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Benjamin Havret

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Université de Paris



Université de Paris
École Doctorale 386, de Sciences Mathématiques de Paris Centre
Laboratoire de Probabilités, Statistique et Modélisation

Thèse de doctorat

Discipline : Mathématiques Appliquées

Présentée par
Benjamin HAVRET

ON THE LYAPUNOV EXPONENT OF RANDOM TRANSFER MATRICES
AND ON PINNING MODELS WITH CONSTRAINTS

Sous la direction de **Giambattista GIACOMIN**

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Présentée et soutenue à Paris le 9 décembre 2019 devant le jury composé de

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Résumé Cette thèse se divise en deux parties indépendantes.

- La première (Chapitres 1 et 2) est consacrée à l’analyse de l’exposant de Lyapunov d’un produit de matrices de transfert aléatoires. Cet exposant de Lyapunov apparaît de multiples fois dans la littérature physique, notamment dans l’analyse du modèle d’Ising dans certains milieux aléatoires. Nous nous intéressons à une prédiction de la littérature physique concernant son comportement singulier et nous proposons une analyse mathématique de cette singularité.
- La seconde partie (Chapitres 3, 4 et 5) porte sur une variation du modèle de Poland-Scheraga pour la dénaturation de l’ADN. Cette variation vise la prise en compte des contraintes géométriques particulières des chaînes d’ADN circulaires. Nous analysons entièrement le modèle homogène: la régularité et le comportement critique de son énergie libre, ainsi que les trajectoires du polymère. Nous nous intéressons enfin au modèle désordonné, pour lequel nous établissons la pertinence du désordre, tant au niveau de l’énergie libre que des trajectoires du système.

Mots-clefs : Produit de matrices aléatoires · Exposant de Lyapunov · Systèmes désordonnés · Comportement critique · Pertinence du désordre · Modèle d’accrochage · Grand saut · Propriétés trajectoires

Abstract This work is made of two independent parts.

- The first (Chapters 1 and 2) is devoted to the study of the Lyapunov exponent of a product of random transfer matrices. This Lyapunov exponent appears repeatedly in the statistical mechanics literature, notably in the analysis of the Ising model in some special disordered environments. The focus is on a singular behaviour that has been pointed out in the physical literature: we provide a mathematical analysis of this singularity.
- In the second part (Chapters 3, 4 and 5) we consider a variation of the Poland-Scheraga model for DNA denaturation. This variation aims at modeling the case of circular DNA. We provide a complete analysis of the homogeneous model, including free energy regularity and critical behaviour, as well as path properties. We also tackle the disordered case, for which we prove relevance of disorder both for the free energy and the trajectories of the system.

Keywords: Product of random matrices · Lyapunov exponent · Disordered systems · Critical behaviour · Relevant disorder · Pinning model · Big jump · Path properties

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Liste des symboles et abréviations

IID : Indépendant·e·s et identiquement distribué·e·s

N : Ensemble des entiers naturels $\{0, 1, 2, \dots\}$

$\mathbf{1}_A$: Fonction indicatrice de l'ensemble A . On notera abusivement $\mathbf{1}_{f(x) \in A}$ pour $\mathbf{1}_{f^{-1}(A)}(x)$

P, E : Probabilité, espérance mathématique. Dans les chapitres 3 et suivants, ces notations sont réservées à l'espérance par rapport au renouvellement, et on utilise \mathbb{E} et \mathbb{P} pour l'espérance et la probabilité associées au désordre (ω_j) .

$x \wedge y$: $\min(x, y)$

$x \vee y$: $\max(x, y)$

x_+ : $\max(x, 0)$

log : Logarithme népérien ($\log(e) = 1$)

$f(x) = o(g(x))$ ou $f(x) \ll g(x)$: $f(x)/g(x) \rightarrow 0$ lorsque $x \rightarrow a$

$f(x) = O(g(x))$: $f(x)/g(x)$ est borné lorsque $x \rightarrow a$

$f(x) \sim g(x)$: $f(x) = g(x)(1 + o(1))$ lorsque $x \rightarrow a$

\simeq, \cong : utilisés lorsqu'on ne souhaite pas être précis.

\mathbf{P}^{d-1} ou $\mathbf{P}^{d-1}(\mathbf{R})$: Espace projectif : ensemble des droites de \mathbf{R}^d

\hat{x} : Un des deux vecteurs de norme 1 et de direction $x \in \mathbf{P}^{d-1}$

Convergence en probabilité : Lorsque μ_n est une suite de mesures de probabilités, X_n une suite de variable aléatoires réelles et c une constante réelle,

$$X_n \xrightarrow[n \rightarrow +\infty]{\mu_n} c$$

signifie que pour tout $\delta > 0$,

$$\mu_n (|X_n - c| \geq \delta) \xrightarrow[n \rightarrow +\infty]{} 0.$$

Part I

ON THE LYAPUNOV EXPONENT OF RANDOM TRANSFER MATRICES

Chapter 1

Produits de matrices aléatoires et Modèles d’Ising désordonnés

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1.1 Motivation

Les produits de matrices aléatoires sont une structure récurrente dans l’étude de certains systèmes désordonnés, parmi lesquels on peut citer le modèle d’Anderson [20, 76] ou les chaînes harmoniques [40, 87]. On s’intéresse ici à des versions désordonnées du modèle d’Ising.

Notations. Commençons par préciser les notations utilisées pour le modèle d'Ising. Le Hamiltonien associé au modèle d'Ising sur un graphe* non orienté $G = (V, E)$ est donné par

$$H(\underline{\sigma}) = - \sum_{(x,y) \in E} J_{x,y} \sigma_x \sigma_y - \sum_{x \in V} h_x \sigma_x, \quad \underline{\sigma} \in \{-1, 1\}^V, \quad (1.1.1)$$

où h_x est le champ magnétique externe au site x , et $J_{x,y}$ est le coefficient de couplage associé à l'arête (x, y) . À une température inverse β , la probabilité d'une configuration $\underline{\sigma}$ est

$$\frac{1}{Z_\beta} e^{-\beta H(\underline{\sigma})}. \quad (1.1.2)$$

On parle de modèle d'Ising homogène lorsque le réseau G est régulier (typiquement \mathbf{Z}^d), et lorsque le champ externe et les coefficients de couplage sont déterministes et spatialement homogènes (c.-à-d. ne dépendent pas du point x). Il s'agit d'un modèle mathématique du ferromagnétisme. Le modèle d'Ising simplifie substantiellement le fonctionnement d'un métal ferromagnétique puisqu'il suppose que :

- Les atomes le constituant sont placés sur un réseau régulier ;
- Leur moment magnétique ne prend que deux états (+1 ou -1) ;
- L'interaction entre atomes au sein du métal est réduite à une interaction entre atomes directement voisins (pas d'interaction à moyenne ou longue portée) et cette interaction est identique pour chaque atome ;
- Le champ externe est spatialement homogène.

Malgré cette série de simplifications, le modèle d'Ising homogène rend compte de la transition ferromagnétique/paramagnétique (typiquement la perte d'aimantation du fer lorsqu'il est chauffé au delà de la température de Curie).

Le fonctionnement d'un matériau ferromagnétique est bien plus complexe et on aimerait relâcher ces simplifications. Les recherches en physique théorique se sont notamment penchées sur des versions désordonnées du modèle d'Ising: l'inhomogénéité du matériau est modélisée par des variables aléatoires (e.g. des champs externes et/ou des coefficients de couplage aléatoires), le plus souvent supposées IID. On pourrait parler de modèle d'Ising en milieu aléatoire.

Nous nous penchons ici sur deux versions désordonnées très particulières du modèle d'Ising (au sens où la géométrie du désordre est spécifique) pour lesquels l'énergie libre associée s'exprime en terme d'un produit de matrices aléatoires. On se pose la question de l'influence de l'introduction du désordre sur l'énergie libre. Plus généralement on peut se demander si les propriétés du modèle à l'échelle macroscopique sont modifiées par l'introduction du désordre (on parle alors de désordre *pertinent* ou *relevant*) ou non (*désordre non pertinent/irrelevant disorder*).

Chaîne d'Ising plongée dans un champ aléatoire. Commençons par un modèle unidimensionnel. On se place sur le réseau torique $\mathbf{Z}/N\mathbf{Z}$, muni de coefficients de couplages J homogènes. En revanche le champ externe est inhomogène et on fait l'hypothèse que $(h_k)_{k \in \mathbf{Z}/N\mathbf{Z}}$ forme une famille IID de variables aléatoires intégrables. Posons $\epsilon = e^{-2\beta J}$

*Dans la suite ce sera toujours le réseau \mathbf{Z}^d ou $\mathbf{Z}/N\mathbf{Z}$.

(quantité déterministe), et $Z_k = e^{-2\beta h_k}$ (variable aléatoire). Alors la fonction de partition s'écrit en termes d'un produit de matrices aléatoires IID :

$$Z_{N,\beta}(\omega) = e^{\beta NJ + \beta \sum_{k=1}^N h_k} \text{Tr}(M_{1,\epsilon} \cdots M_{N,\epsilon}), \quad (1.1.3)$$

où $(M_{k,\epsilon})$ est la famille de matrices aléatoires IID

$$M_{k,\epsilon} = \begin{pmatrix} 1 & \epsilon \\ \epsilon Z_k & Z_k \end{pmatrix}. \quad (1.1.4)$$

Démonstration. Par définition

$$Z_{N,\beta} = \sum_{\sigma \in \{-1,+1\}^{\mathbf{Z}/N\mathbf{Z}}} \exp \left(\beta \sum_{k=1}^N h_k \sigma_k + \beta \sum_{k=1}^N J \sigma_k \sigma_{k+1} \right). \quad (1.1.5)$$

Introduisons la fonction définie sur $\{-1,+1\}^2$ par

$$\ell_k(\sigma, \tau) = \exp(\beta(h_k \sigma + J \sigma \tau)), \quad (1.1.6)$$

qui correspond aux coefficients de la matrice de transfert (aléatoire) au site k :

$$L_k = \begin{pmatrix} e^{\beta(h_k+J)} & e^{\beta(h_k-J)} \\ e^{\beta(-h_k-J)} & e^{\beta(-h_k+J)} \end{pmatrix} = e^{\beta(J+h_k)} M_{k,\epsilon}. \quad (1.1.7)$$

La fonction de partition se réécrit

$$Z_{N,\beta} = \sum_{\sigma_1 \in \{-1,1\}} \cdots \sum_{\sigma_{N+1} \in \{-1,1\}} \mathbf{1}_{\sigma_{N+1}=\sigma_1} \prod_{k=1}^N \ell_k(\sigma_k, \sigma_{k+1}). \quad (1.1.8)$$

C'est-à-dire

$$Z_{N,\beta} = \text{Tr}(L_1 \cdots L_N) = e^{\beta NJ + \beta \sum_{k=1}^N h_k} \text{Tr}(M_{1,\epsilon} \cdots M_{N,\epsilon}). \quad (1.1.9)$$

□

Par suite, l'énergie libre par spin est

$$F(\beta) := \lim_{N \rightarrow +\infty} \frac{1}{N} \log Z_{N,\beta}(\omega) = \beta(J + \mathbf{E}[h_1]) + \lim_{N \rightarrow +\infty} \frac{1}{N} \log \text{Tr}(M_{1,\epsilon} \cdots M_{N,\epsilon}). \quad (1.1.10)$$

L'analyse de l'énergie libre du système se réduit ainsi à celle de la limite (si elle existe)

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \log \text{Tr}(M_{1,\epsilon} \cdots M_{N,\epsilon}), \quad M_{k,\epsilon} = \begin{pmatrix} 1 & \epsilon \\ \epsilon Z_k & Z_k \end{pmatrix}. \quad (1.1.11)$$

L'étude de ce produit de matrices aléatoires est notamment l'objet de l'article de Derrida et Hilhorst [35] (voir aussi [28] et les articles cités). On l'appellera donc modèle de Derrida–Hilhorst et ce sera notre objet d'étude principal (voir les sections suivantes). Leur étude se concentre sur le comportement du système lorsque les interactions deviennent très fortes (c.-à-d. $J \rightarrow +\infty$ ou de façon équivalent $\epsilon \rightarrow 0$), toutes choses égales par ailleurs. Ils se placent dans le régime où $\mathbf{E}[h] > 0$ (c.-à-d. $\mathbf{E}[\log Z] < 0$) et prédisent le phénomène suivant. Dans la limite $J \rightarrow +\infty$, les spins s'alignent majoritairement avec le champ et la proportions de spins -1 tend vers 0. En revanche la structure des configurations diffère selon la loi

précise du champ et deux régimes principaux sont à distinguer. Lorsque $\mathbf{E}[e^{-2\beta h}] < 1$ (c.-à-d. $\mathbf{E}[Z] < 1$), la longueur moyenne des segments de spins -1 dans la chaîne reste bornée. En revanche, dans le cas, statistiquement plus frustré, où $\mathbf{E}[e^{-2\beta h}] > 1$ (c.-à-d. $\mathbf{E}[Z] > 1$), ces segments deviennent infiniment longs. On expliquera dans la section 1.3 que ces deux régimes correspondent à deux comportements asymptotiques de l'énergie libre bien différents.

Observons que la méthode des matrices transferts qui permet l'expression de l'énergie libre grâce à des produits de matrices (aléatoires) ne dépend pas de la forme du désordre et se généralise à d'autres versions désordonnées du modèle d'Ising unidimensionnel. D'autres études sont par exemple développées dans [28].

Modèle de McCoy–Wu. La structure de produits de matrices aléatoires émerge aussi pour des modèles bidimensionnels. Cela repose sur la célèbre méthode d'Onsager pour exprimer l'énergie libre du modèle d'Ising 2D à l'aide de matrices de transfert. Quoique cette dernière repose fortement sur les symétries du système, McCoy et Wu [77] proposent une version désordonnée du modèle d'Ising bidimensionnel, à champ nul ($\forall x, h_x = 0$), pour lequel la méthode d'Onsager reste valable. Il s'agit du modèle suivant. Les coefficients de couplage verticaux sont déterministes et prennent tous la même valeur J_V . Les coefficients de couplage horizontaux sont aléatoires indépendants et de même loi selon chaque ligne, mais sont répétés sur toute la colonne correspondante (Figure 1.1). Soyons plus explicites. Donnons-nous un réel positif J_V , et une suite $(K_i)_{i \in \mathbf{Z}}$ de variable aléatoire positives IID. On pose :

$$\text{pour tout } (i, j) \in \mathbf{Z}^2, \quad \begin{cases} J_{(i,j),(i,j+1)}(\omega) = J_V \\ J_{(i,j),(i+1,j)}(\omega) = K_i(\omega) \end{cases} . \quad (1.1.12)$$

	K_{-1}	K_0	K_1	K_2	
J_V	J_V	J_V	J_V	J_V	J_V
	K_{-1}	K_0	K_1	K_2	
J_V	J_V	J_V	J_V	J_V	J_V
	K_{-1}	K_0	K_1	K_2	
J_V	J_V	J_V	J_V	J_V	J_V
	K_{-1}	K_0	K_1	K_2	
J_V	J_V	J_V	J_V	J_V	J_V
	K_{-1}	K_0	K_1	K_2	

FIGURE 1.1: Les coefficients de couplage du modèle de McCoy et Wu.

Cette définition assure l'invariance de chacune des réalisations par translation verticale, et leur invariance en loi par translation horizontale. Un calcul similaire à celui développé par Onsager fournit l'expression explicite de l'énergie libre par spin. Introduisons, pour $\theta \in \mathbf{R}$, les matrices aléatoires

$$M_{\beta,n}(\theta) := \begin{pmatrix} 1 & \frac{a}{a^2+b^2} \\ \frac{a}{a^2+b^2} \lambda & \frac{\lambda}{a^2+b^2} \end{pmatrix}, \quad (1.1.13)$$

avec

$$a(\theta) = -2z_1 \frac{\sin(\theta)}{|1 + z_1 \exp(i\theta)|^2}, \quad \text{and} \quad b(\theta) = \frac{1 - z_1^2}{|1 + z_1 \exp(i\theta)|^2}, \quad (1.1.14)$$

où

$$z_1 = \tanh(\beta J_V), \quad z_2(n) = \tanh(\beta K_n) \quad \text{and} \quad \lambda = \lambda(n) = z_2^2(n). \quad (1.1.15)$$

McCoy et Wu montrent que l'énergie libre du modèle s'écrit, à une fonction analytique près

$$F_{\text{MW}}(\beta) := \frac{1}{4\pi} \int_{-\pi}^{\pi} \mathcal{L}_{\beta}^{\text{MW}}(\theta) d\theta, \quad (1.1.16)$$

où

$$\mathcal{L}_{\beta}^{\text{MW}}(\theta) = \lim_{N \rightarrow +\infty} \frac{1}{N} \log \|M_{\beta, N}(\theta) \cdots M_{\beta, 1}(\theta)\|. \quad (1.1.17)$$

Les matrices $M_{\beta, n}(\theta)$ apparaissant dans la résolution sont surprenamment similaires de celle du modèle de Derrida-Hilhorst. Ce dernier apparaît ainsi comme un modèle jouet, ou au moins une première étape pour comprendre l'énergie libre $F_{\text{MW}}(\beta)$. Dans [77], l'analyse de $\mathcal{L}_{\beta}^{\text{MW}}(\theta)$ dans une limite de désordre faible (ils considèrent une loi K_1 qui devient déterministe dans la limite $N \rightarrow +\infty$) les conduit à un modèle exactement soluble, pour lequel (1.1.16) devient possible à analyser. Ils prédisent que le désordre est pertinent et que la singularité est complètement lissée par le désordre : l'énergie $F_{\text{MW}}(\beta)$ présente une transition de phase d'ordre infini [42, 68] à une température inverse explicite β_c . C'est-à-dire que $F_{\text{MW}}(\beta)$ est une fonction \mathcal{C}^{∞} de β mais présente une rupture d'analyticité en β_c .

Remarque 1.1.1. Shankar et Murthy [88] proposent une version légèrement différente: les interactions horizontales sont déterministes, et les interactions verticales sont aléatoires dans la direction horizontale, mais sont répétées sur chaque ligne. Quoique le système physique soit plus riches, en ce qu'il autorise la frustration, la même structure émerge.

1.2 Généralités sur l'exposant de Lyapunov

Dans cette section on expose, sans démonstration et sans rentrer dans un grand niveau de détail, une sélection de résultats de la théorie des produits de matrices aléatoires. Beaucoup de résultats généraux peuvent être trouvés dans le livre de Bougerol [20] ou la vue d'ensemble de Viana [91].

Dans cette section, $(M_n)_{n \geq 0}$ désignera une suite IID de matrices aléatoires $d \times d$, de loi commune notée μ . On notera sans distinction $\|\cdot\|$ la norme euclidienne sur \mathbf{R}^d et la norme qui lui est subordonnée sur $\mathcal{M}_d(\mathbf{R})$.

Commençons par la définition de l'exposant de Lyapunov, qui est donnée par le théorème suivant.

Théorème 1.2.1 (Furstenberg–Kesten [46]). *Si $\mathbf{E}[\log_+ \|M_1\|] < +\infty$, alors*

$$\frac{1}{n} \log \|M_n \cdots M_1\| \xrightarrow[n \rightarrow +\infty]{} \mathcal{L}(\mu) \in \mathbf{R} \cup \{-\infty\}, \quad (1.2.1)$$

presque sûrement et dans L^1 , où

$$\mathcal{L}(\mu) = \inf_{n \geq 1} \frac{1}{n} \mathbf{E} \log \|M_n \cdots M_1\| \quad (1.2.2)$$

est appelé exposant de Lyapunov, ou exposant caractéristique de la loi μ .

Exemple 1.2.2 (Matrices triangulaires, [20, Ex. 1.5.3]). Si les matrices M_n sont presque sûrement triangulaires supérieures

$$M_n(\omega) = \begin{pmatrix} \lambda_1^{(n)}(\omega) & & (*) \\ & \ddots & \\ (0) & & \lambda_d^{(n)}(\omega) \end{pmatrix}, \quad (1.2.3)$$

alors

$$\mathcal{L}(\mu) = \max_{1 \leq k \leq d} \mathbf{E}[\log |\lambda_k|]. \quad (1.2.4)$$

Exemple 1.2.3 (Matrice déterministe : Théorème de Gelfand). Lorsque $\mu = \delta_M$, c'est-à-dire que les matrices M_n sont déterministes, l'exposant de Lyapunov s'exprime simplement

$$\mathcal{L}(\delta_M) = \log(\text{rayon spectral de } M). \quad (1.2.5)$$

En dehors de quelques exemples élémentaires, il est difficile d'obtenir une formule explicite de l'exposant de Lyapunov. On peut toutefois, avec des arguments de théorie ergodique, en obtenir une expression implicite, en fonction de la mesure invariante d'une chaîne de Markov.

1.2.1 Expression implicite et mesures invariantes

On suppose dans toute cette partie que μ ne charge que les matrices inversibles, autrement dit $M \in \text{GL}_d(\mathbf{R})$ presque sûrement. Le but de cette section est de donner des expressions implicites de l'exposant de Lyapunov, obtenues notamment par [45, 47, 65]. Elles reposent sur l'écriture « en somme télescopique » suivante : pour tout $y \in \mathbf{R}^d$ non nul,

$$\frac{1}{N} \log \|M_N \cdots M_1 y\| = \frac{1}{N} \sum_{n=1}^N \log \left\| M_n \frac{M_{n-1} \cdots M_1 y}{\|M_{n-1} \cdots M_1 y\|} \right\|, \quad (1.2.6)$$

qui se reformule comme moyenne de Birkhoff d'une chaîne de Markov. Dans cette perspective, il est judicieux d'observer l'action naturelle de $\text{GL}_d(\mathbf{R})$ sur l'espace projectif $\mathbf{P}^{d-1} = \mathbf{P}(\mathbf{R}^d)$: pour toute matrice $A \in \text{GL}_d(\mathbf{R})$ et toute droite $x = \mathbf{R}y \in \mathbf{P}^{d-1}$,

$$A \cdot (\mathbf{R}y) = \mathbf{R}(Ay) \quad (1.2.7)$$

Lorsque $x \in \mathbf{P}^{d-1}$, on notera \hat{x} un des vecteurs de norme 1 (indifféremment) de direction x . Il conviendra de ne pas confondre $M \cdot x$ et $M\hat{x}$: le premier est un élément de l'espace projectif, tandis que le second est un vecteur de l'espace, dont c'est la norme qui nous intéressera. Enfin introduisons la notion de mesure invariante dans ce contexte.

Définition 1.2.4 (Mesure invariante). Une mesure de probabilité ν sur \mathbf{P}^{d-1} est dite invariante par μ lorsque pour toute $f \in \mathcal{C}_b(\mathbf{P}^{d-1})$,

$$\int \mathbf{E}[f(M \cdot x)] d\nu(x) = \int f(x) d\nu(x). \quad (1.2.8)$$

Grâce à des théorèmes ergodiques usuels, l'écriture (1.2.6) fournit une première expression de l'exposant de Lyapunov.

Théorème 1.2.5 (Furstenberg–Kifer [47] et Hennion [65]). *Supposons que μ ne charge que les matrices inversibles, que $\mathbf{E}[\log_+ \|M\|] < \infty$ et que $\mathbf{E}[\log_+ \|M^{-1}\|] < \infty$. Dans ce cas*

$$\mathcal{L}(\mu) = \sup \left\{ \int_{\mathbb{P}^{d-1}} \mathbf{E}[\log \|M\hat{x}\|] d\nu(x) : \nu \text{ invariante par } \mu \right\}, \quad (1.2.9)$$

et ce supremum est atteint.

Quoique le supremum soit atteint, l'expression (1.2.9) est peu aisément manipulable. On aimerait par exemple savoir pour quelle(s) mesure(s) ν il est atteint. Une première idée est de trouver des conditions pour que la mesure invariante soit unique. On cite souvent à ce propos un résultat de Furstenberg [45] (voir aussi [20, Chap. 3]) qui démontre un critère d'unicité de la mesure invariante, sous des hypothèses dynamiques et géométriques sur le support de la loi μ . Malheureusement, nous nous concentrerons précisément sur des limites où ces hypothèses sont mises en défaut: en particulier, on travaillera avec des matrices aléatoires admettant plusieurs mesures invariantes. C'est pourquoi on ne reprend pas ce résultat ici. Au lieu de ça, on s'intéresse à un résultat de Furstenberg et Kifer, qui proposent plutôt une classe de mesures μ pour laquelle le supremum est atteint par n'importe quelle mesure ν . Introduisons à cet effet les deux définitions suivantes.

Définition 1.2.6 (Espace invariant). Si μ est une loi sur $\mathcal{M}_d(\mathbf{R})$, on dit que le sous espace $L \subset \mathbf{R}^d$ est invariant sous l'action de μ si pour μ -presque toute matrice A , on a $AL \subset L$.

Définition 1.2.7 (Irréductibilité). On dit que la probabilité μ sur $\mathcal{M}_d(\mathbf{R})$ est irréductible si les seuls espaces invariants sous l'action de μ sont $\{0\}$ et \mathbf{R}^d .

Théorème 1.2.8 ([47]). *Si μ est une probabilité irréductible sur $\text{GL}_d(\mathbf{R})$ alors pour toute mesure invariante ν ,*

$$\mathcal{L}(\mu) = \int_{\mathbb{P}^{d-1}} \mathbf{E}[\log \|M\hat{x}\|] d\nu(x). \quad (1.2.10)$$

De plus, pour tout $y \in \mathbf{R}^d \setminus \{0\}$, on a la convergence presque sûre

$$\frac{1}{n} \log \|M_n \cdots M_1 y\| \xrightarrow[n \rightarrow +\infty]{} \mathcal{L}(\mu). \quad (1.2.11)$$

Exemple 1.2.9. Soit, pour $p \in (0, 1]$,

$$\mu_p = p \delta_{M_1} + (1-p) \delta_{M_2}, \quad \text{avec} \quad M_1 = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \text{et} \quad M_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (1.2.12)$$

Observons que la probabilité $\nu = \frac{1}{2} \delta_{(0,1)} + \frac{1}{2} \delta_{(1,0)}$ est toujours une mesure invariante. Par ailleurs, lorsque $p \in (0, 1]$, la mesure μ_p est irréductible. Ainsi, grâce au Théorème 1.2.8,

$$\mathcal{L}(\mu_p) = \int_{\mathbb{P}^1} (p \log \|M_1 \hat{x}\| + (1-p) \log \|M_2 \hat{x}\|) d\nu(x) = 0. \quad (1.2.13)$$

Remarque 1.2.10. En général, toutes les mesures invariantes ne maximisent pas l'intégrale dans (1.2.9). L'hypothèse d'irréductibilité est cruciale, et quasiment nécessaire. Par exemple, pour la matrice déterministe diagonale $M = \text{Diag}(1, 2)$, les mesures $\delta_{(1,0)}$ et $\delta_{(0,1)}$ sont

invariantes, mais seule la seconde donne l'exposant de Lyapunov. Les matrices diagonales, et en général les matrices aléatoires ayant des sous-espaces invariants non triviaux, sont pathologiques dans la théorie développée par Furstenberg et al. Ce sont précisément des perturbations de matrices diagonales qui vont nous intéresser dans les sections suivantes, et c'est la raison pour laquelle on s'attend à des comportements singuliers.

On ne développe pas davantage la théorie générale. Ces seuls résultats nous seront suffisants. De nombreux travaux récents en théorie ergodique s'intéressent aux propriétés de l'exposant de Lyapunov (analyticité, harmonicité, etc.) ou de la mesure invariante (existence d'une densité et régularité de celle-ci). Il est à noter que ce sont des questions difficiles même pour des lois aussi élémentaires que $\frac{1}{2}(\delta_A + \delta_B)$ (A, B désignant deux matrices déterministes données).

Dans les deux dernières sous-sections sont exposés quelques résultats de régularité de l'exposant de Lyapunov, en tant que fonction de μ .

1.2.2 Continuité de l'exposant de Lyapunov

Dans la perspective d'étudier des énergies libres définies par un exposant de Lyapunov, la question de la régularité de celui-ci par rapport à la loi de la matrice est cruciale. Commençons par nous intéresser à la continuité de l'application $\mu \mapsto \mathcal{L}(\mu)$, où l'espace des mesures est muni de la distance de la convergence étroite. Signalons que la continuité n'est pas toujours satisfaite, même pour des lois élémentaires. L'exemple suivant l'illustre.

Exemple 1.2.11 (Exemple de discontinuité). Reprenons l'exemple 1.2.9. Clairement $p \mapsto \mu_p$ est continue sur $[0, 1]$ au sens de la convergence étroite. Mais $\mathcal{L}(\mu_p) = 0$ pour $p \in (0, 1]$, tandis que $\mathcal{L}(\mu_0) = \log 2$. Ainsi $p \mapsto \mathcal{L}(\mu_p)$ n'est pas continue en 0.

Toutefois on peut trouver, pour une suite de mesures μ_k convergeant étroitement vers μ , des critères sous lesquels $\mathcal{L}(\mu_k)$ converge vers $\mathcal{L}(\mu)$. Il convient déjà d'imposer une condition d'équintégrabilité.

Définition 1.2.12. On dit qu'une suite de distributions (μ_k) sur $\mathrm{GL}_d(\mathbf{R})$ est équintégrable lorsque

$$\sup_{k \geq 0} \left[\int \log_+ \|A\| \mathbf{1}_{\log_+ \|A\| > T} d\mu_k(A) + \int \log_+ \|A^{-1}\| \mathbf{1}_{\log_+ \|A^{-1}\| > T} d\mu_k(A) \right] \xrightarrow{T \rightarrow \infty} 0. \quad (1.2.14)$$

Proposition 1.2.13 (Kifer [72]). *Donnons-nous une suite équintégrable $(\mu_k)_{k \geq 0}$ de probabilités sur $\mathrm{GL}_d(\mathbf{R})$ convergeant étroitement vers une probabilité μ . Alors*

$$\limsup_{k \rightarrow \infty} \mathcal{L}(\mu_k) \leq \mathcal{L}(\mu). \quad (1.2.15)$$

Si de plus μ est irréductible, alors

$$\mathcal{L}(\mu_k) \xrightarrow{k \rightarrow \infty} \mathcal{L}(\mu). \quad (1.2.16)$$

Remarque 1.2.14. Le cas de mesures irréductibles n'est pas « trop restrictif » : Kifer a montré [72] que l'ensemble des mesures irréductibles est un ouvert dense de l'ensemble des probabilités sur $\mathrm{GL}_d(\mathbf{R})$.

Certes, en un sens il y a beaucoup de mesures irréductibles ; toutefois, dans l'étude de systèmes physiques, notamment des systèmes désordonnés, les lois considérées sont souvent des perturbations d'une matrice déterministe (ou d'une matrice diagonale pour les modèles de Derrida–Hilhorst et McCoy–Wu), qui n'est en particulier que rarement irréductible. Pour ce qui concerne les perturbations d'une matrice déterministe, un critère simple existe.

Théorème 1.2.15 ([72]). *Soit $M \in GL_d(\mathbf{R})$. Il existe $R > 0$ tel que pour toute suite $(\mu_k)_{k \geq 0}$ de probabilités à support dans la boule $\{A \in GL_d(\mathbf{R}) : \|A - M\| \leq R\}$ et convergeant étroitement vers δ_M , on a la convergence des exposants caractéristiques :*

$$\mathcal{L}(\mu_k) \xrightarrow[k \rightarrow \infty]{} \mathcal{L}(\delta_M).$$

1.2.3 Dérivabilité et Analyticité

Pour les modèles auquel on s'intéressera, la continuité par rapport aux paramètres sera presque immédiatement acquise. En revanche son analyticité, ou même sa dérivabilité, sera un problème difficile, et pourtant crucial pour comprendre les transitions de phases des modèles physiques concernés. Même dans un cadre général, de tels résultats sont rares. Nous n'en citerons qu'un, qui aura toutefois des conséquences puissantes sur nos modèles. Il s'agit du théorème de Ruelle [85], repris par Dubois [39], que l'on énonce dans un cadre simplifié. Donnons d'abord une définition de la réelle-analyticité.

Définition 1.2.16. Si V est un espace de Banach, une application $f : V \rightarrow \mathbf{R}$ est dite *analytique réelle* au voisinage du point $x_0 \in V$ s'il existe une suite (a_k) — où pour chaque k , a_k est une forme k -linéaire continue sur V —, ainsi qu'un réel $R > 0$ vérifiant $\sum_{k \geq 0} \|a_k\| R^k < \infty$, et tels que pour tout $x \in B(0, R)$,

$$f(x_0 + x) = \sum_{k \geq 0} a_k(x, x, \dots, x). \quad (1.2.17)$$

On introduit également la notion de cône, utilisée dans le théorème de Ruelle :

Définition 1.2.17. Si V est un espace vectoriel, un *cône* de V est une partie $C \subset V$, stable par multiplication par les scalaires positifs. Il est dit propre s'il est convexe et s'il est contenu dans un demi-espace.

Définition 1.2.18. Une matrice aléatoire M est dite *contractante* s'il existe C_1 et C_2 , deux cônes propres fermés déterministes de \mathbf{R}^d , avec $C_1 \subset \overset{\circ}{C}_2$, tels que presque sûrement,

$$M(C_2) \subset C_1 \cup (-C_1) : \quad (1.2.18)$$

le cône C_2 est *contracté* par M .

Pour une matrice aléatoire M quelconque, définie sur un espace probabilisé Ω , on note, lorsqu'il existe, $\mathcal{L}(M)$ l'exposant de Lyapunov associé. On note aussi $E(M)$ l'espace de Banach des matrices aléatoires A définies sur Ω telles que

$$\|A\|_{E(M)} := \sup_{\omega \in \Omega} \frac{\|A(\omega)\|}{\|M(\omega)\|} < +\infty, \quad (1.2.19)$$

muni de la norme $\|\cdot\|_{E(M)}$.

Théorème 1.2.19 ([39]). *Soit une matrice aléatoire contractante M , telle que $\mathbf{E}[\|\log \|M\|] < +\infty$. L'application $A \mapsto \mathcal{L}(M+A)$ est analytique réelle dans un voisinage de 0 dans l'espace de Banach $E(M)$.*

Exemple 1.2.20. La propriété de contraction est en particulier satisfaite lorsque la matrice M est déterministe et admet une unique valeur propre de module maximal, et que celle-ci est une valeur propre simple et réelle. En particulier, donnons des matrices aléatoires V_k , bornées, et telles que $\sup_{k \geq 1} \|V_k\|_{L^\infty}^{1/k}$ soit fini. Alors l'exposant de Lyapunov de la matrice aléatoire

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k=1}^{+\infty} x^k V_k \quad (1.2.20)$$

est une fonction analytique de x au voisinage de 0.

Signalons, pour ce qui concerne la dérivabilité, les travaux de Hennion [66]. Il établit que pour des matrices à coefficients positifs et $k+2$ fois dérivables, l'exposant de Lyapunov correspondant est k fois dérivable. Entre autres hypothèses techniques, ce résultat requiert que les coefficients de la matrice aléatoire soient, avec une probabilité non nulle, strictement positifs simultanément. Malheureusement, cette dernière restriction l'empêche de fournir des résultats satisfaisants dans les modèles qui nous intéressent. En fait, pour ces modèles physiques, les seuls cas où les travaux de Hennion [66] pourraient s'appliquer, le théorème de Ruelle s'y applique déjà, fournissant un résultat plus fort.

1.3 Le Modèle de Derrida–Hilhorst

Revenons au modèle de Derrida-Hilhorst, introduit dans la section 1.1, et pour lequel les matrices aléatoires sont

$$M_{k,\epsilon} = \begin{pmatrix} 1 & \epsilon \\ \epsilon Z_k & Z_k \end{pmatrix}. \quad (1.3.1)$$

On notera M_ϵ une matrice générique ayant la même loi. Le Théorème 1.2.1 garantit l'existence de la limite

$$\mathcal{L}_Z(\epsilon) = \lim_{N \rightarrow +\infty} \frac{1}{N} \log \|M_{N,\epsilon} \cdots M_{1,\epsilon}\| \quad (1.3.2)$$

pourvu que $\mathbf{E}[\log_+ Z]$ soit finie. Le choix de remplacer la trace de la formule par la norme ne change pas la limite (voir le Lemme 2.2.3). On souhaite comprendre la régularité de $\epsilon \mapsto \mathcal{L}_Z(\epsilon)$. Avant d'aborder le cœur de l'étude, commençons par deux réductions simples. On constate, en factorisant Z dans la matrice M_ϵ , que

$$\mathcal{L}_Z(\epsilon) = \mathbf{E}[\log Z] + \mathcal{L}_{1/Z}(\epsilon). \quad (1.3.3)$$

Par conséquent on supposera toujours, sans perte de généralité, que $\mathbf{E}[\log Z] \leq 0$. Dans ce cas $\mathcal{L}_Z(0) = 0$. De même, une conjugaison par la matrice diagonale $\text{Diag}(-1, 1)$ donne

$$\mathcal{L}_Z(\epsilon) = \mathcal{L}_Z(-\epsilon). \quad (1.3.4)$$

La fonction $\mathcal{L}_Z(\epsilon)$ est paire et l'étude peut donc être restreinte à $\epsilon > 0$.

Lorsque Z a un support compact dans $(0, +\infty)$, observant que pour $\epsilon > 0$, le cône $\{(y_1, y_2) \mid y_1 \geq 0, y_2 \geq 0\}$ est contracté par l'action de M_ϵ , le Théorème 1.2.19 de Ruelle

garantit que $\epsilon \mapsto \mathcal{L}_Z(\epsilon)$ est analytique sur $\mathbf{R} \setminus \{0\}$. Ne reste donc que l'étude de $\mathcal{L}_Z(\epsilon)$ lorsque $\epsilon \searrow 0$. Lorsque le support de Z est $[0, 1 - \delta]$ pour un $\delta \in (0, 1)$, le cône $\{(y_1, y_2) \mid |y_2| \leq y_1\}$ est contracté par M_0 , et, appliquant une nouvelle fois le Théorème 1.2.19, il vient que dans ce cas \mathcal{L}_Z est aussi analytique au voisinage de 0. En dehors de ce cas on s'attend à ce que \mathcal{L}_Z ait un comportement singulier en 0.

L'étude centrale de [35] concerne le cas où $\mathbf{E}[\log Z] < 0$ et $\mathbf{E}[Z] > 1$. (Figure 1.2c) Dans ce cas, par convexité de la fonction $\beta \mapsto \mathbf{E}[Z^\beta]$, il existe un unique $\alpha \in (0, 1)$ tel que $\mathbf{E}[Z^\alpha] = 1$. Derrida et Hilhorst prédisent que

$$\mathcal{L}_Z(\epsilon) \sim C_Z \epsilon^{2\alpha}, \quad (1.3.5)$$

pour une certaine constante $C_Z > 0$ semi-explicite. Il convient de noter que puisque la fonction $\epsilon \mapsto \mathcal{L}_Z(\epsilon)$ est paire (1.3.4), seule les puissances paires de ϵ sont non-singulières en 0. Ainsi le comportement (1.3.5) est singulier même lorsque α est un demi-entier.

Que se passe-t-il dans les autres cas : $\mathbf{E}[Z] \leq 1$ ou $\mathbf{E}[\log Z] = 0$? La littérature physique fournit des prédictions pour ceux-ci, dont l'exposition est facilitée par un paramétrage par l'exposant α , que nous allons décrire. On se place désormais dans le cas où $\mathbf{P}(Z > 1) > 0$, et pour faciliter l'exposition des conjectures et résultats, on supposera que Z est une variable aléatoire bornée.

- Lorsque $\mathbf{E}[\log Z] < 0$, grâce à la convexité de la fonction $\beta \mapsto \mathbf{E}[Z^\beta]$, il existe un unique $\alpha > 0$ tel que $\mathbf{E}[Z^\alpha] = 1$. (Figure 1.2b) Les mêmes auteurs donnent un argument [35, §3] prédisant que $\mathcal{L}_Z(\epsilon)$ a un développement en puissances de ϵ^2 , où le coefficient devant ϵ^{2n} est fini si et seulement si $\mathbf{E}[Z^n] < 1$, c'est-à-dire si et seulement si $n < \alpha$. Par analogie, avec le cas $\alpha \in (0, 1)$ (1.3.5), il est probable que, lorsque $\alpha > 1$, ce soit un terme $\epsilon^{2\alpha}$ qui brise l'analyticité. Ils laissent toutefois ces prédictions au rang de conjectures.
- Lorsque $\mathbf{E}[\log Z] = 0$, $\beta = 0$ est la seule solution de $\mathbf{E}[Z^\beta] = 1$ et on convient de poser $\alpha = 0$ (Figure 1.2a). Pour un désordre Z très spécifique entrant dans ce cadre, Nieuwenhuizen et Luck [79] estiment l'exposant de Lyapunov et prévoient

$$\mathcal{L}_Z(\epsilon) \sim \frac{C_Z}{\log(1/\epsilon)}. \quad (1.3.6)$$

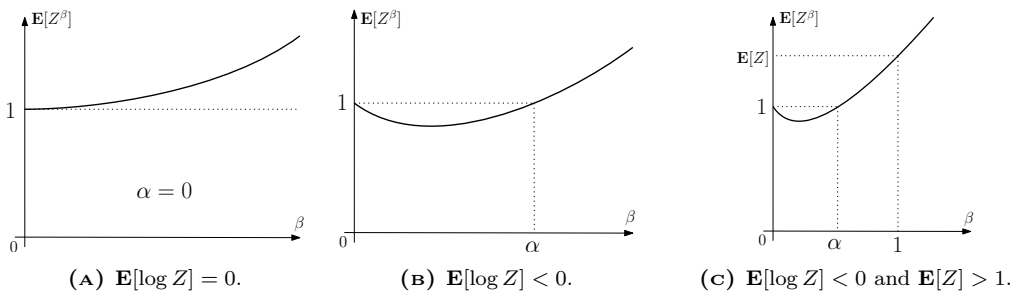


FIGURE 1.2: Définition du paramètre α .

Nous résumons ces prédictions dans la conjecture suivante.

Définition 1.3.1. On dit qu'une variable aléatoire ξ est *arithmétique* lorsqu'il existe une constante $c > 0$ telle que $c\xi \in \mathbf{Z} \cup \{\pm\infty\}$ presque sûrement.

Conjecture 1.3.2. *Supposons que $\log Z$ ne soit pas arithmétique*

1. Si $\alpha \in (0, +\infty) \setminus \mathbf{N}$,

$$\mathcal{L}(\epsilon) = c_1\epsilon^2 + \dots + c_{\lfloor \alpha \rfloor} \epsilon^{2\lfloor \alpha \rfloor} + C_Z \epsilon^{2\alpha} + o(\epsilon^{2\alpha}); \quad (1.3.7)$$

2. Si $\alpha \in \{1, 2, \dots\}$,

$$\mathcal{L}(\epsilon) = c_1\epsilon^2 + \dots + c_{\alpha-1}\epsilon^{2(\alpha-1)} + C_Z \epsilon^{2\alpha} \log \epsilon + o(\epsilon^{2\alpha} \log \epsilon); \quad (1.3.8)$$

3. Si $\alpha = 0$,

$$\mathcal{L}(\epsilon) = \frac{C_Z}{\log(1/\epsilon)} + o((\log \epsilon)^{-1}). \quad (1.3.9)$$

Si $\log Z$ est arithmétique, alors les mêmes développements ont lieu, où la constante C_Z est remplacée par une fonction périodique de $\log \epsilon$.

Remarque 1.3.3. Les coefficients (c_k) peuvent être calculés récursivement (voir Chapitre 2). En revanche, le calcul de la constante C_Z devant la singularité est très difficile [28, §4.2.1], en dehors de cas très particuliers. Dans le cas où $\log Z$ est arithmétique, Derrida et Hilhorst [35] proposent un calcul explicite de la fonction périodique $C_Z(\log \epsilon)$ pour un choix très spécifique (et singulier) de la loi Z .

Plusieurs articles se sont penchés sur cette conjecture ces dernières années, apportant une démonstration rigoureuse à une partie de ces prédictions. Genovese et al. [49] proposent une démonstration rigoureuse de la prédiction (1.3.5) de Derrida et Hilhorst lorsque $\alpha \in (0, 1)$ (c'est le résultat principal de [35]). Observons que la singularité $\epsilon^{2\alpha}$ est dans ce cas la partie dominante de $\mathcal{L}(\epsilon)$. Nous nous sommes penchés sur le cas $\alpha \geq 1$, pour lequel nous obtenons [64] (repris dans le chapitre 2) le développement (1.3.7) jusqu'à la singularité $\epsilon^{2\alpha}$ exclue. En parallèle, Comets et al. [26] se sont penchés sur une limite en désordre faible du modèle de Derrida–Hilhorst. Dans cette limite, le modèle est exactement soluble et l'exposant de Lyapunov correspondant admet une expression explicite. De façon surprenante, le développement (1.3.7) persiste dans la limite de désordre faible, singularité incluse (Sous-section 1.4.2). De plus, ce modèle s'inscrit dans un cadre de perturbation de la matrice identité qui avait déjà suscité de l'intérêt indépendamment (Sous-section 1.4.1); et surtout, il offre un éclairage nouveau sur l'approche de McCoy et Wu. Nous y reviendrons dans les sections suivantes. Pour l'heure, recentrons-nous sur le modèle de Derrida–Hilhorst à proprement parler et détaillons les méthodes de [35, 49, 64].

Ces études reposent sur l'expression semi-explicite (1.2.10) de l'exposant de Lyapunov, qui dans ce contexte peut s'écrire (voir Chapitre 2)

$$\mathcal{L}(\epsilon) = \mathbf{E} \log(1 + \epsilon^2 X_\epsilon) = \int_0^{+\infty} \log(1 + \epsilon^2 x) d\nu_\epsilon(x), \quad (1.3.10)$$

où X_ϵ désigne une variable aléatoire de loi stationnaire, notée ν_ϵ , pour la chaîne de Markov sur \mathbf{R}_+

$$x_{n+1} = Z_{n+1} \frac{1 + x_n}{1 + \epsilon^2 x_n}. \quad (1.3.11)$$

En d'autres termes

$$X_\epsilon \stackrel{(d)}{=} Z \frac{1 + X_\epsilon}{1 + \epsilon^2 X_\epsilon}. \quad (1.3.12)$$

Vue la formule (1.3.10), toute l'inconnue réside dans la mesure ν_ϵ , probabilité invariante d'une suite homographique aléatoire. Elle n'admet aucune expression explicite, et malgré l'apparente simplicité de la chaîne de Markov, comprendre, même grossièrement, la mesure invariante ν_ϵ est malaisé. D'ailleurs, lorsque $\alpha \geq 1$, une compréhension suffisamment fine de celle-ci fait encore défaut, raison pour laquelle la singularité $\epsilon^{2\alpha}$ échappe à l'étude (voir le paragraphe 1.3.2.3).

1.3.1 Développement « régulier » en puissances de ϵ^2

Détaillons un argument heuristique justifiant la prédiction de B. Derrida et H. Hilhorst pour $\alpha > 1$. Ce n'est pas précisément l'argument développé dans [35], quoiqu'il s'en rapproche, mais il fournit le même niveau de prédiction. Celui que l'on développe ici repose presque entièrement sur la simplification de la chaîne (1.3.11) par la chaîne de Markov linéaire

$$x_{n+1} = Z_{n+1}(1 + x_n), \quad (1.3.13)$$

dont on note ν_0 l'unique mesure invariante sur \mathbf{R}_+ , et X_0 la variable aléatoire correspondante. Notre argument repose en grande partie sur le fait que ν_0 constitue une bonne approximation de ν_ϵ , suffisante au moins pour obtenir le développement (1.3.7) jusqu'à l'ordre $[\alpha]$.

Contrairement à ν_ϵ , la mesure invariante ν_0 admet une expression explicite :

$$X_0 \stackrel{(d)}{=} \sum_{n=1}^{+\infty} Z_1 \cdots Z_n. \quad (1.3.14)$$

Malgré l'apparente simplicité de la chaîne, son étude est riche et son analyse peu aisée. L'étude de la queue de X_0 a notamment suscité de l'intérêt (voir notamment [69], ou [70, 71] pour un état de l'art récent). Ces auteurs établissent, sous des hypothèses techniques que l'on omet ici, que

$$\mathbf{P}(X_0 \geq x) \stackrel{x \rightarrow \infty}{\sim} c_\nu x^{-\alpha}. \quad (1.3.15)$$

En fait un comportement aussi précis de la queue ne sera utile que plus tard et pour l'heure, on peut se contenter de lire sur la formule (1.3.14) que, pour $\beta > 0$, le moment $\mathbf{E}[X_0^\beta]$ est fini si et seulement si $\mathbf{E}[Z^\beta] < 1$. Cette observation permet déjà de comprendre, au point de vue heuristique, pourquoi s'attendre au développement (1.3.7) prévu par [35]. Pour cela, utilisons le développement du logarithme dans l'expression (1.3.10). Pour $0 \leq n < \beta \leq n+1$, écrivons formellement

$$\begin{aligned} \mathcal{L}_Z(\epsilon) &= \mathbf{E} \left[\epsilon^2 X_\epsilon - \frac{1}{2} \epsilon^4 X_\epsilon^2 + \dots + \frac{(-1)^{n+1}}{n} \epsilon^{2n} X_\epsilon^n + O(\epsilon^{2\beta} X_\epsilon^\beta) \right] \\ &= \epsilon^2 \mathbf{E}[X_\epsilon] - \frac{1}{2} \epsilon^4 \mathbf{E}[X_\epsilon^2] + \dots + \frac{(-1)^{n+1}}{n} \epsilon^{2n} \mathbf{E}[X_\epsilon^n] + O(\epsilon^{2\beta} \mathbf{E}[X_\epsilon^\beta]) \\ &\simeq \epsilon^2 \mathbf{E}[X_\epsilon] - \frac{1}{2} \epsilon^4 \mathbf{E}[X_\epsilon^2] + \dots + \frac{(-1)^{n+1}}{n} \epsilon^{2n} \mathbf{E}[X_\epsilon^n] + O(\epsilon^{2\beta} \mathbf{E}[X_0^\beta]). \end{aligned} \quad (1.3.16)$$

Le développement est rompu dès que le moment $\mathbf{E}[X_0^\beta]$ n'est plus fini, c'est-à-dire lorsque $\beta \geq \alpha$. Et l'obstacle $\beta = \alpha$ suggère fortement que le terme suivant sera $\epsilon^{2\alpha}$. C'est la

singularité prédite par [35], au moins lorsque α n'est pas un entier (vu le niveau heuristique auquel on se place, permettons-nous d'esquiver la discussion sur le cas où α est entier).

Ces trois lignes de calcul peuvent aisément être rendues rigoureuses, en se contentant de trouver des bornes adéquates pour les moments $\mathbf{E}[X_\epsilon^k]$ (voir chapitre 2). Un pas reste à franchir : les coefficients $\frac{(-1)^{k+1}}{k} \epsilon^{2k} \mathbf{E}[X_\epsilon^k]$, $k < \alpha$ ne sont pas des constantes, et l'approximation $\mathbf{E}[X_\epsilon^n] \simeq \mathbf{E}[X_0^k]$ est trop grossière. Pour parvenir au développement (1.3.7), il faut exhiber un développement en puissances de ϵ^2 de chacun des moments $\mathbf{E}[X_\epsilon^k]$, $k < \alpha$. À cette fin, utilisons l'identité (1.3.12), et de sa conséquence

$$\mathbf{E}[X_\epsilon^k] = \mathbf{E}[Z^k] \mathbf{E} \left[\frac{(1 + X_\epsilon)^k}{(1 + \epsilon^2 X_\epsilon)^k} \right], \quad (1.3.17)$$

que l'on peut développer, comme dans (1.3.16), en puissance de ϵ^2 . Une utilisation de ces identités permet d'obtenir, à travers une procédure récursive, un développement en puissance de ϵ^2 de plus en plus précis des moments $\mathbf{E}[X_\epsilon^k]$, que l'on peut ultimement écrire

$$\mathbf{E}[X_\epsilon^k] = c_{n,0} + c_{n,1} \epsilon^2 + c_{n,2} \epsilon^4 + \dots + O(\epsilon^{2\beta-2k} \mathbf{E}[X_\epsilon^\beta]). \quad (1.3.18)$$

On peut noter que la puissance $\beta = \alpha$ est de nouveau un obstacle pour ce développement. Cependant, s'il est naturel de postuler que le terme suivant est $\epsilon^{2\alpha}$, rien pour l'instant ne vient le conforter. Cette analyse (menée de manière rigoureuse dans le chapitre 2) aboutit à une version faible de la conjecture (1.3.7) :

$$\mathcal{L}(\epsilon) = c_1 \epsilon^2 + \dots + c_{\lfloor \alpha \rfloor} \epsilon^{2\lfloor \alpha \rfloor} + R(\epsilon), \quad \text{où } R(\epsilon) = o(\epsilon^{2\lfloor \alpha \rfloor}). \quad (1.3.19)$$

On aimerait un équivalent précis $R(\epsilon) \sim C_Z \epsilon^{2\alpha}$, que ne fournit pas notre approche. Avançons-nous toutefois un petit peu et voyons où la méthode qui précède faut. L'obtention de ce terme singulier oblige à se confronter aux moments non bornés de X_ϵ , et donc avec sa queue. C'est ici que l'approche précédente est trop faible, car l'approximation $X_\epsilon \simeq X_0$ sur laquelle elle repose devient fort imprécise pour estimer la queue de X_ϵ . Plus précisément, l'approximation

$$\frac{1+x}{1+\epsilon^2 x} \simeq 1+x \quad (1.3.20)$$

est perdue lorsque x est d'ordre ϵ^{-2} . C'est justement l'étude de la queue de X_ϵ , et précisément à l'échelle ϵ^{-2} , qui permet à Derrida et Hilhorst d'obtenir la singularité $\epsilon^{2\alpha}$.

1.3.2 Singularité $\epsilon^{2\alpha}$

1.3.2.1 La stratégie de Derrida et Hilhorst : une approximation à deux échelles de la mesure invariante

L'obtention de la partie régulière de (1.3.7) est peu développée par les auteurs de [35] et leur travail se concentre sur le cas où $\mathbf{E}[Z] > 1$ et $\mathbf{E}[\log Z] < 0$, pour lequel $\alpha \in (0, 1)$ et la singularité $\epsilon^{2\alpha}$ devient le comportement dominant de $\mathcal{L}_Z(\epsilon)$. Leur prédiction prend la forme

$$\mathcal{L}_Z(\epsilon) \stackrel{\epsilon \searrow 0}{\sim} C_Z \epsilon^{2\alpha}. \quad (1.3.21)$$

Pour parvenir à une telle prédiction, Derrida et Hilhorst analysent la loi stationnaire ν_ϵ et en donne une forme très précise, reposant sur une étude à deux échelles. À l'échelle standard, l'approximation $\nu_\epsilon \simeq \nu_0$ vaut. À l'échelle ϵ^{-2} , une autre approche est nécessaire

car la chaîne de Markov changée d'échelle n'a pas, dans la limite $\epsilon \rightarrow 0$, de probabilité invariante non triviale. La loi ν_ϵ y est approchée par une troncature d'une mesure invariante de masse infinie. Un recollement approprié de ces deux échelles fournit selon eux une bonne approximation de la loi invariante ν_ϵ . Cet aperçu grossier étant brossé, entrons dans les détails de leur analyse.

Dans un premier temps, Derrida et Hilhorst font le constat de ce qui nous a déjà servi auparavant : lorsque $\epsilon \rightarrow 0$, la mesure invariante ν_ϵ de la chaîne

$$x_{n+1} = Z_{n+1} \frac{1 + x_n}{1 + \epsilon^2 x_n} \quad (1.3.22)$$

converge vers ν_0 , l'unique probabilité invariante de la chaîne

$$x_{n+1} = Z_{n+1}(1 + x_n). \quad (1.3.23)$$

L'approximation $\nu_\epsilon \simeq \nu_0$ sera utilisée sur cette échelle (régime I).

Ils regardent ensuite la mesure ν_ϵ après avoir « dézoomé » d'un facteur ϵ^{-2} (régime II), c'est-à-dire la mesure invariante ω_ϵ de la chaîne de Markov

$$\sigma_{n+1} := \epsilon^2 x_{n+1} = Z_{n+1} \frac{\epsilon^2 + \sigma_n}{1 + \sigma_n}, \quad (1.3.24)$$

Dans la limite $\epsilon \searrow 0$, cette chaîne devient

$$\sigma_{n+1} = Z_{n+1} \frac{\sigma_n}{1 + \sigma_n}. \quad (1.3.25)$$

En observant que $\sigma_n \leq Z_n \cdots Z_1 \sigma_0$, on se heurte à un premier constat qui peut laisser pessimiste : dans le cadre où $\mathbf{E}[\log Z] < 0$, l'unique probabilité invariante de la chaîne (1.3.25) est la mesure de Dirac δ_0 . Toutefois, le point essentiel est que cette chaîne possède aussi des mesures invariantes (positives) de masse infinie. Celle que Derrida et Hilhorst proposent de choisir peut être construite comme suit. Prenons une mesure invariante ω_ϵ de la chaîne (1.3.24) choisie de telle sorte que $\bar{\omega}_\epsilon([1, +\infty)) = 1$. On peut montrer qu'elle admet une limite (au sens faible), notée ω_0 . Cette mesure ω_0 est une mesure invariante (de masse infinie) de la chaîne (1.3.25). Ils justifient qu'elle doit satisfaire

$$\omega_0([\sigma, +\infty)) \stackrel{\sigma \rightarrow 0}{\sim} c_\omega \sigma^{-\alpha}. \quad (1.3.26)$$

La prédiction de Derrida et Hilhorst est que cette mesure ω_0 décrit, à une renormalisation près, le comportement de la queue de X_ϵ .

Ils prédisent en fait que la mesure ν_ϵ peut-être décrite assez précisément par le recollement à deux échelles, de la mesure ν_0 et de la mesure ω_0 :

$$\nu_\epsilon([x, +\infty)) = \mathbf{P}(X_\epsilon \geq x) \cong \begin{cases} \mathbf{P}(X_0 \geq x) & \text{pour } x \leq \epsilon^{-1} \\ a(\epsilon)\omega_0([\epsilon^2 x, +\infty)) & \text{pour } x \geq \epsilon^{-1} \end{cases} \quad (1.3.27)$$

Pour l'instant nous ne donnons pas de sens précis au symbole \cong . La constante $a(\epsilon)$ est choisie pour que le recollement soit continu. Avec (1.3.15) et (1.3.26), on obtient

$$a(\epsilon) \sim \frac{c_\nu}{c_\omega} \epsilon^{2\alpha}. \quad (1.3.28)$$

En injectant dans l'expression de l'exposant de Lyapunov (1.3.10) la prédiction (1.3.27) faite sur la mesure invariante, il vient précisément

$$\begin{aligned}\mathcal{L}_Z(\epsilon) &= \int_0^{+\infty} \log(1 + \epsilon^2 x) d\nu_\epsilon(x) \\ &\cong \int_0^{\epsilon^{-1}} \log(1 + \epsilon^2 x) d\nu_0(x) + a(\epsilon) \int_{\epsilon^{-1}}^{+\infty} \log(1 + \sigma) d\omega_0(\sigma) \\ &= C_Z \epsilon^{2\alpha} + O(\epsilon^{1+\alpha}), \quad \text{avec } C_Z = \frac{c_\nu}{c_\omega} \int_0^{+\infty} \log(1 + \sigma) d\omega_0(\sigma).\end{aligned}\tag{1.3.29}$$

Remarquons que, lorsque $\alpha \in (0, 1)$, c'est la queue de X_ϵ qui donne la contribution majoritaire à $\mathcal{L}_Z(\epsilon)$. Nommément,

$$a(\epsilon) \int_{\epsilon^{-1}}^{+\infty} \log(1 + \sigma) d\omega_0(\sigma) \sim C_Z \epsilon^{2\alpha},\tag{1.3.30}$$

tandis que

$$\int_0^{\epsilon^{-1}} \log(1 + \epsilon^2 x) d\nu_0(x) = O(\epsilon^{1+\alpha}) = o(\epsilon^{2\alpha}).\tag{1.3.31}$$

C'est aussi ce à quoi l'on s'attend pour $\alpha > 1$ (voir la discussion concluant la sous-section 1.3.1).

1.3.2.2 Contrôler les erreurs avec des normes à poids : l'approche rigoureuse de Genovese et al.

Les prédictions de Derrida et Hilhorst s'avèrent exactes et Genovese, Giacomini et Greenblatt [49] proposent une démonstration mathématique de (1.3.29) sous la forme suivante.

Théorème 1.3.4 (Genovese–Giacomini–Greenblatt [49]). *On suppose*

1. $\mathbf{E}[\log Z] < 0$, et $\mathbf{E}[Z] > 1$;
2. La variable aléatoire Z est à densité par rapport à la mesure de Lebesgue, et cette densité est \mathcal{C}^1 et à support compact $[c_-, c_+] \subset (0, +\infty)$.

