frac{\Delta}{t}\Big)^{\frac{1}{4}}.
\end{aligned}$$

The proof is complete.

Lemma 2.5.17. *Assume $H(q)$ for some $q \geq 1$. For all $t \geq \Delta$, it holds that*

$$\begin{aligned}&\frac{1}{K^2\Delta t}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|A^{N,K}_{\infty,\infty}\Big\{\sum_{j=1}^{N}\Big(c^K_N(j)\Big)^2\sum_{a=0}^{\frac{t}{\Delta}}\int_{a\Delta}^{(a+1)\Delta}\mathbb{E}_\theta[Z^{j,N}_s - Z^{j,N}_{a\Delta}(s)]ds\Big\} - \frac{\Delta t}{2}\Big(A^{N,K}_{\infty,\infty}\Big)^2\Big|\Big]\\ \leq&\frac{C}{\Delta^q}+\frac{C}{t},\end{aligned}$$

where $A^{N,K}_{\infty,\infty} := \sum_{i=1}^{N}\Big(c^K_N(i)\Big)^2\ell_N(i)$.

Proof. Recall that on $\Omega_{N,K}$, $|\ell_N(i)| \leq C$ for all $1 \leq i \leq N$.

$$1 \leq c^K_N(i) \leq 1 + \frac{CK}{N} \text{ when } 1 \leq i \leq K \text{ and } 0 \leq c^K_N(i) \leq \frac{CK}{N} \text{ when } (K+1) \leq i \leq N.$$

By [14, Lemma 16 (ii)], we have

$$\begin{aligned}&\frac{1}{K^2\Delta t}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|A^{N,K}_{\infty,\infty}\Big\{\sum_{j=1}^{N}\Big(c^K_N(j)\Big)^2\sum_{a=0}^{\frac{t}{\Delta}}\int_{a\Delta}^{(a+1)\Delta}\mathbb{E}_\theta[Z^{j,N}_s - Z^{j,N}_{a\Delta}(s)]ds\Big\} - \frac{\Delta t}{2}\Big(A^{N,K}_{\infty,\infty}\Big)^2\Big|\Big]\\ =&\frac{1}{K^2\Delta t}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|A^{N,K}_{\infty,\infty}\Big\{\sum_{j=1}^{N}\Big(c^K_N(j)\Big)^2\sum_{a=0}^{\frac{t}{\Delta}}\int_{a\Delta}^{(a+1)\Delta}\mathbb{E}_\theta\Big[Z^{j,N}_s - Z^{j,N}_{a\Delta}(s) - \mu(s-a\Delta)\ell_N(j)\Big]ds\Big\}\Big|\Big]\\ \leq&\frac{CK^2}{K^2\Delta t}\sum_{a=0}^{\frac{t}{\Delta}}\int_{a\Delta}^{(a+1)\Delta}(1\wedge(a\Delta)^{1-q})ds \leq \frac{C}{\Delta^q}+\frac{C}{t}.\end{aligned}$$

This concludes the proof.

Lemma 2.5.18. *We assume $H(q)$ for some $q \geq 1$. For $0 \leq u \leq 2$,*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|[\mathcal{L}^{t,\Delta}_{N,K}(.),\mathcal{L}^{t,\Delta}_{N,K}(.)]_u - \frac{u(A^{N,K}_{\infty,\infty})^2}{2K^2}\Big|\Big] \leq C\Big(\frac{1}{K\Delta}+\frac{1}{\sqrt{N}}+\Big(\frac{K\sqrt{t}}{\Delta^{q+1}}\Big)^{\frac{1}{2}}+\sqrt{\frac{\Delta}{t}}\Big)$$

Proof. For $s \geq 0$, we introduce $\phi_{t,\Delta}(s) = a\Delta$, where a is the unique integer such that $a\Delta \leq s < (a+1)\Delta$. Then we have

$$\mathcal{L}^{t,\Delta}_{N,K}(u) = \int_0^{tu}\sum_{j=1}^{N}c^K_N(j)(M^{j,N}_s - M^{j,N}_{\phi_{t,\Delta}(s)})\sum_{i=1}^{N}dM^{j,N}_s.$$

So

$$\begin{aligned}[\mathcal{L}^{t,\Delta}_{N,K}(.),\mathcal{L}^{t,\Delta}_{N,K}(.)]_u =&\frac{1}{K^2\Delta t}\int_0^{tu}\Big(\sum_{j=1}^{N}c^K_N(j)(M^{j,N}_s - M^{j,N}_{\phi_{t,\Delta}(s)})\Big)^2\sum_{i=1}^{N}\Big(c^K_N(i)\Big)^2dZ^{i,N}_s\\ =&\mathcal{A}^{u,1}_{N,K}+\mathcal{A}^{u,2}_{N,K}+\mathcal{A}^{u,3}_{N,K},\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}^{u,1}_{N,K} :=&\frac{1}{K^2\Delta t}\int_0^{tu}\Big(\sum_{j=1}^{N}c^K_N(j)(M^{j,N}_s - M^{j,N}_{\phi_{t,\Delta}(s)})\Big)^2\sum_{i=1}^{N}\Big(c^K_N(i)\Big)^2dM^{i,N}_s,\\ \mathcal{A}^{u,2}_{N,K} :=&\frac{1}{K^2\Delta t}\int_0^{tu}\Big(\sum_{j=1}^{N}c^K_N(j)(M^{j,N}_s - M^{j,N}_{\phi_{t,\Delta}(s)})\Big)^2\sum_{i=1}^{N}\Big(c^K_N(i)\Big)^2\Big(\lambda^{i,N}_s - \mu\ell_N(i)\Big)ds,\\ \mathcal{A}^{u,3}_{N,K} :=&\Big[\mu\sum_{i=1}^{N}\Big(c^K_N(i)\Big)^2\ell_N(i)\Big]\frac{1}{K^2\Delta t}\int_0^{tu}\Big(\sum_{j=1}^{N}c^K_N(j)(M^{j,N}_s - M^{j,N}_{\phi_{t,\Delta}(s)})\Big)^2ds.\end{aligned}$$

First we give an upper-bound for $\mathcal{A}^{u,1}_{N,K}$. Recalling (2.8) and that

$$1 \le c_N^K(i) \le 1 + \frac{CK}{N} \text{ when } 1 \le i \le K \text{ and } 0 \le c_N^K(i) \le \frac{CK}{N} \text{ when } (K+1) \le i \le N,$$

and using Doob's inequality and Lemma 2.4.7, we obtain

$$\begin{aligned}
&\mathbb{E}_\theta\Big[\Big(\mathcal{A}^{u,1}_{N,K}\Big)^2\Big] = \frac{1}{K^4(\Delta t)^2}\mathbb{E}_\theta\Big[\int_0^{tu}\Big(\sum_{j=1}^N c_N^K(j)(M_s^{j,N} - M^{j,N}_{\phi_{t,\Delta}(s)})\Big)^4 \sum_{i=1}^N \Big(c_N^K(i)\Big)^4 dZ_s^{i,N}\Big] \\
\le &\frac{C}{K^4(\Delta t)^2}\mathbb{E}_\theta\Big[\max_{a=1,\dots,[\frac{ut}{\Delta}]+1}\sup_{0\le s\le u\Delta}\Big(\sum_{j=1}^N c_N^K(j)(M^{j,N}_{(a-1)\Delta+s} - M^{j,N}_{(a-1)\Delta})\Big)^4 \sum_{j=1}^N \Big(c_N^K(j)\Big)^4 Z_{tu}^{j,N}\Big] \\
\le &\frac{C}{K^4(\Delta t)^2}\mathbb{E}_\theta\Big[\max_{a=1,\dots,[\frac{ut}{\Delta}]+1}\sup_{0\le s\le u\Delta}\Big(\sum_{j=1}^N c_N^K(j)(M^{j,N}_{(a-1)\Delta+s} - M^{j,N}_{(a-1)\Delta})\Big)^8 \\
&\qquad + \Big(\sum_{j=1}^N \Big(c_N^K(j)\Big)^4 Z_{tu}^{j,N}\Big)^2\Big] \\
\le &\frac{C}{K^4(\Delta t)^2}\mathbb{E}_\theta\Big[\sum_{a=1}^{[\frac{ut}{\Delta}]+1}\sup_{0\le s\le u\Delta}\Big(\sum_{j=1}^N c_N^K(j)(M^{j,N}_{(a-1)\Delta+s} - M^{j,N}_{(a-1)\Delta})\Big)^8\Big] + \frac{C}{K^2\Delta^2} \\
\le &\frac{C}{K^4(\Delta t)^2}\mathbb{E}_\theta\Big[\sum_{a=1}^{[\frac{ut}{\Delta}]+1}\Big(\sum_{j=1}^N (c_N^K(j))^2(Z^{j,N}_{a\Delta} - Z^{j,N}_{(a-1)\Delta})\Big)^4\Big] + \frac{C}{K^2\Delta^2} \\
\le &\frac{C\Delta}{t} + \frac{C}{K^2\Delta^2}.
\end{aligned}$$

For the second term, we use Lemma 2.4.6, and we have on $\Omega_{N,K}$

$$\begin{aligned}
&\mathbb{E}_\theta\Big[\Big|\mathcal{A}^{u,2}_{N,K}\Big|\Big] \\
\le &\frac{1}{K^2\Delta t}\int_0^{tu}\mathbb{E}_\theta\Big[\Big(\sum_{j=1}^N c_N^K(j)\Big(M_s^{j,N} - M^{j,N}_{\phi_{t,\Delta}(s)}\Big)\Big)^4\Big]^{\frac12}\mathbb{E}_\theta\Big[\Big|\sum_{i=1}^N \Big(c_N^K(i)\Big)^2\Big(\lambda_s^{i,N} - \mu\ell_N(i)\Big)\Big|^2\Big]^{\frac12} ds \\
\le &\frac{1}{K^2\Delta t}\int_0^{tu}\mathbb{E}_\theta\Big[\Big(\sum_{j=1}^N \Big(c_N^K(j)\Big)^2\Big(Z_s^{j,N} - Z^{j,N}_{\phi_{t,\Delta}(s)}\Big)\Big)^2\Big]^{\frac12}\mathbb{E}_\theta\Big[\Big|\sum_{i=1}^N \Big(c_N^K(i)\Big)^2\Big(\lambda_s^{i,N} - \mu\ell_N(i)\Big)\Big|^2\Big]^{\frac12} ds \\
\le &\frac{1}{K^2\Delta t}\int_0^{tu}\mathbb{E}_\theta\Big[\Big(K\Big(\bar Z_s^{N,K} - \bar Z^{N,K}_{\phi_{t,\Delta}(s)}\Big) + \frac{K^2}{N}\Big(\bar Z_s^N - \bar Z^N_{\phi_{t,\Delta}(s)}\Big)\Big)^2\Big]^{\frac12} \\
&\qquad \times \mathbb{E}_\theta\Big[\Big|\sum_{i=1}^N \Big(c_N^K(i)\Big)^2\Big(\lambda_s^{i,N} - \mu\ell_N(i)\Big)\Big|^2\Big]^{\frac12} ds \\
\le &\frac{C}{Kt}\int_0^{tu}\sum_{i=1}^N \Big(c_N^K(i)\Big)^2\mathbb{E}_\theta\Big[\Big|(\lambda_s^{i,N} - \mu\ell_N(i))\Big|^2\Big]^{\frac12} ds \\
\le &\frac{C}{Kt}\int_0^{t}\sum_{i=1}^K \mathbb{E}_\theta\Big[\Big|\lambda_s^{i,N} - \mu\ell_N(i)\Big|^2\Big]^{\frac12} ds + \frac{C}{Nt}\int_0^{t}\sum_{i=1}^N \mathbb{E}_\theta\Big[\Big|\lambda_s^{i,N} - \mu\ell_N(i)\Big|^2\Big]^{\frac12} ds \\
\le &\frac{C}{\sqrt{N}} + \frac{C}{t^q}.
\end{aligned}$$

For the third term, we write

$$\begin{aligned}
\mathbb{V}ar_\theta(\mathcal{A}^{u,3}_{N,K}) &= \frac{(A^{N,K}_{\infty,\infty})^2}{K^4\Delta^2 t^2}\mathbb{V}ar_\theta\Big[\int_0^{ut}\Big(\sum_{j=1}^N c^K_N(j)(M^{j,N}_s - M^{j,N}_{\phi_{t,\Delta}(s)})\Big)^2 ds\Big] \\
&= \frac{(A^{N,K}_{\infty,\infty})^2}{K^4\Delta^2 t^2}\int_0^{ut}\int_0^{ut}\mathbb{C}ov_\theta\Big[\Big(\sum_{i=1}^N c^K_N(i)(M^{i,N}_s - M^{i,N}_{\phi_{t,\Delta}(s)})\Big)^2, \Big(\sum_{j=1}^N c^K_N(j)(M^{j,N}_{s'} - M^{j,N}_{\phi_{t,\Delta}(s')})\Big)^2\Big] ds ds' \\
&= \frac{(A^{N,K}_{\infty,\infty})^2}{K^4\Delta^2 t^2}\int_0^{ut}\int_0^{ut}\sum_{1\le i,i',j,j'\le N}\mathbb{C}ov_\theta\Big[c^K_N(i)c^K_N(i')(M^{i,N}_s - M^{i,N}_{\phi_{t,\Delta}(s)})(M^{i',N}_s - M^{i',N}_{\phi_{t,\Delta}(s)}), \\
&\qquad\qquad c^K_N(j)c^K_N(j')(M^{j,N}_{s'} - M^{j,N}_{\phi_{t,\Delta}(s')})(M^{j',N}_{s'} - M^{j',N}_{\phi_{t,\Delta}(s')})\Big] ds ds' \\
&= \frac{(A^{N,K}_{\infty,\infty})^2}{K^4\Delta^2 t^2}\int_0^{ut}\int_0^{ut}\Big(\mathbf{1}_{|s-s'|>3\Delta} + \mathbf{1}_{|s-s'|\le 3\Delta}\Big)\sum_{1\le i,i',j,j'\le N} \\
&\qquad\qquad \mathbb{C}ov_\theta\Big[c^K_N(i)c^K_N(i')(M^{i,N}_s - M^{i,N}_{\phi_{t,\Delta}(s)})(M^{i',N}_s - M^{i',N}_{\phi_{t,\Delta}(s)}), \\
&\qquad\qquad c^K_N(j)c^K_N(j')(M^{j,N}_{s'} - M^{j,N}_{\phi_{t,\Delta}(s')})(M^{j',N}_{s'} - M^{j',N}_{\phi_{t,\Delta}(s')})\Big] ds ds'.
\end{aligned}$$

But on $\Omega_{N,K}$, we have

$$\begin{aligned}
&\sum_{i,i',j,j'=1}^N\int_0^{ut}\int_0^{ut}\mathbf{1}_{|s-s'|\le 3\Delta}\mathbb{C}ov_\theta\Big[c^K_N(i)c^K_N(i')(M^{i,N}_s - M^{i,N}_{\phi_{t,\Delta}(s)})(M^{i',N}_s - M^{i',N}_{\phi_{t,\Delta}(s)}), \\
&\qquad\qquad c^K_N(j)c^K_N(j')(M^{j,N}_{s'} - M^{j,N}_{\phi_{t,\Delta}(s')})(M^{j',N}_{s'} - M^{j',N}_{\phi_{t,\Delta}(s')})\Big] ds ds' \\
\le&\int_0^{ut}\int_0^{ut}\mathbf{1}_{|s-s'|\le 3\Delta}\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^N c^K_N(i)(M^{i,N}_s - M^{i,N}_{\phi_{t,\Delta}(s)})\Big)^4\Big]^{\frac12} \\
&\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^N c^K_N(i)(M^{i,N}_{s'} - M^{i,N}_{\phi_{t,\Delta}(s')})\Big)^4\Big]^{\frac12} ds ds' \\
\le&\int_0^{ut}\int_0^{ut}\mathbf{1}_{|s-s'|\le 3\Delta}\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^N (c^K_N(i))^2(Z^{i,N}_s - Z^{i,N}_{\phi_{t,\Delta}(s)})\Big)^2\Big]^{\frac12} \\
&\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^N (c^K_N(i))^2(Z^{i,N}_{s'} - Z^{i,N}_{\phi_{t,\Delta}(s')})\Big)^2\Big]^{\frac12} ds ds' \\
\le& C\int_0^{ut}\int_0^{ut}\mathbf{1}_{|s-s'|\le 3\Delta}\mathbb{E}_\theta\Big[\Big(K\Big(\bar Z^{N,K}_s - \bar Z^{N,K}_{\phi_{t,\Delta}(s)}\Big) + \frac{K^2}{N}\Big(\bar Z^N_s - \bar Z^N_{\phi_{t,\Delta}(s)}\Big)\Big)^2\Big]^{\frac12} \\
&\qquad\qquad \mathbb{E}_\theta\Big[\Big(K\Big(\bar Z^{N,K}_{s'} - \bar Z^{N,K}_{\phi_{t,\Delta}(s')}\Big) + \frac{K^2}{N}\Big(\bar Z^N_{s'} - \bar Z^N_{\phi_{t,\Delta}(s')}\Big)\Big)^2\Big]^{\frac12} ds ds' \\
\le& Ct\Delta^3 K^2
\end{aligned}$$

By [14, Step 6 of the proof of Lemma 30], we already have, when $|s-s'|\ge 3\Delta$, that

$$\begin{aligned}
&\mathbb{C}ov_\theta[(M^{i,N}_s - M^{i,N}_{\phi_{t,\Delta}(s)})(M^{i',N}_s - M^{i',N}_{\phi_{t,\Delta}(s)}), (M^{j,N}_{s'} - M^{j,N}_{\phi_{t,\Delta}(s')})(M^{j',N}_{s'} - M^{j',N}_{\phi_{t,\Delta}(s')})] \\
\le& C(\mathbf{1}_{\{i=i'\}} + \mathbf{1}_{\{j=j'\}})t^{1/2}\Delta^{1-q}.
\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{1}_{\Omega_{N,K}}\sum_{i,i',j,j'=1}^{N}\int_0^t\int_0^t \mathbf{1}_{|s-s'|\geq 3\Delta}\mathbb{C}ov_\theta\Big[c_N^K(i)c_N^K(i')\Big(M_s^{i,N}-M_{\phi_{t,\Delta}(s)}^{i,N}\Big)\Big(M_s^{i',N}-M_{\phi_{t,\Delta}(s)}^{i',N}\Big),\\ c_N^K(j)c_N^K(j')\Big(M_{s'}^{j,N}-M_{\phi_{t,\Delta}(s')}^{j,N}\Big)\Big(M_{s'}^{j',N}-M_{\phi_{t,\Delta}(s')}^{j',N}\Big)\Big]dsds'\\ \leq \mathbf{1}_{\Omega_{N,K}}Ct^{5/2}\Delta^{1-q}\Big(\sum_{i=1}^{N}(c_N^K(i))^2\Big)\Big(\sum_{i=1}^{N}c_N^K(i)\Big)^2\leq CK^3t^{5/2}\Delta^{1-q}.\end{aligned}$$

Overall, we have, on $\Omega_{N,K}$

$$\mathbb{V}ar_\theta(\mathcal{A}_{N,K}^{u,3})\leq\frac{1}{K^4\Delta^2t^2}\Big(A_{\infty,\infty}^{N,K}\Big)^2\Big(\frac{K^3t^{5/2}}{\Delta^{q-1}}+t\Delta^3K^2\Big)\leq C\Big(\frac{K\sqrt{t}}{\Delta^{q+1}}+\frac{\Delta}{t}\Big)$$

Recalling (2.8), by Lemma 2.5.17, we have on $\Omega_{N,K}$,

$$\begin{aligned}&\Big|\mathbb{E}_\theta[\mathcal{A}_{N,K}^{u,3}]-\frac{u(A_{\infty,\infty}^{N,K})^2}{2K^2}\Big|\\ =&\frac{1}{\Delta tK^2}\Big|A_{\infty,\infty}^{N,K}\int_0^{ut}\sum_{j=1}^{N}\Big\{\Big(c_N^K(j)\Big)^2\mathbb{E}_\theta\Big[Z_s^{j,N}-Z_{\phi_{t,\Delta}}(s)\Big]\Big\}ds-\frac{u\Delta t(A_{\infty,\infty}^{N,K})^2}{2}\Big|\\ \leq&\frac{C}{\Delta^q}+\frac{C}{t}.\end{aligned}$$

Gathering the previous results, we obtain:

$$\begin{aligned}&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|\Big[\mathcal{L}_{N,K}^{t,\Delta}(.),\mathcal{L}_{N,K}^{t,\Delta}(.)\Big]_u-\frac{u(A_{\infty,\infty}^{N,K})^2}{2K^2}\Big|\Big]\\ \leq&\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big\{\Big|\mathcal{A}_{N,K}^{u,1}\Big|+\Big|\mathcal{A}_{N,K}^{u,2}\Big|+\mathbb{V}ar_\theta(\mathcal{A}_{N,K}^{u,3})^{\frac12}+\Big|\mathbb{E}_\theta[\mathcal{A}_{N,K}^{u,3}]-\frac{u(A_{\infty,\infty}^{N,K})^2}{2K^2}\Big|\Big\}\Big]\\ \leq&\frac{C}{K\Delta}+C\sqrt{\frac{\Delta}{t}}+\frac{C}{\sqrt{N}}+\frac{C}{t^q}+C\Big(\frac{K\sqrt{t}}{\Delta^{q+1}}+\frac{\Delta}{t}\Big)^{\frac12}+\frac{C}{\Delta^q}+\frac{C}{t}\\ \leq&C\Big(\frac{1}{K\Delta}+\frac{1}{\sqrt{N}}+\Big(\frac{K\sqrt{t}}{\Delta^{q+1}}\Big)^{\frac12}+\sqrt{\frac{\Delta}{t}}\Big).\end{aligned}$$

The proof is finished.

