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# Geometric and probabilistic aspects of Coulomb gases

David García-Zelada

► **To cite this version:**

David García-Zelada. Geometric and probabilistic aspects of Coulomb gases. Mathematical Physics [math-ph]. PSL Research University, 2019. English. NNT: . tel-02499570

**HAL Id: tel-02499570**

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**THÈSE DE DOCTORAT**  
**DE L'UNIVERSITÉ PSL**

Préparée à l'Université Paris Dauphine

**Aspects géométriques et probabilistes des gaz de Coulomb**

Soutenue par

**David GARCÍA-ZELADA**

Le 28 juin 2019

École doctorale n°543

**École Doctorale de Dauphine**

Spécialité

**SCIENCES**

Composition du jury :

Mme Mylène, MAÏDA Professeur, Université de Lille	<i>Présidente</i>
M. Charles, BORDENAVE DR CNRS, Université d'Aix-Marseille	<i>Rapporteur</i>
Mme Sylvia, SERFATY Professor, New York University	<i>Rapporteuse</i>
Mme Laure, DUMAZ CR CNRS, Université Paris-Dauphine	<i>Examinatrice</i>
M. Djalil, CHAFAÏ Professeur, Université Paris-Dauphine	<i>Directeur de thèse</i>



# Acknowledgments

I would like to thank all the wonderful people who have supported me during this adventure which is just beginning.

Thank you Djalil for guiding me through my first steps as a researcher. I will always be grateful to you for having accepted me as a doctoral student. Thank you Charles, Laure, Mylène and Sylvia for being the first persons to read carefully this work and for the interest you put into it. Your work inspire me and I am glad that mine can inspire other mathematicians too.

Thank you Marc Arnaudon, Johel Beltrán, Francisco De Zela, Claudio Landim, Jean-François Le Gall, Alfredo Poirier, Rudy Rosas and Jesús Zapata for all the support and the marvelous discussions. Each of you awaken and feed my never-ending interest in mathematics and physics.

Thank you Juan Carlos Aliaga, Rafael Caballero, Vladimir Calvera, Junior Gonzales, Roberto Guzmán, Claudia Parisuaña and Sebastián Sánchez for the philosophical discussions, the enjoyable ‘physics breakfasts’ and for being my partners in the journey to guide the younger students.

Thank you Sandrine Dallaporta, Maxime Février and Jesús Ramos for your friendship and precious advise. I hope we can collaborate on future projects soon.

Thank you Raphael Butez for the hard work done in our project. I am looking forward to share new ideas and intuitions with you.

And, of course, I could not have done anything without the support of my family. Thank you Luis José García and Anita Zelada for always being there for me. I could not have asked for better parents. Thank you Cristian and Jimena for making my life more interesting. Do not forget that you can always count on me. Thank you Ayerim for taking care of me. I will always be there for you too.

**Résumé.** Nous explorons des modèles probabilistes appelés gaz de Coulomb. Ils apparaissent dans différents contextes comme par exemple dans la théorie des matrices aléatoires, l'effet Hall quantique fractionnaire de Laughlin et les modèles de supraconductivité de Ginzburg-Landau. Dans le but de mieux comprendre le rôle de l'espace ambiant, nous étudions des versions géométriques de ces systèmes. Nous exploitons trois structures sur ces modèles. La première est définie par l'interaction électrostatique provenant de la loi de Gauss. La deuxième est la structure déterminantale disponible que pour des valeurs précises de la température. La troisième est le principe de minimisation de l'énergie libre en physique, qui permet d'étudier des modèles plus généraux. Ces travaux conduisent à des nombreuses questions ouvertes et à une famille de modèles d'intérêt.

*Mots-clés:* Gaz de Coulomb, mesure de Gibbs, grandes déviations, concentration de la mesure, polynôme aléatoire, variété riemannienne, processus ponctuel déterminantal, champ moyen.

**Abstract.** We explore probabilistic models usually called Coulomb gases. They arise naturally in mathematics and physics. We can mention random matrix theory, the Laughlin fractional quantum Hall effect and the Ginzburg-Landau systems of superconductivity. In order to better understand the role of the ambient space, we study geometric versions of such systems. We exploit three structures. The first one comes from the electrostatic nature of the interaction given by Gauss's law. The second one is the determinantal structure which appears only for a specific temperature. The third one is the minimization of the free energy principle, coming from physics which gives us a tool to understand more general models. This work leads to many open questions on a whole family of models which can be of independent interest.

*Keywords:* Coulomb gas, Gibbs measure, large deviations, concentration of measure, random polynomial, Riemannian manifold, determinantal point process, mean field theory.

## Published and accepted articles

- *Concentration for Coulomb gases on compact manifolds.*  
Electronic Communications in Probability **24**, 2019, paper no. 12, 18 pp.
- *A large deviation principle for empirical measures on Polish spaces: Application to singular Gibbs measures on manifolds.*  
To appear in Annales de l'Institut Henri Poincaré, Probabilités et Statistiques.  
arXiv preprint arXiv:1703.02680, first version: March 2017.

## Preprints

- In collaboration with Raphael Butez.  
*Extremal particles of two-dimensional Coulomb gases and random polynomials on a positive background.*  
arXiv preprint arXiv:1811.12225, November 2018.
- *Edge fluctuations for a class of two-dimensional determinantal Coulomb gases.*  
arXiv preprint arXiv:1812.11170, December 2018.



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# Chapter 1

## Introduction (english version)

We have divided the introduction in six parts. The first three parts present the models and emphasize three different kind of structures that are exploited in the following chapters. More precisely, the notion of a Green function is explored in the first section and is the key structure used in Chapter 3. The determinantal structure, presented in the second section, is the key to Chapter 4 and Chapter 5. The physical property of minimization of the free energy may be considered as the main viewpoint of Chapter 6 and it is explained in the third section. The fourth and fifth sections describe the questions we may pose and the main results we obtained. We finish this introductory chapter by stating some open questions that await for an interesting answer.

### 1 Coulomb gases

We will provide the definition of a Green function, the *key structure* used in Chapter 3. For convenience of the reader we will give a short introduction to smooth manifolds and give references for the parts not actually needed. Then we mention the famous example of  $\beta$ -ensembles on the real line and three more examples related to the three two-dimensional homogeneous simply connected geometries. Finally a special Coulomb gas model on the sphere is provided as well as its relation to the usual Coulomb gas on the plane and a particular toy model.

We begin by giving the notion of Coulomb gases on Euclidean space.

#### 1.1 On the Euclidean space

Let  $G : \mathbb{R}^d \times \mathbb{R}^d \rightarrow (-\infty, \infty]$  be defined by

$$G(x, y) = -\frac{1}{2\pi} \log |x - y|$$

if  $d = 2$  and

$$G(x, y) = \frac{c_d}{|x - y|^{d-2}}$$

if  $d \geq 3$  where  $c_d^{-1}$  is  $d - 2$  times the area of  $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ . This can be thought as what the Coulomb two-particle interaction would be if we lived on a space of dimension  $d$ . More precisely, given  $x \in \mathbb{R}^d$ , if  $G_x : \mathbb{R}^d \rightarrow (-\infty, \infty]$  is defined by  $G_x(y) = G(x, y)$  then

$$\Delta G_x = -\delta_x \tag{1}$$

where  $\delta_x$  is the Dirac delta at  $x$  and

$$\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$$

is the usual Laplacian. This is the famous Gauss's law applied to a point charge by choosing the appropriate units. We define the total energy  $H_n : (\mathbb{R}^d)^n \rightarrow (-\infty, \infty]$  of  $n$  particles with charge  $q$  as usual by

$$H_n(x_1, \dots, x_n) = \sum_{i < j} q^2 G(x_i, x_j).$$

Since the total energy is invariant under the action of  $\mathbb{R}^d$  by translations on  $(\mathbb{R}^d)^n$ , any natural probability defined by it should also be invariant under those translations but this does not make sense on the Euclidean space. We need to break the symmetry or, more precisely, we need some confining mechanism. We achieve this by adding an external potential

$$V : \mathbb{R}^d \rightarrow (-\infty, \infty]$$

into the energy to obtain

$$H_n(x_1, \dots, x_n) = \sum_{i < j} q^2 G(x_i, x_j) + \sum_{i=1}^n qV(x_i).$$

So, we are ready to define the Coulomb gas of  $n$  particles.

**Definition 1.1** (Euclidean Coulomb gas). *Let  $V : \mathbb{R}^d \rightarrow (-\infty, \infty]$  be a measurable function and take two positive numbers  $q$  and  $\beta$ . We say that a random element  $(X_1, \dots, X_n)$  of  $(\mathbb{R}^d)^n$  is a Coulomb gas on  $\mathbb{R}^d$  of particles of charge  $q$ , at inverse temperature  $\beta$  and confined by the potential  $V$  if it follows the law  $\mathbb{P}_n$  defined by*

$$d\mathbb{P}_n = \frac{1}{Z_n} e^{-\beta H_n} d\ell_{(\mathbb{R}^d)^n}$$

where  $\ell_{(\mathbb{R}^d)^n}$  denotes the Lebesgue measure on  $(\mathbb{R}^d)^n$  and we require

$$Z_n = \int_{(\mathbb{R}^d)^n} e^{-\beta H_n} d\ell_{(\mathbb{R}^d)^n} \in (0, \infty).$$

## 1.2 On manifolds

Now, we extend this notion to a different kind of space. We will take a more physicist's approach. It is natural to think, at least macroscopically, that the space we live on is Euclidean. Nevertheless, since we can see just a portion of it, all we can really say is that it seems Euclidean near us. This can be formalized through the notion of manifold which we proceed to recall.

Consider a Polish space  $M$ . We say that  $M$  is a *topological manifold* if there exists an open cover  $\{U_\lambda\}_{\lambda \in \Lambda}$  of  $M$  and homeomorphisms

$$\varphi_\lambda : U_\lambda \rightarrow V_\lambda \subset \mathbb{R}^d$$

from  $U_\lambda$  to an open set  $V_\lambda$  of  $\mathbb{R}^d$ . Each map  $\varphi_\lambda$  is usually called a *chart* and the family  $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$  is called an *atlas*. In fact, this Polish space can be seen as a bunch of open sets on  $\mathbb{R}^d$  (the sets  $V_\lambda$ ) with the identifications given by the *transition maps*

$$\varphi_\lambda \circ \varphi_\kappa^{-1} : \varphi_\kappa(U_\lambda \cap U_\kappa) \rightarrow \varphi_\lambda(U_\lambda \cap U_\kappa).$$

This tells us how to compare a region of the space that has at least two possible descriptions. Now, if we want to imitate the description (1) of  $G$  we need to know how to differentiate functions on  $M$ . For this we will look at  $M$  through the charts so that, for instance, some map is differentiable in  $U_\lambda$  if it is differentiable when we identify it with  $V_\lambda$ . A problem appears since we may have two descriptions of the same region and some map can be differentiable under some identification but maybe not under the other. This is where we have to make a choice.

**Definition 1.2** (Smooth manifold). *Let  $M$  be a topological manifold and let  $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$  be an atlas. We say that this atlas is smooth if every transition map  $\varphi_\lambda \circ \varphi_\kappa^{-1}$  is smooth. In this case we call  $(M, \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda})$  a smooth manifold of dimension  $d$ .*

In fact, the usual definition of smooth manifold involves choosing a maximal smooth atlas. Since we will not need it here we will keep using Definition 1.2 and refer to [74, Chapter 1] for the usual definition. A smooth manifold may be thought of as a bunch of open sets on  $\mathbb{R}^d$  glued by the smooth transition maps  $\varphi_\lambda \circ \varphi_\kappa^{-1}$ . So, if  $(M, \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda})$  is a smooth manifold we can define new notions just by saying how these behave under two different charts. We give the examples we will need and refer to [74] for a more complete treatment.

We say that  $f : M \rightarrow \mathbb{R}$  is a *smooth function* if for every  $\lambda \in \Lambda$  the function

$$f^\lambda = f \circ \varphi_\lambda^{-1} : V_\lambda \rightarrow \mathbb{R}$$

is smooth. Notice that this notion makes sense because the atlas is smooth. In fact, if we choose to forget  $M$  and think of it as a bunch of identified open sets we may say that a smooth function is a family  $\{f^\lambda : V_\lambda \rightarrow \mathbb{R}\}_{\lambda \in \Lambda}$  of smooth functions such that

$$f^\kappa = f^\lambda \circ (\varphi_\lambda \circ \varphi_\kappa^{-1}). \quad (2)$$

What would then be the derivative of  $f$ ? Intuitively, it should be given by the family of derivatives  $\{df^\lambda : V_\lambda \rightarrow \mathbb{R}^{d*}\}_{\lambda \in \Lambda}$  where  $\mathbb{R}^{d*}$  denotes the dual of  $\mathbb{R}^d$ . But by differentiating (2) at the point  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$  and denoting  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$  we obtain

$$df_x^\kappa = df_y^\lambda \circ d(\varphi_\lambda \circ \varphi_\kappa^{-1})_x. \quad (3)$$

Objects that transform as in (3) are called 1-forms. More precisely, a family of smooth maps  $\nu = \{\nu^\lambda : V_\lambda \rightarrow \mathbb{R}^{d*}\}_{\lambda \in \Lambda}$  is said to be a *1-form* if for every  $\kappa, \lambda \in \Lambda$

$$\nu_x^\kappa = \nu_y^\lambda \circ d(\varphi_\lambda \circ \varphi_\kappa^{-1})_x \quad (4)$$

whenever  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$  and  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$ . Similarly, a vector field  $X$  is a family of smooth maps  $\{X^\lambda : V_\lambda \rightarrow \mathbb{R}^d\}_{\lambda \in \Lambda}$  such that

$$X^\lambda(y) = \left[ d(\varphi_\lambda \circ \varphi_\kappa^{-1})_x \right] X^\kappa(x).$$

Notice that there is still no notion of gradient of a function. For this we may require a way to identify 1-forms to vector fields. One such way is what we call a Riemannian metric. Let us denote by  $\mathcal{I}$  the space of inner products of  $\mathbb{R}^d$  which can be seen as the open set of symmetric positive definite  $d$ -by- $d$  matrices.

**Definition 1.3** (Riemannian metric). *A family of smooth applications  $\{h^\lambda : V_\lambda \rightarrow \mathcal{I}\}_{\lambda \in \Lambda}$  is said to be a Riemannian metric if*

$$h_y^\lambda = \left[ d(\varphi_\lambda \circ \varphi_\kappa^{-1})_x \right]_* h_x^\kappa$$

whenever  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$  and  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$  where  $*$  denotes the pushforward of metrics. If we think  $\mathcal{I}$  as a set of matrices the condition may be seen as

$$h_y^\lambda = \left[ \text{Jac}(\varphi_\lambda \circ \varphi_\kappa^{-1})^\top(x) \right]^{-1} h_x^\kappa \left[ \text{Jac}(\varphi_\lambda \circ \varphi_\kappa^{-1})(x) \right]^{-1}$$

where  $\text{Jac}$  denotes the Jacobian matrix and  $\top$  denotes the transpose.

A manifold endowed with a Riemannian metric is called a *Riemannian manifold*. One of the purposes of this definition is to make sense of the inner product of two vector fields  $X$  and  $Y$ . Given a Riemannian metric  $h$ , we want the function  $h(X, Y)$  to be defined in the now usual way, using the charts. More precisely, as the family  $\{h(X, Y)^\lambda : V_\lambda \rightarrow \mathbb{R}\}_{\lambda \in \Lambda}$  given by  $h(X, Y)^\lambda(x) = h_x^\lambda(X^\lambda(x), Y^\lambda(x))$ . What is great is that this defines a smooth function in the sense of (2). Moreover, the Riemannian metric allows us, by duality, to define the gradient  $\nabla f$  of a smooth function  $f$ . For more on this we encourage the reader to look at [43].

We have pretty much covered the basics and we may follow the same pattern to define plenty of different objects. Nonetheless, we have not yet arrived to the notion of Laplacian. We will follow a non-standard approach and define first the volume measure associated to a Riemannian metric. The intuition behind a Riemannian metric is that it measures infinitesimal distances. For instance, if  $p \in M$  is seen in coordinates as  $x = \phi^\lambda(p)$  and  $X$  is a vector field, we may think of  $X^\lambda(x)dt$  as an infinitesimal vector that connects  $x$  and  $x + X^\lambda(x)dt$ . Following this, the distance between  $x$  and  $x + X^\lambda(x)dt$  is  $\sqrt{h_x^\lambda(X^\lambda(x), X^\lambda(x))}dt$ . What would an infinitesimal volume be? Well, if  $\{X_1^\lambda(x), \dots, X_m^\lambda(x)\}$  is an orthonormal basis then we could form the ‘infinitesimal’ (hyper)cube induced by the vertex  $x$  and the infinitesimal vectors  $X_1^\lambda(x)dt_1, \dots, X_m^\lambda(x)dt_m$ . Its volume should be  $dt_1 \dots dt_m$ . But the canonical basis  $\{e_1, \dots, e_m\}$  is not necessarily orthonormal so that the volume of the cube induced by the vertex  $x$  and the infinitesimal vectors  $e_1dt_1, \dots, e_mdt_m$  is not  $dt_1 \dots dt_m$ . Instead, it is given by  $\sqrt{\det h_x^\lambda}$  where we are thinking the inner product as a matrix to make sense of the determinant. This leads us to the following definition.

**Definition 1.4** (Volume measure). *The volume measure  $\sigma$  associated to the metric  $h$  is the positive measure on  $M$  such that, for every non-negative smooth function  $f : M \rightarrow \mathbb{R}$ ,*

$$\int_{U_\lambda} f d\sigma = \int_{V_\lambda} f^\lambda(x) \sqrt{\det h_x^\lambda} d\ell_{\mathbb{R}^d}(x).$$

We shall be interested in the compact case where  $\sigma$  can be seen to be a bounded measure which we assume to have total mass one by rescaling the metric by a constant. If the reader wishes to go deeper on the notion of integration on manifolds by using differential forms we recommend [97, Chapter 4]. Having this volume measure we are ready to define the Laplace operator or Laplacian  $\Delta : C^\infty(M) \rightarrow C^\infty(M)$ , where  $C^\infty(M)$  is the space of smooth functions on  $M$ . We will define it by duality, i.e. by saying what

$$\int_M f \Delta g d\sigma$$

should be.

**Definition 1.5** (Laplacian). *The Laplacian is the unique application  $\Delta : C^\infty(M) \rightarrow C^\infty(M)$  such that*

$$\int_M f \Delta g d\sigma = - \int_M h(\nabla f, \nabla g) d\sigma \tag{5}$$

for any pair of smooth functions  $f$  and  $g$  such that  $f$  is compactly supported.

The reader may see [97, Chapter 6] for the related definition involving the Hodge star operator. In a more concrete, but maybe not so enlightening, fashion we can notice that  $\Delta$  is easily given on local coordinates. If  $f$  is a smooth function,  $\Delta f$  is given by

$$(\Delta f)^\lambda = \frac{1}{\sqrt{\det h^\lambda}} \sum_{i,j=1}^m \frac{\partial}{\partial x_i} \left( \sqrt{\det h^\lambda} h_{ij}^\lambda \frac{\partial}{\partial x_j} f^\lambda \right).$$

This can be quickly proved by inserting this formula into (5) and using Stokes’ theorem. We are now ready to define the Coulomb interaction on a compact Riemannian manifold. It will depend on a ‘background’ measure that we will call  $\Lambda$ .

**Definition 1.6** (Green function). *Let  $\Lambda$  be a signed measure on the compact Riemannian manifold  $M$  that has a smooth density with respect to  $\sigma$  and such that  $\Lambda(M) = 1$ . Consider a continuous function  $G : M \times M \rightarrow (-\infty, \infty]$  such that for every  $p \in M$  the function  $G_p : M \rightarrow (-\infty, \infty]$ , given by  $G_p(\tilde{p}) = G(p, \tilde{p})$ , is integrable with respect to  $\sigma$ . If for every smooth function  $f : M \rightarrow \mathbb{R}$*

$$\int_M G_p \Delta f \, d\sigma = -f(p) + \int_M f \, d\Lambda \quad (6)$$

then we say that  $G$  is a Green function associated to  $\Lambda$ .

Given  $\Lambda$ , we will choose any Green function since it is unique up to an additive constant. More information about this object may be found in [10, Chapter 4]. More compactly, we write

$$\Delta G_p = -\delta_p + \Lambda \quad (7)$$

to mean that (6) occurs for every smooth function  $f$ . Indeed, it is easily seen by Definition 1.5 that  $\Delta$  is a symmetric operator so that (7) can be understood as a distributional formulation of (6). It may seem a bit strange not to consider instead the exact analogue of the Euclidean case

$$\Delta G_p = -\delta_p.$$

What happens can be understood in two equivalent ways. The first one is that now we are allowed to use  $f = 1$  in (6) so that  $\Lambda$  is there to make the right-hand side zero. The second one is that the integral of the Laplacian is always zero. This can be seen by using  $f = 1$  in (5).

Define the energy  $H_n : M^n \rightarrow (-\infty, \infty]$  of a system of  $n$  particles of charge  $q$  as

$$H_n(p_1, \dots, p_n) = \sum_{i < j} q^2 G(p_i, p_j).$$

Since  $H_n$  is bounded from below and since we are going to integrate against a finite measure there is no need for a potential. Nevertheless, we can prove that if  $\Lambda_1$  and  $\Lambda_2$  are two signed measures with smooth density and if we denote by  $G_1$  and  $G_2$  the Green functions associated to  $\Lambda_1$  and  $\Lambda_2$  respectively, then there exists a smooth function  $V$  such that

$$G_2(p, \tilde{p}) = G_1(p, \tilde{p}) + V(p) + V(\tilde{p}).$$

In this case the corresponding energies  $H_n^1$  and  $H_n^2$  are related by

$$H_n^2(p_1, \dots, p_n) = H_n^1(p_1, \dots, p_n) + \sum_{i=1}^n (n-1)q^2 V(p_i) = \sum_{i < j} q^2 G_1(p_i, p_j) + \sum_{i=1}^n (n-1)q^2 V(p_i) \quad (8)$$

so, if we choose  $q = 1/(n-1)$  we can interpret  $H_n^2$  as an electrostatic interaction plus a potential.

**Definition 1.7** (Coulomb gas on a compact manifold). *Take two positive numbers  $q$  and  $\beta$ . We say that a random element  $(X_1, \dots, X_n)$  of  $M^n$  is a Coulomb gas on  $M$  of particles of charge  $q$ , at inverse temperature  $\beta$  and confined by the background  $\Lambda$  if it follows the law  $\mathbb{P}_n$  defined by*

$$d\mathbb{P}_n = \frac{1}{\mathcal{Z}_n} e^{-\beta H_n} d\sigma^{\otimes n}$$

where

$$\mathcal{Z}_n = \int_{M^n} e^{-\beta H_n} d\sigma^{\otimes n}.$$

Even though the Laplacian is defined using the metric, we will explain that, in two dimensions, it depends only on its conformal class, defined below. This implies, in particular, that the definition of the Green function requires only a conformal structure to make sense. Let us



consider, then, a two-dimensional smooth manifold  $M$  endowed with a metric  $h$ . The Laplacian  $\Delta f$  of an integrable function  $f \in L^1(M)$  is a distribution on  $M$  that satisfies

$$\forall g \in C^\infty(M), \langle \Delta f, g \rangle = \int_M f \Delta g \, d\sigma.$$

In order not to worry about issues of convergence we will say that  $\Delta$  is an application from  $L^1(M)$  to  $C^\infty(M)^*$ , the algebraic dual of  $C^\infty(M)$ . The remark we would like to verify is that, in dimension two, the operator

$$\Delta : L^1(M) \rightarrow C^\infty(M)^*$$

depends only on what is called the conformal class of  $h$ . To explain what a conformal class is we define the following equivalence relation on the set of metrics. Given two metrics  $h_1$  and  $h_2$ , we write

$$h_1 \sim h_2$$

if and only if there exists a smooth function  $\rho$  such that

$$h_2 = \rho h_1$$

where  $\rho h_1$  is the metric given by the family  $\{\rho^\lambda h_1^\lambda\}_{\lambda \in \Lambda}$ . The equivalence class of a metric  $h$  is called the conformal class of  $h$  and we will denote it by  $\mathbf{C}_h$ . More explicitly, we have

$$\mathbf{C}_h = \{\rho h : \rho \in C^\infty(M) \text{ is positive everywhere}\}.$$

If  $\bar{h} = \rho h \in \mathbf{C}_h$  we may see that the notions we have defined before (volume measure, gradient, Laplacian, etc) change nicely. For instance, if  $\bar{\sigma}$  denotes the volume measure associated to  $\bar{h}$  we can see, by Definition 1.4, that

$$d\bar{\sigma} = \rho \, d\sigma. \tag{9}$$

In particular, we can notice that the space  $L^1(M)$  will depend only on  $\mathbf{C}_h$  (while the actual integration will depend on  $h$ ). For another example, given  $f \in C^\infty(M)$ , if we denote by  $\bar{\nabla} f$  the gradient defined by the metric  $\bar{h}$ , we have

$$\bar{\nabla} f = \rho^{-1} \nabla f.$$

So, by the definition of the Laplacian given by (5), if  $\bar{\Delta} f$  denotes the Laplacian defined by the metric  $\bar{h}$ ,

$$\bar{\Delta} f = \rho^{-1} \Delta f.$$

Finally, let  $\bar{\Delta}$  be the Laplace operator on  $L^1(M)$

$$\bar{\Delta} : L^1(M) \rightarrow C^\infty(M)^*$$

associated to  $\bar{h}$  and notice that for every  $f \in L^1(M)$  and  $g \in C^\infty(M)$

$$\langle \bar{\Delta} f, g \rangle = \int_M f \bar{\Delta} g \, d\bar{\sigma} = \int_M f \rho^{-1} \Delta g \, \rho \, d\sigma = \int_M f \Delta g \, d\sigma = \langle \Delta f, g \rangle$$

so that  $\bar{\Delta} = \Delta$ . This is an amazing fact that involves the notion of complex structure in the standard discussions [12, Chapter 5], but can be also stated on the related framework of conformal structures as we have done here.

### 1.3 Examples from random matrix models

There are some instances of Coulomb gases that appear naturally in the study of random matrices. We will mention a few of them. All these examples may be found in [49].



For a more intrinsic description,  $\mathcal{M}$  has a law proportional to

$$e^{-\text{Tr}(A^*A)} d\ell_{\mathbb{M}_n}(A)$$

or, using the standard notations, it is a Gaussian variable associated to the quadratic form on the vector space  $\mathbb{M}_n$

$$A \mapsto \frac{1}{2} \text{Tr}(A^*A).$$

This example may be found in [49, Subsection 15.1.1]. In this case we have the following result proved by Ginibre [55].

**Proposition 1.9** (Eigenvalues of the Ginibre ensemble). *Suppose  $\mathcal{M}$  belongs to the Ginibre ensemble of size  $n$  and let  $(Z_1, \dots, Z_n) \in \mathbb{C}^n$  be a random vector that follows the law*

$$\frac{1}{\mathcal{Z}_n} \prod_{i < j} |z_i - z_j|^2 e^{-\sum_{i=1}^n |z_i|^2} d\ell_{\mathbb{C}^n}(z_1, \dots, z_n) \quad (11)$$

where  $\mathcal{Z}_n$  is a normalization constant. Then, the empirical measure of the eigenvalues of  $\mathcal{M}$  and the empirical measure of  $(Z_1, \dots, Z_n)$  have the same law. More precisely, if  $\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$  is formed by the eigenvalues of  $\mathcal{M}$ , counting multiplicities,  $\{\lambda_1, \dots, \lambda_n\}$ , then

$$\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i} \sim \frac{1}{n} \sum_{i=1}^n \delta_{Z_i}.$$

This can be stated using the definition of a Coulomb gas. The law given in (11) is the one of a Coulomb gas on  $\mathbb{C}$  of unit charged particles, at inverse temperature  $4\pi$  and confined by the potential

$$V(z) = \frac{|z|^2}{4\pi}.$$

Again, the  $4\pi$  term is due to the normalization we have chosen for the Coulomb gas definition. A spectacular peculiarity of this Coulomb gas is its behavior as the number of particles  $n$  grows to infinity. As will be explained in Subsection 2.4, the point process defined by  $\{Z_1, \dots, Z_n\}$  converges to a determinantal point process invariant under *isometries of the plane*, i.e. invariant under rotations and translations. In this sense, this system of points is naturally attached to the complex plane with its usual Riemannian structure.

### The spherical ensemble

Now, consider  $A$  and  $B$  independent random matrices that belong to the Ginibre ensemble of size  $n$ . We say that a random  $n$ -by- $n$  matrix  $\mathcal{M}$  belongs to the *spherical ensemble* of size  $n$  if

$$\mathcal{M} \sim AB^{-1}.$$

For more information on this we can see [49, Section 15.6]. Krishnapur, [69], realized the following remarkable description.

**Proposition 1.10** (Eigenvalues of the spherical ensemble). *Suppose  $\mathcal{M}$  belongs to the spherical ensemble of size  $n$  and let  $(Z_1, \dots, Z_n) \in \mathbb{C}^n$  be a random vector that follows the law*

$$\frac{1}{\mathcal{Z}_n} \prod_{i < j} |z_i - z_j|^2 \prod_{i=1}^n \frac{1}{(1 + |z_i|^2)^{n+1}} d\ell_{\mathbb{C}^n}(z_1, \dots, z_n) \quad (12)$$

where  $\mathcal{Z}_n$  is a normalization constant. Then, the empirical measure of the eigenvalues of  $\mathcal{M}$  and the empirical measure of  $(Z_1, \dots, Z_n)$  have the same law. More precisely, if  $\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$  is formed by the eigenvalues of  $\mathcal{M}$ , counting multiplicities,  $\{\lambda_1, \dots, \lambda_n\}$ , then

$$\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i} \sim \frac{1}{n} \sum_{i=1}^n \delta_{Z_i}.$$

As the previous case, this can be stated using the concept of a Coulomb gas. The law given in (12) is the one of a Coulomb gas on  $\mathbb{C}$  of particles of charge  $1/(n+1)$ , at inverse temperature  $4\pi(n+1)^2$  and confined by the potential

$$V(z) = \frac{1}{4\pi} \log(1 + |z|^2).$$

Nevertheless, it may not be obvious why the ‘spherical’ is attached to the name. There is a more (I hope) natural interpretation of the spherical ensemble that explains the ‘spherical’ on the name. We shall give a short explanation below. First, let us remark that, by seeing this gas under the stereographic projection<sup>1</sup>, we obtain a Coulomb gas on the round sphere  $S^2$  of unit charged particles, at inverse temperature  $4\pi$  and confined by a uniform background charge. More explicitly, if  $(Z_1, \dots, Z_n)$  follows the law (12) then, by calling  $X_i$  the inverse stereographic projection of  $Z_i$ , the random element  $(X_1, \dots, X_n)$  of  $(S^2)^n$  follows the law

$$\frac{1}{\mathcal{Z}_n} e^{-4\pi \sum_{i < j} G(x_i, x_j)} d\sigma^{\otimes n}(x_1, \dots, x_n)$$

where  $\mathcal{Z}_n$  is a normalization constant,  $\sigma$  is the uniform probability measure on  $S^2$  and  $G$  is the Green function associated to  $\sigma$  which in this case has the simple formula

$$G(x, y) = -\frac{1}{2\pi} \log |x - y|_{\mathbb{R}^3}.$$

This can be seen, for instance, in [49, Section 15.6]. Now, we want to give a nice explanation of the invariance under rotations of this gas without giving the explicit formula for its law. If  $p : \mathbb{C}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{M}_n$  is the application defined by

$$p(z, w) = zA + wB = (A \ B) \begin{pmatrix} z \\ w \end{pmatrix},$$

where  $A$  and  $B$  are independent and belong to the Ginibre ensemble of size  $n$ , then it is not so hard to see that the law of  $p$  is invariant under the action of the unitary group  $U(2)$  on  $\mathbb{C}^2 \setminus \{(0, 0)\}$ . This implies that  $\det p$ , which is explicitly given by

$$\det p(z, w) = \det(zA + wB),$$

is also invariant under unitary transformations so that the complex lines of  $\mathbb{C}^2$  on which  $\det p$  is zero form a point process invariant under unitary transformations in the space of complex lines, known as the projective space  $\mathbb{P}^1$ . By recalling that this space is canonically identified with the sphere and that, under this identification, the unitary transformations acting on  $\mathbb{P}^1$  become the rotations of the sphere we obtain what we wanted: the eigenvalues of a random matrix that belongs to the spherical ensemble have a law invariant under *isometries of the sphere*.

Notice that, by following this construction, we can obtain more general invariant gases by replacing the linear polynomials by polynomials of higher degree. We recently learned by conversations with Carlos Beltrán that this generalization is related to the polynomial eigenvalue problem and that it has not been so much studied except for [7].

### The truncated Haar unitary ensemble

Let  $A$  be an  $(n+1)$ -by- $(n+1)$  random unitary matrix that follows the Haar measure on  $U(n+1)$ . A random  $n$ -by- $n$  matrix  $\mathcal{M}$  belongs to the *truncated Haar unitary ensemble* of size  $n$  if

$$\mathcal{M} \sim (A_{ij})_{i,j \in \{1, \dots, n\}}.$$

For further information see [49, Subsection 15.7.4]. The following was proved in [92] on a greater generality.

---

<sup>1</sup>The application from  $S^2 \setminus \{(0, 0, 1)\}$  to  $\mathbb{C}$  given by  $(x, y, z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z})$ .

**Proposition 1.11** (Eigenvalues of the truncated Haar unitary ensemble). *Suppose  $\mathcal{M}$  belongs to the truncated Haar unitary ensemble of size  $n$ . Denote by  $\mathbb{D}$  the open unit disk on  $\mathbb{C}$  and let  $(Z_1, \dots, Z_n) \in \mathbb{D}^n$  be a random vector that follows the law*

$$\frac{1}{Z_n} \prod_{i < j} |z_i - z_j|^2 d\ell_{\mathbb{D}^n}(z_1, \dots, z_n) \quad (13)$$

where  $\ell_{\mathbb{D}^n}$  denotes the Lebesgue measure on  $\mathbb{C}^n$  restricted to  $\mathbb{D}^n$  and  $Z_n$  is a normalization constant. Then, the empirical measure of the eigenvalues of  $\mathcal{M}$  and the empirical measure of  $(Z_1, \dots, Z_n)$  have the same law. More precisely, if  $\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$  is formed by the eigenvalues of  $\mathcal{M}$ , counting multiplicities,  $\{\lambda_1, \dots, \lambda_n\}$ , then

$$\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i} \sim \frac{1}{n} \sum_{i=1}^n \delta_{Z_i}.$$

As the two previous cases, this can also be stated using the notion of a Coulomb gas. The law given in (13) is the one of a Coulomb gas on  $\mathbb{C}$  of unit charged particles, at inverse temperature  $4\pi$  and confined by the potential

$$V(z) = \begin{cases} 0 & \text{if } |z| \leq 1 \\ \infty & \text{otherwise} \end{cases}.$$

We will see in Subsection 2.4 that, when  $n$  grows to infinity, the point process converges to a determinantal point process invariant under *isometries of the hyperbolic disk*. The truncated Haar unitary ensemble and similar examples that converge to the same process have been barely studied and will be a main topic of Chapter 4 and Chapter 5.

#### 1.4 A Coulomb gas on a curvature background

We describe here a particular kind of Coulomb gases that live on two-dimensional Riemannian manifolds. It appears in Chapter 4 and Chapter 5 on the sphere at a specific temperature and in Chapter 6, when the genus of the surface is greater than one. Since we will deal with the notion of curvature we give here some references. The standard definitions may be found in [43, Chapter 4] for general Riemannian manifolds. For two-dimensional Riemannian manifolds embedded in  $\mathbb{R}^3$ , we can, equivalently, look at the Gaussian curvature whose definition can be found in [44, Section 3.2, Definition 6]. Finally, for the nice formula we will use below in (14) we suggest [12, Chapter 4].

We shall focus here on the case of the sphere since it has a nice interpretation as a Coulomb gas on the plane. We take the sphere  $S^2$  with its usual round metric  $h$  normalized so that  $S^2$  has volume one. Let  $\mathcal{C}$  be the space of metrics conformally equivalent to  $h$  of volume one. More precisely,

$$\mathcal{C} = \left\{ \rho h : \rho \in C^\infty(S^2), \rho \text{ is positive everywhere and } \int_M \rho \, d\sigma = 1 \right\}$$

where  $\sigma$  denotes the volume measure associated to  $h$  or, equivalently, the uniform probability measure on  $S^2$ . By denoting the space of probability measures with smooth and nowhere zero density by  $\mathcal{P}_{C^\infty}(S^2)$  we obtain the identification

$$\mathcal{P}_{C^\infty}(S^2) \rightarrow \mathcal{C} \quad \text{given by } \rho \, d\sigma \mapsto \rho h$$

with inverse

$$\mathcal{C} \rightarrow \mathcal{P}_{C^\infty}(S^2) \quad \text{defined by } \bar{h} \mapsto \text{volume measure of } \bar{h}.$$

So, we may think  $\mathcal{P}_{C^\infty}(S^2)$  as the space of metrics conformally equivalent to  $h$ . As a remark, we must say that these metrics are also understood as the ones compatible with the complex

structure induced by an orientation and  $h$ . Nevertheless, we shall not use that description since it requires concepts not defined here (see, for instance, [12, Chapter 4]).

Given  $\alpha \in \mathcal{P}_{C^\infty}(S^2)$  we consider its scalar curvature (recall that we are identifying  $\alpha$  with a metric)  $R_\alpha$  which is given in the stereographic projection by

$$R_\alpha = -\Delta \log \rho \quad (14)$$

where  $\rho$  is the density of  $\alpha$  with respect to the Lebesgue measure. This fact and a discussion about curvature can be found in [12, Section 4.4]. We will think  $R_\alpha$  as a measure but let us first notice that, by Gauss-Bonnet theorem,

$$\int_{S^2} R_\alpha d\alpha = 8\pi.$$

The normalized version would then be the measure  $\Lambda_\alpha$  defined by

$$d\Lambda_\alpha = \frac{R_\alpha}{8\pi} d\alpha.$$

This will be the charge background of our Coulomb gases. More precisely, if we endow  $S^2$  with the ‘metric’  $\alpha$ , the Coulomb gas on the curvature background will be a Coulomb gas confined by the background  $\Lambda_\alpha$ . In fact, this is very related to a family of Coulomb gases on  $\mathbb{C}$  as we will explain now. Take a measure  $\mu$  of total mass one on  $\mathbb{C}$  and define

$$V^\mu(z) = \frac{1}{2\pi} \int_{\mathbb{C}} \log |z - w| d\mu(w) \quad (15)$$

whenever the integral makes sense. This  $V^\mu$  may be thought as the potential generated by the charge  $-\mu$  since it satisfies

$$\Delta V^\mu = \mu.$$

We claim that the measure  $\sigma_\mu$  given by

$$d\sigma_\mu = \frac{e^{-8\pi V^\mu}}{\mathcal{Z}} d\ell_{\mathbb{C}},$$

where  $\mathcal{Z}$  is a normalization constant, is naturally associated to  $\mu$  in an explicit sense. Indeed, if, by inverse stereographic projection,  $\mu$  defined a measure with smooth density with respect to  $\sigma$  on  $S^2$  we would have

$$\Lambda_{\sigma_\mu} = \mu$$

which can be understood by (14). This is motivation enough in the smooth case. Nevertheless, in the general non-smooth setting we should give a further motivation. The following remark should be enough since it says that the correspondence  $\mu \mapsto \sigma_\mu$  has a nice behavior under conformal transformations (also known as Möbius transformations) and, thus, it is naturally attached to the conformal structure.

**Proposition 1.12** (Behavior of the potential under conformal transformations). *Suppose  $\mu$  is a signed measure of total mass one with finite potential, i.e. (15) is well-defined and finite for every  $z \in \mathbb{C}$ . Then, for every conformal transformation  $T : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$ , the image measure  $T_*\mu$  has finite potential and*

$$\sigma_{T_*\mu} = T_*\sigma_\mu. \quad (16)$$

Moreover, if  $\alpha \in \mathcal{P}_{C^\infty}(S^2)$  and  $G$  is the Green function associated to  $\Lambda_\alpha$  we define

$$U^\mu(x) = - \int_{S^2} G(x, y) d\mu(y) \quad (17)$$

and obtain

$$d\sigma_\mu = \frac{e^{-8\pi U^\mu}}{\mathcal{Z}} d\alpha \quad (18)$$

where  $\mathcal{Z}$  is some normalization constant.

*Proof.* The assertion about the finiteness of the potential can be seen by direct calculation. The proof of (18) is obtained using the formula for  $R_\alpha$  given in (14). Equality (16) is a consequence of (18) and the transformation properties of the metric and its curvature.  $\square$

Additionally, given a measure  $\mu$  of total mass one we consider

$$G^\mu(x, y) = -\frac{1}{2\pi} \log|x - y| + V^\mu(x) + V^\mu(y).$$

This satisfies

$$\Delta G_x^\mu = -\delta_x + \mu,$$

where  $G_x^\mu(y) = G^\mu(x, y)$ , so that we may think  $G^\mu$  as the Green function associated to  $\mu$ . Even if the Laplacian is taken here in  $\mathbb{C}$ , we have the same equality on  $S^2$  essentially because the stereographic projection is a conformal transformation. The Coulomb gas law attached to this data will be

$$\frac{1}{\mathcal{Z}_n} e^{-\beta \sum_{i < j} G^\mu(x_i, x_j)} d\sigma_\mu^{\otimes n}(x_1, \dots, x_n) \quad (19)$$

where  $\mathcal{Z}_n$  is a normalization constant. After developing the appropriate terms, (19) becomes the Coulomb gas law on  $\mathbb{C}$

$$\frac{1}{\mathcal{Z}_n} e^{-\beta \left( -\frac{1}{2\pi} \sum_{i < j} \log|x_i - x_j| + (n-1) \sum_{i=1}^n V^\mu(x_i) \right)} e^{-8\pi V^\mu(x_i)} d\ell_{\mathbb{C}^n}(x_1, \dots, x_n)$$

for some other normalization constant  $\mathcal{Z}_n$ . If  $\beta = 4\pi$  we obtain the determinantal model studied in Chapter 4 and Chapter 5

$$\frac{1}{\mathcal{Z}_n} e^{-4\pi \left( -\frac{1}{2\pi} \sum_{i < j} \log|x_i - x_j| + (n+1) \sum_{i=1}^n V^\mu(x_i) \right)} d\ell_{\mathbb{C}^n}(x_1, \dots, x_n). \quad (20)$$

Since the Laplacian depends only on the conformal class of the metric we can write (19) as

$$\frac{1}{\mathcal{Z}_n} e^{-\beta \left( \sum_{i < j} G(x_i, x_j) + (n-1) \sum_{i=1}^n U^\mu(x_i) \right)} e^{-8\pi U^\mu(x_i)} d\alpha^{\otimes n}(x_1, \dots, x_n)$$

where  $G$  is the Green function associated to  $\Lambda_\alpha$ , the latter being the normalized curvature of  $\alpha \in \mathcal{P}_{C^\infty}(S^2)$ , and  $U^\mu$  is defined by (17). This means that, for any metric we choose on  $S^2$ , the system may be seen as a Coulomb gas on  $S^2$  confined by a potential.

## 1.5 The circle background ensemble and Kac's random polynomials

There is a toy model that we enjoy studying. It is the case where  $\mu$  is the uniform probability measure on the equator or, equivalently, the uniform probability measure on the unit circle. We have

$$V^\mu(z) = \frac{1}{2\pi} \int_{S^1} \log|z - s| ds = \frac{1}{2\pi} \log_+ |z| = \frac{1}{2\pi} \max\{0, \log|z|\},$$

where we use the normalized uniform measure on the circle to perform the integration. We consider the law (20) which we write as

$$\frac{1}{\mathcal{Z}_n} \prod_{i < j} |z_i - z_j|^2 \prod_{i=1}^n \exp \left( \int_{S^1} \log|z_i - s|^2 ds \right)^{-(n+1)} d\ell_{\mathbb{C}^n}(z_1, \dots, z_n).$$

A lot of explicit calculations can and will be done in Chapter 5 for this and a family of Coulomb gases on the plane. There is an analogous system in the context of random polynomials. Let

$\{a_n\}_{n \in \mathbb{N}}$  be a sequence of independent and identically distributed random variables. For each  $n$  define the random polynomial

$$p_n(z) = \sum_{k=0}^n a_k z^k.$$

These are the so-called Kac's random polynomials. If we choose the law of  $a_0$  to be a standard complex Gaussian then we may think  $p_n$  as a Gaussian random element of  $\mathcal{P}_n$ , the space of complex polynomials of degree  $n$ , endowed with the quadratic form

$$p \mapsto \frac{1}{2} \int_{S^1} |p(s)|^2 ds.$$

In fact, an explicit form for the law of the zeros of  $p_n$  has been found in great generality by Zeitouni and Zelditch in [98] (see also [30]) and it is given in this case by

$$\frac{1}{\tilde{Z}_n} \prod_{i < j} |z_i - z_j|^2 \left( \int_{S^1} \prod_{i=1}^n |z_i - s|^2 ds \right)^{-(n+1)} d\ell_{\mathbb{C}^n}(z_1, \dots, z_n).$$

We can see the resemblance between the two models by comparing the two terms

$$\left( \int_{S^1} \prod_{i=1}^n |z_i - s|^2 ds \right)^{-(n+1)} \quad \text{and} \quad \prod_{i=1}^n \exp \left( \int_{S^1} \log |z_i - s|^2 ds \right)^{-(n+1)}$$

the second one being the potential term on the density of the Coulomb gas. It makes us wonder how much the Coulomb gas and the zeros of the Kac's polynomial are alike. It turns out, as we will see in Chapter 5, that the macroscopic limiting behaviors are the same (as far as we know) but the microscopic behavior near the unit circle is pretty different. These models have some interesting generalizations that have been studied in Chapter 4 and Chapter 5.

## 2 Determinantal point processes

We will find a special structure on the Coulomb gases on  $\mathbb{C}$  or the sphere when the temperature is  $\beta = 4\pi$  ( $\beta = 2$  in the usual normalization of random matrix theory). This structure will be a very important tool for Chapter 4 and Chapter 5, since it allows us to do many calculations that are not still available for general  $\beta$ . A standard reference on this subject is [18]. We begin by explaining what a point processes is.

### 2.1 Point processes

Given a Polish space  $M$  we denote by  $\mathcal{M}_+(M)$  the space of positive measures on  $M$ . We define the space of point configurations on  $M$  as

$$\mathcal{C}_M = \{\mu \in \mathcal{M}_+(M) : \mu \text{ is locally finite and take values in } \mathbb{N} \cup \{\infty\}\}$$

where by locally finite we mean that every point  $x \in M$  has an open neighborhood  $U$  such that  $\mu(U) < \infty$ . In fact, a measure  $\mu$  belongs to  $\mathcal{C}_M$  if and only if there exists a discrete and closed set  $D \subset M$  and a function  $c : D \rightarrow \mathbb{N}$  such that

$$\mu = \sum_{x \in D} c(x) \delta_x.$$

In this way  $\mathcal{C}_M$  can be thought of as the space of discrete sets of  $M$  where we allow multiplicities. For simplicity, we will consider the case where  $M$  is also locally compact. We have then the equivalent description

$$\mathcal{C}_M = \{\mu \in \mathcal{M}_+(M) : \forall \text{ compact } K \subset M, \mu(K) \in \mathbb{N}\}.$$



In this case, it makes sense to consider the smallest topology on  $\mathcal{C}_M$  such that the applications  $\hat{f} : \mathcal{C}_M \rightarrow \mathbb{R}$  defined by

$$\hat{f}(\mu) = \mu(f) = \int_M f d\mu$$

are continuous for every compactly supported continuous function  $f : M \rightarrow \mathbb{R}$ . It can be seen, [65, Theorem 4.2], that  $\mathcal{C}_M$  endowed with this topology is a Polish space.

**Definition 2.1** (Point process). *A point process  $\xi$  is a random element of  $\mathcal{C}_M$ .*

## 2.2 Correlation functions

Point processes are not so easy to study so that, analogously to real random variables or vector valued random variables, we need to find a way to characterize their laws. One such way is through the notion of correlation functions.

**Definition 2.2** (Correlation functions). *Let  $(M, \sigma)$  be a Polish locally compact measure space such that  $\sigma$  has no atoms. Let  $\xi$  be a point process on  $M$  and let  $k$  be a positive integer. We say that a non-negative measurable function  $\rho_k : M^k \rightarrow \mathbb{R}$  is a  $k$ -th correlation function if*

$$\mathbb{E}[\xi(A_1) \dots \xi(A_k)] = \int_{A_1 \times \dots \times A_k} \rho_k d\sigma^{\otimes k}$$

for all mutually disjoint measurable sets  $A_1, \dots, A_k$  of  $M$ .

These functions are unique up to a negligible set and behave as the moments of  $\xi$ . Let us explain the relation with the moments. Take tensor powers of  $\xi$  by itself so that we obtain  $\xi^{\otimes k}$ . A usual  $k$ -th moment would be the measure  $\mathbb{E}[\xi^{\otimes k}]$ . Nevertheless, there are extra terms on that expected value that we would like to discard. This can be even understood by taking small values of  $k$  starting by  $k = 1$ .

**Definition 2.3** (Intensity measure). *Let  $M$  be a Polish locally compact space and let  $\xi$  be a point process on  $M$ . The intensity measure is given by  $\alpha_1 = \mathbb{E}[\xi]$ . More precisely,  $\alpha_1$  is the measure on  $M$  that satisfies*

$$\int_M f d\alpha_1 = \mathbb{E}[\xi(f)]$$

for every non-negative measurable function  $f : M \rightarrow \mathbb{R}$ .

In the case of a measure space  $(M, \sigma)$ , if  $\alpha_1$  has a density with respect to  $\sigma$ , this density would be the first correlation function  $\rho_1$ . The second correlation function can be obtained in a similar way but considering  $\xi^{\otimes 2}$  and here is where we can see the moment interpretation appear.

**Definition 2.4** (Second correlation measure). *Let  $M$  be a Polish locally compact space and let  $\xi$  be a point process on  $M$ . Consider the application  $\text{diag} : M \rightarrow M \times M$  given by  $\text{diag}(x) = (x, x)$ . Then, the second correlation measure is given by  $\alpha_2 = \mathbb{E}[\xi \otimes \xi - \text{diag}_* \xi]$ . More precisely  $\alpha_2$  is the measure on  $M \times M$  that satisfies*

$$\int_{M \times M} f(x)g(y) d\alpha_2(x, y) = \mathbb{E}[\xi(f)\xi(g)] - \mathbb{E}[\xi(fg)]$$

for every non-negative measurable functions  $f : M \rightarrow \mathbb{R}$  and  $g : M \rightarrow \mathbb{R}$ .

If we consider  $f$  and  $g$  as the indicator functions of disjoint sets in the previous definition, we see how to recover the second correlation function  $\rho_2$  because the term  $fg$  would be zero. We can similarly define the  $k$ -th correlation measure just by eliminating what would be necessarily singular on  $\mathbb{E}[\xi^{\otimes k}]$ . To have a concrete example of correlation functions in mind we state the following proposition.

**Proposition 2.5** (Fixed number of particles). *Suppose the point process  $\xi$  has exactly  $n$  particles. More precisely, suppose*

$$\xi = \sum_{i=1}^n \delta_{Y_i}$$

where  $(Y_1, \dots, Y_n)$  is a system of  $n$  particles that follows some symmetric law  $\mathbb{P}_n$  given by

$$d\mathbb{P}_n = \rho(y_1, \dots, y_n) d\sigma^{\otimes n}.$$

Then, the  $n$ -th correlation function is given by

$$\rho_n(y_1, \dots, y_n) = n! \rho(y_1, \dots, y_n). \quad (21)$$

Furthermore, the  $k$ -th correlation function can be obtained by

$$\rho_k(y_1, \dots, y_k) = \frac{n!}{(n-k)!} \int_{M^{n-k}} \rho(y_1, \dots, y_n) d\sigma^{\otimes n-k}(y_{k+1}, \dots, y_n) \quad (22)$$

if  $k < n$  and  $\rho_k = 0$  if  $k > n$ .

*Proof.* Notice that

$$\xi(A) = \sum_{i=1}^n 1_A(Z_i)$$

which implies

$$\mathbb{E}[\xi(A_1) \dots \xi(A_n)] = \mathbb{E} \left[ \left( \sum_{i=1}^n 1_{A_1}(Z_i) \right) \dots \left( \sum_{i=1}^n 1_{A_n}(Z_i) \right) \right] = n! \mathbb{P}_n(A_1 \times \dots \times A_n)$$

where we have used that  $A_1, \dots, A_n$  are mutually disjoint. That is the proof of (21). Similarly, if  $k < n$ ,

$$\begin{aligned} \mathbb{E}[\xi(A_1) \dots \xi(A_k)] &= \mathbb{E} \left[ \left( \sum_{i=1}^n 1_{A_1}(Z_i) \right) \dots \left( \sum_{i=1}^n 1_{A_k}(Z_i) \right) \right] \\ &= \frac{n!}{(n-k)!} \mathbb{P}_n(A_1 \times \dots \times A_k \times M \times \dots \times M) \end{aligned}$$

where we have again used that  $A_1, \dots, A_k$  are mutually disjoint and a counting argument. That is the proof of (22). Finally, if  $k > n$  we have that  $\xi(A_1) \dots \xi(A_k) = 0$  whenever  $A_1, \dots, A_k$  are mutually disjoint so that  $\rho_k = 0$ .  $\square$

### 2.3 Determinantal kernel

The point processes we will deal with in Chapter 4 and Chapter 5 are the so-called determinantal point processes. Since our motivation is a specific radial point process, we will start by understanding its structure. Let  $M = \mathbb{C}$  and consider the function on  $\mathbb{C}^n$  defined by

$$F(z_1, \dots, z_n) = \prod_{i < j} |z_i - z_j|^2.$$

This is the common part of the density of the ‘determinantal’ Coulomb gases we will consider. For simplicity think on a radial probability measure  $\sigma$  and think on a system  $(Z_1, \dots, Z_n)$  that follows the law  $\mathbb{P}_n$  given by

$$d\mathbb{P}_n = \frac{1}{Z_n} F(z_1, \dots, z_n) d\sigma^{\otimes n}.$$

where  $Z_n$  is a normalization constant. What is so special is that if we form the Vandermonde matrix

$$\mathcal{V} = \begin{pmatrix} 1 & z_1 & (z_1)^2 & \dots & (z_1)^{n-1} \\ 1 & z_2 & (z_2)^2 & \dots & (z_2)^{n-1} \\ 1 & z_3 & (z_3)^2 & \dots & (z_3)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & (z_n)^2 & \dots & (z_n)^{n-1} \end{pmatrix}$$

then

$$F(z_1, \dots, z_n) = |\det \mathcal{V}|^2 = (\det \mathcal{V}) \overline{(\det \mathcal{V})} = \det(\mathcal{V}\mathcal{V}^*). \quad (23)$$

So  $F$  is, in fact, a determinant. The explicit form of  $\mathcal{V}\mathcal{V}^*$  can be obtained as

$$(\mathcal{V}\mathcal{V}^*)_{ij} = \sum_{l=1}^n (V_{il}) \overline{(V_{jl})} = \sum_{k=0}^{n-1} z_i^k \bar{z}_j^k$$

so that if we define

$$K(z, w) = \sum_{k=0}^{n-1} z^k \bar{w}^k$$

we can write

$$F(z_1, \dots, z_n) = \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, n\}} \right).$$

For a bit of generality we may also consider  $n$  positive numbers  $a_0, \dots, a_{n-1}$  and define

$$K(z, w) = \sum_{k=0}^{n-1} a_k z^k \bar{w}^k$$

to obtain

$$\det \left( K(z_i, z_j)_{i,j \in \{1, \dots, n\}} \right) = a_1 \dots a_n F(z_1, \dots, z_n).$$

Having in mind the radial probability measure  $\sigma$ , a natural choice for  $a_0, \dots, a_{n-1}$  is

$$(a_k)^{-1} = \int_{\mathbb{C}} |z|^{2k} d\sigma(z)$$

so that  $K$  represents the orthogonal projection of  $L^2(\sigma)$  onto the space  $\mathcal{P}_{n-1}$  of polynomials of degree less or equal than  $n-1$ . Since  $K$  represents an orthogonal projection onto a space of dimension  $n$  it has the two nice properties

$$\int_{\mathbb{C}} K(z, x) K(x, w) d\sigma(x) = K(z, w)$$

and

$$\int_{\mathbb{C}} K(x, x) d\sigma(x) = n.$$

These and the formula for the determinant allow us to prove that

$$\int_{\mathbb{C}} \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, k\}} \right) d\sigma(z_k) = (n-k+1) \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, k-1\}} \right).$$

By a straightforward induction we obtain

$$\int_{\mathbb{C}^n} \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, n\}} \right) d\sigma^{\otimes n}(z_1, \dots, z_n) = n!$$

so that  $\frac{1}{n!} \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, n\}} \right)$  is a density function and

$$\frac{1}{n!} \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, n\}} \right) = \frac{1}{Z} F(z_1, \dots, z_n).$$

By Proposition 2.5 we have

$$\rho_k(z_1, \dots, z_k) = \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, k\}} \right)$$

for  $k \leq n$ . For  $k > n$  a decomposition of  $K(z_i, z_j)_{i,j \in \{1, \dots, k\}}$  similar to the one in (23) allows us to say

$$\det \left( K(z_i, z_j)_{i,j \in \{1, \dots, k\}} \right) = 0$$

so that, in general, we obtain the following.

**Proposition 2.6** (Correlations and determinant). *Suppose that  $(Z_1, \dots, Z_n)$  is distributed according to a law proportional to*

$$\prod_{i < j} |z_i - z_j|^2 d\sigma^{\otimes n}(z_1, \dots, z_n).$$

Let  $k \geq 1$ . Then, the  $k$ -th correlation function of the point process  $\sum_{i=1}^n \delta_{Z_i}$  is

$$\rho_k(z_1, \dots, z_k) = \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, k\}} \right)$$

where  $K$  is given by the orthogonal projection onto  $\mathcal{P}_{n-1}$ , i.e.

$$K(z, w) = \sum_{l=0}^{n-1} a_l z^l \bar{w}^l$$

with

$$(a_l)^{-1} = \int_{\mathbb{C}} |z|^{2l} d\sigma(z).$$

This is an amazing remark and there has been built an structure based on it.

**Definition 2.7** (Determinantal point process). *Let  $\xi$  be a point process on a Polish locally compact space  $M$ ,  $\sigma$  be a measure on  $M$  and  $K : M \times M \rightarrow \mathbb{C}$  a continuous function. We say that  $\xi$  is a determinantal point process with kernel  $K : M \times M \rightarrow \mathbb{C}$  with respect to  $\sigma$  if the  $k$ -th correlation functions are*

$$\rho_k(x_1, \dots, x_k) = \det \left( K(x_i, x_j)_{i,j \in \{1, \dots, k\}} \right)$$

for every  $k \geq 1$ .

As a simple remark we should say that if  $\xi$  is a determinantal point process with kernel  $K$  with respect to the measure defined by  $\phi(x)d\sigma(x)$  then  $\xi$  is a determinantal point process with kernel  $(x, y) \mapsto \sqrt{\phi(x)}K(x, y)\sqrt{\phi(y)}$  with respect to  $\sigma$ . Moreover, if  $K$  is a kernel for the determinantal point process  $\xi$  then  $(x, y) \mapsto \psi(x)K(x, y)\psi(y)^{-1}$  works also as a kernel for any function  $\psi : M \rightarrow \mathbb{C} \setminus \{0\}$ .

