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Jessica Guerand

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# THÈSE DE DOCTORAT

de l'Université de recherche Paris Sciences et Lettres  
PSL Research University

Préparée à l'École Normale Supérieure de Paris

*Discontinuous Hamilton-Jacobi equations and parabolic regularity à la De Giorgi*

## Équations de Hamilton-Jacobi discontinues et régularité parabolique à la De Giorgi

École doctorale n°386

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# Introduction générale

Cette thèse est constituée de deux parties. La première porte sur l'étude de questions relatives aux équations de Hamilton-Jacobi. Les questions traitées pendant la thèse sont de plusieurs types : unicité de solutions, sens des conditions au bord et passage à la limite. Dans la seconde partie, nous étudions un résultat de régularité de solutions d'équations paraboliques obtenu par des méthodes de type De Giorgi.

Nous introduisons tout d'abord des éléments de théorie de ces deux sujets, puis nous évoquons les problèmes abordés en rappelant les résultats existants avant de décrire ceux obtenus pendant la thèse.

## 1 Introduction du sujet

### 1.1 Équation de Hamilton-Jacobi

Les équations de Hamilton-Jacobi étudiées pendant la thèse sont des équations du premier ordre de la forme suivante, pour  $T > 0$ ,

$$\partial_t u + H(x, \partial_x u) = 0, \quad (t, x) \in (0, T) \times \Omega, \quad (1)$$

où le domaine d'espace  $\Omega$  peut représenter soit une demi-droite  $(0, +\infty)$ , soit un ouvert régulier borné de  $\mathbb{R}^n$ , soit une jonction composée  $J = \cup_\alpha J_\alpha$  de plusieurs branches  $J_\alpha$  isométriques à  $[0, +\infty)$  (voir Figure 1). Cette équation est soumise à une

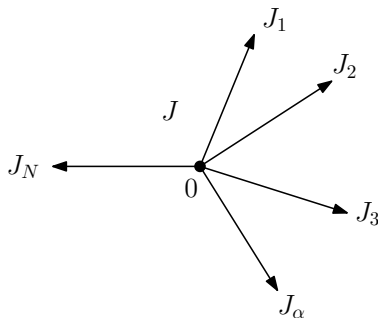


FIGURE 1 : Jonction.

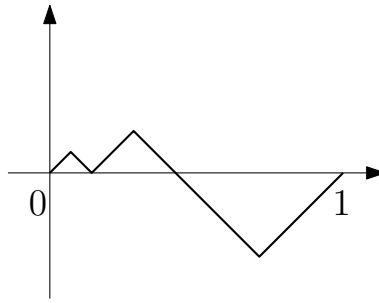


FIGURE 2 : Solutions Lipschitz de (2).

condition de bord de type dynamique de la forme suivante,

$$\partial_t u + F(\partial_x u) = 0, \quad (t, x) \in (0, T) \times \partial\Omega.$$

Ces conditions au bord apparaissent en contrôle optimal et permettent de modéliser des problèmes de trafic routier, de supraconductivité et de mouvements d'interfaces. Dans cette partie, nous introduisons la notion de solution associée aux équations de Hamilton-Jacobi et nous présentons les questions que l'on se pose en général pour ce type d'équations.

La notion de solution en adéquation avec les problèmes physiques décrits par les équations de Hamilton-Jacobi est la notion de solution de viscosité introduite par Crandall et Lions en 1981. Nous résumons ici quelques éléments de cette théorie. Le lecteur trouvera plus de détails dans [38] et [14] dont nous nous inspirons.

### 1.1.1 Motivations des solutions de viscosité

On considère une équation eikonale qui intervient en optique géométrique ainsi que dans des problèmes de contrôle optimal avec temps de sortie. En dimension 1 d'espace, cette équation s'écrit

$$\begin{cases} |u'(x)| = 1, & \forall x \in ]0, 1[ \\ u(0) = u(1) = 0, \end{cases} \quad (2)$$

où on impose une condition au bord de Dirichlet. On cherche une notion de solution qui permet d'avoir un problème bien posé (existence et unicité d'une solution) et qui fournit une solution répondant au problème physique. On se demande alors quelle notion de solutions permet de répondre à ces critères. S'intéresser aux solutions régulières est trop restrictif dans la mesure où le théorème de Rolle assure la non-existence de solutions dérivables. À l'inverse, considérer des solutions lipschitziennes n'est pas assez restrictif, étant donné qu'il existe une infinité de telles solutions vérifiant (2) (voir Figure 2). Cette notion n'est donc pas suffisante pour garantir l'unicité. On introduit alors la notion de solution de viscosité qui permet de sélectionner la solution fournie par la théorie du contrôle optimal.

Une autre idée consiste à ajouter dans le membre de gauche de l'équation (2) le terme  $-\varepsilon u''(x)$  de viscosité évanescence, l'équation possède une unique solution  $u_\varepsilon$ .

Peut-on alors obtenir la convergence de  $u_\varepsilon$  quand  $\varepsilon \rightarrow 0$  vers une unique solution de (2)? Nous allons voir comment la notion de solution de viscosité permet de passer à la limite en un sens précis et donne un bon cadre d'étude qui garantit l'existence et l'unicité.

### 1.1.2 Définition des solutions de viscosité

On considère l'équation de Hamilton-Jacobi suivante,

$$\partial_t u + H(\nabla_x u, D_x^2 u) = 0 \quad (0, T) \times \Omega, \quad (3)$$

où  $\Omega$  est un ouvert de  $\mathbb{R}^n$ . On note  $S_n(\mathbb{R})$  l'ensemble des matrices réelles symétriques de taille  $n \times n$ . On pourrait ajouter une dépendance en  $t$  et  $x$  de l'hamiltonien sans que cela n'ajoute de difficultés techniques pour cette partie introductive. La présence d'un terme de second ordre dans l'équation permet de justifier la définition de solution de viscosité par la condition d'ellipticité suivante vérifiée par l'hamiltonien,

$$H(p, M) \leq H(p, N) \quad \text{si } M \geq N, \quad (4)$$

pour tout  $p \in \mathbb{R}^n$  et  $M, N \in S_n(\mathbb{R})$ . En effet, les solutions régulières de (3) vérifient la propriété suivante, dite du principe du maximum.

**Propriété 1.1** (Principe du Maximum). *Soit  $u \in C^2((0, T) \times \Omega)$ . La fonction  $u$  est solution classique de (3) **si et seulement si**  $u$  vérifie,*

*$\forall \phi \in C_t^1 \cap C_x^2((0, T) \times \Omega)$ , si  $(t_0, x_0)$  est un point de maximum (resp. minimum) local de  $u - \phi$  alors*

$$\partial_t \phi(t_0, x_0) + H(\nabla_x \phi(t_0, x_0), D_x^2 \phi(t_0, x_0)) \leq 0 \quad (\text{resp. } \geq 0).$$

Cette propriété découle du fait qu'en un point de maximum ou de minimum de  $u - \phi$  on a  $\partial_t u(t_0, x_0) = \partial_t \phi(t_0, x_0)$  et  $\nabla_x u(t_0, x_0) = \nabla_x \phi(t_0, x_0)$ . Pour un point de maximum (resp. minimum), on a  $D_x^2 u(t_0, x_0) \leq D_x^2 \phi(t_0, x_0)$  (resp.  $D_x^2 u(t_0, x_0) \geq D_x^2 \phi(t_0, x_0)$ ) et la condition d'ellipticité (4) permet de conclure.

Le principe du maximum permet de donner une caractérisation des solutions qui ne fait pas intervenir les dérivées de la solution. Il est alors possible de donner une notion faible de solution valable pour des fonctions non régulières, où les dérivées sont portées par une fonction test régulière, mais aussi de garantir que les solutions régulières sont solutions de viscosité. Le second ordre permet de choisir le sens des inégalités à vérifier selon que  $u - \phi$  atteignent un maximum ou un minimum. Définissons ainsi les solutions de viscosité.

**Définition 1.2** (Solutions de viscosité). *Soit  $u : (0, T) \times \Omega \rightarrow \mathbb{R}$  une fonction localement bornée.*

- *On dit que la fonction  $u$  est une **sous-solution** de viscosité (resp. une **sur-solution**) de (3) si  $u$  est **semi-continue supérieurement** (resp. **inférieurement**) et si pour tout  $(t_0, x_0) \in (0, T) \times \Omega$  et toute fonction test  $\varphi \in C^2((0, T) \times \Omega)$  telle que  $u - \varphi$  atteigne un **maximum local** (resp. un **minimum local**) en  $(t_0, x_0)$ , on a*

$$\partial_t \phi(t_0, x_0) + H(\nabla_x \phi(t_0, x_0), D_x^2 \phi(t_0, x_0)) \leq 0 \quad (\text{resp. } \leq 0).$$

- Une fonction  $u$  est une solution de viscosité de (3) si elle est à la fois sous-solution et sur-solution de (3).

Ici nous nous plaçons dans un cadre où les solutions sont continues mais nous verrons dans les Chapitres 1, 2, et 3 que nous pouvons définir la notion de solution de viscosité pour des fonctions discontinues seulement localement bornées.

Reprenons l'exemple de l'équation eikonale du paragraphe 1.1.1. La notion de solution de viscosité permet de sélectionner  $u^+(x) = \frac{1}{2} - |x - \frac{1}{2}|$  unique solution de  $|u'(x)| - 1 = 0$  vérifiant la condition au bord  $u(0) = u(1) = 0$ , ainsi que  $u^-(x) = |x - \frac{1}{2}| - \frac{1}{2}$  unique solution de viscosité de l'équation eikonale  $1 - |u'(x)| = 0$  avec la même condition au bord. Attention l'écriture de l'équation n'est pas symétrique, les équations  $(E)$  et  $-(E)$  n'ont pas les mêmes solutions en général.

### 1.1.3 Stabilité et existence des solutions de viscosité

La notion de stabilité est essentielle pour les solutions de viscosité. Elle permet non seulement de donner un sens aux passages à la limite pour les équations de Hamilton-Jacobi mais garantit également l'existence de solutions. Commençons par rappeler deux résultats importants de stabilité avant de décrire un théorème d'existence de solutions de viscosité.

#### Deux résultats de stabilité

Le premier résultat de stabilité permet de répondre à une question posée au paragraphe 1.1.1, celle portant sur la viscosité évanescence. On peut passer à la limite dans les équations de Hamilton-Jacobi au sens suivant : si une suite de solutions d'une suite d'équations converge localement uniformément vers une fonction, alors cette fonction est solution de l'équation limite. Plus précisément, le théorème suivant est démontré dans [14].

**Théorème 1.3** (Stabilité par passage à la limite uniforme). *Pour tout  $\varepsilon > 0$ , soit  $u_\varepsilon$  une sous-solution (resp. sur-solution) de viscosité de l'équation*

$$\partial_t u_\varepsilon + H_\varepsilon(\nabla_x u_\varepsilon, D_x^2 u_\varepsilon) = 0 \quad (0, T) \times \Omega,$$

où  $H_\varepsilon$  est continue et satisfait (4). Si  $u_\varepsilon$  converge localement uniformément vers  $u$  dans  $(0, T) \times \Omega$  et si  $H_\varepsilon$  converge localement uniformément vers  $H$  dans  $\mathbb{R}^n \times S_n(\mathbb{R})$ , alors  $u$  est une sous-solution de viscosité (resp. sur-solution) de

$$\partial_t u + H(\nabla_x u, D_x^2 u) = 0 \quad (0, T) \times \Omega.$$

On déduit donc que dans l'exemple du paragraphe 1.1.1  $u_\varepsilon$  converge localement uniformément vers l'unique solution de viscosité de l'équation (2).

Le deuxième résultat de stabilité permet de prouver l'existence d'une solution de viscosité et utilise les mêmes idées de preuve que le théorème de stabilité précédent.

On définit pour une fonction  $u : (0, T) \times \Omega$  localement bornée son enveloppe semi-continue supérieure (resp. inférieure)  $u^*$  (resp.  $u_*$ ) par

$$u^*(t, x) = \limsup_{(s, y) \rightarrow (t, x)} u(s, y) \quad \left( \text{resp. } u_*(t, x) = \liminf_{(s, y) \rightarrow (t, x)} u(s, y) \right).$$

**Théorème 1.4** (Stabilité par passage au supremum ou infimum). *Soit  $(u_\alpha)_{\alpha \in A}$  une famille quelconque de sous-solutions de viscosité (resp. sur-solution) bornées de (3) alors*

$$u = \left( \sup_{\alpha \in A} u_\alpha \right)^* \quad \left( \text{resp.} \quad \left( \inf_{\alpha \in A} u_\alpha \right)_* \right)$$

*est aussi une sous-solution (resp. sur-solution) de viscosité de (3).*

### Résultat d'existence

La méthode de Perron permet d'obtenir l'existence de solution de viscosité.

**Théorème 1.5** (Méthode de Perron). *Soient  $u$  une sous-solution et  $v$  une sur-solution de (3) telles que*

$$u \leq v \quad \text{dans } (0, T) \times \Omega.$$

*Alors il existe une solution de viscosité  $w$  de (3) telle que  $u \leq w \leq v$ .*

Dans la preuve de ce théorème, on pose  $f$  le supremum de l'ensemble des sous-solutions  $z$  vérifiant  $u \leq z \leq v$ . Il s'agit de montrer que  $f^*$  est une solution de viscosité de (3). Le résultat de stabilité (Théorème 1.4) garantit que  $f^*$  est une sous-solution. On montre par l'absurde que  $f^*$  est également une sur-solution. Si ce n'était pas le cas, il serait possible de construire une sous-solution qui contredirait la maximalité de  $f^*$ . Le lecteur trouvera dans [32] les détails de la preuve.

On peut alors déduire l'existence d'une solution à condition initiale fixée. Considérons l'équation (3) où  $H$  est continue sur  $\mathbb{R}^n \times S_n(\mathbb{R})$  et  $u_0 : \Omega \rightarrow \mathbb{R}$   $L$ -lipschitzienne telle que  $\nabla u_0$  est  $L$ -lipschitzienne. On pose  $C = \max_{|p| \leq L, \|A\| \leq L} |H(p, A)|$ , ainsi que  $u^-(t, x) = u_0(x) - Ct$  et  $u^+(t, x) = u_0(x) + Ct$ . Les fonctions  $u^-$  et  $u^+$  sont respectivement sous et sur-solution donc par la méthode de Perron, on déduit l'existence d'une solution  $u$  comprise entre  $u^-$  et  $u^+$  donc ayant pour condition initiale  $u_0$ .

#### 1.1.4 Questions relatives aux équations de Hamilton-Jacobi

Nous présentons dans cette section trois enjeux majeurs relatifs aux équations de Hamilton-Jacobi. Les principales contributions de cette thèse répondent à des questions liées à ces enjeux.

#### Conditions au bord

Une question majeure pour les équations de Hamilton-Jacobi est de savoir quel sens donner aux conditions de bord.

On considère par exemple une condition au bord de type dynamique, de la forme suivante

$$\partial_t u + F(\partial_x u) = 0, \quad t \in (0, T) \times \Omega. \quad (5)$$

La notion de solutions de viscosité au bord qui paraît la plus simple et la plus naturelle est la suivante : la solution doit vérifier la condition aux limites au sens de viscosité.



**Définition 1.6** (Solution de viscosité au bord au sens fort). *Soit  $u : (0, T) \times \Omega \rightarrow \mathbb{R}$  une fonction localement bornée.*

- *On dit que la fonction  $u$  est une sous-solution de viscosité (resp. une sur-solution) au bord au **sens fort** de (1)-(5) si  $u$  est semi-continue supérieurement (resp. inférieurement) et si pour tout  $(t_0, x_0) \in (0, T) \times \partial\Omega$  et toute fonction test  $\varphi \in C^2((0, T) \times \overline{\Omega})$  telle que  $u - \varphi$  atteigne un maximum local (resp. un minimum local) en  $(t_0, x_0)$ , on a*

$$\partial_t \phi(t_0, x_0) + F(\nabla_x \phi(t_0, x_0)) \leq 0 \quad (\text{resp. } \geq 0).$$

- *Une fonction  $u$  est une solution de viscosité au bord au **sens fort** de (1)-(5) si elle est à la fois sous-solution et sur-solution de (1)-(5).*

Une condition nécessaire pour que le principe du maximum (Propriété 1.1) soit vérifié au bord est que la fonction  $F$  soit **décroissante** par rapport à la composante normale.

Cependant, cette notion de condition aux limites ne garantit pas l'existence de solution de viscosité de (1)-(5). En effet, les deux résultats de stabilité (Théorèmes 1.3 et 1.4) ne sont pas vérifiés pour cette notion de condition de bord (au sens fort). Plus précisément, si on considère une suite de solutions de viscosité de (1)-(5) qui converge localement uniformément, la fonction limite va satisfaire au bord soit (1) soit (5) au sens de viscosité. Pour garantir la stabilité et donc l'existence, il faut ainsi relaxer la notion de condition au bord. C'est pourquoi la définition usuelle de solution de viscosité au bord est la suivante (voir [38, Définition 7.4]).

**Définition 1.7** (Solution de viscosité au bord classique « sens faible »). *Soit  $u : (0, T) \times \Omega \rightarrow \mathbb{R}$  une fonction localement bornée.*

- *On dit que la fonction  $u$  est une sous-solution de viscosité (resp. une sur-solution) au bord de (1)-(5) si  $u$  est semi-continue supérieurement (resp. inférieurement) et si pour tout  $(t_0, x_0) \in (0, T) \times \partial\Omega$  et toute fonction test  $\varphi \in C^2((0, T) \times \overline{\Omega})$  telle que  $u - \varphi$  atteigne un maximum local (resp. un minimum local) en  $(t_0, x_0)$ , on a*

$$\partial_t \phi(t_0, x_0) + F(\nabla_x \phi(t_0, x_0)) \leq 0 \quad (\text{resp. } \geq 0),$$

ou

$$\partial_t \phi(t_0, x_0) + H(\nabla_x \phi(t_0, x_0)) \leq 0 \quad (\text{resp. } \geq 0),$$

- *Une fonction  $u$  est une solution de viscosité au bord de (1)-(5) si elle est à la fois sous-solution et sur-solution de (1)-(5).*

Nous verrons dans le Chapitre 2 qu'il est possible de faire le lien entre ces deux définitions.

### Unicité

Une question centrale concernant les équations de Hamilton-Jacobi est la question de l'unicité des solutions. Contrairement à l'existence qui repose toujours sur la méthode de Perron et ne pose généralement pas de problème, l'unicité est souvent une question difficile.

L'unicité est une conséquence d'un principe de comparaison qui affirme que si une sous-solution  $u$  et une sur-solution  $v$  sont ordonnées au temps initial, c'est-à-dire si

$$u(0, x) \leq v(0, x) \quad \forall x \in \bar{\Omega},$$

alors elles sont ordonnées en tout temps,

$$u(t, x) \leq v(t, x) \quad \forall (t, x) \in [0, T] \times \bar{\Omega}.$$

Tout comme il est plus « facile » d'obtenir l'existence de solutions pour la notion classique de solutions de viscosité au bord (la moins contraignante), il est plus « facile » de montrer l'unicité de solutions pour la notion de solutions de viscosité au bord au sens fort (la plus contraignante). Dans la suite du paragraphe, on considère des sur et sous-solutions qui vérifie la notion « au sens fort » au bord.

Pour comprendre le principe de la preuve, raisonnons dans un premier temps en supposant que  $u$  et  $v$  sont de classe  $C^1$ , respectivement sous et sur-solution de (1)-(5). On pose pour  $\eta > 0$ ,

$$M_\eta = \sup_{(t,x) \in [0,T] \times \bar{\Omega}} \left( u(t, x) - v(t, x) - \frac{\eta}{T-t} \right).$$

Montrons que  $M_\eta \leq 0$ . Le supremum est obligatoirement atteint dans  $[0, T)$ . On note  $(t_\eta, x_\eta)$  un point où le supremum est atteint. Si  $t_\eta \neq 0$  on a  $\partial_t u(t_\eta, x_\eta) = \frac{\eta}{(T-t_\eta)^2} + \partial_t v(t_\eta, x_\eta)$  et  $\partial_x v(t_\eta, x_\eta) = \partial_x u(t_\eta, x_\eta)$ . En prenant les fonctions tests  $\phi_1(t, x) = v(t, x) + \frac{\eta}{T-t}$  et  $\phi_2(t, x) = u(t, x) - \frac{\eta}{T-t}$ , on déduit les inégalités de viscosité suivantes si  $x_\eta \in \Omega$ ,

$$\begin{aligned} \frac{\eta}{(T-t_\eta)^2} + \partial_t v(t_\eta, x_\eta) + H(\partial_x v(t_\eta, x_\eta)) &\leq 0, \\ -\frac{\eta}{(T-t_\eta)^2} + \partial_t u(t_\eta, x_\eta) + H(\partial_x u(t_\eta, x_\eta)) &\geq 0. \end{aligned}$$

De même si  $x_\eta \in \partial\Omega$ , on a les mêmes inégalités avec  $F$  au lieu de  $H$ . Donc en combinant, les deux inégalités, on obtient  $0 \leq -\frac{\eta}{(T-t_\eta)^2}$  qui nous donne une contradiction.

Si cette fois  $u$  et  $v$  ne sont plus supposées régulières, on « dédouble les variables » dans  $M_\eta$  afin d'obtenir des fonctions tests régulières et on ajoute un terme de pénalisation pour forcer les variables dédoublées à être proches. Plus précisément, on introduit

$$M_{\eta,\varepsilon} = \sup_{(t,s,x,y) \in [0,T]^2 \times \bar{\Omega}^2} \left( u(t, x) - v(s, y) - \frac{\eta}{T-t} - \frac{|x-y|^2}{2\varepsilon} - \frac{(t-s)^2}{2\varepsilon} \right),$$

qui va nous permettre de déduire des inégalités de viscosité comme dans le cas régulier et d'aboutir également à une contradiction (voir par exemple [66] pour plus

de détails). Nous étudions dans le Chapitre 2, l'unicité de solutions dans le cadre de l'équation (1)-(5) avec des hypothèses précisées par la suite qui ne permettent pas de conclure avec ce dédoublement des variables. L'idée sera de remplacer le terme  $-\frac{|x-y|^2}{2\varepsilon} - \frac{(t-s)^2}{2\varepsilon}$  par un terme de la forme  $\varepsilon\varphi\left(\frac{(t-s)}{\varepsilon}, \frac{|x-y|}{\varepsilon}\right)$  pour une fonction  $\varphi$  bien choisie.

### Passages à la limite

Un autre enjeu essentiel des équations de Hamilton-Jacobi est la question des passages à la limite. Dans l'exemple du paragraphe 1.1.1 portant sur la viscosité évanescence, on voit que le résultat de stabilité (Théorème 1.3) permet de conclure que  $u_\varepsilon$  tendait localement uniformément vers l'unique solution de viscosité de (2).

Un autre type de passage à la limite est l'étude de convergence et d'estimations d'erreur de schémas numériques pour des équations de Hamilton-Jacobi. Dans le Chapitre 3, nous étudions un schéma numérique qui approche l'équation (1)-(5) dans le cadre d'une jonction.

#### 1.1.5 Contrôle optimal et formule de représentation

Il existe une formule de représentation pour les solutions d'équations de Hamilton-Jacobi quand l'hamiltonien  $H$  est convexe. En effet, dans ce cas le principe de programmation dynamique introduit par Bellman permet de montrer l'existence d'une solution de viscosité en donnant une formule explicite. Cette solution appelée fonction valeur correspond à la minimisation d'un coût d'un problème de contrôle en lien avec  $H$ . De plus, cette fonction est l'unique solution de l'équation de Hamilton-Jacobi.

On définit la transformée de Legendre-Fenchel pour une fonction  $f$  continue, convexe, coercive par

$$f^*(p) = \sup_{q \in \mathbb{R}^n} \{p \cdot q - f(q)\}, \quad \text{pour } p \in \mathbb{R}^n.$$

On appelle  $L = H^*$  le lagrangien associé à  $H$ . L'équation de Hamilton-Jacobi suivante

$$\begin{cases} u_t + H(u_x) = 0, & \text{sur } (0, +\infty) \times \mathbb{R}^n, \\ u(0, x) = u_0(x) & \text{sur } \mathbb{R}^n, \end{cases} \quad (6)$$

se réécrit donc

$$u_t + \sup_{b \in \mathbb{R}^n} \{-u_x \cdot b - L(b)\} = 0.$$

Il s'agit de l'équation de Hamilton-Jacobi-Bellman d'un problème de contrôle optimal en horizon fini. Les trajectoires de ce problème sont solutions de l'EDO suivante

$$\frac{\partial X}{\partial s} = b(s),$$

et le coût instantané est le lagrangien  $L$ . La solution de l'équation (6) est

$$u(t, x) = \inf_{X(\cdot)} \left[ \int_0^t L(\dot{X}(s)) ds + u_0(X(0)) \right],$$

où  $X(\cdot) \in \left\{ X \in W^{1,1}([0, t], \mathbb{R}^N) \mid X(t) = x \right\}$ . Le contrôle constant suivant  $b = \frac{x - X(0)}{t}$  est optimal (voir [14] p.171-172). La trajectoire optimale est donc le segment joignant le point initial  $(0, X(0))$  au point final  $(t, x)$ . On déduit alors la formule de Lax-Oleinik

$$u(t, x) = \inf_{y \in \mathbb{R}^n} \left[ tH^* \left( \frac{x - y}{t} \right) + u_0(y) \right],$$

solution de l'équation de Hamilton-Jacobi (6).

## 1.2 Régularité elliptique et parabolique à la De Giorgi

Dans la deuxième partie de la thèse on s'intéresse à un résultat de régularité de solutions d'équations elliptiques et paraboliques. Plus précisément, la méthode introduite par De Giorgi [39] en 1957 permet de montrer la régularité höldérienne des solutions pour une large classe d'équations elliptiques et paraboliques à coefficients peu réguliers (seulement mesurables et vérifiant des conditions précisées dans la suite). Il a introduit ces techniques pour résoudre le 19<sup>e</sup> problème de Hilbert que nous décrivons dans le paragraphe suivant.

### 1.2.1 Le 19<sup>e</sup> problème de Hilbert

Le 19<sup>e</sup> problème de Hilbert consiste à montrer que les minimiseurs locaux de l'énergie fonctionnelle suivante sont analytiques,

$$\mathcal{E}(w) = \int_{\Omega} F(\nabla w) dx, \quad (7)$$

où  $\Omega$  est un ouvert borné de  $\mathbb{R}^d$  et  $F : \mathbb{R}^d \rightarrow \mathbb{R}$  est une fonction analytique telle que la matrice hessienne de  $f$  (ici  $D^2F(p)$ ) vérifie pour tout  $p \in \mathbb{R}^d$  la condition suivante d'ellipticité,

$$\lambda I \leq A \leq \Lambda I, \quad (8)$$

où  $\lambda, \Lambda > 0$ . Nous allons voir comment le théorème de régularité höldérienne introduit par De Giorgi, lui a permis de résoudre le 19<sup>e</sup> problème de Hilbert.

On procède en trois étapes. Tout d'abord les minimiseurs locaux de l'énergie sont solutions d'une équation d'Euler-Lagrange. Puis on se ramène à montrer que si  $\nabla w \in C^\alpha$  alors par un argument de bootstrap sur une équation elliptique à coefficients  $C^\alpha$ , on a  $w \in C^\infty$  puis analytique. Enfin, on remarque que les dérivées partielles de  $w$  satisfont une équation elliptique qui vérifient les hypothèses du théorème de De Giorgi, ainsi  $\partial_{x_i} w \in C^\alpha$  ce qui permet de conclure par l'étape précédente que  $w \in C^\infty$ . Ces étapes sont inspirées de [39, 97, 49].

### Équation d'Euler-Lagrange

On dit qu'une fonction  $v \in H^1(\Omega)$  est solution au sens faible de

$$-\nabla \cdot (A(x)\nabla v) = 0$$

si pour toute fonction  $\phi \in C_c^\infty(\Omega)$ , on a

$$\int_{\Omega} A \nabla u \cdot \nabla \phi = 0.$$

On montre qu'un minimiseur local de (7) est solution d'une équation d'Euler-Lagrange. En effet, si  $w \in H^1(\Omega)$  est un minimiseur local de (7), alors

$$\int_{\Omega} F(\nabla w + \varepsilon \nabla \phi) dx \geq \int_{\Omega} F(\nabla w) dx,$$

pour  $\varepsilon > 0$  petit et  $\phi \in C_c^\infty(\Omega)$ . Par la formule de Taylor, en utilisant (8) et en faisant tendre  $\varepsilon$  vers 0, on a

$$\int_{\Omega} DF(\nabla w) \cdot \nabla \phi dx \geq 0.$$

Et en considérant  $-\phi$  à la place de  $\phi$ , on obtient l'égalité. Donc  $w$  est solution faible de l'équation d'Euler-Lagrange

$$-\nabla \cdot DF(\nabla w) = 0. \quad (9)$$

**Passage  $\nabla w \in C^\alpha$  à  $w \in C^\infty$**

On suppose ici que  $\nabla w \in C^\alpha(\Omega')$  pour tout  $\Omega' \subset\subset \Omega$  et on montre que  $w \in C^\infty(\Omega)$ .

On introduit la notation suivante pour  $A, B$  des matrices carrées de taille  $d$ .

$$A : B = \sum_{i,j=1}^d A_{i,j} B_{i,j}.$$

On déduit alors en développant (9) que  $w \in C^{1,\alpha}(\Omega')$  est solution de l'équation elliptique suivante

$$A : D^2 w = 0,$$

avec  $A = D^2 F(\nabla w)$  qui vérifie (8) et qui est dans  $C^\alpha(\Omega')$  car  $\nabla w \in C^\alpha(\Omega')$  et  $F$  est suffisamment régulière. La théorie de Schauder (voir [56]) permet alors de d'obtenir que  $w \in C^{2,\alpha}(\Omega')$ . En dérivant cette équation par rapport à  $x_i$  et par le même argument, on montre que  $\partial_{x_i} w \in C^{2,\alpha}(\Omega')$  donc  $w \in C^{3,\alpha}(\Omega')$ . En itérant cette technique, on obtient que  $w \in C^{n,\alpha}(\Omega')$  pour tout  $n \in \mathbb{N}$ . On déduit que  $w \in C^\infty(\Omega')$  pour tout  $\Omega' \subset\subset \Omega$  donc  $w \in C^\infty(\Omega)$ . On savait déjà en 1957 comment obtenir la régularité analytique à partir de la différentiabilité à tout ordre, voir par exemple les travaux de Bernstein [23] et Petrowsky [88].

**Passage  $\nabla w \in H^1$  à  $\nabla w \in C^\alpha$**

Il reste à montrer que  $\nabla w \in C^\alpha(\Omega')$  pour tout  $\Omega' \subset\subset \Omega$ . Le théorème suivant de De Giorgi [39], étudiée plus en détail au Chapitre 4, permet alors de conclure.

**Théorème 1.8** (Régularité höldérienne). *Soit  $u \in H^1(\Omega)$  une solution faible de*

$$-\nabla \cdot (A \nabla u) = 0 \quad x \in \Omega, \quad (10)$$

où  $A = A(x)$  est une fonction mesurable vérifiant (8). Alors  $u \in C^\alpha(\Omega')$  pour tout  $\Omega' \subset\subset \Omega$ .

Un intérêt du théorème de De Giorgi est que les coefficients peuvent être peu réguliers, seulement mesurables vérifiant la condition d'ellipticité (8), donc on peut considérer en particulier  $A = D^2F(\nabla w)$ .

En dérivant (9) par rapport à  $x_i$ , la fonction  $u = \partial_{x_i} w$  est solution de

$$-\nabla \cdot (A\nabla u) = 0,$$

donc  $u = \partial_{x_i} w \in C^\alpha(\Omega')$  et  $\nabla w \in C^\alpha(\Omega')$ . Ce qui permet de conclure que les minimiseurs locaux sont de classe  $C^\infty$  grâce à l'étape précédente, puis analytique.

### 1.2.2 Méthode de De Giorgi

On note  $B_r$  la boule de centre 0 et de rayon  $r$  dans  $\mathbb{R}^d$ . Par argument de remise à l'échelle, il est suffisant d'étudier le cas  $\Omega = B_2$  et  $\Omega' = B_1$ . Dans ce qui suit, une constante universelle désigne une constante qui ne dépend que de  $\lambda, \Lambda, d$ . Pour montrer le Théorème 1.8, De Giorgi a introduit deux outils importants : une estimation  $L^2 - L^\infty$  et un lemme des valeurs intermédiaires.

L'estimation  $L^2 - L^\infty$  est aussi appelé *premier lemme de De Giorgi* et permet d'obtenir qu'une solution (ou sous-solution positive) bornée en norme  $L^2$  dans une grande boule  $B_{r_2}$  est bornée en norme  $L^\infty$  dans une boule plus petite  $B_{r_1}$ .

**Lemme 1.9** (Premier lemme de De Giorgi : estimation  $L^2 - L^\infty$ ). *Il existe  $\delta > 0$  une constante universelle telle que pour toute solution  $u : B_2 \rightarrow \mathbb{R}$  de (10), on a l'implication suivante. Si*

$$\int_{B_{r_2}} u_+^2 \leq \delta,$$

alors

$$u_+ \leq \frac{1}{2} \quad \text{dans } B_{r_1}.$$

Le second outil est le lemme des valeurs intermédiaires appelé parfois *second lemme de De Giorgi*. Ce lemme permet de quantifier (en mesure) le fait qu'une solution ne peut pas faire de saut entre deux valeurs numériques.

**Lemme 1.10** (Second lemme de De Giorgi : lemme des valeurs intermédiaires). *Soit  $u \in H^1(B_1)$ . On a alors*

$$|\{u \leq 0\} \cap B_1| \times |\{u \geq \frac{1}{2}\} \cap B_1| \leq C |\{0 < u < \frac{1}{2}\} \cap B_1|^{\frac{1}{2}} \sqrt{\int_{B_1} \|\nabla u_+(x)\|^2 dx},$$

où  $C$  dépend uniquement de  $d$ .

Ce lemme est valable non seulement pour les solutions de (10) mais également pour toute fonction dans  $H^1$ . Cependant le fait d'être solution de (10) permet d'obtenir que la norme du gradient est bornée par une constante universelle et d'avoir une relation entre les mesures qui ne dépend plus de la solution choisie.

Dans le Chapitre 4, nous verrons que pour montrer le Théorème 1.8, il est suffisant d'obtenir le lemme suivant d'abaissement du maximum : si une solution dans  $B_2$  est plus petite que 1 dans une boule  $B_{3/2}$  et est « suffisamment » plus petite que 0, alors cette solution est loin de 1 dans une plus petite boule  $B_{1/2}$ .

**Lemme 1.11** (Abaissement du maximum). *Il existe  $\mu \in (0, 1)$  qui dépend uniquement de  $\lambda, \Lambda, d$  tel que pour toute solution  $v : B_2 \rightarrow \mathbb{R}$  de (10) qui satisfait*

$$\begin{cases} v \leq 1 \text{ in } B_{\frac{3}{2}} \\ |\{v \leq 0\} \cap B_1| \geq \frac{|B_1|}{2}, \end{cases} \quad (11)$$

on a

$$v \leq 1 - \mu \quad \text{dans } B_{\frac{1}{2}}.$$

Ce lemme permet alors d'obtenir une propriété de décroissance de l'oscillation d'une solution qui sert à montrer la continuité höldérienne des solutions (voir Chapitre 4).

## 2 Contributions de la thèse

Les chapitres de ce manuscrit sont composés des travaux suivants :

- Chapitre 1 : article [61] publié dans *Nonlinear Analysis*.
- Chapitre 2 : article [60] publié dans *Journal of Differential Equation*.
- Chapitre 3 : article [62] écrit en collaboration avec Marwa Koumaiha, prépublication.
- Chapitre 4 : article en préparation.

Dans cette section, nous décrivons les différentes contributions de cette thèse.

### 2.1 Contraintes d'état sur un ouvert borné régulier multidimensionnel

Les conditions au bord de type contraintes d'état pour des équations de Hamilton-Jacobi apparaissent dans des problèmes de contrôle optimal pour lesquels les trajectoires sont contraintes à rester dans un certain domaine  $\Omega$ . Une solution d'un problème de contrainte d'état est solution de viscosité à l'intérieur du domaine et sur-solution de viscosité sur le bord du domaine. On s'intéresse donc à cette équation de Hamilton-Jacobi avec contraintes d'état, dans le cas stationnaire

$$\begin{cases} u + H(\nabla u) = 0 & \text{dans } \Omega \\ u + H(\nabla u) \geq 0 & \text{sur } \partial\Omega, \end{cases} \quad (12)$$

ou dans le cas d'évolution

$$\begin{cases} \partial_t u + H(\nabla u) = 0 & \text{dans } (0, T) \times \Omega \\ \partial_t u + H(\nabla u) \geq 0 & \text{sur } (0, T) \times \partial\Omega. \end{cases} \quad (13)$$

Cette formulation semble ne pas tenir compte du comportement au bord de d'une solution  $u$  en tant que sous-solution. Cependant nous allons voir qu'elle est en réalité équivalente à une formulation avec une condition au bord imposée aux sur et sous-solutions. L'objet du premier chapitre de la thèse est d'établir l'équivalence de trois formulations du problème de contraintes d'état.

### 2.1.1 Résultats existants

Dans le cadre de l'équation de Hamilton-Jacobi-Bellman considérée par Soner [93] et Ishii et Koike [72],  $H$  est un hamiltonien convexe correspondant à un problème de contrôle optimal. Il existe plusieurs résultats d'unicité pour ces problèmes au bord. Soner [93] a montré l'unicité de solutions de viscosité bornées et uniformément continues de (12) pour une équation de type Hamilton-Jacobi-Bellman avec  $H$  convexe. Plus généralement pour un hamiltonien non nécessairement convexe, Capuzzo-Dolcetta et Lions [31] ont montré l'unicité de solutions de viscosité continue de (12). Néanmoins la fonction valeur du problème de contrôle optimal peut être discontinue et ce cas ne rentre pas dans le cadre d'unicité des deux résultats précédents. C'est pourquoi Ishii et Koike [72] proposent une nouvelle formulation des contraintes d'état en ajoutant au bord une condition pour les sous-solutions qui leur permet d'obtenir un résultat d'unicité de solutions de viscosité possiblement discontinues. L'équation qu'ils considèrent est la suivante,

$$\begin{cases} u + H(\nabla u) = 0 & \text{dans } \Omega \\ u + H(\nabla u) \geq 0 & \text{sur } \partial\Omega \\ u + H_{in}(\nabla u) \leq 0 & \text{sur } \partial\Omega, \end{cases} \quad (14)$$

où  $H_{in}$  définie dans [72] est un hamiltonien « rentrant ».

Dans le cas d'un hamiltonien continu, quasi-convexe, coercif, et pour  $\Omega = (a, b)$  un intervalle ouvert borné de  $\mathbb{R}$ , Imbert et Monneau [66] ont montré que la formulation de Soner (12) est équivalente (les deux notions de solutions de viscosité coïncident) au problème suivant,

$$\begin{cases} u + H(\nabla u) = 0 & \text{dans } \Omega \\ u + H^-(\nabla u) \geq 0 & \text{sur } \partial\Omega \\ u + H^-(\nabla u) \leq 0 & \text{sur } \partial\Omega, \end{cases} \quad (15)$$

où  $H^-$  est la partie décroissante de l'hamiltonien selon le vecteur normal entrant. Pour  $H$  convexe, coercive (voir (24)),  $H^-$  correspond à  $H_{in}$  de [72]. Ils ont également montré l'équivalence dans le cas non stationnaire correspondant à l'équation (13) et l'existence et l'unicité de solution pour les équations (15) et (13). Comme la formulation (15) implique la formulation (14) qui implique la formulation (12) de manière directe, le résultat d'équivalence de (15) et (12) permet de déduire l'équivalence des trois formulations. Cela implique que les propriétés des solutions valables pour chaque formulation sont valables pour les trois, comme par exemple l'existence et l'unicité de solutions. En particulier le résultat d'unicité obtenu par Ishii et Koike pour le problème (14) permet de déduire l'unicité de solutions possiblement discontinues de la formulation de Soner (12), dans le cadre des hypothèses de Imbert et Monneau [66].

Plus généralement, dans le cadre quasi-convexe, Imbert et Monneau [66] ont montré l'existence et l'unicité de solutions associées à une classe plus large de conditions au bord. Ce résultat repose sur un outil important, la réduction de l'ensemble des fonctions tests à une seule pente en espace pour la condition au bord. Grâce à cet outil on peut montrer de nombreuses propriétés. Il permet notamment d'obtenir des inégalités de viscosité plus faibles pour des fonctions tests non nécessairement de



classe  $C^1$ . Ces inégalités de viscosité plus faibles sont utiles et nécessaires car elles permettent notamment d'obtenir l'estimation d'erreur du Chapitre 3 [62].

### 2.1.2 Contribution

La première contribution de ce manuscrit est l'extension à un domaine multidimensionnel, borné régulier du résultat de Imbert et Monneau [66] : l'équivalence des trois formulations du problème de contraintes d'état. Ainsi que l'extension, toujours dans le même cadre, de l'outil de réduction de l'ensemble des fonctions tests à une seule pente en espace dans la direction normale.

On considère un ouvert borné  $\Omega$  de  $\mathbb{R}^{d+1}$  avec un bord de classe  $C^1$ . On suppose que l'hamiltonien  $H : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  vérifie les propriétés suivantes

$$\left\{ \begin{array}{l} \text{continuité,} \\ \text{coercivité,} \\ \text{quasi-convexité,} \end{array} \quad \begin{array}{l} \lim_{|p| \rightarrow +\infty} H(p) = +\infty \\ \forall \lambda \in \mathbb{R}, \quad \{H \leq \lambda\} \text{ est convexe.} \end{array} \right. \quad (16)$$

Comme on considère des solutions possiblement discontinues, on doit ajouter une condition sur les sous-solutions. Les sous-solutions doivent être « faiblement continues » au bord c'est-à-dire, que leur enveloppe semi-continue supérieure  $u^*$  doit vérifier

$$\forall (t, x) \in (0, T) \times \partial\Omega \quad u^*(t, x) = \limsup_{y \rightarrow x, s \rightarrow t, y \notin \partial\Omega} u^*(s, y), \quad (17)$$

où  $u^*$  est définie au Chapitre 1. On définit également la partie décroissante de l'hamiltonien  $H^-$  selon le vecteur normal entrant au Chapitre 1. Les notions de solutions de viscosité pour ce problème sont données au Chapitre 1 aux Définitions 2.1 et 2.2. On montre les théorèmes suivants.

**Théorème 2.1** (Reformulation des contraintes d'état, cas d'évolution).

Soit  $H : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  vérifiant (24), et  $u : (0, T) \times \overline{\Omega} \rightarrow \mathbb{R}$ . Alors la fonction  $u$  est une solution de viscosité de

$$\left\{ \begin{array}{ll} u_t + H(\nabla u) \geq 0 & \text{dans } (0, T) \times \overline{\Omega} \\ u_t + H(\nabla u) \leq 0 & \text{dans } (0, T) \times \Omega, \end{array} \right. \quad (18)$$

et vérifie (17) si et seulement si  $u$  est une solution de viscosité de

$$\left\{ \begin{array}{ll} u_t + H(\nabla u) = 0 & \text{dans } (0, T) \times \Omega \\ u_t + H^-(\nabla u) = 0 & \text{sur } (0, T) \times \partial\Omega. \end{array} \right. \quad (19)$$

**Théorème 2.2** (Reformulation des contraintes d'état, cas stationnaire).

Soit  $H : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  vérifiant (24), et  $u : \overline{\Omega} \rightarrow \mathbb{R}$ . Alors la fonction  $u$  est une solution de viscosité de

$$\left\{ \begin{array}{ll} u + H(\nabla u) \geq 0 & \text{dans } \overline{\Omega} \\ u + H(\nabla u) \leq 0 & \text{dans } \Omega, \end{array} \right. \quad (20)$$

et vérifie (17) si et seulement si  $u$  est une solution de viscosité de

$$\left\{ \begin{array}{ll} u + H(\nabla u) = 0 & \text{dans } \Omega \\ u + H^-(\nabla u) = 0 & \text{sur } \partial\Omega. \end{array} \right. \quad (21)$$

## 2.2 Conditions au bord effectives pour des équations de Hamilton-Jacobi d'ordre 1

On étudie l'équation de Hamilton-Jacobi soumise à la condition au bord de type dynamique

$$\begin{cases} u_t + H(u_x) = 0 & \text{pour } t \in (0, T) \text{ et } x > 0 \\ u_t + F(u_x) = 0 & \text{pour } t \in (0, T) \text{ et } x = 0 \end{cases} \quad (22)$$

où  $H$  et  $F$  vérifient des propriétés précisées plus bas.

Le but du Chapitre 2 est d'obtenir un résultat d'équivalence des conditions au bord et de préciser en quel sens elles sont vérifiées. Nous verrons que l'on peut exhiber des classes d'équivalence de conditions au bord au sens où deux conditions au bord sont équivalentes si elles engendrent les mêmes solutions. Nous montrons aussi que dans chaque classe d'équivalence, il existe une unique condition au bord vérifiée au sens fort (Définition 1.6), que l'on appelle *condition au bord effective*. La dénomination « effective » est utilisée dans ce contexte car il s'agit de l'unique condition au bord stable : si on considère une suite de solutions  $u_\varepsilon$  (au sens fort ou faible) associées à une suite d'équations de type (22) avec  $(H_\varepsilon, F_\varepsilon)$  qui converge localement uniformément vers  $u$ , et si  $(H_\varepsilon, F_\varepsilon)$  converge localement uniformément vers  $(\bar{H}, \bar{F})$  alors  $u$  est solution au sens fort de (22) avec  $(\bar{H}, \bar{F})$ .

### 2.2.1 Résultats existants

Il est parfois difficile de montrer qu'un problème est bien posé sur un domaine avec certaines conditions aux limites. Par exemple si on considère le problème de Cauchy suivant,

$$\begin{cases} \partial_t u - \partial_x u = 0, & (t, x) \in (0, T) \times (0, +\infty) \\ u(0, x) = u_0(x), & x \in [0, +\infty), \end{cases}$$

la solution est entièrement déterminée par la condition initiale et vaut  $u(t, x) = u_0(x + t)$ . En particulier, on ne peut pas imposer de condition au bord en  $x = 0$  : l'existence n'est pas satisfaite.

Dans le contexte des lois de conservations scalaires, Bardos, le Roux et Nédélec [12] ont trouvé une formulation faible de la condition de Dirichlet qui garantit l'existence d'une solution entropique. Plus récemment, Andreianov et Sbihi [8, 6, 9] ont exhibé des conditions de bord dites effectives plus générales que celles de Bardos, le Roux et Nédélec qui permettent également d'obtenir l'existence de solutions. Nous verrons que ces conditions de bord ont un lien fort avec les conditions de bord dynamiques que nous étudions dans le cadre des équations de Hamilton-Jacobi.

Concernant les équations de Hamilton-Jacobi, un premier résultat de conditions de bord effectives a été obtenu par Elliott, Giga et Goto [48]. Ils s'intéressent à un problème ayant des applications en supraconductivité et mouvement d'interface. Ils étudient l'équation suivante avec condition de bord dynamique,

$$\begin{cases} u_t - F(t, x)(u_x^2 + \gamma)^{\frac{1}{2}} = 0 & \text{pour } t \in (0, +\infty) \text{ et } x \in (a, b) \\ u_t - F(t, x)\alpha(t, x) = 0 & \text{pour } t \in (0, +\infty) \text{ et } x \in \{a, b\} \end{cases} \quad (23)$$

où  $(a, b)$  est un intervalle borné de  $\mathbb{R}$ ,  $F$  et  $\alpha$  sont des fonctions continues et  $\gamma$  est une constante positive. Ils déterminent des conditions de bord effectives pour cette

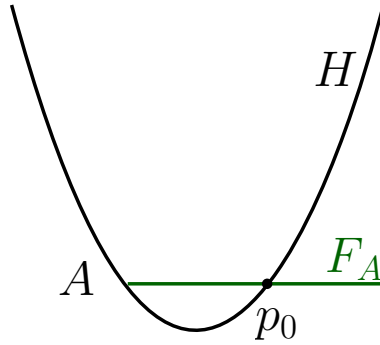


FIGURE 3 : Fonction  $F_A$  dans le cas quasi-convexe.

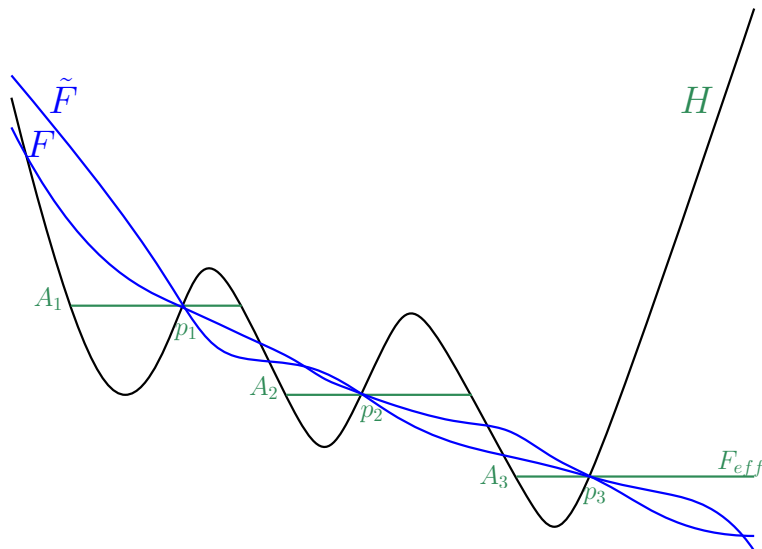


FIGURE 4 : Fonction  $F_{\text{eff}}$  associée à  $F$  et  $\tilde{F}$ .

équation et montrent un résultat d'unicité pour ces nouvelles conditions au bord, ce qui leur permet de déduire l'unicité pour des conditions au bord de (23). Plus récemment, Imbert et Monneau [66] ont déterminé les conditions au bord effectives de l'équation générale (22), dans le cas où l'hamiltonien  $H$  est continu, quasi-convexe et coercif, c'est-à-dire

$$\lim_{|p| \rightarrow +\infty} H(p) = +\infty, \quad (24)$$

et  $F$  est continue décroissante semi-coercive en  $-\infty$ , c'est-à-dire

$$\lim_{p \rightarrow -\infty} F(p) = +\infty. \quad (25)$$

Ils ont obtenu les conditions effectives correspondant à  $F = F_A$  dans (22) (voir Figure 3) avec

$$F_A(p) = \max(A, H^-(p)).$$

Ils ont également montré dans [67] le résultat sur une jonction multidimensionnelle (voir Figure 1). Dans un autre article Imbert et Nguyen [69] ont obtenu le même résultat dans le cas d'une équation parabolique et traitent également le cas d'une condition au bord sans dérivée temporelle.

## 2.2.2 Contribution

Une des contributions du Chapitre 2 a été de traiter le cas d'un hamiltonien non-convexe (aussi continu et coercif). Nous déterminons toutes les conditions de bord effectives correspondantes à (22), ainsi que les propriétés associées (il existe une unique condition au bord effective dans chaque classe d'équivalence qui est vérifiée au sens fort). Nous avons observé que dans le cas d'un hamiltonien quasi-convexe les fonctions  $F_A$  des conditions de bord effectives correspondent exactement aux fonctions flux de Godunov de Bardos, le Roux, Nédélec [12]

$$F_{p_0}(p) = \begin{cases} \sup_{q \in [p, p_0]} H(q) & \text{if } p \leq p_0 \\ \inf_{q \in [p_0, p]} H(q) & \text{if } p \geq p_0. \end{cases} \quad (26)$$

Dans le cas non-convexe, ces fonctions correspondent aussi à des conditions de bord effectives mais ne recouvrent pas toutes les classes d'équivalence. Il y en a beaucoup plus et elles correspondent exactement aux fonctions flux exhibées par Andreianov et Sbihi [8, 6, 9]. On définit alors un ensemble de fonctions  $F$ , appelées non linéarité forte au bord (voir Définition 1.1) correspondant, pour un hamiltonien  $H$  fixé à l'ensemble des fonctions  $F$  continues, décroissantes, semi-coercives en  $-\infty$  (25) qui vérifient la condition suivante :  $F$  est constante sur tout intervalle ouvert où  $F \neq H$ . On montre au Chapitre 2 que cet ensemble correspond exactement à l'ensemble des fonctions effectives nécessaires pour classer les conditions au bord.

On peut les définir comme suit à partir d'un hamiltonien  $H$  fixé. Les fonctions effectives (c'est-à-dire associées à une condition de bord effective) sont toutes les fonctions  $F$  continues, décroissantes, semi-coercives en  $-\infty$  (25) qui vérifient la condition suivante :  $F$  est constante sur tout intervalle ouvert où  $F \neq H$ . De plus on sait qu'il y a existence d'une solution de viscosité au sens faible au bord (Définition 1.7) pour

l'équation (22), voir [66]. Pour chaque fonction  $F$  continue décroissante, on détermine la fonction effective  $F_{\text{eff}}$  qui lui correspond pour laquelle les solutions de viscosité vont être solutions au sens fort au bord. Et pour chaque fonction  $F_{\text{eff}}$  on détermine les fonctions  $F$  qui sont dans sa classe, c'est-à-dire qui vont donner les mêmes solutions. Par exemple sur la Figure 4, on construit plusieurs fonctions  $F$  dans la même classe associées à la fonction  $F_{\text{eff}}$ . On remarque notamment que  $F_{\text{eff}}$  n'est pas une fonction flux de Godunov.

Les notions de solutions de viscosité utilisées sont données dans le Chapitre 2 aux Définitions 2.1 et 2.3. On peut alors énoncer le théorème des conditions au bord effectives.

**Théorème 2.3** (Conditions au bord effectives). *Soient  $H : \mathbb{R} \rightarrow \mathbb{R}$  continue coercive (24) et  $F : \mathbb{R} \rightarrow \mathbb{R}$  continue décroissante, il existe une unique fonction  $F_{\text{eff}}$  telle que  $u$  est solution de viscosité (sens classique) de*

$$\begin{cases} \partial_t u + H(\partial_x u) = 0 & \text{pour } x > 0 \\ \partial_t u + \min(H(\partial_x u), F(\partial_x u)) \leq 0 & \text{en } x = 0 \\ \partial_t u + \max(H(\partial_x u), F(\partial_x u)) \geq 0 & \text{en } x = 0 \end{cases}$$

si et seulement si  $u$  est solution de viscosité (sens fort) de

$$\begin{cases} \partial_t u + H(\partial_x u) = 0 & \text{pour } x > 0 \\ \partial_t u + F_{\text{eff}}(\partial_x u) = 0 & \text{en } x = 0. \end{cases}$$

### 2.2.3 Perspectives

On pourra se demander si le Théorème 2.3 de conditions au bord effectives est encore vrai si l'hamiltonien n'est pas nécessairement coercif (24) mais seulement continu.

## 2.3 Unicité pour une équation de Hamilton-Jacobi d'ordre 1 avec condition au bord dynamique

On s'intéresse dans cette partie à l'unicité de l'équation (22). De nombreux résultats d'unicité pour cette équation existent sous des hypothèses variées de l'hamiltonien  $H$  et de  $F$ . Nous obtenons ici un résultat d'unicité très similaire aux résultats déjà présents dans la littérature, néanmoins avec des hypothèses plus faibles sur la fonction  $F$ .

### 2.3.1 Résultats existants

Le premier résultat d'unicité pour une équation de Hamilton-Jacobi avec condition de Neumann linéaire au bord est obtenu par Lions [78]. Concernant l'équation de Hamilton-Jacobi d'ordre 1 avec condition de Neumann non linéaire, Barles et Lions [17] ont obtenu un principe de comparaison pour l'équation (22) et plus généralement pour un hamiltonien  $H$  dépendant aussi de  $(x, u)$ , sous l'hypothèse suivante pour  $F$ , que l'on écrit ici en dimension 1,

$$F(p - \lambda) - F(p) \geq C\lambda, \quad \text{pour } \lambda \geq 0. \quad (27)$$

De plus, Barles [20, 13] et Ishii [71] ont traité le cas du second ordre. Plus précisément, Barles [20] a montré un principe de comparaison pour des équations paraboliques dégénérées complètement non linéaires sur un domaine  $\Omega$  régulier

$$u_t + H(x, u, Du, D^2u) = 0 \quad \text{dans } \Omega,$$

avec condition de Neumann non linéaire

$$u_t + F(x, u, Du) = 0 \quad \text{sur } \partial\Omega,$$

où  $F$  satisfait (27).

Récemment, Lions et Souganidis [82, 83] ont obtenu un résultat d'unicité dans le cas où le domaine est une jonction. Ils étudient l'équation suivante (que l'on écrit ici dans le cas où le domaine est une demi-droite),

$$\begin{aligned} u_t + H(u_x) &= 0 \quad \text{dans } (0, T) \times (0, +\infty) \\ u_t + H(u_x) &\geq 0 \quad \text{dans } (0, T) \times \{0\}, \end{aligned}$$

où l'hamiltonien est coercif et non nécessairement convexe. Or d'après ce qui a été vu dans la première contribution de la thèse, cette équation avec condition au bord de type contraintes d'état est équivalente à l'équation suivante (voir l'annexe du Chapitre 2 pour le cas  $H$  non-convexe),

$$\begin{aligned} u_t + H(u_x) &= 0 \quad \text{dans } (0, T) \times (0, +\infty) \\ u_t + H^-(u_x) &= 0 \quad \text{dans } (0, T) \times \{0\}, \end{aligned}$$

où  $H^-$  est la partie décroissante de l'hamiltonien définie par

$$H^-(p) = \inf_{q \leq p} H(q).$$

Comme il a été vu dans la seconde contribution,  $H^-$  correspond à une fonction  $F$  dans (22) et appartient à une seule classe d'équivalence. Le résultat de Lions et Souganidis ne permet pas de montrer l'unicité des solutions de (22) pour toutes les fonctions  $F$  décroissantes (plusieurs classe d'équivalence). De même les fonctions  $F_A(p) = \max(A, H^-(p))$  ne suffisent pas à traiter toutes les classes d'équivalence.

### 2.3.2 Contribution

Une des contributions de la thèse est d'avoir obtenu un principe de comparaison de (22) pour un hamiltonien non nécessairement coercif avec une fonction  $F$  qui n'a plus besoin de satisfaire la condition (27) mais seulement  $F$  décroissante telle que  $|F|$  est coercive. Grâce au théorème de conditions de bord effectives (Théorème 2.3), on utilise ce principe de comparaison pour montrer un principe de comparaison pour un hamiltonien coercif, et  $F$  semi-coercif en  $-\infty$  seulement.