By Lemmas 2.3.8, 2.5.16, 2.5.18 and [23, Theorem VIII.3.8], we get the following corollary.

Corollary 2.5.19. *Assume $K\leq N$. For $t\geq 1$, set $\Delta_t=t/(2\lfloor t^{1-4/(q+1)}\rfloor)\sim t^{4/(q+1)}/2$ (for t large). We always work in the asymptotic $(N,K,t)\to(\infty,\infty,\infty)$ and in the regime where $\frac{1}{\sqrt{K}}+\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}+\frac{N}{t\sqrt{K}}+Ne^{-c_{p,\Lambda}K}\to 0$ and where $\frac{K}{N}\to\gamma\leq 1$. It holds true that*

$$(\mathcal{L}_{N,K}^{t,\Delta_t}(u))_{u\geq 0}\xrightarrow{d}\frac{1}{\sqrt{2}}\Big(\frac{1-\gamma}{(1-\Lambda p)}+\frac{\gamma}{(1-\Lambda p)^3}\Big)(B_u)_{u\geq 0}$$

for the Skorokhod topology, where B is a standard Brownian motion.

Next, we are going to give the proof of Lemma 2.5.12. *Proof.*

By Corollaries 2.5.14 and 2.5.19, we conclude that

$$\frac{K}{\sqrt{t\Delta_t}}(\mathbb{X}_{\Delta_t,t,v}^{N,K})_v\xrightarrow{d}\frac{1}{\sqrt{2}}\Big(\frac{1-\gamma}{(1-\Lambda p)}+\frac{\gamma}{(1-\Lambda p)^3}\Big)(B_{2v}-B_v)_v$$

as desired.

2.5.3 Proof of theorem 2.5.1

We notice that: $\mathbb{X}^{N,K}_{2\Delta,t,1} = \mathbb{X}^{N,K}_{\Delta,t,\frac{1}{2}}$. By Lemma 2.5.12, we have

$$\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\frac{N}{t}\Big(\mathbb{X}^{N,K}_{\Delta_t,t,1} + 2\mathbb{X}^{N,K}_{\Delta_t,t,\frac{1}{2}}\Big) \xrightarrow{d} \mathcal{N}\Big(0, \frac{3}{2}\Big(\frac{1-\gamma}{(1-\Lambda p)} + \frac{\gamma}{(1-\Lambda p)^3}\Big)^2\Big).$$

By Lemma 2.5.4, we conclude that

$$\begin{aligned}&\frac{K}{N}\sqrt{\frac{t}{\Delta}}\frac{N}{t}\mathbb{E}\Big[\mathbf{1}_{\Omega_{N,K}}\Big|D^{N,K,3}_{\Delta,t} + 2D^{N,K,3}_{2\Delta,t} - 2\mathbb{X}^{N,K}_{2\Delta,t,1} - \mathbb{X}^{N,K}_{\Delta,t,1}\Big|\Big]\\ \leq&\frac{CK}{N\Delta} + \frac{CKt^{\frac{3}{4}}}{\Delta^{1+\frac{q}{2}}\sqrt{N}} + \frac{C\sqrt{K}}{\sqrt{N\Delta}} + \frac{Ct^{\frac{3}{4}}}{\Delta^{1+\frac{q}{2}}}.\end{aligned}$$

Finally, by Corollary 2.5.3, we have

$$\begin{aligned}&\lim \mathbf{1}_{\Omega_{N,K}}\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\Big(\mathcal{X}^{N,K}_{\Delta_t,t} - \boldsymbol{X}^{N,K}_{\infty,\infty}\Big)\\ =&\lim \mathbf{1}_{\Omega_{N,K}}\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\Big\{D^{N,K,3}_{\Delta_t,t} + 2D^{N,K,3}_{2\Delta_t,t}\Big\}\\ =&\lim \frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\frac{N}{t}\Big(\mathbb{X}^{N,K}_{\Delta_t,t,1} + 2\mathbb{X}^{N,K}_{\Delta_t,t,\frac{1}{2}}\Big),\end{aligned}$$

which goes in distribution to $\mathcal{N}\Big(0, \frac{3}{2}\Big(\frac{1-\gamma}{(1-\Lambda p)} + \frac{\gamma}{(1-\Lambda p)^3}\Big)^2\Big)$.

2.6 The final result in the subcritical case

We can, at last, give the proof of Theorem 2.1.1.

Proof. One can directly check that $\Psi^{(3)}\Big(\frac{\mu}{1-\Lambda p}, \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2}, \frac{\mu}{(1-\Lambda p)^3}\Big) = p$. By the Lagrange mean value theorem, there exist some vectors $\boldsymbol{C^i_{N,K,t}}$ for $i = 1, 2, 3$, lying in the segment joining the points $(\varepsilon^{N,K}_t, \mathcal{V}^{N,K}_t, \mathcal{X}^{N,K}_{\Delta,t})$ and $\boldsymbol{C} := \Big(\frac{\mu}{1-\Lambda p}, \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2}, \frac{\mu}{(1-\Lambda p)^3}\Big)$, such that:

$$\begin{aligned}\hat{p}_{N,K,t} - p =&\Psi^{(3)}(\varepsilon^{N,K}_t, \mathcal{V}^{N,K}_t, \mathcal{X}^{N,K}_{\Delta,t}) - p\\ =&\Psi^{(3)}(\varepsilon^{N,K}_t, \mathcal{V}^{N,K}_t, \mathcal{X}^{N,K}_{\Delta,t}) - \Psi^{(3)}\Big(\frac{\mu}{1-\Lambda p}, \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2}, \frac{\mu}{(1-\Lambda p)^3}\Big)\\ =&\frac{\partial\Psi^{(3)}}{\partial x}(\boldsymbol{C^1_{N,K,t}})\Big(\varepsilon^{N,K}_t - \frac{\mu}{1-\Lambda p}\Big) + \frac{\partial\Psi^{(3)}}{\partial y}(\boldsymbol{C^2_{N,K,t}})\Big(\mathcal{V}^{N,K}_t - \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2}\Big)\\ &+ \frac{\partial\Psi^{(3)}}{\partial z}(\boldsymbol{C^3_{N,K,t}})\Big(\mathcal{X}^{N,K}_{\Delta,t} - \frac{\mu}{(1-\Lambda p)^3}\Big).\end{aligned}$$

From the first paragraph of [26, Section 9], we know that in the asymptotic $(N, K, t) \to (\infty, \infty, \infty)$ and in the regime $\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}} + Ne^{-c_{p,\Lambda}K} \to 0$, it holds that $(\varepsilon^{N,K}_t, \mathcal{V}^{N,K}_t, \mathcal{X}^{N,K}_{\Delta,t}) \to \boldsymbol{C}$ in probability. This implies that the three vectors $\boldsymbol{C}^i_{N,K,t}$, $i = 1, 2, 3$, all converge to $\boldsymbol{C} := \Big(\frac{\mu}{1-\Lambda p}, \frac{(\mu\Lambda)^2 p(1-p)}{(1-\Lambda p)^2}, \frac{\mu}{(1-\Lambda p)^3}\Big)$ in probability in the same regime.

We define from $D' := \{(u, v, w) \in \mathbb{R}^3 : w > u > 0 \text{ and } v > 0\}$ to $\mathbb{R}^3$

$$\Psi^{(1)}(u, v, w) = u\sqrt{\frac{u}{w}}, \quad \Psi^{(2)}(u, v, w) = \frac{v + (u - \Psi^{(1)})^2}{u(u - \Psi^{(1)})}.$$

Then we have $\Psi^{(3)}(u,v,w) = \frac{1-u^{-1}\Psi^{(1)}}{\Psi^{(2)}}$ in D'.

Some tedious but direct computations show that

$$\frac{\partial\Psi^{(1)}}{\partial y}(C) = 0, \quad \frac{\partial\Psi^{(1)}}{\partial z}(C) = \frac{-(1-\Lambda p)^3}{2}, \quad \frac{\partial\Psi^{(2)}}{\partial y}(C) = \frac{(1-\Lambda p)^2}{\mu^2\Lambda p},$$
$$\frac{\partial\Psi^{(2)}}{\partial z}(C) = \Big\{\frac{-2\frac{\partial\Psi^{(1)}}{\partial z}}{u} + \frac{\Psi^{(2)}\frac{\partial\Psi^{(1)}}{\partial z}}{(u-\Psi^{(1)})}\Big\}(C) = \frac{(1-\Lambda p)^4(2p-1)}{2\mu p},$$
$$\frac{\partial\Psi^{(3)}}{\partial y}(C) = -\frac{\frac{\Psi^{(2)}\frac{\partial\Psi^{(1)}}{\partial y}}{u} + (1-\frac{\Phi^{(1)}}{u})\frac{\partial\Psi^{(2)}}{\partial y}}{(\Psi^{(2)})^2}(C) = -\frac{(\Lambda p-1)^2}{\mu^2\Lambda^2},$$
$$\frac{\partial\Psi^{(3)}}{\partial z}(C) = -\frac{\frac{\Psi^{(2)}\frac{\partial\Psi^{(1)}}{\partial z}}{u} + (1-\frac{\Phi^{(1)}}{u})\frac{\partial\Psi^{(2)}}{\partial z}}{(\Psi^{(2)})^2}(C) = \frac{(1-\Lambda p)^4(1-p)}{\mu\Lambda}.$$

Case 1. In the regime with dominating term $\frac{1}{\sqrt{K}}$, i.e. when $[\frac{1}{\sqrt{K}}]/[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}} + \frac{N}{t\sqrt{K}}] \to \infty$, we write

$$\begin{aligned}\sqrt{K}\Big[\hat{p}_{N,K,t} - p\Big] =& \sqrt{K}\frac{\partial\Psi^{(3)}}{\partial x}(\boldsymbol{C^1_{N,K,t}})\Big(\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p}\Big)\\ &+ \sqrt{K}\frac{\partial\Psi^{(3)}}{\partial y}(\boldsymbol{C^2_{N,K,t}})\Big(\mathcal{V}_t^{N,K} - \frac{(\mu\Lambda)^2p(1-p)}{(1-\Lambda p)^2}\Big)\\ &+ \sqrt{K}\frac{\partial\Psi^{(3)}}{\partial z}(\boldsymbol{C^3_{N,K,t}})\Big(\boldsymbol{X}_{t,\Delta_t}^{N,K} - \frac{\mu}{(1-\Lambda p)^3}\Big).\end{aligned}$$

By Lemmas 2.4.1 and 2.3.7 and by Theorem 2.5.1, we have

$$\sqrt{K}\frac{\partial\Psi^{(3)}}{\partial x}(\boldsymbol{C^1_{N,K,t}})\Big(\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p}\Big) + \sqrt{K}\frac{\partial\Psi^{(3)}}{\partial z}(\boldsymbol{C^3_{N,K,t}})\Big(\boldsymbol{X}_{t,\Delta_t}^{N,K} - \frac{\mu}{(1-\Lambda p)^3}\Big) \xrightarrow{d} 0.$$

Next, we notice that

$$\frac{\partial\Psi^{(3)}}{\partial y}(\boldsymbol{C^2_{N,K,t}}) \overset{t,K,N\to\infty}{\longrightarrow} \frac{-(1-\Lambda p)^2}{(\mu\Lambda)^2}.$$

So by Theorems 2.3.2 and 2.4.2, we conclude that

$$\sqrt{K}\frac{\partial\Psi^{(3)}}{\partial y}(\boldsymbol{C^2_{N,K,t}})\Big(\mathcal{V}_t^{N,K} - \frac{(\mu\Lambda)^2p(1-p)}{(1-\Lambda p)^2}\Big) \xrightarrow{d} \mathcal{N}\Big(0, \frac{p^2(1-p)^2}{\mu^4}\Big).$$

Case 2. In the regime with dominating term $\frac{N}{t\sqrt{K}}$, i.e. when $[\frac{N}{t\sqrt{K}}]/[\frac{1}{\sqrt{K}} + \frac{N}{K}\sqrt{\frac{\Delta_t}{t}}] \to \infty$, we write

$$\begin{aligned}\frac{t\sqrt{K}}{N}\Big[\hat{p}_{N,K,t} - p\Big] =& \frac{t\sqrt{K}}{N}\frac{\partial\Psi^{(3)}}{\partial x}(\boldsymbol{C^1_{N,K,t}})\Big(\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p}\Big)\\ &+ \frac{t\sqrt{K}}{N}\frac{\partial\Psi^{(3)}}{\partial y}(\boldsymbol{C^2_{N,K,t}})\Big(\mathcal{V}_t^{N,K} - \frac{(\mu\Lambda)^2p(1-p)}{(1-\Lambda p)^2}\Big)\\ &+ \frac{t\sqrt{K}}{N}\frac{\partial\Psi^{(3)}}{\partial z}(\boldsymbol{C^3_{N,K,t}})\Big(\boldsymbol{X}_{t,\Delta_t}^{N,K} - \frac{\mu}{(1-\Lambda p)^3}\Big).\end{aligned}$$

By Lemmas 2.4.1 and 2.3.7 and by Theorem 2.5.1, we have

$$\frac{t\sqrt{K}}{N}\frac{\partial\Psi^{(3)}}{\partial x}(\boldsymbol{C^1_{N,K,t}})\Big(\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p}\Big) + \frac{t\sqrt{K}}{N}\frac{\partial\Psi^{(3)}}{\partial z}(\boldsymbol{C^3_{N,K,t}})\Big(\boldsymbol{X}_{t,\Delta_t}^{N,K} - \frac{\mu}{(1-\Lambda p)^3}\Big) \xrightarrow{d} 0.$$

Finally, using Theorems 2.3.2 and 2.4.2, we find that

$$\frac{t\sqrt{K}}{N}\Big[\hat{p}_{N,K,t}-p\Big] \xrightarrow{d} \mathcal{N}\Big(0,\frac{2(1-\Lambda p)^2}{\mu^2\Lambda^4}\Big)$$

Case 3. In the regime with dominating term $\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}$, i.e. when $[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]/[\frac{1}{\sqrt{K}}+\frac{N}{t\sqrt{K}}]\to\infty$, $\lim_{N,K\to\infty}\frac{K}{N}=\gamma\le 1$, by Lemmas 2.4.1 and 2.3.7 and Theorem 2.5.1, we have

$$\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}\Big\{\frac{\partial\Psi^{(3)}}{\partial x}(\boldsymbol{C^1_{N,K,t}})\Big(\varepsilon_t^{N,K}-\frac{\mu}{1-\Lambda p}\Big)+\frac{\partial\Psi^{(3)}}{\partial y}(\boldsymbol{C^2_{N,K,t}})\Big(\mathbf{V}_t^{N,K}-\frac{(\mu\Lambda)^2p(1-p)}{(1-\Lambda p)^2}\Big)$$
$$+\frac{\partial\Psi^{(3)}}{\partial z}(\boldsymbol{C^3_{N,K,t}})\Big(\boldsymbol{X}^{N,K}_{\infty,\infty}-\frac{\mu}{(1-\Lambda p)^3}\Big)\Big\}\xrightarrow{d}0.$$

Hence it suffices to study

$$\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}\frac{\partial\Psi^{(3)}}{\partial z}(\boldsymbol{C^3_{N,K,t}})\Big(\boldsymbol{X}^{N,K}_{t,\Delta_t}-\boldsymbol{X}^{N,K}_{\infty,\infty}\Big).$$

But by Theorem 2.5.1, we conclude that

$$\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}\Big[\hat{p}_{N,K,t}-p\Big]\xrightarrow{d}\mathcal{N}\Big(0,\frac{3(1-p)^2}{2\mu^2\Lambda^2}\Big(\big((1-\gamma)(1-\Lambda p)^3+\gamma(1-\Lambda p)\big)^2\Big).$$

Next, we are going to prove the proposition 2.1.3.

Proof. We remark the for the case $p=0$, the result of Theorem 2.5.1 holds true (we do not need the limit of $\frac{K}{N}$ anymore). For matrix, it is easy to check $\bar{\ell}_N^K=1$, $\mathbf{V}_\infty^{N,K}=0$ and $\boldsymbol{X}^{N,K}_{\infty,\infty}=\mu$. We define

$$f(u,v,w):=\frac{u(w-u)}{w+\sqrt{wu}}\qquad \textit{when } u>0,\ w>0\qquad \text{and } f=0 \text{ otherwise.}$$

By [26, Lemma 6.3], we have

$$\lim \mathbf{1}_{\Omega_{N,K}}\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}(\varepsilon_t^{N,K}-\mu)=0.$$

Hence by Theorem 2.5.1, we have:

$$\mathbf{1}_{\Omega_{N,K}}\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}(\mathcal{X}^{N,K}_{\Delta_t,t}-\varepsilon_t^{N,K})\xrightarrow{d}\mathcal{N}\Big(0,\frac{3}{2}\Big).$$

By [26, Lemma 6.3, corollary 8.9], we have in the regime $\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}+\frac{N}{t\sqrt{K}}+Ne^{-c_{p,\Lambda}K}\to 0$,

$$\lim\varepsilon_t^{N,K}=\lim\mathcal{X}^{N,K}_{\Delta_t,t}=\mu\quad \textit{in probability.}$$

Overall, we conclude that

$$\mathbf{1}_{\Omega_{N,K}}\frac{K}{N}\sqrt{\frac{t}{\Delta_t}}f(\varepsilon_t^{N,K},\mathcal{V}_t^{N,K},\mathcal{X}^{N,K}_{t,\Delta_t})\xrightarrow{d}\mathcal{N}\Big(0,\frac{3}{8}\Big).$$

By theorem 2.4.2, we have

$$\mathbf{1}_{\Omega_{N,K}}\frac{t\sqrt{K}}{N}\mathbf{V}_t^{N,K}\xrightarrow{d}\mathcal{N}\Big(0,2\mu^2\Big).$$

Hence in the regime $[\frac{N}{t\sqrt{K}}]/[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]^2 \to \infty$, we have

$$\lim[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]^{-2}\Big|\mathcal{V}_t^{N,K}\Big| = \infty \qquad \textit{in probability}$$

and since $\Psi^{(3)}(u,v,w) = \frac{f^2}{v+f^2}1_{\{v>0\}}$, we deduce that

$$\hat{p}_{N,K,t} = \Psi^{(3)}(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{t,\Delta_t}^{N,K}) \xrightarrow{d} 0.$$

In the regime $[\frac{N}{K}\sqrt{\frac{\Delta_t}{t}}]^2/[\frac{N}{t\sqrt{K}}] \to \infty$, we have

$$\frac{f^2(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{t,\Delta_t}^{N,K})}{\mathcal{V}_t^{N,K} + f^2(\varepsilon_t^{N,K}, \mathcal{V}_t^{N,K}, \mathcal{X}_{t,\Delta_t}^{N,K})} \to 1 \quad \text{in probability}$$

and

$$\lim P\Big(\mathcal{V}_t^{N,K} > 0\Big) = \frac{1}{2}.$$

Therefore

$$\hat{p}_{N,K,t} \xrightarrow{d} X.$$

2.7 Matrix analysis for the supercritical case

We now turn to the supercritical case and thus assume (A).