Alors il existe $\varkappa > 0$ et $C_Z > 0$ tels que

$$\mathcal{L}(\epsilon) = C_Z \epsilon^{2\alpha} + O(\epsilon^{2\alpha+\varkappa}).\tag{1.3.32}$$

Suivant fortement la voie tracée par Derrida et Hilhorst, le travail majeur de [49] est de proposer une version rigoureuse de l'approximation (1.3.27) de la mesure invariante. Sans entrer dans les détails techniques, donnons les grandes lignes de leur approche. La construction de ν_0 et ω_0 ne pose pas de problème et relève de techniques standards. De même que l'estimation de leur queue respective (1.3.15) et (1.3.26), reposant sur l'analyse de transformées de Mellin. Ils proposent ensuite de considérer la mesure, donnée par la fonction de répartition (voir la figure dans [49])

$$\gamma_\epsilon([x, +\infty)) := \begin{cases} \nu_0([x, +\infty)) & \text{pour } x \leq \epsilon^{-1} \\ a(\epsilon)\omega_0([\epsilon^2 x, +\infty)) & \text{pour } x \geq \epsilon^{-1}, \end{cases}\tag{1.3.33}$$

où $a(\epsilon)$ est choisi de telle sorte que la fonction de répartition est continue en ϵ^{-1} . Comme on l'a déjà observé (1.3.29), cette mesure donne le résultat souhaité :

$$\int_0^{+\infty} \log(1 + \epsilon^2 x) d\gamma_\epsilon(x) = C_Z \epsilon^{2\alpha} + O(\epsilon^{1+\alpha}). \quad (1.3.34)$$

Mais il reste à établir que γ_ϵ est proche de ν_ϵ , dans un sens suffisamment fort pour garantir (1.3.32). Une des idées principales des auteurs de [49] est d'introduire des normes à poids sur les mesures signées η :

$$\|\eta\|_\beta = \int_0^{+\infty} x^{\beta-1} |\eta((x, +\infty))| dx, \quad (1.3.35)$$

paramétrées par $\beta \in (0, \alpha)$, qui vont permettre de mesurer la proximité entre ν_ϵ et γ_ϵ . Ces normes présentent un double avantage. D'une part elles suffisent à contrôler l'exposant de Lyapunov :

$$\left| \int_0^{+\infty} \log(1 + \epsilon^2 x) d\nu_1(x) - \int_0^{+\infty} \log(1 + \epsilon^2 x) d\nu_2(x) \right| \leq \epsilon^{2\beta} \|\nu_1 - \nu_2\|_\beta. \quad (1.3.36)$$

Ainsi une majoration adéquate de $\|\nu_\epsilon - \gamma_\epsilon\|$ suffit à assurer (1.3.32). D'autre part, les auteurs montrent que T_ϵ , le noyau de la chaîne de Markov (1.3.22), contracte ces normes. Soyons plus précis. Introduisons l'opérateur T_ϵ défini par

$$\int_0^{+\infty} f(x) dT_\epsilon \eta(x) = \int_0^{+\infty} \mathbf{E} \left[f \left(Z \frac{1+x}{1+\epsilon^2 x} \right) \right] d\eta(x). \quad (1.3.37)$$

Grosso modo, T_ϵ donne la loi transformée après un pas de la chaîne (1.3.22). En particulier la mesure invariante ν_ϵ en est un point fixe $T_\epsilon \nu_\epsilon = \nu_\epsilon$. La propriété de contraction s'écrit

$$\|T_\epsilon \eta_1 - T_\epsilon \eta_2\|_\beta \leq c_\beta \|\eta_1 - \eta_2\|_\beta, \quad c_\beta \in (0, 1), \quad (1.3.38)$$

et par suite

$$\|\nu_\epsilon - \gamma_\epsilon\|_\beta \leq \frac{1}{1 - c_\beta} \|T_\epsilon \gamma_\epsilon - \gamma_\epsilon\|_\beta. \quad (1.3.39)$$

Cette astuce standard sur les opérateurs contractants permet de mesurer la proximité de ν_ϵ et γ_ϵ sans connaître la mesure invariante ν_ϵ . Ainsi, pour montrer que γ_ϵ est proche de la vraie mesure invariante, il suffit de vérifier qu'elle est elle-même « presque invariante », c'est-à-dire que $\|T_\epsilon \gamma_\epsilon - \gamma_\epsilon\|_\beta$ tend suffisamment vite vers 0, ce que les auteurs établissent au terme d'un long calcul technique. Ils obtiennent, pour un certain $\delta > 0$,

$$\|T_\epsilon \gamma_\epsilon - \gamma_\epsilon\|_\beta = O(\epsilon^{2\alpha+\beta}) + O(\epsilon^{\alpha+\beta+\delta}), \quad (1.3.40)$$

ce qui, moyennant un choix approprié de β , conclut leur preuve.

Observons que cette approche n'est pas sans lien avec celle de la sous-section 1.3.1 qui permet d'obtenir la partie régulière du développement. Les normes $\|\cdot\|_\beta$ sont en effet intimement liées au contrôle des moments : lorsque η est la loi d'une variable aléatoire positive Y , on a

$$\|\eta\|_\beta = \mathbf{E}[Y^\beta], \quad \text{et} \quad \left| \mathbf{E}[Y^\beta] - \mathbf{E}[X_\epsilon^\beta] \right| \leq \|\eta - \nu_\epsilon\|_\beta. \quad (1.3.41)$$

1.3.2.3 Une heuristique pour $\alpha > 1$?

L'approche de Genovese et al. n'a permis d'aboutir à la singularité $\epsilon^2\alpha$ que lorsque $\alpha \in (0, 1)$, c'est-à-dire lorsque celle-ci est la partie prépondérante de $\mathcal{L}_Z(\epsilon)$. Voyons pourquoi les tentatives de production d'un résultat analogue pour $\alpha > 1$ échouent. Rappelons la conclusion de la sous-section 1.3.1 :

$$\mathcal{L}(\epsilon) = c_1\epsilon^2 + \cdots + c_{\lfloor \alpha \rfloor}\epsilon^{2\lfloor \alpha \rfloor} + R(\epsilon), \quad \text{où } R(\epsilon) = o(\epsilon^{2\lfloor \alpha \rfloor}). \quad (1.3.42)$$

Pour faciliter l'exposition, on reste dans le cadre $\alpha \notin \mathbf{N}$. Afin d'estimer précisément le terme $R(\epsilon)$, dont on suppose qu'il doit être $C_Z\epsilon^{2\alpha}$, on pourrait être tenté d'élargir l'heuristique proposée par Derrida et Hilhorst au cas $\alpha > 1$. En se souvenant que pour obtenir la partie régulière, on avait besoin de prendre en compte tous les moments de X_ϵ , l'idée pourrait être d'essayer de montrer

$$\nu_\epsilon([x, +\infty)) = \mathbf{P}(X_\epsilon \geq x) \cong \begin{cases} \mathbf{P}(X_\epsilon \geq x) & \text{pour } x \leq q_\epsilon \\ a(\epsilon)\omega_0([\epsilon^2x, +\infty)) & \text{pour } x \geq q_\epsilon \end{cases}, \quad (1.3.43)$$

pour une coupure q_ϵ bien choisie. Avant même de vérifier la validité de cette approximation, on s'aperçoit qu'avec ce découpage, seul le choix de coupure $q_\epsilon = c\epsilon^{-2}$ permet, une fois injecté dans (1.3.29), d'approcher la singularité $\epsilon^{2\alpha}$. Et ce choix de coupure n'aboutit en fait qu'à un encadrement encore imprécis $c_1\epsilon^{2\alpha} \leq R(\epsilon) \leq c_2\epsilon^{2\alpha}$. En fait on observera qu'un tel encadrement est une conséquence (voir Chapitre 2, Remarque 2.1.9) du postulat plus faible

$$\mathbf{P}(X_\epsilon \geq c\epsilon^{-2}) \geq c\mathbf{P}(X_0 \geq c\epsilon^{-2}). \quad (1.3.44)$$

Malheureusement, aucun argument ne permet à ma connaissance de justifier cette dernière estimation. À ce titre il faut constater le fait remarquable que dans [49], la justification de l'approximation

$$\nu_\epsilon([x, +\infty)) \approx a(\epsilon)\omega_0([\epsilon^2x, +\infty)) \quad (1.3.45)$$

pour $x \geq \epsilon^{-1}$ n'est jamais faite individuellement. Certes on peut déduire de leur contrôle de $\|T_\epsilon\gamma_\epsilon - \gamma_\epsilon\|$ que, pour $\alpha \in (0, 1)$,

$$\int_{\epsilon^{-1}}^{+\infty} x^{\beta-1} |a(\epsilon)\omega_0([\epsilon^2x, +\infty)) - \nu_\epsilon([x, +\infty))| dx = O(\epsilon^{2\alpha-\beta}). \quad (1.3.46)$$

Mais cette estimation repose fortement sur la construction *totale* de γ_ϵ , comprenant les deux échelles. C'est une telle construction qui fait précisément défaut pour $\alpha > 1$.

À défaut de l'équivalent désiré $R(\epsilon) \sim C_Z\epsilon^{2\alpha}$, un raffinement de la méthode présentée dans la sous-section 1.3.1 fournit une estimation qualitative de $R(\epsilon)$. En effet les erreurs commises lors des développements successifs en puissance de ϵ^2 (voir (1.3.16) et (1.3.18)), au lieu d'être englobées dans un $O(\epsilon^{2\beta}\mathbf{E}[X_\epsilon^\beta])$, peuvent être précisément encadrées et fournir en définitive un encadrement

$$\epsilon^{2\lceil \alpha \rceil} \ll R(\epsilon) \leq c\epsilon^{2\alpha}. \quad (1.3.47)$$

Cet encadrement amène plusieurs remarques. Déjà il est compatible avec la prédiction $C_Z\epsilon^{2\alpha}$ et la borne supérieure est même du bon ordre de grandeur. Mais aussi, la prépondérance devant $\epsilon^{2\lceil \alpha \rceil}$ nous dit que le développement (1.3.42) ne peut être poursuivi à l'ordre suivant et que $R(\epsilon)$ est nécessairement une singularité en 0 (rappelons que due à la parité de la fonction $\epsilon \mapsto \mathcal{L}_Z(\epsilon)$, seule les puissances paires de ϵ sont non-singulières).

Pour l'heure, aucun encadrement plus précis n'a été démontré. Et malheureusement, aucun argument, même heuristique, ne permet d'accéder précisément à un équivalent $C_Z \epsilon^{2\alpha}$.

1.4 Modèles en désordre faible

Dans cette section nous nous intéressons à des limites en désordre faible des exposants de Lyapunov. Cette digression est motivée par un article récent de Comets, Giacomin et Greenblatt [26] où est étudiée une limite continue et exactement soluble du modèle de Derrida–Hilhorst (sous-section 1.4.2). Un fait remarquable de cet article est que dans la limite continue, l'exposant de Lyapunov exhibe un comportement exactement similaire à ceux de la conjecture 1.3.2, singularité incluse. On commence par un résultat général de Derrida et Zanon, et à une version mathématique de Sadel et Schulz-Baldes, qui s'intéressent à des perturbations de la matrice I_2 (sous-section 1.4.1), dans le cadre duquel peut s'inscrire le modèle de Comets, Giacomin et Greenblatt.

1.4.1 Perturbation faible de la matrice I_2

Dans [92], B. Derrida et N. Zanon considèrent la limite en désordre faible d'une perturbation d'une matrice diagonale déterministe ayant une valeur propre dégénérée. Une première étape est d'étudier une perturbation de la matrice identité 2×2 de la forme

$$M_\eta = I_2 + \eta U + \eta^2 K, \quad (1.4.1)$$

où K est une matrice déterministe et U une matrice aléatoire dont les coefficients sont centrés. Ils prédisent que l'exposant de Lyapunov correspondant s'écrit

$$\mathcal{L}(\eta) = \eta^2 \lambda + O(\eta^3). \quad (1.4.2)$$

Leur approche repose sur un développement perturbatif de la mesure invariante ν_η : ils prévoient que dans la limite $\eta \rightarrow 0$, la mesure ν_η converge vers une mesure à densité, solution d'une équation de Fokker–Planck. C'est cette mesure limite qui produit la contribution principale à l'exposant de Lyapunov $\mathcal{L}(\eta)$. Une version rigoureuse du développement de la mesure invariante est proposée par C. Sadel et H. Schulz-Baldes [86] dans le cadre général d'une perturbation aléatoire de l'opérateur identité, agissant sur une variété compacte. Plus exactement, ces derniers proposent un développement de Taylor des sommes de Birkhoff associées à cette action. Pour faciliter les discussions, nous nous placerons dans le cadre restreint suivant :

$$M_\eta = I_2 + \sum_{k=1}^n \eta^k V_k + \eta^{n+1} W_\eta, \quad \text{avec} \quad \mathbf{E}[V_1] = (0). \quad (1.4.3)$$

On suppose en outre que $n \geq 2$, et que les coefficients de chacune des matrices aléatoires V_k et W_η sont bornés. Enfin on pose $K := \mathbf{E}[V_2]$. Ce cadre réduit permet d'obtenir (1.4.2) (et davantage) sous des hypothèses moins restrictives que celles requises par le résultat de [86]. Le théorème suivant donne une condition pour que les mesures invariantes de M_η sur \mathbb{P}^1 admettent (au sens faible) une décomposition en puissance de η .

Théorème 1.4.1. *Si pour tout $\eta \neq 0$ assez petit, la matrice $\eta V_1 + \eta^2 K$ est irréductible (c'est-à-dire qu'il n'existe pas de direction déterministe qui soit p.s. stable par la matrice), alors il existe des fonctions $\rho_k \in \mathcal{C}^\infty(\mathbb{P}^1)$ telles que pour toute mesure invariante ν_η et pour toute $f \in \mathcal{C}^\infty(\mathbb{P}^1)$,*

$$\int f d\nu_\eta = \sum_{k=0}^{n-2} \eta^k \int f \rho_k d\ell + O(\eta^{n-1}). \quad (1.4.4)$$

Remarque 1.4.2. En particulier $\nu_\eta \rightarrow \rho_0 d\ell$, au sens de la convergence étroite, lorsque $\eta \rightarrow 0$.

Vues la formule intégrale (1.2.10), on obtient naturellement le comportement de $\mathcal{L}(\eta)$ lorsque $\eta \rightarrow 0$.

Théorème 1.4.3. *Si pour tout $\eta \neq 0$ assez petit, la matrice $\eta V_1 + \eta^2 K$ est irréductible, alors il existe des coefficients $(\lambda_k)_{k \geq 2}$, tels que*

$$\mathcal{L}(\eta) = \sum_{k=2}^n \lambda_k \eta^k + O(\eta^{n+1}). \quad (1.4.5)$$

Nous proposons en Annexe 1.A une démonstration de ces résultats, motivée par la restriction des hypothèses des théorèmes.

Remarque 1.4.4. En réalité, les développements à l'ordre $n - 2$ des mesures invariantes et à l'ordre n de l'exposant de Lyapunov exigent uniquement que les queues des coefficients des matrices aléatoires V_k et W_η soient suffisamment contrôlées pour que le développement (1.A.5) du Lemme 1.A.1 ait lieu. Typiquement, pour les matrices aléatoires V_k , des queues exponentielles suffisent.

Peut-on espérer l'analyticité ? Lorsque, par exemple,

$$M_\eta = I_2 + \sum_{k=1}^{+\infty} \eta^k V_k, \quad (1.4.6)$$

le Théorème 1.4.3 fournit un développement de Taylor de $\mathcal{L}(\eta)$ à tout ordre au voisinage de $\eta = 0$. On peut se demander si $\eta \mapsto \mathcal{L}(\eta)$ est une application \mathcal{C}^∞ dans un voisinage de 0 voire s'il y a analyticité. Pour des variables bornées, la question n'a à ma connaissance pas encore de réponse générale. Derrida et Zanon [92] proposent un exemple d'une perturbation gaussienne de I_2 pour laquelle $\mathcal{L}(\eta)$ est indéfiniment différentiable mais présente une rupture d'analyticité en 0.

Expression des coefficients. Si le Théorème 1.4.3 donne l'existence d'un développement de Taylor de l'exposant de Lyapunov, il ne fournit pas d'expression simple des coefficients de ce développement. Intéressons par exemple au premier terme $\lambda_2 = \lim_{\eta \rightarrow 0} \eta^{-2} \mathcal{L}(\eta)$. Les physiciens A. Comtet et al. [27] ont proposé un système de paramètres pour lequel celui-ci s'exprime facilement. Leur analyse repose sur la décomposition d'Iwasawa et sur des outils d'analyse complexe (transformée de Hilbert, intégrales de contour, etc.). En particulier, lorsque

$$V_1 = \begin{pmatrix} 0 & -W \\ W & 0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & 0 \\ 0 & W \end{pmatrix} \quad \text{ou} \quad V_1 = \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix}, \quad (1.4.7)$$

(W désignant une variable aléatoire centrée), ils obtiennent une expression de λ_2 en terme d'une dérivée logarithmique, respectivement, d'une fonction de Bessel de première espèce I_ν , de seconde espèce K_ν , ou de la fonction d'Airy Ai .

1.4.2 Limite continue du modèle de Derrida–Hilhorst : un modèle exactement soluble

L'idée de F. Comets et al. [26] est de considérer une limite continue du produit de matrices aléatoires $M_{n,\epsilon} \cdots M_{1,\epsilon}$. Plus précisément une limite continue du processus de Markov

$$y_{n+1} = \begin{pmatrix} 1 & \epsilon \\ \epsilon Z_n & Z_n \end{pmatrix} y_n = y_n + \begin{pmatrix} 0 & \epsilon \\ \epsilon Z_n & Z_n - 1 \end{pmatrix} y_n. \quad (1.4.8)$$

Formellement, on fixe $\epsilon \neq 0$, $\alpha \in \mathbf{R}$, $\eta > 0$, et des variables gaussiennes standards $(\mathcal{N}_n)_{n \geq 0}$ et on pose

$$Z^\eta(n) = \exp \left(\sigma \eta \mathcal{N}_n - \alpha \frac{\sigma^2}{2} \eta^2 \right). \quad (1.4.9)$$

Il convient de constater que

$$\mathbf{E}[(Z^\eta)^\alpha] = 1. \quad (1.4.10)$$

On considère alors la chaîne de Markov donnée par $X^\eta(0) = X(0) \neq 0$, et

$$X^\eta(n+1) = X^\eta(n) + \begin{pmatrix} 0 & \eta^2 \epsilon \\ \eta^2 \epsilon Z^\eta(n+1) & Z^\eta(n+1) - 1 \end{pmatrix} X^\eta(n) = M_\eta(n) X^\eta(n). \quad (1.4.11)$$

On s'intéresse toujours à l'exposant de Lyapunov associé

$$\mathcal{L}(\eta) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|X^\eta(n)\| = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|M_\eta(n) \cdots M_\eta(1)\|. \quad (1.4.12)$$

Observons que, dans la limite $\eta \rightarrow 0$, la matrice aléatoire M_η admet le développement en puissances de η :

$$M_\eta = I_2 + \eta V_1 + \eta^2 V_2 + O(\eta^3), \quad (1.4.13)$$

où

$$V_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \mathcal{N} \end{pmatrix}, \quad K = \mathbf{E}[V_2] = \begin{pmatrix} 0 & \epsilon \\ \epsilon & \frac{\sigma^2}{2}(1 - \alpha) \end{pmatrix}. \quad (1.4.14)$$

Par ailleurs, pour tout $\eta \neq 0$, la matrice $\eta V_1 + \eta^2 K$ est irréductible, ce qui nous place dans le cadre du Théorème 1.4.3 (plus précisément dans le cadre élargi de la Remarque 1.4.4 car la queue de la variable aléatoire \mathcal{N} est suffisamment décroissante). Ainsi

$$\mathcal{L}(\eta) = \eta^2 \lambda_2 + O(\eta^3), \quad (1.4.15)$$

et le coefficient λ_2 peut se calculer explicitement. Comtet et al. [27] prédisent d'ailleurs qu'il s'exprime comme dérivée logarithmique d'une fonction de Bessel modifiée de seconde espèce (voir le paragraphe closant la sous-section 1.4.1).

F. Comets et al. choisissent une autre voie pour analyser $\mathcal{L}(\eta)$. En trouvant une limite d'échelle au processus $(X^\eta(n))$, ils donnent d'une part une interprétation, en terme de processus, au coefficient λ_2 . Ils établissent que $(X^\eta(\lfloor \eta^{-2} t \rfloor))_{t \geq 0} \Longrightarrow (X(t))_{t \geq 0}$ lorsque

$\eta \rightarrow 0$, au sens de la topologie de Skorokhod, où $X(t)$ est le processus issu de $X(0)$ et obéissant à l'équation différentielle stochastique

$$dX(t) = \begin{pmatrix} 0 & \epsilon dt \\ \epsilon dt & \frac{(1-\alpha)\sigma^2}{2} dt + \sigma dB_t \end{pmatrix} X(t), \quad (1.4.16)$$

avec B . un mouvement brownien standard. En outre

$$\frac{\mathcal{L}(\eta)}{\eta^2} \xrightarrow{\eta \rightarrow 0} \mathcal{L}_{\sigma,\alpha}(\epsilon) := \lim_{t \rightarrow +\infty} \frac{1}{t} \mathbf{E} \log \|X(t)\|. \quad (1.4.17)$$

Le modèle continu est exactement soluble, au sens où la mesure invariante du processus $Y(\cdot) = X_2(\cdot)/X_1(\cdot)$ admet une densité explicite d'où découle une expression de l'exposant de Lyapunov :

$$\mathcal{L}_{\sigma,\alpha}(\epsilon) = \frac{\sigma^2}{4} x \frac{K_{\alpha-1}(x)}{K_\alpha(x)}, \quad \text{où} \quad x = \frac{4\epsilon}{\sigma^2}. \quad (1.4.18)$$

Ils retrouvent ainsi l'expression de λ_2 obtenue par Comtet et al. [27] (voir le paragraphe consacré dans la sous-section 1.4.1). Au vu de la conjecture 1.3.2, le résultat le plus remarquable est le suivant.

Théorème 1.4.5. *Il existe des fonctions rationnelles $c_j(\alpha)$, $j \geq 1$, telles que les développements suivants soient vérifiés, lorsque ϵ tend vers 0.*

1. Si $\alpha \in (0, +\infty) \setminus \mathbf{N}$,

$$\frac{4}{\sigma^2} \mathcal{L}_{\sigma,\alpha}(\epsilon) = c_1 x^2 + \dots + c_{[\alpha]} x^{2[\alpha]} + 2 \frac{\Gamma(1-\alpha)}{4^\alpha \Gamma(\alpha)} x^{2\alpha} + O(x^{\min(2[\alpha], 4\alpha)}). \quad (1.4.19)$$

2. Si $\alpha \in \{1, 2, \dots\}$,

$$\frac{4}{\sigma^2} \mathcal{L}_{\sigma,\alpha}(\epsilon) = c_1 x^2 + \dots + c_{\alpha-1} x^{2(\alpha-1)} + 2 \frac{(-1)^\alpha}{4^\alpha (\alpha-1)!} x^{2\alpha} \log x + O(x^{2\alpha}). \quad (1.4.20)$$

3. Si $\alpha = 0$,

$$\frac{4}{\sigma^2} \mathcal{L}_{\sigma,\alpha}(\epsilon) = \frac{1}{\log(1/x)} + O\left((\log(1/x))^{-2}\right). \quad (1.4.21)$$

4. Si $\alpha < 0$,

$$\frac{4}{\sigma^2} \mathcal{L}_{\sigma,\alpha}(\epsilon) = 2|\alpha| + \frac{4}{\sigma^2} \mathcal{L}_{\sigma,|\alpha|}(\epsilon). \quad (1.4.22)$$

Il est assez frappant que dans la limite continue, qui est une limite de désordre faible, le comportement de l'exposant de Lyapunov coïncide parfaitement, à tous les ordres, avec celui prévu par la conjecture 1.3.2. La richesse de ce comportement est préservée par le passage à la limite. D'autre part, cette étude permet une meilleure compréhension des travaux de McCoy et Wu et rend rigoureuse une partie de leur approche ([26, Appendix]).

1.A Annexe : Démonstration des Théorèmes 1.4.1 et 1.4.3

Les deux prochaines sous-sections (1.A.1 et 1.A.2) reprennent l'approche de Sadel et Schulz-Baldes pour aboutir à la démonstration de ces deux théorèmes. On omettra volontairement dans cette première phase la preuve de trois lemmes techniques. Ce sont ces trois lemmes dont la démonstration dépend des détails du modèle et pour lesquels on justifie (sous-section 1.A.3) que les hypothèses de [86] peuvent être amoindries dans le contexte que l'on considère.

1.A.1 Aperçu de la méthode

Considérons l'opérateur

$$\mathcal{M}_\eta : \mathcal{C}^\infty(\mathbb{P}^1) \longrightarrow \mathcal{C}^\infty(\mathbb{P}^1) \quad (1.A.1)$$

défini par

$$\mathcal{M}_\eta f(x) = \mathbf{E}[f(M_\eta \cdot x)]. \quad (1.A.2)$$

On souhaite établir que toute mesure invariante ν_η peut s'écrire

$$d\nu_\eta \simeq \rho_\eta d\ell, \quad \rho_\eta = \sum_{k=0}^{n-2} \eta^k \rho_k + O(\eta^{n-1}), \quad (1.A.3)$$

où ℓ est la mesure de Haar sur l'espace projectif \mathbb{P}^1 , et les fonctions ρ_k sont des fonctions \mathcal{C}^∞ qu'il convient de déterminer. Pour définir une mesure invariante, la fonction densité ρ_η doit satisfaire $\mathcal{M}_\eta^* \rho_\eta = \rho_\eta$ (où \mathcal{M}_η est l'adjoint, au sens $L^2(\ell)$, de \mathcal{M}_η). Écrivons un développement formel de cette identité par rapport à η afin de déterminer les fonctions ρ_k .

Observons d'abord que l'opérateur \mathcal{M}_η admet un développement par rapport à η :

$$\mathcal{M}_\eta = I + \sum_{k=2}^n \eta^k \mathcal{M}_k + O(\eta^{n+1}). \quad (1.A.4)$$

Ce développement est rigoureux dans le sens suivant.

Lemme 1.A.1. *Il existe une famille d'opérateurs différentiels $\mathcal{M}_k : \mathcal{C}^\infty(\mathbb{P}^1) \rightarrow \mathcal{C}^\infty(\mathbb{P}^1)$, $k \leq n$ tels que pour toute fonction $f \in \mathcal{C}^\infty(\mathbb{P}^1)$,*

$$\mathcal{M}_\eta f = f + \sum_{k=2}^n \eta^k \mathcal{M}_k f + O(\eta^{n+1}). \quad (1.A.5)$$

Le O , dans cette identité et dans toute les suivantes, est uniforme en $x \in \mathbb{P}^1$.

La démonstration est immédiate (il s'agit de faire un développement à l'ordre n de $\mathbf{E}[f(M_\eta \cdot x)]$) et repose sur le fait que les variables V_k sont bornées. On ne la détaille pas ici.

L'identité $\mathcal{M}_\eta^* \rho_\eta = \rho_\eta$ devient, pourvu qu'une permutation des sommes soit possible,

$$\begin{aligned} \rho_\eta &= \left(I + \sum_{k=2}^n \eta^k \mathcal{M}_k^* \right) \rho_\eta + O(\eta^{n+1}) \\ &= \rho_\eta + \sum_{k=2}^n \sum_{l=0}^{n-2} \eta^{k+l} \mathcal{M}_k^* \rho_l + O(\eta^{n+1}) \\ &= \rho_\eta + \sum_{m=2}^n \eta^m \sum_{k=2}^m \mathcal{M}_k^* \rho_{m-k} + O(\eta^{n+1}). \end{aligned} \quad (1.A.6)$$

Ainsi nous obtenons le système d'équations

$$\sum_{k=2}^m \mathcal{M}_k^* \rho_{m-k} = 0, \quad 2 \leq m \leq n. \quad (1.A.7)$$

On peut le récrire sous forme récursive :

$$\mathcal{M}_2^* \rho_0 = 0, \quad (1.A.8)$$

et pour tout $1 \leq m \leq n - 2$,

$$\mathcal{M}_2^* \rho_m = - \sum_{k=1}^m \mathcal{M}_{k+2}^* \rho_{m-k}. \quad (1.A.9)$$

Si les lignes qui précèdent ne sont bien évidemment pas rigoureuses, elles permettent toutefois d'identifier les fonctions ρ_k : ce seront les solutions des équations différentielles (1.A.8) et (1.A.9). Afin que ρ_η soit toujours une densité de probabilité, il conviendra également d'imposer $\rho_0 \geq 0$,

$$\int \rho_0 d\ell = 1, \quad \text{et pour tout } k \geq 1, \quad \int \rho_k d\ell = 0. \quad (1.A.10)$$

Dans les paragraphes suivants, on s'attache à rendre rigoureux cette approche. On commence, en s'appuyant sur l'étude des opérateurs différentiels \mathcal{M}_k , par justifier l'existence d'une famille (ρ_k) solution de (1.A.8) et (1.A.9). Puis on démontre par récurrence le Théorème 1.4.1 grâce à un argument de dualité.

1.A.2 Trois lemmes sur l'opérateur \mathcal{M}_2 et démonstration des théorèmes

On aura besoin pour la démonstration de considérer l'opérateur adjoint de \mathcal{M}_2 au sens L^2 . Cela exigerait habituellement de prolonger \mathcal{M}_2 . Ici, vue la forme de l'opérateur, on peut se contenter de travailler avec des opérateurs agissant sur $\mathcal{C}^\infty(\mathbb{P}^1)$. On introduit à ce propos les notions d'opérateur différentiel et d'adjoint d'un tel opérateur.

Définition 1.A.2 (Opérateur Différentiel – Adjoint). On appellera opérateur différentiel sur $\mathcal{C}^\infty(\mathbb{P}^1)$ tout opérateur s'écrivant $\mathcal{K} = \sum_{i=0}^k \alpha_i D^i$, où D est l'opérateur de dérivation et les α_i sont des applications \mathcal{C}^∞ sur \mathbb{P}^1 . On appellera adjoint de \mathcal{K} et on notera \mathcal{K}^* l'opérateur différentiel sur $\mathcal{C}^\infty(\mathbb{P}^1)$ donné, pour $f \in \mathcal{C}^\infty(\mathbb{P}^1)$, par la formule $\mathcal{K}^* f = \sum_{i=0}^k (-1)^i D^i [\alpha_i f]$.

Lemme 1.A.3. *Il existe une fonction positive $\rho_0 \in \mathcal{C}^\infty(\mathbb{P}^1)$, satisfaisant $\int \rho_0 d\ell = 1$, et telle que*

$$\text{Ker } \mathcal{M}_2^* = \mathbf{R}\rho_0. \quad (1.A.11)$$

Lemme 1.A.4. *Pour tout $k \geq 3$, $\text{Im } \mathcal{M}_k^* \subset \{f : \int f d\ell = 0\} = \text{Im } \mathcal{M}_2^*$.*

Ces deux lemmes garantissent déjà l'existence d'une suite de fonctions $\rho_k \in \mathcal{C}^\infty(\mathbb{P}^1)$ satisfaisant (1.A.8), (1.A.9) et (1.A.10). La preuve du Théorème 1.4.1 nécessite un dernier lemme technique.

Lemme 1.A.5. $(\text{Ker } \mathcal{M}_2^*)^\perp \subset \text{Im } \mathcal{M}_2$.

Avant de démontrer ces trois lemmes, voyons comment ils permettent de démontrer le Théorème 1.4.1.

Démonstration du Théorème 1.4.1. Soit ν_η une mesure invariante. Démontrons par récurrence que pour $-1 \leq r \leq n-2$,

$$\int f d\nu_\eta = \sum_{k=0}^r \eta^k \int f \rho_k d\ell + O(\eta^{r+1}). \quad (1.A.12)$$

Le résultat est immédiat pour $r = -1$. Fixons $0 \leq r \leq n-2$ et supposons que pour toute fonction $g \in \mathcal{C}^\infty(\mathbb{P}^1)$, on ait

$$\int g d\nu_\eta = \sum_{k=0}^{r-1} \eta^k \int g \rho_k d\ell + O(\eta^r) \quad (1.A.13)$$

Soit $f \in \mathcal{C}^\infty(\mathbb{P}^1)$. Grâce au Lemme 1.A.5, on peut décomposer

$$f = \langle f, \rho_0 \rangle + \mathcal{M}_2 h = \int f \rho_0 d\ell + \mathcal{M}_2 h, \quad (1.A.14)$$

pour une certaine fonction $h \in \mathcal{C}^\infty(\mathbb{P}^1)$. Ainsi

$$\int f d\nu_\eta = \int f \rho_0 d\ell + \int \mathcal{M}_2 h d\nu_\eta. \quad (1.A.15)$$

D'autre part, puisque ν_η est une mesure invariante, et en utilisant le Lemme 1.A.1,

$$\int h d\nu_\eta = \int \mathcal{M}_\eta h d\nu_\eta = \int h d\nu_\eta + \sum_{k=2}^{r+2} \eta^k \int \mathcal{M}_k h d\nu_\eta + O(\eta^{r+3}). \quad (1.A.16)$$

Donc

$$\int \mathcal{M}_2 h d\nu_\eta = - \sum_{k=3}^r \eta^{k-2} \int \mathcal{M}_k h d\nu_\eta + O(\eta^{r+1}). \quad (1.A.17)$$

L'hypothèse de récurrence assure que

$$\int \mathcal{M}_2 h d\nu_\eta = - \sum_{k=3}^r \eta^{k-2} \sum_{j=0}^{r-k+2} \eta^j \int \mathcal{M}_k h \rho_j d\ell + O(\eta^{r+1}), \quad (1.A.18)$$

soit, en réarrangeant les sommes ($m := j + k - 2$, $i := k - 2$),

$$\begin{aligned} \int \mathcal{M}_2 h d\nu_\eta &= - \sum_{m=1}^r \eta^m \sum_{i=1}^m \int \mathcal{M}_{i+2} h \rho_{m-i} d\ell + O(\eta^{r+1}) \\ &= - \sum_{m=1}^r \eta^m \sum_{i=1}^m \int h \mathcal{M}_{i+2}^* \rho_{m-i} d\ell + O(\eta^{r+1}) \\ &= - \sum_{m=1}^r \eta^m \int h \times \left(\sum_{i=1}^m \mathcal{M}_{i+2}^* \rho_{m-i} \right) d\ell + O(\eta^{r+1}). \end{aligned} \quad (1.A.19)$$

Avec la définition (1.A.9) des fonctions ρ_m , on obtient

$$\begin{aligned} \int \mathcal{M}_2 h d\nu_\eta &= \sum_{m=1}^r \eta^m \int h (\mathcal{M}_2^* \rho_m) d\ell + O(\eta^{r+1}) \\ &= \sum_{m=1}^r \eta^m \int (\mathcal{M}_2 h) \rho_m d\ell + O(\eta^{r+1}). \end{aligned} \quad (1.A.20)$$

Puisque $\int \rho_m d\ell = 0$ pour tout $m \geq 1$, on a

$$\int (\mathcal{M}_2 h) \rho_m d\ell = \int (f - \langle f, \rho_0 \rangle) \rho_m d\ell = \int f \rho_m d\ell. \quad (1.A.21)$$

Et donc

$$\int \mathcal{M}_2 h d\nu_\eta = \sum_{m=1}^r \eta^m \int f \rho_m d\ell + O(\eta^{r+1}). \quad (1.A.22)$$

Les équations (1.A.15) et (1.A.22) fournissent une preuve de l'hérédité. Le théorème est démontré. \square

Démonstration du Théorème 1.4.3. D'après le Théorème 1.2.5, il existe une mesure invariante ν_η telle que

$$\mathcal{L}(\eta) = \int_{\mathbb{P}^1} \mathbf{E} [\log \|M_\eta \hat{x}\|] d\nu_\eta(x). \quad (1.A.23)$$

Rappelons que \hat{x} désigne un des deux vecteurs de norme 1 et de direction x . D'après le Lemme 1.A.1, il existe des fonctions $H_p \in \mathcal{C}^\infty(\mathbb{P}^1)$ telles que, uniformément en x ,

$$\mathbf{E} [\log \|M_\eta \hat{x}\|] = \sum_{p=0}^n \eta^p H_p(x) + O(\eta^{n+1}). \quad (1.A.24)$$

En fait H_0 est nulle car $M_0 = I_2$, et H_1 aussi car $\mathbf{E}[V_1] = (0)$. Ainsi

$$\mathcal{L}(\eta) = \sum_{p=2}^n \eta^p \int_{\mathbb{P}^1} H_p(x) d\nu_\eta(x) + O(\eta^{n+1}). \quad (1.A.25)$$

Le résultat découle alors immédiatement du Théorème 1.4.1. \square

1.A.3 Étude de l'opérateur différentiel \mathcal{M}_2 et preuves des lemmes

Il s'agit de démontrer les lemmes 1.A.3, 1.A.4 et 1.A.5, sous l'hypothèse que la matrice $\eta V_1 + \eta^2 K$ est irréductible. On commence par donner une expression de l'opérateur différentiel \mathcal{M}_2 . Dorénavant, on utilise la paramétrisation

$$\mathbb{P}^1 = \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] / \left(-\frac{\pi}{2} \sim \frac{\pi}{2} \right) \quad (1.A.26)$$

Écrivons

$$V_1 = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}, \quad K = \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{pmatrix}. \quad (1.A.27)$$

Posons ensuite $T = \tan x$, puis

$$\begin{aligned} \omega_1(x) &= u_3 + u_4 T - u_1 T - u_2 T^2, \\ \omega_2(x) &= \delta_3 + \delta_4 T - \delta_1 T - \delta_2 T^2 + (u_1 + u_2 T)^2 T - (u_1 + u_2 T)(u_3 + u_4 T). \end{aligned} \quad (1.A.28)$$

Alors, notant

$$\begin{aligned} b_1(x) &= \mathbf{E} \left[\frac{\omega_1(x)^2}{2(1+T^2)^2} \right], \\ b_2(x) &= \mathbf{E} \left[\frac{\omega_2(x)(1+T^2) - T\omega_1(x)^2}{(1+T^2)^2} \right], \end{aligned} \quad (1.A.29)$$

on a

$$\mathcal{M}_2 f(x) = b_1(x)f''(x) + b_2(x)f'(x). \quad (1.A.30)$$

Ainsi \mathcal{M}_2 est un opérateur différentiel, et son adjoint s'écrit

$$\mathcal{M}_2^* f(x) = (b_1 f)''(x) - (b_2 f)'(x). \quad (1.A.31)$$

Passons aux démonstrations des trois lemmes. Les preuves vont dépendre de la forme des matrices V_1 et K . Dans l'article [86], C. Sadel, et H. Schulz-Baldes ne traitent que le cas où la matrice aléatoire V_1 est irréductible. En réalité, leur démonstration peut être adaptée sous des hypothèses un peu moins restrictives. Commençons par examiner le champ des possibles. On observe rapidement que, à une conjugaison déterministe près, on est forcément dans l'un des cas suivants

1. V_1 est irréductible;
2. $u_3 = 0$, ${}^t(1, 0)$ est la seule direction invariante, et $\mathbf{P}(u_1 = u_4) < 1$;
3. $u_3 = 0$, ${}^t(1, 0)$ est la seule direction invariante, et $\mathbf{P}(u_1 = u_4) = 1$;
4. $u_2 = u_3 = 0$ et $\mathbf{P}(u_1 = u_4) < 1$;
5. $V_1 = uI_2$, où u est une variable aléatoire réelle.

On traite séparément ces cinq cas, à l'exception du 3^e. Les méthodes sont similaires mais des difficultés techniques apparaissent dans le traitement des 2^e et 4^e cas, qui méritent d'être mentionnées.

Preuve lorsque V_1 est une homothétie. Ce dernier cas est sans doute le plus facile. Puisque $\eta V_1 + \eta^2 K$ est irréductible, c'est que K doit être irréductible et, quitte à conjuguer, on peut supposer que

$$K = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad b > 0. \quad (1.A.32)$$

Alors les fonctions b_1 et b_2 sont constantes, $b_1 = 0$, $b_2 = b$ et $\mathcal{M}_2 f = bf'$. La démonstration des trois lemmes est alors immédiate et $\rho_0 = 1$.

Preuve des trois lemmes lorsque V_1 est irréductible. C'est le cas que traitent C. Sadel et H. Schulz-Baldes. La première observation est que l'application b_1 ne s'annule jamais. En effet pour tout $T \in \mathbf{R} \cup \{\pm\infty\}$,

$$\frac{\omega_1(x)}{1+T^2} = \frac{u_3 + u_4 T - u_1 T - u_2 T^2}{1+T^2} \quad (1.A.33)$$

ne peut pas être presque sûrement nul, sinon la droite ${}^t(1, T)$ (ou ${}^t(0, 1)$ lorsque $T = \infty$) serait stable par V_1 , contredisant l'irréductibilité. Par conséquent pour tout $x \in \mathbf{P}^1$,

$$b_1(x) = \frac{1}{2} \mathbf{E} \left[\left(\frac{\omega_1(x)}{1+T^2} \right)^2 \right] \neq 0. \quad (1.A.34)$$

Démonstration du Lemme 1.A.3. Il s'agit de résoudre

$$(b_1\rho_0)'' - (b_2\rho_0)' = 0. \quad (1.A.35)$$

Après une première intégration on obtient, pour un certain $\alpha \in \mathbf{R}$,

$$(b_1\rho_0)' - b_2\rho_0 = -\alpha. \quad (1.A.36)$$

Soit ϕ une primitive de b_2/b_1 , choisie pour être \mathcal{C}^∞ sur $\mathbf{P}^1 \setminus \{0\}$ (il peut arriver que $\phi(0^-) \neq \phi(0^+)$). Quitte à changer x en $-x$, disons $\phi(0^+) \leq \phi(0^-)$. On a, sur $\mathbf{P}^1 \setminus \{0\}$,

$$(b_1\rho_0 \exp(-\phi))' = -\alpha \exp(-\phi) \quad (1.A.37)$$

Ainsi

$$\rho_0(x) = \frac{1}{b_1(x)} \exp(\phi(x)) \left(-\alpha \int_{0^+}^x \exp(-\phi(t)) d\ell(t) + \beta \right) \quad (1.A.38)$$

Avec cette formule, ρ_0 est \mathcal{C}^∞ sur $\mathbf{P}^1 \setminus \{0\}$. Puis les constantes α et β sont choisies de sorte que

$$\rho_0(0^-) = \rho_0(0^+). \quad (1.A.39)$$

et

$$\int_{\mathbf{P}^1} \rho_0 d\ell = 1. \quad (1.A.40)$$

Pour être explicite, la condition (1.A.39) requiert

$$\alpha = \beta \left(\int_{\mathbf{P}^1} e^{-\phi} \right)^{-1} \left(1 - e^{\phi(0^+) - \phi(0^-)} \right), \quad (1.A.41)$$

c'est-à-dire

$$\rho_0(x) = \frac{\beta}{b_1(x)} \exp(\phi(x)) \left(1 - \frac{\int_{0^+}^x \exp(-\phi(t)) d\ell(t)}{\int_{\mathbf{P}^1} \exp(-\phi) d\ell} \left(1 - e^{\phi(0^+) - \phi(0^-)} \right) \right) \quad (1.A.42)$$

On vérifie ensuite aisément que ρ_0 est partout strictement positive, et que, grâce à (1.A.36), elle est \mathcal{C}^∞ sur \mathbf{P}^1 . Puis la constante β est choisie pour satisfaire (1.A.40). \square

Démonstration du Lemme 1.A.5. Soit $f \in \mathcal{C}^\infty(\mathbf{P}^1)$ telle que $\langle f, \rho_0 \rangle = 0$. On souhaite résoudre

$$b_1 h'' + b_2 h' = f. \quad (1.A.43)$$

En multipliant l'équation par ρ_0 , puis en utilisant (1.A.35) et (1.A.36), on obtient la forme équivalente

$$f\rho_0 = b_1\rho_0 h'' + b_2\rho_0 h' = (b_1\rho_0 h)'' - (b_2\rho_0 h)' + 2\alpha h'. \quad (1.A.44)$$

Soit F une primitive de $f\rho_0$, qui existe et est \mathcal{C}^∞ sur \mathbf{P}^1 car $\langle f, \rho_0 \rangle = 0$. L'équation peut alors être réécrite

$$F + C = (b_1\rho_0 h)' - b_2\rho_0 h + 2\alpha h. \quad (1.A.45)$$

Soit ψ une primitive de $c = b_2/b_1 - 2\alpha/(b_1\rho_0)$ sur $\mathbf{P}^1 \setminus \{0\}$. L'équation devient

$$(F + C) \exp(-\psi) = (b_1\rho_0 h \exp(-\psi))'. \quad (1.A.46)$$

On souhaite définir

$$h(x) = \frac{1}{\rho_0(x)b_1(x)} e^{\psi(x)} \left(\int_{0^+}^x (F(t) + C) e^{-\psi(t)} d\ell(t) + D \right). \quad (1.A.47)$$

C'est une solution de l'équation (1.A.45) sur $\mathbb{P}^1 \setminus \{0\}$. Les constantes C et D doivent être choisies de sorte que $h(0^-) = h(0^+)$. Alors l'équation (1.A.45) assure que $h \in \mathcal{C}^\infty(\mathbb{P}^1)$. Il suffit de prendre D arbitraire puis

$$C = \frac{1}{\int_{\mathbb{P}^1} e^{-\psi} d\ell} \left(D \left(e^{\psi(0^+)-\psi(0^-)} - 1 \right) - \int_{\mathbb{P}^1} F e^{-\psi} d\ell \right). \quad (1.A.48)$$

□

On omet la démonstration du Lemme 1.A.4, qui suit le même schéma.

Preuve lorsque V_1 a une unique direction stable. Les autres cas sont plus techniques mais les méthodes de preuve restent approximativement les mêmes. C'est pourquoi on démontre le Lemme 1.A.3 uniquement. On suppose maintenant que $u_3 = 0$, que ${}^t(1, 0)$ est la seule direction invariante, et que $\mathbf{P}(u_1 = u_4) < 1$. Dans ce cas la fonction b_1 est strictement positive sur $\mathbb{P}^1 \setminus \{0\}$ et a un zéro d'ordre deux au point 0. Par ailleurs, puisque la matrice aléatoire $\eta V_1 + \eta^2 K$ est irréductible, on sait que $\delta_3 \neq 0$ et, quitte à conjuguer encore, on peut supposer $\delta_3 > 0$.

Démonstration du Lemme 1.A.3. On souhaite résoudre

$$(b_1 \rho_0)' - b_2 \rho_0 = -\alpha. \quad (1.A.49)$$

Soit ϕ une primitive de b_2/b_1 qui soit \mathcal{C}^∞ sur $\mathbb{P}^1 \setminus \{0\}$. On a alors

$$(b_1 \rho_0 \exp(-\phi))' = -\alpha \exp(-\phi). \quad (1.A.50)$$

Donc

$$\rho_0(x) = \frac{1}{b_1(x)} \exp(\phi(x)) \left(-\alpha \int \exp(-\phi(t)) dt + \beta \right). \quad (1.A.51)$$

Soit $a, b \in \mathbf{R}$ tels que

$$\frac{b_2(x)}{b_1(x)} = \frac{a}{x^2} + \frac{b}{x} + O(1) \text{ sur } \mathbb{P}^1 \setminus \{0\}. \quad (1.A.52)$$

Le choix $\delta_3 > 0$ garantit que $a > 0$. Ainsi $\phi(x) \rightarrow +\infty$ lorsque $x \rightarrow 0^-$ et donc on est forcé de prendre

$$\rho_0(x) = \frac{\alpha}{b_1(x)} \exp(\phi(x)) \int_x^0 \exp(-\phi(t)) dt, \quad (1.A.53)$$

avec la convention

$$\int_x^0 = \int_x^{\pi/2} + \int_{-\pi/2}^0 \quad (1.A.54)$$

si $x \in (0, \pi/2]$. On définit ainsi une fonction \mathcal{C}^∞ et strictement positive sur $\mathbb{P}^1 \setminus \{0\}$. On peut vérifier que lorsque $x \rightarrow 0^\pm$,

$$\rho_0(x) = \frac{\alpha}{b_2(0)} - \alpha \frac{b_2'(0)}{b_2(0)^2} x + o(x), \quad (1.A.55)$$

puis que ρ_0 admet ainsi un prolongement \mathcal{C}^∞ en 0, toujours strictement positif, et satisfaisant l'équation (1.A.49). On ajuste enfin la constante α de sorte que $\int \rho_0 d\ell = 1$. □

Preuve lorsque V_1 a deux directions stables. De nouveau on ne démontre que le Lemme 1.A.3. On suppose $u_1 = u_2 = u_3 = 0$, et $\mathbf{E}[u_4^2] \neq 0$. Alors la fonction b_1 a un zéro d'ordre deux en 0 et en $\pi/2$, et est strictement positive ailleurs. L'irréductibilité assure que δ_2 et δ_3 ne sont pas nuls. Disons, quitte à conjuguer, que $\delta_3 > 0$. Les deux cas $\delta_2 > 0$ et $\delta_2 < 0$ doivent être traités séparément.

Démonstration du Lemme 1.A.3 lorsque $\delta_2 > 0$. On souhaite toujours résoudre

$$(b_1\rho_0)' - b_2\rho_0 = -\alpha. \quad (1.A.56)$$

Soit ϕ (resp. φ) une primitive de b_2/b_1 qui soit \mathcal{C}^∞ sur $(0, \pi/2)$ (resp. sur $(-\pi/2, 0)$). On a alors, pour $x \in (0, \pi/2)$,

$$(b_1\rho_0 \exp(-\phi))' = -\alpha \exp(-\phi). \quad (1.A.57)$$

Donc

$$\rho_0(x) = \frac{1}{b_1(x)} \exp(\phi(x)) \left(-\alpha \int \exp(-\phi(t)) dt + \beta \right). \quad (1.A.58)$$

Et la même chose sur $(-\pi/2, 0)$, avec φ et d'autres constantes α, β . Soit $a, b \in \mathbf{R}$ tels que

$$\frac{b_2(x)}{b_1(x)} = \frac{a}{x^2} + \frac{b}{x} + O(1). \quad (1.A.59)$$

Le choix $\delta_3 > 0$ garantit que $a > 0$. Ainsi $\phi(x) \rightarrow -\infty$ lorsque $x \rightarrow 0^+$. De même $\phi(x) \rightarrow -\infty$ lorsque $x \rightarrow (\pi/2)^-$, et $\varphi(x) \rightarrow +\infty$ lorsque $x \rightarrow 0^-$ et lorsque $x \rightarrow (-\pi/2)^+$. Ainsi l'unique solution est

$$\rho_0(x) = \frac{\beta}{b_1(x)} \exp(\phi(x)) \mathbf{1}_{x \in (0, \pi/2)}. \quad (1.A.60)$$

On définit ainsi une fonction \mathcal{C}^∞ et positive sur $\mathbf{P}^1 \setminus \{0, \pi/2\}$. On vérifie aisément qu'en fait ρ_0 est \mathcal{C}^∞ sur \mathbf{P}^1 et satisfait l'équation (1.A.56) avec $\alpha = 0$. On ajuste enfin la constante β de sorte que $\int \rho_0 d\ell = 1$. \square

Démonstration du Lemme 1.A.3 lorsque $\delta_2 < 0$. Cette fois $\phi(x) \rightarrow -\infty$ lorsque $x \rightarrow 0^+$, et $\phi(x) \rightarrow +\infty$ lorsque $x \rightarrow (\pi/2)^-$. Et symétriquement $\varphi(x) \rightarrow +\infty$ lorsque $x \rightarrow 0^-$ et $\varphi(x) \rightarrow -\infty$ lorsque $x \rightarrow (-\pi/2)^+$. Ainsi l'unique solution est donnée par

$$\rho_0(x) = \begin{cases} \frac{\alpha}{b_1(x)} \exp(\phi(x)) \int_x^{\pi/2} \exp(-\phi(t)) dt & \text{si } x \in [0, \pi/2] \\ \frac{\alpha}{b_1(x)} \exp(\varphi(x)) \int_x^0 \exp(-\varphi(t)) dt & \text{si } x \in [-\pi/2, 0] \end{cases} \quad (1.A.61)$$

On vérifie la validité des raccords en 0 et $\pi/2$ et qu'on définit ainsi une fonction \mathcal{C}^∞ sur \mathbf{P}^1 et satisfaisant l'équation (1.A.56). On ajuste enfin la constante α de sorte que $\int \rho_0 d\ell = 1$. \square

1.A.4 Questions ouvertes et Commentaires

Que dire lorsque $\eta V_1 + \eta^2 K$ n'est pas irréductible ? La première chose à noter est que la méthode développée précédemment échoue : on ne peut même pas trouver de solution intégrable ρ_0 à l'équation $\mathcal{M}_2^* \rho_0 = 0$. Pour autant, la fonction $\eta \mapsto \mathcal{L}(\eta)$ est-elle

singulière en 0 ? Commençons par le cas caricatural où la matrice $M_\eta = I_2 + \eta V_1 + \eta^2 K$ est presque sûrement triangulaire supérieure. Dans ce cas

$$\begin{aligned} \mathcal{L}(\eta) &= \max \{ \mathbf{E} \log(1 + \eta u_1 + \eta^2 \delta_1), \mathbf{E} \log(1 + \eta u_4 + \eta^2 \delta_4) \} \\ &= \max \left\{ \frac{\eta^2}{2} (2\delta_1 - \langle u_1^2 \rangle) + \frac{\eta^3}{3} \langle u_1^3 \rangle + O(\eta^4), \frac{\eta^2}{2} (2\delta_4 - \langle u_4^2 \rangle) + \frac{\eta^3}{3} \langle u_4^3 \rangle + O(\eta^4) \right\}. \end{aligned} \quad (1.A.62)$$

Ainsi $\mathcal{L}(\eta)$ présente une singularité en 0 lorsque $2\delta_1 - \langle u_1^2 \rangle = 2\delta_4 - \langle u_4^2 \rangle$ et $\langle u_1^3 \rangle \neq \langle u_4^3 \rangle$:

$$\begin{aligned} \mathcal{L}(\eta) &= \frac{\eta^2}{2} (2\delta_1 - \langle u_1^2 \rangle) + \frac{1}{3} \max \{ \eta^3 \langle u_1^3 \rangle, \eta^3 \langle u_4^3 \rangle \} + O(\eta^4) \\ &= \frac{\eta^2}{2} (2\delta_1 - \langle u_1^2 \rangle) + \frac{\eta^3}{6} (\langle u_1^3 \rangle + \langle u_4^3 \rangle) - \frac{1}{6} |\eta|^3 |\langle u_1^3 \rangle - \langle u_4^3 \rangle| + O(\eta^4). \end{aligned} \quad (1.A.63)$$

Se posent alors les questions suivantes, pour lesquelles nous ne connaissons pas d'argument permettant d'y répondre.

- Cette singularité subsiste-t-elle lorsque la partie principale $\eta V_1 + \eta^2 K$ est triangulaire, mais la matrice M_η est irréductible ?
- Si oui, peut-on néanmoins toujours écrire $\mathcal{L}(\eta) = \lambda \eta^2 + O(\eta^3)$?

Que dire si $\mathbf{E}[V_1]$ n'est pas nulle ? Un autre cas qui n'est pas couvert par [86], celui où la matrice V_1 n'est pas centrée. Si toutefois la matrice $K := \mathbf{E}[V_1]$ est irréductible alors, quitte à la conjuguer supposons que

$$K = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \quad (1.A.64)$$

Dans ce cas $\mathcal{M}_\eta = I + \eta \mathcal{M}_1 + \dots$ et $\mathcal{M}_1 = bf'$ et le Théorème 1.4.1 reste vrai, avec $\rho_0 = 1$. Finalement on obtient un développement de la forme

$$\mathcal{L}(\eta) = a\eta + \sum_{k=2}^{n-1} \lambda_k \eta^k + O(\eta^n). \quad (1.A.65)$$

En revanche, si $\mathbf{E}[V_1]$ n'est pas irréductible, la question de la régularité de $\mathcal{L}(\eta)$ n'est pas résolue.

Chapter 2

Regular Expansion for the characteristic exponent of a product of 2×2 random matrices

This chapter is the reproduction of the article [64].

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Abstract We consider a product of 2×2 random matrices which appears in the physics literature in the analysis of some 1D disordered models. These matrices depend on a parameter $\epsilon > 0$ and on a positive random variable Z . Derrida and Hilhorst (J Phys A 16:2641, 1983, §3) conjecture that the corresponding characteristic exponent has a regular expansion with respect to ϵ up to — and not further — an order determined by the distribution of Z . We give a rigorous proof of that statement. We also study the singular term which breaks that expansion.

2.1 Introduction

Random matrix products appeared in the physics literature as a powerful tool to study disordered systems, ranging from Anderson model [20, 76] to disordered harmonic chains

[40, 87] or disordered Ising model (discussed below). Among that wide range of models, the present work focuses on a very specific one, introduced by B. Derrida and H. Hilhorst in [35] to study the strong interaction limit of a 1D disordered Ising model.

Let (Z_n) be iid non-negative and non-deterministic random variables, with law μ . For $\epsilon > 0$, consider the matrices

$$M_{n,\epsilon} = \begin{pmatrix} 1 & \epsilon \\ \epsilon Z_n & Z_n \end{pmatrix}. \quad (2.1.1)$$

We will write Z for a random variable with law μ and M_ϵ for the associated matrix. In fact, we will use Z instead of μ to formulate our assumptions and results. The (leading) Lyapunov exponent — also called characteristic exponent — is the growth rate of their product:

$$\mathcal{L}(\epsilon) = \mathcal{L}_Z(\epsilon) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|M_{n,\epsilon} \cdots M_{1,\epsilon}\|. \quad (2.1.2)$$

We will be particularly interested in the behaviour of $\mathcal{L}(\epsilon)$ in the limit $\epsilon \rightarrow 0$.

2×2 matrices of the form (2.1.1) have appeared several times to express the free energy of the disordered 1D Ising model [28, 30, 35, 79], where the limit $\epsilon \rightarrow 0$ represents a regime of very strong interactions. It is also used in the celebrated work by B. McCoy and T. T. Wu [77] to study a 2D Ising model with 1D disorder, as well as in a similar model proposed by R. Shankar and G. Murthy [88] which includes frustrated interactions.

From a mathematical point of view, a wide literature proposed to study these models and more general matrix products. One should cite the seminal work by H. Furstenberg et al. [45, 46] and Oseledec's theorem [80] (see [91] for a review). Looking at our own task, Furstenberg–Kesten theorem [46] asserts that the limit (2.1.2) exists almost surely and is deterministic, as long as $\mathbf{E}[\log_+ \|M_\epsilon\|]$ is finite (here $\mathbf{E}[\log_+ Z] < +\infty$ suffices). When ϵ vanishes, the matrix $M_{n,\epsilon}$ tends to a diagonal matrix and the Lyapunov exponent can be explicitly computed thanks to the law of large numbers: $\mathcal{L}(0) = \max(0, \mathbf{E}[\log Z])$. However, diagonal matrices are a degenerate case in the theory developed by H. Furstenberg et al. and one expects, in most cases, that $\epsilon \mapsto \mathcal{L}(\epsilon)$ is singular around 0.

It is worth stressing from now that the diagonal matrix $M_{n,0}$ is still random. Therefore we are not in the framework of weak disorder limits such as [21, 86, 92] in which the matrix $M_{n,0}$ is deterministic. The main reference paper for our analysis is rather [35].

2.1.1 General Conjecture and known results

The present work is motivated by the recent mathematical progress by Genovese *et al.* [49]. In this paper some physical predictions, about the limiting behaviour of $\mathcal{L}(\epsilon)$ when ϵ vanishes, mainly stated in [35] (see also [28, 30, 79]), are proven. However the physical predictions go beyond. We now first formulate these predictions in the form of conjectures, which detail the expected limiting behaviour of $\mathcal{L}(\epsilon)$, depending on the distribution of Z . Then we explain what has been proven and what our contribution is.

Definition 2.1.1. A real-valued random variable ξ is said to be *arithmetic* when there exists a constant $c > 0$ such that $c\xi \in \mathbf{Z} \cup \{\pm\infty\}$ almost surely.

Conjecture 2.1.2. Assume that $\log Z$ is nonarithmetic.

1. Suppose in addition that $\mathbf{E}[\log Z] < 0$ and that there exists $\alpha \in (0, +\infty)$ such that $\mathbf{E}[Z^\alpha] = 1$.

– If $\alpha \notin \{1, 2, \dots\}$, then, as ϵ goes to 0,

$$\mathcal{L}(\epsilon) = \sum_{k=1}^{\lfloor \alpha \rfloor} (-1)^{k+1} \ell_k \epsilon^{2k} + (-1)^{\lceil \alpha \rceil + 1} C_Z \epsilon^{2\alpha} + o(\epsilon^{2\alpha}), \quad (2.1.3)$$

where, for $k \leq \lfloor \alpha \rfloor$, ℓ_k is a positive rational function of $\mathbf{E}[Z], \dots, \mathbf{E}[Z^k]$; and C_Z is a positive real number.

– If $\alpha \in \{1, 2, \dots\}$, then

$$\mathcal{L}(\epsilon) = \sum_{k=1}^{\alpha-1} (-1)^{k+1} \ell_k \epsilon^{2k} + (-1)^{\alpha+1} C_Z \epsilon^{2\alpha} \log(1/\epsilon) + o(\epsilon^{2\alpha} \log \epsilon), \quad (2.1.4)$$

where the coefficients (ℓ_k) are the same positive rational functions of Z 's moments as before; and C_Z is still a positive constant.

2. If $\mathbf{E}[\log Z] = 0$ — it is the “ $\alpha = 0$ ” case — then

$$\mathcal{L}(\epsilon) = \frac{C_Z}{\log(1/\epsilon)} + o((\log 1/\epsilon)^{-1}). \quad (2.1.5)$$

The same references motivate further comments.