For more information on this subject we recommend [18, Chapter 4].

## 2.4 Convergence of determinantal point processes

What we enjoy the most is the interaction between the kernel and the process. We will state a very nice continuity property that will be strongly used in Chapter 4 and Chapter 5. This can be found in [90].

**Proposition 2.8** (Continuity). *Suppose that  $\{\xi_n\}_{n \in \mathbb{N}}$  is a sequence of determinantal point processes with a sequence of continuous kernels  $\{K_n : M \times M \rightarrow \mathbb{C}\}_{n \in \mathbb{N}}$  with respect to  $\sigma$ . If*

$$\lim_{n \rightarrow \infty} K_n = K$$

*uniformly on compact spaces then there exists a determinantal point process  $\xi$  with kernel  $K$  with respect to  $\sigma$  and*

$$\lim_{n \rightarrow \infty} \xi_n = \xi$$

*in law.*

For instance, if we take the the eigenvalues of a random matrix that belongs to the Ginibre ensemble we obtain a determinantal point process with kernel

$$K_n(z, w) = \sum_{k=0}^{n-1} \frac{z^k \bar{w}^k}{k!} e^{-|z|^2/2} e^{-|w|^2/2}$$

with respect to  $\ell_{\mathbb{C}}$ . Then, if  $n$  goes to infinity  $K_n$  goes to

$$K(z, w) = e^{z\bar{w}} e^{-|z|^2/2} e^{-|w|^2/2} = e^{-\frac{1}{2}|z-w|^2 + i\Im(z\bar{w})}$$

where  $\Im$  denotes the imaginary part. This limiting  $K$  defines a determinantal point process  $\xi$  that has a law invariant under isometries of the plane. Indeed, if  $|u| = 1$  and  $a \in \mathbb{C}$  then by taking the isometry

$$T : z \mapsto \bar{u}(z - a)$$

the image point process  $T_*\xi$  is determinantal with kernel

$$(z, w) \mapsto K(uz + a, uw + a)$$

with respect to  $\ell_{\mathbb{C}}$  and

$$K(uz + a, uw + a) = e^{i\Im(uz\bar{a})} K(z, w) e^{i\Im(a\bar{u}\bar{w})} = e^{i\Im(zu\bar{a})} K(z, w) e^{-i\Im(wu\bar{a})}.$$

For another example, if we take the eigenvalues of a matrix that belongs to the truncated Haar unitary ensemble we get a determinantal point process with kernel

$$K_n(z, w) = \frac{1}{\pi} \sum_{k=0}^{n-1} (k+1) z^k \bar{w}^k$$

with respect to the Lebesgue measure restricted to the unit disk  $\ell_{\mathbb{D}}$ . As  $n$  goes to infinity this kernel goes to

$$K(z, w) = \frac{1}{\pi} \sum_{k=0}^{\infty} (k+1) z^k \bar{w}^k = \frac{1}{\pi(1 - z\bar{w})^2}.$$

This is known as the Bergman kernel of the unit disk (see [68, Section 1.4]). It defines the orthogonal projection onto the space of square integrable holomorphic functions on  $\mathbb{D}$ . This description allows us to prove the invariance of the limiting process  $\xi$ . Indeed, if  $T : \mathbb{D} \rightarrow \mathbb{D}$  is a Möbius transformations of the disk or, equivalently, an isometry of the hyperbolic disk, then denoting  $T^{-1}(z) = \tilde{z}$  and  $T^{-1}(w) = \tilde{w}$  we have that the following function is a kernel for the image point process  $T_*\xi$

$$(z, w) \mapsto \tilde{K}(z, w) = T'(\tilde{z})^{-1} K(\tilde{z}, \tilde{w}) \overline{T'(\tilde{w})}^{-1}$$

where  $'$  denotes the complex derivative. We can prove that  $\tilde{K}$  also represents the orthogonal projection. Indeed,  $\tilde{K}$  is holomorphic in the first variable, antiholomorphic on the second and if  $f : \mathbb{D} \rightarrow \mathbb{C}$  is a holomorphic function then

$$\begin{aligned} \int_{\mathbb{D}} T'(\tilde{z})^{-1} K(\tilde{z}, \tilde{w}) \overline{T'(\tilde{w})}^{-1} f(w) d\ell_{\mathbb{C}}(w) &= \int_{\mathbb{D}} T'(\tilde{z})^{-1} K(\tilde{z}, \tilde{w}) \overline{T'(\tilde{w})}^{-1} f \circ T(\tilde{w}) |T'(\tilde{w})|^2 d\ell_{\mathbb{C}}(\tilde{w}) \\ &= \int_{\mathbb{D}} T'(\tilde{z})^{-1} K(\tilde{z}, \tilde{w}) (f \circ T(\tilde{w})) T'(\tilde{w}) d\ell_{\mathbb{C}}(\tilde{w}) \\ &= T'(\tilde{z})^{-1} (f \circ T(\tilde{z})) T'(\tilde{z}) \\ &= f \circ T(\tilde{z}) = f(z). \end{aligned}$$

We conclude that  $\tilde{K} = K$  and that the point process  $T_*\xi$  has the same law as the point process  $\xi$  so that the law of  $\xi$  is invariant under isometries of the hyperbolic disk. We explain why this process is naturally attached to the disk as a complex manifold in the next section.

## 2.5 A digression on the Riemann surface case

This subsection is added just for completeness and to try to have a more intrinsic understanding of the kernel  $K$  for the case of the hyperbolic disk.

**Definition 2.9** (Riemann surface). *Let  $M$  be a topological manifold and let  $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$  be an atlas where each chart takes values on  $\mathbb{C}$ . We say that this atlas is holomorphic if every transition map  $\varphi_\lambda \circ \varphi_\kappa^{-1}$  is holomorphic. In this case we call  $(M, \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda})$  a complex manifold of (complex) dimension one or Riemann surface.*

The sphere  $S^2$  is one of our most important examples of Riemann surfaces. The charts considered are the stereographic projection from the north pole and the conjugate of the stereographic projection from the south pole<sup>2</sup>. The transition map will be  $z \mapsto 1/z$ . Moreover, given a Riemann surface we may take an open subset and the restrictions of the charts to obtain new Riemann surfaces, called complex submanifolds, so that, for instance, the unit disk may be seen as a complex submanifold of  $S^2$ .

If  $M$  is a Riemann surface, in analogy to the real case we will say that a family of measurable maps  $\nu = \{\nu^\lambda : V_\lambda \rightarrow \mathbb{C}^*\}_{\lambda \in \Lambda}$  is a (measurable) *complex 1-form* if for every  $\kappa, \lambda \in \Lambda$

$$\nu_x^\kappa = \nu_y^\lambda \circ d(\varphi_\lambda \circ \varphi_\kappa^{-1})_x \quad (24)$$

for every  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$  and  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$ . In fact, in this case, since the derivative of the transition maps is  $\mathbb{C}$ -linear and because the dimension is one we can make a great simplification and say that a family of measurable functions  $F = \{F^\lambda : V_\lambda \rightarrow \mathbb{C}\}_{\lambda \in \Lambda}$  is a *complex 1-form* if for every  $\kappa, \lambda \in \Lambda$

$$F^\kappa(x) = (\varphi_\lambda \circ \varphi_\kappa^{-1})'(x) F^\lambda(y)$$

if  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$  and  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$  where  $'$  denotes the (usual) complex derivative. The great remark is that if  $F$  and  $G$  are two complex 1-forms then the family  $F\bar{G} = \{F^\lambda \bar{G}^\lambda\}_{\lambda \in \Lambda}$  behaves as

$$(F^\kappa \bar{G}^\kappa)(x) = \left| (\varphi_\lambda \circ \varphi_\kappa^{-1})' \right|^2(x) (F^\lambda \bar{G}^\lambda)(y)$$

whenever  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$  and  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$ . This tells us that we are able to define a measure out of  $F\bar{G}$ . We recommend [94, Chapter 5] for the standard approach.

<sup>2</sup>This conjugate stereographic projection from the south pole is the application from  $S^2 \setminus \{(0, 0, -1)\}$  to  $\mathbb{C}$  defined by  $(x, y, z) \mapsto \left(\frac{x}{1+z}, \frac{-y}{1+z}\right)$ .

**Definition 2.10** (Product volume measure). *The product measure of a complex 1-form by itself is the positive measure  $\mu_{F\bar{F}}$  such that for every non-negative smooth function  $f : M \rightarrow \mathbb{R}$*

$$\int_{U_\lambda} f d\mu_{F\bar{F}} = \int_{V_\lambda} f^\lambda(x)(F^\lambda\bar{F}^\lambda)(x) d\ell_{\mathbb{C}}(x).$$

*We say that  $F$  is square integrable if  $\mu_{F\bar{F}}$  is a finite measure.*

*The product volume measure of two square integrable complex 1-forms  $F$  and  $G$  is the complex measure  $\mu_{F\bar{G}}$  such that for every bounded smooth function  $f : M \rightarrow \mathbb{R}$*

$$\int_{U_\lambda} f d\mu_{F\bar{G}} = \int_{V_\lambda} f^\lambda(x)(F^\lambda\bar{G}^\lambda)(x) d\ell_{\mathbb{C}}(x).$$

*The inner product of two square integrable complex 1-forms  $F$  and  $G$  is defined by*

$$\langle F, G \rangle = \mu_{F\bar{G}}(M).$$

What is remarkable about this definition is that we do not need a reference measure for it to make sense. In other words, there is an intrinsic meaning for the product (volume measure and inner product) of two complex 1-forms. We will consider  $L^2(M)$ , the space of square-integrable complex 1-forms where we identify two of them if they are equal almost everywhere on each chart. As usual, this is a Hilbert space, but now it is intrinsically attached to the Riemann surface  $M$ . The notion of a kernel  $K$  turns out to be also well-defined on this particular setting.

**Definition 2.11** (Kernel). *A kernel is a family  $K = \{K^{\lambda, \tilde{\lambda}} : V_\lambda \times V_{\tilde{\lambda}} \rightarrow \mathbb{C}\}_{\lambda, \tilde{\lambda} \in \Lambda}$  such that*

$$K^{\kappa, \tilde{\kappa}}(x, \tilde{x}) = \overline{(\varphi_\lambda \circ \varphi_\kappa^{-1})'(x)} (\varphi_{\tilde{\lambda}} \circ \varphi_{\tilde{\kappa}}^{-1})'(\tilde{x}) K^{\lambda, \tilde{\lambda}}(y, \tilde{y})$$

*whenever  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$ ,  $\tilde{x} \in \varphi_{\tilde{\kappa}}(U_{\tilde{\lambda}} \cap U_{\tilde{\kappa}})$ ,  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$  and  $\tilde{y} = \varphi_{\tilde{\lambda}} \circ \varphi_{\tilde{\kappa}}^{-1}(\tilde{x})$ .*

Following the same pattern, we can take a kernel  $K$ , a 1-form  $F$  and construct the 1-form

$$K \cdot F = \int_M K(x, y) \bar{F}(y)$$

so that  $K$  may define an operator on complex 1-forms. A kernel defines an operator without any need of a reference measure! Moreover, by an analogous procedure we may define  $K(x, x)$  as a measure on  $M$  and  $\det(K(x_i, x_j)_{i, j \in \{1, \dots, k\}})$  as a measure on  $M^k$ . In summary, all the operations necessary to define a determinantal point process makes sense with this new object. There is a particular case of interest for us. We say that the complex 1-form  $F = \{F^\lambda : V_\lambda \rightarrow \mathbb{C}\}_{\lambda \in \Lambda}$  is holomorphic if every  $F^\lambda$  is holomorphic. Let us denote by  $L^2_{\mathcal{H}}(M)$  the space of square integrable holomorphic 1-forms usually called the *Bergman space*. This space is a closed subspace of  $L^2(M)$ , a fact that can be obtained by using the mean value inequalities for subharmonic functions.

**Definition 2.12** (Bergman process). *The kernel  $K$  that defines the orthogonal projection from  $L^2(M)$  onto  $L^2_{\mathcal{H}}(M)$  is called the Bergman kernel and the determinantal point process associated, the Bergman process  $\mathcal{B}_M$ .*

We suggest the reader to look at the first two sections of [67]. It includes the general case of a complex manifold on any dimension.

### 3 A model from physics

Now, we present one of the usual physical motivations for Gibbs measures and explain why we got rid of the kinetic energy term. In particular, we will explain the notion of free energy and state that the Gibbs measures are the minimizers of it. This is a key tool for Chapter 6.

### 3.1 Euclidean framework

Following physicist's notation, we consider  $\mathbb{R}^d$  as the configuration space which we may thought as the space of positions of a particle. To characterize completely its state we need to add what is known as the momentum  $p \in \mathbb{R}^d$  so that the state space, also known as phase space, is  $\mathbb{R}^d \times \mathbb{R}^d$  where the first coordinate is the position and the second is the momentum. Given a potential energy  $U : \mathbb{R}^d \rightarrow (-\infty, \infty]$  we define the total energy as

$$H(x, p) = \frac{1}{2m}p^2 + U(x)$$

where  $m$  denotes the mass of the particle.

Sometimes we may not know the exact position of the system but only have a probability distribution to describe it. When the temperature of the system is known but the total energy is not, a usual guess is the so called Gibbs measure  $\mathbb{P}$  given by

$$d\mathbb{P} = \frac{1}{\mathcal{Z}}e^{-\beta H}d\ell_{\mathbb{R}^{2d}}$$

where  $\beta$  is proportional to the inverse of the temperature and  $\mathcal{Z}$  is the normalization constant usually called partition function. By the definition of  $H$  we notice that  $\mathbb{P}$  is a product measure  $\mathbb{P}_1 \otimes \mathbb{P}_2$  where

$$d\mathbb{P}_1 \text{ is proportional to } e^{-\beta U}d\ell_{\mathbb{R}^d}$$

and

$$d\mathbb{P}_2 \text{ is proportional to } e^{-\beta \frac{p^2}{2m}}d\ell_{\mathbb{R}^d}.$$

The measure  $\mathbb{P}_2$  is a well known Gaussian probability measure so that we are going to be interested only on  $\mathbb{P}_1$ . If we consider  $n$  particles living on  $\mathbb{R}^d$  of mass  $m$  we should replace  $\mathbb{R}^d$  by  $(\mathbb{R}^d)^n$  and  $U$  by  $U_n : (\mathbb{R}^d)^n \rightarrow (-\infty, \infty]$ . The total energy  $H_n : (\mathbb{R}^d \times \mathbb{R}^d)^n \rightarrow (-\infty, \infty]$  of the system would be defined by

$$H_n(x_1, p_1, \dots, x_n, p_n) = \frac{1}{2m} \sum_{i=1}^n p_i^2 + U_n(x_1, \dots, x_n).$$

Again, we define the Gibbs measure  $\mathbb{P}$  by

$$d\mathbb{P} = \frac{1}{\mathcal{Z}_n}e^{-\beta H_n}d\ell_{\mathbb{R}^{2dn}}$$

where  $\mathcal{Z}_n$  is the normalization constant. Its projection  $\mathbb{P}_1$  to the configuration space  $(\mathbb{R}^d)^n$  (space of positions) would be such that

$$d\mathbb{P}_1 \text{ is proportional to } e^{-\beta U_n}d\ell_{\mathbb{R}^{dn}}.$$

In the Riemannian case the only thing to understand would be what momentum is. This may require some definitions that would take us away from the actual articles we want to present. Nevertheless, since we consider this a really nice setting, we will use some lines to mention it and refer the reader to some of the nice books on symplectic manifolds there are (for instance [78] or [9, Chapter 8]).

### 3.2 Riemannian framework

Suppose  $M$  is a Riemannian manifold of dimension  $d$ . If  $U : M \rightarrow (-\infty, \infty]$  is the potential energy, a particle located at  $x$  and with momentum  $p$  will have the total energy

$$H(x, p) = \frac{1}{2m}p^2 + U(x) \tag{25}$$



where  $m$  is the mass of the particle and the first term of the sum is the kinetic energy. Since the particle lives on a manifold, we should first say where the momentum  $p$  really lives. If the particle is located at  $x$  then its velocity lives on  $T_x M$ , the tangent space of  $M$  at  $x$ . This vector space has many different descriptions but we may keep in mind that it is formed by the possible values a vector field may take at  $x$ . The momentum, as being dual to the velocity, should live then on the dual space of  $T_x M$ , denoted by  $T_x^* M$ . Again, this is just formed by the possible values a 1-form may take at  $x$ . We shall give a more precise description. Consider  $\Lambda_x \subset \Lambda$  consisting of all indexes  $\lambda \in \Lambda$  such that  $x \in U_\lambda$ . A functional at  $x$  or a momentum based on  $x$  will be a family  $p = \{p^\lambda\}_{\lambda \in \Lambda_x}$  of elements of  $\mathbb{R}^{d^*}$  such that

$$p^\kappa = p^\lambda \circ d \left( \varphi_\lambda \circ \varphi_\kappa^{-1} \right)_{\varphi_\kappa(x)}$$

for every  $\lambda, \kappa \in \Lambda_x$ . A 1-form, as defined in (4), can be seen as a function from  $M$  to the so-called cotangent bundle  $T^* M = \cup_{x \in M} \{x\} \times T_x^* M$ . The smoothness (and the continuity) of the 1-forms is captured by the natural smooth structure (and topology) on  $T^* M$  given by the atlas  $\{\psi^\lambda : T^* U^\lambda \rightarrow V^\lambda \times \mathbb{R}^{d^*}\}_{\lambda \in \Lambda}$ , where  $T^* U^\lambda = \cup_{x \in U^\lambda} \{x\} \times T_x^* M$ , defined by

$$\psi^\lambda(x, p) = (\varphi^\lambda(x), p^\lambda).$$

If  $h$  is a Riemannian metric on  $M$  we can define, by duality, an inner product on each  $T_x^* M$ . Namely, we define

$$\langle p_1, p_2 \rangle_x = h_{\varphi^\lambda(x)}^\lambda(p_1^\lambda, p_2^\lambda) = (p_1^\lambda)^\top \left( h_{\varphi^\lambda(x)}^\lambda \right)^{-1} p_2^\lambda.$$

where in the right-hand side we are thinking  $h_{\varphi^\lambda(x)}^\lambda$  as a positive definite matrix and the  $p^\lambda$  as column vectors. We can say that by  $p^2$  we meant  $\langle p, p \rangle_x$  in (25) so that  $H$  is a well-defined application on  $T^* M$ . Given  $\beta > 0$  we will consider the Gibbs measure defined by

$$d\bar{\mathbb{P}} = \frac{1}{\mathcal{Z}} e^{-\beta H} d \text{vol}$$

where  $\text{vol}$  denotes the following volume measure on  $T^* M$ .

**Definition 3.1** (Volume measure on the cotangent bundle). *The volume measure  $\text{vol}$  is the positive measure on  $T^* M$  such that, for every non-negative smooth function  $f : T^* M \rightarrow \mathbb{R}$ ,*

$$\int_{T^* U^\lambda} f d \text{vol} = \int_{V^\lambda \times \mathbb{R}^{d^*}} f^\lambda d\ell_{\mathbb{R}^{2d}}.$$

If we insist on the ‘naturalness’ we know there is a standard symplectic structure on  $T^* M$ , the one used to define the dynamics given by a Hamiltonian, so that there exists a natural volume measure which is exactly the one defined above. The nice remark is that, for any  $\kappa > 0$ , the projection of  $\exp(-\kappa p^2) d \text{vol}$  to  $M$  is proportional to the volume measure on  $M$ , associated to the metric, that we shall call  $\sigma$ . So, the projection of  $\bar{\mathbb{P}}$  onto  $M$  is  $\mathbb{P}$  defined by

$$d\mathbb{P} = \frac{1}{\mathcal{Z}} e^{-\beta U} d\sigma.$$

Now, suppose we have a system of  $n$  particles on  $M$ . We may replace  $M$  by  $M^n$ , together with the induced product Riemannian structure, and  $T^* M$  by  $T^*(M^n) = (T^* M)^n$ . The measure induced by the symplectic structure on  $(T^* M)^n$  would be  $\text{vol}^{\otimes n}$ . Replacing  $U$  by  $U_n : M^n \rightarrow (-\infty, \infty]$  we may define the total energy by

$$H_n(x_1, p_1, \dots, x_n, p_n) = \frac{1}{2m} \sum_{i=1}^n p_i^2 + U_n(x_1, \dots, x_n).$$

The projection of the Gibbs measure on  $T^*M^n$  defined by

$$d\bar{\mathbb{P}}_n = \frac{1}{\bar{\mathcal{Z}}_n} e^{-\beta H_n} d\text{vol}^{\otimes n}$$

would be the probability measure  $\mathbb{P}_n$  given by

$$d\mathbb{P}_n = \frac{1}{\mathcal{Z}_n} e^{-\beta U_n} d\sigma^{\otimes n}$$

where  $\sigma$  is the volume measure of  $M$ .

### 3.3 The free energy

In any of those cases or in general if  $(M, \sigma)$  is a Polish measure space and if  $H : M \rightarrow (-\infty, \infty]$  is any (nice enough) measurable function we can define the free energy of a system in the state  $\mu \in \mathcal{P}(M)$ , where  $\mathcal{P}(M)$  denotes the space of probability measures on  $M$ , as

$$F_\beta(\mu) = \mathbb{E}_\mu[H] - \frac{1}{\beta} S(\mu)$$

where  $S(\mu)$  is the physicist's entropy defined by

$$S(\mu) = - \int_M \rho \log \rho d\sigma \quad \text{if} \quad d\mu = \rho d\sigma$$

and

$$S(\mu) = -\infty \quad \text{if it does not make sense.}$$

As a standard principle of thermodynamics, a system at equilibrium tends to minimize the energy and maximize the entropy. At an inverse temperature  $\beta$  this is interpreted as the system wanting to minimize the free energy  $F_\beta$ . One of the possible and really interesting interpretations of the Gibbs measure is that  $F_\beta$  has a unique minimizer which is the Gibbs measure  $\mathbb{P}$  given by

$$d\mathbb{P} = \frac{1}{\mathcal{Z}} e^{-\beta H} d\sigma$$

and the partition function

$$\mathcal{Z} = \int_M e^{-\beta H} d\sigma$$

is related to the minimum  $F_\beta(\mathbb{P})$  by

$$F_\beta(\mathbb{P}) = -\log \mathcal{Z}.$$

This nice idea will be the *key structure* used when studying the large deviations in the setting of Gibbs measures, Chapter 6. Instead of working with general positive measures we will use a probability measure  $\sigma$  and write

$$D(\mu||\sigma) = -S(\mu).$$

This is the usual relative entropy or Kullback-Leibler divergence.

## 4 Questions

Our task will be to describe the limiting behavior of  $\mathbb{P}_n$  as  $n$  goes to infinity. First we will mention what kind of questions we may ask.

## 4.1 Macroscopic behavior

Suppose

$$(X_1, \dots, X_n) \sim \frac{1}{\mathcal{Z}_n} e^{-\beta_n H_n} d\sigma^{\otimes n}.$$

We would like to give conditions on the sequences  $\{H_n\}_{n \in \mathbb{N}}$  and  $\{\beta_n\}_{n \in \mathbb{N}}$  so that we can find a (perhaps random) probability measure  $\mu$  on  $M$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_i} = \mu. \quad (26)$$

The space of probability measures  $\mathcal{P}(M)$  on  $M$  is endowed here with the smallest topology such that the applications

$$\hat{f}(\mu) = \int_M f d\mu$$

are continuous for every bounded continuous function  $f : M \rightarrow \mathbb{R}$ . In the usual case of a deterministic limit we may look for a next-order expansion and search a sequence of positive numbers  $\{\alpha_n\}_{n \in \mathbb{N}}$  such that

$$\alpha_n \left( \frac{1}{n} \sum_{i=1}^n \delta_{X_i} - \mu \right) \text{ converges} \quad (27)$$

in some distributional sense. In Chapter 6 we will see that in a very general setting of non-attracting particles we can prove a large deviation principle that will imply, under some conditions, an almost sure convergence of the empirical measures. This large deviation principle will be a consequence of a Laplace principle which is the main philosophy used in [47]. Our result also applies in the context of random polynomials which was studied in [98] (see also [30]). The first large deviations approach can be found in the work of Ben Arous and Guionnet [16] who relate it to the ideas of free entropy introduced by Voiculescu [96] stressing the analogy to Sanov's theorem. Since then, there has been much activity on that area. We can see for instance [17], [60], [27], [61], [58], [35] and [48] listed in chronological order.

Moreover, in Chapter 3 and for the particular case of a Coulomb gas on a compact manifold we find non-asymptotic bounds to the probability of staying far from the limit, a subject sometimes known as concentration of measure. The Euclidean Coulomb case has been studied in [37] motivated by the one-dimensional logarithmic case from [76].

There has been much activity about the next-order expansion (27). For instance, for a one-dimensional logarithmic interaction it was studied in [64]. The two-dimensional determinantal case was studied first in [83] on the radial setting and then in [3] and [2] on a non-radial general setting. It was extended for general one-dimensional complex manifolds in [22]. Then, in [72] and [14] the two-dimensional Euclidean non-determinantal case was studied.

## 4.2 Outliers behavior

A second question is the following. If we have an open set  $A \subset M$  such that  $\mu(A) = 0$ , where  $\mu$  is obtained in (26), can we find a limiting point process on  $A$ ? More precisely, given a continuous function  $f : M \rightarrow \mathbb{R}$  supported on  $A$ , we would like to find a random variable  $\mathcal{B}_f$  such that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(X_i) = \mathcal{B}_f \quad (28)$$

in law. We will study in Chapter 4 and Chapter 5 the determinantal case in two dimensions and see how the Bergman process considered in [79] and mentioned in Subsection 2.4 appears. Methods of orthogonal polynomials such as the ones used in [4], [5] and [59] can be applied to study the general non-radial case but we wanted to emphasize the simplicity by studying the radial setting.

### 4.3 Microscopic behavior

This third question makes sense on a manifold so that we state it with that generality to stress out that the object obtained naturally lives on the tangent space. We see this only as a nice remark with maybe future importance. Nevertheless, since we are not actually going to use it there is no harm in forgetting  $\varphi$  and read it as if  $M$  where  $\mathbb{R}^d$ .

Consider a Riemannian manifold  $M$ . Take a point  $x \in M$  in the support of  $\mu$ , the limiting measure obtained in (26). Consider a diffeomorphism  $\varphi : U \subset M \rightarrow V \subset T_x M$  such that  $x \in U \mapsto 0 \in V$  and  $d\varphi_x = \text{Id}_{T_x M}$ . The question is to find a sequence  $\alpha_n$  such that for every compactly supported continuous function  $f$  on  $T_x M$  there exists a random variable  $\mathcal{G}_f$  with

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(\alpha_n \varphi(X_i)) = \mathcal{G}_f$$

where we have taken only the  $X_k$  that belong to  $U$ . We may notice that in case this limit exists for every  $f$  then the sequence  $\alpha_n$  is essentially unique as well as the random variables  $\mathcal{G}_f$ . Even though this is the general case, we shall study a system that may be thought as living on the complex plane so that no charts are needed.

We will see plenty of different behaviors in Chapter 4 and Chapter 5. For the first non-trivial microscopic behaviors see [49, Section 15.3]. The infinite Ginibre point process can be found as a microscopic limit in [19] where we can also find interesting generalizations of the Coulomb gas on larger dimensions that involve determinantal point processes. Further behaviors near the boundary of the support of the limiting macroscopic measure are studied in [4], [5] and [59].

## 5 Results

We shall state what we believe are the most important and interesting results of the following chapters. In order for this not to be so repetitive we shall first state a version on the Riemannian case when there is one.

### 5.1 Macroscopic behavior

We will state here some results that can be found in Chapter 6 and Chapter 3. Recall that  $D(\mu \parallel \sigma) = \int_M \rho \log \rho d\sigma$  if  $\rho$  is the density of  $\mu$  with respect to  $\sigma$  and  $D(\mu \parallel \sigma) = \infty$  whenever that does not make sense. Fix a smooth signed measure  $\Lambda$  of total mass one on the compact Riemannian manifold  $M$ . Recall that  $G$  denotes the Green function associated to  $\Lambda$  and that  $\sigma$  denotes the volume measure of  $M$  which we assumed to have total mass one. Consider  $(X_1, \dots, X_n)$  the Coulomb gas on  $M$  of  $n$  particles of charge  $q_n$ , at inverse temperature  $\beta_n$  and confined by the background charge  $\Lambda$ . Part of the following theorem is a consequence of Corollary 4.5 in Chapter 6.

**Theorem 5.1** (Convergence of empirical measures). *If  $n\beta_n q_n^2 \rightarrow \beta \in (0, \infty]$  then, almost surely,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_i} = \mu_{\text{eq}}$$

where  $\mu_{\text{eq}}$  is the minimizer of

$$\mu \in \mathcal{P}(M) \mapsto \frac{1}{2} \int_{M \times M} G(x, y) d\mu(x) d\mu(y) + \frac{1}{\beta} D(\mu \parallel \sigma).$$

*Proof.* It is a consequence of Theorem 5.4 and 5.5 below. To see that it satisfy the assumptions required by those theorems we refer to Corollary 4.5 in Chapter 6. To see that the functional

has a unique minimizer we use that  $D(\cdot\|\sigma)$  is strictly convex in the (convex) set where it is finite and

$$\mu \mapsto \frac{1}{2} \int_{M \times M} G(x, y) d\mu(x) d\mu(y)$$

is also strictly convex in the (convex) set where it is finite. The latter fact is a combination of Proposition 4.8 in Chapter 6 and Lemma 3.2 in Chapter 3.  $\square$

If we assume  $\beta < \infty$  there is an interesting characterization of the minimizer. It can be found in Theorem 4.6 in Chapter 6. We state here its direct consequence.

**Theorem 5.2** (Convergence for positive temperature). *If  $n\beta_n q_n^2 \rightarrow \beta \in (0, \infty)$  then, almost surely,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_i} = \mu_{\text{eq}}$$

where  $\mu_{\text{eq}} \in \mathcal{P}_{C^\infty}(M)$  has a density  $\rho$  with respect to  $\sigma$  that satisfies

$$\frac{1}{\beta} \Delta \log \rho = \mu_{\text{eq}} - \Lambda.$$

*Proof.* It is a consequence of Theorem 5.1 above and Theorem 4.6 in Chapter 6 whose proof works in any dimension.  $\square$

In particular, we know what happens in the curvature background model on two dimensions. More precisely, if  $R$  is the scalar curvature of  $M$ , we choose  $\Lambda$  such that

$$d\Lambda = \frac{R}{4\pi\chi} d\sigma$$

where  $\chi$  denotes the Euler characteristic of  $M$ . If the manifold is the sphere we recover the models defined in Subsection 1.4. If the manifold is different from the sphere and the torus we obtain the following special case which is mentioned as a remark in Chapter 6.

**Corollary 5.3** (Constant curvature limit). *Suppose  $M$  is two-dimensional and that its genus is greater than one. Denote by  $\chi$  its Euler characteristic. If  $n\beta_n q_n^2 \rightarrow -4\pi\chi$  then, almost surely,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_i} = \mu_{\text{eq}}$$

where  $\mu_{\text{eq}} \in \mathcal{P}_{C^\infty}(M)$  has constant curvature.

*Proof.* It is a consequence of Theorem 5.2 and the interpretation in terms of the scalar curvature given in Remark 4.7 in Chapter 6.  $\square$

We consider this a very nice result because it does not depend on the metric we started with. The main theorem that allows us to conclude these convergences is contained in Theorem 1.2 in Chapter 6. The positive temperature part is the following.

**Theorem 5.4** (Laplace principle for positive temperature). *Let  $M$  be a Polish space and consider a sequence  $\{H_n : M^n \rightarrow (-\infty, \infty]\}_{n \in \mathbb{N}}$  of measurable functions uniformly bounded from below. Suppose that*

- for every  $\mu \in \mathcal{P}(M)$  there exists

$$H(\mu) := \lim_{n \rightarrow \infty} \mathbb{E}_{\mu^{\otimes n}} [H_n]$$

- whenever the empirical measures of  $\mathbf{x}_n \in M^n$  converge to some  $\mu \in \mathcal{P}(M)$  we have

$$\liminf_{n \rightarrow \infty} H_n(\mathbf{x}_n) \geq H(\mu).$$

Then, if  $\lim_{n \rightarrow \infty} \beta_n = \beta \in (0, \infty)$  and if  $i_n : M^n \rightarrow \mathcal{P}(M)$  denotes the application defined by  $i_n(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$  we have

$$\lim_{n \rightarrow \infty} \frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f \circ i_n} e^{-n\beta_n H_n} d\sigma^{\otimes n} = - \inf_{\mu \in \mathcal{P}(M)} \left\{ f(\mu) + H(\mu) + \frac{1}{\beta} D(\mu \| \sigma) \right\}$$

for every bounded continuous function  $f : \mathcal{P}(M) \rightarrow \mathbb{R}$ .

This theorem holds whenever  $\beta_n$  has a finite and positive limit. Nevertheless, the usual random matrix models involve a sequence  $\{\beta_n\}_{n \in \mathbb{N}}$  that goes to infinity. This requires two further assumptions as we state now. It is a simplified version of part of Theorem 1.2 in Chapter 6.

**Theorem 5.5** (Laplace principle for zero temperature). *Suppose the same hypotheses as in Theorem 5.4 except that now  $\beta_n \rightarrow \infty$ . Suppose also that*

- for every  $\mu \in \mathcal{P}(M)$  such that  $H(\mu) < \infty$  there exists a sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  that converges to  $\mu$  such that

$$D(\mu_n \| \sigma) < \infty \text{ for every } n \text{ and } \lim_{n \rightarrow \infty} H(\mu_n) = H(\mu)$$

- if  $\mathbf{x}_n$  belongs to  $M^n$  and  $\liminf_{n \rightarrow \infty} H_n(\mathbf{x}_n) < \infty$  then the empirical measures of a subsequence converges.

Then

$$\lim_{n \rightarrow \infty} \frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f \circ i_n} e^{-n\beta_n H_n} d\sigma^{\otimes n} = - \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + H(\mu)\}$$

for every bounded continuous function  $f : \mathcal{P}(M) \rightarrow \mathbb{R}$ .

In fact, a slight modification can be made to be able to treat the case of random polynomials and conditioned Coulomb gases. The precise statement can be found in Chapter 6. We would like to add that the main step in the proof is the property discussed in Subsection 3.3, the characterization of the Gibbs measure as a minimizer of the free energy.

In Chapter 3 we have studied the non-asymptotic behavior for Coulomb gases. Recall that the Kantorovich-Wasserstein distance  $W_1$  is a distance between probability measures defined by

$$W_1(\mu, \nu) = \sup \left\{ \int_M f d\mu - \int_M f d\nu : f \in C^\infty(M) \text{ and } \forall x \in M, h(\nabla f, \nabla f)(x) \leq 1 \right\}.$$

Let  $\mu_{eq}$  be the minimizer of the macroscopic energy  $\mu \mapsto \int_{M \times M} G(x, y) d\mu(x) d\mu(y)$ . We obtain the following as a consequence of Theorem 1.2 in Chapter 3.

**Theorem 5.6** (Concentration inequality). *If the dimension of  $M$  is  $d = 2$ , there exists a constant  $C > 0$  that depends only on the Riemannian manifold  $M$  and on  $\Lambda$  such that*

$$\mathbb{P} \left( W_1 \left( \frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \mu_{eq} \right) \geq r \right) \leq \exp \left( -\beta q^2 n^2 \frac{r^2}{4} + \frac{\beta q^2}{8\pi} n \log(n) + nD(\mu_{eq} \| \sigma) + C\beta q^2 n \right)$$

*If the dimension of  $M$  is  $d \geq 3$ , there exists a constant  $C > 0$  that depends only on the Riemannian manifold  $M$  and on  $\Lambda$  such that*

$$\mathbb{P} \left( W_1 \left( \frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \mu_{eq} \right) \geq r \right) \leq \exp \left( -\beta q^2 n^2 \frac{r^2}{4} + nD(\mu_{eq} \| \sigma) + C\beta q^2 n^{2(d-1)/d} \right).$$

One of the important aspects of this concentration result is that the correction term does not depend on  $r$  and that it has a simple dependence on  $\beta q^2$ .

## 5.2 Outliers behavior

The results stated in this subsection may be found in Chapter 4 and Chapter 5 written on the complex plane. We will start by giving the interpretation on the sphere as a Coulomb gas on the curvature background, a model discussed in Subsection 1.4. Let  $h$  be a Riemannian metric on  $S^2$  of unit volume and invariant under rotations around the  $z$ -axis. Suppose the scalar curvature  $R_h$  satisfies the following hypothesis:

- $R_h \geq 0$  everywhere,
- $R_h = 0$  on the lower hemisphere and
- the support of  $R_h$  contains the equator.

Denote by  $\alpha$  the volume measure associated to  $h$ .

**Theorem 5.7** (Limit on the lower hemisphere). *Let  $(Y_1, \dots, Y_n)$  be a Coulomb gas on  $S^2$  of unit charged particles, at inverse temperature  $\beta = 4\pi$  and confined by the background  $(8\pi)^{-1}R_h d\alpha$ . Denote the open lower hemisphere by  $D$  and take the point process on  $D$  given by*

$$\xi_n = \sum_{i=1}^n 1_{Y_i \in D} \delta_{Y_i}.$$

We have the convergence in law

$$\lim_{n \rightarrow \infty} \xi_n = \mathcal{B}_D$$

where  $\mathcal{B}_D$  is the Bergman process on  $D$  (see Definition 2.12)

This is a reformulation of Theorem 1.5 in Chapter 4 which may be seen as a consequence of the following result in the complex plane.

**Theorem 5.8** (Limit on the disk). *Let  $V : [0, \infty) \rightarrow \mathbb{R}$  be a non-negative measurable function such that*

- $\liminf_{r \rightarrow \infty} \{V(r) - \log r\} > -\infty$ ,
- $V(r) > 0$  if  $r > 1$  and
- $V(r) = 0$  if  $r \leq 1$ .

*If  $(Z_1, \dots, Z_n)$  is a Coulomb gas on  $\mathbb{C}$  of particles of charge  $1/(n+1)$ , at inverse temperature  $4\pi(n+1)$  and confined by the potential  $z \mapsto V(|z|)$  then we have the convergence in law*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n 1_{|Z_i| < 1} \delta_{Z_i} = \mathcal{B}_{\mathbb{D}}$$

where  $\mathbb{D}$  denotes the unit disk and  $\mathcal{B}_{\mathbb{D}}$  the determinantal point process associated to the Bergman kernel on  $\mathbb{D}$  and the Lebesgue measure.

*Proof.* It is a particular case of Theorem 2.1 in Chapter 5. □

This theorem is a consequence of the limiting behavior of the corresponding kernels. Then, as we said before, Theorem 5.7 is obtained if we identify  $S^2$  minus the north pole and  $\mathbb{C}$  so that the lower hemisphere will be the unit disk on  $\mathbb{C}$ . If we used the identification between  $S^2$  minus the south pole and  $\mathbb{C}$  given by the other stereographic projection the lower hemisphere would be the complement of the unit disk. In this way we have a duality between close and far particles. This is very much exploited in Chapter 4 and Chapter 5, and plenty of different

behaviors are obtained for the farthest particle by understanding the behavior of the nearest particle to zero.

We have been very much interested on the degenerate case when  $R_h d\alpha$  would be the uniform measure on the equator mentioned on Subsection 1.5. The macroscopic behavior is well-known and can be obtained from large deviation principles. Suppose  $Z_1, \dots, Z_n$  form either the Coulomb gas or the zeros of the random polynomial of Subsection 1.5 then

$$\frac{1}{n} \sum_{i=1}^n \delta_{Z_i} \rightarrow \text{uniform measure on } S^1.$$

We are able to prove that far from the unit circle their behavior is also the same and involves two independent Bergman processes  $\mathcal{B}_{\mathbb{D}}$  and  $\mathcal{B}_{\mathbb{C} \setminus \overline{\mathbb{D}}}$ . The following result is a simple reformulation of the theorems in Subsection 4.1 of Chapter 5.

**Theorem 5.9** (Limit far from the circle). *Suppose  $Z_1, \dots, Z_n$  form either the determinantal Coulomb gas or the zeros of the random polynomial of Subsection 1.5. Let  $f : \mathbb{C} \rightarrow \mathbb{R}$  be a continuous function with a finite limit at infinity such that  $\text{supp} f \cap S^1 = \emptyset$ . Then*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(Z_i) = \mathcal{B}_{\mathbb{D}}(f|_{\mathbb{D}}) + \mathcal{B}_{\mathbb{C} \setminus \overline{\mathbb{D}}}(f|_{\mathbb{C} \setminus \overline{\mathbb{D}}})$$

in law, where the summands in the right-hand side are independent.

The proof relies on an analysis of the kernels in the Coulomb gas case and on a decomposition of the random polynomials as a sum of two independent random polynomials in the other case.

### 5.3 Microscopic behavior

Continuing with the models mentioned in Subsection 1.5, we can see that they show different behaviors near the unit circle as the next two theorems imply. The following results can be found in Subsection 4.2 of Chapter 5. The first theorem is given just for completeness and because it is a case where very explicit calculations can be made. There are already generalizations in [91].

**Theorem 5.10** (Coulomb gas case). *Suppose  $(Z_1, \dots, Z_n)$  is the determinantal Coulomb gas of Subsection 1.5. Then*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \delta_{n(Z_{i-1})} = X$$

where  $X$  is a determinantal point process on  $\mathbb{C}$  with kernel

$$K(z, w) = \frac{e^{\max\{\Re(z), 0\} + \max\{\Re(w), 0\}}}{\pi} \int_0^1 t(1-t)e^{(z+\bar{w})t} dt,$$

where  $\Re$  denotes the real part, and with respect to the Lebesgue measure on  $\mathbb{C}$ .

**Theorem 5.11** (Random polynomial case). *Suppose  $\{Z_1, \dots, Z_n\}$  is the set of zeros of the random polynomial of Subsection 1.5. Then*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \delta_{n(Z_{i-1})} = X$$

where  $X$  is formed by the zeros of the Gaussian analytic function with covariance given by

$$K(z, w) = \int_0^1 e^{(z+\bar{w})t} dt.$$

The latter theorem is stated in Chapter 5 for not necessarily Gaussian laws for the coefficients and can be easily generalized for other random polynomials. Theorem 5.10 is proved by an analysis of the limiting kernel and Theorem 5.11 is proved by an analysis of the limiting covariances.



## 6 Open questions

To finish this chapter we present some of the questions we would like to find an answer to.

### 6.1 Fluctuations on a uniform background

A question we have not really touched is the one of fluctuations the easiest case of which would be the following. Let  $(Z_1, \dots, Z_n)$  be a Coulomb gas on  $S^2$  of unit charged particles, at (constant) inverse temperature  $\beta$  and confined by the uniform background charge  $\sigma$ . By the large deviation principle we know that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{Z_i} = \sigma$$

almost surely. It would be nice to find the following limit

$$\lim_{n \rightarrow \infty} \left( \sum_{i=1}^n \delta_{Z_i} - n\sigma \right).$$

By intuition and by analogy to the results on [83], [3], [2], [22], [72] and [14] we expect the limit to be related to a Gaussian free field.

### 6.2 Outliers on different spaces and temperatures

In the same spirit as the fluctuations question, we consider, for instance, the Coulomb gas model of Subsection 1.5. We may ask ourselves what could be obtained for

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n 1_{|Z_i| < 1} \delta_{Z_i}$$

in the non-determinantal setting. Moreover, in the usual determinantal setting but in non-radial backgrounds it should be possible to find the limit by studying the asymptotics of the orthogonal polynomials (such as in [4], [5] and [59]). Keeping the same  $\beta$  we hope to find a way to understand what happens in other surfaces where the Coulomb gas is never determinantal.

### 6.3 Macroscopic behavior for negative temperatures

Having in mind Corollary 5.3 we may ask what happens for the sphere  $S^2$ . The problem in that case was that its Euler characteristic is 2 so that the temperature would have to be negative to obtain the constant curvature limit. Nevertheless we can make sense of the system by a simple adaptation of the argument in [33]. We obtain that

$$\int_{(S^2)^n} \exp \left( -\frac{\beta}{n} \sum_{i < j} G(x_i, x_j) \right) d\sigma^{\otimes n} \in (0, \infty) \text{ if and only if } \beta > -8\pi. \quad (29)$$

where  $G(x, y) = -\frac{1}{2\pi} \log |x - y|_{\mathbb{R}^3}$ . So, the task would be to study the Coulomb gas  $(Z_1, \dots, Z_n)$  at inverse temperature  $\beta_n/n$  when  $\beta_n \downarrow -8\pi$ . Since there is not a unique constant curvature metric on a given conformal class<sup>3</sup>, it may not be so obvious what to expect. In fact, by using (29) we can find a sequence  $\beta_n \downarrow -8\pi$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{Z_i} = \delta_Y$$

<sup>3</sup> Indeed, the space of those metrics when seen as a subset of  $\mathcal{P}(S^2)$  is topologically a 3-dimensional open ball.

in law, where  $Y \sim \sigma$ . Moreover, by an adaptation of the arguments of [33] or by the large deviations studied by Berman in [21] the empirical measure still converges to  $\sigma$  when  $\beta > -8\pi$  and so we can find a sequence  $\beta_n \downarrow -8\pi$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{Z_i} = \sigma$$

in law. This confirms that the question may not be a trivial one and that there is not a unique behavior.



# Chapter 2

## Introduction (version française)

Cette introduction se compose de six parties. Les trois premières parties présentent les modèles et mettent en évidence trois structures qui sont exploitées dans les chapitres suivants. Plus précisément, la notion de fonction de Green est explorée dans la première section et constitue la structure clé utilisée dans le chapitre 3. La structure déterminantale, présentée dans la deuxième section, est la clé des chapitres 4 et 5. La propriété physique de minimisation de l'énergie libre peut être considérée comme le point de vue principal du chapitre 6 et elle est présentée dans la troisième section. Les quatrième et cinquième sections décrivent les questions que nous pouvons nous poser et les résultats principaux que nous avons obtenus. Ce chapitre introductif se termine en énonçant quelques questions ouvertes qui pourront faire l'objet de recherches futures.

### 1 Gaz de Coulomb

Nous présentons la fonction de Green, la *structure clé* utilisée dans le chapitre 3. Pour la clarté de l'exposition, nous allons donner une brève introduction aux variétés lisses de dimension finie et des références pour les parties non utilisées. Ensuite, nous présentons les ensembles  $\beta$  sur la droite réelle ainsi que trois autres modèles des matrices aléatoires associés aux trois géométries simplement connexes homogènes de dimension deux. Enfin, un modèle spécial de gaz de Coulomb sur la sphère est présenté, ainsi que sa relation avec le gaz de Coulomb habituel et un cas particulier considéré important car beaucoup de calculs explicites peuvent être faits.

Nous commençons par donner la notion de gaz de Coulomb sur des espaces euclidiens.

#### 1.1 Sur l'espace euclidien

Soit  $G : \mathbb{R}^d \times \mathbb{R}^d \rightarrow (-\infty, \infty]$  défini par

$$G(x, y) = -\frac{1}{2\pi} \log |x - y|$$

si  $d = 2$  et

$$G(x, y) = \frac{c_d}{|x - y|^{d-2}}$$

si  $d \geq 3$  où  $c_d^{-1}$  est  $d-2$  fois l'aire de  $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ . Cette fonction est l'interaction électrostatique en dimension  $d$ . Plus précisément, étant donné  $x \in \mathbb{R}^d$ , nous pouvons définir la fonction  $G_x : \mathbb{R}^d \rightarrow (-\infty, \infty]$  par  $G_x(y) = G(x, y)$ . Cette fonction est solution de

$$\Delta G_x = -\delta_x \tag{1}$$

où  $\delta_x$  est la mesure de Dirac au point  $x$  et

$$\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$$

est le laplacien usuel. Cela est la fameuse loi de Gauss appliquée à une particule ponctuelle chargée (en choisissant les unités appropriées). Nous définissons  $H_n : (\mathbb{R}^d)^n \rightarrow (-\infty, \infty]$ , l'énergie de  $n$  particules, chacune de charge  $q$ , par

$$H_n(x_1, \dots, x_n) = \sum_{i < j} q^2 G(x_i, x_j).$$

Comme l'énergie est invariante sous l'action de  $\mathbb{R}^d$  sur  $(\mathbb{R}^d)^n$ , toute mesure de probabilité naturelle définie par cette énergie devrait également être invariante par ces translations, mais cela n'a aucun sens dans l'espace euclidien. Nous devons briser la symétrie ou, plus précisément, utiliser un mécanisme de confinement. Nous y parvenons en ajoutant un potentiel

$$V : \mathbb{R}^d \rightarrow (-\infty, \infty]$$

dans l'énergie pour obtenir

$$H_n(x_1, \dots, x_n) = \sum_{i < j} q^2 G(x_i, x_j) + \sum_{i=1}^n qV(x_i).$$

Nous pouvons ainsi définir le gaz de Coulomb de  $n$  particules.

**Définition 1.1** (Gaz de Coulomb euclidien). *Soit  $V : \mathbb{R}^d \rightarrow (-\infty, \infty]$  une fonction mesurable et soient  $q, \beta > 0$ . Nous disons qu'un élément aléatoire  $(X_1, \dots, X_n)$  de  $(\mathbb{R}^d)^n$  est un gaz de Coulomb sur  $\mathbb{R}^d$  de particules de charge  $q$ , à la température inverse  $\beta$  et confiné par le potentiel  $V$  s'il suit la loi  $\mathbb{P}_n$  définie par*

$$d\mathbb{P}_n = \frac{1}{\mathcal{Z}_n} e^{-\beta H_n} d\ell_{(\mathbb{R}^d)^n}$$

où  $\ell_{(\mathbb{R}^d)^n}$  est la mesure de Lebesgue sur  $(\mathbb{R}^d)^n$  et

$$\mathcal{Z}_n = \int_{(\mathbb{R}^d)^n} e^{-\beta H_n} d\ell_{(\mathbb{R}^d)^n} \in (0, \infty).$$

## 1.2 Sur une variété

Maintenant, nous étendons cette notion à un autre type d'espace. Nous allons adopter une approche physique pour les définitions. Il est naturel de penser, au moins macroscopiquement, que l'espace dans lequel nous vivons est euclidien. Néanmoins, comme nous ne pouvons en voir qu'une partie, tout ce que nous pouvons dire, c'est que cela semble être euclidien près de nous. Ceci peut être formalisé à travers la notion de variété que nous rappelons.

Considérons un espace polonais  $M$ . Nous disons que  $M$  est une *variété topologique* s'il existe un recouvrement ouvert  $\{U_\lambda\}_{\lambda \in \Lambda}$  de  $M$  et une famille d'homéomorphismes

$$\varphi_\lambda : U_\lambda \rightarrow V_\lambda \subset \mathbb{R}^d$$

de  $U_\lambda$  dans un ensemble ouvert  $V_\lambda$  de  $\mathbb{R}^d$ . Chaque application  $\varphi_\lambda$  est appelée une *carte* et la famille  $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$  s'appelle un *atlas*. En fait, cet espace polonais peut être vu comme des ensembles ouverts de  $\mathbb{R}^d$  (les ensembles  $V_\lambda$ ) avec les identifications données par les *applications de changement de cartes*

$$\varphi_\lambda \circ \varphi_\kappa^{-1} : \varphi_\kappa(U_\lambda \cap U_\kappa) \rightarrow \varphi_\lambda(U_\lambda \cap U_\kappa).$$

Ces applications nous indiquent comment comparer deux descriptions possibles de l'espace. Maintenant, si nous voulons imiter la description (1) de  $G$ , nous devons savoir comment dériver les fonctions sur  $M$ . Pour cela, nous allons regarder  $M$  à travers des cartes afin que, par exemple, une application soit différentiable sur  $U_\lambda$  si elle est différentiable lorsque nous l'identifions avec  $V_\lambda$ . Le problème qui apparaît est qu'une application peut être différentiable sous une identification et pas sous une autre. C'est là que nous devons faire un choix.

**Définition 1.2** (Variété lisse). *Soit  $M$  une variété topologique et  $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$  un atlas. Nous disons que cet atlas est lisse si chaque application de changement de cartes  $\varphi_\lambda \circ \varphi_\kappa^{-1}$  est lisse. Dans ce cas, nous appelons  $(M, \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda})$  une variété lisse de dimension  $d$ .*

En fait, la définition habituelle de variété lisse consiste à choisir un atlas lisse maximal. Comme nous n'en aurons pas besoin ici, nous continuerons d'utiliser la définition 1.2 et nous nous référerons à [74, Chapitre 1] pour la définition habituelle. Une variété lisse peut être considérée comme des ensembles ouverts de  $\mathbb{R}^d$  collés par les applications de changement de cartes lisses  $\varphi_\lambda \circ \varphi_\kappa^{-1}$ . Donc, si  $(M, \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda})$  est une variété lisse, nous pouvons définir de nouvelles notions simplement en utilisant ces cartes. Nous donnons les exemples dont nous aurons besoin et référons à [74] pour une introduction plus complète.

Nous disons que  $f : M \rightarrow \mathbb{R}$  est une *fonction lisse* si pour chaque  $\lambda \in \Lambda$  la fonction

$$f^\lambda = f \circ \varphi_\lambda^{-1} : V_\lambda \rightarrow \mathbb{R}$$

est lisse. Notons que cette notion a du sens car l'atlas est lisse. En fait, si nous choisissons d'oublier  $M$  et que nous le considérons comme des ensembles ouverts identifiés, nous pouvons dire qu'une fonction lisse est une famille  $\{f^\lambda : V_\lambda \rightarrow \mathbb{R}\}_{\lambda \in \Lambda}$  de fonctions lisses telles que

$$f^\kappa = f^\lambda \circ (\varphi_\lambda \circ \varphi_\kappa^{-1}). \quad (2)$$

Quelle serait alors la dérivée de  $f$ ? Intuitivement, elle devrait être donnée par la famille des dérivées  $\{df^\lambda : V_\lambda \rightarrow \mathbb{R}^{d*}\}_{\lambda \in \Lambda}$  où  $\mathbb{R}^{d*}$  désigne le dual de  $\mathbb{R}^d$ . Mais en dérivant (2) dans le point  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$  et si  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$  nous obtenons

$$df_x^\kappa = df_y^\lambda \circ d(\varphi_\lambda \circ \varphi_\kappa^{-1})_x. \quad (3)$$

Les objets satisfaisant la relation (3) sont appelés des 1-formes. Plus précisément, une famille d'applications lisses  $\nu = \{\nu^\lambda : V_\lambda \rightarrow \mathbb{R}^{d*}\}_{\lambda \in \Lambda}$  est dite une *1-forme* si pour chaque  $\kappa, \lambda \in \Lambda$

$$\nu_x^\kappa = \nu_y^\lambda \circ d(\varphi_\lambda \circ \varphi_\kappa^{-1})_x \quad (4)$$

quels que soient  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$  et  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$ . De même, un champ de vecteurs  $X$  est une famille d'applications lisses  $\{X^\lambda : V_\lambda \rightarrow \mathbb{R}^d\}_{\lambda \in \Lambda}$  telle que

$$X^\lambda(y) = \left[ d(\varphi_\lambda \circ \varphi_\kappa^{-1})_x \right] X^\kappa(x).$$

Notez qu'il n'y a toujours pas de notion de gradient d'une fonction. Pour cela, nous avons besoin d'un moyen d'identifier les 1-formes aux champs de vecteurs. Un de ces moyens est ce que nous appelons une métrique riemannienne. Notons par  $\mathcal{I}$  l'espace des produits scalaires de  $\mathbb{R}^d$  qui peut être vu comme l'ensemble ouvert des matrices  $d \times d$  symétriques positives définies.

**Définition 1.3** (Métrique riemannienne). *Une famille d'applications lisses  $\{h^\lambda : V_\lambda \rightarrow \mathcal{I}\}_{\lambda \in \Lambda}$  est une métrique riemannienne si*

$$h_y^\lambda = \left[ d(\varphi_\lambda \circ \varphi_\kappa^{-1})_x \right]_* h_x^\kappa$$

pour chaque  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$  et  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$  où  $*$  désigne l'application induite en  $\mathcal{I}$ . Si nous pensons  $\mathcal{I}$  comme un ensemble de matrices la condition devient

$$h_y^\lambda = \left[ \text{Jac}(\varphi_\lambda \circ \varphi_\kappa^{-1})^\top(x) \right]^{-1} h_x^\kappa \left[ \text{Jac}(\varphi_\lambda \circ \varphi_\kappa^{-1})(x) \right]^{-1}$$

où  $\text{Jac}$  désigne la matrice jacobienne et  $\top$  désigne la transposée.

Une variété dotée d'une métrique riemannienne est appelée une *variété riemannienne*. L'un des objectifs de cette définition est de donner un sens au produit de deux champs de vecteurs  $X$  et  $Y$ . Étant donné une métrique riemannienne  $h$ , nous voulons que la fonction  $h(X, Y)$  soit définie en utilisant les cartes. Plus précisément, comme la famille  $\{h(X, Y)^\lambda : V_\lambda \rightarrow \mathbb{R}\}_{\lambda \in \Lambda}$  donnée par  $h(X, Y)^\lambda(x) = h_x^\lambda(X^\lambda(x), Y^\lambda(x))$ . Cela définit une fonction lisse au sens de (2). De plus, la métrique riemannienne nous permet, par dualité, de définir le gradient  $\nabla f$  d'une fonction lisse  $f$ . Pour plus d'informations à ce sujet, nous encourageons le lecteur à consulter [43].

Nous avons à peu près couvert les bases et nous pouvons suivre le même schéma pour définir de nombreux objets différents. Néanmoins, nous ne sommes pas encore arrivés à la notion de laplacien. Nous allons suivre une approche non standard et définir d'abord la mesure de volume associée à une métrique riemannienne. L'intuition derrière une métrique riemannienne est qu'elle mesure des distances infinitésimales. Par exemple, si  $p \in M$  est vu dans les coordonnées comme  $x = \phi^\lambda(p)$  et que  $X$  est un champ vectoriel, nous pouvons penser à  $X^\lambda(x)dt$  en tant que vecteur infinitésimal reliant  $x$  et  $x + X^\lambda(x)dt$ . Ensuite, la distance entre  $x$  et  $x + X^\lambda(x)dt$  est  $\sqrt{h^\lambda(X^\lambda, X^\lambda)}dt$ . Que serait un volume infinitésimal? Si  $\{X_1^\lambda(x), \dots, X_m^\lambda(x)\}$  est une base orthonormée par rapport à  $h_x^\lambda$ , nous pourrions alors former l'hypercube 'infinitésimal' induit par le sommet  $x$  et les vecteurs infinitésimaux  $X_1^\lambda(x)dt_1, \dots, X_m^\lambda(x)dt_m$ . Son volume devrait être  $dt_1 \dots dt_m$ . Mais la base canonique  $\{e_1, \dots, e_m\}$  n'est pas nécessairement orthonormée par rapport à  $h_x^\lambda$  de sorte que le volume du hypercube induit par le sommet  $x$  et les vecteurs infinitésimaux  $e_1dt_1, \dots, e_mdt_m$  n'est pas  $dt_1 \dots dt_m$ . Au lieu de cela, il est donné par  $\sqrt{\det h_x^\lambda}$  où nous pensons  $h$  en tant que matrice pour donner un sens au déterminant. Cela nous amène à la définition suivante.