**Théorème 2.4** (Principes de comparaison). *Soient  $H : \mathbb{R} \rightarrow \mathbb{R}$  continue et  $F : \mathbb{R} \rightarrow \mathbb{R}$  continue, décroissante et semi-coercive en  $-\infty$  (25). On suppose que la condition initiale  $u_0$  est uniformément continue. De plus, si on a l'une des hypothèses suivantes,*

1. ( $H$  non-coercive et  $F$  « coercive » )

$$\lim_{p \rightarrow +\infty} F(p) = -\infty, \quad (28)$$

2. ( $H$  coercive et  $F$  semi-coercive)

$$\lim_{|p| \rightarrow +\infty} H(p) = +\infty.$$

Alors pour toute sous-solution de viscosité  $u$  et sur-solution de viscosité  $v$  de (22) de condition initiale  $u_0$  vérifiant pour  $T > 0$  et  $C_T > 0$ ,

$$u(t, x) \leq C_T(1 + x), \quad v(t, x) \geq -C_T(1 + x), \quad \forall (t, x) \in (0, T) \times [0, +\infty),$$

on a

$$u \leq v \quad \text{dans} \quad [0, T) \times [0, +\infty).$$

L'idée de la preuve vient de [53] et consiste à coupler le temps et l'espace dans la méthode de dédoublement de variable. Précisément dans [53] pour  $F(p) = -p$ , ils utilisent

$$\frac{(t-s)^2}{2\delta} + \frac{(t-s)}{\delta}(x-y) + \frac{(x-y)^2}{2\delta},$$

au lieu de du terme classique

$$\frac{(t-s)^2}{2\delta} + \frac{(x-y)^2}{2\delta},$$

qui permet d'obtenir directement une contradiction si le supremum du dédoublement des variables est atteint en  $x = 0$  ou  $y = 0$ . Ici on construit une fonction de la forme  $\delta\varphi\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right)$  qui va permettre de traiter toutes les fonctions  $C^1$  telles que  $F' < 0$  et  $|F|$  est coercive (24). Et d'obtenir par un argument de densité le résultat pour  $F$  continues décroissantes telles  $|F|$  est coercive.

*Remarque 2.5.* Dans le cas où  $F$  est une fonction flux de Godunov de la forme (26), en faisant une translation pour ramener le point  $(p_0, H(p_0))$  au point  $(0, 0)$ , le dédoublement des variables classique permet d'obtenir l'unicité. En effet, les inégalités suivantes  $H(p) \leq F(p)$  si  $p \leq 0$  et  $H(p) \geq F(p)$  si  $p \geq 0$  permettent d'obtenir une contradiction avec les inégalités de viscosité. On a vu précédemment que les fonctions flux de Godunov permettaient d'obtenir toutes les classes d'équivalence dans le cas d'un hamiltonien quasi-convexe. Donc grâce au théorème des conditions de bord effectives, on obtient l'unicité pour toutes les fonctions  $F$  dans le cas quasi-convexe. Cependant dans le cas non-convexe, ces fonctions ne sont pas suffisantes pour traiter toutes les classes d'équivalence, et pour les conditions de bord effectives, ce dédoublement des variables ne permet pas de conclure car on a en général plusieurs changements de signe de  $H - F$ . C'est pourquoi l'unicité est plus compliquée à obtenir dans ce cas.

### 2.3.3 Perspectives

On pourra essayer d'appliquer le résultat de conditions au bord effectives pour montrer l'unicité de solutions du problème de Elliott, Giga et Goto [48] dans un domaine multidimensionnel.

## 2.4 Convergence et estimation d'erreur pour une équation de Hamilton-Jacobi sur une jonction

Cette partie est motivée par la modélisation du trafic routier, une principale application de l'équation étudiée ici. Précisément, on s'intéresse à l'approximation des solutions d'une équation de Hamilton-Jacobi discontinue d'ordre 1, posée sur une jonction (voir Figure 1), à l'aide d'un schéma numérique aux différences finies. On note  $J = \bigcup_{\alpha=1}^N J_\alpha$  une jonction constituée d'une réunion finie de branches  $J_\alpha$  (demi-droites) dont l'intersection est  $\{0\}$  le point de jonction. À chaque branches  $J_\alpha$  est associée un hamiltonien  $H_\alpha$  continu quasi-convexe et coercif (16). Le terme  $\frac{\partial u}{\partial x_\alpha}$  désigne la dérivée de  $u$  dans la direction de la branche  $J_\alpha$ . On considère l'équation suivante posée sur  $J$ ,

$$\begin{cases} u_t + H_\alpha(u_x) = 0 & \text{dans } (0, T) \times J_\alpha \setminus \{0\}, \\ u_t + F\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right) = 0 & \text{dans } (0, T) \times \{0\}, \end{cases} \quad (29)$$

On introduit un pas de discrétisation de temps et d'espace  $h = (\Delta t, \Delta x)$ . On note  $\mathcal{G}_h$  la jonction discrétisée et  $(U_i^{\alpha,n})_{i,\alpha,n}$  la solution du schéma aux différences finies suivant,

$$\begin{cases} \frac{U_i^{\alpha,n+1} - U_i^{\alpha,n}}{\Delta t} + \max\{H_\alpha^+(p_{i,-}^{\alpha,n}), H_\alpha^-(p_{i,+}^{\alpha,n})\} = 0, & i \geq 1, \quad \alpha = 1, \dots, N, \\ U_0^{\beta,n} := U_0^n, & i = 0, \quad \beta = 1, \dots, N, \\ \frac{U_0^{n+1} - U_0^n}{\Delta t} + F(p_{0,+}^{1,n}, \dots, p_{0,+}^{N,n}) = 0, \end{cases} \quad (30)$$

où  $p_{i,\pm}^{\alpha,n}$  sont des gradients discrets d'espace définis par

$$p_{i,+}^{\alpha,n} := \frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x}, \quad p_{i,-}^{\alpha,n} := \frac{U_i^{\alpha,n} - U_{i-1}^{\alpha,n}}{\Delta x}, \quad (31)$$

avec la condition initiale

$$U_i^{\alpha,0} = u_0(x_i^\alpha), \quad i \geq 0, \quad \alpha = 1, \dots, N. \quad (32)$$

On montre que  $(U_i^{\alpha,n})_{i,\alpha,n}$  approche la solution continue de (29) au pas de temps  $n\Delta t$  et au pas d'espace  $i\Delta x$ . Crandall, Lions [37] et Barles, Souganidis [18] ont montré qu'un schéma monotone et consistant est convergent. Ici on montre que le schéma est consistant ainsi que monotone si  $\Delta x$  et  $\Delta t$  vérifient une condition CFL précisée au Chapitre 3.

On introduit également les fonctions flux-limitées qui permettent de décrire toutes les classes d'équivalence mentionnées dans la seconde contribution de ce manuscrit,

$$F_A(p) = \max \left( A, \max_{\alpha=1, \dots, N} H_\alpha^-(p_\alpha) \right). \quad (33)$$

On étudie d'une part la convergence pour une condition de jonction générale (avec  $F$ ), et d'autre part une estimation d'erreur de ce schéma pour une condition de jonction avec  $F_A$ .



### 2.4.1 Résultats existants

**Équation de Hamilton-Jacobi sur réseaux.** Il existe de nombreux travaux sur les équations de Hamilton-Jacobi avec hamiltonien discontinu en espace posée sur un réseau. Les premiers résultats ont été obtenu par Schieborn [91] pour l'équation eikonale. Quelques années plus tard, ces résultats ont été étendus dans [92, 2, 68]. Par exemple, Achdou, Camilli, Cutrì et Tchou [2] étudient un problème de contrôle sur réseau. Imbert, Monneau et Zidani [68] montrent un résultat d'unicité, ainsi que de stabilité et d'existence de solution d'une équation de Hamilton-Jacobi posée sur une jonction appliquée à des modèles de trafic routier. Dans [16, 15], les auteurs étudient des problèmes de contrôle à horizon fini avec des dynamiques et coûts instantanés réguliers de chaque côté d'un hyperplan mais discontinus à la traversée de l'hyperplan. Récemment, Imbert et Monneau [66, 67], ainsi que Barles, Briani, Chasseigne et Imbert [21] ont montré des résultats très généraux d'unicité avec des hypothèses affaiblies (hamiltoniens quasi-convexes). Très récemment, Lions et Souganidis [82, 83] ont obtenu un résultat d'unicité en proposant une démonstration plus simple que les précédentes, valables pour des hamiltoniens non nécessairement convexes. Plus précisément, dans [83] ils obtiennent un résultat d'unicité pour des conditions de Kirchoff à la jonction et montrent que ces conditions généralisent les conditions avec une fonction limiteur de flux que l'on étudie dans le Chapitre 3.

**Schémas numériques.** Il existe plusieurs exemples de schémas numériques pour des équations de Hamilton-Jacobi avec un hamiltonien continu (en la variable d'espace). Par exemple,

- les différences finies introduites par Crandall et Lions [37] dans le cadre des équations de Hamilton-Jacobi,
- les schémas semi-lagrangiens [30, 29, 50, 51] qui utilisent le principe de programmation dynamique de Bellman et ne nécessitent pas de condition de Courant-Friedrichs-Lewy (CFL),

Cependant, il existe peu de résultats d'approximation numérique pour des équations de Hamilton-Jacobi sur réseaux. Costeseque, Lebacque et Monneau [36] ont introduit un schéma aux différences finies pour l'équation (29) pour  $F = F_{A_0}$  la fonction flux-limitée minimale. Ils ont montré la convergence de ce schéma. Citons également les travaux de Camilli, Festa et Schieborn [27] qui ont étudié un schéma semi-lagrangien pour des équations eikonales, et Göttlich, Ziegler, Herty [59] qui ont utilisé un schéma adapté de Lax-Friedrichs pour modéliser un problème de trafic routier.

### 2.4.2 Contribution

Nous avons étendu le schéma numérique aux différences finies de Costeseque, Lebacque, Monneau, en prenant une condition de jonction générale (30) (c'est-à-dire une fonction  $F$  générale). Nous avons d'une part montré la convergence de ce schéma et d'autre part obtenu une estimation d'erreur en  $\Delta x^{1/2}$  pour  $A \neq A_0$  où  $F_{A_0}$  est la condition de jonction effective minimale. Énonçons ces deux résultats.

**Théorème 2.6 (Convergence pour une condition de jonction générale).** *Soit  $T > 0$  et  $u_0$  une fonction Lipschitz. Il existe  $\underline{p}_\alpha, \bar{p}_\alpha, p_\alpha^0 \in \mathbb{R}$ ,  $\alpha = 1, \dots, N$ , dépendant uniquement de la condition initiale, des hamiltoniens et de la fonction de jonction  $F$ , telle que, si  $h$  vérifie une condition CFL (voir Chapitre 3 (3.18)), alors la solution numérique  $u^h$  définie par (30)-(32) converge localement uniformément, quand  $h$  tend vers zero, vers l'unique solution de viscosité  $u$  de (29), sur tout compact  $\mathcal{K} \subset [0, T] \times J$ , c'est-à-dire*

$$\limsup_{|h| \rightarrow 0} \sup_{(t,x) \in \mathcal{K} \cap \mathcal{G}_h} |u^h(t, x) - u(t, x)| = 0.$$

**Théorème 2.7 (Estimations d'erreur pour des conditions de jonction à flux-limité).** *Soit  $T > 0$  et  $u_0$  une fonction Lipschitz,  $u^h$  la solution du schéma numérique (30)-(32) et  $u$  la solution de viscosité de (29) pour  $A \in \mathbb{R}$ . Si la condition CFL (voir Chapitre 3 (3.18)) est vérifiée, alors il existe  $C > 0$  (indépendante de  $h$ ) telle que*

$$\sup_{[0, T] \times J \cap \mathcal{G}_h} |u^h(t, x) - u(t, x)| \leq \begin{cases} C(\Delta x)^{1/2} & \text{si } A > A_0, \\ C(\Delta x)^{2/5} & \text{si } A = A_0. \end{cases}$$

Étant donné le lien entre les lois de conservation scalaires et les équations de Hamilton-Jacobi en dimension 1, il est intéressant d'obtenir cette estimation d'erreur en  $\Delta x^{1/2}$  qui est recherchée pour les problèmes de lois de conservation scalaires.

Afin d'obtenir l'estimation d'erreur, dans le cas  $A > A_0$  nous avons étendu la définition la fonction-test utilisée dans [68] pour  $A = A_0$ . Dans le cas  $A = A_0$ , cette fonction n'est pas suffisamment régulière pour obtenir des estimations d'erreur. Mais dans le cas  $A > A_0$ , nous avons étudié sa régularité et montré que le gradient est localement Lipschitz en dehors d'une certaine courbe où cette fonction-test n'est pas  $C^1$ . Nous avons alors utilisé un résultat de Imbert et Monneau [66, Proposition 2.16] qui permet d'obtenir des inégalités de viscosité plus faible dans le cas où la fonction-test n'est pas de classe  $C^1$  en un point. Ces inégalités sont suffisantes pour obtenir l'estimation d'erreur. Nous avons également réalisé des simulations numériques (voir Chapitre 3) qui confirment que l'estimation pour  $A > A_0$  est optimale.

### 2.4.3 Perspectives

Dans le cas  $A = A_0$ , on pourra chercher si l'estimation  $\Delta x^{2/5}$  est optimale ou non. On cherchera à obtenir une estimation d'erreur dans le cadre du Théorème 2.7 dans le cas où les hamiltoniens ne sont pas convexes, en commençant par écrire le schéma monotone aux différences finies correspondant.

## 2.5 Résultat quantitatif pour la régularité parabolique à la De Giorgi

On s'intéresse ici à un résultat de régularité höldérienne de l'équation parabolique suivante,

$$\partial_t u = \nabla_x \cdot (A \nabla_x u) + B \cdot \nabla_x u + s, \quad t \in (T_1, T_2), x \in \Omega, \quad (34)$$

où  $T_1$  et  $T_2$  sont des réels,  $\Omega$  est un ouvert borné de  $\mathbb{R}^d$  et  $A = A(t, x)$ ,  $B = B(t, x)$ ,  $s = s(t, x)$  sont des coefficients peu réguliers (ici seulement mesurables) tel que  $A$

vérifie (8), et  $B, s$  sont bornés. Les techniques de De Giorgi-Nash-Moser permettent de montrer la régularité höldérienne des solutions de cette équation.

### 2.5.1 Résultats existants

En 1957, De Giorgi [39] a introduit des techniques pour montrer la régularité höldérienne des solutions d'équations elliptiques à coefficients peu réguliers. Comme détaillé en section 1.2, ce résultat lui a permis de résoudre le 19<sup>e</sup> problème de Hilbert. En 1958, Nash [86] a prouvé le résultat de régularité höldérienne pour des équations paraboliques avec des techniques différentes. En 1960, Moser [85] a quant à lui montré ces résultats avec une approche différente. Ces méthodes sont depuis appelées techniques de De Giorgi-Nash-Moser.

Il existe de nombreuses utilisations et extensions de ces techniques. Notamment dans des cas dégénérés, par exemple pour le  $p$ -Laplacien, il existe des extensions pour les équations elliptiques [76] puis paraboliques [41]. La méthode a également été étendue pour des opérateurs intégraux avec diffusion fractionnaire [26, 24]. On trouvera plus de détails dans le Chapitre 4 ou dans [97, 58].

Plus récemment, Golse, Imbert, Mouhot, Silvestre et Vasseur [58, 70] ont étendu ces techniques pour montrer la régularité höldérienne ainsi que des inégalités de Harnack pour des équations cinétiques.

Concernant plus particulièrement le lemme des valeurs intermédiaires étudié dans le Chapitre 4 qui est une des étapes clés de la méthode de De Giorgi, plusieurs versions quantitatives ont été obtenues dans le cas des équations elliptiques. De Giorgi [39, 40] a obtenu une version quantitative utilisant un argument d'inégalité isopérimétrique, reprise par DiBenedetto et Vasseur [42, 97]. Récemment, Hou et Niu [64] ont montré une version quantitative de ce lemme à l'aide d'une inégalité de Poincaré. Ces versions sont en réalité valables pour toute fonction dans  $H^1$ . À propos des équations paraboliques, il me semble qu'il n'existe pas de version quantitative. On trouve des versions non quantitatives [97], obtenues par l'absurde et avec un argument de compacité. Cependant il existe une version quantitative de ce lemme pour des équations d'évolution avec un opérateur non local intégral [24] mais qui ne s'applique pas aux équations paraboliques locales.

### 2.5.2 Contribution

Dans ce chapitre, nous obtenons deux versions quantitatives du lemme des valeurs intermédiaires pour des équations paraboliques. On montre une première version pour les solutions de l'équation (34) sans terme d'ordre inférieur (pour  $B = 0$  et  $s = 0$ ) à l'aide d'une inégalité de Poincaré obtenue également dans ce chapitre. On définit pour tout  $\rho > 0$ ,  $Q_\rho = (T_1, T_2) \times B_\rho$ .

**Théorème 2.8** (LVI : cas des solutions sans terme source). *Soit  $u$  une solution de (34) où  $B = 0$  et  $s = 0$  avec  $\Omega = B_{16R}$  telle que  $|u| \leq 1$ . Alors il existe  $\varepsilon > 0$  (dépendant uniquement de  $R, T_1, T_2, \lambda, \Lambda$  et  $d$ ) tel que pour tout  $(k, l) \in \mathbb{R}^2$  vérifiant  $k < l \leq 1$ , on a*

$$|\{f \leq k\} \cap Q_R| |\{f \geq l\} \cap Q_R|^2 \leq C_{l-k} |\{k < f < l\} \cap Q_{2R}|^{\frac{\varepsilon}{2+\varepsilon}},$$

où  $C_{l-k}$  est une constant dépendant uniquement de  $R, T_1, T_2, \lambda, \Lambda, l - k$  et  $d$ .

Une seconde version est obtenue pour les sous-solutions de (34) par une autre méthode qui utilise notamment une version quantitative du lemme des valeurs intermédiaires pour les fonctions  $H^1$ .

**Théorème 2.9** (Lemme des valeurs intermédiaires parabolique). *Soit  $u$  une sous-solution de (34) pour  $\Omega = B_{2R}$  telle que  $f \leq 1$ . Soit  $Q = (T_1, T_2) \times B_R$ ,  $\bar{T} = \frac{T_1 + T_2}{2}$ ,  $Q^- = (T_1, \bar{T}) \times B_R$  et  $Q^+ = (\bar{T}, T_2) \times B_R$ . Alors pour tout  $(k, l) \in \mathbb{R}^2$  tels que  $k < l \leq 1$ , on a*

$$C \frac{l-k}{2-k} |\{f \leq k\} \cap Q^-| |\{f \geq l\} \cap Q^+| \leq |\{k < f < l\} \cap Q|^{\frac{1}{20}},$$

où  $C$  est une constante qui ne dépend que de  $R, T_1, T_2, \lambda, \Lambda$  et  $d$ .

### 2.5.3 Perspectives

Concernant la méthode de De Giorgi pour obtenir un résultat de régularité, on pourra s'intéresser aux questions suivantes. On pourra essayer d'appliquer les idées des preuves des Théorèmes 2.8 et 2.9 pour trouver des résultats de lemmes des valeurs intermédiaires quantitatifs pour plusieurs problèmes,

- pour des équations de Hamilton-Jacobi d'ordre 1 [34],
- pour des équations de Hamilton-Jacobi d'ordre 2 [95],
- pour des équations d'évolutions avec un opérateur non local [26, 24],
- pour des équations cinétiques [58].



# Part I

## Hamilton-Jacobi equations



# Flux-limited solutions and state constraints for quasi-convex Hamilton-Jacobi equations in multidimensional domains

Recently, Imbert and Monneau have introduced the so-called flux-limited formulation of Hamilton-Jacobi equation with state constraint boundary conditions. When the spatial domain is a bounded interval, they proved that the latter formulation is equivalent to the more classical one which was originally introduced by H-M. Soner. In the present paper, we aim to prove the same result for a multidimensional spatial domain. More precisely, we give the proof for a general bounded domain of  $\mathbb{R}^d$  with a  $C^1$  boundary, in both the stationary and evolutive cases. In this setting, we also prove another result given by Imbert and Monneau in dimension one, namely that it is possible to use only a reduced class of test-functions.

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# 1 Introduction

## 1.1 Hamilton-Jacobi equation and state constraint problems

Hamilton-Jacobi equations with state constraint boundary conditions naturally appear in optimal control problems in which trajectories must remain in a closure of a domain  $\Omega$ . Introducing flux-limited solutions, Imbert and Monneau study a more general class of boundary conditions in [66] (respectively [67]) for one-dimensional (respectively multidimensional) spacial domains. They prove existence and uniqueness results for multi-junction domains for quasi-convex Hamiltonians. Recently Lions and Souganidis obtain a comparison principle for non-convex Hamiltonians in [82, 83].

Let us recall Soner's formulation of state constraint (SC) problems, see [93, 94]: a function  $u \in C(\bar{\Omega})$  is a viscosity solution of the SC problem if it satisfies, in the viscosity sense

$$\begin{cases} u + H(\nabla u) = 0 & \text{in } \Omega \\ u + H(\nabla u) \geq 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

This formulation does not seem to take into account the boundary behavior of  $u$  as a sub-solution. However, in [93], Soner proves the uniqueness of bounded uniformly continuous solutions of (1.1) for a convex Hamiltonian  $H$ . Capuzzo-Dolcetta and Lions also prove the uniqueness of continuous solutions of (1.1) in [31] for a Hamiltonian which is non necessarily convex.

In [72], also for a convex Hamiltonian  $H$ , Ishii and Koike show the uniqueness of possibly discontinuous solutions for the problem (1.1) with the following extra inequality for subsolutions,

$$\begin{cases} u + H(\nabla u) = 0 & \text{in } \Omega \\ u + H(\nabla u) \geq 0 & \text{on } \partial\Omega \\ u + H_{in}(\nabla u) \leq 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where  $H_{in}$  is an "inward Hamiltonian" [72].

In [66], for a quasi-convex and continuous coercive Hamiltonian  $H$  and  $\Omega$  a bounded interval, Imbert and Monneau proved that the SC problem (1.1) is equivalent to the flux-limited problem,

$$\begin{cases} u + H(\nabla u) = 0 & \text{in } \Omega \\ u + H^-(\nabla u) \geq 0 & \text{on } \partial\Omega \\ u + H^-(\nabla u) \leq 0 & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where  $H^-$  is the nonincreasing part of the Hamiltonian along the inward vector which will be defined in Section 2. The function  $H^-$  is the same as  $H_{in}$  in [72] if  $H$  is convex and coercive. Imbert and Monneau also proved the result for the evolution type equation. In this paper, we prove the same result for a general  $\mathcal{C}^1$  bounded open set  $\Omega$  of  $\mathbb{R}^d$ .

In some cases, there are links between those three problems. We have the direct following implication: a solution of (1.3) is also a solution of (1.2) which is also a solution of (1.1). When the spatial domain is an open interval and with the same assumptions on  $H$ , Imbert and Monneau show that a function is a solution of (1.1)

if and only if it is a solution of (1.3). We will prove the same result in the multidimensional setting. This result shows that the uniqueness obtained by Ishii and Koike for problem (1.2) implies the uniqueness of possibly discontinuous solutions of (1.1). However, this requires restrictive assumptions: a  $C^1$  domain, a coercive Hamiltonian and a weak continuity condition to be imposed to sub-solutions. The latter weak continuity assumption was already made in [22]. The main contribution of this article is two-fold: on the one hand, we can deal with equations of evolution type; on the other hand, the Hamiltonian is not necessarily convex.

There are other related works. For Hamilton-Jacobi equations posed on two domains separated by an interface where some transmission conditions are imposed, see [89, 90]. In [4, 2], the authors study an optimal control problem where the trajectories of the controlled system must remain in a network.

In this paper, we also prove a useful result about flux-limited solutions in the multidimensional setting: the class of test functions can be significantly reduced. Indeed, this new class contains only test functions with one slope in space. The proof is essentially based on ideas from [66, 67]. In particular it uses a property of critical slopes at the boundary for sub and supersolutions. To obtain this result for a multi-dimensional regular bounded open set, we introduce a local parametrization of the boundary. This result has many applications. For example, Proposition 2.16 in [66] allows to get weaker viscosity inequalities with test functions that do not need to be  $C^1$ . This is useful for proving convergence and error estimates for a numerical scheme in [62]. Also the important theorem of classification of effective boundary conditions, see [66, Theorem 1.1], is a consequence of the result concerning the use of reduced class of test-functions. The theorem of classification says that a general boundary condition is equivalent to a flux-limited one.

**Organization of the paper.** In Section 2, we give the definition of a flux-limited solution by introducing the non-increasing/non-decreasing part of a Hamiltonian along the normal direction on a boundary point. In Section 3, we prove the proposition reducing the class of test-functions. This allows us to prove the reformulation of the state constraint boundary conditions in Section 4. In Section 5, we provide simpler proofs in particular cases. We start with a direct proof for the general super-solution case, then we prove that the critical slope is finite in the stationary case. The reader will notice that the reformulation of the state constraint boundary conditions can be proven either from the result on the reduced class of test function, or more directly from the result of Section 5 about supersolutions, and the critical slope for sub-solution in Section 3.

## 1.2 Main theorem

In order to state our main theorem, let us first recall the notations of articles [66] and [67]. Let  $T$  be a positive number. Let  $\Omega$  be a bounded open set of  $\mathbb{R}^{d+1}$  with a boundary of class  $C^1$ .

The Hamiltonian  $H : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  is assumed to satisfy the following properties:

$$\begin{cases} \text{continuity,} \\ \text{coercivity,} & \lim_{|p| \rightarrow +\infty} H(p) = +\infty \\ \text{quasi-convexity,} & \forall \lambda \in \mathbb{R}, \quad \{H \leq \lambda\} \text{ is convex.} \end{cases} \quad (1.4)$$

We define  $H^- : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  (resp.  $H^+$ ) the nonincreasing (resp. nondecreasing) part of the Hamiltonian  $H$  by introducing some notations. Let  $x \in \partial\Omega$ ,  $p \in \mathbb{R}^{d+1}$  and  $n_x$  be the outward normal vector at  $x$ . The component of  $p$  in the tangent plane to  $\partial\Omega$  at  $x$  is denoted by  $p'$  and the component along  $-n_x = N_x$  is denoted by  $p_N$ , so  $p_N = p \cdot N_x$  and  $p = (p', p_N) \in \mathbb{R}^d \times \mathbb{R}$ . In the same way, the tangential component of  $\nabla u$  is denoted by  $\nabla' u$  and we set  $\partial_N u := \nabla u \cdot N_x$ . Let  $H_x : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  be the Hamiltonian defined with these components, depending on  $x \in \partial\Omega$ . Let  $\pi^0(p') \in \mathbb{R}$  be such that

$$\min_{p_N \in \mathbb{R}} H_x(p', p_N) = H_x(p', \pi^0(p')).$$

*Remark 1.1.* The number  $\pi^0(p')$  may not be unique.

The nonincreasing part of  $H_x$  is defined by

$$H_x^-(p', p_N) = \begin{cases} H_x(p', p_N) & \text{if } p_N \leq \pi^0(p') \\ H_x(p', \pi^0(p')) & \text{otherwise,} \end{cases}$$

and the nondecreasing part of  $H_x$  is defined by

$$H_x^+(p', p_N) = \begin{cases} H_x(p', p_N) & \text{if } p_N \geq \pi^0(p') \\ H_x(p', \pi^0(p')) & \text{otherwise.} \end{cases}$$

*Remark 1.2.* We can show that  $H_x$  does depend continuously on  $x \in \partial\Omega$ , using the fact that the boundary is of class  $\mathcal{C}^1$ . To simplify the notations, in the following we will use  $H$ ,  $H^-$  and  $H^+$  instead of  $H_x$ ,  $H_x^-$  and  $H_x^+$ , but we will keep in mind that  $H^+$ ,  $H^-$  and  $H$  depends on  $x \in \partial\Omega$ .

Here we deal with discontinuous viscosity solutions, introduced by Ishii and Koike in [72]. We recall the definitions in Section 2. The result holds in the case where subsolutions of the state constraint problem are “weakly continuous” at the boundary i.e.,

$$\forall (t, x) \in (0, T) \times \partial\Omega \quad u^*(t, x) = \limsup_{y \rightarrow x, s \rightarrow t, y \notin \partial\Omega} u^*(s, y), \quad (1.5)$$

in which  $u^*$  is defined in Section 2. Let us state our main result.

**Theorem 1.3** (Reformulation of the state constraints boundary conditions, evolutive problem).

Assume that  $H : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  satisfies (1.4), and  $u : (0, T) \times \bar{\Omega} \rightarrow \mathbb{R}$ . Then the function  $u$  is a viscosity solution of

$$\begin{cases} u_t + H(\nabla u) \geq 0 & \text{in } (0, T) \times \bar{\Omega} \\ u_t + H(\nabla u) \leq 0 & \text{in } (0, T) \times \Omega, \end{cases} \quad (1.6)$$

and satisfies (1.5) if and only if  $u$  is a viscosity solution of the flux-limited problem

$$\begin{cases} u_t + H(\nabla u) = 0 & \text{in } (0, T) \times \Omega \\ u_t + H^-(\nabla u) = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad (1.7)$$

where  $H^-$  is the decreasing part of the Hamiltonian along the inward normal vector.

*Remark 1.4.* If  $u$  is a sub-solution of (1.7) then  $u$  satisfies the “weak continuity” (1.5), see [66, Lemma 2.3] with  $F_{A_0} = H^-$ . But if  $u$  is a sub-solution of (1.6) then  $u$  does not necessarily satisfy the “weak continuity”. In fact, without the hypothesis of “weak continuity” (1.5) for sub-solutions, the theorems does not hold. Indeed, let  $x_0 \in \partial\Omega$ , the function

$$u(t, x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{elsewhere,} \end{cases}$$

satisfies (1.6) but does not satisfy (1.7) for  $H(p) = |p|$ .

Using a similar proof we can show the following stationary counterpart of the previous theorem.

**Theorem 1.5** (Reformulation of the state constraints boundary conditions, stationary problem).

Assume that  $H : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  satisfies (1.4) and  $u : \bar{\Omega} \rightarrow \mathbb{R}$ . Then the function  $u$  is a viscosity solution of

$$\begin{cases} u + H(\nabla u) \geq 0 & \text{in } \bar{\Omega} \\ u + H(\nabla u) \leq 0 & \text{in } \Omega, \end{cases} \quad (1.8)$$

and satisfies (1.5) if and only if  $u$  is a viscosity solution of the flux-limited problem

$$\begin{cases} u + H(\nabla u) = 0 & \text{in } \Omega \\ u + H^-(\nabla u) = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.9)$$

*Remark 1.6.* As in Remark 1.4, if  $u$  is a sub-solution of (1.9) then  $u$  satisfies (1.5).

## 2 Definition of flux-limited solutions

We call flux-limited solutions the viscosity solutions of the flux-limited problem. Let us recall the definition of flux-limited solutions given in [66] and used hereafter. The class of test-functions that we use is  $\mathcal{C}^1((0, T) \times \bar{\Omega})$ . The so called flux-limiter function  $A : \mathbb{R}^d \rightarrow \mathbb{R}$  will be continuous and quasi-convex. We define  $F_A$  by

$$F_A(p) = \max(A(p'), H^-(p', p_N)).$$

Note that  $F_A$  depends on  $x \in \partial\Omega$ . We set

$$A_0(p') = \min_{p_N \in \mathbb{R}} H(p', p_N).$$

We recall the definition of the upper and lower semi-continuous envelopes  $u^*$  and  $u_*$  of a (locally bounded) function  $u$  defined on  $[0, T) \times \bar{\Omega}$ ,

$$u^*(t, x) = \limsup_{(s, y) \rightarrow (t, x)} u(s, y) \quad \text{and} \quad u_*(t, x) = \liminf_{(s, y) \rightarrow (t, x)} u(s, y).$$

**Definition 2.1** (Viscosity solutions). *Let  $H$  satisfy (1.4) and  $u : (0, T) \times \bar{\Omega} \rightarrow \mathbb{R}$  be locally bounded. We say that  $u$  is a viscosity sub-solution (resp. viscosity super-solution) at  $(t_0, x_0) \in (0, T) \times \bar{\Omega}$ , of*

$$u_t + H(\nabla u) = 0,$$

*if for all  $\varphi \in C^1((0, T) \times \bar{\Omega})$  such that  $u^* \leq \varphi$  (resp.  $u_* \geq \varphi$ ) in a neighborhood of  $(t_0, x_0) \in (0, T) \times \bar{\Omega}$  with equality at  $(t_0, x_0)$  (we say that  $\varphi$  touch  $u^*$  from above (resp.  $u_*$  from below) in  $(t_0, x_0)$ ), we have*

$$\varphi_t + H(\nabla \varphi) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{at } (t_0, x_0).$$

*We also say that  $u$  satisfies in the viscosity sense at  $(t_0, x_0) \in (0, T) \times \bar{\Omega}$ ,*

$$u_t + H(\nabla u) \leq 0, \quad (\text{resp. } \geq 0).$$

*Moreover, we say that  $u$  is a viscosity solution at  $(t_0, x_0) \in (0, T) \times \bar{\Omega}$ , of*

$$u_t + H(\nabla u) = 0,$$

*if  $u$  is both a viscosity sub and super-solution at  $(t_0, x_0)$ .*

**Definition 2.2** (Flux-limited solutions). *Let  $H$  satisfy (1.4),  $u : (0, T) \times \bar{\Omega} \rightarrow \mathbb{R}$  be locally bounded and  $A : \mathbb{R}^d \rightarrow \mathbb{R}$  be a continuous flux-limiter function such that*

$$\forall p' \in \mathbb{R}^d, \quad A(p') \geq A_0(p'). \quad (1.10)$$

*We say that  $u$  is a  $A$ -flux limited sub-solution (resp.  $A$ -flux limited super-solution) in  $(0, T) \times \bar{\Omega}$  if  $u$  is a viscosity sub-solution (resp. super-solution) of*

$$\begin{cases} u_t + H(\nabla u) = 0 & \text{in } (0, T) \times \Omega \\ u_t + F_A(\nabla u) = 0 & \text{on } (0, T) \times \partial\Omega. \end{cases} \quad (1.11)$$

*Moreover, we say that  $u$  is a  $A$ -flux limited solution of (1.11) if  $u$  is both a  $A$ -flux limited sub-solution and super-solution.*

As it is proven in [66, Lemma 2.3],  $A$ -flux limited sub-solutions always satisfy

$$\forall (t, x) \in (0, T) \times \partial\Omega \quad u^*(t, x) = \limsup_{y \rightarrow x, s \rightarrow t, y \notin \partial\Omega} u^*(s, y). \quad (1.12)$$

*Remark 2.3.* We define stationary solutions by replacing  $u_t$  and  $\phi_t$  by  $u$  and dropping the time dependency.

### 3 A reduced class of test-functions in the case of a $C^1$ domain

We give the results and proofs only in the evolutive case, since the stationary case is treated in a completely similar way.

We define  $\pi^+, \pi^- : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  as in [67] by

$$\pi^+(p', \lambda) = \sup\{p \in \mathbb{R} : H(p', p) = H^+(p', p) = \lambda\},$$

and

$$\pi^-(p', \lambda) = \inf\{p \in \mathbb{R} : H(p', p) = H^-(p', p) = \lambda\}.$$

Let  $X_0 \in \partial\Omega$  be fixed. For all  $X \in \bar{\Omega}$ ,  $X = (X', x^N)$  where  $x^N$  is the component on  $N_{X_0}$  and  $X'$  is the tangential component. Let us give some properties of the functions  $H^-, H^+, \pi^-, \pi^+$  and  $F_A$  which are useful for the following proofs.

**Lemma 3.1.** *Let  $H$  satisfy (1.4) and  $A : \mathbb{R}^d \rightarrow \mathbb{R}$  be a continuous flux-limiter function satisfying (1.10). We have the following properties.*

1.  $H^-$  (resp.  $H^+$ ) is nonincreasing (resp. nondecreasing).
2.  $\pi^-(p', \lambda) \leq \pi^+(p', \lambda)$  for  $p' \in \mathbb{R}^d$  and  $\lambda \geq A_0(p')$ .
3.  $H^\pm(p', \pi^\pm(p', \lambda)) = H(p', \pi^\pm(p', \lambda)) = \lambda$  for  $p' \in \mathbb{R}^d$  and  $\lambda \geq A_0(p')$ .
4.  $H^\pm(p', \pi^\mp(p', \lambda)) = A_0(p')$  for  $p' \in \mathbb{R}^d$  and  $\lambda \geq A_0(p')$ .
5.  $H(p', p_N) \leq A(p') \iff \pi^-(p', A(p')) \leq p_N \leq \pi^+(p', A(p'))$ .
6.  $p_N \rightarrow F_A(p', p_N)$  is nonincreasing for  $p' \in \mathbb{R}^d$ .
7.  $F_{A_0} = H^-$ .
8.  $F_A(p', p_N) = A(p')$  for  $p_N \geq \pi^-(p', A(p'))$ .

*Proof.* The proofs of these properties are direct consequences of the definitions.  $\square$

Following [66, 67] closely, let us give an equivalent definition of viscosity solutions which uses a reduced class of test-functions. In [67], the authors focus on a half-space. We prove the result for a general bounded domain with a  $C^1$  boundary.

**Definition 3.2** (Equivalent definition of viscosity solutions). *Let  $H$  satisfy (1.4) and  $A : \mathbb{R}^d \rightarrow \mathbb{R}$  be a continuous flux-limiter function satisfying (1.10). Let  $u : (0, T) \times \bar{\Omega} \rightarrow \mathbb{R}$  be locally bounded.*

1. *We say that  $u$  is a reduced sub-solution of (1.11) in  $(0, T) \times \bar{\Omega}$  if and only if  $u^*$  satisfies (1.5) and  $u$  is a sub-solution in  $(0, T) \times \Omega$ , and  $\forall \varphi \in C^1((0, T) \times \bar{\Omega})$  such that  $u^* \leq \varphi$  in a neighborhood of  $(t_0, X_0)$  in  $(0, T) \times \partial\Omega$ , with equality at  $(t_0, X_0)$ , of the following form*

$$\varphi(t, X', x^N) = \phi(t, X') + \phi_0(x^N), \quad (1.13)$$

where

$$\begin{cases} \phi \in C^1((0, T) \times \mathbb{R}^d) \\ D'\phi(t_0, X'_0) = p'_0, \end{cases}$$

and

$$\begin{cases} \phi_0 \in C^1(\mathbb{R}) \\ \phi'_0(x_0^N) = \pi^+(p'_0, A(p'_0)), \end{cases}$$

we have

$$\varphi_t + F_A(\nabla\varphi) \leq 0 \quad \text{at } (t_0, X_0).$$

2. We say that  $u$  is a reduced super-solution of (1.11) in  $(0, T) \times \bar{\Omega}$  if and only if  $u$  is a super-solution in  $(0, T) \times \Omega$ , and  $\forall \varphi \in \mathcal{C}^1((0, T) \times \bar{\Omega})$  such that  $u_* \geq \varphi$  in a neighborhood of  $(t_0, X_0)$  in  $(0, T) \times \partial\Omega$ , with equality at  $(t_0, X_0)$ , satisfying (1.13), we have

$$\varphi_t + F_A(\nabla\varphi) \geq 0 \quad \text{at } (t_0, X_0).$$

**Proposition 3.3.** *a) Definitions 2.2 and 3.2 are equivalent. Precisely, we have the two following equivalences.*

- *A function which satisfies (1.5) is a reduced sub-solution of (1.11) if and only if it is a viscosity sub-solution of (1.11).*
- *A function is a reduced super-solution of (1.11) if and only if it is a viscosity super-solution of (1.11).*

*b) If  $u$  is a sub-solution of  $u_t + H(\nabla u) \leq 0$  in  $(0, T) \times \Omega$  satisfying (1.5), then it is a  $A_0$ -flux limited sub-solution.*

To prove this proposition, let us first recall the lemma taken from [66, 67] on the critical slope for sub and super-solutions. To this purpose, let us define a local parametrization of the smooth domain.

Let us define the function  $\psi^N$  on a neighborhood of a fixed point  $X_0 \in \partial\Omega$ , which gives the component along the inward normal vector  $N_{X_0}$  of the projection on the boundary of a point in  $\bar{\Omega}$  (see Figure 1.1).

**Definition 3.4** (Function  $\psi^N$ ). *Let  $X_0$  be on  $\partial\Omega$ . Since  $\Omega$  has a  $\mathcal{C}^1$  boundary, there exists  $r_0 > 0$ ,  $\omega$  an open set of  $T_{X_0}$  the tangent plane at  $X_0$ , and  $\psi^N \in \mathcal{C}^1(\omega)$  such that*

$$\partial\Omega \cap \bar{B}_{r_0}(X_0) = \{(X', x^N) \in \omega \times \mathbb{R}, \quad x^N = \psi^N(X')\}.$$

*Remark 3.5.* Notice that  $\nabla' \psi^N(X'_0) = 0$ .

**Lemma 3.6** (Critical slope for sub-solution). *Let  $u$  be an upper semi-continuous sub-solution of  $u_t + H(\nabla u) = 0$  in  $(0, T) \times \Omega$  which satisfies (1.5), and let  $\varphi$  be a test function touching  $u$  from above at some point  $(t_0, X_0)$ , where  $t_0 \in (0, T)$  and  $X_0 = (X'_0, x_0^N) \in \partial\Omega$ . Let  $r_0$  be given by Definition 3.4. Set*

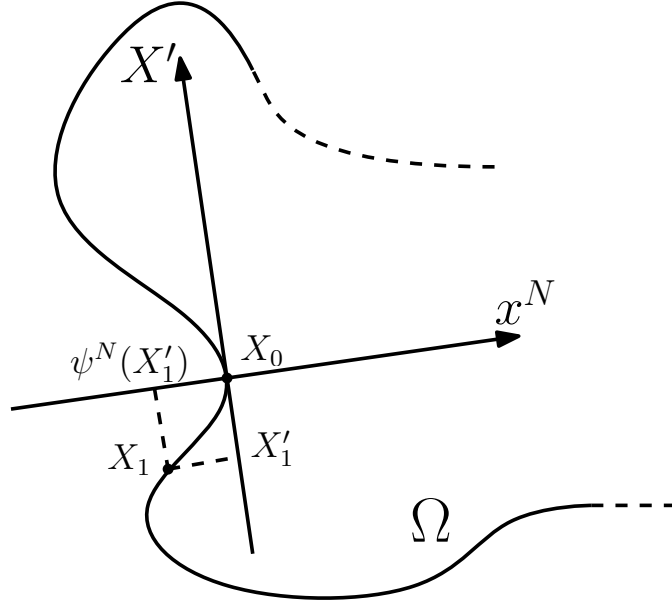
$$B_{t_0, X_0} := \left\{ p \in \mathbb{R} : \exists r \in (0, r_0], \quad \begin{array}{l} \varphi(t, X) + p(x^N - \psi^N(X')) \geq u(t, X), \\ \forall (t, X) \in (t_0 - r, t_0 + r) \times \bar{\Omega} \cap \bar{B}_r(X_0) \end{array} \right\}.$$

*Then the critical slope given by*

$$\bar{p} = \inf B_{t_0, X_0}$$

*is finite, satisfies  $\bar{p} \leq 0$  and*

$$\varphi_t(t_0, X_0) + H(\nabla' \varphi(t_0, X_0), \partial_N \varphi(t_0, X_0) + \bar{p}) \leq 0.$$

Figure 1.1: Function  $\psi^N$  in Definition 3.4

*Proof.* Following the same arguments as in [66], let us show that  $\bar{p}$  is finite in this case.

Let us take any  $p$  belonging to  $B_{t_0, X_0}$ . This implies that there exists  $r > 0$  such that for all  $(t, X) \in D = (t_0 - r, t_0 + r) \times \bar{\Omega} \cap \bar{B}_r(X_0)$ ,

$$u(t, X) \leq \varphi(t, X) + p(x^N - \psi^N(X')).$$

Notice that, replacing  $\varphi$  with  $\varphi + (t - t_0)^2 + \|X - X_0\|^2$  if necessary, we can assume that if  $(t, X) \neq (t_0, X_0)$ , then  $u(t, X) < \varphi(t, X) + p(x^N - \psi^N(X'))$ . In particular, there exists  $\delta > 0$  such that on  $(t_0 - r, t_0 + r) \times \partial B_r(X_0) \cap \bar{\Omega}$ , we have

$$u(t, X) + \delta \leq \varphi(t, X) + p(x^N - \psi^N(X')). \quad (1.14)$$

Since  $u$  satisfies (1.5), there exists  $(t_\epsilon, X_\epsilon) \rightarrow (t_0, X_0)$  such that  $X_\epsilon \notin \partial\Omega$  and  $u(t_0, X_0) = \lim_{\epsilon \rightarrow 0} u(t_\epsilon, X_\epsilon)$ . Let us introduce now the following perturbed test function

$$\Psi(t, X) = \varphi(t, X) + p(x^N - \psi^N(X')) + \epsilon \frac{x_\epsilon^N - \psi^N(X'_\epsilon)}{x^N - \psi^N(X')}.$$

Let  $(s_\epsilon, Y_\epsilon)$  achieve the infimum of  $\Psi - u$  in  $\bar{D}$ . Using the definition of  $\Psi$ , we see that  $Y_\epsilon \notin \partial\Omega$ . Specifically,

$$\varphi(s_\epsilon, Y_\epsilon) + p(y_\epsilon^N - \psi^N(Y'_\epsilon)) - u(s_\epsilon, Y_\epsilon) \leq \Psi(s_\epsilon, Y_\epsilon) - u(s_\epsilon, Y_\epsilon) \leq \Psi(t_\epsilon, X_\epsilon) - u(t_\epsilon, X_\epsilon). \quad (1.15)$$

The last term of the inequality (1.15) goes to 0 when  $\epsilon$  goes to 0. Using (1.14), this implies that  $(s_\epsilon, Y_\epsilon) \rightarrow (t_0, X_0)$ . Since  $\Psi$  is a test function at  $(s_\epsilon, Y_\epsilon)$ , we have



$$\varphi_t(s_\epsilon, Y_\epsilon) + H \left( \nabla \varphi(s_\epsilon, Y_\epsilon) + \begin{pmatrix} -p \nabla' \psi^N(Y'_\epsilon) \\ p \end{pmatrix} - \epsilon \frac{x_\epsilon^N - \psi^N(X'_\epsilon)}{(y_\epsilon^N - \psi^N(Y'_\epsilon))^2} \begin{pmatrix} -\nabla' \psi^N(Y'_\epsilon) \\ 1 \end{pmatrix} \right) \leq 0.$$

We can pass to the limit in the viscosity inequality and find

$$\varphi_t(t_0, X_0) + H \left( \nabla \varphi(t_0, X_0) + \begin{pmatrix} 0 \\ p \end{pmatrix} - \alpha_p \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \leq 0,$$

where

$$\alpha_p = \limsup_{\epsilon \rightarrow 0} \epsilon \frac{x_\epsilon^N - \psi^N(X'_\epsilon)}{(y_\epsilon^N - \psi^N(Y'_\epsilon))^2} \in [0, +\infty].$$

The previous inequality and the fact that  $H$  is coercive implies that  $p - \alpha_p$  is finite, so that  $p$  is bounded since  $\varphi_t(t_0, X_0)$  and  $\nabla \varphi(t_0, X_0)$  are fixed. Thus  $\bar{p}$  is finite.

Now let us prove the second part of the lemma. By definition of  $\bar{p}$ , for all  $\epsilon > 0$  small enough, there exists  $\delta = \delta(\epsilon) \in (0, \epsilon)$  such that  $\partial B_\delta(X_0)$  has exactly two intersection points with  $\delta\Omega$  and such that for all  $(t, X) \in (t_0 - \delta, t_0 + \delta) \times \bar{B}_\delta(X_0) \cap \bar{\Omega}$ ,

$$u(t, X) \leq \varphi(t, X) + (\bar{p} + \epsilon)(x^N - \psi^N(X')), \quad (1.16)$$

and there exists  $(t_\epsilon, X_\epsilon) \in (t_0 - \frac{\delta}{2}, t_0 + \frac{\delta}{2}) \times B_{\frac{\delta}{2}}(X_0)$  such that  $X_\epsilon \notin \partial\Omega$  and

$$u(t_\epsilon, X_\epsilon) > \varphi(t_\epsilon, X_\epsilon) + (\bar{p} - \epsilon)(x_\epsilon^N - \psi^N(X'_\epsilon)).$$

Let  $\Psi : \mathbb{R}^{d+1} \rightarrow [0, 1]$  be a function of class  $\mathcal{C}^\infty$  such that

$$\Psi = \begin{cases} 0 & \text{in } B_{\frac{1}{2}}(0) \\ 1 & \text{outside } B_1(0). \end{cases}$$

We define

$$\Phi(t, X) = \varphi(t, X) + 4\epsilon\delta\Psi \left( \frac{X - X_0}{\delta} \right) + (\bar{p} - \epsilon)(x^N - \psi^N(X')).$$

Thus we have on  $(t_0 - \delta, t_0 + \delta) \times \partial(B_\delta(X_0) \cap \bar{\Omega})$ ,

$$\Phi(t, X) = \begin{cases} \varphi(t, X) + 4\epsilon\delta + (\bar{p} - \epsilon)(x^N - \psi^N(X')) & \text{if } X \in \partial B_\delta(X_0) \\ \varphi(t, X) + 4\epsilon\delta\Psi \left( \frac{X - X_0}{\delta} \right) & \text{if } X \in \partial\Omega. \end{cases}$$

For  $X \in \partial B_\delta(X_0)$ , we have  $|x^N - \psi^N(X')| \leq 2\delta$  so that using (1.16) we deduce

$$\Phi(t, X) \geq u(t, X).$$

Then in all cases, on  $(t_0 - \delta, t_0 + \delta) \times \partial(B_\delta(X_0) \cap \bar{\Omega})$ , there holds

$$\Phi(t, X) \geq u(t, X).$$

We also have

$$\Phi(t_0, X_0) = \varphi(t_0, X_0) = u(t_0, X_0),$$

and

$$\Phi(t_\epsilon, X_\epsilon) = \varphi(t_\epsilon, X_\epsilon) + (\bar{p} - \epsilon)(x_\epsilon^N - \psi^N(X'_\epsilon)) < u(t_\epsilon, X_\epsilon).$$

Thus  $u - \Phi$  reaches his maximum in the interior of  $(t_0 - \delta, t_0 + \delta) \times B_\delta(X_0) \cap \bar{\Omega}$  at some point  $(\bar{t}_\epsilon, \bar{X}_\epsilon)$ . Therefore, since  $\Phi$  is an admissible test function, we find

$$\Phi_t(\bar{t}_\epsilon, \bar{X}_\epsilon) + H(\nabla\Phi(\bar{t}_\epsilon, \bar{X}_\epsilon)) \leq 0,$$

which gives

$$\varphi_t(\bar{t}_\epsilon, \bar{X}_\epsilon) + H\left(\nabla\varphi(\bar{t}_\epsilon, \bar{X}_\epsilon) + 4\epsilon\nabla\Psi\left(\frac{X_\epsilon - X_0}{\delta}\right) + \begin{pmatrix} -(\bar{p} - \epsilon)\nabla'\psi^N(X'_\epsilon) \\ \bar{p} - \epsilon \end{pmatrix}\right) \leq 0.$$

When  $\epsilon$  goes to 0 we deduce that, since  $\nabla'\psi^N(X'_0) = 0$ ,

$$\varphi_t(t_0, X_0) + H(\nabla'\varphi(t_0, X_0), \partial_N\varphi(t_0, X_0) + \bar{p}) \leq 0.$$

□

For the super-solutions, we have a similar lemma, with a similar proof where the fact that the critical slope is finite is now an hypothesis and not a conclusion.

**Lemma 3.7** (Critical slope for super-solution). *Let  $u$  be a lower semi-continuous super-solution of  $u_t + H(\nabla u) = 0$  in  $(0, T) \times \Omega$ , and let  $\varphi$  be a test function touching  $u$  from below at some point  $(t_0, X_0)$ , where  $t_0 \in (0, T)$  and  $X_0 = (X'_0, x_0^N) \in \partial\Omega$ . Let  $r_0$  be given by Definition 3.4. If the critical slope given by*

$$\bar{p} = \sup \left\{ p \in \mathbb{R} : \exists r \in (0, r_0], \quad \begin{array}{l} \varphi(t, X) + p(x^N - \psi^N(X')) \leq u(t, X), \\ \forall (t, X) \in (t_0 - r, t_0 + r) \times \bar{\Omega} \cap \bar{B}_r(X_0) \end{array} \right\}$$

is finite, then it satisfies  $\bar{p} \geq 0$  and

$$\varphi_t(t_0, X_0) + H(\nabla'\varphi(t_0, X_0), \partial_N\varphi(t_0, X_0) + \bar{p}) \geq 0.$$

Now let us prove Proposition 3.3.

*Proof of Proposition 3.3.* In both cases we only need to prove the direct implication since the reverse is obvious. We first prove the results on sub-solutions, and then turn to super-solution.

**Sub-solution.** Let  $u$  be a reduced viscosity sub-solution, as defined in Definition 3.2. Let  $\phi$  be a test function touching  $u^*$  from above at  $(t_0, X_0) \in (0, T) \times \partial\Omega$ , and let  $\lambda = -\phi_t(t_0, X_0)$ . Let  $p' = \nabla'\phi(t_0, X_0)$  and  $p_N = \partial_N\phi(t_0, X_0)$ . We want to show that

$$F_A(p', p_N) \leq \lambda. \tag{1.17}$$

Notice that by Lemma 3.6, there exists  $\bar{p} \leq 0$  such that

$$H(p', p_N + \bar{p}) \leq \lambda,$$

so  $A_0(p') \leq \lambda$ . Since  $H^-$  is non-increasing,

$$\begin{aligned} F_A(p', p_N) &= \max(A(p'), H^-(p', p_N)) \\ &\leq \max(A(p'), H^-(p', p_N + \bar{p})) \\ &\leq \max(A(p'), H(p', p_N + \bar{p})) \\ &\leq \max(A(p'), \lambda). \end{aligned}$$

If  $A = A_0$ , then  $F_A(p', p_N) \leq \lambda$ , so b) is proven. Assume now that  $F_A(p', p_N) > \lambda$ , by contradiction, then we have  $A_0(p') \leq \lambda < A(p')$  and since

$$H(p', p_N + \bar{p}) \leq \lambda < A(p'),$$

then

$$p_N + \bar{p} < \pi^+(p', A(p')).$$

Let us consider the modified test function

$$\varphi(t, X', x^N) = \phi(t, X', \psi^N(X')) - \phi_0(\psi^N(X')) + \phi_0(x^N),$$

where  $\phi_0$  satisfies

$$\begin{cases} \phi_0 \in \mathcal{C}^1(\mathbb{R}) \\ \phi_0'(x_0^N) = \pi^+(p', A(p')). \end{cases}$$

There holds

$$\varphi(t_0, X_0) = \phi(t_0, X_0) = u^*(t_0, X_0).$$

Let us show that

$$\varphi(t, X', x^N) \geq u^*(t, X', x^N), \quad (1.18)$$

on a neighborhood of  $(t_0, X_0)$ . We have

$$\partial_N \phi(t_0, X_0) + \bar{p} < \phi_0'(x_0^N),$$

so there exists  $p_1$  and  $p_2$  such that  $\bar{p} < p_1 < p_2$ , and which satisfy

$$\partial_N \phi(t_0, X_0) + p_i < \phi_0'(x_0^N), \quad \forall i \in \{1, 2\}.$$

As  $\partial_N \phi$  and  $\phi_0'$  are continuous, on a neighborhood of  $(t_0, X_0)$ , we have

$$\partial_N \phi(t, X) + p_i < \phi_0'(x^N), \quad \forall i \in \{1, 2\}.$$

Therefore

$$\begin{aligned} \phi(t, X', x^N) &= \phi(t, X', \psi^N(X')) + \int_{\psi^N(X')}^{x^N} \partial_N \phi(t, X', s) ds \\ &= \varphi(t, X', x^N) + \phi_0(\psi^N(X')) - \phi_0(x^N) + \int_{\psi^N(X')}^{x^N} \partial_N \phi(t, X', s) ds \\ &= \varphi(t, X', x^N) + \int_{\psi^N(X')}^{x^N} (\partial_N \phi(t, X', s) - \phi_0'(s)) ds \\ &\leq \varphi(t, X', x^N) - p_2(x^N - \psi^N(X')), \end{aligned}$$

and by definition of  $\bar{p}$ , there exists a neighborhood  $(t_0 - r, t_0 + r) \times \bar{\Omega} \cap B_r(X_0)$  of  $(t_0, X_0)$ , for some  $r > 0$  such that

$$\begin{aligned} u^*(t, X', x^N) &\leq \phi(t, X', x^N) + p_1(x^N - \psi^N(X')) \\ &\leq \varphi(t, X', x^N) + (p_1 - p_2)(x^N - \psi^N(X')), \end{aligned}$$

so that we find (1.18).

By the definition of reduced sub-solutions, and using that

$$H^-(p', \pi^+(p', A(p'))) = A_0(p') \leq A(p'),$$

we deduce

$$A(p') = F_A(p', \pi^+(p', A(p'))) \leq \lambda.$$

This gives a contradiction. Therefore (1.17) holds true. Let us prove now the super-solution case.

**Super-solution.** Let  $u$  be a reduced viscosity super-solution, as defined in Definition 3.2. Let  $\phi$  be a test function touching  $u_*$  from below at  $(t_0, X_0) \in (0, T) \times \partial\Omega$ . Let  $\lambda = -\phi_t(t_0, X_0)$ . Let  $p' = \nabla' \phi(t_0, X_0)$  and  $p_N = \partial_N \phi(t_0, X_0)$ . We want to show that

$$F_A(p', p_N) \geq \lambda. \quad (1.19)$$

By Lemma 3.7, and since  $H$  is coercive, there exists  $\bar{p}$  finite such that  $\bar{p} \geq 0$  and

$$H(p', p_N + \bar{p}) \geq \lambda. \quad (1.20)$$

Notice that (1.19) holds true if

$$\lambda \leq A(p') \quad \text{or} \quad H^-(p', p_N + \bar{p}) = H(p', p_N + \bar{p}).$$

Assume by contradiction that the inequality (1.19) is not satisfied. Then we have

$$A_0(p') \leq A(p') < \lambda \leq H^+(p', p_N + \bar{p}). \quad (1.21)$$

Since  $H^-(p', p_N + \bar{p}) < H(p', p_N + \bar{p})$ , we deduce in particular that

$$p_N + \bar{p} > \pi^+(p', A(p')) = \phi'_0(x_0^N),$$

because if

$$p_N + \bar{p} \leq \pi^+(p', A(p')),$$

holds true, then

$$\lambda \leq H^+(p', p_N + \bar{p}) \leq H^+(p', \pi^+(p', A(p'))) = A(p'),$$

which is in contradiction with (1.21). As before, with the same proof that in the sub-solution case, we show that the function

$$\varphi(t, X', x^N) = \phi(t, X', \psi^N(X')) - \phi_0(\psi^N(X')) + \phi_0(x^N),$$

is a test function touching  $u$  from below at  $(t_0, X_0)$ , where  $\phi_0$  satisfies

$$\begin{cases} \phi_0 \in \mathcal{C}^1(\mathbb{R}) \\ \phi_0'(x_0^N) = \pi^+(p', A(p')). \end{cases}$$

By definition of reduced super-solutions,

$$A(p') = F_A(p', \pi^+(p', A(p'))) \geq \lambda,$$

which gives a contradiction with (1.21). Therefore (1.19) holds true.  $\square$

We can now prove Theorem 1.3.