Remark 2.1.3. The coefficients (ℓ_k) appearing in the conjecture can be computed recursively. For instance

$$\ell_1 = \frac{\mathbf{E}[Z]}{1 - \mathbf{E}[Z]}, \quad \ell_2 = \frac{(1 + \mathbf{E}[Z])^2 \mathbf{E}[Z^2] + 2\mathbf{E}[Z]^2(1 - \mathbf{E}[Z^2])}{2(1 - \mathbf{E}[Z])^2(1 - \mathbf{E}[Z^2])}. \quad (2.1.6)$$

Precise recursive formulas will be derived in Section 2.3. However it is not clear that a simple closed formula for ℓ_k can be derived. By contrast, apart from a few special situations, the calculation of the constant C_Z is a very hard problem [28, §4.2.1]. On another note, in all the instances developed in Conjecture 2.1.2, the constant C_Z should be replaced by a multiplicatively periodic function of ϵ if $\log Z$ is arithmetic. A precise computation of such a multiplicatively periodic function C_Z is made in [35] for a very specific (and singular) distribution of Z .

Remark 2.1.4. We discuss in this remark the instances which are excluded by the conjecture. The conjecture actually covers almost all the cases where $\mathbf{E}[\log Z] \leq 0$ and $\mathbf{P}(Z > 1) > 0$, except the one discussed in the item 5 of Remark 2.1.7.

1. The case $\mathbf{E}[\log Z] > 0$ (it corresponds to $\alpha < 0$) boils down to $\mathbf{E}[\log Z] < 0$ by factorizing Z in the matrix M_ϵ : $\mathcal{L}_Z(\epsilon) = \mathbf{E}[\log Z] + \mathcal{L}_{1/Z}(\epsilon)$. Similarly, by conjugating by the matrix $\text{Diag}(-1, 1)$, one observes that \mathcal{L} is an even function: $\mathcal{L}(\epsilon) = \mathcal{L}(-\epsilon)$. It implies that the behaviour $\epsilon^{2\alpha}$ is rather $|\epsilon|^{2\alpha}$, so it is actually singular even when α is a half-integer.
2. If $Z \leq 1$ almost surely (that is “ $\alpha = +\infty$ ”), then $\mathcal{L}(\epsilon)$ admits a regular expansion with respect to ϵ^2 up to any order. Is it smooth or analytic in a neighborhood of 0? The problem is still open, except if $Z \in [0, 1 - \eta]$ almost surely, for some $\eta \in (0, 1)$. If so then it is a consequence of a result by D. Ruelle [85] that $\mathcal{L}(\epsilon)$ is a real analytic function of ϵ around 0.

3. That same theorem of D. Ruelle also ensures that the Lyapunov exponent $\mathcal{L}(\epsilon)$ is always an analytic function of ϵ on $(0, +\infty)$.

Very little of Conjecture 2.1.2 has been made mathematically rigorous. To our knowledge only (2.1.3) has been successfully tackled [49], and *only* for $\alpha \in (0, 1)$. When $\alpha \in (0, 1)$, that is $\mathbf{E}[\log Z] < 0$ and $\mathbf{E}[Z] > 1$, the singularity $\epsilon^{2\alpha}$ happens to be the leading behaviour of $\mathcal{L}(\epsilon)$ and (2.1.3) takes the form:

$$\mathcal{L}(\epsilon) \stackrel{\epsilon \searrow 0}{\cong} C_Z \epsilon^{2\alpha} + o(\epsilon^{2\alpha}). \quad (2.1.7)$$

This simplifies in a substantial way the analysis: Derrida and Hilhorst in [35] (see also [30, 35, 79]) explicitly give, based on a two scale argument, a probability measure that is expected to be close, when ϵ is small, to the invariant probability for the action of M_ϵ on the projective space $\mathbf{P}^1(\mathbf{R})$ (that is, the distribution of X_ϵ , in the next paragraph's notations). Then, they use this probability to compute the Lyapunov exponent. This two scale analysis is made rigorous by G. Genovese *et al.* [49], who show that this probability measure is indeed close to the invariant measure in a suitable norm, and this control is sufficiently strong to yield precisely (2.1.7). It appears to be rather challenging to follow the same steps for $\alpha \geq 1$: the guess for the invariant probability would have to be tuned to yield the $[\alpha]$ terms of the regular expansion and the singular $\epsilon^{2\alpha}$ term. Even at a heuristic level, such a construction is lacking. Note, in particular, that in [35], the $\alpha \geq 1$ case is treated in a expedite way, without reference to the invariant probability, and without capturing the singularity $\epsilon^{2\alpha}$.

On the other hand, a weak disorder limit of the model has been investigated. In this limit, the product of random matrices becomes a stochastic differential equation system. An exactly solvable structure emerges from that SDE and the analog of (2.1.3) and (2.1.4) has been shown to hold (see [61] for the case $\alpha \in (0, 2)$ and [26] for the general case). As pointed out in [26], it is rather remarkable that the structure of (2.1.3) and (2.1.4) holds also in the weak disorder limit and this appears to be a rather deep fact. Nonetheless, the fact that the conjecture holds in the weak disorder limit is far from being a mathematical proof of the conjecture for products of matrices.

The main aim of our work is to approach (2.1.3) and (2.1.4). Our results are the following.

1. $\mathcal{L}(\epsilon)$ admits a regular expansion in powers of ϵ^2 , up to order $\epsilon^{2[\alpha]}$, or $\epsilon^{2(\alpha-1)}$ in the integer case (that is the regular part of (2.1.3) and (2.1.4)).
2. We prove that the next order term after this regular part, call it $R(\epsilon)$, satisfies, as ϵ goes to 0, for instance in the non-integer case

$$\epsilon^{2([\alpha]+1)} \ll R(\epsilon) \ll \epsilon^{2[\alpha]}. \quad (2.1.8)$$

Since $\epsilon \mapsto \mathcal{L}(\epsilon)$ is an even function, only even powers of ϵ are non-singular (see Remark 2.1.4 item 1). Hence, $R(\epsilon)$ is necessarily singular. A quantitative and explicit control on this term is given, but it falls short to prove the expected $\epsilon^{2\alpha}$ behaviour of (2.1.3).

2.1.2 Assumptions and Main Result

We will work under the following assumptions, supposed to be satisfied in the whole paper.

Assumptions 2.1.5. The random variable Z is positive, non-deterministic, and

- (a) $\mathbf{E}[\log Z] < 0$ (can be $-\infty$);
- (b) There exists $\delta > 0$ such that $\mathbf{E}[Z^\delta] < +\infty$.

Introduce

$$\mathcal{A} = \{\gamma \in [0, +\infty] \text{ such that } \mathbf{E}[Z^\gamma] < 1\}, \quad (2.1.9)$$

and

$$\alpha = \sup \mathcal{A} \in (0, +\infty]. \quad (2.1.10)$$

The Assumptions 2.1.5, together with a convexity argument, ensure that \mathcal{A} is an interval of positive length. Note that $\alpha = +\infty$ if and only if $Z \leq 1$ almost surely. In any case \mathcal{A} takes the following form: either $\mathbf{E}[Z^\alpha] = 1$ and then $\mathcal{A} = (0, \alpha)$, or $\mathbf{E}[Z^\alpha] < 1$, and then $\mathcal{A} = (0, \alpha]$. In the latter case, necessarily, $\mathbf{E}[Z^\gamma] = +\infty$ for every $\gamma > \alpha$. Here is the main result of this work.

Theorem 2.1.6. *There exist positive coefficients (ℓ_k) , where ℓ_k is a rational function of the moments $\mathbf{E}[Z], \dots, \mathbf{E}[Z^k]$, such that the following expansions hold, as $\epsilon \searrow 0$.*

1. *If $\alpha = +\infty$ (i.e., if $Z \leq 1$ a.s.), then for every $K \geq 0$,*

$$\mathcal{L}(\epsilon) = \sum_{k=1}^K (-1)^{k+1} \ell_k \epsilon^{2k} + \mathcal{O}(\epsilon^{2(K+1)}). \quad (2.1.11)$$

2. *If $\alpha \in \{1, 2, \dots\}$ and if $\mathbf{E}[Z^\alpha] = 1$, then*

$$\mathcal{L}(\epsilon) = \sum_{k=1}^{\alpha-1} (-1)^{k+1} \ell_k \epsilon^{2k} + (-1)^{\alpha+1} R(\epsilon), \quad (2.1.12)$$

where $R(\epsilon)$ is nonnegative and

$$\epsilon^{2\alpha} \ll R(\epsilon) \leq C \epsilon^{2\alpha} \log(1/\epsilon), \quad (2.1.13)$$

for some $C > 0$. The lower bound can be improved if, in addition, Z has a bounded support, to obtain, for some $C \geq c > 0$, the sharper estimate

$$c \leq \frac{R(\epsilon)}{\epsilon^{2\alpha} \log(1/\epsilon)} \leq C. \quad (2.1.14)$$

3. *If $\alpha \in (0, +\infty) \setminus \{1, 2, \dots\}$ and if there exists $\gamma > \alpha$ such that $\mathbf{E}[Z^\gamma]$ is finite, then*

$$\mathcal{L}(\epsilon) = \sum_{k=1}^{\lfloor \alpha \rfloor} (-1)^{k+1} \ell_k \epsilon^{2k} + (-1)^{\lceil \alpha \rceil + 1} R(\epsilon), \quad (2.1.15)$$

where $R(\epsilon)$ is nonnegative and

$$\epsilon^{2\lceil \alpha \rceil} \ll R(\epsilon) \leq C \epsilon^{2\alpha}, \quad (2.1.16)$$

for some $C > 0$. The lower bound can be improved if, in addition, Z has a bounded support: in that case, there exists $\theta \in (\alpha, \lceil \alpha \rceil)$ and $c > 0$ such that $R(\epsilon) \geq c \epsilon^{2\theta}$.

Remark 2.1.7. 1. The constant θ is explicit: $\theta = \lceil \alpha \rceil - \frac{\log \mathbf{E}[Z^{\lceil \alpha \rceil}]}{\log \|Z\|_\infty}$.

2. When $\alpha \in (0, 1)$, then $\lceil \alpha \rceil = 0$, so (2.1.15), and (2.1.16) should be read $\epsilon^2 \ll \mathcal{L}(\epsilon) \leq C\epsilon^{2\alpha}$, which is rough, and of course strongly weaker than the behaviour (2.1.7) obtained by [49].
3. When α is finite, the lower bounds of the error in (2.1.13) and (2.1.16) assert in particular that the regular expansions (2.1.12) and (2.1.15) cannot be continued beyond $K = \lceil \alpha \rceil - 1$: $\mathcal{L}(\epsilon)$ is singular.
4. When α is not an integer, the assumption “there exists $\gamma > \alpha$ such that $\mathbf{E}[Z^\gamma]$ is finite” can be replaced by the weaker assumption “ $\mathbf{E}[Z^\alpha \log_+ Z] < +\infty$ ” (see Remark 2.5.4).
5. Suppose that α is finite and $\mathbf{E}[Z^\alpha] < 1$ (and $\mathbf{E}[Z^\gamma] = +\infty$ for every $\gamma > \alpha$). Whether α is an integer or not, under some technical assumptions on the distribution of Z , the Lyapunov exponent is slightly regularized (see Remark 2.5.4 for a sketch of proof):

$$\mathcal{L}(\epsilon) = \sum_{k=1}^{\lfloor \alpha \rfloor} (-1)^{k+1} \ell_k \epsilon^{2k} + (-1)^{\lfloor \alpha \rfloor + 1} R(\epsilon), \quad \epsilon^{2(\lfloor \alpha \rfloor + 1)} \ll R(\epsilon) \ll \epsilon^{2\alpha}. \quad (2.1.17)$$

2.1.3 Strategy of the proof and structure of the paper

A classical result in the theory of product of random matrices (see e.g. [20, 45]) provides a semi-explicit formula for the Lyapunov exponent, involving an invariant measure for the action of the random matrices on the corresponding projective space. Here, this formula will be obtained in a direct way, and takes the form

$$\mathcal{L}(\epsilon) = \mathbf{E}[\log(1 + \epsilon^2 X_\epsilon)], \quad (2.1.18)$$

where the law of X_ϵ is an invariant measure for the random transformation, on $[0, +\infty)$, $x \mapsto Z \frac{1+x}{1+\epsilon^2 x}$. In other words it satisfies

$$X_\epsilon \stackrel{(d)}{=} Z \frac{1 + X_\epsilon}{1 + \epsilon^2 X_\epsilon}, \quad (2.1.19)$$

where Z is independent of X_ϵ (on the right hand side). Existence and uniqueness of such a random variable X_ϵ will be justified in Section 2.2, as well as formula (2.1.18). A very useful uniform stochastic dominance of the random variables $(X_\epsilon)_{\epsilon > 0}$ will also be proved.

From that point on, the work will only be based on formula (2.1.18) for the Lyapunov exponent and the fixed point equation (2.1.19). Thanks to the former, the problem will readily boil down to studying X_ϵ 's moments. That study can be split into two subproblems. We will know since Section 2.2 which ones of X_ϵ 's moments are bounded as ϵ goes to 0 and which diverge. The two subproblems then are:

- Deriving a regular expansion for X_ϵ 's bounded moments, involving an error in terms of a divergent moment of X_ϵ (Sections 2.3 and 2.4);
- Estimating the divergence speed of X_ϵ 's unbounded moments (Section 2.5).

The former point is addressed in Section 2.3. The analysis is based on a bootstrap procedure, based on recursive uses of the fixed point equation (2.1.19). It gives more and more precise expansions of these moments. Eventually, it will provide the regular expansion (2.1.12) or (2.1.15) with an upper bound on the error $R(\epsilon)$, involving a divergent moment of X_ϵ . That work will be generalized in the appendix 2.A, for matrices of size d , with more general entries.

That same strategy, using a bootstrap procedure to obtain a more and more precise estimate of X_ϵ 's moments, can also provide a lower bound on the error, involving a divergent truncated moment of X_ϵ : Section 2.4 will be devoted to that analysis.

At the end of these sections, the following theorem will be proved, which, unlike Theorem 2.1.6, does not require any extra assumption on Z (apart from Assumptions 2.1.5).

Theorem 2.1.8. *Fix $B > 0$, and an integer $K \in \mathcal{A} \cup \{0\}$. One has, for all $\epsilon > 0$,*

$$\mathcal{L}(\epsilon) = \sum_{k=1}^K (-1)^{k+1} \ell_k \epsilon^{2k} + (-1)^{K+2} R_K(\epsilon), \quad (2.1.20)$$

where, for all $\beta \in (K, K+1]$, and for some positive constants c and C_β ,

$$c\epsilon^{2(K+1)} \mathbf{E}[X_\epsilon^{K+1} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}] \leq R_K(\epsilon) \leq C_\beta \epsilon^{2\beta} \mathbf{E}[X_\epsilon^\beta]. \quad (2.1.21)$$

Remark 2.1.9. The coefficients (ℓ_k) are the same as in Theorem 2.1.6. The neat thing about that theorem is that, unlike the lower bound (2.1.16) of Theorem 2.1.6, the estimate (2.1.21) should be ‘‘sharp’’ in the following sense. If one proves that, as ϵ goes to 0, $\mathbf{P}(X_\epsilon \geq c\epsilon^{-2}) \geq C\epsilon^{2\alpha}$ for some positive constants c and C (the precise analysis of M_ϵ 's invariant measure conducted in [49] provides such an estimate when $\alpha \in (0, 1)$) then (2.1.21) becomes $c\epsilon^{2\alpha} \leq R_K(\epsilon) \leq C\epsilon^{2\alpha}$ (with a log correction if α is an integer). It is the good order of ϵ predicted by Conjecture 2.1.2. Without such an estimate, (2.1.21) is not satisfactory yet for it is not explicit enough.

To obtain the explicit bounds given in Theorem 2.1.6, a study of the divergence speed of X_ϵ 's divergent moments is needed. It is conducted in Section 2.5. The derivation of upper bounds is based a stochastic dominance found in Section 2.2 (namely $X_\epsilon \preceq X_0$), and on renewal theory results describing the limiting behaviour of the tail of X_0 . The lower bounds are only derived when Z is bounded. The analysis is again based on a recursive use of the fixed point equation (2.1.19). It is the point where the sharpness of the lower bound (2.1.21) of the singularity is lost. Theorem 2.1.6 is proved at the end of Section 2.5.

2.2 Existence and First Properties of the invariant measure X_ϵ

In this section we prove the existence of the random variables X_ϵ and derive formula (2.1.18). A first result on X_ϵ 's moments is also proved: it spells out which moments of X_ϵ are bounded as ϵ goes to 0 and which diverge.

We start by introducing an invariant measure of the random matrix M_0 ($\epsilon = 0$). It will play a central role to define the random variables X_ϵ and control their moments. First I need to fix a notation for the stochastic dominance.

Definition 2.2.1. The *stochastic dominance* will be denoted by \preceq . Formally, if X and Y are two real-valued random variables, $X \preceq Y$ means that $\mathbf{P}(X \geq x) \leq \mathbf{P}(Y \geq x)$ for every $x \in \mathbf{R}$. Equivalently, there exist two copies \tilde{X} and \tilde{Y} , of X and Y respectively, such that $\tilde{X} \leq \tilde{Y}$ almost surely.

Lemma 2.2.2. Fix a sequence (Z_n) of iid copies of Z . The series

$$X_0 = \sum_{n=1}^{+\infty} Z_1 \cdots Z_n \quad (2.2.1)$$

converges almost surely. It is the unique random variable (in distribution) satisfying

$$X_0 \stackrel{(d)}{=} Z(1 + X_0), \quad (2.2.2)$$

with Z independent of X_0 . Moreover $\mathbf{E}[\log_+ X_0]$ is finite; and for every $\gamma > 0$,

$$\mathbf{E}[X_0^\gamma] < +\infty \quad \text{if and only if} \quad \mathbf{E}[Z^\gamma] < 1. \quad (2.2.3)$$

Proof. Recall that $\mathbf{E}[\log Z] < 0$. The almost sure convergence of the series follows from the law of large numbers, whereby

$$Z_1 \cdots Z_n = e^{n\mathbf{E}[\log Z] + o(n)} \quad \text{as } n \rightarrow +\infty. \quad (2.2.4)$$

Of course

$$X_0 = \sum_{n=1}^{+\infty} Z_1 \cdots Z_n = Z_1 \left(1 + \sum_{n=2}^{+\infty} Z_2 \cdots Z_n \right) \quad (2.2.5)$$

satisfies the identity (2.2.2). Let's turn to the uniqueness. If \tilde{X}_0 is another random variable satisfying (2.2.2), then, applying this identity N times we get

$$\tilde{X}_0 \stackrel{(d)}{=} \sum_{n=1}^N Z_1 \cdots Z_n + Z_1 \cdots Z_N \tilde{X}_0, \quad (2.2.6)$$

where Z_1, \dots, Z_N are iid copies of Z , independent of \tilde{X}_0 . With (2.2.4), the last term vanishes (in distribution) as N goes to $+\infty$, whereas the first sum converges monotonically towards X_0 . So eventually, $\tilde{X}_0 \stackrel{(d)}{=} X_0$. The uniqueness is proved.

Now fix $\gamma > 0$ such that $\mathbf{E}[Z^\gamma] < 1$. We want to prove that $\mathbf{E}[X_0^\gamma]$ is finite. If $\gamma \geq 1$ we use Minkovsky's inequality:

$$\mathbf{E}[X_0^\gamma]^{1/\gamma} \leq \sum_{n=1}^{+\infty} \mathbf{E}[(Z_1 \cdots Z_n)^\gamma]^{1/\gamma} = \sum_{n=0}^{+\infty} \mathbf{E}[Z^\gamma]^{n/\gamma}. \quad (2.2.7)$$

Thus $\mathbf{E}[X_0^\gamma]$ is finite. On the other hand, if $\gamma \in (0, 1)$, then for all $x, y \geq 0$, $(x + y)^\gamma \leq x^\gamma + y^\gamma$. So

$$\mathbf{E}[X_0^\gamma] \leq \sum_{n=0}^{+\infty} \mathbf{E}[Z^\gamma]^n, \quad (2.2.8)$$

which is again finite. Now, if $\mathbf{E}[Z^\gamma] \geq 1$, then with the identity (2.2.2),

$$\mathbf{E}[X_0^\gamma] = \mathbf{E}[Z^\gamma] \mathbf{E}[(1 + X_0)^\gamma] \geq \mathbf{E}[(1 + X_0)^\gamma], \quad (2.2.9)$$

which can hold only if $\mathbf{E}[X_0^\gamma] = +\infty$ (or $\gamma = 0$). Eventually, pick $\gamma \in \mathcal{A}$ so that $\mathbf{E}[Z^\gamma] < 1$. With the foregoing, we then know that $\mathbf{E}[X_0^\gamma] < +\infty$. Thus, by Jensen's inequality, $\mathbf{E}[\log_+ X_0]$ is finite. \square

The next lemma provides the existence of the random variables X_ϵ and the desired formula for the Lyapunov exponent.

Lemma 2.2.3. *For all $\epsilon > 0$, there exists a non-negative random variable X_ϵ , unique in distribution, such that*

$$X_\epsilon \stackrel{(d)}{=} Z \frac{1 + X_\epsilon}{1 + \epsilon^2 X_\epsilon}, \quad (2.2.10)$$

with Z independent of X_ϵ . Moreover, for every $\epsilon \in (0, 1]$, $Z \preceq X_\epsilon \preceq X_0$. The Lyapunov exponent can be written

$$\mathcal{L}(\epsilon) = \mathbf{E}[\log(1 + \epsilon^2 X_\epsilon)]. \quad (2.2.11)$$

Furthermore, for every $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$ with nonnegative entries,

$$\frac{1}{n} \log \langle \mathbf{x}, M_{n,\epsilon} \cdots M_{1,\epsilon} \mathbf{y} \rangle \xrightarrow{n \rightarrow +\infty} \mathcal{L}(\epsilon) \quad \text{a.s. and in } L^1. \quad (2.2.12)$$

Remark 2.2.4. There could be other distributions, supported on \mathbf{R} , satisfying (2.2.10). We only claim uniqueness for non-negative invariant measure. However, if Z does not have a finite support, then one can prove, using classical results of products of random matrices (see [45] or [20, Chapter 3]), that there exists a unique invariant measure on $\bar{\mathbf{R}}$. With Lemma 2.2.3, we know that it must be supported on \mathbf{R}_+ .

In what follows, X_ϵ will always denote the unique non-negative invariant random variable of Lemma 2.2.3.

Proof. We begin with the proof of the existence, for which we use a standard procedure. Fix an iid sequence (Z_n) of copies of Z , set $x_0 = 0$ and define recursively the random variables

$$x_{n+1} = Z_{n+1} \frac{1 + x_n}{1 + \epsilon^2 x_n}. \quad (2.2.13)$$

Denote by ν_n the distribution of the random variable x_n and consider the measure $\rho_N = \frac{1}{N} \sum_{n=0}^{N-1} \nu_n$. Observe that for any $n \geq 0$, x_n is nonnegative and $x_{n+1} \leq Z_{n+1}(1 + x_n)$. Thus, by an easy induction,

$$0 \leq x_n \leq \sum_{k=0}^{n-1} Z_n \cdots Z_{n-k} \preceq X_0 : \quad (2.2.14)$$

the random variables x_n are uniformly bounded by X_0 . Consequently the sequence (ρ_N) is tight. Pick a limit point ρ_∞ of that sequence and fix a random variable X_ϵ with distribution ρ_∞ . The limit distribution ρ_∞ must be invariant under the random transformation (2.2.13). In other words it must satisfy (2.2.10). The existence of an invariant measure supported on \mathbf{R}_+ is proved. Incidentally we obtained $X_\epsilon \preceq X_0$. As for the stochastic lower bound $X_\epsilon \succcurlyeq Z$, it directly follows from the identity (2.2.10) when $\epsilon \leq 1$.

To deal with the uniqueness, assume that $X_\epsilon^{(0)}$ and $Y_\epsilon^{(0)}$ are two such random variables and fix an iid sequence (Z_n) of copies of Z , independent of $X_\epsilon^{(0)}$ and $Y_\epsilon^{(0)}$. We introduce, for $n \geq 0$,

$$X_\epsilon^{(n+1)} = Z_{n+1} \frac{1 + X_\epsilon^{(n)}}{1 + \epsilon^2 X_\epsilon^{(n)}}, \quad Y_\epsilon^{(n+1)} = Z_{n+1} \frac{1 + Y_\epsilon^{(n)}}{1 + \epsilon^2 Y_\epsilon^{(n)}}. \quad (2.2.15)$$

Observe that, almost surely,

$$|X_\epsilon^{(n+1)} - Y_\epsilon^{(n+1)}| = Z_{n+1} \frac{(1 - \epsilon^2)|X_\epsilon^{(n)} - Y_\epsilon^{(n)}|}{(1 + \epsilon^2 X_\epsilon^{(n)})(1 + \epsilon^2 Y_\epsilon^{(n)})} \leq Z_{n+1} |X_\epsilon^{(n)} - Y_\epsilon^{(n)}|. \quad (2.2.16)$$

Thus, with (2.2.4), $|X_\epsilon^{(n)} - Y_\epsilon^{(n)}|$ vanishes almost surely as n goes to $+\infty$. On the other hand, note that, with the construction (2.2.15), for all $n \geq 0$, $X_\epsilon^{(n)} \stackrel{(d)}{=} X_\epsilon^{(0)}$ and $Y_\epsilon^{(n)} \stackrel{(d)}{=} Y_\epsilon^{(0)}$. The uniqueness follows. Then ρ_N actually converges (without extraction) towards X_ϵ 's distribution.

We are left with the proof of formula (2.2.11). Thanks to a result by H. Hennion [67], since M_ϵ 's entries are positive, the convergence (2.2.12) holds. On the other hand, for every $n \geq 0$,

$$M_{n,\epsilon} \cdots M_{1,\epsilon} \begin{pmatrix} 1 \\ \epsilon X_\epsilon^{(0)} \end{pmatrix} = \left[\prod_{k=0}^{n-1} (1 + \epsilon^2 X_\epsilon^{(k)}) \right] \begin{pmatrix} 1 \\ \epsilon X_\epsilon^{(n)} \end{pmatrix} \quad (2.2.17)$$

So, by taking the log and the expectation,

$$\frac{1}{n} \mathbf{E}[\log \|M_{n,\epsilon} \cdots M_{1,\epsilon}^t(1, \epsilon X_\epsilon^{(0)})\|] = \mathbf{E}[\log(1 + \epsilon^2 X_\epsilon)] + \frac{1}{n} \mathbf{E}[\log \|(1, \epsilon X_\epsilon)\|]. \quad (2.2.18)$$

Since $\mathbf{E}[\log_+ X_\epsilon] \leq \mathbf{E}[\log_+ X_0]$ is finite (Lemma 2.2.2), the last term vanishes as n goes to $+\infty$. On the other hand, one has, for every $n \geq 0$,

$$\begin{aligned} (1, 0) M_{n,\epsilon} \cdots M_{1,\epsilon}^t(1, 0) \\ \leq \|M_{n,\epsilon} \cdots M_{1,\epsilon}^t(1, \epsilon X_\epsilon^{(0)})\| \leq \|M_{n,\epsilon} \cdots M_{1,\epsilon}\| (1 + X_\epsilon^{(0)}). \end{aligned} \quad (2.2.19)$$

Since we know that both the lower and upper bounds goes to $\mathcal{L}(\epsilon)$ (after taking log and expectation) as n goes to $+\infty$, almost surely and in L^1 , we get the result. \square

Remark 2.2.5. Here the most usual result [45] does not apply (no strong irreducibility) and we invoked [67]. Actually, formula (2.2.11) can also be proved using a classical result by H. Furstenberg and Y. Kifer [47, Corollary of Theorem 3.10], which gives an explicit formula for the Lyapunov exponent in terms of invariant measures as soon as M is an invertible random matrix of size $d \times d$ with no deterministic proper invariant subspace. We could also have used the convergence $\rho_n \rightarrow$ law of X_ϵ , to prove (2.2.11) and (2.2.12) without using H. Hennion's results.

Remark 2.2.6. If one notes that the map $\epsilon \mapsto \frac{1+x}{1+\epsilon^2 x}$ is monotone, one obtains, with the previous construction, that the random variables X_ϵ are stochastically decreasing with ϵ : for all $\epsilon' \geq \epsilon > 0$ one has $X_{\epsilon'} \preceq X_\epsilon \preceq X_0$.

Lemma 2.2.7. $X_\epsilon \rightarrow X_0$ in distribution when $\epsilon \rightarrow 0$.

Proof. The stochastic dominance $X_\epsilon \preceq X_0$ ensures that the family of random variables $(X_\epsilon)_{\epsilon > 0}$ is tight. Consider a limit point \tilde{X}_0 of X_ϵ as ϵ goes to 0. Since X_ϵ satisfies the identity (2.2.10), the limit point \tilde{X}_0 must satisfy $\tilde{X}_0 \stackrel{(d)}{=} Z(1 + \tilde{X}_0)$. That means, using Lemma 2.2.2, that X_0 is the only possible limit point of X_ϵ as ϵ goes to 0. The convergence of X_ϵ towards X_0 (in distribution) follows. \square

Using classical integration theorems, one readily obtains the following limiting behaviour of X_ϵ 's moments, or truncated moments, which will be needed in the proof of Theorem 2.1.6.

Corollary 2.2.8. *For any $\gamma > 0$,*

1. *If $\mathbf{E}[Z^\gamma] < 1$ then, as ϵ goes to 0, $\mathbf{E}[X_\epsilon^\gamma] = \mathcal{O}(1)$.*
2. *If $\mathbf{E}[Z^\gamma] \geq 1$ then for any $B > 0$,*

$$\mathbf{E} \left[X_\epsilon^\gamma \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}} \right] \xrightarrow{\epsilon \rightarrow 0} +\infty. \quad (2.2.20)$$

Proof. Recall that $\mathbf{E}[X_0^\gamma]$ is finite if and only if $\mathbf{E}[Z^\gamma] < 1$ (Lemma 2.2.2). With the stochastic dominance $X_\epsilon \preceq X_0$ provided by Lemma 2.2.3, we get $\mathbf{E}[X_\epsilon^\gamma] = \mathcal{O}(1)$ when $\mathbf{E}[Z^\gamma] < 1$. On the other hand, if $\mathbf{E}[Z^\gamma] \geq 1$, then $\mathbf{E}[X_0^\gamma] = +\infty$ (Lemma 2.2.2).

Thanks to Skorokhod's representation theorem and the convergence in distribution provided by Lemma 2.2.7, there exists representatives \tilde{X}_ϵ and \tilde{X}_0 such that $\tilde{X}_\epsilon \rightarrow \tilde{X}_0$ almost surely. And then, with Fatou's lemma, for any $B > 0$.

$$\liminf_{\epsilon \rightarrow 0} \mathbf{E} \left[\tilde{X}_\epsilon^\gamma \mathbf{1}_{\{\epsilon^2 \tilde{X}_\epsilon \leq B\}} \right] \geq \mathbf{E} \left[\liminf_{\epsilon \rightarrow 0} \tilde{X}_\epsilon^\gamma \mathbf{1}_{\{\epsilon^2 \tilde{X}_\epsilon \leq B\}} \right] = \mathbf{E}[X_0^\gamma] = +\infty. \quad (2.2.21)$$

□

2.3 Regular Expansion (Theorem 2.1.8: upper bound)

In this section we prove the existence of a regular expansion for the Lyapunov exponent $\mathcal{L}(\epsilon)$. We also lay out the method, which will be used twice more: for the generalization of this result in Appendix 2.A and in Section 2.4 to obtain the lower bound of the error. It is based on the study of a regular expansion for the moments of X_ϵ which are bounded as ϵ goes to 0. Let us first state the main result of the section.

Proposition 2.3.1. *Pick an integer $K \in \mathcal{A} \cup \{0\}$, and fix $\beta \in [K, K + 1]$. The following expansion holds when ϵ goes to 0,*

$$\mathcal{L}(\epsilon) = \sum_{k=1}^K (-1)^{k+1} \ell_k \epsilon^{2k} + \mathcal{O}(\epsilon^{2\beta} \mathbf{E}[X_\epsilon^\beta]), \quad (2.3.1)$$

where, for $k \leq K$, ℓ_k is a positive rational function of $\mathbf{E}[Z], \dots, \mathbf{E}[Z^k]$.

Remark 2.3.2. With some extra effort, the domination $\mathcal{O}(\epsilon^{2\beta} \mathbf{E}[X_\epsilon^\beta])$ can be replaced by $\mathcal{O}(\mathbf{E}[(\epsilon^2 X_\epsilon)^{\beta_1} \wedge (\epsilon^2 X_\epsilon)^{\beta_2}])$ for any $\beta_1, \beta_2 \in [K, K + 1]$. To achieve this, replace the upper bounds x^β (resp. x^δ) by $x^{\beta_1} \wedge x^{\beta_2}$ (resp. $x^{\delta_1} \wedge x^{\delta_2}$). It will only be needed to explain (see Remark 2.5.4) some generalizations discussed in Remark 2.1.7.

Proof. We use identity $\mathcal{L}(\epsilon) = \mathbf{E}[\log(1 + \epsilon^2 X_\epsilon)]$ (Lemma 2.2.3) and expand the logarithm. There exists $C > 0$ such that for all $x \geq 0$,

$$\left| \log(1 + x) - \sum_{j=1}^K \frac{(-1)^{j+1}}{j} x^j \right| \leq C x^\beta. \quad (2.3.2)$$

Consequently

$$\mathcal{L}(\epsilon) = \sum_{j=1}^K \frac{(-1)^{j+1}}{j} \epsilon^{2j} \mathbf{E}[X_\epsilon^j] + \mathcal{O}\left(\epsilon^{2\beta} \mathbf{E}[X_\epsilon^\beta]\right). \quad (2.3.3)$$

Lemma 2.3.3. *For all $l \leq K$, the following expansion holds,*

$$\mathbf{E}[X_\epsilon^l] = \sum_{k=0}^{K-l} (-1)^k g_{l,k} \epsilon^{2k} + \mathcal{O}(\epsilon^{2(\beta-l)} \mathbf{E}[X_\epsilon^\beta]), \quad (2.3.4)$$

where, for all $l \geq 1$ and for all $k \geq 0$, the coefficient $g_{l,k}$ is a positive rational function of the moments $\mathbf{E}[Z], \dots, \mathbf{E}[Z^{l+k}]$.

We first admit Lemma 2.3.3 and conclude the proof of Proposition 2.3.1. The substitution of (2.3.4) into (2.3.3) yields

$$\mathcal{L}(\epsilon) = \sum_{j=1}^K \sum_{k=0}^{K-j} \frac{(-1)^{j+k+1}}{j} \epsilon^{2(j+k)} g_{j,k} + \mathcal{O}\left(\epsilon^{2\beta} \mathbf{E}[X_\epsilon^\beta]\right). \quad (2.3.5)$$

It can be rewritten

$$\mathcal{L}(\epsilon) = \sum_{s=1}^K (-1)^{s+1} \ell_s \epsilon^{2s} + \mathcal{O}\left(\epsilon^{2\beta} \mathbf{E}[X_\epsilon^\beta]\right), \quad (2.3.6)$$

with

$$\ell_s = \sum_{j=1}^K \sum_{k=0}^{K-j} \frac{g_{j,k}}{j} \mathbf{1}_{\{j+k=s\}}, \quad (2.3.7)$$

and ℓ_s is a positive rational function of $\mathbf{E}[Z], \dots, \mathbf{E}[Z^s]$ by inspection. \square

We are left with the proof of Lemma 2.3.3, for which we briefly explain the strategy. Write the identity

$$\mathbf{E}[X_\epsilon^k] = \mathbf{E}[Z^k] \mathbf{E}\left[\left(\frac{1 + X_\epsilon}{1 + \epsilon^2 X_\epsilon}\right)^k\right]. \quad (2.3.8)$$

Then by expanding the denominator one gets

$$\mathbf{E}[X_\epsilon^k] = \mathbf{E}[Z^k] \sum_{j=0}^n \binom{-k}{j} \epsilon^{2j} \mathbf{E}\left[(1 + X_\epsilon)^k X_\epsilon^j\right] + \text{Remainder}. \quad (2.3.9)$$

It gives a relation between the moments of X_ϵ which will be used via a bootstrap procedure: the substitution of a regular expansion for X_ϵ 's first moments into (2.3.9) will provide a more precise expansion of $\mathbf{E}[X_\epsilon^k]$. That new expansion will in turn be injected into (2.3.9) (for another k), to obtain a more precise regular expansion for that other moment, et cetera. Of course that procedure should be done in a specific order. Doing it rigorously will require a double induction, on k and the length of the expansions. Let's now proceed to the detailed proof.

Proof of Lemma 2.3.3. Set $\delta = \beta - K$. We prove, using a course-of-values double induction with the lexicographic order on (m, j) , that if $j + m \leq K$, then $\mathbf{E}[X_\epsilon^j]$ has an expansion up to the order ϵ^{2m} :

$$\mathbf{E}[X_\epsilon^j] = \sum_{k=0}^m (-1)^k g_{j,k} \epsilon^{2k} + \mathcal{O}\left(\epsilon^{2(m+\delta)} \mathbf{E}[X_\epsilon^\beta]\right), \quad (2.3.10)$$

where for every $j \geq 1$ and $k \geq 0$, the coefficient $g_{j,k}$ is a positive rational function of $\mathbf{E}[Z], \dots, \mathbf{E}[Z^{j+k}]$. Of course $\mathbf{E}[X_\epsilon^0]$ admits such an expansion, up to any order. All that remains is the inductive step. Fix $l \geq 1$ and $n \geq 0$ such that $l + n \leq K$ and suppose that (2.3.10) holds

(A) for all $j \leq K$ and $m \leq (n-1) \wedge (K-j)$;

(B) for all $j \leq l-1$, and $m \leq n$.

We want to show that it also holds for $(j, m) = (l, n)$. To this end, write

$$\mathbf{E}[X_\epsilon^l] = \mathbf{E}\left[\left(Z \frac{1+X_\epsilon}{1+\epsilon^2 X_\epsilon}\right)^l\right] = \mathbf{E}[Z^l] \sum_{r=0}^l \binom{l}{r} \mathbf{E}\left[\frac{X_\epsilon^r}{(1+\epsilon^2 X_\epsilon)^l}\right]. \quad (2.3.11)$$

We want to expand the denominator with respect to ϵ . Let $C > 0$ be such that for any $x \geq 0$ and $l, m \leq K$,

$$\left|\frac{1}{(1+x)^l} - \sum_{i=0}^m \binom{-l}{i} x^i\right| \leq Cx^{m+\delta}. \quad (2.3.12)$$

Thus, for every $r \leq l$,

$$\begin{aligned} \left|\mathbf{E}\left[\frac{X_\epsilon^r}{(1+\epsilon^2 X_\epsilon)^l}\right] - \sum_{i=0}^n \binom{-l}{i} \epsilon^{2i} \mathbf{E}[X_\epsilon^{i+r}]\right| &\leq C\epsilon^{2(n+\delta)} \mathbf{E}[X_\epsilon^{r+n+\delta}] \\ &\leq C\epsilon^{2(n+\delta)} \max_{0 \leq k \leq K} \mathbf{E}[X_\epsilon^{k+\delta}]. \end{aligned} \quad (2.3.13)$$

Actually

$$\max_{k \leq K} \mathbf{E}[X_\epsilon^{k+\delta}] = \mathcal{O}(\mathbf{E}[X_\epsilon^\beta]). \quad (2.3.14)$$

Indeed, if $1 \leq k \leq K-1$, then $\mathbf{E}[Z^{k+\delta}] < 1$, so $\mathbf{E}[X_\epsilon^{k+\delta}] \leq \mathbf{E}[X_0^{k+\delta}] < +\infty$ (Lemmas 2.2.2 and 2.2.3). On the other hand $\mathbf{E}[X_\epsilon^{K+\delta}] = \mathbf{E}[X_\epsilon^\beta] \geq \mathbf{E}[Z^\beta] > 0$ (Lemma 2.2.3). Thus, with (2.3.13) and (2.3.14), we can write, for every $r \leq l$,

$$\mathbf{E}\left[\frac{X_\epsilon^r}{(1+\epsilon^2 X_\epsilon)^l}\right] = \sum_{i=0}^n \binom{-l}{i} \epsilon^{2i} \mathbf{E}[X_\epsilon^{i+r}] + \mathcal{O}(\epsilon^{2(n+\delta)} \mathbf{E}[X_\epsilon^\beta]). \quad (2.3.15)$$

And then, injecting it into (2.3.11), we get

$$\mathbf{E}[X_\epsilon^l] = \mathbf{E}[Z^l] \sum_{r=0}^l \binom{l}{r} \sum_{i=0}^n \binom{-l}{i} \epsilon^{2i} \mathbf{E}[X_\epsilon^{i+r}] + \mathcal{O}(\epsilon^{2(n+\delta)} \mathbf{E}[X_\epsilon^\beta]). \quad (2.3.16)$$

We then isolate the term “ $(i, r) = (0, l)$ ” — that is $\mathbf{E}[Z^l]\mathbf{E}[X_\epsilon^l]$ — on the left-hand side and divide by $1 - \mathbf{E}[Z^l]$, to get

$$\mathbf{E}[X_\epsilon^l] = \frac{\mathbf{E}[Z^l]}{1 - \mathbf{E}[Z^l]} \sum_{\substack{0 \leq r \leq l, 0 \leq i \leq n \\ (i,r) \neq (0,l)}} \binom{l}{r} \binom{-l}{i} \epsilon^{2i} \mathbf{E}[X_\epsilon^{i+r}] + \mathcal{O}(\epsilon^{2(n+\delta)} \mathbf{E}[X_\epsilon^\beta]). \quad (2.3.17)$$

We claim that the induction hypothesis provides expansions for all these terms, up to the required order. The induction hypothesis (2.3.10) on $\mathbf{E}[X_\epsilon^{i+r}]$ (induction hypothesis with $j = i + r$ and $m = n - i$, which is contained in the item (B) if $i = 0$ and in the item (A) if $i \geq 1$), states that

$$\mathbf{E}[X_\epsilon^{i+r}] = \sum_{k=0}^{n-i} \epsilon^{2k} (-1)^k g_{i+r,k} + \mathcal{O}(\epsilon^{2(n-i+\delta)} \mathbf{E}[X_\epsilon^\beta]). \quad (2.3.18)$$

We then inject it into (2.3.17). It yields

$$\begin{aligned} \mathbf{E}[X_\epsilon^l] = \frac{\mathbf{E}[Z^l]}{1 - \mathbf{E}[Z^l]} \sum_{\substack{0 \leq r \leq l, 0 \leq i \leq n \\ (i,r) \neq (0,l)}} \binom{l}{r} \binom{-l}{i} \left(\epsilon^{2i} \sum_{k=0}^{n-i} \epsilon^{2k} (-1)^k g_{i+r,k} \right. \\ \left. + \mathcal{O}(\epsilon^{2(n+\delta)} \mathbf{E}[X_\epsilon^\beta]) \right). \end{aligned} \quad (2.3.19)$$

One can already observe that it is a regular expansion of $\mathbf{E}[X_\epsilon^l]$ up to the order n , as expected. The following lines intend to derive a recursive formula for $g_{l,k}$ so as to check its sign. First note that

$$\binom{-l}{i} = (-1)^i \binom{l+i-1}{i}. \quad (2.3.20)$$

Thus (2.3.19) becomes

$$\begin{aligned} \mathbf{E}[X_\epsilon^l] = \frac{\mathbf{E}[Z^l]}{1 - \mathbf{E}[Z^l]} \sum_{\substack{0 \leq r \leq l, 0 \leq i \leq n \\ (i,r) \neq (0,l) \\ 0 \leq k \leq n-i}} \binom{l}{r} \binom{l+i-1}{i} \epsilon^{2(k+i)} (-1)^{k+i} g_{i+r,k} \\ + \mathcal{O}(\epsilon^{2(n+\delta)} \mathbf{E}[X_\epsilon^\beta]). \end{aligned} \quad (2.3.21)$$

Eventually, it can be written as

$$\mathbf{E}[X_\epsilon^l] = \sum_{s=0}^n (-1)^s g_{l,s} \epsilon^{2s} + \mathcal{O}(\epsilon^{2(n+\delta)} \mathbf{E}[X_\epsilon^\beta]), \quad (2.3.22)$$

with, for every $s \leq n$,

$$g_{l,s} = \frac{\mathbf{E}[Z^l]}{1 - \mathbf{E}[Z^l]} \sum_{\substack{0 \leq r \leq l, 0 \leq i \leq n \\ (i,r) \neq (0,l) \\ 0 \leq k \leq n-i}} \binom{l}{r} \binom{l+i-1}{i} g_{i+r,k} \mathbf{1}_{\{i+k=s\}}. \quad (2.3.23)$$

Thanks to the induction hypothesis, it is a positive rational function of $\mathbf{E}[Z]$, ..., $\mathbf{E}[Z^{l+n}]$. The inductive step is proved, and the lemma follows. \square

2.4 Theorem 2.1.8: lower bound on the error

We prove here the lower bound on the error given in Theorem 2.1.8, formula (2.1.21). We already saw in Proposition 2.3.1's proof, when we studied the signs in front of the coefficients ℓ_k or $g_{l,k}$, that when expanding the algebraic fractions $(1 + \epsilon^2 X_\epsilon)^{-r}$, the term ϵ^{2n} always comes with the sign $(-1)^n$. The same occurs for the error, at each step, at the order ϵ^{K+1} : it comes with the sign $(-1)^{K+1}$. As a result, the error terms, which invariably accumulate with the same sign, effectively add up and cannot offset one another. In practice, these error terms can also be bounded from below. It yields the next result.

Proposition 2.4.1. *Fix an integer $K \in \mathcal{A} \cup \{0\}$ and $B > 0$. There exists $c > 0$ such that, for all $\epsilon > 0$,*

$$(-1)^{K+2} \left[\mathcal{L}(\epsilon) - \sum_{k=1}^K (-1)^{k+1} \ell_k \epsilon^{2k} \right] \geq c \epsilon^{2(K+1)} \mathbf{E}[X_\epsilon^{K+1} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}], \quad (2.4.1)$$

where the coefficients (ℓ_k) are the same as in Proposition 2.3.1.

Unsurprisingly, a similar scheme as in Proposition 2.3.1's proof will be used. We will proceed to a double induction, corresponding to an underlying bootstrap procedure. The only actual difference compared to Section 2.3 is that the estimate (2.3.12) is replaced by the lower bound

$$\frac{1}{(1+x)^m} - \sum_{i=0}^r \binom{-m}{i} x^i \geq C(-x)^{r+1} \mathbf{1}_{\{x \leq B\}}. \quad (2.4.2)$$

We begin with the equivalent of Lemma 2.3.3 in this new perspective.

Lemma 2.4.2. *Fix an integer $K \in \mathcal{A} \cup \{0\}$ and $B > 0$. There exists $c > 0$ such that for all $1 \leq l \leq K$, and $0 \leq n \leq K - l$, the following holds, for the same real coefficients $(g_{l,k})$ as in Lemma 2.3.3:*

$$\mathbf{E}[X_\epsilon^l] - \sum_{k=0}^n (-1)^k g_{l,k} \epsilon^{2k} \geq c (-1)^{n+1} \epsilon^{2(n+1)} \mathbf{E}[X_\epsilon^{l+n} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}] \quad (2.4.3)$$

if $n+1$ is even, and the same with an inequality in the opposite direction if $n+1$ is odd.

Proof. If $K = 0$ the statement is empty, so suppose $K \geq 1$. It will be useful to recall formula (2.3.20). Fix $B > 0$. There exists $C > 0$ such that for all $1 \leq l \leq K+1$ and $n \leq K+1$, and for all $x \geq 0$,

$$\frac{1}{(1+x)^l} - \sum_{i=0}^{n-1} \binom{l+i-1}{i} (-x)^i \begin{cases} \geq C(-x)^n \mathbf{1}_{\{x \leq B\}} & \text{if } n \text{ is even,} \\ \leq C(-x)^n \mathbf{1}_{\{x \leq B\}} & \text{if } n \text{ is odd.} \end{cases} \quad (2.4.4)$$

As in Lemma 2.3.3, we carry out a proof by course-of-values double induction. More precisely, set

$$\tilde{C} := C \min_{1 \leq l \leq K} \frac{\mathbf{E}[Z^l]}{1 - \mathbf{E}[Z^l]}. \quad (2.4.5)$$

We prove that if $j \geq 1$, $m \geq 0$ and $j+m \leq K+1$ then

$$\mathbf{E}[X_\epsilon^j] - \sum_{k=0}^{m-1} \epsilon^{2k} (-1)^k g_{j,k} \geq \tilde{C} (-1)^m \epsilon^{2m} \mathbf{E}[X_\epsilon^{j+m} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}] \quad (2.4.6)$$

if m is even (and of course $(-1)^m = 1$); and the same with an inequality in the opposite direction if m is odd (then $(-1)^m = -1$). The base case $m = 0$ is immediate. For the inductive step, we fix $l \geq 1$, $n \geq 1$ such that $l+n \leq K+1$ and we suppose that (2.4.6) holds for all (j, m) with $m \leq n-1$ and $1 \leq j \leq K+1-m$, and for all (j, n) with $1 \leq j \leq l-1$. We want to prove (2.4.6) for $(j, m) = (l, n)$. For the sake of simplicity, the proof will only be written for n even (inequalities would be in the opposite direction if n is odd). First write the identity

$$\mathbf{E}[X_\epsilon^l] = \mathbf{E} \left[\left(Z \frac{1+X_\epsilon}{1+\epsilon^2 X_\epsilon} \right)^l \right] = \mathbf{E}[Z^l] \sum_{r=0}^l \binom{l}{r} \mathbf{E} \left[\frac{X_\epsilon^r}{(1+\epsilon^2 X_\epsilon)^l} \right]. \quad (2.4.7)$$

Using (2.4.4) we get,

$$\mathbf{E}[X_\epsilon^l] \geq \mathbf{E}[Z^l] \sum_{r=0}^l \left\{ \binom{l}{r} \sum_{i=0}^{n-1} \binom{l+i-1}{i} (-1)^i \epsilon^{2i} \mathbf{E}[X_\epsilon^{i+r}] \right. \\ \left. + C(-1)^n \epsilon^{2n} \mathbf{E}[X_\epsilon^{n+r} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}] \right\}. \quad (2.4.8)$$

We subtract the term $\mathbf{E}[Z^l] \mathbf{E}[X_\epsilon^l]$ (term $(i, r) = (0, l)$) and divide by $1 - \mathbf{E}[Z^l]$ (which is positive) to obtain

$$\mathbf{E}[X_\epsilon^l] \geq \frac{\mathbf{E}[Z^l]}{1 - \mathbf{E}[Z^l]} \sum_{\substack{0 \leq r \leq l, 0 \leq i \leq n-1 \\ (i,r) \neq (0,l)}} \binom{l}{r} \left\{ \binom{l+i-1}{i} (-1)^i \epsilon^{2i} \mathbf{E}[X_\epsilon^{i+r}] \right. \\ \left. + C(-1)^n \epsilon^{2n} \mathbf{E}[X_\epsilon^{n+r} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}] \right\}. \quad (2.4.9)$$

We use the induction hypothesis on $\mathbf{E}[X_\epsilon^{i+r}]$ (induction hypothesis (2.4.6) with $j = i+r$ and $m = n-i$), that is

$$\mathbf{E}[X_\epsilon^{i+r}] - \sum_{k=0}^{n-i-1} \epsilon^{2k} (-1)^k g_{i+r,k} \geq \tilde{C} (-1)^{n-i} \epsilon^{2(n-i)} \mathbf{E}[X_\epsilon^{r+n} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}], \quad (2.4.10)$$

if $n-i$ is even, and the opposite if it is odd. In any case, injecting these lower bounds into (2.4.9) yields

$$\mathbf{E}[X_\epsilon^l] \geq \frac{\mathbf{E}[Z^l]}{1 - \mathbf{E}[Z^l]} \sum_{\substack{0 \leq r \leq l, 0 \leq i \leq n-1 \\ (i,r) \neq (0,l)}} \binom{l}{r} \left\{ \binom{l+i-1}{i} \times \right. \\ \left. \times \left((-1)^i \epsilon^{2i} \sum_{k=0}^{n-i-1} \epsilon^{2k} (-1)^k g_{i+r,k} + \tilde{C} (-1)^n \epsilon^{2n} \mathbf{E}[X_\epsilon^{n+r} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}] \right) \right. \\ \left. + C(-1)^n \epsilon^{2n} \mathbf{E}[X_\epsilon^{n+r} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}] \right\}. \quad (2.4.11)$$

The first terms corresponds to the regular part already found in Lemma 2.3.3 equations (2.3.22) and (2.3.23); the second line contains the ϵ^{2n} -terms which we want to bound from below:

$$\mathbf{E}[X_\epsilon^l] \geq \sum_{s=0}^{n-1} \epsilon^{2s} (-1)^k g_{l,s} + (-1)^n \epsilon^{2n} Q_n, \quad (2.4.12)$$

with

$$Q_n = \frac{\mathbf{E}[Z^l]}{1 - \mathbf{E}[Z^l]} \sum_{\substack{0 \leq r \leq l, 0 \leq i \leq n-1 \\ (i,r) \neq (0,l)}} \binom{l}{r} \left\{ \binom{l+i-1}{i} \tilde{C} + C \right\} \mathbf{E}[X_\epsilon^{n+r} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}]. \quad (2.4.13)$$

Since all the terms in Q_n are non-negative, it is larger than any of them

$$Q_n \geq \frac{\mathbf{E}[Z^l]}{1 - \mathbf{E}[Z^l]} C \mathbf{E}[X_\epsilon^{n+l} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}] \geq \tilde{C} \mathbf{E}[X_\epsilon^{n+l} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}]. \quad (2.4.14)$$

This concludes the proof of the inductive step and thus the proof of the lemma. \square

Proof of Proposition 2.4.1. Let $c' = c'(B, K) > 0$ be such that for all $x \geq 0$,

$$\log(1+x) \geq \sum_{l=1}^K \frac{(-1)^{l+1} x^l}{l} + c' (-1)^{K+2} x^{K+1} \mathbf{1}_{\{x \leq B\}} \quad (2.4.15)$$

if K is even; and the same with an inequality in the opposite direction if K is odd. For the sake of simplicity we suppose that K is even in what follows. Writing $\mathcal{L}(\epsilon) = \mathbf{E} \log(1 + \epsilon^2 X_\epsilon)$, we get

$$\mathcal{L}(\epsilon) \geq \sum_{l=1}^K \frac{(-1)^{l+1}}{l} \epsilon^{2l} \mathbf{E}[X_\epsilon^l] + c' (-1)^{K+2} \epsilon^{2(K+1)} \mathbf{E}[X_\epsilon^{K+1} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}], \quad (2.4.16)$$

and Lemma 2.4.2 provides a lower bound for each term in the sum: for every $1 \leq l \leq K$,

$$\begin{aligned} (-1)^{l+1} \epsilon^{2l} \mathbf{E}[X_\epsilon^l] &\geq (-1)^{l+1} \epsilon^{2l} \sum_{k=0}^{K-l} \epsilon^{2k} (-1)^k g_{l,k} \\ &\quad + c (-1)^{l+1} (-1)^{K+1-l} \epsilon^{2(K+1)} \mathbf{E}[X_\epsilon^{K+1} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}]. \end{aligned} \quad (2.4.17)$$

The conclusion results from the latter two inequalities. \square

2.5 Limiting Behaviour of X_ϵ 's divergent moments

First note that Theorem 2.1.8 is an immediate consequence of Propositions 2.3.1 and 2.4.1. The goal of this section is to obtain estimates of the error $R_K(\epsilon)$, for which we now have

$$c \epsilon^{2(K+1)} \mathbf{E}[X_\epsilon^{K+1} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}] \leq R_K(\epsilon) \leq C_\beta \epsilon^{2\beta} \mathbf{E}[X_\epsilon^\beta]. \quad (2.5.1)$$

In order to give explicit estimates of $R_K(\epsilon)$ in terms of powers of ϵ , one needs to understand the limiting behaviour of X_ϵ 's moments (or truncated moments). The issue was partially addressed by Corollary 2.2.8, which pinpointed the regimes of convergence or divergence

of these moments. Namely $\mathbf{E}[X_\epsilon^\gamma]$ is bounded as ϵ goes to 0 if $\mathbf{E}[Z^\gamma] < 1$ and diverges if $\mathbf{E}[Z^\gamma] \geq 1$. In the following section we address the issue of the divergence speed when $\mathbf{E}[Z^\gamma] \geq 1$.

The first paragraph, based on renewal theory results, describing the heavy tail of X_0 , will provide upper bounds for X_ϵ 's divergent moments. The second paragraph will give lower bounds for these moments under the restriction that Z is bounded.

2.5.1 Upper Bounds

We will need the following result, which combine results by H. Kesten and A. K. Grincevičius depending if $\log Z$ has an arithmetic support or not (see [71, Theorems 1, 3] for a review).

Lemma 2.5.1. *If $\mathbf{E}[Z^\alpha \log_+ Z] < +\infty$, then, as x goes to $+\infty$,*

$$\mathbf{P}(X_0 \geq x) = \mathcal{O}(x^{-\alpha}). \quad (2.5.2)$$

It readily gives the next two results. They provide explicit upper bounds for the speed of divergence of X_ϵ 's moments. If you believe Conjecture 2.1.2, these upper bounds (except the first one when $\mathbf{E}[Z^\alpha] < 1$) are of the good order of ϵ . The first one will be used for $\alpha \in \{1, 2, \dots\}$ whereas the second will be needed when α is not an integer.

Lemma 2.5.2. *If $\mathbf{E}[Z^\alpha \log_+ Z] < +\infty$, then, as ϵ goes to 0,*

$$\mathbf{E}[X_\epsilon^\alpha] = \mathcal{O}(\log(1/\epsilon)). \quad (2.5.3)$$

Proof. The identity $X_\epsilon \stackrel{(d)}{=} Z \frac{1+X_\epsilon}{1+\epsilon^2 X_\epsilon}$ yields, for $\gamma \geq 0$,

$$\begin{aligned} \mathbf{E}[X_\epsilon^\gamma] &= \mathbf{E}[Z^\gamma] \mathbf{E} \left[\left(\frac{1+X_\epsilon}{1+\epsilon^2 X_\epsilon} \right)^\gamma \right] \leq \mathbf{E}[Z^\gamma] \mathbf{E} [((1+X_\epsilon) \wedge \epsilon^{-2})^\gamma] \\ &\leq \mathbf{E}[Z^\gamma] \mathbf{E} [((1+X_0) \wedge \epsilon^{-2})^\gamma]. \end{aligned} \quad (2.5.4)$$

It can be rewritten

$$\mathbf{E}[X_\epsilon^\gamma] \leq \mathbf{E}[Z^\gamma] \left(\gamma \int_0^{\epsilon^{-2}} x^{\gamma-1} \mathbf{P}(X_0 > x-1) dx + \epsilon^{-2\gamma} \mathbf{P}(X_0 \geq \epsilon^{-2}-1) \right). \quad (2.5.5)$$

With $\gamma = \alpha$, Lemma 2.5.1 gives upper bounds for these two terms:

$$\epsilon^{-2\alpha} \mathbf{P}(X_0 \geq \epsilon^{-2}-1) = \mathcal{O}(1) \quad (2.5.6)$$

and

$$\int_0^{\epsilon^{-2}} x^{\alpha-1} \mathbf{P}(X_0 > x-1) dx = \mathcal{O}(\log(1/\epsilon)). \quad (2.5.7)$$

□

Lemma 2.5.3. *Fix $\gamma > \alpha$ and assume that $\mathbf{E}[Z^\gamma]$ is finite. Then, as ϵ goes to 0,*

$$\mathbf{E}[X_\epsilon^\gamma] = \mathcal{O}(\epsilon^{2\alpha-2\gamma}). \quad (2.5.8)$$

Proof. We reuse inequality (2.5.5). Lemma 2.5.1, which applies here, yields

$$\epsilon^{-2\gamma} \mathbf{P}(X_0 \geq \epsilon^{-2} - 1) = \mathcal{O}(\epsilon^{2\alpha-2\gamma}) \quad (2.5.9)$$

and

$$\int_0^{\epsilon^{-2}} x^{\gamma-1} \mathbf{P}(X_0 > x - 1) dx = \mathcal{O}(\epsilon^{2\alpha-2\gamma}). \quad (2.5.10)$$

□

Remark 2.5.4. This remark provides a short explanation for the claims of Remark 2.1.7, items 4 and 5. In both case, only the upper bound on $R(\epsilon)$ requires a specific proof. Recall what was mentioned in Remark 2.3.2: the regular part is derived up to and including order $\epsilon^{2(\lceil\alpha\rceil-1)}$ and the upper bound $\epsilon^\beta \mathbf{E}[X_\epsilon^\beta]$ can be replaced by $\mathbf{E}[(\epsilon^2 X_\epsilon)^\alpha \wedge (\epsilon^2 X_\epsilon)^{\lceil\alpha\rceil}]$. We only need to prove that the upper bound is $\mathcal{O}(\epsilon^{2\alpha})$ for Remark 2.1.7 item 4, and $o(\epsilon^{2\alpha})$ for the item 5.

- For the item 4, note that when $\mathbf{E}[Z^\alpha \log_+ Z] < +\infty$, Lemma 2.5.1, and the techniques used for Lemmas 2.5.2 and 2.5.3, yield $\mathbf{E}[(\epsilon^2 X_\epsilon)^\alpha \wedge (\epsilon^2 X_\epsilon)^{\lceil\alpha\rceil}] = \mathcal{O}(\epsilon^{2\alpha})$.
- For the item 5, an alternative version of Lemma 2.5.1 should be used: when $\mathbf{E}[Z^\alpha] < 1$, and under some extra technical assumptions, the estimate $\mathbf{P}(X_0 \geq x) = o(x^{-\alpha})$ holds (see [70, Theorem 1.3] and [71, Theorem 8]). We deduce $\mathbf{E}[(\epsilon^2 X_\epsilon)^\alpha \wedge (\epsilon^2 X_\epsilon)^{\lceil\alpha\rceil}] = o(\epsilon^{2\alpha})$.