**Définition 1.4** (Mesure de volume). *La mesure de volume  $\sigma$  associée à la métrique  $h$  est la mesure positive sur  $M$  telle que, pour chaque fonction lisse positive  $f : M \rightarrow \mathbb{R}$ ,*

$$\int_{U_\lambda} f d\sigma = \int_{V_\lambda} f^\lambda(x) \sqrt{\det h_x^\lambda} d\ell_{\mathbb{R}^d}(x).$$

Nous nous intéresserons au cas compact où  $\sigma$  est une mesure bornée que nous supposons de masse totale un en multipliant la métrique par une constante. Si le lecteur souhaite approfondir la notion d'intégration sur des variétés en utilisant des formes différentielles, nous recommandons [97, Chapitre 4].

Nous définissons maintenant l'opérateur de Laplace ou laplacien  $\Delta : C^\infty(M) \rightarrow C^\infty(M)$ , où  $C^\infty(M)$  est l'espace des fonctions lisses sur  $M$ . Nous allons le définir par dualité en disant ce que devrait être

$$\int_M f \Delta g d\sigma.$$

**Définition 1.5** (Laplacien). *Le laplacien est l'unique application  $\Delta : C^\infty(M) \rightarrow C^\infty(M)$  telle que*

$$\int_M f \Delta g d\sigma = - \int_M h(\nabla f, \nabla g) d\sigma \tag{5}$$

*pour toute paire de fonctions lisses  $f$  et  $g$  où  $f$  est à support compact.*

Le lecteur peut voir [97, Chapitre 6] pour une définition analogue. De manière plus concrète, mais peut-être moins éclairante, nous pouvons remarquer que  $\Delta$  peut être facilement décrit en coordonnées locales. Si  $f$  est une fonction lisse,  $\Delta f$  est donné par

$$(\Delta f)^\lambda = \frac{1}{\sqrt{\det h^\lambda}} \sum_{i,j=1}^m \frac{\partial}{\partial x_i} \left( \sqrt{\det h^\lambda} h_{ij}^\lambda \frac{\partial}{\partial x_j} f^\lambda \right).$$

Le fait que ces deux définitions coïncident est une conséquence du théorème de Stokes.

Nous présentons à présent l'interaction de Coulomb sur une variété riemannienne compacte. Soit  $\Lambda$  une mesure signée sur la variété riemannienne compacte  $M$  à densité lisse par rapport à  $\sigma$  et de masse totale un.

**Définition 1.6** (Fonction de Green). *Considérons  $G : M \times M \rightarrow (-\infty, \infty]$ , une fonction continue telle que pour chaque  $p \in M$ , la fonction  $G_p : M \rightarrow (-\infty, \infty]$  qui est donnée par  $G_p(\tilde{p}) = G(p, \tilde{p})$ , est intégrable par rapport à  $\sigma$ . Si pour chaque fonction lisse  $f$*

$$\int_M G_p \Delta f \, d\sigma = -f(p) + \int_M f \, d\Lambda \quad (6)$$

nous disons que  $G$  est une fonction de Green associée à  $\Lambda$ .

Étant donné  $\Lambda$ , nous choisirons une fonction de Green quelconque car elle est unique sauf une constante additive (voir [10, Chapitre 4] pour plus d'informations). De manière plus compacte, nous écrivons

$$\Delta G_p = -\delta_p + \Lambda \quad (7)$$

pour signifier que (6) est satisfait pour toute fonction lisse  $f$ . En effet, notre définition permet de voir facilement que  $\Delta$  est un opérateur symétrique, de sorte que (7) puisse être compris comme une formulation distributionnelle de (6). Il peut sembler un peu étrange de ne pas considérer l'analogie exact du cas euclidien

$$\Delta G_p = -\delta_p.$$

Cette définition ne peut pas être adoptée car cette équation n'a pas de solution. Il existe deux manières de voir ce phénomène. Premièrement, si nous utilisons  $f = 1$  dans (6) (ce qui n'était pas autorisé dans l'espace euclidien) nous remarquons que  $\Lambda$  doit être là pour que le terme droit de (6) soit zéro. Nous pouvons aussi noter que l'intégrale du laplacien est toujours égale à zéro. Ceci peut être vu en utilisant  $f = 1$  dans (5). En fait, les deux raisonnements sont l'un le dual de l'autre.

Nous définissons l'énergie  $H_n : M^n \rightarrow (-\infty, \infty]$  d'un système de  $n$  particules de charge  $q$  comme

$$H_n(p_1, \dots, p_n) = \sum_{i < j} q^2 G(p_i, p_j).$$

Comme  $H_n$  est borné inférieurement et que nous allons intégrer par rapport à une mesure finie, nous pouvons définir le gaz de Coulomb sans potentiel confinant. Néanmoins, si  $\Lambda_1$  et  $\Lambda_2$  sont deux mesures signées à densité lisse et si nous désignons par  $G_1$  et  $G_2$  les fonctions de Green associées à  $\Lambda_1$  et  $\Lambda_2$  respectivement, il existe une fonction lisse  $V$  telle que

$$G_2(p, \tilde{p}) = G_1(p, \tilde{p}) + V(p) + V(\tilde{p}).$$

Dans ce cas, les énergies correspondantes  $H_n^1$  et  $H_n^2$  sont liées par

$$H_n^2(p_1, \dots, p_n) = H_n^1(p_1, \dots, p_n) + \sum_{i=1}^n (n-1)q^2 V(p_i) = \sum_{i < j} q^2 G_1(p_i, p_j) + \sum_{i=1}^n (n-1)q^2 V(p_i). \quad (8)$$

Donc, si nous choisissons  $q = 1/(n-1)$ , nous pouvons interpréter  $H_n^2$  comme une interaction électrostatique plus un potentiel.

**Définition 1.7** (Gaz de Coulomb sur une variété compacte). *Soient  $q, \beta > 0$ . Nous disons qu'un élément aléatoire  $(X_1, \dots, X_n)$  de  $M^n$  est un gaz de Coulomb sur  $M$  de particules de charge  $q$ , à la température inverse  $\beta$  et dans un milieu de charge  $\Lambda$  s'il suit la loi  $\mathbb{P}_n$  définie par*

$$d\mathbb{P}_n = \frac{1}{\mathcal{Z}_n} e^{-\beta H_n} d\sigma^{\otimes n}$$

où

$$\mathcal{Z}_n = \int_{M^n} e^{-\beta H_n} d\sigma^{\otimes n}.$$



Même si le laplacien est défini à l'aide de la métrique, nous expliquerons ici que, dans deux dimensions, il ne dépend que de la classe conforme de la métrique, définie ci-dessous. Cela implique notamment que la définition de la fonction de Green ne nécessite qu'une structure conforme pour avoir du sens. Considérons donc une variété lisse  $M$  de dimension deux dotée d'une métrique  $h$ . Le laplacien  $\Delta f$  d'une fonction intégrable  $f \in L^1(M)$  est la distribution sur  $M$  qui satisfait

$$\forall g \in C^\infty(M), \langle \Delta f, g \rangle = \int_M f \Delta g \, d\sigma.$$

Afin de ne pas s'inquiéter des problèmes de convergence, nous dirons que  $\Delta$  est une application de  $L^1(M)$  à  $C^\infty(M)^*$ , le dual algébrique de  $C^\infty(M)$ . La remarque que nous voudrions vérifier est que, si la dimension est deux, l'opérateur

$$\Delta : L^1(M) \rightarrow C^\infty(M)^*$$

ne dépend que de ce qui est appelée la classe conforme de  $h$ . Pour expliquer ce qu'est une classe conforme, nous définissons la relation d'équivalence suivante sur l'ensemble des métriques. Étant donné deux métriques  $h_1$  et  $h_2$ , nous écrivons

$$h_1 \sim h_2$$

si et seulement s'il existe une fonction lisse  $\rho$  telle que

$$h_2 = \rho h_1$$

où  $\rho h_1$  est la métrique donnée par la famille  $\{\rho^\lambda h_1^\lambda\}_{\lambda \in \Lambda}$ . La classe d'équivalence d'une métrique  $h$  s'appelle la *classe conforme* de  $h$  et nous la noterons par  $\mathbf{C}_h$ . Plus explicitement, nous avons

$$\mathbf{C}_h = \{\rho h : \rho \in C^\infty(M) \text{ est strictement positive}\}.$$

Si  $\bar{h} = \rho h \in \mathbf{C}_h$ , nous pouvons voir que les notions que nous avons définies précédemment (la mesure de volume, le gradient, le laplacien, etc) changent de façon agréable. Par exemple, si  $\bar{\sigma}$  dénote la mesure de volume associée à  $\bar{h}$ , nous pouvons voir, par la définition 1.4, que

$$d\bar{\sigma} = \rho \, d\sigma. \tag{9}$$

En particulier, nous pouvons remarquer que l'espace  $L^1(M)$  ne dépend que de la classe conforme  $\mathbf{C}_h$  (alors que l'intégration dépend de  $h$ ). Pour un autre exemple, étant donné  $f \in C^\infty(M)$ , si nous notons  $\bar{\nabla} f$  le gradient défini par la métrique  $\bar{h}$ , nous avons

$$\bar{\nabla} f = \rho^{-1} \nabla f.$$

Ainsi, par la définition du laplacien donnée par (5), si  $\bar{\Delta} f$  désigne le laplacien défini par la métrique  $\bar{h}$ ,

$$\bar{\Delta} f = \rho^{-1} \Delta f.$$

Enfin, si  $\bar{\Delta}$  est l'opérateur de Laplace sur  $L^1(M)$

$$\bar{\Delta} : L^1(M) \rightarrow C^\infty(M)^*$$

associé à  $\bar{h}$ , nous remarquons que si  $f \in L^1(M)$  et  $g \in C^\infty(M)$  alors

$$\langle \bar{\Delta} f, g \rangle = \int_M f \bar{\Delta} g \, d\bar{\sigma} = \int_M f \rho^{-1} \Delta g \, \rho \, d\sigma = \int_M f \Delta g \, d\sigma = \langle \Delta f, g \rangle$$

d'où  $\bar{\Delta} = \Delta$ . Ce fait intéressant fait intervenir la notion de structure complexe dans les discussions habituelles [12, Chapitre 5].



### L'ensemble de Ginibre

Soit  $\mathbb{M}_n$  l'espace des matrices  $n \times n$  à coefficients complexes. Nous disons qu'un élément aléatoire  $\mathcal{M}$  de  $\mathbb{M}_n$  appartient à l'ensemble de Ginibre de taille  $n$  si

- ◇  $\{\mathcal{M}_{ij}\}_{1 \leq i, j \leq n}$  est une famille mutuellement indépendante et
- ◇  $\forall i, j \in \{1, \dots, n\}, \mathcal{M}_{ij} \sim \frac{1}{\pi} e^{-|z|^2} d\ell_{\mathbb{C}}(z)$ .

Autrement dit,  $\mathcal{M}$  suit une loi proportionnelle à

$$e^{-\text{Tr}(A^*A)} d\ell_{\mathbb{M}_n}(A)$$

ou, en utilisant les notations standard,  $\mathcal{M}$  est une variable gaussienne associée à la forme quadratique

$$A \mapsto \frac{1}{2} \text{Tr}(A^*A)$$

de l'espace vectoriel  $\mathbb{M}_n$ . Cet exemple se trouve dans [49, Sous-section 15.1.1]. Dans ce cas, nous avons le résultat suivant démontré par Ginibre [55].

**Proposition 1.9** (Valeurs propres de l'ensemble de Ginibre). *Supposons que  $\mathcal{M}$  appartient à l'ensemble de Ginibre de taille  $n$  et que  $(Z_1, \dots, Z_n) \in \mathbb{C}^n$  est un vecteur aléatoire qui suit la loi*

$$\frac{1}{\mathcal{Z}_n} \prod_{i < j} |z_i - z_j|^2 e^{-\sum_{i=1}^n |z_i|^2} d\ell_{\mathbb{C}^n}(z_1, \dots, z_n) \quad (11)$$

où  $\mathcal{Z}_n$  est une constante de normalisation. Alors, la mesure empirique des valeurs propres de  $\mathcal{M}$  et la mesure empirique de  $(Z_1, \dots, Z_n)$  ont la même loi. Plus précisément, si  $\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$  est formé par les valeurs propres de  $\mathcal{M}$ , comptées avec multiplicité,  $\{\lambda_1, \dots, \lambda_n\}$ , alors

$$\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i} \sim \frac{1}{n} \sum_{i=1}^n \delta_{Z_i}.$$

Ceci peut être reformulé en utilisant la définition d'un gaz de Coulomb. La loi donnée dans (11) est celle d'un gaz de Coulomb sur  $\mathbb{C}$  de particules de charge un, à la température inverse  $4\pi$  et confiné par le potentiel

$$V(z) = \frac{|z|^2}{4\pi}.$$

Encore une fois, le terme  $4\pi$  est dû à la normalisation que nous avons choisie pour la définition du gaz de Coulomb. Une particularité spectaculaire de ce gaz de Coulomb est son comportement lorsque le nombre de particules  $n$  va vers l'infini. Comme nous le verrons dans la sous-section 2.4, le processus ponctuel défini par  $\{Z_1, \dots, Z_n\}$  converge vers un processus ponctuel déterminantal invariant sous les isométries du plan, c'est-à-dire invariant par rotations et translations. En ce sens, ce système de points est naturellement attaché au plan complexe avec sa structure riemannienne habituelle.

### L'ensemble sphérique

Maintenant, considérons deux matrices aléatoires indépendantes  $A$  et  $B$  qui appartiennent à l'ensemble de Ginibre de taille  $n$ . Nous disons qu'une matrice  $n \times n$  aléatoire  $\mathcal{M}$  appartient à l'ensemble sphérique de taille  $n$  si

$$\mathcal{M} \sim AB^{-1}.$$

Pour plus d'informations à ce sujet, le lecteur peut regarder [49, Section 15.6]. Krishnapur, [69], a obtenu la description remarquable suivante.

**Proposition 1.10** (Valeurs propres de l'ensemble sphérique). *Supposons que  $\mathcal{M}$  appartient à l'ensemble sphérique de taille  $n$  et que  $(Z_1, \dots, Z_n) \in \mathbb{C}^n$  est un vecteur aléatoire qui suit la loi*

$$\frac{1}{\mathcal{Z}_n} \prod_{i < j} |z_i - z_j|^2 \prod_{i=1}^n \frac{1}{(1 + |z_i|^2)^{n+1}} d\ell_{\mathbb{C}^n}(z_1, \dots, z_n) \quad (12)$$

où  $\mathcal{Z}_n$  est une constante de normalisation. Alors, la mesure empirique des valeurs propres de  $\mathcal{M}$  et la mesure empirique de  $(Z_1, \dots, Z_n)$  ont la même loi. Plus précisément, si  $\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$  est formé par les valeurs propres de  $\mathcal{M}$ , comptées avec multiplicité,  $\{\lambda_1, \dots, \lambda_n\}$ , alors

$$\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i} \sim \frac{1}{n} \sum_{i=1}^n \delta_{Z_i}.$$

Ceci peut être reformulé en utilisant la définition d'un gaz de Coulomb. La loi donnée dans (12) est celle d'un gaz de Coulomb sur  $\mathbb{C}$  de particules de charge  $1/(n+1)$ , à la température inverse  $4\pi(n+1)^2$  et confiné par le potentiel

$$V(z) = \frac{1}{4\pi} \log(1 + |z|^2).$$

Néanmoins, il se peut que la raison pour laquelle 'sphérique' soit attaché au nom ne soit pas évidente. Il y a une interprétation plus naturelle (nous espérons) de l'ensemble sphérique qui explique la partie 'sphérique' du nom. Nous allons donner une courte explication ci-dessous. Remarquons d'abord que, en regardant ce gaz sous la projection stéréographique<sup>1</sup>, nous obtenons un gaz de Coulomb sur la sphère ronde  $S^2$  de particules de charge un, à la température inverse  $4\pi$  et dans un milieu de charge uniforme. Plus explicitement, si  $(Z_1, \dots, Z_n)$  suit la loi (12), alors en appelant  $X_i$  la projection stéréographique inverse de  $Z_i$ , l'élément aléatoire  $(X_1, \dots, X_n)$  de  $(S^2)^n$  suit la loi

$$\frac{1}{\mathcal{Z}_n} e^{-4\pi \sum_{i < j} G(x_i, x_j)} d\sigma^{\otimes n}(x_1, \dots, x_n)$$

où  $\mathcal{Z}_n$  est une constante de normalisation,  $\sigma$  est la mesure de probabilité uniforme sur  $S^2$  et  $G$  est la fonction de Green associée à  $\sigma$  qui, dans ce cas, a la formule simple

$$G(x, y) = -\frac{1}{2\pi} \log |x - y|_{\mathbb{R}^3}.$$

Ceci peut être vu, par exemple, dans [49, Section 15.6]. Maintenant, nous voulons donner une explication de l'invariance par rotations de ce gaz sans donner la formule explicite de sa loi. Si  $p : \mathbb{C}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{M}_n$  est l'application définie par

$$p(z, w) = zA + wB = (A \ B) \begin{pmatrix} z \\ w \end{pmatrix},$$

où  $A$  et  $B$  sont indépendants et appartiennent à l'ensemble de Ginibre de taille  $n$ , alors la loi de  $p$  est invariante sous l'action du groupe unitaire  $U(2)$  sur  $\mathbb{C}^2 \setminus \{(0, 0)\}$ . Cela implique que  $\det p$ , qui est explicitement donné par

$$\det p(z, w) = \det(zA + wB),$$

est également invariant sous les transformations unitaires de sorte que les droites complexes de  $\mathbb{C}^2$  sur lesquelles  $\det p$  est zéro forme un processus de points invariant sous les transformations unitaires dans l'espace des droites complexes, appelé espace projectif  $\mathbb{P}^1$ . En rappelant que cet espace est canoniquement identifié à la sphère et que, sous cette identification, les transformations unitaires agissant sur  $\mathbb{P}^1$  deviennent les rotations de la sphère nous obtenons ce que nous

<sup>1</sup>L'application de  $S^2 \setminus \{(0, 0, 1)\}$  dans  $\mathbb{C}$  donnée par  $(x, y, z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z})$ .

vouillons: les valeurs propres d'une matrice aléatoire qui appartient à l'ensemble sphérique ont une loi invariante sous les *isométries de la sphère*.

Remarquons qu'en suivant cette construction, nous pouvons obtenir des gaz invariants plus généraux en remplaçant les polynômes linéaires par des polynômes de degré supérieur. Ces polynômes de matrices apparaissent dans le problème de valeurs propres polynômiales, dont le cas aléatoire n'a été que très peu étudié (voir [7]).

### L'ensemble de Haar unitaire tronqué

Soit  $A$  une matrice unitaire  $(n+1) \times (n+1)$  aléatoire qui suit la loi donnée par la mesure de Haar sur  $U(n+1)$ . Une matrice  $n \times n$  aléatoire  $\mathcal{M}$  appartient à l'ensemble de Haar unitaire tronqué de taille  $n$  si

$$\mathcal{M} \sim (A_{ij})_{i,j \in \{1, \dots, n\}}.$$

Pour plus d'informations, voir [49, Sous-section 15.7.4] et [92].

**Proposition 1.11** (Valeurs propres de l'ensemble de Haar unitaire tronqué). *Supposons que  $\mathcal{M}$  appartient à l'ensemble de Haar unitaire tronqué de taille  $n$ . Soit  $\mathbb{D} \subset \mathbb{C}$  le disque unité ouvert et soit  $(Z_1, \dots, Z_n) \in \mathbb{D}^n$  un vecteur aléatoire qui suit la loi*

$$\frac{1}{Z_n} \prod_{i < j} |z_i - z_j|^2 d\ell_{\mathbb{D}^n}(z_1, \dots, z_n) \quad (13)$$

où  $\ell_{\mathbb{D}^n}$  dénote la mesure de Lebesgue sur  $\mathbb{C}^n$  restreint à  $\mathbb{D}^n$  et  $Z_n$  est une constante de normalisation. Alors, la mesure empirique des valeurs propres de  $\mathcal{M}$  et la mesure empirique de  $(Z_1, \dots, Z_n)$  ont la même loi. Plus précisément, si  $\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$  est formé par les valeurs propres de  $\mathcal{M}$ , comptées avec multiplicité,  $\{\lambda_1, \dots, \lambda_n\}$ , alors

$$\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i} \sim \frac{1}{n} \sum_{i=1}^n \delta_{Z_i}.$$

Comme dans les deux cas précédents, cela peut également être reformulé en utilisant la notion de gaz de Coulomb. La loi donnée dans (13) est celle d'un gaz de Coulomb sur  $\mathbb{C}$  de particules de charge un, à la température inverse  $4\pi$  et confiné par le potentiel

$$V(z) = \begin{cases} 0 & \text{si } |z| \leq 1 \\ \infty & \text{sinon} \end{cases}.$$

Nous verrons dans la sous-section 2.4 que, lorsque  $n$  tend vers l'infini, le processus ponctuel converge vers un processus ponctuel déterminantal, invariant sous les *isométries du disque hyperbolique*. L'ensemble de Haar unitaire tronqué et d'autres exemples similaires qui convergent vers le même processus ont été peu étudiés et constitueront un des principaux sujets des chapitres 4 et 5.

## 1.4 Un gaz de Coulomb dans un milieu donné par la courbure

Nous décrivons ici un type particulier de gaz de Coulomb sur des variétés riemanniennes de dimension deux. Il apparaît aux chapitres 4 et 5 sur la sphère à une température donnée et au chapitre 6, lorsque le genre de la surface est supérieur à un. Comme nous allons utiliser la notion de courbure, nous donnons ici quelques références. Les définitions standard peuvent être trouvées dans [43, Chapitre 4] pour les variétés riemanniennes générales. Pour les variétés riemanniennes de dimension deux plongées dans  $\mathbb{R}^3$ , nous pouvons, de manière équivalente, regarder la courbure de Gauss dont la définition peut être trouvée dans [44, Section 3.2, Définition 6]. Enfin, pour la formule que nous allons utiliser dans (14), nous suggérons [12, Chapitre 4].

Nous allons nous concentrer ici sur le cas de la sphère car, par projection stéréographique, nous obtiendrons une belle interprétation en tant que gaz de Coulomb sur le plan. Nous prenons la sphère  $S^2$  avec sa métrique ronde habituelle  $h$  normalisée de sorte que le volume de  $S^2$  soit un. Soit  $\mathcal{C}$  l'espace des métriques dans la classe conforme de  $h$  et de volume un. Plus précisément,

$$\mathcal{C} = \left\{ \rho h : \rho \in C^\infty(S^2), \rho \text{ est strictement positif partout et } \int_M \rho d\sigma = 1 \right\}$$

où  $\sigma$  dénote la mesure de volume associée à  $h$  ou, de manière équivalente, la mesure de probabilité uniforme sur  $S^2$ . En désignant l'espace des mesures de probabilité à densité lisse et strictement positive par  $\mathcal{P}_{C^\infty}(S^2)$ , nous obtenons l'identification

$$\mathcal{P}_{C^\infty}(S^2) \rightarrow \mathcal{C} \text{ donnée par } \rho d\sigma \mapsto \rho h$$

dont l'inverse est

$$\mathcal{C} \rightarrow \mathcal{P}_{C^\infty}(S^2) \text{ définie par } \bar{h} \mapsto \text{mesure de volume de } \bar{h}.$$

Nous pouvons donc penser que  $\mathcal{P}_{C^\infty}(S^2)$  est la classe conforme de  $h$ . A titre de remarque, il faut dire que ces métriques sont également comprises comme celles compatibles avec la structure complexe induite par une orientation et  $h$ . Néanmoins, nous n'utiliserons pas cette description car elle nécessite des concepts non définis ici (voir, par exemple, [12, Chapitre 4]).

Étant donné  $\alpha \in \mathcal{P}_{C^\infty}(S^2)$  nous considérons sa courbure scalaire (rappelons que nous identifions  $\alpha$  avec une métrique)  $R_\alpha$  qui est donnée dans la projection stéréographique par

$$R_\alpha = -\Delta \log \rho \tag{14}$$

où  $\rho$  est la densité de  $\alpha$  par rapport à la mesure de Lebesgue (voir [12, Section 4.4]). Nous allons interpréter  $R_\alpha$  comme une mesure finie, car en vertu du théorème de Gauss-Bonnet

$$\int_{S^2} R_\alpha d\alpha = 8\pi.$$

La version normalisée serait alors la mesure  $\Lambda_\alpha$  définie par

$$d\Lambda_\alpha = \frac{R_\alpha}{8\pi} d\alpha.$$

Ce sera la charge du milieu de nos gaz de Coulomb. Plus précisément, si nous dotons  $S^2$  de la métrique  $\alpha$ , le gaz de Coulomb sur le milieu donné par la courbure sera un gaz de Coulomb dans un milieu de charge  $\Lambda_\alpha$ . En fait, cela est très lié à une famille de gaz de Coulomb sur  $\mathbb{C}$ , comme nous allons l'expliquer maintenant. Prenons une mesure  $\mu$  de masse totale un sur  $\mathbb{C}$  et définissons

$$V^\mu(z) = \frac{1}{2\pi} \int_{\mathbb{C}} \log |z - w| d\mu(w) \tag{15}$$

lorsque c'est possible. Ce  $V^\mu$  peut être considéré comme le potentiel généré par la charge  $-\mu$  car il satisfait

$$\Delta V^\mu = \mu.$$

Nous affirmons que la mesure  $\sigma_\mu$  donnée par

$$d\sigma_\mu = \frac{e^{-8\pi V^\mu}}{\mathcal{Z}} d\ell_{\mathbb{C}},$$

où  $\mathcal{Z}$  est une constante de normalisation, est naturellement associée à  $\mu$  dans un sens explicite. En effet, si, par projection stéréographique inverse,  $\mu$  définissait une mesure à densité lisse par rapport à  $\sigma$  sur  $S^2$ , nous aurions

$$\Lambda_{\sigma_\mu} = \mu$$

qui peut être compris par (14). Néanmoins, dans le contexte non lisse, nous estimons qu'il n'est pas suffisamment motivé. La remarque suivante devrait être une motivation plus forte, car elle dit que la correspondance  $\mu \mapsto \sigma_\mu$  a un comportement agréable sous des transformations conformes (des transformations de Möbius) et alors elle ne dépend que de la structure conforme.

**Proposition 1.12** (Comportement du potentiel sous des transformations conformes). *Supposons que  $\mu$  est une mesure signée de masse totale un et dont le potentiel est fini, c'est-à-dire (15) est bien défini et fini pour chaque  $z \in \mathbb{C}$ .*

*Alors, pour chaque transformation conforme  $T : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$ , la mesure image  $T_*\mu$  a un potentiel fini et*

$$\sigma_{T_*\mu} = T_*\sigma_\mu. \quad (16)$$

*De plus, si  $\alpha \in \mathcal{P}_{C^\infty}(S^2)$  et si  $G$  est la fonction de Green associée à  $\Lambda_\alpha$  nous définissons*

$$U^\mu(x) = - \int_{S^2} G(x, y) d\mu(y) \quad (17)$$

*et obtenons*

$$d\sigma_\mu = \frac{e^{-8\pi U^\mu}}{\mathcal{Z}} d\alpha \quad (18)$$

*où  $\mathcal{Z}$  est une constante de normalisation.*

*Démonstration.* L'affirmation concernant la finitude du potentiel peut être obtenue par calcul direct. La preuve de (18) est obtenue en utilisant la formule (14) de  $R_\alpha$ . L'égalité (16) est une conséquence de (18) et les propriétés de transformation de la métrique et sa courbure.  $\square$

De plus, étant donné la mesure  $\mu$  de masse totale un, considérons

$$G^\mu(x, y) = -\frac{1}{2\pi} \log|x - y| + V^\mu(x) + V^\mu(y).$$

Ceci satisfait

$$\Delta G_x^\mu = -\delta_x + \mu$$

où  $G_x^\mu(y) = G^\mu(x, y)$ . Nous pouvons alors penser  $G^\mu$  en tant que fonction de Green associée à  $\mu$ . Même si le laplacien est pris ici dans  $\mathbb{C}$ , nous avons la même égalité sur  $S^2$  car la projection stéréographique est une transformation conforme. La loi du gaz de Coulomb attachée à ces données sera

$$\frac{1}{\mathcal{Z}_n} e^{-\beta \sum_{i < j} G^\mu(x_i, x_j)} d\sigma_\mu^{\otimes n}(x_1, \dots, x_n) \quad (19)$$

où  $\mathcal{Z}_n$  est une constante de normalisation. Après avoir développé les termes appropriés, (19) devient la loi du gaz de Coulomb sur  $\mathbb{C}$

$$\frac{1}{\mathcal{Z}_n} e^{-\beta \left( -\frac{1}{2\pi} \sum_{i < j} \log|x_i - x_j| + (n-1) \sum_{i=1}^n V^\mu(x_i) \right)} e^{-8\pi V^\mu(x_i)} d\ell_{\mathbb{C}^n}(x_1, \dots, x_n)$$

pour une autre constante de normalisation  $\mathcal{Z}_n$ . Si  $\beta = 4\pi$ , nous obtenons le modèle déterminantal étudié dans les chapitres 4 et 5

$$\frac{1}{\mathcal{Z}_n} e^{-4\pi \left( -\frac{1}{2\pi} \sum_{i < j} \log|x_i - x_j| + (n+1) \sum_{i=1}^n V^\mu(x_i) \right)} d\ell_{\mathbb{C}^n}(x_1, \dots, x_n). \quad (20)$$

Comme le laplacien ne dépend que de la classe conforme de la métrique, nous pouvons écrire (19) sous la forme

$$\frac{1}{\mathcal{Z}_n} e^{-\beta \left( \sum_{i < j} G(x_i, x_j) + (n-1) \sum_{i=1}^n U^\mu(x_i) \right)} e^{-8\pi U^\mu(x_i)} d\alpha^{\otimes n}(x_1, \dots, x_n)$$

où  $G$  est la fonction de Green associée à  $\Lambda_\alpha$ , la courbure normalisée de  $\alpha \in \mathcal{P}_{C^\infty}(S^2)$ , et le potentiel  $U^\mu$  est défini par (17). Cela signifie que, pour toute métrique choisie sur  $S^2$ , le système peut être considéré comme un gaz de Coulomb sur  $S^2$  confiné par un potentiel.

### 1.5 Le milieu circulaire et les polynômes de Kac

Il y a un modèle-jouet que nous aimons étudier. C'est le cas où  $\mu$  est la mesure de probabilité uniforme sur l'équateur de  $S^2$  ou, de manière équivalente, la mesure de probabilité uniforme sur le cercle unité  $S^1$  de  $\mathbb{C}$ . Dans ce cas

$$V^\mu(z) = \frac{1}{2\pi} \int_{S^1} \log |z - s| ds = \frac{1}{2\pi} \log_+ |z| = \frac{1}{2\pi} \max\{0, \log |z|\},$$

où nous utilisons la mesure uniforme normalisée sur le cercle pour effectuer l'intégration. Nous considérons la loi (20) que nous écrivons comme

$$\frac{1}{\mathcal{Z}_n} \prod_{i < j} |z_i - z_j|^2 \prod_{i=1}^n \exp \left( \int_{S^1} \log |z_i - s|^2 ds \right)^{-(n+1)} d\ell_{\mathbb{C}^n}(z_1, \dots, z_n).$$

Beaucoup de calculs explicites peuvent être faits pour ce gaz et pour une famille de gaz de Coulomb sur  $\mathbb{C}$ , ils feront l'objet du chapitre 5. Il existe un système analogue dans le contexte des polynômes aléatoires. Soit  $\{a_n\}_{n \in \mathbb{N}}$  une suite de variables aléatoires indépendantes et identiquement distribuées. Pour chaque  $n$ , prenons le polynôme aléatoire

$$p_n(z) = \sum_{k=0}^n a_k z^k.$$

Ce sont les polynômes aléatoires de Kac. Lorsque  $a_0$  suit une loi gaussienne complexe standard,  $p_n$  est une variable gaussienne à valeurs dans  $\mathcal{P}_n$ , l'espace des polynômes complexes de degré  $n$ , doté de la forme quadratique

$$p \mapsto \frac{1}{2} \int_{S^1} |p(s)|^2 ds.$$

En fait, une forme explicite pour la loi des zéros de  $p_n$  a été trouvée en grande généralité par Zeitouni et Zelditch dans [98] (voir aussi [30]) et elle est donnée, dans le cas précédent, par

$$\frac{1}{\mathcal{Z}_n} \prod_{i < j} |z_i - z_j|^2 \left( \int_{S^1} \prod_{i=1}^n |z_i - s|^2 ds \right)^{-(n+1)} d\ell_{\mathbb{C}^n}(z_1, \dots, z_n).$$

Nous pouvons voir la ressemblance entre les deux modèles en comparant les deux termes

$$\left( \int_{S^1} \prod_{i=1}^n |z_i - s|^2 ds \right)^{-(n+1)} \quad \text{et} \quad \prod_{i=1}^n \exp \left( \int_{S^1} \log |z_i - s|^2 ds \right)^{-(n+1)}$$

le second étant le terme du potentiel dans la densité du gaz de Coulomb. Nous pouvons nous demander à quel point le gaz de Coulomb et les zéros du polynôme de Kac se ressemblent. Comme nous le verrons au chapitre 5, les comportements limites macroscopiques sont les mêmes, mais les comportements microscopiques près du cercle unité sont assez différents. Ces modèles ont des généralisations intéressantes qui ont été étudiées dans les chapitres 4 et 5.

## 2 Processus ponctuel déterminantal

Lorsque  $\beta = 4\pi$  ( $\beta = 2$  dans la normalisation habituelle de la théorie des matrices aléatoires) les gaz de Coulomb sur  $\mathbb{C}$  ou la sphère possèdent une structure spéciale qui sera un outil très important dans les chapitres 4 et 5. Cette structure nous permet de faire de nombreux calculs qui ne sont pas encore disponibles pour  $\beta$  général. Une référence standard sur ce sujet est [18]. Nous commençons par expliquer ce qu'est un processus ponctuel.



## 2.1 Processus ponctuel

Soit  $M$  un espace polonais. Nous notons  $\mathcal{M}_+(M)$  l'espace des mesures positives sur  $M$ . Nous définissons l'espace des configurations de points sur  $M$  comme

$$\mathcal{C}_M = \{\mu \in \mathcal{M}_+(M) : \mu \text{ est localement fini et prend des valeurs dans } \mathbb{N} \cup \{\infty\}\}$$

où par localement fini nous entendons que chaque point  $x \in M$  a un voisinage ouvert  $U$  tel que  $\mu(U) < \infty$ . En fait, une mesure  $\mu$  appartient à  $\mathcal{C}_M$  si, et seulement si, il existe un ensemble discret et fermé  $D \subset M$  et une fonction  $c : D \rightarrow \mathbb{N}$  tels que

$$\mu = \sum_{x \in D} c(x) \delta_x.$$

De cette façon,  $\mathcal{C}_M$  peut être considéré comme l'espace des ensembles discrets de  $M$  où nous autorisons les multiplicités. Pour simplifier, considérons le cas où  $M$  est aussi localement compact. Nous avons alors la description équivalente

$$\mathcal{C}_M = \{\mu \in \mathcal{M}_+(M) : \forall \text{ compact } K \subset M, \mu(K) \in \mathbb{N}\}.$$

Dans ce cas, il est logique de considérer la plus petite topologie de  $\mathcal{C}_M$  telle que les applications  $\hat{f} : \mathcal{C}_M \rightarrow \mathbb{R}$  définies par

$$\hat{f}(\mu) = \mu(f) = \int_M f d\mu$$

sont continues pour chaque fonction continue à support compact  $f : M \rightarrow \mathbb{R}$ . Nous pouvons voir, [65, Theorem 4.2], que  $\mathcal{C}_M$  doté de cette topologie est un espace polonais.

**Définition 2.1** (Processus ponctuel). *Un processus ponctuel  $\xi$  est un élément aléatoire de  $\mathcal{C}_M$ .*

## 2.2 Fonctions de corrélation

Les processus ponctuels ne sont pas si faciles à étudier que nous devons trouver un moyen de caractériser leurs lois, de manière analogue aux variables aléatoires réelles ou vectorielles. L'un des moyens consiste à utiliser la notion de fonction de corrélation.

**Définition 2.2** (Fonctions de corrélation). *Soit  $(M, \sigma)$  un espace mesuré polonais localement compact tel que  $\sigma$  n'a pas d'atomes. Soit  $\xi$  un processus ponctuel sur  $M$  et soit  $k$  un entier strictement positif. Nous disons qu'une fonction mesurable positive  $\rho_k : M^k \rightarrow \mathbb{R}$  est la  $k$ -ème fonction de corrélation si*

$$\mathbb{E}[\xi(A_1) \dots \xi(A_k)] = \int_{A_1 \times \dots \times A_k} \rho_k d\sigma^{\otimes k}$$

pour tous les  $k$ -uplets d'ensembles mesurables deux à deux disjoints  $A_1, \dots, A_k$  de  $M$ .

Ces fonctions sont uniques modulo l'égalité presque partout et se comportent comme les moments de  $\xi$ . Expliquons la relation avec les moments. Prenons le produit de  $\xi$  par lui-même pour obtenir  $\xi^{\otimes k}$ . Le  $k$ -ème moment usuel serait la mesure  $\mathbb{E}[\xi^{\otimes k}]$ . Néanmoins, il existe des termes supplémentaires dans cette espérance que nous aimerions supprimer. Ceci peut même être compris en prenant de petites valeurs de  $k$ . Prenons, d'abord,  $k = 1$ .

**Définition 2.3** (Mesure d'intensité). *Soit  $M$  un espace polonais localement compact et soit  $\xi$  un processus ponctuel sur  $M$ . La mesure d'intensité est donnée par  $\alpha_1 = \mathbb{E}[\xi]$ . Plus précisément,  $\alpha_1$  est la mesure sur  $M$  qui satisfait*

$$\int_M f d\alpha_1 = \mathbb{E}[\xi(f)]$$

pour chaque fonction mesurable positive  $f : M \rightarrow \mathbb{R}$ .

Dans le cas d'un espace mesuré  $(M, \sigma)$ , si  $\alpha_1$  a une densité par rapport à  $\sigma$ , cette densité serait la première fonction de corrélation  $\rho_1$ . La deuxième fonction de corrélation peut être obtenue de la même manière mais en considérant  $\xi^{\otimes 2}$ .

**Définition 2.4** (Deuxième mesure de corrélation). *Soit  $M$  un espace polonais localement compact et soit  $\xi$  un processus ponctuel sur  $M$ . Considérons l'application  $\text{diag} : M \rightarrow M \times M$  donnée par  $\text{diag}(x) = (x, x)$ . Ensuite, la deuxième mesure de corrélation est donnée par  $\alpha_2 = \mathbb{E}[\xi \otimes \xi - \text{diag}_* \xi]$ . Plus précisément,  $\alpha_2$  est la mesure sur  $M \times M$  qui satisfait*

$$\int_{M \times M} f(x)g(y) d\alpha_2(x, y) = \mathbb{E}[\xi(f)\xi(g)] - \mathbb{E}[\xi(fg)]$$

pour toutes les fonctions mesurables positives  $f : M \rightarrow \mathbb{R}$  et  $g : M \rightarrow \mathbb{R}$ .

Si nous considérons  $f$  et  $g$  comme des fonctions indicatrices d'ensembles disjoints dans la définition précédente, nous voyons comment récupérer la deuxième fonction de corrélation  $\rho_2$  car le terme  $fg$  s'annule. De même, nous pouvons définir la  $k$ -ème mesure de corrélation en éliminant simplement ce qui serait nécessairement singulier dans  $\mathbb{E}[\xi^{\otimes k}]$ . Pour avoir un exemple concret de fonction de corrélation dans l'esprit, formulons la proposition suivante.

**Proposition 2.5** (Nombre fixe de particules). *Supposons que le processus ponctuel  $\xi$  a exactement  $n$  particules. Plus précisément, supposons que*

$$\xi = \sum_{i=1}^n \delta_{Y_i}$$

où  $(Y_1, \dots, Y_n)$  est un système de  $n$  particules qui suit une loi symétrique  $\mathbb{P}_n$  donné par

$$d\mathbb{P}_n = \rho(y_1, \dots, y_n) d\sigma^{\otimes n}.$$

Ensuite, la  $n$ -ème fonction de corrélation est donnée par

$$\rho_n(y_1, \dots, y_n) = n! \rho(y_1, \dots, y_n). \quad (21)$$

De plus, la  $k$ -ème fonction de corrélation peut être obtenue avec

$$\rho_k(y_1, \dots, y_k) = \frac{n!}{(n-k)!} \int_{M^{n-k}} \rho(y_1, \dots, y_n) d\sigma^{\otimes n-k}(y_{k+1}, \dots, y_n) \quad (22)$$

si  $k < n$  et  $\rho_k = 0$  si  $k > n$ .

*Démonstration.* Notons que

$$\xi(A) = \sum_{i=1}^n 1_A(Z_i),$$

ce qui implique

$$\mathbb{E}[\xi(A_1) \dots \xi(A_n)] = \mathbb{E} \left[ \left( \sum_{i=1}^n 1_{A_1}(Z_i) \right) \dots \left( \sum_{i=1}^n 1_{A_n}(Z_i) \right) \right] = n! \mathbb{P}_n(A_1 \times \dots \times A_n)$$

où nous avons utilisé que  $A_1, \dots, A_n$  sont deux à deux disjoints. Nous avons démontré (21). De même, si  $k < n$ ,

$$\begin{aligned} \mathbb{E}[\xi(A_1) \dots \xi(A_k)] &= \mathbb{E} \left[ \left( \sum_{i=1}^n 1_{A_1}(Z_i) \right) \dots \left( \sum_{i=1}^n 1_{A_k}(Z_i) \right) \right] \\ &= \frac{n!}{(n-k)!} \mathbb{P}_n(A_1 \times \dots \times A_k \times M \times \dots \times M) \end{aligned}$$

où nous avons encore utilisé que  $A_1, \dots, A_k$  sont deux à deux disjoints et un argument de comptage. Nous avons démontré (22). Enfin, si  $k > n$ , nous avons que  $\xi(A_1) \dots \xi(A_k) = 0$  car  $A_1, \dots, A_k$  sont deux à deux disjoints. Alors  $\rho_k = 0$ .  $\square$

### 2.3 Noyau déterminantal

Les processus ponctuels que nous utiliserons dans les chapitres 4 et 5 est ce que nous appelons les processus ponctuels déterminantaux. Comme notre motivation est un processus ponctuel radial spécifique, nous commencerons par comprendre sa structure. Soit  $M = \mathbb{C}$  et considérons la fonction sur  $\mathbb{C}^n$  définie par

$$F(z_1, \dots, z_n) = \prod_{i < j} |z_i - z_j|^2.$$

C'est la partie commune de la densité des gaz de Coulomb 'déterminantaux' que nous étudierons. Par souci de simplicité, prenons une mesure de probabilité radiale  $\sigma$  et un vecteur aléatoire  $(Z_1, \dots, Z_n)$  qui suit la loi  $\mathbb{P}_n$  donnée par

$$d\mathbb{P}_n = \frac{1}{Z_n} F(z_1, \dots, z_n) d\sigma^{\otimes n}.$$

où  $Z_n$  est une constante de normalisation. Ce qui est si spécial, c'est que si nous formons la matrice de Vandermonde

$$\mathcal{V} = \begin{pmatrix} 1 & z_1 & (z_1)^2 & \dots & (z_1)^{n-1} \\ 1 & z_2 & (z_2)^2 & \dots & (z_2)^{n-1} \\ 1 & z_3 & (z_3)^2 & \dots & (z_3)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & (z_n)^2 & \dots & (z_n)^{n-1} \end{pmatrix}$$

nous aurons

$$F(z_1, \dots, z_n) = |\det \mathcal{V}|^2 = (\det \mathcal{V})(\overline{\det \mathcal{V}}) = \det(\mathcal{V}\mathcal{V}^*). \quad (23)$$

Donc,  $F$  est, en fait, un déterminant. La forme explicite de  $\mathcal{V}\mathcal{V}^*$  peut être obtenue sous la forme

$$(\mathcal{V}\mathcal{V}^*)_{ij} = \sum_{l=1}^n (V_{il})(\overline{V_{jl}}) = \sum_{k=0}^{n-1} z_i^k \bar{z}_j^k$$

de sorte que si nous définissons

$$K(z, w) = \sum_{k=0}^{n-1} z^k \bar{w}^k$$

nous pouvons écrire

$$F(z_1, \dots, z_n) = \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, n\}} \right).$$

Pour un peu de généralité, nous pouvons aussi considérer  $n$  nombres positifs  $a_0, \dots, a_{n-1}$  et définir

$$K(z, w) = \sum_{k=0}^{n-1} a_k z^k \bar{w}^k$$

pour obtenir

$$\det \left( K(z_i, z_j)_{i,j \in \{1, \dots, n\}} \right) = a_1 \dots a_n F(z_1, \dots, z_n).$$

Comme la mesure de probabilité  $\sigma$  est radiale, un choix naturel pour  $a_0, \dots, a_{n-1}$  est

$$(a_k)^{-1} = \int_{\mathbb{C}} |z|^{2k} d\mu(z)$$

de sorte que  $K$  représente la projection orthogonale de  $L^2(\mu)$  sur l'espace  $\mathcal{P}_{n-1}$  des polynômes de degré inférieur ou égal à  $n-1$ . Comme  $K$  représente une projection orthogonale sur un espace de dimension  $n$ , il a les deux propriétés intéressantes

$$\int_{\mathbb{C}} K(z, x) K(x, w) d\sigma(x) = K(z, w)$$

et

$$\int_{\mathbb{C}} K(x, x) d\sigma(x) = n.$$

Celles-ci et la formule du déterminant nous permettent de démontrer que

$$\int_{\mathbb{C}} \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, k\}} \right) d\sigma(z_k) = (n - k + 1) \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, k-1\}} \right).$$

Par une récurrence, nous obtenons

$$\int_{\mathbb{C}^n} \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, n\}} \right) d\sigma^{\otimes n}(z_1, \dots, z_n) = n!$$

ce qui implique que  $\frac{1}{n!} \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, n\}} \right)$  est une fonction de densité et

$$\frac{1}{n!} \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, n\}} \right) = \frac{1}{Z} F(z_1, \dots, z_n).$$

Par la proposition 2.5 nous avons

$$\rho_k(z_1, \dots, z_k) = \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, k\}} \right)$$

pour  $k \leq n$ . Pour  $k > n$ , une décomposition de  $K(z_i, z_j)_{i,j \in \{1, \dots, k\}}$  similaire à celle de (23) nous permet de conclure que

$$\det \left( K(z_i, z_j)_{i,j \in \{1, \dots, k\}} \right) = 0$$

de sorte qu'en général, nous obtenons ce qui suit.

**Proposition 2.6** (Corrélations et déterminants). *La  $k$ -ème fonction de corrélation du processus ponctuel  $\sum_{i=1}^n \delta_{Z_i}$  où  $(Z_1, \dots, Z_n)$  suit une loi proportionnelle à*

$$\prod_{i < j} |z_i - z_j|^2 d\sigma^{\otimes n}(z_1, \dots, z_n)$$

est

$$\rho_k(z_1, \dots, z_k) = \det \left( K(z_i, z_j)_{i,j \in \{1, \dots, k\}} \right)$$

où  $K$  est donné par la projection orthogonale sur  $\mathcal{P}_{n-1}$ , c'est-à-dire

$$K(z, w) = \sum_{l=0}^{n-1} a_l z^l \bar{w}^l$$

avec

$$(a_l)^{-1} = \int_{\mathbb{C}} |z|^{2l} d\sigma(z).$$

C'est une remarque vraiment jolie et une structure basée sur celle-ci a été construite.

**Définition 2.7** (Processus ponctuel déterminantal). *Soient  $\xi$  un processus ponctuel sur un espace polonais localement compact  $M$ ,  $\sigma$  une mesure sans atomes sur  $M$  et  $K : M \times M \rightarrow \mathbb{C}$  une fonction continue. Nous disons que  $\xi$  est un processus ponctuel déterminantal de noyau  $K : M \times M \rightarrow \mathbb{C}$  par rapport à  $\sigma$  si la  $k$ -ème fonction de corrélation est*

$$\rho_k(x_1, \dots, x_k) = \det \left( K(x_i, x_j)_{i,j \in \{1, \dots, k\}} \right)$$

pour  $k \geq 1$ .

À titre de remarque, nous devrions dire que si  $\xi$  est un processus ponctuel déterminantal de noyau  $K$  par rapport à la mesure définie par  $\phi(x)d\sigma(x)$  alors  $X$  est un processus ponctuel déterminantal de noyau  $(x, y) \mapsto \sqrt{\phi(x)}K(x, y)\sqrt{\phi(y)}$  par rapport à  $\sigma$ . De plus, si  $K$  est un noyau pour le processus ponctuel déterminantal  $\xi$  alors  $(x, y) \mapsto \psi(x)K(x, y)\psi(y)^{-1}$  fonctionne également en tant que noyau pour toute fonction  $\psi : M \rightarrow \mathbb{C} \setminus \{0\}$ .

Pour plus d'informations à ce sujet, nous vous recommandons [18, Chapitre 4].

## 2.4 Convergence des processus ponctuels déterminantaux

Ce que nous apprécions le plus, c'est l'interaction entre le noyau et le processus. Nous allons énoncer une très belle propriété de continuité qui sera fortement utilisée dans les chapitres 4 et 5. Cela se trouve dans [90].

**Proposition 2.8** (Continuité). *Supposons que  $\{\xi_n\}_{n \in \mathbb{N}}$  est une suite de processus ponctuels déterminantaux de noyaux continus  $\{K_n : M \times M \rightarrow \mathbb{C}\}_{n \in \mathbb{N}}$  par rapport à  $\sigma$ . Si*

$$\lim_{n \rightarrow \infty} K_n = K$$

*uniformément sur les espaces compacts de  $M \times M$ , il existe un processus ponctuel déterminantal  $\xi$  de noyau  $K$  par rapport à  $\sigma$  et*

$$\lim_{n \rightarrow \infty} \xi_n = \xi$$

*en loi.*

Par exemple, si nous prenons les valeurs propres d'une matrice aléatoire appartenant à l'ensemble de Ginibre, nous obtenons un processus ponctuel déterminantal de noyau

$$K_n(z, w) = \sum_{k=0}^{n-1} \frac{z^k \bar{w}^k}{k!} e^{-|z|^2/2} e^{-|w|^2/2}$$

par rapport à  $\ell_{\mathbb{C}}$ . Ensuite, si  $n$  tend vers l'infini,  $K_n$  tend vers

$$K(z, w) = e^{z\bar{w}} e^{-|z|^2/2} e^{-|w|^2/2} = e^{-\frac{1}{2}|z-w|^2 + i\Im(z\bar{w})}$$

où  $\Im$  désigne la partie imaginaire. Ce noyau  $K$  définit un processus ponctuel déterminantal  $\xi$  qui a une loi invariante sous les isométries du plan. En effet, si  $|u| = 1$  et  $a \in \mathbb{C}$ , alors, en prenant l'isométrie

$$T : z \mapsto \bar{u}(z - a),$$

le processus ponctuel image  $T_*\xi$  est déterminantal de noyau

$$(z, w) \mapsto K(uz + a, uw + a)$$

par rapport à  $\ell_{\mathbb{C}}$  et

$$K(uz + a, uw + a) = e^{i\Im(uz\bar{a})} K(z, w) e^{i\Im(a\bar{u}\bar{w})} = e^{i\Im(zu\bar{a})} K(z, w) e^{-i\Im(wu\bar{a})}.$$

Pour un autre exemple, si nous prenons les valeurs propres d'une matrice appartenant à l'ensemble de Haar unitaire tronqué, nous obtenons un processus ponctuel déterminantal de noyau

$$K_n(z, w) = \frac{1}{\pi} \sum_{k=0}^{n-1} (k+1) z^k \bar{w}^k$$

par rapport à la mesure de Lebesgue restreinte au disque unité  $\ell_{\mathbb{D}}$ . Quand  $n$  tend vers l'infini, ce noyau tend vers

$$K(z, w) = \frac{1}{\pi} \sum_{k=0}^{\infty} (k+1) z^k \bar{w}^k = \frac{1}{\pi(1-z\bar{w})^2}.$$

Ceci est appelé le noyau de Bergman du disque unité (voir [68, Section 1.4]). Il définit la projection orthogonale sur l'espace des fonctions holomorphes de carré intégrable sur  $\mathbb{D}$ . Cette description nous permet de démontrer l'invariance du processus limite  $\xi$ . En effet, si  $T : \mathbb{D} \rightarrow \mathbb{D}$  est une transformation de Möbius du disque ou, de manière équivalente, une isométrie du disque

hyperbolique, en utilisant la notation  $T^{-1}(z) = \tilde{z}$  et  $T^{-1}(w) = \tilde{w}$ , la fonction suivante est un noyau pour le processus ponctuel image  $T_*\xi$

$$(z, w) \mapsto \tilde{K}(z, w) = T'(\tilde{z})^{-1}K(\tilde{z}, \tilde{w})\overline{T'(\tilde{w})}^{-1}$$

où ' désigne la dérivée complexe. Nous pouvons démontrer que  $\tilde{K}$  représente également la projection orthogonale sur l'espace des fonctions holomorphes de carré intégrable sur  $\mathbb{D}$ . En effet,  $\tilde{K}$  est holomorphe par rapport à la première variable, antiholomorphe par rapport à la seconde et si  $f : \mathbb{D} \rightarrow \mathbb{C}$  est une fonction holomorphe alors

$$\begin{aligned} \int_{\mathbb{D}} T'(\tilde{z})^{-1}K(\tilde{z}, \tilde{w})\overline{T'(\tilde{w})}^{-1}f(w)d\ell_{\mathbb{D}}(w) &= \int_{\mathbb{D}} T'(\tilde{z})^{-1}K(\tilde{z}, \tilde{w})\overline{T'(\tilde{w})}^{-1}f \circ T(\tilde{w})|T'(\tilde{w})|^2d\ell_{\mathbb{D}}(\tilde{w}) \\ &= \int_{\mathbb{D}} T'(\tilde{z})^{-1}K(\tilde{z}, \tilde{w})(f \circ T(\tilde{w}))T'(\tilde{w})d\ell_{\mathbb{D}}(\tilde{w}) \\ &= T'(\tilde{z})^{-1}(f \circ T(\tilde{z}))T'(\tilde{z}) \\ &= f \circ T(\tilde{z}) = f(z). \end{aligned}$$

Nous concluons que  $\tilde{K} = K$  et que le processus ponctuel  $T_*\xi$  a la même loi que le processus ponctuel  $\xi$ , de sorte que la loi de  $\xi$  est invariante sous les isométries du disque hyperbolique. Nous expliquons pourquoi ce processus est naturellement lié au disque en tant que variété complexe dans la section suivante.

## 2.5 Une digression sur les surfaces de Riemann

Cette sous-section est ajoutée uniquement par souci d'exhaustivité et pour tenter de mieux comprendre le noyau  $K$  dans le cas du disque hyperbolique.

**Définition 2.9** (Surface de Riemann). *Soit  $M$  une variété topologique et  $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$  un atlas où chaque carte prend des valeurs dans  $\mathbb{C}$ . Nous disons que cet atlas est holomorphe si chaque application de changement de cartes  $\varphi_\lambda \circ \varphi_\kappa^{-1}$  est holomorphe. Dans ce cas, nous appelons  $(M, \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda})$  une variété complexe de dimension (complexe) 1 ou une surface de Riemann.*

La sphère  $S^2$  est l'un de nos exemples les plus importants de surfaces de Riemann. Les cartes considérées sont la projection stéréographique à partir du pôle nord et le conjugué de la projection stéréographique à partir du pôle sud<sup>2</sup>. L'application de changement de cartes sera  $z \mapsto 1/\bar{z}$ . D'autres exemples de surfaces de Riemann, appelées sous-variétés complexes, peuvent être obtenus en considérant des sous-ensembles ouverts dotés de l'atlas formé par les restrictions des cartes. Par exemple, le disque unité (l'hémisphère sud) est une sous-variété complexe de la surface de Riemann  $S^2$ .

Si  $M$  est une surface de Riemann, par analogie avec le cas réel, nous dirons qu'une famille d'applications mesurables  $\nu = \{\nu^\lambda : V_\lambda \rightarrow \mathbb{C}^*\}_{\lambda \in \Lambda}$  est une *1-forme complexe* (mesurable) si quels que soient  $\kappa, \lambda \in \Lambda$

$$\nu_x^\kappa = \nu_y^\lambda \circ d(\varphi_\lambda \circ \varphi_\kappa^{-1})_x \quad (24)$$

pour chaque  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$  et  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$ . En fait, dans ce cas, comme les dérivées des applications de changement de cartes sont  $\mathbb{C}$ -linéaires et que la dimension est 1, nous pouvons faire une grande simplification et nous considérons une *1-forme complexe* comme une famille de fonctions mesurables  $F = \{F^\lambda : V_\lambda \rightarrow \mathbb{C}\}_{\lambda \in \Lambda}$  telle que pour chaque  $\kappa, \lambda \in \Lambda$

$$F^\kappa(x) = (\varphi_\lambda \circ \varphi_\kappa^{-1})'(x) F^\lambda(y)$$

<sup>2</sup>Cette projection stéréographique conjuguée à partir du pôle sud est l'application de  $S^2 \setminus \{(0, 0, -1)\}$  dans  $\mathbb{C}$  définie par  $(x, y, z) \mapsto (\frac{x}{1+z}, \frac{-y}{1+z})$ .

si  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$  et  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$  où ' désigne la dérivée complexe. La grande remarque est que si  $F$  et  $G$  sont deux 1-formes complexes, alors la famille  $F\bar{G} = \{F^\lambda \bar{G}^\lambda\}_{\lambda \in \Lambda}$  satisfait

$$(F^\kappa \bar{G}^\kappa)(x) = \left| (\varphi_\lambda \circ \varphi_\kappa^{-1})' \right|^2 (x) (F^\lambda \bar{G}^\lambda)(y)$$

pour chaque  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$  et  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$ . Cela nous indique que nous pouvons définir une mesure à partir de  $F\bar{G}$ . Nous recommandons [94, Chapitre 5] pour l'approche standard.

**Définition 2.10** (Mesure de volume produit). *La mesure de volume produit d'une 1-forme complexe par elle-même est la mesure positive  $\mu_{F\bar{F}}$  telle que pour chaque fonction lisse positive  $f : M \rightarrow \mathbb{R}$*

$$\int_{U_\lambda} f d\mu_{F\bar{F}} = \int_{V_\lambda} f^\lambda(x) (F^\lambda \bar{F}^\lambda)(x) d\ell_{\mathbb{C}}(x).$$

Nous disons que  $F$  est de carré intégrable si  $\mu_{F\bar{F}}$  est une mesure finie.

La mesure de volume produit de deux 1-formes complexes de carré intégrables  $F$  et  $G$  est la mesure complexe  $\mu_{F\bar{G}}$  telle que pour chaque fonction lisse bornée  $f : M \rightarrow \mathbb{R}$

$$\int_{U_\lambda} f d\mu_{F\bar{G}} = \int_{V_\lambda} f^\lambda(x) (F^\lambda \bar{G}^\lambda)(x) d\ell_{\mathbb{C}}(x).$$

Le produit scalaire de deux 1-formes complexes de carré intégrables  $F$  et  $G$  est défini par

$$\langle F, G \rangle = \mu_{F\bar{G}}(M).$$

Ce qui est remarquable à propos de cette définition, c'est que nous n'avons pas besoin d'une mesure de référence pour qu'elle ait un sens. En d'autres termes, il existe une signification intrinsèque du produit (mesure de volume et produit scalaire) de deux 1-formes complexes. Nous allons considérer  $L^2(M)$ , l'espace des 1-formes complexes de carré intégrable, où nous identifions deux si elles sont égales presque partout sur chaque carte. Comme d'habitude, il s'agit d'un espace de Hilbert, mais il est maintenant intrinsèquement lié à la surface de Riemann  $M$ . La notion de noyau  $K$  s'avère également bien définie dans ce contexte.