## 4 Proof of Theorem 1.3

Proposition 3.3 allows us to reduce the set of test functions. Now let us prove the theorem using the same ideas as in [66].

*Proof of Theorem 1.3.* We do the proof in three steps. In the first two steps, we prove that (1.6) implies (1.7).

**1st step:** Let us prove that

$$u_t + H(\nabla u) \leq 0 \quad \text{in } (0, T) \times \Omega,$$

implies

$$u_t + H^-(\nabla u) \leq 0 \quad \text{on } (0, T) \times \partial\Omega.$$

Using b) of Proposition 3.3,  $u$  is an  $A_0$ -flux limited sub-solution, so that

$$u_t + F_{A_0}(\nabla u) \leq 0 \quad \text{on } (0, T) \times \partial\Omega.$$

Since  $F_{A_0}(\nabla u) = H^-(\nabla u)$ , we have

$$u_t + H^-(\nabla u) \leq 0 \quad \text{on } (0, T) \times \partial\Omega.$$

**2nd step:** Let us prove that

$$u_t + H(\nabla u) \geq 0 \quad \text{in } (0, T) \times \bar{\Omega},$$

implies

$$u_t + H^-(\nabla u) \geq 0 \quad \text{on } (0, T) \times \partial\Omega.$$

Let  $\varphi$  be a test function touching  $u_*$  from below at  $(t_0, X_0) \in (0, T) \times \partial\Omega$ . Using Proposition 3.3, we assume that

$$\varphi(t, X) = \phi(t, X') + \phi_0(x^N),$$

where

$$\phi \in \mathcal{C}^1((0, T) \times \mathbb{R}^d), \quad \nabla' \phi(t_0, X'_0) = p'_0,$$

and

$$\phi_0 \in C^1(\mathbb{R}), \quad \phi'_0(x_0^N) = \pi^+(p'_0, A_0(p'_0)).$$

As

$$H(\nabla\varphi) = H^+(\nabla\varphi) = A_0(p'_0) = H^-(\nabla\varphi),$$

and as by hypothesis  $\varphi_t + H(\nabla\varphi) \geq 0$ , we deduce

$$\varphi_t + H^-(\nabla\varphi) \geq 0.$$

**3rd step:** In this step we prove that (1.7) implies (1.6). We only need to prove that

$$u_t + H(\nabla u) \geq 0 \quad \text{in } (0, T) \times \partial\Omega. \quad (1.22)$$

By assumption we have

$$u_t + H^-(\nabla u) \geq 0 \quad \text{in } (0, T) \times \partial\Omega.$$

Since  $H^- \leq H$ , we deduce (1.22).  $\square$

## 5 Simpler proofs in particular cases

We give some direct proofs in particular cases without relying on Proposition 3.3.

### 5.1 Multi-dimensional case for super-solutions

Let us give a direct proof of the following theorem which is a part of the general reformulation of state constraints theorem.

**Theorem 5.1.** *If  $u$  satisfies*

$$u_t + H(\nabla u) \geq 0 \quad \text{on } (0, T) \times \partial\Omega$$

*then*

$$u_t + H^-(\nabla u) \geq 0 \quad \text{on } (0, T) \times \partial\Omega.$$

*Proof.* Let  $\varphi$  be a test function touching  $u_*$  from below at  $(t_0, X_0) \in (0, T) \times \partial\Omega$ , so that

$$\varphi_t + H(\nabla'\varphi, \partial_N\varphi) \geq 0 \quad \text{at } (t_0, X_0),$$

and let  $\lambda = -\varphi_t(t_0, X_0)$ , so

$$\partial_N\varphi \leq \pi^-(\nabla'\varphi, \lambda) \quad \text{or} \quad \partial_N\varphi \geq \pi^+(\nabla'\varphi, \lambda).$$

If  $\partial_N\varphi(t_0, X_0) \leq \pi^-(\nabla'\varphi, \lambda)$ , then

$$H^-(\nabla\varphi) = H(\nabla\varphi),$$

giving

$$\varphi_t + H^-(\nabla'\varphi, \partial_N\varphi) \geq 0.$$

If  $\partial_N \varphi(t_0, X_0) \geq \pi^+(\nabla' \varphi, \lambda)$ , we consider

$$\phi(t, X) = \varphi(t, X) + (\pi^-(\nabla' \varphi, A_0(\nabla' \varphi)) - \partial_N \varphi(t_0, X_0))(x^N - \psi^N(X')),$$

where  $\psi^N$  is the function defined previously, so that  $\phi$  is defined on a neighborhood  $(0, T) \times \bar{B}_{r_0}(X_0) \cap \bar{\Omega}$  of  $(t_0, X_0)$ . Using that

$$\pi^-(\nabla' \varphi, A_0(\nabla' \varphi)) \leq \pi^+(\nabla' \varphi, \lambda)$$

and

$$\forall X \in \bar{B}_{r_0}(X_0) \cap \bar{\Omega}, \quad x^N \geq \psi^N(X'),$$

we have

$$\forall (t, X) \in (0, T) \times \bar{B}_{r_0}(X_0) \cap \bar{\Omega}, \quad \phi(t, X) \leq \varphi(t, X),$$

thus  $\phi$  is a test function such that

$$\partial_N \phi(t_0, X_0) = \pi^-(\nabla' \varphi, A_0(\nabla' \varphi)),$$

$$\nabla' \phi(t_0, X_0) = \nabla' \varphi(t_0, X_0),$$

as  $\nabla' \psi^N(X'_0) = 0$  and

$$\phi(t_0, X_0) = \varphi(t_0, X_0).$$

Therefore  $H(\nabla' \phi, \partial_N \phi(t_0, X_0)) = A_0(\nabla' \phi)$ , and as  $\phi$  is a test function,

$$\phi_t(t_0, X_0) + A_0(\nabla' \phi) \geq 0,$$

and we find

$$\varphi_t(t_0, X_0) + H^-(\nabla' \varphi, \partial_N \varphi) \geq \varphi_t(t_0, X_0) + A_0(\nabla' \varphi) \geq 0. \quad \square$$

## 5.2 The stationary case: finiteness of the critical slope in Lemma 3.6

Let us give a simpler proof of the fact that the critical slope  $\bar{p}$  is finite in the stationary case in Lemma 3.6.

**Lemma 5.2.** *Let  $u$  be an upper semi-continuous sub-solution of  $u + H(\nabla u) = 0$  in  $\Omega$  which satisfies (1.5), and let  $\varphi$  be a test function touching  $u$  from above at some point  $X_0$  and  $X_0 = (X'_0, x_0^N) \in \partial\Omega$ . Let  $r_0$  be given by Definition 3.4. Then the critical slope given by*

$$\bar{p} = \inf \left\{ p \in \mathbb{R} : \exists r \in (0, r_0], \quad \varphi(X) + p(x^N - \psi^N(X')) \geq u(X), \quad \forall X \in \bar{\Omega} \cap \bar{B}_r(X_0) \right\}$$

is finite.

*Proof.* The function  $u$  is a sub-solution and is therefore bounded on the compact set  $\bar{\Omega}$ . Let  $M > 0$  be such that  $|u| \leq M$ . Thus for all test function  $\phi$  touching from above at  $X \in \Omega$ , we have

$$H(\nabla \phi) \leq -u(X) \leq M.$$

As  $H$  is coercive,  $|\nabla\phi| \leq C$  for some  $C > 0$ , so  $u$  is  $C$ -lipschitz on  $\Omega$  by Lemma 2.2 of [1, page 148]. Using the assumption (1.5), for  $(x, y) \in \overline{\Omega}^2$ , there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\Omega$  which tends to  $x$ , such that  $\lim_{n \rightarrow +\infty} u(x_n) = u(x)$ , and a sequence  $(y_n)_{n \in \mathbb{N}}$  in  $\Omega$  which tends to  $y$ , such that  $\lim_{n \rightarrow +\infty} u(y_n) = u(y)$ . Then

$$|u(x_n) - u(y_n)| \leq C|x_n - y_n|,$$

and letting  $n$  go to  $+\infty$ , we have

$$|u(x) - u(y)| \leq C|x - y|,$$

and  $u$  is  $C$ -lipschitz on  $\overline{\Omega}$ . Let  $\varphi$  be a test function touching  $u$  from above at  $X_0 \in \partial\Omega$ , (see Definition 2.2). As the function  $\varphi$  is of class  $\mathcal{C}^1$ , it is  $C'$ -lipschitz on  $\overline{\Omega}$ , so for  $p$  satisfying

$$\varphi(X) - \varphi(X_0) + p(x^N - \psi^N(X')) \geq u(X) - u(X_0),$$

on  $B_r(X_0)$ , we deduce that

$$C'|X - X_0| + p(x^N - \psi^N(X')) \geq -C|X - X_0|.$$

Taking  $X' = X'_0$  et  $x^N \geq x_0^N$ , we find

$$C'(x^N - x_0^N) + p(x^N - x_0^N) \geq -C(x^N - x_0^N).$$

Then  $p \geq -(C + C')$ , and  $\bar{p}$  is finite. □





# Effective nonlinear Neumann boundary conditions for 1D nonconvex Hamilton-Jacobi equations

We study Hamilton-Jacobi equations in  $[0, +\infty)$  of evolution type with nonlinear Neumann boundary conditions in the case where the Hamiltonian is not necessarily convex with respect to the gradient variable. In this paper, we give two main results. First, we prove for a nonconvex and coercive Hamiltonian that general boundary conditions in a relaxed sense are equivalent to *effective ones* in a strong sense. Here, we exhibit the *effective* boundary conditions while for a quasi-convex Hamiltonian, we already know them (Imbert and Monneau 2017). Second, we give a comparison principle for a nonconvex and nonnecessarily coercive Hamiltonian where the boundary condition can have constant parts.

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## 1 Introduction

Let us consider the following Hamilton-Jacobi equation in  $(0, T) \times [0, +\infty)$ , where  $T > 0$

$$\begin{cases} u_t + H(u_x) = 0 & \text{for } t \in (0, T) \text{ and } x > 0 \\ u_t + F(u_x) = 0 & \text{for } t \in (0, T) \text{ and } x = 0 \end{cases} \quad (2.1)$$

subject to the initial condition

$$u(0, x) = u_0(x) \quad \text{for } x \geq 0. \quad (2.2)$$

### 1.1 Main theorems

In order to state our first main theorem, we first define strong boundary nonlinearities associated to a Hamiltonian.

**Definition 1.1** (Strong boundary nonlinearity). *Let  $H : \mathbb{R} \rightarrow \mathbb{R}$  be continuous and coercive, i.e.,*

$$\lim_{|p| \rightarrow +\infty} H(p) = +\infty. \quad (2.3)$$

*A function  $F : \mathbb{R} \rightarrow \mathbb{R}$  is called a strong boundary nonlinearity for  $H$ , if  $F$  is continuous, non-increasing and semi-coercive, i.e.,*

$$\lim_{p \rightarrow -\infty} F(p) = +\infty. \quad (2.4)$$

*and satisfies*

$$F(p_0) \neq H(p_0) \implies F = F(p_0) \quad \text{on a neighborhood of } p_0.$$

*Remark 1.2.* The hypothesis “ $F$  is non-increasing” is necessary for the maximum principle to hold true.

Our first main theorem is about exhibiting *equivalent classes* of boundary conditions and a representative for each class, the *effective* boundary condition. These *effective* boundary conditions are exactly the strong boundary nonlinearities and are the only one to be satisfied in a strong sense.

**Theorem 1.3** (Effective boundary conditions). *Assume that the Hamiltonian  $H : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and coercive (2.3) and the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, non-increasing and semi-coercive (2.4). Then there exists a unique strong boundary nonlinearity  $F_{\text{eff}}$  such that a function  $u$  is a viscosity solution of (2.1) with  $F$  if and only if  $u$  is a strong viscosity solution of (2.1) with  $F_{\text{eff}}$ .*

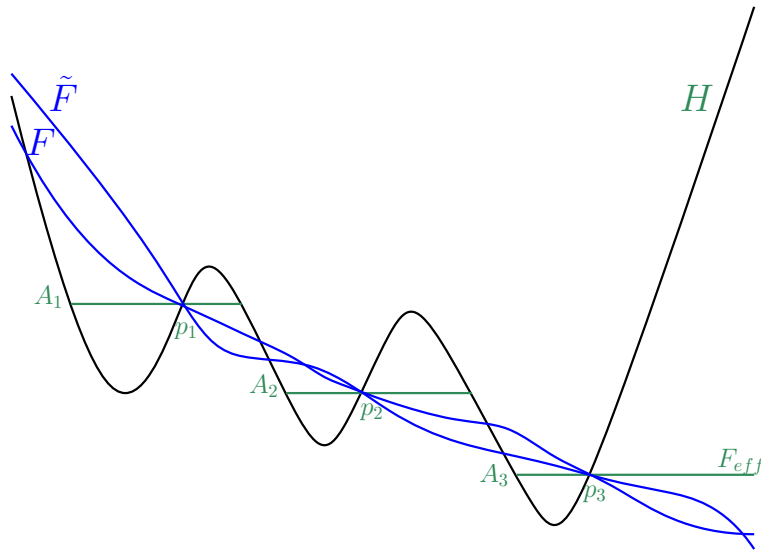


Figure 2.1: Illustration of a function  $F_{\text{eff}}$  associated to  $F$  and  $\tilde{F}$  in Theorem 1.3.

*Remark 1.4.* The definition of viscosity solutions and strong viscosity solutions are given in section 2. More precisely in the theorem, a viscosity solution of

$$u_t + H(u_x) = 0 \quad \text{in } (0, T) \times (0, +\infty),$$

satisfies at the boundary (i.e., at  $x = 0$ ) the following inequalities in the viscosity sense (relaxed sense)

$$\begin{cases} u_t + \min(H(u_x), F(u_x)) \leq 0, \\ u_t + \max(H(u_x), F(u_x)) \geq 0, \end{cases}$$

if and only if it satisfies at the boundary the following equality in the viscosity sense (strong sense)

$$u_t + F_{\text{eff}}(u_x) = 0.$$

*Remark 1.5.* Many functions  $F$  are associated to the same  $F_{\text{eff}}$ . Precisely, only the Hamiltonian  $H$  and few points of the function  $F$  characterize the effective  $F_{\text{eff}}$ . The set of these points is referred to as the *set of effective points* and we define it in section 3.

*Remark 1.6.* This theorem is the nonconvex counterpart of [66]. Monneau [84] is developing independently a different approach for multi-dimensional junctions [67].

To understand what the set of effective points is, we comment it on an example, see Figure 2.1. In the general case,  $F_{\text{eff}}$  is a non-increasing function which is “almost” the function  $H$  where each non-decreasing part are replaced by the “right constant”. In the particular case of Figure 2.1, the “right constants” are given by the intersections of  $F$  and the non-decreasing parts of  $H$ . So here the set of effective points associated to  $F$  is  $A_F = \{p_1, p_2, p_3\}$  and the constants are  $A_i = H(p_i)$  for  $i = 1, 2, 3$ . Theorem 1.3 implies here that taking another function  $\tilde{F}$  instead of  $F$  having the same intersection with the non-decreasing parts of  $H$  gives the same viscosity solutions of (2.1). In other words, all these boundary conditions with  $F, \tilde{F}$  and  $F_{\text{eff}}$  are in the same equivalence class.

*Remark 1.7.* In fact, we prove a more general result in section 3 (see Proposition 3.23). Theorem 1.3 is true if  $F$  is only continuous and non-increasing i.e., not necessarily semi-coercive (2.4), providing that the solution satisfies the “weak continuity” condition (2.12). Assuming that  $F$  is semi-coercive (2.4) implies that the solution satisfies (2.12), see Lemma 2.5. Without the “weak continuity” condition, Theorem 1.3 does not hold true. Indeed, let

$$u(t, x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{elsewhere.} \end{cases}$$

The function  $u$  does not satisfy (2.12). It is a viscosity solution of (2.1) with  $H(p) = |p|$  and  $F(p) = 0$  but it is not a strong viscosity solution of the corresponding (2.1) with  $F_{\text{eff}}$  (see section 3 for the construction of  $F_{\text{eff}}$ ), here

$$F_{\text{eff}}(p) = \begin{cases} -p & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

Our second main result deals with comparison principles.

**Theorem 1.8** (Comparison principles). *Assume that the Hamiltonian  $H : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, non-increasing and semi-coercive (2.4) and the initial datum  $u_0$  is uniformly continuous. Moreover, assume that we have one of the following assumptions,*

1. (a noncoercive  $H$  and a “coercive”  $F$ )

$$\lim_{p \rightarrow +\infty} F(p) = -\infty, \tag{2.5}$$

2. (a coercive  $H$  and a semi-coercive  $F$ )

$$\lim_{|p| \rightarrow +\infty} H(p) = +\infty.$$

Then for all viscosity sub-solution  $u$  and viscosity super-solution  $v$  of (2.1)-(2.2) satisfying for some  $T > 0$  and  $C_T > 0$ ,

$$u(t, x) \leq C_T(1 + x), \quad v(t, x) \geq -C_T(1 + x), \quad \forall (t, x) \in (0, T) \times [0, +\infty),$$

we have

$$u \leq v \quad \text{in} \quad [0, T) \times [0, +\infty).$$

*Remark 1.9.* In fact, we have  $u^* \leq v_*$  (see section 2) but since  $u \leq u^*$  and  $v_* \leq v$ , we get  $u \leq v$ . So in all the following proofs, we assume that  $u$  is upper semi-continuous and  $v$  is lower semi-continuous.

*Remark 1.10.* As for Theorem 1.3, we can prove a more general result for the second part of the theorem. The second part is true if  $F$  is only continuous and non-increasing i.e., not necessarily semi-coercive (2.4) providing that sub-solutions satisfy (2.12).

As far as existence results are concerned, the proof of [66, Theorem 2.14] prove also the existence of a solution in our case, for a nonconvex and noncoercive Hamiltonian.

## 1.2 Comparison with known results

First we review known results about comparison principles. There exist many results for Hamilton-Jacobi equations with boundary conditions of Neumann type. In [78], the author studied the case of linear Neumann boundary condition. For first-order Hamilton-Jacobi equations, Barles and Lions prove a comparison principle result in [17] under a nondegeneracy condition on the boundary nonlinearity (see (2.6) below). The second-order case was treated by Ishii and Barles in [71, 13, 20]. More precisely, Barles proves in [20] a comparison principle for fully non linear second order, degenerate, parabolic equations, in a smooth subset  $\Omega$  of  $\mathbb{R}^N$ , i.e.,

$$u_t + H(x, u, Du, D^2u) = 0 \text{ in } \Omega,$$

with a nonlinear Neumann boundary condition satisfying the same nondegeneracy as in [17] where it is studied for,

$$u_t + F(x, u, Du) = 0 \text{ in } \partial\Omega.$$

In this paper, we restrict ourselves to the case where  $H$  and  $F$  only depends on the gradient variable. In [20, 17], considering only the gradient variable dependence, the boundary condition satisfies (in dimension 1)

$$F(p - \lambda) - F(p) \geq C\lambda, \quad \text{for } \lambda > 0. \quad (2.6)$$

Here we assume a more general boundary condition,  $F$  is non-increasing, possibly with constant parts, and satisfies

$$\lim_{p \rightarrow -\infty} F(p) = +\infty \quad \text{and} \quad \lim_{p \rightarrow +\infty} F(p) = -\infty. \quad (2.7)$$

For example, the function  $F(p) = -\text{argsh}(p)$  does not satisfy the first condition but satisfies the second one. Moreover, condition (2.6) is too restrictive to modify  $F$  to make it non-increasing by a density argument as in Theorem 4.1.

Dealing with convex Hamiltonians, Soner [93] and Ishii and Koike [72] prove a comparison principle for state constraint problems. For a quasi-convex Hamiltonian  $H$ , in [66] the authors prove that the following state constraint problem,

$$\begin{aligned} u_t + H(u_x) &= 0 & \text{in } (0, T) \times (0, +\infty) \\ u_t + H(u_x) &\geq 0 & \text{in } (0, T) \times \{0\}, \end{aligned} \quad (2.8)$$

is equivalent to

$$\begin{aligned} u_t + H(u_x) &= 0 & \text{in } (0, T) \times (0, +\infty) \\ u_t + H^-(u_x) &= 0 & \text{in } (0, T) \times \{0\}, \end{aligned} \quad (2.9)$$

where  $H^-$  is the decreasing part of the Hamiltonian defined by

$$H^-(p) = \inf_{q \leq p} H(q), \quad (2.10)$$

see also [61] for the multidimensional case, and they prove a comparison principle for (2.9). More generally, they give a comparison principle for (2.1) with a quasi-convex  $H$  and  $F$  a non-increasing function. In [80], the authors deal with nonconvex

coercive Hamiltonians on junctions. In particular, they prove a comparison principle for (2.8) with  $H$  nonconvex. One can prove the equivalence between (2.8) and (2.9) for  $H$  nonconvex using the same methods as in [66, 61] and results of this paper (see Section 5). For a junction with many branches, one can get the same kind of equivalence of equations with the same tools. In this paper, still in the nonconvex case, we get a comparison principle for (2.9) which is equivalent to (2.8), and also more generally for (2.1) with  $F$  a non-increasing function (i.e., not only for  $F = H^-$ ).

As far as effective boundary conditions are concerned, in a pioneer work Andreianov and Sbihi [8, 6, 9] are able to describe effective boundary conditions for scalar conservation laws. Concerning the Hamilton-Jacobi framework, first results were obtained by Elliott, Giga and Goto [48] for a specific Hamiltonian on a bounded interval, for a specific dynamic boundary condition which has applications in superconductivity and interface evolution. Then more recently Imbert and Monneau can deal with general quasi-convex Hamiltonians. They treat the problem on a junction with several branches in 1D [66] and in the multi-dimensional case [67]. They prove that the effective boundary conditions are flux-limited ones

$$F_A(p) = \max(A, H^-(p)), \quad (2.11)$$

where  $H^-$  is the non-increasing part of the Hamiltonian  $H$  defined in (2.10). Still in a quasi-convex framework, the authors in [69] exhibit effective boundary conditions for degenerate parabolic equations. The nonconvex case has been out of reach so far. In this paper, we describe effective boundary conditions for a nonconvex Hamiltonian in 1D on the half-line. Monneau [84] mentioned to us that he developed a different approach dealing with  $N$  branches in the multi-dimensional case.

After Imbert and Monneau [66, 67], many papers deal with the flux-limited formulation (i.e., we take  $F = F_A$  in (2.1)) and results associated to the reduction of the set of test functions. These problems show the relevance of considering a more general class of boundary conditions than the classical state constraint problem [93, 72] (i.e. considering  $F_A$  that is more general than  $H^-$ ). Homogenization results using the flux-limited formulation have been recently obtained in [55, 54]. Moreover, there have been numerical results for a quasi-convex Hamiltonian and a flux-limited function at the junction point. There is a convergence result for a flux-limited function at the junction point in [36].

In [65], the authors find an error estimate of order  $\Delta x^{\frac{1}{3}}$  of the same scheme as in [36], and prove a convergence result for a general junction function at the junction point. This error estimate has been improved in [62] to order  $\Delta x^{\frac{1}{2}}$  if  $A > A_0$  where  $F_{A_0}$  is the minimal flux-limited function. There are also applications in optimal control, for example in [3] where the authors study problem related to flux-limited functions.

### 1.3 Comments and difficulties

For the effective boundary condition result, the main difficulty was to find the good definition of strong boundary nonlinearity  $F_{\text{eff}}$  for a nonconvex coercive Hamiltonian. In [66], for a quasi-convex Hamiltonian, Imbert and Monneau prove that the effective

boundary conditions are the flux-limited functions of the following form (see figure 2.2)

$$F_A(p) = \max(A, H^-(p)),$$

which are also BLN flux functions (see [12]) defined as, for  $p_0 \in \mathbb{R}$ ,

$$F_{p_0}(p) = \begin{cases} \sup_{q \in [p, p_0]} H(q) & \text{if } p \leq p_0 \\ \inf_{q \in [p_0, p]} H(q) & \text{if } p \geq p_0. \end{cases}$$

The BLN flux functions can be defined for nonconvex Hamiltonians. However, in the nonconvex case, BLN flux functions are effective boundary conditions but are not enough to cover all the effective boundary conditions as we see in section 3. For example, for an Hamiltonian with two minima (see figure 2.3), we need functions with two constant parts  $A_1$  and  $A_2$  like in figure 2.3, but this function is not a BLN flux function. However, it is locally a BLN function. In fact, it is exactly the “effective” boundary condition introduced in [8, 6, 9]. Since we only have a comparison result for the half line case, we only give the proof of the effective boundary condition result in this setting.

For the comparison principle, we tried to generalize the idea of Imbert and Monneau in [66] of the “vertex test function”. In their comparison principle, they replaced the classical term  $\frac{(x-y)^2}{2\varepsilon}$  by a function  $G$  called the “vertex test function” which satisfies (almost) the following condition

$$H(y, -G_y) \leq H(x, G_x),$$

which gives a contradiction combining the two viscosity inequalities. But for nonconvex Hamiltonians even for a junction with only one branch, it is very difficult to find such a “vertex test function”. However, we follow the idea of coupling time and space in the doubling variable method in [53]. For example for the boundary condition  $F(p) = H(0, p) = -p$ , taking

$$\frac{(t-s)^2}{2\delta} + \frac{(t-s)}{\delta}(x-y) + \frac{(x-y)^2}{2\delta},$$

instead of the classical term

$$\frac{(t-s)^2}{2\delta} + \frac{(x-y)^2}{2\delta},$$

allows to get rid of the case  $x = 0$  or  $y = 0$  in the viscosity inequalities. In this paper, we give an example of such a function coupling time and space of the form  $\delta\varphi\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right)$  which solves the problem for all boundary conditions satisfying,  $F$  is of class  $C^1$  such that  $F' < 0$  and  $|F|$  is coercive (2.7). With a density argument, we deduce a comparison principle for all  $F$  continuous, non-increasing and which satisfies (2.7).



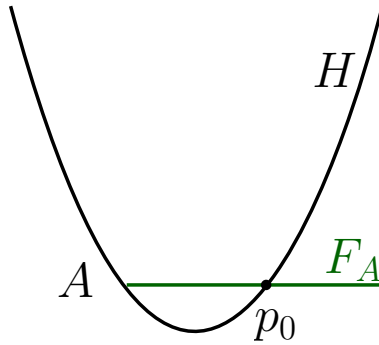


Figure 2.2: Illustration of the function  $F_A$  in the convex case.

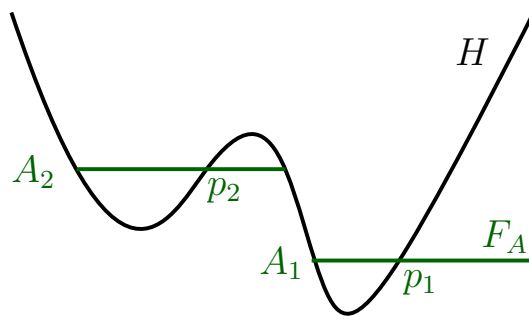


Figure 2.3: Illustration of a function  $F_A$  in the nonconvex case.

## 1.4 Organization of the paper

In section 2, we give the definition of viscosity and strong viscosity solutions. In section 3, as in [66] for quasi-convex Hamiltonians, we prove first that boundary conditions can be reduced to the *effective ones* for a nonconvex coercive Hamiltonian. Precisely, we exhibit *equivalent classes* for boundary conditions where the representative of the class is a strong boundary nonlinearity  $F_{\text{eff}}$ . Moreover, we prove that for these *effective boundary conditions* (i.e., with  $F_{\text{eff}}$ ), viscosity solutions of (2.1) are solutions in a stronger sense: they are also strong viscosity solutions (see Definition 2.3), and this property is only true for these strong boundary nonlinearity  $F_{\text{eff}}$  (see Lemmas 3.24 and 3.25). At the end of the section, we prove the associated comparison principle. In section 4, we prove a comparison principle for a nonconvex and noncoercive Hamiltonian where the boundary condition can have constant parts.

## 2 Viscosity solutions

In this section, we give the definitions of viscosity solutions and strong viscosity solutions and we recall that we have a weak continuity condition for sub-solutions when  $F$  is semi-coercive (2.4).

A test function is a  $\mathcal{C}^1$  function  $\phi : (0, T) \times [0, +\infty) \rightarrow \mathbb{R}$  which touches a function  $u$  from below (resp. from above) at  $(t, x)$ , i.e.,  $u - \phi$  reaches a local minimum (resp. maximum) at  $(t, x)$ .

We recall the definition of upper and lower semi-continuous envelopes  $u^*$  and  $u_*$  of a (locally bounded) function  $u$  defined on  $[0, T) \times [0, +\infty)$ ,

$$u^*(t, x) = \limsup_{(s, y) \rightarrow (t, x)} u(s, y) \quad \text{and} \quad u_*(t, x) = \liminf_{(s, y) \rightarrow (t, x)} u(s, y).$$

**Definition 2.1** (Viscosity solutions). *Let  $u : [0, T) \times [0, +\infty) \rightarrow \mathbb{R}$ .*

- i) We say that  $u$  is a viscosity sub-solution (resp. viscosity super-solution) of (2.1) in  $(0, T) \times [0, +\infty)$  if for all test function  $\phi \in \mathcal{C}^1$  touching  $u^*$  (resp.  $u_*$ ) from above (resp. from below) at  $(t_0, x_0)$ , we have if  $x_0 > 0$ ,*

$$\phi_t(t_0, x_0) + H(\phi_x(t_0, x_0)) \leq 0 \quad (\text{resp. } \geq 0)$$

*if  $x_0 = 0$ ,*

$$\begin{aligned} & \text{either } \phi_t(t_0, 0) + H(\phi_x(t_0, 0)) \leq 0 \quad (\text{resp. } \geq 0) \\ & \text{or } \phi_t(t_0, 0) + F(\phi_x(t_0, 0)) \leq 0 \quad (\text{resp. } \geq 0). \end{aligned}$$

- ii) We say that  $u$  is a viscosity sub-solution (resp. viscosity super-solution) of (2.1)-(2.2) on  $[0, T) \times [0, +\infty)$  if additionally*

$$u^*(0, x) \leq u_0(x) \quad (\text{resp. } u_*(0, x) \geq u_0(x)) \quad \forall x \in [0, +\infty).$$

- iii) We say that  $u$  is a viscosity solution if  $u$  is both a viscosity sub-solution and super-solution.*

*Remark 2.2.* It is well-known that the boundary condition has to be considered in a relaxed sense, see [78, 20].

Let us give the definition of strong viscosity solutions.

**Definition 2.3** (Strong viscosity solutions). *Let  $u : [0, T) \times [0, +\infty) \rightarrow \mathbb{R}$ .*

- i) We say that  $u$  is a strong viscosity sub-solution (resp. strong viscosity super-solution) of (2.1) in  $(0, T) \times [0, +\infty)$  if for all test function  $\phi \in \mathcal{C}^1$  touching  $u^*$  (resp.  $u_*$ ) from above (resp. from below) at  $(t_0, x_0)$ , we have if  $x_0 > 0$ ,*

$$\phi_t(t_0, x_0) + H(\phi_x(t_0, x_0)) \leq 0 \quad (\text{resp. } \geq 0)$$

*if  $x_0 = 0$ ,*

$$\phi_t(t_0, 0) + F(\phi_x(t_0, 0)) \leq 0 \quad (\text{resp. } \geq 0).$$

- ii) We say that  $u$  is a strong viscosity sub-solution (resp. strong viscosity super-solution) of (2.1)-(2.2) on  $[0, T) \times [0, +\infty)$  if additionally*

$$u^*(0, x) \leq u_0(x) \quad (\text{resp. } u_*(0, x) \geq u_0(x)) \quad \forall x \in [0, +\infty).$$

- iii) We say that  $u$  is a strong viscosity solution if  $u$  is both a strong viscosity sub-solution and a strong viscosity super-solution.*

*Remark 2.4.* A strong viscosity sub-solution (resp. super-solution) is obviously a viscosity sub-solution (resp. super-solution).

For the same reason as in [66], we need a weak continuity condition for sub-solutions to get the effective boundary condition result in section 3. More precisely, let us recall that any viscosity sub-solution satisfies automatically the “weak continuity” condition if the function  $F$  is semi-coercive (2.4). In fact, we recall [66, Lemma 2.3] without proving it since the proof is the same in our case.

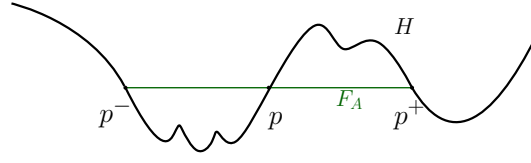
**Lemma 2.5** (“Weak continuity” condition). *Assume that the Hamiltonian  $H : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and coercive (2.3), the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, non-increasing and semi-coercive (2.4). Then any viscosity sub-solution  $u$  of (2.1) satisfies for all  $t \in (0, T)$*

$$u^*(t, 0) = \limsup_{(s,y) \rightarrow (t,0), y>0} u^*(s, y). \quad (2.12)$$

### 3 Effective boundary conditions

In this section, we see that only the Hamiltonian  $H$  and few points of the function  $F$  characterize the boundary conditions. First, we characterize strong boundary nonlinearities by exhibiting this important set of point, the set of effective points  $A$ . We obtain a result of reduction of the set of test functions as [66] for the strong boundary nonlinearities. Then we prove the effective boundary condition theorem. Using the result of the fourth section, we prove that the viscosity solution of the problem (2.1)-(2.2) is unique.

In this section, the Hamiltonian  $H : \mathbb{R} \rightarrow \mathbb{R}$  is assumed to be continuous and coercive (2.3).

Figure 2.4: Illustration of  $p^-$  and  $p^+$  in Definition 3.1

### 3.1 Set of effective points

In this subsection, we exhibit a set of points which characterizes the strong boundary nonlinearities. This characterization is more practical for the following proofs. First, let us give some definitions and lemmas which are used to define this set.

#### 3.1.1 Numbers $p^-$ and $p^+$

**Definition 3.1** (Numbers  $p^-$  and  $p^+$ ). *Let  $p \in \mathbb{R}$ . We define*

$$p^- = \sup \{q < p \mid H(q) \geq H(p)\},$$

and

$$p^+ = \inf \{q > p \mid H(q) \leq H(p)\},$$

with the convention  $\inf \emptyset = +\infty$ .

*Remark 3.2.* Since the Hamiltonian  $H$  is coercive,  $p^-$  is the supremum of a nonempty set.

*Remark 3.3.* Let  $[a, b]$  be an interval of  $\mathbb{R}$ . Notice that if  $H$  is non-increasing on  $[a, b]$ , then for all  $p \in [a, b]$ ,  $p^- = p = p^+$ .

We deduce the following lemma from the definition.

**Lemma 3.4.** *For all  $p \in \mathbb{R}$  such that  $p^+ < +\infty$ , we have*

$$H(p^-) = H(p) = H(p^+).$$

Moreover, we have

$$\forall q \in ]p^-, p[, \quad H(q) < H(p), \quad (2.13)$$

and

$$\forall q \in ]p, p^+[ , \quad H(q) > H(p). \quad (2.14)$$

*Proof of Lemma 3.4.* The second part of the lemma is a consequence of the definition of  $p^-$  and  $p^+$ . Let us prove the first part. By definition, we have  $H(p^-) \geq H(p)$  and for all  $q \in ]p^-, p[$ ,  $H(q) < H(p)$ . Sending  $q \rightarrow p^-$  and by continuity of  $H$ , we deduce  $H(p^-) \leq H(p)$  so  $H(p^-) = H(p)$ . By the same arguments, we have  $H(p) = H(p^+)$ .  $\square$

On Figure 2.4, the position of  $H$  compared to  $H(p)$  is illustrated.

Let us give the following useful lemma.

**Lemma 3.5.** *We have the following properties.*

1. *Assume  $]p^-, p[\cap]q^-, q[\neq \emptyset$ . We have  $H(p) \leq H(q)$  if and only if  $[p^-, p] \subset [q^-, q]$  i.e.,  $q^- \leq p^- < p \leq q$ .*
2. *Assume  $]p, p^+[\cap]q, q^+[\neq \emptyset$ . We have  $H(p) \leq H(q)$  if and only if  $[q, q^+] \subset [p, p^+]$  i.e.,  $p \leq q < q^+ \leq p^+$ .*
3. *If  $]p^-, p[\cap]q, q^+[\neq \emptyset$ , then  $H(p) > H(q)$ .*

*Proof of Lemma 3.5.* Let us prove the first point. The second point is very similar to the first one so we skip the proof. Assume that  $H(p) \leq H(q)$ . If by contradiction  $p > q$ , then since  $]p^-, p[\cap]q^-, q[\neq \emptyset$ , we have  $p^- < q < p$ . We deduce that

$$H(q) < H(p) \leq H(q)$$

which gives a contradiction. So we deduce that  $p \leq q$ . Moreover, since  $]p^-, p[\cap]q^-, q[\neq \emptyset$ , we have  $q^- < p \leq q$ . Assume by contradiction that  $p^- < q^-$ , then

$$H(p^-) = H(p) \leq H(q) = H(q^-),$$

but  $q^- \in ]p^-, p[$ , which gives a contradiction with Lemma 3.4. So we deduce that  $[p^-, p] \subset [q^-, q]$ . Assume now that  $[p^-, p] \subset [q^-, q]$ . In particular we have  $p \in [q^-, q]$ , hence  $H(p) \leq H(q)$ .

Let us prove the third point. Assume that

$$]p^-, p[\cap]q, q^+[\neq \emptyset, \tag{2.15}$$

then we have  $q \leq p$ . Necessarily by Lemma 3.4, we have  $H(p) \geq H(q)$ . If by contradiction, we have  $H(p) = H(q)$ , then either  $q = p$  so  $q^- = p^-$  or  $q \leq p^-$  so  $q^+ \leq p^-$ . But these two cases gives a contradiction with (2.15). So we deduce that  $H(p) > H(q)$ .  $\square$

### 3.1.2 Set of effective points and $A$ -strong boundary nonlinearity

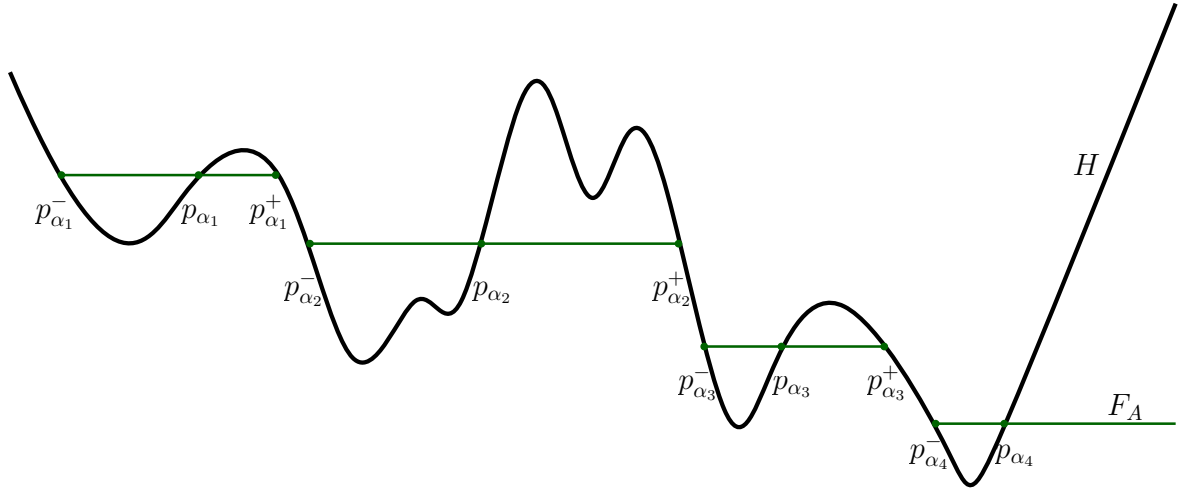
**Definition 3.6** (Set of effective points  $A$ ). *The set  $A$  is called a set of effective points if  $A$  is a set of points of  $\mathbb{R}$  indexed by  $I$ ,  $A = (p_\alpha)_{\alpha \in I}$ , such that*

1.  $\forall \alpha \in I, p_\alpha^- \neq p_\alpha^+$ ,
2. For  $\alpha_1, \alpha_2 \in I$ , if  $p_{\alpha_1} < p_{\alpha_2}$  then  $H(p_{\alpha_1}) \geq H(p_{\alpha_2})$ ,
3.
  - $\forall p \in \mathbb{R}$  such that  $p^- < p$ ,  $\exists \alpha \in I$  such that  $]p^-, p[\cap]p_\alpha^-, p_\alpha^+[\neq \emptyset$ ,
  - $\forall p \in \mathbb{R}$  such that  $p < p^+$ ,  $\exists \alpha \in I$  such that  $]p, p^+[\cap]p_\alpha^-, p_\alpha^+[\neq \emptyset$ .

*Remark 3.7.*  $A$  is not empty since the Hamiltonian  $H$  is coercive and  $A$  is countable.

We deduce the following lemma which allows to define the  $A$ -strong boundary nonlinearity.

**Lemma 3.8.** *If  $p_1 < p_2$  and  $H(p_1) \geq H(p_2)$  then we have  $]p_1^-, p_1^+[\cap]p_2^-, p_2^+[\neq \emptyset$ . In particular, the intervals  $]p_\alpha^-, p_\alpha^+[\$  for  $\alpha \in I$  are disjoint.*

Figure 2.5: Illustration of a function  $F_A$  in Definition 3.9

*Proof of Lemma 3.8.* This lemma is a direct consequence of Lemma 3.5.  $\square$

Now we can define the  $A$ -strong boundary nonlinearity which is a characterization of strong boundary nonlinearities associated to the parameter  $A$ .

**Definition 3.9** ( $A$ -strong boundary nonlinearity  $F_A$ ). *Let  $A$  be a set of effective points. The function  $F_A : \mathbb{R} \rightarrow \mathbb{R}$  defined by*

$$F_A(p) = \begin{cases} H(p_\alpha) & \text{if } p \in [p_\alpha^-, p_\alpha^+], \text{ for } \alpha \in I \\ H(p) & \text{elsewhere} \end{cases}$$

*is called a  $A$ -strong boundary nonlinearity.*

**Proposition 3.10.** *The function  $F_A$  is well-defined, continuous, non-increasing and semi-coercive (2.4). Moreover,  $F_A$  is a strong boundary nonlinearity (see Definition 1.1).*

We give an example of a  $A$ -strong boundary nonlinearity in Figure 2.5.

*Proof of Proposition 3.10.* Lemma 3.8 ensures that the function  $F_A$  is well-defined and Lemma 3.4 ensures that  $F_A$  is continuous. Let us prove that  $F_A$  is non-increasing. Assume by contradiction that there exists  $p < q$  such that  $F_A(p) < F_A(q)$ . Without loss of generality, we assume that  $p < q$  such that  $H(p) = F_A(p) < F_A(q) = H(q)$ . Indeed, if we have  $p \in [p_\alpha^-, p_\alpha^+]$  for  $\alpha \in I$ , we also have  $p_\alpha < q$  and  $H(p_\alpha) = F_A(p_\alpha) = F_A(p) < F_A(q)$ . We can use the same argument for  $q$ , if  $q \in [p_{\alpha'}^-, p_{\alpha'}^+]$  for  $\alpha' \in I$ .

Let  $p_1 = \inf \left\{ r \geq p \mid H(r) = \frac{H(p)+H(q)}{2} \right\}$  and  $q_1 = \sup \left\{ r \leq q \mid H(r) = \frac{H(p)+H(q)}{2} \right\}$ . We have

$$p_1^- < p < p_1 \leq q_1 < q < q_1^+,$$

and

$$H(p) < H(p_1) = H(q_1) < H(q). \quad (2.16)$$

Using 3. of Definition 3.6, there exists  $\alpha \in I$  such that

$$]p_1^-, p_1[\cap]p_\alpha^-, p_\alpha^+[\neq \emptyset.$$

We distinguish two cases.

If  $]p_1^-, p_1[\cap]p_\alpha^-, p_\alpha^+[\neq \emptyset$ , then using 1. of Lemma 3.5, we deduce  $H(p_\alpha) < H(p_1)$  and  $p_\alpha < p_1$ . Indeed, if by contradiction we have  $H(p_\alpha) \geq H(p_1)$ , then by 1. of Lemma 3.5, we deduce that  $p \in ]p_1^-, p_1[ \subset ]p_\alpha^-, p_\alpha^+]$ . Hence, we have

$$H(p) = F_A(p) = F_A(p_\alpha) = H(p_\alpha) \geq H(p_1),$$

which gives a contradiction with (2.16). We deduce that

$$H(p_\alpha) = F_A(p_\alpha) < H(p_1)$$

and  $]p_\alpha^-, p_\alpha^+[\subset ]p_1^-, p_1[$  with 1. of Lemma 3.5, hence  $p_\alpha < p_1$ .

If  $]p_1^-, p_1[\cap]p_\alpha^-, p_\alpha^+[\neq \emptyset$ , then  $p_\alpha < p_1$  and using 3. of Lemma 3.5, we deduce that

$$H(p_\alpha) = F_A(p_\alpha) < H(p_1).$$

By symmetric arguments, we also have  $\alpha' \in I$  such that

$$H(p_{\alpha'}) = F_A(p_{\alpha'}) > H(q_1),$$

and  $q_1 < p_{\alpha'}$ .

Combining these conclusions, we deduce that

$$p_\alpha < p_1 < q_1 < p_{\alpha'},$$

and

$$H(p_\alpha) < H(p_1) = H(q_1) < H(p_{\alpha'}),$$

which gives a contradiction with 2. of Definition 3.6. We deduce that  $F_A$  is non-increasing.

Assume by contradiction that  $F_A$  is not semi-coercive. Since  $H$  is semi-coercive, it exists  $p$  such that for all  $q < p$ ,  $F_A(q) < H(q)$ . Let  $q_1 < p$  then by definition of  $F_A$ , it exists  $\alpha \in I$  such that  $q_1 \in ]p_\alpha^-, p_\alpha^+]$  (since  $H$  is above  $F_A$  in this set). But by definition of  $F_A$ , we have  $F_A(p_\alpha) = H(p_\alpha)$  and since  $p_\alpha < p$  we get a contradiction. So  $F_A$  is semi-coercive.

Now by definition,  $F_A$  is clearly a strong boundary nonlinearity.  $\square$

**Lemma 3.11.** *Let  $A_1$  and  $A_2$  be two sets of effective points. If  $A_1 \neq A_2$  then  $F_{A_1} \neq F_{A_2}$ .*

*Proof of Lemma 3.11.* Assume that  $p_{\alpha_1} \in A_1$  but  $p_{\alpha_1} \notin A_2$ . Then  $p_{\alpha_1}^- \neq p_{\alpha_1}^+$ . By symmetry, assume that  $p_{\alpha_1}^- < p_{\alpha_1}$ . By 3. of Definition 3.6, it exists  $p_{\alpha_2} \in A_2$  such that  $]p_{\alpha_1}^-, p_{\alpha_1}[\cap]p_{\alpha_2}^-, p_{\alpha_2}^+[\neq \emptyset$ . So we have two cases either  $]p_{\alpha_1}^-, p_{\alpha_1}[\cap]p_{\alpha_2}^-, p_{\alpha_2}^+[\neq \emptyset$  or  $]p_{\alpha_1}^-, p_{\alpha_1}[\cap]p_{\alpha_2}^-, p_{\alpha_2}^+[\neq \emptyset$ . In the first case, let  $p \in ]p_{\alpha_1}^-, p_{\alpha_1}[\cap]p_{\alpha_2}^-, p_{\alpha_2}^+]$ . By 3. of Lemma 3.5, we deduce

$$F_{A_1}(p) = H(p_{\alpha_1}) < H(p_{\alpha_2}) = F_{A_2}(p).$$

So  $F_{A_1} \neq F_{A_2}$ . In the second case, let  $p \in ]p_{\alpha_1}^-, p_{\alpha_1}[\cap]p_{\alpha_2}^-, p_{\alpha_2}[$ . Assume by contradiction that  $F_{A_1}(p) = F_{A_2}(p)$ . Then

$$H(p_{\alpha_1}) = F_{A_1}(p_{\alpha_1}) = F_{A_1}(p) = F_{A_2}(p) = F_{A_2}(p_{\alpha_2}) = H(p_{\alpha_2}).$$

So by 1. of Lemma 3.5, necessarily  $p_{\alpha_1} = p_{\alpha_2}$  which gives a contradiction. We deduce that  $F_{A_1} \neq F_{A_2}$ .  $\square$

Now let us prove the characterisation of strong boundary nonlinearity with the set  $A$ : a strong boundary nonlinearity is in fact a  $A$ -strong boundary nonlinearity for some  $A$ .

**Proposition 3.12.** *Let  $F$  be a strong boundary nonlinearity. There exists a unique set of effective points  $A$  such that,  $F = F_A$ , i.e.,  $F$  is a  $A$ -strong boundary nonlinearity.*

*Proof.* The uniqueness is a direct consequence of Lemma 3.11. Let us define the following set

$$A = \left\{ p \in \mathbb{R} \mid F(p) = H(p) \quad \text{and} \quad p^- \neq p^+ \right\}.$$

Let us prove that

$$F(q) = \begin{cases} H(p) & \text{if } q \in [p^-, p^+], \text{ for } p \in A \\ H(q) & \text{elsewhere} \end{cases} \quad (2.17)$$

before showing that  $A$  is a set of effective points.

First, we prove that for all  $p \in A$ ,  $F$  is constant on  $[p^-, p^+]$ . Let  $p$  be in  $A$ . Since  $F$  is non-increasing and since  $H(p^-) = H(p^+) = H(p) = F(p)$ , we only have to prove that  $F(p^-) = H(p^-)$  and  $F(p^+) = H(p^+)$ . By symmetry we only prove the first equality. Assume by contradiction that  $F(p^-) \neq H(p^-)$ . Then since  $F$  is non-increasing, necessarily  $F(p^-) > F(p) = H(p^-)$ . Let  $p_1 = \inf \{s > p^- \mid F(s) < F(p^-)\}$ , then by continuity of  $F$ ,

$$F(p_1) = F(p^-) > F(p).$$

So  $p^- < p_1 < p$  and

$$F(p_1) > F(p) = H(p) > H(p_1).$$

But  $F$  is non constant on a neighborhood of  $p_1$ , which gives a contradiction with Definition 1.1. We deduce that  $F$  is constant on  $[p^-, p^+]$ .

For  $q \notin \bigcup_{p \in A} ]p^-, p^+[$ , let us prove that  $F(q) = H(q)$ . Let  $q \notin \bigcup_{p \in A} ]p^-, p^+[$ . Assume by contradiction that  $F(q) \neq H(q)$ . By symmetry we assume that  $F(q) < H(q)$ . We define

$$a = \sup \{s < q \mid F(s) = H(s)\},$$

and

$$b = \inf \{s > q \mid F(s) = H(s)\}.$$

Notice that  $a$  is finite but  $b$  can be  $+\infty$ . Indeed, since  $F$  is semi-coercive (2.4), it cannot be constant on  $(-\infty, q)$ , so it exists  $q_1 \in (-\infty, q)$  such that  $F$  is not constant on a neighborhood of  $q_1$  and  $H(q_1) = F(q_1)$  by Definition 1.1 so  $a$  must be finite.



Necessarily,  $F = F(q) < H$  on  $(a, b)$  with  $F(a) = H(a)$  and  $a < q < a^+ = b$ . So we have  $a \in A$  and  $q \in \bigcup_{p \in A} ]p^-, p^+[$  which gives a contradiction. So we deduce (2.17).

Now, let us prove that  $A$  is a set of effective points. It is clear that  $A$  satisfies 1. and 2. of Definition 3.6 since  $F$  is non-increasing. Assume by contradiction that  $A$  does not satisfy 3. of Definition 3.6. Then it exists  $q$  such that  $]q^-, q[ \cap \bigcup_{p \in A} ]p^-, p^+[ = \emptyset$  (or  $]q, q^+[\cap \bigcup_{p \in A} ]p^-, p^+[ = \emptyset$ , by symmetry we only treat the first case). So  $F = H$  on  $]q^-, q[$  by (2.17). For  $p \in ]q^-, q[$  we have  $F(p) = H(p) < H(q) = F(q)$ , by continuity of  $F$ . But  $F$  is non-increasing which gives a contradiction. We deduce that  $A$  is a set of effective points.  $\square$

*Remark 3.13.* Since strong boundary nonlinearities and  $A$ -strong boundary nonlinearities are the same, we uses  $A$ -strong boundary nonlinearities in the following subsections.

We give the following lemma which is useful for the next subsection.

**Lemma 3.14.** *The function  $F_A$  satisfies the following properties,*

1. for  $\alpha \in I$ ,  $\forall p \in ]p_\alpha^-, p_\alpha[$ ,  $F_A(p) > H(p)$ ,
2. for  $\alpha \in I$ ,  $\forall p \in ]p_\alpha, p_\alpha^+[$ ,  $F_A(p) < H(p)$ ,
3. If  $p \notin \bigcup_{\alpha \in I} ]p_\alpha^-, p_\alpha[ \cup ]p_\alpha, p_\alpha^+[$ , then  $F_A(p) = H(p)$ .

*Proof.* This result is a direct consequence of Lemma 3.4 and Definition 3.9.  $\square$

The following lemma ensures uniqueness of a strong boundary nonlinearity in each class of equivalence.

**Lemma 3.15.** *Let  $A_1$  and  $A_2$  be two set of effective points. If*

$$\{u \mid u \text{ solution of (2.1) with } F = F_{A_1}\} = \{u \mid u \text{ solution of (2.1) with } F = F_{A_2}\},$$

then

$$A_1 = A_2.$$

*Proof.* Assume by contradiction that  $A_1 \neq A_2$ . Then let  $p_{\alpha_1} \in A_1$  such that  $p_{\alpha_1} \notin A_2$ . By 3. of Definition 3.6, there exists  $p_{\alpha_2} \in A_2$  such that

$$]p_{\alpha_1}^-, p_{\alpha_1}^+[\cap ]p_{\alpha_2}^-, p_{\alpha_2}^+[\neq \emptyset. \quad (2.18)$$

We have two cases, either  $p_{\alpha_1}^- \in ]p_{\alpha_2}^-, p_{\alpha_2}^+[$  or  $p_{\alpha_2}^- \in ]p_{\alpha_1}^-, p_{\alpha_1}^+[$ . By symmetry, one can suppose that  $p_{\alpha_1}^- \in ]p_{\alpha_2}^-, p_{\alpha_2}^+[$ . Consider  $u(t, x) = -H(p)t + px$ , with  $p = p_{\alpha_1}^-$ . For  $x > 0$ , the first equation in (2.1) is clearly satisfied. For  $x = 0$ , we notice that  $F_{A_1}(p) = H(p_{\alpha_1})$  by Definition 3.9 of  $F_{A_1}$ . It follows that the second equation in (2.1) holds with  $F = F_{A_1}$ . Therefore  $u$  is solution of (2.1) with  $F = F_{A_1}$ . So by assumption, it is also solution of (2.1) with  $F = F_{A_2}$ . Writing the second equation in (2.1) with  $F = F_{A_2}$ , we deduce  $F_{A_2}(p) = H(p)$ . And by Definition 3.9,  $F_{A_2}(p) = H(p_{\alpha_2})$  since  $p \in ]p_{\alpha_2}^-, p_{\alpha_2}^+[$ . It follows  $H(p_{\alpha_2}) = H(p) = H(p_{\alpha_1})$ . Necessarily, since  $p_{\alpha_1} \neq p_{\alpha_2}$ , Lemma 3.8 gives a contradiction with (2.18). We deduce that  $A_1 = A_2$ .

### 3.2 Reducing the set of test functions

With this definition of  $F_A$ , as in [66, 67, 61], we can prove a theorem for reducing the set of test functions for the  $A$ -strong boundary nonlinearity. We consider functions satisfying a Hamilton-Jacobi equation in  $(0, +\infty)$ , solution of

$$u_t + H(u_x) = 0 \quad \forall (t, x) \in (0, T) \times (0, +\infty). \quad (2.19)$$

**Theorem 3.16** (Reduced set of test functions). *Assume that the Hamiltonian  $H$  is continuous and coercive (2.3). Let  $A$  be a set of effective points. For all point  $\alpha \in A$ , let us fix any time independent test function  $\phi_\alpha(x)$  satisfying*

$$\phi'_\alpha(0) = p_\alpha.$$

*Given a function  $u : (0, T) \times J \rightarrow \mathbb{R}$ , the following properties hold true. Let  $t_0$  be in  $(0, T)$ .*

- i) If  $u$  is an upper semi-continuous sub-solution of (2.19) and satisfies (2.12) at  $t = t_0$  and if for any test function  $\varphi$  touching  $u$  from above at  $(t_0, 0)$  with*

$$\varphi(t, x) = \psi(t) + \phi_\alpha(x) \quad (2.20)$$

*where  $\psi \in C^1(0, +\infty)$  and where  $\alpha \in I$  is such that  $p_\alpha^- \neq p_\alpha$ , we have*

$$\varphi_t + F_A(\varphi_x) \leq 0 \quad \text{at} \quad (t_0, 0),$$

*then  $u$  is a strong viscosity sub-solution at  $(t_0, 0)$  for  $F = F_A$ .*

- ii) If  $u$  is a lower semi-continuous super-solution of (2.19) and if for any test function  $\varphi$  touching  $u$  from below at  $(t_0, 0)$  with*

$$\varphi(t, x) = \psi(t) + \phi_\alpha(x)$$

*where  $\psi \in C^1(0, +\infty)$  and where  $\alpha \in I$  is such that  $p_\alpha \neq p_\alpha^+$ , we have*

$$\varphi_t + F_A(\varphi_x) \geq 0 \quad \text{at} \quad (t_0, 0),$$

*then  $u$  is a strong viscosity super-solution at  $(t_0, 0)$  for  $F = F_A$ .*

*Remark 3.17.* We only need to consider test functions satisfying  $p_\alpha^- \neq p_\alpha$  (resp.  $p_\alpha \neq p_\alpha^+$ ) for the sub-solution (resp. super-solution) case. Indeed in  $[p_\alpha, p_\alpha^+]$  (resp.  $[p_\alpha^-, p_\alpha]$ ), the function  $F_A$  is lower (resp. upper) than  $H$  that gives directly the result, using the following Lemmas (see the following proof). We recover the result [66, Theorem 2.7 i)] for a quasi-convex Hamiltonian and for  $F = F_{A_0} = H^-$  the decreasing part of the Hamiltonian. In [66],  $\pi^+(A)$  is the supremum of intersection points between  $A$  and the nondecreasing part of  $H$ . In this case, the set of effective points is  $A = \{\pi^+(A_0)\}$  where  $H(\pi^+(A_0)) = A_0$  the minimum of  $H$ , we have  $(\pi^+(A_0))^- = \pi^+(A_0)$ . That is why the author don't need any test function for this case in [66, Theorem 2.7 i)].