2.5.2 Lower Bounds when Z is bounded

We start with a quite general, albeit quite complex, lower bound for X_ϵ 's moments.

Lemma 2.5.5. Fix $\gamma \geq 1$, $C > 0$, $B > 1$ and $N \in \mathbf{N}$ and set $\tau = \frac{\gamma}{B-1} \left(\frac{B}{B-1} + C \right)$. One has

$$\mathbf{E}[X_\epsilon^\gamma] \geq \sum_{k=1}^N \mathbf{E}[Z^\gamma \mathbf{1}_{\{Z \leq B\}}]^k \exp(-\tau \epsilon^2 B^k) \mathbf{P}(X_\epsilon \leq C). \quad (2.5.11)$$

Proof. Let $X_\epsilon^{(N)}$ be a copy of X_ϵ and (Z_k) be iid copies of Z , independent of $X_\epsilon^{(N)}$. Define recursively, for $0 \leq k \leq N-1$,

$$X_\epsilon^{(k)} = Z_{k+1} \frac{1 + X_\epsilon^{(k+1)}}{1 + \epsilon^2 X_\epsilon^{(k+1)}}. \quad (2.5.12)$$

For every $k \leq N$, one has $X_\epsilon^{(k)} \stackrel{(d)}{=} X_\epsilon$. On the other hand, one can derive the following lower bounds:

$$(X_\epsilon^{(0)})^\gamma = Z_1^\gamma \frac{(1 + X_\epsilon^{(1)})^\gamma}{(1 + \epsilon^2 X_\epsilon^{(1)})^\gamma} \geq \frac{Z_1^\gamma}{(1 + \epsilon^2 X_\epsilon^{(1)})^\gamma} + \frac{Z_1^\gamma}{(1 + \epsilon^2 X_\epsilon^{(1)})^\gamma} (X_\epsilon^{(1)})^\gamma. \quad (2.5.13)$$

Here the condition $\gamma \geq 1$ is used through the convexity inequality $(1+x)^\gamma \geq 1+x^\gamma$. Then, inductively,

$$\begin{aligned} (X_\epsilon^{(0)})^\gamma &\geq \frac{Z_1^\gamma}{(1 + \epsilon^2 X_\epsilon^{(1)})^\gamma} + \frac{Z_1^\gamma}{(1 + \epsilon^2 X_\epsilon^{(1)})^\gamma} \frac{Z_2^\gamma}{(1 + \epsilon^2 X_\epsilon^{(2)})^\gamma} \\ &\quad + \dots + \prod_{j=1}^N \frac{Z_j^\gamma}{(1 + \epsilon^2 X_\epsilon^{(j)})^\gamma}. \end{aligned} \quad (2.5.14)$$

By taking the expectation we get

$$\mathbf{E}[X_\epsilon^\gamma] \geq \sum_{k=1}^N \mathbf{E} \left[\prod_{j=1}^k \frac{Z_j^\gamma}{(1 + \epsilon^2 X_\epsilon^{(j)})^\gamma} \right]. \quad (2.5.15)$$

If $X_\epsilon^{(k)} \leq C$ and $Z_k \leq B$, then, with definition (2.5.12), $X_\epsilon^{(k-1)} \leq B(1+C)$. So, inductively, if $X_\epsilon^{(k)} \leq C$, and $Z_0, \dots, Z_k \leq B$, then, for every $j \leq k$,

$$X_\epsilon^{(k-j)} \leq \sum_{i=1}^j B^i + B^j C \leq B^j \left(\frac{B}{B-1} + C \right) = B^j \sigma, \quad (2.5.16)$$

with $\sigma = \frac{B}{B-1} + C$. Thus,

$$\prod_{j=1}^k \frac{Z_j^\gamma}{(1 + \epsilon^2 X_\epsilon^{(j)})^\gamma} \geq \left[\prod_{j=1}^k \frac{Z_j^\gamma \mathbf{1}_{\{Z_j \leq B\}}}{(1 + \sigma \epsilon^2 B^{k-j})^\gamma} \right] \mathbf{1}_{\{X_\epsilon^{(k)} \leq C\}}. \quad (2.5.17)$$

We compute

$$\begin{aligned} \prod_{j=1}^k \frac{1}{(1 + \sigma \epsilon^2 B^{k-j})^\gamma} &\geq \exp \left(-\sigma \gamma \epsilon^2 \sum_{j=1}^k B^{k-j} \right) \\ &\geq \exp \left(-\frac{\sigma \gamma}{B-1} \epsilon^2 B^k \right) = \exp \left(-\tau \epsilon^2 B^k \right). \end{aligned} \quad (2.5.18)$$

Taking the expectation in (2.5.17) and using that $X_\epsilon^{(k)}$ and Z_1, \dots, Z_k are independent, we obtain

$$\mathbf{E} \left[\prod_{j=1}^k \frac{Z_j^\gamma}{(1 + \epsilon^2 X_\epsilon^{(j)})^\gamma} \right] \geq \mathbf{E} [Z^\gamma \mathbf{1}_{\{Z \leq B\}}]^k \exp \left(-\tau \epsilon^2 B^k \right) \mathbf{P}(X_\epsilon \leq C). \quad (2.5.19)$$

The conclusion follows by injecting this lower bound into (2.5.15). \square

One could expect to use this general lower bound for any given Z . However it only gives satisfactory results when Z is bounded. In that case we can get rid of the indicator $\mathbf{1}_{\{Z \leq B\}}$ in (2.5.11).

Lemma 2.5.6. *If Z has a bounded support then $X_\epsilon \leq \epsilon^{-2} \|Z\|_{L^\infty}$ almost surely.*

Proof. It is an immediate consequence of the invariance identity $X_\epsilon \stackrel{(d)}{=} Z \frac{1+X_\epsilon}{1+\epsilon^2 X_\epsilon}$ and of the inequality $\frac{1+x}{1+\epsilon^2 x} \leq \epsilon^{-2}$, which holds for every $x \geq 0$. \square

Lemma 2.5.6 justifies that we only study X_ϵ 's moments instead of its truncated moments: as long as B is chosen larger than $\|Z\|_{L^\infty}$ one has

$$\mathbf{E} [X_\epsilon^{K+1} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}] = \mathbf{E} [X_\epsilon^{K+1}]. \quad (2.5.20)$$

In the next two lemmas we give a lower bound for X_ϵ 's moments when Z is bounded. In that instance, note that $\mathbf{E}[Z^\alpha] = 1$: the set \mathcal{A} cannot take the form $\mathcal{A} = (0, \alpha]$. Lemma 2.5.7 will be used if α is an integer, and Lemma 2.5.9 when α is not an integer. However, both of them hold true regardless of the nature of α .

Lemma 2.5.7. *For $\alpha \geq 1$, if Z has a bounded support then, for some $c > 0$, and ϵ sufficiently small,*

$$\mathbf{E}[X_\epsilon^\alpha] \geq c \log(1/\epsilon). \quad (2.5.21)$$

Proof. Recall that since Z is bounded, $\mathbf{E}[Z^\alpha] = 1$. Choose $\gamma = \alpha$ and $B = \|Z\|_\infty$ in Lemma 2.5.5 to get

$$\begin{aligned} \mathbf{E}[X_\epsilon^\alpha] &\geq \sum_{k=1}^N \mathbf{E}[Z^\alpha]^k \exp(-\tau \epsilon^2 B^k) \mathbf{P}(X_\epsilon \leq C) \\ &\geq N \exp(-\tau \epsilon^2 B^N) \mathbf{P}(X_\epsilon \leq C). \end{aligned} \quad (2.5.22)$$

First note that, thanks to Lemma 2.2.7,

$$\mathbf{P}(X_\epsilon \leq C) \rightarrow \mathbf{P}(X_0 \leq C), \quad (2.5.23)$$

which is positive if C is large enough. Choosing $N = N_\epsilon = \lfloor 2 \frac{1}{\log B} \log \frac{1}{\epsilon} \rfloor$, we obtain

$$\mathbf{E}[X_\epsilon^\alpha] \geq N_\epsilon \exp(-\tau) \mathbf{P}(X_\epsilon \leq C) \geq c \log(1/\epsilon). \quad (2.5.24)$$

□

Remark 2.5.8. If Z is not bounded but $\mathbf{E}[Z^\kappa] < +\infty$ for some $\kappa > \alpha$ then, with another choice of B_ϵ and N_ϵ , one can get the slightly weaker lower bound $\mathbf{E}[X_\epsilon^\alpha] \geq c \frac{\log(1/\epsilon)}{\log \log(1/\epsilon)}$.

Lemma 2.5.9. *If Z is bounded, and if $\gamma \geq 1$ is such that $\mathbf{E}[Z^\gamma] > 1$, then, for some $c > 0$, and for ϵ sufficiently small,*

$$\mathbf{E}[X_\epsilon^\gamma] \geq c \epsilon^{-2\eta}, \quad \text{where} \quad \eta = \frac{\log \mathbf{E}[Z^\gamma]}{\log \|Z\|_\infty} \in (0, \gamma - \alpha). \quad (2.5.25)$$

Proof. Set $B = \|Z\|_{L^\infty}$ in Lemma 2.5.5 to get

$$\begin{aligned} \mathbf{E}[X_\epsilon^\gamma] &\geq \sum_{k=1}^N \mathbf{E}[Z^\gamma]^k \exp(-\tau \epsilon^2 B^k) \mathbf{P}(X_\epsilon \leq C) \\ &\geq \mathbf{E}[Z^\gamma]^N \exp(-\tau \epsilon^2 B^N) \mathbf{P}(X_\epsilon \leq C). \end{aligned} \quad (2.5.26)$$

Choosing again $N_\epsilon = \lfloor 2 \frac{1}{\log B} \log \frac{1}{\epsilon} \rfloor$, we obtain

$$\mathbf{E}[X_\epsilon^\gamma] \geq \mathbf{E}[Z^\gamma]^{N_\epsilon} \exp(-\tau) \mathbf{P}(X_\epsilon \leq C) \geq c \epsilon^{-2\eta}. \quad (2.5.27)$$

□

2.5.3 Proof of Theorem 2.1.6

Proof. We recall here the upper and lower bounds provided by Theorem 2.1.8:

$$c \epsilon^{2(K+1)} \mathbf{E}[X_\epsilon^{K+1} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}] \leq R_K(\epsilon) \leq C_\beta \epsilon^{2\beta} \mathbf{E}[X_\epsilon^\beta], \quad (2.5.28)$$

If $\alpha = +\infty$, then, thanks to Lemma 2.2.3

$$R_K(\epsilon) \leq C_{K+1} \epsilon^{2(K+1)} \mathbf{E}[X_\epsilon^{K+1}] \leq \epsilon^{2(K+1)} \mathbf{E}[X_0^{K+1}]. \quad (2.5.29)$$

Since $\mathbf{E}[X_0^{K+1}]$ is finite (Lemma 2.2.2), the result (2.1.11) follows.

From now on we suppose that α is finite and $\mathbf{E}[Z^\alpha] = 1$ and we set $K = \lceil \alpha \rceil - 1$. By Corollary 2.2.8,

$$R(\epsilon) \geq c\epsilon^{2(K+1)} \mathbf{E}[X_\epsilon^{K+1} \mathbf{1}_{\{\epsilon^2 X_\epsilon \leq B\}}] \gg \epsilon^{2(K+1)}. \quad (2.5.30)$$

If α is an integer then the lower and upper bounds given by (2.1.13) or (2.1.14) follow from Lemmas 2.5.2 and 2.5.7. If α is not an integer then the lower and upper bounds (2.1.16) given by Theorem 2.1.6 are a consequence of Lemmas 2.5.3 (with γ such that $\mathbf{E}[Z^\gamma] < +\infty$) and 2.5.9 (with $\gamma = K + 1$). \square

2.A Appendix: Generalization to higher dimension

The techniques developed in the previous sections are sufficiently robust to be used in more general settings. We apply them to a square matrix of size $d + 1$ which is a perturbation of a matrix alike $\text{Diag}(1, Z)$, which still have a preferred direction. Since the proofs are only slightly different from the previous sections, they will be only sketched in this appendix. We will just point out the arguments that must be adapted and many details will be omitted.

We now consider the $(d + 1) \times (d + 1)$ matrix

$$M_\epsilon = \begin{pmatrix} 1 & \epsilon L_\epsilon \\ \epsilon C_\epsilon & N_\epsilon \end{pmatrix}, \quad (2.A.1)$$

where L_ϵ and C_ϵ are random vectors of size d , and N_ϵ is a random matrix, of size $d \times d$. We are still interested in the Lyapunov exponent, defined by the limit

$$\mathcal{L}(\epsilon) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|M_{n,\epsilon} \cdots M_{1,\epsilon}\|, \quad (2.A.2)$$

where $(M_{k,\epsilon})_{k \geq 1}$ are iid copies of M_ϵ . This limit exists almost surely and is deterministic (see again [46]) as soon as for every $\epsilon > 0$, $\mathbf{E}[\log_+ \|M_\epsilon\|] < +\infty$.

We derive in this section a regular expansion for $\mathcal{L}(\epsilon)$, alike the expansion provided by Proposition 2.3.1 in the previous setting. However, no lower bound on the error will be given here. We start by deriving a formula alike “ $\mathcal{L}(\epsilon) = \mathbf{E}[\log(1 + \epsilon^2 X_\epsilon)]$ ” (Lemma 2.A.3).

In the whole section $\|\cdot\|$ will denote a given norm on \mathbf{R}^d or \mathbf{R}^{d+1} , as well as the induced operator norm on $\mathcal{M}_d(\mathbf{R})$ or $\mathcal{M}_{d+1}(\mathbf{R})$. On another note, if $x, y \in \mathbf{R}^d$, we will write $x \leq y$ if the inequality holds coordinatewise. Similarly the stochastic dominance \preceq will be extended to random vectors: $X \preceq Y$ means that there exists a copy \tilde{X} of X and a copy \tilde{Y} of Y satisfying $\tilde{X} \leq \tilde{Y}$ almost surely (coordinatewise).

Let’s introduce the assumptions under which we will work in the section. Observe that under these assumptions, the condition $\mathbf{E}[\log_+ \|M_\epsilon\|] < +\infty$ is fulfilled so the Lyapunov exponent is well defined.

Assumptions 2.A.1. We assume that the following holds, for every $\epsilon \in (0, \epsilon_0)$.

- (a) The random matrix M_ϵ has non-negative entries. And, almost surely, there exists $N \geq 1$ such that the product $M_{N,\epsilon} \cdots M_{1,\epsilon}$ has positive entries.

(b) There exists $\delta_\epsilon > 0$ such that $\mathbf{E}[\|N_\epsilon\|^{\delta_\epsilon}] < 1$ and $\mathbf{E}[\|C_\epsilon\|^{\delta_\epsilon}] < +\infty$.

(c) $\mathbf{E}[\log_+ \|L_\epsilon\|] < +\infty$.

Before deriving the formula for the Lyapunov exponent, we introduce the random vector Y_ϵ , which will play the same role as X_0 in our new setting (except that here it will depend on ϵ). Namely it will be used through stochastic dominances.

Lemma 2.A.2. *Fix $\epsilon \in (0, \epsilon_0)$ and let $(N_{\epsilon,k}, C_{\epsilon,k})$ be iid copies of (N_ϵ, C_ϵ) . The series*

$$Y_\epsilon = \sum_{n=0}^{+\infty} N_{\epsilon,1} \dots N_{\epsilon,n-1} C_{\epsilon,n} \quad (2.A.3)$$

converges almost surely. Moreover $\mathbf{E}[\log_+ \|Y_\epsilon\|]$ is finite. If, in addition,

$$\limsup_{\epsilon \rightarrow 0} \mathbf{E}[\|N_\epsilon\|^\beta] < 1 \quad \text{and} \quad \limsup_{\epsilon \rightarrow 0} \mathbf{E}[\|C_\epsilon\|^\beta] < +\infty, \quad (2.A.4)$$

then $\mathbf{E}[\|Y_\epsilon\|^\beta] = \mathcal{O}(1)$ as ϵ goes to 0.

Proof. Since all the entries of M_ϵ are non-negative, the sum (2.A.3) is always defined. A priori, some of its entries could be $+\infty$. Denote by Y_ϵ the random vector defined by this infinite sum. Using Minkowski's inequality or another convexity inequality as for Lemma 2.2.2, one proves, under Assumption 2.A.1 (b), that $\mathbf{E}[\|Y_\epsilon\|^{\delta_\epsilon}]$ is finite. So Y_ϵ 's entries are almost surely finite. With the same technique, we prove the rest of the lemma. \square

The next lemma provides the desired formula for $\mathcal{L}(\epsilon)$.

Lemma 2.A.3. *There exists a random vector $X_\epsilon \in \mathbf{R}^d$, with non-negative entries, satisfying*

$$\begin{pmatrix} 1 \\ \epsilon X_\epsilon \end{pmatrix} \stackrel{(d)}{=} \begin{pmatrix} 1 & \epsilon L_\epsilon \\ \epsilon C_\epsilon & N_\epsilon \end{pmatrix} \begin{pmatrix} 1 \\ \epsilon X_\epsilon \end{pmatrix} \quad \text{in the projective space } \mathbf{P}^d(\mathbf{R}), \quad (2.A.5)$$

or equivalently,

$$X_\epsilon \stackrel{(d)}{=} \frac{C_\epsilon + N_\epsilon X_\epsilon}{1 + \epsilon^2 L_\epsilon X_\epsilon}, \quad (2.A.6)$$

where C_ϵ, N_ϵ and L_ϵ are the blocks of the random matrix M_ϵ , independent of X_ϵ . One has $X_\epsilon \preceq Y_\epsilon$. Moreover the Lyapunov exponent can be written as

$$\mathcal{L}(\epsilon) = \mathbf{E}[\log(1 + \epsilon^2 L_\epsilon X_\epsilon)]. \quad (2.A.7)$$

And for every $\mathbf{x}, \mathbf{y} \in \mathbf{R}_+^{d+1}$,

$$\mathcal{L}(\epsilon) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \langle \mathbf{x}, M_{n,\epsilon} \dots M_{1,\epsilon} \mathbf{y} \rangle. \quad (2.A.8)$$

Proof. The method is the same as in Lemma 2.2.3's proof for 2×2 matrices. We fix iid copies $(M_{\epsilon,n})$ of M_ϵ and set $x_0 = \mathbf{0}_{\mathbf{R}^d}$. Then define inductively, for $n \geq 0$, the random variables

$$x_{n+1} = \frac{C_{\epsilon,n} + N_{\epsilon,n} x_n}{1 + \epsilon^2 L_{\epsilon,n} x_n}. \quad (2.A.9)$$

Observe that since all the vectors have non-negative entries, one can write, coordinatewise,

$$x_{n+1} \leq C_{\epsilon,n} + N_{\epsilon,n}x_n. \quad (2.A.10)$$

So, by an easy induction, $x_n \preceq Y_\epsilon$ for every $n \geq 0$. The end of the proof is the same as for Lemma 2.2.3. We do not reiterate all the details here. Just note that we do not claim the uniqueness of a non-negative solution to (2.A.6) and that Assumption 2.A.1 (a) is a sufficient condition for H. Hennion's result to apply. \square

To state our main result, and more precisely to formulate its premises, some multi-index notations will be required, which we set in the next lines. The norm of a multi-index $\lambda \in \mathbf{N}^d$ will be denoted by $|\lambda|$:

$$|\lambda| := \lambda_1 + \dots + \lambda_d. \quad (2.A.11)$$

For every $l \geq 0$, there are $\binom{l+d-1}{d-1}$ multi-indices with norm l : it is the number of (weak) compositions of l into d non-negative integers. For a vector $x \in \mathbf{R}^d$ and a multi-index $\lambda \in \mathbf{N}^d$, we define the multi-index power

$$x^\lambda = x_1^{\lambda_1} \times \dots \times x_d^{\lambda_d}. \quad (2.A.12)$$

Similarly, for a matrix $A \in \mathcal{M}_d(\mathbf{R})$ and a multi-index $\omega \in \mathbf{N}^{d^2} \simeq \mathcal{M}_d(\mathbf{N})$, define

$$A^\omega = \prod_{i,j} (A_{i,j})^{\omega_{i,j}} \quad \text{and} \quad |\omega| = \sum_{i,j} \omega_{i,j}. \quad (2.A.13)$$

There should be no confusion with a standard matrix power since ω is a multi-index.

For $l \geq 0$, consider the square matrix $G^{(l)}$ with size $\binom{l+d-1}{d-1}$, whose elements are

$$G_{\lambda,\lambda'}^{(l)} = \sum_{\substack{\omega \in \mathbf{N}^{d^2} \\ \sum_j \omega_{i,j} = \lambda_i \\ \sum_i \omega_{i,j} = \lambda'_j}} \lim_{\epsilon \rightarrow 0} \mathbf{E}[N_\epsilon^\omega], \quad \text{for } \lambda, \lambda' \in \mathbf{N}^d \text{ such that } |\lambda| = |\lambda'| = l. \quad (2.A.14)$$

Note that all the multi-indices ω in the sum have norm $|\omega| = l$. The matrix $G^{(l)}$ will play a similar role as $\mathbf{E}[Z^l]$ in this generalized context. Of course these matrices, which require the existence of $\lim_{\epsilon \rightarrow 0} \mathbf{E}[N_\epsilon^\omega]$, are not always defined.

We have set enough notations to state the generalization of Proposition 2.3.1, giving a regular expansion of the Lyapunov exponent $\mathcal{L}(\epsilon)$.

Proposition 2.A.4. *Fix $K \geq 0$ and $\beta \in (K, K+1]$. Suppose that*

1. *For all multi-indices $\lambda, \mu \in \mathbf{N}^d$, $\omega \in \mathbf{N}^{d^2}$ such that $l = |\lambda| + |\mu| + |\omega| \leq K$, $\mathbf{E}[L_\epsilon^\lambda C_\epsilon^\mu N_\epsilon^\omega]$ is finite and admits a regular expansion, as ϵ goes to 0, up to the order $2(K-l)$:*

$$\mathbf{E}[L_\epsilon^\lambda C_\epsilon^\mu N_\epsilon^\omega] = \sum_{r=0}^{2(K-l)} c_{\lambda,\mu,\omega,r} \epsilon^r + \mathcal{O}(\epsilon^{2(\beta-l)}); \quad (2.A.15)$$

2. *For all $1 \leq l \leq K$, the matrix $I - G^{(l)}$ is invertible;*
3. *$\limsup_{\epsilon \rightarrow 0} \mathbf{E}[\|L_\epsilon\|^\beta]$ is finite.*

Then there exist real coefficients q_2, \dots, q_{2K} such that, as ϵ goes to 0,

$$\mathcal{L}(\epsilon) = \sum_{k=2}^{2K} q_k \epsilon^k + \mathcal{O}(\epsilon^{2\beta} \mathbf{E}[1 + \|X_\epsilon\|^\beta]). \quad (2.A.16)$$

Remark 2.A.5. For Proposition 2.A.4 to be usable, one needs to control $\mathbf{E}[\|X_\epsilon\|^\beta]$. With Lemmas 2.A.2 and 2.A.3, one has $\mathbf{E}[\|X_\epsilon\|^\beta] = \mathcal{O}(1)$ as ϵ goes to 0 as soon as (2.A.4) holds.

Remark 2.A.6. One could be surprised that the upper bound involves $\mathbf{E}[1 + \|X_\epsilon\|^\beta]$ instead of $\mathbf{E}[\|X_\epsilon\|^\beta]$. Such a caution was not necessary in the previous context since the latter was bounded from below as ϵ goes to 0. Here, a priori, it could happen that $\mathbf{E}[\|X_\epsilon\|^\beta]$ vanishes as ϵ goes to 0.

Remark 2.A.7. The existence of $G^{(l)}$, for $l \leq K$, is ensured by the assumption (2.A.15), which gives $\lim_{\epsilon \rightarrow 0} \mathbf{E}[N_\epsilon^\omega] = c_{\mathbf{0}, \mathbf{0}, \omega, 0}$. The invertibility of $I - G^{(l)}$ is the counterpart of the assumption “ $\mathbf{E}[Z^l] < 1$ ” in Proposition 2.3.1.

Proof. The same proof as for Proposition 2.3.1 works: one expands the logarithm:

$$\begin{aligned} \mathbf{E}[\log(1 + \epsilon^2 L_\epsilon X_\epsilon)] &= \sum_{k=0}^K \frac{(-1)^{k+1}}{k} \epsilon^{2k} \mathbf{E}[(L_\epsilon X_\epsilon)^k] + \mathcal{O}(\epsilon^{2\beta} \mathbf{E}[(L_\epsilon X_\epsilon)^\beta]) \\ &= \sum_{k=0}^K \frac{(-1)^{k+1}}{k} \epsilon^{2k} \sum_{1 \leq r_1, \dots, r_k \leq d} \mathbf{E} \left[\prod_{i=1}^k (L_\epsilon)_{r_i} \right] \mathbf{E} \left[\prod_{i=1}^k (X_\epsilon)_{r_i} \right] \\ &\quad + \mathcal{O}(\epsilon^{2\beta} \mathbf{E}[(L_\epsilon X_\epsilon)^\beta]), \end{aligned} \quad (2.A.17)$$

where $x^{(r)}$ stands for the r^{th} coordinate of x . Note that

$$\mathbf{E}[(L_\epsilon X_\epsilon)^\beta] \leq \mathbf{E}[\|L_\epsilon\|^\beta] \mathbf{E}[\|X_\epsilon\|^\beta] \leq C \mathbf{E}[\|X_\epsilon\|^\beta], \quad (2.A.18)$$

and that for any r_1, \dots, r_k there exists $\lambda \in \mathbf{N}^d$, with norm k such that $\mathbf{E} \left[\prod_{i=1}^k X_\epsilon^{(r_i)} \right] = \mathbf{E}[X_\epsilon^\lambda]$. Thus we need expansions for X_ϵ 's moments. They are given by the next lemma. By substituting the regular expansion (2.A.19), given in Lemma 2.A.8, in the expansion (2.A.17) of $\mathcal{L}(\epsilon)$, the proof of Proposition 2.A.4 will be complete. \square

Lemma 2.A.8. *Under Proposition 2.A.4's premises, for all $l \leq K$, and $\lambda \in \mathbf{N}^d$, such that $|\lambda| = l$, the following expansion holds, for some real coefficients $(g_{\lambda, k})$:*

$$\mathbf{E}[X_\epsilon^\lambda] = \sum_{k=0}^{2(K-l)} \epsilon^k g_{\lambda, k} + \mathcal{O}(\epsilon^{2(\beta-l)} \mathbf{E}[1 + \|X_\epsilon\|^\beta]). \quad (2.A.19)$$

Sketch of proof of Lemma 2.A.8. We can follow the same proof as for Lemma 2.3.3. We go back to that proof to understand how the present one must be adjusted. The only point which merits special attention is the line (2.3.17) where the term $\mathbf{E}[Z^l] \mathbf{E}[X_\epsilon^l]$ is isolated on the left-hand side. That line could be summarized as follow: we wrote

$$\mathbf{E}[X_\epsilon^l] = \mathbf{E}[Z^l] \mathbf{E}[X_\epsilon^l] + (\diamond_l), \quad (2.A.20)$$

where (\diamond_l) stands for all the terms in the expansion of $\mathbf{E}[X_\epsilon^l]$ for which the induction hypothesis provided an expansion up to the required order. To be explicit,

$$(\diamond_l) = \mathbf{E}[Z^l] \sum_{\substack{0 \leq j \leq l, 0 \leq i \leq n \\ (i,j) \neq (0,l)}} \binom{l}{j} \binom{-l}{i} \epsilon^{2i} \mathbf{E}[X_\epsilon^{i+j}] + \mathcal{O}(\epsilon^{2(n+\delta)} \mathbf{E}[X_\epsilon^\beta]). \quad (2.A.21)$$

Then we could conclude by writing

$$\mathbf{E}[X_\epsilon^l] = \frac{1}{1 - \mathbf{E}[Z^l]} (\diamond_l), \quad (2.A.22)$$

and applying the induction hypothesis. That is where was used the condition “ $\mathbf{E}[Z^l] < 1$ ” (actually $\mathbf{E}[Z^l] \neq 1$ was enough), and this is where will be used the invertibility of $1 - G^{(l)}$.

In our generalized setting, we still carry out an induction on $(n, l = |\lambda|)$ (equipped with the lexicographic order). For the inductive step, there are a lot of multi-indices with given norm l . They will be solved simultaneously, by writing a joint system satisfied by all these multi-indices moments $\mathbf{E}[X_\epsilon^\lambda]$ with $|\lambda| = l$. To this end, use the identity

$$\mathbf{E}[X_\epsilon^\lambda] = \mathbf{E} \left[\left(\frac{C_\epsilon + N_\epsilon X_\epsilon}{1 + \epsilon^2 L_\epsilon X_\epsilon} \right)^\lambda \right] = \mathbf{E} \left[\frac{(C_\epsilon + N_\epsilon X_\epsilon)^\lambda}{(1 + \epsilon^2 L_\epsilon X_\epsilon)^l} \right]. \quad (2.A.23)$$

Then develop the denominator

$$\mathbf{E}[X_\epsilon^\lambda] = \mathbf{E} \left[(C_\epsilon + N_\epsilon X_\epsilon)^\lambda \left(\sum_{j=0}^n \binom{-l}{j} \epsilon^{2j} (L_\epsilon X_\epsilon)^j + \epsilon^{2(n+\delta)} \mathcal{O}((L_\epsilon X_\epsilon)^{n+\delta}) \right) \right]. \quad (2.A.24)$$

Eventually, after manipulation, that moment takes the form

$$\mathbf{E}[X_\epsilon^\lambda] = \sum_{\lambda': |\lambda'|=l} G_{\lambda, \lambda'}^{(l)} \mathbf{E}[X_\epsilon^{\lambda'}] + (\diamond_\lambda), \quad (2.A.25)$$

where, again, (\diamond_λ) stands for all the term in the expansion of $\mathbf{E}[X_\epsilon^\lambda]$ for which the induction hypothesis, and the premise (2.A.15) of Proposition 2.A.4, provide an expansion up to the required order. Then, since $I - G^{(l)}$ is invertible, one can solve that joint system satisfied by the family $(\mathbf{E}[X_\epsilon^\lambda])$:

$$\mathbf{E}[X_\epsilon^\lambda] = \left[(I - G^{(l)})^{-1} (\diamond) \right]_\lambda = \sum_{\lambda': |\lambda'|=l} \left((I - G^{(l)})^{-1} \right)_{\lambda, \lambda'} (\diamond_{\lambda'}). \quad (2.A.26)$$

That concludes the proof of the induction step and thus the proof of the lemma. \square

Remark 2.A.9. The same methods as in Section 2.4 can produce the lower bound on the error

$$(-1)^{K+2} R_K(\epsilon) \geq c \epsilon^{2(K+1)} \mathbf{E} \left[(L_\epsilon X_\epsilon)^{K+1} \mathbf{1}_{\{L_\epsilon X_\epsilon \leq B\}} \right] + \mathcal{O}(\epsilon^{2(K+1)}), \quad (2.A.27)$$

as long as (2.A.15) holds with $\beta = K + 1$.

Application to a 1D Ising model. The product of random matrices considered in the first sections appeared in [35] to express the free energy of the nearest-neighbour Ising model on the line with inhomogeneous magnetic field. The generalization considered in this appendix allows finite range interactions to be included. Let us be more precise. Consider the Ising model on $\mathbf{T}_N := \mathbf{Z}/N\mathbf{Z}$, with homogeneous interactions up to the distance d and inhomogeneous magnetic field (h_k) . It is the spin model with configurations* $\underline{\sigma} \in \{0, 1\}^{\mathbf{T}_N}$ whose Hamiltonian is

$$H(\underline{\sigma}) = \sum_{k \in \mathbf{T}_N} \left(h_k \sigma_k + \sum_{l=1}^d \alpha_l \mathbf{1}_{\{\sigma_k \neq \sigma_{k+l}\}} \right). \quad (2.A.28)$$

The magnetic field $(h_k)_{k \in \mathbf{T}_N}$ is supposed to be iid. Thanks to a transfer matrix approach, the free energy in the thermodynamic limit can be expressed through a random matrix products:

$$f(T) = \lim_{N \rightarrow +\infty} \frac{1}{N} \log \text{Tr} \left(\prod_{n=1}^N A_n \right), \quad (2.A.29)$$

where A_n is a $2^d \times 2^d$ sparse matrix (two non-zero entries on each line and each column) whose entries are the following. If $\underline{\tau}, \underline{v} \in \{0, 1\}^d$, which represent the partial configuration $(\sigma_n, \dots, \sigma_{n+d-1})$ and its shift $(\sigma_{n+1}, \dots, \sigma_{n+d})$, then

$$A_n(\underline{\tau}, \underline{v}) = \exp \left(-\frac{1}{T} \tau_1 h_n - \frac{1}{T} \sum_{l=1}^d \alpha_l \tau_l v_l \right) \mathbf{1}_{\{\tau_2=v_1, \dots, \tau_{d-1}=v_d\}}. \quad (2.A.30)$$

One can check that Assumption 2.A.1 (a) holds with $N = d$. Proposition 2.A.4 provides an expansion for the free energy $f(T)$ when the coupling constants α_l tend to be very large. Set $Z_n = \exp(-h_n/T)$ and $\epsilon_l = \exp(-\alpha_l/T)$ for every $l \leq d$. The parameters ϵ_l vanish when the coupling constants α_l tend to be very large. Then A_n is a random perturbation of $\text{Diag}(1, 0, \dots, 0, Z_n)$ if one writes the configurations $\underline{\tau}, \underline{v}$ in lexicographic order. Thus, Proposition 2.A.4 yields

$$f(T) = \sum_{\lambda \in \mathbf{N}^d: |\lambda| < \beta} c_\lambda \epsilon_1^{\lambda_1} \dots \epsilon_d^{\lambda_d} + \mathcal{O} \left(\sum_{l=1}^d \epsilon_l^\beta \right), \quad (2.A.31)$$

as soon as $\mathbf{E}[Z^\beta] < 1$ (note that β is not the inverse temperature here).

Remark 2.A.10. Similarly, the results apply for an Ising model on a strip of finite width s (i.e. $[N] \times [s]$), or a cylinder $([N] \times \mathbf{Z}/s\mathbf{Z})$ with an inhomogeneous magnetic field and finite-range interactions, with free, fixed or periodic boundary conditions.

*We choose $\{0, 1\}^{\mathbf{T}_N}$ instead of $\{-1, 1\}^{\mathbf{T}_N}$ to simplify the formulas. They are equivalent by easy manipulations.

Part II

PINNING MODELS WITH CONSTRAINTS

Chapter 3

An introduction to homogeneous and disordered Poland–Scheraga model

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Effective models for DNA denaturation. Since 1953, after the famous discovery by Francis Crick and James Watson [84], it is known that DNA is a molecule composed of two chains – or DNA strands – that coil around each other to form a double helix. Each chain contains the genetic code used in the development, functioning and for reproduction in all known living organisms (this includes many, but not all, viruses). The two DNA strands are composed of simpler monomeric units called nucleotides. The nucleotides are almost identical and they just differ for one part that is called nucleobase, or simply base. There are four bases: C=cytosine, G=guanine, A=adenine and T=thymine. While the nucleotides within one strand are attached to each other by strong *covalent bonds*, the connection between the two strands relies on the weaker *hydrogen bonds* between bases. The two strands, at least in living beings, are complementary according to the base pairing rules “A with T and C with G”. The two strands of a double stranded DNA are said to be in their *native state* when all bases are paired together. This is what happens notably when the DNA is folded into the very complex structure that fits in the cell nucleus. But DNA needs to open for its two main functions: coding for proteins, and replication.

Coding and replication are extremely complex processes. But since several decades researcher study in detail also the process of DNA denaturation (Figure 3.1), i.e. the unbinding of the two strands, that happens at high temperatures or that happens because a force is applied to the extremities of the strands: this second procedure can be performed thanks to the great progress that have been made in single molecule experiments. A number of models have been therefore developed to analyse these phenomena or procedures.

A very simple model, introduced by D. Poland and H. A. Scheraga [81, 82, 83] in the 60s, has finally turned out to be a reference model, even at a quantitative level. In the Poland–Scheraga (PS) model, the double strand structure is forgotten and double stranded DNA is modeled by a linear sequence of bound pairs and loops (see Figure 3.2 in Section 3.1). The

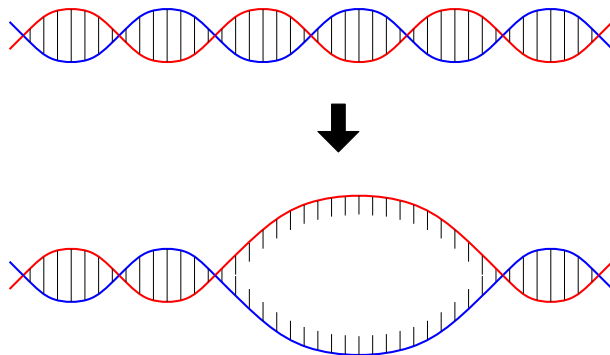


FIGURE 3.1: In the top figure, DNA is in its *native state*: all bases are paired together. In the bottom, a loop opens and DNA is *denatured* at this spot. In practice – and above all approaching denaturation – due to thermal fluctuations, there are several loops even when the two strands are still together. We will talk about denaturation only when the two chains are completely detached, except at some bases at finite distance of the extremities (in fact, we will always impose that the extremities are attached, and the thermal fluctuations at the boundary lead to having some contacts of the two strands, essentially only at a finite distance from the extremities). By passing to the complementary, also the notion of native state that we will use will be somewhat generalized.

loops are the regions in which the two strands are unbound. The parameters that enter the definition of the model are the energetic gain of the bound pairs and the entropic term associated to the fluctuation freedom of the loop sections. We stress that the rich DNA physical structure is reduced, in the PS model, to bond energies and loop energy+entropy term.

Aside for experimental applications, a major atout of PS model is its solvable character when one neglects that the A-T bonds are weaker than G-C bonds. With this simplification, which can viewed as looking at an homogenous version of the model, one obtains an almost explicit formula for the free energy density from which it is rather straightforward to infer the existence of a transition – the denaturation transition – along with the precise value of the critical point and the critical behaviour (Section 3.1). Such a complete solution of the homogeneous model has been crucial in obtaining then results also about the case in which one keeps the inhomogeneous nature of the original model (Section 3.2), that is, in which the A-T hydrogen bonds are weaker than G-C bonds. In physical terms, the sequence of bases can be seen as a *disorder*.

3.1 The Homogeneous Poland-Scheraga Model

In the beginning of the 80s there has been a wave of interest in the physical community for the exactly solvable character of the homogeneous PS model and a number of approaches, motivated both by different physical applications, and by *somewhat different* methods of solution. In 1984 Michael Fisher [43] observed that the solvable nature of the model can be seen in full generality in terms of a rather simple computation. In reality, as it has been pointed out in [57, App. A], *Fisher’s simple computation* was a well-known fact in mathematics from the end of the 40s: the homogeneous PS model is just a rewriting of the basic discrete renewal process. And renewal theory greatly developed in the 50s and 60s, so the full solution of the PS model is just a corollary of known mathematics. Let us present the mathematical formalism that is commonly used for the model. This requires a short introduction on renewal processes.

Discrete renewal processes.

Definition 3.1.1. A process $\tau = (\tau_k)_{k \geq 0}$ is a *discrete renewal process* – its law is denoted by \mathbf{P} – when $\tau_0 = 0$, and the increments $(\eta_k)_{k \geq 1} = (\tau_k - \tau_{k-1})_{k \geq 1}$ are IID random variables taking values in $\mathbf{N}^* \cup \{\infty\} = \{1, 2, \dots\} \cup \{\infty\}$.

We will use the notation $K(n) = \mathbf{P}(\eta_1 = n)$: by definition, $\sum_{n \in \mathbf{N}^*} K(n) + K(\infty) = 1$. Note that if $\eta_k = \infty$ for one k (that is if $K(\infty) > 0$), then $\tau_k = \tau_{k+1} = \dots = \infty$. In this case the renewal is called *transient* or *terminating*. Instead, if $K(\infty) = 0$, then $\tau_k < \infty$ for all k , and the renewal is called *persistent*. The points $(\tau_k)_{k \geq 0}$ will be called *pinned points* or *contact points* and the renewal process will often be seen as a subset τ of the discrete line $\mathbf{N} \cup \{\infty\}$. If the renewal is terminating, then τ is a finite set and it contains the point ∞ , whereas if the renewal is persistent, then τ is an infinite subset of \mathbf{N} .

Moreover, let us recall from now the classical fundamental *Renewal Theorem* formula* (see e.g. [41, § XI.1])

$$\lim_{n \rightarrow +\infty} \mathbf{P}(n \in \tau) = \lim_{n \rightarrow +\infty} \mathbf{P}(\text{there exists } j \text{ such that } \tau_j = n) = \frac{1}{\mathbf{E}[\eta_1]} \in [0, 1], \quad (3.1.1)$$

as soon as the renewal does not live on a sub-lattice of \mathbf{N} (*aperiodicity condition*). Aperiodicity is going to be satisfied in the models we consider and, in any case, (3.1.1) is easily generalized in the periodic case. So, if τ is terminating, we have $\mathbf{E}[\eta_1] = +\infty$ and $\lim_n \mathbf{P}(n \in \tau) = 0$, but it may happen that $\mathbf{E}[\eta_1] = +\infty$ also in the persistent case. This leads to the further classification into *null persistent* (when $K(\infty) = 0$ and $\mathbf{E}[\eta_1] = +\infty$) and *positive persistent* (when $\mathbf{E}[\eta_1] < +\infty$). Note also that in the terminating and in the null persistent case, the Renewal Theorem (3.1.1) is not fully satisfactory (for one would want to know more precisely in which way the renewal function tends to zero): for this we refer to the caption of Figure 3.3. For the nomenclature, $n \mapsto \mathbf{P}(n \in \tau)$ is called *renewal function*.

It is an elementary fact, see e.g. [50, 52], that there is no loss of generality in assuming that $K(\infty) = 0$ for the PS model, but we will quickly see that, nevertheless, the terminating case is going to have a central role in the arguments and in the results. In the physical literature $K(n)$ is often chosen equal to $1/n^c$, up to the normalizing constant, and theoretical arguments are set forth in favor of a value of c around 2.15 for comparison with real experiments [16], but from a theoretical viewpoint any value of $c > 1$ goes. We will make the more general assumption that $K(\cdot)$ is *regularly varying* with index $c = 1 + \alpha \geq 1$: there exists n_0 such that for $n \geq n_0$,

$$K(n) = \frac{\varphi(n)}{n^{1+\alpha}}, \quad (3.1.2)$$

with $\varphi(n)$ a *slowly varying function*:

Definition 3.1.2. A function $\varphi : (0, +\infty) \rightarrow (0, +\infty)$ is called *slowly varying* if it is measurable and if for any $a > 0$,

$$\lim_{x \rightarrow +\infty} \frac{\varphi(ax)}{\varphi(x)} = 1. \quad (3.1.3)$$

We refer to [14] for the many properties of slowly varying functions. We observe that the fact that $K(\cdot)$ is eventually positive implies that the renewal is aperiodic.

*The reader can start appreciating from (3.1.1) the notational advantage of looking at τ as a set and not as a sequence.

A formal definition of the PS model. The PS model, corresponding to a DNA chain of length N (Figure 3.2, is introduced by modifying the distribution of the renewal via an energy reward $h \in \mathbf{R}$ for each contact point before and including N . Explicitly, the new law on the point set τ is given by

$$\nu_{N,h}(A) = \frac{1}{Z_{N,h}} \mathbf{E} \left[e^{h|\tau \cap (0,N]|} \mathbf{1}_A(\tau) \mathbf{1}_{N \in \tau} \right], \quad (3.1.4)$$

where A is an arbitrary collection of point set configurations, i.e. A is an element of power set of \mathbf{N} and $Z_{N,h}$ is the normalization and it is usually called *partition function*:

$$Z_{N,h} = \mathbf{E} \left[e^{h|\tau \cap (0,N]|} \mathbf{1}_{N \in \tau} \right] = \sum_{m=1}^N e^{hm} \mathbf{P}(|\tau \cap (0, N]| = m; N \in \tau) = \sum_{m=1}^N e^{hm} \mathbf{P}(\tau_m = N). \quad (3.1.5)$$

When the contact energy h is large, the contact points are favored. Conversely, when it is very negative, the loops are favored. A competition between entropy and energy occurs and results, for $N \rightarrow \infty$, in a phase transition between a delocalized phase, where there is a unique very large loop[†] and very few contact points, and a localized phase where the number of contact points is of order N and the loops are small. Everything can be made very explicit by analyzing the partition function $Z_{N,h}$ and this is what we explain next.

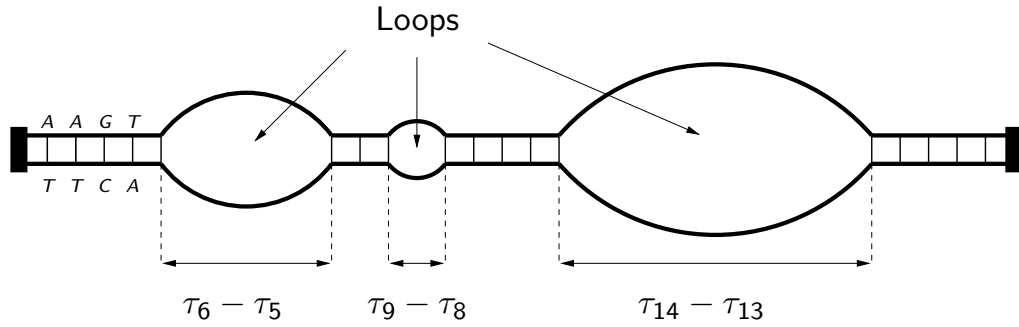


FIGURE 3.2: A schematic view [52] of the correspondence between DNA and renewal process: the paired bases in DNA are the contact points of the renewal, and the loops correspond to inter-arrival of length 2 or more. The coiling structure is forgotten.

An exact computation of the free energy. By restarting from the right-most expression in (3.1.5) and by decomposing the event $\{|\tau \cap (0, N]| = m; N \in \tau\}$ on all the possible η configurations, in an elementary fashion, we obtain

$$Z_{N,h} = \sum_{m=1}^N e^{hm} \sum_{\substack{\ell \in (\mathbf{N}^*)^m: \\ \sum_{j=1}^m \ell_j = N}} \prod_{j=1}^m \mathbf{K}(\ell_j) = \sum_{m=1}^N \sum_{\substack{\ell \in (\mathbf{N}^*)^m: \\ \sum_{j=1}^m \ell_j = N}} \prod_{j=1}^m e^{h\ell_j} \mathbf{K}(\ell_j). \quad (3.1.6)$$

Now there are two possibilities:

[†]In the technical part of the work we will prefer the mathematical terminology of *big jump*, with respect to *large loop*, but this is of course just a nomenclature issue, related to the fact that this phenomenon already appears in the random walk and renewal theory literature.

1. Either $\sum_{\ell \in \mathbf{N}^*} e^h \mathbf{K}(\ell) \leq 1$ (i.e., $h \leq 0$), and in this case we set $\bar{\mathbf{K}}_h(\ell) := e^h \mathbf{K}(\ell)$ for $\ell \in \mathbf{N}$, and $\bar{\mathbf{K}}_h(\infty) = 1 - e^h \geq 0$;
2. Or $\sum_{\ell \in \mathbf{N}^*} e^h \mathbf{K}(\ell) > 1$ (i.e., $h > 0$) and in this case, by monotonicity, there exists a unique positive solution \mathbb{F} to the equation

$$\sum_{\ell \in \mathbf{N}^*} e^{h - \mathbb{F}\ell} \mathbf{K}(\ell) = 1. \quad (3.1.7)$$

This defines $\mathbb{F} = \mathbb{F}(h)$ and we set $\bar{\mathbf{K}}_h(\ell) := e^{h - \mathbb{F}(h)\ell} \mathbf{K}(\ell)$ for $\ell \in \mathbf{N}$.

We set $\mathbb{F}(h) = 0$ for $h \leq 0$ so that we have $\bar{\mathbf{K}}_h(\ell) = e^{h - \mathbb{F}(h)\ell} \mathbf{K}(\ell)$ for every $h \in \mathbf{R}$ and every $\ell \in \mathbf{N}$. The crucial observation is that $\bar{\mathbf{K}}_h(\cdot)$ defines a new renewal process $\tau^{(h)}$ that

- is terminating if $h < 0$;
- coincides with τ if $h = 0$ (and it may be null or positive persistent);
- is positive persistent if $h > 0$.

This observation is crucial because it leads to

$$Z_{N,h} = \exp(\mathbb{F}(h)N) \mathbf{P}\left(N \in \tau^{(h)}\right), \quad (3.1.8)$$

as a direct consequence of (3.1.6) and of the fact that, for a general renewal, we have

$$\mathbf{P}(N \in \tau) = \sum_{m=1}^N \sum_{\substack{\ell \in (\mathbf{N}^*)^m \\ \sum_{j=1}^m \ell_j = N}} \prod_{j=1}^m \mathbf{K}(\ell_j). \quad (3.1.9)$$

Equation (3.1.8) is therefore telling us that estimating $Z_{N,h}$ boils down to the classical issue of estimating the renewal function; and therefore it will come with no surprise that sharp estimates on $Z_{N,h}$ are in the renewal process literature (see [23, 50, 52] for statements and survey of the literature). Sharp estimates on $Z_{N,h}$ have as important consequence the full control of the path properties of the point process $\nu_{N,h}$ in the $N \rightarrow \infty$ limit (see Figure 3.3).

But even without resorting to sharp estimates for $h \leq 0$ it is very easy to see from (3.1.8) that

$$\mathbb{F}(h) = \lim_{N \rightarrow +\infty} \frac{1}{N} \log Z_{N,h}, \quad (3.1.10)$$

so $\mathbb{F}(h)$ is the *free energy (density)* of the model. The proof of (3.1.10) follows from the Renewal Theorem formula (3.1.1) if $h > 0$. If $h \leq 0$, it suffices to observe that $e^h \mathbf{K}(n) \leq \mathbf{P}(n \in \tau^{(h)}) \leq 1$.

The free energy $\mathbb{F}(\cdot)$ carries already a lot of information. Let us first observe that it is a convex function (this can be seen from its definition, but it is probably more easily seen from (3.1.10) since $h \mapsto \log Z_{N,h}$ is convex) and it is non decreasing. The analytic Implicit Function Theorem [73, Ch. 1] implies that it is real analytic for $h > 0$ and this is of course obvious for $h < 0$. Therefore $h = 0$ is a non analyticity point – in fact, the only one. In the statistical mechanics language, it is the *critical point*, and the system exhibits a *phase transition* at this point. The nature of this transition can also be read out of the

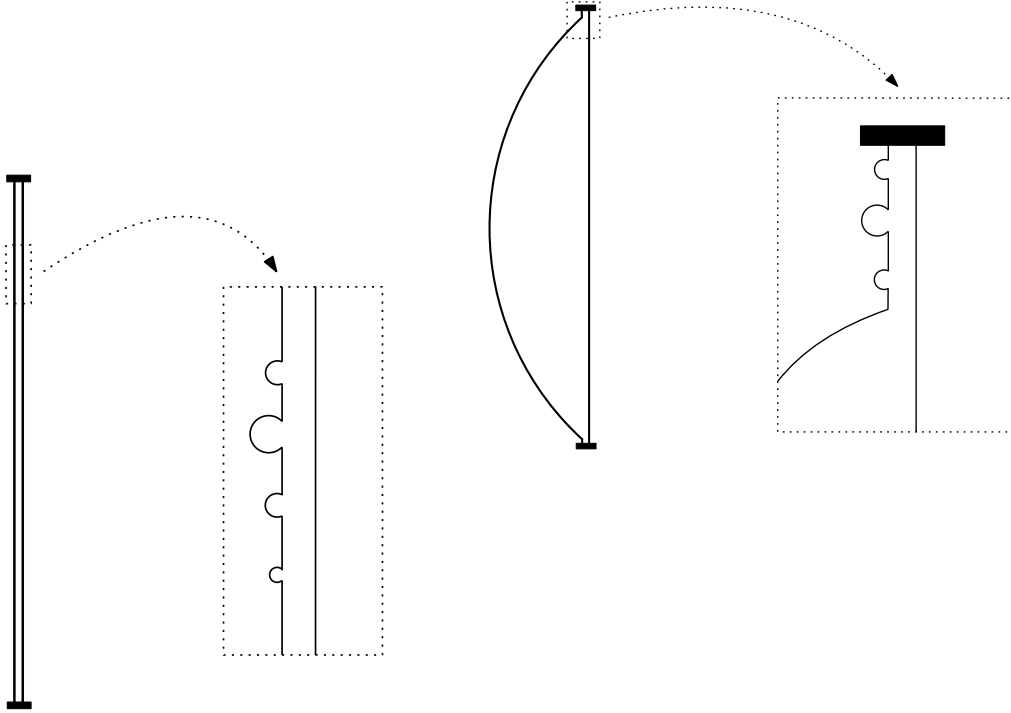


FIGURE 3.3: By differentiating the free energy with respect to h one readily sees (3.1.11) that for $h < 0$ the density of contact sites is zero, while for $h > 0$ such a density is positive. But thanks to (3.1.8) one can go much beyond just by exploiting estimates on the renewal function (see [23, 50, 52]). In fact $\nu_{N,h}(A)$ can be always expressed in terms of ratios of partition functions. What one can show is that, in the limit $N \rightarrow \infty$, the point process converges (in more than one sense, see again [23, 50, 52]) to the renewal process $\tau^{(h)}$ defined by the jump probability $\bar{K}_h(\cdot)$ (see the list after (3.1.6)). For $h > 0$, the Renewal Theorem formula (3.1.1) is sufficient to establish such a result, but sharp estimates on how the real function vanishes at ∞ are needed to recover such a result in the $h < 0$ case. For $h > 0$ (figure on the left), all loops are essentially of finite size (more precisely: the largest has size $O(\log(N))$ and this logarithmic behaviour just reflects the exponentially decreasing behaviour of $\bar{K}_h(n)$ for $n \rightarrow \infty$). On the other hand, if $h < 0$ (figure on the right) the renewal points are finitely many and localized near the extremities: one huge loop, of size that differs from N only by a finite (random) number, occupies the whole system. At criticality, $h = 0$, all depends on $K(\cdot)$: we invite the interested reader to look at [23, 50, 52].

free energy. By basic properties of convex functions, taking the derivative with respect to h can be exchanged with the limit $N \rightarrow \infty$ in (3.1.10), as long as the $F(\cdot)$ is differentiable. So, if we exclude $h = 0$, this operation is safe and we obtain that the *contact fraction* coincides with $F'(h)$:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}_{\nu_{N,h}}[|\tau \cap (0, N)|] = \lim_{N \rightarrow \infty} \frac{1}{N} \frac{\mathbf{E}[|\tau \cap (0, N)| e^{h|\tau \cap (0, N)|} \mathbf{1}_{N \in \tau}]}{\mathbf{E}[e^{h|\tau \cap (0, N)|} \mathbf{1}_{N \in \tau}]} = F'(h), \quad (3.1.11)$$

and therefore it is zero for $h < 0$ and it is positive for $h > 0$ (see Figure 3.4). Thus, the transition at $h = 0$ is between a *delocalized* regime in which there is zero density of contact points and a *localized* regime in which there is a positive density of contact points.

As announced, the sharp estimates on the partition function allow to go much further and precise statements can be made that we have informally summed up into Figure 3.3 and its caption (references in the caption).

We are left with the important issue of characterizing the free energy singularity at $h = 0$: in physical terms this corresponds to identifying the critical behaviour (of the free

energy). This just follows from an accurate analysis of $\sum_n e^{-Fn}K(n)$ for $F \searrow 0$ and by applying the asymptotic inversion results for regularly varying functions [14, §1.5.7]. The final result of this rather explicit analysis is [50, 52]

$$F(h) \underset{h \searrow 0}{\sim} \begin{cases} c_K h & \text{if } \tau \text{ is finite recurrent, i.e. if } \sum_n nK(n) < \infty, \\ \tilde{\varphi}_\alpha(1/h)h^{1/\alpha} & \text{if } \tau \text{ is null recurrent, i.e. if } \sum_n nK(n) = \infty, \end{cases} \quad (3.1.12)$$

where $c_K = 1/\mathbf{E}[\eta_1] > 0$ and $\tilde{\varphi}_\alpha(\cdot)$ is a suitable slowly varying function (Figure 3.4). The most prominent fact in (3.1.12) is that the *critical exponent* of the free energy is $\max(1, 1/\alpha)$: the whole spectrum of possible singularities can be achieved by playing on the value of α . Also the case $\alpha = 0$ is included, and one can in fact show that in that case the free energy in that case goes to zero faster than any power of h (but of course sharp results can be obtained, see for example [2]).

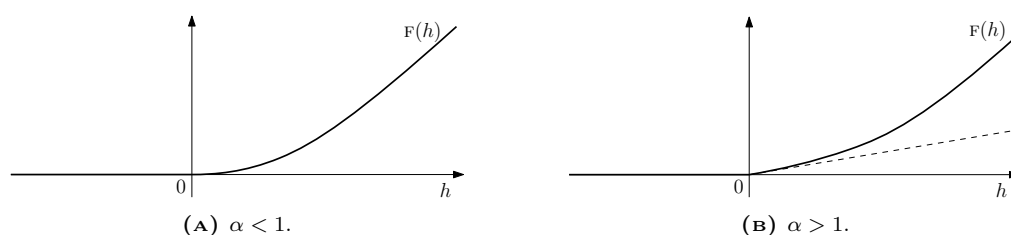


FIGURE 3.4: Free energy $h \mapsto F(h)$ and its critical behaviour: when $\alpha > 1$, the transition is first order ($\partial_h F(h)$ is discontinuous at $h = 0$) whereas it is at least second order when $\alpha < 1$.

Remark 3.1.3. We observe that in the physical literature we speak of transition of order $k = 1, 2, \dots$ if the free energy is \mathcal{C}^{k-1} , but not \mathcal{C}^k , at criticality. Therefore (3.1.12) strongly suggests that the order of the transition in the PS model is given by the integer part of $1/\alpha$ (see [52, Sec. A.2.3] for a proof). Moreover, a transition is of infinite order if the free energy is \mathcal{C}^∞ , but (of course) not analytical, and, for the PS model, the free energy vanishes faster than any power of h minus its critical value (which is zero in this specific case).

3.2 The Disordered Poland-Scheraga Model

If we want to use the PS model for modeling real DNA denaturation we would need to use a model whose partition function is for example (with T the temperature)

$$\mathbf{E} \left[e^{-\frac{1}{T} \sum_{j=1}^N E_j \delta_j} \mathbf{1}_{N \in \tau} \right], \quad (3.2.1)$$

where $\delta_j = \mathbf{1}_{j \in \tau}$ and E_j are the (negative) energies that correspond to the A-T and G-C bonds: we have therefore a deterministic inhomogeneous sequence (E_j) , with entries that take two values. In reality, the situation is definitely more complex than this and some generalized models that take into account the possibility of mismatches or of non perfect complementarity of the two DNA strands have been used in applications and studied theoretically (see Section 3.3). But even sticking to the very basic model defined by (3.2.1), it is not easy to extract mathematical results for given deterministic sequences. The theoretical research (not only in mathematics, but also in physics) has then focused

on the case in which $\{\mathbf{E}_j\}_{j=1,2,\dots}$ is the realization of a sequence of random variables and, in most of the cases, these random variable are IID. We use the notation

$$Z_N^{h,\omega,\beta} = \mathbf{E} \left[e^{\sum_{j=1}^N (\beta\omega_j + h)\delta_j} \mathbf{1}_{N \in \tau} \right], \quad (3.2.2)$$

which just corresponds to $-\mathbf{E}_j/T = \beta\omega_j + h$: $\{\omega_j\}_{j \in \mathbf{N}^*}$ is a sequence of IID random variables, independent of τ and whose law is denoted by \mathbb{P} , on which we assume $\mathbb{E}[\exp(t\omega_1)] < \infty$ for every $t \in \mathbf{R}$. Moreover, without loss of generality we can and do assume that $\mathbb{E}[\omega_1] = 0$ and $\mathbb{E}[\omega_1^2] = 1$ because we have the two parameters $\beta \geq 0$ that plays the role of standard deviation and $h \in \mathbf{R}$ that plays the role of the mean-value.

Remark that now $Z_N^{h,\omega,\beta}$ is a random variable: we define the *quenched* free energy as

$$\mathbb{F}(\beta, h) := \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\log Z_N^{h,\omega,\beta} \right]. \quad (3.2.3)$$

The existence of this limit follows by standard super-additive arguments. We can of course apply these arguments also to the non disordered case ($\mathbb{F}(0, h) = \mathbb{F}(h)$!), but with $\beta = 0$ we had a rather explicit solution that is no longer available. Nevertheless $(\beta, h) \mapsto \log Z_{N,\omega,\beta,h}$ is still convex, therefore so is $(\beta, h) \mapsto \mathbb{F}(\beta, h)$, and the function $h \mapsto \mathbb{F}(\beta, h)$ is non decreasing. Moreover

Proposition 3.2.1. *For every $\beta > 0$ we have that*

$$\mathbb{F}(0, h) \leq \mathbb{F}(\beta, h) \leq \mathbb{F}(\beta, h + \lambda(\beta)), \quad (3.2.4)$$

with $\lambda(\beta) = \log \mathbb{E}[\exp(\beta\omega_1)]$.