We recall that the matrix A_N is defined by $A_N(i,j) := N^{-1}\theta_{ij}$ for $i,j \in \{1,...,N\}$. We assume here that $p \in (0,1]$ and we introduce the events:

$$\Omega_N^2 := \Big\{\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N A_N(i,j) > \frac{p}{2} \quad \text{and} \quad |NA_N^2(i,j) - p^2| < \frac{p^2}{2N^{3/8}} \quad \text{for all } i,j = 1,...,N \Big\},$$

$$\Omega_N^{K,2} := \Big\{\frac{1}{K}\sum_{i=1}^K\sum_{j=1}^N A_N(i,j) > \frac{p}{2}\Big\} \cap \Omega_N^2.$$

We know from [26, Lemma 10.1] that

Lemma 2.7.1. *There are some constants $C, c > 0$ such that for all $1 \le K \le N$, we have*

$$P(\Omega_N^{K,2}) \ge 1 - Ce^{-cN^{1/4}}.$$

Next, we recall the definition of ρ_N of [14, Lemma 34].

Lemma 2.7.2. *Assume that $p \in (0,1]$. On Ω_N^2, the spectral radius ρ_N of A_N is a simple eigenvalue of A_N and $\rho_N \in [p(1-1/(2N^{3/8})), p(1+1/(2N^{3/8}))]$. There is a unique eigenvector $\boldsymbol{V}_N \in (\mathbb{R}_+)^N$ of A_N for the eigenvalue ρ_N such that $||V_N||_2 = \sqrt{N}$. We also have $V_N(i) > 0$ for all $i = 1,\dots,N$.*

Remark 2.7.3. On Ω_N^2, we define α_N as the unique number such that $\rho_N \int_0^\infty e^{-\alpha_N s}\phi(s)ds = 1$. From $\rho_N \in [p(1-1/(2N^{3/8})), p(1+1/(2N^{3/8}))]$, we conclude that $\lim_{N\to\infty}\alpha_N = \alpha_0$, where $\alpha_0 > 0$ is defined by $p\int_0^\infty e^{-\alpha_0 s}\phi(s)ds = 1$. Recall that $\Lambda p = p\int_0^\infty \phi(s)ds > 1$ by A. Furthermore, under assumption (A), we have $\alpha_N = \rho_N - b$ and $\alpha_0 = p - b$.

We introduce $\boldsymbol{V}_N^K := I_K \boldsymbol{V}_N$, as well as $\bar{V}_N^K = \frac{1}{K}\sum_{i=1}^K V_N(i)$ and we write $\bar{V}_N := \bar{V}_N^N$. We first recall the following result, see [26, Proposition 10.6].

Lemma 2.7.4. *There is $N_0 \geq 1$ and $C > 0$ (depending only on p) such that for all $N \geq N_0$ and all $K \in \{1, \dots, N\}$,*

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}}\Big|\mathcal{U}_\infty^{N,K} - \Big(\frac{1}{p} - 1\Big)\Big|\Big] \leq \frac{C}{\sqrt{K}},$$

where $\mathcal{U}_\infty^{N,K} := \frac{N}{K(\bar{V}_N^K)^2}\sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2$.

Next we state some properties of the vector $\boldsymbol{V}_N^K$.

Lemma 2.7.5. *Assume that $p \in (0, 1]$. The following properties hold true.*

(i) $\lim_{N,K\to\infty} \mathbf{1}_{\Omega_N^{K,2}}\sqrt{K}\bar{V}_N^K/\|\boldsymbol{V}_N^K\|_2 = 1$ *a.s. For any fixed* $i \in \mathbb{N}$, $\lim_{N\to\infty} \mathbf{1}_{\Omega_N^{K,2}} V_N(i) = 1$ *a.s.*

(ii) There is $C > 0$ such that on $\Omega_N^{K,2}$, we have $\frac{N}{K}\sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2 \leq CN^{\frac{1}{4}}$ a.s.

(iii) There is $C > 0$ such that $\mathbb{E}[\mathbf{1}_{\Omega_N^{K,2}}\frac{N}{K}\sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2] \leq C$.

Proof. By [14, Step 1 of the proof of Lemma 39], on $\Omega_N^{K,2} \subset \Omega_N^2$, we have, for any $i = 1, ..., N$,

$$V_N(i) \in [\kappa_N(1 - 1/(2N^{3/8})), \kappa_N(1 + 1/(2N^{3/8}))], \tag{2.19}$$

where $\kappa_N = p^2\rho_N^{-2}N^{-1}\sum_{j=1}^N V_N(j)$. Then the first result of (i) is obvious, by Lemma 2.7.1. For the second result in (i), we only need to verify that $\lim_{N\to\infty}\kappa_N = 1$. But we easily deduce from (2.19) that $\lim_{N\to\infty}\kappa_N = \lim_{N\to\infty}\bar{V}_N$, so that $\lim_{N\to\infty}\kappa_N = \lim_{N\to\infty}\|\boldsymbol{V}_N\|_2/\sqrt{N} = 1$.

By [14, Lemma 35-(ii)], we already know that $V_N(i) \in [\frac{1}{2}, 2]$, for any $i = 1, ..., N$, on Ω_N^2. By (2.19), we deduce that $\kappa_N \in [\frac{1}{4}, 4]$. Using again (2.19) gives us (ii).

For (iii), since $\bar{V}_N^K$ is bounded by some constant C (on $\Omega_N^{K,2}$) and by Lemma 2.7.4, we have

$$\mathbb{E}[\mathbf{1}_{\Omega_N^{K,2}}\frac{N}{K}\sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2] = \mathbb{E}[\mathbf{1}_{\Omega_N^{K,2}}(\bar{V}_N^K)^2\mathcal{U}_\infty^{N,K}] \leq C\mathbb{E}[\mathbf{1}_{\Omega_N^{K,2}}\mathcal{U}_\infty^{N,K}] \leq C$$

as desired.

Lemma 2.7.6. *Assume that $p \in (0, 1]$. On $\Omega_N^{K,2}$, there exists some constant N_0, such that for all $N \geq N_0$, all $K \in \{1, \dots, N\}$, for all $n \geq 2$,*

$$||I_K A_N^n \mathbf{1}_N||_2 \in \Big[\|\boldsymbol{V}_N^K\|_2\rho_N^n e^{-CN^{-\frac{3}{16}}}, \|\boldsymbol{V}_N^K\|_2\rho_N^n e^{CN^{-\frac{3}{16}}}\Big].$$

Proof.

We write $A_N^n\mathbf{1}_N = ||A_N^n\mathbf{1}_N||_2(\|\boldsymbol{V}_N\|_2^{-1}\boldsymbol{V}_N + \mathbf{Z}_{N,n})$,
where $\mathbf{Z}_{N,n} = ||A_N^n\mathbf{1}_N||_2^{-1}A_N^n\mathbf{1}_N - \|\boldsymbol{V}_N\|_2^{-1}\boldsymbol{V}_N$.
By [14, Lemma 35-(v)] (with $r = 2$), we know that $||\mathbf{Z}_{N,n}||_2 \leq 3(2N^{-3/8})^{\lfloor n/2\rfloor+1}$. We next write, for each $n \geq 0$, $A_N^{n+1}\mathbf{1}_N = ||A_N^n\mathbf{1}_N||_2(\|\boldsymbol{V}_N\|_2^{-1}\rho_N\boldsymbol{V}_N + A_N\mathbf{Z}_{N,n})$ (recall that $A_N\boldsymbol{V}_N = \rho_N\boldsymbol{V}_N$). Using the fact that $|||A_N|||_2 \leq 1$, we conclude that

$$\big|||A_N^{n+1}\mathbf{1}_N||_2 - \rho_N||A_N^n\mathbf{1}_N||_2\big| \leq 3||A_N^n\mathbf{1}_N||_2||A_N\mathbf{Z}_{N,n}||_2 \leq 3||A_N^n\mathbf{1}_N||_2(2N^{-3/8})^{\lfloor n/2\rfloor+1}.$$

We now set $x_n = ||A_N^n\mathbf{1}_N||_2/(\|\boldsymbol{V}_N\|_2\rho_N^n)$. Dividing the previous inequality by $\rho_N^{n+1}\|\boldsymbol{V}_N\|_2$ and using that $\rho_N \geq p/2$ on Ω_N^2, see Lemma 2.7.2, we have, for all $n \geq 0$,

$$|x_{n+1} - x_n| \leq 3x_n(2N^{-3/8})^{\lfloor n/2\rfloor+1}/\rho_N \leq 6x_n(2N^{-3/8})^{\lfloor n/2\rfloor+1}/p.$$

We easily conclude, using that $x_0 = 1$, that for all $n \geq 0$,

$$x_n \in \Big[\prod_{k=1}^n \Big(1 - 6(2N^{-3/8})^{\lfloor k/2\rfloor+1}/p\Big), \prod_{k=1}^n \Big(1 + 6(2N^{-3/8})^{\lfloor k/2\rfloor+1}/p\Big)\Big].$$

We conclude that there exists a constant C such that for all N large enough, for all $n \geq 0$,

$$e^{-CN^{-\frac{3}{16}}} \leq \prod_{k=1}^{n} \Big(1 - 6(2N^{-3/8})^{\lfloor k/2 \rfloor + 1}/p\Big) \leq \prod_{k=1}^{n} \Big(1 + 6(2N^{-3/8})^{\lfloor k/2 \rfloor + 1}/p\Big) \leq e^{CN^{-\frac{3}{16}}}.$$

We obtain that for all N large enough, on Ω_N^2, for all $n \geq 0$, $x_n \in [e^{-CN^{-\frac{3}{16}}}, e^{CN^{-\frac{3}{16}}}]$. We also have $x_n \in [1/2, 2]$ for all $n \geq 0$. By definition of x_n, we conclude that for all $n \geq 0$,

$$||A_N^n \mathbf{1}_N||_2 \in [\|\boldsymbol{V}_N\|_2 \rho_N^n e^{-CN^{-\frac{3}{16}}}, \|\boldsymbol{V}_N\|_2 \rho_N^n e^{CN^{-\frac{3}{16}}}].$$

Moreover, by [26, end of the proof of Lemma 10.3-(vii)], we know that for all N large enough, all $K \in \{1, \dots, N\}$, on $\Omega_N^{K,2}$, for all $n \geq 0$,

$$\Big| \frac{||I_K A_N^n \mathbf{1}_N||_2}{||A_N^n \mathbf{1}_N||_2} - \frac{\|\boldsymbol{V}_N^K\|_2}{\|\boldsymbol{V}_N\|_2} \Big| \leq 3(2N^{-3/8})^{\lfloor n/2 \rfloor + 1}.$$

Gathering the two previous estimates and using the fact that $||A_N^n \mathbf{1}_N||_2 \leq N^{1/2}$, we thus conclude that, still on $\Omega_N^{K,2}$, for all $n \geq 0$,

$$||I_K A_N^n \mathbf{1}_N||_2 \in \Big[\|\boldsymbol{V}_N^K\|_2 \rho_N^n e^{-CN^{-\frac{3}{16}}} - 3N^{1/2}(2N^{-3/8})^{\lfloor n/2 \rfloor + 1}, \\ \|\boldsymbol{V}_N^K\|_2 \rho_N^n e^{CN^{-\frac{3}{16}}} + 3N^{1/2}(2N^{-3/8})^{\lfloor n/2 \rfloor + 1}\Big].$$

The conclusion easily follows, because one can find a constant $C > 0$ such that, on $\Omega_N^{K,2}$,

$$3N^{1/2}(2N^{-3/8})^{\lfloor n/2 \rfloor + 1} \leq C\|\boldsymbol{V}_N^K\|_2 \rho_N^n (1 + CN^{-3/16}).$$

Indeed, $\|\boldsymbol{V}_N^K\|_2 \geq 1/2$ (because, as already seen, $V_N(1) \geq 1/2$), and $\rho_N \geq p/2 \geq 2N^{-3/8}$, see Lemma 2.7.2. Since $n \geq 2$ (recall the statement), this is sufficient.

Next, we define

$$v_t^{N,K} := \mu \sum_{n \geq 0} \frac{\|I_K A_N^n \mathbf{1}_N\|_2}{\|\boldsymbol{V}_N^K\|_2} \int_0^t s\phi^{*n}(t-s)ds.$$

and we recall the definition of α_N, see Remark 2.7.3.

Corollary 2.7.7. *Assume (A). There are some constants $C > 0$ and $N_0 \geq 1$, such that for all $N \geq N_0$, $t \geq 1$ on $\Omega_N^{K,2}$,*

$$v_t^{N,K} \in \Big[\mu e^{-CN^{-\frac{3}{16}}} \frac{\rho_N}{(\alpha_N)^2}(e^{\alpha_N t} - 1) - Ct, \mu e^{CN^{-\frac{3}{16}}} \frac{\rho_N}{(\alpha_N)^2}(e^{\alpha_N t} - 1) + Ct\Big].$$

Proof. For $n = 0, 1$ we always have, on $\Omega_N^{K,2}$, since $\|I_K A_N^n \mathbf{1}_N\|_2 \leq \sqrt{K}$, $\|\boldsymbol{V}_N^K\|_2 \geq c\sqrt{K}$ (since $V_N(i) \geq 1/2$ as already seen) and $\rho_N \leq 2p$ (see Lemma 2.7.2)

$$\Big| \frac{\|I_K A_N^n \mathbf{1}_N\|_2}{\|\boldsymbol{V}_N^K\|_2} \int_0^t s\phi^{*n}(t-s)ds \Big| + \Big| \mu e^{-CN^{-\frac{3}{16}}} \rho_N^n \int_0^t s\phi^{*n}(t-s)ds \Big| \leq Ct.$$

Hence by Lemma 2.7.6, we have

$$v_t^{N,K} \in \Big[\mu e^{-CN^{-\frac{3}{16}}} \sum_{n \geq 2} \rho_N^n \int_0^t s\phi^{*n}(t-s)ds - Ct, \mu e^{CN^{-\frac{3}{16}}} \sum_{n \geq 2} \rho_N^n \int_0^t s\phi^{*n}(t-s)ds + Ct\Big].$$

But, recalling Assumption A, we have $\phi^{*n}(t) = t^{n-1}e^{-bt}/(n-1)!$ for $n \geq 1$, which implies that (recall that $\alpha_N = \rho_N - b$)

$$\sum_{n \geq 2} \rho_N^n \int_0^t s\phi^{*n}(t-s)ds = \rho_N \int_0^t se^{-b(t-s)}(e^{\rho_N(t-s)} - 1)ds = \frac{\rho_N}{(\alpha_N)^2}(e^{\alpha_N t} - 1) - \frac{\rho_N}{b^2}(1 - e^{-bt}).$$

Since $\frac{\rho_N}{b^2}(1 - e^{-bt}) \leq C$, the conclusion follows.

It is easy to deduce the following statement.

Corollary 2.7.8. *Assume (A). There are some constants $C, c > 0$, $t_0 \geq 1$ and $N_0 \geq 1$, such that for all $N \geq N_0$ and $t \geq t_0$, on $\Omega_N^{K,2}$,*

$$ce^{\alpha_N t} \leq v_t^{N,K} \leq Ce^{\alpha_N t}.$$

Proof. We work on $\Omega_N^{K,2}$. From Remark 2.7.3, we already have $\alpha_N - \alpha_0 = \rho_N - p$ and $p/2 \leq \rho_N \leq 2p$. By Lemma 2.7.2, we conclude that

$$\alpha_N \in \Big[\alpha_0 - \frac{C}{N^{3/8}}, \alpha_0 + \frac{C}{N^{3/8}}\Big].$$

By Assumption (A), we know that $\alpha_0 = p - b > 0$. We can choose N_0 large enough so that for all $N \geq N_0$, $\alpha_N > \frac{\alpha_0}{2} > 0$ and $\alpha_N < \frac{3\alpha_0}{2}$. So there are some constants $C > 0$, $c > 0$ and $t_0 \geq 1$ such that for all $t \geq t_0$,

$$\mu e^{CN^{-\frac{3}{16}}} \frac{\rho_N}{(\alpha_N)^2}(e^{\alpha_N t} - 1) + Ct \leq Ce^{\alpha_N t}$$
$$ce^{\alpha_N t} \leq \mu e^{-CN^{-\frac{3}{16}}} \frac{\rho_N}{(\alpha_N)^2}(e^{\alpha_N t} - 1) - Ct.$$

Then the result follows from Corollary 2.7.7.

2.8 Analysis of the process in the supercritical case

The aim of this subsection is to give some analysis of the process $\boldsymbol{Z}_t^{N,K}$, which will be used in the proof of the main theorem in supercritical case. Recalling [26, Equations (20)-(21)], we write

$$\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}] = \mu \sum_{n\geq 0} \Big[\int_0^t s\phi^{*n}(t-s)ds\Big] I_K A_N^n \mathbf{1}_N = v_t^{N,K} \boldsymbol{V}_N^K + \boldsymbol{I}_t^{N,K}, \tag{2.20}$$

$$\boldsymbol{U}_t^{N,K} = \boldsymbol{Z}_t^{N,K} - \mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}] = \sum_{n\geq 0} \int_0^t \phi^{*n}(t-s) I_K A_N^n \boldsymbol{M}_s^N ds = \boldsymbol{M}_t^{N,K} + \boldsymbol{J}_t^{N,K}, \tag{2.21}$$

where

$$v_t^{N,K} = \mu \sum_{n\geq 0} \frac{\|I_K A_N^n \mathbf{1}_N\|_2}{\|\boldsymbol{V}_N^K\|_2} \int_0^t s\phi^{*n}(t-s)ds, \tag{2.22}$$

$$\boldsymbol{I}_t^{N,K} = \mu \sum_{n\geq 0} \Big[\int_0^t s\phi^{*n}(t-s)ds\Big] \Big[I_K A_N^n \mathbf{1}_N - \frac{\|I_K A_N^n \mathbf{1}_N\|_2}{\|\boldsymbol{V}_N^K\|_2} \boldsymbol{V}_N^K\Big], \tag{2.23}$$

$$\boldsymbol{J}_t^{N,K} = \sum_{n\geq 1} \int_0^t \phi^{*n}(t-s) I_K A_N^n \boldsymbol{M}_s^N ds. \tag{2.24}$$

As usual, we denote by $I_t^{i,N}$ and $J_t^{i,N}$ the coordinates of $\mathbf{I}_t^{N,K}$ and $\mathbf{J}_t^{N,K}$ and by $\bar{I}_t^{N,K}$ and $\bar{J}_t^{N,K}$ their empirical mean.