To prove this result, we need the two following lemmas already proven in [66, 67, 61]. Here we skip the proof on these lemmas.

**Lemma 3.18** (Critical slope for sub-solution [66]). *Let  $u$  be an upper semi-continuous sub-solution of (2.19) which satisfies (2.12) and let  $\varphi$  be a test function touching  $u$  from above at some point  $(t_0, 0)$  where  $t_0 \in (0, T)$ . Then the critical slope given by*

$$\bar{p} = \inf \{p \in \mathbb{R} : \exists r > 0, \quad \varphi(t, x) + px \geq u(t, x), \quad \forall (t, x) \in (t_0 - r, t_0 + r) \times [0, r]\}$$

*is finite, satisfies  $\bar{p} \leq 0$  and*

$$\varphi_t(t_0, 0) + H(\varphi_x(t_0, 0) + \bar{p}) \leq 0.$$

*Remark 3.19.* We need the “weak continuity” of sub-solutions to prove that  $\bar{p}$  is finite, and we need  $\bar{p}$  to be finite for the proof of Theorem 3.16.

**Lemma 3.20** (Critical slope for super-solution [66]). *Let  $u$  be a lower semi-continuous super-solution of (2.19) and let  $\varphi$  be a test function touching  $u$  from below at some point  $(t_0, 0)$  where  $t_0 \in (0, T)$ . If the critical slope given by*

$$\bar{p} = \sup \{p \in \mathbb{R} : \exists r > 0, \quad \varphi(t, x) + px \leq u(t, x), \quad \forall (t, x) \in (t_0 - r, t_0 + r) \times [0, r]\}$$

*is finite, then it satisfies  $\bar{p} \geq 0$  and we have*

$$\varphi_t(t_0, 0) + H(\varphi_x(t_0, 0) + \bar{p}) \geq 0.$$

*Proof of Proposition 3.16.* We first prove the results concerning sub-solutions.

**Sub-solution.** Let  $\phi$  be a test function touching  $u$  from above at  $(t_0, 0)$  and let  $\lambda = -\phi_t(t_0, 0)$ . Let  $p = \phi_x(t_0, 0)$ . We want to show that

$$F_A(p) \leq \lambda. \tag{2.21}$$

Notice that by lemma 3.18, there exists  $\bar{p} \leq 0$  such that

$$H(p + \bar{p}) \leq \lambda.$$

Since  $F_A$  is non-increasing, we have

$$F_A(p) \leq F_A(p + \bar{p})$$

and using Lemma 3.14, if  $p + \bar{p} \notin \bigcup_{\alpha \in I} ]p_\alpha^-, p_\alpha[$  we have

$$F_A(p) \leq F_A(p + \bar{p}) \leq H(p + \bar{p}) \leq \lambda,$$

which proves the result.

Now if  $p + \bar{p} \in ]p_\alpha^-, p_\alpha[$  for some  $\alpha \in I$  such that  $p_\alpha^- \neq p_\alpha$ , then

$$p + \bar{p} < p_\alpha = \phi'_\alpha(0).$$

Let us consider the modified test function

$$\varphi(t, x) = \phi(t, 0) + \phi_\alpha(x) - \phi_\alpha(0).$$

We have

$$\varphi(t_0, 0) = \phi(t_0, 0) = u(t_0, 0).$$

Let us show that

$$\varphi(t, x) \geq u(t, x), \quad (2.22)$$

on a neighborhood of  $(t_0, 0)$ . We have

$$p + \bar{p} = \phi_x(t_0, 0) + \bar{p} < \phi'_\alpha(0),$$

so there exists  $p_1$  and  $p_2$  such that  $\bar{p} < p_1 < p_2$  and which satisfy

$$p + p_i = \phi_x(t_0, 0) + p_i < \phi'_\alpha(0), \quad \forall i \in \{1, 2\}.$$

Since  $\phi_x$  and  $\phi'_\alpha$  are continuous, on a neighborhood of  $(t_0, 0)$ , we have

$$\phi_x(t, x) + p_i < \phi'_\alpha(x), \quad \forall i \in \{1, 2\}.$$

So we have on a neighborhood of  $(t_0, 0)$ ,

$$\begin{aligned} \phi(t, x) &= \phi(t, 0) + \int_0^x \phi_x(t, y) dy \\ &= \varphi(t, x) + \phi_\alpha(0) - \phi_\alpha(x) + \int_0^x \phi_x(t, y) dy \\ &= \varphi(t, x) + \int_0^x (\phi_x(t, y) - \phi'_\alpha(y)) dy \\ &\leq \varphi(t, x) - p_2 x, \end{aligned}$$

and by definition of  $\bar{p}$ , there exists a neighborhood  $(t_0 - r, t_0 + r) \times [0, r)$  of  $(t_0, 0)$ , for some  $r > 0$  such that

$$\begin{aligned} u(t, x) &\leq \phi(t, x) + p_1 x \\ &\leq \varphi(t, x) + (p_1 - p_2)x, \\ &\leq \varphi(t, x) \end{aligned}$$

so we get (2.22).

This test function satisfies in particular (2.20) so we deduce that

$$-\lambda + F_A(p_\alpha) \leq 0,$$

so we have since  $p + \bar{p} \in ]p_\alpha^-, p_\alpha[$  and  $F_A$  is constant in this interval,

$$F_A(p) \leq F_A(p + \bar{p}) = F_A(p_\alpha) \leq \lambda.$$

Therefore (2.21) holds true.

Let us prove now the super-solution case.

**Super-solution.** Let  $\phi$  be a test function touching  $u$  from below at  $(t_0, 0)$ . Let  $\lambda = -\phi_t(t_0, 0)$ , and  $p = \phi_x(t_0, 0)$ . We want to show that

$$F_A(p) \geq \lambda. \quad (2.23)$$

By Lemma 3.20, if  $\bar{p}$  is finite, then  $\bar{p} \geq 0$  and

$$H(p + \bar{p}) \geq \lambda. \quad (2.24)$$

If  $\bar{p} = +\infty$  then since  $H$  is coercive, the inequality (2.24) is true replacing  $\bar{p}$  with some large  $\tilde{p}$ . To simplify the notations,  $\bar{p}$  will denote the real number satisfying the inequality (2.24) in the first or the second case.

Since  $F_A$  is non-increasing, we have

$$F_A(p) \geq F_A(p + \bar{p})$$

and using Lemma 3.14, if  $p + \bar{p} \notin \bigcup_{\alpha \in I} ]p_\alpha, p_\alpha^+[$  we have

$$F_A(p) \geq F_A(p + \bar{p}) \geq H(p + \bar{p}) \geq \lambda,$$

which prove the result. Now if  $p + \bar{p} \in ]p_\alpha, p_\alpha^+[$  for some  $\alpha \in I$  such that  $p_\alpha \neq p_\alpha^+$ , then

$$p + \bar{p} > p_\alpha = \phi'_\alpha(0).$$

As for the sub-solution case, let us consider the modified test function

$$\varphi(t, x) = \phi(t, 0) + \phi_\alpha(x) - \phi_\alpha(0).$$

Arguing as in the subsolution case, we can show that  $\varphi$  touches  $u$  from below at  $(t_0, 0)$ .

This test function satisfies in particular (2.20) so we deduce that

$$-\lambda + F_A(p_\alpha) \geq 0,$$

so we have

$$F_A(p + \bar{p}) = F_A(p_\alpha) \geq \lambda.$$

Therefore (2.23) holds true.  $\square$

### 3.3 Proof of the effective boundary condition result

To prove Theorem 1.3, we first have to define the set  $A_F$  associated to the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  continuous, non-increasing and semi-coercive (2.4). In fact, we don't need  $F$  to be semi-coercive, if we assume that all sub-solutions satisfy (2.12), see Remark 1.7. So for Definition 3.21 and Proposition 3.22, which don't involve solutions, we don't assume that  $F$  is semi-coercive.

**Definition 3.21** (Set of effective points  $A_F$ ). *The set  $A_F$  is the set of points  $p \in \mathbb{R}$  such that either*

$$\left\{ \begin{array}{l} (i) \quad p^- \neq p, \\ (ii) \quad F(p) \geq H(p), \\ (iii) \quad \forall q \in \mathbb{R} \text{ such that } F(q) \geq H(q) \text{ and } ]q^-, q^+[ \cap ]p^-, p[ \neq \emptyset, \\ \quad \quad \text{we have } H(q) \leq H(p), \end{array} \right. \quad (2.25)$$

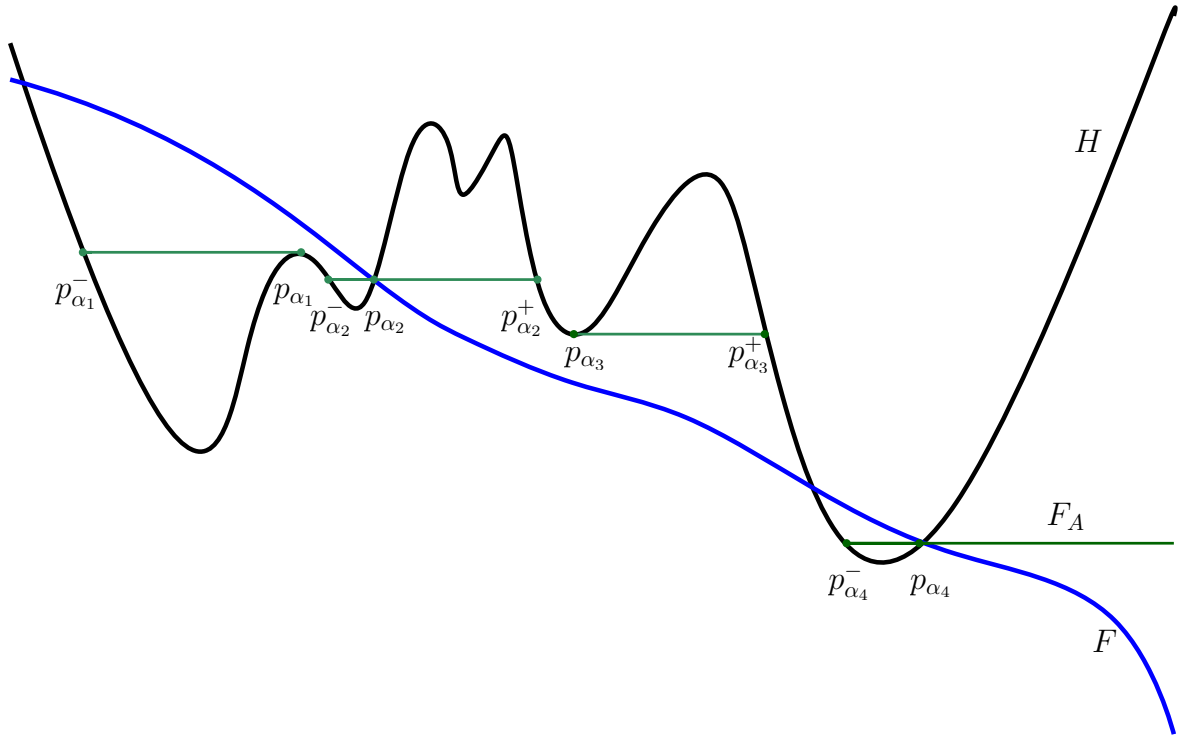


Figure 2.6: Illustration of a function  $F_{A_F}$  in Definition 3.21

or

$$\left\{ \begin{array}{l} (i) \quad p^+ \neq p, \\ (ii) \quad F(p) \leq H(p), \\ (iii) \quad \forall q \in \mathbb{R} \text{ such that } F(q) \leq H(q) \text{ and } ]q^-, q^+[\cap ]p, p^+[\neq \emptyset, \\ \quad \quad \text{we have } H(q) \geq H(p). \end{array} \right. \quad (2.26)$$

Notice that since  $H$  is coercive,  $A_F$  is not empty. We give an example of a  $A_F$ -strong boundary nonlinearity in Figure 2.6. To illustrate the set  $A_F$ , one can see that in the sets where  $F \geq H$ , the points of  $A_F$  satisfying (2.25) are local maximas of  $H$ . In the sets where  $F \leq H$ , the points of  $A_F$  satisfying (2.26) are local minimas of  $H$ . The points of  $A_F$  satisfying (2.25) and (2.26) are intersection points of  $F$  with non-decreasing part of  $H$  if  $H$  has a finite number of minimas (see Figure 2.6). We now show that  $p^- \neq p$  or  $p^+ \neq p$  for  $p \in A_F$  characterizes the fact that  $p$  satisfies (2.25) or (2.26).

**Proposition 3.22.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be continuous, non-increasing, and  $A_F$  be defined as in Definition 3.21, then  $A_F$  is a set of effective points. Moreover  $A_F$  satisfies the following property. If  $p \in A_F$  and  $p^- \neq p$  (resp.  $p^+ \neq p$ ) then  $p$  satisfies (2.25) (resp. (2.26)). In particular, if  $p^- < p < p^+$ , then  $F(p) = H(p)$ .*

*Proof.* Let us prove that  $A_F$  is a set of effective points. The set  $A_F$  satisfies 1. of Definition 3.6 since either  $p^- \neq p$  or  $p^+ \neq p$ . Let us prove that it satisfies 2. and 3. of Definition 3.6.

**Step 1:  $A_F$  satisfies 2. of Definition 3.6.**

Assume by contradiction that there exists  $p_1, p_2 \in A_F$  such that  $p_1 < p_2$  and  $H(p_1) < H(p_2)$ . We distinguish four cases.

**Case 1:  $p_1$  satisfies (2.26),  $p_2$  satisfies (2.25)**

We have

$$F(p_1) \leq H(p_1) < H(p_2) \leq F(p_2).$$

But  $F$  is non-increasing, so we get a contradiction and we have  $H(p_1) \geq H(p_2)$ .

**Case 2:  $p_1, p_2$  satisfy (2.25)**

Let  $p = \inf \{q > p_1 \mid H(q) \geq H(p_2)\}$ . We have

$$p^- < p_1^- < p_1 < p \leq p_2$$

and

$$F(p) \geq F(p_2) \geq H(p_2) = H(p) > H(p_1).$$

So  $p_1$  does not satisfy (2.25) (iii) with  $p$ , that gives a contradiction.

**Case 3:  $p_1, p_2$  satisfy (2.26)**

Let  $p = \sup \{q < p_2 \mid H(q) \leq H(p_1)\}$ . By symmetry with case 2, we prove that  $p_2$  does not satisfy (2.26) (iii) and get a contradiction.

**Case 4:  $p_1$  satisfies (2.25),  $p_2$  satisfies (2.26)**

We have  $F(p_1) \geq H(p_1)$  and  $F(p_2) \leq H(p_2)$ . Let us define

$$q_1 = \inf \{q \geq p_1 \mid H(q) = F(q)\},$$

$$r_1 = \inf \{q \geq p_1 \mid H(q) = H(q_1)\},$$

and

$$q_2 = \sup \{q \leq p_2 \mid H(q) = F(q)\},$$

$$r_2 = \sup \{q \leq p_2 \mid H(q) = H(q_2)\}.$$

Then if  $H(r_1) = H(q_1) > H(p_1)$ , we have

$$r_1^- < p_1^- < p_1 < r_1$$

and  $F(r_1) \geq F(q_1) = H(q_1) = H(r_1)$ . So  $p_1$  does not satisfy (2.25) (iii) with  $r_1$  that gives a contradiction. We deduce that  $H(q_1) \leq H(p_1)$ , so

$$H(r_2) = H(q_2) = F(q_2) \leq F(q_1) = H(q_1) \leq H(p_1) < H(p_2)$$

and we have

$$r_2 < p_2 < p_2^+ < r_2^+,$$

and  $F(r_2) \leq F(q_2) = H(q_2) = H(r_2)$ . So  $p_2$  does not satisfy (2.26) (iii) with  $r_2$  that gives a contradiction.

**Step 2:  $A_F$  satisfies 3. of Definition 3.6.**

Let  $p \in \mathbb{R}$  such that  $p^- \neq p^+$ . We distinguish four cases.

**Case 1:**  $p^- \neq p$  and  $F(p) < H(p)$ .

Let  $p_1 = \sup \{q \leq p \mid H(q) = F(q)\}$  and  $p_2 = \sup \left\{ q \in [p_1, p] \mid H(q) = \min_{s \in [p_1, p]} H(s) \right\}$ .

The number  $p_1$  could be  $-\infty$  but since  $H$  is coercive,  $p_2 < +\infty$ .

We are going to prove that  $p_2 \in A_F$  and  $]p^-, p^+[\cap]p_2^-, p_2^+[\neq \emptyset$ . Observe first that  $p_2$  satisfies (2.26) (i), (ii). Let us prove that it satisfies (2.26) (iii). Assume by contradiction that there exists  $q \in \mathbb{R}$  such that

$$F(q) \leq H(q), \quad (2.27)$$

$$]q^-, q^+[\cap]p_2^-, p_2^+[\neq \emptyset \quad (2.28)$$

and

$$H(q) < H(p_2). \quad (2.29)$$

We distinguish three possibilities for  $q$ . If  $q < p_1$  then using (2.27) and (2.29), we have  $F(q) < H(p_2) \leq H(p_1) \leq F(p_1)$ , that gives a contradiction with the fact that  $F$  is non-increasing. If  $q \in [p_1, p]$  then by definition of  $p_2$ ,  $H(p_2) \leq H(q)$  that gives a contradiction with (2.29). If  $q > p$  then using (2.29), we deduce that  $q^- \geq p_2^+$  that gives a contradiction with (2.28). We deduce that  $p_2 \in A_F$ . Moreover,  $p_2$  satisfies

$$]p^-, p[\cap]p_2^-, p_2^+[\neq \emptyset. \quad (2.30)$$

Indeed, we have for  $r \in ]p^-, p[$ ,  $H(r) < H(p)$  by Lemma 3.4, so  $H(p_2) < H(p)$  and  $p_2 < p < p_2^+$ .

**Case 2:**  $p^- \neq p$  and  $F(p) \geq H(p)$ .

Let  $p_1 = \inf \{q \geq p \mid H(q) = F(q)\}$  and  $p_2 = \inf \left\{ q \in [p, p_1] \mid H(q) = \max_{s \in [p, p_1]} H(s) \right\}$ . We are going to prove that  $p_2 \in A_F$  and satisfies (2.30). We have

$$p_2^- \leq p^- < p \leq p_2 \leq p_1,$$

so we deduce that  $p_2$  satisfies (2.25) (i) and by definition, we deduce that  $p_2$  satisfies (2.25) (ii). Let us prove that it satisfies (2.25) (iii). Assume by contradiction that there exists  $q \in \mathbb{R}$  such that

$$F(q) \geq H(q), \quad (2.31)$$

$q$  satisfies

$$]q^-, q^+[\cap]p_2^-, p_2^+[\neq \emptyset \quad (2.32)$$

and

$$H(q) > H(p_2). \quad (2.33)$$

We distinguish three possibilities for  $q$ . If  $q > p_1$  then using (2.31) and (2.33), we have  $F(q) > F(p_1)$ , that gives a contradiction with the fact that  $F$  is non-increasing. If  $q \in [p^-, p_1]$  then  $H(p_2) \geq H(q)$  that gives a contradiction with (2.33). If  $q < p^-$



then  $q^+ \leq p_2^-$  that gives a contradiction with (2.32). We deduce that  $p_2 \in A_F$  and satisfies (2.30).

**Case 3:**  $p \neq p^+$  and  $F(p) \leq H(p)$ .

Using the same arguments as in cases 1 and 2 with  $p_1 = \sup \{q \leq p \mid H(q) = F(q)\}$  and  $p_2 = \sup \left\{ q \in [p_1, p] \mid H(q) = \min_{s \in [p_1, p]} H(s) \right\}$ , we deduce that  $p_2 \in A_F$  and satisfies

$$]p, p^+[\cap]p_2^-, p_2^+[\neq \emptyset. \quad (2.34)$$

**Case 4:**  $p \neq p^+$  and  $F(p) > H(p)$ .

Using the same arguments as in cases 1 and 2 with  $p_1 = \inf \{q \geq p \mid H(q) = F(q)\}$  and  $p_2 = \inf \left\{ q \in [p, p_1] \mid H(q) = \max_{s \in [p, p_1]} H(s) \right\}$ , we deduce that  $p_2 \in A_F$  and satisfies (2.34).

Now let us prove the property of  $A_F$ . We only prove the result for  $p^+ \neq p$  since it is very similar for  $p^- \neq p$ . If  $p$  satisfies (2.26), we are done. If  $p$  satisfies (2.25), let us prove that it also satisfies (2.26) in this case. By hypothesis, it satisfies (2.26) (i). Let us prove that it satisfies (2.26) (ii). Assume by contradiction that  $F(p) > H(p)$ . Consider  $p_2$  defined in Step 2 Case 2. Then  $p_2$  gives a contradiction with (2.25) (iii), so  $p$  satisfies (2.26) (ii) and  $F(p) = H(p)$ .

Now let us prove that  $p$  satisfies (2.26) (iii). Assume by contradiction that there exists  $q \in \mathbb{R}$  such that

$$]q^-, q^+[\cup]p, p^+[\neq \emptyset, \quad (2.35)$$

$$F(q) \leq H(q) \quad (2.36)$$

and

$$H(q) < H(p). \quad (2.37)$$

We have that (2.36), (2.37) implies  $H(p) = F(p) > H(q) \geq F(q)$ . So since  $F$  is non-increasing, we have  $q > p$  and Lemma 3.8 gives a contradiction with (2.35). We deduce the result.  $\square$

$\square$

Now we can deduce the main theorem 1.3 from the following proposition.

**Proposition 3.23** (General Neumann boundaries reduce to strong boundary nonlinearities). *Assume that the Hamiltonian  $H : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and coercive, the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, non-increasing. Then the set of effective points  $A_F$  is such that*

- any viscosity super-solution of (2.1) with  $F$  is a strong viscosity super-solution of (2.1) with  $F_{A_F}$ ;

- any viscosity sub-solution of (2.1) with  $F$  such that for all  $t \in (0, T)$  we have (2.12), is a strong viscosity sub-solution with  $F_{A_F}$ ;
- any strong viscosity sub-solution (resp. super-solution) of (2.1) with  $F_{A_F}$  is a viscosity sub-solution (resp. super-solution) of (2.1) with  $F$ .

*Proof of Theorem 3.23.* We first prove that viscosity sub-solutions satisfying (2.12) are strong viscosity sub-solutions. We only do the proof for sub-solutions since it is very similar for super-solutions. Let  $u$  be a viscosity sub-solution. Thanks to Theorem 3.16, it is enough to show that for all  $\varphi$  touching  $u^*$  from above at  $(t, 0)$  such that  $\varphi_x(t, 0) = p \in A_F$ , and  $p^- \neq p$ , we have

$$\varphi_t(t, 0) + H(p) \leq 0.$$

Let  $\varphi$  be such a test function. Since  $u$  is a viscosity sub-solution, we have

$$\varphi_t + \min(F(p), H(p)) \leq 0.$$

Since  $p^- \neq p$ , Proposition 3.22 implies  $F(p) \geq H(p)$  so we deduce the result.

The third point of the theorem is a direct consequence of the inequality

$$\min(F, H) \leq F_{A_F} \leq \max(F, H).$$

Indeed, if  $p \in [p_\alpha^-, p_\alpha]$  where  $p_\alpha \in A_F$  and  $p_\alpha^- \neq p_\alpha$ , using Proposition 3.22, and (2.25) (ii), we have

$$F(p) \geq F(p_\alpha) \geq H(p_\alpha) = F_{A_F}(p) \geq H(p).$$

If  $p \in [p_\alpha, p_\alpha^+]$  where  $p_\alpha \in A_F$  and  $p_\alpha^+ \neq p_\alpha$ , using Proposition 3.22, and (2.26) (ii), we have

$$F(p) \leq F(p_\alpha) \leq H(p_\alpha) = F_{A_F}(p) \leq H(p).$$

If  $p \notin \bigcup_{\alpha \in I} [p_\alpha^-, p_\alpha^+]$ , then  $H(p) = F_{A_F}(p)$ . □

*Proof of Theorem 1.3.* Apply Proposition 3.23 and Lemma 3.15. □

Let us prove in two lemmas that for a  $A$ -strong boundary nonlinearity  $F_A$ , viscosity solutions and strong viscosity solutions are the same and this property is only true for  $A$ -strong boundary nonlinearities.

**Lemma 3.24.** *Let  $A$  be a set of effective points. The set  $A_{F_A}$  is the set  $A$ . In particular, a viscosity sub-solution (resp. super-solution) of (2.1) for  $F = F_A$  is a strong viscosity sub-solution (resp. super-solution) for  $F = F_A$ .*

*Proof of Lemma 3.24.* Let us prove that  $A \subset A_{F_A}$ . Let  $p \in A$ . Without loss of generality, assume that  $p^- \neq p$ , so  $p$  satisfies (i) of Definition 3.21. By definition of  $F_A$ , we have  $F_A(p) = H(p)$ , so  $p$  satisfies (ii) of Definition 3.21. Let us prove that  $p$  satisfies (iii) of Definition 3.21. Assume by contradiction that there exists  $q$  such that  $F_A(q) \geq H(q)$  and

$$]q^-, q^+[\cap ]p^-, p[ \neq \emptyset, \tag{2.38}$$

and

$$H(p) < H(q). \quad (2.39)$$

Then we deduce that

$$F_A(p) = H(p) < H(q) \leq F_A(q),$$

so  $q < p$ . We distinguish two cases, either  $q \in ]p^-, p[$ , or  $q < p^-$ . The first case is not possible since  $q$  satisfies (2.39) which gives a contradiction with Lemma 3.4. So we have  $q < p^-$ . But (2.39) and Lemma 3.4 imply that  $q^+ < p^-$ , that gives a contradiction with (2.38). So we have  $A \subset A_{F_A}$ . Using Proposition 3.22,  $A_{F_A}$  is a set of effective points. Notice that if we add (resp. remove) an element to (resp. from) a set of effective points, this new set is not a set of effective points anymore. So necessarily,  $A = A_{F_A}$  and we get the result.  $\square$

Now we prove that  $A$ -strong boundary nonlinearities  $F_A$  are the only continuous and non-increasing functions  $F$  such that any viscosity solutions of (2.1) with  $F$  satisfying (2.12) is in fact a strong viscosity solution of (2.1) with  $F$ .

**Lemma 3.25.** *Assume that  $F$  is continuous, non-increasing and semi-coercive (2.4), and assume  $F$  is not a  $A$ -strong boundary nonlinearity, then*

$$\{u \mid u \text{ strong visc. super-sol. of (2.1) with } F\} \subsetneq \{u \mid u \text{ visc. super-sol. of (2.1) with } F\}.$$

*Proof of Lemma 3.25.* The inclusion is obvious. Let us prove that the two sets are not equal. The function  $F$  is not a  $A$ -strong boundary nonlinearity, so it exists  $q \in \mathbb{R}$  such that  $F(q) \neq F_{A_F}(q)$ . Let us consider two cases, the first one  $F(q) < F_{A_F}(q) \leq H(q)$  or  $H(q) = F_{A_F}(q) < F(q)$  and the second one  $H(q) < F_{A_F}(q) < F(q)$ . In the first case, consider  $u(t, x) = -H(q)t + qx$ . In the second case  $q \in (p_\alpha^-, p_\alpha)$  where  $p_\alpha \in A_F$ , so consider  $u(t, x) = -H(p_\alpha^-)t + p_\alpha^-x$ . The function  $u$  is a viscosity solution of (2.1) with  $F_{A_F}$ , so by Proposition 3.23,  $u$  is a viscosity solution of (2.1) with  $F$ . But  $u$  is not a strong viscosity solution of (2.1) with  $F$ . So we deduce the result.  $\square$

### 3.4 Comparison principle for a coercive Hamiltonian

Using 1. of Theorem 1.8 and Proposition 3.23, we can deduce a comparison principle for a coercive Hamiltonian, but for  $F$  only semi-coercive. Although the proof of Part 1 of Theorem 1.8 is in the next section, we prove Part 2 in this section because it is the comparison principle associated to Theorem 1.3. It implies that it exists a unique viscosity solution  $u$  of (2.1)-(2.2) satisfying  $|u(t, x)| \leq C_T(1 + x)$  which is also the unique strong viscosity solution of (2.1)-(2.2) with  $F = F_{A_F}$ .

*Proof of 2. of Theorem 1.8.* We assume here that  $F$  is semi-coercive (2.4). If  $F < H$  let  $p$  be any real number else we define

$$p = \sup \{q \in \mathbb{R} \mid H(q) = F(q)\},$$

and  $G : \mathbb{R} \rightarrow \mathbb{R}$  a continuous function such that  $G(x) \rightarrow -\infty$  when  $x \rightarrow +\infty$ ,  $G$  satisfies  $G \leq F$  on  $[p, +\infty[$  and  $G(p) = F(p)$ . We define the function  $\tilde{F} : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\tilde{F} = \begin{cases} F & \text{on } ]-\infty, p] \\ G & \text{on } [p, +\infty[. \end{cases}$$

We have  $A_F = A_{\tilde{F}}$ . Indeed, notice that we have the following equivalences for  $F$  and  $\tilde{F}$ ,

$$H(p) \leq F(p) \iff H(p) \leq \tilde{F}(p)$$

and

$$H(p) \geq F(p) \iff H(p) \geq \tilde{F}(p).$$

Since in the definition of  $A_F$ , only the relative position between  $F$  and  $H$  takes the function  $F$  into account, the previous equivalences give the result. So we deduce using Proposition 3.23 that a function  $u$  is a viscosity sub-solution (resp. super-solution) for  $F$  if and only if  $u$  is a strong viscosity sub-solution (resp. super-solution) for  $F_{A_F}$ , if and only if  $u$  is a viscosity sub-solution (resp. super-solution) for  $\tilde{F}$ . We deduce the comparison principle for  $F$  using the comparison principle for  $\tilde{F}$  (1. of Theorem 1.8).  $\square$

*Remark 3.26.* In Remark 1.10, we say that we don't need  $F$  to be semi-coercive (2.4), providing that sub-solutions satisfy (2.12). Using the same arguments as in the previous proof, if  $F$  is not semi-coercive, we define  $\bar{p} \leq p$  such that if  $F < H$ ,  $\bar{p}$  is any real number satisfying  $\bar{p} \leq p$ , else

$$\bar{p} = \inf \{q \in \mathbb{R} \mid H(q) = F(q)\}.$$

We define  $\tilde{F} : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\tilde{F} = \begin{cases} \bar{G} & \text{on } ]-\infty, \bar{p}] \\ F & \text{on } [\bar{p}, p] \\ G & \text{on } [p, +\infty[ \end{cases}$$

where  $\bar{G} : \mathbb{R} \rightarrow \mathbb{R}$  a continuous function such that  $\bar{G}(x) \rightarrow +\infty$  when  $x \rightarrow -\infty$ ,  $\bar{G}$  satisfies  $F \leq \bar{G} \leq H$  on  $[p, +\infty[$  and  $\bar{G}(\bar{p}) = F(\bar{p})$ . Then the following of the proof is the same as the previous proof.

## 4 Comparison principle for nonconvex and noncoercive Hamilton-Jacobi equations allowing constant parts

In this section, we prove the first main comparison principle 1. of Theorem 1.8 for a nonconvex and noncoercive Hamiltonian where the boundary condition allows constant parts. The proof follows the idea of coupling time and space in the doubling variable method in [53]. First, we give a restricted version of the theorem which easily implies the main theorem. Then we prove the theorem for a class of test function which satisfy some properties. Finally, we give an example of such a test function so that the theorem is proven.

## 4.1 Simplification of the theorem

Let us prove a restricted version of 1. of Theorem 1.8 where the function  $F$  satisfies more hypotheses.

**Theorem 4.1** (Restricted comparison principle).

*Assume that the Hamiltonian  $H : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  is of class  $\mathcal{C}^1$  and satisfies  $F' < 0$ ,  $F(0) = 0$  and (2.4)-(2.5), and the initial datum  $u_0$  is uniformly continuous. Then for all viscosity sub-solution  $u$  and viscosity super-solution  $v$  of (2.1)-(2.2) satisfying for some  $T > 0$  and  $C_T > 0$ ,*

$$u(t, x) \leq C_T(1 + x), \quad v(t, x) \geq -C_T(1 + x), \quad \forall (t, x) \in (0, T) \times [0, +\infty),$$

we have

$$u \leq v \quad \text{in} \quad [0, T) \times [0, +\infty).$$

*Proof of 1. of Theorem 1.8 using Theorem 4.1.* It is enough to assume  $F(0) = 0$  as in [66, Lemma 3.1], by defining

$$u(t, x) = \tilde{u}(t, x) - tF(0) \quad \text{and} \quad v(t, x) = \tilde{v}(t, x) - tF(0)$$

and  $\tilde{F} = F - F(0)$ ,  $\tilde{H} = H - F(0)$ . The function  $u$  (resp.  $v$ ) is a sub-solution (resp. super-solution) of (2.1) if and only if  $\tilde{u}$  (resp.  $\tilde{v}$ ) is a sub-solution (resp. super-solution) of (2.1) replacing  $H$  by  $\tilde{H}$  and  $F$  by  $\tilde{F}$ . Let the function  $F$  be such that  $F(0) = 0$  and satisfy the hypothesis of 1. of Theorem 1.8, i.e. a continuous and non-increasing function which satisfies (2.4)-(2.5). By density, one can approximate  $F$  by a sequence  $F_n$  satisfying

$$\|F_n - F\|_\infty \leq \frac{1}{n} \quad \forall n \in \mathbb{N}^*,$$

with the hypothesis of Theorem 4.1, i.e. of class  $\mathcal{C}^1$  and decreasing such that  $F' < 0$  which satisfies (2.4)-(2.5). Let  $u$  be a sub-solution of (2.1) with the function  $F$ . Let us define  $u_n = u(x) - \frac{t}{n}$  which is a sub-solution of (2.1) with the function  $F_n$  and  $v_n = v(x) + \frac{t}{n}$  which is a super-solution of (2.1) with the function  $F_n$ . Using Theorem 4.1, we deduce

$$u(t, x) - \frac{t}{n} \leq v(t, x) + \frac{t}{n} \quad \forall (t, x) \in [0, T) \times [0, +\infty).$$

Sending  $n$  to  $+\infty$ , we deduce the result.  $\square$

## 4.2 The coupling time and space test function

Let us define the norm  $|(., .)| : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$|(t, x)| = \sqrt{t^2 + x^2}.$$

**Theorem 4.2** (Coupling time and space test function). *Assume the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  is of class  $\mathcal{C}^1$  and satisfies  $F' < 0$ ,  $F(0) = 0$  and (2.4)-(2.5). Then there exists a function  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  of class  $\mathcal{C}^1$  which satisfies the following properties.*

1. (Superlinearity)

$$\lim_{|(t,x)| \rightarrow +\infty} \frac{\varphi(t,x)}{1 + |(t,x)|} = +\infty, \quad (2.40)$$

2. (Bounded from below)

$$\forall (t,x) \neq (0,0), \quad \varphi(t,x) > \varphi(0,0) = 0. \quad (2.41)$$

3. (Differential inequalities) For all  $t \in \mathbb{R}$ ,

$$\begin{cases} \varphi_t(t,x) + F(\varphi_x(t,x)) \geq 0 & \text{if } x \leq 0, \\ \varphi_t(t,x) + F(\varphi_x(t,x)) \leq 0 & \text{if } x \geq 0. \end{cases} \quad (2.42)$$

*Remark 4.3.* We first admit this theorem to prove the comparison principle and we show it in the next subsection. The idea of the proof is to replace in the doubling variable method, the usual term  $\frac{(t-s)^2}{2\delta} + \frac{(x-y)^2}{2\varepsilon}$  by  $\delta\varphi\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right)$  which prevents the following supremum to be reached at the boundary.

### 4.3 Proof of the comparison principle

Let us recall [66, Lemma 3.4] since we use it in the proof. The proof of this lemma is exactly the same as in [66] so we skip it.

**Lemma 4.4** (A priori control). *Let  $T > 0$  and let  $u$  be a sub-solution and  $v$  be a super-solution as in Theorem 4.1. Then there exists a constant  $C = C(T) > 0$  such that for all  $(t,x), (s,y) \in [0,T] \times [0,+\infty)$ , we have*

$$u(t,x) \leq v(s,y) + C(1 + |x - y|).$$

*Proof of Theorem 4.1.* The proof proceeds in several steps.

**Step 1: Penalization procedure.** As it is explained in Remark 1.9, we assume that  $u$  is upper semi-continuous and  $v$  is lower semi-continuous. We want to prove that

$$M = \sup_{(t,x) \in [0,T] \times [0,+\infty)} (u(t,x) - v(t,x)) \leq 0.$$

Assume by contradiction that  $M > 0$ . Let us define

$$M_{\delta,\alpha} = \sup_{(t,x),(s,y) \in [0,T] \times [0,+\infty)} \left\{ u(t,x) - v(s,y) - \delta\varphi\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right) - \frac{\eta}{T-t} - \frac{\eta}{T-s} - \frac{\alpha x^2}{2} \right\}$$

where  $\delta, \eta, \alpha$  are positive constants. Then for  $\alpha, \eta$  small enough, we have  $M_{\delta,\alpha} \geq \frac{M}{2} > 0$ . Indeed, by definition of the supremum  $M$ , there exists  $(t_0, x_0) \in [0, T] \times [0, +\infty)$  such that

$$u(t_0, x_0) - v(t_0, x_0) \geq \frac{3M}{4},$$

so

$$M_{\delta,\alpha} \geq u(t_0, x_0) - v(t_0, x_0) - \frac{2\eta}{T-t_0} - \alpha \frac{x_0^2}{2} \geq \frac{M}{2},$$

for  $\alpha, \eta$  small enough. We want to show that this supremum is reached. For all  $x, y, t, s$  such that

$$0 < \frac{M}{2} \leq u(t, x) - v(s, y) - \delta \varphi \left( \frac{t-s}{\delta}, \frac{x-y}{\delta} \right) - \frac{\eta}{T-t} - \frac{\eta}{T-s} - \alpha \frac{x^2}{2}, \quad (2.43)$$

by Lemma 4.4, we have

$$0 < \frac{M}{2} \leq C_T(1 + |x-y|) - \delta \varphi \left( \frac{t-s}{\delta}, \frac{x-y}{\delta} \right) - \frac{\eta}{T-t} - \frac{\eta}{T-s} - \alpha \frac{x^2}{2}, \quad (2.44)$$

so we deduce that

$$\delta \varphi \left( \frac{t-s}{\delta}, \frac{x-y}{\delta} \right) \leq C_T(1 + |x-y|), \quad (2.45)$$

and that

$$\frac{(\alpha x)^2}{2} \leq \alpha C_T(1 + |x-y|) \quad (2.46)$$

By dividing (2.45) by  $1 + |(t-s, x-y)|$ , the property (2.40) of  $\varphi$  implies that  $x-y$  and  $t-s$  are bounded, independently of  $\alpha$ , for  $x, y, t, s$  satisfying (2.43). So using (2.46),  $x, y, t, s$  are in a compact set so the supremum  $M_{\delta, \alpha}$  is reached at some point  $(t, x, s, y) = (t_\delta, x_\delta, s_\delta, y_\delta)$ . Moreover, for  $\delta \rightarrow 0$ , using any converging subsequence and (2.45) dividing by  $1 + |(t-s, x-y)|$ , using the property (2.40) and (2.41), we deduce that,  $t_\delta - s_\delta$  and  $x_\delta - y_\delta$  go to 0.

**Step 2: Use of the initial condition.** If  $t_\delta$  or  $s_\delta = 0$  along a subsequence then  $t_\delta, s_\delta \rightarrow 0$  since  $\delta \rightarrow 0$  by the previous step so, up to extract once again,  $(x_\delta, y_\delta) \rightarrow (x_0, x_0)$ . So we get from (2.43),

$$0 < \frac{M}{2} \leq u(t_\delta, x_\delta) - v(s_\delta, y_\delta).$$

So letting  $\delta \rightarrow 0$ , the limit superior of the right hand side is smaller than  $u_0(x_0) - u_0(x_0) = 0$  and we get a contradiction.

**Step 3: Use of viscosity inequalities.** We can now assume that  $t_\delta > 0$  and  $s_\delta > 0$  and write the viscosity inequalities at  $(t, x, s, y) = (t_\delta, x_\delta, s_\delta, y_\delta)$ .

**Case 1:** If  $x = 0$  and  $\min(H, F) = F$  at  $\varphi_x \left( \frac{t-s}{\delta}, \frac{-y}{\delta} \right)$ .

The inequality for the sub-solution is

$$\frac{\eta}{(T-t)^2} + \varphi_t \left( \frac{t-s}{\delta}, \frac{-y}{\delta} \right) + F \left( \varphi_x \left( \frac{t-s}{\delta}, \frac{-y}{\delta} \right) \right) \leq 0.$$

Using property (2.42), we get a positive left-hand side which gives a contradiction.

**Case 2:** If  $y = 0$  and  $\max(H, F) = F$  at  $\varphi_x \left( \frac{t-s}{\delta}, \frac{x}{\delta} \right)$ .

The inequality for the super-solution is

$$-\frac{\eta}{(T-s)^2} + \varphi_t \left( \frac{t-s}{\delta}, \frac{x}{\delta} \right) + F \left( \varphi_x \left( \frac{t-s}{\delta}, \frac{x}{\delta} \right) \right) \geq 0.$$

Using property (2.42), we get a negative left-hand side which gives a contradiction.

**Case 3:** Other cases.

The inequality for the sub-solution is

$$\frac{\eta}{(T-t)^2} + \varphi_t\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right) + H\left(\varphi_x\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right) + \alpha x\right) \leq 0,$$

and the inequality for the super-solution is

$$-\frac{\eta}{(T-s)^2} + \varphi_t\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right) + H\left(\varphi_x\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right)\right) \geq 0.$$

Subtracting these inequalities, we get

$$\frac{2\eta}{T^2} \leq H\left(\varphi_x\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right)\right) - H\left(\varphi_x\left(\frac{t-s}{\delta}, \frac{x-y}{\delta}\right) + \alpha x\right). \quad (2.47)$$

Since  $t-s$  and  $x-y$  are bounded independently of  $\alpha$  and since  $\alpha x$  goes to 0 when  $\alpha \rightarrow 0$ , thanks to (2.46), using the fact that  $H$  is uniformly continuous in compact subsets, the right hand side of (2.47) goes to 0 when  $\alpha \rightarrow 0$ , we get a contradiction. The proof is now complete.  $\square$

#### 4.4 Construction of the test function

The idea is to construct a test function coupling time and space, of the form

$$\varphi(t, x) = f(t) + g(x) + xE(t),$$

where the functions  $f, g, E : \mathbb{R} \rightarrow \mathbb{R}$  are of class  $\mathcal{C}^1$ . In this section, the function  $F$  satisfies the hypothesis of Theorem 4.1. Let us first define a function  $G$ , we will next use it to define the function  $E$ .

**Definition 4.5** (Function  $G$ ). *Let  $G$  be a continuous function such that*

- $G \geq \max((( -F)^{-1})', (-2F)^{-1}) > 0$ ,
- $G$  is even i.e.  $\forall t \in \mathbb{R}, \quad G(-t) = G(t)$ ,
- $G$  is non-increasing in  $(-\infty, 0]$  and non-decreasing on  $[0, +\infty)$ .

*Remark 4.6.* The function  $G$  exists since  $\max((( -F)^{-1})', (-2F)^{-1})$  is continuous and  $(( -F)^{-1})'$  is positive. Moreover, we have

$$\lim_{x \rightarrow \pm\infty} G(x) = +\infty,$$

since  $(-2F)^{-1}$  is increasing and goes to  $+\infty$  at  $+\infty$ .



**Proposition 4.7** (Function  $E$ ). *Assume  $F$  is of class  $\mathcal{C}^1$  and satisfies  $F' < 0$ ,  $F(0) = 0$  and (2.4)-(2.5). Then there exists a function  $E$  of class  $\mathcal{C}^1$  solution of the ODE*

$$\begin{cases} E' = \frac{1}{G(-2F(E))} \\ E(0) = 0, \end{cases} \quad (2.48)$$

which satisfies the same properties as  $-F$ , i.e.,  $E' > 0$ ,  $E(0) = 0$  and

$$\lim_{x \rightarrow -\infty} E(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow +\infty} E(x) = +\infty. \quad (2.49)$$

Moreover, we have

$$\lim_{x \rightarrow \pm\infty} E'(x) = 0. \quad (2.50)$$

*Proof of Proposition 4.7.* The existence of a solution for (2.48) is given by Cauchy-Peano-Arzela global existence theorem. Indeed, since  $0 < ((-F)^{-1})'(0) \leq G$ , we have  $0 < \frac{1}{G} \leq \frac{1}{((-F)^{-1})'(0)}$  so the function

$$\frac{1}{G(-2F)}$$

is bounded and continuous. Moreover, since  $G \geq ((-F)^{-1})' > 0$ , we have  $E' > 0$ . Let us prove that  $E$  satisfies (2.49) by contradiction. If  $E$  has a finite limit then using (2.48),  $E'$  has a finite limit  $L > 0$  so

$$E(t) \sim Lt$$

and  $E$  has an infinite limit which is a contradiction. We deduce (2.50) using (2.48).  $\square$

Let us define the function  $f$ .

**Definition 4.8** (Function  $f$ ). *Let  $f$  be the function of class  $\mathcal{C}^1$  such that  $f'(t) = -F(E(t))$  and  $f(0) = 0$ .*

Let us define the function  $g$ . First, we define some functions  $\psi$ ,  $\psi_1$  and  $\psi_2$ ,

$$\begin{aligned} \psi(t, x) &= (-F)^{-1}(xE'(t) - F(E(t))) - E(t), \\ \psi_1(x) &= \sup_{t \in \mathbb{R}} \psi(t, x), \\ \psi_2(x) &= \inf_{t \in \mathbb{R}} \psi(t, x). \end{aligned}$$

**Proposition 4.9.** *The function  $\psi_1$  is lower semi-continuous and locally bounded in  $[0, +\infty)$ , continuous at 0 and satisfies  $\psi_1(0) = 0$ . The function  $\psi_2$  is upper semi-continuous and locally bounded in  $(-\infty, 0]$ , continuous at 0 and satisfies  $\psi_2(0) = 0$ .*

*Proof of Proposition 4.9.* The function  $\psi_1$  (resp.  $\psi_2$ ) is lower (resp. upper) semi-continuous because it is a supremum (resp. infimum) of continuous functions.

Let us prove that  $\psi_1$  and  $\psi_2$  are locally bounded and continuous at 0. By using the Taylor expansion of the function  $(-F)^{-1}$  of class  $\mathcal{C}^1$ , there exists  $\theta : \mathbb{R}^2 \rightarrow [0, 1]$  such that

$$\psi(t, x) = xE'(t)((-F)^{-1})'(-F(E(t))) + \theta(t, x)x^2E'(t).$$

If  $0 \leq x \leq R$ , for  $R > 0$ , since  $G \geq ((-F)^{-1})' > 0$ , we have

$$\begin{aligned} 0 \leq \psi(t, x) &\leq xE'(t)G(-F(E(t)) + \theta(t, x)xE'(t)) \\ &\leq xE'(t)G(-F(E(t)) + RE'(t)). \end{aligned} \quad (2.51)$$

Let us prove that the continuous function  $h : t \rightarrow E'(t)G(-F(E(t)) + RE'(t))$  is bounded in  $\mathbb{R}$ . Since  $h$  is continuous, we only need to prove that  $h$  is bounded for  $|t|$  big enough. Using (2.50), for  $t \geq 0$  big enough, we have  $RE'(t) \leq 1$  and  $-F(E(t)) + 1 \leq -2F(E(t))$ . Using that  $G$  is non-decreasing in  $[0, +\infty)$ , we deduce from (2.48) that

$$0 \leq h(t) \leq E'(t)G(-F(E(t)) + 1) \leq \frac{G(-F(E(t)) + 1)}{G(-2F(E(t)))} \leq 1.$$

By the same argument, for  $t \leq 0$  small enough, we have  $RE'(t) \geq -1$  and  $-F(E(t)) - 1 \geq -2F(E(t))$ . So since  $G$  is non-increasing in  $(-\infty, 0]$ , we deduce with (2.48) that

$$0 \leq h(t) \leq E'(t)G(-F(E(t)) - 1) \leq 1.$$

We deduce from (2.51) that  $\psi_1$  is locally bounded in  $[0, +\infty)$  and that  $\psi_1(0) = 0$ . By the same arguments, we also deduce that  $\psi_2$  is locally bounded in  $(-\infty, 0]$  and that  $\psi_2(0) = 0$ . The proof is now complete.  $\square$

**Lemma 4.10** (Function  $g$ ). *Let  $g$  be a function of class  $\mathcal{C}^1$  such that  $g(0) = 0$  and such that  $g'$  satisfies  $g'(0) = 0$  and*

$$g'(x) \geq \max(2x, \psi_1(x)) \quad \text{for } x \geq 0,$$

and

$$g'(x) \leq \min(2x, \psi_2(x)) \quad \text{for } x \leq 0.$$

*Proof.* The construction of the function  $g'$  is a consequence of the fact that  $\psi_1$  and  $\psi_2$  are locally bounded and continuous at 0.  $\square$

Now, we can prove that the function  $\varphi$  defined by  $\varphi(t, x) = f(t) + g(x) + xE(t)$  satisfies (2.42).

**Proposition 4.11.** *The function  $\varphi(t, x) = f(t) + g(x) + xE(t)$  satisfies (2.42).*

*Proof of Proposition 4.11.* Since the function  $g$  satisfies for all  $t \in \mathbb{R}$ ,

$$g'(x) \geq \psi_1(x) \geq \psi(t, x) = ((-F)^{-1})(xE'(t) - F(E(t))) - E(t) \quad \text{for } x \geq 0,$$

and

$$g'(x) \leq \psi_2(x) \leq \psi(t, x) = ((-F)^{-1})(xE'(t) - F(E(t))) - E(t) \quad \text{for } x \leq 0,$$

and since  $(-F)^{-1}$  is increasing, we deduce that

$$-F(E(t)) + xE'(t) + F(g'(x) + E(t)) \leq 0 \quad \text{for } x \geq 0,$$

and

$$-F(E(t)) + xE'(t) + F(g'(x) + E(t)) \geq 0 \quad \text{for } x \leq 0.$$

These inequalities are exactly (2.42).  $\square$

Let us prove that the function  $\varphi$  satisfies (2.40) and (2.41).

**Proposition 4.12.** *The function  $\varphi$  is of class  $\mathcal{C}^1$  and superlinear (2.40).*

*Proof of Proposition 4.12.* By construction, the function  $\varphi$  is of class  $\mathcal{C}^1$ . With the definition of  $g$  in hand, we deduce that  $g(x) \geq x^2$ . Using that

$$|xE(t)| \leq \frac{x^2}{2} + \frac{E(t)^2}{2},$$

we deduce that

$$\begin{aligned} \varphi(t, x) &\geq f(t) + x^2 - \frac{E(t)^2}{2} - \frac{x^2}{2}, \\ &= f(t) - \frac{E(t)^2}{2} + \frac{x^2}{2}. \end{aligned} \quad (2.52)$$

Let us prove that  $\frac{E^2}{2f}$  goes to 0 when  $|t| \rightarrow +\infty$ . We first compare their derivative which are simpler. We have

$$\begin{aligned} \frac{2f'(t)}{(E^2)'(t)} = \frac{-F(E(t))}{E'(t)E(t)} &= \frac{-F(E(t))G(-2F(E(t)))}{E(t)}, \\ &\geq \frac{-F(E(t))(-2F)^{-1}(-2F(E(t)))}{E(t)} \\ &\geq -F(E(t)). \end{aligned} \quad (2.53)$$

where the last term goes to  $+\infty$  since  $t$  goes to  $+\infty$ . We have the same result for  $t \leq 0$  using the same argument and the fact that  $G$  is even,

$$\frac{2f'(t)}{(E^2)'(t)} \geq F(E(t)),$$

where the last term goes to  $+\infty$  since  $t$  goes to  $-\infty$ . We deduce that

$$\frac{(E^2)'(t)}{f'(t)} \rightarrow 0 \quad \text{for } t \rightarrow \pm\infty.$$

Since  $\int_0^t E^2'(s) ds = E^2(t)$  diverges when  $t \rightarrow \pm\infty$ , we have

$$\frac{\int_0^t (E^2)'(s) ds}{\int_0^t f'(s) ds} \rightarrow 0,$$

so

$$\frac{E(t)^2}{f(t)} \rightarrow 0 \quad \text{for } t \rightarrow \pm\infty.$$

And since  $f$  is superlinear (2.40),  $t \rightarrow f(t) - \frac{E(t)^2}{2}$  is superlinear. We deduce, from (2.52) that  $\varphi$  satisfies (2.40).  $\square$

**Proposition 4.13.** *The function  $\varphi$  satisfies (2.41).*

*Proof of Proposition 4.13.* The function  $\varphi$  is of class  $\mathcal{C}^1$ , satisfies  $\varphi(0, 0) = 0$  and is superlinear (2.40) in  $(t, x)$ . Let us prove that its local extremum is reached only at the point  $(0, 0)$  and this implies (2.41). Let  $(t, x) \in \mathbb{R}^2$  satisfy,

$$\begin{cases} \varphi_t(t, x) = -F(E(t)) + xE'(t) = 0 \\ \varphi_x(t, x) = g'(x) + E(t) = 0. \end{cases} \quad (2.54)$$

First, we notice that for  $(t, x)$  satisfying (2.54),  $t = 0$  if and only if  $x = 0$ . Assume by contradiction that  $(t, x) \neq (0, 0)$ . Let us prove that  $t = 0$  as soon as  $x > 0$  and  $(t, x)$  satisfies (2.54). If  $x > 0$ , we have taking  $s = 0$

$$\begin{aligned} -E(t) = g'(x) &\geq \sup_{s \in \mathbb{R}} \{((-F)^{-1}(xE'(s)) - F(E(s))) - E(s)\} \\ &\geq (-F)^{-1}(xE'(0)), \end{aligned}$$

so we have

$$E(t) \leq F^{-1}(xE'(0)).$$

And we also have, since  $F$  is decreasing,

$$xE'(t) = F(E(t)) \geq F(F^{-1}(xE'(0))) = xE'(0).$$

If  $t \geq 0$ , since  $E'$  is non-increasing in  $[0, +\infty)$ , we deduce that  $t \leq 0$  so  $t = 0$  and  $x = 0$ , which gives a contradiction. If  $t \leq 0$ , since  $E'$  is non-decreasing, we deduce that  $t \geq 0$  so  $t = 0$  and  $x = 0$ , which also gives a contradiction. The case  $x < 0$  is similar so we skip it. This ends the proof.  $\square$

*Proof of Theorem 4.2.* Combine Propositions 4.11, 4.12 and 4.13.  $\square$

## 5 Appendix

### Reformulation of state constraints

Let us prove the reformulation of state constraint result in the case where the Hamiltonian is not necessarily convex.

**Theorem 5.1** (Reformulation of state constraints). *Assume  $H : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and coercive (2.3) and  $u : (0, T) \times [0, +\infty) \rightarrow \mathbb{R}$  satisfies (2.12) then  $u$  is a viscosity solution of*

$$\begin{cases} u_t + H(u_x) = 0 & \text{in } (0, T) \times (0, +\infty) \\ u_t + H(u_x) \geq 0 & \text{in } (0, T) \times \{0\}, \end{cases} \quad (2.55)$$

if and only if  $u$  is a viscosity solution of

$$\begin{cases} u_t + H(u_x) = 0 & \text{in } (0, T) \times (0, +\infty) \\ u_t + H^-(u_x) = 0 & \text{on } (0, T) \times \{0\}, \end{cases} \quad (2.56)$$

where  $H^-$  is the decreasing part of the Hamiltonian defined by

$$H^-(p) = \inf_{q \leq p} H(q).$$

First we prove that  $F_{A_{H^-}} = H^-$  that allows us to use Theorem 3.16 of reduction of the set of test functions.

**Definition 5.2** (Set of effective points  $A_0$ ). *Let  $H : \mathbb{R} \rightarrow \mathbb{R}$  be continuous and coercive (2.3). The set of effective points  $A_0$  is the set of points  $p \in \mathbb{R}$  such that*

- $p^- = p < p^+$ ,
- $\forall q \in \mathbb{R}$  such that  $]q^-, q^+[\cap]p, p^+[ \neq \emptyset$ , we have  $H(q) \geq H(p)$ .

**Lemma 5.3.** *We have  $A_{H^-} = A_0$ .*

*Proof of Lemma 5.3.* Notice first that  $H^- \leq H$  and that  $H^-$  is non-increasing. Using Definition 3.21, it only remains to prove that for all  $p \in A_{H^-}$  we have  $p^- = p$ . Assume by contradiction that there exists  $p \in A_{H^-}$  such that  $p^- < p$ . Then using Proposition 3.22 we deduce that  $p$  satisfies (ii) of (2.25) so  $H(p) = H^-(p)$ . We deduce from Lemma 3.4 that

$$\forall q \in ]p^-, p[ \quad H^-(q) \leq H(q) < H(p) = H^-(p),$$

but  $H^-$  is non-increasing which gives a contradiction. So we have  $p^- = p$ . We deduce that  $A_{H^-} = A_0$ .  $\square$

**Lemma 5.4.** *We have  $F_{A_{H^-}} = F_{A_0} = H^-$ .*

*Proof of Lemma 5.4.* From Lemma 5.3, we deduce that  $F_{H^-} = F_{A_0}$ . Let us prove that  $F_{A_0} = H^-$ . Notice first that

$$F_{A_0} \leq H. \tag{2.57}$$

Let  $p \in \mathbb{R}$ .

If there exists  $p_\alpha \in A_0$  such that  $p \in ]p_\alpha, p_\alpha^+[$  then we have

$$H^-(p) \leq F_{A_0}(p) = H(p_\alpha).$$

Moreover, from Lemma 3.4 we have

$$\forall q \in ]p_\alpha, p[ \quad H(p_\alpha) < H(q)$$

and since  $F_{A_0}$  is non-increasing and by (2.57), we have also

$$\forall q \leq p_\alpha \quad H(p_\alpha) = F_{A_0}(p_\alpha) \leq F_{A_0}(q) \leq H(q).$$

So we have

$$H^-(p) = \inf_{q \leq p} H(q) = H(p_\alpha) = F_{A_0}(p).$$

If  $p \notin \bigcup_{p_\alpha \in A_0} ]p_\alpha, p_\alpha^+[$ , then

$$F_{A_0}(p) = H(p) \geq H^-(p).$$

Moreover, since  $F_{A_0}$  is non-increasing and by (2.57), we have

$$\forall q \leq p \quad H(p) = F_{A_0}(p) \leq F_{A_0}(q) \leq H(q).$$

So  $F_{A_0}(p) = H(p) = H^-(p)$ . We deduce that  $F_{A_0} = H^-$ .  $\square$

The proof is exactly the same as in [61, 66].

*Proof of Theorem 5.1.* We do the proof in three steps.