Proof. The upper bound, called *annealed* bound, is a consequence of Jensen's inequality:

$$\mathbb{E} \log Z_N^{h,\omega,\beta} \leq \log \mathbb{E}[Z_N^{h,\omega,\beta}] = \log Z_{N,h+\lambda(\beta)}. \quad (3.2.5)$$

The lower bound is a consequence of the monotonicity of $\beta \mapsto \mathbb{F}(\beta, h)$ for $\beta \geq 0$ (which follows from convexity in β and from the fact that $\partial_\beta \mathbb{E} \log Z_N^{h,\omega,\beta}$ is zero at $\beta = 0$). \square

Therefore the graph of $h \mapsto \mathbb{F}(\beta, h)$ looks like as explained in Figure 3.5 and, like for the $\beta = 0$ case, we necessarily have a transition at

$$h_c(\beta) := \inf\{h \in \mathbf{R} : \mathbb{F}(\beta, h) > 0\}. \quad (3.2.6)$$

By the very same arguments as for the $\beta = 0$ case, we realize that we are dealing again with a transition between a delocalized regime and a localized one. Remark also that a direct consequence of Proposition 3.2.1 is that

$$-\lambda(\beta) \leq h_c(\beta) \leq 0, \quad (3.2.7)$$

and in full generality one can show that the second inequality is strict: $h_c(\beta) < 0$ (see [50, Th. 5.2] and [32, Ch. 7]). The first inequality is in some cases an equality, as we will see (cf. Theorem 3.2.2).

We refer to [52] for a review on the literature on the several results on the trajectories of the model that go beyond what one obtains by differentiating the free energy. Here we focus on the issue of whether or not the disorder affects the critical behaviour.

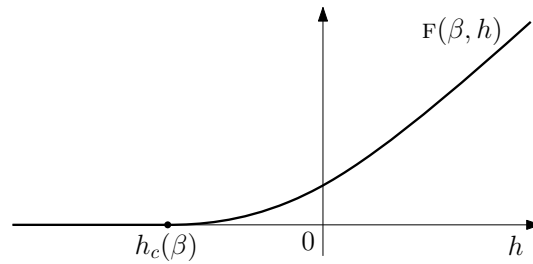


FIGURE 3.5: Free energy $h \mapsto F(\beta, h)$ for $\beta > 0$.

Since the solvability of the model is lost, a priori this looks like a daunting question (and in fact, to a certain extent, it is still very open). There is nevertheless a remarkable approach in physics, due to A. B. Harris [63], and based on the renormalization group approach (*Harris criterion*) that, for the PS model [34, 44], leads to the prediction that

- if $\alpha < 1/2$ then, at least for β not too large, disorder is *irrelevant* which, in general, means that the critical exponent of the $\beta > 0$ case – more precisely $\beta \in (0, \beta_0)$ for some $\beta_0 > 0$ – is the same as for the $\beta = 0$ case. For the PS model the prediction is also that $h_c(\beta) = -\lambda(\beta)$, if disorder is irrelevant.
- if $\alpha > 1/2$ disorder is relevant, so the critical exponent is predicted to be different from the $\beta = 0$ case for every $\beta > 0$. Moreover $h_c(\beta)$ is predicted to be larger than $-\lambda(\beta)$.

The Harris criterion is normally stated in the elegant and compact form: disorder is irrelevant if

$$\nu d > 2, \quad (3.2.8)$$

where ν is the critical exponent of the correlation length of the non disordered system and d is the dimension. There are several ways to define the correlation length in a statistical mechanics system, but any reasonable definition should yield the same critical exponent (at least for non disordered systems [24, 52]). For example, for the PS model we can take as the correlation between sites j and k simply the correlation between the random variables δ_j and δ_k in the $N \rightarrow \infty$ limit and one can show that such a correlation decays for $|j - k| \rightarrow \infty$ like $\exp(-|j - k|/L_\beta(h))$ [51, 56, 89] for a certain quantity $L_\beta(h)$. This $L_\beta(h)$ is therefore a natural correlation length and $\log L_\beta(h) \sim \nu_\beta \log(1/(h - h_c(\beta)))$ for $h \searrow h_c(\beta)$. For pinning models it turns out that $\log L_\beta(h) \sim \log(1/F(\beta, h))$ approaching criticality, see for example [34, 44] for the physical prediction and [51, 52, 56, 89] for proofs. Therefore $\nu = \nu_0 = \max(1, 1/\alpha)$ and, since $d = 1$, (3.2.8) becomes $\max(1, 1/\alpha) > 2$, that is $\alpha < 1/2$. Harris' arguments are essentially a linear stability analysis for the renormalization transformation: $\nu d > 2$ is the stability condition for the non disordered model, and the renormalization flow is supposed to damp the disorder on large scale (irrelevant disorder). The renormalization group is instead *unstable* for $\max(1, 1/\alpha) > 2$ and there is no reason to believe that the large scale behaviour of the disordered system has anything to do with the non disorder system behaviour (relevant disorder). But Harris criterion predicts nothing about the critical behaviour of the disordered system when disorder is relevant.

Mathematically results go very far for the role of disorder in PS model. In particular, the irrelevant disorder regime is fully under control and disorder relevance is rigorously established. We sum up the most relevant results:

Theorem 3.2.2. *For every $\beta > 0$ the map $h \mapsto F(\beta, h)$ is \mathcal{C}^∞ on $(h_c(\beta), +\infty)$ [56] (we have already seen that it is convex and increasing). Moreover*

1. *if $\alpha < 1/2$ then $h_c(\beta) = -\lambda(\beta)$ and*

$$F(\beta, h_c(\beta) + \epsilon) \stackrel{\epsilon \searrow 0}{\sim} F(0, \epsilon) \stackrel{\epsilon \searrow 0}{\sim} \epsilon^{\frac{1}{\alpha}} \bar{\varphi}(1/\epsilon), \quad (3.2.9)$$

for a suitable slowly varying function $\bar{\varphi}(\cdot)$ [1, 59, 74, 90];

2. *if $\alpha > 1/2$ then $h_c(\beta) > -\lambda(\beta)$ and thanks to the smoothing bound [22, 57, 58]*

$$F(\beta, h_c(\beta) + \epsilon) \stackrel{\epsilon \searrow 0}{\asymp} O(\epsilon^2), \quad (3.2.10)$$

that holds for every α , we see that the critical behaviour for $\beta > 0$ is different than the critical behaviour for $\beta = 0$ if $\alpha > 1/2$:

$$\liminf_{\epsilon \searrow 0} \frac{\log F(\beta, h_c(\beta) + \epsilon)}{\log \epsilon} \geq 2 > \max\left(1, \frac{1}{\alpha}\right). \quad (3.2.11)$$

Moreover (3.2.10) implies that $\partial_h F(\beta, \cdot)$ is continuous on \mathbf{R} for every value of α .

The delicate *marginal disorder* case is treated in [12, 54] to which we refer also for the related literature. In particular, the slowly varying function $\varphi(\cdot)$ is crucial to determine whether disorder is relevant or irrelevant.

Many questions concerning the complete behaviour of the disordered PS free energy are still open, in particular:

- is $F(\beta, h)$ analytic for $h > h_c(\beta)$ (and, of course, $\beta > 0$)?
- when disorder is relevant, in particular therefore for $\alpha > 1/2$, it is not known if a critical exponent γ exists and, of course, we do not know its value (as we have seen, the smoothing bound implies $\gamma \geq 2$). The predictions in the physical literature are not fully consistent, but the prediction $\gamma = \infty$ has been set forth (see [52, Ch. 5] for a review of the literature on this issue) and recently this has been proven for two very special simplifications/variations of the PS model [11, 25, 36]

We end this section by signaling also the Large Deviations approach to PS models and to more general disordered pinning models developed in [15, 17, 18] and references therein.

3.3 Beyond the Poland-Scheraga model

Generalized Poland-Scheraga. It is impossible to give here a proper account of the several models that go toward a more realistic, or at least more detailed, modeling of the DNA structure. But one direction, that we have already briefly mentioned, is the one of trying to account for mismatches and slips in the matching of the two strands, or to account for non exact complementary of the two strands, or even possibly for different length of the two strands. The natural generalization of the PS model, proposed in [48, 78] and mathematically studied in [9, 53], accounts for the richness of the possible DNA structures that we just outlined and, very remarkably, keeps the solvable character of the homogeneous

version of the model. This new model is called Generalized Poland-Scheraga (GPS) model. We refer to [53] for a mathematical introduction to this model and we just quickly mention that the solvable nature is once again connected to a renewal structure embodied in the model, but this time the renewal is two dimensional. The GPS model is of interest to us because of its richer phenomenology: we will find it also in the models that we will consider. More precisely, the GPS has a localization/delocalization transition that corresponds to DNA denaturation, but it has also other transitions that happen in the localized regime, that is when the two strands are in their native state. We stress from now that these new transitions have been identified rigorously only in absence of disorder, so, till specified otherwise, we limit ourselves to the non disordered case. What happens in the GPS model if the two strands have different lengths $M > N$ (and if the strands are bound together at both extremities: we make this choice for ease of exposition, see [9, 53] for the general case) is that they can bind together in two different ways:

1. the two strands fully bind together (Figure 3.6) in the sense that essentially all loops have finite length (more precisely, the largest loop is of length $O(\log N)$ for $N \rightarrow \infty$): this is possible because asymmetric loops are allowed in the GPS model, that is loops that involve a different number of bases for the two stands;
2. the two strands bind together in such a way that essentially all the $M - N$ bases that are in excess are absorbed by one large loop (Figure 3.7): again, this is possible because asymmetric loops are allowed. This appearance of a large loop has been presented in then physical literature as a *condensation phenomenon*. In [9, 53] the transition between localization without a large loop and with a large loop is viewed from Large Deviations perspective as the passage from Cramer Deviations (i.e. deviations that can be captured by an exponential tilt of the measure with which one modifies the expectation of the underlying jumps) to non Cramer Deviations, in which no exponential tilt of the measure can lead to an expectation of the underlying jumps compatible with the constraint that the two chains meet at the extremities. The study of this phenomenon, even beyond Large Deviations and the Cramer versus non Cramer behaviour, is already present in the mathematical literature on random walks and renewal processes and the keyword is: *big jump* regime.

A mathematical analysis of the denaturation transition in the disordered case has been developed in [10, 13] and no work has been done on the effect of disorder on the large loop transitions (one can find some numerics in [78]).

A model for circular DNA. The generalization of the PS model we are interested in goes *a priori* toward a different direction. One physical aspect of DNA that is taken into account only in a very indirect way both in the PS and GPS models is the fact that the helix structure involves a global rotation effect when a loop opens. This is treated as a minor effect in the PS and GPS models because the helix is free to *rotate*. But DNA appears in nature also in a different geometry for which one cannot neglect the effect we just mentioned: the circular geometry. We talk about circular DNA when each of the two DNA strands close, thus forming two rings, entangled by their coiling around each other. This is not an exceptional geometry: it is the standard one for the so called *plasmids* that are very actively studied for several reasons, notably their involvement in important pathologies and their use as vehicle of genetic code in genetical engineering, see for example

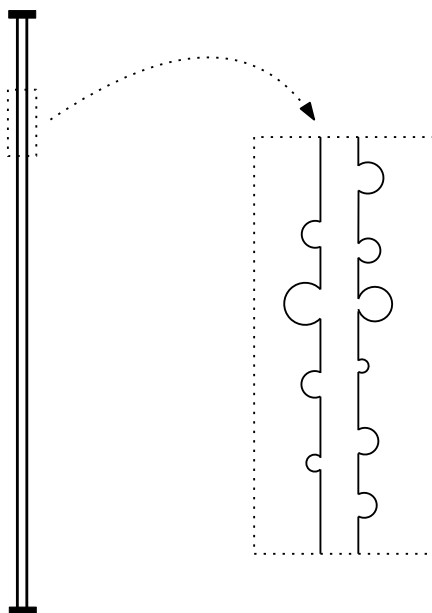


FIGURE 3.6: Schematic view of the localized phase of GPS model when the two strands fully bind together (item 1). On the right a zoom on the local structure: the jumps are $O(\log N)$ and, of course, the loops must be asymmetric to allow such a localized behaviour despite the fact that the two strands have different lengths $M > N$.

Plasmids on the Scitable of Nature Education (www.nature.com/scitable). So, in a double stranded DNA with a ring structure, one can no longer neglect the winding of the chain: one can easily set up a macroscopic experiment with two pieces of rope and see what happens when one tries to separate the two ropes.

Here is a very informal view of what happens in circular double stranded DNA:

- the two strands can no longer get completely separated;
- if a loop opens, then the winding lost in that area (that can be quantified in a number) must be absorbed somewhere else in the chain, for example in the backbone of the two strands (with an energetic cost) or in some regions in which the two strands curl around each other and bend in an atypical way (again with an energetic cost).

Our purpose is to tackle mathematically the models proposed in [6, 7] for circular DNA: the variety corresponds to different mechanisms for absorbing the excess winding. For all of them, it is pointed out that there can be a condensation phenomenon like for the GPS model: the two strands cannot fully separate, but it might happen that a large loop, i.e. of a size proportional to the length of the chains, appears. We will present in Chapter 4 a general framework that

- includes the circular DNA models of [6, 7], but it includes also models in which there can be also a denaturation transition;
- can be seen as a toy model for the GPS, and for transitions of the type *large loop* in the localized regime. The reason is simply that the large loop phenomenon emerges because of the same *mechanism*: the big jump one.

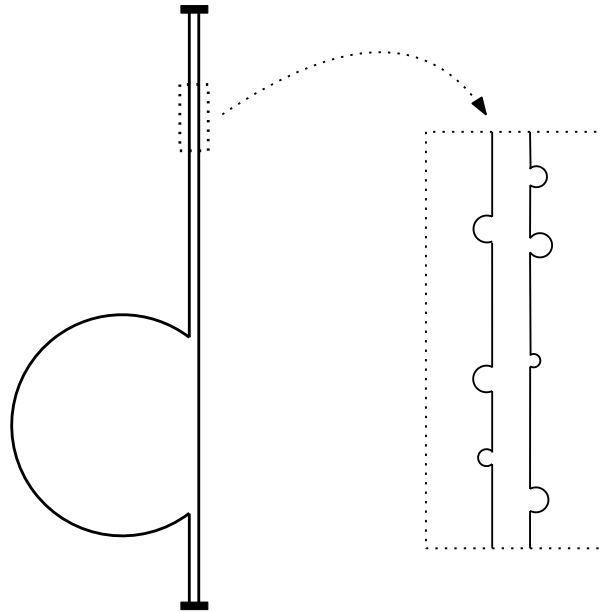


FIGURE 3.7: Schematic view of the *big jump* regime in the GPS model (item 2): a unique large loop emerges, the other loops are essentially small (see the zoomed figure on the right).

In this general framework we will give a full analysis of the homogeneous models: this includes a complete understanding of the transitions and critical phenomena, along with precise results on the behaviour of the trajectories of the polymer in all the regimes, including the critical one.

But we will also tackle the disordered case (Chapter 5) and we will present results that, unlike what is predicted in the physical literature [7, §IV], show that disorder suppresses the large loop/big jump phenomenon.

Chapter 4

Homogeneous Circular DNA

The development of this chapter has greatly benefited from exchanges and suggestions by Quentin Berger.

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Introduction

As we have seen in the previous chapter, generalizations of the Poland-Scheraga (PS) model have been proposed to go beyond the rigid constraints of the PS model, notably the exact symmetry of the two strands, and to allow for richer geometrical structures (see

Section 3.3). In this chapter, we focus on models of circular DNA, that is cases in which the DNA consists of two rings, entangled by their coiling around each other. In the end, the annulus structure turns out to be a real obstacle to a full denaturation transition: unlike the PS model (in which the two strands are free to rotate), in circular DNA the linking number must be conserved. Thus, a loop opening creates an excess of winding somewhere else in the chains. In particular the two strands cannot fully separate. Various models have been proposed in the biophysical literature for circular DNA corresponding to different mechanisms for absorbing the excess winding. We focus on the biophysics works [6, 7] and we propose a general mathematical model (or class of models), based on discrete renewal processes, that includes all the models in [6, 7]. In this framework, the large loop phenomenon, predicted by [6, 7], is directly related to the big jump phenomenon for the renewal and the general model we present can also be seen as toy model for the GPS model [9, 10, 53] mentioned in Section 3.3.

4.1 Circular DNA model(s)

Let us start by laying out the model proposed by Bar, Kabakçioğlu and Mukamel [6, 7]. It is an homogeneous model as it neglects that the A-T bonds in the strand pairing are weaker than the G-C bonds. A double stranded circular DNA of length N is modeled by a sequence of bond segments, supercoil segments and loops. The contribution of these three types of segments to the energy and to the entropy are the following.

- (i) A bond segment of length k forms with probability proportional to $e^{\beta k E_b} = \omega_b^k$.
- (ii) A supercoil segment of length k forms with probability proportional to $e^{\beta k E_s} = \omega_s^k$. These internal energies typically satisfy $E_b < E_s < 0$.
- (iii) A loop of length k forms with probability proportional to $\frac{\omega_\ell^k}{k^c}$.

A supercoil segment corresponds to area where the DNA double helix tangles around itself (as the cord on landline telephones sometimes does). Denote by L_b , L_s and L_ℓ the total length of bonds, supercoils and loops (L_b is the sum of the lengths of all bond segments, etc.). Three mechanisms have mainly been considered in the biophysics literature to model the winding constraint in circular DNA.

1. In the first model, the excess of winding created by a loop opening is compensated by supercoils and an overtwist, somewhere else in the chain. The supercoils can exactly compensate for the open loops, but if not, the excess linking number $L_\ell - L_s$ causes an overtwist, supported by the $L_b + L_s$ contact points between the two strands of the DNA. The average stacking angle per contact point is $\Delta\theta = (L_\ell - L_s)/(L_b + L_s)$ and the overtwist induces an energy penalty per contact point proportional to $(\Delta\theta)^2$, say $\chi \cdot (\Delta\theta)^2$. The total energy penalty due to overtwisting is therefore

$$(L_b + L_s) \cdot \chi \cdot (\Delta\theta)^2 = \chi \frac{(L_\ell - L_s)^2}{L_b + L_s}. \quad (4.1.1)$$

This model will be referred to as the model with *overtwist and supercoils*. Two other models has also been considered in the literature, which are actually degenerate cases of this first one.

2. In the second model, the supercoil phenomenon does not exist ($\omega_s = 0$ and $L_s = 0$). The energy cost of overtwisting is then $\chi L_\ell^2/L_b$. This model will be called the *overtwist model*.
3. In the third model, instead of an energy cost $\chi \cdot (\Delta\theta)^2$, the winding constraint is rigid and no overtwist is allowed. It can be recovered by taking $\chi = \infty$ in the first model. This means that only the configurations in which $L_s = L_\ell$ are possible. It will be referred to as the model with *strong constraint and supercoils*.

For these models a large loop transition is predicted in the physics literature [6, 7]. At low temperature, *full localization* is observed: bonds, loops and supercoils (except in the second model, of course) coexist (the lengths L_b, L_s, L_ℓ are of order N) but the loops are typically small. Above a critical temperature, a macroscopic (i.e., of size proportional to N) loop forms, but, unlike a delocalized phase, the two strands are not completely separated: the density of contact points stays positive (the lengths L_b and L_s are also of order N): this is the *big jump phase*.

We stress that the big jump phenomenon – also called condensation phenomenon or non Cramer Large Deviations regime in different contexts discussed in the previous chapter – is not delocalization and the transition is not a localization/delocalization transition: this will be seen notably in the different critical behaviour and it will become even more evident in the next chapter where the disordered version of the model is introduced. Nevertheless, the general framework we are going to study can and does accommodate localization/delocalization transitions, providing therefore a full analogy with the GPS model of Section 3.3. We will however minimize the analysis of this aspect of the model, for sake of clarity of the presentation.

4.1.1 A mathematical model

From a mathematical perspective, the mechanisms involved in circular DNA will be studied using a modification of the Poland-Scheraga (PS) model. In this model, the sequence of loops is represented by a discrete renewal process (see Section 3.1). We recall the main definitions and notations.

Definition 4.1.1. A process $\tau = (\tau_k)_{k \geq 0}$ is a *discrete renewal process* under distribution \mathbf{P} when $\tau_0 = 0$, and the increments $(\eta_k)_{k \geq 1} = (\tau_k - \tau_{k-1})_{k \geq 1}$ are IID random variables taking values in $\mathbf{N}^* = \{1, 2, \dots\}$. We will use the notation $\mathbf{K}(n) = \mathbf{P}(\eta_1 = n)$. The points $(\tau_k)_{k \geq 0}$ will be called pinned points or contact points and the renewal process will often be seen as a subset of the discrete line \mathbf{N} . For convenience, we use a slightly different notation with respect to Chapter 3: when the length N of the DNA strands is fixed, the number of contact points is

$$|\tau| = \sup\{k \geq 1 \mid \tau_k \leq N\}, \quad (4.1.2)$$

it plays the role of what was denoted by $|\tau \cap (0, N]|$ in Chapter 3. Similarly, the set of *contact points* is restricted to $(0, N]$:

$$\tau = \{\tau_k : 1 \leq k \leq |\tau|\} \subset (0, N]. \quad (4.1.3)$$

In that formalism, the loops are the segment between two contact points which are not immediately adjacent in the DNA strand, i.e., the segments for which $\tau_k - \tau_{k-1} \geq 2$. The total length of loops is

$$L_\ell = N - |\tau|. \quad (4.1.4)$$

In the PS model, the distribution of the renewal is modified by giving an energy reward $h \in \mathbf{R}$ for each contact point, yielding a new distribution (see Section 3.1)

$$\nu_{N,h}(A) = \frac{1}{Z_{N,h}} \mathbf{E} \left[e^{h|\tau|} \mathbf{1}_A(\tau) \mathbf{1}_{N \in \tau} \right], \quad (4.1.5)$$

where A is an arbitrary collection of point set configurations, i.e. A is an element of power set of $(0, N]$. Formally, $\nu_{N,h}$ is a measure on the power set of $((0, N])$, equipped with its discrete σ -algebra \mathcal{F} .

To model bonds and supercoils, consider, independently of the renewal, IID fair Bernoulli random variables v_1, \dots, v_N taking value in $\{b, s\}$: that is $\mathbf{P}(v_1 = b) = \mathbf{P}(v_1 = s) = 1/2$. The contact points tagged with b are the bonds and the contact points tagged with s are the supercoils. The sets of bonds and supercoils are respectively

$$\mathbf{B} = \{j \in \tau \mid v_j = b\}, \quad \mathbf{S} = \{j \in \tau \mid v_j = s\} \quad (4.1.6)$$

and their cardinals

$$L_b = |\mathbf{B}| = \sum_{j \in \tau} \mathbf{1}_{v_j=b}, \quad L_s = |\mathbf{S}| = \sum_{j \in \tau} \mathbf{1}_{v_j=s}, \quad (4.1.7)$$

are the total length of bonds and the total length of supercoils. Note that $\mathbf{B} \sqcup \mathbf{S} = \tau$ and $L_b + L_s = |\tau|$. Before explaining our general framework, we detail what the first model becomes in that mathematical formalism.

An example: the model with supercoils and overtwisting. We assign a reward h for each contact point and a reward w for each supercoil. The Boltzmann weight associated to a configuration τ is

$$\exp \left\{ h(L_b + L_s) + wL_s - \chi \frac{(L_b - L_s)^2}{L_b + L_s} \right\}. \quad (4.1.8)$$

Therefore, the distribution of the renewal is modified in the following way: for each $A \in \mathcal{F}$,

$$\xi_N^{\text{os},h}(A) = \frac{1}{Z_N^{\text{os},h}} \mathbf{E} \left[\exp \left\{ h(L_b + L_s) + wL_s - \chi \frac{(L_b - L_s)^2}{L_b + L_s} \right\} \mathbf{1}_A(\tau) \mathbf{1}_{N \in \tau} \right]. \quad (4.1.9)$$

If we decompose this expectation depending on the numbers $m = |\tau|$ and $p = L_s$, we get

$$\begin{aligned} & \mathbf{E} \left[\exp \left\{ h(L_b + L_s) + wL_s - \chi \frac{(L_b - L_s)^2}{L_b + L_s} \right\} \mathbf{1}_A \mathbf{1}_{N \in \tau} \right] \\ &= \sum_{m=1}^N \sum_{p=0}^m \mathbf{E} \left[\exp \left\{ hm + pw - \chi \frac{(N - m - p)^2}{m} \right\} \mathbf{1}_A \mathbf{1}_{\tau_m=N} \mathbf{1}_{L_s=p} \right] \\ &= \sum_{m=1}^N \sum_{p=0}^m \exp \left\{ hm + pw - \chi \frac{(N - m - p)^2}{m} \right\} \mathbf{P}(A; \tau_m = N) \mathbf{P}(\text{Bin}(m, 1/2) = p) \\ &= \sum_{m=1}^N e^{hm} \Psi_{\text{os}}(m, N) \mathbf{P}(A; \tau_m = N) \\ &= \mathbf{E} \left[e^{h|\tau|} \Psi_{\text{os}}(|\tau|, N) \mathbf{1}_A \mathbf{1}_{N \in \tau} \right], \end{aligned} \quad (4.1.10)$$

where

$$\Psi_{\text{os}}(m, N) = \sum_{p=0}^m \exp \left\{ pw - \chi \frac{(N - m - p)^2}{m} \right\} 2^{-m} \binom{m}{p}. \quad (4.1.11)$$

The partition function can also be rewritten

$$Z_N^{\text{os},h} = \mathbf{E} \left[e^{h|\tau|} \Psi_{\text{os}}(|\tau|, N) \mathbf{1}_{N \in \tau} \right] = \sum_{m=1}^N e^{hm} \Psi_{\text{os}}(m, N) \mathbf{P}(\tau_m = N). \quad (4.1.12)$$

The same type of decomposition holds for the PS model (3.1.5). But here the energy factor e^{hm} is replaced by a more complex factor $e^{hm} \Psi_{\text{os}}(m, N)$ which takes into account the geometric constraint of circular DNA. The same structure emerges for the two other models detailed in the biophysics introduction and a general convolution formalism can be set up to include all of them.

General framework. The next definition provides a general framework to study geometric mechanisms such as those involved in circular DNA. It takes the form of a slightly more general modification of the renewal distribution:

$$\xi_N^{\Psi,h}(A) = \frac{1}{Z_N^{\Psi,h}} \mathbf{E} \left[e^{h|\tau|} \Psi(|\tau|, N) \mathbf{1}_A \mathbf{1}_{N \in \tau} \right] \quad (4.1.13)$$

First observe that the measure and the partition function can be decomposed depending on the number of contact points:

$$Z_N^{\Psi,h} = \sum_{m=1}^N \mathbf{E} \left[e^{h|\tau|} \Psi(|\tau|, N) \mathbf{1}_{|\tau|=m} \mathbf{1}_{N \in \tau} \right] = \sum_{m=1}^N e^{hm} \Psi(m, N) \mathbf{P}(\tau_m = N). \quad (4.1.14)$$

The competition between the energy factor e^{hm} and the entropy $\mathbf{P}(\tau_m = N)$ which occurs in the PS model is replaced by a competition between a more complex factor $e^{hm} \Psi(m, N)$, and the entropy $\mathbf{P}(\tau_m = N)$. These new models appear as a convolution between a PS model with a fixed number of contact points, and a nonnegative kernel Ψ representing the geometric mechanism. That's why these models will be called *kernel models*. The required properties on the kernel Ψ will be detailed later. Before, let us consider again the biophysics models of the introduction.

Example 4.1.2. The three biophysics models introduced page 80 (items 1, 2 and 3) to model circular DNA can be written and studied in this general framework. For each one we give the corresponding measure, for $A \in \mathcal{F}$, and the kernel Ψ .

1. (Overtwist and supercoils)

$$\begin{aligned} \xi_N^{\Psi_{\text{os},h}}(A) &= \frac{1}{Z_N^{\Psi_{\text{os},h}}} \mathbf{E} \left[\exp \left\{ h(L_b + L_s) + wL_s - \chi \frac{(L_\ell - L_s)^2}{L_b + L_s} \right\} \mathbf{1}_A(\tau) \mathbf{1}_{N \in \tau} \right], \\ \Psi_{\text{os}}(m, N) &= \sum_{p=0}^m \exp \left\{ pw - \chi \frac{(N - m - p)^2}{m} \right\} 2^{-m} \binom{m}{p}. \end{aligned} \quad (4.1.15)$$

Note that if $\chi = 0$, then it boils down to a (shifted) standard Poland-Scheraga model: $\Psi_{\text{os}}(m, N) = e^{mh_0}$ with $h_0 = \log(e^w + 1) - \log 2$.

2. (Overtwist)

$$\begin{aligned}\xi_N^{\Psi_o, h}(A) &= \frac{1}{Z_N^{\Psi_o, h}} \mathbf{E} \left[\exp \left\{ hL_b - \chi \frac{L_\ell^2}{L_b} \right\} \mathbf{1}_A(\tau) \mathbf{1}_{N \in \tau} \right], \\ \Psi_o(m, N) &= \exp \left\{ -\chi \frac{(N-m)^2}{m} \right\}.\end{aligned}\tag{4.1.16}$$

3. (Strong constraint and supercoils) It is the limit $\chi \rightarrow +\infty$ of the model with overtwist and supercoils:

$$\begin{aligned}\xi_N^{\Psi_s, h}(A) &= \frac{1}{Z_N^{\Psi_s, h}} \mathbf{E} \left[e^{h(L_b + L_s)} \mathbf{1}_A(\tau) \mathbf{1}_{N \in \tau} \mathbf{1}_{L_\ell = L_s} \right], \\ \Psi_s(m, N) &= 2^{-m} \binom{m}{N-m} \mathbf{1}_{N \leq 2m}.\end{aligned}\tag{4.1.17}$$

The big jump transition described in the physics literature [6, 7] and the corresponding singularity of the free energy will be recovered, under some hypothesis on the distribution of the renewal. For the circular DNA models it is the only phase transition. But, in addition, certain classes of kernels Ψ (without physical interpretation if we restrict to [6, 7]) can yield other types of phase transitions, including a delocalization transition if $\Psi(\varrho N, N)$ is smooth around $\varrho = 0$. Those will be discussed in subsection 4.2.4.

Given the convolution representation (4.1.14), a first step to tackle these models is to study the PS model with a given number of contact points. Not only will it be useful to compute their partition function and their free energy but also to understand these new models from a trajectory viewpoint.

For $1 \leq L \leq N$, we consider the measure

$$\mu_{N,L}(A) = \mathbf{P}(\tau \in A | N \in \tau; |\tau| = L) = \mathbf{P}(\tau \in A | \tau_L = N).\tag{4.1.18}$$

The associated partition function is $\mathbf{P}(\tau_L = N)$. With a tiny change of viewpoint, $\mu_{N,L}$ is also the equilibrium measure of an irreducible zero-range process with L sites and N particles (see [5]). In that framework, many path properties of this model have been investigated, which we will translate in our setting. This model will be called *constrained PS model*, or PS model with a fixed density, or a fixed number of contact points.

4.1.2 Notations and Main Assumptions

- The letters c, C, \dots will denote positive constants, whose value does not matter and can change from one line to another.
- Similarly, $\bar{\varphi}$ will be a generic slowly varying function (see [14, §1.2.1]) which can change from one line to another, whereas φ will be the one defining K (see (4.1.19)).
- N, L will be two integers, with $L \leq N$.

About the renewal. The hypotheses on the renewal τ are the same as the usual assumptions made for the study of the PS model (see Chapter 3). Though, for convenience, we will use slightly different notations.

Assumptions 4.1.3 (on the renewal distribution). The renewal distribution K regularly varies with n : there exists $\alpha \geq 0$ and a slowly varying function φ such that

$$K(n) = \frac{\varphi(n)}{n^{1+\alpha}}, \quad n \geq 1. \quad (4.1.19)$$

We also suppose $K(1) > 0$.

We define $K(u)$ for $u \in (0, +\infty)$ with the same formula. Set

$$\mathbf{E}[\eta] = \sum_n nK(n) \in (1, +\infty], \quad \varrho_c = \frac{1}{\mathbf{E}[\eta]} \in [0, 1), \quad \sigma^2 = \text{Var}(\eta). \quad (4.1.20)$$

Of course $\mathbf{E}[\eta] = +\infty$ if $\alpha < 1$ and it is finite if $\alpha > 1$. As in the study of the PS model, we will need to consider tilted measures. For $x \geq 0$, consider the measure \mathbf{P}_x under which the renewal has distribution

$$K_x(n) = \frac{K(n)e^{-xn}}{\mathbf{E}[e^{-x\eta}]}. \quad (4.1.21)$$

The corresponding variance is

$$\sigma_x^2 = \text{Var}_x(\eta) = \mathbf{E}_x[\eta^2] - \mathbf{E}_x[\eta]^2. \quad (4.1.22)$$

We will also need to consider, for any $N \leq L\mathbf{E}[\eta]$, the unique $x_L \geq 0$ (which actually also depends on N) such that

$$\mathbf{E}_{x_L}[\eta] = \frac{N}{L}. \quad (4.1.23)$$

Last but not least, the α -stable random variable X_α and the sequences a_L and b_L (which also depend on α) are defined in the appendix (Lemma 4.A.4 and Proposition 4.A.7).

About the kernel Ψ . The assumptions are a bit technical. In a word, we want $\Psi(m, N) \simeq \exp(NH(m/N))$ for a certain concave function H .

Assumptions 4.1.4 (on the kernel Ψ). The function $\Psi(m, N)$ can be written

$$\Psi(m, N) = \exp\left(NH\left(\frac{m}{N}\right)\right) Q(m, N) \geq 0, \quad (4.1.24)$$

where H and Q satisfy the following properties.

- (i) (a) The function $H : [0, 1] \rightarrow \mathbf{R} \cup \{-\infty\}$ is strictly concave. Let $\mathcal{I} = \{H > -\infty\}$, which is an interval, whose endpoints will be denoted by $a = \inf \mathcal{I} \in [0, 1]$ and $b = \sup \mathcal{I} \in [0, 1]$, with $a < b$ to avoid trivialities. We extend H' on \mathcal{I}^c by setting $H'(\varrho) = +\infty$ for $\varrho \in [0, a] \setminus \mathcal{I}$, and $H'(\varrho) = -\infty$ for $\varrho \in [b, 1] \setminus \mathcal{I}$.
- (b) For every $\varrho \in (a, b)$, $H''(\varrho) < 0$.
- (c) H is analytic on (a, b) .
- (d) Either $H(a) = -\infty$ or $H(a) = \lim_{y \searrow a} H(y)$. The same for b .

(ii) (a) For every $\varrho \in (a, b)$,

$$\sup_{|m-\varrho N| \leq \sqrt{N} \log N} \left| \frac{Q(m, N)}{Q(\lfloor \varrho N \rfloor, N)} - 1 \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (4.1.25)$$

(b) (Bounds for Q in (a, b)) For any $[u, v] \subset (a, b)$, there exists $c_1, c_2 > 0$ such that

$$c_1 N^{-c_1} \leq Q(m, N) \leq c_2 N^{c_2}, \quad \text{for } \frac{m}{N} \in [u, v]. \quad (4.1.26)$$

(c) (Upper bound for Ψ) For any $B, \delta > 0$, there exists $c > 0$ such that for every $N \geq m \geq 1$,

$$\Psi(m, N) \leq c N^c e^{NH(\frac{m}{N}) + \delta N} + c N^c e^{-BN}. \quad (4.1.27)$$

Note that these assumptions imply that for any $\varrho \in (a, b)$,

$$\frac{1}{N} \log \Psi(m, N) \xrightarrow[\frac{m}{N} \rightarrow \varrho]{N \rightarrow +\infty} H(\varrho). \quad (4.1.28)$$

These assumptions are fulfilled by the three models introduced before. A formula can be given for the function H for both of them. Actually for none of them the term e^{-BN} is needed. In fact, the term e^{-BN} is only useful for some models for which, for instance, $a > 0$ (and $H(\varrho) = -\infty$ for $\varrho < a$) but $\Psi(N, m)$ is not exactly zero for $m/N < a$ (it is just very small).

Example 4.1.5. With Stirling's approximation, we get the following estimates of H and Q for each of the three models of Example 4.1.2.

1. (Overtwist and supercoils)

$$H_{\text{os}}(\varrho) = \sup_{\varsigma \in [0, \varrho]} \psi(\varsigma, \varrho) = \psi(\varsigma_0(\varrho), \varrho), \quad (4.1.29)$$

where

$$\psi(\varsigma, \varrho) = \varsigma w - \varrho \log 2 - \chi \frac{(1 - \varrho - \varsigma)^2}{\varrho} + \varrho \log \varrho - \varsigma \log(\varsigma) - (\varrho - \varsigma) \log(\varrho - \varsigma). \quad (4.1.30)$$

And one can derive the asymptotic behaviour of Q :

$$Q_{\text{os}}(m, N) \sim q\left(\frac{m}{N}\right), \quad \text{where} \quad q(\varrho) = \sqrt{\frac{\varrho}{\varsigma_0(\varrho)(\varrho - \varsigma_0(\varrho)) |\partial_{\varsigma}^2 \psi(\varsigma_0(\varrho), \varrho)|}}. \quad (4.1.31)$$

The function ψ is concave on the convex domain $\{0 \leq \varsigma \leq \varrho \leq 1\}$. Thus, H_{os} is also concave. Moreover H_{os} is analytic on $(0, 1)$, with $H_{\text{os}}(0) = -\infty$.

2. (Overtwist)

$$H_{\text{o}}(\varrho) = -\chi \frac{(1 - \varrho)^2}{\varrho}, \quad Q_{\text{o}}(m, N) = 1. \quad (4.1.32)$$

3. (Strong constraint and supercoils)

$$H_{\text{s}}(\varrho) = \begin{cases} -\varrho \log 2 + \varrho \log \varrho - (1 - \varrho) \log(1 - \varrho) - (2\varrho - 1) \log(2\varrho - 1) & \text{if } \varrho \in [\frac{1}{2}, 1], \\ -\infty & \text{if } \varrho \in [0, \frac{1}{2}), \end{cases} \quad (4.1.33)$$

and

$$Q_{\text{s}}(m, N) \sim \sqrt{\frac{m}{2\pi(N - m)(2m - N)}}. \quad (4.1.34)$$

4.2 Main Results

We discuss the main results about the constrained PS model $\mu_{N,L}$ (its free energy and how the conditioning $\tau_L = N$ affects the size of the largest jump), and detail the consequences on the kernel model (free energy and path properties).

4.2.1 Free Energy

4.2.1.1 Free energy for the constrained PS model

We start by giving a formula for the free energy G associated to the PS model with a fixed density $\mu_{N,L}$ and the basic properties which derive from it. The result are summarized in Figure 4.1.

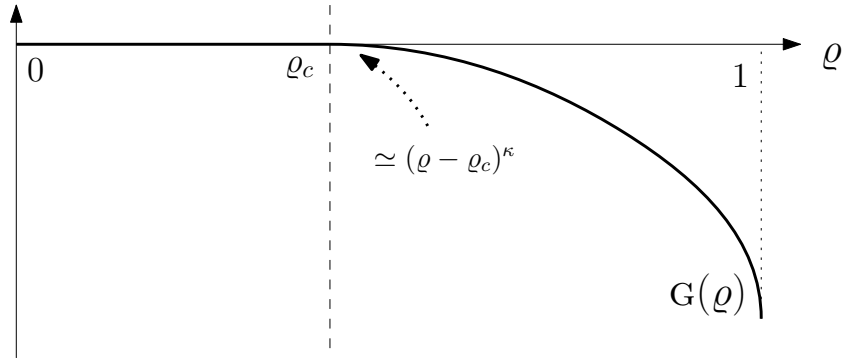


FIGURE 4.1: Look of the function $\rho \mapsto G(\rho)$ when $\mathbf{E}[\eta] < +\infty$. The dashed line separates the big jump phase on the left side, and the localized phase on the right side.

Lemma 4.2.1 (Basic properties of G). *For any $\rho \in [0, 1]$,*

$$\frac{1}{N} \log \mathbf{P}(\tau_L = N) \xrightarrow[N \rightarrow +\infty]{\frac{L}{N} \rightarrow \rho} G(\rho) := \inf_{x \in \mathbf{R}_+} \{x + \rho \log \mathbf{E}[e^{-x\eta}]\}. \quad (4.2.1)$$

1. *The function $\rho \mapsto G(\rho)$ is continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$;*
2. *$G(\rho) = 0$ for $\rho \in [0, \rho_c]$ and it is negative, decreasing, concave, and analytic on $(\rho_c, 1]$, with $G'(\rho) \rightarrow -\infty$ when $\rho \rightarrow 1$.*

Remark 4.2.2. Recall that $\rho_c = 0$ when $\mathbf{E}[\eta] = +\infty$, in particular when $\alpha < 1$.

The next lemma gives the critical exponent of the free energy G for what will be the big jump/localization phase transition.

Proposition 4.2.3 (Singular behaviour of G). *If $\alpha \neq 1$ then, as $\rho \searrow \rho_c$ (whether $\rho_c > 0$ or not),*

$$G(\rho) \sim -(\rho - \rho_c)^\kappa \bar{\varphi}((\rho - \rho_c)^{-1}), \quad (4.2.2)$$

where $\bar{\varphi}$ is a slowly varying function (details in Lemma 4.3.2), and

$$\kappa = \begin{cases} \frac{1}{1-\alpha} & \text{if } \alpha \in [0, 1) \\ \frac{\alpha}{\alpha-1} & \text{if } \alpha \in (1, 2] \\ 2 & \text{if } \alpha \geq 2. \end{cases} \quad (4.2.3)$$

If $\alpha = 1$ then G vanishes faster than any power of $(\varrho - \varrho_c)$:

$$\frac{\log |G(\varrho)|}{\log(\varrho - \varrho_c)} \xrightarrow{\varrho \searrow \varrho_c} +\infty, \quad (4.2.4)$$

and formally, $\kappa = +\infty$.

4.2.1.2 Free energy for the kernel models

We start by the definition the free energy F_H , which is naturally expressed with a variational formula.

Lemma 4.2.4. *For any $h \in \mathbf{R}$,*

$$\frac{1}{N} \log Z_N^{\Psi, h} \xrightarrow{N \rightarrow +\infty} F_H(h) = \sup_{\varrho \in [0,1]} \{\varrho h + H(\varrho) + G(\varrho)\}. \quad (4.2.5)$$

Now we discuss the regularity and the singular behaviour of the free energy F_H . Recall the conventions of Assumptions 4.1.4 for H' . First, and most importantly, the singularity of G at ϱ_c leads to a singularity at point

$$h_c := -H'(\varrho_c) \quad (4.2.6)$$

for the kernel model free energy F_H . Depending on H , this point may not exist (i.e., be infinite): see Example 4.2.5 for the three circular DNA models. An important result, we will see, is that the singular behaviour of F_H at h_c is precisely the same as the singular behaviour of G at ϱ_c . In particular the critical exponent is also κ . In fact, the corresponding phase transition on trajectories is the same: it is the big jump/localization phase transition, which kernel models will also exhibit (see the next subsection). For the three biophysical models of Example 4.1.2, it is the only singular point. But for some smoother kernels (with no physical significance, e.g., $H(x) = x(1-x)$), there can be two other critical points

$$h_\star = -H'(a) - G'(a), \quad h^\star = -H'(b) - G'(b), \quad (4.2.7)$$

where F_H exhibits a singular behaviour. Once again, depending on H , these points may not exist (i.e., be infinite), and the actual set of critical points, in all generality, is

$$\mathcal{C} = \{h_c, h_\star, h^\star\} \cap \mathbf{R}. \quad (4.2.8)$$

As said, the point h_c corresponds to the big jump/localization phase transition whereas h_\star and h^\star (when they are finite) correspond to other types of phase transition, which will be briefly discussed in Subsection 4.2.4. First let's see what happens for the three circular DNA models.

Example 4.2.5. In the models of Example 4.1.2, H has infinite derivative on the boundary of its domain. Thus $h_\star = -\infty$ and $h^\star = +\infty$. Only the critical point h_c can exist. We detail it for each of these models.

1. (Overtwist and supercoils) If $\mathbf{E}[\eta] < +\infty$ then $h_c \in \mathbf{R}$; otherwise $h_c = -\infty$.
2. (Overtwist) If $\mathbf{E}[\eta] < +\infty$ then $h_c = -\chi(1 - \varrho_c^2)/\varrho_c^2$; otherwise $h_c = -\infty$.

3. (Strong constraint and supercoils) If $\mathbf{E}[\eta] < 2$ then $h_c = \log \frac{2(2\varrho_c - 1)^2}{\varrho_c(1 - \varrho_c)}$; otherwise $h_c = -\infty$.

For the sake of completeness, the proofs will be written in all generality. But, at first reading, and for the applications, one may forget the points h_\star and h^\star , or suppose $h_\star = -\infty$ and $h^\star = +\infty$. The first result states that h_c , h_\star and h^\star are indeed the only possible critical points.

Proposition 4.2.6. *The free energy $h \mapsto \mathbb{F}_H(h)$ is convex and continuously differentiable on \mathbf{R} and it is analytic on $\mathbf{R} \setminus \mathcal{L}$.*

Remark 4.2.7. A quick study will tell that \mathbb{F}_H is linear for $h \leq h_\star$ and $h \geq h^\star$ with slopes a and b , and that $\mathbb{F}'_H(h) \in (a, b)$ for $h \in (h_\star, h^\star)$. The critical point h_c is precisely the solution of $\mathbb{F}'_H(h) = \varrho_c$.

Here is the main result, stating, in a nutshell, that the critical exponent at h_c is κ .

Proposition 4.2.8 (Singular behaviour of \mathbb{F}_H : big jump transition). *Suppose $a < \varrho_c < b$. Then the function \mathbb{F}_H exhibits at h_c the same singularity as \mathbb{G} at ϱ_c . More precisely, set*

$$(\mathbb{F}_H)_{reg}(h) = \sup_{\varrho \in [0,1]} \{\varrho h + H(\varrho)\}, \quad (4.2.9)$$

which is analytic on (h_\star, h^\star) , and $\mathbb{F}_H(h) = (\mathbb{F}_H)_{reg}(h)$ for $h \leq h_c$. Then, if $\alpha \neq 1$, there exists $c > 0$ such that

$$\mathbb{F}_H(h) - (\mathbb{F}_H)_{reg}(h) \underset{h \searrow h_c}{\sim} c \mathbb{G} \left(\varrho_c \frac{h}{h_c} \right) \underset{h \searrow h_c}{\sim} -(h - h_c)^\kappa \bar{\varphi}((h - h_c)^{-1}); \quad (4.2.10)$$

and if $\alpha = 1$,

$$\lim_{h \searrow h_c} \frac{\log(\mathbb{F}_H(h) - (\mathbb{F}_H)_{reg}(h))}{\log(h - h_c)} = +\infty. \quad (4.2.11)$$

This result motivates further remarks.

Remark 4.2.9.

- Since \mathbb{F}_H is non-trivial on both the left side and the right side of h_c , one needs to subtract a regular part (here the left part) to exhibit the singularity.
- The constant c in (4.2.10) can be made explicit and be expressed with the parameters of the model $(h_c, \varrho_c, \kappa, H''(\varrho_c)$ and $\sigma^2)$ (see the proof).
- When $\alpha = 1$, the critical exponents are formally $+\infty$ for both \mathbb{F}_H and \mathbb{G} , but the asymptotic equivalence (4.2.10) is not necessarily recovered.
- The singular behaviour when h approaches h_\star or h^\star (when they are finite) will be briefly discussed in Subsection 4.2.4.

It is interesting to compare with the PS free energy

$$\mathbb{F}(h) = \lim_{N \rightarrow +\infty} \frac{1}{N} \log Z_{N,h} = \sup_{\varrho \in [0,1]} \{\varrho h + \mathbb{G}(\varrho)\}, \quad (4.2.12)$$

for which the unique phase transition occurs at $h_c = 0$, and the critical exponent is $\max(1, 1/\alpha)$ (see (3.1.12)). In fact, this can be recovered in our setting, by using the variational formula (4.2.12) and the properties of \mathbb{G} .

4.2.2 Path Properties for the kernel models

In this subsection, we state formally the results about the big jump phase. The relevant indicator is the size of the largest jump of the renewal

$$M_{|\tau|} = \max_{1 \leq k \leq |\tau|} \eta_k = \max_{1 \leq k \leq |\tau|} (\tau_k - \tau_{k-1}), \quad (4.2.13)$$

under the measure $\xi_N^{\Psi, h}$ and then under the measure $\mu_{N, L}$. Depending if the density ϱ is smaller or larger than ϱ_c , it will behave drastically differently. In some cases, looking at the size of the second largest jump,

$$M'_{|\tau|} = \max_{1 \leq k_1 \neq k_2 \leq |\tau|} \min(\eta_{k_1}, \eta_{k_2}), \quad (4.2.14)$$

can be relevant. Here are the main results concerning the kernel models path properties.

We recall* that for a given sequence of probability measures μ_n , and a sequence of real-valued random variables X_n , and a constant c ,

$$X_n \xrightarrow[n \rightarrow +\infty]{\mu_n} c$$

means that for any $\delta > 0$,

$$\mu_n (|X_n - c| \geq \delta) \xrightarrow[n \rightarrow +\infty]{} 0.$$

Proposition 4.2.10 (Laws of large numbers for the contact density and the largest jump).
For any $h \in \mathbf{R}$,

$$\frac{|\tau|}{N} \xrightarrow[N \rightarrow +\infty]{\xi_N^{\Psi, h}} \varrho = F'_H(h), \quad (4.2.15)$$

and

$$\frac{M_{|\tau|}}{N} \xrightarrow[N \rightarrow +\infty]{\xi_N^{\Psi, h}} \left(1 - \frac{\varrho}{\varrho_c}\right)_+. \quad (4.2.16)$$

Remark 4.2.11. When $\varrho_c = 0$ (i.e., $\mathbf{E}[\eta] = +\infty$), the limit in (4.2.16) becomes $\mathbf{1}_{\varrho=0}$.

Before discussing what it tells us about typical paths of the kernel models, let us give a refinement about the largest jump(s) behaviour. Recall the definition of a_N (Lemma 4.A.4), which is roughly $N^{\frac{1}{\alpha}}$.

Proposition 4.2.12.

- When $h < h_c$, the second largest is power law: $a_N^{-1} M'_{|\tau|}$ converges in distribution to a nontrivial random variable.
- When $h = h_c \in (h_*, h^*)$, the largest jump has a polynomial size: $a_N^{-1} M_{|\tau|}$ is tight.
- When $h > h_c$, the largest jump has logarithmic size: there exists $c > 0$ such that

$$\frac{M_{|\tau|}}{c \log N} \xrightarrow[N \rightarrow +\infty]{\xi_N^{\Psi, h}} 1. \quad (4.2.17)$$

*See the list of symbols and abbreviations.

These two propositions described the big jump/localization phase transition. In a nutshell, when $h < h_c$, there is a unique “big jump”, with linear size (the other jumps are $o(N)$), whereas when $h > h_c$, all jumps have at most logarithmic size. We finish with a fluctuation result.

Proposition 4.2.13 (Fluctuations of the contact density). *If $h_\star < h < h^\star$, then $|\tau|$ has Gaussian fluctuations: there exists $\tilde{\sigma} > 0$ (explicit) such that*

$$\frac{|\tau| - \varrho N}{\tilde{\sigma}\sqrt{N}} \xrightarrow[N \rightarrow \infty]{\xi_N^{\Psi, h}} \mathcal{N}(0, 1). \quad (4.2.18)$$

Fluctuations of the largest jump $M_{|\tau|}$ can also be obtained. They derive from the next subsection’s results, dealing with the asymptotic behaviour of the largest jump under the constrained measure $\mu_{N,L}$, together with Proposition 4.5.2, which provides an approximation of $\xi_N^{\Psi, h}$ as a mixing of the measures $\mu_{N,L}$ in a rather explicit and more usable fashion than the original (4.1.14). We refer to 4.5.4 for some details.

Remark 4.2.14. For $h \notin (h_\star, h^\star)$, we only gave the law of large number for $|\tau|$ and $M_{|\tau|}$. Fluctuations or more precise path properties under $\xi_N^{\Psi, h}$ when $h \notin (h_\star, h^\star)$ depend on the behaviour of H and Ψ on the boundary of \mathcal{S} . We do not investigate it deeper since the phases $h \notin (h_\star, h^\star)$ are actually excluded in the three physical models we consider, but the proofs below can easily be adjusted.

4.2.3 Largest Jump for the constrained PS model

In the following paragraphs, we investigate the size M_L of the largest jump for the constrained PS model $\mu_{N,L}$, that is the PS model with a fixed number of contact points. Under the measure \mathbf{P} , that is for L independent jumps with distribution K , the size of the largest jump is power law, roughly $N^{1/\alpha}$ (Proposition 4.A.6). How does the conditioning $\tau_L = N$ affects it? Of course, the behaviour depends on the value of the asymptotic density $\varrho = \lim \frac{L}{N}$: whether it is smaller or larger than ϱ_c .

The question has been deeply studied by Armendariz et al. (see for instance [4, 5]) from a slightly different point of view: the measure $\mu_{N,L}$ is also the equilibrium measure of a particular zero-range process. In this setting, the big jump transition corresponds to a condensation phenomenon. In a word:

Theorem 4.2.15. *For any $\varrho \in [0, 1]$,*

$$\frac{M_L}{N} \xrightarrow[L/N \rightarrow \varrho]{\mu_{N,L}, N \rightarrow +\infty} \begin{cases} \left(1 - \frac{\varrho}{\varrho_c}\right) & \text{if } \varrho < \varrho_c, \\ 0 & \text{if } \varrho > \varrho_c. \end{cases} \quad (4.2.19)$$

The second largest jump has size $o(N)$ in probability.

The result is not fully satisfactory, particularly for $\varrho > \varrho_c$. In fact, the investigation conducted in [4, 5] goes way deeper, providing refined limit theorems and fluctuations results for $M_{|\tau|}$. The critical regime $\varrho = \varrho_c$ is the most subtle and it requires a more difficult investigation.

4.2.3.1 Subcritical and supercritical density

We start with the subcritical regime ($\varrho < \varrho_c$), for which the event $\tau_L = N$ corresponds to non Cramer deviations for the random walk (τ_n) and produces a condensation phenomenon (or big jump phenomenon, depending on the perspective), that is, the first line in (4.2.19).

Proposition 4.2.16 (Subcritical density regime, [5]). *Suppose $\mathbf{E}[\eta] < +\infty$ and $\frac{L}{N} \rightarrow \varrho \in [0, \varrho_c)$. Then the largest jump has linear size*

$$\frac{M_L}{N - L\mathbf{E}[\eta]} \xrightarrow{\mu_{N,L}} 1. \quad (4.2.20)$$

And, if $L \rightarrow +\infty$, its fluctuations satisfy

$$\frac{M_L - (N - L\mathbf{E}[\eta])}{b_L} \xrightarrow{\mu_{N,L}} -X_\alpha. \quad (4.2.21)$$

As for the second largest jump, for any $t \geq 0$, we have

$$\mu_{N,L} \left(\frac{M'_L}{a_L} \leq t \right) \sim \mathbf{P} \left(\frac{M_L}{a_L} \leq t \right) \xrightarrow{L \rightarrow +\infty} \exp \left(-\frac{1}{\alpha t^\alpha} \right). \quad (4.2.22)$$

Remark 4.2.17. Note that convergence (4.2.20) still holds when L stays bounded.

Contrariwise, the supercritical regime corresponds to Cramer deviations of (τ_n): the atypical event $\tau_L = N$ is typically accomplished when all the jumps are small (actually the event is typical for jumps with a distribution tilted by an exponential decay). Then the largest jump has logarithmic size instead of power law:

Proposition 4.2.18 (Supercritical density regime). *Suppose $\alpha \geq 0$ and $\frac{L}{N} \rightarrow \varrho \in (\varrho_c, 1)$. Consider the unique $x > 0$ such that $\mathbf{E}_x[\eta] = \varrho$. There exists a sequence of integers (z_L), with $z_L \sim \frac{\log L}{x}$, and a bounded sequence $s_L \geq c_\varrho > 0$ (these sequences also depend on N), such that, for any $k \in \mathbf{Z}$,*

$$\mu_{N,L}(M_L - z_L \leq k) \sim \exp(-s_L e^{-xk}). \quad (4.2.23)$$

Remark 4.2.19. Of course (4.2.23) is equivalent to

$$\mu_{N,L}(M_L - z_L = k) \sim \exp(-s_L e^{-xk}) - \exp(-s_L e^{-x(k-1)}). \quad (4.2.24)$$

The sequences z_L, s_L are explicit (see the proof). The sequence (s_L) does not necessarily converge: if not then neither does $\mu_{N,L}(M_L - z_L \leq k)$. Actually, considering the unique $x_L > 0$ such that $\mathbf{E}_{x_L}[\eta] = \frac{N}{L}$ (note that $x_L \rightarrow x$), one has

$$\mu_{N,L}(M_L - z_L \leq k) \sim \mathbf{P}_{x_L}(M_L - z_L \leq k). \quad (4.2.25)$$

In other words, once the distribution of the jumps is tilted, the conditioning does not affect the largest jump.

4.2.3.2 Critical regime

This paragraph is devoted to the critical regime, that is

$$\frac{N}{L} \longrightarrow \mathbf{E}[\eta]. \quad (4.2.26)$$

Some of the following results are from [4]. The critical regime is the most demanding, and different behaviours emerge, depending on the parameter α . We will need to distinguish upside moderate deviations (when $N \geq L\mathbf{E}[\eta]$) and downside moderate deviations (when $N \leq L\mathbf{E}[\eta]$).

Proposition 4.2.20 (Critical regime, $\alpha \in (1, 2)$). *Write $N = L\mathbf{E}[\eta] + u_L a_L$.*

1. *If $u_L \rightarrow u \in \mathbf{R}$ then for any $w > 0$,*

$$\mu_{N,L} \left(\frac{M_L}{a_L} \leq w \right) \xrightarrow{L \rightarrow +\infty} e^{-\frac{1}{\alpha w^\alpha} \frac{g_{\alpha,w} \left(u + \frac{1}{(\alpha-1)w^{\alpha-1}} \right)}{g_\alpha(u)}}. \quad (4.2.27)$$

where g_α is the density of X_α and, for $w > 0$, $g_{\alpha,w}$ the density of the random variable $X_{\alpha,w}$ whose characteristic function is

$$\mathbf{E}[e^{itX_{\alpha,w}}] = \exp \left(\int_0^w \frac{e^{ity} - 1 - ity}{y^{1+\alpha}} dy \right). \quad (4.2.28)$$

2. *If $u_L \rightarrow +\infty$, then (4.2.20), (4.2.21) and (4.2.22) hold.*

Remark 4.2.21. We leave open the question for $u_L \rightarrow -\infty$. Note that then $x_L \rightarrow 0$ (see Def. (4.1.22)) and there exists an explicit sequence $A_L \rightarrow +\infty$ such that

$$\mathbf{P}_{x_L}(x_L(M_L - A_L) \leq t) \xrightarrow{L \rightarrow +\infty} e^{-e^{-t}}. \quad (4.2.29)$$

We expect it to also be the right answer for $\mu_{N,L}$, but $\mathbf{P}_{x_L}(\tau_L = N)$ and $\hat{\mathbf{P}}_{x_L}(\tau_L = N)$ are not under control.

Note that when $\alpha \in (1, 2)$, the size of the largest jump M_L and the deviations of τ_L are of the same order under \mathbf{P} . That's why the conditioning $\tau_L = N$ affects the size of the largest jump, even for moderate deviations. It is not the case when $\alpha > 2$, for the largest jump size is much smaller than the typical deviations of τ_L . The two following results are from [4], who only investigates the regime $\alpha > 2$. The first one concerns upside deviations whereas the second one, which can be slightly extended, concerns downside deviations. We provide a proof for the second one, for which one case was not entirely addressed in [4].

Proposition 4.2.22 (Critical regime, $\alpha > 2$: upside deviations). *Suppose $N \geq L\mathbf{E}[\eta]$ and let us write*

$$N = L\mathbf{E}[\eta] + \sigma \sqrt{(\alpha - 2)L \log L} \left(1 + \frac{\alpha + 1}{2(\alpha - 2)} \frac{\log \log L}{\log L} - \frac{\log \varphi(\sqrt{L \log L})}{(\alpha - 2) \log L} + \frac{u_L}{\log L} \right), \quad (4.2.30)$$

which defines u_L

1. *If $u_L \rightarrow +\infty$, then (4.2.20), (4.2.21) and (4.2.22) hold.*

2. If $u_L \rightarrow -\infty$, then

$$\mu_{N,L} \left(\frac{M_L}{a_L} \leq t \right) \sim \mathbf{P} \left(\frac{M_L}{a_L} \leq t \right) \xrightarrow{L \rightarrow +\infty} \exp \left(-\frac{1}{\alpha t^\alpha} \right). \quad (4.2.31)$$

3. If $u_L \rightarrow u \in \mathbf{R}$ then there exists $p_u \in (0, 1)$ such that

$$\frac{M_L}{N - L\mathbf{E}[\eta]} \xrightarrow{L \rightarrow +\infty} \text{Ber}(p_u), \quad (4.2.32)$$

with $p_u \rightarrow 0$ when $u \rightarrow -\infty$ and $p_u \rightarrow 1$ when $u \rightarrow +\infty$.