**Définition 2.11** (Noyau). *Un noyau est une famille  $K = \{K^{\lambda, \bar{\lambda}} : V_\lambda \times V_{\bar{\lambda}} \rightarrow \mathbb{C}\}_{\lambda, \bar{\lambda} \in \Lambda}$  telle que*

$$K^{\kappa, \bar{\kappa}}(x, \tilde{x}) = \overline{(\varphi_\lambda \circ \varphi_\kappa^{-1})'(x)} (\varphi_{\bar{\lambda}} \circ \varphi_{\bar{\kappa}}^{-1})'(\tilde{x}) K^{\lambda, \bar{\lambda}}(y, \tilde{y})$$

quels que soient  $x \in \varphi_\kappa(U_\lambda \cap U_\kappa)$ ,  $\tilde{x} \in \varphi_{\bar{\kappa}}(U_{\bar{\lambda}} \cap U_{\bar{\kappa}})$ ,  $y = \varphi_\lambda \circ \varphi_\kappa^{-1}(x)$  et  $\tilde{y} = \varphi_{\bar{\lambda}} \circ \varphi_{\bar{\kappa}}^{-1}(\tilde{x})$ .

En suivant le même schéma, nous pouvons prendre un noyau  $K$ , une 1-forme  $F$  et construire la 1-forme

$$K \cdot F = \int_M K(x, y) \bar{F}(y).$$

Un noyau définit un opérateur sans avoir besoin d'une mesure de référence! De plus, nous pouvons définir  $K(x, x)$  comme une mesure sur  $M$  et  $\det(K(x_i, x_j)_{i, j \in \{1, \dots, k\}})$  comme une mesure sur  $M^k$  par une procédure analogue. En résumé, toutes les opérations nécessaires pour définir un processus ponctuel déterminantal ont un sens avec ce nouvel objet. Il y a un cas d'intérêt particulier pour nous. Nous disons que la 1-forme complexe  $F = \{F^\lambda : V_\lambda \rightarrow \mathbb{C}\}_{\lambda \in \Lambda}$  est holomorphe si chaque  $F^\lambda$  est holomorphe. Notons  $L^2_{\mathcal{H}}(M)$  l'espace des 1-formes holomorphes de carré intégrable habituellement appelé l'espace de Bergman. Cet espace est un sous-espace fermé de  $L^2(M)$ , ce qui peut être obtenu en utilisant la propriété de sous-moyenne locale pour les fonctions sous-harmoniques.

**Définition 2.12** (Processus de Bergman). *Le noyau  $K$  qui définit la projection orthogonale de  $L^2(M)$  sur  $L^2_{\mathcal{H}}(M)$  est appelé le noyau de Bergman de  $M$  et le processus ponctuel déterminantal associé s'appelle processus de Bergman  $\mathcal{B}_M$ .*

Nous suggérons au lecteur de regarder les deux premières sections de [67]. Cela inclut le cas général d'une variété complexe dans n'importe quelle dimension.

### 3 Un modèle physique

Maintenant, nous présentons l'une des motivations physiques habituelles pour considérer des mesures de Gibbs et expliquons pourquoi nous nous sommes débarrassés du terme d'énergie cinétique. En particulier, nous allons expliquer la notion d'énergie libre et affirmer que les mesures de Gibbs en sont les minimiseurs. C'est un outil essentiel pour le chapitre 6.

#### 3.1 Le cadre euclidien

Nous considérons  $\mathbb{R}^d$  comme l'espace de configuration. Nous pourrions le penser comme l'espace des positions d'une particule. Pour caractériser complètement son état, nous devons ajouter la quantité de mouvement  $p \in \mathbb{R}^d$ . L'espace d'états, également appelé espace de phases, sera  $\mathbb{R}^d \times \mathbb{R}^d$  où la première coordonnée est la position et la seconde la quantité de mouvement. Soit  $U : \mathbb{R}^d \rightarrow (-\infty, \infty]$  une fonction mesurable pensée comme l'énergie potentielle. Nous définissons l'énergie totale

$$H(x, p) = \frac{1}{2m} p^2 + U(x)$$

où  $m$  désigne la masse de la particule. Parfois, nous pouvons ne pas connaître la position exacte du système mais nous disposons d'une distribution de probabilité pour le décrire. Lorsque la température du système est connue mais que l'énergie totale ne l'est pas nous choisissons la mesure de Gibbs  $\mathbb{P}$  donnée par

$$d\mathbb{P} = \frac{1}{\mathcal{Z}} e^{-\beta H} d\ell_{\mathbb{R}^{2d}}$$

où  $\beta$  est (proportionnel à) l'inverse de la température et  $\mathcal{Z}$  est la constante de normalisation généralement appelée fonction de partition. Par la définition de  $H$ , nous remarquons que  $\mathbb{P}$  est une mesure produit  $\mathbb{P}_1 \otimes \mathbb{P}_2$  où

$$d\mathbb{P}_1 \text{ est proportionnel à } e^{-\beta U} d\ell_{\mathbb{R}^d}$$

et

$$d\mathbb{P}_2 \text{ est proportionnel à } e^{-\beta \frac{p^2}{2m}} d\ell_{\mathbb{R}^d}.$$

La mesure  $\mathbb{P}_2$  est une mesure de probabilité gaussienne, de sorte que nous ne nous intéresserons qu'à  $\mathbb{P}_1$ . Si nous considérons  $n$  particules sur  $\mathbb{R}^d$  de masse  $m$ , nous devrions remplacer  $\mathbb{R}^d$  par  $(\mathbb{R}^d)^n$  et  $U$  par  $U_n : (\mathbb{R}^d)^n \rightarrow (-\infty, \infty]$ . L'énergie totale  $H_n : (\mathbb{R}^d \times \mathbb{R}^d)^n \rightarrow (-\infty, \infty]$  du système serait définie par

$$H_n(x_1, p_1, \dots, x_n, p_n) = \frac{1}{2m} \sum_{i=1}^n p_i^2 + U_n(x_1, \dots, x_n).$$

De nouveau, nous définissons la mesure de Gibbs  $\mathbb{P}$  par

$$d\mathbb{P} = \frac{1}{\mathcal{Z}_n} e^{-\beta H_n} d\ell_{\mathbb{R}^{2dn}}$$

où  $\mathcal{Z}_n$  est la constante de normalisation. Sa projection  $\mathbb{P}_1$  sur l'espace de configuration  $(\mathbb{R}^d)^n$  (espace des positions) serait telle que

$$d\mathbb{P}_1 \text{ est proportionnel à } e^{-\beta U_n} d\ell_{\mathbb{R}^{dn}}.$$

Dans le cas riemannien, la seule chose à comprendre serait ce qu'est la quantité de mouvement. Cela peut nécessiter des définitions qui nous éloigneraient des travaux que nous voulons présenter. Néanmoins, comme nous considérons cela comme un cadre très intéressant, nous utiliserons quelques lignes pour le mentionner et renverrons au lecteur aux quelques-uns des livres très intéressants sur les variétés symplectiques (par exemple, [78] ou [9, Chapitre 8]).



### 3.2 Le cadre riemannien

Soit  $M$  une variété riemannienne de dimension  $d$ . Si  $U : M \rightarrow (-\infty, \infty]$  est l'énergie potentielle, une particule située à  $x$  et de quantité de mouvement  $p$  aura l'énergie totale

$$H(x, p) = \frac{1}{2m} p^2 + U(x) \quad (25)$$

où  $m$  est la masse de la particule et le premier terme de la somme est l'énergie cinétique. Comme la particule vit sur une variété, nous devrions d'abord préciser où réside réellement la quantité de mouvement  $p$ . Si la particule est située dans  $x$  alors sa vitesse vit sur  $T_x M$ , l'espace tangent de  $M$  en  $x$ . Cet espace vectoriel a de nombreuses descriptions différentes mais nous pouvons garder à l'esprit qu'il est formé par les valeurs possibles qu'un champ de vecteur peut prendre en  $x$ . La quantité de mouvement, en tant que dual de la vitesse, devrait alors vivre sur le dual de  $T_x M$ , noté  $T_x^* M$ . Encore une fois, ceci est simplement formé par les valeurs possibles qu'une 1-forme peut prendre en  $x$ . Nous allons donner une description plus précise. Considérons  $\Lambda_x \subset \Lambda$  composé de tous les index  $\lambda \in \Lambda$  tels que  $x \in U_\lambda$ . Une forme linéaire dans  $x$  ou une quantité de mouvement basée sur  $x$  sera une famille  $p = \{p^\lambda\}_{\lambda \in \Lambda_x}$  d'éléments de  $\mathbb{R}^{d^*}$  telle que

$$p^\kappa = p^\lambda \circ d \left( \varphi_\lambda \circ \varphi_\kappa^{-1} \right)_{\varphi_\kappa(x)}$$

pour chaque  $\lambda, \kappa \in \Lambda_x$ . Une 1-forme telle que définie dans (4) peut être vue comme une fonction de  $M$  à valeurs dans le fibré cotangent  $T^* M = \cup_{x \in M} \{x\} \times T_x^* M$ . La régularité (et la continuité) des 1-formes est capturée par la structure de variété lisse (et la topologie) naturelle de  $T^* M$  donnée par l'atlas  $\{\psi^\lambda : T^* U^\lambda \rightarrow V^\lambda \times \mathbb{R}^{d^*}\}_{\lambda \in \Lambda}$ , où  $T^* U^\lambda = \cup_{x \in U^\lambda} \{x\} \times T_x^* M$  et

$$\psi^\lambda(x, p) = (\varphi^\lambda(x), p^\lambda).$$

Si  $h$  est une métrique riemannienne sur  $M$ , nous pouvons définir, par dualité, un produit scalaire sur chaque  $T_x^* M$ . À savoir, nous définissons

$$\langle p_1, p_2 \rangle_x = h_{\varphi^\lambda(x)}^\lambda(p_1^\lambda, p_2^\lambda) = (p_1^\lambda)^\top \left( h_{\varphi^\lambda(x)}^\lambda \right)^{-1} p_2^\lambda.$$

où à droite nous pensons  $h_{\varphi^\lambda(x)}^\lambda$  comme une matrice définie positive et  $p^\lambda$  comme un vecteur colonne. Si nous définissons, dans (25),  $p^2$  comme  $\langle p, p \rangle_x$  alors  $H$  est une application bien définie sur  $T^* M$ . Étant donné  $\beta > 0$  nous allons considérer la mesure de Gibbs  $\bar{\mathbb{P}}$  définie par

$$d\bar{\mathbb{P}} = \frac{1}{\mathcal{Z}} e^{-\beta H} d \text{vol}$$

où  $\text{vol}$  dénote la mesure suivante sur  $T^* M$ .

**Définition 3.1** (La mesure de volume du fibré cotangent). *La mesure de volume  $\text{vol}$  est la mesure positive sur  $T^* M$  telle que, pour chaque fonction lisse positive  $f : T^* M \rightarrow \mathbb{R}$ ,*

$$\int_{T^* U^\lambda} f d \text{vol} = \int_{V^\lambda \times \mathbb{R}^{d^*}} f^\lambda d\ell_{\mathbb{R}^{2d}}.$$

Si nous voulons une formulation plus intrinsèque, nous pouvons utiliser la structure symplectique standard sur  $T^* M$ . Celle-ci est utilisée pour définir la dynamique donnée par un hamiltonien et elle induit aussi une mesure naturelle qui est exactement celle obtenue dans la définition 3.1. La remarque intéressante est que, pour tout  $\kappa > 0$ , la projection de  $\exp(-\kappa p^2) d \text{vol}$  sur  $M$  est proportionnelle à la mesure de volume sur  $M$ , associée à la métrique, que nous appellerons  $\sigma$ . Ainsi, la projection de  $\bar{\mathbb{P}}$  sur  $M$  est  $\mathbb{P}$ , donnée par

$$d\mathbb{P} = \frac{1}{\mathcal{Z}} e^{-\beta U} d\sigma.$$

Supposons maintenant que nous avons un système de  $n$  particules sur  $M$ . Nous pouvons remplacer  $M$  par  $M^n$ , doté de la structure riemannienne produit induite, et  $T^*M$  par  $T^*(M^n)$  qui est canoniquement égal à  $(T^*M)^n$ . La mesure induite par la structure symplectique sur  $(T^*M)^n$  serait  $\text{vol}^{\otimes n}$ . En remplaçant  $U$  par  $U_n : M^n \rightarrow (-\infty, \infty]$ , nous pouvons définir l'énergie totale par

$$H_n(x_1, p_1, \dots, x_n, p_n) = \frac{1}{2m} \sum_{i=1}^n p_i^2 + U_n(x_1, \dots, x_n).$$

La projection de la mesure de Gibbs sur  $T^*M^n$  définie par

$$d\bar{\mathbb{P}}_n = \frac{1}{\bar{\mathcal{Z}}_n} e^{-\beta H_n} d\text{vol}^{\otimes n}$$

serait la mesure de probabilité  $\bar{\mathbb{P}}_n$  donnée par

$$d\mathbb{P}_n = \frac{1}{\mathcal{Z}_n} e^{-\beta U_n} d\sigma^{\otimes n}.$$

où  $\sigma$  est la mesure de volume sur  $M$ .

### 3.3 L'énergie libre

Dans tous ces cas ou en général si  $(M, \sigma)$  est un espace mesuré polonais et si  $H : M \rightarrow (-\infty, \infty]$  est une fonction mesurable, nous pouvons définir l'énergie libre dans l'état  $\mu \in \mathcal{P}(M)$ , où  $\mathcal{P}(M)$  est l'espace des mesures de probabilité, comme

$$F_\beta(\mu) = \mathbb{E}_\mu[H] - \frac{1}{\beta} S(\mu)$$

où  $S(\mu)$  est l'entropie physique définie par

$$S(\mu) = - \int_M \rho \log \rho d\sigma \quad \text{si } d\mu = \rho d\sigma$$

et

$$S(\mu) = -\infty \quad \text{si l'intégral n'est pas bien définie.}$$

Un principe standard de la thermodynamique nous dit qu'un système à l'équilibre tend à minimiser l'énergie et à maximiser l'entropie. À une température inverse  $\beta$  cela est interprété comme le système souhaitant minimiser l'énergie libre  $F_\beta$ . Une des interprétations possibles et vraiment intéressantes de la mesure de Gibbs est que  $F_\beta$  a un minimiseur unique qui est la mesure de Gibbs  $\mathbb{P}$  donnée par

$$d\mathbb{P} = \frac{1}{\mathcal{Z}} e^{-\beta H} d\sigma$$

et que la fonction de partition

$$\mathcal{Z} = \int_M e^{-\beta H} d\sigma$$

est liée au minimum  $F_\beta(\mathbb{P})$  par

$$F_\beta(\mathbb{P}) = -\log \mathcal{Z}.$$

Cette idée sera la *structure clé* utilisée pour étudier les grandes déviations dans le cadre des mesures de Gibbs (voir chapitre 6). Au lieu de travailler avec des mesures positives générales, nous allons utiliser une mesure de probabilité  $\sigma$  et écrire

$$D(\mu \parallel \sigma) = -S(\mu).$$

Ceci est l'entropie relative habituelle ou divergence de Kullback-Leibler.

## 4 Questions

Notre tâche consistera à décrire le comportement limite de  $\mathbb{P}_n$  quand  $n$  tend vers l'infini. Nous allons d'abord mentionner le type de questions que nous pouvons nous poser.

### 4.1 Comportement macroscopique

Supposons

$$(X_1, \dots, X_n) \sim \frac{1}{Z_n} e^{-\beta_n H_n} d\sigma^{\otimes n}.$$

Nous voudrions donner des conditions aux suites  $\{H_n\}_{n \in \mathbb{N}}$  et  $\{\beta_n\}_{n \in \mathbb{N}}$  afin que nous puissions trouver une (qui peut être aléatoire) mesure de probabilité  $\mu$  sur  $M$  telle que

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_i} = \mu. \quad (26)$$

L'espace des mesures de probabilité  $\mathcal{P}(M)$  sur  $M$  est doté ici de la plus petite topologie telle que les applications

$$\hat{f}(\mu) = \int_M f d\mu$$

soient continues pour chaque fonction continue bornée  $f : M \rightarrow \mathbb{R}$ . Dans le cas habituel d'une limite déterministe  $\mu$ , nous pouvons aussi chercher une suite de nombres positifs  $\{\alpha_n\}_{n \in \mathbb{N}}$  telle que

$$\alpha_n \left( \frac{1}{n} \sum_{i=1}^n \delta_{X_i} - \mu \right) \text{ converge} \quad (27)$$

en un sens à préciser. Dans le chapitre 6, nous verrons que dans un cadre très général de particules non-attirantes, nous pouvons démontrer un principe de grandes déviations qui implique, dans certaines conditions, une convergence presque sûre. Ce principe de grandes déviations sera une conséquence du principe de Laplace, philosophie principale utilisée dans [47]. Notre résultat s'applique également dans le contexte des polynômes aléatoires étudiés dans [98] (voir aussi [30]). La première approche de grandes déviations peut être trouvée dans les travaux de Ben Arous et Guionnet [16] qui la relie aux idées d'entropie libre introduites par Voiculescu [96] en soulignant l'analogie avec le théorème de Sanov. Depuis lors, il y a eu beaucoup d'activité dans ce domaine. Nous pouvons voir par exemple [17], [60], [27], [61], [58], [35] et [48] cités par ordre chronologique.

De plus, dans le chapitre 3 et dans le cas particulier d'un gaz de Coulomb sur une variété compacte, nous trouvons des bornes non asymptotiques à la probabilité de rester loin de la limite, sujet parfois appelé concentration de la mesure. Certains gaz de Coulomb euclidiens ont été étudiés dans [37] motivés par le cas logarithmique unidimensionnel de [76].

Il y a eu beaucoup d'activités concernant la question (27). Par exemple, dans le cas d'une interaction logarithmique unidimensionnelle, nous référons à [64]. Le cas déterminantal bidimensionnel a été étudié d'abord dans [83] dans le cas radial, puis dans [3] et [2] dans un contexte non radial. Il a été étendu pour des variétés complexes unidimensionnelles générales dans [22]. Ensuite, dans [72] et [14], le cas non déterminantal de dimension deux a été étudié.

### 4.2 Comportement des outliers

Une deuxième question est la suivante. Si nous avons un ensemble ouvert  $A \subset M$  tel que  $\mu(A) = 0$ , où  $\mu$  est obtenu dans (26), trouver un processus ponctuel (limite) sur  $A$ . Plus précisément, étant donné une fonction continue  $f : M \rightarrow \mathbb{R}$  à support sur  $A$ , nous voudrions

trouver une variable aléatoire  $\mathcal{B}_f$  telle que

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(X_i) = \mathcal{B}_f. \quad (28)$$

Nous étudierons aux chapitres 4 et 5 le cas déterminantal de dimension deux et nous obtiendrons le processus de Bergman qui est considéré dans [79] et mentionné dans la sous-section 2.4. Les méthodes de polynômes orthogonaux telles que celles utilisées dans [4], [5] et [59] peuvent être appliquées pour étudier le cas non radial, mais nous voulions souligner la simplicité en étudiant le cadre radial.

### 4.3 Comportement microscopique

Cette troisième question a du sens sur une variété, et nous la posons avec cette généralité afin de souligner que l'objet obtenu serait défini naturellement sur l'espace tangent. Nous voyons cela seulement comme une remarque intéressante avec peut-être une importance future. Néanmoins, comme nous n'allons pas vraiment l'utiliser, il n'y a aucun mal à oublier  $\varphi$  et à prendre  $M = \mathbb{R}^d$ .

Considérons une variété riemannienne  $M$ . Prenons un point  $x \in M$  dans le support de  $\mu$ , la mesure limite obtenue dans (26). Considérons un difféomorphisme  $\varphi : U \subset M \rightarrow V \subset T_x M$  tel que  $x \in U \mapsto 0 \in V$  et  $d\varphi_x = \text{Id}_{T_x M}$ . La question est de trouver une suite  $\{\alpha_n\}_{n \in \mathbb{N}}$  telle que pour chaque fonction continue à support compact  $f$  sur  $T_x M$ , il existe une variable aléatoire  $\mathcal{G}_f$  telle que

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(\alpha_n \varphi(X_i)) = \mathcal{G}_f$$

où nous avons pris uniquement les  $X_k$  qui appartiennent à  $U$ . Nous pouvons remarquer que dans le cas où cette limite existe pour chaque  $f$ , la suite  $\{\alpha_n\}_{n \in \mathbb{N}}$  est essentiellement unique, de même que les variables aléatoires  $\mathcal{G}_f$ . Bien que ce soit le cas général, nous étudierons un système sur la sphère ou, par la projection stéréographique, sur le plan complexe, de sorte qu'aucune carte n'est nécessaire.

Nous verrons beaucoup de comportements différents dans les chapitres 4 et 5. Pour les premiers comportements microscopiques non triviaux, voir [49, Section 15.3]. Le processus ponctuel infini de Ginibre peut être trouvé comme une limite microscopique dans [19] où nous pouvons également trouver des généralisations intéressantes des gaz de Coulomb dans un cadre déterminantal sur des variétés complexes de dimension quelconque. D'autres comportements proches du bord du support de la mesure macroscopique limite sont étudiés dans [4], [5] et [59].

## 5 Résultats

Nous énoncerons ce que nous croyons être les résultats les plus intéressants des chapitres suivants. Pour que cela ne soit pas aussi répétitif, nous allons d'abord énoncer une version du cas riemannien lorsqu'il en existe une.

### 5.1 Comportement macroscopique

Rappelons que  $D(\mu \parallel \sigma) = \int_M \rho \log \rho \, d\sigma$  si  $\rho$  est la densité de  $\mu$  par rapport à  $\sigma$  et  $D(\mu \parallel \sigma) = \infty$  si  $\mu$  n'est pas absolument continu par rapport à  $\sigma$ . Fixons une mesure signée lisse  $\Lambda$  de masse totale un sur la variété riemannienne compacte  $M$ . Rappelons que  $G$  est la fonction de Green associée à  $\Lambda$  et que  $\sigma$  est la mesure de volume de  $M$  de masse total un. Considérons  $(X_1, \dots, X_n)$  le gaz de Coulomb sur  $M$  de  $n$  particules de charge  $q_n$ , à la température inverse  $\beta_n$  et dans un milieu de charge  $\Lambda$ . Le théorème suivant est une conséquence du corollaire 4.5 du chapitre 6.

**Théorème 5.1** (Convergence des mesures empiriques). *Si  $n\beta_n q_n^2 \rightarrow \beta \in (0, \infty]$  alors*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_i} = \mu_{\text{eq}}$$

*presque sûrement où  $\mu_{\text{eq}}$  est le minimiseur de*

$$\mu \in \mathcal{P}(M) \mapsto \frac{1}{2} \int_{M \times M} G(x, y) d\mu(x) d\mu(y) + \frac{1}{\beta} D(\mu \| \sigma).$$

*Démonstration.* C'est une conséquence du théorème 5.4 et 5.5 ci-dessous. Pour voir qu'il satisfait aux conditions requises par ces théorèmes, nous renvoyons au chapitre 3. Pour voir que le fonctionnel a un minimiseur unique, nous utilisons que  $D(\cdot \| \sigma)$  est strictement convexe dans l'ensemble (convexe) où il est fini et

$$\mu \mapsto \frac{1}{2} \int_{M \times M} G(x, y) d\mu(x) d\mu(y)$$

est également strictement convexe dans l'ensemble (convexe) où il est fini. Ce dernier fait est une combinaison de la proposition 4.8 du chapitre 6 et du lemme 3.2 du chapitre 3.  $\square$

Si nous supposons que  $\beta < \infty$ , il existe une caractérisation intéressante du minimiseur. Nous pouvons la trouver dans le théorème 4.6 au chapitre 6. Nous énonçons ici sa conséquence immédiate.

**Théorème 5.2** (Convergence pour une température positive). *Si  $n\beta_n q_n^2 \rightarrow \beta \in (0, \infty)$  alors*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_i} = \mu_{\text{eq}}$$

*presque sûrement où  $\mu_{\text{eq}} \in \mathcal{P}_{C^\infty}(M)$  a une densité  $\rho$  par rapport à  $\sigma$  qui satisfait*

$$\frac{1}{\beta} \Delta \log \rho = \mu_{\text{eq}} - \Lambda.$$

*Démonstration.* C'est une conséquence du théorème 5.1 ci-dessus et du théorème 4.6 du chapitre 6 dont la preuve marche dans n'importe quelle dimension.  $\square$

En particulier, nous savons ce qui se passe si le milieu est donné par la courbure en dimension deux. Plus précisément, si  $R$  est la courbure scalaire de  $M$ , choisissons  $\Lambda$  tel que

$$d\Lambda = \frac{R}{4\pi\chi} d\sigma$$

où  $\chi$  désigne la caractéristique d'Euler de  $M$ . Si la variété est la sphère, nous récupérons les modèles définis dans la sous-section 1.4. Si la variété est différente de la sphère et du tore, nous obtenons le cas spécial suivant qui est mentionné comme une remarque au chapitre 6.

**Corollaire 5.3** (Limite de courbure constante). *Supposons que  $M$  soit de dimension deux et que son genre soit strictement supérieur à 1. Notons par  $\chi$  sa caractéristique d'Euler. Si  $n\beta_n q_n^2 \rightarrow -4\pi\chi$  alors*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_i} = \mu_{\text{eq}}$$

*presque sûrement où  $\mu_{\text{eq}} \in \mathcal{P}_{C^\infty}(M)$  a courbure constante.*

*Démonstration.* C'est une conséquence du théorème 5.2 et de l'interprétation de la courbure scalaire donnée dans la remarque 4.7 du chapitre 6.  $\square$

Nous considérons cela comme un fait fascinant car il ne dépend pas de la métrique avec laquelle nous avons commencé. Le résultat principal qui permet de conclure ces convergences se trouve dans le théorème 1.2 du chapitre 6. La partie sur la température positive est la suivante.

**Théorème 5.4** (Principe de Laplace pour une température positive). *Soit  $M$  un espace polonais et considérons une suite  $\{H_n : M^n \rightarrow (-\infty, \infty]\}_{n \in \mathbb{N}}$  de fonctions mesurables uniformément bornées inférieurement. Supposons que*

- pour chaque  $\mu \in \mathcal{P}(M)$  existe

$$H(\mu) := \lim_{n \rightarrow \infty} \mathbb{E}_{\mu^{\otimes n}}[H_n]$$

- chaque fois que les mesures empiriques de  $\mathbf{x}_n \in M^n$  convergent vers certain  $\mu \in \mathcal{P}(M)$  nous avons

$$\liminf_{n \rightarrow \infty} H_n(\mathbf{x}_n) \geq H(\mu).$$

Alors, si  $\lim_{n \rightarrow \infty} \beta_n = \beta \in (0, \infty)$  et si  $i_n : M^n \rightarrow \mathcal{P}(M)$  désigne l'application définie par  $i_n(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ , nous avons

$$\lim_{n \rightarrow \infty} \frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f \circ i_n} e^{-n\beta_n H_n} d\sigma^{\otimes n} = - \inf_{\mu \in \mathcal{P}(M)} \left\{ f(\mu) + H(\mu) + \frac{1}{\beta} D(\mu \| \sigma) \right\}$$

pour toute fonction continue et bornée  $f : M \rightarrow \mathbb{R}$ .

Ce théorème est valable si  $\beta_n$  a une limite finie et positive. Néanmoins, les modèles de matrices aléatoires usuels ont une suite  $\{\beta_n\}_{n \in \mathbb{N}}$  qui tend vers l'infini. Cela nécessite deux autres hypothèses, comme nous le précisons ci-dessous. C'est une version simplifiée d'une partie du théorème 1.2 du chapitre 6.

**Théorème 5.5** (Principe de Laplace pour une température nulle). *Supposons les mêmes hypothèses que dans le théorème 5.4, sauf que maintenant  $\beta_n \rightarrow \infty$ . Supposons également que*

- pour chaque mesure  $\mu \in \mathcal{P}(M)$ , telle que  $H(\mu) < \infty$ , existe une suite  $\{\mu_n\}_{n \in \mathbb{N}}$  qui converge vers  $\mu$  telle que

$$D(\mu_n \| \sigma) < \infty \text{ pour chaque } n \text{ et } \lim_{n \rightarrow \infty} H(\mu_n) = H(\mu)$$

- si  $\mathbf{x}_n$  appartient à  $M^n$  et  $\liminf_{n \rightarrow \infty} H_n(\mathbf{x}_n) < \infty$  alors les mesures empiriques d'une sous-suite convergent.

Alors, nous avons la limite suivante

$$\lim_{n \rightarrow \infty} \frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f \circ i_n} e^{-n\beta_n H_n} d\sigma^{\otimes n} = - \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + H(\mu)\}$$

pour toute fonction continue et bornée  $f : M \rightarrow \mathbb{R}$ .

En fait, une légère modification doit être faite pour pouvoir traiter le cas des polynômes aléatoires et des gaz de Coulomb conditionnés. Les conditions se trouvent dans le chapitre 6. Nous voudrions ajouter que l'étape principale de la preuve est la propriété décrite dans la sous-section 3.3, la caractérisation de la mesure de Gibbs comme le minimiseur de l'énergie libre.

Dans le chapitre 3, nous avons étudié le comportement non asymptotique des gaz de Coulomb. Rappelons que la distance de Kantorovich-Wasserstein  $W_1$  est une distance entre les mesures de probabilité définie par

$$W_1(\mu, \nu) = \sup \left\{ \int_M f d\mu - \int_M f d\nu : f \in C^\infty(M) \text{ et } \forall x \in M, h(\nabla f, \nabla f)(x) \leq 1 \right\}.$$

Soit  $\mu_{eq}$  le minimiseur de l'énergie macroscopique  $\mu \mapsto \int_{M \times M} G(x, y) d\mu(x) d\mu(y)$ . Nous obtenons ce qui suit comme conséquence du théorème 1.2 au chapitre 3.

**Théorème 5.6** (Inégalité de concentration). *Si la dimension de  $M$  est  $d = 2$ , il existe une constante  $C > 0$  qui dépend uniquement de la variété riemannienne  $M$  et de  $\Lambda$  telle que*

$$\mathbb{P} \left( W_1 \left( \frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \mu_{eq} \right) \geq r \right) \leq \exp \left( -\beta q^2 n^2 \frac{r^2}{4} + \frac{\beta q^2}{8\pi} n \log(n) + nD(\mu_{eq} \parallel \sigma) + C\beta q^2 n \right)$$

*Si la dimension de  $M$  est  $d \geq 3$ , il existe une constante  $C > 0$  qui dépend uniquement de la variété riemannienne  $M$  et de  $\Lambda$  telle que*

$$\mathbb{P} \left( W_1 \left( \frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \mu_{eq} \right) \geq r \right) \leq \exp \left( -\beta q^2 n^2 \frac{r^2}{4} + nD(\mu_{eq} \parallel \sigma) + C\beta q^2 n^{2(d-1)/d} \right).$$

L'un des aspects importants de ce résultat de concentration est que le terme de correction ne dépend pas de  $r$  et qu'il a une dépendance simple de  $\beta q^2$ .

## 5.2 Comportement des outliers

Les résultats présentés dans cette sous-section se trouvent dans les chapitres 4 et 5. Nous allons commencer par donner l'interprétation de ces résultats en relation au modèle présenté dans la sous-section 1.4 sur la sphère.

Soit  $h$  une métrique riemannienne sur  $S^2$  de volume un et invariante par les rotations autour de l'axe  $z$ . Supposons que la courbure scalaire  $R_h$  vérifie les hypothèses suivantes:

- $R_h \geq 0$  partout,
- $R_h = 0$  sur l'hémisphère inférieur et
- le support de  $R_h$  contient l'équateur.

Notons par  $\alpha$  la mesure de volume associée à  $h$ .

**Théorème 5.7** (Limite sur l'hémisphère inférieur). *Soit  $(Y_1, \dots, Y_n)$  un gaz de Coulomb sur  $S^2$  de particules de charge un, à la température inverse  $\beta = 4\pi$  et dans un milieu de charge  $(8\pi)^{-1} R_h d\alpha$ . Désignons l'hémisphère inférieur ouvert par  $D$  et prenons le processus ponctuel sur  $D$  donné par*

$$\xi_n = \sum_{i=1}^n 1_{Y_i \in D} \delta_{Y_i}.$$

*Nous avons la convergence en loi suivante*

$$\lim_{n \rightarrow \infty} \xi_n = \mathcal{B}_D$$

où  $\mathcal{B}_D$  est le processus de Bergman sur  $D$  (voir la définition 2.12)

Il s'agit d'une reformulation du théorème 1.5 du chapitre 4 qui peut être considérée comme une conséquence du résultat suivant dans le plan complexe.

**Théorème 5.8** (Limite sur le disque). *Soit  $V : [0, \infty) \rightarrow \mathbb{R}$  une fonction mesurable positive telle que*

- $\liminf_{r \rightarrow \infty} \{V(r) - \log r\} > -\infty$ ,
- $V(r) > 0$  si  $r > 1$  et
- $V(r) = 0$  si  $r \leq 1$ .

*Si  $(Z_1, \dots, Z_n)$  est un gaz de Coulomb sur  $\mathbb{C}$  de particules de charge  $1/(n+1)$ , à la température inverse  $4\pi(n+1)$  et confiné par le potentiel  $z \mapsto V(|z|)$  alors nous avons la convergence en loi*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n 1_{|Z_i| < 1} \delta_{Z_i} = \mathcal{B}_{\mathbb{D}}$$

*où  $\mathbb{D}$  désigne le disque unité et  $\mathcal{B}_{\mathbb{D}}$  le processus ponctuel déterminantal associé au noyau de Bergman sur  $\mathbb{D}$  et par rapport à la mesure de Lebesgue.*

*Démonstration.* Voir le théorème 2.1 au chapitre 5. □

Ce théorème est une conséquence du comportement limite des noyaux correspondants. Ensuite, comme nous l'avons déjà dit, le théorème 5.7 est obtenu si nous identifions  $S^2$  moins le pôle nord et  $\mathbb{C}$ , de sorte que l'hémisphère inférieur est le disque unité de  $\mathbb{C}$ . Si nous utilisons l'identification entre  $S^2$  moins le pôle sud et  $\mathbb{C}$  donnée par l'autre projection stéréographique, l'hémisphère inférieur serait le complément du disque unité. De cette façon, nous avons une dualité entre les particules proches et éloignées de l'origine. Ceci est très exploité dans les chapitres 4 et 5, et beaucoup de comportements différents sont obtenus pour la particule la plus éloignée en comprenant le comportement de la particule la plus proche à zéro.

Nous nous sommes beaucoup intéressés au cas dégénéré où  $R_h d\alpha$  serait la mesure uniforme sur l'équateur mentionné dans la sous-section 1.5. Le comportement macroscopique est bien connu et peut être obtenu à partir des principes de grandes déviations. Supposons que  $Z_1, \dots, Z_n$  forment le gaz de Coulomb ou les zéros du polynôme aléatoire de la sous-section 1.5, puis

$$\frac{1}{n} \sum_{i=1}^n \delta_{Z_i} \rightarrow \text{mesure uniforme sur } S^1.$$

Nous pouvons démontrer que, loin du cercle unité, leur comportement est identique et il est relié à deux processus de Bergman indépendants  $\mathcal{B}_{\mathbb{D}}$  et  $\mathcal{B}_{\mathbb{C} \setminus \mathbb{D}}$ . Le résultat suivant est une reformulation simple des théorèmes de la sous-section 4.1 du chapitre 5.

**Théorème 5.9** (Limite loin du cercle). *Supposons que  $Z_1, \dots, Z_n$  forment le gaz de Coulomb déterminantal ou les zéros du polynôme aléatoire de la sous-section 1.5. Soit  $f : \mathbb{C} \rightarrow \mathbb{R}$  une fonction continue avec une limite finie à l'infini telle que son support et le cercle unité sont disjoints. Alors*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(Z_i) = \mathcal{B}_{\mathbb{D}}(f|_{\mathbb{D}}) + \mathcal{B}_{\mathbb{C} \setminus \mathbb{D}}(f|_{\mathbb{C} \setminus \mathbb{D}})$$

*en loi, où  $\mathcal{B}_{\mathbb{D}}$  et  $\mathcal{B}_{\mathbb{C} \setminus \mathbb{D}}$  sont indépendants.*

La preuve repose sur une analyse des noyaux dans le cas du gaz de Coulomb et sur une décomposition des polynômes aléatoires comme la somme de deux polynômes aléatoires indépendants dans l'autre cas.



### 5.3 Comportement microscopique

En continuant avec les modèles mentionnés dans la sous-section 1.5, nous pouvons démontrer qu'ils présentent des comportements différents près du cercle unité, comme l'impliquent les deux théorèmes suivants. Ces résultats peuvent être trouvés dans la sous-section 4.2 du chapitre 5. Le premier théorème est donné par souci d'exhaustivité et parce que c'est un cas où des calculs très explicites peuvent être faits. Il existe déjà des généralisations dans [91].

**Théorème 5.10** (Cas du gaz de Coulomb). *Supposons que  $(Z_1, \dots, Z_n)$  est le gaz de Coulomb déterminantal de la sous-section 1.5. Alors*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \delta_{n(Z_i-1)} = X$$

où  $X$  est un processus ponctuel déterminantal sur  $\mathbb{C}$  de noyau

$$K(z, w) = \frac{e^{\max\{\Re(z), 0\} + \max\{\Re(w), 0\}}}{\pi} \int_0^1 t(1-t)e^{(z+\bar{w})t} dt,$$

où  $\Re$  désigne la partie réelle, et par rapport à la mesure de Lebesgue sur  $\mathbb{C}$ .

**Théorème 5.11** (Cas du polynôme aléatoire). *Supposons que  $\{Z_1, \dots, Z_n\}$  est l'ensemble des zéros du polynôme aléatoire de la sous-section 1.5. Alors,*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \delta_{n(Z_i-1)} = X$$

où  $X$  est formé par les zéros de la fonction gaussienne analytique de covariance donnée par

$$K(z, w) = \int_0^1 e^{(z+\bar{w})t} dt.$$

Ce dernier théorème est énoncé au chapitre 5 pour des lois non nécessairement gaussiennes et pourrait être facilement généralisé pour d'autres polynômes aléatoires. Le théorème 5.10 est démontré par une analyse du noyau limite et le théorème 5.11 est démontré par une analyse de la covariance limite.

## 6 Questions ouvertes

Pour terminer ce chapitre, nous présentons certaines des questions auxquelles nous aimerions trouver une réponse.

### 6.1 Fluctuations sur un milieu chargé uniformément

Une question que nous n'avons pas vraiment abordée est celle des fluctuations dont le cas le plus simple serait le suivant. Soit  $(Z_1, \dots, Z_n)$  un gaz de Coulomb sur  $S^2$  de particules de charge un, à la température inverse (constante)  $\beta$  et dans un milieu de charge uniforme  $\sigma$ . Selon le principe de grandes déviations, nous savons que

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{Z_i} = \sigma$$

presque sûrement. Il serait intéressant d'étudier la limite suivante

$$\lim_{n \rightarrow \infty} \left( \sum_{i=1}^n \delta_{Z_i} - n\sigma \right).$$

Par intuition et par analogie avec les résultats dans [83], [3], [2], [22], [72] et [14] nous nous attendons à ce que la limite soit liée à un champ libre gaussien.

## 6.2 Outliers sur des différents espaces et températures

Dans le même esprit que la question des fluctuations, nous considérons, par exemple, le modèle de gaz de Coulomb de la sous-section 1.5. Nous pouvons nous demander ce qui pourrait être obtenu pour

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n 1_{|Z_i| < 1} \delta_{Z_i}$$

dans le cadre non déterminantal. De plus, dans le cadre déterminantal mais dans des milieux non radiaux, il devrait être possible de déterminer la limite en étudiant l'asymptotique des polynômes orthogonaux (comme dans [4], [5] et [59]). En gardant le même  $\beta$ , nous espérons trouver un moyen de comprendre ce qui se passe sur d'autres surfaces où le gaz de Coulomb n'est jamais déterminantal.

## 6.3 Comportement macroscopique pour des températures négatives

En gardant à l'esprit le corollaire 5.3, nous pouvons nous demander ce qui se passe pour la sphère  $S^2$ . Le problème dans ce cas est que la caractéristique d'Euler de  $S^2$  est 2, de sorte que la température devrait être négative pour obtenir la limite de courbure constante. Mais nous pouvons donner un sens au système par une simple adaptation de l'argument dans [33]. Nous obtenons que

$$\int_{(S^2)^n} \exp\left(-\frac{\beta}{n} \sum_{i < j} G(x_i, x_j)\right) d\sigma^{\otimes n} \in (0, \infty) \text{ si et seulement si } \beta > -8\pi. \quad (29)$$

où  $G(x, y) = -\frac{1}{2\pi} \log|x - y|_{\mathbb{R}^3}$ . Le but serait donc de trouver ce qui se passe lorsque  $\beta_n \downarrow -8\pi$  pour le gaz de Coulomb  $(Z_1, \dots, Z_n)$  à la température inverse  $\beta_n/n$ . Comme la métrique de courbure constante n'est pas unique sur une classe conforme<sup>3</sup>, ce n'est pas évident à quoi s'attendre.

En fait, en utilisant (29), nous pouvons trouver une suite  $\beta_n \downarrow -8\pi$  telle que

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{Z_i} = \delta_Y$$

en loi, où  $Y \sim \sigma$ . De plus, par une adaptation des arguments de [33] ou par les grandes déviations étudiées par Berman dans [21], la mesure empirique converge toujours vers  $\sigma$  lorsque  $\beta > -8\pi$  et nous pouvons trouver une suite  $\beta_n \downarrow -8\pi$  telle que

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{Z_i} = \sigma$$

en loi. Cela confirme que la question peut ne pas être triviale et qu'il n'y a pas de comportement unique.

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<sup>3</sup>En effet l'espace de ces métriques est topologiquement une boule ouverte de dimension trois.



## Chapter 3

# Concentration for Coulomb gases

*This chapter corresponds to [52].*

**Abstract.** We study the non-asymptotic behavior of a Coulomb gas on a compact Riemannian manifold. This gas is a symmetric  $n$ -particle Gibbs measure associated to the two-body interaction energy given by the Green function. We encode such a particle system by using an empirical measure. Our main result is a concentration inequality in Kantorovich-Wasserstein distance inspired from the work of Chafaï, Hardy and Maïda on the Euclidean space. Their proof involves large deviation techniques together with an energy-distance comparison and a regularization procedure based on the superharmonicity of the Green function. This last ingredient is not available on a manifold. We solve this problem by using the heat kernel and its short-time asymptotic behavior.

**Résumé.** Nous étudions le comportement non asymptotique d'un gaz de Coulomb sur une variété riemannienne compacte. Ce gaz est une mesure de Gibbs symétrique associée à l'énergie d'interaction à deux corps donnée par la fonction de Green. Nous regardons la mesure empirique de ce système de particules. Notre principal résultat est une inégalité de concentration pour la distance de Kantorovich-Wasserstein du système à l'équilibre inspirée par les travaux de Chafaï, Hardy et Maïda sur l'espace euclidien. Leur preuve repose sur des techniques des grandes déviations, une comparaison entre l'énergie et la distance et une procédure de régularisation qu'utilise la surharmonicité de la fonction Green. Ce dernier ingrédient n'est pas disponible sur une variété compacte. Nous résolvons ce problème en utilisant le noyau de la chaleur et son comportement asymptotique en temps petit.

*2010 MSC:* 60B05; 26D10; 35K05.

*Keywords:* Gibbs measure; Green function; Coulomb gas; empirical measure; concentration of measure; interacting particle system; singular potential; heat kernel.

*Mots-clés:* Mesure de Gibbs; fonction de Green; gaz de Coulomb; mesure empirique; concentration de la mesure; système de particules en interaction; potentiel singulière; noyau de la chaleur.

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## 1 Introduction

We shall consider the model of a Coulomb gas on a Riemannian manifold introduced in [51, Subsection 4.1] and study its non-asymptotic behavior by obtaining a concentration inequality for the empirical measure around its limit. Let us describe the model and the main theorem of this article.

Let  $(M, g)$  be a compact Riemannian manifold of volume form  $\pi$ . We suppose, for simplicity, that  $\pi(M) = 1$  so that  $\pi \in \mathcal{P}(M)$  where  $\mathcal{P}(M)$  denotes the space of probability measures

on  $M$ . We endow  $\mathcal{P}(M)$  with the topology of weak convergence, i.e. the smallest topology such that  $\mu \rightarrow \int_M f d\mu$  is continuous for every continuous function  $f : M \rightarrow \mathbb{R}$ . Denote by  $\Delta : C^\infty(M) \rightarrow C^\infty(M)$  the Laplace-Beltrami operator on  $(M, g)$ . We shall say that

$$G : M \times M \rightarrow (-\infty, \infty]$$

is a Green function for  $\Delta$  if it is a symmetric continuous function such that for every  $x \in M$  the function  $G_x : M \rightarrow (-\infty, \infty]$  defined by  $G_x(y) = G(x, y)$  is integrable and

$$\Delta G_x = -\delta_x + 1 \tag{1}$$

in the distributional sense. It can be proven that such a  $G$  is integrable with respect to  $\pi \otimes \pi$  and that if  $f \in C^\infty(M)$  then  $\psi : M \rightarrow \mathbb{R}$ , defined by

$$\psi(x) = \int_M G(x, y)f(y)d\pi(y),$$

satisfies that

$$\psi \in C^\infty(M) \quad \text{and} \quad \Delta\psi = -f + \int_M f(y)d\pi(y). \tag{2}$$

In particular,  $\int_M G_x d\pi$  does not depend on  $x \in M$  and the Green function is unique up to an additive constant. See [10, Chapter 4] for a proof of these results. We will denote by  $G$  the Green function for  $\Delta$  such that

$$\int_M G_x d\pi = 0 \tag{3}$$

for every  $x \in M$ .

For  $x \in M$  the function  $G_x$  may be thought of as the potential generated by the distribution of charge  $\delta_x - 1$ . This would represent a unit charged particle located at  $x \in M$  and a negatively charged background of unit density. The total energy of a system of  $n$  particles of charge  $1/n$  (each particle coming with a negatively charged background) would be  $H_n : M^n \rightarrow (-\infty, \infty]$  defined by

$$H_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i < j} G(x_i, x_j).$$

Take a sequence  $\{\beta_n\}_{n \geq 2}$  of non-negative numbers and consider the sequence of Gibbs probability measures

$$d\mathbb{P}_n(x_1, \dots, x_n) = \frac{1}{Z_n} e^{-\beta_n H_n(x_1, \dots, x_n)} d\pi^{\otimes n}(x_1, \dots, x_n) \tag{4}$$

where  $Z_n$  is such that  $\mathbb{P}_n(M^n) = 1$ . This can be thought of as the Riemannian generalization of the usual Coulomb gas model described in [89] or [37]. In the particular case of the round two-dimensional sphere, it is known (see [69]) that if  $\beta_n = 4\pi n^2$  the probability measure  $\mathbb{P}_n$  describes the eigenvalues of the quotient of two independent  $n \times n$  matrices with independent Gaussian entries. Define  $H : \mathcal{P}(M) \rightarrow (-\infty, \infty]$  by

$$H(\mu) = \frac{1}{2} \int_{M \times M} G(x, y) d\mu(x) d\mu(y).$$

This is a convex lower semicontinuous function. We can see [51, Subsection 4.1] for a proof of these properties and [75, Chapter 9] for a short introduction and further information in the Euclidean setting. Let  $i_n : M^n \rightarrow \mathcal{P}(M)$  be defined by

$$i_n(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}.$$

If  $\beta_n/n \rightarrow \infty$ , the author in [51] tells us that  $\{i_{n*}(\mathbb{P}_n)\}_{n \geq 2}$ , the sequence of image measures of  $\mathbb{P}_n$  by  $i_n$ , satisfies a large deviation principle with speed  $\beta_n$  and rate function  $H - \inf H$ .

In particular, if  $F$  is a closed set of  $\mathcal{P}(M)$  we have

$$\limsup_{n \rightarrow \infty} \frac{1}{\beta_n} \log \mathbb{P}_n(i_n^{-1}(F)) \leq - \inf_{\mu \in F} (H(\mu) - \inf H).$$

or, equivalently

$$\mathbb{P}_n(i_n^{-1}(F)) \leq \exp \left( -\beta_n \inf_{\mu \in F} (H(\mu) - \inf H) + o(\beta_n) \right). \quad (5)$$

The aim of this article is to understand the  $o(\beta_n)$  term for some family of closed sets  $F$ . Suppose we choose some metric  $d$  in  $\mathcal{P}(M)$  that induces the topology of weak convergence. Since the unique minimizer of  $H$  is  $\mu_{\text{eq}} = \pi$  (see Theorem 3.1) a nice family of closed sets are the sets

$$F_r = \{\mu \in \mathcal{P}(M) : d(\mu, \mu_{\text{eq}}) \geq r\}$$

for  $r > 0$ . Instead of writing  $\mathbb{P}_n(i_n^{-1}(F_r))$  we shall write  $\mathbb{P}_n(d(i_n, \mu_{\text{eq}}) \geq r)$ , in other words, when we write  $\{d(i_n, \mu_{\text{eq}}) \geq r\}$  we mean the set  $i_n^{-1}(F_r) = \{\vec{x} \in M^n : d(i_n(\vec{x}), \mu_{\text{eq}}) \geq r\}$ . Since  $H$  is lower semicontinuous we have that  $\inf_{\mu \in F_r} (H(\mu) - \inf H)$  is strictly positive and the large deviation inequality is not vacuous. We would like a simple expression in terms of  $r$  for the leading term, so instead of using  $\inf_{\mu \in F_r} (H(\mu) - \inf H)$  we will use a simple function of  $r$ .

Let  $d_g$  denote the Riemannian distance. The metric we shall use on  $\mathcal{P}(M)$  is the function  $W_1 : \mathcal{P}(M) \times \mathcal{P}(M) \rightarrow [0, \infty)$  defined by

$$W_1(\mu, \nu) = \inf \left\{ \int_{M \times M} d_g(x, y) d\Pi(x, y) : \Pi \text{ is a coupling between } \mu \text{ and } \nu \right\} \quad (6)$$

which is known as the Wasserstein or Kantorovich metric. See [95, Theorem 7.12] for a proof that it metrizes the topology of weak convergence. The main result of this article is the following.

**Theorem 1.1** (Concentration inequality for Coulomb gases). *Let  $m$  be the dimension of  $M$ . If  $m = 2$  there exists a constant  $C > 0$  that does not depend on the sequence  $\{\beta_n\}_{n \geq 2}$  such that for every  $n \geq 2$  and  $r \geq 0$*

$$\mathbb{P}_n(W_1(i_n, \pi) \geq r) \leq \exp \left( -\beta_n \frac{r^2}{4} + \frac{\beta_n \log(n)}{8\pi n} + C \frac{\beta_n}{n} \right).$$

*If  $m \geq 3$  there exists a constant  $C > 0$  that does not depend on the sequence  $\{\beta_n\}_{n \geq 2}$  such that for every  $n \geq 2$  and  $r \geq 0$*

$$\mathbb{P}_n(W_1(i_n, \pi) \geq r) \leq \exp \left( -\beta_n \frac{r^2}{4} + C \frac{\beta_n}{n^{2/m}} \right).$$

In fact, by a slight modification we will also prove the following generalization. Denote by  $D(\cdot \| \pi) : \mathcal{P}(M) \rightarrow (-\infty, \infty]$  the relative entropy of  $\mu$  with respect to  $\pi$ , also known as the Kullback–Leibler divergence, i.e.  $D(\mu \| \pi) = \int_M \rho \log \rho d\pi$  if  $d\mu = \rho d\pi$  and  $D(\cdot \| \pi)$  is infinity when  $\mu$  is not absolutely continuous with respect to  $\pi$ .

**Theorem 1.2** (Concentration inequality for Coulomb gases in a potential). *Take a twice continuously differentiable function  $V : M \rightarrow \mathbb{R}$  and define*

$$H_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i < j} G(x_i, x_j) + \frac{1}{n} \sum_{i=1}^n V(x_i) \quad \text{and}$$

$$H(\mu) = \frac{1}{2} \int_{M \times M} G(x, y) d\mu(x) d\mu(y) + \int_M V(x) d\mu(x).$$

Then  $H$  has a unique minimizer that will be called  $\mu_{eq}$ . Suppose  $\mathbb{P}_n$  is defined by (1) and let  $m$  be the dimension of  $M$ . If  $m = 2$  there exists a constant  $C > 0$  that does not depend on the sequence  $\{\beta_n\}_{n \geq 2}$  such that for every  $n \geq 2$  and  $r \geq 0$

$$\mathbb{P}_n(W_1(i_n, \mu_{eq}) \geq r) \leq \exp\left(-\beta_n \frac{r^2}{4} + \frac{\beta_n \log(n)}{8\pi n} + nD(\mu_{eq} \|\pi) + C \frac{\beta_n}{n}\right).$$

If  $m \geq 3$  there exists a constant  $C > 0$  that does not depend on the sequence  $\{\beta_n\}_{n \geq 2}$  such that for every  $n \geq 2$  and  $r \geq 0$

$$\mathbb{P}_n(W_1(i_n, \mu_{eq}) \geq r) \leq \exp\left(-\beta_n \frac{r^2}{4} + nD(\mu_{eq} \|\pi) + C \frac{\beta_n}{n^{2/m}}\right).$$

*Remark 1.3* (About the sharpness). As we will see below it can be proved that

$$\mathbb{P}_n(W_1(i_n, \pi) \geq r) \leq \exp\left(-\beta_n \frac{r^2}{2} + o(\beta_n)\right)$$

and the natural question would be to find an explicit next order  $o(\beta_n)$ . In the two theorems above we have relaxed this inequality to

$$\mathbb{P}_n(W_1(i_n, \pi) \geq r) \leq \exp\left(-\beta_n \frac{r^2}{4} + o(\beta_n)\right)$$

and obtained a bound to  $o(\beta_n)$  that does not depend on  $r$ . In this relaxed inequality and at a fixed  $r > 0$  the next order terms cannot be exact. Indeed, strictly speaking we have

$$\mathbb{P}_n(W_1(i_n, \pi) \geq r) \leq \exp\left(-\beta_n \frac{r^2}{4} + \eta(\beta_n)\right)$$

where  $\eta(\beta)/\beta \rightarrow -r^2/2$  as  $\beta$  goes to infinity. Nevertheless, the importance of our result lies on the lack of dependence on  $r$  and the explicitness of the terms.

To prove Theorem 1.1 we follow [37] in turn inspired by [76] (see also [84]). We proceed in three steps. The first part, described in Section 2, may be used in any measurable space but it demands an energy-distance comparison and a regularization procedure. The energy-distance comparison will be explained in Section 3 and it may be extended to include Green functions of some Laplace-type operators. The regularization by the heat kernel, in Section 4, will use a short time asymptotic expansion. It may apply to more general kind of energies where a short-time asymptotic expansion of their heat kernel is known. Having acquired all the tools, Section 5 will complete the proof of Theorem 1.1 and, by a slight modification, Theorem 1.2.

## 2 Energy-distance comparison and regularization

In this section  $M$  may be any measurable space,  $\pi$  may be any probability measure on  $M$  and  $H_n : M^n \rightarrow (-\infty, \infty]$ , any measurable function bounded from below. Given  $\beta_n > 0$  we define the Gibbs probability measure by (1). Let  $H : \mathcal{P}(M) \rightarrow (-\infty, \infty]$  be any function that has a unique minimizer  $\mu_{eq} \in \mathcal{P}(M)$ . This shall be thought of as a rate function of some Laplace principle as in [51]. Consider a metric

$$d : \mathcal{P}(M) \times \mathcal{P}(M) \rightarrow [0, \infty)$$

on  $\mathcal{P}(M)$  that induces the topology of weak convergence and define

$$F_r = \{\mu \in \mathcal{P}(M) : d(\mu, \mu_{eq}) \geq r\}$$

for  $r > 0$ . We want to understand a non-asymptotic inequality similar to (5) with an explicit  $o(\beta_n)$  term. For this, we consider the following assumption.

**Assumption 1.** We will say that an increasing convex function  $f : [0, \infty) \rightarrow [0, \infty)$  satisfies Assumption A if, for all  $\mu \in \mathcal{P}(M)$ ,

$$f(d(\mu, \mu_{\text{eq}})) \leq H(\mu) - H(\mu_{\text{eq}}). \quad (\text{A})$$

Under Assumption A, (5) implies

$$\mathbb{P}_n(i_n^{-1}(F_r)) \leq \exp(-\beta_n f(r) + o(\beta_n)). \quad (7)$$

This  $o(\beta_n)$  term may depend on  $r$ . We will prove that if we relax the inequality (7) to

$$\mathbb{P}_n(i_n^{-1}(F_r)) \leq \exp(-\beta_n 2f(r/2) + o(\beta_n))$$

we can find bounds of the  $o(\beta_n)$  term that do not depend on  $r$ . To properly use Assumption A when  $\mu$  is an empirical measure  $\frac{1}{n} \sum_{i=1}^n \delta_{x_i}$  we will have to regularize  $\mu$ . The reason behind this is that when  $\mu$  is an empirical measure we usually obtain  $H(\mu) = \infty$  by the self-interactions of the particles with themselves. In the Euclidean setting this regularization is obtained by a convolution with a radial distribution while in the Riemannian setting this will be obtained by a diffusion using the heat kernel of the Laplacian which in the Euclidean case may be seen as a convolution by a Gaussian function. The following result is the general concentration inequality we get and it is the first part of the method mentioned in Section 1.

**Theorem 2.1** (General concentration inequality). *Suppose we have two real numbers  $a_n$  and  $b_n$  such that there exists a measurable function  $R : M^n \rightarrow \mathcal{P}(M)$  with the following property*

- for every  $\vec{x} = (x_1, \dots, x_n) \in M^n$  we have

$$H_n(x_1, \dots, x_n) \geq H(R(\vec{x})) - a_n, \quad \text{and} \quad d(R(\vec{x}), i_n(\vec{x})) \leq b_n.$$

Let us denote  $e_n = \int_{M^n} H_n d\mu_{\text{eq}}^{\otimes n}$  and  $e = H(\mu_{\text{eq}})$ . If  $f : [0, \infty) \rightarrow [0, \infty)$  is an increasing convex function that satisfies Assumption A then

$$\mathbb{P}_n(d(i_n, \mu_{\text{eq}}) \geq r) \leq \exp\left(-\beta_n 2f\left(\frac{r}{2}\right) + nD(\mu_{\text{eq}} \parallel \pi) + \beta_n(e_n - e) + \beta_n a_n + \beta_n f(b_n)\right).$$

*Proof.* We first prove the two following results. The first lemma we state is the analogue of [37, Lemma 4.1].