**1st step:** Let us prove that

$$u_t + H(u_x) \leq 0 \quad \text{in } (0, T) \times (0, +\infty),$$

implies

$$u_t + H^-(u_x) \leq 0 \quad \text{on } (0, T) \times \{0\}.$$

Since  $\forall p_\alpha \in A_0$ ,  $p_\alpha^- = p_\alpha$ , using Theorem 3.16, we deduce that  $u$  is a strong viscosity sub-solution with  $F_{A_0}$ , so

$$u_t + F_{A_0}(u_x) \leq 0 \quad \text{on } (0, T) \times \{0\}.$$

Since  $F_{A_0}(u_x) = H^-(u_x)$ , we have

$$u_t + H^-(u_x) \leq 0 \quad \text{on } (0, T) \times \{0\}.$$

**2nd step:** Let us prove that

$$u_t + H(u_x) \geq 0 \quad \text{in } (0, T) \times [0, +\infty),$$

implies

$$u_t + H^-(u_x) \geq 0 \quad \text{on } (0, T) \times \{0\}.$$

Let  $\varphi$  be a test function touching  $u_*$  from below at  $(t_0, 0)$ . Using Theorem 3.16, we assume that

$$\varphi(t, x) = \psi(t) + \phi_\alpha(x),$$

where  $\psi \in \mathcal{C}^1((0, T))$  and

$$\phi_\alpha \in \mathcal{C}^1([0, +\infty)), \quad \phi'_\alpha(0) = p_\alpha.$$

We have  $\varphi_x(t_0, 0) = p_\alpha$  and

$$H(\varphi_x(t_0, 0)) = H(p_\alpha) = F_{A_0}(p_\alpha) = H^-(p_\alpha) = H^-(\varphi_x(t_0, 0)),$$

so by hypothesis, we have  $\varphi_t + H(\varphi_x(t_0, 0)) \geq 0$ . We deduce that

$$\varphi_t + H^-(\varphi_x(t_0, 0)) \geq 0.$$

**3rd step:** The reverse come from the fact that  $H^- \leq H$ . □



# Error estimates for finite difference schemes associated with Hamilton-Jacobi equations on a junction

In collaboration with Marwa Koumaiha

This paper is concerned with monotone (time-explicit) finite difference schemes associated with first order Hamilton-Jacobi equations posed on a junction. They extend the schemes recently introduced by Costeseque, Lebacque and Monneau (2013) to general junction conditions. On the one hand, we prove the convergence of the numerical solution towards the viscosity solution of the Hamilton-Jacobi equation as the mesh size tends to zero for general junction conditions. On the other hand, we derive optimal error estimates of order  $(\Delta x)^{\frac{1}{2}}$  in  $L_{loc}^\infty$  for junction conditions of optimal-control type at least if the flux is “strictly limited”.

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## 1 Introduction

This paper is concerned with numerical approximation of first order Hamilton-Jacobi equations posed on a junction, that is to say a network made of one node and a finite number of edges.

The theory of viscosity solutions for such equations on such domains has reached maturity by now [91, 92, 2, 68, 66]. In particular, it is now understood that general junction conditions reduce to special ones of optimal-control type [66]. Roughly speaking, it is proved in [66] that imposing a junction condition ensuring the existence of a continuous viscosity solution and a comparison principle is equivalent to imposing a junction condition obtained by “limiting the flux” at the junction point.

For the “minimal” *flux-limited* junction conditions, Costeseque, Lebacque and Monneau [36] introduced a monotone numerical scheme and proved its convergence. Their scheme can be naturally extended to general junction conditions and our first contribution is to introduce it and to prove its convergence.

Our second and main result is an error estimate à la Crandall-Lions [37] in the case of flux-limited junction conditions. It is explained in [37] that the proof of the comparison principle between sub- and super-solutions of the continuous Hamilton-Jacobi equation can be adapted in order to derive error estimates between the numerical solution associated with monotone (stable and consistent) schemes and the continuous solution. In the Euclidian case, the comparison principle is proved thanks to the technique of doubling variables; it relies on the classical penalisation term  $\varepsilon^{-1}|x - y|^2$ . Such a penalisation procedure is known to fail in general if the equation is posed in a

junction; it is explained in [66] that it has to be replaced with a *vertex test function*. Here we replace it by the *reduced minimal action* introduced in [68] for the “minimal” *flux-limited* junction conditions (i.e. for  $F_{A_0}$ , see (3.4)-(3.6)). We study and use it in the case where the flux is “strictly” limited (i.e. for  $A > A_0$ ).

In order to derive error estimates as in [37], it is important to study the regularity of the test function. More precisely, we prove (Proposition 5.12) that its gradient is locally Lipschitz continuous, at least if the flux is “strictly limited” and far away from a special curve. But we also see that the reduced minimal action (defined in (3.42)) is not of class  $C^1$  on this curve. However we can get “weaker” viscosity inequalities thanks to a result in [66] (see Proposition 2.6). Such a regularity result is of independent interest.

## 1.1 Hamilton-Jacobi equations posed on junctions

A *junction* is a network made of one node and a finite number of infinite edges. It can be viewed as the set of  $N$  distinct copies ( $N \geq 1$ ) of the half-line which are glued at the origin. For  $\alpha = 1, \dots, N$ , each branch  $J_\alpha$  is assumed to be isometric to  $[0, +\infty)$  and

$$J = \bigcup_{\alpha=1, \dots, N} J_\alpha \quad \text{with} \quad J_\alpha \cap J_\beta = \{0\} \quad \text{for} \quad \alpha \neq \beta$$

where the origin 0 is called the *junction point*. For points  $x, y \in J$ ,  $d(x, y)$  denotes the geodesic distance on  $J$  defined as

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \text{ belong to the same branch,} \\ |x| + |y| & \text{if } x, y \text{ belong to different branches.} \end{cases}$$

With such a notation in hand, we consider the following Hamilton-Jacobi equation posed on the junction  $J$ ,

$$\begin{cases} u_t + H_\alpha(u_x) = 0 & \text{in } (0, T) \times J_\alpha \setminus \{0\}, \\ u_t + F\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right) = 0 & \text{in } (0, T) \times \{0\}, \end{cases} \quad (3.1)$$

submitted to the initial condition

$$u(0, x) = u_0(x), \quad \text{for } x \in J \quad (3.2)$$

where  $u_0$  is globally Lipschitz in  $J$ . The second equation in (3.1) is referred to as the *junction condition*. Let us recall that  $H_\alpha$  is quasi-convex if the sub-level sets  $\{p : H_\alpha(p) \leq \lambda\}$  are convex. We consider the important case of quasi-convex Hamiltonians  $H_\alpha$  satisfying the following conditions:

$$\text{There exists } p_0^\alpha \in \mathbb{R} \text{ such that } \begin{cases} H_\alpha \in C^2(\mathbb{R}) \text{ and } H_\alpha''(p_0^\alpha) > 0 \\ \pm H_\alpha'(p) > 0 \text{ for } \pm(p - p_0^\alpha) > 0 \\ \lim_{|p| \rightarrow +\infty} H_\alpha(p) = +\infty. \end{cases} \quad (3.3)$$

In particular  $H_\alpha$  is non-increasing in  $(-\infty, p_0^\alpha]$  and non-decreasing in  $[p_0^\alpha, +\infty)$ , and we set

$$H_\alpha^-(p) = \begin{cases} H_\alpha(p) & \text{for } p \leq p_0^\alpha \\ H_\alpha(p_0^\alpha) & \text{for } p \geq p_0^\alpha \end{cases} \quad \text{and} \quad H_\alpha^+(p) = \begin{cases} H_\alpha(p_0^\alpha) & \text{for } p \leq p_0^\alpha \\ H_\alpha(p) & \text{for } p \geq p_0^\alpha \end{cases}$$

where  $H_\alpha^-$  is non-increasing and  $H_\alpha^+$  is non-decreasing.

We next introduce a one-parameter family of junction conditions: given a *flux limiter*  $A \in \mathbb{R} \cup \{-\infty\}$ , the  $A$ -limited flux junction function is defined for  $p = (p_1, \dots, p_N)$  as,

$$F_A(p) = \max \left( A, \max_{\alpha=1, \dots, N} H_\alpha^-(p_\alpha) \right) \quad (3.4)$$

for some given  $A \in \mathbb{R} \cup \{-\infty\}$  where  $H_\alpha^-$  is non-increasing part of  $H_\alpha$ .

We now consider the following important special case of (3.1),

$$\begin{cases} u_t + H_\alpha(u_x) = 0 & \text{in } (0, T) \times J_\alpha \setminus \{0\}, \\ u_t + F_A\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right) = 0 & \text{in } (0, T) \times \{0\}. \end{cases} \quad (3.5)$$

We point out that for all  $A \in [-\infty, A_0]$ , we have  $F_A = F_{A_0}$  if one chooses

$$A_0 = \max_{\alpha=1, \dots, N} \min_{\mathbb{R}} H_\alpha. \quad (3.6)$$

As far as general junction conditions are concerned, we assume that the junction function  $F : \mathbb{R}^n \mapsto \mathbb{R}$  satisfies

$$\begin{cases} F \text{ is continuous and piecewise } C^1(\mathbb{R}^n), \\ \forall \alpha, \forall p = (p_1, \dots, p_N) \in \mathbb{R}^N, \frac{\partial F}{\partial p_\alpha}(p) < 0, \\ F(p_1, \dots, p_N) \rightarrow +\infty \text{ as } \min_{i \in \{1, \dots, N\}} p_i \rightarrow -\infty. \end{cases} \quad (3.7)$$

**Hypothesis in the following of the paper:**  $p_0^\alpha = 0$ . Without loss of generality (see [66, Lemma 3.1]), we consider in this paper that  $p_0^\alpha = 0$  for  $\alpha = 1, \dots, N$ , i.e.,

$$\min_{\lambda \in \mathbb{R}} H_\alpha(\lambda) = H_\alpha(0). \quad (3.8)$$

Indeed,  $u$  solves (3.5) if and only if  $\tilde{u}(t, x) := u(t, x) - p_0^\alpha x$  for  $x \in J_\alpha$  solves the same equation in which  $H_\alpha$  is replaced by  $\tilde{H}_\alpha(p) = H_\alpha(p + p_0^\alpha)$ . We have the same result for  $u^h$  the solution of the scheme (3.15).

**The optimal control framework.** It is well known that the Legendre-Fenchel conjugate is crucial in establishing a link between the general Cauchy problem (3.5)-(3.2) and a control problem [81]. Through this link, we obtain the representation formula for the exact solution. Before treating the case where the Hamiltonians  $H_\alpha$  satisfy (3.3), we first consider the case of Hamiltonians satisfying the hypotheses of [68] i.e.,

$$\begin{cases} \text{(Regularity)} & H_\alpha \text{ is of class } C^2 \\ \text{(Coercivity)} & \lim_{|p| \rightarrow +\infty} H_\alpha(p) = +\infty \\ \text{(Convexity)} & H_\alpha \text{ is convex and is the Legendre Fenchel transform of } L_\alpha \\ & \text{where } L_\alpha \text{ is of class } C^2 \text{ and satisfies (B0)}. \end{cases} \quad (3.9)$$

We recall that

$$H_\alpha(p) = L_\alpha^*(p) = \sup_{q \in \mathbb{R}} (pq - L_\alpha(q)). \quad (3.10)$$

We consider the following hypothesis for  $L_\alpha$ ,

**(B0)** There exists a constant  $\gamma > 0$  such that for all  $\alpha = 1, \dots, N$ , the  $C^2(\mathbb{R})$  functions  $L_\alpha$  satisfy  $L''_\alpha \geq \gamma > 0$ .

An optimal control interpretation of the Hamilton-Jacobi equation (3.5) is given in [19, 11, 81, 79]. We define the set of admissible controls at a point  $x \in J$  by

$$\mathcal{U}(x) = \begin{cases} \mathbb{R}e_{\alpha_0} & \text{if } x \in J_{\alpha_0}^*, \\ \cup_{\alpha=1, \dots, N} \mathbb{R}^+ e_\alpha & \text{if } x = 0. \end{cases}$$

For  $(s, y), (t, x) \in [0, T] \times J$  with  $s \leq t$ , we define the set of admissible trajectories from  $(s, y)$  to  $(t, x)$  by

$$\mathcal{A}(s, y; t, x) = \left\{ X \in W^{1,1}([s, t], \mathbb{R}^2) : \begin{array}{l} X(\tau) \in J \quad \text{for all } \tau \in (s, t) \\ \dot{X}(\tau) \in \mathcal{U}(X(\tau)) \quad \text{for a.e } \tau \in (s, t) \\ X(s) = y \quad \text{and } X(t) = x \end{array} \right\}. \quad (3.11)$$

For  $P = pe_i \in \mathcal{U}(x)$  with  $p \in \mathbb{R}$ , we define the Lagrangian on the junction

$$L(x, p) = \begin{cases} L_\alpha(p) & \text{if } x \in J_\alpha^*, \\ L_A(p) & \text{if } x = 0, \end{cases} \quad (3.12)$$

with

$$L_A(p) = \min \left( -A, \min_{\alpha=1, \dots, N} L_\alpha(p) \right). \quad (3.13)$$

The Hopf-Lax representation formula of the solution of (3.5)-(3.2) is given in [68, 5] by

$$u_{oc}(t, x) = \inf_{y \in J} \{ u_0(y) + \mathcal{D}(0, y; t, x) \} \quad (3.14)$$

with

$$\mathcal{D}(0, y; t, x) = \inf_{X \in \mathcal{A}(0, y; t, x)} \left\{ \int_0^t L(X(\tau), \dot{X}(\tau)) d\tau \right\}.$$

## 1.2 Presentation of the scheme

The domain  $(0, +\infty) \times J$  is discretized with respect to time and space. We choose a regular grid in order to simplify the presentation but it is clear that more general meshes could be used here. The space step is denoted by  $\Delta x$  and the time step by  $\Delta t$ . If  $h$  denotes  $(\Delta t, \Delta x)$ , the mesh (or grid)  $\mathcal{G}_h$  is chosen as

$$\mathcal{G}_h = \{n\Delta t : n \in \mathbb{N}\} \times J^{\Delta x}$$

where

$$J^{\Delta x} = \bigcup_{\alpha=1, \dots, N} J_\alpha^{\Delta x} \quad \text{with} \quad J_\alpha \supset J_\alpha^{\Delta x} \simeq \{i\Delta x : i \in \mathbb{N}\}.$$

It is convenient to write  $x_i^\alpha$  for  $i\Delta x \in J_\alpha$ .

A numerical approximation  $u^h$  of the solution  $u$  of the Hamilton-Jacobi equation is defined in  $\mathcal{G}_h$ ; the quantity  $u^h(n\Delta t, x_i^\alpha)$  is simply denoted by  $U_i^{\alpha, n}$ . We want it to

be an approximation of  $u(n\Delta t, x_i^\alpha)$  for  $n \in \mathbb{N}$ ,  $i \in \mathbb{N}$ , where  $\alpha$  stands for the index of the branch.

We consider the following time-explicit scheme: for  $n \geq 0$ ,

$$\begin{cases} \frac{U_i^{\alpha, n+1} - U_i^{\alpha, n}}{\Delta t} + \max\{H_\alpha^+(p_{i,-}^{\alpha, n}), H_\alpha^-(p_{i,+}^{\alpha, n})\} = 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ U_0^{\beta, n} := U_0^n, & i = 0, \quad \beta = 1, \dots, N \\ \frac{U_0^{n+1} - U_0^n}{\Delta t} + F(p_{0,+}^{1,n}, \dots, p_{0,+}^{N,n}) = 0, \end{cases} \quad (3.15)$$

where  $p_{i,\pm}^{\alpha, n}$  are the discrete (space) gradients defined by

$$p_{i,+}^{\alpha, n} := \frac{U_{i+1}^{\alpha, n} - U_i^{\alpha, n}}{\Delta x}, \quad p_{i,-}^{\alpha, n} := \frac{U_i^{\alpha, n} - U_{i-1}^{\alpha, n}}{\Delta x} \quad (3.16)$$

with the initial condition

$$U_i^{\alpha, 0} = u_0(x_i^\alpha), \quad i \geq 0, \quad \alpha = 1, \dots, N. \quad (3.17)$$

The following Courant-Friedrichs-Lewy (CFL) condition ensures that the explicit scheme is monotone,

$$\frac{\Delta x}{\Delta t} \geq \max \left\{ \max_{\substack{i \geq 0, \\ \alpha=1, \dots, N, \\ 0 \leq n \leq n_T}} |H'_\alpha(p_{i,+}^{\alpha, n})|; \max_{0 \leq n \leq n_T} \left\{ (-\nabla \cdot F)(p_{0,+}^{1,n}, \dots, p_{0,+}^{N,n}) \right\} \right\} \quad (3.18)$$

where the integer  $n_T$  is the integer part of  $\frac{T}{\Delta t}$  for a given  $T > 0$ .

### 1.3 Main results

As previously noticed in [36] in the special case  $F = F_{A_0}$ , it is not clear that the time step  $\Delta t$  and space step  $\Delta x$  can be chosen in such a way that the CFL condition (3.18) holds true since the discrete gradients  $p_{i,+}^{\alpha, n}$  depend itself on  $\Delta t$  and  $\Delta x$  (through the numerical scheme). We thus impose a more stringent CFL condition,

$$\frac{\Delta x}{\Delta t} \geq \max \left\{ \max_{\substack{\alpha=1, \dots, N, \\ \underline{p}_\alpha \leq p \leq \bar{p}_\alpha}} |H'_\alpha(p)|; \max_{\substack{\alpha=1, \dots, N, \\ \underline{p}_\alpha^0 \leq p_\alpha \leq \bar{p}_\alpha}} \left\{ (-\nabla \cdot F)(p_1, \dots, p_N) \right\} \right\} \quad (3.19)$$

for some  $\underline{p}_\alpha, \bar{p}_\alpha, \underline{p}_\alpha^0 \in \mathbb{R}$  to be fixed (only depending on  $u_0$ ,  $H$ , and  $F$ ). We can argue as in [36] and prove that  $\underline{p}_\alpha, \bar{p}_\alpha, \underline{p}_\alpha^0 \in \mathbb{R}$  can be chosen in such a way that the CFL condition (3.19) implies (3.18) and, in turn, the scheme is monotone (Lemma 4.1 in Section 4). We will also see that it is stable (Lemma 4.5) and consistent (Lemma 4.6). It is thus known that it converges [37, 18]. Notice that taking  $F = F_A$ , gives the following CFL condition

$$\frac{\Delta x}{\Delta t} \geq \max_{\substack{\alpha=1, \dots, N, \\ \underline{p}_\alpha \leq p \leq \bar{p}_\alpha}} |H'_\alpha(p)|. \quad (3.20)$$

**Theorem 1.1 (Convergence for general junction conditions).** *Let  $T > 0$  and  $u_0$  be Lipschitz continuous. There exist  $\underline{p}_\alpha, \bar{p}_\alpha, \underline{p}_\alpha^0 \in \mathbb{R}$ ,  $\alpha = 1, \dots, N$ , depending only*

on the initial data, the Hamiltonians and the junction function  $F$ , such that, if  $h$  satisfies the CFL condition (3.19), then the numerical solution  $u^h$  defined by (3.15)-(3.17) converges locally uniformly as  $h$  goes to zero to the unique relaxed viscosity solution  $u$  of (3.1)-(3.2), on any compact set  $\mathcal{K} \subset [0, T] \times J$ , i.e.

$$\limsup_{|h| \rightarrow 0} \sup_{(t,x) \in \mathcal{K} \cap \mathcal{G}_h} |u^h(t, x) - u(t, x)| = 0. \quad (3.21)$$

*Remark 1.2.* We know from [66] that the equation (3.1)-(3.2) may have no viscosity solution in “strong sense” but always a unique relaxed viscosity solution. Notice that the scheme has a junction condition which is not relaxed. However the solution of the scheme converges to the unique relaxed solution of the associated Hamilton-Jacobi equation.

The main result of this paper lies in getting error estimates in the case of flux-limited junction conditions.

**Theorem 1.3 (Error estimates for flux-limited junction conditions).** *Let  $T > 0$  and  $u_0$  be Lipschitz continuous,  $u^h$  be the solution of the associated numerical scheme (3.15)-(3.17) and  $u$  be the viscosity solution of (3.5)-(3.2) for some  $A \in \mathbb{R}$ . If the CFL condition (3.20) is satisfied, then there exists  $C > 0$  (independent of  $h$ ) such that*

$$\sup_{[0,T] \times J \cap \mathcal{G}_h} |u^h(t, x) - u(t, x)| \leq \begin{cases} C(\Delta x)^{1/2} & \text{if } A > A_0, \\ C(\Delta x)^{2/5} & \text{if } A = A_0. \end{cases} \quad (3.22)$$

*Remark 1.4.* In the case  $A > A_0$ , this error estimate is optimal as we can deduce in the numerical simulations (see subsection 6.2).

## 1.4 Related results

**Numerical schemes for Hamilton-Jacobi equations on networks.** The discretization of viscosity solutions of Hamilton-Jacobi equations posed on networks has been studied in a few papers only. Apart from [36] mentioned above, we are only aware of two other works. A convergent semi-Lagrangian scheme is introduced in [27] for equations of eikonal type. In [59], an adapted Lax-Friedrichs scheme is used to solve a traffic model; it is worth mentioning that this discretization implies to pass from the scalar conservation law to the associated Hamilton-Jacobi equation at each time step.

For optimal control problems, the numerical approximation of (HJ) has already been studied using schemes based on the discrete dynamic programming principle. Essentially, these schemes are built by replacing the continuous optimal control problem by its discrete time version. We refer to Capuzzo Dolcetta [29], Capuzzo Dolcetta-Ishii [30] for the results concerning the convergence of  $u_h$  to  $u$  and the a priori estimates (of order  $\Delta x$ ), in the  $L^\infty$ , giving the order of convergence of the discrete-time approximation. We refer to Falcone [50] for the results related to the order of convergence of the fully discrete (i.e. in space and time) approximation and for the construction of the algorithm, we mention that under a semiconcavity assumption the rate of convergence is of order 1. We cite also [51] and references therein for discrete time high order schemes for Hamilton Jacobi Bellman equations.

**Link with monotone schemes for scalar conservation laws.** We first follow [36] by emphasizing that the convergence result, Theorem 1.1, implies the convergence of a monotone scheme for scalar conservation laws (in the sense of distributions).

In order to introduce the scheme, it is useful to introduce a notation for the numerical Hamiltonian  $\mathcal{H}_\alpha$ ,

$$\mathcal{H}_\alpha(p^+, p^-) = \max\{H_\alpha^-(p^+), H_\alpha^+(p^-)\}.$$

The discrete solution  $(V^n)$  of the scalar conservation law is defined as follows,

$$V_{i+\frac{1}{2}}^{\alpha,n} = \begin{cases} \frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x} & \text{if } i \geq 1 \\ \frac{U_1^{\alpha,n} - U_0^n}{\Delta x} & \text{if } i = 0. \end{cases}$$

In view of (3.15), it satisfies for all  $\alpha = 1, \dots, N$ ,

$$\begin{cases} \frac{V_{i+\frac{1}{2}}^{\alpha,n+1} - V_{i+\frac{1}{2}}^{\alpha,n}}{\Delta t} + (\Delta x)^{-1} \left( \mathcal{H}_\alpha(V_{i+\frac{3}{2}}^{\alpha,n}, V_{i+\frac{1}{2}}^{\alpha,n}) - \mathcal{H}_\alpha(V_{i+\frac{1}{2}}^{\alpha,n}, V_{i-\frac{1}{2}}^{\alpha,n}) \right) = 0, & i \geq 1, \\ \frac{V_{\frac{1}{2}}^{\alpha,n+1} - V_{\frac{1}{2}}^{\alpha,n}}{\Delta t} + (\Delta x)^{-1} \left( \mathcal{H}_\alpha(V_{\frac{3}{2}}^{\alpha,n}, V_{\frac{1}{2}}^{\alpha,n}) - F(V_{\frac{1}{2}}^{1,n}, \dots, V_{\frac{1}{2}}^{N,n}) \right) = 0. \end{cases}$$

submitted to the initial condition

$$V_{i+\frac{1}{2}}^{\alpha,0} = \frac{u_0(x_i^\alpha) - u_0(0)}{\Delta x}, \quad i \geq 0, \quad \alpha = 1, \dots, N.$$

In view of Theorem 1.1, we thus can conclude that the discrete solution  $v^h$  constructed from  $(V^n)$  converges towards  $u_x$  in the sense of distributions, at least far from the junction point.

**Scalar conservation laws with Dirichlet boundary conditions and constrained fluxes.** We would like next to explain why our result can be seen as the Hamilton-Jacobi counterpart of the error estimates obtained by Ohlberger and Vovelle [87] for scalar conservation laws submitted to Dirichlet boundary conditions.

On the one hand, it is known since 1979 and Bardos, Le Roux and Nedelec [12] that Dirichlet boundary conditions imposed to scalar conservation laws should be understood in a generalized sense. This can be seen by studying the parabolic regularization of the problem. A boundary layer analysis can be performed for systems if the solution of the conservation law is smooth; see for instance [57, 63]. Depending on the fact that the boundary is characteristic or not, the error is  $h^{\frac{1}{2}}$  or  $h$ . In the scalar case, it is proved in [46] that the error between the solution of the regularized equation with a vanishing viscosity coefficient equal to  $h$  and the entropy solution of the conservation law (which is merely of bounded variation in space) is of order  $h^{1/3}$  (in  $L_t^\infty L_x^1$  norm). In [87], the authors derive error estimates for finite volume schemes associated with such boundary value problems and prove that it is of order  $(\Delta x)^{1/6}$  (in  $L_{t,x}^1$  norm). More recently, scalar conservation laws with flux constraints were studied [35, 33] and some finite volume schemes were built [7]. In [28], assuming that the flux is bell-shaped, that is to say the opposite is quasi-convex, it is proved that

the error between the finite volume scheme and the entropy solution is of order  $(\Delta x)^{\frac{1}{3}}$  and that it can be improved to  $(\Delta x)^{\frac{1}{2}}$  under an additional condition on the traces of the BV entropy solution. It is not known if the estimates from [28] are optimal or not.

On the other hand, the derivative of a viscosity solution of a Hamilton-Jacobi equation posed on the real line is known to coincide with the entropy solution of the corresponding scalar conservation law. It is therefore reasonable to expect that the error between the viscosity solution of the Hamilton-Jacobi equation and its approximation is as good as the one obtained between the entropy solution of the scalar conservation law and its approximation.

Moreover, it is explained in [68] that the junction conditions of optimal-control type are related to the BLN condition mentioned above; such a correspondance is recalled in Section 7.3. It is therefore interesting to get an error estimate of order  $(\Delta x)^{1/2}$  for the Hamilton-Jacobi problem.

## 1.5 Open problems

Let us first mention that it is not known if the error estimate between the (entropy) solution of the scalar conservation law with Dirichlet boundary condition and the solution of the parabolic approximation [46] or with the numerical scheme [87] is optimal or not. Here, we prove an optimal error estimate for  $A > A_0$  but we do not know if our error estimate is optimal or not for  $A = A_0$ .

Deriving error estimates for general junction conditions seems difficult to us. The main difficulty is the singular geometry of the domain. The test function, used in deducing the error estimates with flux limited solutions, is designed to compare flux limited solutions. Consequently, when applying the reasoning of Section 6, the discrete viscosity inequality cannot be combined with the continuous one. We expect that a layer develops between the continuous solution and the discrete scheme at the junction point.

**Organization of the article.** The remaining of the paper is organized as follows. In Section 2, we recall definitions and results from [66] about viscosity solutions for (3.1)-(3.2). Section 3 is dedicated to the derivation of discrete gradient estimates for the numerical scheme. In Section 4, the convergence result, Theorem 1.1 is proved. In Section 5, we study the reduced minimal action for a “strictly” limited flux and prove that the gradient is locally Lipschitz continuous (at least if the flux is strictly limited). The final section, Section 6, is dedicated to the proof of the error estimates.

## 2 Preliminaries

### 2.1 Viscosity solutions

We introduce the main definitions related to viscosity solutions for Hamilton-Jacobi equations that are used in the remaining. For a more general introduction to viscosity solutions, the reader could refer to Barles [19] and to Crandall, Ishii, Lions [38].



**Space of test functions.** For a smooth real valued function  $u$  defined on  $J$ , we denote by  $u^\alpha$  the restriction of  $u$  to  $(0, T) \times J_\alpha$ .

Then we define the natural space of functions on the junction:

$$C^1(J_T) = \{u \in C(J_T) : \forall \alpha = 1, \dots, N, u^\alpha \in C^1((0, T) \times J_\alpha)\}.$$

**Viscosity solutions.** In order to define classical viscosity solutions, we recall the definition of upper and lower semi-continuous envelopes  $u^*$  and  $u_*$  of a (locally bounded) function  $u$  defined on  $[0, T) \times J$ :

$$u^*(t, x) = \limsup_{(s, y) \rightarrow (t, x)} u(s, y) \quad u_*(t, x) = \liminf_{(s, y) \rightarrow (t, x)} u(s, y).$$

**Definition 2.1 (Viscosity solution).** Assume that the Hamiltonians satisfy (3.3) and that  $F$  satisfies (3.7) and let  $u: (0, T) \times J \rightarrow \mathbb{R}$ .

(i) We say that  $u$  is a sub-solution (resp. super-solution) of (3.1) in  $(0, T) \times J$  if for all test function  $\varphi \in C^1(J_T)$  such that

$$u^* \leq \varphi \quad (\text{resp. } u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in J_T$$

with equality at  $(t_0, x_0)$  for some  $t_0 > 0$ , we have

$$\varphi_t + H_\alpha(\varphi_x) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{at } (t_0, x_0) \in (0, T) \times J_\alpha$$

if  $x_0 \neq 0$ , else

$$\varphi_t + F_A\left(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_N}\right) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{at } (t_0, x_0) = (t_0, 0).$$

(ii) We say that  $u$  is a sub-solution (resp. super-solution) of (3.1)-(3.2) on  $[0, T) \times J$  if additionally

$$u^*(0, x) \leq u_0(x) \quad (\text{resp. } u_*(0, x) \geq u_0(x)) \quad \text{for all } x \in J.$$

(iii) We say that  $u$  is a (viscosity) solution if  $u$  is both a sub-solution and a super-solution.

As explained in [66], it is difficult to construct viscosity solutions in the sense of Definition 2.1 because of the junction condition. It is possible in the case of the flux-limited junction conditions  $F_A$ . For general junction conditions, the Perron process generates a viscosity solution in the following relaxed sense [66].

**Definition 2.2 (Relaxed viscosity solution).** Assume that the Hamiltonians satisfy (3.3) and that  $F$  satisfies (3.7) and let  $u: (0, T) \times J \rightarrow \mathbb{R}$ .

(i) We say that  $u$  is a relaxed sub-solution (resp. relaxed super-solution) of (3.1) in  $(0, T) \times J$  if for all test function  $\varphi \in C^1(J_T)$  such that

$$u^* \leq \varphi \quad (\text{resp. } u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in J_T$$

with equality at  $(t_0, x_0)$  for some  $t_0 > 0$ , we have

$$\varphi_t + H_\alpha(\varphi_x) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{at } (t_0, x_0) \in (0, T) \times J_\alpha$$

if  $x_0 \neq 0$ , else

$$\begin{cases} \text{either } \varphi_t + F(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_N}) \leq 0 \quad (\text{resp. } \geq 0) & \text{at } (t_0, x_0) = (t_0, 0) \\ \text{or } \varphi_t + H_\alpha(\frac{\partial \varphi}{\partial x_\alpha}) \leq 0 \quad (\text{resp. } \geq 0) & \text{at } (t_0, x_0) = (t_0, 0) \quad \text{for some } \alpha. \end{cases}$$

(ii) We say that  $u$  is a relaxed (viscosity) solution of (3.1) if  $u$  is both a sub-solution and a super-solution.

Let us recall some theorems in [66].

**Theorem 2.3 (Comparison principle on a junction).** *Let  $A \in \mathbb{R} \cup \{-\infty\}$ . Assume that the Hamiltonians satisfy (3.3) and the initial datum  $u_0$  is uniformly continuous. Then for all sub-solution  $u$  and super-solution  $v$  of (3.5)-(3.2) satisfying for some  $T > 0$  and  $C_T > 0$*

$$u(t, x) \leq C_T(1 + d(0, x)), \quad v(t, x) \geq -C_T(1 + d(0, x)), \quad \text{for all } (t, x) \in [0, T) \times J,$$

we have

$$u \leq v \quad \text{in } [0, T) \times J.$$

**Theorem 2.4 (General junction conditions reduce to flux-limited ones).** *Assume that the Hamiltonians satisfy (3.3) and that  $F$  satisfies (3.7). Then there exists  $A_F \in \mathbb{R}$  such that any relaxed viscosity (sub-/super-)solution of (3.1) is in fact a viscosity (sub-/super-)solution of (3.5) with  $A = A_F$ .*

**Theorem 2.5 (Existence and uniqueness on a junction).** *Assume that the Hamiltonians satisfy (3.3) and that  $F$  satisfies (3.7) and that the initial datum  $u_0$  is Lipschitz continuous. Then there exists a unique relaxed viscosity solution  $u$  of (3.1)-(3.2), such that*

$$|u(t, x) - u_0(x)| \leq Ct \quad \text{for all } (t, x) \in [0, T) \times J$$

for some constant  $C$  only depending on  $H$  and  $u_0$ . Moreover, it is Lipschitz continuous with respect to time and space, in particular,

$$\|\nabla u\|_\infty \leq C.$$

The following proposition is a main tool in the proof of error estimates. Indeed, we use a test function which is not  $C^1$  with respect to the gradient variable at one point and this proposition allows us to get a “weak viscosity inequality”. We don’t give the proof since it is the same as the proof of [66, Proposition 2.16].

**Proposition 2.6** (Non  $C^1$  test function at one point [66]). *Assume that  $H$  satisfies (3.3) and let  $u$  be a solution of*

$$u_t + H_\alpha(u_x) = 0 \quad \text{in } (0, T) \times J_\alpha \setminus \{0\}.$$

*For all  $x_0 \in J_\alpha \setminus \{0\}$  and all test function  $\varphi \in C^1((0, T) \times J_\alpha \setminus \{0, x_0\}) \cap C^0((0, T) \times J_\alpha)$*

$$u^* \leq \varphi \quad (\text{resp. } u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in (0, T) \times J_\alpha \setminus \{0\}$$

*with equality at  $(t_0, x_0)$ , we have*

$$\varphi_t(t_0, x_0) + \max \left\{ H_\alpha^+(\varphi_x(t_0, x_0^-), H_\alpha^-(\varphi_x(t_0, x_0^+)) \right\} \leq 0 \quad (\text{resp. } \geq 0),$$

*where*

$$\varphi_x(t_0, x_0^-) = \lim_{\substack{x \rightarrow x_0, \\ x < x_0}} \varphi_x(t_0, x) \quad \text{and} \quad \varphi_x(t_0, x_0^+) = \lim_{\substack{x \rightarrow x_0, \\ x > x_0}} \varphi_x(t_0, x).$$

### 3 Discrete gradient estimates

This section is devoted to the proofs of the discrete (time and space) gradient estimates. These estimates ensure the monotonicity of the scheme and, in turn, its convergence. The discrete time derivative is defined as

$$W_i^{\alpha, n} := \frac{U_i^{\alpha, n+1} - U_i^{\alpha, n}}{\Delta t}.$$

**Theorem 3.1 (Discrete gradient estimates).** *If  $u^h = (U_i^{\alpha, n})$  is the numerical solution of (3.15)-(3.17) and if the CFL condition (3.19) is satisfied and if*

$$m^0 = \inf_{\substack{\beta=1, \dots, N, \\ i \in \mathbb{N}}} W_i^{\beta, 0} \tag{3.23}$$

*is finite, then the following two properties hold true for any  $n \geq 0$ .*

(i) (Gradient estimate) *There exist  $\underline{p}_\alpha, \bar{p}^\alpha, \underline{p}_\alpha^0$  (only depending on  $H_\alpha, u_0$  and  $F$ ) such that*

$$\begin{cases} \underline{p}_\alpha \leq p_{i,+}^{\alpha, n} \leq \bar{p}^\alpha & i \geq 1, \alpha = 1, \dots, N, \\ \underline{p}_\alpha^0 \leq p_{0,+}^{\alpha, n} \leq \bar{p}^\alpha & i = 0, \alpha = 1, \dots, N. \end{cases} \tag{3.24}$$

(ii) (Time derivative estimate) *The discrete time derivative  $W_i^{\alpha, n}$  satisfies*

$$m^0 \leq m^n \leq m^{n+1} \leq M^{n+1} \leq M^n \leq M^0$$

*where*

$$m^n := \inf_{\alpha, i} W_i^{\alpha, n}, \quad M^n := \sup_{\alpha, i} W_i^{\alpha, n}.$$

In the proofs of discrete gradient estimates, “generalized” inverse functions of  $H_\alpha^\pm$  are needed; they are defined as follows:

$$\begin{cases} \pi_\alpha^+(a) := \sup\{p : H_\alpha^+(p) = \max(a, A_\alpha)\} \\ \pi_\alpha^-(a) := \inf\{p : H_\alpha^-(p) = \max(a, A_\alpha)\} \end{cases} \quad (3.25)$$

with the additional convention that  $(H_\alpha^\pm)^{-1}(+\infty) = \pm\infty$ , where

$$A_\alpha := \min_{\mathbb{R}} H_\alpha.$$

In order to define a “generalized” inverse function of  $F$ , we remark that (3.7) implies that for all  $K \in \mathbb{R}$ , there exists  $\underline{\rho}(K) = (\rho_1(K), \dots, \rho_N(K)) \in \mathbb{R}^N$  such that

$$F(p_1, \dots, p_N) \leq K \Rightarrow p_\alpha \geq \underline{\rho}_\alpha(K).$$

Remark that the functions  $\underline{\rho}_\alpha$  can be chosen non-increasing.

*Remark 3.2.* The quantities  $\underline{p}_\alpha, \bar{p}_\alpha, \underline{p}_\alpha^0$  are defined as follows

$$\begin{cases} \underline{p}_\alpha = \begin{cases} \pi_\alpha^-(-m^0) & \text{if } -m_0 > A_\alpha \\ \pi_\alpha^-(-m^0 + 1) & \text{if } -m_0 = A_\alpha \end{cases} \\ \bar{p}_\alpha = \begin{cases} \pi_\alpha^+(-m^0) & \text{if } -m_0 > A_\alpha \\ \pi_\alpha^+(-m^0 + 1) & \text{if } -m_0 = A_\alpha \end{cases} \\ \underline{p}_\alpha^0 = \begin{cases} \underline{\rho}_\alpha(-m^0) & \text{if } \underline{\rho}_\alpha(-m^0) < \bar{p}_\alpha \\ \underline{\rho}_\alpha(-m^0 + 1) & \text{if } \underline{\rho}_\alpha(-m^0) = \bar{p}_\alpha \end{cases} \end{cases} \quad (3.26)$$

where  $m^0$  is defined in (3.23).

In order to establish Theorem 3.1, we first prove two auxiliary results. In order to state them, some notation should be introduced.

### 3.1 Discrete time derivative estimates

In order to state the first one, Proposition 3.3 below, we introduce some notation. For  $\sigma \in \{+, -\}$ , we set

$$I_{i,\sigma}^{\alpha,n} := [\min(p_{i,\sigma}^{\alpha,n}, p_{i,\sigma}^{\alpha,n+1}), \max(p_{i,\sigma}^{\alpha,n}, p_{i,\sigma}^{\alpha,n+1})]$$

with  $p_{i,\sigma}^{\alpha,n}$  defined in (3.16) and

$$D_{i,+}^{\alpha,n} := \sup \left\{ \sup_{p_\alpha \in I_{i,+}^{\alpha,n}} |H'_\alpha(p_\alpha)|, \sup_{p_\alpha \in I_{i,+}^{\alpha,n}} \left\{ -(\nabla \cdot F)(p_1, \dots, p_N) \right\} \right\}. \quad (3.27)$$

The following proposition asserts that if the discrete space gradients enjoy suitable estimates, then the discrete time derivative is controlled.

**Proposition 3.3 (Discrete time derivative estimate).** *Let  $n \geq 0$  be fixed and  $\Delta x, \Delta t > 0$ . Let us consider  $(U_{i,\alpha}^{\alpha,n})_{\alpha,i}$  satisfying for some constant  $C^n > 0$ :*

$$|p_{i,+}^{\alpha,n}| \leq C^n \quad \text{for } i \geq 0, \alpha = 1, \dots, N.$$

We also consider  $(U_i^{\alpha,n+1})_{\alpha,i}$  and  $(U_i^{\alpha,n+2})_{\alpha,i}$  computed using the scheme (3.15). If

$$D_{i,+}^{\alpha,n} \leq \frac{\Delta x}{\Delta t} \quad \text{for } i \geq 0, \alpha = 1, \dots, N, \quad (3.28)$$

then

$$m^n \leq m^{n+1} \leq M^{n+1} \leq M^n.$$

**Proof.** For  $\sigma = +$  (resp.  $\sigma = -$ ),  $-\sigma$  denotes  $-$  (resp.  $+$ ). We introduce for  $n \geq 0$ ,  $\alpha \in \{1, \dots, N\}$ ,  $i \in \{1, \dots, N\}$ ,  $\sigma \in \{+, -\}$ ,

$$\begin{aligned} C_{i,\sigma}^{\alpha,n} &:= -\sigma \int_0^1 (H_\alpha^{-\sigma})' \left( p_{i,\sigma}^{\alpha,n+1} + \tau(p_{i,\sigma}^{\alpha,n} - p_{i,\sigma}^{\alpha,n+1}) \right) d\tau \geq 0, \\ C_{0,+}^{\alpha,n} &:= -\int_0^1 \frac{\partial F}{\partial p_\alpha} \left( \{p_{0,+}^{\beta,n+1} + \tau(p_{0,+}^{\beta,n} - p_{0,+}^{\beta,n+1})\}_\beta \right) d\tau \geq 0. \end{aligned} \quad (3.29)$$

Notice that for  $i \geq 1$ ,  $C_{i,\sigma}^{\alpha,n}$  is defined as the integral of  $(H_\alpha^{-\sigma})'$  over a convex combination of  $p \in I_{i,\sigma}^{\alpha,n}$ . Similarly for  $C_{0,+}^{\alpha,n}$  which is defined as the integral of  $F'$  on a convex combination of  $p \in I_{0,+}^{\alpha,n}$ . Hence, in view of (3.28), we have for any  $n \geq 0$ ,  $\alpha = 1, \dots, N$  and for any  $\sigma \in \{+, -\}$  or for  $i = 0$  and  $\sigma = +$ , we can check that

$$\begin{cases} C_{i,\sigma}^{\alpha,n} \leq \frac{\Delta x}{\Delta t} & \text{if } i \geq 1, \sigma \in \{-, +\} \\ \sum_{\beta=1}^N C_{0,+}^{\beta,n} \leq \frac{\Delta x}{\Delta t}. \end{cases} \quad (3.30)$$

We can also underline that for any  $n \geq 0$ ,  $\alpha = 1, \dots, N$  and for any  $i \geq 1$ ,  $\sigma \in \{+, -\}$  or for  $i = 0$  and  $\sigma = +$ , we have the following relationship

$$\frac{p_{i,\sigma}^{\alpha,n} - p_{i,\sigma}^{\alpha,n+1}}{\Delta t} = -\sigma \frac{W_{i+\sigma}^{\alpha,n} - W_i^{\alpha,n}}{\Delta x}. \quad (3.31)$$

Let  $n \geq 0$  be fixed and consider  $(U_i^{\alpha,n})_{\alpha,i}$  with  $\Delta x, \Delta t > 0$  given. We compute  $(U_i^{\alpha,n+1})_{\alpha,i}$  and  $(U_i^{\alpha,n+2})_{\alpha,i}$  using the scheme (3.15).

**Step 1:  $(m^n)_n$  is non-decreasing.** We want to show that  $W_i^{\alpha,n+1} \geq m^n$  for  $i \geq 0$  and  $\alpha = 1, \dots, N$ . Let  $i \geq 0$  be fixed and let us distinguish two cases.

**Case 1:  $i \geq 1$ .** Let a branch  $\alpha$  be fixed and let  $\sigma(i, \alpha, n+1) = \sigma \in \{+, -\}$  be such that

$$\max \left\{ H_\alpha^+(p_{i,-}^{\alpha,n+1}), H_\alpha^-(p_{i,+}^{\alpha,n+1}) \right\} = H_\alpha^{-\sigma}(p_{i,\sigma}^{\alpha,n+1}). \quad (3.32)$$

We have

$$\begin{aligned}
\frac{W_i^{\alpha,n+1} - W_i^{\alpha,n}}{\Delta t} &= \frac{1}{\Delta t} \left( \max \left\{ H_\alpha^+(p_{i,-}^{\alpha,n}), H_\alpha^-(p_{i,+}^{\alpha,n}) \right\} - \max \left\{ H_\alpha^+(p_{i,-}^{\alpha,n+1}), H_\alpha^-(p_{i,+}^{\alpha,n+1}) \right\} \right) \\
&\geq \frac{1}{\Delta t} \left( H_\alpha^{-\sigma}(p_{i,\sigma}^{\alpha,n}) - H_\alpha^{-\sigma}(p_{i,\sigma}^{\alpha,n+1}) \right) \\
&= \int_0^1 (H_\alpha^{-\sigma})'(p_{i,\sigma}^{\alpha,n+1} + \tau(p_{i,\sigma}^{\alpha,n} - p_{i,\sigma}^{\alpha,n+1})) \left( \frac{p_{i,\sigma}^{\alpha,n} - p_{i,\sigma}^{\alpha,n+1}}{\Delta t} \right) d\tau \\
&= C_{i,\sigma}^{\alpha,n} \left( \frac{W_{i+\sigma}^{\alpha,n} - W_i^{\alpha,n}}{\Delta x} \right)
\end{aligned}$$

where we used (3.29) and (3.31) in the last line. Using (3.30), we thus get

$$\begin{aligned}
W_i^{\alpha,n+1} &\geq \left( 1 - C_{i,\sigma}^{\alpha,n} \frac{\Delta t}{\Delta x} \right) W_i^{\alpha,n} + C_{i,\sigma}^{\alpha,n} \frac{\Delta t}{\Delta x} W_{i+\sigma}^{\alpha,n} \\
&\geq m^n.
\end{aligned}$$

**Case 2:**  $i = 0$ . We recall that in this case, we have  $U_0^{\beta,n} := U_0^n$  and  $W_0^{\beta,n} := W_0^n = \frac{U_0^{n+1} - U_0^n}{\Delta t}$  for any  $\beta = 1, \dots, N$ . We compute in this case:

$$\begin{aligned}
\frac{W_0^{n+1} - W_0^n}{\Delta t} &= \frac{1}{\Delta t} \left( -F(\{p_{0,+}^{\alpha,n+1}\}_\alpha) + F(\{p_{0,+}^{\alpha,n}\}_\alpha) \right) \\
&= \frac{1}{\Delta t} \int_0^1 \sum_{\beta=1}^N p_\beta \frac{\partial F}{\partial p_\beta} (\{p_{0,+}^{\alpha,n+1} + \tau p_\alpha\}_\alpha) d\tau \quad \text{with } p = (\{p_{0,+}^{\alpha,n} - p_{0,+}^{\alpha,n+1}\}_\alpha) \\
&= - \int_0^1 \sum_{\beta=1}^N \frac{\partial F}{\partial p_\beta} (\{p_{0,+}^{\alpha,n+1} + \tau p_\alpha\}_\alpha) d\tau \left( \frac{W_1^{\beta,n} - W_0^n}{\Delta x} \right) \\
&= \sum_{\beta=1}^N C_{0,+}^{\beta,n} \left( \frac{W_1^{\beta,n} - W_0^n}{\Delta x} \right).
\end{aligned}$$

Using (3.30), we argue like in Case 1 and get

$$W_0^{n+1} \geq m^n.$$

**Step 2:**  $(M^n)_n$  is non-increasing. We want to show that  $W_i^{\alpha,n+1} \leq M^n$  for  $i \geq 0$  and  $\alpha = 1, \dots, N$ . We argue as in Step 1 by distinguishing two cases.

**Case 1:**  $i \geq 1$ . We simply choose  $\sigma = \sigma(i, \alpha, n)$  (see (3.32)) and argue as in Step 1.

**Case 2:**  $i = 0$ . Using (3.28), we can argue exactly as in Step 1. The proof is now complete.  $\square$

### 3.2 Gradient estimates

The second result needed in the proof of Theorem 3.1 is the following one. It asserts that if the discrete time derivative is controlled from below, then a discrete gradient estimate holds true.

**Proposition 3.4 (Discrete gradient estimate).** *Let  $n \geq 0$  be fixed, consider that  $(U_i^{\alpha,n})_{\alpha,i}$  is given and compute  $(U_i^{\alpha,n+1})_{\alpha,i}$  using the scheme (3.15)-(3.16). If there exists a constant  $K \in \mathbb{R}$  such that for any  $i \geq 0$  and  $\alpha = 1, \dots, N$ ,*

$$K \leq W_i^{\alpha,n} := \frac{U_i^{\alpha,n+1} - U_i^{\alpha,n}}{\Delta t}$$

then

$$\begin{cases} \pi_\alpha^-(-K) \leq p_{i,+}^{\alpha,n} \leq \pi_\alpha^+(-K), & \alpha = 1, \dots, N, \quad i \geq 1, \\ \underline{\rho}_\alpha(-K) \leq p_{0,+}^{\alpha,n} \leq (H_\alpha^+)^{-1}(-K), & \alpha = 1, \dots, N \end{cases}$$

where  $p_{i,+}^{\alpha,n}$  is defined in (3.16) and  $\pi_\alpha^\pm$  and  $\underline{p}$  are the “generalized” inverse functions of  $H_\alpha$  and  $F$ , respectively.

**Proof.** Let  $n \geq 0$  be fixed and consider  $(U_i^{\alpha,n})_{\alpha,i}$  with  $\Delta x, \Delta t > 0$  given. We compute  $(U_i^{\alpha,n+1})_{\alpha,i}$  using the scheme (3.15). Let us consider any  $i \geq 0$  and  $\alpha = 1, \dots, N$ .

If  $i \geq 1$ , the result follows from

$$K \leq W_i^{\alpha,n} = - \max_{\sigma=+,-} H_\alpha^\sigma(p_{i,\sigma}^{\alpha,n}).$$

If  $i = 0$ , the results follows from

$$K \leq W_0^n = -F\left(\{p_{0,+}^{\alpha,n}\}_\alpha\right).$$

This achieves the proof of Proposition 3.4 □

### 3.3 Proof of gradient estimates

**Proof of Theorem 3.1.** The idea of the proof is to introduce new Hamiltonians  $\tilde{H}_\alpha$  and a new junction function  $\tilde{F}$  for which it is easier to derive gradient estimates but whose corresponding numerical scheme in fact coincide with the original one.

**Step 1: Modification of the Hamiltonians and the junction function.** Let the new Hamiltonians  $\tilde{H}_\alpha$  for all  $\alpha = 1, \dots, N$  be defined as

$$\tilde{H}_\alpha(p) = \begin{cases} H_\alpha(\underline{p}_\alpha) - \frac{C_\alpha}{2}(p - \underline{p}_\alpha) & \text{if } p \leq \underline{p}_\alpha \\ H_\alpha(p) & \text{if } p \in [\underline{p}_\alpha, \bar{p}_\alpha] \\ H_\alpha(\bar{p}_\alpha) + \frac{C_\alpha}{2}(p - \bar{p}_\alpha) & \text{if } p \geq \bar{p}_\alpha \end{cases} \quad (3.33)$$

where  $\underline{p}_\alpha$  and  $\bar{p}_\alpha$  are defined in (3.26) respectively, and

$$C_\alpha = \sup_{p_\alpha \in [\underline{p}_\alpha, \bar{p}_\alpha]} |H'_\alpha(p_\alpha)|.$$

These new Hamiltonians are now globally Lipschitz continuous: their derivatives are bounded. More precisely, the  $\tilde{H}_\alpha$  satisfy (3.3) and

$$\tilde{H}_\alpha \equiv H_\alpha \text{ in } [p_\alpha, \bar{p}_\alpha]$$

and

$$\forall p \in \mathbb{R}, \quad |\tilde{H}'_\alpha(p)| \leq \sup_{p_\alpha \in [p_\alpha, \bar{p}_\alpha]} |H'_\alpha(p_\alpha)|. \quad (3.34)$$

Let the new  $\tilde{F}$  satisfy (3.7), be such that

$$\tilde{F} \equiv F \text{ in } Q_0 := \prod_{\alpha=1}^N [p_\alpha^0, \bar{p}_\alpha]$$

and (See Section 7.2)

$$\forall p \in \mathbb{R}^N, \quad (-\nabla \cdot \tilde{F})(p) \leq \sup_{Q_0} (-\nabla \cdot F). \quad (3.35)$$

In the remainder of the proof, when notation contains a tilde, it is associated with the new Hamiltonians  $\tilde{H}_\alpha$  and the new non-linearity  $\tilde{F}$ . We then consider the new numerical scheme

$$\begin{cases} \frac{\tilde{U}_i^{\alpha,n+1} - \tilde{U}_i^{\alpha,n}}{\Delta t} + \max\{\tilde{H}_\alpha^+(\tilde{p}_{i,-}^{\alpha,n}), \tilde{H}_\alpha^-(\tilde{p}_{i,+}^{\alpha,n})\} = 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ \tilde{U}_0^{\beta,n} := U_0^n, & i = 0, \quad \beta = 1, \dots, N \\ \frac{\tilde{U}_0^{n+1} - \tilde{U}_0^n}{\Delta t} + \tilde{F}(\tilde{p}_{0,+}^{1,n}, \tilde{p}_{0,+}^{2,n}, \dots, \tilde{p}_{0,+}^{N,n}) = 0 \end{cases}$$

with the same initial condition, namely,

$$\tilde{U}_i^{\alpha,0} = u_0^\alpha(i\Delta x), \quad i \geq 0, \quad \alpha = 1, \dots, N.$$

In view of (3.34) and (3.35), the CFL condition (3.19) gives that for any  $i \geq 0$ ,  $n \geq 0$ , and  $\alpha = 1, \dots, N$

$$\tilde{D}_{i,+}^{\alpha,n} \leq \sup \left\{ \sup_{p_\alpha \leq p \leq \bar{p}_\alpha} |H'_\alpha(p)|; \sup_{\tilde{I}_{0,+}^{\alpha,n}} (-\nabla \cdot F) \right\} \leq \frac{\Delta x}{\Delta t} \quad (3.36)$$

where  $\tilde{D}_{i,+}^{\alpha,n}$  is given by (3.27) after replacing  $H_\alpha$  and  $F$  with  $\tilde{H}_\alpha$  and  $\tilde{F}$ .

**Step 2: First gradient bounds.** Let  $n \geq 0$  be fixed. If  $\tilde{m}^n$  and  $\tilde{M}^n$  are finite, we have

$$\tilde{m}^n \leq \tilde{W}_i^{\alpha,n} \quad \text{for any } i \geq 0, \quad \alpha = 1, \dots, N.$$

Proposition 3.4 implies that

$$\begin{cases} \tilde{\pi}_\alpha^-( -\tilde{m}^n ) \leq \tilde{p}_{i,+}^{\alpha,n} \leq \tilde{\pi}_\alpha^+( -\tilde{m}^n ), & i \geq 1, \quad \alpha = 1, \dots, N, \\ \tilde{\rho}_\alpha^-( -\tilde{m}^n ) \leq \tilde{p}_{0,+}^{\alpha,n} \leq \tilde{\pi}_\alpha^+( -\tilde{m}^n ), & i \geq 0, \quad \alpha = 1, \dots, N. \end{cases}$$



In particular, we get that

$$|\tilde{p}_{i,+}^{\alpha,n}| \leq C^n \quad \text{for } i \geq 0, \quad \alpha = 1, \dots, N$$

with

$$C^n = \max_{\alpha} \left( \max \left( |\tilde{\pi}_{\alpha}^{-}(-\tilde{m}^n)|, |\tilde{\pi}_{\alpha}^{+}(-\tilde{m}^n)|, |\tilde{\rho}_{\alpha}(-\tilde{m}^n)| \right) \right).$$

In view of (3.36), Proposition 3.3 implies that

$$\tilde{m}^n \leq \tilde{m}^{n+1} \leq \tilde{M}^{n+1} \leq \tilde{M}^n \quad \text{for any } n \geq 0. \quad (3.37)$$

In particular,  $\tilde{m}^{n+1}$  is also finite. Since  $\tilde{m}^0 = m^0$  and  $\tilde{M}^0 = M^0$  are finite, we conclude that  $\tilde{m}^n$  and  $\tilde{M}^n$  are finite for all  $n \geq 0$  and for all  $n \geq 0$ ,

$$m^0 \leq \tilde{m}^n \leq \tilde{M}^n \leq M^0. \quad (3.38)$$

**Step 3: Time derivative and gradient estimates.** Now we can repeat the same reasoning but applying Proposition 3.4 with  $K = m^0$  and get

$$\begin{cases} \underline{p}_{\alpha} \leq \tilde{p}_{i,+}^{\alpha,n} \leq \bar{p}_{\alpha}, & i \geq 1, \quad \alpha = 1, \dots, N, \\ \underline{p}_{\alpha}^0 \leq \tilde{p}_{0,+}^{\alpha,n} \leq \bar{p}_{\alpha}, & i \geq 0, \quad \alpha = 1, \dots, N. \end{cases} \quad (3.39)$$

This implies that  $\tilde{U}_i^{\alpha,n} = U_i^{\alpha,n}$  for all  $i \geq 0$ ,  $n \geq 0$ ,  $\alpha = 1, \dots, N$ . In view of (3.37), (3.38) and (3.39), the proof is now complete.  $\square$

## 4 Convergence for general junction conditions

This section is devoted to the convergence of the scheme defined by (3.15)-(3.16). In order to do so, we first make precise how to choose  $\bar{p}_{\alpha}, \underline{p}_{\alpha}$  and  $\underline{p}_{\alpha}^0$  in the CFL condition (3.19).

### 4.1 Monotonicity of the scheme

In order to prove the convergence of the numerical solution as the mesh size tends to zero, we need first to prove a monotonicity result. It is common to write the scheme defined by (3.15)-(3.16) under the compact form

$$u^h(t + \Delta t, x) = S^h[u^h(t)](x)$$

where the operator  $S^h$  is defined on the set of functions defined in  $J^h$ . The scheme is monotone if

$$u \leq v \quad \Rightarrow \quad S^h[u] \leq S^h[v].$$

In our cases, if  $t = n\Delta t$  and  $x = i\Delta x \in J^{\alpha}$  and  $U(t, x) = U_i^{\alpha,n}$  for  $x \in J^{\alpha}$ , then  $S^h[U]$  is defined as follows,

$$\begin{cases} U_i^{\alpha,n+1} = S_{\alpha}[U_{i-1}^{\alpha,n}, U_i^{\alpha,n}, U_{i+1}^{\alpha,n}] \text{ for } i \geq 1, \alpha = 1, \dots, N, \\ U_0^{n+1} = S_0[U_0^n, (U_1^{\beta,n})_{\beta=1, \dots, N}] \end{cases}$$

where

$$\begin{cases} S_\alpha[U_{i-1}^{\alpha,n}, U_i^{\alpha,n}, U_{i+1}^{\alpha,n}] := U_i^{\alpha,n} - \Delta t \max \left\{ H_\alpha^+ \left( \frac{U_i^{\alpha,n} - U_{i-1}^{\alpha,n}}{\Delta x} \right), H_\alpha^- \left( \frac{U_{i+1}^{\alpha,n} - U_i^{\alpha,n}}{\Delta x} \right) \right\}, \\ S_0[U_0^n, (U_1^{\beta,n})_{\beta=1,\dots,N}] := U_0^n - \Delta t F(p_{0,+}^{1,n}, \dots, p_{0,+}^{N,n}) \end{cases} \quad (3.40)$$

Checking the monotonicity of the scheme reduces to checking that  $S_\alpha$  and  $S_0$  are non-decreasing in all their variables.