Proposition 4.2.23 (Critical regime, $\eta \in \mathbf{L}^2$: downside deviations). Suppose $\alpha \geq 2$ and $\sigma^2 = \text{Var}(\eta) < +\infty$. Write $N = L\mathbf{E}[\eta] - \theta_L$, with $0 \leq \theta_L \ll L$.

1. If $\theta_L \ll \frac{L}{a_L}$, then

$$\mu_{N,L} \left(\frac{M_L}{a_L} \leq t \right) \sim \mathbf{P} \left(\frac{M_L}{a_L} \leq t \right) \xrightarrow{L \rightarrow +\infty} \exp \left(-\frac{1}{\alpha t^\alpha} \right). \quad (4.2.33)$$

2. If $\frac{a_L \theta_L}{\sigma^2 L} \rightarrow \omega \in (0, +\infty)$, then

$$\mu_{N,L} \left(\frac{M_L}{a_L} \leq t \right) \xrightarrow{L \rightarrow +\infty} \exp \left\{ -\int_t^{+\infty} e^{-\omega s} s^{-1-\alpha} ds \right\}. \quad (4.2.34)$$

3. If $\theta_L \gg \frac{L}{a_L}$, there exists a sequence $A_L \rightarrow +\infty$ and $y_L = \frac{\theta_L}{\sigma^2 L} \rightarrow 0$ such that

$$\mu_{N,L} (y_L(M_L - A_L) \leq t) \xrightarrow{L \rightarrow +\infty} e^{-e^{-t}}. \quad (4.2.35)$$

Remark 4.2.24. We do not treat the marginal cases $\alpha \in \{1, 2\}$. When $\alpha = 2$ and $\text{Var}(\eta) < +\infty$, the result for upside moderate deviations should be the same as Proposition 4.2.22, replacing (4.2.30) by

$$N = L\mathbf{E}[\eta] + \sqrt{L}K_L \left(1 + \frac{u_L}{K_L^2} \right), \quad (4.2.36)$$

for a non-explicit sequence $1 \ll K_L \ll \sqrt{\log L}$ which depends on the slowly varying function φ .

When $\alpha \in \{1, 2\}$ with $\text{Var}(\eta) = +\infty$, a more subtle analysis should be conducted, to obtain an analogous of Proposition 4.2.20 with a presumably different scaling. However we leave these predictions as conjectures.

4.2.4 Discussion: phases and phase transitions in the kernel models

For the constrained PS model, as well as for the convoluted models, the phase transition between the big jump phase and the localized phase has been fully described, and the free energy critical exponent computed in the previous subsections. We mentioned in the introduction other types of phase transition for the kernel models. They can occur if h_* or h^* is finite. We will not go through much details since these do not happen in the physical models. Though let us sketch what phenomena can emerge. Four types of phases can occur, corresponding to different values of the contact density ϱ (see Proposition 4.2.10).

- (i) Delocalized phase (Figure 4.3): when $\varrho = 0$. The number of contact points is $o(N)$. There is a unique macroscopic loop of size N . This occurs if and only if $a = 0$ and $H'(0) < +\infty$.
- (ii) Big jump phase (Figure 4.4): when $0 < \varrho < \varrho_c$. The number of contact points is of order N . There is a unique big loop of size $c(\varrho)N$, the size of the other loops is power law.
- (iii) Localized phase (Figure 4.2): when $\varrho > \varrho_c$. The number of contact points is of order N , as well as the total loop length, but the loops are typically small (logarithmic size).
- (iv) Density saturation: when $\varrho = a \in (0, 1)$ (resp. $\varrho = b \in (0, 1)$). It is a phase where the density of contact points stays constant. It can happen in the localized phase (if $a > \varrho_c$) or in the jump phase (when $a < \varrho_c$), and in this case the normalized size of the big jump $(1 - a/\varrho_c)$ is also saturated. This phase exists when $H'(a) < +\infty$ (resp. $H'(b) > -\infty$).

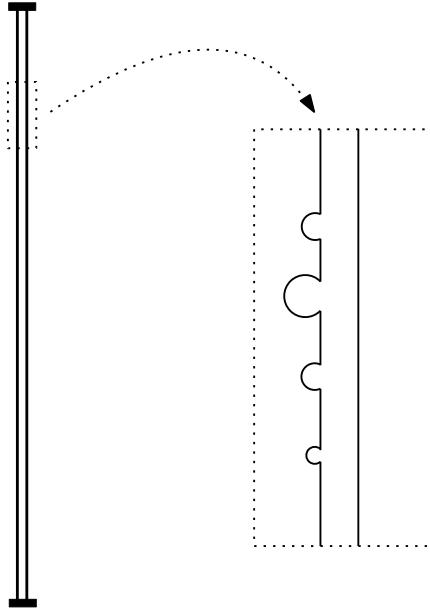


FIGURE 4.2: A schematic view of the localized phase for the kernel models, completely similar to the PS localized phase. The right line is the defect line. On the right, a zoom show the local structure of the renewal: all jumps are typically finite, the largest jump has size $O(\log N)$.

Depending on Ψ , and particularly on the smoothness of H , different phase transitions between the phases described above occur, and the corresponding critical exponent for F_H can be computed in each case. Let us give one example when $h_\star > -\infty$ (for instance if $H(x) = x(1 - x)$). Define $U(x) = G(x) + H(x)$ and suppose that when $x \searrow a$, one has $U'(x) = U'(a) - c(x - a)^\lambda(1 + o(1))$ for some $c, \lambda > 0$ (typically $\lambda = 1$). Then, for $h \leq h_\star$,

$$F_H(h) = U(a) + ah. \quad (4.2.37)$$

And for $h \searrow h_\star$,

$$F_H(h) = U(a) + ah + C(h - h_\star)^{\frac{\lambda+1}{\lambda}} \mathbf{1}_{h \geq h_\star} (1 + o(1)). \quad (4.2.38)$$

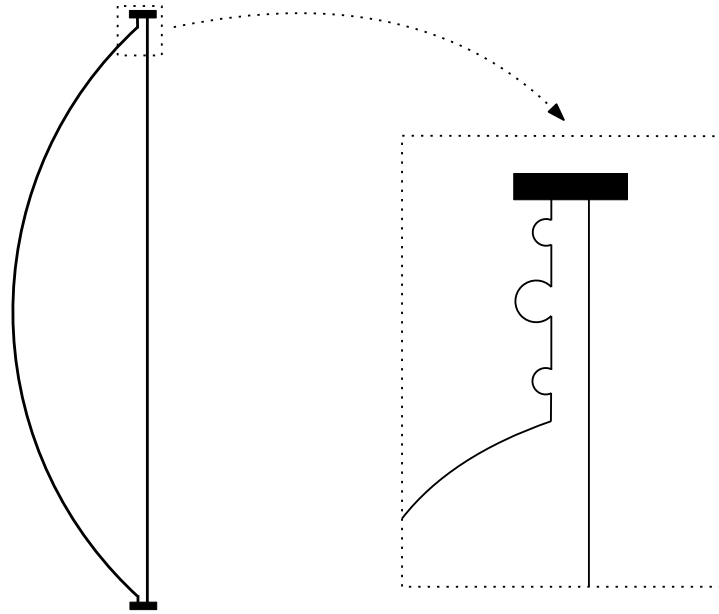


FIGURE 4.3: Delocalized phase: only a few sites ($o(N)$) are pinned near the the extremities.

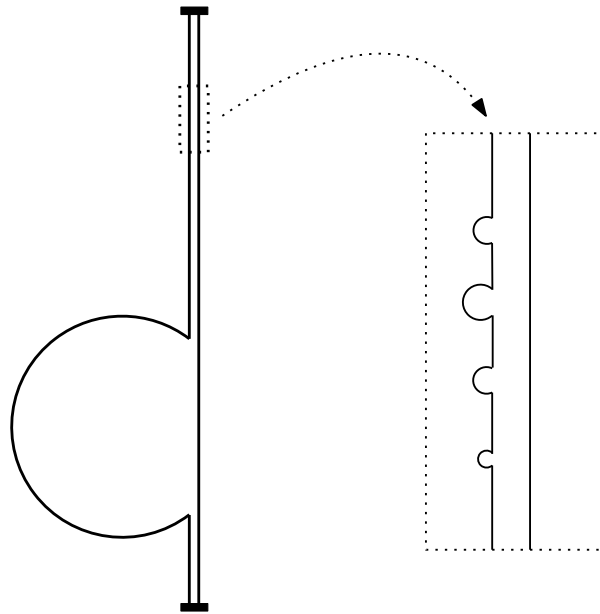


FIGURE 4.4: Big jump phase: a unique large loop emerges, its size is proportional to N . The rest of the renewal is “typical” and the other loops have finite length (more precisely: the largest (once the big jump is excluded) has a power law size, which just reflects the power law behaviour of $K(n)$ for $n \rightarrow \infty$).

The corresponding phase transition for the trajectories depends on a and α .

- When $a > 0$, it is a contact density saturation.
- When $a = 0$ and $\mathbf{E}[\eta] < +\infty$, it is a the delocalization/big jump phase transition.
- When $a = 0$ and $\alpha < 1$, then $h_c = h_* \in \mathbf{R}$ and it is a delocalization/localization phase transition. Writing $H'(x) = H'(0) - cx^\theta(1+o(1))$ as $x \searrow 0$, one has $\lambda = \min(\theta, \kappa - 1)$

and the critical exponent for F_H is $\min(1 + 1/\theta, 1/\alpha)$.

4.3 Proofs: free energy

4.3.1 Free Energy $G(\varrho)$

Proof of Lemmas 4.2.1. – We leave for now the convergence (4.2.1), and we define G by the infimum in (4.2.1). For any $\varrho \in (0, 1)$, the function $g_\varrho(x) = x + \varrho \log \mathbf{E}[e^{-x\eta}]$ is strictly convex on \mathbf{R}_+ . Furthermore, with $\mathbf{E}[e^{-\eta x}] \sim \mathbf{P}(\eta = 1)e^{-x}$, we get $\lim_{x \rightarrow +\infty} g_\varrho(x) = +\infty$. Therefore its minimum is reached at a unique point $x_\varrho \in \mathbf{R}_+$. We compute the derivative:

$$g'_\varrho(x) = 1 - \varrho \frac{\mathbf{E}[\eta e^{-x\eta}]}{\mathbf{E}[e^{-x\eta}]} = 1 - \varrho \mathbf{E}_x[\eta]. \quad (4.3.1)$$

If $\varrho \leq \varrho_c = 1/\mathbf{E}[\eta]$ then g_ϱ is nondecreasing and $G(\varrho) = g_\varrho(0) = 0$. If $\varrho > \varrho_c$ then the minimum is reached at a point $x_\varrho > 0$, which is the only zero of g'_ϱ on $(0, +\infty)$. Since $(\varrho, x) \mapsto g'_\varrho(x)$ is \mathcal{C}^1 on $\mathbf{R} \times \mathbf{R}_+$, and analytic on $\mathbf{R} \times \mathbf{R}_+$, with the implicit function theorem, $\varrho \mapsto x_\varrho$ is continuous, and analytic on $(\varrho_c, 1)$, and so is G . Moreover, $\varrho \mapsto x_\varrho$ is increasing on $(\varrho_c, 1)$, and $x_\varrho \rightarrow +\infty$ when $\varrho \rightarrow 1$. Then G is decreasing on $(\varrho_c, 1)$, and $G'(\varrho) = \log \mathbf{E}[e^{-x_\varrho \eta}] \rightarrow -\infty$ when $\varrho \rightarrow 1$.

- We finish with the convergence (4.2.1). When $\varrho < \varrho_c$, it is a consequence of (4.A.46), whereas for $\varrho > \varrho_c$, it can be deduced from Lemma 4.4.3 (see also [19]). For $\varrho = \varrho_c$, it is slightly more technical and we admit for now. If $\varrho_c > 0$, fix $\epsilon > 0$ sufficiently small. Then let L_1, L_2, N_1, N_2 such that $L_1 + L_2 = L$, $N_1 + N_2 = N$, $\frac{L_1}{N_1} \rightarrow \varrho(1 - \epsilon)$, $\frac{L_2}{N_2} \rightarrow \varrho(1 + \epsilon)$, $\frac{L_i}{L} \rightarrow \frac{1}{2}$, and $\frac{N_i}{N} \rightarrow \frac{1}{2}$. Observe that

$$\mathbf{P}(\tau_L = N) \geq \mathbf{P}(\tau_{L_1} = N_1) \mathbf{P}(\tau_{L_2} = N_2). \quad (4.3.2)$$

Thus

$$\liminf \mathbf{P}(\tau_L = N) \geq \frac{1}{2} G(\varrho_c + \epsilon), \quad (4.3.3)$$

which vanishes when $\epsilon \searrow 0$. Thus we get the convergence (4.2.1). On the other hand, when $\varrho_c = 0$, and $\frac{L}{N} \rightarrow 0$, just write $\log \mathbf{P}(\tau_L = N) \geq (L - 1) \log K(1) + \log K(N - L - 1)$. □

We add some useful properties to Lemma 4.2.1.

Lemma 4.3.1. – *If $\alpha > 1$ and $\text{Var}(\eta) = +\infty$, then G is \mathcal{C}^2 on $[0, 1)$, with $G''(\varrho_c) = 0$.*

- *If $\varrho = \varrho_c$ with $\sigma^2 = \text{Var}(\eta) < +\infty$, then $\lim_{\varrho \searrow \varrho_c} G''(\varrho) = -(\varrho_c^3 \sigma^2)^{-1}$.*
- *If $\varrho > \varrho_c$ then let $x = x_\varrho$ be such that $\mathbf{E}_x[\eta] = \frac{1}{\varrho}$, and $\sigma_x^2 = \text{Var}_x(\eta)$. Then $G''(\varrho) = -(\varrho^3 \sigma_x^2)^{-1}$.*

Proof. We know that G is analytic on $[0, 1) \setminus \{\varrho_c\}$. We compute $G''(\varrho)$ for $\varrho > \varrho_c$. When $\varrho > \varrho_c$, since x_ϱ solves $\mathbf{E}_{x_\varrho}[\eta] = 1/\varrho$, one gets

$$\partial_\varrho x_\varrho = -\frac{1}{\varrho^2} \frac{1}{(\partial_x \mathbf{E}_x[\eta])|_{x_\varrho}} = \frac{-1}{\varrho^2 (-\text{Var}_{x_\varrho}(\eta))} = \frac{1}{\varrho^2 \sigma_x^2}. \quad (4.3.4)$$

Hence,

$$G''(\varrho) = \partial_{\varrho} x_{\varrho} \times (\partial_x \log \mathbf{E}[e^{-x\eta}]|_{x_{\varrho}}) = \frac{1}{\varrho^2 \sigma_x^2} (-\mathbf{E}_{x_{\varrho}}[\eta]) = \frac{-1}{\varrho^3 \sigma_x^2}. \quad (4.3.5)$$

And this goes to $\frac{-1}{\varrho^3 \sigma_x^2}$ as $\varrho \searrow \varrho_c$. \square

Instead of Proposition 4.2.3, we prove the following stronger result.

Lemma 4.3.2. *The following hold as $\varrho \searrow \varrho_c$, for a certain slowly varying function $\tilde{\varphi}$ (depending on α and φ).*

1. If $\alpha \in [0, 1)$,

$$G(\varrho) \sim -\varrho^{\frac{1}{1-\alpha}} \tilde{\varphi}(1/\varrho), \quad G'(\varrho) \sim -\frac{1}{1-\alpha} \varrho^{\frac{\alpha}{1-\alpha}} \tilde{\varphi}(1/\varrho). \quad (4.3.6)$$

2. If $\alpha = 1$,

$$\lim_{\varrho \searrow \varrho_c} \frac{\log G(\varrho)}{\log(\varrho - \varrho_c)} = +\infty, \quad \lim_{\varrho \searrow \varrho_c} \frac{\log G'(\varrho)}{\log(\varrho - \varrho_c)} = +\infty. \quad (4.3.7)$$

3. If $\alpha \in (1, 2]$ and $\sigma^2 = +\infty$,

$$G(\varrho) \sim -(\varrho - \varrho_c)^{\frac{\alpha}{\alpha-1}} \tilde{\varphi}((\varrho - \varrho_c)^{-1}), \quad G'(\varrho) \sim -\frac{\alpha}{\alpha-1} (\varrho - \varrho_c)^{\frac{1}{\alpha-1}} \tilde{\varphi}((\varrho - \varrho_c)^{-1}). \quad (4.3.8)$$

And, if $\alpha = 2$, $\tilde{\varphi}(u) \rightarrow 0$ when $u \rightarrow +\infty$.

4. If $\alpha \geq 2$ and $\sigma^2 < +\infty$ then

$$G(\varrho) \sim -C(\varrho - \varrho_c)^2, \quad G'(\varrho) \sim -2C(\varrho - \varrho_c). \quad (4.3.9)$$

Proof. We know that

$$G(\varrho) = g_{\varrho}(x_{\varrho}) = x_{\varrho} + \varrho \log \mathbf{E}[e^{-x_{\varrho}\eta}], \quad G'(\varrho) = \partial_{\varrho} g_{\varrho}(x_{\varrho}) = \log \mathbf{E}[e^{-x_{\varrho}\eta}], \quad (4.3.10)$$

where, when $\varrho > \varrho_c$, x_{ϱ} is the unique positive solution of

$$\frac{\mathbf{E}[\eta e^{-x_{\varrho}\eta}]}{\mathbf{E}[e^{-x_{\varrho}\eta}]} = \frac{1}{\varrho}, \quad (4.3.11)$$

which vanishes as $\varrho \searrow \varrho_c$. We write $\varrho = \varrho_c + \delta$ and use the estimates of Lemma 4.A.5 to obtain a equivalent of x_{ϱ} . The asymptotic behaviour of G' will immediately follow thanks to

$$G'(\varrho) \sim \mathbf{E}[e^{-x_{\varrho}\eta}] - 1. \quad (4.3.12)$$

First note that when $\varrho_c > 0$, $\mathbf{E}[\eta] - \frac{1}{\varrho} \sim \mathbf{E}[\eta]^2 \delta$.

1. If $\text{Var}(\eta) < +\infty$ then

$$\frac{\mathbf{E}[\eta e^{-x\eta}]}{\mathbf{E}[e^{-x\eta}]} = \frac{\mathbf{E}[\eta] - \mathbf{E}[\eta^2]x(1 + o(1))}{1 - x\mathbf{E}[\eta](1 + o(1))} = \mathbf{E}[\eta] - x \text{Var}(\eta)(1 + o(1)). \quad (4.3.13)$$

Thus $x_{\varrho} \sim \frac{\mathbf{E}[\eta]^2}{\text{Var}(\eta)} \delta$. We find the desired expansion by injecting it in the expression of G' .

2. If $\alpha \in [1, 2]$ with $\mathbf{E}[\eta] < +\infty$ and $\text{Var}(\eta) = +\infty$ then

$$\frac{\mathbf{E}[\eta e^{-x\eta}]}{\mathbf{E}[e^{-x\eta}]} = \frac{\mathbf{E}[\eta] - x^{\alpha-1}\bar{\varphi}(1/x)(1+o(1))}{1 - x\mathbf{E}[\eta](1+o(1))} = \mathbf{E}[\eta] - x^{\alpha-1}\bar{\varphi}(1/x)(1+o(1)) \quad (4.3.14)$$

with $\bar{\varphi}(u) \rightarrow 0$ when $\alpha = 1$ and $\bar{\varphi}(u) \rightarrow +\infty$ when $\alpha = 2$ as $u \rightarrow +\infty$. Thus,

$$x_\varrho^{\alpha-1}\bar{\varphi}(1/x_\varrho) \sim \mathbf{E}[\eta]^2\delta. \quad (4.3.15)$$

Therefore, with [14, Th. 1.5.12], $x_\varrho \sim \delta^{\frac{1}{\alpha-1}}\varphi_0(1/\delta)$ for a certain slowly varying function φ_0 as long as $\alpha \neq 1$.

$$G'(\varrho) \sim -x_\varrho\mathbf{E}[\eta] \sim -\mathbf{E}[\eta]\delta^{\frac{1}{\alpha-1}}\varphi_0(1/\delta). \quad (4.3.16)$$

When $\alpha = 2$, $\varphi_0(u) \rightarrow 0$ as $u \rightarrow +\infty$.

When $\alpha = 1$, we have

$$\bar{\varphi}(1/x_\varrho) \sim \mathbf{E}[\eta]^2\delta \quad (4.3.17)$$

with $\bar{\varphi}(u) \rightarrow 0$. We know [14, Prop. 1.3.6], that for any $a > 0$, $x^a \ll \varphi(1/x)$ as $x \searrow 0$. So, for any $a > 0$, $x_\varrho \ll \delta^{\frac{1}{a}}$:

$$\lim_{\varrho \searrow \varrho_c} \frac{\log x_\varrho}{\log(\varrho - \varrho_c)} = +\infty. \quad (4.3.18)$$

And the same goes for G and G' since

$$G(\varrho) = x_\varrho - \varrho x_\varrho \bar{\varphi}(1/x_\varrho)(1+o(1)), \quad G'(\varrho) = -x_\varrho \bar{\varphi}(1/x_\varrho)(1+o(1)). \quad (4.3.19)$$

3. If $\alpha \in [0, 1]$ and $\mathbf{E}[\eta] = +\infty$ then

$$\frac{\mathbf{E}[\eta e^{-x\eta}]}{\mathbf{E}[e^{-x\eta}]} \sim x^{\alpha-1}\bar{\varphi}(1/x), \quad (4.3.20)$$

with $\bar{\varphi}(u) \rightarrow 0$ when $\alpha = 0$ and $\bar{\varphi}(u) \rightarrow +\infty$ when $\alpha = 1$ as $u \rightarrow +\infty$. And $1/\varrho = \delta^{-1}$. Therefore, with [14, Th. 1.5.12], $x_\varrho \sim \delta^{\frac{1}{1-\alpha}}\varphi_0(1/\delta)$ for a certain slowly varying function φ_0 as long as $\alpha \neq 1$. And

$$G'(\varrho) \sim -x_\varrho^\alpha \bar{\varphi}(1/x_\varrho) \sim -\delta^{\frac{\alpha}{1-\alpha}}\bar{\varphi}(1/\delta). \quad (4.3.21)$$

For $\alpha = 1$ the discussion is the same as before since $\frac{1}{\bar{\varphi}(1/x_\varrho)} \sim \delta$, with $\frac{1}{\bar{\varphi}(u)} \rightarrow 0$.

When $\alpha \neq 1$, to obtain the asymptotic behaviour of G from what we found for G' , we use [14, Prop. 1.5.10]. \square

4.3.2 Free Energy $F_H(h)$

The proof of Lemma 4.2.4 is postponed to section 4.5 (see the proof of Lemma 4.5.1). Until there, F_H will be the function

$$F_H(h) = \sup_{\varrho \in [0,1]} \{\varrho h + G(\varrho) + H(\varrho)\}. \quad (4.3.22)$$

Recall that the definition of a, b and \mathcal{I} can be found in the Assumptions 4.1.4. Define

$$U : \varrho \mapsto H(\varrho) + G(\varrho). \quad (4.3.23)$$

Lemma 4.3.3. *For any $h \in \mathbf{R}$, the function $\varrho \mapsto \varrho h + U(\varrho)$ reaches its maximum at a unique maximum point $\varrho(h) \in \mathcal{I} \setminus \{1\}$. The function $h \mapsto \varrho(h)$ is continuous and nondecreasing, with $\varrho(h) = a$ for $h \leq h_*$, $\varrho(h) = b$ for $h \geq h^*$, and $\varrho(h_c) = \varrho_c$. The function $\varrho(h)$ is analytic as soon as $h \notin \mathcal{C}$.*

The free energy F_H is analytic on $\mathbf{R} \setminus \mathcal{C}$.

Proof. Both H (Assumptions 4.1.4) and G (Lemma 4.2.1) are concave on $[0, 1]$, and H is strictly concave. Therefore, so is U . As a strictly concave function, $\varrho \mapsto \varrho h + U(\varrho)$ has a unique maximum point $\varrho(h) \in \mathcal{I}$. Since its left derivative at $\varrho = 1$ is $-\infty$ (Lemma 4.2.1), the minimum is actually reached in $\mathcal{I} \setminus \{1\}$.

- When $h \leq h_*$, the function $\varrho \mapsto \varrho h + U(\varrho)$ is decreasing on $\mathcal{I} \setminus \{1\}$. So $\varrho(h) = a$ and $F_H(h) = ha + U(a)$: it is analytic on $(-\infty, h_*)$.
- Similarly, when $h \geq h^*$, $\varrho \mapsto \varrho h + U(\varrho)$ is increasing on $\mathcal{I} \setminus \{1\}$, so $\varrho(h) = b$ and $F_H(h) = hb + U(b)$.
- When $h \in (h_*, h^*)$, $\varrho(h) \in (a, b)$: it solves $U'(\varrho(h)) = -h$. Since U' is nonincreasing, $h \mapsto \varrho(h)$ is nondecreasing.
- The function $(h, \varrho) \mapsto h\varrho + U(\varrho)$ is continuously differentiable on $\mathbf{R} \times (a, b)$, and analytic on $\mathbf{R} \times ((a, b) \setminus \{\varrho_c\})$ (Lemma 4.2.1). Thus, by the implicit function theorem, $h \mapsto \varrho(h)$ is continuous, and it is analytic on (h_*, h^*) as long as $\varrho(h) \neq \varrho_c$, that is, since $\varrho(h)$ solves $U'(\varrho(h)) = -h$, as long as $h \neq h_c := -U'(\varrho_c)$.

□

Proof of Proposition 4.2.8. Recall the definitions

$$F_H(h) = \sup_{\varrho \in [0,1]} \{\varrho h + H(\varrho) + G(\varrho)\}, \quad (F_H)_{\text{reg}}(h) = \sup_{\varrho \in [0,1]} \{\varrho h + H(\varrho)\}. \quad (4.3.24)$$

1. **When $\text{Var}(\eta) < \infty$.** We look at the expansions of the interesting quantities around ϱ_c . Set $\epsilon = \varrho - \varrho_c$. The next expansions are for $\epsilon \searrow 0$. Recall (Lemma 4.3.2) that

$$G(\varrho) = -B\epsilon^2 \mathbf{1}_{\epsilon \geq 0} + o(\epsilon^2), \quad (4.3.25)$$

and write the expansion of H

$$H(\varrho) = H(\varrho_c) + H'(\varrho_c)\epsilon - A\epsilon^2 + o(\epsilon^2), \quad (4.3.26)$$

with $A, B > 0$ by concavity, and $H'(\varrho_c) = -h_c$ by definition. Hence the function U admits the following expansion:

$$U(\varrho) = H(\varrho_c) - h_c\epsilon - A\epsilon^2 - B\epsilon^2 \mathbf{1}_{\epsilon \geq 0} + o(\epsilon^2). \quad (4.3.27)$$

Similarly, for the derivative,

$$U'(\varrho) = -h_c - 2A\epsilon - 2B\epsilon \mathbf{1}_{\epsilon \geq 0} + o(\epsilon). \quad (4.3.28)$$

It yields

$$h + U'(\varrho) = (h - h_c) - 2A\epsilon - 2B\epsilon \mathbf{1}_{\epsilon \geq 0} + o(\epsilon). \quad (4.3.29)$$

Recall that $\varrho(h)$ solves $h + U'(\varrho) = 0$. Therefore, when h is close to h_c , the difference $\hat{\epsilon} = \hat{\epsilon}(h) := \varrho(h) - \varrho_c$ has the sign of $g := h - h_c$. More precisely, as $g \rightarrow 0$,

$$\hat{\epsilon} = \frac{g}{2A + 2B\mathbf{1}_{g \geq 0} + o(1)} = \frac{g}{2A + 2B\mathbf{1}_{g \geq 0}} + o(g). \quad (4.3.30)$$

Rewrite the free energy as

$$\begin{aligned} \mathbb{F}_H(h) &= h\varrho(h) + U(\varrho(h)) \\ &= h\varrho(h) + H(\varrho_c) - h_c\hat{\epsilon} - (A + B\mathbf{1}_{\hat{\epsilon} \geq 0})\hat{\epsilon}^2 + o(\hat{\epsilon}^2) \\ &= (h_c + g)(\varrho_c + \hat{\epsilon}) + H(\varrho_c) - h_c\hat{\epsilon} - (A + B\mathbf{1}_{\hat{\epsilon} \geq 0})\hat{\epsilon}^2 + o(\hat{\epsilon}^2) \\ &= h_c\varrho_c + H(\varrho_c) + \varrho_c g + g\hat{\epsilon} - (A + B\mathbf{1}_{\hat{\epsilon} \geq 0})\hat{\epsilon}^2 + o(\hat{\epsilon}^2). \end{aligned} \quad (4.3.31)$$

We also know that $h_c\varrho_c + H(\varrho_c) = \mathbb{F}_H(\varrho_c)$. Then, with the expansion (4.3.30) of $\hat{\epsilon}(h)$,

$$\begin{aligned} \mathbb{F}_H(h) &= \mathbb{F}_H(\varrho_c) + \varrho_c g + \frac{1}{2}g^2 \frac{1}{2A + 2B\mathbf{1}_{g \geq 0}} + o(g^2) \\ &= \mathbb{F}_H(\varrho_c) + \varrho_c g + \frac{1}{4A}g^2 - \frac{B}{4A(A+B)}g^2\mathbf{1}_{g \geq 0} + o(g^2). \end{aligned} \quad (4.3.32)$$

We get the expected result, with $c = (h_c/\varrho_c)^2(4A(A+B))^{-1}$, where $2A = H''(\varrho_c)$, and $2B = \lim_{\varrho \searrow \varrho_c} G''(\varrho)$ (which can be made explicit, see Lemma 4.3.1).

2. When $\alpha = 2$ and $\text{Var}(\eta) = \infty$. Recalling Lemma 4.3.2,

$$U(\varrho) = H(\varrho_c) - h_c\epsilon - A\epsilon^2 - \epsilon^2\tilde{\varphi}(1/\epsilon)\mathbf{1}_{\epsilon \geq 0}(1 + o(1)) + O(\epsilon^3), \quad (4.3.33)$$

$$U'(\varrho) = -h_c - 2A\epsilon - 2\epsilon\tilde{\varphi}(1/\epsilon)\mathbf{1}_{\epsilon \geq 0}(1 + o(1)) + O(\epsilon^2), \quad (4.3.34)$$

where $\tilde{\varphi}$ slowly varies and $\tilde{\varphi} \rightarrow 0$. We know that, when $g \searrow 0$,

$$g = h - h_c = 2A\hat{\epsilon} + 2\hat{\epsilon}\tilde{\varphi}(1/\hat{\epsilon})(1 + o(1)). \quad (4.3.35)$$

Thus

$$\hat{\epsilon} = \frac{g}{2A} \left(1 - \frac{1}{2}\tilde{\varphi}(1/g)(1 + o(1)) \right). \quad (4.3.36)$$

At the end of the day, when $h \searrow h_c$,

$$\begin{aligned} \mathbb{F}_H(h) &= h_c\varrho_c + H(\varrho_c) + \varrho_c g + g\hat{\epsilon} - A\hat{\epsilon}^2 - \hat{\epsilon}^2\tilde{\varphi}(1/\hat{\epsilon})(1 + o(1)) \\ &= \mathbb{F}_H(\varrho_c) + \varrho_c g + \frac{1}{4A}g^2 - \frac{1}{4A^2}g^2\tilde{\varphi}(1/g)(1 + o(1)). \end{aligned} \quad (4.3.37)$$

And similarly, when $h \rightarrow h_c$,

$$(\mathbb{F}_H)_{\text{reg}}(h) = \mathbb{F}_H(\varrho_c) + \varrho_c g + \frac{1}{4A}g^2 + O(g^3). \quad (4.3.38)$$

Eventually, recall (Lemma 4.3.2)

$$G(\varrho) \sim -(\varrho - \varrho_c)^{\frac{\alpha}{\alpha-1}}\tilde{\varphi}((\varrho - \varrho_c)^{-1}), \quad (4.3.39)$$

(here $\kappa = 2$ (4.2.3)). Therefore,

$$\mathbb{F}_H(h) - (\mathbb{F}_H)_{\text{reg}}(h) \sim -\frac{1}{4A^2}(h - h_c)^2\tilde{\varphi}(1/(h - h_c)) \sim cG\left(\varrho_c \frac{h}{h_c}\right). \quad (4.3.40)$$

with $c = (h_c/(2A\varrho_c))^2$.

3. **When $\alpha \in (1, 2)$.** The method is the same, only the form of the expansions of U and U' change. Set again $\epsilon = \varrho - \varrho_c$. We work in the limit $\epsilon \searrow 0$. Recall the critical behaviour of G and its derivative when $\epsilon > 0$:

$$G(\varrho) = -\epsilon^\kappa \tilde{\varphi}(1/\epsilon), \quad G'(\varrho) = -\kappa \epsilon^{\kappa-1} \tilde{\varphi}(1/\epsilon)(1 + o(1)). \quad (4.3.41)$$

Write the expansion of H and H' :

$$H(\varrho) = H(\varrho_c) + H'(\varrho_c)\epsilon - a_2\epsilon^2 - a_3\epsilon^3 - \dots - a_k\epsilon^k - a_{k+1}\epsilon^{k+1} + O(\epsilon^{k+2}), \quad (4.3.42)$$

$$H'(\varrho) = H'(\varrho_c) - 2a_2\epsilon - 3a_3\epsilon^2 - \dots - ka_k\epsilon^{k-1} - (k+1)a_{k+1}\epsilon^k + O(\epsilon^{k+1}), \quad (4.3.43)$$

where $k+1 = \lceil \kappa \rceil$ and $a_2 > 0$. Recall also that $H'(\varrho_c) = -h_c$. Thus, when $\kappa \in \mathbf{N}$,

$$U'(\varrho) = -h_c - 2a_2\epsilon - 3a_3\epsilon^2 - \dots - ka_k\epsilon^{k-1} - \kappa\epsilon^{\kappa-1}(a_\kappa + \tilde{\varphi}(1/\epsilon)(1 + o(1))) + O(\epsilon^\kappa). \quad (4.3.44)$$

When $\kappa \notin \mathbf{N}$, then $k+1 = \kappa + \delta$ and

$$U'(\varrho) = -h_c - 2a_2\epsilon - 3a_3\epsilon^2 - \dots - ka_k\epsilon^{k-1} - \kappa\epsilon^{\kappa-1}\tilde{\varphi}(1/\epsilon)(1 + o(1)) + O(\epsilon^{\kappa-1+\delta}). \quad (4.3.45)$$

In both cases it can be written

$$U'(\varrho) = -h_c - 2a_2\epsilon - 3a_3\epsilon^2 - \dots - ka_k\epsilon^{k-1} - \kappa\epsilon^{\kappa-1}\psi(1/\epsilon) + O(\epsilon^{\kappa-1+\delta}), \quad (4.3.46)$$

with $\psi(1/\epsilon) = o(1/\epsilon^{\delta'})$ for any $\delta' > 0$. Note that when $\kappa \in \mathbf{N}$, $\psi = a_\kappa + \tilde{\varphi}$ is not necessary slowly varying for it can vanish. Similarly,

$$U(\varrho) = H(\varrho_c) - h_c\epsilon - a_2\epsilon^2 - a_3\epsilon^3 - \dots - a_k\epsilon^k - \epsilon^\kappa(a_\kappa + \tilde{\varphi}(1/\epsilon)) + O(\epsilon^{\kappa+\delta}), \quad (4.3.47)$$

with $a_\kappa = 0$ when $\kappa \notin \mathbf{N}$. Then the optimizer $\hat{\epsilon} = \hat{\epsilon}(h)$, which solves $h + U'(\varrho) = 0$, has an expansion:

$$\hat{\epsilon} = c_1g + \dots + c_{k-1}g^{k-1} + g^{\kappa-1}\tilde{\psi}(1/g), \quad (4.3.48)$$

where $g = h - h_c$. $\tilde{\psi}$ does not necessarily slowly vary, but it satisfies $\tilde{\psi}(1/g) = o(1/g^{\delta'})$ for any $\delta' > 0$. By injecting it into the equation $h + U'(\varrho(h)) = 0$ we get the coefficient $c_1 = \frac{1}{2a_2}$. Then write again

$$\begin{aligned} \mathbb{F}_H(h) &= h\varrho(h) + U(\varrho(h)) \\ &= h\varrho(h) + H(\varrho_c) - h_c\hat{\epsilon}(h) - a_2\hat{\epsilon}^2 - a_3\hat{\epsilon}^3 - \dots - a_k\hat{\epsilon}^k - \hat{\epsilon}^\kappa(a_\kappa + \tilde{\varphi}(1/\hat{\epsilon})) + O(\hat{\epsilon}^{\kappa+\delta}) \\ &= h_c\varrho_c + H(\varrho_c) + \varrho_c g + g\hat{\epsilon} - a_2\hat{\epsilon}^2 - a_3\hat{\epsilon}^3 - \dots - a_k\hat{\epsilon}^k - \hat{\epsilon}^\kappa(a_\kappa + \tilde{\varphi}(1/\hat{\epsilon})) + O(\hat{\epsilon}^{\kappa+\delta}). \end{aligned} \quad (4.3.49)$$

Eventually, by injecting the expansion (4.3.48) of $\hat{\epsilon}$, we get

$$\begin{aligned} \mathbb{F}_H(h) &= \mathbb{F}_H(\varrho_c) + T(g) + [\tilde{\psi}(1/g) - 2a_2c_1\tilde{\psi}(1/g) - c_1^\kappa(a_\kappa + \tilde{\varphi}(1/\hat{\epsilon}))]g^\kappa + O(g^{\kappa+\delta}) \\ &= \mathbb{F}_H(\varrho_c) + T(g) - c_1^\kappa g^\kappa(a_\kappa + \tilde{\varphi}(1/g) + O(g^{\kappa+\delta})), \end{aligned} \quad (4.3.50)$$

where T is a polynomial. To conclude, just note that the same method for $(\mathbb{F}_H)_{\text{reg}}$ yields

$$(\mathbb{F}_H)_{\text{reg}}(h) = \mathbb{F}_H(\varrho_c) + T(g) - c_1^\kappa a_\kappa g^\kappa + O(g^{\kappa+\delta}). \quad (4.3.51)$$

So

$$\mathbb{F}_H(h) - (\mathbb{F}_H)_{\text{reg}}(h) = -c_1^\kappa g^\kappa \tilde{\varphi}(1/\hat{\epsilon}) + O(g^{\kappa+\delta}) = -c_1^\kappa g^\kappa \tilde{\varphi}(1/g)(1 + o(1)). \quad (4.3.52)$$

Recalling (4.3.41), we get the result.

4. **When $\alpha = 1$ and $\mathbf{E}[\eta] < +\infty$.** For any $k \geq 1$, the expansions (4.3.42) and (4.3.43) hold, and, with Lemma 4.3.2, $G(\varrho) = O(\epsilon^{k+2})$ and $G'(\varrho) = O(\epsilon^{k+1})$. Thus, following the same scheme, one obtains $F_H(h) - (F_H)_{\text{reg}}(h) = O(\epsilon^{k+1})$. Thus the critical exponent for F_H is $\kappa = +\infty$, which is the desired result. \square

4.4 Proofs: largest jump for the constrained PS model

4.4.1 Proof in the “big jump domain”

We start by what is the main result of [5], whose proof can be adapted to cover some cases in the critical regime. Recall Lemma 4.A.11: suppose $\mathbf{E}[\eta] < +\infty$ and let c_N be such that

$$\begin{cases} c_N = \gamma_N b_N & \text{if } \alpha \in [1, 2) \\ c_N = qN, \quad q \in (0, 1) & \text{if } \alpha = 2 \\ c_N = \sigma \sqrt{(\alpha - 2) \varrho_c N \log N} \left(1 + \frac{\alpha + 1}{2(\alpha - 2)} \frac{\log \log N}{\log N} - \frac{\log \varphi(\sqrt{N \log N})}{(\alpha - 2) \log N} + \frac{\gamma_N}{\log N} \right) & \text{if } \alpha > 2 \end{cases} \quad (4.4.1)$$

with $\gamma_N \rightarrow +\infty$. Set $V_N = \varrho_c(N - c_N)$. Then

$$\sup_{L \leq V_N} \left| \frac{\mathbf{P}(\tau_L = N)}{\text{LK}(N - L\mathbf{E}[\eta])} - 1 \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (4.4.2)$$

Denote by $\pi_{N,L}$ the distribution, under \mathbf{P} , of

$$\mathfrak{s} \left(\left(\eta_1, \dots, \eta_{L-1}, N - \sum_{j=1}^{L-1} \eta_j \right) \right), \quad (4.4.3)$$

where \mathfrak{s} is the random permutation that exchanges the L^{th} coordinate with a uniformly chosen coordinate among $\{1, \dots, L\}$.

Proposition 4.4.1. *Let c_N be such that[†] $c_N \gg b_N$ and such that (4.4.2) holds, with $V_N = \varrho_c(N - c_N)$. Then*

$$\limsup_{N \rightarrow +\infty} \sup_{1 \leq L \leq V_N} \|\mu_{N,L} - \pi_{N,L}\|_{t.v.} = 0, \quad (4.4.4)$$

where $\|\cdot\|_{t.v.}$ is the total variation distance:

$$\|\mu_{N,L} - \pi_{N,L}\|_{t.v.} = \sup_{C \subset \mathbf{N}^L} |\mu_{N,L}(C) - \pi_{N,L}(C)|. \quad (4.4.5)$$

Remark 4.4.2. In particular it holds for the sequences (4.4.1).

Proof. Let T be the map that exchanges the biggest coordinate and the last one:

$$T : \begin{cases} \mathbf{N}^L & \longrightarrow & \mathbf{N}^L \\ (x_i)_{i=1}^L & \longmapsto & (y_i)_{i=1}^L \end{cases}, \quad (4.4.6)$$

[†]See Proposition 4.A.7 for the definition of b_N .

with $y_L = \max x_i = x_{i_0}$ (with i_0 minimal if the maximum is not unique) and $y_{i_0} = x_L$. In this way the biggest coordinate of $T((x_i)_{i=1}^L)$ is the last one. Let $C \subset \{\eta_L > \max_{j \leq L-1} \eta_j\}$. By symmetry,

$$\mu_{N,L} \circ T^{-1}(C) = L\mu_{N,L}(C). \quad (4.4.7)$$

The same goes for $\pi_{N,L}$:

$$\pi_{N,L} \circ T^{-1}(C) = L\pi_{N,L}(C). \quad (4.4.8)$$

Now, let t_N be such that $b_N \ll t_N < c_N/2$ and $t_N \ll N$. Introduce the set

$$D_{N,L} = \{r \in \mathbf{N} : |N - L\mathbf{E}[\eta] - r| \leq t_N\}, \quad (4.4.9)$$

and consider the event

$$B_{N,L} = \left\{ \eta_L \in D_{N,L} ; \max_{1 \leq j \leq L-1} \eta_j < t_N \right\} \subset \{ \eta_L > \max_{j \leq L-1} \eta_j \}. \quad (4.4.10)$$

Let $A \in \mathcal{B}(\mathbf{R}^{L-1})$. To simplify, we will also write A for the event $\{(\eta_j)_{j=1}^{L-1} \in A\}$. One has

$$\mu_{N,L} \circ T^{-1}(A \cap B_{N,L}) = L\mu_{N,L}(A \cap B_{N,L}) = L \frac{\mathbf{P}(A \cap B_{N,L} \cap \{\tau_L = N\})}{\mathbf{P}(\tau_L = N)}. \quad (4.4.11)$$

By assumption, we know that

$$\sup_{L \leq V_N} \left| \frac{\mathbf{P}(\tau_L = N)}{LK(N - L\mathbf{E}[\eta])} - 1 \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (4.4.12)$$

As a consequence, as $N \rightarrow +\infty$,

$$\sup_{1 \leq L \leq V_N} \sup_{A \in \mathcal{B}(\mathbf{R}^{L-1})} \left| \mu_{N,L} \circ T^{-1}(A \cap B_{N,L}) - \frac{\mathbf{P}(A \cap B_{N,L} \cap \{\tau_L = N\})}{K(N - L\mathbf{E}[\eta])} \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (4.4.13)$$

We compute the probability

$$\begin{aligned} & \mathbf{P}(A \cap B_{N,L} \cap \{\tau_L = N\}) \\ &= \sum_{r \in D_{N,L}} K(r) \mathbf{P} \left(A \cap \left\{ \sum_{j=1}^{L-1} \eta_j = N - r \right\} \cap \left\{ \max_{1 \leq j \leq L-1} \eta_j < t_N \right\} \right). \end{aligned} \quad (4.4.14)$$

With Lemma 4.A.1, we have

$$\sup_{L \leq V_N} \sup_{r \in D_{N,L}} \left| \frac{K(r)}{K(N - L\mathbf{E}[\eta])} - 1 \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (4.4.15)$$

Hence, as $N \rightarrow +\infty$, and uniformly in $L \leq \varrho_c(N - c_N)$ and $A \in \mathcal{B}(\mathbf{R}^{L-1})$,

$$\begin{aligned} \frac{\mathbf{P}(A \cap B_{N,L} \cap \{\tau_L = N\})}{K(N - L\mathbf{E}[\eta])} &\sim \sum_{r \in D_{N,L}} \mathbf{P} \left(A \cap \left\{ \sum_{j=1}^{L-1} \eta_j = N - r \right\} \cap \left\{ \max_{1 \leq j \leq L-1} \eta_j < t_N \right\} \right) \\ &= \mathbf{P} \left(A \cap \left\{ \left| \sum_{j=1}^{L-1} \eta_j - L\mathbf{E}[\eta] \right| \leq t_N \right\} \cap \left\{ \max_{1 \leq j \leq L-1} \eta_j < t_N \right\} \right). \end{aligned} \quad (4.4.16)$$

The last two events (between braces) occur with high probability as $N \rightarrow +\infty$. It is a consequence of Proposition 4.A.7 (recall $t_N \gg b_N$):

$$\inf_{L \leq \varrho_c(N-c_N)} \mathbf{P} \left(\left| \sum_{j=1}^{L-1} \eta_j - L\mathbf{E}[\eta] \right| \leq t_N \right) \xrightarrow{N \rightarrow +\infty} 1; \quad (4.4.17)$$

and Proposition 4.A.6:

$$\inf_{L \leq \varrho_c(N-c_N)} \mathbf{P} \left(\max_{1 \leq j \leq L-1} \eta_j < t_N \right) \xrightarrow{N \rightarrow +\infty} 1. \quad (4.4.18)$$

Eventually,

$$\sup_{1 \leq L \leq \varrho_c(N-c_N)} \sup_{A \in \mathcal{B}(\mathbf{R}^{L-1})} \left| \mu_{N,L} \circ T^{-1}(A \cap B_{N,L}) - \mathbf{P}(A) \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (4.4.19)$$

By taking $A = \mathbf{R}^{L-1}$ we get

$$\sup_{1 \leq L \leq \varrho_c(N-c_N)} \left| \mu_{N,L} \circ T^{-1}(B_{N,L}) - 1 \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (4.4.20)$$

Also with Corollary 4.A.7,

$$\sup_{1 \leq L \leq \varrho_c(N-c_N)} \left| \pi_{N,L} \circ T^{-1}(B_{N,L}) - 1 \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (4.4.21)$$

Eventually, with the same uniformity as in the previous lines,

$$\mu_{N,L} \circ T^{-1}(A \cap B_{N,L}) \simeq \mu_{N,L} \circ T^{-1}(A) \simeq \pi_{N,L} \circ T^{-1}(A \cap B_{N,L}) \simeq \pi_{N,L} \circ T^{-1}(A). \quad (4.4.22)$$

The statement of the proposition is a consequence of the symmetry of the L coordinates (η_j) (see (4.4.7)). \square

As a corollary we get Proposition 4.2.16, Proposition 4.2.20 item 2, and Proposition 4.2.22 item 1. We prove it briefly.

Proof. The proof is the same for these three cases. The only point to observe is that in both of them, $N - L\mathbf{E}[\eta]$ is, by assumption, much larger than a_L and b_L . Denote by $\tilde{\pi}_{N,L}$ the distribution, under \mathbf{P} , of

$$\left(\eta_1, \dots, \eta_{L-1}, N - \sum_{j=1}^{L-1} \eta_j \right), \quad (4.4.23)$$

and by $\bar{\eta}_L$ its last coordinate. With Proposition 4.4.1, and since the random permutation \mathfrak{s} does not change M_L and M'_L , it is sufficient to prove (4.2.20), (4.2.21) and (4.2.22) for $\tilde{\pi}_{N,L}$. Consider the event

$$A = \left\{ \frac{\bar{\eta}_L - (N - L\mathbf{E}[\eta])}{b_L} \leq t \right\} = \left\{ -\frac{\sum_{j=1}^{L-1} \eta_j - L\mathbf{E}[\eta]}{b_L} \leq t \right\}. \quad (4.4.24)$$

Proposition 4.A.7 yields $\tilde{\pi}_{N,L}(A) \rightarrow \mathbf{P}(-X_\alpha \leq t)$, which is (4.2.21). The law of large numbers (4.2.20) is a direct consequence for $N - L\mathbf{E}[\eta] \gg b_L$ (and in particular $M_L = \bar{\eta}_L$ under $\tilde{\pi}_{N,L}$). Eventually, observe that

$$\tilde{\pi}_{N,L}(M'_L \leq xa_L) \sim \tilde{\pi}_{N,L}\left(\max_{j \leq L-1} \eta_j \leq xa_L\right) = \mathbf{P}(M_{L-1} \leq xa_L) \rightarrow \exp(-1/(\alpha x^\alpha)). \quad (4.4.25)$$

The last line is a consequence of Proposition 4.A.6 (and Lemma 4.A.4 which gives $a_L \sim a_{L-1}$). \square

4.4.2 Proof in the other regimes

4.4.2.1 General strategy and notations

We use the same strategy as [4], starting from the elementary formula

$$\mu_{N,L}(M_L \leq c_L) = \frac{\mathbf{P}(\tau_L = N, M_L \leq c_L)}{\mathbf{P}(\tau_L = N)} = \frac{\mathbf{P}(\tau_L = N | \forall i, \eta_i \leq c_L)}{\mathbf{P}(\tau_L = N)} \mathbf{P}(\eta \leq c_L)^L. \quad (4.4.26)$$

When the density $\frac{L}{N}$ is sufficiently large, we will rather use the tilted measure and the identity

$$\mu_{N,L}(M_L \leq c_L) = \frac{\mathbf{P}_{x_L}(\tau_L = N | \forall i, \eta_i \leq c_L)}{\mathbf{P}_{x_L}(\tau_L = N)} \mathbf{P}_{x_L}(\eta \leq c_L)^L, \quad (4.4.27)$$

with x_L such that $\mathbf{E}_{x_L}[\eta] = \frac{N}{L}$. Then, using that

$$\mathbf{P}_{x_L}(\eta \leq c_L)^L = \exp\left\{-L\mathbf{P}_{x_L}(\eta > c_L)(1 + o(1))\right\}, \quad (4.4.28)$$

the first step is to find $c_L \rightarrow +\infty$ such that $L\mathbf{P}_{x_L}(\eta > c_L)$ is bounded, and bounded away from 0. The second (and a bit harder) step is to prove a local limit theorem to estimate $\mathbf{P}_{x_L}(\tau_L = N | \forall i, \eta_i \leq c_L)$ and $\mathbf{P}_{x_L}(\tau_L = N)$.

Before going through the proofs, let's introduce some notations. Besides the tilted measure \mathbf{P}_x , we will use

$$\hat{\mathbf{P}} = \hat{\mathbf{P}}^{(c_L)} := \mathbf{P}(\cdot | \forall i, \eta_i \leq c_L), \quad \hat{\mathbf{P}}_x = \hat{\mathbf{P}}_x^{(c_L)} := \mathbf{P}_x(\cdot | \forall i, \eta_i \leq c_L). \quad (4.4.29)$$

The superscript c_L will always be omitted. Under these laws, the renewal distribution is respectively

$$\hat{\mathbf{K}}(n) = \frac{\mathbf{K}(n)}{\mathbf{P}(\eta \leq c_L)} \mathbf{1}_{n \leq c_L}, \quad \hat{\mathbf{K}}_x(n) = \frac{\mathbf{K}_x(n)}{\mathbf{P}_x(\eta \leq c_L)} \mathbf{1}_{n \leq c_L}. \quad (4.4.30)$$

4.4.2.2 Estimation of $\mathbf{P}_x(\tau_L = N)$ and $\hat{\mathbf{P}}_x(\tau_L = N)$

The next lemma (which already appears in [4]) allows asymptotic estimates of $\mathbf{P}_{x_L}(\tau_L = k)$ and $\hat{\mathbf{P}}_{x_L}(\tau_L = k)$ when $\eta \in L^2$ or when $\varrho > \varrho_c$.

Lemma 4.4.3. *Suppose either $\frac{L}{N} \rightarrow \varrho \in (\varrho_c, 1)$ or $\frac{L}{N} \rightarrow \varrho_c$ with $N \leq L\mathbf{E}[\eta]$ and $\text{Var}(\eta) < +\infty$. Observe that $x_L \rightarrow x$ which solves $\mathbf{E}_x[\eta] = \frac{1}{\varrho}$, and that $\sigma_x^2 = \lim_{L \rightarrow \infty} \text{Var}_{x_L}(\eta)$. Then,*

$$\sup_{k \in \mathbf{Z}} \left| \sqrt{\sigma_x^2 L} \mathbf{P}_{x_L}(\tau_L = k) - f_{\mathcal{N}}\left(\frac{k - L\mathbf{E}_{x_L}[\eta]}{\sqrt{\sigma_x^2 L}}\right) \right| \rightarrow 0. \quad (4.4.31)$$

And, for any $c_L \rightarrow +\infty$, such that $L\mathbf{P}(\eta \geq c_L)$ is bounded,

$$\sup_{k \in \mathbf{Z}} \left| \sqrt{\sigma_x^2 L} \hat{\mathbf{P}}_{x_L}(\tau_L = k) - f_{\mathcal{N}} \left(\frac{k - L\mathbf{E}_{x_L}[\eta]}{\sqrt{\sigma_x^2 L}} \right) \right| \rightarrow 0, \quad (4.4.32)$$

where $f_{\mathcal{N}}$ is the standard Gaussian density function.