Lemma 2.8.1. *Assume (A). There are some constants $C > 0$ and $N_0 \geq 1$ such that for any $N \geq N_0$, any $K \in \{1, \ldots, N\}$, any $t \geq 0$,*

$$\mathbf{1}_{\Omega_N^{K,2}} \|\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}]\|_\infty \leq Ce^{\alpha_N t}.$$

Proof. By [14, Lemma 35-(vii)], we already have, for all $n \geq 0$, $||A_N^n \mathbf{1}_N||_\infty \leq C\rho_N^n$. Then by (2.20), on $\Omega_N^{K,2}$,

$$\begin{aligned}||\mathbb{E}_\theta[\mathbf{Z}_t^{N,K}]||_\infty \leq& \mu \sum_{n\geq 0} \Big[\int_0^t s\phi^{*n}(t-s)ds\Big] \|I_K A_N^n \mathbf{1}_N\|_\infty \\ \leq& \mu \sum_{n\geq 0} \Big[\int_0^t s\phi^{*n}(t-s)ds\Big] \|A_N^n \mathbf{1}_N\|_\infty \\ \leq& C\sum_{n\geq 0} \rho_N^n \int_0^t s\phi^{\star n}(t-s)ds.\end{aligned}$$

Recalling that $\phi^{\star n}(s) = s^{n-1}e^{-bs}/(n-1)!$,

$$\begin{aligned}||\mathbb{E}_\theta[\mathbf{Z}_t^{N,K}]||_\infty \leq& Ct + C\rho_N \int_0^t se^{\alpha_N(t-s)}ds \\ =& C\Big(t + \frac{\rho_N}{(\alpha_N)^2}(e^{\alpha_N t} - 1)\Big) \leq Ce^{\alpha_N t}.\end{aligned}$$

For the last inequality, we used that $\rho_N/\alpha_N^2 \leq C$ (for $N \geq N_0$), as in the proof of Corollary 2.7.8.

Next, we give the bound of $\|\boldsymbol{I}_t^{N,K}\|_2$.

Lemma 2.8.2. *Assume (A). There exists $N_0 \geq 1$ and $C > 0$ such that for all $N \geq N_0$, on $\Omega_N^{K,2}$, we have for all $t \geq 0$*

$$\|\boldsymbol{I}_t^{N,K}\|_2 \leq Ct\sqrt{K}N^{-\frac{3}{8}}.$$

Proof. See [26, Lemma 11.1].

Corollary 2.8.3. *Assume (A). There exists $N_0 \geq 1$ and $C > 0$ such that for all $N \geq N_0$, on $\Omega_N^{K,2}$, we have, for all $t \geq 0$,*

$$(\bar{I}_t^{N,K})^2 \leq Ct^2 N^{-\frac{3}{4}}.$$

Proof. By using the Cauchy-Schwarz inequality and Lemma 2.8.2, we directly have:

$$(\bar{I}_t^{N,K})^2 \leq K^{-1}\|\boldsymbol{I}_t^{N,K}\|_2^2 \leq Ct^2 N^{-\frac{3}{4}}$$

which completes the proof.

Corollary 2.8.4. *Assume (A). There exist $N_0 \geq 1$, $t_0 \geq 1$, and $C > c > 0$ such that for all $N \geq N_0$, on the set $\Omega_N^{K,2}$, we have, for all $t \geq t_0$,*

$$ce^{\alpha_N t} \leq \mathbb{E}_\theta[\bar{Z}_t^{N,K}] \leq Ce^{\alpha_N t}.$$

Proof. By (2.20), we have $\mathbb{E}_\theta[\bar{Z}_t^{N,K}] = v_t^{N,K}\bar{V}_N^K + \bar{I}_t^{N,K}$. By [14, Lemma 35-(ii)], we have $V_N(i) \in [\frac{1}{2}, 2]$ for all $i = 1, ..., N$, whence $\bar{V}_N^K \in [\frac{1}{2}, 2]$. Then the conclusion foolows from Corollaries 2.7.8 and 2.8.3.

Lemma 2.8.5. *Assume (A). There exist $N_0 \geq 1$ and $C > 0$ such that for any $N \geq N_0$, any $t \geq 0$ and any $i = 1, \ldots, N$,*

(i) $\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta[(J_t^{i,N})^2] \leq CN^{-1}e^{2\alpha_N t}$ *and* $\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta[(Z_t^{i,N})^2] \leq Ce^{2\alpha_N t}$,

(ii) $\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta[(U_t^{i,N})^2] \leq C[N^{-1}e^{2\alpha_N t} + e^{\alpha_N t}]$ *and* $\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta[(J_t^{i,N})^4] \leq Ce^{4\alpha_N t}$,

(iii) $\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta[(Z_t^{i,N})^4] \leq Ce^{4\alpha_N t}$ *and* $\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta[(U_t^{i,N})^4] \leq Ce^{4\alpha_N t}$.

Proof. First we prove for any $i = 1, \dots, N$, $\mathbf{1}_{\Omega_N^{K,2}} \mathbb{E}_\theta[(J_t^{i,N})^2] \leq CN^{-1} e^{2\alpha_N t}$. By [14, Remark 10], we already have

$$\mathbb{E}_\theta[M_r^{j,N} M_s^{k,N}] = \mathbf{1}_{\{j=k\}} \mathbb{E}_\theta[Z_{r\wedge s}^{j,N}]. \tag{2.25}$$

Recalling (2.24), we deduce that

$$\begin{aligned}
\mathbb{E}_\theta[(J_t^{i,N})^2] &= \sum_{m,n\geq 1} \int_0^t \int_0^t \phi^{\star m}(t-r)\phi^{\star n}(t-s) \sum_{j,k=1}^N A_N^m(i,j) A_N^n(i,k) \mathbb{E}_\theta[M_r^{j,N} M_s^{k,N}] dr ds \\
&= \sum_{m,n\geq 1} \int_0^t \int_0^t \phi^{\star m}(t-r)\phi^{\star n}(t-s) \sum_{j=1}^N A_N^m(i,j) A_N^n(i,j) \mathbb{E}_\theta[Z_{r\wedge s}^{j,N}] dr ds.
\end{aligned}$$

And by [14, Lemma 35-(iv)], for all $n \geq 2$, we have $A_N^n(i,j) \in [\rho_N^n/(3N), 3\rho_N^n/N]$ on $\Omega_N^{K,2}$, while by Lemma 2.8.1, we know that $\mathbb{E}_\theta[Z_{r\wedge s}^{j,N}] \leq Ce^{\alpha_N(r\wedge s)}$. Hence

$$\begin{aligned}
\mathbb{E}_\theta[(J_t^{i,N})^2] \leq & 9N^{-1} \sum_{m,n\geq 1} \int_0^t \int_0^t \phi^{\star m}(t-r)\phi^{\star n}(t-s) \rho_N^{m+n} e^{\alpha_N(r\wedge s)} dr ds \\
\leq & 9N^{-1} \sum_{m,n\geq 1} \int_0^t \int_0^t \phi^{\star m}(t-r)\phi^{\star n}(t-s) \rho_N^{m+n} e^{\alpha_N(\frac{r+s}{2})} dr ds \\
= & 9N^{-1} \Big(\sum_{m\geq 1} \int_0^t \phi^{\star m}(t-r) \rho_N^m e^{\frac{\alpha_N r}{2}} dr \Big)^2 \\
= & 9N^{-1} \Big(\rho_N \int_0^t e^{\alpha_N(t-r)} e^{\frac{\alpha_N r}{2}} dr \Big)^2 \leq CN^{-1} e^{2\alpha_N t},
\end{aligned}$$

since $\rho_N \leq 2p$, see Remark 2.7.3. This finishes the proof of the first part of (i).

By (2.25) and Lemma 2.8.1, $\mathbf{1}_{\Omega_N^{K,2}} \mathbb{E}_\theta[(M_t^{i,N})^2] \leq \mathbf{1}_{\Omega_N^{K,2}} \|\mathbb{E}_\theta[\boldsymbol{Z}_t^N]\|_\infty \leq Ce^{\alpha_N t}$, whence, recall (2.21),

$$\mathbf{1}_{\Omega_N^{K,2}} \mathbb{E}_\theta[(U_t^{i,N})^2] \leq 2\mathbf{1}_{\Omega_N^{K,2}} \Big\{ \mathbb{E}_\theta[(J_t^{i,N})^2] + \mathbb{E}_\theta[(M_t^{i,N})^2] \Big\} \leq C[N^{-1} e^{2\alpha_N t} + e^{\alpha_N t}],$$

By (2.21), we write $Z_t^{i,N} = \mathbb{E}_\theta[Z_t^{i,N}] + U_t^{i,N}$. And by Lemma 2.8.1, we have $\mathbf{1}_{\Omega_N^{K,2}} \mathbb{E}_\theta[Z_t^{i,N}]^2 \leq Ce^{2\alpha_N t}$, whence

$$\mathbf{1}_{\Omega_N^{K,2}} \mathbb{E}_\theta[(Z_t^{i,N})^2] \leq \mathbf{1}_{\Omega_N^{K,2}} 2 \Big\{ \mathbb{E}_\theta[Z_t^{i,N}]^2 + \mathbb{E}_\theta[(U_t^{i,N})^2] \Big\} \leq Ce^{2\alpha_N t}.$$

We have finished the proof of (i) and of the first part of (ii).

To verify the second part of (ii), we use that by Doob's inequality,

$$\mathbf{1}_{\Omega_N^{K,2}} \mathbb{E}_\theta[(M_s^{i,N})^4] \leq \mathbf{1}_{\Omega_N^{K,2}} \mathbb{E}_\theta[(Z_t^{i,N})^2] \leq Ce^{2\alpha_N t}. \tag{2.26}$$

Then by Minkowski's inequality, we have, on $\Omega_N^{K,2}$, (recalling (2.24) for the expression of $J_t^{i,N}$)

$$\begin{aligned}
\mathbb{E}_\theta[(J_t^{i,N})^4]^{\frac{1}{4}} \leq & \sum_{n\geq 1} \int_0^t \phi^{\star n}(t-s) \sum_{k=1}^N A_N^n(i,k) \mathbb{E}_\theta[(M_s^{k,N})^4]^{\frac{1}{4}} ds \\
\leq & C \sum_{n\geq 1} \int_0^t \rho_N^n \phi^{\star n}(t-s) e^{\frac{\alpha_N s}{2}} ds \\
= & C\rho_N \int_0^t e^{\alpha_N(t-s)} e^{\frac{\alpha_N s}{2}} ds \leq Ce^{\alpha_N t},
\end{aligned}$$

which ends the proof of (ii).

Point (iii) easily follows from (ii) and (2.26), using (2.21) and Lemma 2.8.1:

$$\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta[(U_t^{i,N})^4] \le \mathbf{1}_{\Omega_N^{K,2}} 4\Big\{\mathbb{E}_\theta[(J_t^{i,N})^4] + \mathbb{E}_\theta[(M_t^{i,N})^4]\Big\} \le C[e^{4\alpha_N t} + e^{2\alpha_N t}]$$
$$\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta[(Z_t^{i,N})^4] \le \mathbf{1}_{\Omega_N^{K,2}} 4\Big\{\mathbb{E}_\theta[Z_t^{i,N}]^4 + \mathbb{E}_\theta[(U_t^{i,N})^4]\Big\} \le Ce^{4\alpha_N t}.$$

We finished the proof.

Lemma 2.8.6. *Assume (A). There exist $C > 0$ and $N_0 \ge 1$ such that for all $N \ge N_0$, all $K = 1, \dots, N_0$, all $t \ge 0$*

$$\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta[|\bar{U}_t^{N,K}|^2] \le CK^{-1}e^{2\alpha_N t}.$$

Proof. By (2.21), Lemma 2.8.5 and the Cauchy-Schwarz inequality, we have

$$\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta[|\bar{U}_t^{N,K}|^2] \le \mathbf{1}_{\Omega_N^{K,2}} 2\Big\{\mathbb{E}_\theta[(\bar{J}_t^{N,K})^2] + \mathbb{E}_\theta[(\bar{M}_t^{N,K})^2]\Big\} \le C\Big[\frac{e^{2\alpha_N t}}{N} + \frac{e^{\alpha_N t}}{K}\Big] \le \frac{Ce^{2\alpha_N t}}{K}$$

Lemma 2.8.7. *Assume (A). There are $C > 0$ and $N_0 \ge 1$ and $t_0 \ge 1$ such that, for all $N \ge N_0$, all $K = 1, \dots, N$, on the set $\Omega_N^{K,2}$, for all $t \ge t_0$,*

$$P_\theta\Big(\bar{Z}_t^{N,K} \le \frac{1}{4}v_t^{N,K}\Big) \le C\Big(\frac{1}{\sqrt{K}} + \frac{t}{e^{\alpha_N t}}\Big).$$

Proof. We work on $\Omega_N^{K,2}$. By [14, Lemma 35-(ii)], we have $\bar{V}_N^K \ge \frac{1}{2}$. Thus $\bar{Z}_t^{N,K} \le v_t^{N,K}/4$ implies that $|\frac{\bar{Z}_t^{N,K}}{v_t^{N,K}} - \bar{V}_N^K| \ge \frac{1}{4}$, so that

$$P_\theta\Big(\bar{Z}_t^{N,K} \le \frac{1}{4}v_t^{N,K}\Big) \le 4\mathbb{E}_\theta\Big[\Big|\frac{\bar{Z}_t^{N,K}}{v_t^{N,K}} - \bar{V}_N^K\Big|\Big] \le 4\mathbb{E}_\theta\Big[\frac{|\bar{U}_t^{N,K}| + |\bar{I}_t^{N,K}|}{v_t^{N,K}}\Big]$$

by (2.20) and (2.21). The conclusion follows, since $v_t^{N,K} \ge ce^{\alpha_N t}$ by Corollary 2.7.8, $\mathbb{E}_\theta[|\bar{I}_t^{N,K}|] \le CtN^{-3/8}$ by Corollary 2.8.3, and $\mathbb{E}_\theta[|\bar{U}_t^{N,K}|] \le CK^{-1/2}e^{\alpha_N t}$ by Lemma 2.8.6.

The following statement is then clear from Lemmas 2.7.1 and 2.8.7 and Corollary 2.7.8.

Corollary 2.8.8. *Assume (A). It holds that,*

$$\lim_{(N,K,t)\to(\infty,\infty,\infty)} P\Big(\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \ge \frac{1}{4}v_t^{N,K} > 0\}\Big) = 1.$$

We conclude the subsection with the following statement.

Lemma 2.8.9. *We assume (A). In the regime where $(N, K, t) \to (\infty, \infty, \infty)$ with $1 \le K \le N$ and*

$$\frac{N}{\sqrt{K}e^{\alpha_0 t}} + \frac{1}{\sqrt{K}} \to 0 \quad \text{with} \quad \frac{N}{e^{\alpha_0 t}} \to \infty, \tag{2.27}$$

we have $\lim \mathbf{1}_{\Omega_N^{K,2}}(\frac{N}{\sqrt{K}e^{\alpha_N t}} + \frac{1}{\sqrt{K}}) = 0$ and $\lim \mathbf{1}_{\Omega_N^{K,2}} \frac{N}{e^{\alpha_N t}} = \infty$ a.s.

Proof. On $\Omega_N^{K,2}$, we have $\alpha_N = \rho_N - b$ and $\alpha_0 = p - b$ (see Remark 2.7.3), whence $\frac{e^{\alpha_N t}}{e^{\alpha_0 t}} = e^{(\rho_N - p)t}$ so that $\frac{e^{\alpha_N t}}{e^{\alpha_0 t}} \in [e^{-\frac{p}{2N^{3/8}}t}, e^{\frac{p}{2N^{3/8}}t}]$ by Lemma 2.7.2. The conclusion follows.

2.9 Proof of the main result in the supercritical case

We recall that $\mathcal{U}_t^{N,K}$ was defined in (2.3). The main result, in the supercritical case, will easily follow from the following statement.

Proposition 2.9.1. We assume (A). In the regime (2.27), it holds

$$\mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}}\frac{e^{\alpha_N t}\sqrt{K}}{N}\Big[\mathcal{U}_t^{N,K}-\Big(\frac{1}{p}-1\Big)\Big]\xrightarrow{d}\mathcal{N}\Big(0,\frac{2(\alpha_0)^4}{(\mu p)^2}\Big).$$

We set $\mathcal{D}_t^{N,K}=\mathcal{U}_t^{N,K}-(\frac{1}{p}-1)$ and we use the following decomposition, on the event $\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}$:

$$\mathcal{D}_t^{N,K}=\mathcal{D}_t^{N,K,1}+\mathcal{D}_t^{N,K,2}+\mathcal{D}_t^{N,K,3} \tag{2.28}$$

where, recalling that $\mathcal{U}_\infty^{N,K}$ was introduced in Lemma 2.7.4 and that $v_t^{N,K}$ was defined in (2.22),

$$\begin{aligned}
\mathcal{D}_t^{N,K,1}=&\frac{1}{(\bar{Z}_t^{N,K})^2}\Big(\frac{N}{K}\|\boldsymbol{Z}_t^{N,K}-\bar{Z}_t^{N,K}\mathbf{1}_K\|_2^2-N\bar{Z}_t^{N,K}-(v_t^{N,K})^2\frac{N}{K}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2\Big),\\
\mathcal{D}_t^{N,K,2}=&\frac{N}{K}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2\Big[\Big(\frac{v_t^{N,K}}{\bar{Z}_t^{N,K}}\Big)^2-\frac{1}{(\bar{V}_N^K)^2}\Big],\\
\mathcal{D}^{N,K,3}=&\Big[\mathcal{U}_\infty^{N,K}-\Big(\frac{1}{p}-1\Big)\Big].
\end{aligned}$$

Lemma 2.9.2. *We assume (A). In the regime (2.27), in probability,*

$$\lim\mathbf{1}_{\Omega_N^{K,2}}\frac{e^{\alpha_N t}\sqrt{K}}{N}|\mathcal{D}^{N,K,3}|=0.$$

Proof. It suffices to gather Lemma 2.8.9, from which $\mathbf{1}_{\Omega_N^{K,2}}e^{\alpha_N t}/N\to 0$ a.s. and Lemma 2.7.4, from which $\sqrt{K}\mathbf{1}_{\Omega_N^{K,2}}|\mathcal{U}_\infty^{N,K}-(\frac{1}{p}-1)|$ is bounded in L^1.

Next, we consider the term $\mathcal{D}_t^{N,K,2}$.

Lemma 2.9.3. *Assume (A). There are some constants $C>0$, $t_0\geq 1$ and $N_0\geq 1$, such that for all $N\geq N_0$ and $t\geq t_0$, on the event $\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}$,*

$$\mathbb{E}_\theta[|\mathcal{D}_t^{N,K,2}|]\leq C\frac{N}{K}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2\Big(\frac{1}{\sqrt{K}}+\frac{t}{N^{\frac{3}{8}}e^{\alpha_N t}}\Big).$$

Proof. We work on $\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}$. Recalling (2.20) and (2.21), we can write

$$\Big|\bar{Z}_t^{N,K}(v_t^{N,K})^{-1}-\bar{V}_N^K\Big|\leq(v_t^{N,K})^{-1}\Big(|\bar{I}_t^{N,K}|+|\bar{U}_t^{N,K}|\Big).$$

According to Corollary 2.7.8, there exists some positive constant c such that $ce^{\alpha_N t}\leq v_t^{N,K}$. On the event $\Omega_N^{K,2}$, we already have $\bar{V}_N^K\geq\frac{1}{2}$ by [14, Lemma 35-(ii)]. Since $|\frac{1}{x^2}-\frac{1}{y^2}|=|\frac{(x-y)(x+y)}{x^2y^2}|\leq 128|x-y|$, for $x,y\geq\frac{1}{4}$, it holds that

$$|\mathcal{D}_t^{N,K,2}|\leq\frac{128N}{K}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2\frac{|\bar{I}_t^{N,K}|+|\bar{U}_t^{N,K}|}{v_t^{N,K}}\leq\frac{CNe^{-\alpha_N t}}{K}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2(|\bar{I}_t^{N,K}|+|\bar{U}_t^{N,K}|).$$

By Corollary 2.8.3 and Lemma 2.8.6, we finally obtain

$$\mathbb{E}_\theta[|\mathcal{D}_t^{N,K,2}|]\leq C\frac{N}{K}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2\Big(\frac{t}{N^{\frac{3}{8}}e^{\alpha_N t}}+\frac{1}{\sqrt{K}}\Big)$$

which completes the result.