**Lemma 4.1 (Monotonicity of the numerical scheme).** *Let  $(U^n) := (U_i^{\alpha,n})_{\alpha,i}$  the numerical solution of (3.15)-(3.17). Under the CFL condition (3.18) the scheme is monotone.*

**Proof.** We distinguish two cases.

**Case 1:**  $i \geq 1$ . It is straightforward to check that, for any  $\alpha = 1, \dots, N$ , the function  $S_\alpha$  is non-decreasing with respect to  $U_{i-1}^{\alpha,n}$  and  $U_{i+1}^{\alpha,n}$ . Moreover,

$$\frac{\partial S_\alpha}{\partial U_i^{\alpha,n}} = \begin{cases} 1 - \frac{\Delta t}{\Delta x} (H_\alpha^+)'(p_{i,-}^{\alpha,n}) & \text{if } \max\{H_\alpha^+(p_{i,-}^{\alpha,n}), H_\alpha^-(p_{i,+}^{\alpha,n})\} = H_\alpha^+(p_{i,-}^{\alpha,n}) \\ 1 + \frac{\Delta t}{\Delta x} (H_\alpha^-)'(p_{i,+}^{\alpha,n}) & \text{if } \max\{H_\alpha^+(p_{i,-}^{\alpha,n}), H_\alpha^-(p_{i,+}^{\alpha,n})\} = H_\alpha^-(p_{i,+}^{\alpha,n}) \end{cases}$$

which is non-negative if the CFL condition (3.18) is satisfied.

**Case 2:**  $i = 0$ . Similarly it is straightforward to check that  $S_0$  is non-decreasing with respect to  $U_1^{\beta,n}$  for  $\beta = 1, \dots, N$ . Moreover,

$$\frac{\partial S_0}{\partial U_0^n} = 1 + \frac{\Delta t}{\Delta x} \sum_{\beta=1}^N \frac{\partial F}{\partial p_\beta} \{(p_{0,+}^{\alpha,n})_{\alpha=1}^N\}$$

which is non-negative due to the CFL condition. The proof is now complete.  $\square$

A direct consequence of the previous lemma is the following elementary but useful discrete comparison principle.

**Lemma 4.2 (Discrete Comparison Principle).** *Let  $(U^n) := (U_i^{\alpha,n})_{\alpha,i}$  and  $(V^n) := (V_i^{\alpha,n})_{\alpha,i}$  be such that*

$$\forall n \geq 1, \quad U^{n+1} \leq S^h[U^n] \quad \text{and} \quad V^{n+1} \geq S^h[V^n].$$

*If the CFL condition (3.18) is satisfied and if  $U^0 \leq V^0$ , then  $U^n \leq V^n$  for all  $n \in \mathbb{N}$ .*

*Remark 4.3.* The discrete function  $(U^n)$  (resp.  $(V^n)$ ) can be seen as a *sub-scheme* (resp. *super-scheme*).

We finally recall how to derive discrete viscosity inequalities for monotone schemes.

**Lemma 4.4 (Discrete viscosity inequalities).** *Let  $u^h$  be a solution of (3.15)-(3.17) with  $F = F_A$  defined in (3.4). If  $u^h - \varphi$  has a global maximum (resp. global minimum) on  $\mathcal{G}_h$  at  $(\bar{t} + \Delta t, \bar{x})$ , then*

$$\delta_t \varphi(\bar{t}, \bar{x}) + \mathcal{H}(\bar{x}, D_+ \varphi(\bar{t}, \bar{x}), D_- \varphi(\bar{t}, \bar{x})) \leq 0. \quad (\text{resp. } \geq 0)$$

where

$$\mathcal{H}(x, p_+, p_-) = \begin{cases} \max\{H_\alpha^+(p_-), H_\alpha^-(p_+)\} & \text{if } \bar{x} \neq 0 \\ \max\{A, \max_\alpha H_\alpha^-(p_\alpha^+)\} & \text{if } \bar{x} = 0 \end{cases}$$

and

$$\begin{aligned} D_+ \varphi(\bar{t}, \bar{x}) &= \begin{cases} \frac{1}{\Delta x} \{\varphi(\bar{t}, \bar{x} + \Delta x) - \varphi(\bar{t}, \bar{x})\} & \text{if } \bar{x} \neq 0 \\ \left( \frac{1}{\Delta x} \{\varphi^\alpha(\bar{t}, \Delta x) - \varphi^\alpha(\bar{t}, 0)\} \right)_\alpha & \text{if } \bar{x} = 0 \end{cases} \\ D_- \varphi(\bar{t}, \bar{x}) &= \frac{1}{\Delta x} \{\varphi(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x} - \Delta x)\} \\ \delta_t \varphi(\bar{t}, \bar{x}) &= \frac{1}{\Delta t} \{\varphi(\bar{t} + \Delta t, \bar{x}) - \varphi(\bar{t}, \bar{x})\}. \end{aligned}$$

## 4.2 Stability and Consistency of the scheme

We first derive a local  $L^\infty$  bound for the solution of the scheme.

**Lemma 4.5 (Stability of the numerical scheme).** *Assume that the CFL condition (3.19) is satisfied and let  $u^h$  be the solution of the numerical scheme (3.15)-(3.17). There exists a constant  $C_0 > 0$ , such that for all  $(t, x) \in \mathcal{G}_h$ ,*

$$|u^h(t, x) - u_0(x)| \leq C_0 t. \quad (3.41)$$

*In particular, the scheme is (locally) stable.*

**Proof.** If  $C_0$  large enough so that

$$\begin{cases} C_0 + \max\{H_\alpha^+(p_{i,-}^{\alpha,0}), H_\alpha^-(p_{i,+}^{\alpha,0})\} \geq 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ C_0 + F(p_{0,+}^{1,0}, p_{0,+}^{2,0}, \dots, p_{0,+}^{N,0}) \geq 0, \end{cases}$$

and

$$\begin{cases} -C_0 + \max\{H_\alpha^+(p_{i,-}^{\alpha,0}), H_\alpha^-(p_{i,+}^{\alpha,0})\} \leq 0, & i \geq 1, \quad \alpha = 1, \dots, N \\ -C_0 + F(p_{0,+}^{1,0}, p_{0,+}^{2,0}, \dots, p_{0,+}^{N,0}) \leq 0, \end{cases}$$

then  $\bar{U}_i^{\alpha,n} = U_i^{\alpha,0} + C_0 n \Delta t$  is a super-scheme and  $\bar{U}_i^{\alpha,n} = U_i^{\alpha,0} - C_0 n \Delta t$  is a sub-scheme (see Remark 4.3). The discrete comparison principle, Lemma 4.2, then implies

$$|U_i^{\alpha,n} - U_i^{\alpha,0}| \leq C_0 n \Delta t$$

which is the desired inequality. This achieves the proof.  $\square$

Another condition to satisfy convergence of the numerical scheme (3.15) towards the continuous solution of (3.5) is the consistency of the scheme (which is obvious in our case).

**Lemma 4.6 (Consistency of the numerical scheme).** *Under the assumptions on the Hamiltonians (3.3), the finite difference scheme is consistent with the continuous problem (3.5), that is to say for any smooth function  $\varphi(t, x)$ , we have locally uniformly*

$$\frac{S^h[\varphi](s, y) - \varphi(s, y)}{\Delta t} \rightarrow H_\alpha(\varphi_x(t, x)) \quad \text{as } \mathcal{G}_h \ni (s, y) \rightarrow (t, x)$$

if  $x \in J_\alpha \setminus \{0\}$ , and

$$\frac{S^h[\varphi](s, y) - \varphi(s, y)}{\Delta t} \rightarrow F\left(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_N}(t, 0)\right) \quad \text{as } \mathcal{G}_h \ni (s, y) \rightarrow (t, 0).$$

### 4.3 Convergence of the numerical scheme

In this subsection, we present a sketch of the proof of Theorem 1.1.

*Sketch of the proof of Theorem 1.1.* Let  $T > 0$  and  $h := (\Delta t, \Delta x)$  satisfying the CFL condition (3.19). We recall that

$$u^h(0, x) = u(0, x) \quad \text{for } x \in \mathcal{G}_h.$$

We consider  $\bar{u}$  and  $\underline{u}$  respectively defined as

$$\bar{u}(t, y) = \limsup_{\substack{h \rightarrow 0 \\ \mathcal{G}_h \ni (t', y') \rightarrow (t, y)}} u^h(t', y'), \quad \underline{u}(t, y) = \liminf_{\substack{h \rightarrow 0 \\ \mathcal{G}_h \ni (t', y') \rightarrow (t, y)}} u^h(t', y').$$

By construction, we have  $\underline{u} \leq \bar{u}$ . Since the scheme is monotone (Lemma 4.1), stable (Lemma 4.5) and consistent (Lemma 4.6), we can follow [37, 18, 36] we can show that  $\underline{u}$  (resp.  $\bar{u}$ ) is a relaxed viscosity super-solution (resp. viscosity sub-solution) of equation (3.1)-(3.2), see for example the proof of [36, Theorem 1.8]. Using Theorem 2.4, we know that  $\underline{u}$  (resp.  $\bar{u}$ ) is a viscosity super-solution (resp. sub-solution) of (3.5)-(3.2). Moreover, (3.41) implies that

$$\bar{u}(0, x) \leq u_0(x) \leq \underline{u}(0, x).$$

The comparison principle (see Theorem 2.3) then implies that

$$\bar{u} \leq u \leq \underline{u}$$

which achieves the proof. □

## 5 Study of the reduced minimal action

In this section, we consider that the Hamiltonians  $H_\alpha$  satisfy (3.9). We study the reduced minimal action  $\mathcal{D}_0$  which replace the classical term  $\frac{(x-y)^2}{2\epsilon}$  in the doubling variable method. This function allows us to prove that the error estimate is of order  $(\Delta x)^{\frac{1}{2}}$ .

## 5.1 Reduction of the study

We start this section by the following remark, the analysis can be reduced to the case  $(s, t) = (0, 1)$ . Precisely, using the fact that the Hamiltonian does not depend on time and is homogeneous with respect to the state, the reader can check that a change of variables in time yields the following Lemma.

**Lemma 5.1.** *For all  $y, x \in J$  and  $s < t$ , we have*

$$\mathcal{D}(s, y; t, x) = (t - s)\mathcal{D}\left(0, \frac{y}{t - s}; 1, \frac{x}{t - s}\right).$$

where

$$\mathcal{D}(s, y; t, x) = \inf_{X \in \mathcal{A}(s, y; t, x)} \left\{ \int_s^t L(X(\tau), \dot{X}(\tau)) d\tau \right\}.$$

This is the reason why we consider the reduced minimal action  $\mathcal{D}_0 : J^2 \rightarrow \mathbb{R}$  defined by

$$\mathcal{D}_0(y, x) = \mathcal{D}(0, y; 1, x). \quad (3.42)$$

We also need the following lower bound on  $\mathcal{D}$ .

**Lemma 5.2.** *Assume (B0). Then*

$$\mathcal{D}(s, y; t, x) \geq \frac{\gamma}{2(t - s)} d^2(x, y) - A(t - s)$$

where  $\gamma$  is defined in (B0).

Moreover,

$$\mathcal{D}(s, x; t, x) \leq L_A(0)(t - s).$$

*Proof.* We only prove the first inequality since the other inequality is elementary. As  $L'_\alpha(0) = 0$ , and  $L_\alpha(0) \geq L_A(0) = -A$ , we have

$$L_\alpha(p) \geq \frac{\gamma}{2} p^2 + L'_\alpha(0)p + L_\alpha(0) \geq \frac{\gamma}{2} p^2 - A.$$

Thus, we can write for  $X(\cdot) \in \mathcal{A}(s, y; t, x)$ ,

$$\int_s^t L(X(\tau), \dot{X}(\tau)) d\tau \geq -A(t - s) + \frac{\gamma}{2} \int_s^t (\dot{X}(\tau))^2 d\tau.$$

Then Jensen's inequality allows us to conclude.  $\square$

## 5.2 Piecewise linear trajectories

We are going to see that the infimum defining the minimal action can be computed among piecewise linear trajectories. In order to state a precise statement, we first introduce that optimal curves are of two types depending on the position of  $y$  and  $x$  on the same branch or not: if they are, then the trajectories are of two types: either they reach the junction point, or they stay in a branch and are straight lines. For  $y \in J_\beta$ ,  $x \in J_\alpha$  with  $\beta \neq \alpha$ , the trajectories can spend some time at the junction point.

**Lemma 5.3.** *The infimum defining the reduced minimal action  $\mathcal{D}_0$  can be computed among piecewise linear trajectories; more precisely for all  $y, x \in J$ ,*

$$\mathcal{D}_0(y, x) = \begin{cases} \mathcal{D}_{junction}(y, x) & \text{if } \alpha \neq \beta, \\ \min(L_\alpha(x - y), \mathcal{D}_{junction}(y, x)) & \text{if } \alpha = \beta, \end{cases} \quad (3.43)$$

where for  $x \in J_\alpha, y \in J_\beta$

$$\mathcal{D}_{junction}(y, x) = \inf_{0 \leq t_1 \leq t_2 \leq 1} \left\{ t_1 L_\beta \left( \frac{-y}{t_1} \right) + (t_2 - t_1) L_A(0) + (1 - t_2) L_\alpha \left( \frac{x}{1 - t_2} \right) \right\}, \quad (3.44)$$

where  $L_A$  is defined in (3.13).

*Proof.* We write  $\mathcal{D}_0 = \inf_{X \in \mathcal{A}_0(y, x)} \Lambda(X)$ , where  $\Lambda(X) = \int_0^1 L(X(\tau), \dot{X}(\tau)) d\tau$ . In order to prove the lemma, it is enough to consider a curve  $X \in \mathcal{A}(0, y; 1, x)$  and prove that

$$\Lambda(X) \geq \min(L_\alpha(x - y), \mathcal{D}_{junction}(y, x)).$$

For  $\alpha \neq \beta$ , the trajectories can spend some time at the junction point, hence we can write

$$\begin{aligned} \mathcal{D}_0(y, x) &= \inf_{\substack{X(0)=y \\ X(1)=x}} \left\{ \int_0^{t_1} L_\beta(\dot{X}(\tau)) d\tau + \int_{t_1}^{t_2} L(X(\tau), \dot{X}(\tau)) d\tau + \int_{t_2}^1 L_\alpha(\dot{X}(\tau)) d\tau \right\} \\ &\geq \inf_{0 \leq t_1 \leq t_2 \leq 1} \left\{ \inf_{\substack{X(0)=y \\ X(t_1)=x}} \int_0^{t_1} L_\beta(\dot{X}(\tau)) d\tau + \inf_{\substack{X(t_1)=0 \\ X(t_2)=0}} \int_{t_1}^{t_2} L(X(\tau), \dot{X}(\tau)) d\tau \right. \\ &\quad \left. + \inf_{\substack{X(t_2)=0 \\ X(1)=x}} \int_{t_2}^1 L_\alpha(\dot{X}(\tau)) d\tau \right\} \end{aligned}$$

then using that  $L \geq L_A$  for the second term and Jensen's inequality for all terms, we conclude that

$$\mathcal{D}_0(y, x) \geq \mathcal{D}_{junction}(y, x).$$

Now for  $\alpha = \beta$ , we can deduce from the preceding that

$$\mathcal{D}_0(y, x) \geq \min \left( \mathcal{D}_{junction}(y, x), \inf_{\substack{X(0)=y \\ X(1)=x}} \int_0^1 L_\alpha(\dot{X}(\tau)) d\tau \right).$$

Then, by Jensen's inequality once again, we can deduce (3.43). This ends the proof.  $\square$

In view of (3.43), we see that the study of  $\mathcal{D}_0$  can now be reduced to the study of  $\mathcal{D}_{junction}$ .

### 5.3 Study of $\mathcal{D}_{junction}$

We introduce a simpler notation of  $\mathcal{D}_{junction}$  defined in (3.44),

$$\mathcal{D}_{junction}(y, x) = \inf_{0 \leq t_1 \leq t_2 \leq 1} G(t_1, t_2, y, x), \quad (3.45)$$

where

$$G(t_1, t_2, y, x) = t_1 L_\beta \left( \frac{-y}{t_1} \right) + (t_2 - t_1) L_A(0) + (1 - t_2) L_\alpha \left( \frac{x}{1 - t_2} \right).$$

As in [68], for  $(y, x) \in J_\beta^* \times J_\alpha^*$  the function  $(t_1, t_2) \rightarrow G(t_1, t_2, y, x)$  is strictly convex on  $(0, 1) \times (0, 1)$ . Indeed, for  $t_1, t_2 \in (0, 1)$ , we compute

$$D^2 G(t_1, t_2, y, x) = \frac{L''_\beta \left( \frac{-y}{t_1} \right)}{t_1} V_y^T V_y + \frac{L''_\alpha \left( \frac{x}{1-t_2} \right)}{1-t_2} V_x^T V_x \geq 0,$$

where  $V_y = \left( \frac{-y}{t_1}, 0, 1, 0 \right)$  and  $V_x = \left( 0, \frac{x}{1-t_2}, 0, 1 \right)$  and in particular, we have

$$\frac{\partial^2}{\partial t_1^2} G(t_1, t_2, y, x) = \frac{y^2}{t_1^3} L''_\beta \left( \frac{-y}{t_1} \right) > 0,$$

and

$$\frac{\partial^2}{\partial t_2^2} G(t_1, t_2, y, x) = \frac{x^2}{(1-t_2)^3} L''_\alpha \left( \frac{x}{1-t_2} \right) > 0.$$

So we deduce that for  $(y, x) \in J_\beta^* \times J_\alpha^*$ , if the function  $(t_1, t_2) \rightarrow G(t_1, t_2, y, x)$  admits a critical point, then it reaches its infimum at this point, else it reaches its infimum at the boundary.

**Lemma 5.4.** *Let  $(y, x) \in J$ , and  $\mathcal{D}_{junction}(y, x)$  as in (3.44). We have the following equivalences for the infimum,*

$$\begin{cases} x = 0 \Leftrightarrow t_2 = 1, \\ y = 0 \Leftrightarrow t_1 = 0. \end{cases}$$

*Proof.* It is a direct consequence of the expression (3.44). □

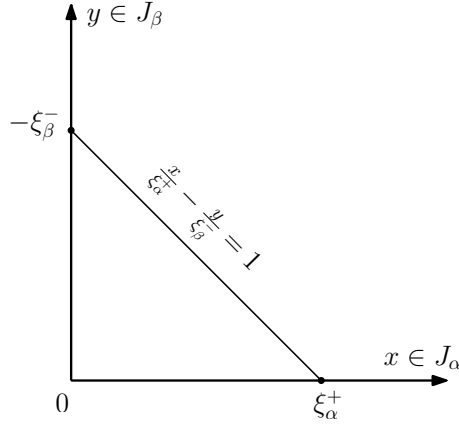
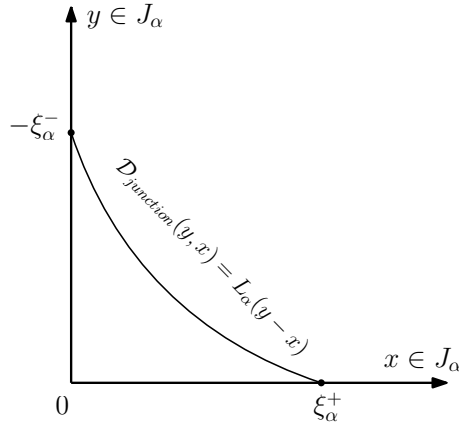
**Definition 5.5** (Numbers  $\xi_l^+, \xi_l^-$ ). *We define  $\xi_l^-, \xi_l^+$  thanks to the following function (for  $l \in \{1, \dots, N\}$ )*

$$K_l(x) = L_l(x) - xL'_l(x) - L_A(0). \quad (3.46)$$

*We define  $(K_l^-)^{-1}$  (resp.  $(K_l^+)^{-1}$ ) as the inverse of the function  $K_l$  restricted to  $(-\infty, 0]$  (resp.  $[0, +\infty)$ ), in fact one can write*

$$K'_l(x) = -xL''_l(x) < 0 \text{ on } (0, +\infty) \quad (\text{resp. } > 0 \text{ on } (-\infty, 0)).$$

*More precisely, we define  $\xi_l^\pm = (K_l^\pm)^{-1}(0)$ .*

Figure 3.1: Illustration of the several subsets for  $\mathcal{D}_{junction}$  for  $\alpha \neq \beta$ .Figure 3.2: Illustration of the several subsets for  $\mathcal{D}_0$  for  $\alpha = \beta$ .

**Lemma 5.6 (Explicit expression of  $\mathcal{D}_{junction}(y, x)$ ).** *It exists a unique function  $\tau : J \times J \rightarrow (0, 1)$  of class  $\mathcal{C}^1$  such that for  $(y, x) \in J_\beta \times J_\alpha$ , we have*

$$\mathcal{D}_{junction}(y, x) = \begin{cases} \tau(y, x)L_\beta\left(\frac{-y}{\tau(y, x)}\right) + (1 - \tau(y, x))L_\alpha\left(\frac{x}{1 - \tau(y, x)}\right) & \text{if } (y, x) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}, \\ -yL'_\beta(\xi_\beta^-) + xL'_\alpha(\xi_\alpha^+) + L_A(0) & \text{if } (y, x) \in \Delta_{\beta\alpha}, \\ L_\alpha(x) & \text{if } y = 0 \text{ and } x > \xi_\alpha^+, \\ L_\beta(-y) & \text{if } x = 0 \text{ and } y > -\xi_\beta^-, \end{cases} \quad (3.47)$$

where

$$\Delta_{\beta\alpha} = \left\{ (y, x) \in J_\beta \times J_\alpha, \quad \frac{x}{\xi_\alpha^+} - \frac{y}{\xi_\beta^-} \leq 1 \right\}.$$

We have a different expression of  $\mathcal{D}_{junction}$  on each subset of the previous Lemma (see Figure 3.1).

*Proof.* Writing the optimal conditions of  $G$  associated with the infimum in (3.45), we



have

$$\begin{cases} \frac{y}{t_1} L'_\beta \left( \frac{-y}{t_1} \right) - L_A(0) + L_\beta \left( \frac{-y}{t_1} \right) = 0, \\ -\frac{x}{1-t_2} L'_\alpha \left( \frac{x}{1-t_2} \right) - L_A(0) + L_\alpha \left( \frac{x}{1-t_2} \right) = 0, \end{cases} \quad (3.48)$$

where  $t_1$  and  $t_2$  are the quantities realizing the minimum. Hence from (3.48), we deduce

$$K_\beta \left( -\frac{y}{t_1} \right) = 0 = K_\alpha \left( \frac{x}{1-t_2} \right).$$

But  $K_\beta$  is a bijection on  $(-\infty, 0)$ , and so is  $K_\alpha$  on  $(0, +\infty)$ . Therefore, setting  $(K_\beta^-)^{-1}(0) := \xi_\beta^-$ , and  $(K_\alpha^+)^{-1}(0) := \xi_\alpha^+$ , we deduce for  $(y, x) \in \Delta_{\beta\alpha} \setminus \{xy = 0\}$ ,

$$\begin{aligned} \mathcal{D}_{\text{junction}}(y, x) &= \frac{-y}{\xi_\beta^-} L_\beta(\xi_\beta^-) + \frac{x}{\xi_\alpha^+} L_\alpha(\xi_\alpha^+) + \left( 1 - \frac{x}{\xi_\alpha^+} + \frac{y}{\xi_\beta^-} \right) L_A(0) \\ &= -y L'_\beta(\xi_\beta^-) + x L'_\alpha(\xi_\alpha^+) + L_A(0). \end{aligned}$$

Now, for  $x = 0$  and  $y < -\xi_\beta^-$ , using the first condition of (3.48), we deduce that

$$\mathcal{D}_{\text{junction}}(y, 0) = -y L'_\beta(\xi_\beta^-) + L_A(0).$$

For  $x = 0$  and  $y \geq -\xi_\beta^-$ , we deduce from Lemma 5.4, that  $t_2 = 1$ . Using the first optimal condition in (3.48), we have  $K_\beta \left( \frac{-y}{t_1} \right) = 0$  so  $t_1 = \frac{-y}{\xi_\beta^-} \geq 1$ . We deduce that the optimal condition must be satisfied at the boundary of the set  $\{0 \leq t_1 \leq 1\}$ . Here using (3.44), we have  $t_1 = 1$ , so

$$\mathcal{D}_{\text{junction}}(y, 0) = L_\beta(-y).$$

Similarly, for  $y = 0$  and  $x < \xi_\alpha^+$ ,

$$\mathcal{D}_{\text{junction}}(y, x) = x L'_\alpha(\xi_\alpha^+) + L_A(0).$$

For  $y = 0$  and  $x \geq \xi_\alpha^+$ , we deduce that

$$\mathcal{D}_{\text{junction}}(0, x) = L_\alpha(x).$$

In all other cases, that is to say for  $(y, x) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}$ , the infimum of  $G$  is attained at the boundary of  $\{0 \leq t_1 \leq t_2 \leq 1\}$ , here for some  $t_1 = t_2 = \tau \in (0, 1)$ . Hence we have

$$\mathcal{D}_{\text{junction}}(y, x) = \inf_{0 < \tau < 1} \left\{ \tau L_\beta \left( \frac{-y}{\tau} \right) + (1 - \tau) L_\alpha \left( \frac{x}{1 - \tau} \right) \right\}$$

Once again, writing the optimal conditions for  $G(\tau, \tau, y, x)$ , we deduce that

$$K_\beta \left( \frac{-y}{\tau} \right) = K_\alpha \left( \frac{x}{1 - \tau} \right). \quad (3.49)$$

We define

$$\tilde{G}(\tau, y, x) = K_\beta \left( \frac{-y}{\tau} \right) - K_\alpha \left( \frac{x}{1 - \tau} \right).$$

Deriving

$$\frac{\partial \tilde{G}}{\partial \tau} = K'_\beta \left( \frac{-y}{\tau} \right) \frac{y}{\tau^2} - K'_\alpha \left( \frac{x}{1-\tau} \right) \frac{x}{(1-\tau)^2} > 0 \quad \text{for } (y, x) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha},$$

by implicit function theorem, we deduce that there exists a unique  $\tilde{\tau} \in C^1(0, 1)$  satisfying  $\tilde{G}(\tilde{\tau}, y, x) = 0$ . The proof is thus complete.  $\square$

**Lemma 5.7 (Continuity of  $\mathcal{D}_{\text{junction}}$ ).** *The function  $\mathcal{D}_{\text{junction}}$  is continuous in  $J^2$ .*

*Proof.* From (3.47), we already know that  $\mathcal{D}_{\text{junction}} \in C((J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}) \cup C(\Delta_{\beta\alpha} \cup \{x = 0\} \cup \{y = 0\})$ . Therefore in order to prove that  $\mathcal{D}_{\text{junction}} \in C(J_\beta \times J_\alpha)$ , it is sufficient to prove that for any given sequence  $(y^k, x^k) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}$  such that  $(y^k, x^k) \rightarrow (y, x)$ , where  $(y, x) \in \bar{\Delta} := \left\{ \frac{x}{\xi_\alpha^+} - \frac{y}{\xi_\beta^-} = 1 \right\} \cup \{x \geq \xi_\alpha^+\} \cup \{y \geq -\xi_\beta^-\}$ , we have

$$\mathcal{D}_{\text{junction}}(y^k, x^k) \rightarrow \mathcal{D}_{\text{junction}}(y, x).$$

Since the sequence  $\{\tau(y^k, x^k)\}$  is bounded, we can deduce that there exists a subsequence such that  $\tau(y^k, x^k) \rightarrow \tau^0$ . We distinguish the following cases.

**Case 1:**  $\tau^0 \in (0, 1)$ . By continuity of  $K_l$ , we have

$$K_\alpha \left( \frac{x}{1-\tau^0} \right) = K_\beta \left( \frac{-y}{\tau^0} \right). \quad (3.50)$$

If  $x = 0$ , we have as  $K_\alpha(0) > 0$  and  $(K_\beta^-)^{-1}$  is increasing

$$\frac{y}{\tau^0} = -(K_\beta^-)^{-1}(K_\alpha)(0) < -(K_\beta^-)^{-1}(0) = -\xi_\beta^-,$$

hence deduce that  $(y, 0) \notin \bar{\Delta}$ , so this case is not possible.

Similarly, if  $y = 0$ , we have

$$\frac{x}{1-\tau^0} = (K_\alpha^+)^{-1}(K_\beta)(0) < (K_\alpha^+)^{-1}(0) = \xi_\alpha^+,$$

hence deduce that  $(0, x) \notin \bar{\Delta}$ , so this case is not possible.

Now if  $(y, x) \in (J_\beta^* \times J_\alpha^*) \cap \bar{\Delta}$ , then  $\frac{x}{\xi_\alpha^+} - \frac{y}{\xi_\beta^-} = 1$  and passing to the limit, we have (3.50). We know that  $K_\alpha(\xi_\alpha^+) = K_\beta(\xi_\beta^-) = 0$ , so if we set  $\bar{\tau} = -\frac{y}{\xi_\beta^-} = 1 - \frac{x}{\xi_\alpha^+}$  so  $1 - \bar{\tau} = \frac{x}{\xi_\alpha^+}$ , we have

$$K_\beta \left( \frac{-y}{\bar{\tau}} \right) = 0 = K_\alpha \left( \frac{x}{1-\bar{\tau}} \right). \quad (3.51)$$

By uniqueness of  $\tau$  satisfying (3.49), we deduce that  $\tau^0 = \bar{\tau}$ . So we have

$$\mathcal{D}_{\text{junction}}(y^k, x^k) \rightarrow -yL'_\beta(\xi_\beta^-) + xL'_\alpha(\xi_\alpha^+) + L_A(0) = \mathcal{D}_{\text{junction}}(y, x).$$

**Case 2:**  $\tau^0 = 0$ . In this case using Lemma 5.4,  $y^k \rightarrow y = 0$ , so  $x \geq \xi_\alpha^+$  and with (3.49) we deduce that

$$\frac{-y^k}{\tau(y^k, x^k)} = (K_\beta^-)^{-1} \left( K_\alpha \left( \frac{x^k}{1-\tau(y^k, x^k)} \right) \right) \rightarrow (K_\beta^-)^{-1}(K_\alpha(x)). \quad (3.52)$$

Therefore  $\mathcal{D}_{\text{junction}}(y^k, x^k) \rightarrow L_\alpha(x) = \mathcal{D}_{\text{junction}}(0, x)$ .

**Case 3:**  $\tau^0 = 1$ . In this case,  $x^k \rightarrow x = 0$ . Arguing as in Case 2, we deduce that  $y \geq \xi_\beta^-$ , and

$$\frac{x^k}{1 - \tau(y^k, x^k)} = (K_\alpha^+)^{-1} \left( K_\beta \left( \frac{-y^k}{\tau(y^k, x^k)} \right) \right) \rightarrow (K_\alpha^+)^{-1} (K_\beta(-y)). \quad (3.53)$$

Therefore,  $\mathcal{D}_{junction}(y^k, x^k) \rightarrow L_\beta(-y) = \mathcal{D}_{junction}(y, x)$ .

The proof is thus complete.  $\square$

**Lemma 5.8.** *The function  $\mathcal{D}_{junction}$  is  $C^1$  in  $J^2$  and for  $(y, x) \in J_\beta \times J_\alpha$ , we have*

$$\partial_x \mathcal{D}_{junction}(y, x) = \begin{cases} L'_\alpha \left( \frac{x}{1-\tau} \right) & \text{if } (y, x) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}, \\ L'_\alpha(\xi_\alpha^+) & \text{if } (y, x) \in \Delta_{\beta\alpha}, \\ L'_\alpha(x) & \text{if } y = 0 \text{ and } x > \xi_\alpha^+, \\ L'_\alpha \circ (K_\alpha^+)^{-1} \circ K_\beta(-y) & \text{if } x = 0 \text{ and } y > -\xi_\beta^-, \end{cases} \quad (3.54)$$

and

$$\partial_y \mathcal{D}_{junction}(y, x) = \begin{cases} -L'_\beta \left( \frac{-y}{\tau} \right) & \text{if } (y, x) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}, \\ -L'_\beta(\xi_\beta^-) & \text{if } (y, x) \in \Delta_{\beta\alpha}, \\ -L'_\beta \circ (K_\beta^-)^{-1} \circ K_\alpha(x) & \text{if } y = 0 \text{ and } x > \xi_\alpha^+, \\ -L'_\beta(-y) & \text{if } x = 0 \text{ and } y > -\xi_\beta^-. \end{cases} \quad (3.55)$$

*Proof.* We compute the partial derivatives in domains where the function is naturally of class  $C^1$  using that the function  $\tau$  is continuously differentiable in  $(0, 1)^2$  and using (3.49). We prove the continuity of the partial derivatives using the same proof as Lemma 5.7.  $\square$

## 5.4 Compatibility condition

In this subsection, we prove a compatibility result, which will be used in deriving error estimates. Let us introduce the following shorthand notation

$$H(x, p) = \begin{cases} H_\alpha(p) & \text{if } x \in J_\alpha^* \\ F_A(p) & \text{if } x = 0. \end{cases}$$

*Remark 5.9.* In  $J_\alpha \times J_\alpha$ , we give a description of  $\{\mathcal{D}_{junction}(y, x) = L_\alpha(x - y)\} \cap \Delta_{\alpha\alpha}$  using [68], see Figure 3.2. We have

$$\begin{cases} \mathcal{D}_{junction}(0, \xi_\alpha^+) = \xi_\alpha^+ L'_\alpha(\xi_\alpha^+) + L_A(0) = L_\alpha(\xi_\alpha^+), \\ \mathcal{D}_{junction}(-\xi_\alpha^-, 0) = \xi_\alpha^- L'_\alpha(\xi_\alpha^-) + L_A(0) = L_\alpha(\xi_\alpha^-). \end{cases}$$

This means that the functions  $\mathcal{D}_{junction}$  and  $(y, x) \rightarrow L_\alpha(x - y)$  coincide at the same points  $X_\alpha = (0, \xi_\alpha^+)$  and  $Y_\alpha = (-\xi_\alpha^-, 0)$ . Therefore we have

$$L_\alpha(x - y) < \mathcal{D}_{junction}(y, x) \quad \text{on the open line segment } ]X_\alpha, Y_\alpha[$$

because  $\mathcal{D}_{junction}$  is linear and  $L_\alpha$  is strictly convex as a function of  $y - x$ .

The function  $(y, x) \mapsto L_\alpha(x - y) - \mathcal{D}_{\text{junction}}(y, x)$  being convex because  $\mathcal{D}_{\text{junction}}(y, x)$  is linear, we can consider the convex set

$$K^\alpha = \{(y, x) \in J_\alpha \times J_\alpha, \quad L_\alpha(x - y) \leq \mathcal{D}_{\text{junction}}(y, x)\}.$$

Then the set

$$\Gamma^\alpha = \{(y, x) \in \Delta_{\alpha\alpha}, \quad \mathcal{D}_{\text{junction}}(y, x) = L_\alpha(x - y)\}$$

is contained in the boundary of the convex set  $K^\alpha$ . More precisely, we have

$$\Gamma^\alpha = ((\partial K^\alpha) \cap \Delta_{\alpha\alpha}) \subset J_\alpha \times J_\alpha$$

which shows that  $\Gamma^\alpha$  is a curve which contains the points  $X_\alpha$  and  $Y_\alpha$ .

**Theorem 5.10.** *Assume the Hamiltonians are convex, with Legendre Fenchel transform satisfying (B0). Then for all  $(x, y) \in J \times J \setminus \bigcup_{\alpha \in \{1, \dots, N\}} \Gamma^\alpha$ , (i.e., everywhere except on the curves where  $\mathcal{D}_0$  is not  $C^1$ ), we have*

$$H(y, -\partial_y \mathcal{D}_0(y, x)) = H(x, \partial_x \mathcal{D}_0(y, x)).$$

*Proof of Theorem 5.10.* First, notice that in the interior of  $K^\alpha$  (i.e., in the regions where  $\mathcal{D}_0(y, x) = L_\alpha(x - y)$ ), we have the result as

$$H(y, -\partial_y \mathcal{D}_0(y, x)) = H_\alpha(L'_\alpha(x - y)) = H(x, \partial_x \mathcal{D}_0(y, x)).$$

Now we prove the result in the regions where  $\mathcal{D}_0 = \mathcal{D}_{\text{junction}}$ . We distinguish different regions of  $J_\beta \times J_\alpha$ , defined in the expressions of  $\partial_x \mathcal{D}_{\text{junction}}$  and  $\partial_y \mathcal{D}_{\text{junction}}$  in (3.54)-(3.55). Let us first point out that we have the following assertion

$$H_\alpha(p) + L_\alpha(q) = pq \Leftrightarrow q \in \partial H_\alpha(p), \quad (3.56)$$

where  $\partial H_\alpha(p)$  is the convex subdifferential of  $H_\alpha(p)$ .

We distinguish several cases.

**Case 1**  $(y, x) \in (J_\beta^* \times J_\alpha^*) \setminus \Delta_{\beta\alpha}$ . From (3.56), on one hand, and from (3.55) we have

$$H_\beta \left( L'_\beta \left( \frac{-y}{\tau} \right) \right) = \frac{-y}{\tau} L'_\beta \left( \frac{-y}{\tau} \right) - L_\beta \left( \frac{-y}{\tau} \right).$$

From (3.46), we have then  $H_\beta \left( L'_\beta \left( \frac{-y}{\tau} \right) \right) = -K_\beta \left( \frac{-y}{\tau} \right) - L_A(0)$ .

On the other hand, and from (3.54)

$$H_\alpha \left( L'_\alpha \left( \frac{x}{1-\tau} \right) \right) = \frac{x}{1-\tau} L'_\alpha \left( \frac{x}{1-\tau} \right) - L_\alpha \left( \frac{x}{1-\tau} \right),$$

similarly, from (3.46), we deduce that  $H_\alpha \left( L'_\alpha \left( \frac{x}{1-\tau} \right) \right) = -K_\alpha \left( \frac{x}{1-\tau} \right) - L_A(0)$ . Hence, from (3.49), the compatibility condition.

**Case 2**  $(y, x) \in (J_\beta^* \times J_\alpha^*) \cap \Delta_{\beta\alpha}$ . We argue as in Case 1, one can deduce that

$$\begin{aligned} H_\beta(L'_\beta(\xi_\beta^-)) &= -K_\beta(\xi_\beta^-) - L_A(0) = A \\ H_\alpha(L'_\alpha(\xi_\alpha^+)) &= -K_\alpha(\xi_\alpha^+) - L_A(0) = A \end{aligned}$$

From the definition of  $\xi_\alpha^+$  and  $\xi_\beta^-$ , one can deduce the compatibility condition.

*Remark 5.11.* We deduce that the functions  $\pi_\alpha^+, \pi_\beta^-$  defined in [66] satisfy

$$\pi_\alpha^+(A) = L'_\alpha(\xi_\alpha^+) \quad \text{and} \quad \pi_\beta^-(A) = L'_\beta(\xi_\beta^-).$$

**Case 3**  $y = 0$  and  $x > \xi_\alpha^+$ . Let us check the following equality

$$\max \left( A, \max_{\beta=1, \dots, N} H_\beta^- \left( L'_\beta \left( (K_\beta^-)^{-1} \circ K_\alpha(x) \right) \right) \right) = H_\alpha(L'_\alpha(x)).$$

On one hand, from the definition of  $K_\beta^-$ , we deduce that

$$H_\beta^- \left( L'_\beta \left( (K_\beta^-)^{-1} \circ K_\alpha(x) \right) \right) = H_\beta \left( L'_\beta \left( (K_\beta^-)^{-1} \circ K_\alpha(x) \right) \right),$$

and arguing as previously, we deduce that

$$H_\beta \left( L'_\beta \left( (K_\beta^-)^{-1} \circ K_\alpha(x) \right) \right) = -K_\beta \left( (K_\beta^-)^{-1} \circ K_\alpha(x) \right) - L_A(0) = -K_\alpha(x) - L_A(0).$$

On the other hand from (3.56), we have  $H_\alpha(L'_\alpha(x)) = -K_\alpha(x) - L_A(0)$ .

And for  $x > \xi_\alpha^+$ , we have  $H_\alpha(L'_\alpha(x)) > H_\alpha(L'_\alpha(\xi_\alpha^+)) = H_\alpha(\pi_\alpha^+(A)) = A$ . So one can deduce the compatibility condition.

**Case 4**  $x = 0$  and  $y > -\xi_\beta^-$ . Let us check the following equality

$$\max \left( A, \max_{\substack{\alpha=1, \dots, N \\ \alpha \neq \beta}} H_\alpha^- \left( L'_\alpha \left( (K_\alpha^+)^{-1} \circ K_\beta(-y) \right) \right), H_\beta^- (L'_\beta(-y)) \right) = H_\beta (L'_\beta(-y)).$$

Similarly, as in the previous case, one can deduce that

$$\max_{\substack{\alpha=1, \dots, N \\ \alpha \neq \beta}} H_\alpha^- \left( L'_\alpha \left( (K_\alpha^+)^{-1} \circ K_\beta(-y) \right) \right) = A_0 \leq A.$$

And for  $y > \xi_\beta^-$ , we have  $H_\beta^- (L'_\beta(-y)) > H_\beta^- (\pi_\beta^-(A)) = A$ .

**Case 5**  $y = 0$  and  $0 < x \leq \xi_\alpha^+$ . Let us check the following equality

$$\max(A, \max_{\beta=1, \dots, N} H_\beta^- (L'_\beta(\xi_\beta^-))) = H_\alpha(L'_\alpha(\xi_\alpha^+)).$$

On one hand, from (3.56)

$$H_\alpha(L'_\alpha(\xi_\alpha^+)) = -K_\alpha(\xi_\alpha^+) - L_A(0) = -L_A(0) = A.$$

On the other hand,

$$\max_{\beta=1, \dots, N} H_\beta^- (L'_\beta(\xi_\beta^-)) = \max_{\beta=1, \dots, N} H_\beta^- (L'_\beta(\xi_\beta^-)) = \max_{\beta=1, \dots, N} H_\beta^- (\pi_\beta^-(A)) = A.$$

**Case 6**  $x = 0$  and  $0 < y \leq -\xi_\beta^-$ . Let us check the following equality

$$\max(A, \max_{\alpha=1, \dots, N} H_\alpha^-(L'_\alpha(\xi_\alpha^+)) = H_\beta(L'_\beta(\xi_\beta^-)).$$

Similarly, as in Case 5, one can deduce the compatibility condition.

**Case 7**  $x = 0$  and  $y = 0$ . Let us check the following equality

$$\max(A, \max_{\beta=1, \dots, N} H_\beta^-(L'_\beta(\xi_\beta^-)) = \max(A, \max_{\alpha=1, \dots, N} H_\alpha^-(L'_\alpha(\xi_\alpha^+)).$$

In fact, it follows directly from Case 5 and Case 6.

The proof is thus complete.  $\square$

### 5.5 $C^{1,1}$ estimates for the reduced minimal action

In this section, we study the Lipschitz regularity of the gradient of the reduced minimal action  $\mathcal{D}_0$ . It turns out that its gradient is indeed Lipschitz if the flux limiter  $A$  is not equal to  $A_0$ , the minimal flux limiter. Such a technical result will be used when deriving error estimates. It is also of independent interest.

**Proposition 5.12** ( *$C^{1,1}$  estimates for the reduced minimal action*). *Let  $\rho > 0$  and assume that the Hamiltonians satisfy (3.9) and (3.8). The function  $\mathcal{D}_0$  associated with the flux limiter  $A_0 + \rho$  can be chosen  $C^{1,1}(J_K^2)$  for any  $K > 0$  where  $J_K^2 = \{(x, y) \in J^2 : d(0, x) \leq K \text{ and } d(0, y) \leq K\}$ . Moreover, there exists  $C_K$  and  $C'_K$  such that*

$$\|\partial_{xx}\mathcal{D}_{junction}\|_{L^\infty(J_K^2)} \leq \frac{C_K}{\min(1, \rho)}; \tag{3.57}$$

and

$$\|H'_\alpha(\partial_x\mathcal{D}_{junction})\partial_{xx}\mathcal{D}_{junction}\|_{L^\infty(J_K^2)} \leq \frac{C'_K}{\min(1, \sqrt{\rho})}. \tag{3.58}$$

the constants  $C_K$  and  $C'_K$  depends only on  $K$  and (3.9).

Moreover, in the case where for all  $\alpha \in \{1, \dots, N\}$ ,  $\min H_\alpha = A_0$ , we have

$$\|\partial_{xx}\mathcal{D}_{junction}\|_{L^\infty(J_K^2)} \leq C_K. \tag{3.59}$$

**Proof.** In the following  $A$  denotes  $A_0 + \rho$ . Using (3.54), we see that  $\partial_{xx}\mathcal{D}_{junction} = 0$  on  $\Delta_{\beta\alpha}$  for all  $(\beta, \alpha) \in \{1, \dots, N\}^2$  and  $\partial_{xx}\mathcal{D}_{junction}(y, x) = L''_\alpha(x)$  on  $\{0\} \times \{x \in J_\alpha \mid x > \xi_\alpha^+\}$ . So it is sufficient to prove (3.57) and (3.58) on  $T := J_\beta^* \times J_\alpha^* \setminus \Delta_{\beta\alpha}$  for all  $(\beta, \alpha) \in \{1, \dots, N\}^2$ . By (3.54), we deduce that on  $T$ ,

$$\partial_{xx}\mathcal{D}_{junction}(y, x) = \left( \frac{1}{1 - \tau(y, x)} + \frac{x}{(1 - \tau(y, x))^2} \frac{\partial\tau}{\partial x}(y, x) \right) L''_\alpha \left( \frac{x}{1 - \tau(y, x)} \right).$$

Let us compute also  $\frac{\partial\tau}{\partial x}$  using (3.49),

$$\frac{\partial\tau}{\partial x}(y, x) = \frac{\frac{1}{1 - \tau(y, x)} K'_\alpha \left( \frac{x}{1 - \tau(y, x)} \right)}{\frac{y}{\tau(y, x)^2} K'_\beta \left( \frac{-y}{\tau(y, x)} \right) - \frac{x}{(1 - \tau(y, x))^2} K'_\alpha \left( \frac{x}{1 - \tau(y, x)} \right)}.$$

And as  $K'_\beta\left(\frac{-y}{\tau}\right) = \frac{y}{\tau}L''_\beta\left(\frac{-y}{\tau}\right) \geq 0$  and  $K'_\alpha\left(\frac{x}{1-\tau}\right) = \frac{-x}{1-\tau}L''_\alpha\left(\frac{x}{1-\tau}\right) \leq 0$  we deduce that

$$\frac{\partial\tau}{\partial x}(y, x) = \frac{\frac{-x}{(1-\tau(y, x))^2}L''_\alpha\left(\frac{x}{1-\tau(y, x)}\right)}{\frac{y^2}{\tau(y, x)^3}L''_\beta\left(\frac{-y}{\tau(y, x)}\right) + \frac{x^2}{(1-\tau(y, x))^3}L''_\alpha\left(\frac{x}{1-\tau(y, x)}\right)}. \quad (3.60)$$

So we have on  $T$

$$\partial_{xx}\mathcal{D}_{junction}(y, x) = \frac{\frac{y^2}{(1-\tau(y, x))\tau(y, x)^3}L''_\alpha\left(\frac{x}{1-\tau(y, x)}\right)L''_\beta\left(\frac{-y}{\tau(y, x)}\right)}{\frac{y^2}{\tau(y, x)^3}L''_\beta\left(\frac{-y}{\tau(y, x)}\right) + \frac{x^2}{(1-\tau(y, x))^3}L''_\alpha\left(\frac{x}{1-\tau(y, x)}\right)} \geq 0. \quad (3.61)$$

As the denominator is a sum of two positive functions,  $\partial_{xx}\mathcal{D}_{junction}$  from above by the same numerator over only one term of the denominator. We deduce in these two cases that,

$$\partial_{xx}\mathcal{D}_{junction}(y, x) \leq \begin{cases} 2L''_\alpha\left(\frac{x}{1-\tau(y, x)}\right) & \text{if } \tau(y, x) \leq \frac{1}{2} \\ \frac{8y^2}{\left(\frac{x}{1-\tau(y, x)}\right)^2}L''_\beta\left(\frac{-y}{\tau(y, x)}\right) & \text{if } \tau(y, x) \geq \frac{1}{2}. \end{cases} \quad (3.62)$$

Moreover, we have on  $T$ ,

$$H'_\alpha(\partial_x\mathcal{D}_{junction}(y, x)) = H'_\alpha\left(L'_\alpha\left(\frac{x}{1-\tau(y, x)}\right)\right) = \frac{x}{1-\tau(y, x)},$$

and

$$\frac{x}{1-\tau(y, x)}\partial_{xx}\mathcal{D}_{junction}(y, x) \leq \begin{cases} 4x^2L''_\alpha\left(\frac{x}{1-\tau(y, x)}\right) & \text{if } \tau(y, x) \leq \frac{1}{2} \\ \frac{8y^2}{\frac{x}{1-\tau(y, x)}}L''_\beta\left(\frac{-y}{\tau(y, x)}\right) & \text{if } \tau(y, x) \geq \frac{1}{2}, \end{cases}$$

In the case  $\tau(y, x) \leq \frac{1}{2}$ , as  $0 \leq \frac{x}{1-\tau(y, x)} \leq 2x$ , we get the inequality (3.57) and (3.58). Let us prove the following lower bound for  $(y, x) \in T$ ,

$$\frac{x}{1-\tau(y, x)} \geq \xi_\alpha^+, \quad (3.63)$$

which helps us for the second case. For  $y \in J_\beta$ , we see that  $x \rightarrow \frac{x}{1-\tau(y, x)}$  has a non-negative derivative using (3.60), so it is a non-decreasing function. Therefore to prove (3.63), it is sufficient to show it on  $\partial T$ . Let  $(y, x)$  be in  $\partial T$ . We distinguish three cases.

In the case where  $y = 0$ , necessarily  $x \geq \xi_\alpha^+$  and as  $\tau(y, x) \in [0, 1]$ , we deduce (3.63).

In the case where  $y \in ]0, -\xi_\beta^-]$ , we have  $(y, x) \in \left\{ (y, x) \in J_\beta \times J_\alpha, \quad \frac{x}{\xi_\alpha^+} - \frac{y}{\xi_\beta^-} = 1 \right\}$ .

So by (3.51) we deduce that  $\frac{x}{1-\tau(y, x)} = \xi_\alpha^+$ .

In the case where  $y \geq -\xi_\beta^-$ , we have  $x = 0$ . It is enough to prove that

$$\liminf_{x' \rightarrow 0} \frac{x'}{1-\tau(y, x')} \geq \xi_\alpha^+. \quad (3.64)$$

We have for  $(y, x') \in T$ ,

$$K_\alpha \left( \frac{x'}{1 - \tau(y, x')} \right) = K_\beta \left( \frac{-y}{\tau(y, x')} \right) \leq K_\beta \left( \frac{-\xi_\beta^-}{\tau(y, x')} \right),$$

as  $K_\beta$  is non-decreasing on  $] -\infty, 0]$ . We deduce that

$$\frac{x'}{1 - \tau(y, x')} \geq (K_\alpha^+)^{-1} \circ K_\beta \left( \frac{-\xi_\beta^-}{\tau(y, x')} \right),$$

as  $(K_\alpha^+)^{-1}$  is non-increasing. As  $\lim_{x' \rightarrow 0} \tau(y, x') = 1$ , taking the limit inferior in the preceding inequality gives (3.64). So we deduce (3.63) and

$$\partial_{xx} \mathcal{D}_{\text{junction}}(y, x) \leq \frac{8y^2}{(\xi_\alpha^+)^2} L_\beta'' \left( \frac{-y}{\tau(y, x)} \right) \quad \text{if } \tau(y, x) \geq \frac{1}{2},$$

$$\frac{x}{1 - \tau(y, x)} \partial_{xx} \mathcal{D}_{\text{junction}}(y, x) \leq \frac{8y^2}{\xi_\alpha^+} L_\beta'' \left( \frac{-y}{\tau(y, x)} \right) \quad \text{if } \tau(y, x) \geq \frac{1}{2}.$$

If  $\xi_\alpha^+ > 1$ , we deduce (3.57). If  $\xi_\alpha^+ \leq 1$ , let us prove that it exists a constant  $C > 0$  only depending on (3.9) such that

$$(\xi_\alpha^+)^2 \geq C\rho. \quad (3.65)$$

As  $A = A_0 + \rho$  we have

$$K_\alpha(\xi) = L_\alpha(\xi) - \xi L_\alpha'(\xi) + A_0 + \rho,$$

and

$$K_\alpha'(\xi) = -\xi L_\alpha''(\xi).$$

The function  $L_\alpha''$  is bounded on  $[0, 1]$ , it exists  $M > 0$  such that

$$\gamma \leq L_\alpha'' \leq M.$$

So we have  $K_\alpha'(\xi) \geq -M\xi$ . We integrate from 0 to  $\xi \geq 0$  and get

$$K_\alpha(\xi) - K_\alpha(0) \geq -M \frac{\xi^2}{2}. \quad (3.66)$$

Taking  $\xi = \xi_\alpha^+$ , as  $K_\alpha(\xi_\alpha^+) = 0$  and as  $L_\alpha(0) + A_0 \geq 0$ , we deduce that

$$(\xi_\alpha^+)^2 \geq \frac{2}{M} (L_\alpha(0) + A_0 + \rho) \geq \frac{2}{M} \rho.$$

So we get (3.65) and we deduce (3.57) and (3.58).

In the case where for all  $\alpha \in \{1, \dots, N\}$ ,  $\min H_\alpha = A_0$ , we only have to consider the case  $\tau(y, x) \geq \frac{1}{2}$  in (3.62) since the case  $\tau(y, x) \leq \frac{1}{2}$  gives already the bound (3.62). In order to get a bound for the term  $\frac{8y^2}{\left(\frac{x}{1-\tau(y,x)}\right)^2} = \frac{8y^2}{\left((K_\alpha^+)^{-1} \circ K_\beta\left(-\frac{y}{\tau(y,x)}\right)\right)^2}$ , let us prove that for all  $\xi \in [-2K, 2K]$ , we have

$$\frac{\xi^2}{\left((K_\alpha^+)^{-1} \circ K_\beta(-\xi)\right)^2} \leq C_{2K}, \quad (3.67)$$



where  $C_{2K} > 0$  is a constant which depends on  $K$ . Let  $M_{2K}$  be such that on  $[-2K, 2K]$  we have for all  $\alpha \in \{1, \dots, N\}$ ,

$$\gamma \leq L''_{\alpha} \leq M_{2K}.$$

Replacing  $\xi$  by  $(K_{\alpha}^+)^{-1}(\xi)$  in (3.66), we deduce that

$$M_{2K} \frac{((K_{\alpha}^+)^{-1}(\xi))^2}{2} \geq -\xi + K_{\alpha}(0).$$

So we have

$$M_{2K} \frac{((K_{\alpha}^+)^{-1} \circ K_{\beta}(-\xi))^2}{2} \geq -K_{\beta}(-\xi) + K_{\alpha}(0).$$

As for (3.66), we have the following inequality

$$K_{\beta}(0) - K_{\beta}(-\xi) \geq \gamma \frac{\xi^2}{2}.$$

So as  $K_{\alpha}(0) = K_{\beta}(0) = \rho$  we deduce that

$$M_{2K} \frac{((K_{\alpha}^+)^{-1} \circ K_{\beta}(-\xi))^2}{2} \geq \gamma \frac{\xi^2}{2} + K_{\alpha}(0) - K_{\beta}(0) \geq \gamma \frac{\xi^2}{2}.$$

That gives (3.67) and we deduce (3.59).  $\square$

## 6 Error estimates

### 6.1 Proof of the error estimates

To prove Theorem 1.3, we will need the following result whose classical proof is given in Appendix for the reader's convenience.

**Lemma 6.1 (A priori control).** *Let  $T > 0$  and let  $u^h$  be a sub-solution of the numerical scheme (3.15)-(3.17) and  $u$  a super-solution of (3.1)-(3.2) satisfying for some  $C_T > 0$ ,*

$$u(t, x) \geq -C_T(1 + d(0, x)) \quad \text{for } t \in (0, T).$$

*Then there exists a constant  $C = C(T) > 0$  such that for all  $(t, x) \in \mathcal{G}_h$ ,  $t \leq T$ , and  $(s, y) \in [0, T) \times J$ , we have*

$$u^h(t, x) \leq u(s, y) + C(1 + d(x, y)). \quad (3.68)$$

We also need the following result [66, Lemma 4.4] where the proof is given in [66].

**Lemma 6.2 (From quasi-convex to convex Hamiltonians).** *Let  $K \in (0, +\infty)$ . Given Hamiltonians  $H_{\alpha} : [-K, K] \rightarrow \mathbb{R}$  satisfying (3.3), there exists a function  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  such that the functions  $\beta \circ H_{\alpha}$  satisfy (3.9) for  $\alpha = 1, \dots, N$ . Moreover, we can choose  $\beta$  such that  $\beta \in C^2(\mathbb{R})$  and  $\beta' > 1$ .*

*Remark 6.3.* In [66, Lemma 4.4], the functions  $\beta \circ H_\alpha$  satisfy in fact the following assumptions

$$\begin{cases} H_\alpha \in C^2(\mathbb{R}) \text{ with } H_\alpha'' > 0 \text{ on } \mathbb{R}, \\ H_\alpha' < 0 \text{ on } (-\infty, 0) \text{ and } H_\alpha' > 0 \text{ on } (0, +\infty), \\ \lim_{|p| \rightarrow +\infty} \frac{H_\alpha(p)}{|p|} = +\infty. \end{cases} \quad (3.69)$$

which implies (3.9). Indeed, in the next proof on error estimates, we only need to consider Hamiltonians on a compact set which only depends on  $u_0$  and the Hamiltonians  $H_\alpha$ , thanks to the fact that the solution is Lipschitz continuous, see Theorem 2.5 and (3.24). So on  $[-K, K]$ , the functions  $(\beta \circ H_\alpha)''$  are bounded by some constant  $C > 0$ . We deduce that the functions  $L_\alpha$  are of class  $C^2$  and satisfy  $L_\alpha'' \geq \gamma = \frac{1}{C}$ . Indeed, from the relation  $H_\alpha(p) + L_\alpha(q) = pq$  with  $q = H_\alpha'(p)$ , one can deduce that  $L_\alpha'(H_\alpha'(q)) = q$ , so

$$L_\alpha''(q) = \frac{1}{H_\alpha'' \circ (H_\alpha')^{-1}(q)} \geq \gamma.$$

We now turn to the proof of the error estimates in the case of flux-limited junction conditions.