Proof. Thanks to [29, Th. 1.2], it is sufficient to prove a central limit theorem for \mathbf{P}_{x_L} and $\hat{\mathbf{P}}_{x_L}$. Let us prove it briefly for $\hat{\mathbf{P}}_{x_L}$ (the other case is recovered by taking $c_L = \infty$). With dominated convergence, since $c_L \rightarrow +\infty$, and $x_L \rightarrow x$, we obtain

$$\hat{\mathbf{E}}_{x_L}[\eta] \rightarrow \mathbf{E}_x[\eta] \quad \text{and} \quad \hat{\mathbf{E}}_{x_L}[\eta^2] \rightarrow \mathbf{E}_x[\eta^2]. \quad (4.4.33)$$

Therefore,

$$\hat{\mathbf{E}}_{x_L} [(\eta - \mathbf{E}_{x_L}[\eta])^2] \rightarrow \sigma_x^2. \quad (4.4.34)$$

On the other hand, with Schwarz inequality, one has

$$\sqrt{L}(\hat{\mathbf{E}}_{x_L}[\eta] - \mathbf{E}_{x_L}[\eta]) = \sqrt{L}\mathbf{E}_{x_L}[\eta \mathbf{1}_{\eta > c_L}] \leq ([L\mathbf{P}_{x_L}(\eta > c_L)]\mathbf{E}_{x_L}[\eta^2 \mathbf{1}_{\eta > c_L}])^{1/2} \xrightarrow{L \rightarrow +\infty} 0, \quad (4.4.35)$$

because the first factor is bounded, by assumption; and the second, by dominated convergence, vanishes as $L \rightarrow +\infty$. We then compute the characteristic function, which, thanks to (4.4.34), can be developed

$$\begin{aligned} \hat{\mathbf{E}}_{x_L} \left[\exp \left(it \frac{\eta - \mathbf{E}_{x_L}[\eta]}{\sqrt{\sigma_x^2 L}} \right) \right] &= 1 + \frac{it}{\sqrt{\sigma_x^2 L}} (\hat{\mathbf{E}}_{x_L}[\eta] - \mathbf{E}_{x_L}[\eta]) \\ &\quad - \frac{t^2}{2\sigma_x^2 L} \hat{\mathbf{E}}_{x_L} [(\eta - \mathbf{E}_{x_L}[\eta])^2] (1 + o(1)). \end{aligned} \quad (4.4.36)$$

And combining these three lines, we get

$$\hat{\mathbf{E}}_{x_L} \left[\exp \left(it \frac{\eta - \mathbf{E}_{x_L}[\eta]}{\sqrt{\sigma_x^2 L}} \right) \right]^L \rightarrow \exp(-t^2/2). \quad (4.4.37)$$

□

4.4.2.3 Supercritical regime (Proof of Proposition 4.2.18)

Proof of Proposition 4.2.18. First note that $x_L \rightarrow x$ as $L \rightarrow \infty$, where x solves $\mathbf{E}_x[\eta] = \frac{1}{\varrho}$. We compute

$$L\mathbf{P}_{x_L}(\eta > k) = \frac{L}{\mathbf{E}[e^{-x_L \eta}]} \sum_{n=k+1}^{+\infty} \varphi(n) \frac{e^{-x_L n}}{n^{1+\alpha}} \underset{L \rightarrow \infty}{\sim} \frac{L}{\mathbf{E}[e^{-x \eta}]} \sum_{n=k+1}^{+\infty} \varphi(n) \frac{e^{-x n}}{n^{1+\alpha}}. \quad (4.4.38)$$

Introduce

$$A_L = \frac{1}{x_L} (\log L - (1 + \alpha) \log \log L + (1 + \alpha) \log x + \log \varphi(\log L) - \log \mathbf{E}[e^{-x \eta}]). \quad (4.4.39)$$

In particular $A_L \rightarrow +\infty$ and it is chosen to satisfy

$$\frac{L\varphi(A_L)e^{-xLA_L}}{A_L^{1+\alpha}\mathbf{E}[e^{-x\eta}]} \xrightarrow{L \rightarrow +\infty} 1. \quad (4.4.40)$$

So, for any $k \in \mathbf{Z}$,

$$\begin{aligned} L\mathbf{P}_{x_L}(\eta > \lfloor A_L \rfloor + k) &\underset{L \rightarrow \infty}{\sim} \frac{L}{\mathbf{E}[e^{-x\eta}]} \sum_{n=k+1}^{+\infty} \varphi(\lfloor A_L \rfloor + n) \frac{e^{-xL(\lfloor A_L \rfloor + n)}}{(\lfloor A_L \rfloor + n)^{1+\alpha}} \\ &\sim \frac{L\varphi(A_L)e^{-xLA_L}}{A_L^{1+\alpha}\mathbf{E}[e^{-x\eta}]} \sum_{n=k+1}^{+\infty} \frac{\varphi(\lfloor A_L \rfloor + n)}{\varphi(A_L)} \frac{e^{-xL(n - \{A_L\})}}{(1 + \frac{n}{A_L})^{1+\alpha}} \\ &\sim \sum_{n=k+1}^{+\infty} e^{-x(n - \{A_L\})} = \frac{e^{-x(k+1 - \{A_L\})}}{1 - e^{-x}}. \end{aligned} \quad (4.4.41)$$

Eventually, with Lemma 4.4.3,

$$\frac{\hat{\mathbf{P}}_{x_L}(\tau_L = N)}{\mathbf{P}_{x_L}(\tau_L = N)} \xrightarrow{L \rightarrow \infty} 1. \quad (4.4.42)$$

Combining these estimates, we get the result:

$$\mu_{N,L}(M_L - \lfloor A_L \rfloor \leq k) \sim e^{-\frac{e^{-x(k+1 - \{A_L\})}}{1 - e^{-x}}} \sim \mathbf{P}_{x_L}(M_L - \lfloor A_L \rfloor = k), \quad (4.4.43)$$

which is the statement of the proposition, with $z_L = \lfloor A_L \rfloor$ and $s_L = (1 - e^{-x})^{-1}e^{x(\{A_L\} - 1)}$. \square

4.4.2.4 Critical behaviour for $\alpha \in (1, 2)$ (Proof of Proposition 4.2.20 item 1)

Proposition 4.4.4 (Local limit theorem for the truncated random variables). *Let $c_L \sim wa_L$. One has*

$$\frac{\tau_L - L\hat{\mathbf{E}}[\eta]}{a_L} \xrightarrow[L \rightarrow +\infty]{\hat{\mathbf{P}}} X_{\alpha,w}, \quad (4.4.44)$$

where

$$\mathbf{E}[e^{itX_{\alpha,w}}] = \exp\left(\int_0^w \frac{e^{ity} - 1 - ity}{y^{1+\alpha}} dy\right). \quad (4.4.45)$$

Its continuous density will be denoted by $g_{\alpha,w}$. Uniformly in w in any compact subset of $(0, +\infty)$, and uniformly in $k \in \mathbf{N}$,

$$\left| a_L \hat{\mathbf{P}}(\tau_L = k) - g_{\alpha,w}\left(\frac{k - L\hat{\mathbf{E}}[\eta]}{a_L}\right) \right| \xrightarrow[L \rightarrow +\infty]{} 0. \quad (4.4.46)$$

Proof. For convenience we write $c_L = wa_L$. We compute the characteristic function

$$\hat{\mathbf{E}}[e^{it(\tau_L - L\hat{\mathbf{E}}[\eta])}] = \hat{\mathbf{E}}[e^{it(\eta - \hat{\mathbf{E}}[\eta])}]^L. \quad (4.4.47)$$

One has

$$\begin{aligned}\hat{\mathbf{E}} \left[e^{it(\eta - \hat{\mathbf{E}}[\eta])} \right] &= \sum_{n=1}^{wa_L} \mathbf{K}(n) e^{it(n - \hat{\mathbf{E}}[\eta])} \frac{1}{\mathbf{P}(\eta \leq wa_L)} \\ &= 1 + \frac{1}{\mathbf{P}(\eta \leq wa_L)} \sum_{n=1}^{wa_L} \frac{\varphi(n)}{n^{1+\alpha}} (e^{it(n - \hat{\mathbf{E}}[\eta])} - 1 - it(n - \hat{\mathbf{E}}[\eta])).\end{aligned}\tag{4.4.48}$$

Thus

$$\begin{aligned}\hat{\mathbf{E}} \left[e^{\frac{it}{a_L}(\tau_L - L\hat{\mathbf{E}}[\eta])} \right] &= \left[1 + \frac{1}{\mathbf{P}(\eta \leq wa_L)} \sum_{n=1}^{wa_L} \frac{\varphi(n)}{n^{1+\alpha}} \left(e^{it \frac{n - \hat{\mathbf{E}}[\eta]}{a_L}} - 1 - it \frac{n - \hat{\mathbf{E}}[\eta]}{a_L} \right) \right]^L \\ &\sim \left[1 + \frac{\varphi(a_L)}{a_L^\alpha} \frac{1}{a_L} \sum_{n=1}^{wa_L} \frac{\varphi(n)}{\varphi(a_L)} \left(\frac{n}{a_L} \right)^{-(1+\alpha)} \left(e^{it \frac{n - \hat{\mathbf{E}}[\eta]}{a_L}} - 1 - it \frac{n - \hat{\mathbf{E}}[\eta]}{a_L} \right) \right]^L \\ &\rightarrow \exp \left(\int_0^w \frac{e^{ity} - 1 - ity}{y^{1+\alpha}} dy \right) =: v_{\alpha, w}(t).\end{aligned}\tag{4.4.49}$$

Therefore the convergence (4.4.44) is proved. One readily see that $v_{\alpha, w} \in L^1$, so the limit $X_{\alpha, w}$ has a continuous density $g_{\alpha, w}$. Then we need to prove the local limit theorem. Let

$$f(t) = \mathbf{E}[e^{it\eta}], \quad \psi(t) = \mathbf{E}[e^{it(\eta - \mathbf{E}[\eta])}],\tag{4.4.50}$$

and

$$\hat{f}(t) = \hat{\mathbf{E}}[e^{it\eta}], \quad \hat{\psi}(t) = \hat{\mathbf{E}}[e^{it(\eta - \hat{\mathbf{E}}[\eta])}],\tag{4.4.51}$$

and

$$v_{\alpha, w}(t) = \mathbf{E}[e^{itX_{\alpha, w}}], \quad g_{\alpha, w}(z) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-itz} v_{\alpha, w}(t) dt.\tag{4.4.52}$$

The same proof as for Proposition 4.A.8 works: write

$$2\pi a_L \hat{\mathbf{P}}(\tau_L = k) = \int_{-\pi a_L}^{\pi a_L} e^{-izu} \hat{\psi} \left(\frac{u}{a_L} \right)^L du,\tag{4.4.53}$$

where $z = z_{k, L} = \frac{k - L\hat{\mathbf{E}}[\eta]}{a_L}$. The only point that must be adapted is the control of

$$J = \int_{A \leq |u| \leq \pi a_L} e^{-izu} \hat{\psi} \left(\frac{u}{a_L} \right)^L du.\tag{4.4.54}$$

For $t \in [-\pi, \pi]$, and L sufficiently large,

$$\begin{aligned}|\hat{\psi}(t)| = |\hat{f}(t)| &= \frac{1}{\mathbf{P}(\eta \leq wa_L)} \left| \sum_{n=1}^{+\infty} e^{int} \mathbf{K}(n) - \sum_{n > wa_L} e^{int} \mathbf{K}(n) \right| \\ &\leq \frac{1}{1 - \frac{Cw^{-\alpha}}{L}} \left(|f(t)| + \frac{Cw^{-\alpha}}{L} \right) \\ &\leq \frac{1}{1 - \frac{Cw^{-\alpha}}{L}} \left(|f(t)| \vee \frac{1}{2} \right) \left(1 + \frac{2Cw^{-\alpha}}{L} \right) \\ &\leq \left(|f(t)| \vee \frac{1}{2} \right) \left(1 + \frac{\tilde{C}w^{-\alpha}}{L} \right).\end{aligned}\tag{4.4.55}$$

Thus, for any $t \in [-\pi, \pi]$,

$$|\hat{\psi}(t)|^L \leq \left(|f(t)| \vee \frac{1}{2} \right)^L e^{\tilde{C}w^{-\alpha}} \leq \left(|f(t)|^L + \frac{1}{2^L} \right) e^{\tilde{C}w^{-\alpha}}. \quad (4.4.56)$$

Then,

$$\begin{aligned} |J| &\leq \int_{A \leq |u| \leq \pi a_L} \left(\left| f\left(\frac{u}{a_L}\right) \right|^L + \frac{1}{2^L} \right) e^{\tilde{C}w^{-\alpha}} du \\ &\leq e^{\tilde{C}w^{-\alpha}} \left(\int_{A \leq |u| \leq \pi a_L} \left| f\left(\frac{u}{a_L}\right) \right|^L du + 2 \frac{\pi a_L}{2^L} \right). \end{aligned} \quad (4.4.57)$$

This integral was controlled in the proof of Proposition 4.A.8 and the second term vanishes when L goes to $+\infty$. Eventually

$$\limsup_{L \rightarrow +\infty} 2\pi \left| a_L \hat{\mathbf{P}}(\tau_L = k) - g_{\alpha, w}(z) \right| = 0. \quad (4.4.58)$$

The local limit theorem is proved. \square

Proof of Proposition 4.2.20 item 1. Recall that

$$\mu_{N,L}(M_L \leq wa_L) = \frac{\hat{\mathbf{P}}(\tau_L = N)}{\mathbf{P}(\tau_L = N)} \mathbf{P}(\eta \leq wa_L)^L, \quad (4.4.59)$$

and that $u_L = \frac{N - L\mathbf{E}[\eta]}{a_L} \rightarrow u$. With Proposition 4.A.8, we know that

$$a_L \mathbf{P}(\tau_L = N) = g_\alpha \left(\frac{N - L\mathbf{E}[\eta]}{a_L} \right) + o(1) \xrightarrow{L \rightarrow +\infty} g_\alpha(u). \quad (4.4.60)$$

On the other hand, with Proposition 4.4.4,

$$a_L \hat{\mathbf{P}}(\tau_L = N) = g_{\alpha, w} \left(\frac{N - L\hat{\mathbf{E}}[\eta]}{a_L} \right) + o(1), \quad (4.4.61)$$

with

$$\frac{N - L\hat{\mathbf{E}}[\eta]}{a_L} = u_L + L \frac{\mathbf{E}[\eta] - \hat{\mathbf{E}}[\eta]}{a_L}. \quad (4.4.62)$$

We readily see that

$$\lim_{L \rightarrow +\infty} L \frac{\mathbf{E}[\eta] - \hat{\mathbf{E}}[\eta]}{a_L} = \lim_{L \rightarrow +\infty} \frac{L}{a_L} \mathbf{E}[\eta \mathbf{1}_{\eta > wa_L}] = \frac{1}{(\alpha - 1)w^{\alpha-1}}. \quad (4.4.63)$$

So

$$a_L \hat{\mathbf{P}}(\tau_L = N) \xrightarrow{L \rightarrow +\infty} g_{\alpha, w} \left(u + \frac{1}{(\alpha - 1)w^{\alpha-1}} \right). \quad (4.4.64)$$

On the other hand,

$$\mathbf{P}(\eta \leq wa_L)^L = \left(1 - \frac{\varphi(a_L)}{\alpha w^\alpha a_L^\alpha} (1 + o(1)) \right)^L \xrightarrow{L \rightarrow +\infty} e^{-\frac{1}{\alpha w^\alpha}}. \quad (4.4.65)$$

Eventually,

$$\mu_{N,L}(M_L \leq wa_L) \xrightarrow{L \rightarrow +\infty} e^{-\frac{1}{\alpha w^\alpha}} \frac{g_{\alpha, w} \left(u + \frac{1}{(\alpha - 1)w^{\alpha-1}} \right)}{g_\alpha(u)}. \quad (4.4.66)$$

\square

4.4.2.5 Critical behaviour for $\eta \in L^2$ (Proof of Proposition 4.2.23)

Proof of Proposition 4.2.23. Note that, with Lemma 4.A.5, as $x \searrow 0$.

$$\mathbf{E}_x[\eta] = \mathbf{E}[\eta] - x\sigma^2(1 + o(1)). \quad (4.4.67)$$

Thus, x_L satisfies

$$x_L \sim y_L = \frac{\theta_L}{\sigma^2 L}. \quad (4.4.68)$$

We want

$$LP_{x_L}(\eta \geq c_L) \sim L \sum_{n \geq c_L} \varphi(n) \frac{e^{-nx_L}}{n^{1+\alpha}} \quad (4.4.69)$$

to be of order one. Then the result will follow from Lemma 4.4.3.

1. If $\frac{a_L \theta_L}{\sigma^2 L} \rightarrow \omega \in [0, +\infty)$, then

$$\begin{aligned} LP_{x_L}(\eta \geq ta_L) &\sim \frac{L\varphi(a_L)}{a_L^\alpha} \frac{1}{a_L} \sum_{n \geq ta_L} \frac{\varphi(n)}{\varphi(a_L)} \frac{e^{-\frac{n}{a_L} a_L x_L}}{\left(\frac{n}{a_L}\right)^{1+\alpha}} \\ &\rightarrow \int_t^{+\infty} e^{-\omega s} s^{-1-\alpha} ds, \end{aligned} \quad (4.4.70)$$

and we get the first two cases of the proposition.

2. If $\frac{a_L \theta_L}{\sigma^2 L} \rightarrow +\infty$, then set

$$\beta_L = \log L + \alpha \log x_L + \log \varphi\left(\frac{1}{x_L}\right), \quad (4.4.71)$$

$$B_L = \frac{1}{x_L} \left(\beta_L - (1 + \alpha) \log \beta_L + \log \frac{\varphi\left(\frac{\beta_L}{x_L}\right)}{\varphi\left(\frac{1}{x_L}\right)} \right). \quad (4.4.72)$$

With the definition of a_L (Lemma 4.A.4),

$$e^{\beta_L} = Lx_L^\alpha \varphi(1/x_L) \sim (a_L x_L)^\alpha \frac{\varphi(1/x_L)}{\varphi(a_L)}. \quad (4.4.73)$$

Therefore, since $a_L x_L \rightarrow +\infty$ we get, with Potter's bound (Lemma 4.A.2), $e^{\beta_L} \rightarrow +\infty$ and $\beta_L \gg \log \frac{\varphi\left(\frac{\beta_L}{x_L}\right)}{\varphi\left(\frac{1}{x_L}\right)}$. Then

$$LP_{x_L}(x_L(\eta - B_L) \geq t) \sim \frac{Le^{-x_L B_L} \varphi(B_L)}{x_L B_L^{\alpha+1}} x_L \sum_{n \geq B_L + t/x_L} \frac{\varphi(n)}{\varphi(B_L)} \frac{e^{-(n-B_L)x_L}}{\left(1 + \frac{(n-B_L)x_L}{x_L B_L}\right)^{1+\alpha}}. \quad (4.4.74)$$

Note that $x_L B_L \rightarrow +\infty$ and

$$\frac{Le^{-x_L B_L} \varphi(B_L)}{x_L B_L^{\alpha+1}} \rightarrow 1. \quad (4.4.75)$$

Thus

$$LP_{x_L}(x_L(\eta - B_L) \geq t) \rightarrow \int_t^{+\infty} e^{-s} ds = e^{-t}. \quad (4.4.76)$$

□

4.5 Proofs: path properties for the models with a kernel

In this section we set $\varrho = \varrho(h) \in [a, b]$, the maximum point of $\varrho \mapsto \varrho h + G(\varrho) + H(\varrho)$. Recall that

$$Z_N = Z_N^{\Psi, h} = \sum_{m=0}^N e^{mh} \Psi(m, N) \mathbf{P}(\tau_m = N). \quad (4.5.1)$$

Lemma 4.5.1. *For any $\epsilon > 0$,*

$$Z_N^{\Psi, h} \underset{N \rightarrow +\infty}{\sim} \hat{Z}_N := \sum_{|m - \varrho N| \leq \epsilon N} e^{mh} \Psi(m, N) \mathbf{P}(\tau_m = N). \quad (4.5.2)$$

Proof. Fix $\epsilon > 0$. First we get a lower bound on Z_N . Observe that for any $y \in (a, b)$,

$$\frac{1}{N} \log Z_N \geq \frac{1}{N} (h[yN] + \log \Psi([yN], N) + \log \mathbf{P}(\tau_{[yN]} = N)), \quad (4.5.3)$$

where $[yN] = \lfloor yN \rfloor \vee 1$. With (4.1.28) and Lemma 4.2.1, we readily get

$$\liminf_{N \rightarrow +\infty} \frac{1}{N} \log Z_N \geq \sup_{y \in (a, b)} \{hy + H(y) + G(y)\}. \quad (4.5.4)$$

Since G is continuous on $[a, b]$ (Lemma 4.2.1) and given the assumption 4.1.4 (item (i)(d)) on H ,

$$\liminf_{N \rightarrow +\infty} \frac{1}{N} \log Z_N \geq \sup_{y \in [a, b]} \{hy + H(y) + G(y)\} = F_H(h). \quad (4.5.5)$$

On the other hand we try to obtain an upper bound on $Z_N - \hat{Z}_N$. For any $N \geq m \geq 1$,

$$\mathbf{P}(\tau_m = N) = e^{N G(\frac{m}{N})} \mathbf{P}_{x_m}(\tau_m = N) \leq e^{N G(\frac{m}{N})}, \quad (4.5.6)$$

where $x_m = 0$ if $\frac{m}{N} \leq \varrho_c$ and x_m solves $\mathbf{E}_{x_m}[\eta] = \frac{N}{m}$ otherwise. Thus

$$e^{mh} e^{N H(\frac{m}{N})} \mathbf{P}(\tau_m = N) \leq e^{N(H(\frac{m}{N}) + G(\frac{m}{N}) + h \frac{m}{N})}. \quad (4.5.7)$$

Let $B, \delta > 0$ (to be chosen later). We know (Assumptions 4.1.4) that for any $N \geq m \geq 1$,

$$\Psi(m, N) \leq cN^c \left[e^{N(H(\frac{m}{N}) + \delta)} + e^{-NB} \right], \quad (4.5.8)$$

for a certain $c > 0$. Therefore

$$Z_N - \hat{Z}_N \leq \sum_{|m - \varrho N| > \epsilon N} cN^c \left[e^{\delta N} e^{N(H(\frac{m}{N}) + G(\frac{m}{N}) + h \frac{m}{N})} + e^{mh} e^{-NB} \right] \leq 2cN^{c+1} e^{KN}, \quad (4.5.9)$$

where

$$K = \max \left(|h| - B; \delta + \sup_{|y - \varrho| \geq \epsilon} \{yh + G(y) + H(y)\} \right). \quad (4.5.10)$$

Since $y \mapsto yh + G(y) + H(y)$ is strictly concave, one can choose δ and ϵ small enough, and B large enough, so that $K < F_H(h)$, and therefore

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \log(Z_N - \hat{Z}_N) < F_H(h). \quad (4.5.11)$$

Using the same method, we can obtain, for any $\delta, B > 0$,

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \log \hat{Z}_N \leq \max \left(|h| - B; \delta + \sup_{|y-\varrho| \leq \epsilon} \{yh + G(y) + H(y)\} \right), \quad (4.5.12)$$

and therefore

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \log \hat{Z}_N \leq F_H(h). \quad (4.5.13)$$

Combining (4.5.5), (4.5.11) and (4.5.13), we get

$$Z_N \sim \hat{Z}_N = \sum_{|m-\varrho N| \leq \epsilon N} e^{mh} \Psi(m, N) \mathbf{P}(\tau_m = N), \quad (4.5.14)$$

and

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \log Z_N = F_H(h), \quad (4.5.15)$$

which is the statement of Lemma 4.2.4. \square

Proof of Propositions 4.2.10 and 4.2.12. The law of large numbers (4.2.15) is an immediate consequence of Lemma 4.5.1: for any $\epsilon > 0$,

$$\xi_N^{\Psi, h}(|\tau| - \varrho N \leq \epsilon N) \underset{N \rightarrow +\infty}{\sim} \frac{1}{Z_N} \sum_{|m-\varrho N| \leq \epsilon N} e^{mh} \Psi(m, N) \mathbf{P}(\tau_m = N) \xrightarrow{N \rightarrow +\infty} 1. \quad (4.5.16)$$

The continuity and properties of $h \mapsto \varrho(h)$ can be found in Lemma 4.3.3. Now look at the largest jump(s).

1. When $h < h_c$, we know that $\varrho(h) < \varrho_c$. From Lemma 4.5.1, we know that, for any $\epsilon > 0$,

$$Z_N \sim \hat{Z}_N = \sum_{|m-\varrho N| \leq \epsilon N} e^{mh} \Psi(m, N) \mathbf{P}(\tau_m = N). \quad (4.5.17)$$

Consider the event

$$A = \{|\tau| - \varrho N \leq \epsilon N\} \cap \{|M_{|\tau|} - (N - |\tau|/\varrho_c)| \leq \epsilon N\}. \quad (4.5.18)$$

We compute

$$\xi_{N, h}(A) \sim \frac{1}{\hat{Z}_N} \sum_{|m-\varrho N| \leq \epsilon N} e^{mh} \Psi(m, N) \mathbf{P}(\tau_m = N) \mu_{N, m}(|M_m - (N - m/\varrho_c)| \leq \epsilon N). \quad (4.5.19)$$

From the results of subsection 4.2.3, we know that

$$\mu_{N, m}(|M_m - (N - m/\varrho_c)| \leq \epsilon N) \rightarrow 1. \quad (4.5.20)$$

and the convergence can be made uniform on $\{|m - \varrho N| \leq \epsilon N\}$. Thus,

$$\xi_{N, h}(A) \sim \frac{1}{\hat{Z}_N} \sum_{|m-\varrho N| \leq \epsilon N} e^{mh} \Psi(m, N) \mathbf{P}(\tau_m = N) \times 1 = 1. \quad (4.5.21)$$

We get the convergence (4.2.16).

2. For the second largest jump when $h < h_c$, consider the event

$$B = \{|\tau| - \varrho N \leq \epsilon N\} \cap \left\{ a_{|\tau|}^{-1} M'_{|\tau|} \leq x \right\}. \quad (4.5.22)$$

With the same method as before and with Proposition 4.2.16, we get $\xi_N^{\Psi, h}(B) \rightarrow \psi(x)$ (a nontrivial cumulative distribution function). In other words, $a_{|\tau|}^{-1} M'_{|\tau|}$ converges in distribution to a non-trivial random variable. Then

$$a_N^{-1} M'_{|\tau|} = \frac{a_{|\tau|}}{a_N} a_{|\tau|}^{-1} M'_{|\tau|}. \quad (4.5.23)$$

The first ratio converges (in probability) to a constant because $|\tau|/N \rightarrow \varrho$ in probability and a_L is regularly varying. And we conclude with Slutsky's theorem.

The same methods gives the logarithmic asymptotic behaviour of the largest jump when $h > h_c$. \square

Now we want to prove Proposition 4.2.13 about Gaussian fluctuations for the number of contact points. In fact, we prove a slightly stronger result. From the decomposition (4.1.14), we know that $\xi_N^{\Psi, h}$ is a mixing of the measures $\mu_{N, L}$. Thanks to Lemma 4.5.1, we know that this mixing can be restricted to $L = \varrho N \pm \epsilon N$. In fact, when $h \in (h_*, h^*)$, it can be restricted to $L = \varrho N + O(\sqrt{N})$ and the next proposition makes it rather explicit. It basically states that $\xi_N^{\Psi, h} \simeq \mu_{N, \underline{L}}$, where the number of contact point \underline{L} is random, concentrated around ϱN with approximately Gaussian deviations.

Proposition 4.5.2. *Suppose $h_* < h < h^*$ and define $\tilde{\sigma} = \tilde{\sigma}(h)$ such that*

$$\frac{1}{\tilde{\sigma}(h)^2} = -G''(\varrho) - H''(\varrho), \quad (4.5.24)$$

(at ϱ_c take $G''(\varrho_c) = \lim_{\varrho \searrow \varrho_c} G''(\varrho)$) and

$$\pi_N = \sum_{\ell \geq 1} p_\ell \mu_{N, \ell}, \quad \text{where} \quad p_\ell = \mathbf{P} \left(\left\lfloor \varrho N + \sqrt{N} \tilde{\sigma} \mathcal{N} \right\rfloor \vee 1 = \ell \right), \quad (4.5.25)$$

(\mathcal{N} is a standard gaussian). Then

$$\|\xi_N^{\Psi, h} - \pi_N\|_{t.v.} \xrightarrow{N \rightarrow +\infty} 0. \quad (4.5.26)$$

Remark 4.5.3. Here the total variation distance can be written as a supremum over the Borel sets:

$$\|\xi_N^{\Psi, h} - \pi_N\|_{t.v.} = \sup_{C \in \mathcal{B}(\cup_{L \geq 1} \mathbf{N}^L)} \left| \xi_N^{\Psi, h}(C) - \pi_N(C) \right|. \quad (4.5.27)$$

Proof. Here $\varrho \in (a, b)$ and it satisfies $h + G'(\varrho) + H'(\varrho) = 0$. Fix $\epsilon > 0$ such that $a < \varrho - \epsilon$ and $\varrho + \epsilon < b$. We already know (Lemma 4.5.1) that $Z_N \sim \hat{Z}_N$, where \hat{Z}_N is Z_N with the summation restricted to $|m - \varrho N| \leq \epsilon N$. Here we first and mainly want to prove that if B is large enough,

$$\hat{Z}_N \sim T_N := \sum_{|m - \varrho N| \leq B\sqrt{N \log N}} e^{mh} \Psi(m, N) \mathbf{P}(\tau_m = N). \quad (4.5.28)$$

We start by getting an upper bound on $\hat{Z}_N - T_N$.

Thanks to Taylor-Lagrange inequality, there exists $A > 0$ such that for every $1 \leq m \leq N$,

$$\begin{aligned} mh + NH\left(\frac{m}{N}\right) &\leq N(\varrho h + H(\varrho)) + (m - \varrho N)(h + H'(\varrho)) - A\frac{(m - \varrho N)^2}{N} \\ &= N(\mathbb{F}_H(h) - \mathbb{G}(\varrho)) - (m - \varrho N)\mathbb{G}'(\varrho) - A\frac{(m - \varrho N)^2}{N}. \end{aligned} \quad (4.5.29)$$

When $\varrho \leq \varrho_c$, $\mathbb{G}(\varrho) = \mathbb{G}'(\varrho) = 0$ and set $x = 0$. When $\varrho > \varrho_c$, let $x > 0$ such that $\mathbf{E}_x[\eta] = \frac{1}{\varrho}$. In both cases, remember that $\mathbb{G}(\varrho) = x + \varrho \log \mathbf{E}[e^{-x\eta}]$ and $\mathbb{G}'(\varrho) = \log \mathbf{E}[e^{-x\eta}]$ (see the proof of Lemma 4.2.1). We get

$$mh + NH\left(\frac{m}{N}\right) \leq N\mathbb{F}_H(h) - A\frac{(m - \varrho N)^2}{N} - Nx - m \log \mathbf{E}[e^{-x\eta}]. \quad (4.5.30)$$

Recall that

$$e^{-Nx - m \log \mathbf{E}[e^{-x\eta}]} \mathbf{P}(\tau_m = N) = \mathbf{P}_x(\tau_m = N) \leq 1, \quad (4.5.31)$$

to obtain, for every $1 \leq m \leq N$,

$$e^{mh} e^{NH(\frac{m}{N})} \mathbf{P}(\tau_m = N) \leq e^{N\mathbb{F}_H(h)} e^{-A\frac{(m - \varrho N)^2}{N}}. \quad (4.5.32)$$

We know (Assumptions 4.1.4) that for $|m - \varrho N| \leq \epsilon N$,

$$\Psi(m, N) \leq cN^c e^{NH(\frac{m}{N})}, \quad (4.5.33)$$

for a certain $c > 0$. Thus we get the upper bound

$$\hat{Z}_N - T_N \leq \sum_{|m - \varrho N| > B\sqrt{N \log N}} cN^c e^{N\mathbb{F}_H(h)} e^{-A\frac{(m - \varrho N)^2}{N}} \leq cN^{c+1-AB^2} e^{N\mathbb{F}_H(h)}. \quad (4.5.34)$$

Now we estimate \hat{Z}_N . The function H is analytic in a neighbourhood of ϱ . Thus, uniformly in m satisfying $|m - \varrho N| \leq B\sqrt{N \log N}$, one has

$$\begin{aligned} mh + NH\left(\frac{m}{N}\right) &= N(\varrho h + H(\varrho)) + (m - \varrho N)(h + H'(\varrho)) \\ &\quad + \frac{(m - \varrho N)^2}{2N} H''(\varrho) + O\left(\frac{(m - \varrho N)^3}{N^2}\right) \\ &= N(\varrho h + H(\varrho)) + (m - \varrho N)(h + H'(\varrho)) \\ &\quad + \frac{(m - \varrho N)^2}{2N} H''(\varrho) + O(N^{-1/4}). \end{aligned} \quad (4.5.35)$$

With the same computation as before (see (4.5.32)) and then with Assumptions 4.1.4, we get

$$\begin{aligned} e^{mh} \Psi(m, N) \mathbf{P}(\tau_m = N) &\sim Q(m, N) e^{N\mathbb{F}_H(h)} e^{\frac{(m - \varrho N)^2}{2N} H''(\varrho)} \mathbf{P}_x(\tau_m = N) \\ &\sim Q(\lfloor \varrho N \rfloor, N) e^{N\mathbb{F}_H(h)} e^{\frac{(m - \varrho N)^2}{2N} H''(\varrho)} \mathbf{P}_x(\tau_m = N). \end{aligned} \quad (4.5.36)$$

We now need to analyze the asymptotic behaviour of $\mathbf{P}_x(\tau_m = N)$.

1. If $h < h_c$ then $\varrho < \varrho_c$, and $x = 0$. And we know (see (4.A.46)) that, uniformly in m satisfying $|m - \varrho N| \leq B\sqrt{N \log N}$,

$$\mathbf{P}_x(\tau_m = N) = \mathbf{P}(\tau_m = N) \sim \varrho N K(N - \varrho N \mathbf{E}[\eta]). \quad (4.5.37)$$

2. If $h = h_c$ and $\text{Var}(\eta) = +\infty$, then $\varrho = \varrho_c$, and once again $x = 0$. With Proposition 4.A.8, and recalling that $b_N \gg \sqrt{N}$, we have

$$e^{\frac{(m-\varrho N)^2}{2N} H''(\varrho)} \mathbf{P}(\tau_m = N) \sim \frac{g_\alpha(0)}{b_m} e^{\frac{(m-\varrho N)^2}{2N} H''(\varrho)} \sim \frac{g_\alpha(0)}{\varrho^{\frac{1}{\alpha}} b_N} e^{\frac{(m-\varrho N)^2}{2N} H''(\varrho)}, \quad (4.5.38)$$

uniformly in m satisfying $|m - \varrho N| \leq C\sqrt{N}$. And

$$\limsup \left[b_N e^{\frac{(m-\varrho N)^2}{2N} H''(\varrho)} \mathbf{P}(\tau_m = N) \right] \leq e^{C^2 H''(\varrho)} \sup |g_\alpha|, \quad (4.5.39)$$

($H''(\varrho) < 0$), uniformly in m satisfying $|m - \varrho N| \geq C\sqrt{N}$. Combining these two lines, we get

$$\sup_{|m-\varrho N| \leq B\sqrt{N \log N}} \left| b_N e^{\frac{(m-\varrho N)^2}{2N} H''(\varrho)} \mathbf{P}(\tau_m = N) - \frac{g_\alpha(0)}{\varrho^{\frac{1}{\alpha}}} e^{\frac{(m-\varrho N)^2}{2N} H''(\varrho)} \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (4.5.40)$$

3. If either $h = h_c$ and $\text{Var}(\eta) < +\infty$ or $h > h_c$. Recall $\sigma_x^2 = \text{Var}_x(\eta)$. One has, still with Proposition 4.A.8,

$$\mathbf{P}_x(\tau_m = N) \sim e^{-\frac{(N-m/\varrho)^2}{2\sigma_x^2 m}} \frac{1}{\sqrt{2\pi\sigma_x^2 m}} \sim e^{-\frac{(m-N\varrho)^2}{2\varrho^3\sigma_x^2 N}} \frac{1}{\sqrt{2\pi\sigma_x^2 \varrho N}}. \quad (4.5.41)$$

Remember (Lemma 4.3.1) that $G''(\varrho) = -(\varrho^3\sigma_x^2)^{-1}$. With the same reasoning as before we obtain

$$\sup_{\substack{|m-\varrho N| \\ \leq B\sqrt{N \log N}}} \left| \sqrt{2\pi\sigma_x^2 \varrho N} e^{\frac{(m-\varrho N)^2}{2N} H''(\varrho)} \mathbf{P}_x(\tau_m = N) - e^{\frac{(m-\varrho N)^2}{2N} (H''(\varrho) + G''(\varrho))} \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (4.5.42)$$

Recall that $G''(\varrho) = 0$ in the first two cases (Lemma 4.3.1). Therefore we get in the three cases

$$\sup_{\substack{|m-\varrho N| \\ \leq B\sqrt{N \log N}}} \left| R(N)^{-1} e^{\frac{(m-\varrho N)^2}{2N} H''(\varrho)} \mathbf{P}_x(\tau_m = N) - e^{\frac{(m-\varrho N)^2}{2N} (H''(\varrho) + G''(\varrho))} \right| \xrightarrow{N \rightarrow +\infty} 0, \quad (4.5.43)$$

where $R(N)$ is regularly varying with N : $R(N) = \varrho N K(N(1 - \varrho/\varrho_c))$ in the first case, $R(N) = \frac{g_\alpha(0)}{\varrho^{\frac{1}{\alpha}} b_N}$ in the second, and $b_N = (2\pi\varrho N \sigma_x^2)^{-1/2}$ in the third one. Recall that

$H''(\varrho) + G''(\varrho) = -\tilde{\sigma}^{-2}$. We obtain

$$\begin{aligned}
\hat{Z}_N &\sim Q(\lfloor \varrho N \rfloor, N) e^{N_{\text{FH}}(h)} \sum_{|m-\varrho N| \leq B\sqrt{N \log N}} e^{\frac{(m-\varrho N)^2}{2N} H''(\varrho)} \mathbf{P}_x(\tau_m = N) \\
&\sim R(N) e^{N_{\text{FH}}(h)} Q(\lfloor \varrho N \rfloor, N) \sum_{|m-\varrho N| \leq B\sqrt{N \log N}} e^{-\frac{(m-\varrho N)^2}{2\tilde{\sigma}^2 N}} \\
&\sim R(N) e^{N_{\text{FH}}(h)} Q(\lfloor \varrho N \rfloor, N) \sqrt{2\pi N} \tilde{\sigma} \sum_{|m-\varrho N| \leq B\sqrt{N \log N}} \mathbf{P}(\lfloor \varrho N + \sqrt{N} \tilde{\sigma} \mathcal{N} \rfloor = m). \\
&\sim R(N) e^{N_{\text{FH}}(h)} Q(\lfloor \varrho N \rfloor, N) \sqrt{2\pi N} \tilde{\sigma}.
\end{aligned} \tag{4.5.44}$$

With the upper bound (4.5.34) and the estimation (4.5.44) we get the asymptotic equivalence (4.5.28) (as soon as B is large enough).

Moreover, if $I = \left\{ r \in \mathbf{N}^* \mid x\sqrt{N} \leq r - \varrho N \leq y\sqrt{N} \right\}$, and if A is an event for the renewal, then

$$\begin{aligned}
\xi_N^{\Psi, h}(A, L \in I) &= \frac{1}{Z_N} \sum_{r \in I} e^{rh} \Psi(N, r) \mathbf{P}(A, \tau_r = N) \\
&= \frac{1}{Z_N} \sum_{r \in I} e^{rh} \Psi(N, r) \mathbf{P}(\tau_r = N) \mu_{N, r}(A).
\end{aligned} \tag{4.5.45}$$

We got in the previous lines a uniform asymptotic equivalence for $e^{rh} \Psi(N, r) \mathbf{P}(\tau_r = N)$:

$$\begin{aligned}
e^{rh} \Psi(N, r) \mathbf{P}(\tau_r = N) &\sim R(N) e^{N_{\text{FH}}(h)} Q(\lfloor \varrho N \rfloor, N) \sqrt{2\pi N} \tilde{\sigma} \mathbf{P}\left(\lfloor \varrho N + \sqrt{N} \tilde{\sigma} \mathcal{N} \rfloor = r\right) \\
&\sim \hat{Z}_N \mathbf{P}\left(\lfloor \varrho N + \sqrt{N} \tilde{\sigma} \mathcal{N} \rfloor = r\right) \\
&\sim Z_N \mathbf{P}\left(\lfloor \varrho N + \sqrt{N} \tilde{\sigma} \mathcal{N} \rfloor = r\right).
\end{aligned} \tag{4.5.46}$$

Thus,

$$\xi_N^{\Psi, h}(A, L \in I) \sim \sum_{r \in I} \mathbf{P}\left(\lfloor \varrho N + \sqrt{N} \tilde{\sigma} \mathcal{N} \rfloor = r\right) \mu_{N, r}(A). \tag{4.5.47}$$

The last expression is precisely $\pi_N(A, L \in I)$. \square

Remark 4.5.4. From Proposition 4.5.2 and the result of Subsection 4.2.3, dealing with the size of the largest jump for the constrained PS model $\mu_{N, L}$ (L fixed) in the different regimes, fluctuations results for the largest jump $M_{|\tau|}$ in the kernel models can be derived when $h \in (h_*, h^*)$. It would be cumbersome to detail all the cases. But let us give one example: when $h_* < h < h_c$,

$$\left(\frac{|\tau| - \varrho N}{\tilde{\sigma} \sqrt{N}}, \frac{M_{|\tau|} - (N - |\tau|L)}{b_{|\tau|}} \right) \xrightarrow[N \rightarrow \infty]{\xi_N^{\Psi, h}} (Y, -X_\alpha), \tag{4.5.48}$$

where $Y \sim \mathcal{N}(0, 1)$ is independent of the stable random variable X_α (see Proposition 4.A.7 for the definitions b_L and X_α).

4.A Appendix: Results for heavy tail random walks

4.A.1 Slowly Varying Functions and Laplace Transform Estimates

A complete reference for slowly varying functions is [14]. We recall some of the results which are useful in this work.

Lemma 4.A.1 (Uniform convergence Theorem [14, Th. 1.2.1]). *For any $0 < a < b < +\infty$,*

$$\sup_{an \leq m \leq bn} \left| \frac{\varphi(m)}{\varphi(n)} - 1 \right| \xrightarrow{n \rightarrow +\infty} 0. \quad (4.A.1)$$

Lemma 4.A.2 (Potter's bound, [14, Th. 1.5.6]). *If φ is slowly varying then for any chosen constants $A > 1$, $\delta > 0$ there exists $X = X(1, \delta)$ such that for any $x, y \geq X$,*

$$\frac{\varphi(y)}{\varphi(x)} \leq A \max \left(\left(\frac{y}{x} \right)^\delta, \left(\frac{y}{x} \right)^{-\delta} \right). \quad (4.A.2)$$

Lemma 4.A.3. *Let $g : (0, +\infty) \rightarrow \mathbf{C}$ be a piecewise continuous function such that the integrals $\int_1^{+\infty} |g(y)|y^\delta dy$ and $\int_0^1 |g(y)|y^{-\delta} dy$ are finite, for some $\delta > 0$. Then*

$$x \sum_{n=1}^{+\infty} \frac{\varphi(n)}{\varphi(1/x)} g(xn) \xrightarrow{x \searrow 0} \int_0^{+\infty} g(y) dy. \quad (4.A.3)$$

Lemma 4.A.4 ([14, Th. 1.5.12]). *There exists a sequence a_L , regularly varying with index $\frac{1}{\alpha}$, such that*

$$\frac{L\varphi(a_L)}{a_L^\alpha} \xrightarrow{L \rightarrow +\infty} 1. \quad (4.A.4)$$

4.A.2 Laplace Transform

We will need the precise behaviour of K Laplace transform at 0. The following lemma is a consequence of [14, § 1.5.6 and Cor. 1.7.3].

Lemma 4.A.5. *The following behaviours hold as $x \searrow 0$. The function φ is the slowly varying function of the renewal and φ_1 is, in each integer case, an other slowly varying function.*

1. *If $\alpha = 0$ then*

$$\begin{aligned} \mathbf{E}[e^{-x\eta}] &= 1 - \varphi_1(1/x)(1 + o(1)), \\ \mathbf{E}[\eta e^{-x\eta}] &= \frac{\varphi(1/x)}{x}(1 + o(1)), \end{aligned} \quad (4.A.5)$$

and $\varphi_1 \gg \varphi$.

2. *If $\alpha \in (0, 1)$ then*

$$\begin{aligned} \mathbf{E}[e^{-x\eta}] &= 1 - \frac{1}{\alpha} \Gamma(1 - \alpha) x^\alpha \varphi(1/x)(1 + o(1)), \\ \mathbf{E}[\eta e^{-x\eta}] &= \Gamma(1 - \alpha) x^{\alpha-1} \varphi(1/x)(1 + o(1)). \end{aligned} \quad (4.A.6)$$

3. If $\alpha = 1$ and $\mathbf{E}[\eta] = +\infty$ then

$$\begin{aligned}\mathbf{E}[e^{-x\eta}] &= 1 - x\varphi_1(1/x)(1 + o(1)), \\ \mathbf{E}[\eta e^{-x\eta}] &= \varphi_1(1/x)(1 + o(1)),\end{aligned}\tag{4.A.7}$$

with $\varphi_1 \gg \varphi$ and $\lim_{+\infty} \varphi_1 = +\infty$.

4. If $\alpha = 1$ and $\mathbf{E}[\eta] < +\infty$ then

$$\begin{aligned}\mathbf{E}[e^{-x\eta}] &= 1 - x\mathbf{E}[\eta] + x\varphi_1(1/x)(1 + o(1)), \\ \mathbf{E}[\eta e^{-x\eta}] &= \mathbf{E}[\eta] - \varphi_1(1/x)(1 + o(1)),\end{aligned}\tag{4.A.8}$$

with $\varphi_1 \gg \varphi$ and $\lim_{+\infty} \varphi_1 = 0$.

5. If $\alpha \in (1, 2)$ then

$$\begin{aligned}\mathbf{E}[e^{-x\eta}] &= 1 - x\mathbf{E}[\eta] + \Gamma(-\alpha)x^\alpha\varphi(1/x)(1 + o(1)), \\ \mathbf{E}[\eta e^{-x\eta}] &= \mathbf{E}[\eta] - \alpha\Gamma(-\alpha)x^{\alpha-1}\varphi(1/x)(1 + o(1)).\end{aligned}\tag{4.A.9}$$

6. If $\alpha = 2$ and $\text{Var}(\eta) = +\infty$ then

$$\begin{aligned}\mathbf{E}[e^{-x\eta}] &= 1 - x\mathbf{E}[\eta] + \frac{x^2}{2}\varphi_1(1/x)(1 + o(1)), \\ \mathbf{E}[\eta e^{-x\eta}] &= \mathbf{E}[\eta] - x\varphi_1(1/x)(1 + o(1)),\end{aligned}\tag{4.A.10}$$

with $\varphi_1 \gg \varphi$ and $\lim_{+\infty} \varphi_1 = +\infty$.

7. If $\alpha \geq 2$ and $\text{Var}(\eta) < +\infty$ then

$$\begin{aligned}\mathbf{E}[e^{-x\eta}] &= 1 - x\mathbf{E}[\eta] + \frac{\mathbf{E}[\eta^2]}{2}x^2(1 + o(1)), \\ \mathbf{E}[\eta e^{-x\eta}] &= \mathbf{E}[\eta] - \mathbf{E}[\eta^2]x(1 + o(1)).\end{aligned}\tag{4.A.11}$$

4.A.3 Limit Theorems

Recall that

$$M_L = \max_{1 \leq k \leq L} \eta_k = \max_{1 \leq k \leq L} (\tau_k - \tau_{k-1}).\tag{4.A.12}$$

Proposition 4.A.6 (Limit theorem for M_L). *For any $\alpha > 0$, and any $t \geq 0$,*

$$\mathbf{P}\left(\frac{M_L}{a_L} \leq t\right) \xrightarrow{L \rightarrow +\infty} \exp\left(-\frac{1}{\alpha t^\alpha}\right).\tag{4.A.13}$$

Proposition 4.A.7 (Limit theorems for τ_L , [60, Ch. 7]). *For any $\alpha > 0$ the following convergence holds*

$$\frac{\tau_L - \mu_L}{b_L} \xrightarrow{L \rightarrow +\infty} X_\alpha,\tag{4.A.14}$$

for sequences μ_L, b_L and an α -stable random variable X_α , which are detailed for each case.

– $\mu_L = 0$ when $\alpha \in (0, 1)$; $\mu_L = L\mathbf{E}[\eta \mathbf{1}_{\eta \leq a_L}]$ when $\alpha = 1$; and $\mu_L = L\mathbf{E}[\eta]$ when $\alpha > 1$.

- If $\alpha \in (0, 2) \setminus \{1\}$, then $b_L = a_L$.
If $\alpha \in \{1, 2\}$ with $\text{Var}(\eta) = +\infty$ then b_L is regularly varying with index $\frac{1}{\alpha}$. Furthermore $b_L \gg \sqrt{L}$ and $b_L \gg a_L$.
If $\text{Var}(\eta) < +\infty$, then $b_L = \sqrt{L \text{Var}(\eta)} \gg a_L$.
- X_α is an α -stable random variable, whose continuous and bounded density function is denoted by g_α .

$$\mathbf{E}[e^{itX_\alpha}] = \exp\left(|t|^\alpha e^{-i \text{sgn}(t) \frac{\alpha\pi}{2}} \Gamma(-\alpha)\right) \quad \alpha \in (0, 2) \setminus \{1\}; \quad (4.A.15)$$

$$\mathbf{E}[e^{itX_\alpha}] = \exp\left(-|t| \left(\frac{\pi}{2} - i \text{sgn}(t) \log \frac{\pi|t|}{2}\right)\right) \quad \alpha = 1; \quad (4.A.16)$$

$$\mathbf{E}[e^{itX_\alpha}] = \exp\left(-\frac{t^2}{2}\right), \quad \text{i.e., } X_\alpha = \mathcal{N}(0, 1), \quad \alpha \geq 2. \quad (4.A.17)$$

4.A.4 Local Limit Theorems

We start with the most standard local limit theorem for discrete random variables, for “moderate deviations”.

Proposition 4.A.8 (Local limit theorem). *For any $\alpha > 0$,*

$$\sup_{k \in \mathbf{N}} \left| b_L \mathbf{P}(\tau_L = k) - g_\alpha \left(\frac{k - \mu_L}{b_L} \right) \right| \xrightarrow{L \rightarrow +\infty} 0. \quad (4.A.18)$$

The definitions of μ_L , b_L and g_α are in Proposition 4.A.7. One can find a proof in [60, Ch. 9 § 49 and § 50], except for the case $\alpha = 2$ and $\text{Var}(\eta) = +\infty$, for which we refer to [37]. Note that it also holds for any sum of IID random variables with finite variance. For it is useful in the proof of Proposition 4.2.20, we give, following [60], a complete proof of this result when $\alpha \in (1, 2)$.

Proof when $\alpha \in (1, 2)$. We compute the characteristic function

$$\mathbf{E}[e^{it(\tau_L - L\mathbf{E}[\eta])}] = \mathbf{E}[e^{it(\eta - \mathbf{E}[\eta])}]^L. \quad (4.A.19)$$

One has

$$\begin{aligned} \mathbf{E} \left[e^{it(\eta - \mathbf{E}[\eta])} \right] &= \sum_{n=1}^{xa_L} \mathbf{K}(n) e^{it(n - \mathbf{E}[\eta])} \\ &= 1 + \sum_{n=1}^{+\infty} \frac{\varphi(n)}{n^{1+\alpha}} (e^{it(n - \mathbf{E}[\eta])} - 1 - it(n - \mathbf{E}[\eta])). \end{aligned} \quad (4.A.20)$$

Thus

$$\begin{aligned} \mathbf{E} \left[e^{\frac{it}{a_L}(\tau_L - L\mathbf{E}[\eta])} \right] &= \left[1 + \sum_{n=1}^{+\infty} \frac{\varphi(n)}{n^{1+\alpha}} \left(e^{it \frac{n - \mathbf{E}[\eta]}{a_L}} - 1 - it \frac{n - \mathbf{E}[\eta]}{a_L} \right) \right]^L \\ &= \left[1 + \frac{\varphi(a_L)}{a_L^\alpha} \frac{1}{a_L} \sum_{n=1}^{+\infty} \frac{\varphi(n)}{\varphi(a_L)} \left(\frac{n}{a_L} \right)^{-(1+\alpha)} \left(e^{it \frac{n - \mathbf{E}[\eta]}{a_L}} - 1 - it \frac{n - \mathbf{E}[\eta]}{a_L} \right) \right]^L \\ &\rightarrow \exp \left(\int_0^{+\infty} \frac{e^{ity} - 1 - ity}{y^{1+\alpha}} dy \right). \end{aligned} \quad (4.A.21)$$

Therefore the convergence of Proposition 4.A.7 is proved. Then we need to prove the local version. Let

$$f(t) = \mathbf{E}[e^{it\eta}], \quad \psi(t) = \mathbf{E}[e^{it(\eta - \mathbf{E}[\eta])}], \quad (4.A.22)$$

and

$$v_\alpha(t) = \mathbf{E}[e^{itX_\alpha}], \quad g_\alpha(z) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-itz} v_\alpha(t) dt. \quad (4.A.23)$$

We will need the following two lemmas, whose proofs are postponed after the proposition's proof.

Lemma 4.A.9. $v_\alpha \in L^1$.

Lemma 4.A.10. *There exists $\epsilon > 0$ and $c > 0$ such that for any $t \in [-\epsilon, \epsilon]$,*

$$|f(t)| \leq e^{-c|t|^\alpha \varphi(1/|t|)}. \quad (4.A.24)$$

First observe that

$$2\pi \mathbf{P}(\tau_L = k) = \int_{-\pi}^{\pi} e^{-ikt} \mathbf{E}[e^{it\tau_L}] dt = \int_{-\pi}^{\pi} e^{-ikt} f(t)^L dt = \int_{-\pi}^{\pi} e^{-i(k-L\mathbf{E}[\eta])t} \psi(t)^L dt. \quad (4.A.25)$$

With the change of variable $u = ta_L$ we get

$$2\pi a_L \mathbf{P}(\tau_L = k) = \int_{-\pi a_L}^{\pi a_L} e^{-izu} \psi\left(\frac{u}{a_L}\right)^L du, \quad (4.A.26)$$

where $z = z_{k,L} = \frac{k-L\mathbf{E}[\eta]}{a_L}$. Then we can write

$$2\pi (a_L \mathbf{P}(\tau_L = k) - g_\alpha(z)) = I_1 - I_2 + I_3 + I_4, \quad (4.A.27)$$

where

$$\begin{aligned} I_1 &= \int_{-A}^A e^{-izu} \left(\psi\left(\frac{u}{a_L}\right)^L - v_\alpha(u) \right) du, & I_2 &= \int_{|u| \geq A} e^{-izu} v_\alpha(u) du, \\ I_3 &= \int_{A \leq |u| \leq \epsilon a_L} e^{-izu} \psi\left(\frac{u}{a_L}\right)^L du, & I_4 &= \int_{\epsilon a_L \leq |u| \leq \pi a_L} e^{-izu} \psi\left(\frac{u}{a_L}\right)^L du. \end{aligned} \quad (4.A.28)$$

We start with I_3 . Fix $\epsilon \in (0, 1)$ such as in Lemma 4.A.10. Then, for $|u| \leq \epsilon a_L$,

$$\begin{aligned} \left| f\left(\frac{u}{a_L}\right) \right|^L &\leq \exp\left(-c \frac{L\varphi(a_L/|u|)}{a_L^\alpha} |u|^\alpha\right), \\ &= \exp\left(-c|u|^\alpha \frac{L\varphi(a_L)}{a_L^\alpha} \frac{\varphi(a_L/|u|)}{\varphi(a_L)}\right). \end{aligned} \quad (4.A.29)$$

The Potter's bound (Lemma 4.A.2) ensures that, even if it means taking ϵ smaller, one has, for L sufficiently large and $A \leq |u| \leq \epsilon a_L$,

$$\frac{\varphi(a_L/|u|)}{\varphi(a_L)} \geq 2|u|^{-\frac{\alpha}{2}}. \quad (4.A.30)$$

On the other hand, if L is sufficiently large,

$$\frac{L\varphi(a_L)}{a_L^\alpha} \geq \frac{1}{2}. \quad (4.A.31)$$

Eventually,

$$|I_3| \leq 2 \int_A^{\epsilon a_L} \exp\left(-c|u|^{\frac{\alpha}{2}}\right) du \leq 2 \int_A^{+\infty} \exp\left(-c|u|^{\frac{\alpha}{2}}\right) du. \quad (4.A.32)$$

For I_4 , set

$$r = r(\epsilon) = \sup_{\epsilon \leq |t| \leq \pi} |f(t)| \in (0, 1). \quad (4.A.33)$$

Then,

$$|I_4| \leq 2\pi a_L r^L. \quad (4.A.34)$$

Fix $\delta > 0$ and let A be large enough. We have

$$|I_2| \leq \int_{|u| \geq A} |v_\alpha(u)| du \leq \delta, \quad 2 \int_A^{+\infty} e^{-c|u|^{\frac{\alpha}{2}}} du \leq \delta. \quad (4.A.35)$$

We know that for any $u \in \mathbf{R}$,

$$\psi\left(\frac{u}{a_L}\right)^L \xrightarrow{L \rightarrow +\infty} v_\alpha(u). \quad (4.A.36)$$

Hence, one A is fixed,

$$I_1 = \int_{-A}^A e^{-izu} \left(\psi\left(\frac{u}{a_L}\right)^L - v_\alpha(u) \right) du \xrightarrow{L \rightarrow +\infty} 0 \quad (4.A.37)$$

by dominated convergence. On the other hand,

$$2\pi a_L r^L \xrightarrow{L \rightarrow +\infty} 0. \quad (4.A.38)$$

Therefore

$$\limsup_{L \rightarrow +\infty} 2\pi |a_L \mathbf{P}(\tau_L = k) - g_\alpha(z)| \leq 2\delta. \quad (4.A.39)$$

The local limit theorem is proved. \square

Proof of Lemma 4.A.9. We know the expression of $v_\alpha(t)$:

$$|v_\alpha(t)| = \left| \exp\left(\int_0^{+\infty} \frac{e^{ity} - 1 - ity}{y^{1+\alpha}} dy\right) \right| = \exp\left(\int_0^{+\infty} \frac{\cos(ty) - 1}{y^{1+\alpha}} dy\right) = \exp(-c|t|^\alpha), \quad (4.A.40)$$

where $c = \int_0^{+\infty} \frac{1 - \cos(z)}{z^{1+\alpha}} dz$. Thus, v_α is integrable. \square

Proof of Lemma 4.A.10. First we estimate $1 - f(t)$:

$$\begin{aligned} 1 - f(t) &= \sum_{n=1}^{+\infty} \frac{\varphi(n)}{n^{1+\alpha}} (1 - e^{itn}) \\ &= |t|^\alpha \varphi\left(\frac{1}{|t|}\right) |t| \sum_{n=1}^{+\infty} \frac{\varphi(n|t|/|t|)}{\varphi(1/|t|)} \frac{1}{(n|t|)^{1+\alpha}} (1 - e^{itn}) \\ &\underset{t \rightarrow 0}{\sim} |t|^\alpha \varphi\left(\frac{1}{|t|}\right) \int_0^{+\infty} \frac{1 - e^{iy \operatorname{sgn}(t)}}{y^{1+\alpha}} dy. \end{aligned} \quad (4.A.41)$$

Hence,

$$|f(t)| = 1 - |t|^\alpha \varphi \left(\frac{1}{|t|} \right) \int_0^{+\infty} \frac{1 - \cos(y)}{y^{1+\alpha}} dy (1 + o_{t \rightarrow 0}(1)). \quad (4.A.42)$$

If ϵ is small enough, then for every $|t| \leq \epsilon$,

$$|f(t)| \leq 1 - c|t|^\alpha \varphi \left(\frac{1}{|t|} \right) \leq e^{-c|t|^\alpha \varphi(1/|t|)}. \quad (4.A.43)$$

□

Lemma 4.A.11 (Local limit theorem: big jump domain). *Suppose $\mathbf{E}[\eta] < +\infty$. Let c_N be such that*

$$\begin{cases} c_N = \gamma_N b_N & \text{if } \alpha \in [1, 2) \\ c_N = qN, \quad q \in (0, 1) & \text{if } \alpha = 2 \\ c_N = \sigma \sqrt{(\alpha - 2) \varrho_c N \log N} \left(1 + \frac{\alpha+1}{2(\alpha-2)} \frac{\log \log N}{\log N} - \frac{\log \varphi(\sqrt{N \log N})}{(\alpha-2) \log N} + \frac{\gamma_N}{\log N} \right) & \text{if } \alpha > 2 \end{cases} \quad (4.A.44)$$

with $\gamma_N \rightarrow +\infty$. Set $V_N = \varrho_c(N - c_N)$. Then

$$\sup_{L \leq V_N} \left| \frac{\mathbf{P}(\tau_L = N)}{LK(N - L\mathbf{E}[\eta])} - 1 \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (4.A.45)$$

In particular for any $\varrho < \varrho_c$,

$$\sup_{L \leq \varrho N} \left| \frac{\mathbf{P}(\tau_L = N)}{LK(N - L\mathbf{E}[\eta])} - 1 \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (4.A.46)$$

Proof. The main references are [33] and [8], for which we refer to the corresponding results. For $\alpha \in (1, 2)$ see [33, Th.2.1 and 9.1]. For $\alpha = 2$ see [33, Cor.2.1] (the same reference contains sharper results for this case). For $\alpha = 1$, see [8, Th.2.4]. For $\alpha > 2$, it is a consequence of the next result (Lemma 4.A.12). □

Lemma 4.A.12 ([38, Th.2]). *If $\alpha > 2$ and if $z = \frac{N - L\mathbf{E}[\eta]}{\sqrt{L}} \rightarrow +\infty$, then*

$$\mathbf{P}(\tau_L = N) = \frac{1}{\sqrt{2\pi\sigma^2 L}} e^{-\frac{z^2}{2\sigma^2 L}} (1 + o(1)) + LK(N - L\mathbf{E}[\eta])(1 + o(1)). \quad (4.A.47)$$

Chapter 5

Disordered Circular DNA

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In this chapter, we consider a disordered version of the fixed contact number and kernel PS models introduced in Chapter 4. Like in the disordered PS model (Section 3.2), the inhomogeneity of the interactions between the two strands (A-T bonds are weaker than G-C bonds) is modeled by IID random variables $(\omega_j)_{j=1,2,\dots}$, whose law is denoted by \mathbb{P} ; the disorder is independent of the renewal randomness. For the sake of conciseness, we assume that the disorder is bounded (Hypothesis 5.1.1).

Our main result is that disorder suppresses the big jump phenomenon and we will argue that both at the free energy level and by analyzing the trajectories of the model.

5.1 Models and Assumptions

The setting is the same as in Chapter 4, except that the contact interaction h at site j is replaced by $\beta\omega_j + h$, where β is a nonnegative parameter (of course the homogeneous model is recovered by taking $\beta = 0$). We still work under Assumptions 4.1.3 and 4.1.4, and we make the additional assumption on the disorder:

Assumptions 5.1.1. The random variables ω_j are normalized

$$\mathbb{E}[\omega_j] = 0, \quad \mathbb{E}[\omega_j^2] = 1, \tag{5.1.1}$$

and bounded

$$|\omega_j| \leq C \quad \mathbb{P}\text{-a.s.} \tag{5.1.2}$$

We are interested in the two following models.

Disordered fixed contact number model. For any $1 \leq L \leq N$, we consider the random measure

$$\mu_{N,L}^{\omega,\beta}(A) = \frac{1}{Z_{N,L}^{\omega,\beta}} \mathbf{E} \left[e^{\sum_{j=1}^N \beta \omega_j \delta_j} \mathbf{1}_A \mathbf{1}_{\tau_L=N} \right], \quad (5.1.3)$$

where $A \in \mathcal{F}$ (see (4.1.5)), $\delta_j = \mathbf{1}_{j \in \tau}$, and the partition function is the random variable

$$Z_{N,L}^{\omega,\beta} = \mathbf{E} \left[\exp \left(\sum_{j=1}^N \beta \omega_j \delta_j \right) \mathbf{1}_{\tau_L=N} \right]. \quad (5.1.4)$$

Disordered kernel model. The kernel model becomes in presence of disorder:

$$\xi_N^{\Psi,h,\omega,\beta}(A) = \frac{1}{Z_N^{\Psi,h,\omega,\beta}} \mathbf{E} \left[e^{\sum_{j=1}^N (\beta \omega_j + h) \delta_j} \Psi(|\tau|, N) \mathbf{1}_A(\tau) \mathbf{1}_{N \in \tau} \right], \quad A \in \mathcal{F}. \quad (5.1.5)$$

Once again the partition function can be decomposed depending on the number of contact points:

$$Z_N^{\Psi,h,\omega,\beta} = \sum_{m=1}^N e^{hm} \Psi(m, N) Z_{N,m}^{\omega,\beta}. \quad (5.1.6)$$

For simplicity, the results will only be stated under the following assumption, which is fulfilled by the circular DNA mechanisms introduced in Chapter 4 (Example 4.1.2).

Assumptions 5.1.2. $H'(a) = +\infty$ and $H'(b) = -\infty$.

By taking formally $\Psi = 1$ (which does not fulfill Assumptions 4.1.4), we recover the standard disordered PS model, defined by

$$\nu_N^{h,\omega,\beta}(A) = \frac{1}{Z_N^{h,\omega,\beta}} \mathbf{E} \left[e^{\sum_{j=1}^N (\beta \omega_j + h) \delta_j} \mathbf{1}_A(\tau) \mathbf{1}_{N \in \tau} \right]. \quad (5.1.7)$$

and whose partitions function is:

$$Z_N^{h,\omega,\beta} = \mathbf{E} \left[\exp \left(\sum_{j=1}^N (\beta \omega_j + h) \delta_j \right) \mathbf{1}_{N \in \tau} \right] = \sum_{m=1}^N e^{hm} Z_{N,m}^{\omega,\beta}. \quad (5.1.8)$$

5.2 Main Results

We investigate two aspects of these models. First the free energy properties (smoothness and critical exponents), second the typical trajectories, depending on the parameters. For the disordered PS model $\nu_N^{h,\omega,\beta}$, these have been deeply studied and a summary of the known results can be found in Section 3.2. These results are important for us both for comparison with some of the new results we obtain in this chapter and because we are going to exploit some of them in the arguments of proof.

5.2.1 Free Energy: existence and regularity

The first result gives the existence of the free energies (that is the convergence of the partition functions), and provides variational formulas relating them.

Proposition 5.2.1. *For any $\beta \geq 0$, $h \in \mathbf{R}$ and $\varrho \in [0, 1]$, in the case of the fixed contact number model we have, almost surely and in \mathbf{L}^1 , the existence of the limit*

$$\lim_{\substack{N \rightarrow +\infty \\ \frac{L}{N} \rightarrow \varrho}} \frac{1}{N} \log \mathcal{Z}_{N,L}^{\omega, \beta} =: \mathbb{G}(\beta, \varrho), \quad (5.2.1)$$

while for the kernel model in the same sense we have the existence of the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log Z_N^{\Psi, h, \omega, \beta} =: \mathbb{F}_H(\beta, h), \quad (5.2.2)$$

and we recall that for the standard disordered PS model

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log Z_N^{h, \omega, \beta} =: \mathbb{F}(\beta, h). \quad (5.2.3)$$

The free energies $\mathbb{G}(\beta, \cdot)$, $\mathbb{F}(\beta, \cdot)$ and $\mathbb{F}_H(\beta, \cdot)$ are finite, non random, and obey the conjugate variational formulas

$$\mathbb{F}(\beta, h) = \sup_{\varrho \in [0, 1]} \{h\varrho + \mathbb{G}(\beta, \varrho)\}, \quad \mathbb{G}(\beta, \varrho) = \inf_{h \in \mathbf{R}} \{\mathbb{F}(\beta, h) - \varrho h\}, \quad (5.2.4)$$

and

$$\mathbb{F}_H(\beta, h) = \sup_{\varrho \in [0, 1]} \{h\varrho + H(\varrho) + \mathbb{G}(\beta, \varrho)\}, \quad \mathbb{G}(\beta, \varrho) = \inf_{h \in \mathbf{R}} \{\mathbb{F}_H(\beta, h) - \varrho h - H(\varrho)\}. \quad (5.2.5)$$

From now on, unless otherwise specified, we suppose $\beta > 0$. Our approach is to exploit what is known about $\mathbb{F}(\beta, h)$ to extract information about $\mathbb{G}(\beta, \varrho)$, thanks to the variational formula (5.2.4), and then employ this information to establish results about $\mathbb{F}_H(\beta, h)$.