**Lemma 2.2** (Lower bound of the partition function). *We have the following lower bound.*

$$Z_n \geq \exp(-\beta_n e_n - nD(\mu_{\text{eq}} \parallel \pi)).$$

*Proof.* If  $d\mu_{\text{eq}} = \rho_{\text{eq}} d\pi$  we have

$$\begin{aligned} Z_n &= \int_{M^n} e^{-\beta_n H_n(x_1, \dots, x_n)} d\pi^{\otimes n}(x_1, \dots, x_n) \\ &\geq \int_{M^n} e^{-\beta_n H_n(x_1, \dots, x_n)} e^{-\sum_{i=1}^n 1_{\rho_{\text{eq}} > 0}(x_i) \log \rho_{\text{eq}}(x_i)} d\mu_{\text{eq}}^{\otimes n}(x_1, \dots, x_n) \\ &\geq \int_{M^n} e^{-\beta_n H_n(x_1, \dots, x_n) - \sum_{i=1}^n 1_{\rho_{\text{eq}} > 0}(x_i) \log \rho_{\text{eq}}(x_i)} d\mu_{\text{eq}}^{\otimes n}(x_1, \dots, x_n) \\ &\geq e^{-\int_{M^n} (\beta_n H_n(x_1, \dots, x_n) + \sum_{i=1}^n 1_{\rho_{\text{eq}} > 0}(x_i) \log \rho_{\text{eq}}(x_i)) d\mu_{\text{eq}}^{\otimes n}(x_1, \dots, x_n)} \\ &= e^{-\beta_n e_n - nD(\mu_{\text{eq}} \parallel \pi)} \end{aligned}$$

where we have used Jensen's inequality to get the last inequality.  $\square$



The second lemma will help us in the step of regularization.

**Lemma 2.3** (Comparison). *Take  $\vec{x} = (x_1, \dots, x_n) \in M^n$ . If  $d(R(\vec{x}), i_n(\vec{x})) \leq b_n$  then*

$$f(d(R(\vec{x}), \mu_{\text{eq}})) \geq 2f\left(\frac{d(i_n(\vec{x}), \mu_{\text{eq}})}{2}\right) - f(b_n).$$

*Proof.* Since

$$d(i_n(\vec{x}), \mu_{\text{eq}}) \leq d(i_n(\vec{x}), R(\vec{x})) + d(R(\vec{x}), \mu_{\text{eq}})$$

we have that

$$\begin{aligned} f\left(\frac{d(i_n(\vec{x}), \mu_{\text{eq}})}{2}\right) &\leq f\left(\frac{1}{2}d(i_n(\vec{x}), R(\vec{x})) + \frac{1}{2}d(R(\vec{x}), \mu_{\text{eq}})\right) \\ &\leq \frac{1}{2}f(d(i_n(\vec{x}), R(\vec{x}))) + \frac{1}{2}f(d(R(\vec{x}), \mu_{\text{eq}})) \\ &\leq \frac{1}{2}f(b_n) + \frac{1}{2}f(d(R(\vec{x}), \mu_{\text{eq}})) \end{aligned}$$

where we have used that  $f$  is increasing and convex. □

Now, define

$$A_r = i_n^{-1}(F_r) = \{\vec{x} \in M^n : d(i_n(\vec{x}), \mu_{\text{eq}}) \geq r\}.$$

Then

$$\begin{aligned} \mathbb{P}_n(A_r) &= \frac{1}{Z_n} \int_{A_r} e^{-\beta_n H_n(x_1, \dots, x_n)} d\pi^{\otimes n}(x_1, \dots, x_n) \\ &\leq e^{\beta_n e_n + nD(\mu_{\text{eq}} \parallel \pi)} \int_{A_r} e^{-\beta_n H(R(\vec{x})) + \beta_n a_n} d\pi^{\otimes n}(x_1, \dots, x_n) \\ &\leq e^{\beta_n e_n + \beta_n a_n + nD(\mu_{\text{eq}} \parallel \pi)} \int_{A_r} e^{-\beta_n H(R(\vec{x}))} d\pi^{\otimes n}(x_1, \dots, x_n) \\ &\stackrel{(*)}{\leq} e^{\beta_n(e_n - e) + \beta_n a_n + nD(\mu_{\text{eq}} \parallel \pi)} \int_{A_r} e^{-\beta_n f(d(R(\vec{x}), \mu_{\text{eq}}))} d\pi^{\otimes n}(x_1, \dots, x_n) \\ &\stackrel{(**)}{\leq} e^{\beta_n(e_n - e) + \beta_n a_n + nD(\mu_{\text{eq}} \parallel \pi)} \int_{A_r} e^{-\beta_n 2f\left(\frac{d(i_n(\vec{x}), \mu_{\text{eq}})}{2}\right) + \beta_n f(b_n)} d\pi^{\otimes n}(x_1, \dots, x_n) \\ &\stackrel{(***)}{\leq} e^{\beta_n(e_n - e) + \beta_n a_n + nD(\mu_{\text{eq}} \parallel \pi)} e^{-\beta_n 2f\left(\frac{r}{2}\right) + \beta_n f(b_n)} \\ &\leq e^{-\beta_n 2f\left(\frac{r}{2}\right) + nD(\mu_{\text{eq}} \parallel \pi) + \beta_n(e_n - e) + \beta_n a_n + \beta_n f(b_n)} \end{aligned}$$

where in (\*) we have used Assumption A, in (\*\*) we have used Lemma 2.3 and in (\*\*\*) we have used the monotonicity of  $f$ . □

In the next section we return to the case of a compact Riemannian manifold and study a energy-distance comparison that will imply Assumption A.

### 3 Energy-distance comparison in compact manifolds

We take the notation used in Section 1. The Kantorovich metric  $W_1$  defined in (6) can be written as

$$W_1(\mu, \nu) = \sup \left\{ \int_M f d\mu - \int_M f d\nu : \|f\|_{\text{Lip}} \leq 1 \right\}$$

where

$$\|f\|_{\text{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_g(x, y)}.$$

This result is known as the Kantorovich-Rubinstein theorem (see [95, Theorem 1.14]). In the case of a Riemannian manifold, by a smooth approximation argument such as the one in [11], we can prove that

$$W_1(\mu, \nu) = \sup \left\{ \int_M f \, d\mu - \int_M f \, d\nu : f \in C^\infty(M) \text{ and } \|\nabla f\|_\infty \leq 1 \right\}.$$

The next theorem gives the energy-distance comparison required to satisfy Assumption A. This is the analogue of [76, Theorem 1.3] and [37, Lemma 3.1].

**Theorem 3.1** (Comparison between distance and energy). *Suppose that  $\mu_{\text{eq}} \in \mathcal{P}(M)$  is a probability measure on  $M$  such that  $H(\mu_{\text{eq}}) \leq H(\mu)$  for every  $\mu \in \mathcal{P}(M)$ . Then*

$$\frac{1}{2}W_1(\mu, \mu_{\text{eq}})^2 \leq H(\mu) - H(\mu_{\text{eq}}) \quad (8)$$

for every  $\mu \in \mathcal{P}(M)$ . This implies, in particular, that  $H$  has a unique minimizer and that Assumption A is satisfied by  $f(r) = \frac{r^2}{2}$ . Furthermore,  $\mu_{\text{eq}} = \pi$ .

Let  $\mathcal{F}$  be the space of finite signed measures  $\mu$  on  $M$  such that  $\int_{M \times M} G \, d|\mu|^{\otimes 2} < \infty$ . For convenience we shall define  $\mathcal{E} : \mathcal{F} \rightarrow (-\infty, \infty]$  by

$$\mathcal{E}(\mu) = \int_{M \times M} G(x, y) \, d\mu(x) \, d\mu(y)$$

so that  $\mathcal{E}(\mu) = 2H(\mu)$  whenever  $\mu \in \mathcal{P}(M) \cap \mathcal{F}$ . We can also notice that if  $\mu, \nu \in \mathcal{P}(M)$  are such that  $H(\mu)$  and  $H(\nu)$  are finite then, due to the convexity of  $H$ , we have  $\int_{M \times M} G \, d\mu^{\otimes 2} < \infty$ , the measure  $\mu - \nu$  belongs to  $\mathcal{F}$  and

$$\mathcal{E}(\mu - \nu) = \mathcal{E}(\mu) + \mathcal{E}(\nu) - 2 \int_{M \times M} G(x, y) \, d\mu(x) \, d\nu(y). \quad (9)$$

We begin by proving the following result that may be seen as a comparison of distances where the ‘energy distance’ between two probability measures  $\mu, \nu \in \mathcal{P}(M)$  of finite energy is defined as  $\sqrt{\mathcal{E}(\mu - \nu)}$ . This is the analogue of [37, Theorem 1.1].

**Lemma 3.2** (Comparison of distances). *Let  $\mu, \nu \in \mathcal{P}(M)$  such that  $H(\mu)$  and  $H(\nu)$  are finite. Then  $\mathcal{E}(\mu - \nu) \geq 0$  and*

$$W_1(\mu, \nu) \leq \sqrt{\mathcal{E}(\mu - \nu)}.$$

*Proof.* First suppose  $\mu$  and  $\nu$  differentiable, i.e. suppose they have a differentiable density with respect to  $\pi$ . Define  $U : M \rightarrow \mathbb{R}$  by

$$U(x) = \int_M G(x, y) \, (d\mu(y) - d\nu(y)).$$

Then, as remarked in (2), we know that  $U$  is differentiable and

$$\Delta U = -(\mu - \nu).$$

Take  $f \in C^\infty(M)$  such that  $\|\nabla f\|_\infty \leq 1$ . We can see that

$$\int_M f \, (d\mu - d\nu) = - \int_M f \, \Delta U = \int_M \langle \nabla f, \nabla U \rangle \, d\pi \leq \|\nabla f\|_2 \|\nabla U\|_2 \leq \|\nabla f\|_\infty \|\nabla U\|_2.$$

We also know that

$$(\|\nabla U\|_2)^2 = \int_M \langle \nabla U, \nabla U \rangle d\pi = - \int_M U \Delta U = \int_M U (d\mu - d\nu) = \mathcal{E}(\mu - \nu).$$

Then,

$$\int_M f (d\mu - d\nu) \leq \|\nabla f\|_\infty \|\nabla U\|_2 \leq \|\nabla f\|_\infty \sqrt{\mathcal{E}(\mu - \nu)}.$$

This implies that

$$W_1(\mu, \nu) \leq \sqrt{\mathcal{E}(\mu - \nu)}.$$

In general, let  $\mu, \nu \in \mathcal{P}(M)$  such that  $H(\mu)$  and  $H(\nu)$  are finite. Take two sequences  $\{\mu_n\}_{n \in \mathbb{N}}$  and  $\{\nu_n\}_{n \in \mathbb{N}}$  of differentiable probability measures that converge to  $\mu$  and  $\nu$  respectively and such that  $\mathcal{E}(\mu_n) \rightarrow \mathcal{E}(\mu)$  and  $\mathcal{E}(\nu_n) \rightarrow \mathcal{E}(\nu)$  (see [15] for a proof of their existence) and proceed by a limit argument. □

The next step to prove Theorem 3.1 is a fact that works for general two-body interactions i.e.  $G$  is not necessarily a Green function.

**Lemma 3.3** (Comparison of energies). *Suppose that  $\mu_{\text{eq}}$  is a probability measure such that  $H(\mu_{\text{eq}}) \leq H(\mu)$  for every  $\mu \in \mathcal{P}(M)$ . Then, for every  $\mu \in \mathcal{P}(M)$  such that  $H(\mu) < \infty$ , we have*

$$\mathcal{E}(\mu - \mu_{\text{eq}}) \leq \mathcal{E}(\mu) - \mathcal{E}(\mu_{\text{eq}}).$$

*Proof.* Since  $H(\mu)$  and  $H(\mu_{\text{eq}})$  are finite we use (9) to notice that the assertion

$$\mathcal{E}(\mu - \mu_{\text{eq}}) \leq \mathcal{E}(\mu) - \mathcal{E}(\mu_{\text{eq}})$$

is equivalent to

$$\int_{M \times M} G(x, y) d\mu(x) d\mu_{\text{eq}}(y) \geq \mathcal{E}(\mu_{\text{eq}}).$$

But, if

$$\int_{M \times M} G(x, y) d\mu(x) d\mu_{\text{eq}}(y) < \mathcal{E}(\mu_{\text{eq}})$$

were true then, defining  $\mu_t = (1 - t)\mu_{\text{eq}} + t\mu = \mu_{\text{eq}} + t(\mu - \mu_{\text{eq}})$ , we would see that the linear term of  $\mathcal{E}(\mu_t)$  is  $\int_{M \times M} G(x, y) d\mu(x) d\mu_{\text{eq}}(y) - \mathcal{E}(\mu_{\text{eq}}) < 0$ . This means that  $\mathcal{E}(\mu_t) < \mathcal{E}(\mu_{\text{eq}})$  for  $t > 0$  small which is a contradiction. □

Now we may complete the proof of Theorem 3.1.

*Proof of Theorem 3.1.* Let  $\mu_{\text{eq}}$  be a minimizer of  $H$  and let  $\mu \in \mathcal{P}(M)$  be a probability measure on  $M$ . If  $H(\mu)$  is infinite there is nothing to prove. If it is not, by Lemma 3.2 and 3.3 we conclude (8).

To prove that  $H$  has a unique minimizer suppose  $\tilde{\mu}_{\text{eq}}$  is another minimizer and use Inequality (8) with  $\mu = \tilde{\mu}_{\text{eq}}$  to get  $W_1(\tilde{\mu}_{\text{eq}}, \mu_{\text{eq}}) = 0$  and, thus,  $\tilde{\mu}_{\text{eq}} = \mu_{\text{eq}}$ .

Finally, to see that  $\mu_{\text{eq}} = \pi$  we use (3). Then  $\mathcal{E}(\mu - \pi) = \mathcal{E}(\mu) - \mathcal{E}(\pi)$  when  $\mu$  has finite energy. But by Lemma 3.2 we know that  $\mathcal{E}(\mu - \pi) \geq 0$  and then  $\mathcal{E}(\mu) \geq \mathcal{E}(\pi)$  for every  $\mu \in \mathcal{P}(M)$  of finite energy. □

In the next section we study a way to regularize the empirical measures in the sense of the hypotheses of Theorem 2.1.

## 4 Heat kernel regularization of the energy

In this section the main tool is the heat kernel for  $\Delta$ . A proof of the following proposition may be found in [40, Chapter VI].

**Proposition 4.1** (Heat kernel). *There exists a unique smooth function  $p : (0, \infty) \times M \times M \rightarrow \mathbb{R}$  such that*

$$\frac{\partial}{\partial t} p_t(x, y) = \Delta_y p_t(x, y) \quad \text{and} \quad \lim_{t \rightarrow 0} p_t(x, \cdot) = \delta_x$$

for every  $x, y \in M$  and  $t > 0$ . Such a function will be called the heat kernel for  $\Delta$ . It is non-negative, it is mass preserving, i.e.

$$\int_M p_t(x, y) d\pi(y) = 1$$

for every  $x \in M$  and  $t > 0$ , it is symmetric, i.e.

$$p_t(x, y) = p_t(y, x)$$

for every  $x, y \in M$  and  $t > 0$  and it satisfies the semigroup property i.e.

$$\int_M p_t(x, y) p_s(y, z) d\pi(y) = p_{t+s}(x, z)$$

for every  $x, y \in M$  and  $t, s > 0$ . Furthermore,

$$\lim_{t \rightarrow \infty} p_t(x, y) = 1$$

uniformly on  $x$  and  $y$ .

Let  $p$  be the heat kernel associated to  $\Delta$ . For each point  $x \in M$  and  $t > 0$  define the probability measure  $\mu_x^t \in \mathcal{P}(M)$  by

$$d\mu_x^t = p_t(x, \cdot) d\pi, \tag{10}$$

or, more precisely,  $d\mu_x^t(y) = p_t(x, y) d\pi(y)$ . Then we define  $R_t : M^n \rightarrow \mathcal{P}(M)$  by

$$R_t(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n \mu_{x_i}^t$$

and we want to find  $a_n$  and  $b_n$  of the hypotheses of Theorem 2.1 for  $R = R_t$ .

We begin by looking for  $b_n$ .

### 4.1 Distance to the regularized measure

**Proposition 4.2** (Distance to the regularized measure). *There exists a constant  $C > 0$  such that for all  $t > 0$  and  $\vec{x} \in M^n$*

$$W_1(R_t(\vec{x}), i_n(\vec{x})) \leq C\sqrt{t}.$$

*Proof.* The following arguments are very similar to those in [73] and they will be repeated for convenience of the reader. Since  $W_1 : \mathcal{P}(M) \times \mathcal{P}(M) \rightarrow [0, \infty)$  is the supremum of linear functions, it is convex. So

$$W_1(R_t(\vec{x}), i_n(\vec{x})) \leq \frac{1}{n} \sum_{i=1}^n W_1(\mu_{x_i}^t, \delta_{x_i}).$$

Then, we will try to find a constant  $C > 0$  such that  $W_1(\mu_x^t, \delta_x) \leq C\sqrt{t}$  for every  $x \in M$ . Since the only coupling between  $\delta_x$  and  $\mu_x^t$  is their product we see that

$$W_1(\mu_x^t, \delta_x) = \int_M d_g(x, y) d\mu_x^t(y).$$

In fact, we will study the 2-Kantorovich distance between  $\delta_x$  and  $\mu_x^t$

$$\begin{aligned} D_t(x) &= \int_M d_g(x, y)^2 d\mu_x^t(y) \\ &= \int_M d_g(x, y)^2 p_t(x, y) d\pi(y). \end{aligned}$$

If we prove that there exists a constant  $C > 0$  such that for every  $x \in M$

$$D_t(x) \leq C^2 t \tag{11}$$

we may conclude that  $W_1(\delta_x, \mu_x^t) \leq C\sqrt{t}$  for every  $x \in M$  by Jensen's inequality. To obtain (11) we use the following lemma whose proof may be found in [62, Section 3.4] and [62, Theorem 3.5.1].

**Lemma 4.3** (Radial process representation). *Take  $x \in M$ . Let  $X$  be the Markov process with generator  $\Delta$  starting at  $x$  (i.e.  $X_t = B_{2t}$  where  $B$  is a Brownian motion on  $M$  starting at  $x$ ). Define  $r : M \rightarrow [0, \infty)$  by  $r(y) = d_g(x, y)$ . Then  $r$  is differentiable  $\pi$ -almost everywhere and there exists a non-decreasing process  $L$  and a one-dimensional Euclidean Brownian motion  $\beta$  such that*

$$r(X_t) = \beta_{2t} + \int_0^t \Delta r(X_s) ds - L_t$$

for every  $t \geq 0$  where  $\Delta r$  is the  $\pi$ -almost everywhere defined Laplacian of  $r$ .

Applying Lemma 4.3 and Itô's formula and then taking expected values we get

$$\mathbb{E}[r(X_t)^2] = 2 \int_0^t \mathbb{E}[r(X_s) \Delta r(X_s)] ds - \mathbb{E} \left[ 2 \int_0^t r(X_s) dL_s \right] + 2t \leq \int_0^t 2\mathbb{E}[r(X_s) \Delta r(X_s)] ds + 2t$$

where we are using the notation of Lemma 4.3. By [62, Theorem 3.4.2] we know that  $r\Delta r$  is bounded in  $M$  with a bound that does not depend on  $x$  and since  $D_t(x) = \mathbb{E}[r(X_t)^2]$  we obtain (11) where the constant  $C$  does not depend on  $x$ .  $\square$

Now we will look for  $a_n$  of the hypotheses of Theorem 2.1.

## 4.2 Comparison between the regularized and the non-regularized energy

**Theorem 4.4** (Comparison between the regularized and the non-regularized energy). *Let  $m$  be the dimension of  $M$ . If  $m = 2$  there exists a constant  $C > 0$  such that, for every  $n \geq 2$ ,  $t \in (0, 1]$  and  $\vec{x} \in M^n$ ,*

$$H_n(\vec{x}) \geq H(R_t(\vec{x})) - t + \frac{1}{8\pi n} \log(t) - \frac{C}{n}.$$

*If  $m > 2$  there exists a constant  $C > 0$  such that, for every  $n \geq 2$ ,  $t \in (0, 1]$  and  $\vec{x} \in M^n$ ,*

$$H_n(\vec{x}) \geq H(R_t(\vec{x})) - t - \frac{C}{nt^{\frac{m}{2}-1}}.$$

The terms  $\frac{1}{8\pi} \log(t) - C$  and  $-1/t^{m/2-1}$  come from the self-interaction of the regularized punctual charges  $\frac{1}{8\pi}$  while the term  $-t$  comes from the negatively charged background. In the Euclidean setting, since there is no charged background, the  $\frac{1}{8\pi} \log(t) - C$  and  $-1/t^{m/2-1}$  terms arise from the self-interactions without potential and the  $-t$  term arise from the regularization of the potential. The proof may be adapted to treat two-body interactions by the Green function of different Markov processes where the short-time asymptotic behavior is known.

To compare  $H(R_t(\vec{x}))$  and  $H_n(\vec{x})$  we will write, for  $\vec{x} = (x_1, \dots, x_n) \in M^n$ ,

$$H(R_t(\vec{x})) = \frac{1}{n^2} \sum_{i < j} \int_{M \times M} G(\alpha, \beta) d\mu_{x_i}^t(\alpha) d\mu_{x_j}^t(\beta) + \frac{1}{2n^2} \sum_{i=1}^n \int_{M \times M} G(\alpha, \beta) d\mu_{x_i}^t(\alpha) d\mu_{x_i}^t(\beta).$$

Let us define

$$\begin{aligned} G_t(x, y) &= \int_{M \times M} G(\alpha, \beta) d\mu_x^t(\alpha) d\mu_y^t(\beta) \\ &= \int_{M \times M} G(\alpha, \beta) p_t(x, \alpha) d\pi(\alpha) p_t(y, \beta) d\pi(\beta). \end{aligned}$$

Then we may write

$$H(R_t(\vec{x})) = \frac{1}{n^2} \sum_{i < j} G_t(x_i, x_j) + \frac{1}{2n^2} \sum_{i=1}^n G_t(x_i, x_i).$$

So we want to compare  $G_t$  and  $G$ . The idea we shall use is that if  $G$  is the kernel of the operator  $\bar{G}$  and  $p_t$  is the kernel of the operator  $\bar{P}_t$  then  $G_t$  is the kernel of the operator  $\bar{P}_t \bar{G} \bar{P}_t$ . But using the eigenvector decomposition we can see that

$$\bar{G} = \int_0^\infty (\bar{P}_s - e_0 \otimes e_0^*) ds \quad (12)$$

where  $e_0$  is the eigenvector of eigenvalue 0, i.e. the constant function equal to one. Then

$$\bar{P}_t \bar{G} \bar{P}_t = \int_0^\infty (\bar{P}_{2t+s} - e_0 \otimes e_0^*) ds \quad (13)$$

where we have used the semigroup property of  $t \mapsto \bar{P}_t$ , the fact that  $\bar{P}_t e_0 = e_0$  and  $\bar{P}_t^* = \bar{P}_t$ . Notice that this representation can also be obtained when  $G$  is the Green function of different Markov processes.

We will prove the previous idea in a somehow different but very related way. We begin by proving the analogue of (12).

**Proposition 4.5** (Integral representation of the Green function). *For every pair of different points  $x, y \in M$  the function  $t \mapsto p_t(x, y) - 1$  is integrable. For every  $x \in M$  the negative part of the function  $t \mapsto p_t(x, x) - 1$  is integrable. Moreover, we have the following integral representation of the Green function. For every  $x, y \in M$*

$$G(x, y) = \int_0^\infty (p_t(x, y) - 1) dt.$$

*Proof.* To prove the integrability of  $t \mapsto p_t(x, y) - 1$  we will need to know the behavior of  $p_t$  for large and short  $t$ . For the large-time behavior we have the following result.

**Lemma 4.6** (Large-time behavior). *There exists  $\lambda > 0$  such that for every  $T > 0$ ,  $s \geq 0$  and  $x, y \in M$*

$$|p_{T+s}(x, y) - 1| \leq e^{-\lambda s} \sqrt{|p_T(x, x) - 1| |p_T(y, y) - 1|}. \quad (14)$$

*Proof.* We follow the same arguments as in the proof of [56, Corollary 3.17]. By the semigroup property, the symmetry of  $p_t$  and Cauchy-Schwarz inequality we get

$$\begin{aligned} |p_{T+s}(x, y) - 1| &= \left| \int_M \left( p_{\frac{T+s}{2}}(x, z) - 1 \right) \left( p_{\frac{T+s}{2}}(z, y) - 1 \right) d\pi(z) \right| \\ &\leq \left\| p_{\frac{T+s}{2}}(x, \cdot) - 1 \right\|_{L^2} \left\| p_{\frac{T+s}{2}}(y, \cdot) - 1 \right\|_{L^2}. \end{aligned} \quad (15)$$

If  $\lambda$  is the first strictly positive eigenvalue of  $-\Delta$  and if  $f \in L^2(M)$  we get

$$\left\| \int_M (p_{\frac{s}{2}}(\cdot, z) - 1) f(z) d\pi(z) \right\|_{L^2} \leq e^{-\lambda \frac{s}{2}} \left\| f - \int_M f d\pi \right\|_{L^2}.$$

If we choose  $f = p_{\frac{T}{2}}(x, \cdot) - 1$  we obtain

$$\left\| p_{\frac{T+s}{2}}(x, \cdot) - 1 \right\|_{L^2} \leq e^{-\lambda \frac{s}{2}} \left\| p_{\frac{T}{2}}(x, \cdot) - 1 \right\|_{L^2} = e^{-\lambda \frac{s}{2}} \sqrt{p_T(x, x) - 1} \quad (16)$$

where we have used the semigroup property for the last equality. Similarly, we get

$$\left\| p_{\frac{T+s}{2}}(y, \cdot) - 1 \right\|_{L^2} \leq e^{-\lambda \frac{s}{2}} \sqrt{p_T(y, y) - 1}. \quad (17)$$

By (15), (16) and (17) we may conclude (14).  $\square$

For the short-time behavior, [62, Theorem 5.3.4] implies the following lemma.

**Lemma 4.7** (Short-time behavior). *Let  $m$  be the dimension of  $M$ . Then there exist two positive constants  $C_1$  and  $C_2$  such that for every  $t \in (0, 1)$  and  $x, y \in M$  we have*

$$\frac{C_1}{t^{\frac{m}{2}}} e^{-\frac{d_g(x, y)^2}{4t}} \leq p_t(x, y) \leq \frac{C_2}{t^{m-\frac{1}{2}}} e^{-\frac{d_g(x, y)^2}{4t}}.$$

The integrability of  $t \mapsto p_t(x, y) - 1$  when  $x \neq y$  and the fact that  $\int_0^\infty (p_t(x, x) - 1) dt = \infty$  for every  $x \in M$  can be obtained from Lemma 4.7 and Lemma 4.6.

Using Lemma 4.6 and the dominated convergence theorem we obtain the continuity of the function given by  $(x, y) \mapsto \int_1^\infty (p_t(x, y) - 1) dt$  at any  $(x, y) \in M \times M$ . By the dominated convergence theorem and Lemma 4.7 we obtain the continuity of the function given by  $(x, y) \mapsto \int_0^1 (p_t(x, y) - 1) dt$  for  $x \neq y$ . Using Fatou's lemma we obtain the continuity of  $(x, y) \mapsto \int_0^1 (p_t(x, y) - 1) dt$  at  $(x, y)$  such that  $x = y$ . So, we get that the function  $K : M \times M \rightarrow (-\infty, \infty]$  defined by

$$K(x, y) = \int_0^\infty (p_t(x, y) - 1) dt$$

is continuous. The following lemma assures that  $K(x, \cdot)$  is integrable for every  $x \in M$ .

**Lemma 4.8** (Global integrability). *For every  $x \in M$*

$$\int_0^\infty \int_M |p_t(x, y) - 1| d\pi(y) dt < \infty.$$

*Proof.* Take  $T > 0$ . By Lemma 4.6 we obtain that

$$\int_T^\infty \int_M |p_t(x, y) - 1| d\pi(y) dt < \infty.$$

On the other hand we have

$$\int_0^T \int_M |p_t(x, y) - 1| d\pi(y) dt \leq \int_0^T \int_M (p_t(x, y) + 1) d\pi(y) dt = 2T < \infty.$$

$\square$

Let  $0 = \lambda_0 < \lambda_1 \leq \lambda_2, \dots$  be the sequence of eigenvalues of  $-\Delta$  and  $e_0, e_1, e_2, \dots$  the sequence of respective eigenfunctions. Then, for every  $\psi \in C^\infty(M)$

$$\sum_{n=0}^{\infty} \exp(-\lambda_n t) |\langle e_n, \psi \rangle|^2 = \langle \psi, e^{t\Delta} \psi \rangle = \int_{M \times M} \psi(x) p_t(x, y) \psi(y) d\pi(x) d\pi(y).$$

Equivalently we have

$$\sum_{n=1}^{\infty} \exp(-\lambda_n t) |\langle e_n, \psi \rangle|^2 = \int_{M \times M} \psi(x) (p_t(x, y) - 1) \psi(y) d\pi(x) d\pi(y)$$

and integrating in  $t$  from zero to infinity we obtain

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} |\langle e_n, \psi \rangle|^2 = \int_{M \times M} \psi(x) K(x, y) \psi(y) d\pi(x) d\pi(y).$$

By a polarization identity we have that, for every  $\phi, \psi \in C^\infty(M)$ ,

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \langle \psi, e_n \rangle \langle e_n, \phi \rangle = \int_{M \times M} \psi(x) K(x, y) \phi(y) d\pi(x) d\pi(y).$$

Taking  $\phi = \Delta \alpha$  we get

$$\langle \psi, \alpha \rangle - \int_M \psi d\pi \int_M \alpha d\pi = \sum_{n=1}^{\infty} \langle \psi, e_n \rangle \langle e_n, \alpha \rangle = \int_{M \times M} \psi(x) K(x, y) \Delta \alpha(y) d\pi(x) d\pi(y).$$

By definition of the Green function we know that  $\int_M G(x, y) \Delta \alpha(y) d\pi(y) = -\alpha(x) + \int_M \alpha d\pi$  and thus

$$\int_{M \times M} \psi(x) G(x, y) \Delta \alpha(y) d\pi(x) d\pi(y) = \int_{M \times M} \psi(x) K(x, y) \Delta \alpha(y) d\pi(x) d\pi(y).$$

Since  $\int_M K(x, y) d\pi(y) = 0 = \int_M G(x, y) d\pi(y)$  and by the continuity of  $K$  and  $G$  we obtain  $G(x, y) = K(x, y)$  for every  $x, y \in M$ . □

Now we will state and prove (13).

**Proposition 4.9** (Integral representation of the regularized Green function). *For every  $t > 0$  and  $x, y \in M$*

$$G_t(x, y) = \int_{2t}^{\infty} (p_s(x, y) - 1) ds.$$

*Proof.* Take the time (i.e. with respect to  $t$ ) derivative (denoted by a dot above the function)

$$\dot{G}_t(x, y) = \int_{M \times M} \dot{p}_t(x, \alpha) G(\alpha, \beta) p_t(y, \beta) d\pi(\alpha) d\pi(\beta) + \int_{M \times M} \dot{p}_t(x, \alpha) G(\alpha, \beta) \dot{p}_t(y, \beta) d\pi(\alpha) d\pi(\beta).$$

We will study the first term of the sum (the second being analogous).

$$\begin{aligned} \int_{M \times M} \dot{p}_t(x, \alpha) G(\alpha, \beta) p_t(y, \beta) d\pi(\alpha) d\pi(\beta) &= \int_{M \times M} \Delta_\alpha p_t(x, \alpha) G(\alpha, \beta) p_t(y, \beta) d\pi(\alpha) d\pi(\beta) \\ &= \int_M \left( \int_M \Delta_\alpha p_t(x, \alpha) G(\alpha, \beta) d\pi(\alpha) \right) p_t(y, \beta) d\pi(\beta) \\ &= \int_M \left( \int_M p_t(x, \alpha) \Delta_\alpha G(\alpha, \beta) d\pi(\alpha) \right) p_t(y, \beta) d\pi(\beta) \\ &= \int_M \left( \int_M p_t(x, \alpha) (-\delta_\beta(\alpha) + 1) d\pi(\alpha) \right) p_t(y, \beta) d\pi(\beta) \\ &= \int_M (-p_t(x, \beta) + 1) p_t(y, \beta) d\pi(\beta) \\ &= -p_{2t}(x, y) + 1 \end{aligned}$$



where in the last line we have used the symmetry and the semigroup property of  $p$ . Using again the symmetry of  $p$  we get

$$\dot{G}_t(x, y) = -2p_{2t}(x, y) + 2,$$

and by integrating we obtain

$$G_t(x, y) - G_s(x, y) = \int_s^t (-2p_{2u}(x, y) + 2) du = \int_{2s}^{2t} (-p_s(x, y) + 1) ds$$

for every  $0 < s < t < \infty$ . As a consequence of the uniform convergence of Proposition 4.1 we can see that  $\mu_x^t$  and  $\mu_y^t$  defined in (10) converge to  $\pi$  as  $t$  goes to infinity. Fix any  $T > 0$ . Since  $G_{T+s}(x, y) = \int_{M \times M} G_T(\alpha, \beta) d\mu_x^s(\alpha) d\mu_y^s(\beta)$  for any  $s > 0$  and since  $G_T$  is continuous we obtain  $\lim_{t \rightarrow \infty} G_t(x, y) = \int_{M \times M} G_T(x, y) d\pi(x) d\pi(y) = 0$  and then

$$G_t(x, y) = \int_{2t}^{\infty} (p_s(x, y) - 1) ds.$$

□

Using Proposition 4.5 and 4.9 we conclude the following inequality. We can find an analogous result in [71, Lemma 5.2].

**Corollary 4.10** (Off-diagonal behavior). *For every  $n \geq 2$ ,  $t > 0$  and  $(x_1, \dots, x_n) \in M^n$*

$$\sum_{i < j} G(x_i, x_j) \geq \sum_{i < j} G_t(x_i, x_j) - tn^2.$$

*Proof.* Since the heat kernel is non-negative, by Proposition 4.5 and 4.9 we have that, for every  $x, y \in M$ ,

$$G(x, y) - G_t(x, y) = \int_0^{2t} (p_s(x, y) - 1) ds \geq -2t.$$

Then, if  $(x_1, \dots, x_n) \in M^n$ ,

$$\sum_{i < j} G(x_i, x_j) \geq \sum_{i < j} G_t(x_i, x_j) - tn(n-1) \geq \sum_{i < j} G_t(x_i, x_j) - tn^2.$$

□

What is left to understand is  $\sum_{i=1}^n G_t(x_i, x_i)$ . This will be achieved using Proposition 4.9 and the short-time asymptotic expansion of the heat kernel. A particular case is mentioned in [71, Lemma 5.3].

**Proposition 4.11** (Diagonal behavior). *Let  $m$  be the dimension of  $M$ . If  $m = 2$  there exists a constant  $C > 0$  such that for every  $t \in (0, 1]$  and  $x \in M$*

$$G_t(x, x) \leq -\frac{1}{4\pi} \log(t) + C.$$

*If  $m > 2$  there exists a constant  $C > 0$  such that for every  $t \in (0, 1]$  and  $x \in M$*

$$G_t(x, x) \leq \frac{C}{t^{\frac{m}{2}-1}}.$$

*Proof.* By the asymptotic expansion of the heat kernel (see for instance [40, Chapter VI.4]) we have that there exists a constant  $\tilde{C} > 0$  (independent of  $x$  and  $t$ ) such that, for  $t \leq 1$ ,

$$\left| p_t(x, x) - \frac{1}{(4\pi t)^{\frac{m}{2}}} \right| \leq \tilde{C} t^{-\frac{m}{2}+1}.$$

Then,

$$p_t(x, x) \leq \frac{1}{(4\pi t)^{\frac{m}{2}}} + \tilde{C}t^{-\frac{m}{2}+1}. \quad (18)$$

We know by Proposition 4.9 that

$$\begin{aligned} G_t(x, x) &= \int_{2t}^{\infty} (p_s(x, x) - 1) ds \\ &= \int_{2t}^2 (p_s(x, x) - 1) ds + \int_2^{\infty} (p_s(x, x) - 1) ds \\ &\leq \int_{2t}^2 \left[ \frac{1}{(4\pi s)^{\frac{m}{2}}} + \tilde{C}s^{-\frac{m}{2}+1} - 1 \right] ds + \int_2^{\infty} (p_s(x, x) - 1) ds \\ &= \int_{2t}^2 \left[ \frac{1}{(4\pi s)^{\frac{m}{2}}} + \tilde{C}s^{-\frac{m}{2}+1} \right] ds + G_2(x, x). \end{aligned}$$

In the case  $m = 2$  we obtain that, for  $t \in (0, 1]$ ,

$$G_t(x, x) \leq -\frac{1}{4\pi} \log(t) + C$$

where  $C$  is  $2\tilde{C}$  plus a bound for  $G_2(x, x)$  independent of  $x$ . In the case  $m > 2$  we use that  $s^{-m/2+1} \leq 2s^{-m/2}$  for  $s \in (0, 1]$  and that  $G_2(x, x)$  is bounded from above to obtain a constant  $C$  such that, for  $t \in (0, 1]$ ,

$$G_t(x, x) \leq \frac{C}{t^{\frac{m}{2}-1}}.$$

□

Knowing the diagonal and off-diagonal behavior of the regularized Green function we can proceed to prove Theorem 4.4.

*Proof of Theorem 4.4.* Take  $\vec{x} = (x_1, \dots, x_n) \in M^n$ . Then if  $m = 2$  we have

$$\begin{aligned} H_n(\vec{x}) &\geq \frac{1}{n^2} \sum_{i < j} G_t(x_i, x_j) - t \\ &\geq \frac{1}{n^2} \sum_{i < j} G_t(x_i, x_j) - t + \frac{1}{2n^2} \sum_{i=1}^n G_t(x_i, x_i) + \frac{1}{8\pi n} \log(t) - \frac{1}{2n} C \\ &= H(R_t(\vec{x})) - t + \frac{1}{8\pi n} \log(t) - \frac{1}{2n} C \end{aligned}$$

where we have used Corollary 4.10 and Proposition 4.11. If  $m > 2$  we proceed in the same way to get

$$\begin{aligned} H_n(\vec{x}) &\geq \frac{1}{n^2} \sum_{i < j} G_t(x_i, x_j) - t \\ &\geq \frac{1}{n^2} \sum_{i < j} G_t(x_i, x_j) - t + \frac{1}{2n^2} \sum_{i=1}^n G_t(x_i, x_i) + \frac{C}{2nt^{\frac{m}{2}-1}} \\ &= H(R_t(\vec{x})) - t + \frac{C}{2nt^{\frac{m}{2}-1}}. \end{aligned}$$

□

*Remark 4.12* (Euclidean setting). Let us give a quick explanation of the regularization of the energy in the Euclidean case. Define the two-body interaction  $G$  by

$$G(x, y) = \begin{cases} -\log|x - y| & \text{if } m = 2 \\ |x - y|^{2-m} & \text{if } m > 2 \end{cases}.$$

Suppose  $\mu$  is a radial probability measure on  $\mathbb{R}^m$  of finite energy, i.e.  $\int_{\mathbb{R}^m \times \mathbb{R}^m} |G| d\mu^{\otimes 2} < \infty$ . For  $\varepsilon > 0$  define  $S_\varepsilon : \mathbb{R}^m \rightarrow \mathbb{R}^m$  by

$$S_\varepsilon(x) = \varepsilon x$$

and for  $x \in \mathbb{R}^m$  define  $T_x : \mathbb{R}^m \rightarrow \mathbb{R}^m$  by

$$T_x(\alpha) = \alpha + x.$$

The regularization of the punctual charge at  $x \in \mathbb{R}^m$  will be  $\mu_x^\varepsilon = (T_x \circ S_\varepsilon)_* \mu$  where the subindex  $*$  is used to denote the image measure. Define the two-body regularized interaction  $G_\varepsilon$  by

$$G_\varepsilon(x, y) = \int_{\mathbb{R}^m \times \mathbb{R}^m} G(\alpha, \beta) d\mu_x^\varepsilon(\alpha) d\mu_y^\varepsilon(\beta).$$

The analogue of Corollary 4.10 would be

$$\sum_{i < j} G(x_i, x_j) \geq \sum_{i < j} G_\varepsilon(x_i, x_j)$$

which is a consequence of the superharmonicity of  $G(x, \cdot)$ . The analogue of Proposition 4.11 would be

$$G_\varepsilon(x, x) = -\log \varepsilon - \int_{\mathbb{R}^m \times \mathbb{R}^m} \log|\alpha - \beta| d\mu(\alpha) d\mu(\beta)$$

when  $m = 2$  and

$$G_\varepsilon(x, x) = \varepsilon^{2-m} \int_{\mathbb{R}^m \times \mathbb{R}^m} |\alpha - \beta|^{2-m} d\mu(\alpha) d\mu(\beta)$$

when  $m > 2$ . This is a straightforward application of the change-of-variables formula. Finally, if we define  $R_\varepsilon(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n \mu_{x_i}^\varepsilon$ , the analogue of Proposition 4.2 would be

$$W_1(R_\varepsilon(\vec{x}), i_n(\vec{x})) \leq \varepsilon \int_{\mathbb{R}^m} |y| d\mu(y).$$

Having acquired all the tools to apply Theorem 2.1 to the case of a Coulomb gas on a compact Riemannian manifold, the next section will be devoted to prove the main theorem and its almost immediate extension.

## 5 Proof of the concentration inequality for Coulomb gases

*Proof of Theorem 1.1.* First, we notice that  $e_n = \int_M H_n d\mu_{\text{eq}} = \frac{n-1}{n} e = 0$ . To use Theorem 2.1 we define

$$f(r) = \frac{r^2}{2} \text{ and } R = R_t \text{ for } t = n^{-\frac{2}{m}}.$$

In this case, Proposition 4.2 tells us that  $W_1(R(\vec{x}), i_n(\vec{x})) \leq C/n^{1/m}$  for some  $C > 0$  independent of  $\vec{x}$  and  $n$ . This may be considered as the natural choice since  $1/n^{1/m}$  is the ‘closest’ a fixed probability measure absolutely continuous with respect to  $\pi$  can get to an arbitrary empirical measure of  $n$  points.

If  $m = 2$ , by Theorem 4.4 and Proposition 4.2, we have that there exists a constant  $\tilde{C} > 0$  such that

$$H_n(\vec{x}) \geq H(R(\vec{x})) - \frac{1}{8\pi n} \log(n) - \frac{\tilde{C}}{n}$$

$$W_1(R(\vec{x}), i_n(\vec{x})) \leq \frac{\tilde{C}}{\sqrt{n}}$$

for every  $\vec{x} \in M^n$  and  $n \geq 2$  so we can apply Theorem 2.1 to obtain the desired result with  $C = \frac{\tilde{C}^2}{2} + \tilde{C}$ . Similarly, if  $m > 2$ , by Theorem 4.4 and Proposition 4.2, we have that there exists a constant  $\tilde{C} > 0$  such that

$$H_n(\vec{x}) \geq H(R(\vec{x})) - \frac{\tilde{C}}{n^{\frac{2}{m}}}$$

$$W_1(R(\vec{x}), i_n(\vec{x})) \leq \frac{\tilde{C}}{n^{\frac{1}{m}}}$$

for every  $\vec{x} \in M^n$  and  $n \geq 2$  so we we can apply Theorem 2.1 to obtain the desired result with  $C = \frac{\tilde{C}^2}{2} + \tilde{C}$ .  $\square$

Finally we present the proof of Theorem 1.2.

*Proof of Theorem 1.2.* To apply Theorem 2.1 we notice that Assumption A is satisfied by  $f(r) = \frac{r^2}{2}$ . Indeed, Theorem 3.1 is still true for this new  $H$  except for the characterization of the minimizer. In particular,  $H$  has a unique minimizer. By a calculation we can obtain that  $e - e_n = \frac{1}{2n} \int_{M \times M} G(x, y) d\mu_{eq}(x) d\mu_{eq}(y)$  which is of order  $\frac{1}{n}$  and will be absorbed by the constant  $C$ . To meet the hypotheses of Theorem 2.1, we need to compare

$$\frac{1}{n} \sum_{i=1}^n V(x_i) \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n \int_M V d\mu_{x_i}^t.$$

By using the relation

$$\mathbb{E}[V(X_t)] = V(x) + \int_0^t \mathbb{E}[\Delta f(X_s)] ds$$

where  $X_t$  is the Markov process with generator  $\Delta$  starting at  $x$  we obtain

$$|\mathbb{E}[V(X_t)] - V(x)| \leq \hat{C}t$$

where  $\hat{C}$  is some upper bound to  $\Delta V$  and thus

$$\left| \frac{1}{n} \sum_{i=1}^n \int_M V d\mu_{x_i}^t - \frac{1}{n} \sum_{i=1}^n V(x_i) \right| \leq \hat{C}t.$$

In conclusion, if we choose  $R = R_{n-\frac{2}{m}}$ , there still exists a constant  $C > 0$  such that

$$H_n(\vec{x}) \geq H(R(\vec{x})) - \frac{1}{8\pi n} \log(n) - \frac{C}{n}$$

in dimension two and

$$H_n(\vec{x}) \geq H(R(\vec{x})) - \frac{C}{n^{\frac{2}{m}}}$$

in dimension  $m > 2$  so that we can apply Theorem 2.1.  $\square$



# Chapter 4

## Extremals on background ensembles

*This chapter corresponds to [32]. It is a joint work with Raphael Butez.*

**Abstract.** We study the extremal particles of a class of two-dimensional determinantal Coulomb gases and zeros of random polynomials. More precisely, the systems we consider are related to radial positive unit charge backgrounds. Different behaviors arise for the farthest particles depending on the background at infinity. In the particular case of a compactly supported background, we obtain that the farthest particles of the Coulomb gases converge towards the Bergman point process of the complement of the unit disc. We also obtain that the farthest zeros of random polynomials converge towards the zero set of a random analytic function, which coincides with the Bergman point process when the polynomials are Gaussian.

**Résumé.** Nous étudions les particules extrémales d'une famille de gaz de Coulomb, dans le cadre déterminantal, et des zéros d'une famille des polynômes aléatoires. Plus précisément, les systèmes que nous considérons habitent dans des milieux invariants par rotation et chargés positivement. Différents comportements apparaissent, et dépendent du comportement du milieu à l'infini. Dans le cas particulier d'un milieu dont la charge est à support compact nous obtenons que les particules les plus éloignées des gaz de Coulomb convergent vers le processus de Bergman du complément du disque unité. Nous obtenons aussi que les zéros les plus éloignés des polynômes aléatoires correspondants convergent vers l'ensemble des zéros d'une fonction analytique aléatoire, qui coïncide avec le processus de Bergman si les polynômes sont gaussiens.

*2010 MSC:* 60G55; 60F05; 60K35; 82C22; 30C15

*Keywords:* Gibbs measure; Coulomb gas; random polynomial; interacting particle system; determinantal point process.

*Mots-clés:* Mesure de Gibbs; gaz de Coulomb; polynôme aléatoire; système de particules en interaction; processus ponctuel déterminantal.

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### 1 Introduction

#### 1.1 Coulomb gases and random polynomials

A Coulomb gas in the complex plane is, in this article, a system of  $n$  interacting particles in equilibrium located at  $x_1, \dots, x_n \in \mathbb{C}$  with joint distribution

$$\frac{1}{Z_n} \exp \left( -\beta \left[ -\sum_{i < j} \log |x_i - x_j| + (n+1) \sum_{i=1}^n V(x_i) \right] \right) d\ell_{\mathbb{C}^n}(x_1, \dots, x_n) \quad (1)$$

where  $Z_n$  is a normalizing constant,  $\beta$  is a real number usually interpreted as the inverse of the temperature,  $V$  is a continuous real valued function, called the confining potential and  $\ell_{\mathbb{C}^n}$  is the Lebesgue measure on  $\mathbb{C}^n$ . If we define the Hamiltonian

$$(x_1, \dots, x_n) \mapsto - \sum_{i < j} \log |x_i - x_j| + (n+1) \sum_{i=1}^n V(x_i)$$

then the joint distribution (1) is the canonical Gibbs measure at inverse temperature  $\beta$  associated to this Hamiltonian. Coulomb gases model the positions of  $n$  electrons with unit charge, at inverse temperature  $\beta$  and in the presence of a confining potential  $V$ . This model is well defined as soon as the potential  $V$  satisfies

$$\liminf_{|z| \rightarrow \infty} \{V(z) - \log |z|\} > -\infty. \quad (2)$$

In this article, we will only consider potentials of the form

$$V^\nu(z) = \int_{\mathbb{C}} \log |z - w| d\nu(w) \quad (3)$$

for a rotationally invariant probability measure  $\nu$  such that its potential  $V^\nu$  is well defined and we will treat only the specific inverse temperature

$$\beta = 2.$$

For this particular inverse temperature, the Coulomb gas forms a determinantal point process which is the key to our approach. Our choice of potential corresponds to the physical situation where  $n$  electrons with unit charge are attracted by a positive distribution  $(n+1)\nu$  with total charge  $n+1$ . On this model, Frostman's criterion [85] implies that

$$\frac{1}{n} \sum_{k=1}^n \delta_{x_k} \xrightarrow[n \rightarrow \infty]{} \nu.$$

**Assumption 2.** *In all this article, the probability measure  $\nu$  is rotationally invariant and satisfies*

$$\int_{\mathbb{C}} |\log |z|| d\nu(z) < \infty.$$

*This implies in particular that  $V^\nu$  is finite everywhere and that*

$$\mathcal{E}(\nu) = - \int_{\mathbb{C}} V^\nu(z) d\nu(z) = - \int_{\mathbb{C}^2} \log |z - w| d\nu(z) d\nu(w) \in \mathbb{R}.$$

Given such a measure  $\nu$ , one can also consider random polynomials of the form

$$P_n(z) = \sum_{k=0}^n a_k R_k(z) \propto \prod_{k=1}^n (z - z_k) \quad (4)$$

where  $(R_k)_{k \in \{0, \dots, n\}}$  is an orthonormal basis of  $\mathbb{C}_n[X]$  for the inner product

$$\langle P, Q \rangle_n = \int_{\mathbb{C}} P(z) \overline{Q(z)} e^{-2nV^\nu(z)} d\nu(z) \quad (5)$$

and where the  $a_k$ 's are i.i.d. random variables. We will assume  $R_k$  are monomials. This can be done due to the invariance under rotations of  $\nu$ . If  $\nu$  is non-radial, for standard complex Gaussian coefficients, the law of the random polynomial is independent of the choice of orthonormal basis, but this is not true for non-Gaussian coefficients, and the basis would have to be introduced as additional data.

This model covers the classical random polynomials ensembles, namely Kac polynomials, Elliptic polynomials and (nearly) Weyl polynomials for specific choices of  $\nu$ .

Model	Basis	Measure	Potential
Kac	$X^k$	$\nu_{S^1}$ uniform on $S^1$	$V^{\nu_{S^1}}(z) = \max(\log  z , 0)$
Elliptic	$\sqrt{\binom{n}{k}} X^k$	$d\nu_{FS}(z) = \frac{d\ell_{\mathbb{C}}(z)}{\pi(1+ z ^2)^2}$	$V^{\omega_{FS}}(z) = \frac{1}{2} \log(1 +  z ^2)$
Nearly Weyl	$\frac{\sqrt{n^k} X^k}{\sqrt{k! - \int_n^\infty r^k e^{-r} dr}}$	$d\nu_D(z) = \frac{1_{ z <1} d\ell_{\mathbb{C}}(z)}{\pi}$	$V^{\nu_D}(z) = \begin{cases} \frac{1}{2}( z ^2 - 1) & \text{if }  z  < 1 \\ \log  z  & \text{if }  z  \geq 1 \end{cases}$

The random polynomials that we called ‘‘Nearly Weyl’’ polynomials are not exactly the classical rescaled Weyl polynomials, which are usually defined as

$$P_n^{\text{Rescaled Weyl}}(z) = \sum_{k=0}^n \frac{\sqrt{n^k}}{\sqrt{k!}} a_k z^k$$

and which would be related to the uniform measure on the plane.

## 1.2 Results on Coulomb gases

In this section we state three theorems on the extremal particles of Coulomb gases associated to a positive background  $\nu$ . First, when  $\nu$  is compactly supported, we study the point process of the particle outside of the support. Then, when the support of  $\nu$  is included in the complement of a disk, we analyze the particles inside this disk. In fact, we will see that these behaviors are the inverse of each other. Finally, when  $\nu$  has mass near infinity (respectively, near the origin), we rescale the farthest particles (resp. closest particles) to study the convergence of these point processes.

Before giving the statements of the theorems, we introduce the Bergman point process of the unit disk which will appear in the Coulomb gas setting and in the random polynomials case.

**Definition 1.1** (Bergman point process in the disk). *The Bergman point process  $\mathcal{B}$  is the determinantal point process in  $\mathbb{D}$ , the open unit disk of  $\mathbb{C}$ , with kernel  $K$  given by the Bergman kernel of the disk, i.e.*

$$\forall z, w \in \mathbb{D} \quad K(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}.$$

The point process  $\mathcal{B}$  can be seen as the zeros of a Gaussian analytic function by the work of Peres and Virág [79]. This point process is invariant under any conformal map of  $\mathbb{D}$ , it has an infinite number of points and every point of the unit circle is almost surely an accumulation point of this process.

**Definition 1.2** (Inversion map). *We define the inversion map  $i : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$  by*

$$i(z) = 1/z$$

In this article, we will use  $i_*\nu$  to denote the pushforward of the measure  $\nu$  by the inversion map. None of the measures studied in this article have an atom at the origin, hence this pushforward is well-defined.

**Theorem 1.3** (Farthest particles). *Let  $(x_1, \dots, x_n)$  be a Coulomb gas with joint distribution (1), associated to  $\nu$  satisfying Assumption 2 and*

$$S^1 \subset \text{supp } \nu \subset \bar{\mathbb{D}}.$$

*Let  $\mathcal{O}_n = \{x_k \text{ such that } |x_k| > 1\}$  be the point process of the particle outside of the support of  $\nu$ , then the following holds.*



1. The point process  $\mathcal{O}_n$  converges weakly as  $n$  goes to infinity towards the image under inversion of the Bergman point process  $i(\mathcal{B}) = \{i(x), x \in \mathcal{B}\}$ .
2. The sequence of random variables  $(\max_{k \in \{1, \dots, n\}} |x_k|)_{n \in \mathbb{Z}_+}$  converges in distribution towards a limiting random variable  $x_\infty$ , supported on  $[1, \infty)$ , with cumulative distribution function

$$\mathbb{P}(x_\infty < t) = \prod_{k=1}^{\infty} (1 - t^{-2k}).$$

The topology on (deterministic) point process is reminded in Section 4 for convenience of the reader. The point process  $i(\mathcal{B})$  can be seen as the determinantal point process in  $(\bar{\mathbb{D}})^c$  associated to its Bergman kernel. One can easily check that the limiting random variable  $x_\infty$  has infinite variance, as well as the variable  $\max_{k \in \{1, \dots, n\}} |x_k|$  for any  $n$ . This tells us that  $\max_{k \in \{1, \dots, n\}} |x_k|$  does not converge to 1 as  $n$  goes to infinity. This is a very different behavior from what was known in the context of strongly confining potentials [38]. Since the Bergman point process has finitely many points in any disk with radius strictly smaller than one, the inverse of this process has finitely many points in the complement of any disk of radius greater than one.

*Remark 1.4* (Universal behavior of the outer process). The limiting distribution for the modulus of the farthest particle and the limiting point process do not depend on the choice of the background  $\nu$ . Since most of the particles fill the unit disk according to the measure  $\nu$ , the outer particles "see" two canceling effects: on one hand they are attracted by the positive background, but they are repelled by the negative charges which have nearly the same effect as the background. The universal behavior of the outer point process is a consequence of this competition. Nevertheless, the behavior depends on the coefficient  $(n+1)$  at the left side of  $V$  on (1) as can be seen in [53].

**Theorem 1.5** (Closest particles). *Let  $(x_1, \dots, x_n)$  be a Coulomb gas with joint distribution (1), associated to  $\nu$  satisfying Assumption 2 and*

$$S^1 \subset \text{supp } \nu \subset \mathbb{D}^c.$$

If  $\mathcal{I}_n = \{x_k \text{ such that } |x_k| < 1\}$  then the following holds.

1. The point process  $\mathcal{I}_n$  converges weakly as  $n$  goes to infinity towards the Bergman point process  $\mathcal{B}$ .
2. The sequence of random variables  $(\min_{k \in \{1, \dots, n\}} |x_k|)_{n \in \mathbb{Z}_+}$  converges in distribution towards a limiting random variable  $y_\infty$ , supported on  $[0, 1]$ , with cumulative distribution function

$$\mathbb{P}(y_\infty < t) = 1 - \prod_{k=1}^{\infty} (1 - t^{2k}).$$

*Remark 1.6* (Comment on Figure 1.2). In order to simulate realizations of the Coulomb gas (1) with potential associated to the uniform measure on the unit circle

$$V(z) = \frac{1}{2\pi} \int_0^{2\pi} \log |z - e^{i\theta}| d\theta = \max(\log |z|, 0) \quad (6)$$

we used the algorithm introduced by Chafaï and Ferré [36]. The uniform measure on the unit circle is the only probability measure that strictly satisfies both theorems 1.3 and 1.5. This figure also illustrates the convergence of  $\max |x_k|$  towards  $x_\infty = 1/y_\infty$  since  $\max |x_k|$  has the same distribution as  $1/\min |x_k|$ .

Like in Theorem 1.3, the limiting point process and the limiting distribution for the closest particle to the origin are universal.

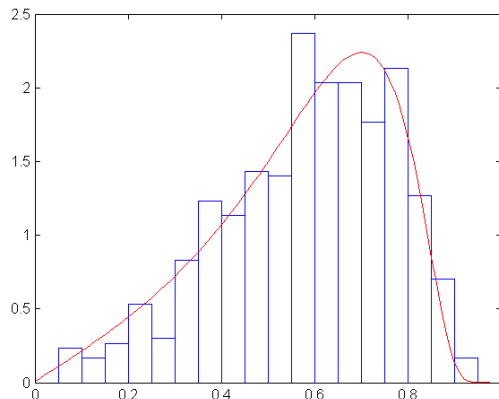


Figure 4.1: Histogram for the lowest modulus in the gas associated to the circular potential (6) with  $n = 50$ , and density of the random variable  $y_\infty$ .