**Proof of Theorem 1.3.** We assume that the Hamiltonians  $H_\alpha$  satisfy (3.3). Let  $u$  be the solution of (3.5) and  $u^h$  the solution of the corresponding scheme (3.15) with  $F = F_A$ .

In order to get (3.22), we only prove that

$$u^h(t, x) - u(t, x) \leq \begin{cases} C_T(\Delta x)^{1/2} & \text{if } A > A_0, \\ C_T(\Delta x)^{2/5} & \text{if } A = A_0 \end{cases} \quad \text{in } [0, T] \times J \cap \mathcal{G}_h$$

since the proof of the other inequality is very similar. We are going to prove that

$$u^h(t, x) - u(t, x) \leq \begin{cases} \mathcal{O}\left(\frac{\Delta t}{\nu}\right) + \mathcal{O}\left(\frac{\Delta x}{\varepsilon}\right) + \mathcal{O}(\varepsilon) + \mathcal{O}(\nu) & \text{if } A > A_0, \\ \mathcal{O}\left(\frac{\Delta t}{\nu}\right) + \mathcal{O}\left(\frac{\Delta x}{\varepsilon\sqrt{\rho}}\right) + \mathcal{O}\left(\frac{(\Delta x)^2}{(\varepsilon\rho)^2}\right) + \mathcal{O}(\rho) + \mathcal{O}(\varepsilon) + \mathcal{O}(\nu) & \text{if } A = A_0. \end{cases} \quad (3.70)$$

which yields the desired inequality by minimizing the right hand side with respect to  $\varepsilon$  and  $\nu$  in the case  $A > A_0$  and with respect to  $\rho, \varepsilon$  and  $\nu$  in the case  $A = A_0$ . Let  $\beta$  be the function defined in Lemma 6.2 such that the functions  $\beta \circ H_\alpha$  satisfy (3.9). In the following, we consider that the function  $\mathcal{D}_0$  is associated to the Hamiltonians  $\beta \circ H_\alpha$  and to the flux limiter  $\beta(A)$  which satisfies  $\beta(A) > \beta(A_0)$  in the case  $A > A_0$ .

The remaining of the proof proceeds in several steps.

**Step 1: Penalization procedure.** Using the expression of  $\mathcal{D}_0$  in (3.43) and  $\mathcal{D}_{junction}$  in (3.47), we deduce that it exists  $C > 0$ , such that  $\forall x \in J$

$$\mathcal{D}_0(0, 0) = L_A(0) = -A \leq \mathcal{D}_0(x, x) \leq C.$$

Let  $\tilde{\mathcal{D}}_0 = \mathcal{D}_0 + A$ , we have that

$$0 \leq \tilde{\mathcal{D}}_0(x, x) \leq C + A.$$

For  $\eta, \delta, \varepsilon, \nu$  positive constants, let us define

$$M_{\varepsilon, \delta} = \sup_{\substack{(t, x) \in \mathcal{G}_h, \\ (s, y) \in [0, T] \times J}} \left\{ u^h(t, x) - u(s, y) - \varepsilon \tilde{\mathcal{D}}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \frac{(t-s)^2}{2\nu} - \frac{\delta}{2} d^2(y, 0) - \frac{\eta}{T-s} \right\} \quad (3.71)$$

where the test function  $\mathcal{D}_0$  is given in (3.43). In this step, we assume that  $M_{\varepsilon, \delta} > 0$ . Thanks to Lemma 6.1 and the superlinearity of  $\mathcal{D}_0$  (see Lemma 5.2), we deduce that for  $(x, y)$  such that the quantity in the supremum is larger than  $\frac{M_{\varepsilon, \delta}}{2}$ , we have

$$0 < \frac{M_{\varepsilon, \delta}}{2} \leq C(1 + d(y, x)) - \varepsilon \frac{\gamma}{2} d^2 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \frac{(t-s)^2}{2\nu} - \frac{\delta}{2} d^2(y, 0) - \frac{\eta}{T-s}$$

which implies in particular

$$\frac{\gamma}{2\varepsilon} d^2(y, x) \leq C(1 + d(y, x)),$$

and

$$\frac{\delta}{2} d^2(y, 0) \leq C(1 + d(y, x)).$$

Notice that in the following, we use the notation  $\mathcal{D}_0$  instead of  $\tilde{\mathcal{D}}_0$ . Indeed we deal only with partial derivatives of  $\mathcal{D}_0$  which are equal to partial derivatives of  $\tilde{\mathcal{D}}_0$  and differences between two values of  $\mathcal{D}_0$  at two points which are equal to differences between two values of  $\tilde{\mathcal{D}}_0$  at these two points.

We deduce from the two last inequalities that  $d(y, x)$  is bounded and  $d(y, 0)$  is bounded, so the supremum is reached at some point  $(t, x, s, y)$  where  $y \in J_\beta$  and  $x \in J_\alpha$ . This estimate together with the fact that  $-\partial_y \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \delta d(y, 0)$  lies in the viscosity subdifferential of  $u(t, \cdot)$  at  $x$  and the fact that  $\delta d(y, 0)$  is bounded, implies that there exists  $K > 0$  only depending on  $\|\nabla u\|_\infty$  (see Theorem 2.5) such that the point  $(t, x, s, y)$  realizing the maximum satisfies

$$\left| \partial_y \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right| \leq K. \quad (3.72)$$

If  $\alpha = \beta$ , for  $\frac{y}{\varepsilon}$  or  $\frac{x}{\varepsilon}$  large, then (3.72) implies

$$\left| L'_\alpha \left( \frac{y}{\varepsilon} - \frac{x}{\varepsilon} \right) \right| \leq K.$$

As  $L_\alpha$  is superlinear, it implies that  $d \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \leq C$ , for  $C > 0$  which is sufficient for the use in step 2 of the  $C^{1,1}$  estimates as  $\mathcal{D}_0$  only depends on  $d \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right)$  for  $\frac{y}{\varepsilon}$  or  $\frac{x}{\varepsilon}$  large. If  $\alpha \neq \beta$ , assume by contradiction that  $\frac{y}{\varepsilon}$  or  $\frac{x}{\varepsilon}$  are not bounded when  $\varepsilon \rightarrow 0$ . Then using (3.55) and (3.49) we get a contradiction with (3.72). So  $\frac{y}{\varepsilon}$  and  $\frac{x}{\varepsilon}$  are bounded by a constant which only depends on  $\|\nabla u\|_\infty$  and the Hamiltonians  $H_\alpha$ .

We want to prove that for  $\eta > \eta^*$  (to be determined) the supremum in (3.71) is attained for  $t = 0$  or  $s = 0$ . We assume that  $t > 0$  and  $s > 0$  and we prove that  $\eta \leq \eta^*$ .

**Step 2: Viscosity inequalities.** Since  $t > 0$  and  $s > 0$ , we can use Lemma 4.4 and get the following viscosity inequalities.

If  $x \neq 0$ , then

$$\frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left\{ H_\alpha^- \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x+\Delta x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right\} \right), \right. \\ \left. H_\alpha^+ \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon} \right) \right\} \right) \right\} \leq 0.$$

If  $x = 0$ , then

$$\frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left( A, \max_\beta \left\{ H_\beta^- \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{\Delta x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, 0 \right) \right\} \right) \right\} \right) \leq 0.$$

If  $y \neq 0$ , then

$$-\frac{\eta}{(T-s)^2} + \frac{t-s}{\nu} + H_\alpha \left( -\partial_y \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \delta d(y, 0) \right) \geq 0.$$

If  $y = 0$ , then

$$-\frac{\eta}{(T-s)^2} + \frac{t-s}{\nu} + F_A \left( -\partial_y \mathcal{D}_0 \left( 0, \frac{x}{\varepsilon} \right) \right) \geq 0.$$

We now distinguish the case  $A > A_0$  and  $A = A_0$ .

**Case  $A > A_0$ .** Thanks to the  $C^{1,1}$  regularity of the function  $\mathcal{D}_0$ , see Proposition 5.12, and the fact that the functions  $H_\alpha^\pm, H_\alpha$  are locally Lipschitz we obtain, for  $x \in J_\alpha$  and  $y \in J_\beta$  with  $\alpha \neq \beta$  (i.e. for  $\mathcal{D}_0 = \mathcal{D}_{junction}$ ),

$$\text{if } x \neq 0, \quad \frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + H_\alpha \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right) + O \left( \frac{\Delta x}{\varepsilon} \right) \leq 0 \quad (3.73)$$

$$\text{if } x = 0, \quad \frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + F_A \left( \partial_x \mathcal{D}_0 \left( \frac{y}{\varepsilon}, 0 \right) \right) + O \left( \frac{\Delta x}{\varepsilon} \right) \leq 0 \quad (3.74)$$

$$\text{if } y \neq 0, \quad \frac{t-s}{\nu} + H_\beta \left( -\partial_y \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right) + O(\sqrt{\delta}) \geq \frac{\eta}{2T^2} \quad (3.75)$$

$$\text{if } y = 0, \quad \frac{t-s}{\nu} + F_A \left( -\partial_y \mathcal{D}_0 \left( 0, \frac{x}{\varepsilon} \right) \right) \geq \frac{\eta}{2T^2}. \quad (3.76)$$

Now for  $(y, x) \in J_\alpha \times J_\alpha$ , from (3.43) and (3.47), one can deduce that  $\mathcal{D}_0$  is in fact  $C^2$  far away from the curve  $\Gamma^\alpha$  defined in Remark 5.9, hence the viscosity inequalities (3.73)-(3.76) remain true.

Now we treat the case where  $(\frac{y}{\varepsilon}, \frac{x}{\varepsilon})$  is near the curve  $\Gamma^\alpha$ , but not on it.

First if  $(\frac{y}{\varepsilon}, \frac{x}{\varepsilon})$  is such that  $(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}) \in K^\alpha \setminus \Gamma^\alpha$  and  $(\frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon}) \notin K^\alpha$ , we have

$$\mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon} \right) \leq L_\alpha \left( \frac{x-\Delta x-y}{\varepsilon} \right).$$

So as  $H_\alpha^+$  is non-decreasing, we deduce that

$$H_\alpha^+ \left( \frac{\varepsilon}{\Delta x} \left\{ L_\alpha \left( \frac{x-y}{\varepsilon} \right) - L_\alpha \left( \frac{x-\Delta x-y}{\varepsilon} \right) \right\} \right) \leq H_\alpha^+ \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon} \right) \right\} \right).$$

Hence the viscosity inequalities (3.73)-(3.76) remain true. If  $(\frac{y}{\varepsilon}, \frac{x}{\varepsilon})$  is such that  $(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}) \notin K^\alpha$  and  $(\frac{y}{\varepsilon}, \frac{x+\Delta x}{\varepsilon}) \in K^\alpha \setminus \Gamma^\alpha$ , we have

$$\mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x+\Delta x}{\varepsilon} \right) \leq \mathcal{D}_{junction} \left( \frac{y}{\varepsilon}, \frac{x+\Delta x}{\varepsilon} \right).$$

So as  $H_\alpha^-$  is non-increasing, we deduce that

$$\begin{aligned} H_\alpha^- \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_{junction} \left( \frac{y}{\varepsilon}, \frac{x+\Delta x}{\varepsilon} \right) - \mathcal{D}_{junction} \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right\} \right) \\ \leq H_\alpha^- \left( \frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x+\Delta x}{\varepsilon} \right) - \mathcal{D}_0 \left( \frac{y}{\varepsilon}, \frac{x}{\varepsilon} \right) \right\} \right). \end{aligned} \quad (3.77)$$

Hence the viscosity inequalities (3.73)-(3.76) remain true.

Now for  $(\frac{y}{\varepsilon}, \frac{x}{\varepsilon})$  on the curve  $\Gamma^\alpha$ , we get the following viscosity inequalities, using Proposition 2.6.

If  $x \neq 0$ , then

$$\begin{aligned} \frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left\{ H_\alpha^- \left( \frac{\varepsilon}{\Delta x} \left\{ L_\alpha \left( \frac{x+\Delta x-y}{\varepsilon} \right) - L_\alpha \left( \frac{x-y}{\varepsilon} \right) \right\} \right), \right. \\ \left. H_\alpha^+ \left( \frac{\varepsilon}{\Delta x} \left\{ L'_\alpha(\xi_\alpha^+) \frac{x}{\varepsilon} - L'_\alpha(\xi_\alpha^+) \left( \frac{x-\Delta x}{\varepsilon} \right) \right\} \right) \right\} \leq 0. \end{aligned}$$

If  $x = 0$ , then

$$\frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left( A, \max_\alpha \left\{ H_\alpha^- \left( \frac{\varepsilon}{\Delta x} \left\{ L_\alpha \left( \frac{\Delta x-y}{\varepsilon} \right) - L_\alpha \left( -\frac{y}{\varepsilon} \right) \right\} \right) \right\} \right) \leq 0.$$

If  $y \neq 0$ , then

$$-\frac{\eta}{(T-s)^2} + \frac{t-s}{\nu} + \max \left\{ H_\alpha^- \left( L'_\alpha \left( \frac{x-y}{\varepsilon} \right) - \delta d(y, 0) \right), H_\alpha^+ \left( L'_\alpha(\xi_\alpha^-) - \delta d(y, 0) \right) \right\} \geq 0.$$

If  $y = 0$ , then

$$-\frac{\eta}{(T-s)^2} + \frac{t-s}{\nu} + \max \left( A, \max_\alpha H_\alpha^- \left( L'_\alpha \left( \frac{x}{\varepsilon} \right) \right) \right) \geq 0.$$

We now simplify the above inequalities,

$$\text{if } x \neq 0, \quad \frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left\{ H_\alpha^- \left( L'_\alpha \left( \frac{x-y}{\varepsilon} \right) \right), H_\alpha^+ \left( L'_\alpha(\xi_\alpha^+) \right) \right\} + O\left(\frac{\Delta x}{\varepsilon}\right) \leq 0 \quad (3.78)$$

$$\text{if } x = 0, \quad \frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + \max \left( A, \max_\alpha H_\alpha^- \left( L'_\alpha \left( -\frac{y}{\varepsilon} \right) \right) \right) + O\left(\frac{\Delta x}{\varepsilon}\right) \leq 0 \quad (3.79)$$

$$\text{if } y \neq 0, \quad \frac{t-s}{\nu} + \max \left\{ H_\alpha^- \left( L'_\alpha \left( \frac{x-y}{\varepsilon} \right) \right), H_\alpha^+ \left( L'_\alpha(\xi_\alpha^-) \right) \right\} + O(\sqrt{\delta}) \geq \frac{\eta}{2T^2} \quad (3.80)$$

$$\text{if } y = 0, \quad \frac{t-s}{\nu} + \max \left( A, \max_\alpha H_\alpha^- \left( L'_\alpha \left( \frac{x}{\varepsilon} \right) \right) \right) \geq \frac{\eta}{2T^2}. \quad (3.81)$$

Combining these viscosity inequalities and using Theorem 5.10 with the Hamiltonians  $\beta \circ H_\alpha$ , we deduce the same equalities for the Hamiltonians  $H_\alpha$  as  $\beta$  is a bijection. We use also the fact that  $H_\alpha^+(L'_\alpha(\xi_\alpha^+)) = A$  and  $H_\alpha^+(L'_\alpha(\xi_\alpha^-)) = A_0$ , we get in all cases

$$\eta \leq \mathcal{O}\left(\frac{\Delta t}{\nu}\right) + \mathcal{O}\left(\frac{\Delta x}{\varepsilon}\right) + \mathcal{O}(\sqrt{\delta}) =: \eta^*.$$

**Case  $A = A_0$ .** In this case the function  $\mathcal{D}_{junction}$  is not of class  $C^{1,1}$ , see Proposition 5.12. So we consider the function  $\mathcal{D}_0$  associated with  $A = A_0 + \rho$  where  $\rho$  is a small parameter. The main difference with the case  $A > A_0$  is in the case  $x \in J_\alpha$  and  $y \in J_\beta$  with  $\alpha \neq \beta$ . We only treat the case  $x \in J_\alpha \setminus \{0\}$  and  $y \in J_\beta$  with  $\alpha \neq \beta$  since in the other cases the arguments are the same as in the proof of the case  $A > A_0$ . Since  $\mathcal{D}_0(\frac{y}{\varepsilon}, \cdot)$  is nondecreasing and  $H_\alpha^-(p) = A_0$  for  $p \geq 0$ , and  $H_\alpha^+(p) = H_\alpha(p)$  for  $p \geq 0$ , we have

$$\frac{t-s}{\nu} - \frac{\Delta t}{2\nu} + H_\alpha\left(\frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right) - \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon}\right) \right\}\right) \leq 0. \quad (3.82)$$

By using the Taylor expansion of the function  $\mathcal{D}_0(\frac{y}{\varepsilon}, \cdot)$  of class  $C^1$ , there exists  $\theta_1 \in [0, 1]$  such that

$$H_\alpha\left(\frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right) - \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon}\right) \right\}\right) = H_\alpha\left(\partial_x \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right) - \frac{\Delta x}{2\varepsilon} \partial_{xx} \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\theta_1 \Delta x}{\varepsilon}\right)\right).$$

Using now a Taylor expansion of the function  $H_\alpha$  of class  $C^2$ , there exists  $\theta_2 \in [0, 1]$  such that

$$\begin{aligned} & H_\alpha\left(\frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right) - \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon}\right) \right\}\right) = \\ & H_\alpha\left(\partial_x \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right) - \frac{\Delta x}{2\varepsilon} \partial_{xx} \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\theta_1 \Delta x}{\varepsilon}\right)\right) H'_\alpha\left(\partial_x \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right)\right) \\ & + \frac{1}{8} \left(\frac{\Delta x}{\varepsilon}\right)^2 \partial_{xx} \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\theta_1 \Delta x}{\varepsilon}\right)^2 H''_\alpha\left(\partial_x \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right) - \frac{\theta_2 \Delta x}{2\varepsilon} \partial_{xx} \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\theta_1 \Delta x}{\varepsilon}\right)\right). \end{aligned} \quad (3.83)$$

Using Taylor expansion for  $\partial_x \mathcal{D}_0(\cdot, \frac{y}{\varepsilon})$  and  $H'_\alpha$  of class  $C^1$  there exists  $\theta_3, \theta_4 \in [0, 1]$  such that

$$\begin{aligned} H'_\alpha\left(\partial_x \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right)\right) &= H'_\alpha\left(\partial_x \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\theta_1 \Delta x}{\varepsilon}\right) + \theta_1 \frac{\Delta x}{\varepsilon} \partial_{xx} \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\theta_3 \Delta x}{\varepsilon}\right)\right) \\ &= H'_\alpha\left(\partial_x \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\theta_1 \Delta x}{\varepsilon}\right)\right) \\ &+ \theta_1 \frac{\Delta x}{\varepsilon} \partial_{xx} \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\theta_3 \Delta x}{\varepsilon}\right) H''_\alpha\left(\partial_x \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\theta_1 \Delta x}{\varepsilon}\right) + \theta_4 \frac{\Delta x}{\varepsilon} \partial_{xx} \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\theta_3 \Delta x}{\varepsilon}\right)\right). \end{aligned} \quad (3.84)$$

Notice that the terms in  $H''_\alpha$  are bounded since  $\frac{x}{\varepsilon}$ ,  $\frac{y}{\varepsilon}$  and  $\frac{\Delta x}{\varepsilon \rho}$  are bounded independently of  $\Delta x \leq 1$  as we take  $\varepsilon = \rho = \Delta x^{\frac{2}{5}}$ .

So combining (3.83) and (3.84), thanks to the  $C^{1,1}$  regularity of the function  $\mathcal{D}_0$ , see Proposition 5.12 we deduce that

$$H_\alpha\left(\frac{\varepsilon}{\Delta x} \left\{ \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right) - \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x-\Delta x}{\varepsilon}\right) \right\}\right) = H_\alpha\left(\partial_x \mathcal{D}_0\left(\frac{y}{\varepsilon}, \frac{x}{\varepsilon}\right)\right) + \mathcal{O}\left(\frac{\Delta x}{\varepsilon \sqrt{\rho}}\right) + \mathcal{O}\left(\left(\frac{\Delta x}{\varepsilon \rho}\right)^2\right).$$

So combining the viscosity inequality and using the fact that  $|F_A - F_{A_0}| \leq \rho$  we have

$$\eta \leq \mathcal{O}\left(\frac{\Delta t}{\nu}\right) + \mathcal{O}\left(\frac{\Delta x}{\varepsilon\sqrt{\rho}}\right) + \mathcal{O}\left(\left(\frac{\Delta x}{\varepsilon\rho}\right)^2\right) + \mathcal{O}(\sqrt{\delta}) + \rho =: \eta^*. \quad (3.85)$$

**Step 3: Estimate of the supremum.** We proved in the previous step that, if  $\eta > \eta^*$ , then either  $M_{\varepsilon,\delta} \leq 0$  or  $M_{\varepsilon,\delta}$  is reached either for  $t = 0$  or  $s = 0$ .

If  $t = 0$ , then using Theorem 2.5, we have

$$M_{\varepsilon,\delta} \leq u_0(x) - u_0(y) - \frac{\gamma}{2\varepsilon}d^2(y, x) + C_T s - \frac{s^2}{2\nu}.$$

Using the fact that  $u_0$  is  $L_0$ -Lipschitz, one can deduce

$$\begin{aligned} M_{\varepsilon,\delta} &\leq \sup_{r \geq 0} \left( L_0 r - \frac{\gamma}{2\varepsilon} r^2 \right) + \sup_{r > 0} \left( C r - \frac{r^2}{2\nu} \right) \\ &\leq O(\varepsilon) + O(\nu). \end{aligned}$$

If  $s = 0$ , then we can argue similarly (by using the stability of the numerical scheme Lemma 4.5 and get

$$M_{\varepsilon,\delta} \leq O(\varepsilon) + O(\nu).$$

**Step 4: Conclusion.** We proved that for  $\eta > \eta^*$ ,  $M_{\varepsilon,\delta} \leq O(\varepsilon) + O(\nu)$ . This implies that for all  $(t, x) \in \mathcal{G}_h$ ,  $t \leq T/2$ , we have

$$u^h(t, x) - u(t, x) \leq \varepsilon \tilde{\mathcal{D}}_0 \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon} \right) + \frac{\delta}{2} d^2(x, 0) + \frac{2\eta}{T} + O(\varepsilon) + O(\nu)$$

Replacing  $\eta$  by  $2\eta^*$  and recalling that  $\tilde{\mathcal{D}}_0(x, x) \leq C + A$  for all  $x \in J$ , we deduce that for  $(t, x) \in \mathcal{G}_\varepsilon$  and  $t \leq T/2$  (after letting  $\delta \rightarrow 0$ ),

$$u^h(t, x) - u(t, x) \leq O\left(\frac{\Delta t}{\nu}\right) + O\left(\frac{\Delta x}{\varepsilon}\right) + O(\varepsilon) + O(\nu).$$

Using the CFL condition (3.19) and optimizing with respect to  $\varepsilon$  and  $\nu$  yields the desired result on  $[0, \frac{T}{2}]$ . Doing the whole proof with  $u$  the solution of (3.5) and  $u^h$  the solution of the corresponding scheme (3.15) with  $F = F_A$  on  $[0, 2T]$  yields the desired result on  $[0, T]$ . □

*Remark 6.4.* If for all  $\alpha \in \{1, \dots, N\}$ ,  $\min H_\alpha = A_0$ , then in the case where  $A = A_0$ , thanks to the  $C^{1,1}$  regularity of the function  $D_0$ , see Proposition 5.12, we can conclude as the case  $A > A_0$  that the error estimate is of order  $\Delta x^{\frac{1}{2}}$ .

$\Delta x$	$\ u(T, \cdot) - u^h(T, \cdot)\ _\infty$
0.00250	$1,192 \times 10^{-4}$
0.00100	$0,753 \times 10^{-4}$
0.00075	$0,644 \times 10^{-4}$
0.00050	$0,503 \times 10^{-4}$
0.00025	$0,329 \times 10^{-4}$

Figure 3.3: Error estimates for  $A = A_0 = 0$ 

$\Delta x$	$\ u(T, \cdot) - u^h(T, \cdot)\ _\infty$
0.00250	$1,266 \times 10^{-4}$
0.00100	$0,719 \times 10^{-4}$
0.00075	$0,616 \times 10^{-4}$
0.00050	$0,511 \times 10^{-4}$
0.00025	$0,350 \times 10^{-4}$

Figure 3.4: Error estimates for  $A = 0.1 > A_0$ 

## 6.2 Numerical simulations

In this subsection, we give a numerical example which illustrates the convergence rate we obtained in the previous subsection. In the case  $A > A_0$ , we get an optimal error estimate of order  $\Delta x^{\frac{1}{2}}$ . But in the case  $A = A_0$  we only have examples with an error estimate of order  $\Delta x^{\frac{1}{2}}$  when in the proof we have  $\Delta x^{\frac{2}{5}}$ . So we wonder if the error estimate obtained in the proof is optimal for the case  $A = A_0$ .

Here we consider a junction with two branches  $J_1 = J_2 = [0, X]$ . We have the two following Hamiltonians,

$$\begin{aligned} H_1(p) &= p^2, \\ H_2(p) &= p^2 - 1, \end{aligned}$$

and the initial data

$$u_0(x) = \begin{cases} \sin(0.2x) & \text{if } x \in J_1, \\ \sin(x) & \text{if } x \in J_2. \end{cases}$$

In the simulation we take  $X = 0.1$  and we give the error  $\|u(T, \cdot) - u^h(T, \cdot)\|_\infty$  at time  $T = 0.01$ . Here we have  $A_0 = 0$ .

For  $A_0 = 0$ ,  $A = 0.1 > A_0$  and  $\Delta t = \frac{\Delta x}{10}$  we get the following result, see Figure 3.5 and 3.6 plotted in logarithmic scale and the error values in Table 3.3 and 3.4.

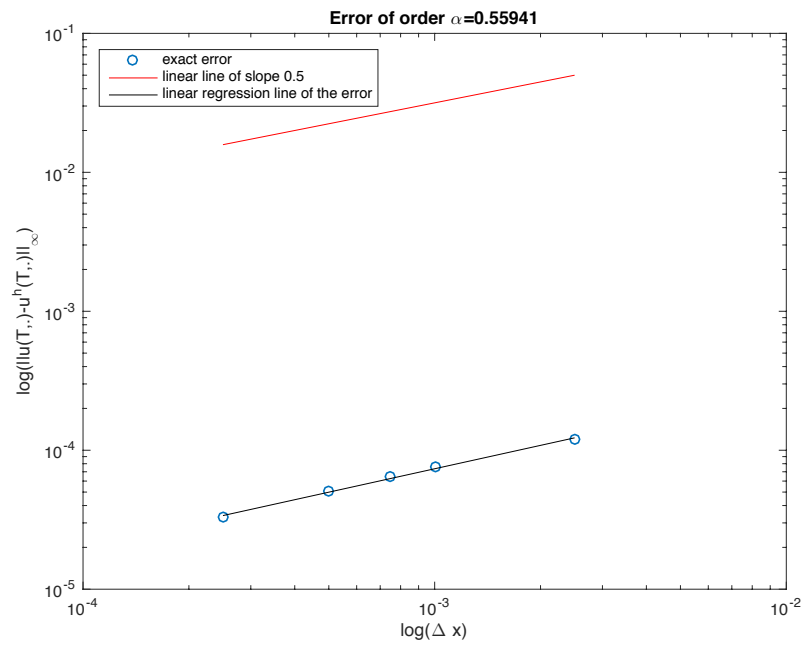
## 7 Appendix

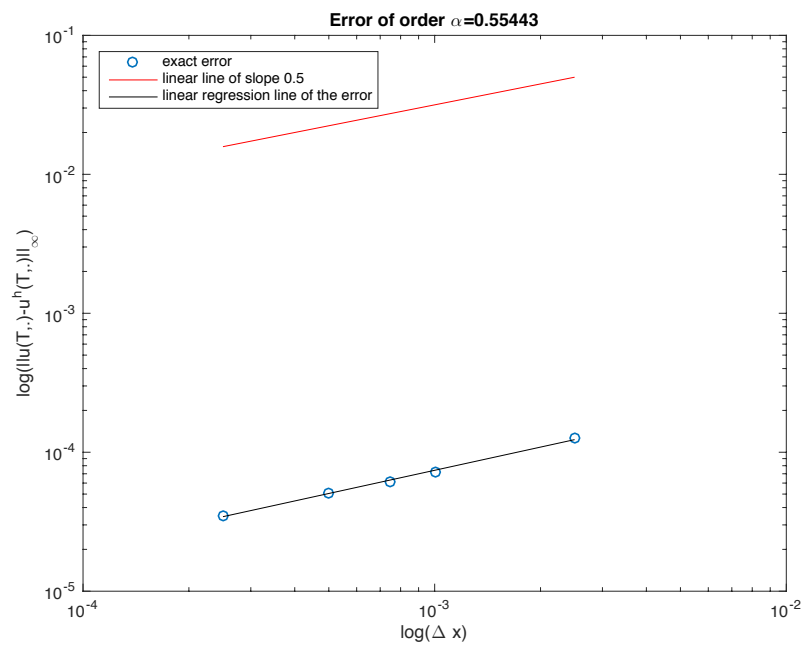
In this section we prove some technical results.

### 7.1 Proof of a priori control

In order to prove Lemma 6.1, we need the following one.



Figure 3.5: Error estimates for  $A = A_0 = 0$

Figure 3.6: Error estimates for  $A = 0.1 > A_0$

**Lemma 7.1 (A priori control at the same time).** *Assume that  $u_0$  is Lipschitz continuous. Let  $T > 0$  and let  $u^h$  be a sub-solution of (3.15)-(3.17) and  $u$  be a super-solution of (3.1)-(3.2). Then there exists a constant  $C = C_T > 0$  such that for all  $(t, x) \in \mathcal{G}_h$ ,  $t \leq T$ ,  $y \in J$ , we have*

$$u^h(t, x) \leq u(t, y) + C_T(1 + d(x, y)). \quad (3.86)$$

We first derive Lemma 6.1 from Lemma 7.1.

**Proof of Lemma 6.1.** Let us fix some  $h$  and let us consider the sub-solution  $u^-$  of (3.15) and the super-solution  $u^+$  of (3.1) defined as :

$$u^+(t, x) = u_0(x) + C_0 t$$

$$u^-(n\Delta t, i\Delta x) = u_0(i\Delta x) - C_0 n\Delta t$$

where

$$C_0 = \max \left\{ |A|, \max_{\alpha=1, \dots, N} \max_{|p_\alpha| \leq L_0} |H_\alpha(p_\alpha)|; \max_{|p_\alpha| \leq L_0} F(p_1, \dots, p_N) \right\}$$

and  $L_0$  denotes the Lipschitz constant of  $u_0$ . We have for all  $(t, x) \in \mathcal{G}_h$ , with  $t \leq T$ ,  $(s, y) \in [0, T) \times J$

$$u^-(t, x) - u^+(s, y) \leq 2C_0 T + L_0 d(x, y).$$

We first apply Lemma 7.1 to control  $u^h(t, x) - u^-(t, x)$  and then apply Lemma 6.1 to control  $u^+(s, y) - u(s, y)$ . Finally we get the control on  $u^h(t, x) - u(s, y)$ .  $\square$

We can now prove Lemma 7.1.

**Proof of Lemma 7.1.** We define  $\varphi$  in  $J^2$  as

$$\varphi(x, y) = \sqrt{1 + d^2(x, y)}.$$

Since,

$$d^2(x, y) = \begin{cases} (x - y)^2 & \text{if } (x, y) \in J_\alpha \times J_\alpha \\ (x + y)^2 & \text{if } (x, y) \in J_\alpha \times J_\beta \text{ with } \alpha \neq \beta \end{cases}$$

we see that  $d^2$  (and consequently  $\varphi$ ) is in  $C^{1,1}$  in  $J^2$ . Moreover  $\varphi$  satisfies

$$|\varphi_x(x, y)|, |\varphi_y(x, y)| \leq 1. \quad (3.87)$$

Recalling that there exists  $C > 0$  such that

$$|u^h(t, x) - u_0(x)| \leq Ct \quad \text{and} \quad |u(t, y) - u_0(y)| \leq Ct$$

(see Theorem 2.5 and (3.41)) and using that  $u_0$  is  $L_0$ -Lipschitz continuous we deduce that for all  $(t, x) \in \mathcal{G}_h$ ,  $t \leq T$ ,  $y \in J$ ,

$$u^h(t, x) - u(t, y) \leq 2Ct + L_0 \varphi(x, y),$$

which yields the desired result.  $\square$

## 7.2 Construction of $\tilde{F}$

**Lemma 7.2.** *There exists  $\tilde{F}$ , such that*

1.  $\tilde{F}$  satisfies (3.7);
2.  $F = \tilde{F}$  in  $Q_0$ ;
3. For  $p \in \mathbb{R}^N$ ,  $(-\nabla \cdot \tilde{F})(p) \leq \sup_{Q_0}(-\nabla \cdot F)$ .

*Proof.* Let  $I_\alpha$  denote  $[\underline{p}_\alpha^0; \bar{p}_\alpha]$  so that  $Q_0 = \prod_\alpha I_\alpha$ . We then define

$$\tilde{F}(p) = F(\mathcal{P}_1(p_1), \dots, \mathcal{P}_N(p_N)) - \sum_{\alpha=1}^N C_\alpha (p_\alpha - \mathcal{P}_\alpha(p_\alpha)),$$

where

$$C_\alpha = \min_{p \in Q_0} \left( -\frac{\partial F}{\partial p_\alpha}(p_1, \dots, p_N) \right),$$

and

$$\mathcal{P}_\alpha(r) = \begin{cases} \underline{p}_\alpha^0 & \text{if } r < \underline{p}_\alpha^0, \\ r & \text{if } r \in I_\alpha, \\ \bar{p}_\alpha & \text{if } r > \bar{p}_\alpha. \end{cases}$$

Remark that in view of the assumptions made on  $F$ , we have  $C_\alpha > 0$  which will ensure that (3.7) holds true. It is now easy to check that (3.7) and Item 3 are satisfied. This ends the proof of the Lemma.  $\square$

## 7.3 Relation between the junction and BLN conditions

Consider the following scalar conservation law posed on  $(0, +\infty)$ ,

$$\begin{cases} \partial_t v + \partial_x(H(v)) = 0, & t > 0, x > 0, \\ v(t, 0) = v_b(t), & t > 0, \\ v(0, x) = v_0(x), & x > 0. \end{cases}$$

The usual BLN condition asserts that the trace  $v_\tau$  of the entropy solution at  $x = 0$  (if it exists) of the previous scalar conservation law should satisfy

$$\forall \kappa \in [\min(v_b, v_\tau), \max(v_b, v_\tau)], \quad \text{sgn}(v_\tau - v_b)(H(v_\tau) - H(\kappa)) \leq 0.$$

If  $H$  is quasi-convex, this reduces to

$$H(v_\tau) = \max(H^-(v_\tau), H^+(v_b)).$$

This corresponds to a flux limiter  $A = H^+(v_b)$ .



## Part II

# Parabolic regularity à la De Giorgi



# Quantitative intermediate value lemmas for parabolic De Giorgi regularity

In this chapter, we deal with the De Giorgi Hölder regularity theory for parabolic equations with rough coefficients. We give a quantitative version of the intermediate value lemma, one of the two main step of the De Giorgi method sometimes called “second lemma of De Giorgi”.

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# 1 Introduction

Let us first introduce the main results and a historical overview of the elliptic regularity theory of De Giorgi [39].

## 1.1 Main results

We give two quantitative versions of the intermediate value lemma for parabolic equations. We first consider the following parabolic equation

$$\partial_t f = \nabla_x \cdot (A \nabla_x f), \quad t \in (T_1, T_2), x \in \Omega, \quad (4.1)$$

where  $T_1$  and  $T_2$  are real numbers,  $d$  is a positive integer,  $\Omega$  is an open set of  $\mathbb{R}^d$ ,  $f$  is a real-valued function of  $(t, x)$  and  $A = A(t, x)$  a  $d \times d$  bounded measurable matrix and  $A$  satisfies an ellipticity condition for two positive constants  $\lambda, \Lambda$ ,

$$0 < \lambda I \leq A \leq \Lambda I. \quad (4.2)$$

Let  $Q_1 = (T_1, T_2) \times \Omega$  and  $C_c^\infty(Q_1)$  denotes the set of smooth functions compactly supported in  $Q_1$ . We say that  $f$  is a *weak solution* of (4.1) if  $f \in L^\infty(T_1, T_2; L^2(\Omega))$  such that  $\nabla_x f \in L^2(Q_1)$  and  $\partial_t f \in L^2(T_1, T_2; H^{-1}(\Omega))$ , and for all  $\varphi \in C_c^\infty(Q_1)$  we have

$$-\int_{Q_1} f \partial_t \varphi + \int_{Q_1} A \nabla_x f \cdot \nabla_x \varphi = 0.$$

Let  $B_R$  be the ball centered at 0 of radius  $R > 0$  in  $\mathbb{R}^N$ . For  $X = (x_1, \dots, x_N) \in \mathbb{R}^N$ , we define  $|X| := \sqrt{x_1^2 + \dots + x_N^2}$  and for  $\rho > 0$ , the cylinder  $Q_\rho = (T_1, T_2) \times B_\rho$ . We say that a constant is universal if it depends only on  $R, T_1, T_2, \lambda, \Lambda$ , and  $d$ . Our first main theorem is the following intermediate value lemma.

**Theorem 1.1** (Parabolic intermediate value lemma: Solution case without source term). *Let  $f$  be a solution of (4.1) for  $\Omega = B_{16R}$  such that  $|f| \leq 1$ . Then there exists  $\varepsilon > 0$  universal such that for all  $(k, l) \in \mathbb{R}^2$  satisfying  $k < l \leq 1$ , we have*

$$|\{f \leq k\} \cap Q_R| |\{f \geq l\} \cap Q_R|^2 \leq C_{l-k} |\{k < f < l\} \cap Q_{2R}|^{\frac{\varepsilon}{2+\varepsilon}}, \quad (4.3)$$

where  $C_{l-k}$  is a constant only depending on  $R, T_1, T_2, \lambda, \Lambda, l - k$  and  $d$ .

Then, we consider a more general parabolic equation with a source term,

$$\partial_t f = \nabla_x \cdot (A \nabla_x f) + B \cdot \nabla_x f + s, \quad t \in (T_1, T_2), x \in \Omega, \quad (4.4)$$

where  $s = s(t, x)$ ,  $B = B(t, x)$  are bounded measurable coefficients,  $A = A(t, x)$  as before satisfies (4.2) and  $B, s$  satisfy,

$$\begin{cases} |B| \leq \Lambda, \\ |s| \leq 1. \end{cases} \quad (4.5)$$

We say that  $f$  is a *weak subsolution* of (4.4) if  $f \in L^\infty(T_1, T_2; L^2(\Omega))$  such that  $\nabla_x f \in L^2(Q_1)$  and  $\partial_t f \in L^2(T_1, T_2; H^{-1}(\Omega))$ , and for all  $\varphi \in C_c^\infty(Q_1)$  nonnegative we have

$$-\int_{Q_1} f \partial_t \varphi + \int_{Q_1} A \nabla_x f \cdot \nabla_x \varphi - \int_{Q_1} B \cdot \nabla_x f \varphi - \int_{Q_1} s \varphi \leq 0.$$

We have the following intermediate value lemma.

**Theorem 1.2** (Parabolic intermediate value lemma). *Let  $f$  be a subsolution of (4.4) for  $\Omega = B_{2R}$  such that  $f \leq 1$ . Let  $\bar{T} = \frac{T_1+T_2}{2}$ ,  $Q_R^- = (T_1, \bar{T}) \times B_R$  and  $Q_R^+ = (\bar{T}, T_2) \times B_R$ . Then for all  $(k, l) \in \mathbb{R}^2$  such that  $k < l \leq 1$ , we have*

$$C \frac{l-k}{2-k} |\{f \leq k\} \cap Q_R^-| |\{f \geq l\} \cap Q_R^+| \leq |\{k < f < l\} \cap Q_R|^{\frac{1}{20}}, \quad (4.6)$$

where  $C$  is universal.

*Remark 1.3.* Theorem 1.2 is a step to obtain Hölder regularity with the De Giorgi method (see subsection 2.2). It is also useful to derive Harnack inequalities as in [58]. The inequality in the solution case (Theorem 1.1) is stronger than the inequality in the subsolution case as the intervals of time do not have to be disjoint. But the inequality for subsolutions is valid for a larger class of parabolic equations with lower order terms. In fact, in the subsection 3.4, we will see that the intervals of time must be disjoint in the subsolution because there exists counterexamples if they are not (see subsection 3.4.1).

## 1.2 Historical overview

De Giorgi [39, 40] introduced techniques in 1957 to solve 19<sup>th</sup> Hilbert problem about the analytic regularity of local minimizers of an energy functional. In fact, these minimizers are solutions of quasilinear Euler-Lagrange equations. The idea of De Giorgi was to see quasilinear elliptic equations as linear elliptic equation with merely measurable coefficients. Thus he proved the Hölder regularity of solutions of elliptic equations with rough coefficients. In 1958, Nash [86] got the result with different techniques for both elliptic and parabolic equations. Then, Moser [85] proved in 1960 the Hölder regularity with a different approach. These methods are now called the De Giorgi-Nash-Moser techniques.

In his paper [39], De Giorgi exhibited a class of functions that satisfy energy estimates and showed that any function in this class is locally bounded and Hölder continuous. Ladyzhenskaya and Ural'tseva [75] extended his ideas to linear parabolic equations with lower order terms and to quasilinear parabolic equations. They introduced the corresponding De Giorgi classes and proved that Hölder estimate holds when  $\pm u$  are both in a De Giorgi class. One can find more details in [74] and in Chapter 6 of [77].

There are extensions of the method in degenerate cases, like the  $p$ -Laplacian, by Ladyzhenskaya and Ural'tseva [76] in the elliptic case. Then DiBenedetto [41] covered the degenerate parabolic cases, see also DiBenedetto, Gianazza and Vesprì [43, 44, 45].

Concerning nonlinear nonlocal time-dependent variational problems, Caffarelli and Vasseur [26] on the first hand and Caffarelli, Chan and Vasseur [24] on the second hand extended the method of De Giorgi to nonlocal parabolic equations and got a Hölder regularity result for solutions of problems with translation invariant kernels. Also Caffarelli, Soria, Vázquez [25] used the De Giorgi method to prove Hölder continuity of solutions of a porous medium equation with nonlocal diffusion effects. This kind of equation has also been studied earlier by Kassmann [73] using Moser's

techniques where he got local regularity results and by Kassmann and Felsinger [52] where they obtained a weak Harnack inequality .

Recently, Golse, Imbert, Mouhot, Silvestre and Vasseur proved the Hölder regularity and obtained Harnack inequalities for kinetic equations. More precisely, the Fokker-Planck kinetic equation with rough coefficients was studied by Golse, Imbert, Mouhot, Vasseur [58] and provides the results for the Landau equation. Imbert and Silvestre [70] studied a class of kinetic integro-differential equations and deduced the results for the inhomogeneous Boltzmann equation without cut-off.

### 1.3 Contribution of the chapter and comparison with existing result

Here we study one of the two main steps of the De Giorgi method, sometimes called second lemma of De Giorgi or intermediate value lemma. In the elliptic case there are many quantitative versions of this lemma. De Giorgi [39, 40] obtained a quantitative version using an isoperimetric inequality argument, taken up by DiBenedetto [42] and Vasseur [97]. Recently, Hou and Niu [64] prove a quantitative version of this lemma using a Poincaré inequality. These versions are actually valid for any function in  $H^1$ . About parabolic equations, quantitative version of this lemma does not seem to exist. One can find nonquantitative versions, for example in [97], obtained by contradiction with a compactness argument. However, there exists a quantitative version of this lemma for nonlocal time-dependent integral operator [24] that does not apply for local parabolic equations.

In this chapter, we derive two quantitative versions of the intermediate value lemma. A first one for solutions with no lower order term. And a second one for sub-solutions of parabolic equations with lower order terms and a source term.

### 1.4 Organization of the chapter

In Section 2, we recall the main steps of the De Giorgi method for the Hölder regularity of elliptic and parabolic equations in order to understand the use of the intermediate value lemma. In Section 3, we recall a proof of the intermediate value lemma in the elliptic case and prove Theorems 1.1 and 1.2, the two parabolic versions.

## 2 De Giorgi method for elliptic and parabolic equations

In this section, we give the main steps of the De Giorgi method for elliptic and parabolic equations. In this section, a universal constant is a constant which only depends on  $\lambda, \Lambda$  and  $d$ .

### 2.1 Elliptic equation

Let us introduce the theorem of Hölder regularity obtained by De Giorgi [39] and recall the main tools for the proof.

### 2.1.1 De Giorgi Theorem

We consider the following elliptic equation

$$-\nabla \cdot (A \nabla u) = 0 \quad \text{for } x \in B_2, \quad (4.7)$$

where  $A = A(x)$  a  $d \times d$  bounded measurable matrix satisfying (4.2). We say that  $u$  is a *weak solution* of (4.7), if  $u \in H^1(B_2)$  and for all  $\varphi \in H_0^1(B_2)$  we have

$$\int_{B_2} A \nabla u \cdot \nabla \varphi = 0.$$

In the following, we will use the word “solution” instead of “weak solution”. De Giorgi proved the following Hölder regularity theorem.

**Theorem 2.1** (Hölder continuity: elliptic case). *Let  $u : B_2 \rightarrow \mathbb{R}$  be a solution of (4.7). Then  $u \in C^\alpha(B_1)$  with*

$$\|u\|_{C^\alpha(B_1)} \leq C \|u\|_{L^2(B_2)},$$

where  $C > 0$  and  $\alpha$  are universal constants.

*Remark 2.2.* Thanks to the scaling property, Theorem 2.1 holds true for all  $\Omega'$  and  $\Omega$  such that  $\Omega' \subset\subset \Omega$  where  $\Omega$  is a bounded subset of  $\mathbb{R}^d$ , instead of  $B_1$  and  $B_2$ . Indeed, let  $B, C, D$  be real constants and  $x \in \Omega'$ , the function  $\bar{u}(y) = Bu(x + Cy) + D$  is also a solution of (4.7) for a matrix  $\bar{A}$  which satisfies the same ellipticity condition (4.2) as the initial matrix  $A$ .

Let us introduce the positive part of a function  $f_+ = \max(f, 0)$  and its oscillation on a set  $E$ ,

$$\operatorname{osc}_E f(x) = \sup_{x \in E} f(x) - \inf_{x \in E} f(x).$$

We are going to give the main steps of the De Giorgi method to prove this theorem. We first explain how we can reduce Theorem 2.1 to a lemma called “lowering of the maximum”. This lemma states that if a solution is below 1 and mostly below zero, then it is far from 1 in a smaller ball. To prove this lemma we need two essential results. The first result is a  $L^2 - L^\infty$  estimate, called sometimes the first lemma of De Giorgi, which states that a solution bounded in  $L^2$  is in fact bounded in  $L^\infty$  in a smaller ball. The second one is the intermediate value lemma which quantifies in measure the fact that solutions of these equations cannot make a jump between two numerical values.

Before describing the method, two following lemmas will be needed.

The first lemma is a consequence of the  $L^2 - L^\infty$  estimate we will present next in the De Giorgi method.

**Lemma 2.3** ( $L^2 - L^\infty$  bound). *Let  $u : B_2 \rightarrow \mathbb{R}$  be a solution of (4.7). Then we have*

$$\|u\|_{L^\infty(B_{3/2})} \leq C \|u\|_{L^2(B_2)},$$

where  $C$  is a universal constant.

We also need the following lemma which is a consequence of an energy estimate.

**Lemma 2.4** (Caccioppoli estimate). *Let  $u : B_2 \rightarrow \mathbb{R}$  be a solution of (4.7). Then we have*

$$\|\nabla u_+\|_{L^2(B_1)} \leq C \|u_+\|_{L^2(B_{3/2})},$$

where  $C$  is a universal constant.

*Proof of Lemma 2.4.* The function  $u_+$  is a subsolution of (4.7) by Stampacchia's theorem. Let  $\varphi = u_+ \phi^2$  where  $\phi$  is a nonnegative  $C^\infty$  cut-off function equal to 1 in  $B_1$  and to 0 outside  $B_{\frac{3}{2}}$  such that  $0 \leq \phi \leq 1$  and  $|\nabla \phi| \leq 4$ . Then we have

$$\int_{B_2} A \nabla u_+ \cdot \nabla (u_+ \phi^2) \leq -2 \int_{B_2} A(\phi \nabla u_+) \cdot (u_+ \nabla \phi).$$

So we have

$$\begin{aligned} \lambda \int_{B_2} |\nabla u_+|^2 \phi^2 &\leq 2\Lambda \int_{B_2} |\phi \nabla u_+| |u_+ \nabla \phi| \\ &\leq \frac{\lambda}{2} \int_{B_2} |\nabla u_+|^2 \phi^2 + \frac{2\Lambda^2}{\lambda} \int_{B_2} |\nabla \phi|^2 u_+^2. \end{aligned}$$

Then

$$\int_{B_2} |\nabla u_+|^2 \phi^2 \leq \frac{4\Lambda^2}{\lambda^2} \int_{B_2} |\nabla \phi|^2 u_+^2,$$

and we deduce the result.  $\square$

### 2.1.2 De Giorgi method

Now we can introduce the steps of the proof.

#### Preliminary step: Reduction of the problem.

We first prove that one can reduce Theorem 2.1. The Hölder continuity is a consequence of the following lemmas.

**Lemma 2.5** (Traduction of the definition). *Let  $u : B_2 \rightarrow \mathbb{R}$  be a solution of (4.7). Then  $u$  satisfies*

$$\forall x_0 \in B_1, \forall r \in \left(0, \frac{1}{2}\right), \quad \text{osc}_{B_r(x_0)} u \leq Cr^\alpha \|u\|_{L^2(B_2)},$$

where  $C, \alpha$  are universal constants.

We assume that Lemma 2.5 is true and prove Theorem 2.1.

*Proof of Theorem 2.1.* Let  $(x, y) \in B_1^2$ ,  $x_0 = \frac{x+y}{2}$ ,  $x_1 = \frac{x+x_0}{2}$  and  $y_1 = \frac{x_0+y}{2}$ . We choose  $r = |x - x_1| = \frac{|x-y|}{4}$  and define  $x_2 = \frac{x+x_1}{2}$ . Then we have

$$|f(x) - f(x_1)| \leq \text{osc}_{B_r(x_2)} u \leq \frac{C}{4^\alpha} \|u\|_{L^2(B_2)} |x - y|^\alpha.$$

We compute the same inequality with  $|f(x_1) - f(x_0)|$ ,  $|f(x_0) - f(y_1)|$  and  $|f(y_1) - f(y)|$  and deduce the result thanks to a triangular inequality.  $\square$

In order to get the algebraic decay of the oscillation, it is enough to prove that the rate of decay of the oscillation between the ball of radius  $3/2$  and the ball of radius  $1/2$  is universally bounded by a constant strictly smaller than 1.

**Lemma 2.6** (Local decrease of the oscillation). *Let  $u : B_2 \rightarrow \mathbb{R}$  be a solution of (4.7). Then there exists a universal constant  $\theta \in (0, 1)$  such that*

$$\operatorname{osc}_{B_{\frac{1}{2}}} u \leq \theta \operatorname{osc}_{B_{\frac{3}{2}}} u.$$

We assume that Lemma 2.6 is true and prove Lemma 2.5.

*Proof of Lemma 2.5.* Let us define a sequence  $u_n(y) = u(x_0 + \frac{y}{3^n})$  of solutions of (4.7) with different matrices which satisfy the ellipticity condition (4.2). Then using Lemma 2.6, we have

$$\operatorname{osc}_{B_{\frac{3}{2}}} u_n = \operatorname{osc}_{B_{\frac{1}{2}}} u_{n-1} \leq \theta \operatorname{osc}_{B_{\frac{3}{2}}} u_{n-1}.$$

By induction and using Lemma 2.3 we deduce for all  $n \geq 1$

$$\operatorname{osc}_{B_{\frac{3}{2} \frac{1}{3^n}}(x_0)} u = \operatorname{osc}_{B_{\frac{3}{2}}} u_n \leq \theta^{n-1} \operatorname{osc}_{B_{\frac{3}{2}}} u_1 \leq \theta^{n-1} \operatorname{osc}_{B_{\frac{3}{2}}} u \leq \theta^{n-1} C \|u\|_{L^2(B_2)}.$$

We choose  $\alpha \in (0, 1)$  such that  $\theta = \frac{1}{3^\alpha}$ . For all  $r \in (0, \frac{1}{2})$ , there exists  $n \in \mathbb{N} - \{0\}$  such that  $\frac{3}{2} \frac{1}{3^{n+1}} \leq r \leq \frac{3}{2} \frac{1}{3^n}$ . We deduce from the previous inequality that

$$\operatorname{osc}_{B_r(x_0)} u \leq \operatorname{osc}_{B_{\frac{3}{2} \frac{1}{3^n}}(x_0)} u \leq \left(\frac{1}{3^n}\right)^\alpha C 3^\alpha \|u\|_{L^2(B_2)} \leq r^\alpha C 9^\alpha \|u\|_{L^2(B_2)}.$$

This achieves the proof of the lemma. □

The local decrease of the oscillation is a consequence of the following result.

**Lemma 2.7** (Lowering the maximum). *There exists a universal constant  $\mu \in (0, 1)$  such that for any solution  $v : B_2 \rightarrow \mathbb{R}$  of (4.7) satisfying*

$$\begin{cases} v \leq 1 \text{ in } B_{\frac{3}{2}} \\ |\{v \leq 0\} \cap B_1| \geq \frac{|B_1|}{2}, \end{cases} \quad (4.8)$$

we have

$$v \leq 1 - \mu \quad \text{in } B_{\frac{1}{2}}.$$

We assume that Lemma 2.7 is true and prove Lemma 2.6.

*Proof of Lemma 2.6.* We set  $v = \frac{2}{\operatorname{osc}_{B_{3/2}} u} \left( u - \frac{\sup u + \inf u}{2} \right)$ , where the supremum and the infimum are taken in  $B_{3/2}$ . Thus  $-1 \leq v \leq 1$  in  $B_{3/2}$  and either  $v$  or  $-v$  satisfy (4.8). We deduce that

$$\operatorname{osc}_{B_{1/2}} v \leq 2 - \mu,$$

and

$$\operatorname{osc}_{B_{1/2}} u \leq \left(1 - \frac{\mu}{2}\right) \operatorname{osc}_{B_{3/2}} u.$$

This implies Lemma 2.6 with  $\theta = 1 - \frac{\mu}{2}$ . □

**Proof of Lemma 2.7.**

Let us introduce two main results for the proof of Lemma 2.7.

**Lemma 2.8** (First lemma of De Giorgi:  $L^2 - L^\infty$  estimate). *There exists a positive universal constant  $\delta$  such that for any solution  $u : B_2 \rightarrow \mathbb{R}$  of (4.7) the following implication holds true. If*

$$\int_{B_1} u_+^2 \leq \delta,$$

then we have

$$u_+ \leq \frac{1}{2} \quad \text{in } B_{1/2}.$$

*Remark 2.9.* The previous lemma is a consequence of energy estimates and Sobolev inequalities. Here we admit it. One can find the proof in [97]. We have the same result if  $B_1$  and  $B_{1/2}$  are replaced by  $B_2$  and  $B_{3/2}$ , so applying this lemma to  $u$  and  $-u$ , we deduce Lemma 2.3.

We also have the following result thanks to Hou and Niu [64] and DiBenedetto [42] called sometimes the second lemma of De Giorgi.

**Lemma 2.10** (Second lemma of De Giorgi: intermediate value lemma in  $H^1$ ). *Let  $u \in H^1(B_1)$ . Then we have*

$$|\{u \leq 0\} \cap B_1| \times |\{u \geq \frac{1}{2}\} \cap B_1| \leq C |\{0 < u < \frac{1}{2}\} \cap B_1|^{\frac{1}{2}} \sqrt{\int_{B_1} |\nabla u_+(x)|^2 dx},$$

where  $C$  only depends on  $d$ .

*Remark 2.11.* One can find a slightly different version of Lemma 2.10 in [97] which uses an isoperimetric argument instead of a Poincaré inequality as in the proof of Lemma 3.1.

*Proof of Lemma 2.10.* We apply Lemma 3.1 with  $k = 0$  and  $l = \frac{1}{2}$ . □

This lemma is true not only for solutions of (4.7) but also for all functions in  $H^1$ . Now it remains to prove Lemma 2.7.

*Proof of Lemma 2.7.* We introduce a sequence of solutions of (4.7),

$$\begin{cases} v_0 = v \\ v_k = 2 \left( v_{k-1} - \frac{1}{2} \right). \end{cases}$$

So we have  $v_k = 2^k (v - (1 - 2^{-k}))$ . We consider two cases. Either there exists  $k_0$  such that  $\int_{B_1} (v_{k_0})_+^2 \leq \delta$ . By Lemma 2.8 we have  $(v_{k_0})_+ \leq \frac{1}{2}$  in  $B_{1/2}$  so that  $v \leq 1 - \frac{1}{2^{k_0+1}}$  in  $B_{1/2}$ . Or for all  $k \in \mathbb{N}$ , we have  $\int_{B_1} (v_k)_+^2 > \delta$ . In this case, we deduce that

$$|\{v_k \geq \frac{1}{2}\} \cap B_1| = |\{v_{k+1} \geq 0\} \cap B_1| \geq \int_{B_1} (v_{k+1})_+^2 > \delta,$$

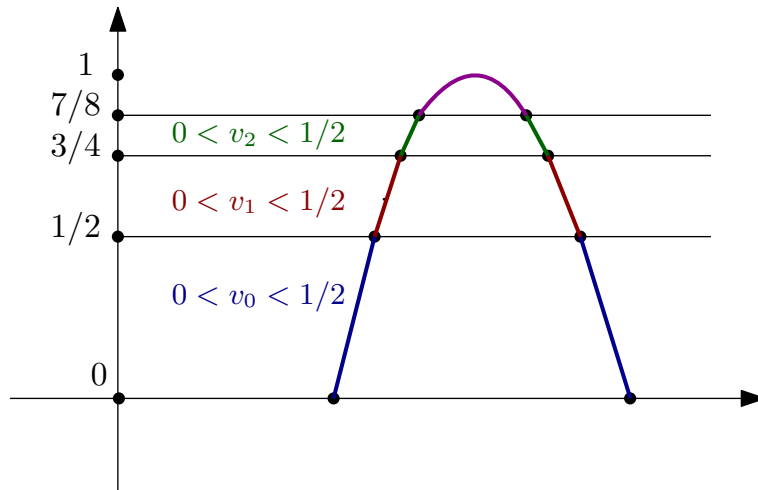


Figure 4.1: Intermediate set at each step.

and

$$|\{v_k \leq 0\} \cap B_1| \geq |\{v \leq 0\} \cap B_1| \geq \frac{|B_1|}{2}.$$

The fact that  $v_k$  is a solution of (4.7) and that  $0 \leq (v_k)_+ \leq 1$  implies thanks to Lemma 2.4, that  $\sqrt{\int_{B_1} |\nabla(v_k)_+(x)|^2 dx} \leq C$  where  $C$  is universal. So thanks to Lemma 2.10, there exists a universal constant  $\gamma > 0$  such that

$$|\{0 < v_k < \frac{1}{2}\} \cap B_1| \geq \gamma. \tag{4.9}$$

In Figure 4.1, one can see that the intermediate sets at each step are disjoint and because of (4.9), there cannot exist infinitely many steps. We deduce it by induction that

$$\begin{aligned} |B_1| \geq |\{v_k \leq 0\} \cap B_1| &\geq |\{v_{k-1} \leq 0\} \cap B_1| + |\{0 < v_{k-1} < \frac{1}{2}\} \cap B_1| \\ &\geq |\{v_{k-1} \leq 0\} \cap B_1| + \gamma \\ &\geq \frac{|B_1|}{2} + k\gamma, \end{aligned}$$

that gives a contradiction for  $k$  big enough. So only the first case holds true and we deduce the result. Note that  $k_0 \leq \frac{|B_1|}{\gamma}$  so  $v \leq 1 - \frac{1}{2^{k_0+1}} \leq 1 - \frac{1}{2^{\frac{|B_1|}{\gamma}+1}}$  and we choose

$$\mu = \frac{1}{2^{\frac{|B_1|}{\gamma}+1}}. \quad \square$$

## 2.2 Parabolic equation

Let us introduce the theorem of Hölder regularity for parabolic equations and recall the main tools for the proof.