Corollary 2.9.4. *We assume* (A)*. In the regime* (2.27)*, in probability,*

$$\lim \mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}}\frac{e^{\alpha_N t}\sqrt{K}}{N}|\mathcal{D}_t^{N,K,2}|=0.$$

Proof. By Lemma 2.9.3, we have $\mathbb{E}_\theta[|\mathcal{D}_t^{N,K,2}|]\leq C\frac{N}{K}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2(\frac{t}{N^{\frac{3}{8}}e^{\alpha_N t}}+\frac{1}{\sqrt{K}})$ on the event $\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}$. By Lemma 2.7.5-(iii), we know that $\mathbf{1}_{\Omega_N^{K,2}}\frac{N}{K}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2$ is bounded in L^1, whence the conclusion, since $\frac{e^{\alpha_N t}\sqrt{K}}{N}(\frac{t}{N^{\frac{3}{8}}e^{\alpha_N t}}+\frac{1}{\sqrt{K}})\mathbf{1}_{\Omega_N^{K,2}}\to 0$ by Lemma 2.8.9.

Next, we consider the term $\mathcal{D}_t^{N,K,1}$, starting from

$$\mathcal{D}_t^{N,K,1}=\mathcal{D}_t^{N,K,11}+\mathcal{D}_t^{N,K,12}+2\mathcal{D}_t^{N,K,13}+2\mathcal{D}_t^{N,K,14},$$

where

$$\begin{aligned}
\mathcal{D}_t^{N,K,11}=&\frac{N}{K(\bar{Z}_t^{N,K})^2}\|\boldsymbol{I}_t^{N,K}-\bar{I}_t^{N,K}\mathbf{1}_K+\boldsymbol{J}_t^{N,K}-\bar{J}_t^{N,K}\mathbf{1}_K\|_2^2,\\
\mathcal{D}_t^{N,K,12}=&\frac{N}{K(\bar{Z}_t^{N,K})^2}\Big[\|\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\|_2^2-NZ_t^{N,K}\Big],\\
\mathcal{D}_t^{N,K,13}=&\frac{N}{K(\bar{Z}_t^{N,K})^2}\Big(\boldsymbol{I}_t^{N,K}-\bar{I}_t^{N,K}\mathbf{1}_K+\boldsymbol{J}_t^{N,K}-\bar{J}_t^{N,K}\mathbf{1}_K,\\
&\qquad v_t^{N,K}(V_N^K-\bar{V}_N^K\mathbf{1}_K)+\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\Big),\\
\mathcal{D}_t^{N,K,14}=&\frac{N}{K(\bar{Z}_t^{N,K})^2}v_t^{N,K}\Big(\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K,\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\Big).
\end{aligned}$$

First, we study $\mathcal{D}_t^{N,K,11}$. In order to obtain its limit theorem, we need the following lemme.

Lemma 2.9.5. *Assume* (A)*. There exist* $N_0\geq 1$ *and* $C>0$ *such that for all* $N\geq N_0$*, any* $K\in\{1,\dots,N\}$*, on the set* $\Omega_N^{K,2}$*, for any* $t\geq 0$*, we have*

$$\begin{aligned}
&(i)\quad \mathbb{E}_\theta[\|\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\|_2^2]\leq CKe^{\alpha_N t},\\
&(ii)\quad \mathbb{E}_\theta[\|\boldsymbol{J}_t^{N,K}-\bar{J}_t^{N,K}\mathbf{1}_K\|_2^2]^{\frac{1}{2}}\leq C\sqrt{\frac{K}{N}}\Big[e^{\frac{1}{2}\alpha_N t}+\frac{\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2}{\|\boldsymbol{V}_N^K\|_2}e^{\alpha_N t}\Big],\\
&(iii)\quad \mathbb{E}_\theta[\|\boldsymbol{U}_t^{N,K}-\bar{U}_t^{N,K}\mathbf{1}_K\|_2^2]\leq C\Big(Ke^{\alpha_N t}+\frac{e^{2\alpha_N t}}{N}\|\boldsymbol{V}_N^K-\bar{V}_N^K\mathbf{1}_K\|_2^2\Big).
\end{aligned}$$

Proof. We work on $\Omega_N^{K,2}$. Recalling (2.25), we write

$$\mathbb{E}_\theta[\|\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\|_2^2]=\sum_{i=1}^K\mathbb{E}_\theta[(M_t^{i,N})^2]-K\mathbb{E}_\theta[(\bar{M}_t^{N,K})^2]=\sum_{i=1}^K\mathbb{E}_\theta[Z_t^{i,N}]-K\mathbb{E}_\theta[(\bar{M}_t^{N,K})^2].$$

Hence we deduce from Lemma 2.8.1 that

$$\mathbb{E}_\theta[\|\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\|_2^2]\leq\sum_{i=1}^K\mathbb{E}_\theta[Z_t^{i,N}]\leq CKe^{\alpha_N t},$$

which proves (i). For (ii), in view of (2.24), by the Minkowski inequality, we have

$$\mathbb{E}_\theta\Big[\|\boldsymbol{J}_t^{N,K}-\bar{J}_t^{N,K}\mathbf{1}_K\|_2^2\Big]^{\frac{1}{2}}\leq\sum_{n\geq 1}\int_0^t\phi^{*n}(t-s)\mathbb{E}_\theta\Big[\|I_KA_N^n\boldsymbol{M}_s^N-\overline{I_KA_N^n\boldsymbol{M}_s^N}\mathbf{1}_K\|_2^2\Big]^{\frac{1}{2}}ds,$$

where $\overline{I_K A_N^n \boldsymbol{M}_s^N} := \frac{1}{K}\sum_{j=1}^N \sum_{i=1}^K A_N^n(i,j) M_s^{j,N}$. Using again (2.25), we see that

$$\begin{aligned}\mathbb{E}_\theta\Big[\|I_K A_N^n \boldsymbol{M}_s^N - \overline{I_K A_N^n \boldsymbol{M}_s^N}\mathbf{1}_K\|_2^2\Big] &= \sum_{i=1}^K \mathbb{E}_\theta\Big[\Big(\sum_{j=1}^N A_N^n(i,j)M_s^{j,N} - \frac{1}{K}\sum_{k=1}^K\sum_{j=1}^N A_N^n(k,j)M_s^{j,N}\Big)^2\Big] \\ &= \sum_{i=1}^K\sum_{j=1}^N \Big(A_N^n(i,j) - \frac{1}{K}\sum_{k=1}^K A_N^n(k,j)\Big)^2 \mathbb{E}_\theta[Z_s^{j,N}] \\ &\le Ce^{\alpha_N s}\sum_{j=1}^N \|I_K A_N^n \boldsymbol{e}_j - \overline{I_K A_N^n \boldsymbol{e}_j}\mathbf{1}_K\|_2^2.\end{aligned}$$

For the last inequality, we used that $\max_{i=1,\dots,N}\mathbb{E}_\theta[(Z_t^{i,N})^2] \le Ce^{2\alpha_N t}$ by Lemma 2.8.5 and we introduced $\boldsymbol{e}_j \in \mathbb{R}^N$ with coordinates $\boldsymbol{e}_j(i) = \mathbf{1}_{\{i=j\}}$. Using the inequality $||\boldsymbol{x} - \bar{x}\mathbf{1}_N||_2 - ||\boldsymbol{y} - \bar{y}\mathbf{1}_N||_2 \le ||\boldsymbol{x}-\boldsymbol{y}||_2$ for all $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^N$,

$$\begin{aligned}&\|I_K A_N^n \boldsymbol{e}_j - \overline{I_K A_N^n \boldsymbol{e}_j}\mathbf{1}_K\|_2 \\ \le&\Big\|I_K A_N^n \boldsymbol{e}_j - \frac{1}{\|\boldsymbol{V}_N^K\|_2}\|I_K A_N^n \boldsymbol{e}_j\|_2 \boldsymbol{V}_N^K\Big\|_2 + \frac{\|I_K A_N^n \boldsymbol{e}_j\|_2}{\|\boldsymbol{V}_N^K\|_2}\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2 \\ =&\|I_K A_N^n \boldsymbol{e}_j\|_2\Big(\Big\|\frac{I_K A_N^n \boldsymbol{e}_j}{\|I_K A_N^n \boldsymbol{e}_j\|_2} - \frac{\boldsymbol{V}_N^K}{\|\boldsymbol{V}_N^K\|_2}\Big\|_2 + \frac{\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2}{\|\boldsymbol{V}_N^K\|_2}\Big) \\ \le&C\|I_K A_N^n \boldsymbol{e}_j\|_2\Big(N^{-\frac{3}{8}\lfloor\frac{n}{2}\rfloor} + \frac{\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2}{\|\boldsymbol{V}_N^K\|_2}\Big).\end{aligned}$$

by [26, Lemma 10.3-(viii)] with $r=2$. From [26, Lemma 10.3-(iv)], for all $n\ge 2$, we have

$$\|I_K A_N^n \boldsymbol{e}_j\|_2 \le \Big[\sum_{i=1}^K (A_N^n(i,j))^2\Big]^{\frac{1}{2}} \le \frac{3\sqrt{K}}{N}\rho_N^n.$$

We then conclude that

$$\mathbb{E}_\theta[||I_K A_N^n M_s^N - \overline{I_K A_N^n M_s^N}\mathbf{1}_K||_2^2]^{1/2} \le Ce^{\alpha_N s/2}\sqrt{\frac{K}{N}}\rho_N^n\Big((2N^{-3/8})^{\lfloor n/2\rfloor} + \frac{\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2}{\|\boldsymbol{V}_N^K\|_2}\Big).$$

So on the event $\Omega_N^{K,2}$,

$$\mathbb{E}_\theta[\|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K\|_2^2]^{\frac{1}{2}} \le C\sqrt{\frac{K}{N}}\sum_{n\ge1}\rho_N^n\Big[(2N^{-\frac{3}{8}})^{\lfloor\frac{n}{2}\rfloor} + \frac{\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2}{\|\boldsymbol{V}_N^K\|_2}\Big]\int_0^t \phi^{*n}(t-s)e^{\frac{\alpha_N s}{2}}ds.$$

Using [14, Lemma 43-(iii)], we conclude

$$\sum_{n\ge1}\rho_N^n(2N^{-\frac{3}{8}})^{\lfloor\frac{n}{2}\rfloor}\int_0^t \phi^{*n}(t-s)e^{\frac{\alpha_N s}{2}}ds \le e^{\frac{\alpha_N t}{2}}\sum_{n\ge1}\rho_N^n(2N^{-\frac{3}{8}})^{\lfloor\frac{n}{2}\rfloor}\int_0^t \phi^{*n}(t-s)ds \le Ce^{\frac{\alpha_N t}{2}}.$$

And we can compute directly, recalling that $\phi(u) = e^{-bu}$ and that $\rho_N = \alpha_N + b$, that

$$\sum_{n\ge1}\rho_N^n\int_0^t \phi^{*n}(t-s)e^{\frac{\alpha_N s}{2}}ds = \rho_N\int_0^t e^{\alpha_N(t-s)}e^{\frac{\alpha_N s}{2}}ds \le \frac{2\rho_N}{\alpha_N}e^{\alpha_N t} \le Ce^{\alpha_N t}$$

by Remark 2.7.3. All in all,

$$\mathbb{E}_\theta[\|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K\|_2^2]^{\frac{1}{2}} \le C\sqrt{\frac{K}{N}}\Big[e^{\frac{\alpha_N t}{2}} + \frac{\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2}{\|\boldsymbol{V}_N^K\|_2}e^{\alpha_N t}\Big].$$

This completes the proof of (ii). For (iii), we recall (2.21) and write $\boldsymbol{U}_t^{N,K} = \boldsymbol{M}_t^{N,K} + \boldsymbol{J}_t^{N,K}$, whence

$$\mathbb{E}_\theta[\|\boldsymbol{U}_t^{N,K} - \bar{U}_t^{N,K}\mathbf{1}_K\|_2^2] \leq 2\Big(\mathbb{E}_\theta[\|\boldsymbol{M}_t^{N,K} - \bar{M}_t^{N,K}\mathbf{1}_K\|_2^2] + \mathbb{E}_\theta[\|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K\|_2^2]\Big).$$

By (i), we have $\mathbb{E}_\theta[\|\boldsymbol{M}_t^{N,K} - \bar{M}_t^{N,K}\mathbf{1}_K\|_2^2] \leq CKe^{\alpha_N t}$. By (ii), we have $\mathbb{E}_\theta[\|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K\|_2^2] \leq C(\frac{K}{N}e^{\alpha_N t} + \frac{e^{2\alpha_N t}}{N}\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2^2)$. The conclusion follows, since, as already seen, we have $V_N(i) \geq \frac{1}{2}$, whence $\|\boldsymbol{V}_N^K\|_2 \geq c\sqrt{K}$, by [14, Lemma 35-(ii)].

Lemma 2.9.6. *Assume* (A)*. In the regime* (2.27)*, in probability*

$$\lim \mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K} \geq \frac{1}{4}v_t^{N,K} > 0\}} \frac{e^{\alpha_N t}\sqrt{K}}{N}|\mathcal{D}_t^{N,K,11}| = 0.$$

Proof. By Corollary 2.7.8, we easily deduce that

$$\mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K} \geq \frac{1}{4}v_t^{N,K} > 0\}}|\mathcal{D}_t^{N,K,11}| \leq \mathbf{1}_{\Omega_N^{K,2}}\frac{CN}{Ke^{2\alpha_N t}}\Big\{\|\boldsymbol{I}_t^{N,K}\|_2^2 + \|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K\|_2^2\Big\}.$$

By Lemma 2.8.2, we have

$$\mathbf{1}_{\Omega_N^{K,2}}\frac{N}{Ke^{2\alpha_N t}}\|\boldsymbol{I}_t^{N,K}\|_2^2 \leq \mathbf{1}_{\Omega_N^{K,2}}\frac{N}{Ke^{2\alpha_N t}}t^2\frac{K}{N^{\frac{3}{4}}} = \mathbf{1}_{\Omega_N^{K,2}}\frac{t^2N^{\frac{1}{4}}}{e^{2\alpha_N t}}.$$

As seen in the proof of Corollary 2.7.8, on $\Omega_N^{K,2}$, $\alpha_N > \alpha_0/2$ (if N is large enough), so that $\lim \mathbf{1}_{\Omega_N^{K,2}}\frac{t^2}{e^{\alpha_N t}} = 0$. And $\mathbf{1}_{\Omega_N^{K,2}}\frac{N}{\sqrt{K}e^{\alpha_N t}} = 0$, see Lemma 2.8.9, implies that $\mathbf{1}_{\Omega_N^{K,2}}\frac{N^{\frac{1}{4}}}{e^{\alpha_N t}} = 0$. Hence, we obtain $\lim \mathbf{1}_{\Omega_N^{K,2}}\frac{N}{Ke^{2\alpha_N t}}\|\boldsymbol{I}_t^{N,K}\|_2^2 = 0$.

From Lemma 2.9.5-(ii) and since $||\boldsymbol{V}_N^K||_2 \geq c\sqrt{K}$ on $\Omega_N^{K,2}$ (see the end of the proof of the previous lemma), $\mathbb{E}_\theta[\|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K\|_2^2] \leq C(\frac{K}{N}e^{\alpha_N t} + \frac{e^{2\alpha_N t}}{N}\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2^2)$. Hence by Lemma 2.7.5-$(ii)$ and since $\alpha_N > \alpha_0/2$, we have:

$$\mathbf{1}_{\Omega_N^{K,2}}\frac{N}{Ke^{2\alpha_N t}}\mathbb{E}_\theta[\|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K\|_2^2] \leq C\mathbf{1}_{\Omega_N^{K,2}}\Big(\frac{1}{e^{\alpha_N t}} + \frac{1}{N^{\frac{3}{4}}}\Big) \leq C\mathbf{1}_{\Omega_N^{K,2}}\Big(\frac{1}{e^{\frac{\alpha_0 t}{2}}} + \frac{1}{N^{\frac{3}{4}}}\Big).$$

Finally we have $\lim \mathbb{E}[\mathbf{1}_{\Omega_N^{K,2}}\frac{N}{Ke^{2\alpha_N t}}\mathbb{E}_\theta[\|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K\|_2^2]] = 0$ which complete the proof.

Lemma 2.9.7. *Assume* (A)*. In the regime* (2.27)*, in probability*

$$\lim \mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K} \geq \frac{1}{4}v_t^{N,K} > 0\}} \frac{e^{\alpha_N t}\sqrt{K}}{N}|\mathcal{D}_t^{N,K,14}| = 0.$$

Proof. By (2.25) and Lemma 2.8.5, we have

$$\begin{aligned}\mathbb{E}_\theta\Big[\Big(\boldsymbol{M}_t^{N,K} - \bar{M}_t^{N,K}\mathbf{1}_K, \boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\Big)^2\Big] &= \mathbb{E}_\theta\Big[\Big(\boldsymbol{M}_t^{N,K}, \boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\Big)^2\Big] \\ &= \sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2\mathbb{E}_\theta[Z_t^{i,N}] \\ &\leq C\|\boldsymbol{V}_N^K - \bar{V}_N^K\|_2^2 e^{\alpha_N t}.\end{aligned}$$

By Corollary 2.7.8, we knwo that $ce^{\alpha_N t} \leq \frac{1}{4}v_t^{N,K} \leq \bar{Z}_t^{N,K}$ for t big enough, on the event $\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \geq \frac{1}{4}v_t^{N,K} > 0\}$. By definition of $\mathcal{D}_t^{N,K,14}$,

$$\begin{aligned}&\mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K} \geq \frac{1}{4}v_t^{N,K} > 0\}}\frac{e^{\alpha_N t}\sqrt{K}}{N}\mathbb{E}_\theta[|\mathcal{D}_t^{N,K,14}|] \\ \leq &\frac{C}{\sqrt{K}}\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta\Big[\Big(\boldsymbol{M}_t^{N,K} - \bar{M}_t^{N,K}\mathbf{1}_K, \boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\Big)^2\Big]^{\frac{1}{2}}\Big] \\ \leq &\mathbf{1}_{\Omega_N^{K,2}}\frac{Ce^{\frac{\alpha_N t}{2}}}{\sqrt{N}}\sqrt{\frac{N}{K}}\|\boldsymbol{V}_N^K - \bar{V}_N^K\|_2.\end{aligned}$$

It suffices to gather Lemma 2.8.9, from which $\mathbf{1}_{\Omega_N^{K,2}} e^{\alpha_N t}/N \to 0$ in probability and Lemma 2.7.5 (iii), from which $\mathbf{1}_{\Omega_N^{K,2}} \frac{N}{K} \|\boldsymbol{V}_N^K - \bar{V}_N^K\|_2^2$ is bounded in L^1.