**Theorem 1.7** (Extremal particles in the support). *Let  $\nu$  be a measure satisfying Assumption 2 and let  $(x_1, \dots, x_n)$  be the Coulomb gas associated to the background  $\nu$ .*

1. *If there exists  $\alpha > 0$  and  $\lambda > 0$  such that*

$$\lim_{r \rightarrow 0} \frac{\nu(D_r)}{r^\alpha} = \lambda,$$

*where  $D_r$  denotes the open disk of radius  $r$ , then the point process  $\{x_1, \dots, x_n\}$  converges weakly towards the determinantal point process in  $\mathbb{C}$  associated to the kernel*

$$K(z, w) = \sum_{k=0}^{\infty} b_k z^k \bar{w}^k e^{-\gamma|z|^\alpha} e^{-\gamma|w|^\alpha} \quad (7)$$

*where*

$$\gamma = \frac{\lambda}{\alpha} \quad \text{and} \quad b_k^{-1} = 2\pi \int_0^\infty r^{2k+1} e^{-2\gamma r^\alpha} dr.$$

*This implies that*

$$n^{1/\alpha} \min_{k \in \{1, \dots, n\}} |x_k| \xrightarrow[n \rightarrow \infty]{\mathcal{L}} z_\infty$$

*where the random variable  $z_\infty$  has cumulative distribution function*

$$F(y) = 1 - \prod_{k=0}^{\infty} \frac{\Gamma\left(\frac{2k+2}{\alpha}, \frac{2\lambda}{\alpha} y^\alpha\right)}{\Gamma\left(\frac{2k+2}{\alpha}\right)} \quad (8)$$

*and the  $\Gamma$  with two arguments denotes the upper incomplete gamma function while the  $\Gamma$  with one argument denotes the gamma function.*

2. *If there exists  $\alpha > 0$  and  $\lambda > 0$  such that*

$$\lim_{r \rightarrow \infty} r^\alpha \nu(D_r^c) = \lambda,$$

*where  $D_r^c$  is the complement of the open disk of radius  $r$ , then the rescaled point process  $\{n^{-1/\alpha} x_1, \dots, n^{-1/\alpha} x_n\}$ , seen as a point process in  $\mathbb{C} \setminus \{0\}$ , converges weakly towards the inverse of the determinantal point process associated to (7). Furthermore, we have that*

$$\frac{1}{n^{1/\alpha}} \max_{k \in \{1, \dots, n\}} |x_k| \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \frac{1}{z_\infty}$$

*where, as before,  $z_\infty$  has cumulative distribution given by (8).*

This theorem extends the corresponding result of Jiang and Qi [63, Theorem 1] on the spherical ensemble. Note that if  $\nu$  has a positive density at the origin then the previous result applies with  $\alpha = 2$  and  $\lambda$  being the density at the origin.

### 1.3 Results on random polynomials

In this section, we present results on the extremal zeros of random polynomials associated to a background measure  $\nu$  which are the counterparts of the results obtained for Coulomb gases in the previous section. The results are very close to what was obtained before and are presented in the same order.

**Theorem 1.8** (Farthest roots of random polynomials). *Let  $\nu$  be a probability measure satisfying Assumption 2 and*

$$S^1 \subset \text{supp } \nu \subset \bar{\mathbb{D}}.$$

Let  $(a_k)_{k \in \mathbb{N}}$  be a sequence of i.i.d. random variables such that

$$\mathbb{E}(\log(1 + |a_0|)) < \infty.$$

Let  $P_n$  be the random polynomial defined by (4) and  $\{z_1, \dots, z_n\}$  the zeros of  $P_n$ . If we denote  $\mathcal{O}_n^{Pol} = \{z_k \text{ such that } |z_k| > 1\}$  then the following holds.

1.  $\mathcal{O}_n^{Pol}$  converges in law towards the zero set of the random series  $\sum_{k=0}^{\infty} a_k \frac{1}{z^k}$ .

When the coefficients are  $\mathcal{N}_{\mathbb{C}}(0, 1)$  random variables, this zero set has the same distribution as  $i(\mathcal{B})$ .

2. The sequence of random variables  $(\max_{k \in \{1, \dots, n\}} |z_k|)_{n \in \mathbb{Z}_+}$  converges in law towards the random variable  $x_{\infty} = \max \left\{ |z| \text{ such that } \sum_{k=0}^{\infty} a_k \frac{1}{z^k} = 0 \right\}$ .

When the coefficients are  $\mathcal{N}_{\mathbb{C}}(0, 1)$  random variables, the cumulative distribution function of  $x_{\infty}$  is

$$\mathbb{P}(x_{\infty} < t) = \prod_{k=1}^{\infty} (1 - t^{-2k}).$$

**Theorem 1.9** (Closest roots of random polynomials). *Let  $\nu$  be a probability measure satisfying Assumption 2 and*

$$S^1 \subset \text{supp } \nu \subset \mathbb{D}^c.$$

Let  $(a_k)_{k \in \mathbb{N}}$  be a sequence of i.i.d. random variables such that

$$\mathbb{E}(\log(1 + |a_0|)) < \infty.$$

Let  $P_n$  be the random polynomial defined by (4) and  $\{z_1, \dots, z_n\}$  the zeros of  $P_n$ . If we denote  $\mathcal{I}_n^{Pol} = \{z_k \text{ such that } |z_k| < 1\}$  then the following holds.

1.  $\mathcal{I}_n^{Pol}$  converges almost surely towards the zero set of  $\sum_{k=0}^{\infty} a_k z^k$ .

When the coefficients are  $\mathcal{N}_{\mathbb{C}}(0, 1)$  random variables, this zero set has the same distribution as  $\mathcal{B}$ .

2. The sequence of random variables  $(\min_{k \in \{1, \dots, n\}} |z_k|)_{n \in \mathbb{Z}_+}$  converges almost surely towards the random variable  $y_{\infty} = \min \left\{ |z| \text{ such that } \sum_{k=0}^{\infty} a_k z^k = 0 \right\}$ .

When the coefficients are  $\mathcal{N}_{\mathbb{C}}(0, 1)$  random variables, the cumulative distribution function of  $y_{\infty}$  is

$$\mathbb{P}(y_{\infty} < t) = 1 - \prod_{k=1}^{\infty} (1 - t^{2k}).$$

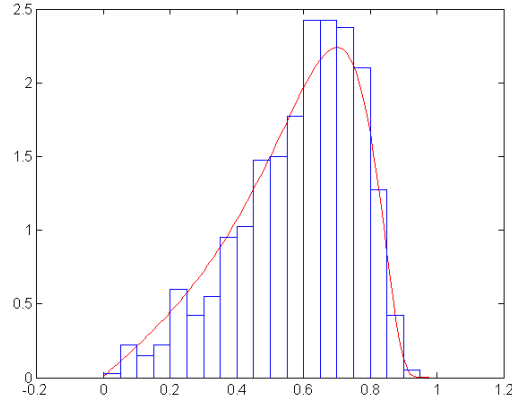


Figure 4.2: Histogram for the smallest modulus among the roots of random Kac polynomials with degree 200 and density of  $y_{(\infty)}$ .

Theorems 1.8 and 1.9 are the exact analogs of Theorems 1.3 and 1.5 for Coulomb gases. The fact that we recover the Bergman point process is exactly the result of Peres and Virág [79].

In the case where the support of  $\nu$  is unbounded or contains the origin, we observe a phenomenon similar to the results of Theorem 1.7 but for which the limiting random variable may differ of the Coulomb case.

**Theorem 1.10** (Rescaled extremal roots of random polynomials). *Let  $\nu$  be a measure satisfying Assumption 2. Let  $(a_k)_{k \in \mathbb{N}}$  be a sequence of i.i.d. random variables such that*

$$\mathbb{E}(\log(1 + |a_0|)) < \infty.$$

Let  $P_n$  be the random polynomial defined by (4) and  $\{z_1, \dots, z_n\}$  the zeros of  $P_n$ .

1. If there exists  $\alpha > 0$  and  $\lambda > 0$  such that

$$\lim_{r \rightarrow 0} \frac{\nu(D_r)}{r^\alpha} = \lambda$$

then the point process  $\{n^{1/\alpha} z_1, \dots, n^{1/\alpha} z_n\}$  converges almost surely towards the roots of the following random entire function

$$f_{\alpha, \lambda}(z) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(1 + \frac{2k}{\alpha})^{1/2}} \left(\frac{\lambda}{\alpha}\right)^{k/\alpha} z^k$$

sometimes known as the Mittag-Leffler random function. As a consequence,

$$n^{1/\alpha} \min |z_k| \xrightarrow[n \rightarrow \infty]{a.s.} m_\infty = \inf\{|z| \mid f_{\alpha, \lambda}(z) = 0\}.$$

2. If there exists  $\alpha > 0$  and  $\lambda > 0$  such that

$$\lim_{r \rightarrow \infty} r^\alpha \nu(D_r^c) = \lambda$$

then the point process  $\{n^{-1/\alpha} z_1, \dots, n^{-1/\alpha} z_n\}$ , seen as a point process in  $\mathbb{C} \setminus \{0\}$ , converges towards the inverse of the point process of the zeros of  $f_{\alpha, \lambda}$ . Furthermore, we have that

$$\frac{1}{n^{1/\alpha}} \max |z_k| \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \frac{1}{m_\infty}.$$

Notice that the coefficients of  $f_{\alpha,\lambda}$  can be written using the gamma function. Notice also that when  $\alpha = 2$  and  $\lambda = 2$ , which occurs for elliptic polynomials, the function  $f_{\alpha,\lambda}$  is the Planar Gaussian Analytic function

$$f_{2,2}(z) = \sum_{k=0}^{\infty} a_k \frac{1}{\sqrt{k!}} z^k.$$

For complex Gaussian coefficients, the random Mittag-Leffler functions are studied for the rigidity of their zero set, [66]. There does not seem to exist good reasons for  $z_{\infty}$  and  $m_{\infty}$  to be similar. Figure 1.3 suggests they are not.

*Remark 1.11.* In the special case where  $\nu = \nu_{S^1}$  is the uniform measure on the unit disk, both theorems reduce to the result of Peres and Virág [79]. For other random polynomials such as Weyl polynomials, this result is new. There is no hope that this result holds for other distribution on the coefficients. This remark can be found, for instance, in [93, Section 5] or in [26]. See [31] for a discussion of this non-universality in the case of the Kac polynomials.

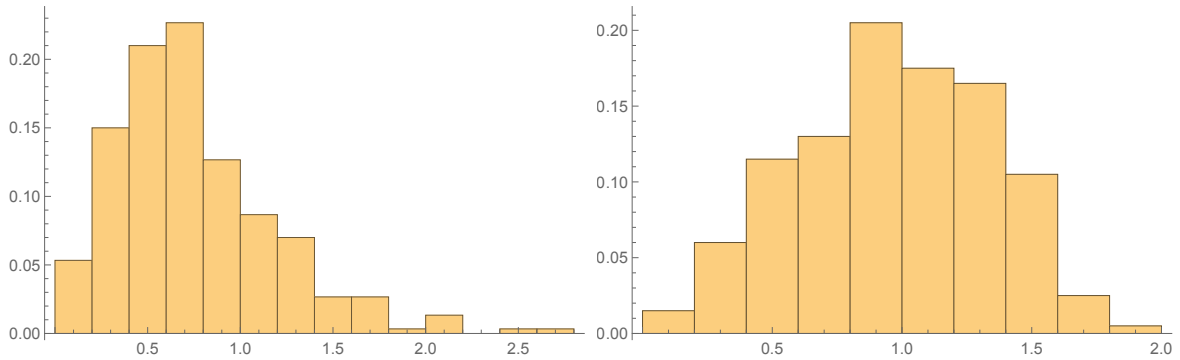


Figure 4.3: Approximations of the histograms of the random variables  $z_{\infty}$  and  $m_{\infty}$ . On the left we simulated the eigenvalues of the Forrester-Krishnapur ensemble and on the right we simulated the zeros of Weyl polynomials, both with degree 100.

It is possible to extend Theorem 1.8 to the case where  $\mu(\mathbb{C}) > 1$ . Indeed, the same methods that will be used to prove Theorem 1.8 would allow us to prove the straightforward generalization. Since it escapes the main models of interest in this article we will only state the case of Weyl polynomials which have an easier and shorter proof.

**Theorem 1.12** (Farthest particles for the Weyl polynomials). *Let  $(a_k)_{k \in \mathbb{N}}$  be a sequence of i.i.d. random variables such that*

$$\mathbb{E}(\log(1 + |a_0|)) < \infty.$$

Let

$$P_n(z) = \sum_{k=0}^n \frac{\sqrt{n^k}}{\sqrt{k!}} a_k z^k.$$

Then, the same results of Theorem 1.8 apply. In particular, when the coefficients  $a_k$  are  $\mathcal{N}_{\mathbb{C}}(0, 1)$  random variables,

$$\lim_{n \rightarrow \infty} \mathbb{P}(\max\{|z| : P_n(z) = 0\} \leq t) = \prod_{k=1}^{\infty} (1 - t^{-2k}).$$

for every  $t \in [1, \infty)$ .

## 2 Comments and perspectives

### 2.1 Related results

For Coulomb gases at inverse temperature  $\beta = 2$  and associated to a strongly confining, radially symmetric potential  $V$ , Chafaï and P  ch   [38] showed that all the particles converge towards the support of the limiting measure and that the farthest particle shows Gumbel fluctuations. Their result is, to our knowledge, the only situation where the outliers of Coulomb gases were studied. Recently, Lacroix-A-Chez-Toine, Grabsch, Majumdar and Schehr [70] showed a universal intermediate deviations regime for the largest particles of some Coulomb gases at  $\beta = 2$ .

The farthest particles of further models are studied in [87], [57] and [39], in link with the spectral radii of some non Hermitian matrix models. We recover and generalize some of their results on Coulomb gases. Some other weakly and strongly confining models are studied by one of the authors in [53] by analyzing the behavior of the kernel when it is feasible.

### 2.2 Open questions

If we only assume the background  $\nu$  to be compactly supported with a finite logarithmic energy, i.e.  $\nu$  not necessarily radial, we expect to observe a point process outside of the support of the equilibrium measure, with accumulation points only on the edge of the support of  $\nu$ . We also expect the largest particle of this point process to have infinite variance such as in the radial case. For domain reasons, the possible limiting processes cannot be the Bergman point process, but could be a deformed version of it.

If we do not assume  $\beta = 2$ , all the results presented in this article collapse. We hope that similar results hold for any inverse temperature  $\beta$ . In dimension one, the Sine point process and the Airy point process have  $\beta$  counterparts which generalizes them to any temperature. We can dream of a generalization to any  $\beta$  of the Bergman point process.

For random polynomials, the study of the outliers in the case where  $\nu$  is not radial seems hard to study. One may try to understand the behavior of the orthogonal polynomials  $R_k$  associated to an inner product

$$\langle P, Q \rangle = \int P(z) \overline{Q(z)} e^{-2nV^\nu(z)} d\nu(z)$$

which is in general a difficult question. After it is understood, the study of the outliers could be carried out by studying the asymptotics of the covariance kernel of the Gaussian field  $(P_n(z))_{z \in \mathbb{C}}$

$$K_n(z, w) = \sum_{k=0}^n R_k(z) \overline{R_k(w)}$$

outside of the support of  $\nu$ .

## 3 Proof of the results

First, we prove the results on Coulomb gases, then we prove the corresponding results on random polynomials in the same order.

### 3.1 Results on Coulomb gases

We start with the proof of Theorem 1.5, then we deduce Theorem 1.3 and we finish with the proof of Theorem 1.7.

**Proof of Theorem 1.5**

**Lemma 3.1** (Useful formula for the potential). *Let  $\nu \in \mathcal{P}(\mathbb{C})$  be a rotationally invariant probability measure such that  $\int_{D_1^c} \log |x| d\nu(x) < \infty$ . Then*

$$V^\nu(z) = \int_1^{|z|} \frac{\nu(D_r)}{r} dr + \int_{D_1^c} \log |x| d\nu(x)$$

where  $D_r$  denotes the open disk of radius  $r$ .

This formula is known as Jensen's formula and can be seen as a consequence of the Poisson-Jensen formula [85, Theorem II.4.10].

*Remark 3.2* (The potential is defined up to a constant.). Note that one can choose to add a constant to the potential  $V^\nu$  without changing the joint distribution (1). Adding a constant will only change the normalizing constant  $Z_N$ . We can modify the potential  $V^\nu$  so that it is constant equal to zero on the unit circle and then

$$\forall z \in \mathbb{C} \quad V^\nu(z) = \int_1^{|z|} \frac{\nu(D_r)}{r} dr.$$

In fact,  $V^\nu$  defined in this way satisfies Poisson's equation with source  $\nu$  even if the actual logarithmic potential (3) does not make sense. Nevertheless, it is only when the condition of Lemma 3.1 is satisfied that  $V^\nu$  satisfies condition (2). We emphasize that this representation of the potential will be very helpful in the rest of the article.

*Proof of Lemma 3.1.* Since  $\nu$  is radial, the disintegration theorem<sup>1</sup> [1, Theorem 5.3.1] allows us to write

$$\nu = \int_0^\infty l_r d\mu(r)$$

where  $l_r \in \mathcal{P}(\mathbb{C})$  is the uniform probability measure on  $C_r$ , the circle centered at 0 of radius  $r$ , and  $\mu$  is a probability measure on  $[0, \infty)$  characterized by

$$\forall a, b > 0 \quad \mu((a, b)) = \nu(\{a < |z| < b\}).$$

This decomposition of the measure means that for any  $f$  positive measurable function or integrable with respect to  $\nu$  we have

$$\int_{\mathbb{C}} f(z) d\nu(z) = \int_0^\infty \left( \int_{C_r} f(w) dl_r(w) \right) d\mu(r).$$

Using this relation to compute the potential of the measure  $\nu$  gives for every  $z \in \mathbb{C}$

$$\begin{aligned} V^\nu(z) &= \int_{\mathbb{C}} \log |z - w| d\nu(w) = \int_0^\infty \left( \int_{C_r} \log |z - w| dl_r(w) \right) d\mu(r) \\ &= \int_0^\infty V^{l_r}(z) d\mu(r). \end{aligned}$$

where  $V^{l_r}$  is the potential of the uniform measure on  $C_r$ . In fact, since the integrability of  $\log |z - \cdot|$  is not yet known, we may proceed by a limiting argument by first integrating over the complement of an open annulus that contains  $z$ . But, since  $V^{l_r}$  can be computed explicitly and is equal to [81, p.29]

$$V^{l_r}(z) = \begin{cases} \log r & \text{if } |z| \leq r \\ \log |z| & \text{if } |z| > r \end{cases}$$

<sup>1</sup>Conditional expectation in probabilist language.

we are able to complete the limiting argument. Hence we obtain

$$\begin{aligned} V^\nu(z) &= \int_{[|z|, \infty)} \log r \, d\mu(r) + \int_{[0, |z|)} \log |z| \, d\mu(r) \\ &= \int_{[|z|, \infty)} \log r \, d\mu(r) + \log |z| \nu(D_{|z|}) \end{aligned} \quad (9)$$

where the last term is not there if  $z = 0$ . Notice that by (9) the lemma is already proven for  $|z| = 1$ . Suppose  $z \neq 0$ . Let us notice that Fubini's theorem implies

$$\begin{aligned} \int_{|z|}^{\infty} \frac{1}{r} \nu(D_r^c) \, dr &= \int_{\mathbb{R}^+} 1_{r \geq |z|} \frac{1}{r} \left( \int_{\mathbb{R}^+} 1_{s \geq r} \, d\mu(s) \right) \, dr \\ &= \int_{\mathbb{R}^+} 1_{s \geq |z|} \left( \int_{|z|}^s \frac{1}{r} \, dr \right) \, d\mu(s) \\ &= \int_{[|z|, \infty)} \log s \, d\mu(s) - \log |z| \nu(D_{|z|}^c). \end{aligned}$$

Then, by replacing this equality in (9), we obtain

$$\begin{aligned} V^\nu(z) &= \log |z| (\nu(D_{|z|}) + \nu(D_{|z|}^c)) + \int_{|z|}^{\infty} \frac{\nu(D_r^c)}{r} \, dr \\ &= \log |z| + \int_{|z|}^1 \frac{\nu(D_r^c)}{r} \, dr + \int_1^{\infty} \frac{\nu(D_r^c)}{r} \, dr \\ &= \log |z| + \int_{|z|}^1 \frac{1 - \nu(D_r)}{r} \, dr + \int_1^{\infty} \frac{\nu(D_r^c)}{r} \, dr \\ &= \int_1^{|z|} \frac{\nu(D_r)}{r} \, dr + \int_1^{\infty} \frac{\nu(D_r^c)}{r} \, dr \\ &= \int_1^{|z|} \frac{\nu(D_r)}{r} \, dr + \int_{[1, \infty)} \log r \, d\mu(r) \end{aligned}$$

where the last equality is obtained by taking  $|z| = 1$ . The case  $z = 0$  follows the same argument.  $\square$

*Proof of Theorem 1.5. Proof of 1.* We begin by proving the first item.

**Step 1: Kernel of the Coulomb gas**

Let  $\nu$  be a measure satisfying *Assumption 2* such that  $S^1 \subset \text{supp } \nu \subset \mathbb{D}^c$ . Let  $(x_1, \dots, x_n)$  be the Coulomb gas associated to the background potential  $V^\nu$  where we assume, by adding a constant to  $V^\nu$  if necessary, that  $V^\nu = 0$  inside  $\mathbb{D}$ . Since  $\beta = 2$ , the point process  $\{x_1, \dots, x_n\}$  is determinantal with kernel  $K_n : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$  defined by

$$K_n(z, w) = \sum_{k=0}^{n-1} b_{k,n} z^k \bar{w}^k e^{-(n+1)V^\nu(z)} e^{-(n+1)V^\nu(w)}$$

where

$$(b_{k,n})^{-1} = \int_{\mathbb{C}} |z|^{2k} e^{-2(n+1)V^\nu(z)} \, d\ell_{\mathbb{C}}(z).$$

The point process  $\mathcal{I}_n = \{x_k \mid |x_k| < 1\}$  is the restriction to the open unit disk of the point process  $\{x_1, \dots, x_n\}$ , and is also a determinantal point process, with kernel given by the restriction on  $\mathbb{D} \times \mathbb{D}$  of  $K_n$ . We will also write this kernel  $K_n$ . In order to prove the convergence of the sequence of point processes  $(\mathcal{I}_n)_{n \in \mathbb{N}}$  towards the Bergman point process, it is enough to prove that the sequence of kernels  $(K_n)_{n \in \mathbb{N}}$  converges uniformly on compact subsets of  $\mathbb{D} \times \mathbb{D}$  towards the Bergman kernel of the disk

$$K(z, w) = \frac{1}{\pi} \frac{1}{(1 - z\bar{w})^2} = \sum_{k=0}^{\infty} \frac{1}{\pi} (k+1) z^k \bar{w}^k.$$



In fact, the proof will work, and thus the theorem is true, as soon as the radial potential  $V$  satisfying (2) is zero inside of the closed unit disk and positive outside of it.

**Step 2: Convergence of the coefficients**

First, let us notice that

$$\begin{aligned} (b_{k,n})^{-1} &= \int_{\mathbb{C}} |z|^{2k} e^{-2(n+1)V^\nu(z)} d\ell_{\mathbb{C}}(z) = 2\pi \int_0^\infty r^{2k+1} e^{-2(n+1)V^\nu(r)} dr \\ &= 2\pi \int_0^1 r^{2k+1} e^{-2(n+1)V^\nu(r)} dr + 2\pi \int_1^\infty r^{2k+1} e^{-2(n+1)V^\nu(r)} dr, \end{aligned}$$

where we use  $V^\nu(r)$  to denote  $V^\nu$  evaluated at any point of norm  $r$ . Since the potential  $V^\nu$  is equal to 0 inside the unit disk, one can compute the first term

$$2\pi \int_0^1 r^{2k+1} e^{-2(n+1)V^\nu(r)} dr = 2\pi \int_0^1 r^{2k+1} dr = \frac{\pi}{k+1}.$$

For the second term, let us prove that

$$\lim_{n \rightarrow \infty} \int_1^\infty r^{2k+1} e^{-2(n+1)V^\nu(r)} dr = 0.$$

But this is a consequence of Lebesgue's dominated convergence theorem where we use the bound

$$r^{2k+1} e^{-2(n+1)V^\nu(r)} \leq r^{2k+1} e^{-2(k+2)V^\nu(r)}$$

for  $k \leq n-1$ . The fact that  $r^{2k+1} e^{-2(n+1)V^\nu(r)}$  goes to zero when  $r > 1$  can be seen from the fact that  $V^\nu > 0$  which in turn can be seen from the formula in Lemma 3.1 as follows. Let  $r > 0$  and write

$$V^\nu(r) = \int_1^r \frac{\nu(D_s)}{s} ds.$$

If  $V^\nu(r)$  were zero the integrand  $\nu(D_s)$  would be zero for almost every  $s \in [1, r]$  which is impossible because  $\nu(D_s) > 0$  for  $s > 1$ .

In summary, we obtain  $\lim_{n \rightarrow \infty} b_{k,n} = \frac{k+1}{\pi}$ .

**Step 3: Convergence of the kernels**

Let us fix  $\rho \in (0, 1)$  then for any  $z, w$  inside the disk of radius  $\rho$  we have

$$|K_n(z, w) - K(z, w)| \leq \sum_{k=0}^{\infty} \left| b_{k,n} - \frac{k+1}{\pi} \right| |z|^k |w|^k \leq \sum_{k=0}^{\infty} \left| b_{k,n} - \frac{k+1}{\pi} \right| \rho^{2k}.$$

The right-hand term converges to zero as  $n$  goes to infinity by an application of Lebesgue's dominated convergence theorem, noticing that

$$\forall k, n \quad 0 \leq b_{k,n} \leq \frac{k+1}{\pi}.$$

which implies

$$\left| b_{k,n} - \frac{k+1}{\pi} \right| \rho^{2k} \leq 2 \left( \frac{k+1}{\pi} \right) \rho^{2k}.$$

**Step 4: Convergence of the point process and the minima**

By [90, Proposition 3.10] the point process  $\mathcal{I}_n$  converges towards the Bergman point process  $\mathcal{B}$  and we have completed the proof of the first item.

**Proof of 2.** Since  $\mathcal{I}_n$  converges towards  $\mathcal{B}$ , by the continuity of the minimum (Lemma 4.1) we obtain that the minimum of the norms of  $\mathcal{I}_n$  converges to the minimum of the norms of  $\mathcal{B}$ . But since the minimum of the norms of  $\mathcal{B}$  is different from 1 almost surely ( $\mathcal{B}$  always has an infinite number of points), the limit of the minimum of the norms of  $\mathcal{I}_n$  coincides with the limit of the minimum of  $\{|x_1|, \dots, |x_n|\}$ .

Thanks to [18, Theorem 4.7.1], the set of the absolute values of the point of the Bergman point process have the same distribution as  $\{U_k^{1/2k}, k \in \mathbb{N}\}$ , with the  $U_k$ 's being independent uniform random variables on  $(0, 1)$ . This immediately implies that

$$\mathbb{P}(\#(\mathcal{B} \cap D_t) = 0) = 1 - \prod_{k=0}^{\infty} (1 - t^{2k})$$

which completes the proof of this Theorem. □

**Proof of Theorem 1.3**

The main idea of this proof is that Coulomb gases are stable under inversion. More precisely, if  $(x_1, \dots, x_n)$  is a Coulomb gas of the form (1), with potential generated by a background measure  $\nu$  satisfying *Assumption 2* and  $S^1 \subset \text{supp } \nu \subset \mathbb{D}$ , then  $(\frac{1}{x_1}, \dots, \frac{1}{x_n})$  forms a Coulomb gas with potential generated by the background measure  $i_*\nu$  which satisfies *Assumption 2* and  $S^1 \subset \text{supp } i_*\nu \subset \mathbb{D}^c$ .

**Lemma 3.3** (Coulomb gas are stable under inversion). *Let  $V$  be a continuous real valued function on  $\mathbb{C}$  such that (2) is satisfied. Define  $\tilde{V}$  on  $\mathbb{C} \setminus \{0\}$  by*

$$\tilde{V}(z) = V\left(\frac{1}{z}\right) + \log|z|$$

Let  $(x_1, \dots, x_n)$  be a random element of  $\mathbb{C}^n$  distributed according to the law

$$\frac{1}{Z_n} \exp\left(-2\left[-\sum_{i<j} \log|x_i - x_j| + (n+1)\sum_{i=1}^n V(x_i)\right]\right) d\ell_{\mathbb{C}^n}(x_1, \dots, x_n) \tag{10}$$

where  $Z_n$  is a normalization constant. Then the random vector  $(\frac{1}{x_1}, \dots, \frac{1}{x_n})$  is almost surely well-defined and it is distributed according to the law

$$\frac{1}{Z_n} \exp\left(-2\left[-\sum_{i<j} \log|x_i - x_j| + (n+1)\sum_{i=1}^n \tilde{V}(x_i)\right]\right) d\ell_{\mathbb{C}^n}(x_1, \dots, x_n). \tag{11}$$

In particular, if  $\nu$  satisfies *Assumption 2* and  $(x_1, \dots, x_n)$  is the Coulomb gas associated to the background  $\nu$ , then  $(\frac{1}{x_1}, \dots, \frac{1}{x_n})$  is the Coulomb gas associated to the background  $i_*\nu$ .

*Proof of Lemma 3.3.* This lemma can be proved by a direct calculation. Instead, we would like to point out another proof. Define

$$G^V(x, y) = -\log|x - y| + V(x) + V(y)$$

and define the positive measure  $\pi$  by  $d\pi = e^{-4V} d\ell_{\mathbb{C}}$ . Using these definitions we can write

$$\begin{aligned} & \exp\left(-2\left[-\sum_{i<j} \log|x_i - x_j| + (n+1)\sum_{i=1}^n V(x_i)\right]\right) d\ell_{\mathbb{C}^n}(x_1, \dots, x_n) \\ &= e^{-2\sum_{i<j} G^V(x_i, x_j)} d\pi^{\otimes n}(x_1, \dots, x_n). \end{aligned}$$

By a straightforward calculation we obtain

$$G^V(i(x), i(y)) = -\log|x - y| + \tilde{V}(x) + \tilde{V}(y) =: G^{\tilde{V}}(x, y)$$

and that  $\tilde{\pi} := i(\pi)$  is given by

$$d\tilde{\pi} = e^{-4\tilde{V}(x)} d\ell_{\mathbb{C}}(x).$$

In summary, the ‘inverse’ of  $e^{-2\sum_{i<j} G^V(x_i, x_j)}$  is  $e^{-2\sum_{i<j} G^{\tilde{V}}(x_i, x_j)}$  and the ‘inverse’ of  $\pi$  is  $\tilde{\pi}$  from which we obtain the result.

Let  $\nu$  be a probability measure. We can prove that

$$\forall z \in \mathbb{C}, \quad V^{i_*\nu}(z) = V^\nu\left(\frac{1}{z}\right) + \log|z| - \int_{\mathbb{C}} \log|w| d\nu(w).$$

Indeed, let us fix  $z \in \mathbb{C} \setminus \{0\}$ . Then

$$\begin{aligned} V^{i_*\nu}\left(\frac{1}{z}\right) &= \int_{\mathbb{C}} \log\left|\frac{1}{z} - \frac{1}{w}\right| d\nu(w) \\ &= \int_{\mathbb{C}} \log|z - w| d\nu(w) - \int_{\mathbb{C}} \log|z| d\nu(w) - \int_{\mathbb{C}} \log|w| d\nu(w) \\ &= \int_{\mathbb{C}} \log|z - w| d\nu(w) - \log|z| - \int_{\mathbb{C}} \log|w| d\nu(w) \end{aligned}$$

which gives the desired formula.

This proves that if  $(x_1, \dots, x_n)$  is a Coulomb gas with background generated by  $\nu$ , then  $(\frac{1}{x_1}, \dots, \frac{1}{x_n})$  is a Coulomb gas associated to the potential

$$\tilde{V}(z) = V^{i_*\nu}(z) - \int_{\mathbb{C}} \log|w| d\nu(w).$$

One can remove the constant  $\int_{\mathbb{C}} \log|w| d\nu(w)$  from the definition of the potential since it may enter into the normalizing constant associated to this model. Hence, the image by inversion of the Coulomb gas associated to  $V^\nu$  is the Coulomb gas associated to  $V^{i_*\nu}$ . That  $i_*\nu$  is radially symmetric if  $\nu$  is radially symmetric is a consequence of the fact that  $T \circ i \circ T = i$  for every rotation  $T$ . That the integral of  $|\log|z||$  with respect to  $i_*\nu$  is finite if the integral of  $|\log|z||$  with respect to  $\nu$  is finite is a consequence of the change-of-variables formula. □

*Proof of Theorem 1.3. Proof of 1.* Let  $(x_1, \dots, x_n)$  be a Coulomb gas associated to a background potential  $V^\nu$ , with the measure  $\nu$  satisfying *Assumption 2* and

$$S^1 \subset \text{supp } \nu \subset \bar{\mathbb{D}}.$$

Lemma 3.3 implies that  $(\frac{1}{x_1}, \dots, \frac{1}{x_n})$  is almost surely well-defined and is the Coulomb gas associated to the background potential  $V^{i_*\nu}$ , with the probability measure  $i_*\nu$  satisfying *Assumption 2* and

$$S^1 \subset \text{supp } \nu \subset \mathbb{D}^c.$$

Thanks to Theorem 1.5, the point process  $\{\frac{1}{x_1}, \dots, \frac{1}{x_n}\}$  converges weakly towards the Bergman point process. This implies that the point process  $\{x_1, \dots, x_n\}$  converges weakly towards the image under inversion of the Bergman point process.

**Proof of 2.** Let  $t \geq 1$  be fixed, then using Theorem 1.5 we obtain

$$\begin{aligned} \mathbb{P}(\max |x_i| < t) &= \mathbb{P}(\forall i \in \{1, \dots, n\}, |x_i| < t) \\ &= \mathbb{P}\left(\forall i \in \{1, \dots, n\}, \frac{1}{|x_i|} > \frac{1}{t}\right) \\ &= \mathbb{P}\left(\min \left|\frac{1}{x_i}\right| > \frac{1}{t}\right) \xrightarrow{n \rightarrow \infty} \prod_{k=1}^{\infty} \left(1 - \frac{1}{t^{2k}}\right). \end{aligned}$$

□

**Proof of Theorem 1.7**

*Proof. Proof of 1.* We begin by the proof of the first item.

**Step 1: Kernel of the rescaled Coulomb gas**

Let  $(x_1, \dots, x_n)$  be the Coulomb gas associated to the potential  $V^\nu$ , then the point process  $\left\{\frac{x_1}{n^{1/\alpha}}, \dots, \frac{x_n}{n^{1/\alpha}}\right\}$  is a determinantal point process associated to the kernel

$$\begin{aligned}\tilde{K}_n(z, w) &= \frac{1}{n^{2/\alpha}} K_n\left(\frac{z}{n^{1/\alpha}}, \frac{w}{n^{1/\alpha}}\right) \\ &= \sum_{k=0}^n b_{k,n} z^k \bar{w}^k e^{-(n+1)V^\nu\left(\frac{z}{n^{1/\alpha}}\right)} e^{-(n+1)V^\nu\left(\frac{w}{n^{1/\alpha}}\right)}\end{aligned}$$

where

$$b_{k,n}^{-1} = 2\pi \int_0^\infty r^{2k+1} e^{-2(n+1)V^\nu\left(\frac{r}{n^{1/\alpha}}\right)} dr.$$

We will prove that the sequence of kernels  $\tilde{K}_n$  converges uniformly on compact subsets of  $\mathbb{C} \times \mathbb{C}$  towards the kernel

$$K(z, w) = \sum_{k=0}^{\infty} b_k z^k \bar{w}^k e^{-\gamma|z|^\alpha} e^{-\gamma|w|^\alpha} \quad (12)$$

where

$$\gamma = \frac{\lambda}{\alpha} \quad \text{and} \quad b_k^{-1} = 2\pi \int_0^\infty r^{2k+1} e^{-2\gamma r^\alpha} dr.$$

To prove this convergence, we will first prove that  $b_{k,n} \rightarrow b_k$ , then we will find a sequence  $(B_k)_{k \in \mathbb{N}}$  such that  $\sum B_k r^k$  has an infinite radius of convergence and  $b_{k,n} \leq B_k$  for every  $n$ . This will imply the uniform convergence of the kernels.

**Step 2: Properties satisfied by the potential**

Let  $\nu$  be a rotationally invariant probability measure such that there exists  $\alpha > 0$  and  $\lambda > 0$  such that

$$\lim_{r \rightarrow 0} \frac{\nu(D_r)}{r^\alpha} = \lambda.$$

Since the potential of  $\nu$  can be written as

$$V^\nu(z) = \int_1^{|z|} \frac{\nu(D_r)}{r} dr$$

we obtain that

$$\frac{V^\nu(r) - V^\nu(0)}{r^\alpha} \xrightarrow{r \rightarrow 0} \frac{\lambda}{\alpha}.$$

and that  $V^\nu(r) > V^\nu(0)$  for every  $r > 0$ . From now on, we will assume that  $V^\nu(0) = 0$ , since adding a constant to the potential  $V^\nu$  does not change the distribution of the associated Coulomb gas. Using this new convention, we have

$$\lim_{r \rightarrow 0} \frac{V^\nu(r)}{r^\alpha} = \frac{\lambda}{\alpha} =: \gamma \quad (13)$$

and

$$V^\nu(r) > 0 \quad \text{for every } r > 0. \quad (14)$$

In fact, those two properties of the potential are the only properties needed, apart from (2), for the theorem to be true.

**Step 3: Convergence of the coefficients**

We prove that  $b_{k,n}$  converges to  $b_k$  as  $n$  goes to infinity or, equivalently,

$$\lim_{n \rightarrow \infty} \int_0^\infty r^{2k+1} e^{-2(n+1)V^\nu\left(\frac{r}{n^{1/\alpha}}\right)} dr = \int_0^\infty r^{2k+1} e^{-2\gamma r^\alpha} dr$$

We divide the integral in three parts.

$$\begin{aligned} & \int_0^\infty r^{2k+1} e^{-2(n+1)V^\nu\left(\frac{r}{n^{1/\alpha}}\right)} dr \\ &= \int_0^{n^{1/\alpha}\varepsilon} r^{2k+1} e^{-2(n+1)V^\nu\left(\frac{r}{n^{1/\alpha}}\right)} dr \\ &+ \int_{n^{1/\alpha}\varepsilon}^{n^{1/\alpha}M} r^{2k+1} e^{-2(n+1)V^\nu\left(\frac{r}{n^{1/\alpha}}\right)} dr \\ &+ \int_{n^{1/\alpha}M}^\infty r^{2k+1} e^{-2(n+1)V^\nu\left(\frac{r}{n^{1/\alpha}}\right)} dr \end{aligned}$$

where we choose  $\varepsilon > 0$  such that  $\frac{\gamma}{2}r^\alpha \leq V^\nu(r)$  for  $|r| \leq \varepsilon$  and  $M > \varepsilon$  such that  $\frac{1}{2} \log |r| \leq V^\nu(r)$  whenever  $|r| \geq M$ . We also know, by the continuity and the positivity outside 0 of  $V$  that there exists a constant  $C > 0$  such that  $C \leq V(r)$  for  $r \in [\varepsilon, M]$ . Since

$$e^{-2(n+1)V\left(\frac{r}{n^{1/\alpha}}\right)} \mathbf{1}_{[0, n^{1/\alpha}\varepsilon]}(r) \leq e^{-\frac{(n+1)}{n}\gamma r^\alpha} \leq e^{-\gamma r^\alpha}$$

we can use Lebesgue's dominated convergence theorem for the first term. The second term is bounded by

$$\int_{n^{1/\alpha}\varepsilon}^{n^{1/\alpha}M} r^{2k+1} e^{-2(n+1)C} dr$$

which goes exponentially fast to zero when  $n \rightarrow \infty$ . The last integral is bounded by

$$\begin{aligned} \int_{n^{1/\alpha}M}^\infty r^{2k+1} e^{-2(n+1)\log\left(\frac{r}{n^{1/\alpha}}\right)} dr &= n^{(2k+2)/\alpha} \int_M^\infty \rho^{2k+1} e^{-2(n+1)\log \rho} d\rho \\ &= n^{(2k+2)/\alpha} \int_M^\infty \rho^{2k-2n-1} d\rho \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

#### Step 4: Convergence of the kernels

Notice that, uniformly on compact sets,

$$(n+1)V\left(\frac{|z|}{n^{1/\alpha}}\right) \xrightarrow{n \rightarrow \infty} \gamma|z|^\alpha$$

due to (13). Then, it is left to prove that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n b_{k,n} z^k \bar{w}^k = \sum_{k=0}^\infty b_k z^k \bar{w}^k$$

uniformly on compact sets of  $\mathbb{C} \times \mathbb{C}$ .

Take  $\varepsilon > 0$  such that  $2\lambda r^\alpha \geq V(r)$  for  $|r| \leq \varepsilon$ . Then

$$\begin{aligned} \int_0^\infty r^{2k+1} e^{-2(n+1)V\left(\frac{r}{n^{1/\alpha}}\right)} dr &\geq \int_0^{n^{1/\alpha}\varepsilon} r^{2k+1} e^{-4\frac{n+1}{n}\gamma r^\alpha} dr \\ &\geq \int_0^{n^{1/\alpha}\varepsilon} r^{2k+1} e^{-8\gamma r^\alpha} dr \\ &\geq \int_0^{k^{1/\alpha}\varepsilon} r^{2k+1} e^{-8\gamma r^\alpha} dr. \end{aligned}$$

So, if we define  $B_k$  by

$$(B_k)^{-1} = \int_0^{k^{1/\alpha}\varepsilon} r^{2k+1} e^{-8\gamma r^\alpha} dr = k^{(2k+2)/\alpha} \int_0^\varepsilon \rho^{2k+1} e^{-8\gamma k \rho^\alpha} d\rho = k^{(2k+2)/\alpha} \int_0^\varepsilon e^{k(2\log \rho - 8\gamma \rho^\alpha)} \rho d\rho.$$

By the root test, for  $\sum_{k=0}^{\infty} B_k x^k$  to converge for every  $x > 0$ , we need that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \left[ (B_k)^{-1} \right] = \infty.$$

We know that  $\lim_{k \rightarrow \infty} \frac{1}{k} \log \left[ k^{(2k+2)/\alpha} \right] = \infty$  so that it would be enough to prove that

$$\frac{1}{k} \log \int_0^\varepsilon e^{k(2 \log \rho - 8\gamma\rho^\alpha)} \rho \, d\rho$$

is bounded from below. In fact, by the Laplace method we know that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \int_0^\varepsilon e^{k(2 \log \rho - 8\gamma\rho^\alpha)} \rho \, d\rho = \sup_{\rho \in [0, \varepsilon]} \{2 \log \rho - 8\gamma\rho^\alpha\} > -\infty.$$

Take  $R > 0$  and suppose  $|z|, |w| \leq R$ .

$$\left| \sum_{k=0}^{n-1} b_{k,n} z^k \bar{w}^k - \sum_{k=0}^{\infty} b_k z^k \bar{w}^k \right| \leq \sum_{k=0}^{\infty} |b_{k,n} - b_k| |z|^k |\bar{w}|^k \leq \sum_{k=0}^{\infty} |b_{k,n} - b_k| R^{2k}$$

where we have defined  $b_{k,n} = 0$  for  $k \geq n$ . Since  $|b_{k,n} - b_k| R^{2k}$  is bounded by  $2B_k R^{2k}$  we can use the dominated convergence theorem to conclude.

**Step 5: Convergence of the point process and the minima**

By [90, Proposition 3.10] the point process  $\left\{ \frac{x_1}{n^{1/\alpha}}, \dots, \frac{x_n}{n^{1/\alpha}} \right\}$  converges to a determinantal point process  $P$  associated to the kernel  $K$  defined in (12). By the continuity of the minimum (Lemma 4.1) we obtain that  $\left\{ \frac{x_1}{n^{1/\alpha}}, \dots, \frac{|x_n|}{n^{1/\alpha}} \right\}$  converges to the minimum of the norms of  $P$ .

**Step 6: Analysis of the limiting distribution**

Let  $\{Y_k\}_{k \geq 0}$  be a sequence of positive independent random variables such that  $Y_k$  follows the law

$$\frac{r^{2k+1} e^{-2\gamma r^\alpha} \, dr}{\int_0^\infty s^{2k+1} e^{-2\gamma s^\alpha} \, ds}.$$

If  $P$  is the determinantal point process associated to the kernel  $K$  then, by [18, Theorem 4.7.1], the law of  $\{|z| \mid z \in P\}$  is the same as the law of the point process defined by  $\{Y_k\}_{k \geq 0}$ . So the infimum has cumulative distribution function

$$\begin{aligned} \mathbb{P}(\inf\{Y_k\} \leq y) &= 1 - \mathbb{P}(\inf\{Y_k\} > y) = 1 - \prod_{k=0}^{\infty} \mathbb{P}(Y_k > y) \\ &= 1 - \prod_{k=0}^{\infty} (1 - \mathbb{P}(Y_k \leq y)). \end{aligned}$$

But, by a change of variables we may see that

$$\begin{aligned} \int_y^\infty r^{2k+1} e^{-2\gamma r^\alpha} \, dr &= \frac{1}{\alpha(2\gamma)^{(2k+2)/\alpha}} \int_{2\gamma y^\alpha}^\infty \rho^{(2k+2)/\alpha-1} e^{-\rho} \, d\rho \\ &= \frac{1}{\alpha(2\gamma)^{(2k+2)/\alpha}} \Gamma\left(\frac{2k+2}{\alpha}, 2\gamma y^\alpha\right) \end{aligned}$$

so that

$$\mathbb{P}(Y_k \leq y) = \frac{\int_0^y r^{2k+1} e^{-2\gamma r^\alpha} \, dr}{\int_0^\infty s^{2k+1} e^{-2\gamma s^\alpha} \, ds} = \frac{\Gamma\left(\frac{2k+2}{\alpha}, 2\gamma y^\alpha\right)}{\Gamma\left(\frac{2k+2}{\alpha}\right)}$$

from which we may conclude the proof of the theorem.

**Proof of 2.** The proof follows the same argument as the proof of Theorem 1.3 along with the fact that if  $\nu$  satisfies  $\lim_{r \rightarrow \infty} r^\alpha \nu(D_r^c) = \lambda$ , then its pushforward by the inversion,  $i_* \nu$ , satisfies  $\lim_{r \rightarrow 0} i_* \nu(D_r) / r^\alpha = \lambda$ .

□

### 3.2 Results on random polynomials

We prove the results on random polynomials in the same order as we did for Coulomb gases: we prove Theorem 1.9, then Theorem 1.8 and finally Theorem 1.10.

First we recall that if  $(a_k)_{k \in \mathbb{N}}$  are i.i.d. random variables satisfying

$$\mathbb{E}(\log(1 + |a_0|)) < +\infty,$$

then the random power series  $\sum_{k=0}^{\infty} a_k z^k$  has almost surely a radius of convergence equal to one. In fact, the following lemma immediately implies the general statement in Corollary 3.5.

**Lemma 3.4** (Arnold [8]). *Let  $(a_k)_{k \in \mathbb{N}}$  be i.i.d. random variables. Fix  $\varepsilon > 0$ . Then*

$$\sup_{k \in \mathbb{N}} \frac{|a_k|}{e^{\varepsilon k}} < \infty \text{ a.s.} \iff \mathbb{E}(\log(1 + |a_0|)) < \infty.$$

*Proof of the lemma.* For every non negative random variable  $X$  we have:

$$\sum_{k=1}^{\infty} \mathbb{P}(X \geq k) \leq \mathbb{E}(X) \leq \sum_{k=0}^{\infty} \mathbb{P}(X \geq k).$$

Those inequalities come from the relation:  $\mathbb{E}(X) = \int_{\mathbb{R}^+} \mathbb{P}(X \geq x) dx$ . Now we apply this inequality to the non-negative random variable

$$X = \frac{1}{\varepsilon} \log(1 + |a_0|).$$

We deduce that

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\frac{1 + |a_k|}{e^{\varepsilon k}} \geq 1\right) < \infty$$

Borel-Cantelli's lemma implies that  $\limsup_{k \in \mathbb{N}} \frac{|a_k|}{e^{\varepsilon k}} \leq 1$ , which implies  $\sup_{k \in \mathbb{N}} \frac{|a_k|}{e^{\varepsilon k}} < \infty$ . The reverse implication relies on a similar reasoning, but will not be used in this article.  $\square$

**Corollary 3.5** (Radius of convergence of a random power series). *Let  $(\alpha_k)_{k \in \mathbb{N}}$  be a (deterministic) sequence of complex numbers. Suppose that  $(a_k)_{k \in \mathbb{N}}$  is a sequence of i.i.d. complex random variables such that*

$$\mathbb{E}(\log(1 + |a_0|)) < \infty.$$

*If  $a_0$  is not zero (i.e. the law of  $a_0$  is not the Dirac delta at 0) then the radius of convergence of the random power series  $\sum_{k=0}^{\infty} \alpha_k a_k z^k$  is almost surely equal to the radius of convergence of the deterministic power series  $\sum_{k=0}^{\infty} \alpha_k z^k$ .*

*Proof.* Lemma 3.4 implies that

$$\limsup_{k \rightarrow \infty} |a_k|^{1/k} \leq 1 \text{ a.s.} \iff \mathbb{E}(\log(1 + |a_0|)) < \infty.$$

Since

$$\limsup_{k \rightarrow \infty} |\alpha_k|^{1/k} \limsup_{k \rightarrow \infty} |a_k|^{1/k} \geq \limsup_{k \rightarrow \infty} |\alpha_k a_k|^{1/k}$$

we obtain that the radius of convergence of  $\sum_{k=0}^{\infty} \alpha_k a_k z^k$  is greater or equal than the radius of convergence of  $\sum_{k=0}^{\infty} \alpha_k z^k$ . On the other hand, if  $\alpha_k z^k$  is not bounded, then  $\alpha_k a_k z^k$  is not bounded. This is a consequence of the second Borel-Cantelli lemma since we can find  $\varepsilon > 0$  such that

$$\mathbb{P}(|a_0| > \varepsilon) > 0$$

which implies that, almost surely,

$$|a_k| > \varepsilon \text{ for infinitely many } k.$$

This concludes the proof.  $\square$

**Proof of Theorem 1.9**

*Proof. Proof of 1.* Let  $\nu$  be a probability measure satisfying Assumption 2 and satisfying  $S^1 \subset \text{supp } \nu \subset \mathbb{D}^c$ . Since this measure is rotationally invariant, the random polynomials  $P_n$  are

$$P_n(z) = \sum_{k=0}^n a_k \frac{1}{\|X^k\|_n} z^k.$$

We define

$$c_n = \int_{\mathbb{C}} e^{-2nV^\nu(z)} d\nu(z),$$

and we would like to prove that  $(c_n P_n)_{n \in \mathbb{N}}$  converges uniformly towards  $z \mapsto \sum a_k z^k$  almost surely on any compact set included in the open unit disk. Afterwards, we conclude by Lemma 4.2.

Let  $\nu_n$  be the probability measure defined by

$$d\nu_n(z) = \frac{1}{c_n} e^{-2nV^\nu(z)} d\nu(z).$$

We begin by proving that

$$\nu_n \xrightarrow[n \rightarrow \infty]{\text{weakly}} \nu_{S^1}$$

where  $\nu_{S^1}$  denotes the uniform measure on the unit circle. This will be a consequence of Laplace's method, some simple tightness property, and the invariance under rotations of  $\nu_n$ . Then, we expect that  $\|X^k\|_n/c_n = \int_{\mathbb{C}} |z^k| d\nu_n(z)$  converges to  $\int_{\mathbb{C}} |z^k| d\nu_{S^1}(z)$ . This may be a consequence of the weak convergence if  $|z^k|$  were bounded. By Laplace's method, we prove that the integral of  $|z^k|$  outside a large open disk converges to zero and then we may consider  $z^k$  as bounded.

**Step 1: Convergence of the measures**

In this step we prove that

$$\nu_n \xrightarrow[n \rightarrow \infty]{\text{weakly}} \nu_{S^1}.$$

First, let us show that

$$\lim_n \frac{1}{n} \log c_n = 0.$$

This may be seen as an application of Laplace's method but we write the proof for the reader's convenience. Since  $V^\nu$  is non-negative,  $c_n \leq 1$ , which implies

$$\overline{\lim}_n \frac{1}{n} \log c_n \leq 0.$$

Let  $\varepsilon > 0$  fixed. Then, since  $V^\nu$  is continuous and equals 0 on the unit circle, there exists  $\varepsilon'$  such that  $V^\nu(z) \leq \varepsilon$  for all  $z \in D_{1+\varepsilon'}$ . This implies that

$$c_n = \int_{\mathbb{C}} e^{-2nV^\nu(z)} d\nu(z) \geq \int_{D_{1+\varepsilon'}} e^{-2nV^\nu(z)} d\nu(z) \geq e^{-2n\varepsilon} \nu(D_{1+\varepsilon'}).$$

Taking the logarithm and the limit inferior we get, since  $\nu(D_{1+\varepsilon'}) > 0$ ,

$$\underline{\lim}_n \frac{1}{n} \log c_n \geq -2\varepsilon.$$

Since this can be done for every  $\varepsilon > 0$  we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log c_n = 0.$$



This behavior along with the fact that for any closed  $A \subset \{z \mid |z| > 1\}$

$$\frac{1}{n} \log \int_A e^{-2nV^\nu(z)} d\nu(z) \leq - \inf_{z \in A} V^\nu(z) < 0$$

imply that for any  $r > 1$

$$\nu_n(D_r^c) \rightarrow 0. \quad (15)$$

This last fact also implies that the sequence is tight. The rotational invariance of  $\nu_n$  and (15) implies that the sequence of measures  $\nu_n$  has a unique limit point  $\nu_{S^1}$  so that

$$\nu_n \xrightarrow[n \rightarrow \infty]{\text{weakly}} \nu_{S^1}.$$

### Step 2: Convergence of the integrals

Now let us prove that for any fixed non-negative integer  $k$  we have

$$\frac{\|X^k\|_n}{c_n} = \int_{\mathbb{C}} |z|^{2k} d\nu_n(z) \xrightarrow[n \rightarrow \infty]{} \int_{\mathbb{C}} |z|^k d\nu_{S^1}(z) = 1.$$

For  $A > 1$  we write

$$\int_{\mathbb{C}} |z|^{2k} d\nu_n(z) = \int_{D_A} |z|^{2k} d\nu_n(z) + \int_{D_A^c} |z|^{2k} d\nu_n(z).$$

First, we notice that the convergence of  $\nu_n$  towards  $\nu_{S^1}$  implies that

$$\int_{D_A} |z|^{2k} d\nu_n(z) \xrightarrow[n \rightarrow \infty]{} \int_{D_A} |z|^k d\nu_{S^1}(z) = 1.$$

By a direct application of Laplace's method we may see that

$$\int_{D_A^c} |z|^{2k} e^{-2nV^\nu(z)} d\nu(z) \leq e^{-n \inf_{z \in D_A^c} V^\nu(z) + o(n)}$$

which together with the behavior of  $c_n$  implies that

$$\frac{1}{c_n} \int_{D_A^c} |z|^{2k} e^{-2nV^\nu(z)} d\nu(z) \xrightarrow[n \rightarrow \infty]{} 0.$$

For convenience of the reader we will proceed in a somewhat more explicit way. Since  $V^\nu(z) \sim_{|z| \rightarrow \infty} \log |z|$ , we may have chosen  $A > 1$  such that for  $|z| > A$  we have  $V(z) \geq 1/2 \log |z|$ . We obtain

$$\begin{aligned} \int_{D_A^c} |z|^{2k} e^{-2nV^\nu(z)} d\nu(z) &\leq \int_{D_A^c} |z|^{2k} e^{-n \log |z|} d\nu(z) \\ &= \int_{D_A^c} |z|^{2k-n} d\nu(z) \\ &\leq A^{2k-n} \end{aligned}$$

if  $n \geq 2k$ . This entails that

$$\overline{\lim} \frac{1}{n} \log \frac{1}{c_n} \int_{D_A^c} |z|^{2k} e^{-2nV^\nu(z)} d\nu(z) \leq -A$$

which, using the behavior of  $c_n$ , implies that

$$\frac{1}{c_n} \int_{D_A^c} |z|^{2k} e^{-2nV^\nu(z)} d\nu(z) \xrightarrow[n \rightarrow \infty]{} 0.$$

Hence, we obtained that for any fixed  $k$ ,

$$\frac{\|X^k\|_n}{c_n} \xrightarrow{n \rightarrow \infty} 1.$$

**Step 3: Uniform convergence of the polynomials**

Let  $\rho \in (0, 1)$ , then for any  $z \in D_\rho$  we have

$$\left| c_n P_n(z) - \sum_{k=0}^n a_k z^k \right| \leq \sum_{k=0}^n \left| \frac{c_n}{\|X^k\|_n} - 1 \right| |a_k| \rho^k.$$

This implies the almost sure uniform convergence of  $(c_n P_n)_{n \in \mathbb{N}}$  towards  $z \mapsto \sum_{k=0}^{\infty} a_k z^k$  on any compact of the open unit disk if we notice that

$$\frac{c_n}{\|X^k\|_n} \leq 1$$

which allows us to use the dominated convergence theorem.

**Step 4: Convergence of the point process**

To complete the proof, we use Hurwitz's Theorem, detailed in Lemma 4.2, which gives the almost sure convergence of the point process.

**Proof of 2.** The proof follows the same argument as the proof of the second item of Theorem 1.5.  $\square$

**Proof of Theorem 1.8**

The following Lemma is the analog of Lemma 3.3 for random polynomials.

**Lemma 3.6** (Inversion on the roots of random polynomials). *Let  $P_n$  be a random polynomial of degree less or equal than  $n$  associated to  $\nu$ . Then the random polynomial  $Q_n$  defined by*

$$Q_n(z) = z^n P_n\left(\frac{1}{z}\right)$$

*is a random polynomial of degree less or equal than  $n$  associated to the measure  $i_*\nu$ .*

*Proof.* It is enough to notice that the application  $*$  :  $\mathbb{C}_n[X] \rightarrow \mathbb{C}_n[X]$  defined by

$$P^*(z) = z^n P(1/z)$$

is an isometry between  $\mathbb{C}_n[X]$  with the inner product defined by

$$\langle P, Q \rangle_{n, \nu} = \int_{\mathbb{C}} P \bar{Q} e^{-2nV^\nu} d\nu$$

and  $\mathbb{C}_n[X]$  with the inner product defined by

$$\langle P, Q \rangle_{n, i_*\nu} = \int_{\mathbb{C}} P \bar{Q} e^{-2nV^{i_*\nu}} di_*\nu$$

which may be seen by the change-of-variables formula.  $\square$

*Proof of Theorem 1.8.* Let  $\nu$  be a probability measure satisfying *Assumption 2* and such that  $S^1 \subset \text{supp } \nu \subset \mathbb{D}$ . Let

$$P_n(z) = \sum_{k=0}^n a_k R_k(z) = \frac{a_n}{\|R_n\|_{n, \nu}} \prod_{k=1}^n (z - z_k)$$

be the random polynomial associated to  $\nu$ . Lemma 3.6 implies that

$$Q_n = z^n P_n \left( \frac{1}{z} \right) = \sum_{k=0}^n a_k R_{n-k}(z) = P_n(0) \prod_{k=1}^n \left( z - \frac{1}{z_k} \right)$$

is the random polynomial associated to the measure  $i_*\nu$ . In addition, the measure  $i_*\nu$  satisfies *Assumption 2* and  $S^1 \subset \text{supp } \nu \subset \mathbb{D}^c$ . Theorem 1.9 implies that the point process

$$\mathcal{I}_n = \left\{ \frac{1}{z_k}, \frac{1}{|z_k|} < 1 \right\}$$

converges towards the Bergman point process and that  $(\min_{k \in \{1, \dots, n\}} 1/|z_k|)_{n \in \mathbb{N}}$  converges in distribution towards the random variable  $y_\infty$ . This implies that

$$\mathcal{O}_n = \{z_k, |z_k| > 1\}$$

converges towards the inverse of the Bergman point process and that the sequence of random variables  $(\max_{k \in \{1, \dots, n\}} |z_k|)_{n \in \mathbb{N}}$  converges in distribution towards the random variable  $x_\infty$ .  $\square$

### Proof of Theorem 1.10

*Proof. Proof of 1.* Let us define the positive measure  $\mu_\alpha$  by

$$d\mu_\alpha(z) = \frac{\alpha}{2\pi} |z|^{\alpha-2} d\ell_{\mathbb{C}}(z)$$

which is the only measure satisfying that for every  $r > 0$ ,  $\mu(D_r) = r^\alpha$ . A simple calculation shows that

$$\int_{\mathbb{C}} |z|^{2k} e^{-\frac{\lambda}{\alpha}|z|^\alpha} \lambda d\mu_\alpha(z) = \alpha \left( \frac{\alpha}{\lambda} \right)^{2k/\alpha} \Gamma \left( 1 + \frac{2k}{\alpha} \right). \quad (16)$$

Using Lemma 4.2 and Lemma 4.1 it is enough to prove that

$$\frac{1}{\sqrt{n}} P_n \left( \frac{z}{n^{1/\alpha}} \right) = \sum_{k=0}^n a_k \frac{1}{\sqrt{n}} \frac{1}{n^{k/\alpha} \|X^k\|_n} z^k$$

converges uniformly on compact sets of  $\mathbb{C}$  towards

$$f_{\alpha, \lambda}(z) = \sum_{k=0}^{\infty} \frac{a_k}{\left( \int_{\mathbb{C}} |z|^{2k} e^{-\frac{\lambda}{\alpha}|z|^\alpha} \lambda d\mu_\alpha(z) \right)^{1/2}} z^k.$$

We start by recalling some properties of the potential  $V^\nu$ . Indeed for the convergence to hold we assume  $V^\nu(0) = 0$  which can be done by adding a constant. Then we prove that for any  $k$ ,

$$n^{1+2k/\alpha} \|X^k\|_n^2 = n^{2k/\alpha} \int_{\mathbb{C}} |z|^{2k} e^{-2nV^\nu(z)} n d\nu(z) \xrightarrow{n \rightarrow \infty} \int_{\mathbb{C}} |z|^{2k} e^{-\frac{\lambda}{\alpha}|z|^\alpha} \lambda d\mu_\alpha(z). \quad (17)$$

The idea is quite simple. If  $T_n : z \mapsto n^{1/\alpha} z$  then

$$n^{2k/\alpha} \int_{\mathbb{C}} |z|^{2k} e^{-2nV^\nu(z)} n d\nu(z) = \int_{\mathbb{C}} |z|^{2k} e^{-2nV^\nu \left( \frac{z}{n^{1/\alpha}} \right)} n dT_n(\nu)(z)$$

where  $T_n(\nu)$  denotes the image measure of  $\nu$  by  $T_n$ . By the hypotheses we should have that  $nT_n(\nu)$  converges towards  $\lambda\mu_\alpha$  and  $nV^\nu \left( \frac{z}{n^{1/\alpha}} \right)$  converges towards  $(\lambda/\alpha)|z|^\alpha$  in some sense what would imply (17). Finally, we find a sequence  $B_k$  such that for any  $n \in \mathbb{N}$ ,

$$\frac{1}{n^{1+k/\alpha} \|X^k\|_n} \leq B_k$$

with  $\sum_{k=0}^{\infty} a_k B_k z^k$  having an infinite radius of convergence.