### 2.2.1 Hölder regularity Theorem

Let us introduce the parabolic cylinder  $Q_r = (-r^2, 0) \times B_r$ . We study the following parabolic equation

$$\partial_t f = \nabla_x \cdot (A \nabla_x f) + B \cdot \nabla_x f + s, \quad (t, x) \in Q_2 \tag{4.10}$$



where  $s = s(t, x)$ ,  $B = B(t, x)$  are bounded measurable coefficients,  $A = A(t, x)$  as before satisfies (4.2) and  $B, s$  satisfy (4.5). Let us state the Hölder regularity theorem in the parabolic case.

**Theorem 2.12** (Hölder continuity: parabolic case). *Let  $u : Q_2 \rightarrow \mathbb{R}$  be a solution of (4.10). Then  $u \in C^\alpha(Q_1)$  with*

$$\|u\|_{C^\alpha(Q_1)} \leq C \left( \|u\|_{L^2(Q_2)} + 1 \right),$$

where  $C$  and  $\alpha$  are universal constants.

One can find the full proof in [58].

*Remark 2.13.* For the same reasons as in the elliptic case (the scaling property), Theorem 2.12 holds true for all  $Q' = (s, T) \times \Omega'$  and  $Q = (0, T) \times \Omega$  such that  $\Omega' \subset\subset \Omega$  and  $0 < s < T$ , instead of  $Q_1$  and  $Q_2$ .

We have the same two lemmas as the elliptic case. Here we admit them as the proofs are very similar.

**Lemma 2.14** ( $L^2 - L^\infty$  bound). *Let  $u : Q_2 \rightarrow \mathbb{R}$  be a solution of (4.10). Then we have*

$$\|u\|_{L^\infty(Q_{3/2})} \leq C \|u\|_{L^2(Q_2)}.$$

**Lemma 2.15** (Gradient  $L^2$  bound). *Let  $u : Q_2 \rightarrow \mathbb{R}$  be a solution of (4.10). Then we have*

$$\|\nabla u_+\|_{L^2(Q_1)} \leq C \left( \|u_+\|_{L^2(Q_{3/2})} + 1 \right).$$

## 2.2.2 De Giorgi method

Let us consider the following hypothesis for the source term

$$|s| \leq \beta, \tag{4.11}$$

where  $\beta = \frac{1}{4^{\frac{|\Omega_1|}{\gamma}}}$  is a universal constant smaller than 1, with  $\gamma = \left(\frac{C\delta|\Omega_1|}{2}\right)^{20}$  where  $\delta$  is the universal constant given in Lemma 2.19 and  $C$  a universal constant corresponding to  $C^{\frac{l-k}{2-k}}$  in Theorem 1.2 (here  $k = 0$  and  $l = 1/2$ ).

Now we can introduce the steps of the proof.

### Preliminary step: Reduction of the problem.

We first prove that one can reduce Theorem 2.12. The Hölder continuity is a consequence of the following Lemmas.

**Lemma 2.16** (Traduction of the definition). *Let  $u : Q_2 \rightarrow \mathbb{R}$  be a solution of (4.10) such that the source term  $s$  satisfies (4.11). Then  $u$  satisfies*

$$\forall (t_0, x_0) \in Q_1, \forall r \in \left(0, \frac{1}{2}\right), \quad \text{osc}_{Q_r(t_0, x_0)} u \leq Cr^\alpha \left( \|u\|_{L^2(Q_2)} + 1 \right),$$

where  $C$  and  $\alpha$  are universal constants.

We assume that Lemma 2.16 is true and prove Theorem 2.12.

*Proof of Theorem 2.12.* The function  $\beta u$  is a solution of (4.10) which satisfies (4.11). The proof of Lemma 2.5 remains true in this case by replacing  $B_1$  and  $B_2$  by  $Q_1$  and  $Q_2$ . So we get the result for  $\beta u$  and then for  $u$ .  $\square$

The previous lemma is a consequence of the following oscillation decrease. This version of the Lemma is slightly different from the elliptic case because of the source term.

**Lemma 2.17** (Local decrease of the oscillation). *Let  $u : Q_2 \rightarrow \mathbb{R}$  be a solution of (4.10) such that the source term  $s$  satisfies (4.11). Then there exists a universal constant  $\theta \in (\frac{1}{2}, 1)$  such that*

- if  $\operatorname{osc}_{Q_{3/2}} u \geq 2$ , then  $\operatorname{osc}_{Q_{1/2}} u \leq \theta \operatorname{osc}_{Q_{3/2}} u$ ,
- if  $\operatorname{osc}_{Q_{3/2}} u \leq 2$ , then  $\operatorname{osc}_{Q_{1/2}} u \leq 2\theta$ .

We assume that Lemma 2.17 is true and prove Lemma 2.16.

*Proof of Lemma 2.16.* Let us define a sequence of solutions of (4.10) with a source term  $s$  satisfying (4.11) (since  $\frac{1}{9\theta} < 1$ ),

$$u_n(\tau, y) = \frac{2\theta^{1-n}}{\max(2, \operatorname{osc}_{Q_{3/2}} u)} u\left(t_0 + \frac{\tau}{9^n}, x_0 + \frac{y}{3^n}\right).$$

By induction let us prove that for all  $n \in \mathbb{N} - \{0\}$ ,

$$\operatorname{osc}_{Q_{1/2}} u_n \leq 2\theta. \tag{4.12}$$

Indeed for  $n = 1$ , we have (4.12) thanks to Lemma 2.17. Assuming that  $\operatorname{osc}_{Q_{1/2}} u_{n-1} \leq 2\theta$  and using Lemma 2.17, we distinguish two cases. If  $\operatorname{osc}_{Q_{3/2}} u_n \leq 2$ , we have (4.12). If

$\operatorname{osc}_{Q_{3/2}} u_n \geq 2$ , we have

$$\operatorname{osc}_{Q_{1/2}} u_n \leq \theta \operatorname{osc}_{Q_{3/2}} u_n = \operatorname{osc}_{Q_{1/2}} u_{n-1} \leq 2\theta,$$

and we deduce (4.12). So using (4.12) for  $n - 1$  we have,

$$\operatorname{osc}_{Q_{3/2}} u_n = \frac{1}{\theta} \operatorname{osc}_{Q_{1/2}} u_{n-1} \leq 2.$$

Thus we deduce by induction and using Lemma 2.14 that for all  $n \geq 1$ ,

$$\operatorname{osc}_{Q_{\frac{3}{2} \frac{1}{3^n}}(x_0)} u = \frac{\max(2, \operatorname{osc}_{Q_{3/2}} u) \theta^{n-1}}{2} \operatorname{osc}_{Q_{\frac{3}{2}}} u_n \leq \theta^{n-1} \max(2, \operatorname{osc}_{Q_{\frac{3}{2}}} u) \leq \theta^{n-1} C \left( \|u\|_{L^2(Q_2)} + 1 \right).$$

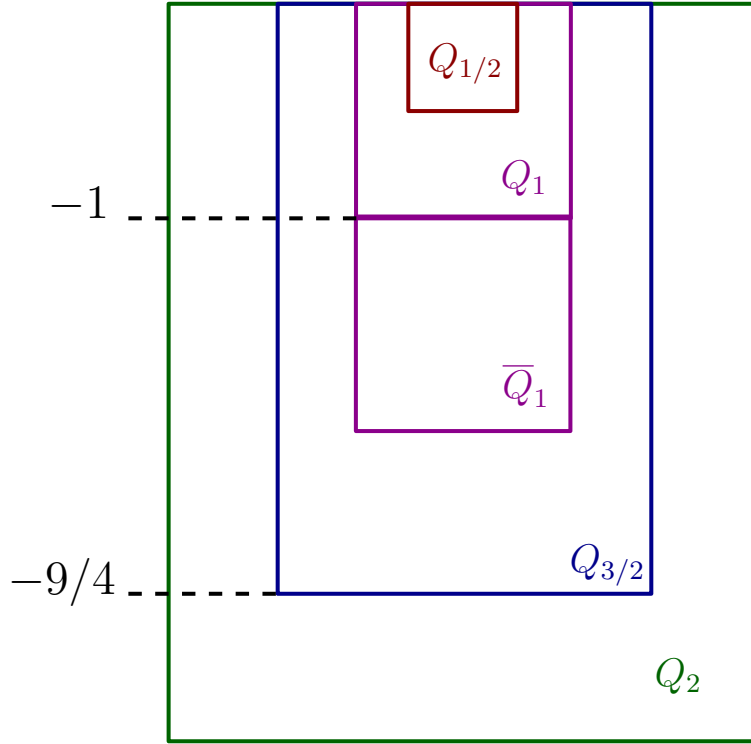


Figure 4.2: Parabolic cylinders.

We choose  $\alpha \in (0, 1)$  such that  $\theta = \frac{1}{3^\alpha}$ . Let  $r \in (0, \frac{1}{2})$ . In particular there exists  $n \in \mathbb{N} - \{0\}$  such that  $\frac{3}{2} \frac{1}{3^{n+1}} \leq r \leq \frac{3}{2} \frac{1}{3^n}$ . So we deduce that

$$\operatorname{osc}_{Q_r(x_0)} u \leq \operatorname{osc}_{Q_{\frac{3}{2} \frac{1}{3^n}}(x_0)} u \leq \left(\frac{1}{3^n}\right)^\alpha C 3^\alpha (\|u\|_{L^2(Q_2)} + 1) \leq r^\alpha C 9^\alpha (\|u\|_{L^2(Q_2)} + 1).$$

□

We define  $\bar{Q}_r = (-2r^2, -r^2] \times B_r$ . The local decrease of the oscillation is a consequence of the following result.

**Lemma 2.18** (Lowering the maximum). *There exists a universal constant  $\mu \in (0, 1)$  such that for any solution  $v : Q_2 \rightarrow \mathbb{R}$  of (4.10) with a source term  $s$  satisfying (4.11), if  $v$  verifies*

$$\begin{cases} v \leq 1 \text{ in } Q_{\frac{3}{2}} \\ |\{v \leq 0\} \cap \bar{Q}_1| \geq \frac{|\bar{Q}_1|}{2}, \end{cases} \quad (4.13)$$

then

$$v \leq 1 - \mu \quad \text{in } Q_{\frac{1}{2}}.$$

These cylinders are represented in Figure 4.2. We assume that Lemma 2.18 is true and prove Lemma 2.17.

*Proof of Lemma 2.17.* We distinguish two cases: either  $\operatorname{osc}_{Q_{3/2}} u \geq 2$  or  $\operatorname{osc}_{Q_{3/2}} u \leq 2$ . In the first case, we set  $v = \frac{2}{\operatorname{osc}_{Q_{3/2}} u} \left( u - \frac{\sup u + \inf u}{2} \right)$ , where the supremum and the infimum are taken in  $Q_{3/2}$ . So  $v$  is a solution of (4.10) with  $\bar{A}, \bar{B}$  and  $\bar{s}$  satisfying (4.2), (4.5) and (4.11). And following the same steps as the proof of Lemma 2.6, we deduce the result. In the second case, we set  $v = u - \frac{\sup u + \inf u}{2}$ . The function  $v$  or  $-v$  satisfies (4.13). So we have

$$\operatorname{osc}_{Q_{1/2}} u = \operatorname{osc}_{Q_{1/2}} v \leq 2 - \mu.$$

In both cases we deduce the result taking  $\theta = \max(1/2, 1 - \mu/2)$ . □

**Proof of Lemma 2.18.**

Let us introduce two main results for the proof of Lemma 2.18 where we consider solutions such that the source term  $s$  satisfies  $|s| \leq 1$  independantly of the universal constant  $\beta$ .

**Lemma 2.19** (First lemma of De Giorgi:  $L^2 - L^\infty$  estimate). *There exists a positive universal constant  $\delta$  such that for any solution  $u : Q_2 \rightarrow \mathbb{R}$  of (4.10) the following implication holds true. If*

$$\int_{Q_1} u_+^2 \leq \delta,$$

*then we have*

$$u_+ \leq \frac{1}{2} \quad \text{in } Q_{1/2}.$$

*Remark 2.20.* The previous lemma is a consequence of energy estimates and Sobolev inequalities. Here we admit it. One can find the proof in [58, Theorem 12]. We have the same result for  $Q_2$  and  $Q_{3/2}$ , so applying this lemma to  $u$  and  $-u$ , we deduce Lemma 2.14.

Here, the second result is the parabolic intermediate value lemma, Theorem 1.2, sometimes called second lemma of De Giorgi.

Now let us prove Lemma 2.18.

*Proof of Lemma 2.18.* We introduce a sequence of solutions of (4.10),

$$\begin{cases} v_0 = v \\ v_k = 2 \left( v_{k-1} - \frac{1}{2} \right). \end{cases}$$

Here  $v$  is a solution of (4.10) where the source term satisfies (4.11) and the functions  $v_k$  are solutions of (4.10) where the source term satisfies  $|s| \leq 1$  (it will be explain at the end of the proof). So we have  $v_k = 2^k \left( v - (1 - 2^{-k}) \right)$ . We consider two cases. Either there exists  $k_0$  such that  $\int_{Q_1} (v_{k_0})_+^2 \leq \delta$ . By Lemma 2.19 we have  $(v_{k_0})_+ \leq \frac{1}{2}$  in  $Q_{1/2}$  so that  $v \leq 1 - \frac{1}{2^{k_0+1}}$  in  $Q_{1/2}$ . Or for all  $k \in \mathbb{N}$ , we have  $\int_{Q_1} (v_k)_+^2 > \delta$ . In this case, we deduce that

$$|\{v_k \geq \frac{1}{2}\} \cap Q_1| = |\{v_{k+1} \geq 0\} \cap Q_1| \geq \int_{Q_1} (v_{k+1})_+^2 > \delta,$$

and

$$|\{v_k \leq 0\} \cap \bar{Q}_1| \geq |\{v \leq 0\} \cap \bar{Q}_1| \geq \frac{|\bar{Q}_1|}{2}.$$

We define  $\tilde{Q} = Q_1 \cup \bar{Q}_1$ . So thanks to Theorem 1.2, there exists  $\gamma > 0$  only depending on  $\lambda, \Lambda$  and  $d$  such that

$$|\{0 < v_k < \frac{1}{2}\} \cap \tilde{Q}| \geq \gamma.$$

Then we deduce recursively that

$$\begin{aligned} |\tilde{Q}| \geq |\{v_k \leq 0\} \cap \tilde{Q}| &\geq |\{v_{k-1} \leq 0\} \cap \tilde{Q}| + |\{0 < v_{k-1} < \frac{1}{2}\} \cap \tilde{Q}| \\ &\geq |\{v_{k-1} \leq 0\} \cap \tilde{Q}| + \gamma \\ &\geq \frac{|\bar{Q}_1|}{2} + k\gamma, \end{aligned}$$

which gives a contradiction for  $k$  big enough. So only the first case holds true and we deduce the result. Note that  $k_0 \leq 2^{\frac{|\bar{Q}_1|}{\gamma}}$  so  $v \leq 1 - \frac{1}{2^{k_0+1}} \leq 1 - \frac{1}{2^{\frac{2|\bar{Q}_1|}{\gamma}+1}}$  and we choose  $\mu = \frac{1}{2^{\frac{2|\bar{Q}_1|}{\gamma}+1}}$ . And the universal constant  $\beta = \frac{1}{2^{\frac{2|\bar{Q}_1|}{\gamma}}}$  implies that  $v_k$  are solutions of (4.10) with  $|s| \leq 1$ .  $\square$

### 3 Intermediate value lemmas

In this section, we deal with intermediate value lemmas for elliptic and parabolic equations. We first recall the lemma in the elliptic case in order to understand the proofs of the parabolic cases. Then we prove Theorem 1.1, an intermediate value lemma for solutions of parabolic equation without lower order terms. Finally, we give the proof of Theorem 1.2, a version for subsolution of parabolic equation with lower order terms.

#### 3.1 Functions in $H^1$

We give a simpler proof for Lemma [64, Theorem 2.9], about an intermediate value lemma for functions which are bounded in the Sobolev space  $H^1$ . This lemma in an alternative version of the De Giorgi isoperimetric inequality [97, Lemma 10]. As we previously see, it is a crucial tool in the De Giorgi proof of the Hölder regularity for solutions of elliptic equations.

The measure of the sets  $\{x \in B_R, u(x) \leq k\}$ ,  $\{x \in B_R, u(x) \geq l\}$  and  $\{x \in B_R, k < u(x) < l\}$  are respectively denoted by  $|u \leq k|$ ,  $|u \geq l|$  and  $|k < u < l|$ .

**Lemma 3.1** (Intermediate value lemma in  $H^1$ ). *Let  $u \in H^1(B_R)$ . Then for all  $(k, l) \in \mathbb{R}^2$  such that  $k \leq l$ , we have*

$$(l-k) \left| \{u \leq k\} \cap B_R \right| \times \left| \{u \geq l\} \cap B_R \right| \leq C \left| \{k < u < l\} \cap B_R \right|^{\frac{1}{2}} \sqrt{\int_{B_R} |\nabla(u-k)_+(x)|^2 dx}, \quad (4.14)$$

where  $C$  a universal constant.

*Proof.* We define the following truncated function

$$v(x) = \begin{cases} 0 & \text{if } u(x) \leq k, \\ u(x) - k & \text{if } k < u(x) < l, \\ l - k & \text{if } u(x) \geq l. \end{cases} \quad (4.15)$$

We have  $v \in H^1(B_R)$  by Stampacchia theorem [47]. By Poincaré inequality since  $v \in W^{1,1}(B_R)$ , see for example [49, Theorem 2, p.293], we have

$$\int_{B_R} |v(x) - \bar{v}| dx \leq CR \int_{B_R} |\nabla v(x)| dx, \quad (4.16)$$

where  $\bar{v} = \frac{1}{|B_R|} \int_{B_R} v(x) dx$ . The sets  $\{x \in B_R, v(x) = 0\}$ , and  $\{x \in B_R, v(x) = l - k\}$  are respectively denoted by  $\{v = 0\}$  and  $\{v = l - k\}$  and their measures by  $|v = 0|$  and  $|v = l - k|$ . We have the following inequalities

$$\frac{(l - k)}{|B_R|} |v = 0| |v = l - k| \leq \int_{\{v=0\}} \bar{v} dx \leq \int_{\{v=0\}} |v(x) - \bar{v}| dx \leq \int_{B_R} |v(x) - \bar{v}| dx, \quad (4.17)$$

and by Cauchy-Schwarz inequality

$$\int_{B_R} |\nabla v(x)| dx = \int_{\{k < u < l\}} |\nabla v(x)| dx \leq \sqrt{\int_{B_R} |\nabla(u - k)_+(x)|^2 dx} |k < u < l|^{\frac{1}{2}}. \quad (4.18)$$

Using (4.16), (4.17) and (4.18) and the equalities  $|v = 0| = |u \leq k|$  and  $|v = l - k| = |u \geq l|$ , we deduce (4.14).  $\square$

### 3.2 Solutions of parabolic equations

In this subsection, we prove Theorem 1.1. We need the result of gain of integrability for the gradient proven in [96, Theorem 2.1]. It states that there exists universal constants  $0 < \varepsilon < 1$  and  $C > 0$  (depending only on  $R, T_1, T_2, \lambda, \Lambda$  and  $d$ ) such that for any solution  $f$  of (4.1), we have

$$\int_{T_1}^{T_2} \int_{B_{2R}} |\nabla_x f(t, x)|^{2+\varepsilon} dx dt \leq C \left( \int_{T_1}^{T_2} \int_{B_{8R}} |\nabla_x f(t, x)|^2 dx dt \right)^{\frac{2+\varepsilon}{2}}. \quad (4.19)$$

The proof of Theorem 1.1 relies on a Poincaré type inequality, in the spirit of the Aramaki's one [10, Theorem 2.2],

$$\int_{Q_R} |f - \bar{f}|^2 dt dx \leq C \int_{Q_{2R}} |\nabla_x f|^2 dt dx,$$

for any solution  $f$  of (4.1) in  $Q_{2R}$  where  $\bar{f} = \frac{1}{|Q_R|} \int_{Q_R} f(t, x) dt dx$  and  $C$  is a universal constant.

**Theorem 3.2** (Poincaré inequality). *Let  $f$  be a weak solution of (4.1) for  $\Omega = B_{2R}$ . Let  $G : \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^\infty$  function such that  $G, G'$  and  $G''$  are bounded by  $C_G$ . Then there exists a constant  $C > 0$  depending only on  $R, T_1, T_2, \lambda, \Lambda, d$  and  $C_G$  and a universal constant  $\varepsilon$  such that*

$$\begin{aligned} \int_{T_1}^{T_2} \int_{B_R} |G(f)(t, x) - \overline{G(f)}|^2 dx dt &\leq C \int_{T_1}^{T_2} \int_{B_{2R}} (|\nabla_x G(f)(t, x)|^2 + |\nabla_x G(f)(t, x)|) dx dt \\ &\quad + C \int_{T_1}^{T_2} \int_{B_{8R}} |\nabla_x f(t, x)|^2 dx dt \left( \int_{T_1}^{T_2} \int_{B_{2R}} |G''(f)(t, x)|^{1+\frac{2}{\varepsilon}} dx dt \right)^{\frac{\varepsilon}{2+\varepsilon}}, \end{aligned}$$

where  $\overline{G(f)} = \frac{1}{(T_2 - T_1)|B_R|} \int_{T_1}^{T_2} \int_{B_R} G(f)(t, x) dx dt$ .

In the following the constant  $C > 0$  will denote any constant only depending on  $R, T_1, T_2, \lambda, \Lambda, d$  and  $C_G$ .

*Proof of Theorem 3.2.* Notice first that  $G(f)$  satisfies the following equation

$$\partial_t G(f) = \nabla_x \cdot (A \nabla_x G(f)) - G''(f) A \nabla_x f \cdot \nabla_x f, \quad (4.20)$$

i.e for all  $\eta \in C_c^\infty(Q_{16R})$  we have

$$-\int_{Q_{16R}} f \partial_t \eta + \int_{Q_{16R}} A \nabla_x f \cdot \nabla_x \eta + \int_{Q_{16R}} G''(f) A \nabla_x f \cdot (\nabla_x f) \eta = 0.$$

Let  $\phi$  be a smooth cut-off function such that  $0 \leq \phi \leq 1$ ,  $|\nabla_x \phi| \leq \frac{2}{R}$  and

$$\phi(x) = \begin{cases} 1 & \text{in } B_R \\ 0 & \text{outside } B_{2R}. \end{cases}$$

Let us define two functions

$$\psi(t, x) = \left( G(f)(t, x) - \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} G(f)(\tau, x) d\tau \right) \phi(t, x)$$

and

$$\varphi(t, x) = \int_{T_1}^t \psi(\tau, x) d\tau - \frac{t - T_1}{T_2 - T_1} \int_{T_1}^{T_2} \psi(\tau, x) d\tau.$$

The function  $\varphi$  is an admissible test-function so we get

$$\begin{aligned} \int_{T_1}^{T_2} \int_{B_{2R}} G(f)(t, x) \partial_t \varphi(t, x) dx dt &= \int_{T_1}^{T_2} \int_{B_{2R}} A(t, x) \nabla_x G(f)(t, x) \cdot \nabla_x \varphi(t, x) dx dt \\ &\quad + \int_{T_1}^{T_2} \int_{B_{2R}} G''(f)(t, x) A(t, x) \nabla_x f(t, x) \cdot \nabla_x f(t, x) \varphi(t, x) dx dt \quad (4.21) \end{aligned}$$

For the term of the left hand side of (4.21), we have

$$\begin{aligned} &\int_{T_1}^{T_2} \int_{B_{2R}} G(f)(t, x) \partial_t \varphi(t, x) dx dt \\ &= \int_{T_1}^{T_2} \int_{B_{2R}} G(f)(t, x) \left( \psi(t, x) - \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \psi(\tau, x) d\tau \right) dx dt \\ &= \int_{T_1}^{T_2} \int_{B_{2R}} \left( G(f)(t, x) - \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} G(f)(\tau, x) d\tau \right) \psi(t, x) dx dt \\ &= \int_{T_1}^{T_2} \int_{B_{2R}} \left( G(f)(t, x) - \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} G(f)(\tau, x) d\tau \right)^2 \phi(t, x) dx dt. \end{aligned} \quad (4.22)$$

For the first term of the right hand side of (4.21), we have

$$\begin{aligned}
& \left| \int_{T_1}^{T_2} \int_{B_{2R}} A(t, x) \nabla_x(G(f))(t, x) \cdot \nabla_x \varphi(t, x) \, dx dt \right| \\
& \leq \Lambda \int_{T_1}^{T_2} \int_{B_{2R}} |\nabla_x(G(f))(t, x)| \left( \int_{T_1}^{T_2} |\nabla_x \psi(\tau, x)| \, d\tau + \frac{t - T_1}{T_2 - T_1} \int_{T_1}^{T_2} |\nabla_x \psi(\tau, x)| \, d\tau \right) \, dx dt \\
& \leq C \int_{T_1}^{T_2} \int_{B_{2R}} |\nabla_x(G(f))(t, x)| \left( \int_{T_1}^{T_2} |\nabla_x(G(f))(\tau, x)| \, d\tau \right) \, dx dt \\
& + CC_G \int_{T_1}^{T_2} \int_{B_{2R}} |\nabla_x(G(f))(t, x)| \left( \int_{T_1}^{T_2} |\nabla_x \phi(\tau, x)| \, d\tau \right) \, dx dt.
\end{aligned}$$

Then using a Cauchy-Schwarz inequality and a Jensen inequality, we get

$$\begin{aligned}
& \left| \int_{T_1}^{T_2} \int_{B_{2R}} A(t, x) \nabla_x(G(f))(t, x) \cdot \nabla_x \varphi(t, x) \, dx dt \right| \\
& \leq C(1 + C_G) \int_{T_1}^{T_2} \int_{B_{2R}} (|\nabla_x(G(f))(t, x)|^2 + |\nabla_x(G(f))(t, x)|) \, dx dt. \quad (4.23)
\end{aligned}$$

For the second term of the right hand side of (4.21), we have using a Hölder inequality and (4.19),

$$\begin{aligned}
& \left| \int_{T_1}^{T_2} \int_{B_{2R}} G''(f)(t, x) A(t, x) \nabla_x f(t, x) \cdot \nabla_x f(t, x) \varphi(t, x) \, dx dt \right| \\
& \leq C \int_{T_1}^{T_2} \int_{B_{2R}} |G''(f)(t, x)| |\nabla_x f(t, x)|^2 \, dx dt \\
& \leq C \left( \int_{T_1}^{T_2} \int_{B_{2R}} |\nabla_x f(t, x)|^{2+\varepsilon} \, dx dt \right)^{\frac{2}{2+\varepsilon}} \left( \int_{T_1}^{T_2} \int_{B_{2R}} |G''(f)(t, x)|^{1+\frac{2}{\varepsilon}} \, dx dt \right)^{\frac{\varepsilon}{2+\varepsilon}} \\
& \leq C \left( \int_{T_1}^{T_2} \int_{B_{8R}} |\nabla_x f(t, x)|^2 \, dx dt \right) \left( \int_{T_1}^{T_2} \int_{B_{2R}} |G''(f)(t, x)|^{1+\frac{2}{\varepsilon}} \, dx dt \right)^{\frac{\varepsilon}{2+\varepsilon}} \quad (4.24)
\end{aligned}$$

Then combining (4.21), (4.22), (4.23) and (4.24) we deduce that

$$\begin{aligned}
& \int_{T_1}^{T_2} \int_{B_R} \left( G(f)(t, x) - \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} G(f)(\tau, x) \, d\tau \right)^2 \, dx dt \\
& \leq C \int_{T_1}^{T_2} \int_{B_{2R}} |\nabla_x(G(f))(t, x)|^2 \, dx dt + C \int_{T_1}^{T_2} \int_{B_{2R}} |\nabla_x(G(f))(t, x)| \, dx dt \\
& + C \int_{T_1}^{T_2} \int_{B_{8R}} |\nabla_x f(t, x)|^2 \, dx dt \left( \int_{T_1}^{T_2} \int_{B_{2R}} |G''(f)(t, x)|^{1+\frac{2}{\varepsilon}} \, dx dt \right)^{\frac{\varepsilon}{2+\varepsilon}}. \quad (4.25)
\end{aligned}$$

And since we have

$$\begin{aligned}
& \int_{T_1}^{T_2} \int_{B_R} |G(f)(t, x) - \overline{G(f)}|^2 \, dx dt \leq C \int_{T_1}^{T_2} \int_{B_R} \left( G(f)(t, x) - \frac{1}{T} \int_{T_1}^{T_2} G(f)(\tau, x) \, d\tau \right)^2 \, dx dt \\
& + C \int_{T_1}^{T_2} \int_{B_R} \left( \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} G(f)(\tau, x) \, d\tau - \overline{G(f)} \right)^2 \, dx dt, \quad (4.26)
\end{aligned}$$



we can use the bound (4.25) for the first term of the right hand side of (4.26).

For the second term of the right hand side of (4.26), we apply Jensen inequality and Poincaré inequality, and get

$$\begin{aligned}
& \int_{T_1}^{T_2} \int_{B_R} \left( \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} G(f)(\tau, x) \, d\tau - \overline{G(f)} \right)^2 \, dxdt \\
& \leq \int_{T_1}^{T_2} \int_{B_R} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \left( G(f)(\tau, x) - \frac{1}{|B_R|} \int_{B_R} G(f)(\tau, y) \, dx \right)^2 \, d\tau dxdt \\
& \leq C \int_{T_1}^{T_2} \int_{B_R} |\nabla_x(G(f))(t, x)|^2 \, dxdt. \tag{4.27}
\end{aligned}$$

We get the desired result by combining (4.25), (4.26) and (4.27).  $\square$

Let us prove the intermediate value lemma.

*Proof of Theorem 1.1.* We apply Theorem 3.2 to the solution  $f - k$  with  $G : \mathbb{R} \rightarrow \mathbb{R}$  smooth such that  $0 < G < l - k$  on  $(0, l - k)$ ,  $|G'| \leq \frac{2}{l-k}$  and

$$G(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0] \\ l - k & \text{if } x \in [l - k, +\infty). \end{cases}$$

Using a Cacciopoli inequality (see step 1 of the proof of Theorem 1.2 for more details) and that  $|f| \leq 1$ , we get

$$\int_{T_1}^{T_2} \int_{B_{8R}} |\nabla_x f(t, x)|^2 \, dxdt \leq C \int_{T_1}^{T_2} \int_{B_{16R}} f(t, x)^2 \, dxdt \leq C. \tag{4.28}$$

Let us recall that  $Q_R = (T_1, T_2) \times B_R$ . The left hand side of the Poincaré inequality is estimated from below as follows

$$\begin{aligned}
\int_{T_1}^{T_2} \int_{B_R} |G(f)(t, x) - \overline{G(f)}|^2 \, dxdt & \geq \int \int_{\{G(f)=0\} \cap Q_R} \overline{G(f)}^2 \, dxdt \\
& \geq C(l-k)^2 |\{G(f) = 0\} \cap Q_R| |\{G(f) \geq l-k\} \cap Q_R|^2 \\
& \geq C(l-k)^2 |\{f \leq k\} \cap Q_R| |\{f \geq l\} \cap Q_R|^2. \tag{4.29}
\end{aligned}$$

Using a Hölder inequality,  $\nabla_x G(f) = G'(f) \nabla_x f$ , (4.19) and (4.28), the first term of the right hand side of the Poincaré inequality from Theorem 3.2 gives

$$\begin{aligned}
& C \int_{T_1}^{T_2} \int_{B_{2R}} |\nabla_x G(f)(t, x)|^2 \, dxdt \\
& \leq C \left( \int_{T_1}^{T_2} \int_{B_{2R}} |\nabla_x G(f)(t, x)|^{2+\varepsilon} \, dxdt \right)^{\frac{2}{2+\varepsilon}} \left( \int_{T_1}^{T_2} \int_{B_{2R}} \mathbb{1}_{\{k < f(t) < l\}} \, dxdt \right)^{\frac{\varepsilon}{2+\varepsilon}} \\
& \leq CC_G^2 |\{k < f < l\} \cap Q_{2R}|^{\frac{\varepsilon}{2+\varepsilon}} \int_{T_1}^{T_2} \int_{B_{8R}} |\nabla_x f(t, x)|^2 \, dxdt \\
& \leq CC_G^2 |\{k < f < l\} \cap Q_{2R}|^{\frac{\varepsilon}{2+\varepsilon}}. \tag{4.30}
\end{aligned}$$

Using a Hölder inequality, the second term gives

$$\begin{aligned}
& C \int_{T_1}^{T_2} \int_{B_{2R}} |\nabla_x G(f)(t, x)| \, dx dt \\
& \leq C \left( \int_{T_1}^{T_2} \int_{B_{2R}} |\nabla_x G(f)(t, x)|^2 \, dx dt \right)^{\frac{1}{2}} \left( \int_{T_1}^{T_2} \int_{B_{2R}} \mathbb{1}_{\{k < f(t) < l\}} \, dx dt \right)^{\frac{1}{2}} \\
& \leq CC_G |\{k < f < l\} \cap Q_{2R}|^{\frac{1}{2}} \left( \int_{T_1}^{T_2} \int_{B_{2R}} |\nabla_x f(t, x)|^2 \, dx dt \right)^{\frac{1}{2}} \\
& \leq CC_G |\{k < f < l\} \cap Q_{2R}|^{\frac{1}{2}}. \tag{4.31}
\end{aligned}$$

The third term gives

$$\begin{aligned}
& C \int_{T_1}^{T_2} \int_{B_{8R}} |\nabla_x f(t, x)|^2 \, dx dt \left( \int_{T_1}^{T_2} \int_{B_{2R}} |G''(f)(t, x)|^{1+\frac{2}{\varepsilon}} \, dx dt \right)^{\frac{\varepsilon}{2+\varepsilon}} \\
& \leq C \left( \int_{T_1}^{T_2} \int_{B_{2R}} |G''(f)(t, x)|^{1+\frac{2}{\varepsilon}} \, dx dt \right)^{\frac{\varepsilon}{2+\varepsilon}} \\
& \leq CC_G |\{k < f < l\} \cap Q_{2R}|^{\frac{\varepsilon}{2+\varepsilon}}. \tag{4.32}
\end{aligned}$$

We get the desired result by combining Theorem 3.2, (4.29), (4.30), (4.31) and (4.32).  $\square$

### 3.3 Subsolutions of parabolic equations

In this section, we prove Theorem 1.2. We first introduce two lemmas. The first lemma is a Cacciopoli inequality already obtained in [97, 58].

**Lemma 3.3** (Caccioppoli inequality). *Let  $u : Q_{2R} \rightarrow \mathbb{R}$  be a non-negative subsolution of (4.4) bounded by a constant  $M > 0$ . Then there exists a universal constant  $C > 0$  such that*

$$\int_{T_1}^{T_2} \int_{B_R} |\nabla_x u(\tau, x)|^2 \, dx d\tau \leq CM^2.$$

*Proof of Lemma 3.3.* Let  $\phi$  be a smooth cut-off function such that  $0 \leq \phi \leq 1$ ,  $|\nabla_x \phi| \leq \frac{2}{R}$  and

$$\phi(x) = \begin{cases} 1 & \text{in } B_R \\ 0 & \text{outside } B_{2R}. \end{cases}$$

We multiply the inequation by  $u\phi^2$  which is non negative, and integrate over  $(t_1, t_2) \times B_{2R}$  with  $T_1 \leq t_1 \leq t_2 \leq T_2$ . We get for almost every  $t_1, t_2$

$$\begin{aligned}
& \int_{B_{2R}} u^2(t_2, x) \phi^2(x) \, dx \leq \int_{B_{2R}} u^2(t_1, x) \phi^2(x) \, dx - \int_{t_1}^{t_2} \int_{B_{2R}} A(\tau, x) \nabla_x u(\tau, x) \cdot \nabla_x (u\phi^2)(\tau, x) \, dx d\tau \\
& + \int_{t_1}^{t_2} \int_{B_{2R}} B(\tau, x) \cdot \nabla_x u(\tau, x) \phi^2(x) u(\tau, x) \, dx d\tau + \int_{t_1}^{t_2} \int_{B_{2R}} |s(\tau, x)| u^2(\tau, x) \, dx d\tau.
\end{aligned}$$

So using the bounds on  $A$  and  $B$  we have,

$$\begin{aligned} \lambda \int_{t_1}^{t_2} \int_{B_{2R}} |\nabla_x u(\tau, x)|^2 \phi^2(x) \, dx d\tau &\leq \int_{B_{2R}} u^2(0, x) \phi^2(x) \, dx \\ &+ 2\Lambda \int_{t_1}^{t_2} \int_{B_{2R}} \phi(x) |\nabla_x u(\tau, x)| \times |\nabla_x \phi(\tau, x)| u(\tau, x) \, dx d\tau \\ &+ \Lambda \int_{t_1}^{t_2} \int_{B_{2R}} |\nabla_x u(\tau, x)| \phi^2(x) u(\tau, x) \, dx d\tau + \int_{t_1}^{t_2} \int_{B_{2R}} u^2(\tau, x) \, dx d\tau. \end{aligned}$$

Then using that  $0 \leq u \leq M$  on  $B_{2R}$  and a Young inequality on the second and the third terms of the right hand side, we get

$$\begin{aligned} \lambda \int_{t_1}^{t_2} \int_{B_{2R}} |\nabla_x u(\tau, x)|^2 \phi^2(x) \, dx d\tau &\leq CM^2 + \frac{\lambda}{4} \int_{t_1}^{t_2} \int_{B_{2R}} |\nabla_x u(\tau, x)|^2 \phi^2(x) \, dx d\tau \\ &+ \frac{4\Lambda}{\lambda^2} \int_{t_1}^{t_2} \int_{B_{2R}} |\nabla_x \phi(\tau, x)|^2 u^2(\tau, x) \, dx d\tau \\ &+ \frac{\lambda}{4} \int_{t_1}^{t_2} \int_{B_{2R}} |\nabla_x u(\tau, x)|^2 \phi^2(x) \, dx d\tau \\ &+ \frac{\Lambda}{\lambda^2} \int_{t_1}^{t_2} \int_{B_{2R}} |\phi(\tau, x)|^2 u^2(\tau, x) \, dx d\tau, \end{aligned}$$

where  $C$  is a universal constant. Thus we deduce the desired result.  $\square$

The second lemma is a first step for the proof of Theorem 1.2. It gives ‘‘almost’’ an intermediate value lemma with an error which is small for close times. The measure of the following sets  $\{(t, x) \in (\tau_1, \tau_2) \times B_R, f(t, x) < l\}$ ,  $\{(t, x) \in (\tau_1, \tau_2) \times B_R, f(t, x) \leq l\}$  and  $\{(t, x) \in (\tau_1, \tau_2) \times B_R, k < f(t, x) < l\}$  are respectively denoted by  $|f < l, (\tau_1, \tau_2)|$ ,  $|f \leq l, (\tau_1, \tau_2)|$  and  $|k < f < l, (\tau_1, \tau_2)|$ .

**Lemma 3.4** (A key inequality for close times). *Let  $f : Q_{2R} \rightarrow \mathbb{R}$  be a subsolution of (4.4) such that  $f \leq 1$ . Then for all  $(k, l) \in \mathbb{R}^2$  such that  $k < l \leq 1$  and for all  $(t_1, t_2, \tau) \in (T_1, T_2)^3$  such that  $T_1 < t_1 < \tau < t_2 < T_2$ , we have*

$$C \frac{l-k}{2-k} |f < l, (t_1, \tau)| |f \geq l, (\tau, t_2)| \leq (t_2 - t_1) |k < f < l, (t_1, \tau)|^{\frac{1}{4}} + (t_2 - t_1)^{\frac{9}{4}}.$$

*Proof of Lemma 3.4.* Let  $u = (f - k)_+$ . The function  $u$  is a subsolution of the following equation

$$\partial_t u = \nabla_x \cdot (A \nabla_x u) + B \cdot \nabla_x u + \mathbb{1}_{\{u > 0\}} s.$$

Let  $\varphi(x) = \left(1 - \frac{|x|^2}{R^2}\right)^{\frac{3}{4}}$ . We multiply the inequation by  $\varphi$  which is non-negative on  $B_R$ , and integrate over  $(s, t) \times B_R$  where  $T_1 \leq s < t \leq T_2$ . We get for almost  $s, t$

$$\begin{aligned} \int_{B_R} u(t, x) \varphi(x) \, dx &\leq \int_{B_R} u(s, x) \varphi(x) \, dx - \int_s^t \int_{B_R} A(\tau, x) \nabla_x u(\tau, x) \cdot \nabla_x \varphi(x) \, dx d\tau \\ &+ \int_s^t \int_{B_R} B(\tau, x) \cdot \nabla_x u(\tau, x) \varphi(x) \, dx d\tau + \int_s^t \int_{B_R} |s(\tau, x)| \, dx d\tau. \end{aligned} \tag{4.33}$$

By using a reverse Hölder inequality, we find a lower bound for the left hand side of (4.33),

$$\begin{aligned} \int_{B_R} u(t, x) \varphi(x) \, dx &\geq (l - k) \int_{\{f(t, x) \geq l\}} \varphi(x) \, dx \\ &\geq (l - k) \left( \int_{B_R} \mathbb{1}_{\{f(t) \geq l\}} \, dx \right)^2 \left( \int_{B_R} \frac{1}{\varphi(x)} \, dx \right)^{-1} \\ &\geq C(l - k) |f(t) \geq l|^2, \end{aligned} \quad (4.34)$$

where  $C$  is a constant which depends only on  $R$ ,  $\{f(t) \geq l\}$  denotes the set of points  $x \in B_R$  which satisfy  $f(t, x) \geq l$  and  $|f(t) \geq l|$  its measure. Also  $|k < f(s) < l|$  and  $|f(s) \leq k|$  will denote respectively the measures of the set of points  $x \in B_R$  which satisfy  $k < f(s, x) < l$  and  $f(s, x) \leq k$ . For the first term of the right hand side of (4.33), we have

$$\begin{aligned} \int_{B_R} u(s, x) \varphi(x) \, dx &\leq (1 - k) \left( \int_{B_R} \mathbb{1}_{\{k < f(s) < l\}} + \mathbb{1}_{\{f(s) \geq l\}} \, dx \right) \\ &\leq (1 - k) (|k < f(s) < l| + |f(s) \geq l|). \end{aligned} \quad (4.35)$$

For the second term of the right hand side of (4.33), using Cauchy-Schwarz inequality and Lemma 3.3, we get

$$\begin{aligned} - \int_s^t \int_{B_R} A(\tau, x) \nabla_x u(\tau, x) \cdot \nabla_x \varphi(x) \, dx d\tau &\leq \Lambda \int_s^t \int_{B_R} |\nabla_x u(\tau, x)| \times |\nabla_x \varphi(x)| \, dx d\tau \\ &\leq \Lambda \sqrt{\int_s^t \int_{B_R} |\nabla_x u|^2} \sqrt{\int_s^t \int_{B_R} |\nabla_x \varphi|^2} \\ &\leq C(1 - k) \sqrt{t - s}. \end{aligned} \quad (4.36)$$

With the same arguments, the third term is bounded by  $C(1 - k)\sqrt{t - s}$ . The last term in (4.33) is bounded by  $C(t - s)$ . By combining (4.34), (4.35) and (4.36), we deduce,

$$C(l - k) |f(t) \geq l|^2 \leq (1 - k) (|k < f(s) < l| + |u(s) \geq l - k|) + C(1 - k) \sqrt{t - s} + C(t - s). \quad (4.37)$$

For almost every  $s \in [0, t)$ ,  $f(s) \in H^1(B_R)$  so multiplying (4.37) by  $(l - k) |f(s) \leq k| = (l - k) |u(s) \leq 0|$  and applying Lemma 3.1, we get

$$\begin{aligned} C(l - k)^2 |f(s) \leq k| |f(t) \geq l|^2 &\leq (1 - k)^2 |\{k < f(s) < l\}| \\ &\quad + (1 - k) |k < f(s) < l|^{\frac{1}{2}} \sqrt{\int_{B_R} |\nabla u(s, x)|^2 \, dx} + (1 - k)^2 \sqrt{t - s}. \end{aligned} \quad (4.38)$$

We integrate (4.38) in  $s \in [t_1, \tau]$  and  $t \in [\tau, t_2]$  with  $T_1 \leq t_1 < \tau < t_2 \leq T_2$ . By Jensen inequality, we have

$$\int_\tau^{t_2} |f(t) \geq l|^2 \, dt \geq \frac{1}{t_2 - \tau} \left( \int_\tau^{t_2} |f(t) \geq l| \, dt \right)^2 = \frac{1}{t_2 - \tau} |f \geq l, (\tau, t_2)|^2. \quad (4.39)$$

By Cauchy-Schwarz inequality and Lemma 3.3, we get

$$\begin{aligned} & \int_{t_1}^{\tau} \left( |k < f(s) < l|^{\frac{1}{2}} \sqrt{\int_{B_R} |\nabla u(s, x)|^2 dx} \right) ds \\ & \leq |k < f < l, (t_1, \tau)|^{\frac{1}{2}} \sqrt{\int_{t_1}^{\tau} \int_{B_R} |\nabla u(s, x)|^2 dx ds} \\ & \leq C |k < f < l, (t_1, \tau)|^{\frac{1}{2}}. \end{aligned} \quad (4.40)$$

Using (4.38), (4.39), (4.40), we deduce from (4.37) that

$$C(l-k)^2 |f \leq k, (t_1, \tau)| |f \geq l, (\tau, t_2)|^2 \leq (1+(1-k)^2) \left( (t_2 - t_1) |k < f < l, (t_1, \tau)|^{\frac{1}{2}} + (t_2 - t_1)^{\frac{7}{2}} \right).$$

So we have,

$$C \frac{(l-k)^2}{1+(1-k)^2} |f \leq k, (t_1, \tau)| |f \geq l, (\tau, t_2)|^2 \leq (t_2 - t_1) |k < f < l, (t_1, \tau)|^{\frac{1}{2}} + (t_2 - t_1)^{\frac{7}{2}},$$

which gives also, taking the square root of the last inequality and using  $|f \leq k, (t_1, \tau)| \leq |B_R|^{\frac{1}{2}} (\tau - t_1)^{\frac{1}{2}} |f \leq k, (t_1, \tau)|^{\frac{1}{2}}$ ,

$$C \frac{l-k}{2-k} |f \leq k, (t_1, \tau)| |f \geq l, (\tau, t_2)| \leq (t_2 - t_1) |k < f < l, (t_1, \tau)|^{\frac{1}{4}} + (t_2 - t_1)^{\frac{9}{4}}. \quad (4.41)$$

Using the equality  $|f \leq k, (t_1, \tau)| = |f < l, (t_1, \tau)| - |k < f < l, (t_1, \tau)|$ , we deduce from (4.41) the desired result.  $\square$

Now let us prove Theorem 1.2. The idea of the proof is to understand that the “error” term  $(t_2 - t_1)^{\frac{9}{4}}$  in Lemma 3.4 is negligible compared to the other terms when  $t_2 - t_1$  is small and when the intervals are well-chosen.

*Proof of Theorem 1.2.* Let  $n \in \mathbb{N}^*$ ,  $\alpha_n = \frac{T_2 - T_1}{2n}$ ,  $\bar{T} = \frac{T_2 + T_1}{2}$  and  $t_k = k\alpha_n$ . Necessarily, there exists  $i \in [1, n]$  such that

$$|f < l, (t_{i-1}, t_i)| \geq \frac{|f \leq k, (T_1, \bar{T})|}{2n}, \quad (4.42)$$

and there exists  $j \in [n, 2n - 1]$  such that

$$|f \geq l, (t_j, t_{j+1})| \geq \frac{|f \geq l, (\bar{T}, T_2)|}{2n}. \quad (4.43)$$

We distinguish two cases, either there exists  $m \in [1, 2n - 1]$  such that  $m + 1$  does not satisfy (4.42) (i.e., (4.42) is false for  $i = m + 1$ ), or for all  $m \in [1, 2n - 1]$ ,  $m + 1$  does satisfy (4.42). In the first case, letting  $p$  be the first integer  $m$  satisfying “ $m + 1$  does not satisfy (4.42)”, we have

$$|f < l, (t_p, t_{p+1})| < \frac{|f \leq k, (T_1, \bar{T})|}{2n},$$

so

$$|f \geq l, (t_p, t_{p+1})| \geq |B_R| \alpha_n - \frac{|f \leq k, (T_1, \bar{T})|}{2n} \geq \frac{|f \geq l, (\bar{T}, T_2)|}{2n}$$

and

$$|f < l, (t_{p-1}, t_p)| \geq \frac{|f \leq k, (T_1, \bar{T})|}{2n}.$$

In the second case, let  $p = j$ . Then in all cases, using Lemma 3.4 we have,

$$\begin{aligned} C \frac{l-k}{2-k} \frac{|f \leq k, (T_1, \bar{T})|}{2n} \frac{|f \geq l, (\bar{T}, T_2)|}{2n} &\leq C \frac{l-k}{2-k} |f < l, (t_{p-1}, t_p)| |f \geq l, (t_p, t_{p+1})| \\ &\leq \frac{|k < f < l, (T_1, T_2)|^{\frac{1}{4}}}{n} + n^{-\frac{9}{4}}. \end{aligned}$$

Thus, we have

$$C \frac{l-k}{2-k} |f \leq k, (T_1, \bar{T})| |f \geq l, (\bar{T}, T_2)| \leq n |k < f < l, (T_1, T_2)|^{\frac{1}{4}} + n^{-\frac{1}{4}}.$$

So necessarily  $|k < f < l, (T_1, T_2)| > 0$ . And taking  $n$  such that  $n^{-\frac{1}{4}} \leq \frac{n |k < f < l, (T_1, T_2)|^{\frac{1}{4}}}{2}$ , for example  $n = \left\lfloor \frac{2}{|k < f < l, (T_1, T_2)|^{\frac{1}{5}}} \right\rfloor + 1$ , we get

$$C \frac{l-k}{2-k} |f \leq k, (T_1, \bar{T})| |f \geq l, (\bar{T}, T_2)| \leq |k < f < l, (T_1, T_2)|^{\frac{1}{20}}.$$

This achieves the proof of the theorem. □

### 3.4 Remarks and counterexamples

In this subsection we exhibit some counterexamples which help us to understand some issues.

#### 3.4.1 Parabolic intermediate value lemma

Theorem 1.1 is false for subsolutions. For example, for  $T_1 = 0$  and  $T_2 = 1$  the function

$$f(t, x) = \begin{cases} 1 & \text{for } t \in \left(0, \frac{1}{2}\right] \\ 0 & \text{for } t \in \left(\frac{1}{2}, 1\right), \end{cases}$$

is a subsolution of (4.1) but does not satisfy Lemma 1.1. That is why we have a weaker version of the intermediate value lemma for subsolutions which takes into account the disjoint intervals of time in a specific order (see Lemma 1.2).

#### 3.4.2 Extension to kinetic equations ?

Let us consider the following kinetic Fokker-Planck equation of [58],

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + s, \quad (t, v, x) \in Q_{2R}, \quad (4.44)$$

where  $Q_R = (-R^2, 0) \times B_R \times B_{R^3}$ . We define  $Q_R^+ = (-R^2, 0) \times B_R \times B_{R^3}$  and  $Q_R^- = (-2R^2, -R^2) \times B_R \times B_{R^3}$ . In [58], the authors prove a non-quantitative version of an intermediate value lemma. Trying to extend Theorem 1.2 to subsolutions of (4.44), some problems are encountered. First, subsolutions of (4.44) are not  $H^1$  in the

variable  $x$ . Second, the following relevant generalization of the parabolic intermediate value lemma to the kinetic one,

$$|\{f \leq k\} \cap Q_R^-|^\alpha |\{f \geq l\} \cap Q_R^+|^\beta \leq C |\{k < f < l\} \cap Q_R|^\gamma, \quad (4.45)$$

for some universal constants  $\alpha, \beta, \gamma$  and  $C$  which depends also on  $k, l$ , is not true since there exists counterexamples. For example, let  $d = 1$ , the function

$$f(t, x, v) = \begin{cases} 1 & \text{for } x + 2Rt < -2R^3 \\ 0 & \text{for } x + 2Rt \geq -2R^3, \end{cases}$$

is a subsolution of (4.44) but does not satisfy (4.45). In fact, for some parameters  $c > R$  (to have a subsolution) and  $a \in \mathbb{R}$ ,

$$f_{a,c}(t, x, v) = \begin{cases} 1 & \text{for } x + ct < a \\ 0 & \text{for } x + ct \geq a, \end{cases}$$

is also a subsolution of (4.44). Drawing many lines of discontinuity  $x + ct = a$ , we notice that to find a valid intermediate value inequality, we must consider two cylinders which cannot be both crossed by the same line of discontinuity  $x + ct = a$ . More precisely, we must have a “gap” in time between the two cylinders of the same size (or at least not smaller) than the two cylinders. Let us define  $\bar{Q}_R = (-3R^2, -2R^2) \times B_R \times B_{R^3}$ . The two domains  $Q_R$  and  $\bar{Q}_R$  are never both crossed by any possible line of discontinuity  $x + ct = a$ . That is why this intermediate value inequality seems to be more accurate,

$$|\{f \leq k\} \cap \bar{Q}_R|^\alpha |\{f \geq l\} \cap Q_R|^\beta \leq C |\{k < f < l\} \cap Q_R|^\gamma.$$

So it seems necessary to add new arguments which take into account a “good gap” between the two cylinders to find a quantitative version of the intermediate value lemma.

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## Résumé

Cette thèse est constituée de deux parties. Une première partie est consacrée à l'étude des équations de Hamilton-Jacobi du premier ordre. Ces équations apparaissent en contrôle optimal et permettent de modéliser des problèmes de trafic routier, de supraconductivité et de mouvements d'interface. Le premier chapitre de la thèse présente un résultat d'équivalence de conditions au bord de type contraintes d'état. On obtient l'équivalence de trois formulations de ces conditions au bord. Ceci permet notamment de déduire que les résultats d'existence et d'unicité valables pour l'une des trois formulations sont valables pour les trois. Le second chapitre porte principalement sur un résultat d'équivalence de conditions au bord de type dynamique ; il est complété par un résultat d'unicité pour ce problème. En considérant la relation «avoir les mêmes solutions», on peut regrouper les conditions aux limites (vérifiées en un sens faible) en classe d'équivalence. Nous montrons que dans chaque classe il y a une unique condition vérifiée en un sens fort. Le troisième chapitre est consacrée à l'étude d'un schéma monotone aux différences finies pour une équation de Hamilton-Jacobi posée sur une jonction. Une jonction est un réseau formé d'un seul nœud et d'un nombre fini d'arrêtes infinies. La convergence du schéma vers l'unique solution a été montrée par Costesèque, Lebacque, Monneau dans le cas d'une condition de jonction de type «flux limité minimal». Nous présenterons un résultat de convergence pour une condition de jonction générale, ainsi qu'une estimation d'erreur dans le cas d'une condition de jonction de type «flux limité» (pas forcément minimal). Une seconde partie porte sur la régularité höldérienne pour une large classe d'équations paraboliques à coefficients peu réguliers par la méthode introduite par De Giorgi en 1957. Le quatrième chapitre contient un résultat quantitatif d'une des deux grandes étapes de la méthode de De Giorgi : le lemme des valeurs intermédiaires. Ce lemme permet de quantifier (en mesure) le fait que les solutions de ces équations ne peuvent pas faire de saut entre deux valeurs numériques. Deux versions quantitatives de ce lemme sont présentées.

## Mots Clés

Équations de Hamilton-Jacobi, Solutions de viscosité, Conditions de bord effectives, Régularité elliptique à la De Giorgi, Lemme des valeurs intermédiaires.

## Abstract

This thesis contains two parts. The first part is devoted to the study of first order Hamilton-Jacobi equations. These equations appear in optimal control and enable to model road traffic, superconductivity and interface motion problems. The first chapter of the thesis presents an equivalence result of state constraint boundary conditions. Three equivalent formulations of these conditions are obtained. In particular, it can be inferred that the existence and uniqueness results valid for one of the three formulations are valid for all three. The second chapter deals mainly with an equivalence result of dynamic boundary conditions; it is completed by a result of uniqueness for this problem. By considering the equivalence relation "to have the same solutions", boundary conditions (satisfied in a weak sense) can be grouped into equivalence classes. We show that in each class there is a single condition verified in a strong sense. The third chapter is devoted to the study of a monotone finite difference scheme for a Hamilton-Jacobi equation on a junction. A junction is a network formed by a single node and a finite number of infinite edges. The convergence of the scheme towards the single solution was shown by Costesèque, Lebacque, Monneau in the case of a "minimal flux-limited" type junction condition. We will present a convergence result for a general junction condition, as well as an error estimate in the case of a "flux-limited" (not necessarily minimal) junction condition. A second part deals with Hölder regularity for a large class of parabolic equations with rough coefficients using the method introduced by De Giorgi in 1957. The fourth chapter contains a quantitative result of one main steps of De Giorgi's method: the intermediate value lemma. This lemma quantifies (in measure) the fact that solutions of these equations cannot make a jump between two numerical values. Two quantitative versions of this lemma are presented.

## Keywords

Hamilton-Jacobi equations, Viscosity solutions, Effective boundary conditions, De Giorgi elliptic regularity, Intermediate value lemma.