Next, we rewrite $\mathcal{D}_t^{N,K,13} = \mathcal{D}_t^{N,K,131} + \mathcal{D}_t^{N,K,132} + \mathcal{D}_t^{N,K,133}$, where

$$\begin{aligned}
\mathcal{D}_t^{N,K,131} =& \frac{N}{K(\bar{Z}_t^{N,K})^2}\Big(\boldsymbol{I}_t^{N,K} - \bar{I}_t^{N,K}\mathbf{1}_K + \boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K, v_t^{N,K}(V_N^K - \bar{V}_N^K\mathbf{1}_K)\Big),\\
\mathcal{D}_t^{N,K,132} =& \frac{N}{K(\bar{Z}_t^{N,K})^2}\Big(\boldsymbol{I}_t^{N,K} - \bar{I}_t^{N,K}\mathbf{1}_K, \boldsymbol{M}_t^{N,K} - \bar{M}_t^{N,K}\mathbf{1}_K\Big),\\
\mathcal{D}_t^{N,K,133} =& \frac{N}{K(\bar{Z}_t^{N,K})^2}\Big(\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K, \boldsymbol{M}_t^{N,K} - \bar{M}_t^{N,K}\mathbf{1}_K\Big).
\end{aligned}$$

Lemma 2.9.8. *We assume* (A)*. In the regime* (2.27)*, in probability,*

$$\lim \mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K} \geq \frac{1}{4}v_t^{N,K}>0\}} \frac{e^{\alpha_N t}\sqrt{K}}{N}|\mathcal{D}_t^{N,K,131}| = 0.$$

Proof. By Corollary 2.7.8, we have $v_t^{N,K} \geq ce^{\alpha_N t}$ on the event $\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \geq \frac{1}{4}v_t^{N,K} > 0\}$ whence, by definition of $\mathcal{D}_t^{N,K,131}$,

$$\begin{aligned}
&\mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K} \geq \frac{1}{4}v_t^{N,K}>0\}}|\mathcal{D}_t^{N,K,131}|\\
\leq& \mathbf{1}_{\Omega_N^{K,2}} \frac{N}{Ke^{\alpha_N t}}\Big(\|\boldsymbol{I}_t^{N,K} - \bar{I}_t^{N,K}\mathbf{1}_K\|_2 + \|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K\|_2\Big)\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K} \geq \frac{1}{4}v_t^{N,K}>0\}} \frac{e^{\alpha_N t}\sqrt{K}}{N}|\mathcal{D}_t^{N,K,131}|\\
\leq& \mathbf{1}_{\Omega_N^{K,2}} \frac{C}{\sqrt{N}}\Big[\|\boldsymbol{I}_t^{N,K}\|_2 + \|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K\|_2\Big]\Big(\sqrt{\frac{N}{K}}\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2\Big).
\end{aligned}$$

By Lemma 2.8.2, we have

$$\mathbf{1}_{\Omega_N^{K,2}} \frac{C}{\sqrt{N}}\|\boldsymbol{I}_t^{N,K}\|_2\Big(\sqrt{\frac{N}{K}}\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2\Big) \leq \mathbf{1}_{\Omega_N^{K,2}} \frac{Ct\sqrt{K}}{N^{\frac{7}{8}}}\Big(\sqrt{\frac{N}{K}}\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2\Big).$$

By Lemma 2.7.5-(iii), we know that $\mathbf{1}_{\Omega_N^{K,2}} \frac{N}{K}\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2^2$ is bounded in L^1. In the regime (2.27), we have $\lim \mathbf{1}_{\Omega_N^{K,2}} \frac{Ct\sqrt{K}}{N^{\frac{7}{8}}} = 0$. So $\lim \mathbf{1}_{\Omega_N^{K,2}} \frac{C}{\sqrt{N}}\|\boldsymbol{I}_t^{N,K}\|_2\Big(\sqrt{\frac{N}{K}}\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2\Big) = 0$ in probability.

By Lemma 2.9.5-(ii), we conclude that:

$$\frac{1}{\sqrt{N}}\mathbb{E}[\mathbf{1}_{\Omega_N^{K,2}}\|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K\|_2^2]^{\frac{1}{2}} \leq \frac{C\sqrt{K}}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}}\Big\{e^{\frac{1}{2}\alpha_N t} + \frac{\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2}{\|\boldsymbol{V}_N^K\|_2}e^{\alpha_N t}\Big\}\Big].$$

As already seen, on $\Omega_N^{K,2}$, we have $V_N(i) \geq \frac{1}{2}$, whence $\|\boldsymbol{V}_N^K\|_2 \geq c\sqrt{K}$, by [14, Lemma 35-(ii)]. And by Remark 2.7.3, we see on $\Omega_N^{K,2}$, $e^{\alpha_N t} \leq e^{\alpha_0 t}e^{\frac{p}{2N^{3/8}}t}$. Hence

$$\frac{1}{\sqrt{N}}\mathbb{E}[\mathbf{1}_{\Omega_N^{K,2}}\|\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}\mathbf{1}_K\|_2^2]^{\frac{1}{2}} \leq \frac{Ce^{\frac{1}{2}\alpha_0 t}e^{\frac{pt}{2N^{3/8}}}\sqrt{K}}{N} + \frac{e^{\alpha_0 t}e^{\frac{pt}{2N^{3/8}}}}{N}\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}}\|\boldsymbol{V}_N^K - \bar{V}_N^K\mathbf{1}_K\|_2\Big].$$

In view of (2.27), we know that $\lim e^{\frac{pt}{2N^{3/8}}} = 1$ a.s., $\lim \frac{e^{\alpha_0 t}}{N} = 0$. By Lemma 2.7.5-(iii), from which $\mathbf{1}_{\Omega_N^{K,2}} \frac{N}{K}\|\boldsymbol{V}_N^K - \bar{V}_N^K\|_2^2$ is bounded in L^1. Then we finish the proof.

Lemma 2.9.9. *We assume* (A)*. In the regime* (2.27)*, in probability,*

$$\lim \mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}}\frac{e^{\alpha_N t}\sqrt{K}}{N}|\mathcal{D}_t^{N,K,132}|=0.$$

Proof. By Corollary 2.7.8, we have $ce^{\alpha_N t}\leq \bar{Z}_t^{N,K}$ on the event $\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}$. By the Cauchy-Schwarz inequality,

$$\mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}}\frac{e^{\alpha_N t}\sqrt{K}}{N}|\mathcal{D}_t^{N,K,132}|\leq \mathbf{1}_{\Omega_N^{K,2}}\frac{C}{e^{\alpha_N t}\sqrt{K}}\|\boldsymbol{I}_t^{N,K}\|_2\|\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\|_2.$$

By Lemmas 2.8.2 and 2.9.5-(i),

$$\mathbf{1}_{\Omega_N^{K,2}}\frac{C}{e^{\alpha_N t}\sqrt{K}}\mathbb{E}_\theta[\|\boldsymbol{I}_t^{N,K}\|_2\|\boldsymbol{M}_t^{N,K}-\bar{M}_t^{N,K}\mathbf{1}_K\|_2]\leq \mathbf{1}_{\Omega_N^{K,2}}\frac{Ct\sqrt{K}}{e^{\frac{\alpha_N t}{2}}N^{\frac{3}{8}}}.$$

By Remark 2.7.3, we obtain

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}}\frac{e^{\alpha_N t}\sqrt{K}}{N}|\mathcal{D}^{N,K,132}|\Big]\leq \mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}}\frac{Ct\sqrt{K}}{e^{\frac{\alpha_N t}{2}}N^{\frac{3}{8}}}\Big]\leq e^{\frac{pt}{2N^{3/8}}}\frac{Ct\sqrt{K}}{e^{\frac{\alpha_0 t}{2}}N^{\frac{3}{8}}}.$$

In the regime (2.27), we have $\lim e^{\frac{pt}{2N^{3/8}}}=1$ a.s. and $\lim \frac{t\sqrt{K}}{e^{\frac{\alpha_0 t}{2}}N^{\frac{3}{8}}}=0$ a.s., which ends the proof.

Lemma 2.9.10. *Assume* (A)*. There are some constants* $C>0$*,* $t_0\geq 1$ *and* $N_0\geq 1$*, such that for all* $N\geq N_0$*, on the event, for all* $t\geq t_0$*,* $\Omega_N^{K,2}$*,*

$$\mathbb{E}_\theta\Big[(\boldsymbol{J}_t^{N,K}-\bar{J}_t^{N,K},\boldsymbol{M}_t^{N,K})^2\Big]^{\frac{1}{2}}\leq Cte^{\alpha_N t}\sqrt{\frac{K}{N}}+\frac{Cte^{\frac{3}{2}\alpha_N t}\sqrt{K}}{N^{\frac{7}{8}}}.$$

Proof. In view of [14, Proof of Lemma 35, Step 5], we already know that on Ω_N^2, for all $n\geq 2$, $j=1,...,N$,

$$\frac{\max_i A_N^n(i,j)}{\min_i A_N^n(i,j)}\leq (1+2N^{-3/8})^2(1+8N^{-3/8})\leq 1+8N^{-3/8}.$$

By [14, Lemma 35-(iv)], we know that for all $n\geq 2$, $A_N^n(i,j)\in[\rho_N^n/(3N),3\rho_N^n/N]$. Hence we deduce that : for all $n\geq 2$, $i,\ j=1,...,N$,

$$\Big|A_N^n(i,j)-\frac{1}{K}\sum_{k=1}^K A_N^n(k,j)\Big|\leq\frac{C\rho_N^n}{N^{1+\frac{3}{8}}}.$$

We then write, for $n\geq 1$,

$$\begin{aligned}\mathbb{E}_\theta\Big[\Big(I_K A_N^n\boldsymbol{M}_s^N-\overline{I_K A_N^n\boldsymbol{M}_s^N}\mathbf{1}_K,\boldsymbol{M}_s^N\Big)^2\Big]=&\sum_{i,i'=1}^K\sum_{j,j'=1}^N\Big(A_N^n(i,j)-\frac{1}{K}\sum_{k=1}^K A_N^n(k,j)\Big)\\&\Big(A_N^n(i',j')-\frac{1}{K}\sum_{k=1}^K A_N^n(k,j')\Big)\mathbb{E}_\theta[M_s^{j,N}M_t^{i,N}M_s^{j',N}M_t^{i',N}].\end{aligned}$$

By Lemma 2.10.3-(iii)-(iv) in the appendix, we conclude that, for $n\geq 2$,

$$\begin{aligned}\mathbb{E}_\theta\Big[\Big(I_K A_N^n\boldsymbol{M}_s^N-\overline{I_K A_N^n\boldsymbol{M}_s^N}\mathbf{1}_K,\boldsymbol{M}_s^N\Big)^2\Big]\leq&\frac{C\rho_N^{2n}}{N^{2+\frac{3}{4}}}\sum_{i,i'=1}^K\sum_{j,j'=1}^N|\mathbb{E}_\theta[M_s^{j,N}M_t^{i,N}M_s^{j',N}M_t^{i',N}]|\\\leq&\frac{CNK}{N^{2+\frac{3}{4}}}\rho_N^{2n}e^{\alpha_N(t+s)}+\frac{CN^2Kt^2}{N^2N^{2+\frac{3}{4}}}\rho_N^{2n}e^{\alpha_N t}\\\leq&\frac{CKt^2}{N^{1+\frac{3}{4}}}\rho_N^{2n}e^{\alpha_N(t+s)}.\end{aligned}$$

For $n=1$, we just use $A_N(i,j)\leq \frac{1}{N}$ to write,

$$\begin{aligned}\mathbb{E}_\theta\Big[\Big(I_K A_N \boldsymbol{M}_s^N - \overline{I_K A_N \boldsymbol{M}_s^N}\mathbf{1}_K, \boldsymbol{M}_s^N\Big)^2\Big] \leq& \frac{C}{N^2}\sum_{i,i'=1}^K\sum_{j,j'=1}^N |\mathbb{E}_\theta[M_s^{j,N}M_t^{i,N}M_s^{j',N}M_t^{i',N}]|\\ \leq& \frac{CNK}{N^2}e^{\alpha_N(t+s)} + \frac{Ct^2N^2K}{N^2N^2}e^{\alpha_N t}\\ \leq& \frac{CKt^2}{N}e^{\alpha_N(t+s)}\end{aligned}$$

by Lemma 2.10.3-(iii)-(iv) again. Then we conclude, recalling (2.24),

$$\begin{aligned}&\mathbb{E}_\theta\Big[(\boldsymbol{J}_t^{N,K} - \bar{J}_t^{N,K}, \boldsymbol{M}_t^{N,K})^2\Big]^{\frac{1}{2}}\\ \leq& \sum_{n\geq 1}\int_0^t \phi^{*n}(t-s)\mathbb{E}_\theta\Big[\Big(I_K A_N^n \boldsymbol{M}_s^N - \overline{I_K A_N^n \boldsymbol{M}_s^N}\mathbf{1}_K, \boldsymbol{M}_s^N\Big)^2\Big]^{\frac{1}{2}}ds\\ \leq& Cte^{\frac{\alpha_N t}{2}}\Big\{\sqrt{\frac{K}{N}}\int_0^t e^{\frac{\alpha_N s}{2}}\phi(t-s)ds + \frac{C\sqrt{K}}{N^{\frac{7}{8}}}\sum_{n\geq 1}\int_0^t \rho_N^n e^{\frac{\alpha_N s}{2}}\phi^{*n}(t-s)ds\Big\}\\ =& Cte^{\frac{\alpha_N t}{2}}\Big\{\sqrt{\frac{K}{N}}\int_0^t e^{\frac{\alpha_N s}{2}}e^{-b(t-s)}ds + \frac{C\rho_N\sqrt{K}}{N^{\frac{7}{8}}}\sum_{n\geq 0}\int_0^t \frac{\rho_N^n(t-s)^n}{n!}e^{\frac{\alpha_N s}{2}}e^{-b(t-s)}ds\Big\}\\ \leq& Cte^{\alpha_N t}\sqrt{\frac{K}{N}} + \frac{Cte^{\frac{3}{2}\alpha_N t}\sqrt{K}}{N^{\frac{7}{8}}}.\end{aligned}$$

We used that $\phi^{*n}(t) = t^{n-1}e^{-bt}/(n-1)!$ for all $n\geq 1$.

Lemma 2.9.11. *We assume* (A)*. In the regime* (2.27)*, in probability,*

$$\lim \mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq \frac{1}{4}v_t^{N,K}>0\}}\frac{e^{\alpha_N t}\sqrt{K}}{N}|\mathcal{D}_t^{N,K,133}| = 0.$$

Proof. By Corollary 2.7.8, we know that on the event $\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq \frac{1}{4}v_t^{N,K}>0\}$, we always have $ce^{\alpha_N t}\leq \bar{Z}_t^{N,K}$. With Lemma 2.9.10, we conclude that, by Definition of $\mathcal{D}_t^{N,K,133}$,

$$\mathbf{1}_{\Omega_{N,K}\cap\{\bar{Z}_t^{N,K}\geq \frac{1}{4}v_t^{N,K}>0\}}\frac{e^{\alpha_N t}\sqrt{K}}{N}\mathbb{E}_\theta[|\mathcal{D}_t^{N,K,133}|] \leq \frac{Ct}{\sqrt{N}} + \frac{Cte^{\frac{\alpha_N t}{2}}}{N^{\frac{7}{8}}} \leq \frac{Ct}{\sqrt{N}} + e^{\frac{pt}{2N^{3/8}}}\frac{Cte^{\frac{\alpha_0 t}{2}}}{N^{\frac{7}{8}}}.$$

In view of (2.27), the proof is complete.

Summarizing Lemmas 2.9.8, 2.9.9 and 2.9.11, we have the following corollary:

Corollary 2.9.12. *Assume* (A)*. In the regime* (2.27)*, in probability*

$$\lim \mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq \frac{1}{4}v_t^{N,K}>0\}}\frac{e^{\alpha_N t}\sqrt{K}}{N}|\mathcal{D}_t^{N,K,13}| = 0.$$

By Corollaries 2.9.4, 2.9.6, 2.9.7 2.9.12 and Lemma 2.9.2, we conclude:

Corollary 2.9.13. *Assume* (A)*. In the regime* (2.27)*, in probability*

$$\lim \frac{e^{\alpha_N t}\sqrt{K}}{N}\Big\{|\mathcal{D}_t^{N,K,2}| + |\mathcal{D}_t^{N,K,3}| + |\mathcal{D}_t^{N,K,11}| + |\mathcal{D}_t^{N,K,13}| + |\mathcal{D}_t^{N,K,14}|\Big\} = 0$$

It only remains to study $\mathcal{D}_t^{N,K,12}$. We need some preparation.

Lemma 2.9.14. *Assume (A). In the regime (2.27), in probability*

$$\lim \mathbf{1}_{\Omega_N^{K,2}} \frac{\mathbb{E}_\theta[\bar{Z}_t^{N,K}]}{e^{\alpha_N t}} = \frac{\mu p}{(\alpha_0)^2}, \quad \lim \mathbf{1}_{\Omega_N^{K,2}} \frac{\bar{Z}_t^{N,K}}{e^{\alpha_N t}} = \frac{\mu p}{(\alpha_0)^2}, \quad \lim \mathbf{1}_{\Omega_N^{K,2}} \frac{\sum_{i=1}^K (Z_t^{i,N})^2}{K e^{2\alpha_N t}} = \frac{(\mu p)^2}{(\alpha_0)^4}.$$

Proof. Corollary 2.8.3 tells us that $\lim \mathbf{1}_{\Omega_N^{K,2}} \frac{\mathbb{E}_\theta[|\bar{I}_t^{N,K}|]}{e^{\alpha_N t}} = 0$ in probability. In view of (2.20), we have, in probability,

$$\lim \mathbf{1}_{\Omega_N^{K,2}} \frac{\mathbb{E}_\theta[\bar{Z}_t^{N,K}]}{e^{\alpha_N t}} = \lim \mathbf{1}_{\Omega_N^{K,2}} \frac{v_t^{N,K} \bar{V}_N^K}{e^{\alpha_N t}}.$$

From (2.19) (and a few lines after) $\lim \mathbf{1}_{\Omega_N^{K,2}} \bar{V}_N^K = \lim \mathbf{1}_{\Omega_N^{K,2}} \kappa_N = 1$. From Remark 2.7.3, we already have $\lim \alpha_N = \alpha_0$ and $\lim \rho_N = p$. And by Corollary 2.7.7, we know that $\lim \mathbf{1}_{\Omega_N^{K,2}} \frac{v_t^{N,K}}{e^{\alpha_N t}} = \lim \mathbf{1}_{\Omega_N^{K,2}} \frac{\mu \rho_N}{(\alpha_N)^2} = \frac{\mu p}{(\alpha_0)^2}$. This finishes the proof of the first part.