**Step 1: Properties of the potential**

Let  $\nu$  be a rotationally invariant probability measure such that there exists  $\alpha > 0$  and  $\lambda > 0$  such that

$$\lim_{r \rightarrow 0} \frac{\nu(D_r)}{r^\alpha} = \lambda.$$

We will assume that  $V^\nu(0) = 0$ , since adding a constant to the potential  $V^\nu$  only changes the polynomials  $R_k$  by a multiplicative constant which has no impact on the zeros of  $P_n$ . So, using Lemma 3.1, we can write

$$V^\nu(z) = \int_0^{|z|} \frac{\nu(D_r)}{r} dr.$$

We obtain that

$$\frac{V^\nu(r)}{r^\alpha} \xrightarrow{r \rightarrow 0} \frac{\lambda}{\alpha} =: \gamma$$

and that  $V^\nu(r) > 0$  for every  $r > 0$ . We may also obtain a useful lower bound for  $V^\nu$ . If  $\delta \in (0, 1)$  and if  $|z| \geq \delta$  we can write

$$\begin{aligned} V^\nu(z) &= \int_0^\delta \frac{\nu(D_r)}{r} dr + \int_\delta^{|z|} \frac{\nu(D_r)}{r} dr \\ &\geq \nu(D_\delta) \log |z| - \nu(D_\delta) \log \delta \\ &\geq \nu(D_\delta) \log |z| \end{aligned}$$

where we have used that  $\log \delta < 0$ .

**Step 2: Convergence of the coefficients**

Let us define  $T_n : z \mapsto n^{1/\alpha} z$ . Then for any  $r > 0$  we have

$$nT_n(\nu)(D_r) = n\nu(D_{r/n^{1/\alpha}}) \xrightarrow{n \rightarrow \infty} \lambda r^\alpha = \lambda \mu_\alpha(D_r).$$

In particular,  $nT_n(\nu)(D_K)$  converges towards  $\lambda \mu_\alpha(D_K)$  and the cumulative distribution function of  $nT_n(\nu)/nT_n(\nu)(D_K)$ , which is a probability measure on  $D_K$ , converges pointwise towards the cumulative distribution function of  $\lambda \mu_\alpha/\lambda \mu_\alpha(D_K)$ . This implies that, for any  $K > 0$  and any bounded continuous function  $g$  on  $D_K$  we have

$$\int_{D_K} g n dT_n(\nu) \xrightarrow{n \rightarrow \infty} \lambda \int_{D_K} g d\mu_\alpha.$$

Let  $\varepsilon > 0$ . There exists  $\delta \in (0, 1)$  such that for any  $z \in D_\delta$  we have

$$(1 - \varepsilon)\gamma|z|^\alpha \leq V(z) \leq (1 + \varepsilon)\gamma|z|^\alpha$$

and for any  $r \in (0, 2\delta)$  we have

$$\nu(D_r) \leq 2\lambda r^\alpha. \tag{18}$$

Let us decompose the integral that interests us, the left-hand side of (17), as the sum

$$\begin{aligned} \int_{\mathbb{C}} n^{2k/\alpha} |z|^{2k} e^{-2nV^\nu(z)} n d\nu(z) &= \int_{D_\delta} n^{2k/\alpha} |z|^{2k} e^{-2nV^\nu(z)} n d\nu(z) \\ &\quad + \int_{D_\delta^c} n^{2k/\alpha} |z|^{2k} e^{-2nV^\nu(z)} n d\nu(z). \end{aligned}$$

From the lower bound found in Step 1, we know that if we denote  $\kappa = \nu(D_\delta)$  then  $V^\nu(z) \geq \kappa \log |z|$  for  $|z| \geq \delta$ . This implies that

$$0 \leq \int_{D_\delta^c} n^{2k/\alpha} |z|^{2k} e^{-2nV^\nu(z)} n d\nu(z) \leq n^{2k/\alpha+1} \delta^{2k-2\kappa n}$$

if  $n \geq k/\kappa$  and then

$$\int_{D_\delta^\varepsilon} n^{2k/\alpha} |z|^{2k} e^{-2nV^\nu(z)} n d\nu(z) \xrightarrow{n \rightarrow \infty} 0.$$

To study the first term, the integral over  $D_\delta$ , we start by noticing that

$$\int_{D_\delta} n^{2k/\alpha} |z|^{2k} e^{-2n\gamma(1+\varepsilon)|z|^\alpha} n d\nu(z) \leq \int_{D_\delta} n^{2k/\alpha} |z|^{2k} e^{-2nV^\nu(z)} n d\nu(z)$$

and

$$\int_{D_\delta} n^{2k/\alpha} |z|^{2k} e^{-2nV^\nu(z)} n d\nu(z) \leq \int_{D_\delta} n^{2k/\alpha} |z|^{2k} e^{-2n\gamma(1-\varepsilon)|z|^\alpha} n d\nu(z).$$

If we prove that for any  $\gamma' > 0$ , we have

$$\int_{D_\delta} n^{2k/\alpha} |z|^{2k} e^{-2n\gamma'|z|^\alpha} n d\nu(z) \xrightarrow{n \rightarrow \infty} \int_{\mathbb{C}} |z|^{2k} e^{-\gamma'|z|^\alpha} \lambda d\mu_\alpha(z) \quad (19)$$

then we will obtain that for any  $\varepsilon > 0$

$$\overline{\lim}_n \int_{D_\delta} n^{2k/\alpha} |z|^{2k} e^{-2n\gamma(1+\varepsilon)|z|^\alpha} n d\nu(z) \leq \int_{\mathbb{C}} |z|^{2k} e^{-\gamma(1-\varepsilon)|z|^\alpha} \lambda d\mu_\alpha(z)$$

and

$$\underline{\lim}_n \int_{D_\delta} n^{2k/\alpha} |z|^{2k} e^{-2n\gamma(1+\varepsilon)|z|^\alpha} n d\nu(z) \geq \int_{\mathbb{C}} |z|^{2k} e^{-\gamma(1+\varepsilon)|z|^\alpha} \lambda d\mu_\alpha(z).$$

Taking the limit as  $\varepsilon$  goes to zero will complete the proof.

To prove (19) let us write

$$\begin{aligned} \int_{D_\delta} n^{2k/\alpha} |z|^{2k} e^{-2n\gamma'|z|^\alpha} n d\nu(z) &= \int_{D_{\delta n^{1/\alpha}}} |x|^{2k} e^{-2n\gamma'|x/n^{1/\alpha}|^\alpha} n dT_n(\nu)(x) \\ &= \int_{D_{\delta n^{1/\alpha}}} |x|^{2k} e^{-2\gamma'|x|^\alpha} n dT_n(\nu)(x). \end{aligned}$$

For any fixed integer  $K > 0$ , the weak convergence of the measures  $(nT_n(\nu))_{n \in \mathbb{N}}$  toward  $\lambda\mu_\alpha$  implies that

$$\int_{D_K} |x|^{2k} e^{-2\gamma'|x|^\alpha} n dT_n(\nu)(x) \xrightarrow{n \rightarrow \infty} \int_{D_K} |x|^{2k} e^{-2\gamma'|x|^\alpha} \lambda d\mu_\alpha(\nu)(x). \quad (20)$$

If we are able to find a function  $h(K)$  going to zero as  $K$  goes to infinity for which

$$0 \leq \int_{D_{\delta n^{1/\alpha}} \setminus D_K} |x|^{2k} e^{-2\gamma'|x|^\alpha} n dT_n(\nu)(x) \leq h(K)$$

for  $n$  large enough then (19) would be established. To this aim, we write

$$\begin{aligned} \int_{D_{\delta n^{1/\alpha}} \setminus D_K} |x|^{2k} e^{-2\gamma'|x|^\alpha} n dT_n(\nu)(x) &\leq \sum_{j=K}^{\lfloor \delta n^{1/\alpha} \rfloor} \int_{D_{j+1} \setminus D_j} |x|^{2k} e^{-2\gamma'|x|^\alpha} n dT_n(\nu)(x) \\ &\leq \sum_{j=K}^{\lfloor \delta n^{1/\alpha} \rfloor} (j+1)^{2k} e^{-2\gamma'j^\alpha} n T_n(\nu)(D_{j+1}) \\ &\leq \sum_{j=K}^{\lfloor \delta n^{1/\alpha} \rfloor} (j+1)^{2k} e^{-2\gamma'j^\alpha} n \nu(D_{(j+1)/n^{1/\alpha}}) \\ &\leq \sum_{j=K}^{\lfloor \delta n^{1/\alpha} \rfloor} (j+1)^{2k} e^{-2\gamma'j^\alpha} \lambda 2(j+1)^\alpha \\ &\leq \sum_{j=K}^{\infty} (j+1)^{2k} e^{-2\gamma'j^\alpha} \lambda 2(j+1)^\alpha = h(K) \end{aligned}$$

where we have used that  $[\delta n^{1/\alpha}] + 1 \leq 2\delta n^{1/\alpha}$  for  $n$  large enough to apply (18). This ends the proof of this step.

**Step 3: Dominated convergence**

To obtain the uniform convergence of  $P_n(\cdot/n^{1/\alpha})/\sqrt{n}$  towards  $f_{\alpha,\lambda}$ , it is sufficient to find a sequence  $b_k$ , independent of  $n$ , such that for any  $k$  and  $n$

$$\frac{1}{n^{1+k\alpha}\|X^k\|_n^2} \leq b_k$$

and with the power series  $\sum_{k=0}^{\infty} b_k z^k$  having an infinite radius of convergence.

Let  $\varepsilon > 0$  such that for any  $r \in [0, \varepsilon]$

$$\frac{9\lambda}{10}\mu_\alpha(D_r) \leq \nu(D_r) \leq \frac{11\lambda}{10}\mu_\alpha(D_r) \tag{21}$$

and for any  $Z \in D_\varepsilon$

$$\frac{\gamma}{2}|z|^\alpha \leq V^\nu(z) \leq \frac{3\gamma}{2}|z|^\alpha.$$

Such an  $\varepsilon$  exists due to the properties stated at Step 0.

$$\begin{aligned} \int_{\mathbb{C}} n^{2k/\alpha} |z|^{2k} e^{-2nV^\nu(z)} n d\nu(z) &\geq \int_{D_\varepsilon} |n^{1/\alpha} z|^{2k} e^{-3\gamma|zn^{1/\alpha}|^\alpha} d\nu(z) \\ &\geq \int_{D_{\varepsilon n^{1/\alpha}}} |x|^{2k} e^{-3\gamma|x|^\alpha} dT_n(\nu)(x) \\ &\geq \int_{D_{\varepsilon k^{1/\alpha}} \setminus D_{\varepsilon k^{1/\alpha}/2}} |x|^{2k} e^{-3\gamma|x|^\alpha} dT_n(\nu)(x) \\ &\geq \left(\frac{\varepsilon}{2} k^{1/\alpha}\right)^{2k} e^{-3\gamma\varepsilon^\alpha k} \left(T_n(\nu)(D_{\varepsilon k^{1/\alpha}}) - T_n(\nu)(D_{\varepsilon k^{1/\alpha}/2})\right). \end{aligned}$$

Due to the inequality (21), we deduce

$$\begin{aligned} T_n(\nu)(D_{\varepsilon k^{1/\alpha}}) - T_n(\nu)(D_{\varepsilon k^{1/\alpha}/2}) &= \nu(D_{\varepsilon(k/n)^{1/\alpha}}) - \nu(D_{\varepsilon(k/n)^{1/\alpha}/2}) \\ &\geq \frac{11}{10}\lambda n\mu(D_{\varepsilon(k/n)^{1/\alpha}}) - \frac{9}{10}\lambda\mu(D_{\varepsilon(k/n)^{1/\alpha}/2}) \\ &\geq \frac{11}{10}\lambda\varepsilon^\alpha k - \frac{9}{10}\lambda\left(\frac{\varepsilon}{2}\right)^\alpha k \\ &\geq k\frac{13}{20}\lambda\varepsilon^\alpha. \end{aligned}$$

We can now define

$$b_k^{-1} = \left(\frac{\varepsilon}{2} k^{1/\alpha}\right)^{2k} e^{-3\gamma\varepsilon^\alpha k} k \frac{13}{20} \lambda \varepsilon^\alpha$$

and we check easily that

$$\frac{1}{k} \log b_k \xrightarrow[k \rightarrow \infty]{} -\infty$$

which implies that  $\sum_{k=0}^{\infty} b_k z^k$  has an infinite radius of convergence. Using the dominated convergence theorem, we obtain that the following convergence holds uniformly on compact sets of  $\mathbb{C}$

$$\frac{1}{\sqrt{n}} P_n \left( \frac{z}{n^{1/\alpha}} \right) \xrightarrow[n \rightarrow \infty]{} \sum_{k=0}^{\infty} \frac{a_k}{\left( \int_{\mathbb{C}} |z|^{2k} e^{-\frac{\lambda}{\alpha}|z|^\alpha} \lambda d\mu_\alpha(z) \right)^{1/2}} z^k$$

which, as said before, is exactly  $f_{\alpha,\lambda}$  after computation of the coefficients.

**Proof of 2.** By the same argument as in the proof of 1.8, it is an immediate consequence of the first point, using the fact that if  $\nu$  satisfies  $\lim_{r \rightarrow \infty} r^\alpha \nu(D_r^c) = \lambda$ , then its pushforward by the inversion,  $i_*\nu$ , satisfies  $\lim_{r \rightarrow 0} i_*\nu(D_r)/r^\alpha = \lambda$ .  $\square$

**Proof of Theorem 1.12**

As promised, this will be a very short proof which will use the same ideas of the previous proofs. Notice that  $P_n$  has the same law as

$$\tilde{P}_n(z) = \sum_{k=0}^n \frac{\sqrt{n^k}}{\sqrt{k!}} a_{n-k} z^k.$$

Consider

$$Q_n(z) = \frac{\sqrt{n!}}{\sqrt{n^n}} z^n \tilde{P}_n(1/z) = \sum_{k=0}^n \frac{\sqrt{n(n-1)\dots(n-k+1)}}{\sqrt{n^k}} a_k z^k.$$

Notice that

$$\frac{\sqrt{n(n-1)\dots(n-k+1)}}{\sqrt{n^k}} \leq 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\sqrt{n(n-1)\dots(n-k+1)}}{\sqrt{n^k}} = 1$$

for any  $k$ . By a dominated convergence argument and since the series

$$\sum_{k=0}^{\infty} a_k z^k \tag{22}$$

has a radius of convergence one, we obtain that, almost surely,  $Q_n$  converges uniformly on compact sets of  $\mathbb{D}$  towards the random analytic function defined by (22). The proof is completed by the Hurwitz's continuity (Lemma 4.2), the continuity of the minimum (Lemma 4.1) and the structure of the zeros when the coefficients are standard complex Gaussians.

**4 Appendix: Point processes**

We remind some definitions and properties of point processes that are used in the article. Let  $X$  be a Polish space. We denote by  $\mathcal{C}_X$  the space of locally finite positive measures  $P$  on  $X$  such that  $P(A)$  is a non-negative integer or infinity for every measurable set  $A \subset X$ . It is not hard to see that for every  $P$  there exists a countable family  $(x_\lambda)_{\lambda \in \Lambda}$  of elements of  $X$  such that every  $x \in X$  has an open neighborhood  $U \subset X$  for which the cardinal of  $\{\lambda \in \Lambda : x_\lambda \in U\}$  is finite and such that

$$P = \sum_{\lambda \in \Lambda} \delta_{x_\lambda}.$$

Indeed, we could have defined  $\mathcal{C}_X$  more loosely by saying

$$\mathcal{C}_X = \{P \subset X : P \text{ is locally finite and admits multiplicities}\}$$

and the measure version  $P$  would count the number of points inside a set. This set notation shall be used along the article and sometimes we will use  $\#(P \cap A)$  to denote  $P(A)$ . We will endow  $\mathcal{C}_X$  with a topology. Let  $f : X \rightarrow \mathbb{R}$  be a continuous function with compact support. Define  $\hat{f} : \mathcal{C}_X \rightarrow \mathbb{R}$  by

$$\hat{f}(P) = \sum_{x \in P} f(x) = \int_X f dP$$

where in the sum we count  $x$  with multiplicity. Notice that  $\hat{f}$  makes sense since  $P$  is locally finite and  $f$  is compactly supported. Then we endow  $\mathcal{C}_X$  with the smallest topology such that  $\hat{f}$  is continuous for every continuous function  $f : X \rightarrow \mathbb{R}$  with compact support. Notice that this topology is the vague topology if  $\mathcal{C}_X$  is seen as a subspace of the space of Radon measures on  $X$ . In particular, since the space of Radon measures on a Polish space is Polish [13] it can be proved that  $\mathcal{C}_X$ , since it is a closed subset of this space, is a Polish space too.

We shall be mainly interested in the cases where  $X$  is  $\mathbb{R}^+ = [0, \infty)$ ,  $X$  is  $[0, 1)$  or  $X$  is an open subset of  $\mathbb{C}$ .

**Lemma 4.1** (Continuity of the minimum). *The application  $\min : \mathcal{C}_{\mathbb{R}^+} \rightarrow [0, \infty]$ , that to each  $P \in \mathcal{C}_{\mathbb{R}^+}$  associates its minimum or, in measure terms, the infimum of its support, is continuous. Similarly, the application that to each  $P \in \mathcal{C}_{[0,1]}$  associates its minimum in  $[0, 1]$  is continuous.*

*Proof.* We will only give the proof of the continuity of  $\min$  since the proof of the continuity of the second application follows the same steps. Let  $P \in \mathcal{C}_{\mathbb{R}^+}$  and consider a sequence  $\{P_n\}_{n \in \mathbb{N}}$  that converges to  $P$ .

**Suppose**  $\min P < \infty$ . Take  $\varepsilon > 0$  and any positive continuous function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  supported on  $[0, \min P + \varepsilon]$  such that  $f(\min P) > 0$ . Since  $\hat{f}(P_n) \rightarrow \hat{f}(P)$  we have that  $\hat{f}(P_n) > 0$  for  $n$  large enough and then  $\min P_n \leq \min P + \varepsilon$  for  $n$  large enough so that

$$\overline{\lim}_{n \rightarrow \infty} \min P_n \leq \min P + \varepsilon.$$

Since this can be done for every  $\varepsilon > 0$  we obtain  $\overline{\lim}_{n \rightarrow \infty} \min P_n \leq \min P$ .

Now take  $a \in \mathbb{R}^+$  such that  $a < \min P$ . Consider any continuous function  $f$  supported on  $[0, \min P]$  such that  $f(r) = 1$  if  $r \leq a$ . Since  $\hat{f}(P_n) \rightarrow \hat{f}(P) = 0$  we have that  $\hat{f}(P_n) < 1$  for  $n$  large enough. Then,  $P_n([0, a]) = 0$  and, thus,  $\min P_n \geq a$  for  $n$  large enough. So,

$$a \leq \underline{\lim}_{n \rightarrow \infty} \min P_n.$$

Since this can be done for every  $a < \min P$  we obtain  $\min P \leq \underline{\lim}_{n \rightarrow \infty} \min P_n$  and we may conclude.

**Suppose**  $\min P = \infty$ , i.e.  $P(\mathbb{R}^+) = 0$ . Take  $M > 0$  and consider a non-negative continuous function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  with compact support such that  $f(y) = 1$  if  $y \leq M$ . Then, since  $\hat{f}(P_n) \rightarrow \hat{f}(P) = 0$ , we have that  $\hat{f}_n(P) < 1$  for  $n$  large enough. In particular  $P_n([0, M]) = 0$  for  $n$  large enough which implies that  $\min P_n > M$  for those  $n$ . Since this can be done for every  $M > 0$  we obtain  $\lim_{n \rightarrow \infty} \min P_n = \infty$  by definition. □

**Lemma 4.2** (Hurwitz's continuity). *Consider an open connected subset  $\Omega$  of  $\mathbb{C}$  and denote by  $\mathcal{O}(\Omega)$  the space of not identically zero holomorphic functions endowed with the compact-open topology (the topology of uniform convergence on compact sets). Then the application  $\text{zero} : \mathcal{O}(\Omega) \rightarrow \mathcal{C}_\Omega$  defined by*

$$\text{zero}(p) = \sum_{p(z)=0} \delta_z,$$

*where the zeros are counted with multiplicity, is continuous.*

*Proof.* Let  $(p_n)_{n \in \mathbb{N}}$  be a sequence of elements in  $\mathcal{O}(\Omega)$  that has a limit  $p \in \mathcal{O}(\Omega)$ . Let  $f : \Omega \rightarrow \mathbb{R}$  be a continuous function with compact support. Denote by  $z_1, \dots, z_l$  the zeros of  $P$  inside  $\text{supp}(f)$ . Denote by  $L$  the total number of zeros of  $P$  counted with multiplicity. Take  $\varepsilon > 0$ . We will find  $N > 0$  such that

$$\left| \sum_{p_n(z)=0} f(z) - \sum_{p(z)=0} f(z) \right| < \varepsilon$$

for  $n > N$  where the zeros are, again, counted with multiplicity in the sums.

By the continuity of  $f$  we can choose  $\tilde{\delta} > 0$  such that for every  $k \in \{1, \dots, l\}$  we have  $|f(z) - f(z_k)| < \varepsilon/L$  for every  $z \in D_{\tilde{\delta}}(z_k)$ . By Hurwitz's theorem, since  $(p_n)_{n \in \mathbb{N}}$  converges to  $p$  uniformly on compact sets, there exists  $\delta > 0$  and  $\tilde{N} > 0$  such that  $\delta < \tilde{\delta}$  and for every  $k \in \{1, \dots, l\}$  the number of zeros of  $p_n$  inside  $D_\delta(z_k)$  counted with multiplicity is exactly the same as the multiplicity the zero  $z_k$  of  $p$  for  $n \geq \tilde{N}$ . Define

$$K = \text{supp} f \cap D_\delta(z_1)^c \cap \dots \cap D_\delta(z_l)^c.$$



Because of the uniform convergence on  $K$  and because  $p(z) \neq 0$  for every  $z \in K$  we can take  $N > 0$  such that  $N > \tilde{N}$  and

$$\sup_{z \in K} |p_n(z) - p(z)| < \inf_{z \in K} |p(z)|.$$

This implies, in particular, that  $p_n(z) \neq 0$  for every  $n \geq N$  and  $z \in K$ . We may conclude by saying that

$$\begin{aligned} \left| \sum_{p_n(z)=0} f(z) - \sum_{p(z)=0} f(z) \right| &\leq \sum_{k=1}^l \left( \sum_{\substack{p_n(z)=0 \\ z \in D_\delta(z_k)}} |f(z) - f(z_k)| \right) \\ &< \sum_{k=1}^l \left( \sum_{\substack{p_n(z)=0 \\ z \in D_\delta(z_k)}} \varepsilon/L \right) = \varepsilon. \end{aligned}$$

□

# Chapter 5

## Edge fluctuations

*This chapter corresponds to [53].*

**Abstract.** We study the fluctuations of the maxima of some classes of two-dimensional determinantal Coulomb gases. Different behaviors are given when the uniform measure on the circle is the equilibrium measure. This includes exponential fluctuations at quadratic speed and Gumbel fluctuations at linear speed. We also obtain the limiting kernel of certain two-dimensional determinantal Coulomb gases at the origin and at the unit circle. Finally, we explore the relations between Kac polynomials and a particular Coulomb gas. We show the independence of their limiting inner and outer process and obtain, in this way, that their limiting behavior is the same far from the unit circle. On the other hand, we characterize the limiting behaviors at the unit circle and remark, in particular, that they are different.

**Résumé.** Nous étudions les fluctuations de la particule la plus éloignée dans certaines classes de gaz de Coulomb déterminantaux de dimension deux. Différents comportements sont obtenus lorsque la mesure d'équilibre est la mesure uniforme sur le cercle. Nous obtenons en particulier des fluctuations exponentielles à une vitesse quadratique et des fluctuations Gumbel à une vitesse linéaire. Nous obtenons également le noyau limite de quelques gaz de Coulomb déterminantaux à l'origine et au cercle unité. Finalement, nous explorons les relations entre les polynômes de Kac et un gaz de Coulomb particulier. Nous démontrons l'indépendance entre le processus intérieur et le processus extérieur quand le nombre de particules tend vers l'infini. Nous obtenons, ainsi, que le comportement limite de ces deux processus est le même loin du cercle unité. D'autre part, nous caractérisons les comportements limites près du cercle unité et nous remarquons, en particulier, qu'ils sont différents.

*2010 MSC:* 60G55; 60F05; 60K35; 82C22; 30C15

*Keywords:* Gibbs measure; interacting particle system; determinantal point process; Coulomb gas; random polynomial.

*Mots-clés:* Mesure de Gibbs; système des particules en interaction; processus ponctuel déterminantal; gaz de Coulomb; polynôme aléatoire.

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### 1 Introduction

We will be interested in the fluctuations of the maxima of particles distributed according to two-dimensional determinantal Coulomb gases defined in (1) below. The first result we are aware of is the Gumbel fluctuations at speed  $\sqrt{n \log n}$  obtained by Rider [82] for the farthest particle of the Ginibre ensemble. Later, Chafai and Pécché [38] generalized this result obtaining Gumbel fluctuations at the same speed for a class of strongly confining potentials that includes

the Ginibre ensemble. Then, Seo [87] considered the hard edge version of this result by proving exponential fluctuations at speed  $n$ . On a series of articles, [63, 57, 39], Qi and his collaborators have studied different cases related to matrix models which includes truncated circular unitary matrices and products of matrices from the spherical and from the Ginibre ensemble. Very recently, Butez and the author [32] studied a class of weakly confining potentials generated by probability measures. Fluctuations of the maxima for Coulomb gases have also attracted physicists attention as we can see, for instance, in the work of Lacroix-A-Chez-Toine, Grabsch, Majumdar and Schehr [70] where even an intermediate deviation regime is explored. The farthest particle has also been of interest for fermionic systems as in the work of Dean, Le Doussal, Majumdar and Schehr [42]. Despite these efforts, there is not yet a complete classification of the possible limiting behaviors.

In this article we show different classes of universality disproving, for instance, the common belief that ‘strongly confining’ may be seen as a universality class. In Section 2 we consider potentials that have the uniform measure on the unit circle as the limit of the empirical measures and we show that different behaviors may arise. We consider weakly confining potentials in Subsection 2.1 except for Theorem 2.4 where the potential may be weakly or strongly confining. The proofs use the results stated and proved in Section 5 about the behavior of the point process near zero. In Subsection 2.2 we consider strongly confining potentials and obtain Gumbel fluctuations at linear speed. The beauty of this model is its integrability since very explicit calculations can be made. In Subsection 2.3 we consider a hard edge potential. The proof involves the calculation of the limiting kernel at the unit circle as in [87]. Once more, we are delighted by the integrability of this model. This even helps us obtain in Theorem 6.13 the limiting kernel at the unit circle for some non-radial processes. The proofs of the results stated in Section 2 are found in Section 6. In Section 3 we deal briefly with two classes of potentials generated by positive measures. One of them is an example of application for some theorems stated in Section 2 and the other one has a proof that uses the same techniques. In Section 4 we consider the relation between Kac polynomials and a particular Coulomb gas. One of the main results of this section is the independence between the inner point process and the outer point process as the number of particles goes to infinity, which implies, in particular, the independence of the minimum and the maximum. The second main result is that both processes differ near the unit circle. The proofs may be found in Section 7 and 8. Finally, we include two short appendices in Section 9 that have independent interest and are useful for the proof of Theorem 4.1 and Theorem 4.4 and a third appendix that makes a link between the weakly and the strongly confining case.

The general radial determinantal Coulomb gas is given by a positive number  $\chi > 0$  and a continuous function  $V : [0, \infty) \rightarrow \mathbb{R}$  bounded from below. It is the system of particles  $(x_1^{(n)}, \dots, x_n^{(n)})$  that follows the law proportional to

$$\prod_{i < j} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{\infty} V(|x_i|)} d\ell_{\mathbb{C}}(x_1) \dots d\ell_{\mathbb{C}}(x_n) \quad (1)$$

where  $\ell_{\mathbb{C}}$  denotes the Lebesgue measure on  $\mathbb{C}$ . For the integral of (1) to be finite we shall assume that

$$\liminf_{r \rightarrow \infty} \{V(r) - \log r\} > \infty. \quad (2)$$

If we wish to take  $\chi = 0$  in (1) we can assume that the potential is *strongly confining*, i.e.

$$\lim_{r \rightarrow \infty} \{V(r) - \log r\} = \infty. \quad (3)$$

If (3) is not satisfied we say that the potential is *weakly confining*. We shall also consider some degenerate cases such as the *hard edge* radial determinantal Coulomb gases. They are given by a continuous function  $V : [0, 1] \rightarrow \mathbb{R}$  and a real number  $\chi \in \mathbb{R}$ . The only difference is that the

system of particles  $(x_1^{(n)}, \dots, x_n^{(n)})$  lives in  $\bar{D}_1$ , the closed unit disk centered at zero, and follows the law proportional to

$$\prod_{i < j} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{\infty} V(|x_i|)} d\ell_{\bar{D}_1}(x_1) \dots d\ell_{\bar{D}_1}(x_n) \quad (4)$$

where  $\ell_{\bar{D}_1}$  denotes the Lebesgue measure restricted to  $\bar{D}_1$ . It may be thought as a particular case of (1) where we let  $V(r) = \infty$  for  $r > 1$ .

The usual motivations for these models are random matrix theory, the fractional quantum Hall effect and the Ginzburg-Landau model. We refer to [89] for further motivations.

## 2 Circle potentials

This article is mainly focused on what we call *circle potentials*. These are potentials for which the corresponding empirical measures converge to the uniform measure on the unit circle. More precisely, we will say that a continuous function  $V : [0, \infty) \rightarrow [0, \infty)$  (or  $V : [0, 1] \rightarrow [0, \infty)$  in the hard edge case) satisfies *the circle conditions* or that  $V$  is a *circle potential* if

$$V(1) = 0 \text{ and } V(r) \geq \max\{0, \log(r)\} \text{ for every } r \geq 0 \quad (5)$$

In this case, by Frostman's conditions and well-known large deviation principles (see, for instance, [58], [35] or [51]) we know that the sequence of empirical measures  $\frac{1}{n} \sum_{k=1}^n \delta_{x_k^{(n)}}$  will converge towards the uniform measure on the unit circle. In fact, Frostman's conditions are exactly conditions (5).

In Subsection 2.1 we state four theorems. On the first three,  $V$  is necessarily a weakly confining potential. On the fourth one  $V$  may be weakly or strongly confining. In Subsection 2.2 we recover the Gumbel distribution at linear speed for a family of potentials. Finally, in Subsection 2.3, one theorem about a hard edge potential, where we recover an exponential distribution, is stated.

### 2.1 Weakly confining circle potentials

In this subsection, the potentials of the first three theorems will satisfy

$$\lim_{r \rightarrow \infty} \{V(r) - \log(r)\} = 0$$

as a consequence of the hypotheses. The potential treated on the fourth theorem can be strongly or weakly confining. The proofs follow the same methods Butez and the author recently used in [32]. In the first theorem we find a generalization of the Bergman process (case  $\chi = 1$ ). The second theorem tells us that the limiting process may have a finite number of particles. The third theorem is the infinite particle counterpart of the second theorem. The fourth and final theorem of this subsection is an example of a potential that may be strongly confining and whose maximum does not have Gumbel fluctuations.

**Theorem 2.1** (Very weak confinement). *Suppose that  $V$  satisfies the circle conditions (5). Suppose there exists  $R \geq 1$  such that*

$$V(r) = \log(r) \text{ for every } r \geq R \text{ and } V(r) > \log(r) \text{ for every } r \in (1, R).$$

Then

$$\lim_{n \rightarrow \infty} \{x_k^{(n)} : k \in \{1, \dots, n\} \text{ and } |x_k^{(n)}| > R\} = \mathbb{B}^R$$

where  $\mathbb{B}^R$  is the determinantal point process in the complement of the closed disk of radius  $R$  associated to the Lebesgue measure and to the kernel

$$K_{\mathbb{B}^R}(z, w) = \frac{1}{\pi |z\bar{w}|^{\chi+1}} \sum_{k=0}^{\infty} \frac{(k+\chi)R^{2k+2\chi}}{(z\bar{w})^k}.$$

Furthermore, the maximum of  $|x_1^{(n)}|, \dots, |x_n^{(n)}|$  converges in law to the maximum of  $\mathbb{B}^R$ . More explicitly,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \max\{|x_1^{(n)}|, \dots, |x_n^{(n)}|\} \leq t \right) = \prod_{k=0}^{\infty} \left( 1 - (R/t)^{2k+2\chi} \right)$$

for every  $t \geq R$ .

In the last appendix in Proposition 9.4 we will describe the behavior as  $\chi$  goes to infinity of the limiting variable obtained in Theorem 2.1. It may be seen as a connection between the weakly and the strongly confining case.

**Theorem 2.2** (Finite limiting process). *Suppose that  $V$  satisfies the circle conditions (5). Take  $\alpha \geq 2\chi$ . Suppose that*

$$V(r) > \log(r) \text{ for every } r \geq 1 \text{ and } \lim_{r \rightarrow \infty} r^\alpha (V(r) - \log(r)) = \gamma \in (0, \infty).$$

Suppose also that there exists  $L_+ > 0$  and  $L_- \geq 1$  such that

$$\lim_{r \rightarrow 1^+} \frac{V(r)}{r-1} = L_+ + 1 \quad \text{and} \quad \lim_{r \rightarrow 1^-} \frac{V(r)}{1-r} = L_- - 1.$$

Then

$$\lim_{n \rightarrow \infty} \{n^{-1/\alpha} x_k^{(n)} : k \in \{1, \dots, n\}\} = \mathbb{F}_{L_-, L_+}^\alpha$$

where  $\mathbb{F}_{L_-, L_+}^\alpha$  is a determinantal point process on  $\mathbb{C} \setminus \{0\}$  associated to the Lebesgue measure and to the kernel

$$K_{\mathbb{F}_{L_-, L_+}^\alpha}(z, w) = \frac{1}{|z\bar{w}|^{\chi+1}} \sum_{k=0}^{\infty} \frac{a_k}{(z\bar{w})^k} e^{-\gamma/|z|^\alpha} e^{-\gamma/|w|^\alpha}$$

where

$$(a_k)^{-1} = \begin{cases} \infty & 2k + 2\chi > \alpha \\ 2\pi \int_0^\infty r^{2k+2\chi-1} e^{-2\gamma r^\alpha} dr & 2k + 2\chi < \alpha \\ \pi \left( \frac{1}{\alpha\gamma} + \frac{1}{L_+} + \frac{1}{L_-} \right) & 2k + 2\chi = \alpha \end{cases}.$$

This point process has  $\lfloor \alpha/2 - \chi \rfloor$  particles if  $\alpha/2 - \chi$  is not an integer. If  $N = \alpha/2 - \chi$  is an integer, this point process has  $N$  particles with probability  $\pi(1/L_+ + 1/L_-)a_N$  and  $N+1$  particles with probability  $1 - \pi(1/L_+ + 1/L_-)a_N$ .

Furthermore,  $n^{-1/\alpha}$  times the maximum of  $|x_1^{(n)}|, \dots, |x_n^{(n)}|$  converges in law to the maximum of  $\mathbb{F}_{L_-, L_+}^\alpha$ . More explicitly, if  $\alpha/2 - \chi$  is not an integer,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( n^{-1/\alpha} \max\{|x_1^{(n)}|, \dots, |x_n^{(n)}|\} \leq t \right) = \prod_{k=0}^{\lfloor \alpha/2 - \chi \rfloor} \frac{\Gamma\left(\frac{2k+2\chi}{\alpha}, 2\gamma t^{-\alpha}\right)}{\Gamma\left(\frac{2k+2\chi}{\alpha}\right)}$$

and if  $\alpha/2 - \chi$  is an integer

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P} \left( n^{-1/\alpha} \max\{|x_1^{(n)}|, \dots, |x_n^{(n)}|\} \leq t \right) \\ &= \prod_{k=0}^{\alpha/2 - \chi - 1} \frac{\Gamma\left(\frac{2k+2\chi}{\alpha}, 2\gamma t^{-\alpha}\right)}{\Gamma\left(\frac{2k+2\chi}{\alpha}\right)} \left[ e^{-2\gamma t^{-\alpha}} + (1 - e^{-2\gamma t^{-\alpha}}) \left( \frac{1}{L_+} + \frac{1}{L_-} \right) \left( \frac{1}{\alpha\gamma} + \frac{1}{L_+} + \frac{1}{L_-} \right)^{-1} \right]. \end{aligned}$$

Notice that in the extreme case  $\alpha = 2\chi$  the limiting distribution of the maxima is a convex combination of a Fréchet distribution and a Dirac measure at zero. The same techniques can be used to generalize this theorem to a case where  $V(r)$  has different kind of singularities at  $r = 1$ . In those generalizations actual Fréchet distributions can be obtained. An extreme case is considered in the following theorem where there is a very strong singularity which results in an infinite number of particles.

**Theorem 2.3** (Strong singularity at the unit circle). *Suppose that  $V$  satisfies the circle conditions (5). Take  $\alpha > 0$ . Suppose that*

$$V(r) > \log(r) \text{ for every } r \geq 1 \text{ and } \lim_{r \rightarrow \infty} r^\alpha (V(r) - \log(r)) = \gamma \in (0, \infty).$$

Suppose also that

$$\lim_{r \rightarrow 1} \frac{V(r)}{|r - 1|^k} = \infty \text{ for every } k > 0.$$

Then

$$\lim_{n \rightarrow \infty} \{n^{-1/\alpha} x_k^{(n)} : k \in \{1, \dots, n\}\} = \mathbb{I}^\alpha$$

where  $\mathbb{I}^\alpha$  is a determinantal point process in  $\mathbb{C} \setminus \{0\}$  associated to the Lebesgue measure and to the kernel

$$K_{\mathbb{I}^\alpha}(z, w) = \frac{1}{|z\bar{w}|^{\chi+1}} \sum_{k=0}^{\infty} \frac{a_k}{(z\bar{w})^k} e^{-\gamma/|z|^\alpha} e^{-\gamma/|w|^\alpha}, \quad (a_k)^{-1} = 2\pi \int_0^\infty r^{2k+2\chi-1} e^{-2\gamma r^\alpha} dr$$

Furthermore,  $n^{-1/\alpha}$  times the maximum of  $|x_1^{(n)}|, \dots, |x_n^{(n)}|$  converges in law to the maximum of  $\mathbb{I}^\alpha$ . More explicitly,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( n^{-1/\alpha} \max\{|x_1^{(n)}|, \dots, |x_n^{(n)}|\} \leq t \right) = \prod_{k=0}^{\infty} \frac{\Gamma\left(\frac{2k+2\chi}{\alpha}, 2\gamma t^{-\alpha}\right)}{\Gamma\left(\frac{2k+2\chi}{\alpha}\right)}.$$

The following result involves a potential that does not need to be weakly confining. It tells us that the fact of being strongly confining does not immediately imply a Gumbel fluctuation of the maximum. The limiting process has an infinite number of particles on an annulus that are accumulated in the unit circle.

**Theorem 2.4** (Particles on an annulus). *Suppose that  $V$  satisfies the circle conditions (5). Suppose there exists  $R > 0$  such that*

$$V(r) = \log(r) \text{ for every } r \in [1, R] \text{ and } V(r) > \log(r) \text{ for every } r \in (R, \infty).$$

Then

$$\lim_{n \rightarrow \infty} \{x_k^{(n)} : k \in \{1, \dots, n\} \text{ and } |x_k^{(n)}| > 1\} = \mathbb{A}^R$$

where  $\mathbb{A}^R$  is the determinantal point process in the complement of the closed unit disk with kernel

$$K_{\mathbb{A}^R}(z, w) = \begin{cases} \frac{1}{\pi|z\bar{w}|^{\chi+1}} \sum_{k=0}^{\infty} \frac{k+\chi}{(1-R^{-(2k+2\chi)})(z\bar{w})^k} & \text{if } |z|, |w| \leq R \\ 0 & \text{otherwise} \end{cases}.$$

Furthermore, the maximum of  $|x_1^{(n)}|, \dots, |x_n^{(n)}|$  converges in law to the maximum of  $\mathbb{A}^R$ . More explicitly,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \max\{|x_1^{(n)}|, \dots, |x_n^{(n)}|\} \leq t \right) = \prod_{k=0}^{\infty} \left( \frac{1 - t^{-2k-2\chi}}{1 - R^{-2k-2\chi}} \right)$$

for every  $t \in [1, R]$ .

Notice that  $\mathbb{A}^R$  of Theorem 2.4 is  $\mathbb{B}^1$  of Theorem 2.1 conditioned to live in the annulus  $\{x \in \mathbb{C} : 1 < |x| < R\}$  and, in particular, the limit of the maxima in Theorem 2.4 is the limit of the maxima in Theorem 2.1 conditioned to live in  $\{x \in \mathbb{C} : 1 < |x| < R\}$ .

## 2.2 Strongly confining circle potentials

Here we consider a potential for which we can make explicit calculations. We obtain the expected Gumbel distribution fluctuation but with a different speed of convergence than the one in [38]. We would like to remark once more that the beauty of this model is the explicitness of the calculations and that it may be considered as a toy model where many conjectures could be tested.

**Theorem 2.5** (Strongly confining potential). *Take  $q > 1$  and  $R > 1$ . Suppose  $V$  is such that*

$$V(r) \geq q \log(r) \text{ for every } r > 1 \text{ and } V(r) = \max\{0, q \log(r)\} \text{ for every } r \in [0, R].$$

*Define  $\varepsilon_n > 0$  as the unique solution to*

$$e^{2(q-1)n\varepsilon_n} \varepsilon_n = 1.$$

*Then*

$$n \left( \max \left( |x_1^{(n)}|, \dots, |x_n^{(n)}| \right) - 1 - \varepsilon_n \right) \rightarrow G$$

*where  $G$  has a non-standard Gumbel distribution that satisfies*

$$\mathbb{P}(G \leq a) \rightarrow e^{-\frac{1}{2q} \frac{1}{e^{2(q-1)a}}} = e^{-e^{-(q-1)\left(a - \frac{\log(2q)}{q-1}\right)}}$$

*for every  $a \in \mathbb{R}$ .*

## 2.3 Hard edge circle potentials

Here we restrict the system of particles to lie on the unit disk. The proof will involve a limit kernel calculation at the edge, such as the one in [87].

**Theorem 2.6** (Hard edge potential). *Suppose  $V$  is such that  $V(r) = \infty$  for every  $r > 1$  and suppose there exists  $R \in (0, 1)$  such that  $V(r) = 0$  for every  $r \in [R, 1]$ . Then*

$$n^2 \left( 1 - \max \left( |Y_1^{(n)}|, \dots, |Y_n^{(n)}| \right) \right) \rightarrow E$$

*where  $E$  follows a standard exponential distribution, i.e.*

$$\mathbb{P}(E \leq t) = 1 - e^{-t}$$

*for every  $t \geq 0$ .*

## 3 Related positive background models

For completeness we shall give positive background model examples for two of our results. Given a positive radial measure  $\nu$  in  $\mathbb{C}$  we define

$$V^\nu(r) = \int_1^r \frac{\nu(D_s)}{s} ds. \tag{6}$$

It can be proved that the Laplacian of  $z \in \mathbb{C} \mapsto V^\nu(|z|)$  is  $2\pi\nu$  so that this potential can be thought as some sort of electrostatic potential generated by the charge  $-\nu$ .

**Theorem 3.1** (An example of background model circle potentials). *Let  $q > 1$ . Suppose that  $\nu = q\delta_{S^1} + \bar{\nu}$  where  $\delta_{S^1}$  denotes the uniform measure on the unit circle and  $\bar{\nu}$  is a positive measure supported on the complement of a disk  $D_R$  with  $R > 1$  but whose support contains  $\partial D_R$ . Then,  $V^\nu$  satisfies the conditions of Theorem 2.4 if  $q = 1$  and it satisfies the conditions of Theorem 2.5 if  $q > 1$ .*

*Proof.* It is a consequence of the formula (6) for  $V^\nu$ .  $\square$

In fact, Theorem 2.4 admits the following extension.

**Theorem 3.2** (Particles on an annulus for positive background potentials). *Suppose  $\nu(D_1) = 1$  and suppose there exists  $R > 1$  such that  $\nu(D_R) = 1$  and  $\partial D_R$  is contained in the support of  $\nu$ . Then the same conclusions as in Theorem 2.4 hold for  $V^\nu$ .*

*Proof.* The proof follows the same steps as the proof of Theorem 2.4. In particular it is a consequence of formula (6) and Theorem 5.3 below.  $\square$

## 4 The standard circle potential and Kac polynomials

The most interesting case for us is the extreme case of  $V : [0, \infty) \rightarrow \mathbb{R}$  defined by

$$V(r) = \max\{0, \log(r)\} \quad (7)$$

and  $\chi = 1$ . This is an example of a positive background model (6) where  $\nu$  is the uniform probability measure on the unit circle. It is known that the asymptotic of this model has some similarity with the asymptotic of the zeros of standard Gaussian Kac polynomials. We can see, for instance, [32]. In Subsection 4.1 we show a further similarity while in Subsection 4.2 we show an compelling difference.

### 4.1 Inner and outer independence

By Proposition 6.1 below, the Coulomb gas model associated to 7 and  $\chi = 1$  is invariant under the inversion  $z \mapsto 1/z$ . As such, we know that the inner point process defined by  $\{x_k^{(n)} : k \in \{1, \dots, n\} \text{ and } |x_k^{(n)}| < 1\}$  converges to the Bergman process on the unit disk and the outer process  $\{x_k^{(n)} : k \in \{1, \dots, n\} \text{ and } |x_k^{(n)}| > 1\}$  converges to the Bergman process on the complement of the unit disk. A natural question to ask is about the joint limit distribution of the inner and the outer process. This is solved on a greater generality in the next theorem.

**Theorem 4.1** (Inner and outer independence for background Coulomb gases). *Suppose that  $\nu$  is a radial probability measure on  $\mathbb{C}$  such that its support is contained in  $\{x \in \mathbb{C} : R \leq |x| \leq \bar{R}\}$  for some  $R, \bar{R} > 0$ . Suppose that  $R$  and  $\bar{R}$  are the optimal numbers such that this happens, i.e. suppose the support of  $\nu$  contains  $\partial D_R$  and  $\partial D_{\bar{R}}$ . Consider  $V^\nu$  defined by (6). Denote by  $\mathcal{B}$  the Bergman process in the unit disk. More precisely, let  $\mathcal{B}$  be the determinantal point process on the unit disk associated to the Lebesgue measure and to the kernel*

$$K_{\mathcal{B}}(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}. \quad (8)$$

*Let  $\bar{\mathcal{B}}$  be an independent copy of  $\mathcal{B}$ . Let  $I_n = \{x_k^{(n)}/R : k \in \{1, \dots, n\} \text{ and } |x_k^{(n)}| < R\}$  and  $O_n = \{\bar{R}/x_k^{(n)} : k \in \{1, \dots, n\} \text{ and } |x_k^{(n)}| > \bar{R}\}$  be the inner and the outer processes. Then*

$$\lim_{n \rightarrow \infty} (I_n, O_n) = (\mathcal{B}, \bar{\mathcal{B}}).$$

It is a natural question to ask if this also happens in the case of Kac polynomials. We answer affirmatively.

**Theorem 4.2** (Inner and outer independence for Kac polynomials). *Let  $\{a_k\}_{k \in \mathbb{N}}$  be an independent sequence of standard complex Gaussian random variables. Consider the Gaussian random polynomials  $p_n$  defined by*

$$p_n(z) = \sum_{k=0}^n a_k z^k.$$



Let  $\mathcal{B}$  and  $\bar{\mathcal{B}}$  be two independent copies of the Bergman process on the unit disk, i.e. the determinantal process in the unit disk associated to Lebesgue measure and to the kernel (8). Let  $I_n = \{z \in \mathbb{C} : p_n(z) = 0 \text{ and } |z| < 1\}$  and  $O_n = \{1/z \in \mathbb{C} : p_n(z) = 0 \text{ and } |z| > 1\}$  be the inner and the outer processes. Then

$$\lim_{n \rightarrow \infty} (I_n, O_n) = (\mathcal{B}, \bar{\mathcal{B}}).$$

In fact, the same result holds when the coefficients are not Gaussian if we replace  $\mathcal{B}$  and  $\bar{\mathcal{B}}$  by the independent copies of the same limiting process.

## 4.2 Point process at the unit circle

Having seen that the point processes inside and outside of the unit disk have the same limiting behavior, it may be natural to ask if the behavior in the circle is the same. We answer negatively by describing the limits. The first limit is a well-known result. Indeed, the proof is a straightforward calculation and a nice generalization can be found in [91, Theorem 5]. We include the proof in this article for completeness.

**Theorem 4.3** (Coulomb gas at the unit circle). *Define  $R : \mathbb{C} \rightarrow \mathbb{R}$  by  $R(x) = \min\{\Re(x), 0\}$ . If  $V$  is defined by (7) then*

$$\lim_{n \rightarrow \infty} \{n(1 - x_k^{(n)}) : k \in \{1, \dots, n\}\} = \mathcal{E}$$

where  $\mathcal{E}$  is a determinantal point process on  $\mathbb{C}$  associated to the Lebesgue measure and to the kernel

$$K_{\mathcal{E}}(\alpha, \beta) = \frac{1}{\pi(\alpha + \bar{\beta})^2} \left(1 + e^{-(\alpha + \bar{\beta})}\right) e^{R(\alpha) + R(\bar{\beta})} + \frac{2}{\pi(\alpha + \bar{\beta})^3} \left(e^{-(\alpha + \bar{\beta})} - 1\right) e^{R(\alpha) + R(\bar{\beta})}.$$

**Theorem 4.4** (Random zeros at the unit circle). *Let  $\{a_k\}_{k \in \mathbb{N}}$  be an independent sequence of identically distributed complex centered random variables with variance half the identity. More precisely,  $\mathbb{E}[a_k] = 0$  and  $\mathbb{E}[|a_k|^2] = 1$ . Consider the random polynomials  $p_n$  defined by*

$$p_n(z) = \sum_{k=0}^n a_k z^k.$$

Then

$$\lim_{n \rightarrow \infty} \{n(1 - z) \in \mathbb{C} : p_n(z) = 0\} = \{z \in \mathbb{C} : \mathcal{F}(z) = 0\}$$

where  $\mathcal{F}$  is the Gaussian analytic function with covariance given by

$$K_{\mathcal{F}}(z, w) = \frac{1 - e^{-(z + \bar{w})}}{z + \bar{w}}.$$

That the limiting point processes in Theorem 4.3 and Theorem 4.4 are not the same can be seen by calculating the first intensities  $\rho_{\mathcal{E}}$  and  $\rho_{\mathcal{F}}$ . For  $\mathcal{E}$  we have  $\rho_{\mathcal{E}}(x) = K_{\mathcal{E}}(x, x)$  and for  $\mathcal{F}$  we have  $\rho_{\mathcal{F}}(z) = \frac{1}{4\pi} \Delta K_{\mathcal{F}}(z, z)$  by the Edelman-Kostlan formula. See [18, Section 2.4] for a proof of this formula.

## 5 Results about the minima

The proof of some of our results (namely Theorems 2.1, 2.2, 2.3 and 2.4) use the behavior near zero of an inverted model. The driving idea is that the maximum and the minimum are indistinguishable on the sphere. In fact, Lemma 6.1 is motivated by the regular case where the Laplacian of  $V$  is thought as a  $(1, 1)$ -form and  $e^{-2V}$  is thought as a metric on the tautological

line bundle on the sphere. These objects can be found in the work of Berman [20] who consider analogous processes on complex manifolds. We emphasize that no complex geometry is needed in this article but that the ideas fit nicely in that context.

Having  $\chi > 0$  fixed, we will consider a system of particles  $(x_1^{(n)}, \dots, x_n^{(n)})$  distributed according to the law proportional to

$$\prod_{i < j} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{\infty} V(|x_i|)} d\Lambda_{\chi}(x_1) \dots d\Lambda_{\chi}(x_n)$$

where

$$d\Lambda_{\chi}(x) = |x|^{2(\chi-1)} d\ell_{\mathbb{C}}(x).$$

**Theorem 5.1** (Finite limiting process at zero). *Suppose  $V(r)$  is strictly positive as soon as  $r \in (0, 1) \cup (1, \infty)$ . Take  $\alpha > 0$ . Suppose that*

$$\lim_{r \rightarrow 0} \frac{1}{r^{\alpha}} V(r) = \lambda \in (0, \infty), \quad \lim_{r \rightarrow 1^+} \frac{V(r)}{r-1} = l_+ \in (0, \infty) \quad \text{and} \quad \lim_{r \rightarrow 1^-} \frac{V(r)}{1-r} = l_- \in (0, \infty).$$

Then

$$\lim_{n \rightarrow \infty} \{n^{1/\alpha} x_k^{(n)} : k \in \{1, \dots, n\}\} = \mathbb{G}_{l_+, l_-}^{\alpha}$$

where  $\mathbb{G}_{l_+, l_-}^{\alpha}$  is the determinantal point process in  $\mathbb{C}$  associated to the reference measure  $\Lambda_{\chi}$  and to the kernel

$$K_{\mathbb{G}_{l_+, l_-}^{\alpha}}(z, w) = \sum_{k=0}^{\infty} a_k z^k \bar{w}^k e^{-\lambda|z|^{\alpha}} e^{-\lambda|w|^{\alpha}}$$

where

$$(a_k)^{-1} = \begin{cases} \infty & 2k + 2\chi > \alpha \\ 2\pi \int_0^{\infty} r^{2k+2\chi-1} e^{-2\lambda r^{\alpha}} dr & 2k + 2\chi < \alpha \\ \pi \left( \frac{1}{\alpha\lambda} + \frac{1}{l_+} + \frac{1}{l_-} \right) & 2k + 2\chi = \alpha \end{cases}$$

Notice that  $\mathbb{G}_{l_+, l_-}^{\alpha}$  has a finite number of particles. In fact, the number of particles belongs to the interval  $[\alpha/2 - \chi, \alpha/2 - \chi + 1]$  and it can be thought as a finite Coulomb gas with power potential. More precisely,  $\mathbb{G}_{l_+, l_-}^{\alpha}$  has  $\lceil \alpha/2 - \chi \rceil$  particles if  $\alpha/2 - \chi$  is not an integer. If  $N = \alpha/2 - \chi$  is an integer,  $\mathbb{G}_{l_+, l_-}^{\alpha}$  has  $N$  particles with probability  $\pi(1/l_+ + 1/l_-)a_N$  and  $N + 1$  particles with probability  $1 - \pi(1/l_+ + 1/l_-)a_N$ .

*Proof.* Notice that  $\{n^{1/\alpha} x_k^{(n)} : k \in \{1, \dots, n\}\}$  is a determinantal point process associated to the kernel

$$K_n(z, w) = \sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k e^{-(n+\chi)V\left(\frac{|z|}{n^{1/\alpha}}\right)} e^{-(n+\chi)V\left(\frac{|w|}{n^{1/\alpha}}\right)}$$

with respect to  $\Lambda_{\chi}$  where

$$(a_k^{(n)})^{-1} = \int_{\mathbb{C}} |z|^{2k} e^{-2(n+\chi)V\left(\frac{|z|}{n^{1/\alpha}}\right)} d\Lambda_{\chi}(z) = 2\pi \int_0^{\infty} r^{2k+2\chi-1} e^{-2(n+\chi)V\left(\frac{r}{n^{1/\alpha}}\right)} dr.$$

By [90, Proposition 3.10], our objective is to prove that  $K_n$  converges uniformly on compact sets to  $K_{\mathbb{G}_{l_+, l_-}^{\alpha}}$ . Since  $\lim_{r \rightarrow 0} \frac{1}{r^{\alpha}} V(r) = \lambda \in (0, \infty)$  we already have that

$$(n + \chi)V\left(\frac{|z|}{n^{1/\alpha}}\right) \rightarrow \lambda|z|^{\alpha}$$

uniformly on compact sets. Then what is left to prove is that

$$\sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k \rightarrow \sum_{k=0}^{\infty} a_k z^k \bar{w}^k$$

uniformly on compact sets. We will proceed by the following steps.

1. Notice that  $\lim_{n \rightarrow \infty} a_k^{(n)} = a_k$ .
2. Find a sequence  $\{A_k\}_{k \in \mathbb{N}}$  such that  $\sum_{k=0}^{\infty} A_k r^k$  converges for every  $r \geq 0$  and such that  $a_k^{(n)} \leq A_k$  for every  $n$  and  $k < n$ .
3. Use Lebesgue's dominated convergence theorem to conclude.