The disorder smooths the fixed contact number model transition. We are now going to explain how the big jump transition mechanism at the free energy level for $\beta = 0$ does not exist anymore for $\beta > 0$.

We first present the argument in an informal way, carrying along the parallel between $\beta = 0$ and $\beta > 0$, and then we will give two precise statements. So the point is to argue that from the variational formula

$$\mathbb{G}(\beta, \varrho) = \inf_{h \in \mathbf{R}} \{\mathbb{F}(\beta, h) - \varrho h\}, \quad (5.2.6)$$

and by using the smoothing bound (3.2.10) one readily obtains that the optimization procedure changes radically when switching from $\beta = 0$ to $\beta > 0$.

First of all for the homogeneous model ($\beta = 0$) the situation is:

H1: If $\mathbf{E}[\tau_1] = +\infty$, then $h \mapsto \partial_h \mathbb{F}_h(0, h)$ is continuous on \mathbf{R} and the equation $\partial_h \mathbb{F}(0, h) = \varrho$ is solvable for every $\varrho \in (0, 1)$: the solution is unique, positive and depends analytically on ϱ (Figure 5.1).

H2: If $\mathbf{E}[\tau_1] < +\infty$ then

$$\partial_h F_h(0, h) \stackrel{h \searrow 0}{\sim} \varrho_c h, \quad (5.2.7)$$

and the equation $\partial_h F(0, h) = \varrho$ is solvable only for $\varrho \geq \varrho_c > 0$, with analytic dependence for $\varrho > \varrho_c$, and the variational problem is solved by $h = 0$ for $\varrho < \varrho_c$ (and of course $G(0, \varrho) = 0$ then). Therefore $G(0, \varrho) = 0$ for $\varrho \in [0, \varrho_c]$ and $G(0, \varrho) < 0$ for $\varrho > \varrho_c$ (Figure 5.2). This transition is the big jump transition.

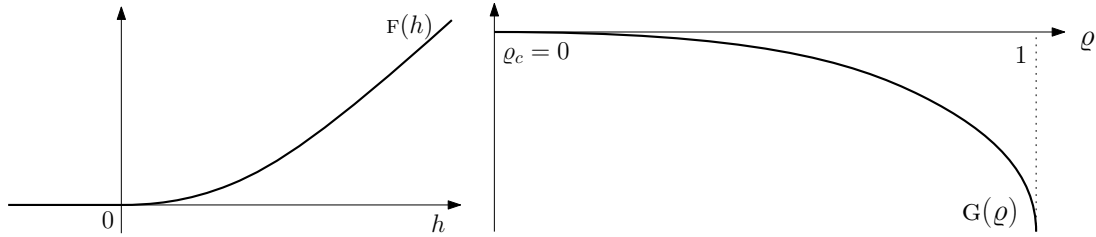


FIGURE 5.1: The case $\mathbf{E}[\tau_1] = +\infty$ (item H1): The function $h \mapsto F(h)$ is continuously differentiable and G is analytic on $(0, 1)$.

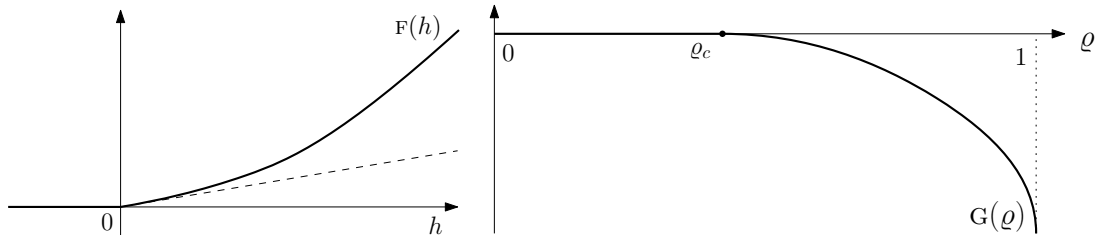


FIGURE 5.2: The case $\mathbf{E}[\tau_1] < +\infty$ (item H2): The function $h \mapsto \partial_h F(h)$ is discontinuous at $h = 0$ and G is not analytic at $\varrho_c = \partial_h^+ F(0)$.

Let us switch the disorder on ($\beta > 0$):

- the smoothing bound (3.2.10) implies continuity of $\partial_h F(\beta, \cdot)$. Hence the singularity arising from the mechanism in H2 is no longer present and the optimization procedure becomes the one in H1, even when $\mathbf{E}[\tau_1]$ is finite. That is, the equation $\partial_h F(\beta, h) = \varrho$ has a unique solution for every $\varrho \in (0, 1)$: there is no switch (unlike in H2) between a regime in which $\partial_h F(\beta, h) = \varrho$ is solvable and one in which it is not solvable, like for the big jump transition (Figure 5.3).
- However, we cannot conclude, like in H1, that the free energy is analytical. The main obstruction is that we do not know whether $F(\beta, \cdot)$ is analytic in the localized regime.

Nevertheless, the arguments we have just outlined strongly suggest that the big jump transition is suppressed by the disorder. And we will actually show that this is the case by analyzing the trajectories of the disordered system, in Section 5.2.2.

On a more subordinate note, the behaviour of $G(\beta, \varrho)$ when $\varrho \searrow 0$ can be deduced from $F(\beta, h)$ critical behaviour (see Proposition 5.3.5). In a nutshell, $G(\beta, \varrho)$ should look like Figure 5.3. Once the behaviour of $G(\beta, \varrho)$ is (partially) understood, it can be injected into the variational formula (5.2.5) to obtain properties of $F_H(\beta, h)$.

Let us make more precise this informal discussion:

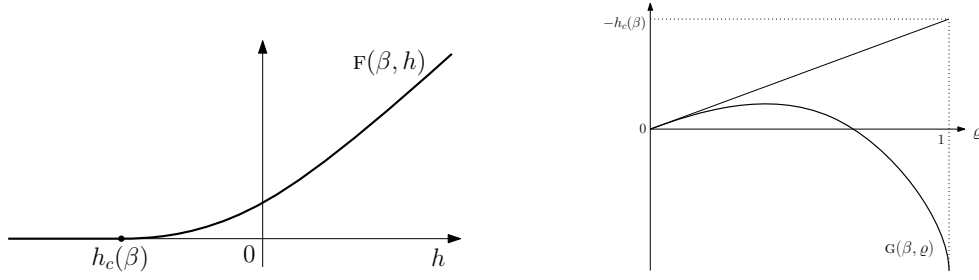


FIGURE 5.3: The case $\beta > 0$: the free energy $F(\beta, \cdot)$ is continuously differentiable.

Theorem 5.2.2.

1. $G(\beta, \cdot)$ is continuous and strictly concave on $[0, 1]$. It is infinitely differentiable on $(0, 1)$.
2. $F_H(\beta, \cdot)$ is convex and infinitely differentiable on \mathbf{R} , with

$$\partial_h F_H(\beta, h) \xrightarrow{h \rightarrow -\infty} a \quad \text{and} \quad \partial_h F_H(\beta, h) \xrightarrow{h \rightarrow +\infty} b. \quad (5.2.8)$$

3. Moreover the map $\varrho \mapsto G(\beta, \varrho)$ is analytic on $(0, 1)$ if and only if $h \mapsto F(\beta, h)$ is analytic on $(h_c(\beta), +\infty)$. In this case, also $F_H(\beta, \cdot)$ is analytic on \mathbf{R} .

The step from smoothness to analyticity in disordered systems is obstructed by the so called *Griffiths singularities*: we refer to [56] for a discussion and for relevant literature on this issue which, however, is essentially fully open for the PS model.

5.2.2 Path Properties

Like in the homogeneous models, the contact density satisfies a law of large number.

Proposition 5.2.3. *For any $h \in \mathbf{R}$, one has, \mathbb{P} -almost surely, the convergence in probability of contact density: for the disordered PS model*

$$\frac{|\tau|}{N} \xrightarrow[N \rightarrow +\infty]{\nu_N^{h, \omega, \beta}} \partial_h F(\beta, h) \in [0, 1], \quad (5.2.9)$$

and for the kernel models

$$\frac{|\tau|}{N} \xrightarrow[N \rightarrow +\infty]{\xi_N^{\Psi, h, \omega, \beta}} \partial_h F_H(\beta, h) \in (a, b). \quad (5.2.10)$$

From our perspective, the largest jump plays a crucial role:

$$M_{|\tau|} = \max_{1 \leq k \leq |\tau|} (\tau_k - \tau_{k-1}). \quad (5.2.11)$$

We are in fact going to show that when $\beta > 0$, unlike for the homogeneous models and whatever the value of α is, all the jumps are small.

Theorem 5.2.4 (No big jump for the fixed contact number and Kernel models). *Fix* $\beta > 0$.

1. For every $\varrho \in (0, 1]$,

$$\frac{M_{|\tau|}}{N} \xrightarrow[N \rightarrow +\infty]{\substack{\mu_{N,L}^{\omega,\beta} \\ \frac{L}{N} \rightarrow \varrho}} 0 \quad \mathbb{P}\text{-a.s.} \quad (5.2.12)$$

2. For every $h \in \mathbf{R}$,

$$\frac{M_{|\tau|}}{N} \xrightarrow[N \rightarrow +\infty]{\xi_N^{\Psi,h,\omega,\beta}} 0 \quad \mathbb{P}\text{-a.s.} \quad (5.2.13)$$

Of course Theorem 5.2.4 should be compared with Theorem 4.2.15 and with Proposition 4.2.10: the disorder has wiped out the big jump transition! This contrasts with [7, §IV] where disorder irrelevance is predicted (at least for $\alpha \in (1, 2)$) based on a *direct* application of Harris criterion (see Section 3.2) that we can easily reproduce here. Their argument relies on the *hyperscaling relation* (see [52, §5.3.1]), which in our context can be written $\nu d = \kappa$, where κ is the critical exponent for the free energy $F_H(h)$ (see Proposition 4.2.8 and definition (4.2.3)). Therefore, Harris criterion states that disorder is irrelevant for $\alpha \in (1, 2)$, and marginal for $\alpha \geq 2$. It would be extremely interesting to be able to adapt or to make sense of the Harris argument for fixed contact number and Kernel models, even just at a heuristic level.

Remark 5.2.5. For completeness we recall that for the disordered PS in the localized phase, all jumps have finite length (more precisely, the largest jump is of length $O(\log N)$: see [56]): in particular we have

$$\frac{M_{|\tau|}}{N} \xrightarrow[N \rightarrow +\infty]{\nu_N^{h,\omega,\beta}} 0 \quad \mathbb{P}\text{-a.s.} \quad \text{for } h > h_c(\beta). \quad (5.2.14)$$

We mention that the delocalized phase* there are only very few contacts and close to the boundary [3, 55] and

$$\frac{M_{|\tau|}}{N} \xrightarrow[N \rightarrow +\infty]{\nu_N^{h,\omega,\beta}} 1 \quad \text{in } \mathbb{P} \text{ probability} \quad \text{for } h < h_c(\beta). \quad (5.2.15)$$

But this is a delocalization transition and it has nothing to do with a big jump transition.

We complete this section by signaling that in the zero range context the effect of disorder has been considered in [31, 62, 75]: the rates are chosen in such a way that the invariant measure of the process is, like in the non disordered case, given by a product measure. The particle conservation law of the dynamics imposes a global constraint like in the fixed contact number case we consider. The disorder is however of a rather different nature as it is indexed by the excursions: in our language, the disorder is introduced as a modification of the renewal probability $K(\cdot)$ that become themselves random. We refer to the original publications for the details, but we stress that also for some of the models in [31, 62, 75] the suppression of the big jump phenomenon due to the disorder has been established.

*Sometimes the critical regime is included inside the delocalized phase: by delocalized here we mean non critical

5.3 Proofs: Free energies

5.3.1 Free energy $G(\beta, \varrho)$

We start with the existence of $G(\beta, \varrho)$. It is based on a standard super-additivity argument. Some care is needed though. We simplify the notation $\mathcal{Z}_{N,L}^{\omega,\beta} = \mathcal{Z}_{N,L,\omega}$ and set, for $\varrho \in (0, 1]$,

$$\tilde{\mathcal{Z}}_{N,\varrho N,\omega} := \min_{m \in \{\lceil \varrho N \rceil, \lceil \varrho N \rceil\}} \mathcal{Z}_{N,m,\omega}, \quad (5.3.1)$$

where the set over which the minimum is taken reduces to one point if ϱN is an integer.

Lemma 5.3.1. *($\log \tilde{\mathcal{Z}}_{N,\varrho N,\omega})_{N \in \mathbf{N}}$ is super-additive: more precisely, for every $N, M \in \mathbf{N}$ we have*

$$\log \tilde{\mathcal{Z}}_{N+M,\varrho(N+M),\omega} \geq \log \tilde{\mathcal{Z}}_{N,\varrho N,\omega} + \log \tilde{\mathcal{Z}}_{M,\varrho M,\theta^N \omega}. \quad (5.3.2)$$

Proof. First of all, given $b, c \geq 0$

$$\lfloor b \rfloor + \lfloor c \rfloor \leq \lfloor b + c \rfloor \leq \lceil b + c \rceil \leq \lceil b \rceil + \lceil c \rceil. \quad (5.3.3)$$

Therefore both lower and upper integer part of $b + c$ can be written as sum of lower and/or upper integer parts of b and c .

The basic observation is that

$$\log \mathcal{Z}_{N+M,n+m,\omega} \geq \log \mathcal{Z}_{N,n,\omega} + \log \mathcal{Z}_{M,m,\theta^N \omega}, \quad (5.3.4)$$

which is obtained by inserting the event that $\tau_n = N$ and using the independence of the increments of τ . Since $\lceil \varrho(N + M) \rceil$ is either $\lceil \varrho N \rceil + \lceil \varrho M \rceil$ or $\lceil \varrho N \rceil + \lfloor \varrho M \rfloor$ we readily obtain that

$$\log \mathcal{Z}_{N+M,\lceil \varrho(N+M) \rceil,\omega} \geq \log \tilde{\mathcal{Z}}_{N,\varrho N,\omega} + \log \tilde{\mathcal{Z}}_{M,\varrho M,\theta^N \omega}, \quad (5.3.5)$$

and a strictly analogous argument holds for $\log \mathcal{Z}_{N+M,\lfloor \varrho(N+M) \rfloor,\omega}$, obtaining exactly the same lower bound. Therefore (5.3.2) is proven. \square

Proposition 5.3.2. *For any $\varrho \in [0, 1]$, there exists a deterministic quantity $G(\beta, \varrho)$ such that*

$$\frac{1}{N} \log \mathcal{Z}_{N,L,\omega} \xrightarrow[\frac{L}{N} \rightarrow \varrho]{N \rightarrow +\infty} G(\beta, \varrho) \quad (5.3.6)$$

almost surely and in \mathbf{L}^1 . Furthermore, for any $\varrho \in [0, 1]$, there exists $c = c(\varrho) > 0$ such that, for any $\epsilon > 0$,

$$\limsup_{N \rightarrow +\infty} \sup_{|L - \varrho N| \leq \epsilon N} \left| \frac{1}{N} \log \mathcal{Z}_{N,L,\omega} - G(\beta, \varrho) \right| \leq c\epsilon, \quad (5.3.7)$$

almost surely and in \mathbf{L}^1 . For $\varrho = 1$,

$$\limsup_{N \rightarrow +\infty} \sup_{0 \leq N - L \leq \epsilon N} \left| \frac{1}{N} \log \mathcal{Z}_{N,L,\omega} - G(\beta, 1) \right| \leq c\epsilon + G(1 - \epsilon) - G(1), \quad (5.3.8)$$

where $G(\beta, 1) = G(1) = \log K(1)$.

Proof. Fix $\varrho \in (0, 1]$. First of all, in order to apply Kingman's sub-additive ergodic Theorem we have to manage the fact that $\tilde{\mathcal{Z}}_{N, \varrho N, \omega}$ becomes zero when $\varrho N < 1$. But Kingman's Theorem can be applied to $\{\log \tilde{\mathcal{Z}}_{nN_0, \varrho nN_0, \omega}\}_{n=1,2,\dots}$, for $N_0 = \lceil 1/\varrho \rceil$, so we can set

$$G(\beta, \varrho) := \lim_{n \rightarrow \infty} \frac{1}{nN_0} \mathbb{E} \log \tilde{\mathcal{Z}}_{nN_0, \varrho nN_0, \omega} = \sup_{n \rightarrow \infty} \frac{1}{nN_0} \mathbb{E} \log \tilde{\mathcal{Z}}_{nN_0, \varrho nN_0, \omega}. \quad (5.3.9)$$

And one has

$$G(\beta, \varrho) = \lim_{n \rightarrow \infty} \frac{1}{nN_0} \log \tilde{\mathcal{Z}}_{nN_0, \varrho nN_0, \omega}, \quad (5.3.10)$$

almost surely and in \mathbf{L}^1 . The quantity $G(\beta, \varrho)$ is defined for $\varrho \in (0, 1]$.

Now we want to prove (5.3.7) and (5.3.8), from which (5.3.6) derives. Fix $\varrho \in (0, 1)$ and $\epsilon > 0$. Thanks to the super-additivity of $\log \mathcal{Z}$, we get the lower bound

$$\begin{aligned} \log \mathcal{Z}_{N, L, \omega} &\geq \log \mathcal{Z}_{N - \delta_N, m, \omega} + \log \mathcal{Z}_{\delta_N, L - m, \theta^{(N - \delta_N)} \omega} \\ &\geq \log \tilde{\mathcal{Z}}_{N - \delta_N, \varrho(N - \delta_N), \omega} + \log \mathcal{Z}_{\delta_N, L - m, \theta^{(N - \delta_N)} \omega}, \end{aligned} \quad (5.3.11)$$

where $m = \lfloor \varrho(N - \delta_N) \rfloor$ and δ_N is a well chosen sequence. Choose δ_N such that $N - \delta_N$ is a multiple of N_0 and such that for every L satisfying $|L - \varrho N| \leq \epsilon N$, one has

$$1 \leq \lfloor \varrho(N - \delta_N) \rfloor \leq N - \delta_N, \quad \text{and} \quad 1 \leq L - \lfloor \varrho(N - \delta_N) \rfloor \leq \delta_N, \quad (5.3.12)$$

One readily check that these inequalities are satisfied as soon as

$$\delta_N \geq \max \left\{ \frac{2 + \epsilon N}{\varrho}, \frac{1 + \epsilon N}{1 - \varrho} \right\}. \quad (5.3.13)$$

Thus, in addition, we can assume (and we do) that

$$\limsup_{N \rightarrow +\infty} \frac{\delta_N}{N} \leq \frac{\epsilon}{\min(\varrho, 1 - \varrho)}. \quad (5.3.14)$$

Going back to (5.3.11), the first term satisfies, with (5.3.10),

$$\frac{1}{N - \delta_N} \log \tilde{\mathcal{Z}}_{N - \delta_N, \varrho(N - \delta_N), \omega} \longrightarrow G(\beta, \varrho). \quad (5.3.15)$$

Thus, with the assumption on δ_N we get

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \log \tilde{\mathcal{Z}}_{N - \delta_N, \varrho(N - \delta_N), \omega} - G(\beta, \varrho) \geq -\frac{\epsilon G(\beta, \varrho)}{\min(\varrho, 1 - \varrho)} =: -c'\epsilon. \quad (5.3.16)$$

The second term can be bounded from below (recall the assumption $K(1) > 0$)

$$\begin{aligned} \log \mathcal{Z}_{\delta_N, L - m, \theta^{N - \delta_N} \omega} &\geq \log \mathcal{Z}_{\delta_N - (L - m - 1), 1, \theta^{(N - \delta_N)} \omega} + \sum_{j=1}^{L - m - 1} \log \mathcal{Z}_{1, 1, \theta^{N - j} \omega} \\ &\geq \log K(\delta_N - (L - m - 1)) + \beta \omega_{N - \delta_N + 1} \\ &\quad + (L - m - 1) \log K(1) + \beta \sum_{j=0}^{L - m - 2} \omega_{N - j}. \end{aligned} \quad (5.3.17)$$

With the assumption on δ_N , we check that

$$\limsup_{N \rightarrow +\infty} \sup_{|L - \varrho N| \leq \epsilon N} \frac{L - m}{N} \leq \epsilon \left(1 + \frac{\varrho}{\min(\varrho, 1 - \varrho)} \right) =: c\epsilon. \quad (5.3.18)$$

Therefore, (5.3.17) yields

$$\liminf_{N \rightarrow +\infty} \sup_{|L - \varrho N| \leq \epsilon N} \frac{1}{N} \left| \log \mathcal{Z}_{\delta_N, L - m, \theta^{N - \delta_N} \omega} \right| \leq c\epsilon (-\log K(1) + \beta \|\omega\|_{\mathbf{L}^\infty}). \quad (5.3.19)$$

At the end of the day,

$$\liminf_{N \rightarrow +\infty} \inf_{|L - \varrho N| \leq \epsilon N} \frac{1}{N} \log \mathcal{Z}_{N, L, \omega} - G(\beta, \varrho) \geq -c\epsilon (-\log K(1) + \beta \|\omega\|_{\mathbf{L}^\infty}) - c'\epsilon \quad (5.3.20)$$

\mathbb{P} -a.s. and in \mathbf{L}^1 . In the same way we get an upper bound:

$$\log \mathcal{Z}_{N, L, \omega} \leq \log \tilde{\mathcal{Z}}_{N + \delta_N, \varrho(N + \delta_N), \omega} - \min_{m \in \{m_+, m_-\}} \log \mathcal{Z}_{\delta_N, m - L, \theta^{N \omega}}, \quad (5.3.21)$$

where $m_- = \lfloor \varrho(N + \delta_N) \rfloor$ and $m_+ = \lceil \varrho(N + \delta_N) \rceil$. In the same way as in (5.3.17), the second term can be bounded from below by some $c''\epsilon N$, thus yielding

$$\limsup_{N \rightarrow +\infty} \sup_{|L - \varrho N| \leq \epsilon N} \frac{1}{N} \log \mathcal{Z}_{N, L, \omega} - G(\beta, \varrho) \leq c''\epsilon. \quad (5.3.22)$$

Combining (5.3.20) and (5.3.22), we get (5.3.7). For $\varrho = 0$, it is easier: when $1 \leq L \leq \epsilon N$,

$$\begin{aligned} |\log \mathcal{Z}_{N, L, \omega}| &\leq \beta \|\omega\|_{\mathbf{L}^\infty} L + |\log \mathbf{P}(\tau_L = N)| \\ &\leq \beta \|\omega\|_{\mathbf{L}^\infty} \epsilon N + \epsilon N \log K(1) + \log K(N - L + 1), \end{aligned} \quad (5.3.23)$$

and we get (5.3.7), with $G(\beta, 0) = 0$. Finally, for $\varrho = 1$, and $0 \leq N - L \leq \epsilon N$,

$$\begin{aligned} \mathcal{Z}_{N, L, \omega} &= \mathbf{E} \left[e^{\sum_{j=1}^N \beta \omega_j \delta_j} \mathbf{1}_{\tau_L = N} \right] \\ &\leq e^{\|\omega\|_{\mathbf{L}^\infty} \beta(N-L)} \mathbf{E} \left[e^{\sum_{j=1}^N \beta \omega_j} \mathbf{1}_{\tau_L = N} \right] \\ &\leq e^{\|\omega\|_{\mathbf{L}^\infty} \beta \epsilon N} e^{\sum_{j=1}^N \beta \omega_j} \mathbf{P}(\tau_L = N). \end{aligned} \quad (5.3.24)$$

Then, with the law of large numbers and Lemma 4.2.1, we get \mathbb{P} -a.s.,

$$\limsup_{N \rightarrow +\infty} \sup_{0 \leq N - L \leq \epsilon N} \frac{1}{N} \log \mathcal{Z}_{N, L, \omega} \leq \|\omega\|_{\mathbf{L}^\infty} \beta \epsilon + G(1 - \epsilon). \quad (5.3.25)$$

Similarly,

$$\liminf_{N \rightarrow +\infty} \inf_{0 \leq N - L \leq \epsilon N} \frac{1}{N} \log \mathcal{Z}_{N, L, \omega} \geq -\|\omega\|_{\mathbf{L}^\infty} \beta \epsilon + G(1). \quad (5.3.26)$$

Thus we obtain $G(\beta, 1) = G(1)$, as well as (5.3.8). \square

Lemma 5.3.3. *The map $\varrho \mapsto G(\beta, \varrho)$ is continuous and concave on $[0, 1]$, with $G(\beta, 0) = 0$ and $G(\beta, 1) = G(0, 1) = \log K(1)$, and $\partial_\varrho G(\beta, \varrho) \rightarrow -\infty$ as $\varrho \rightarrow 1$.*

Proof. We proceed by steps.

- Fix $0 \leq \varrho_1 < \varrho_2 \leq 1$ and $\lambda \in (0, 1)$. For any N sufficiently large,

$$\begin{aligned} \frac{1}{N} \mathbb{E} \log \mathcal{Z}_{N, \lceil (\lambda \varrho_1 + (1-\lambda) \varrho_2) N \rceil, \omega} &\geq \frac{1}{N} \mathbb{E} \log \mathcal{Z}_{\lceil \lambda N \rceil, \lceil \lambda \varrho_1 N \rceil, \omega} \\ &\quad + \frac{1}{N} \mathbb{E} \log \mathcal{Z}_{N - \lceil \lambda N \rceil, \lceil (\lambda \varrho_1 + (1-\lambda) \varrho_2) N \rceil - \lceil \lambda \varrho_1 N \rceil, \omega}. \end{aligned} \quad (5.3.27)$$

Thanks to the convergence (5.2.1), we get in the limit

$$G(\beta, \lambda \varrho_1 + (1 - \lambda) \varrho_2) \geq \lambda G(\beta, \varrho_1) + (1 - \lambda) G(\beta, \varrho_2). \quad (5.3.28)$$

Therefore, $G(\beta, \cdot)$ is concave.

- Recall that the random variables ω_j are centered and bounded ($|\omega_j| \leq C$). Therefore, for any $\beta \geq 0$ and $1 \leq L \leq N$, and for any realization of the environment ω ,

$$e^{-C\beta(N-L)} e^{\sum_{j=1}^N \beta \omega_j} \leq e^{\sum_{j=1}^N \beta \omega_j \delta_j} \leq e^{C\beta(N-L)} e^{\sum_{j=1}^N \beta \omega_j} \quad (5.3.29)$$

As a consequence, for any $\varrho \in (0, 1]$,

$$|G(\beta, \varrho) - G(\varrho)| \leq C\beta(1 - \varrho) + G(\varrho). \quad (5.3.30)$$

Then,

$$G(\beta, 1) - G(\beta, \varrho) = G(1) - G(\beta, \varrho) \leq G(1) - G(\varrho) + C\beta(1 - \varrho). \quad (5.3.31)$$

Thanks to Lemma 4.2.1, we get

$$\lim_{\varrho \nearrow 1} \frac{G(\beta, 1) - G(\beta, \varrho)}{1 - \varrho} = -\infty. \quad (5.3.32)$$

With the concavity we get $\lim_{\varrho \nearrow 1} \partial_\varrho G(\beta, \varrho) = -\infty$.

- The continuity of $G(\beta, \cdot)$ on $(0, 1)$ is a consequence of concavity. The continuity at 1 is a consequence of (5.3.30) and of continuity of $G(\cdot)$ (Lemma 4.2.1). A similar argument yields $|G(\beta, \varrho) - G(\varrho)| \leq C\beta\varrho$, and thus the continuity at 0.

□

5.3.2 Free energy $\mathbf{F}(\beta, h)$ and $\mathbf{F}_H(\beta, h)$

Proof of Proposition 5.2.1. The convergence (5.2.1) has been proved in the previous subsection. Let us prove the convergence towards $\mathbf{F}_H(\beta, h)$. For $\mathbf{F}(\beta, h)$ it is identical: change Ψ in 1 and H in 0. For any $y \in (a, b)$ one has

$$\frac{1}{N} \log Z_N^{\Psi, \beta, \omega, h} \geq \frac{1}{N} \left(h[yN] + \log \Psi([yN], N) + \log Z_N^{\beta, \omega, m} \right), \quad (5.3.33)$$

where $[yN] = \lfloor yN \rfloor \vee 1$. With (4.1.28) and Proposition 5.3.2, we readily get

$$\liminf_{N \rightarrow +\infty} \frac{1}{N} \log Z_N^{\Psi, \beta, \omega, h} \geq \sup_{y \in (a, b)} \{hy + H(y) + G(\beta, y)\}. \quad (5.3.34)$$

Since $G(\beta, \cdot)$ is continuous on $[a, b]$ (Lemma 5.3.3) and given the assumption 4.1.4 (item (i)(d)) on H ,

$$\liminf_{N \rightarrow +\infty} \frac{1}{N} \log Z_N^{\Psi, \beta, \omega, h} \geq \sup_{y \in [0, 1]} \{hy + H(y) + G(\beta, y)\} = F_H(\beta, h). \quad (5.3.35)$$

On the other hand, fix $\delta, B > 0$

$$\begin{aligned} \frac{1}{N} \log Z_N^{\Psi, \beta, \omega, h} &\leq cN^c \sum_{m=0}^N e^{hm} \left(e^{NH(\frac{m}{N}) + \delta} + e^{-NB} \right) Z_N^{\beta, \omega, m} \\ &\leq 2cN^{c+1} \exp \left\{ N \max \left\{ \max_{1 \leq m \leq N} \left[H\left(\frac{m}{N}\right) + h\frac{m}{N} + \frac{1}{N} \log Z_N^{\beta, \omega, m} \right] + \delta; \right. \right. \\ &\quad \left. \left. \max_{1 \leq m \leq N} \left[h\frac{m}{N} + \frac{1}{N} \log Z_N^{\beta, \omega, m} \right] - B \right\} \right\}. \end{aligned} \quad (5.3.36)$$

One readily sees, dividing $[0, 1]$ into small segments and using (5.3.7), that \mathbb{P} -a.s.,

$$\begin{aligned} \limsup_{N \rightarrow +\infty} \max_{1 \leq m \leq N} \left[H\left(\frac{m}{N}\right) + h\frac{m}{N} + \frac{1}{N} \log Z_N^{\beta, \omega, m} \right] &\leq \sup_{\varrho \in [0, 1]} \{\varrho h + H(\varrho) + G(\beta, \varrho)\} + \delta \\ &= F_H(\beta, h) + \delta. \end{aligned} \quad (5.3.37)$$

The same goes for the second maximum in (5.3.36), and we obtain

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \log Z_N^{\Psi, \beta, \omega, h} \leq \max \left\{ F_H(\beta, h) + 2\delta; \sup_{y \in [0, 1]} [hy + G(\beta, y)] + \delta - B \right\}. \quad (5.3.38)$$

Letting $\delta \rightarrow 0$ and $B \rightarrow +\infty$ we get the result:

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \log Z_N^{\Psi, \beta, \omega, h} \leq F_H(\beta, h). \quad (5.3.39)$$

The convergence (5.2.2) is proved. As a supremum of linear functions, $h \mapsto F_H(\beta, h)$ is convex. And, since $G(\beta, \cdot)$ is concave (Lemma 5.3.3), we get the reciprocal variation formula. \square

5.3.3 Free energy regularity

Recall that $\beta > 0$ is fixed. The aim of this section is to exploit the variational formulas (5.2.4) to derive results on $G(\beta, \cdot)$ from $F(\beta, \cdot)$'s properties. The next lemma explains at which point the infimum/supremum in these variational formulas are reached.

Lemma 5.3.4.

1. For any $h \in \mathbf{R}$ the point $\varrho(\beta, h) := \partial_h F(\beta, h)$ is the unique point where the supremum $\sup_{\varrho \in [0, 1]} \{G(\beta, \varrho) + \varrho h\}$ is reached.
2. The function $G(\beta, \cdot)$ is strictly concave and continuously differentiable. For any $\varrho \in [0, 1)$, the point $h(\beta, \varrho) := -\partial_\varrho G(\beta, \varrho)$ is the only point where the infimum $\inf_{h \in \mathbf{R}} \{F(\beta, h) - \varrho h\}$ is reached.

3. The function $F_H(\beta, \cdot)$ is continuously differentiable and, for any $h \in \mathbf{R}$, $\varrho_H(\beta, h) := \partial_h F_H(\beta, h)$ is the unique point where the supremum $\sup_{\varrho \in [0,1]} \{G(\beta, \varrho) + H(\varrho) + \varrho h\}$ is reached.

Proof of Lemma 5.3.4 and Theorem 5.2.2. Fix $\beta > 0$.

- We start with a basic observation: since $F(\beta, h) = 0$ for $h \leq h_c(\beta)$, one has, for $\varrho \in [0, 1]$,

$$G(\beta, \varrho) \stackrel{(5.2.4)}{=} \inf_{h \in \mathbf{R}} \{F(\beta, h) - \varrho h\} = \inf_{h \geq h_c(\beta)} \{F(\beta, h) - \varrho h\}. \quad (5.3.40)$$

The function $F(\beta, \cdot)$ is convex on $[h_c(\beta), +\infty)$; its derivative $\partial_h F(\beta, \cdot)$ is continuous and increasing, with $\partial_h F(\beta, h_c(\beta)) = 0$ and $\partial_h F(\beta, h) \rightarrow 1$ when $h \rightarrow +\infty$ (by using $F(\beta, h) \geq G(\beta, 1) + h = \log K(1) + h$). Therefore for any $\varrho \in [0, 1]$, the infimum (5.3.40) is reached at the unique $h \geq h_c(\beta)$ such that $\partial_h F(\beta, h) = \varrho$.

We know (Theorem 5.5.1) that $\partial_h^2 F(\beta, h) > 0$ for every $h > h_c(\beta)$, and that $F(\beta, \cdot)$ is \mathcal{C}^∞ on $(h_c(\beta), +\infty)$ (Theorem 3.2.2). Therefore, by the implicit function theorem, the function $\partial_h F(\beta, \cdot)$ admits a \mathcal{C}^∞ increasing inverse $\varrho \in (0, 1) \mapsto h(\beta, \varrho) \in (h_c(\beta), +\infty)$. Then,

$$G(\beta, \varrho) = F(\beta, h(\beta, \varrho)) - \varrho h(\beta, \varrho), \quad (5.3.41)$$

which is thus infinitely differentiable on $(0, 1)$, with $\partial_\varrho G(\beta, \varrho) = -h(\beta, \varrho)$. And $G(\beta, \cdot)$ is strictly concave on $(0, 1)$.

- As for the supremum

$$F(\beta, h) = \sup_{\varrho \in [0,1]} \{G(\beta, \varrho) + \varrho h\}, \quad (5.3.42)$$

it is reached when $F(\beta, h) - \varrho h = G(\beta, \varrho)$, that is, with the previous item, when $\varrho = \partial_h F(\beta, h)$.

- If $F(\beta, \cdot)$ is analytic on $(h_c(\beta), +\infty)$, then so is $\varrho \in (0, 1) \mapsto h(\beta, \varrho) \in (h_c(\beta), +\infty)$. And therefore $G(\beta, \cdot)$ (and similarly $F_H(\beta, \cdot)$) is also analytic.
- With the same method, we obtain item 3 and $F_H(\beta, \cdot)$'s properties.

Eventually, observe that since $\partial_h F(\beta, h)$ is positive when $h > h_c(\beta)$ and vanishes as $h \searrow h_c(\beta)$, we have

$$\inf_{h \geq h_c(\beta)} \{F(\beta, h) - \varrho h\} = \inf_{h \in [h_c(\beta), h_c(\beta) + \delta]} \{F(\beta, h) - \varrho h\} \quad (5.3.43)$$

as soon as $\varrho \leq c_\delta$ sufficiently small. □

Proposition 5.3.5 (Behaviour of $G(\beta, \cdot)$ at small density). *There exists $C > 0$ such that for any ϱ sufficiently small,*

$$G(\beta, \varrho) \leq -\varrho h_c(\beta) - C\varrho^2. \quad (5.3.44)$$

If $F(\beta, h_c(\beta) + \epsilon) \sim \epsilon^\gamma \bar{\varphi}(1/\epsilon)$ when $\epsilon \searrow 0$, for some slowly varying function $\bar{\varphi}$ and $\gamma \geq 2$, then

$$G(\beta, \varrho) \stackrel{\varrho \searrow 0}{\sim} -\varrho h_c(\beta) - \varrho^{\frac{\gamma}{\gamma-1}} \varphi_1(1/\varrho)(1 + o(1)), \quad (5.3.45)$$

where φ_1 also slowly varies.

Proof of Proposition 5.3.5. With (5.3.43), for ϱ sufficiently small,

$$G(\beta, \varrho) = \inf_{h \in [h_c(\beta), h_c(\beta) + \delta]} \{F(\beta, h) - \varrho h\} = -\varrho h_c(\beta) + \inf_{\epsilon \in [0, \delta]} \{F(\beta, h_c(\beta) + \epsilon) - \varrho \epsilon\}. \quad (5.3.46)$$

Suppose $F(\beta, h_c(\beta) + \epsilon) \sim c\epsilon^\gamma$ for $c > 0$ and $\gamma \geq 2$. Fix $\eta > 0$ and $\delta > 0$ such that for $\epsilon \in [0, \delta]$,

$$c(1 - \eta)\epsilon^\gamma \leq F(\beta, h_c(\beta) + \epsilon) \leq c(1 + \eta)\epsilon^\gamma. \quad (5.3.47)$$

Then, when ϱ is sufficiently small,

$$\inf_{\epsilon \in [0, \delta]} \{c(1 - \eta)\epsilon^\gamma - \varrho \epsilon\} \leq G(\beta, \varrho) + \varrho h_c(\beta) \leq \inf_{\epsilon \in [0, \delta]} \{c(1 + \eta)\epsilon^\gamma - \varrho \epsilon\}. \quad (5.3.48)$$

These are easy to compute and one gets

$$-\frac{\gamma - 1}{\gamma} (\gamma c(1 - \eta))^{-\frac{1}{\gamma-1}} \varrho^{\frac{\gamma}{\gamma-1}} \leq G(\beta, \varrho) + \varrho h_c(\beta) \leq -\frac{\gamma - 1}{\gamma} (\gamma c(1 + \eta))^{-\frac{1}{\gamma-1}} \varrho^{\frac{\gamma}{\gamma-1}}. \quad (5.3.49)$$

At the end of the day,

$$G(\beta, \varrho) + \varrho h_c(\beta) \stackrel{e \searrow 0}{\sim} -\frac{\gamma - 1}{\gamma} (\gamma c)^{-\frac{1}{\gamma-1}} \varrho^{\frac{\gamma}{\gamma-1}}. \quad (5.3.50)$$

Similarly, if $F(\beta, h_c(\beta) + \epsilon) \sim \epsilon^\gamma \bar{\varphi}(1/\epsilon)$ (Theorem 3.2.2 for $\alpha < 1/2$), then

$$G(\beta, \varrho) + \varrho h_c(\beta) \stackrel{e \searrow 0}{\sim} -\varrho^{\frac{\gamma}{\gamma-1}} \varphi_1(1/\varrho). \quad (5.3.51)$$

If $F(\beta, h_c(\beta) + \epsilon) \leq c\epsilon^2$, for ϵ small (Theorem 3.2.2), then

$$G(\beta, \varrho) \leq -\varrho h_c(\beta) + \inf_{\epsilon \in [0, \delta]} \{c\epsilon^2 - \varrho \epsilon\} = -\varrho h_c(\beta) - \frac{\varrho^2}{4c} \quad (5.3.52)$$

for ϱ small. □

Remark 5.3.6. When Assumptions 5.1.2 are not fulfilled, the same phenomenon as for the homogeneous model emerge: the continuous differentiability is only true (a priori) on an interval $(h_*(\beta), h^*(\beta))$, where

$$h_*(\beta) = \lim_{\varrho \searrow a} (-\partial_\varrho G(\beta, \varrho) - H'(\varrho)), \quad h^*(\beta) = \lim_{\varrho \nearrow b} (-\partial_\varrho G(\beta, \varrho) - H'(\varrho)). \quad (5.3.53)$$

The behaviour of $F_H(\beta, h)$ when h approaches $h_*(\beta)$ or $h^*(\beta)$ can be deduced from the behaviour of $G(\beta, \varrho)$ and $H(\varrho)$ as ϱ approaches a or b .

5.4 Proofs: path properties

Proof of Proposition 5.2.3. We prove it for the kernel model. The proof is the same for the disordered PS model (replace H with 0). Fix $\epsilon > 0$, $h \in \mathbf{R}$ and set $\varrho = \partial_h F_H(\beta, h)$. We prove that

$$Z_N := Z_N^{\Psi, \beta, \omega, h} \sim \hat{Z}_N, \quad (5.4.1)$$

where \hat{Z}_N is Z_N with the sum restricted to $|m - \varrho N| \leq \epsilon N$. Of course $Z_N \geq \hat{Z}_N$. More precisely, we prove that there exist $c_\epsilon > 0$, and a sequence $\eta_N(\omega)$, which vanishes \mathbb{P} -a.s. when $N \rightarrow +\infty$, such that for every $N \geq 1$,

$$0 \leq \frac{Z_N - \hat{Z}_N}{Z_N} \leq e^{-c_\epsilon(1+\eta_N)N}. \quad (5.4.2)$$

Using the same method as in the proof of Proposition 5.2.1, we obtain, almost surely,

$$\limsup \frac{1}{N} \log (Z_N - \hat{Z}_N) \leq \sup_{|y-\varrho| \geq \epsilon} \{yh + H(y) + G(\beta, y)\} < F_H(\beta, h), \quad (5.4.3)$$

(see Lemma 5.3.4 for the strict inequality) and

$$\liminf \frac{1}{N} \log \hat{Z}_N \geq \sup_{|y-\varrho| \leq \epsilon} \{yh + H(y) + G(\beta, y)\} = F_H(\beta, h), \quad (5.4.4)$$

from which (5.4.2), and then (5.4.1), follow. Then, writing

$$\xi_N^{\Psi, h, \omega, \beta}(|\tau| - \varrho N \leq \epsilon N) = \frac{\hat{Z}_N}{Z_N}, \quad (5.4.5)$$

we get the statement of Proposition 5.2.3. \square

Proof of Theorem 5.2.4. Like for the free energy, the strategy is to use what is known about the disordered PS process to infer results on the disordered fixed contact number PS model, and then on the disordered kernel models. The localization phenomena is known for the standard disordered Poland-Scheraga model [56]. We will actually reprove it (with similar arguments) for we need an almost sure exponential decay for

$$\nu_N^{h, \omega, \beta}(M_{|\tau|} \geq \delta N). \quad (5.4.6)$$

Then the result is used to obtain a similar result, first for $\mu_{N,L}^{\omega, \beta}$ and then for $\xi_N^{\Psi, h, \omega, \beta}$. This is achieved by using that $\nu_N^{h, \omega, \beta}$ and $\xi_N^{\Psi, h, \omega, \beta}$ can be written as a mixing of measures $\mu_{N,L}^{\omega, \beta}$.

Proof of (5.2.14). Fix $\beta > 0$ and $h > h_c(\beta)$, so that $F(\beta, h) > 0$. To simplify, we will omit the superscript h, ω, β for $\nu_N^{h, \omega, \beta}$ and $Z_N^{h, \omega, \beta}$. Fix $\delta > 0$. First we use an union bound, decomposing the event depending on the extremities of the largest jump

$$\nu_N(M_{|\tau|} \geq \delta N) \leq \sum_{\substack{0 \leq x \leq y \leq N \\ y-x \geq \delta N}} \nu_N(E_{x,y}), \quad (5.4.7)$$

where, for $1 \leq x \leq y \leq N$,

$$E_{x,y} = \{[x, y] \text{ is a jump}\} = \{x \in \tau\} \cap \{y \in \tau\} \cap \{\forall x < z < y, z \notin \tau\}. \quad (5.4.8)$$

Now we want an upper bound for $\nu_N(E_{x,y})$. To this end, introduce, for $1 \leq x \leq y \leq N$,

$$Z_{[x,y]} = \mathbf{E} \left[e^{\sum_{j=x+1}^y (\beta \omega_j + h) \delta_j} \mathbf{1}_{y \in \tau} \mid x \in \tau \right]. \quad (5.4.9)$$

By definition,

$$\nu_N(E_{x,y}) = \frac{Z_{[0,x]} K(y-x) e^{\beta\omega_y + h} Z_{[y,N]}}{Z_N}. \quad (5.4.10)$$

By inserting the event $\{y \in \tau\}$ and using the independence of the renewal increments, we get

$$Z_N \geq Z_{[0,y]} Z_{[y,N]}. \quad (5.4.11)$$

Therefore

$$\nu_N(E_{x,y}) \leq K(y-x) e^{\beta\omega_y + h} \frac{Z_{[0,x]}}{Z_{[0,y]}}. \quad (5.4.12)$$

Since the ω_j are bounded one has, uniformly in x, y ,

$$\nu_N(E_{x,y}) \leq C \frac{Z_{[0,x]}}{Z_{[0,y]}}. \quad (5.4.13)$$

By injecting it into (5.4.7), we get

$$\nu_N(M_{|\tau|} \geq \delta N) \leq CN^2 \sup_{\substack{0 \leq x \leq y \leq N \\ y-x \geq \delta N}} \frac{Z_{[0,x]}}{Z_{[0,y]}}. \quad (5.4.14)$$

We know that \mathbb{P} -almost surely,

$$\frac{1}{n} \log Z_n \xrightarrow{n \rightarrow +\infty} F(\beta, h). \quad (5.4.15)$$

Therefore, \mathbb{P} -almost surely,

$$\sup_{\substack{0 \leq x \leq y \leq N \\ y-x \geq \delta N}} \left(\frac{1}{N} \log Z_{[0,x]} - \frac{1}{N} \log Z_{[0,y]} \right) = -F(\beta, h)\delta, \quad (5.4.16)$$

and finally

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \log \nu_N(M_{|\tau|} \geq \delta N) \leq -F(\beta, h)\delta. \quad (5.4.17)$$

In particular we get (5.2.14). \square

Proof of Theorem 5.2.4, part 1. If $h \in \mathbf{R}$, we know that for any event A for the renewal,

$$\nu_N^{h,\omega,\beta}(A) = \frac{1}{Z_N^{h,\omega,\beta}} \sum_{m=1}^N e^{hm} \mathcal{Z}_{N,m}^{\omega,\beta} \mu_{N,m}^{\omega,\beta}(A), \quad (5.4.18)$$

Consequently, for any $\varrho \in (0, 1)$ and $\epsilon > 0$, one has

$$\sup_{|m-\varrho N| \leq \epsilon N} \mu_{N,m}^{\omega,\beta}(A) \leq \left(\frac{Z_N^{h,\omega,\beta}}{\inf_{|m-\varrho N| \leq \epsilon} e^{hm} \mathcal{Z}_{N,m}^{\omega,\beta}} \right) \nu_N^{h,\omega,\beta}(A). \quad (5.4.19)$$

Fix $\varrho \in (0, 1)$, and h such that $\partial_h F(\beta, h) = \varrho$. Then, \mathbb{P} -a.s.,

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \log Z_N^{h,\omega,\beta} = F(\beta, h) = G(\beta, \varrho) + h\varrho, \quad (5.4.20)$$

and, with (5.3.7),

$$\limsup_{N \rightarrow +\infty} \inf_{|m - \varrho N| \leq \epsilon} \frac{1}{N} \log \left(e^{hm} \mathcal{Z}_{N,m}^{\omega, \beta} \right) \geq h\varrho - h\epsilon + G(\beta, \varrho) - \|\omega\|_{L^\infty} \beta \epsilon. \quad (5.4.21)$$

Thus, \mathbb{P} -a.s.,

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \log \sup_{|m - \varrho N| \leq \epsilon N} \mu_{N,m}^{\omega, \beta}(A) \leq (h + \|\omega\|_{L^\infty} \beta) \epsilon + \limsup_{N \rightarrow +\infty} \frac{1}{N} \log \nu_N^{h, \omega, \beta}(A). \quad (5.4.22)$$

In particular, for the event $A = \{M_{|\tau|} \geq \delta N\}$, we obtain, thanks to (5.4.17),

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \log \sup_{|m - \varrho N| \leq \epsilon N} \mu_{N,m}^{\omega, \beta}(A) \leq (h + \|\omega\|_{L^\infty} \beta) \epsilon - F(\beta, h) \delta, \quad (5.4.23)$$

\mathbb{P} -almost surely. This lim sup is negative if ϵ is chosen sufficiently small. \square

Proof of Theorem 5.2.4, part 2. Without Assumption 5.1.2, $h \in \mathbf{R}$ is replaced by the condition $h > -H'(0) - \partial_\varrho G(\beta, 0)$, so that $\partial_h F_H(\beta, h) \in (0, 1)$. Fix such an h , then $\varrho_H = \partial_h F_H(\beta, h) \in (0, 1)$ and $\epsilon < \varrho$. To simplify, set $Z_N = Z_N^{\Psi, h, \omega, \beta}$ and $\mathcal{Z}_{N,m} = \mathcal{Z}_{N,m}^{\omega, \beta}$. By the proof of Proposition 5.2.3, there exists $c_\epsilon > 0$ and $\eta_N(\omega)$, which vanishes a.s. as $N \rightarrow +\infty$, such that

$$0 \leq \frac{Z_N - \hat{Z}_N}{Z_N} \leq e^{-c_\epsilon(1+\eta_N)N}. \quad (5.4.24)$$

On the other hand, for any event A for the renewal,

$$\begin{aligned} \xi_N^{\Psi, h, \omega, \beta}(A) &\leq \frac{1}{Z_N} \sum_{|m - \varrho_H N| \leq \epsilon N} \Psi(N, m) e^{mh} \mathcal{Z}_{N,m}^{\omega, \beta}(A) + \frac{Z_N - \hat{Z}_N}{Z_N} \\ &\leq \sup_{|m - \varrho_H N| \leq \epsilon N} \mu_{N,m}^{\omega, \beta}(A) \left(\frac{1}{Z_N} \sum_{|m - \varrho_H N| \leq \epsilon N} \Psi(N, m) e^{mh} \mathcal{Z}_{N,m} \right) + \frac{Z_N - \hat{Z}_N}{Z_N} \\ &\leq \sup_{|m - \varrho_H N| \leq \epsilon N} \mu_{N,m}^{\omega, \beta}(A) + \frac{Z_N - \hat{Z}_N}{Z_N}. \end{aligned} \quad (5.4.25)$$

Eventually, fix \tilde{h} such that $\varrho_H = \partial_h F(\beta, \tilde{h})$. Therefore, with (5.4.22) and (5.4.24), \mathbb{P} -a.s.,

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \log \xi_N^{\Psi, h, \omega, \beta}(A) \leq \max \left(C\epsilon + \limsup_{N \rightarrow +\infty} \frac{1}{N} \log \nu_N^{\tilde{h}, \omega, \beta}(A), -c_\epsilon \right). \quad (5.4.26)$$

Once again, the result follows from (5.4.17). \square

5.5 On strict convexity for the PS free energy

The proof of the following result stems from a suggestion by Hubert Lacoin.

Theorem 5.5.1. *For every $\beta > 0$ and $h > h_c(\beta)$,*

$$\partial_h^2 F(\beta, h) > 0. \quad (5.5.1)$$

Proof. The argument is close in its basic idea to what one finds in [50, Ch. 5, Sec. 2]), that is the conditioning on what happens on even sites. To simplify the exposition we assume like in [50] that both $K(1) > 0$ and $K(2) > 0$: the argument easily generalizes, but at the expense of heavier notations (they are developed for example in [32, Ch. 7]). We know [56, Proof of Th. 2.1] that

$$\partial_h^2 F(\beta, h) = \lim_{N \rightarrow +\infty} \frac{1}{N} \mathbb{E} \operatorname{Var}_{\nu_N^{h,\omega,\beta}} \left(\sum_{j=1}^{N-1} \delta_j \right). \quad (5.5.2)$$

So it suffices to find a lower bound on the variance: without loss of generality, let us make this estimate with N replaced by $2N$. To simplify the notations we set $\nu_{2N,\omega} := \nu_{2N}^{h,\omega,\beta}$: the disordered PS measure. We start by introducing the σ -algebra \mathcal{F}_e generated by δ_j for j even. Let us remark that, with Jensen's inequality,

$$\begin{aligned} \operatorname{Var}_{\nu_{2N,\omega}} \left(\sum_{j=1}^{2N-1} \delta_j \right) &\geq \mathbf{E}_{\nu_{2N,\omega}} \left[\operatorname{Var}_{\nu_{2N,\omega}(\cdot|\mathcal{F}_e)} \left(\sum_{j=1}^{2N-1} \delta_j \right) \right] \\ &= \mathbf{E}_{\nu_{2N,\omega}} \left[\operatorname{Var}_{\nu_{2N,\omega}(\cdot|\mathcal{F}_e)} \left(\sum_{j=1}^N \delta_{2j-1} \right) \right]. \end{aligned} \quad (5.5.3)$$

Now we consider the conditional variance on the event $E_\sigma := \{\tau \mid \delta_{2j} = \sigma_j\}$ for every given configuration $\sigma \in \{0,1\}^{\{0,1,\dots,N\}}$, and we can of course restrict to $\sigma_0 = \sigma_N = 1$. Given a configuration σ , we set $n(\sigma)$ (the number of even pinned sites), and we introduce, for $j = 0, 1, \dots, n(\sigma)$ the even *contact locations* by setting $\ell_0 = 0$ and $\ell_{j+1} = 2 \min\{\ell > \ell_j \mid \sigma_\ell = 1\}$. Of course, $\ell_{n(\sigma)} = 2N$. If we write

$$\sum_{j=1}^N \delta_{2j-1} = \sum_{k=1}^{n(\sigma)} \sum_{j=1+\ell_{k-1}/2}^{\ell_k/2} \delta_{2j-1}, \quad (5.5.4)$$

we remark that on E_σ (we are working with the probability $\nu_{2N,\omega}(\cdot|\mathcal{F}_e)$) the random variables $\left(\sum_{j=1+\ell_{k-1}/2}^{\ell_k/2} \delta_{2j-1} \right)_{k=1,\dots,n(\sigma)}$ are independent. Therefore, on E_σ ,

$$\begin{aligned} \operatorname{Var}_{\nu_{2N,\omega}(\cdot|\mathcal{F}_e)} \left(\sum_{j=1}^N \delta_{2j-1} \right) &= \sum_{k=1}^{n(\sigma)} \operatorname{Var}_{\nu_{2N,\omega}(\cdot|\mathcal{F}_e)} \left(\sum_{j=1+\ell_{k-1}/2}^{\ell_k/2} \delta_{2j-1} \right) \\ &\geq \sum_{\substack{k=1,\dots,n(\sigma) \\ \ell_k - \ell_{k-1} = 2}} \operatorname{Var}_{\nu_{2N,\omega}(\cdot|\mathcal{F}_e)} (\delta_{\ell_k-1}). \end{aligned} \quad (5.5.5)$$

But δ_{ℓ_k-1} is just a Bernoulli random variable whose success probability depends on the disorder variable $\omega = \omega_{\ell_k-1}$ and it is equal to

$$\frac{K(1)^2 \exp(h + \beta\omega)}{K(1)^2 \exp(h + \beta\omega) + K(2)}. \quad (5.5.6)$$

Therefore the variance of this random variable is bounded below by

$$\sigma_{\beta C}^2(h) := \min_{s=\pm 1} \frac{K(1)^2}{K(1)^2 + K(2) \exp(-h + s\beta C)} \left(1 - \frac{K(1)^2}{K(1)^2 + K(2) \exp(-h + s\beta C)} \right), \quad (5.5.7)$$

with $C = \|\omega\|_{\mathbf{L}^\infty}$. So

$$\sum_{\substack{k=1, \dots, n(\sigma) \\ \ell_k - \ell_{k-1} = 2}} \text{Var}_{\nu_{2N, \omega}(\cdot | \mathcal{F}_e)}(\delta_{\ell_k - 1}) \geq \sigma_{\beta C}^2(h) |\{k = 1, \dots, n(\sigma) \mid \ell_k - \ell_{k-1} = 2\}|. \quad (5.5.8)$$

Putting together (5.5.3), (5.5.5) and (5.5.8), we have

$$\text{Var}_{\nu_{2N, \omega}} \left(\sum_{j=1}^{2N-1} \delta_j \right) \geq \sigma_{\beta C}^2(h) \mathbf{E}_{\nu_{2N, \omega}} \left[\sum_{k=0}^{N-1} \delta_{2k} \delta_{2(k+1)} \right]. \quad (5.5.9)$$

Therefore it suffices to find a lower bound on

$$\liminf_{N \rightarrow +\infty} \frac{1}{2N} \sum_{k=0}^{N-1} \mathbb{E} \mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k} \delta_{2(k+1)}]. \quad (5.5.10)$$

We claim that, for $k = 0, \dots, N$,

$$\mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k} \delta_{2(k+1)}] \geq c \mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k}], \quad (5.5.11)$$

and in the same spirit, a lower bound which will be used later:

$$\mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k} \delta_{2k+1}] \geq c \mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k+1}]. \quad (5.5.12)$$

Here and in the rest of the proof, c is a positive constant, depending only on h , $K(\cdot)$ and $C = \|\omega\|_\infty$, whose value may change from one line to another. With (5.5.11), we are therefore left with finding a lower bound for

$$\liminf_{N \rightarrow +\infty} \frac{1}{2N} \sum_{k=0}^{N-1} \mathbb{E} \mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k}]. \quad (5.5.13)$$

Observe that, with (5.5.12),

$$\mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k}] \geq \mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k} \delta_{2k+1}] \geq c \mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k+1}]. \quad (5.5.14)$$

Therefore,

$$\mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k}] \geq \frac{c}{1+c} (\mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k}] + \mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k+1}]), \quad (5.5.15)$$

and thus,

$$\frac{1}{2N} \sum_{k=0}^{N-1} \mathbf{E}_{\nu_{2N, \omega}} [\delta_{2k}] \geq \frac{c}{1+c} \times \frac{1}{2N} \sum_{k=0}^{2N-1} \mathbf{E}_{\nu_{2N, \omega}} [\delta_k]. \quad (5.5.16)$$

With Proposition 5.2.4 and by dominated convergence,

$$\frac{1}{2N} \sum_{k=0}^{2N-1} \mathbb{E} \mathbf{E}_{\nu_{2N, \omega}} [\delta_k] \xrightarrow{N \rightarrow +\infty} \partial_h F(\beta, h), \quad (5.5.17)$$

which is positive for $h > h_c(\beta)$. The proof is therefore complete once (5.5.11) is established and this is what we do next.

We introduce the notation

$$Z_{[x,y]} = \mathbf{E} \left[e^{\sum_{j=x+1}^y (h+\beta\omega_j)\delta_j} \mid x \in \tau \right], \quad (5.5.18)$$

for $0 \leq x \leq y \leq N$, with the convention $Z_{[x,x]} = 1$. By definition,

$$\mathbf{E}_{\nu_{2N,\omega}} [\delta_{2k}\delta_{2(k+1)}] = \frac{Z_{[0,2k]}Z_{[2k,2(k+1)]}Z_{[2(k+1),2N]}}{Z_{[0,2N]}} \geq \frac{Z_{[0,2k]}K(2)e^{h-\beta C}Z_{[2(k+1),2N]}}{Z_{[0,2N]}}. \quad (5.5.19)$$

We want to prove that

$$Z_{[2(k+1),2N]} \geq cZ_{[2k,2N]}, \quad (5.5.20)$$

which will yield

$$\mathbf{E}_{\nu_{2N,\omega}} [\delta_{2k}\delta_{2(k+1)}] \geq \frac{Z_{[0,2k]}Z_{[2k,2N]}}{Z_{[0,2N]}} = c\mathbf{E}_{\nu_{2N,\omega}} [\delta_{2k}], \quad (5.5.21)$$

that is, (5.5.11). To prove (5.5.20), we start from the right-hand side and sum on every possible first jump

$$Z_{[2k,2N]} = \sum_{j=1}^{2(N-k)} K(j)e^{h+\beta\omega_{2k+j}} Z_{[2k+j,2N]}. \quad (5.5.22)$$

For $j \geq 3$,

$$\begin{aligned} K(j)Z_{[2k+j,2N]} &\leq \frac{K(j)}{K(j-2)}K(j-2)Z_{[2k+j,2N]} \\ &\leq cK(j-2)Z_{[2k+j,2N]}, \end{aligned} \quad (5.5.23)$$

where we used the regular variation of K to obtain that $\sup_{j \geq 3} K(j)/K(j-2)$ is finite. Thus,

$$\begin{aligned} \sum_{j=3}^{2(N-k)} K(j)e^{h+\beta\omega_{2k+j}} Z_{[2k+j,2N]} &\leq c \sum_{j=3}^{2(N-k)} K(j-2)e^{h+\beta\omega_{2k+j}} Z_{[2k+j,2N]} \\ &= cZ_{[2k+2,2N]}. \end{aligned} \quad (5.5.24)$$

For $j = 2$,

$$K(2)e^{h+\beta\omega_{2k+2}} Z_{[2k+2,2N]} \leq cZ_{[2k+2,2N]}. \quad (5.5.25)$$

The term $j = 1$ also deserves a special treatment and we need to sum on every possible second jump:

$$Z_{[2k+1,2N]} = \sum_{j=1}^{2(N-k)-1} K(j)e^{h+\beta\omega_{2k+1+j}} Z_{[2k+j+1,2N]}. \quad (5.5.26)$$

And with the same method we readily get

$$Z_{[2k+1,2N]} \leq cZ_{[2k+2,2N]}. \quad (5.5.27)$$

At the end of the day, we obtain (5.5.20), thus, completing the proof. \square

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