By Lemma 2.8.6, we have $\lim \mathbf{1}_{\Omega_N^{K,2}} \frac{\mathbb{E}_\theta[|\bar{U}_t^{N,K}|]}{e^{\alpha_N t}} = 0$ in L^1. Then we conclude that, in probability,

$$\begin{aligned}\lim \mathbf{1}_{\Omega_N^{K,2}} \bar{Z}_t^{N,K}/e^{\alpha_N t} &= \lim \mathbf{1}_{\Omega_N^{K,2}} \Big(\bar{U}_t^{N,K} + \mathbb{E}_\theta[\bar{Z}_t^{N,K}] \Big)/e^{\alpha_N t} \\ &= \lim \mathbf{1}_{\Omega_N^{K,2}} \mathbb{E}_\theta[\bar{Z}_t^{N,K}]/e^{\alpha_N t} = (\mu p)/(\alpha_0)^2.\end{aligned}$$

For the third part, we start from:

$$\frac{\sum_{i=1}^K (Z_t^{i,N})^2}{K e^{2\alpha_N t}} = \frac{\sum_{i=1}^K \mathbb{E}_\theta[Z_t^{i,N}]^2 + \sum_{i=1}^K (U_t^{i,N})^2 + 2\sum_{i=1}^K \mathbb{E}_\theta[Z_t^{i,N}] U_t^{i,N}}{K e^{2\alpha_N t}}.$$

As seen in the proof of Corollary 2.7.8, on $\Omega_N^{K,2}$, $\alpha_N > \alpha_0/2$ (if N is large enough). By Lemma 2.8.5-(ii), we have:

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}} \frac{\sum_{i=1}^K (U_t^{i,N})^2}{K e^{2\alpha_N t}} \Big] \le \mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}} \Big(\frac{C}{N} + \frac{C}{e^{\alpha_N t}} \Big) \Big] \le \frac{C}{N} + \frac{C}{e^{\frac{1}{2}\alpha_0 t}},$$

which implies $\lim \mathbf{1}_{\Omega_N^{K,2}} \frac{\sum_{i=1}^K (U_t^{i,N})^2}{K e^{2\alpha_N t}} = 0$ in probability. Recall Lemma 2.8.1: we already have $\mathbf{1}_{\Omega_N^{K,2}} \|\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}]\|_\infty \le C e^{\alpha_N t}$. Hence, by Lemma 2.8.5-(ii), we have

$$\begin{aligned}\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}} \frac{\sum_{i=1}^K \mathbb{E}_\theta[Z_t^{i,N}] |U_t^{i,N}|}{K e^{2\alpha_N t}} \Big] &\le \mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}} \frac{\sum_{i=1}^K |U_t^{i,N}|}{K e^{\alpha_N t}} \Big] \\ &\le \mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}} \Big(\frac{C}{\sqrt{N}} + \frac{C}{e^{\frac{1}{2}\alpha_N t}} \Big) \Big] \le \frac{C}{\sqrt{N}} + \frac{C}{e^{\frac{1}{4}\alpha_0 t}},\end{aligned}$$

which tends to 0 in probability. By Lemma 2.7.5-(i), we have $\lim \frac{\|\boldsymbol{V}_N^K\|_2^2}{K} = \lim (\bar{V}_N^K)^2 = 1$. By Lemma 2.8.2, we see that $\lim \frac{\|\boldsymbol{I}_t^{N,K}\|_2^2}{K e^{2\alpha_N t}} = 0$. Hence, in view of (2.20) by Lemma 2.7.7,

$$\lim \frac{\sum_{i=1}^K \mathbb{E}_\theta[Z_t^{i,N}]^2}{K e^{2\alpha_N t}} = \lim \frac{(v_t^{N,K})^2 \|\boldsymbol{V}_N^K\|_2^2 + \|\boldsymbol{I}_t^{N,K}\|_2^2 + 2 v_t^{N,K} (\boldsymbol{V}_N^K, \boldsymbol{I}_t^{N,K})}{K e^{2\alpha_N t}} = \frac{(\mu p)^2}{(\alpha_0)^4},$$

which complete the proof.

Next, we consider the term $\mathcal{D}_t^{N,K,12}$. By Itô's formula and (2.25), we have:

$$\sum_{i=1}^K (M_t^{i,N} - \bar{M}_t^{N,K})^2 - K \bar{Z}_t^{N,K} = \sum_{i=1}^K 2 \int_0^t M_{s-}^{i,N} dM_s^{i,N} - K (\bar{M}_t^{N,K})^2.$$

On $\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \ge \frac{1}{4} v_t^{N,K} > 0\}$, we write $\mathcal{D}_t^{N,K,12} = 2\mathcal{D}_t^{N,K,121} - \mathcal{D}_t^{N,K,122}$, where

$$\mathcal{D}_t^{N,K,121} = \frac{N}{K(\bar{Z}_t^{N,K})^2} \sum_{i=1}^K \int_0^t M_{s-}^{i,N} dM_s^{i,N},$$

$$\mathcal{D}_t^{N,K,122} = \frac{N(\bar{M}_t^{N,K})^2}{(\bar{Z}_t^{N,K})^2}.$$

Lemma 2.9.15. *Assume (A). In the regime (2.27), in probability*

$$\lim \mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}}\frac{e^{\alpha_N t}\sqrt{K}}{N}|\mathcal{D}_t^{N,K,122}| = 0.$$

Proof. In view of (2.25), by Lemma 2.8.1, we have

$$\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta[(\bar{M}_t^{N,K})^2] = \mathbf{1}_{\Omega_N^{K,2}}\frac{1}{K^2}\sum_{i=1}^K\mathbb{E}_\theta[Z_t^{i,N}] \leq \frac{C}{K}\mathbf{1}_{\Omega_N^{K,2}}e^{\alpha_N t}.$$

By Corollary 2.7.8,$\bar{Z}_t^{N,K} \geq ce^{\alpha_N t}$, fro t large enough, on the event $\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}$. Hence

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq\frac{1}{4}v_t^{N,K}>0\}}e^{\alpha_N t}\frac{\sqrt{K}(\bar{M}_t^{N,K})^2}{(\bar{Z}_t^{N,K})^2}\Big] \leq \frac{C}{\sqrt{K}}.$$

This completes the proof.

For $t \geq 1$ and $1 \leq K \leq N$, we set, for $m \in [0,1]$,

$$\mathcal{E}_{N,K}^t(m) := e^{-2\alpha_N t}\sum_{i=1}^K\int_0^{\varphi_t(m)} M_{s-}^{i,N}dM_s^{i,N} \quad \text{where} \quad \varphi_t(m) = t + \frac{1}{2\alpha_0}\log[(1-e^{-2\alpha_0 t})m + e^{-2\alpha_0 t}],$$

so that $\mathcal{E}_{N,K}^t$ is a martingale (in the filtration $\mathcal{G}_m = \boldsymbol{F}_{\varphi_t(m)}$) issued from 0. Recalling the definition of $\mathcal{D}_t^{N,K,121}$, we are interested in the convergence of $\mathcal{E}_{N,K}^t(1)$.

Lemma 2.9.16. *Assume (A). In the regime (2.27), in probability,*

$$\lim \mathbf{1}_{\Omega_N^{K,2}}\frac{e^{\alpha_N t}}{\sqrt{K}}\sup_{0\leq m\leq 1}\Big|\mathcal{E}_{N,K}^t(m) - \mathcal{E}_{N,K}^t(m-)\Big| = 0.$$

Proof. Recall that $\alpha_0 > 0$. By Doob's inequality and Lemma 2.8.1, we have:

$$\mathbf{1}_{\Omega_N^{K,2}}\max_{i=1,\dots,K}\mathbb{E}_\theta\Big[\sup_{[0,t]}(M_s^{i,N})^2\Big] \leq C\mathbf{1}_{\Omega_N^{K,2}}\max_{i=1,\dots,K}\|\mathbb{E}_\theta[\boldsymbol{Z}_t^{N,K}]\|_\infty \leq Ce^{\alpha_N t}.$$

Hence, since the jumps of all our martingales have size 1 and since they never jump simultaneously,

$$\begin{aligned}
\mathbf{1}_{\Omega_N^{K,2}}\mathbb{E}_\theta\Big[\sup_{0\leq m\leq 1}\Big|\mathcal{E}_{N,K}^t(m) - \mathcal{E}_{N,K}^t(m-)\Big|\Big] \leq& \mathbf{1}_{\Omega_N^{K,2}}e^{-2\alpha_N t}\mathbb{E}_\theta\Big[\sup_{0\leq s\leq t}\max_{i=1,\dots,K}|M_s^{i,N}|\Big]\\
\leq& \mathbf{1}_{\Omega_N^{K,2}}e^{-2\alpha_N t}\mathbb{E}_\theta\Big[\sqrt{\sup_{0\leq s\leq t}\sum_{i=1}^K|M_s^{i,N}|^2}\Big]\\
\leq& \mathbf{1}_{\Omega_N^{K,2}}e^{-2\alpha_N t}\mathbb{E}_\theta\Big[\sup_{0\leq s\leq t}\sum_{i=1}^K|M_s^{i,N}|^2\Big]^{\frac{1}{2}}\\
\leq& C\mathbf{1}_{\Omega_N^{K,2}}\sqrt{K}e^{\frac{-3\alpha_N t}{2}}.
\end{aligned}$$

As seen in the proof of Corollary 2.7.8, on $\Omega_N^{K,2}$, $\alpha_N > \alpha_0/2$ (if N is large enough). We finally conclude that

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_N^{K,2}}\frac{e^{\alpha_N t}}{\sqrt{K}}\sup_{0\leq m\leq 1}\Big|\mathcal{E}_{N,K}^t(m) - \mathcal{E}_{N,K}^t(m-)\Big|\Big] \leq Ce^{-\frac{\alpha_0 t}{4}},$$

which ends the proof.

Lemma 2.9.17. *We assume (A). In the regime (2.27), it holds*

$$\mathbf{1}_{\Omega_N^{K,2}}\Big(\frac{e^{\alpha_N t}\mathcal{E}^t_{N,K}(m)}{\sqrt{K}}\Big)_{0\le m\le 1} \xrightarrow{d} \frac{\mu p}{\sqrt{2}(\alpha_0)^2}(B_m)_{0\le m\le 1}$$

for the Skorohod topology, where B is a standard Brownian motion.

Proof. By [23, Chapter VIII, Theorem 3.8-(b)] and thanks to Lemma 2.9.16, it suffices to verify that $\lim \mathbf{1}_{\Omega_N^{K,2}} C^t_{N,K}(m) = \frac{(\mu p)^2}{2(\alpha_0)^4} m$ in probability, for all $0 \le m \le 1$, where

$$C^t_{N,K}(m) = \Big[\frac{e^{\alpha_N t}\mathcal{E}^t_{N,K}(.)}{\sqrt{K}}, \frac{e^{\alpha_N t}\mathcal{E}^t_{N,K}(.)}{\sqrt{K}}\Big]_m.$$

We start from

$$C^t_{N,K}(m) = \frac{e^{2\alpha_N t}}{K}\Big[\mathcal{E}^t_{N,K}(.), \mathcal{E}^t_{N,K}(.)\Big]_m = \frac{1}{Ke^{2\alpha_N t}}\sum_{i=1}^K \Big[\int_0^{\cdot} M^{i,N}_{s-}dM^{i,N}_s, \int_0^{\cdot} M^{i,N}_{s-}dM^{i,N}_s\Big]_{\varphi_t(m)}$$

by (2.25), from which we also have

$$C^t_{N,K}(m) = \frac{1}{Ke^{2\alpha_N t}}\sum_{i=1}^K \int_0^{\varphi_t(m)} (M^{i,N}_{s-})^2 dZ^{i,N}_s.$$

Using now Itô's formula, we find

$$\begin{aligned}
C^t_{N,K}(m) =& \frac{1}{Ke^{2\alpha_N t}}\sum_{i=1}^K \Big[2\int_0^{\varphi_t(m)}\Big(\int_0^{s-} M^{i,N}_{l-}dM^{i,N}_l\Big)dZ^{i,N}_s + \int_0^{\varphi_t(m)} Z^{i,N}_{s-}dZ^{i,N}_s\Big]\\
=& \frac{2}{Ke^{2\alpha_N t}}\sum_{i=1}^K \int_0^{\varphi_t(m)}\Big(\int_0^{s-} M^{i,N}_{l-}dM^{i,N}_l\Big)dZ^{i,N}_s + \frac{1}{2Ke^{2\alpha_N t}}\sum_{i=1}^K\Big[[Z^{i,N}_{\varphi_t(m)}]^2 - Z^{i,N}_{\varphi_t(m)}\Big]\\
=& C^{t,1}_{N,K}(m) + C^{t,2}_{N,K}(m),
\end{aligned}$$

the last equality standing for a definition.

Step 1. In this step, we prove that $\lim \mathbf{1}_{\Omega_N^{K,2}} C^{t,1}_{N,K}(m)$ tends to 0 in probability, for each fixed $m \in [0,1]$. By integration by parts, we have

$$\begin{aligned}
&\int_0^u\Big(\int_0^{s-} M^{i,N}_{l-}dM^{i,N}_l\Big)dZ^{i,N}_s\\
=& Z^{i,N}_u\int_0^u M^{i,N}_{l-}dM^{i,N}_l - \int_0^u M^{i,N}_{l-}Z^{i,N}_{l-}dM^{i,N}_l - \int_0^u M^{i,N}_{l-}dZ^{i,N}_l\\
=& U^{i,N}_u\int_0^u M^{i,N}_{l-}dM^{i,N}_l + \int_0^u M^{i,N}_{l-}(\mathbb{E}_\theta[Z^{i,N}_u] - Z^{i,N}_{l-})dM^{i,N}_l - \int_0^u M^{i,N}_{l-}dZ^{i,N}_l.
\end{aligned}$$

We write $C^{t,1}_{N,K}(m) = C^{t,11}_{N,K}(m) + C^{t,12}_{N,K}(m) + C^{t,13}_{N,K}(m)$ accordingly.

Step 1.1. Concerning $C^{t,11}_{N,K}(m)$, we first observe, using again an integration by parts, that

$$\Big|U^{i,N}_u\int_0^u M^{i,N}_{l-}dM^{i,N}_l\Big| = \frac{1}{2}\Big|U^{i,N}_u\Big((M^{i,N}_u)^2 - Z^{i,N}_u\Big)\Big| \le Ct\Big(U^{i,N}_u\Big)^2 + \frac{C}{t}\Big((M^{i,N}_u)^4 + (Z^{i,N}_u)^2\Big).$$

As seen in the proof of Corollary 2.7.8, on $\Omega_N^{K,2}$, $\alpha_N > \alpha_0/2$ (if N is large enough). By Lemma 2.8.5-(ii), we have: for $0 \le u \le t$,

$$\mathbf{1}_{\Omega_N^{K,2}}\frac{t}{Ke^{2\alpha_N t}}\sum_{i=1}^K \mathbb{E}_\theta\Big[\Big(U^{i,N}_u\Big)^2\Big] \le C\mathbf{1}_{\Omega_N^{K,2}}\Big(\frac{t}{N} + \frac{t}{e^{\alpha_N t}}\Big) \le C\Big(\frac{t}{N} + \frac{t}{e^{\frac{1}{2}\alpha_0 t}}\Big).$$

In view of (2.25), by Lemma 2.8.5-(i), we conclude that, for $u \in [0,t]$,

$$\mathbf{1}_{\Omega_N^{K,2}} \frac{1}{Ke^{2\alpha_N t}} \sum_{i=1}^K \mathbb{E}_\theta\Big[\frac{C}{t}\Big((M_u^{i,N})^4 + (Z_u^{i,N})^2\Big)\Big] \le \frac{C}{t}.$$

This concludes the sub-step.

Step 1.2 To study $C_{N,K}^{t,12}(m)$, we first use (2.25): on $\Omega_N^{K,2}$,

$$\begin{aligned}
&\mathbb{E}_\theta\Big[\Big(\sum_{i=1}^K \int_0^u M_{l-}^{i,N}(\mathbb{E}_\theta[Z_u^{i,N}] - Z_{l-}^{i,N}) dM_l^{i,N}\Big)^2\Big] \\
=&\mathbb{E}_\theta\Big[\sum_{i=1}^K \int_0^u \Big(\mathbb{E}_\theta[Z_u^{i,N}] - Z_{l-}^{i,N}\Big)^2 \Big(M_{l-}^{i,N}\Big)^2 dZ_l^{i,N}\Big] \\
\le&\sum_{i=1}^K \mathbb{E}_\theta\Big[\Big(\mathbb{E}_\theta[Z_u^{i,N}] + Z_u^{i,N}\Big)^2 \int_0^u \Big(M_{l-}^{i,N}\Big)^2 dZ_l^{i,N}\Big] \\
\le&2\sum_{i=1}^K \mathbb{E}_\theta\Big[\Big(\mathbb{E}_\theta[Z_u^{i,N}] + Z_u^{i,N}\Big)^4 + \Big(\int_0^u \Big(M_{l-}^{i,N}\Big)^2 dZ_l^{i,N}\Big)^2\Big] \\
\le&C\sum_{i=1}^K \mathbb{E}_\theta\Big[\Big(Z_u^{i,N}\Big)^4 + \Big(\sup_{0\le l\le u}\Big(M_l^{i,N}\Big)^2 Z_u^{i,N}\Big)^2\Big] \\
\le&C\sum_{i=1}^K \mathbb{E}_\theta\Big[\Big(Z_u^{i,N}\Big)^4 + \sup_{0\le l\le u}\Big(M_l^{i,N}\Big)^8 + \Big(Z_u^{i,N}\Big)^4\Big].
\end{aligned}$$

By Doob's inequality, we have on $\Omega_N^{K,2}$, $\mathbb{E}_\theta[\sup_{0\le l\le u}(M_l^{i,N})^8] \le C\mathbb{E}_\theta[(Z_u^{i,N})^4] \le CKe^{4\alpha_N t}$ by Lemma 2.8.5 (iii). We conclude that on $\Omega_N^{K,2}$, for any $m \in [0,1]$,

$$\mathbb{E}_\theta[C_{N,K}^{t,12}(m)] \le \frac{C}{\sqrt{K}}.$$

Step 1.3 We next write, for $u \in [0,t]$,

$$\Big|\int_0^u M_{l-}^{i,N} dZ_l^{i,N}\Big| \le Z_u^{i,N} \sup_{0\le s\le u} |M_s^{i,N}| \le t^{-1}(Z_u^{i,N})^2 + t \sup_{0\le s\le u} (M_s^{i,N})^2.$$

In view of (2.25), we have, on $\Omega_N^{K,2}$,

$$\sum_{i=1}^K \mathbb{E}_\theta\Big[\Big|\int_0^u M_{l-}^{i,N} dZ_l^{i,N}\Big|\Big] \le t^{-1}\mathbb{E}_\theta[(Z_u^{i,N})^2] + Ct\mathbb{E}[Z_u^{i,N}] \le \frac{CKe^{2\alpha_N t}}{t} + CKte^{\alpha_N t}$$

by Lemma 2.8.5-(i). This implies that, still on $\Omega_N^{K,2}$,

$$\mathbb{E}_\theta[|C_{N,K}^{t,13}(m)|] \le C\Big(\frac{1}{t} + \frac{t}{e^{\alpha_N t}}\Big),$$

which tends to 0.

Step 2. We next study $C_{N,K}^{t,2}(m)$. Using Lemma 2.9.14 and that $\varphi_t(m) \in [0,t]$, we conclude that, in probability

$$\lim \mathbf{1}_{\Omega_N^{K,2}} \frac{\bar{Z}_{\varphi_t(m)}^{N,K}}{e^{2\alpha_N t}} = 0.$$

in probability. Since furthermore, by Corollary 2.8.8, we have $\lim \mathbf{1}_{\Omega_N^{K,2}} = 1$ in probability, we are reduced to check that, for all $m \in (0,1]1$,

$$\lim \frac{1}{2Ke^{2\alpha_N t}} \sum_{i=1}^{K} [Z^{i,N}_{\varphi_t(m)}]^2 = \frac{(\mu p)^2}{2\alpha_0^4} m.$$

We write

$$\lim \frac{1}{2Ke^{2\alpha_N t}} \sum_{i=1}^{K} [Z^{i,N}_{\varphi_t(m)}]^2 = \lim \Big(\frac{1}{2Ke^{2\alpha_N \varphi_t(m)}} \sum_{i=1}^{K} [Z^{i,N}_{\varphi_t(m)}]^2 \Big) e^{2\alpha_N(\varphi_t(m)-t)} = \frac{(\mu p)^2}{2\alpha_0^4} m$$

by Lemma 2.9.14 and since $\varphi_t(m) = t + \frac{1}{2\alpha_0} \log(m(1-e^{-2\alpha_0 t}) + e^{-2\alpha_0 t})$ and since $\alpha_N \to \alpha_0$, see Remark 2.7.3. The proof is complete.

Corollary 2.9.18. *Assume (A). In the regime (2.27), we have the following convergence in distribution*

$$\mathbf{1}_{\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \geq \frac{1}{4} v_t^{N,K} > 0\}} \frac{e^{\alpha_N t} \sqrt{K} \mathcal{D}_t^{N,K,12}}{N} \to \mathcal{N}\Big(0, \frac{2(\alpha_0)^4}{(\mu p)^2}\Big).$$

Proof. First, we know from Corollary 2.8.8 that $\lim \mathbf{1}_{\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \geq \frac{1}{4} v_t^{N,K} > 0\}} = 1$ in probability. Also, we recall that $\mathcal{D}_t^{N,K,12} = 2\mathcal{D}_t^{N,K,121} - \mathcal{D}_t^{N,K,122}$ and that by Lemma 2.9.15,

$$\lim \mathbf{1}_{\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \geq \frac{1}{4} v_t^{N,K} > 0\}} \frac{e^{\alpha_N t} \sqrt{K}}{N} |\mathcal{D}_t^{N,K,122}| = 0$$

in probability. Since next

$$\mathcal{D}_t^{N,K,121} = \frac{N}{K(\bar{Z}_t^{N,K})^2} \sum_{i=1}^{K} \int_0^t M_{s-}^{i,N} dM_s^{i,N} = \frac{N}{K} \frac{e^{2\alpha_N t}}{(\bar{Z}_t^{N,K})^2} \mathcal{E}_{N,K}^t(1)$$

and since $\lim \frac{e^{2\alpha_N t}}{(\bar{Z}_t^{N,K})^2} = (\alpha_0^2/(\mu p))^2$ by Lemma 2.9.14, we conclude from Lemma 2.9.17 that

$$\lim \mathbf{1}_{\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \geq \frac{1}{4} v_t^{N,K} > 0\}} \frac{e^{\alpha_N t} \sqrt{K} \mathcal{D}_t^{N,K,12}}{N} \to \frac{\sqrt{2}\alpha_0^2}{\mu p} B_1$$

in distribution.

Proof. [Proof of Proposition 2.9.1] We recall that we have written

$$\mathcal{D}_t^{N,K} = \mathcal{D}_t^{N,K,11} + \mathcal{D}_t^{N,K,12} + \mathcal{D}_t^{N,K,13} + \mathcal{D}_t^{N,K,14} + \mathcal{D}_t^{N,K,2} + \mathcal{D}_t^{N,K,3}.$$

Gathering Corollaries 2.9.13 and 2.9.18 ends the proof.

Proof. [Proof of Theorem 2.1.4] We work in the regime (2.27). By Corollary 2.8.8, we know that, in probability, $\lim \mathbf{1}_{\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \geq \frac{1}{4} v_t^{N,K} > 0\}} = 1$. Also, we know from Proposition 2.9.1 that

$$\lim \mathbf{1}_{\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \geq \frac{1}{4} v_t^{N,K} > 0\}} \mathcal{U}_t^{N,K} = \frac{1}{p} - 1.$$

in probability. Since $p < 1$, we deduce that $\lim \mathbf{1}_{\{\mathcal{U}_t^{N,K} \geq 0\}} \mathbf{1}_{\Omega_N^{K,2} \cap \{\bar{Z}_t^{N,K} \geq \frac{1}{4} v_t^{N,K} > 0\}} = 1$ in probability. In view of (2.4), we have

$$\Big(\mathcal{P}_t^{N,K} - p\Big) \mathbf{1}_{\{\mathcal{U}_t^{N,K} \geq 0\}} = \frac{-p\Big[\mathcal{U}_t^{N,K} - \Big(\frac{1}{p} - 1\Big)\Big]}{\mathcal{U}_t^{N,K} + 1} \mathbf{1}_{\{\mathcal{U}_t^{N,K} \geq 0\}}.$$

Hence

$$\lim \frac{e^{\alpha_0 t}\sqrt{K}}{N}\Big(\mathcal{P}_t^{N,K} - p\Big)$$
$$= \lim \mathbf{1}_{\Omega_N^{K,2}\cap\{\bar{Z}_t^{N,K}\geq \frac{1}{4}v_t^{N,K}>0\}\cap\{\mathcal{U}_t^{N,K}\geq 0\}} \frac{-pe^{(\alpha_0-\alpha_N)t}e^{\alpha_N t}\sqrt{K}}{N}\frac{\mathcal{U}_t^{N,K} - \Big(\frac{1}{p}-1\Big)}{\mathcal{U}_t^{N,K}+1}.$$

By Remark 2.7.3, we have $|\alpha_N - \alpha_0| = |\rho_N - p| \leq \frac{C}{N^{\frac{3}{8}}}$. Since $\lim \frac{Ct}{N^{\frac{3}{8}}} = 0$ in the regime (2.27), we conclude that $\lim e^{(\alpha_0-\alpha_N)t} = 1$. Finally, by Proposition 2.9.1, we deduce that

$$\lim \frac{e^{\alpha_0 t}\sqrt{K}}{N}\Big(\mathcal{P}_t^{N,K} - p\Big) = \lim \frac{-pe^{\alpha_N t}\sqrt{K}}{N}\frac{\mathcal{U}_t^{N,K} - \Big(\frac{1}{p}-1\Big)}{\mathcal{U}_t^{N,K}+1} \xrightarrow{d} \mathcal{N}\Big(0, \frac{2(\alpha_0)^4 p^2}{\mu^2}\Big).$$

This ends the proof.

2.10 Appendix

We first write down two lemmas concerning the convolution of the function ϕ that will be useful in the subcritical case.

Lemma 2.10.1. *We consider $\phi : [0,\infty) \to [0,\infty)$ such that $\Lambda = \int_0^\infty \phi(s)ds < \infty$ and, for some $q \geq 1$, $\int_0^\infty s^q\phi(s)ds < \infty$. Then, for all $n \geq 1$ and $r \geq 1$,*

$$\int_r^\infty \sqrt{s}\phi^{*n}(s)ds \leq C\Lambda^n n^q r^{\frac{1}{2}-q} \quad \text{and} \quad \int_0^\infty \sqrt{s}\phi^{*n}(s)ds \leq \sqrt{n}\Lambda^n.$$

Proof. We introduce some i.i.d. random variables $X_1, X_2, \dots$ with density $\Lambda^{-1}\phi$ and set $S_0 = 0$ as well as $S_n = X_1 + \cdots + X_n$ for all $n \geq 1$. We observe that

$$\int_r^\infty \sqrt{s}\phi^{*n}(s)ds = \Lambda^n\mathbb{E}[\sqrt{S_n}\mathbf{1}_{S_n\geq r}] \leq \Lambda^n r^{\frac{1}{2}-q}\mathbb{E}[S_n^q\mathbf{1}_{S_n\geq r}] \leq \Lambda^n n^q r^{\frac{1}{2}-q}\mathbb{E}[X_1^q] \leq C\Lambda^n n^q r^{\frac{1}{2}-q}.$$

We used the Minkowski inequality and that $\mathbb{E}[X_1^q] = \Lambda^{-1}\int_0^\infty s^q\phi(s)ds < \infty$ by assumption. For the second part, we write

$$\int_0^\infty \sqrt{s}\phi^{*n}(s)ds = \Lambda^n\mathbb{E}[\sqrt{S_n}] \leq \sqrt{n}\Lambda^n\sqrt{\mathbb{E}[X_1]} \leq \sqrt{n}\Lambda^n$$

by the Cauchy-Schwarz inequality.

Lemma 2.10.2. *Under the same conditions as in Lemma 2.10.1, we have, for $n \in \mathbb{N}_+$ and $r \geq 1$,*

$$\Big|\int_0^t \phi^{*n}(s)ds - \Lambda^n\Big| \leq n\Lambda^{n-1}\int_{\frac{t}{n}}^\infty \phi(s)ds.$$

Proof. Then consider n i.i.d random variables $\{X_i\}_{i=1,\dots,n}$ with density $\phi(s)/\Lambda$ and write

$$\Big|\int_0^t \phi^{*n}(s)ds - \Lambda^n\Big| = \Lambda^n P\Big(\sum_{i=1}^n X_i \geq t\Big) \leq \Lambda^n P\Big(\max_{i=1,\dots,n} X_i \geq t/n\Big) \leq n\Lambda^n P(X_1 \geq t/n),$$

which complete the proof.

We next adopt the notation of the supercritical case and study the martingales $M_t^{i,N}$.

Lemma 2.10.3. *Assume (A). For any $s \in [0,t]$, $i \neq k$, on $\Omega_N^{K,2}$,*

(i) $\mathbb{E}_\theta[M_t^{i,N} M_s^{j,N} M_t^{i',N} M_{s'}^{j',N}] = 0$ *if* $\#\{i,j,i',j'\} = 4$.

(ii) $|\mathbb{E}_\theta[M_t^{i,N} M_s^{i,N} M_t^{k,N}]| \leq \frac{Cse^{\alpha_N s}}{N}$ *if* $i \neq k$.

(iii) $|\mathbb{E}_\theta[M_t^{i,N} M_s^{j,N} M_t^{i',N} M_s^{j',N}]| \leq \frac{C(e^{\alpha_N t} t^2)}{N^2}$ *if* $\#\{i,j,i',j'\} = 3$.

(iv) $|\mathbb{E}_\theta[M_t^{i,N} M_s^{j,N} M_t^{i',N} M_s^{j',N}]| \leq Ce^{\alpha_N (t+s)}$ *without any conditions.*

Proof. First, we recall that under (A), we have $\phi^{*n}(t) = t^{n-1}e^{-bt}/(n-1)!$ for all $n \geq 1$.

Point (i) follows from the fact that, since when i,j,i',j' are pairwise different, the martingales $M^{i,N}$, $M^{i',N}$, $M^{j,N}$ and $M^{j',N}$ are orthogonal by (2.25).

For point (ii), we first use that $M^{i,N}$ and $M^{k,N}$ are orthogonal and that $t \geq s$ to write

$$\mathbb{E}_\theta[M_t^{i,N} M_s^{i,N} M_t^{k,N}] = \mathbb{E}_\theta[M_s^{i,N} \mathbb{E}_\theta[M_t^{i,N} M_t^{k,N} | \boldsymbol{F}_s]] = \mathbb{E}_\theta[(M_s^{i,N})^2 M_s^{k,N}].$$

Since $[M^{i,N}, M^{i,N}]_s = Z_s^{i,N}$, it holds that $(M_s^{i,N})^2 = 2\int_0^s M_{r-}^{i,N} dM_r^{i,N} + Z_s^{i,N}$. Using that $\int_0^\cdot M_{r-}^{i,N} dM_r^{i,N}$ and $M^{k,N}$ are orthogonal, we deduce that $\mathbb{E}_\theta[(M_s^{i,N})^2 M_s^{k,N}]| = \mathbb{E}_\theta[Z_s^{i,N} M_s^{k,N}]$ whence, by (2.21),

$$\begin{aligned}
|\mathbb{E}_\theta[(M_s^{i,N})^2 M_s^{k,N}]| =& |\mathbb{E}_\theta[U_s^{i,N} M_s^{k,N}]| \\
&\leq \sum_{n\geq 0} \sum_{j=1}^N \int_0^s \phi^{*n}(s-l) A_N^n(i,j) |\mathbb{E}_\theta[M_l^{j,N} M_s^{k,N}]| dl \\
&= \sum_{n\geq 0} \int_0^s \phi^{*n}(s-l) A_N^n(i,k) \mathbb{E}_\theta[Z_l^{k,N}] dl
\end{aligned}$$

by (2.25). By [14, Lemma 35-(iv)], we have $A_N^n(i,k) \leq \frac{C\rho_N^n}{N}$ for all $n \geq 1$ and $A_N^n(i,k) = 0$ for $n = 0$ (since $i \neq k$). We also know that $\mathbb{E}_\theta[Z_l^{k,N}] \leq Ce^{\alpha_N l}$ (on $\Omega_N^{K,2}$), so that

$$\begin{aligned}
|\mathbb{E}_\theta[(M_s^{i,N})^2 M_s^{k,N}]| \leq& \frac{C}{N} \sum_{n\geq 1} \int_0^s e^{-b(s-l)} \frac{(\rho_N)^n (s-l)^{n-1}}{(n-1)!} e^{\alpha_N l} dl \\
=& \frac{C\rho_N}{N} \int_0^s e^{(\rho_N - b)(s-l)} e^{\alpha_N l} dl = \frac{C\rho_N s e^{(\rho_N - b)s}}{N} \leq \frac{Cse^{(\rho_N - b)s}}{N},
\end{aligned}$$

since $\alpha_N = \rho_N - b$ and $\rho_N \leq 2p$ on $\Omega_N^{K,2}$ by Remark 2.7.3.

For point (iii), we first consider the case $j = j'$ (and i, i', j are pairwise different). We have $\mathbb{E}_\theta[M_t^{i,N} M_s^{j,N} M_t^{i',N} M_s^{j,N}] = \mathbb{E}_\theta[M_s^{i,N} M_s^{i',N} (M_s^{j,N})^2]$ because $t \geq s$ and $M^{i,N}$ and $M^{i',N}$ are orthogonal. Using the Itô formula as in (ii), we find $(M_s^{j,N})^2 = 2\int_0^s M_{r-}^{j,N} dM_r^{j,N} + Z_s^{j,N}$, with $\int_0^\cdot M_{r-}^{i,N} dM_r^{i,N}$ orthogonal to $M^{i,N} M^{i',N}$. As a consequence, $\mathbb{E}_\theta[M_t^{i,N} M_s^{j,N} M_t^{i',N} M_s^{j,N}] = \mathbb{E}_\theta[M_s^{i,N} M_s^{i',N} Z_s^{j,N}] = \mathbb{E}_\theta[M_s^{i,N} M_s^{i',N} U_s^{j,N}]$, recall (2.21), so that

$$\begin{aligned}
&|\mathbb{E}_\theta[M_t^{i,N} M_s^{j,N} M_t^{i',N} M_s^{j,N}]| \\
&\leq \sum_{n\geq 0} \sum_{q=1}^N \int_0^s \phi^{*n}(s-l) A_N^n(j,q) |\mathbb{E}_\theta[M_s^{i,N} M_s^{i',N} M_l^{q,N}]| dl \\
&= \sum_{n\geq 0} \int_0^s \phi^{*n}(s-l) \Big\{ A_N^n(j,i) |\mathbb{E}_\theta[M_s^{i,N} M_s^{i',N} M_l^{i,N}]| + A_N^n(j,i') |\mathbb{E}_\theta[M_s^{i,N} M_s^{i',N} M_l^{i',N}]| \Big\} dl \\
&\leq \frac{C}{N^2} \sum_{n\geq 1} \int_0^s \frac{\rho_N^n (s-l)^{n-1}}{(n-1)!} e^{-b(s-l)} l e^{\alpha_N l} dl
\end{aligned}$$

by point (ii) and since, as in the proof of (ii), $A_N^n(j,i) \le \frac{C\rho_N^n}{N}$ for all $n \ge 1$ and $A_N^n(j,i) = 0$ for $n = 0$ (since $i \ne j$), and similar considerations for i'. Using as usual that, on $\Omega_N^{K,2}$, we have $\rho_N \le 2p$ and $\alpha_N = \rho_N - b$, we easily conclude that

$$|\mathbb{E}_\theta[M_t^{i,N} M_s^{j,N} M_t^{i',N} M_s^{j,N}]| = \frac{C\rho_N}{N^2} \int_0^s l e^{\alpha_N s} dl \le \frac{Cs^2 e^{\alpha_N s}}{N^2}.$$

For the case where $i = i'$ (and where i, j, j' are pairwise different), we have, proceeding as previously, $\mathbb{E}_\theta[M_t^{i,N} M_s^{j,N} M_t^{i,N} M_s^{j',N}] = \mathbb{E}_\theta[Z_t^{i,N} M_s^{j,N} M_s^{j',N}] = \mathbb{E}_\theta[U_t^{i,N} M_s^{j,N} M_s^{j',N}]$. Hence, using (2.21),

$$\begin{aligned}
&|\mathbb{E}_\theta[M_t^{i,N} M_s^{j,N} M_t^{i,N} M_s^{j',N}]| \\
&\le \sum_{n\ge 0} \sum_{q=1}^N \int_0^t \phi^{*n}(t-l) A_N^n(i,q) |\mathbb{E}_\theta[M_l^{q,N} M_s^{j,N} M_s^{j',N}]| dl \\
&\le \sum_{n\ge 0} \int_0^t \phi^{*n}(t-l) \Big\{ A_N^n(i,j) |\mathbb{E}_\theta[M_l^{j,N} M_s^{j,N} M_s^{j',N}]| + A_N^n(i,j') |\mathbb{E}_\theta[M_l^{j',N} M_s^{j,N} M_s^{j',N}]| \Big\} dl.
\end{aligned}$$

If $l \ge s$, we see that $\mathbb{E}_\theta[M_l^{j,N} M_s^{j,N} M_s^{j',N}] = \mathbb{E}_\theta[M_s^{j,N} M_s^{j,N} M_s^{j',N}]$. So in any case, we can apply point (ii) and we find

$$|\mathbb{E}_\theta[M_t^{i,N} M_s^{j,N} M_t^{i,N} M_s^{j',N}]| \le \frac{C}{N} \sum_{n\ge 0} \int_0^t \phi^{*n}(t-l)[A_N^n(i,j) + A_N^n(i,j')](l \wedge s) e^{\alpha_N (l\wedge s)} dl.$$

Using the same argument as in the previous case, we conclude that

$$|\mathbb{E}_\theta[M_t^{i,N} M_s^{j,N} M_t^{i,N} M_s^{j',N}]| \le \frac{C}{N^2} \sum_{n\ge 1} \int_0^t \frac{l(\rho_N)^n (t-l)^{n-1}}{(n-1)!} e^{-b(t-l)} e^{\alpha_N l} dl \le \frac{Ct^2 e^{\alpha_N t}}{N^2}$$

as usual. Finally, if $i = j$ (and i, i', j' are pairwise different), we first write, using that the three involved martingales are orthogonal, $\mathbb{E}_\theta[M_t^{i,N} M_s^{i,N} M_t^{i',N} M_s^{j',N}] = \mathbb{E}_\theta[M_s^{i,N} M_s^{i,N} M_s^{i',N} M_s^{j',N}]$, so that, arguing as in the previous cases, $\mathbb{E}_\theta[M_t^{i,N} M_s^{i,N} M_t^{i',N} M_s^{j',N}] = \mathbb{E}_\theta[U_s^{i,N} M_s^{i',N} M_s^{j',N}]$. It then suffices to copy the previous case (when $t = s$ and replacing j by i') to find

$$|\mathbb{E}_\theta[M_t^{i,N} M_s^{i,N} M_t^{i',N} M_s^{j',N}]| \frac{Cs^2 e^{\alpha_N s}}{N^2}$$

This completes the proof of (iii).

By Doob's inequality, (2.25) and Lemma 2.8.5-(i), we have $\mathbb{E}_\theta[(M_t^{i,N})^4] \le C\mathbb{E}_\theta[(Z_t^{i,N})^2] \le Ce^{2\alpha_N t}$, whence

$$|\mathbb{E}_\theta[M_t^{i,N} M_s^{j,N} M_t^{i',N} M_s^{j',N}]| \le \mathbb{E}_\theta[(M_t^{i,N})^4]^{\frac14} \mathbb{E}_\theta[(M_s^{j,N})^4]^{\frac14} \mathbb{E}_\theta[(M_t^{i',N})^4]^{\frac14} \mathbb{E}_\theta[(M_s^{j',N})^4]^{\frac14}$$

is bounded by $Ce^{\alpha_N (t+s)}$ as desired.

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