**Step 1.** We want to find the limit, as  $n$  goes to infinity, of

$$\left(2\pi a_k^{(n)}\right)^{-1} = \int_0^{\infty} r^{2k+2\chi-1} e^{-2(n+\chi)V\left(\frac{r}{n^{1/\alpha}}\right)} dr = n^{(2k+2\chi)/\alpha} \int_0^{\infty} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr.$$

We divide the integral in plenty of intervals

$$[0, \infty) = [0, \varepsilon) \cup [\varepsilon, \varepsilon^*) \cup [\varepsilon^*, 1) \cup [1, M^*) \cup [M^*, M) \cup [M, \infty)$$

where we have chosen

- $\varepsilon > 0$  such that  $\frac{\lambda}{2} r^\alpha \leq V(r)$  for  $r \leq \varepsilon$ ,
- $\varepsilon^* \in (\varepsilon, 1)$  such that  $\frac{l_-}{2}(1-r) \leq V(r) \leq 2l_-(1-r)$  for  $r \in [\varepsilon^*, 1]$ ,
- $M^* > 1$  such that  $\frac{l_+}{2}(r-1) \leq V(r) \leq 2l_+(r-1)$  for  $r \in [1, M^*]$  and
- $M > M^*$  such that  $\frac{1}{2} \log r \leq V(r)$  for  $r \geq M$ .

We study the integrals in order.

**Integral over  $[0, \varepsilon)$ .** Since  $e^{-2(n+\chi)V\left(\frac{r}{n^{1/\alpha}}\right)} 1_{[0, n^{1/\alpha}\varepsilon]}(r) \leq e^{-\frac{(n+\chi)}{n}\lambda r^\alpha} \leq e^{-\lambda r^\alpha}$  we can use Lebesgue's dominated convergence theorem to conclude that

$$\begin{aligned} n^{(2k+2\chi)/\alpha} \int_0^\varepsilon r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \\ = \int_0^{n^{1/\alpha}\varepsilon} r^{2k+2\chi-1} e^{-2(n+\chi)V\left(\frac{r}{n^{1/\alpha}}\right)} dr \rightarrow \int_0^\infty r^{2k+2\chi-1} e^{-2\lambda r^\alpha} dr. \end{aligned}$$

**Integral over  $[\varepsilon, \varepsilon^*)$ .** Since  $V$  is positive lower semicontinuous on  $[\varepsilon, \varepsilon^*]$  there exists  $C > 0$  such that  $C \leq V(r)$  for  $r \in [\varepsilon, \varepsilon^*]$ . Then

$$n^{(2k+2\chi)/\alpha} \int_\varepsilon^{\varepsilon^*} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \leq n^{(2k+2\chi)/\alpha} e^{-2(n+\chi)C} \rightarrow 0.$$

**Integral over  $[\varepsilon^*, 1)$ .** We write

$$\begin{aligned} n^{(2k+2\chi)/\alpha} \int_{\varepsilon^*}^1 r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr &= n^{(2k+2\chi)/\alpha} \int_0^{1-\varepsilon^*} (1-r)^{2k+2\chi-1} e^{-2(n+\chi)V(1-r)} dr \\ &= \frac{n^{(2k+2\chi)/\alpha}}{n} \int_0^{n(1-\varepsilon^*)} \left(1 - \frac{r}{n}\right)^{2k+2\chi-1} e^{-2(n+\chi)V\left(1-\frac{r}{n}\right)} dr. \end{aligned}$$

We use

$$\left(1 - \frac{r}{n}\right)^{2k+2\chi-1} e^{-2(n+\chi)V\left(1-\frac{r}{n}\right)} 1_{[0, n(1-\varepsilon^*)]}(r) \leq e^{-\frac{(n+\chi)}{n}l_- r} \leq e^{-l_- r}$$

to apply Lebesgue's dominated convergence theorem and obtain that

$$\int_0^{n(1-\varepsilon^*)} \left(1 - \frac{r}{n}\right)^{2k+2\chi-1} e^{-2(n+\chi)V\left(1-\frac{r}{n}\right)} dr \rightarrow \int_0^\infty e^{-2l_- r} dr = \frac{1}{2l_-}.$$

Then

$$n^{(2k+2\chi)/\alpha} \int_{\varepsilon^*}^1 r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \rightarrow \begin{cases} \infty & \text{if } 2k+2\chi > \alpha \\ \frac{1}{2l_-} & \text{if } 2k+2\chi = \alpha \\ 0 & \text{if } 2k+2\chi < \alpha \end{cases}$$

**Integral over**  $[1, M^*]$ . We write

$$\begin{aligned} n^{(2k+2\chi)/\alpha} \int_1^{M^*} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr &= n^{(2k+2\chi)/\alpha} \int_0^{M^*-1} (1+r)^{2k+2\chi-1} e^{-2(n+\chi)V(1+r)} dr \\ &= \frac{n^{(2k+2\chi)/\alpha}}{n} \int_0^{n(M^*-1)} \left(1 + \frac{r}{n}\right)^{2k+2\chi-1} e^{-2(n+\chi)V(1+\frac{r}{n})} dr. \end{aligned}$$

We use

$$\left(1 + \frac{r}{n}\right)^{2k+2\chi-1} e^{-2(n+\chi)V(1+\frac{r}{n})} 1_{[0, n(M^*-1)]}(r) \leq (M^*)^{2k+2\chi-1} e^{-\frac{(n+\chi)}{n}l_+r} \leq (M^*)^{2k+2\chi-1} e^{-l_+r}$$

to apply Lebesgue's dominated convergence theorem and obtain that

$$\int_0^{n(M^*-1)} \left(1 + \frac{r}{n}\right)^{2k+2\chi-1} e^{-2(n+\chi)V(1+\frac{r}{n})} dr \rightarrow \int_0^\infty e^{-2l_+r} dr = \frac{1}{2l_+}.$$

Then

$$n^{(2k+2\chi)/\alpha} \int_1^{M^*} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \rightarrow \begin{cases} \infty & \text{if } 2k+2\chi > \alpha \\ \frac{1}{2l_+} & \text{if } 2k+2\chi = \alpha \\ 0 & \text{if } 2k+2\chi < \alpha \end{cases}$$

**Integral over**  $[M^*, M]$ . Since  $V$  is positive lower semicontinuous on  $[M^*, M]$  there exists  $C > 0$  such that  $C \leq V(r)$  for  $r \in [M^*, M]$ . Then

$$n^{(2k+2\chi)/\alpha} \int_{M^*}^M r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \leq n^{(2k+2\chi)/\alpha} e^{-2(n+\chi)C} (M - M^*) \rightarrow 0.$$

**Integral over**  $[M, \infty)$ .

$$n^{(2k+2\chi)/\alpha} \int_M^\infty r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \leq n^{(2k+2\chi)/\alpha} \int_M^\infty r^{2k+2\chi-1} e^{-(n+\chi) \log r} dr \rightarrow 0.$$

In summary, we have obtained that

$$\lim_{n \rightarrow \infty} (a_k^{(n)})^{-1} = \begin{cases} \infty & 2k+2 > \alpha \\ 2\pi \int_0^\infty r^{2k+2\chi-1} e^{-2\lambda r^\alpha} dr & 2k+2 < \alpha \\ 2\pi \int_0^\infty r^{2k+2\chi-1} e^{-2\lambda r^\alpha} dr + \pi \left(\frac{1}{l_+} + \frac{1}{l_-}\right) & 2k+2 = \alpha \end{cases}$$

**Step 2.** Take  $\varepsilon > 0$  such that  $2\lambda r^\alpha \geq V(r)$  for  $r \leq \varepsilon$ . Then, since

$$2(n+\chi)V\left(\frac{r}{n^{1/\alpha}}\right) \leq 4\frac{n+\chi}{n}\lambda r^\alpha \leq 4(1+\chi)\lambda r^\alpha$$

if  $r \leq k^{1/\alpha}\varepsilon \leq n^{1/\alpha}\varepsilon$ , we obtain

$$\int_0^\infty r^{2k+2\chi-1} e^{-2(n+\chi)V\left(\frac{r}{n^{1/\alpha}}\right)} dr \geq \int_0^{k^{1/\alpha}\varepsilon} r^{2k+2\chi-1} e^{-4(1+\chi)\lambda r^\alpha} dr.$$

So, if we define  $A_k$  by

$$\begin{aligned} (A_k)^{-1} &= 2\pi \int_0^{k^{1/\alpha}\varepsilon} r^{2k+2\chi-1} e^{-4(1+\chi)\lambda r^\alpha} dr = 2\pi k^{(2k+2\chi)/\alpha} \int_0^\varepsilon r^{2k+2\chi-1} e^{-4(1+\chi)\lambda k r^\alpha} dr \\ &= 2\pi k^{(2k+2\chi)/\alpha} \int_0^\varepsilon e^{k(2\log r - 4(1+\chi)\lambda r^\alpha)} r^{2\chi-1} dr \end{aligned}$$

we get  $a_k^{(n)} \leq A_k$ . An infinite radius of convergence for the power series  $\sum_{k=0}^{\infty} A_k x^k$  is obtained if and only if

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \left[ (A_k)^{-1} \right] = \infty.$$

This can be seen by noticing that  $\lim_{k \rightarrow \infty} \frac{1}{k} \log \left[ 2\pi k^{(2k+2)/\alpha} \right] = \infty$  and that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \int_0^\varepsilon e^{k(2 \log r - 4(1+\chi)\lambda r^\alpha)} r^{2\chi-1} dr = \sup_{r \in [0, \varepsilon]} \{2 \log r - 4(1+\chi)\lambda r^\alpha\} > -\infty$$

where the last equality is obtained by Laplace's method.

**Step 3.** If  $R > 0$  and  $|z|, |w| \leq R$  we have

$$\left| \sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k - \sum_{k=0}^{\infty} a_k z^k \bar{w}^k \right| \leq \sum_{k=0}^{\infty} |a_k^{(n)} - a_k| |z|^k |w|^k \leq \sum_{k=0}^{\infty} |a_k^{(n)} - a_k| R^{2k}$$

where  $a_k^{(n)}$  is zero if  $k \geq n$ . By noticing that  $|a_k^{(n)} - a_k| R^{2k} \leq 2A_k R^{2k}$  we apply Lebesgue's dominated convergence theorem to conclude.

**Number of particles.** The assertion about the number of particles is an immediate consequence of [18, Theorem 4.5.3] since  $K_{\mathbb{G}_{+,i_-}^\alpha}$  defines a projection onto a space of dimension  $[\alpha/2 - \chi]$  if  $\alpha/2 - \chi$  is not an integer and it is almost a projection with only one eigenvalue less than one if  $\alpha/2 - \chi$  is an integer.  $\square$

**Theorem 5.2** (A strong singularity). *Suppose  $V(r)$  is strictly positive if  $r \in (0, 1) \cup (1, \infty)$ . Take  $\alpha \in (0, \infty)$ . Suppose that*

$$\lim_{r \rightarrow 0} \frac{1}{r^\alpha} V(r) = \lambda \in (0, \infty) \quad \text{and} \quad \lim_{r \rightarrow 1} \frac{V(r)}{|r-1|^p} = \infty \text{ for every } p > 0.$$

Then

$$\lim_{n \rightarrow \infty} \{n^{1/\alpha} x_k^{(n)} : k \in \{1, \dots, n\}\} = \mathbb{G}^\alpha$$

where  $\mathbb{G}^\alpha$  is the determinantal point process in  $\mathbb{C}$  associated to the reference measure  $\Lambda_\chi$  and to the kernel

$$K_{\mathbb{G}^\alpha}(z, w) = \sum_{k=0}^{\infty} a_k z^k \bar{w}^k e^{-\lambda|z|^\alpha} e^{-\lambda|w|^\alpha}, \quad (a_k)^{-1} = 2\pi \int_0^\infty r^{2k+2\chi-1} e^{-2\lambda r^\alpha} dr$$

*Proof.* The proof follows exactly the same steps as the proof of Theorem 5.1 except for the convergence of  $a_k^{(n)}$ . Choose  $p > 0$  such that  $(2k+2\chi)/\alpha < 1/p$ . As in the proof of Theorem 5.1 we decompose the integral defining  $a_k^{(n)}$  in plenty of intervals

$$[0, \infty) = [0, \varepsilon) \cup [\varepsilon, \varepsilon^*) \cup [\varepsilon^*, M^*) \cup [M^*, M) \cup [M, \infty)$$

where we chose  $\varepsilon$  and  $M$  as before but  $\varepsilon^* \in (\varepsilon, 1)$  and  $M^* \in (1, M)$  are chosen such that  $V(r) \geq |r-1|^p$  for every  $r \in [\varepsilon^*, M^*]$ . The integral on every interval is dealt in the same way except for the interval  $[\varepsilon^*, M^*)$  where a slight change is made.

**Integral over  $[\varepsilon^*, M^*)$ .** We write

$$\begin{aligned} n^{(2k+2\chi)/\alpha} \int_{\varepsilon^*}^{M^*} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr &\leq n^{(2k+2\chi)/\alpha} \int_{\varepsilon^*}^{M^*} r^{2k+2\chi-1} e^{-2(n+\chi)|r-1|^p} dr \\ &\leq n^{(2k+2\chi)/\alpha} (M^*)^{2k+2\chi-1} \int_{-\infty}^{\infty} e^{-2(n+\chi)|r-1|^p} dr \\ &\leq \frac{n^{(2k+2\chi)/\alpha}}{(n+\chi)^{1/p}} (M^*)^{2k+2\chi-1} \int_{-\infty}^{\infty} e^{-2|r|^p} dr \\ &\rightarrow 0 \end{aligned}$$

Finally, we obtain

$$\lim_{n \rightarrow \infty} (a_k^{(n)})^{-1} = 2\pi \int_0^\infty r^{2k+2\chi-1} e^{-2\lambda r^\alpha} dr$$

and we conclude the proof following the steps of the proof of Theorem 5.1.  $\square$

In the following theorem we must allow  $V$  to have a singularity at zero. In fact, we only need  $V$  to be lower semicontinuous.

**Theorem 5.3** (Particles at zero potential). *Suppose  $V : [0, \infty) \rightarrow [0, \infty]$  is non-negative and lower semicontinuous. Denote  $A = \{r \geq 0 : V(r) = 0\}$  and  $R = \text{ess sup } A$ , i.e.  $R$  is such that the Lebesgue measure of  $A \cap (R, \infty)$  is zero but the Lebesgue measure of  $A \cap (\bar{R}, \infty)$  is different from zero for every  $\bar{R} < R$ . Then, if  $(x_1^{(n)}, \dots, x_n^{(n)})$  follows a law proportional to (1), we have*

$$\lim_{n \rightarrow \infty} \{x_k^{(n)} : k \in \{1, \dots, n\} \text{ and } |x_k^{(n)}| < R\} = \mathbb{M}^A$$

where  $\mathbb{M}^A$  is the (inclusion into the open unit disk of radius  $R$  of the) determinantal point process in  $\{x \in \mathbb{C} : |x| < R \text{ and } V(|x|) = 0\}$  associated to the reference measure  $\Lambda_\chi$  and to the kernel

$$K_{\mathbb{M}^A}(z, w) = \sum_{k=0}^{\infty} a_k z^k \bar{w}^k, \quad (a_k)^{-1} = 2\pi \int_A r^{2k+2\chi-1} dr.$$

*Proof.* Notice that  $\{x_k^{(n)} : k \in \{1, \dots, n\} \text{ and } |x_k^{(n)}| < R\}$  is a determinantal point process associated to the kernel

$$K_n(z, w) = \sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k e^{-(n+\chi)V(|z|)} e^{-(n+\chi)V(|w|)}$$

with respect to  $\Lambda_\chi$  where

$$(a_k^{(n)})^{-1} = 2\pi \int_0^\infty r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr.$$

Denote

$$Z = \{x \in \mathbb{C} : |x| < R \text{ and } V(|x|) = 0\}.$$

We will prove that

$$\{x_k^{(n)} : k \in \{1, \dots, n\} \text{ and } x_k^{(n)} \in Z\} \rightarrow \mathbb{M}^A$$

and that

$$\#\{x_k^{(n)} : k \in \{1, \dots, n\}, |x_k^{(n)}| \leq \bar{R} \text{ and } x_k^{(n)} \notin Z\} \rightarrow 0$$

in distribution for every  $\bar{R} < R$ . Then we conclude by the following lemma.

**Lemma 5.4** (Union with an empty point process). *Let  $X$  be a Polish space and let  $C \subset X$  be a closed subset of  $X$ . Suppose we have a sequence of random point processes  $\{P_n\}_{n \in \mathbb{N}}$  and a random point process  $P$  on  $C$  such that we have the following convergences in distributions*

$$P_n \cap C \rightarrow P \quad \text{and} \quad \#(P_n \cap K \cap C^c) \rightarrow 0 \text{ for every compact set } K \subset X.$$

Then

$$P_n \rightarrow P$$

in distribution where  $P$  is seen as a random point process in  $X$  by the natural inclusion.

*Proof.* By [65, Theorem 4.11] we have to prove that

$$\sum_{x \in P_n} f(x) \rightarrow \sum_{x \in P} f(x)$$

weakly for every continuous function  $f : X \rightarrow \mathbb{R}$  with compact support. We already know that

$$\sum_{x \in P_n \cap C} f(x) \rightarrow \sum_{x \in P} f(x)$$

so that it is enough, by Slutsky's theorem, to prove that

$$\sum_{x \in P_n \cap C^c} f(x) \rightarrow 0.$$

Let  $K = \text{supp } f$ . Then, by hypothesis,  $\#(P_n \cap K \cap C^c) \rightarrow 0$ . We can use that

$$\left| \sum_{x \in P_n \cap C^c} f(x) \right| \leq \#(P_n \cap K \cap C^c) \|f\|_\infty$$

to conclude. □

Our first objective is to prove that  $K_n$  converges uniformly on compact sets of  $Z \times Z$  to  $K_{\mathbb{M}^A}$  which would imply, by [90, Proposition 3.10], that

$$\{x_k^{(n)} : k \in \{1, \dots, n\} \text{ and } x_k^{(n)} \in Z\} \rightarrow \mathbb{M}^A.$$

In fact we can prove that

$$\sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k \rightarrow \sum_{k=0}^{\infty} a_k z^k \bar{w}^k$$

uniformly on compact sets of  $D_R \times D_R$ . We will proceed by the following steps.

1. Notice that  $\lim_{n \rightarrow \infty} a_k^{(n)} = a_k$ .
2. Find a sequence  $\{A_k\}_{k \in \mathbb{N}}$  such that  $\sum_{k=0}^{\infty} A_k r^k$  converges for every  $r \in [0, R^2)$  and such that  $a_k^{(n)} \leq A_k$  for every  $n$  and  $k < n$ .
3. Use Lebesgue's dominated convergence theorem to conclude.

**Step 1.** We want to find the limit, as  $n$  goes to infinity, of

$$\left(2\pi a_k^{(n)}\right)^{-1} = \int_0^\infty r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr.$$

By Lebesgue's dominated convergence theorem, using the bound

$$r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} \leq r^{2k+2\chi-1} e^{-2(k+1+\chi)V(r)},$$

we obtain that

$$\int_0^\infty r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \rightarrow \int_A r^{2k+2\chi-1} dr.$$

**Step 2.** By definition of  $A$  we have

$$\int_0^\infty r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \geq \int_A r^{2k+2\chi-1} dr.$$

So, we define  $A_k$  by

$$(A_k)^{-1} = 2\pi \int_A r^{2k+2\chi-1} dr$$

and notice, by Laplace's method, that

$$\frac{1}{k} \log \int_A r^{2k+2\chi-1} dr = \frac{1}{k} \log \int_A e^{k \log r^2} r^{2\chi-1} dr \rightarrow \sup\{\log r^2\}$$

where the supremum is taken over the support of the Lebesgue measure on  $A$ . By the definition of  $R$  this supremum is  $\log R^2$  and the radius of convergence of  $\sum_{k=0}^{\infty} A_k r^k$  is  $R^2$ .

**Step 3.** Take  $r \in [0, R)$  and suppose that  $|z|, |w| \leq r$ . Then

$$\left| \sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k - \sum_{k=0}^{\infty} a_k z^k \bar{w}^k \right| \leq \sum_{k=0}^{\infty} |a_k^{(n)} - a_k| |z|^k |\bar{w}|^k \leq \sum_{k=0}^{\infty} |a_k^{(n)} - a_k| r^{2k}$$

where we have defined  $a_k^{(n)} = 0$  for  $k \geq n$ . Since  $|a_k^{(n)} - a_k| r^{2k}$  is bounded by  $2A_k r^{2k}$  we can use Lebesgue's dominated convergence theorem to conclude.

Then, to prove that

$$\#\{x_k^{(n)} : k \in \{1, \dots, n\}, |x_k^{(n)}| \leq \bar{R} \text{ and } x_k^{(n)} \notin Z\} \rightarrow 0$$

we notice that

$$\mathbb{E} \left[ \#\{x_k^{(n)} : k \in \{1, \dots, n\}, |x_k^{(n)}| \leq \bar{R} \text{ and } x_k^{(n)} \notin Z\} \right] = \int_{Z^c \cap \bar{D}_{\bar{R}}} K_n(z, z) d\ell_{\mathbb{C}}(z).$$

Since  $K_n(z, z)$  is bounded by  $\sum_{k=0}^{n-1} a_k^{(n)} |z|^{2k}$ , which we know converges uniformly on  $D_{\bar{R}}$ , we can use Lebesgue's dominated convergence theorem to conclude that

$$\mathbb{E} \left[ \#\{x_k^{(n)} : k \in \{1, \dots, n\}, |x_k^{(n)}| \leq \bar{R} \text{ and } x_k^{(n)} \notin Z\} \right] \rightarrow 0$$

and then

$$\{x_k^{(n)} : k \in \{1, \dots, n\}, |x_k^{(n)}| \leq \bar{R} \text{ and } x_k^{(n)} \notin Z\} \rightarrow 0$$

in distribution. □

## 6 Proofs of the circle potential theorems

### 6.1 The weakly confining potentials

The main approach to obtain the results of Subsection 2.1 can be seen in [32]. Here an inversion  $z \mapsto 1/z$  is made and we may use the corresponding results of Section 5 along with the following lemma.

**Lemma 6.1** (Inversion of Coulomb gases). *Let  $V : \mathbb{C} \rightarrow (-\infty, \infty]$  be a measurable function. Define  $\tilde{V} : \mathbb{C} \setminus \{0\} \rightarrow (-\infty, \infty]$  by*

$$\tilde{V}(x) = V\left(\frac{1}{x}\right) + \log|x|.$$

*Then, the image of the measure*

$$\prod_{i < j} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{\infty} V(x_i)} d\ell_{\mathbb{C}}(x_1) \dots d\ell_{\mathbb{C}}(x_n)$$

*under the application  $(x_1, \dots, x_n) \mapsto (1/x_1, \dots, 1/x_n)$  is the measure*

$$\prod_{i < j} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{\infty} \tilde{V}(x_i)} d\Lambda_{\chi}(x_1) \dots d\Lambda_{\chi}(x_n)$$

*where*

$$d\Lambda_{\chi}(x) = |x|^{2(\chi-1)} d\ell_{\mathbb{C}}(x)$$



*Proof.* To avoid possible mistakes, we divide the change of variables in two steps. Consider the function  $G^V : \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\} \rightarrow (-\infty, \infty]$  and the positive measure  $\pi$  defined by

$$G^V(x, y) = -\log|x - y| + V(x) + V(y) \quad \text{and} \quad d\pi = e^{-2(\chi+1)V} d\ell_{\mathbb{C}}.$$

Then we may write

$$\begin{aligned} & \prod_{i < j} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{\infty} V(x_i)} d\ell_{\mathbb{C}}(x_1) \dots d\ell_{\mathbb{C}}(x_n) \\ &= \exp \left( -2 \left[ -\sum_{i < j} \log|x_i - x_j| + (n + \chi) \sum_{i=1}^n V(x_i) \right] \right) d\ell_{\mathbb{C}}(x_1) \dots d\ell_{\mathbb{C}}(x_n) \\ &= e^{-2 \sum_{i < j} G^V(x_i, x_j)} d\pi(x_1) \dots d\pi(x_n). \end{aligned}$$

It is enough, then, to notice that the image of  $G^V$  and  $\pi$  under the inversion are  $G^{\tilde{V}}$  and  $\tilde{\pi}$ , respectively, defined by

$$G^{\tilde{V}}(x, y) = -\log|x - y| + \tilde{V}(x) + \tilde{V}(y) \quad \text{and} \quad d\tilde{\pi} = e^{-2(\chi+1)V} d\Lambda_{\chi}.$$

□

Then Theorem 2.1 and Theorem 2.4 are consequences of Theorem 5.3 by an inversion. Similarly, Theorem 2.2 is a consequence of Theorem 5.1 and Theorem 2.3 is a consequence of Theorem 5.2. For further details we refer to [32].

## 6.2 The strongly confining case

*Proof of Theorem 2.5.* For each natural  $n$  define  $n$  independent non-negative random variables  $X_0^{(n)}, \dots, X_{n-1}^{(n)}$  such that the law of  $X_k^{(n)}$  is proportional to

$$r^{2k+1} e^{-2(n+\chi)V(r)} dr.$$

It is known (see for instance [38, Theorem 1.2]) that the law of the point process defined by  $\{|x_1^{(n)}|, \dots, |x_n^{(n)}|\}$  is the same as the law of the point process defined by  $\{X_0^{(n)}, \dots, X_{n-1}^{(n)}\}$ . Define  $M_n = \max\{X_0^{(n)}, \dots, X_{n-1}^{(n)}\}$  which has the same law as  $\max\{|x_1^{(n)}|, \dots, |x_n^{(n)}|\}$ .

Let  $q > 1$  and define  $V_q(r) = \max\{0, q \log(r)\}$ . We will first study this potential.

**Case  $V = V_q$ .** Suppose  $V = V_q$ . Let  $m \geq 0$  and let us calculate  $\mathbb{P}(M_n \leq m)$ . By the independence we can see that

$$\mathbb{P}(M_n \leq m) = \prod_{k=0}^{n-1} \mathbb{P}(X_k^{(n)} \leq m)$$

so that we should calculate  $\mathbb{P}(X_k^{(n)} \leq m)$ .

If we suppose  $m \geq 1$  we have

$$\begin{aligned}
\int_0^m r^{2k+1} e^{-2q(n+\chi)V(r)} dr &= \int_0^1 r^{2k+1} dr + \int_1^m r^{2k+1} e^{-2q(n+\chi) \log(r)} dr \\
&= \int_0^1 r^{2k+1} dr + \int_1^m r^{2k+1-2q(n+\chi)} dr \\
&= \frac{1}{2k+2} + \int_1^m r^{2k+1-2q(n+\chi)} dr \\
&= \frac{1}{2k+2} + \frac{m^{2k+2-2q(n+\chi)}}{2k+2-2q(n+\chi)} - \frac{1}{2k+2-2q(n+\chi)} \\
&= \frac{1}{2k+2} - \frac{m^{2k+2-2q(n+\chi)}}{2(q(n+\chi) - (k+1))} + \frac{1}{2(q(n+\chi) - (k+1))} \\
&= \frac{1}{2} \left[ \frac{1}{k+1} - \frac{m^{2k+2-2q(n+\chi)}}{q(n+\chi) - (k+1)} + \frac{1}{q(n+\chi) - (k+1)} \right] \\
&= \frac{1}{2} \left[ \frac{q(n+\chi) - (k+1)m^{2k+2-2q(n+\chi)}}{(k+1)(q(n+\chi) - (k+1))} \right].
\end{aligned}$$

In particular

$$\int_0^\infty r^{2k+1} e^{-2q(n+\chi)V(r)} dr = \frac{1}{2} \left[ \frac{q(n+\chi)}{(k+1)(q(n+\chi) - (k+1))} \right]$$

and we get

$$\mathbb{P}(X_k^{(n)} \leq m) = \frac{q(n+\chi) - (k+1)m^{2k+2-2q(n+\chi)}}{q(n+\chi)} = 1 - \frac{(k+1)m^{2k+2-2q(n+\chi)}}{q(n+\chi)}$$

so that we obtain the following cumulative distribution function of  $M_n$ .

**Proposition 6.2** (A formula for the cumulative distribution function).

$$\mathbb{P}(M_n \leq m) = \prod_{k=0}^{n-1} \left( 1 - \frac{(k+1)m^{2k+2-2q(n+\chi)}}{q(n+\chi)} \right).$$

Suppose  $\{m_n\}_{n \in \mathbb{N}}$  is a sequence of numbers greater than one such that  $m_n \rightarrow 1$ . We hope to find the right sequence such that  $\lim_{n \rightarrow \infty} \mathbb{P}(M_n \leq m_n)$  is not trivial. But, instead of calculating  $\lim_{n \rightarrow \infty} \mathbb{P}(M_n \leq m_n)$  we will calculate  $\lim_{n \rightarrow \infty} \log \mathbb{P}(M_n \leq m_n)$ . We know that

$$\log \mathbb{P}(M_n \leq m_n) = \sum_{k=0}^{n-1} \log \left( 1 - \frac{(k+1)m_n^{2k+2-2q(n+\chi)}}{q(n+\chi)} \right).$$

If  $\{m_n\}_{n \in \mathbb{N}}$  is such that  $m_n^{-n} \rightarrow 0$  then, by using that  $\log(1+x) = x + o(x)$ , we can prove that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \log \left( 1 - \frac{(k+1)m_n^{2k+2-2q(n+\chi)}}{q(n+\chi)} \right) = - \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \frac{(k+1)m_n^{2k+2-2q(n+\chi)}}{q(n+\chi)}.$$

So, we should study

$$\sum_{k=0}^{n-1} \frac{(k+1)m_n^{2k+2-2q(n+\chi)}}{q(n+\chi)} = \frac{m_n^2}{q(n+\chi)m_n^{2q(n+\chi)}} \sum_{k=0}^{n-1} (k+1)(m_n^2)^k \sim \frac{1}{q n m_n^{2qn}} \sum_{k=0}^{n-1} (k+1)(m_n^2)^k.$$

Define

$$f(x) = \sum_{k=0}^{n-1} (k+1)x^k = \frac{(n+1)x^n}{x-1} + \frac{x^{n+1} - 1}{(x-1)^2}.$$

By further simplifications we may obtain the following equivalence.

**Proposition 6.3** (An equivalence for the cumulative distribution function). *If  $m_n^n \rightarrow \infty$  then*

$$\log \mathbb{P}(M_n \leq m_n) \sim -\frac{1}{2q} \frac{1}{m_n^{2(q-1)n} (m_n - 1)}.$$

*Proof.* We have already seen that if  $m_n^n \rightarrow \infty$  then

$$\log \mathbb{P}(M_n \leq m_n) \sim -\frac{1}{q n m_n^{2qn}} f(m_n^2).$$

Write  $f(m_n^2) = \theta_n + \gamma_n$  where

$$\theta_n = \frac{(n+1)m_n^{2n}}{m_n^2 - 1} \quad \text{and} \quad \gamma_n = \frac{m_n^{2(n+1)} - 1}{(m_n^2 - 1)^2}.$$

Since  $m_n \rightarrow 1$  and if  $m_n^n \rightarrow \infty$  we have

$$\theta_n \sim \frac{n m_n^{2n}}{2(m_n - 1)} \quad \text{and} \quad \gamma_n \sim \frac{m_n^{2n}}{4(m_n - 1)^2}$$

If  $m_n \rightarrow 1$  we have that  $\log(m_n) \sim m_n - 1$  and then  $m_n^n \rightarrow \infty$  holds if and only if  $n(m_n - 1) \rightarrow \infty$  holds. We deduce that

$$\gamma_n = o(\theta_n)$$

and conclude the proof of the proposition.  $\square$

Now let us try to understand the term  $m_n^{2(q-1)n}$ . We have

$$m_n^{2(q-1)n} = e^{2(q-1)n \log(m_n)} = e^{2(q-1)n[(m_n - 1) + O(m_n - 1)^2]}.$$

So, we obtain the following further simplification.

**Proposition 6.4** (A further equivalence for the cumulative distribution function). *If  $m_n^n \rightarrow \infty$  and  $n(m_n - 1)^2 \rightarrow 0$  then*

$$\log \mathbb{P}(M_n \leq m_n) \sim -\frac{1}{2q} \frac{1}{e^{2(q-1)n(m_n - 1)} (m_n - 1)}.$$

Take  $a > 0$  and define

$$m_n = \frac{a}{n} + \varepsilon_n + 1.$$

We notice the following result.

**Proposition 6.5** (Properties of epsilon). *The following assertions are true.*

- $\varepsilon_n \rightarrow 0$ ,
- $n\varepsilon_n \rightarrow \infty$ , and
- $n\varepsilon_n^{1+k} \rightarrow 0$  for every  $k > 0$  or equivalently  $n^p \varepsilon_n \rightarrow 0$  for every  $p \in [0, 1)$ .

*Proof.* Taking the logarithm in the definition of  $\varepsilon_n$  we get  $2(q-1)n\varepsilon_n = -\log(\varepsilon_n)$ . From this we get

$$\varepsilon_n \rightarrow 0$$

and

$$n\varepsilon_n \rightarrow \infty.$$

Then, multiply  $2(q-1)n\varepsilon_n = -\log(\varepsilon_n)$  by  $\varepsilon_n^k$  for  $k > 0$  we get  $2(q-1)n\varepsilon_n^{1+k} = -\varepsilon_n^k \log(\varepsilon_n)$  and, taking the limit, we get that

$$n\varepsilon_n^{1+k} \rightarrow 0$$

for every  $k > 0$  or, equivalently,

$$n^p \varepsilon_n \rightarrow 0$$

for every  $p < 1$ .  $\square$

This implies the following properties of  $m_n$ .

**Proposition 6.6** (Properties of  $m_n$ ).

$$m_n - 1 \sim \varepsilon_n$$

In particular, the following assertions are true.

- $m_n \rightarrow 1$ ,
- $n(m_n - 1) \rightarrow \infty$ , and
- $n(m_n - 1)^{1+k} \rightarrow 0$  for every  $k > 0$  or equivalently  $n^p(m_n - 1) \rightarrow 0$  for every  $p \in [0, 1)$ .

*Proof.* That  $m_n - 1 \sim \varepsilon_n$  is a consequence of  $n\varepsilon_n \rightarrow \infty$ . The other assertions follow from the previous proposition.  $\square$

Finally, we have

$$\begin{aligned} \log \mathbb{P}(M_n \leq m_n) &\sim -\frac{1}{2q} \frac{1}{e^{2(q-1)n(m_n-1)} (m_n - 1)} \\ &= -\frac{1}{2q} \frac{1}{e^{2(q-1)a} e^{2(q-1)\varepsilon_n} (m_n - 1)} \\ &\sim -\frac{1}{2q} \frac{1}{e^{2(q-1)a} e^{2(q-1)\varepsilon_n \varepsilon_n}} \\ &= -\frac{1}{2q} \frac{1}{e^{2(q-1)a}} \end{aligned}$$

which is the result we were looking for.

**A hard edge case.** Consider  $R > 1$  and define

$$V(r) = \max\{0, q \log(r)\} + \infty 1_{(R, \infty)}(r) = \begin{cases} \infty & \text{if } r > R \\ V_q(r) & \text{if } r \leq R \end{cases}.$$

If  $M_n$  denotes the maximum of the moduli, we want to understand the limit of

$$\mathbb{P}(M_n \leq m_n) = \prod_{k=0}^{n-1} \frac{\int_0^{m_n} r^{2k+1} e^{-2(n+\chi)V(r)} dr}{\int_0^\infty r^{2k+1} e^{-2(n+\chi)V(r)} dr}. \quad (9)$$

By the case  $V_q$  we already know the limit of

$$\prod_{k=0}^{n-1} \frac{\int_0^{m_n} r^{2k+1} e^{-2(n+\chi)V_q(r)} dr}{\int_0^\infty r^{2k+1} e^{-2(n+\chi)V_q(r)} dr}. \quad (10)$$

So, we would like to prove that the limit of the quotient of (10) and (9) is equal to one. Since  $\int_0^{m_n} r^{2k+1} e^{-2(n+\chi)V(r)} dr = \int_0^{m_n} r^{2k+1} e^{-2(n+\chi)V_q(r)} dr$  for  $n$  large enough, the limit of this quotient becomes the limit of

$$\prod_{k=0}^{n-1} \frac{\int_0^\infty r^{2k+1} e^{-2(n+\chi)V(r)} dr}{\int_0^\infty r^{2k+1} e^{-2(n+\chi)V_q(r)} dr} = \prod_{k=0}^{n-1} \frac{\int_0^R r^{2k+1} e^{-2(n+\chi)V_q(r)} dr}{\int_0^\infty r^{2k+1} e^{-2(n+\chi)V_q(r)} dr}.$$

But this is the probability that the maximum, for the case  $V_q$ , is less or equal than  $R$  which, since the maximum converges in law to 1, goes to 1.

In other words, since the Coulomb gas defined by  $V$  is the Coulomb gas defined by  $V_q$  conditioned to live in the disk with center 0 and radius  $R$  and since the probability that the particles lie inside this disk goes to one, the fluctuations are the same.

**End of the proof.** Take  $V : [0, \infty) \rightarrow [0, \infty]$  such that  $V(r) \geq q \log(r)$  for every  $r > 1$  and suppose there exists  $R > 1$  such that  $V(r) = \max\{0, q \log(r)\}$  for every  $r \in [0, R]$ .

Since  $\max\{0, q \log(r)\} \leq V \leq \max\{0, q \log(r)\} + \infty 1_{[R, \infty)}(r)$  and since the three potentials are the same for  $r \leq R$  we can use a comparison argument to conclude.  $\square$

*Remark 6.7* (A slight generalization). We are able to follow the previous proof to study the potentials  $V : [0, \infty) \rightarrow [0, \infty]$  defined by

$$V(r) = \begin{cases} -\tilde{q} \log(r) & \text{if } r \leq 1 \\ q \log(r) & \text{if } r > 1 \end{cases}$$

for some  $q \in (1, \infty)$  and  $\tilde{q} \in [0, \infty]$ . We would obtain that

$$\lim_{n \rightarrow \infty} \log \mathbb{P}(M_n \leq m_n) = -\frac{1}{2} \frac{\tilde{q} + 1}{\tilde{q} + q} \frac{1}{e^{2(q-1)a}}.$$

### 6.3 The hard edge case

Following the ideas of [87] we use the following theorem, which can be obtained by a straightforward explicit calculation of the kernel. A generalization may be found in [91, Theorem 5].

**Theorem 6.8** (Limiting kernel near the circle). *Take  $q \in [1, \infty]$  and define  $V_q : [0, \infty) \rightarrow [0, \infty]$  by*

$$V_q(r) = \max\{0, q \log(r)\}.$$

Define

$$K_n(z, w) = \sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k, \quad \left(a_k^{(n)}\right)^{-1} = 2\pi \int_0^\infty r^{2k+1} e^{-2(n+\chi)V_q(r)} dr.$$

Then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\pi}{n^2} K_n \left(1 - \frac{\alpha}{n}, 1 - \frac{\beta}{n}\right) \\ &= \frac{1}{\alpha + \beta} e^{-(\alpha + \bar{\beta})} \left(\frac{1}{q} - 1\right) + \frac{1}{(\alpha + \bar{\beta})^2} \left(1 + e^{-(\alpha + \bar{\beta})} \left(\frac{2}{q} - 1\right)\right) + \frac{2}{q(\alpha + \bar{\beta})^3} \left(e^{-(\alpha + \bar{\beta})} - 1\right) \end{aligned}$$

uniformly on the compact sets of  $\mathbb{C} \times \mathbb{C}$ .

*Proof.* Let us calculate  $a_k^{(n)}$ .

$$\begin{aligned} \left(a_k^{(n)}\right)^{-1} &= 2\pi \int_0^\infty r^{2k+1} e^{-2(n+\chi)V_q(r)} dr \\ &= 2\pi \int_0^1 r^{2k+1} dr + 2\pi \int_1^\infty r^{2k+1} e^{-2q(n+\chi) \log(r)} dr \\ &= 2\pi \int_0^1 r^{2k+1} dr + 2\pi \int_1^\infty r^{2k+1-2q(n+\chi)} dr \\ &= \frac{\pi}{k+1} + 2\pi \int_1^\infty r^{2k+1-2q(n+\chi)} dr \\ &= \frac{\pi}{k+1} - \frac{\pi}{k+1 - q(n+\chi)} \\ &= \frac{\pi}{k+1} + \frac{\pi}{q(n+\chi) - (k+1)} \\ &= \frac{q(n+\chi)\pi}{(k+1)(q(n+\chi) - (k+1))} \end{aligned}$$

so that

$$a_k^{(n)} = \frac{1}{\pi} \left[ (k+1) - \frac{(k+1)^2}{q(n+\chi)} \right].$$

Now, let us define

$$F_n(x) = \sum_{k=0}^{n-1} \pi a_k^{(n)} x^k.$$

We would like to prove that the sequence of functions  $\{\bar{F}_n : \mathbb{C} \rightarrow \mathbb{R}\}_{n \in \mathbb{N}}$  defined by

$$\bar{F}_n(x) = \frac{1}{n^2} F_n \left( 1 - \frac{x}{n} \right)$$

converges uniformly on compact sets of  $\mathbb{C}$  to

$$\bar{F}_\infty(x) = \frac{1}{x} e^{-x} \left( \frac{1}{q} - 1 \right) + \frac{1}{x^2} \left( 1 + e^{-x} \left( \frac{2}{q} - 1 \right) \right) + \frac{2}{qx^3} (e^{-x} - 1). \quad (11)$$

In fact, we can find a closed-form expression for  $F_n$ . To simplify the calculation define, for  $|x| < 1$ ,

$$f_0(x) = \sum_{k=0}^{\infty} x^k, \quad f_1(x) = \sum_{k=0}^{\infty} (k+1)x^k, \quad \text{and} \quad f_2(x) = \sum_{k=0}^{\infty} (k+1)^2 x^k.$$

Then, we can write

$$\begin{aligned} \sum_{k=0}^{n-1} (k+1)x^k &= \sum_{k=0}^{\infty} (k+1)x^k - \sum_{k=n}^{\infty} (k+1)x^k = f_1(x) - x^n \sum_{k=0}^{\infty} (n+k+1)x^k \\ &= f_1(x) - x^n (nf_0(x) + f_1(x)) \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{n-1} (k+1)^2 x^k &= \sum_{k=0}^{\infty} (k+1)^2 x^k - \sum_{k=n}^{\infty} (k+1)^2 x^k = f_2(x) - x^n \sum_{k=0}^{\infty} (n+k+1)^2 x^k \\ &= f_2(x) - x^n (n^2 f_0(x) + 2nf_1(x) + f_2(x)). \end{aligned}$$

So, we obtain

$$\begin{aligned} F_n(x) &= \sum_{k=0}^{n-1} (k+1)x^k - \frac{1}{q(n+\chi)} \sum_{k=0}^{n-1} (k+1)^2 x^k \\ &= f_1(x) - x^n (nf_0(x) + f_1(x)) - \frac{1}{q(n+\chi)} f_2(x) + \frac{1}{q(n+\chi)} x^n (n^2 f_0(x) + 2nf_1(x) + f_2(x)) \\ &= f_0(x) x^n \left( \frac{n^2}{q(n+\chi)} - n \right) + f_1(x) \left( 1 + x^n \left( \frac{2n}{q(n+\chi)} - 1 \right) \right) + f_2(x) (x^n - 1) \frac{1}{q(n+\chi)}. \end{aligned}$$

In fact, we can obtain a closed-form expression for  $f_0$ ,  $f_1$  and  $f_2$ . Namely,

$$f_0(x) = \frac{1}{1-x}, \quad f_1(x) = \frac{1}{(1-x)^2}, \quad \text{and} \quad f_2(x) = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}.$$

Since these closed-form expressions are holomorphic functions of  $x \in \mathbb{C}$  except at  $x = 1$  the formula found for  $F_n$  also works when  $|x| \geq 1$  (this could have also been done by a straightforward calculation but we consider it is clearer and less messy this way). It is enough then to calculate

$$\lim_{n \rightarrow \infty} \frac{1}{n} f_0 \left( 1 - \frac{x}{n} \right) = \frac{1}{x}, \quad \lim_{n \rightarrow \infty} \frac{1}{n^2} f_1 \left( 1 - \frac{x}{n} \right) = \frac{1}{x^2}, \quad \lim_{n \rightarrow \infty} \frac{1}{n^3} f_2 \left( 1 - \frac{x}{n} \right) = \frac{2}{x^3}$$

and

$$\lim_{n \rightarrow \infty} \left( 1 - \frac{x}{n} \right)^n = e^{-x}$$

to obtain the convergence of  $\bar{F}_n(x)$  towards  $\bar{F}_\infty(x)$  defined in (11) at least when  $x \neq 0$ . What is left to prove is that this convergence is uniform. This can be done directly but it is easier for us to notice that the sequence  $\bar{F}_n$  is a normal family of holomorphic functions. This can be

done, for instance, by noticing that  $|\bar{F}_n(x)| \leq F_n(-R)$  if  $|x| \leq R$ . The pointwise convergence implies then the uniform convergence on compact sets. Finally, we define  $L : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$  by

$$L_n(\alpha, \beta) = n \left( 1 - \left( 1 - \frac{\alpha}{n} \right) \left( 1 - \frac{\bar{\beta}}{n} \right) \right).$$

which converges uniformly on compact sets of  $\mathbb{C} \times \mathbb{C}$  to  $L_\infty(\alpha, \bar{\beta}) = \alpha + \bar{\beta}$  and deduce that  $\bar{F}_n \circ L_n$  converges uniformly on compact sets of  $\mathbb{C} \times \mathbb{C}$  to  $\bar{F}_\infty \circ L_\infty$ .  $\square$

*Remark 6.9* (Integral representation). The limit of

$$\frac{1}{n^2} \sum_{k=0}^{n-1} a_k^{(n)} \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\bar{\beta}}{n} \right)^k$$

where

$$a_k^{(n)} = \frac{1}{\pi} \left[ (k+1) - \frac{(k+1)^2}{q(n+\chi)} \right].$$

can be also obtained by a Riemann sum approximation. Using this we would obtain the limit

$$\frac{1}{\pi} \int_0^1 (t - t^2/q) e^{-(\alpha+\bar{\beta})t} dt$$

which is equal but has a simpler form than the one in Theorem 6.8.

In this subsection, we shall be interested in the infinite  $q$  case. We will extend Theorem 6.8 to include potentials such as the ones in the hypotheses of Theorem 2.6.

**Theorem 6.10** (Limiting kernel). *Suppose  $V : [0, \infty) \rightarrow [0, \infty]$  is a measurable function such that  $V(r) = \infty$  for every  $r > 1$  and suppose there exists  $R \in (0, 1)$  such that  $V(r) = 0$  for every  $r \in [R, 1]$ . Define*

$$K_n(z, w) = \sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k, \quad (a_k^{(n)})^{-1} = 2\pi \int_0^\infty r^{2k+1} e^{-2(n+\chi)V(r)} dr.$$

Then

$$\lim_{n \rightarrow \infty} \frac{\pi}{n^2} K_n \left( 1 - \frac{\alpha}{n}, 1 - \frac{\bar{\beta}}{n} \right) = \frac{1}{(\alpha + \bar{\beta})^2} \left( 1 - e^{-(\alpha + \bar{\beta})} \right) - \frac{1}{\alpha + \bar{\beta}} e^{-(\alpha + \bar{\beta})}$$

uniformly on the compact sets of  $\mathbb{C} \times \mathbb{C}$ .

*Proof.* If we take  $a_k = (k+1)/\pi$  and  $\tilde{a}_k = (k+1)/(\pi(1-R^{2k}))$  it is true that

$$a_k \leq a_k^{(n)} \leq \tilde{a}_k \tag{12}$$

which can be obtained by noticing that

$$0 \leq V \leq \infty 1_{[0, R)}$$

in  $[0, 1]$  and by taking the respective integrals. By Theorem 6.8 we only need to prove that

$$D_n(\alpha, \beta) = \frac{1}{n^2} \sum_{k=0}^{n-1} (a_k^{(n)} - a_k) \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\bar{\beta}}{n} \right)^k \rightarrow 0$$

uniformly on compact sets of  $\mathbb{C} \times \mathbb{C}$ . By the Cauchy-Schwarz inequality we get

$$|D_n(\alpha, \beta)| \leq D_n(\alpha, \alpha)^{1/2} D_n(\beta, \beta)^{1/2}$$

where we have taken advantage of the fact that  $a_k^{(n)} - a_k \geq 0$  to simplify notation. So, it would be enough to prove that  $D_n(\alpha, \alpha) \rightarrow 0$  uniformly on compact sets of  $\mathbb{C}$ . Furthermore, we obtain, by (12), that

$$D_n(\alpha, \alpha) \leq \frac{1}{n^2} \sum_{k=0}^{n-1} (\tilde{a}_k - a_k) \left(1 - \frac{\alpha}{n}\right)^k \left(1 - \frac{\bar{\alpha}}{n}\right)^k$$

so that it is enough to consider the model  $V = \infty 1_{[0,R)}$ . Take any  $\varepsilon > 0$  and consider  $N$  such that

$$\left| \frac{1}{1 - R^{2k}} - 1 \right| \leq \varepsilon \text{ for every } k \geq N.$$

Then, if  $n \geq N$ ,

$$\begin{aligned} & \frac{1}{n^2} \sum_{k=0}^{n-1} (\tilde{a}_k - a_k) \left(1 - \frac{\alpha}{n}\right)^k \left(1 - \frac{\bar{\alpha}}{n}\right)^k \\ &= \frac{1}{n^2} \sum_{k=0}^{N-1} (\tilde{a}_k - a_k) \left(1 - \frac{\alpha}{n}\right)^k \left(1 - \frac{\bar{\alpha}}{n}\right)^k + \frac{1}{n^2} \sum_{k=N}^{n-1} (\tilde{a}_k - a_k) \left(1 - \frac{\alpha}{n}\right)^k \left(1 - \frac{\bar{\alpha}}{n}\right)^k \\ &\leq \frac{1}{n^2} \sum_{k=0}^{N-1} (\tilde{a}_k - a_k) \left(1 - \frac{\alpha}{n}\right)^k \left(1 - \frac{\bar{\alpha}}{n}\right)^k + \frac{\varepsilon}{n^2} \sum_{k=N}^{n-1} a_k \left(1 - \frac{\alpha}{n}\right)^k \left(1 - \frac{\bar{\alpha}}{n}\right)^k \\ &\leq \frac{1}{n^2} \sum_{k=0}^{N-1} (\tilde{a}_k - a_k) \left(1 - \frac{\alpha}{n}\right)^k \left(1 - \frac{\bar{\alpha}}{n}\right)^k + \frac{\varepsilon}{n^2} \sum_{k=0}^{n-1} a_k \left(1 - \frac{\alpha}{n}\right)^k \left(1 - \frac{\bar{\alpha}}{n}\right)^k. \end{aligned}$$

But we know that, as  $n \rightarrow \infty$  ( $N$  fixed),

$$\frac{1}{n^2} \sum_{k=0}^{N-1} (\tilde{a}_k - a_k) \left(1 - \frac{\alpha}{n}\right)^k \left(1 - \frac{\bar{\alpha}}{n}\right)^k \rightarrow 0$$

uniformly on compact sets of  $\mathbb{C}$  and that

$$\frac{1}{n^2} \sum_{k=0}^{n-1} a_k \left(1 - \frac{\alpha}{n}\right)^k \left(1 - \frac{\bar{\alpha}}{n}\right)^k$$

converges uniformly on compact sets to some continuous function in  $\mathbb{C}$ . Varying  $\varepsilon$  this implies the required assertion.  $\square$

If we define  $\rho_n : [0, \infty) \rightarrow [0, \infty)$

$$\rho_n(r) = \frac{1}{n^2} K_n \left(1 - \frac{r}{n}, 1 - \frac{r}{n}\right)$$

we obtain the following consequence.

**Corollary 6.11** (Limiting first intensity). *Suppose  $V$  satisfies the conditions of Theorem 6.10. Then*

$$\lim_{n \rightarrow \infty} \rho_n(r) = -\frac{1}{2\pi r} e^{-2r} + \frac{1}{4\pi r^2} (1 - e^{-2r})$$

*uniformly on compact sets of  $[0, \infty)$ .*

Now let us prove Theorem 2.6.

*Proof of Theorem 2.6.* Let  $X_0^{(n)}, \dots, X_{n-1}^{(n)}$  be  $n$  independent random variables taking values in  $[0, 1]$  such that the law of  $X_k^{(n)}$  is proportional to

$$r^{2k+1} e^{-2(n+\chi)V(r)} dr.$$



As explained in the proof of Theorem 2.5, it is known that the law of  $M_n = \max\{X_0^{(n)}, \dots, X_{n-1}^{(n)}\}$  is the same as the law of  $\max\{|x_1^{(n)}|, \dots, |x_n^{(n)}|\}$  (see for instance [38, Theorem 1.2]). Let us study the cumulative distribution function of  $M_n$ . First, by independence,

$$\mathbb{P}(M_n \leq m) = \prod_{k=0}^{n-1} \left(1 - \mathbb{P}\left(m < X_k^{(n)}\right)\right).$$

Then, by using that  $V = 0$  in  $[R, 1]$  for the numerator and by comparing to the potential  $\infty 1_{[0, R)}$  for the denominator, we can see that, for  $m \geq R$ ,

$$\mathbb{P}\left(m < X_k^{(n)}\right) = \frac{\int_m^1 r^{2k+1} e^{-2(n+\chi)V(r)} dr}{\int_0^1 r^{2k+1} e^{-2(n+\chi)V(r)} dr} \leq \frac{1 - m^{2(k+1)}}{1 - R^{2(k+1)}}.$$

Since

$$\frac{1 - m^{2(k+1)}}{1 - R^{2(k+1)}} \leq \frac{1 - m^{2n}}{1 - R^2},$$

we notice that, as soon as  $m_n^{2n} \rightarrow 1$  and by using that  $\log(1 - x) \sim -x + o(x)$ , we have

$$\log \mathbb{P}(M_n \leq m_n) \sim - \sum_{k=0}^{n-1} \mathbb{P}\left(m_n < X_k^{(n)}\right).$$

But

$$\mathbb{P}\left(m_n < X_k^{(n)}\right) = a_k^{(n)} 2\pi \int_{m_n}^1 r^{2k+1} e^{-2(n+\chi)V(r)} dr$$

which implies that

$$\begin{aligned} \sum_{k=0}^{n-1} \mathbb{P}\left(m_n < X_k^{(n)}\right) &= \int_{m_n}^1 \sum_{k=0}^{n-1} a_k^{(n)} r^{2k} e^{-2(n+\chi)V(r)} 2\pi r dr \\ &= \int_{m_n}^1 n^2 \rho_n(n(1-r)) 2\pi r dr \\ &= \int_0^{1-m_n} \rho_n(nr) 2\pi(1-r) d(n^2 r) \\ &= \int_0^{n^2(1-m_n)} \rho_n\left(\frac{r}{n}\right) 2\pi \left(1 - \frac{r}{n^2}\right) dr. \end{aligned}$$

So that, if we consider  $m_n$  such that  $n^2(1 - m_n) = a$  we obtain

$$\sum_{k=0}^{n-1} \mathbb{P}\left(m_n < X_k^{(n)}\right) = \int_0^a \rho_n\left(\frac{y}{n}\right) 2\pi \left(1 - \frac{y}{n^2}\right) dy.$$

Since  $\rho_n(r) \rightarrow -\frac{1}{2\pi r} e^{-2r} + \frac{1}{4\pi r^2} (1 - e^{-2r})$  uniformly on  $[0, a]$  and since this limit is a continuous function that takes the value  $1/(2\pi)$  at  $r = 0$  we obtain that

$$\sum_{k=0}^{n-1} \mathbb{P}\left(m_n < X_k^{(n)}\right) \rightarrow a$$

which completes the proof of the theorem.  $\square$

*Remark 6.12* (Distance to the unit circle). By the same method, the limiting kernel at the unit circle for  $V_q$  with  $q < \infty$  allows us to find the fluctuations of the distance to the unit circle. The speed would be quadratic and the fluctuation would be a multiple of the exponential distribution.

We would like to point out that a similar argument as the one in the proof of Theorem 6.10 allows us to treat a general compactly supported measurable function  $V : D_1 \rightarrow \mathbb{C}$  defined on the open unit disk  $D_1$ . We explain how in the following theorem.

**Theorem 6.13** (Point process at the circle for a non-radial potential). *Let  $V : D_1 \rightarrow [0, \infty]$  be a non-negative measurable function on the open unit disk  $D_1$  with compact support. Denote the space of complex polynomials of degree less or equal than  $n - 1$  by  $\mathcal{P}_{n-1}$ . Consider  $\{p_k^{(n)}\}_{k \in \{0, \dots, n-1\}}$  any orthonormal basis of  $\mathcal{P}_{n-1}$  with respect to the inner product*

$$\langle f, g \rangle_n = \int_{D_1} \bar{f} g e^{-2(n+\chi)V} d\ell_{\mathbb{C}}.$$

Define

$$K_n(z, w) = \sum_{k=0}^{n-1} p_k(z) \bar{p}_k(w).$$

Then

$$\lim_{n \rightarrow \infty} \frac{\pi}{n^2} K_n \left( 1 - \frac{\alpha}{n}, 1 - \frac{\beta}{n} \right) = \frac{1}{(\alpha + \bar{\beta})^2} \left( 1 - e^{-(\alpha + \bar{\beta})} \right) - \frac{1}{\alpha + \bar{\beta}} e^{-(\alpha + \bar{\beta})}$$

uniformly on the compact sets of  $\mathbb{C} \times \mathbb{C}$ .

*Proof.* First, we would like to notice that

$$K_n(z, z) = \sup_{p \in \mathcal{P}_{n-1}} \frac{|p(z)|^2}{\langle p, p \rangle_n}.$$

Indeed, using the reproducing property of  $K_n$  we have that for every  $p \in \mathcal{P}_{n-1}$

$$\int_{D_1} K_n(z, w) p(w) e^{-2(n+\chi)V(w)} d\ell_{\mathbb{C}}(w) = p(z).$$

By the Cauchy-Schwarz inequality we obtain

$$|p(z)|^2 \leq \int_{D_1} |K_n(z, w)|^2 e^{-2(n+\chi)V(w)} d\ell_{\mathbb{C}}(w) \int_{D_1} |f(w)|^2 e^{-2(n+\chi)V(w)} d\ell_{\mathbb{C}}(w).$$

But since  $|K_n(z, w)|^2 = K_n(z, w) K_n(w, z)$ , we obtain

$$|p(z)|^2 \leq K_n(z, z) \int_{D_1} |f(w)|^2 e^{-2(n+\chi)V(w)} d\ell_{\mathbb{C}}(w).$$

If we choose  $p = K_n(\cdot, z)$  we obtain

$$|p(z)|^2 = K_n(z, z) \int_{D_1} |f(w)|^2 e^{-2(n+\chi)V(w)} d\ell_{\mathbb{C}}(w)$$

so that

$$K_n(z, z) = \sup_{p \in \mathcal{P}_{n-1}} \frac{|p(z)|^2}{\langle p, p \rangle_n}.$$

But there exists  $R \in (0, 1)$  such that  $V(z) = 0$  if  $|z| \geq R$ . Then

$$\int_{D_1 \setminus D_R} |p|^2 d\ell_{\mathbb{C}} \leq \int_{D_1} |p|^2 e^{-2(n+\chi)V} d\ell_{\mathbb{C}} \leq \int_{D_1} |p|^2 d\ell_{\mathbb{C}}$$

for every  $p \in \mathcal{P}_{n-1}$ . This implies, in particular, that  $K_n(z, z)$  takes values between two functions that, after the appropriate recentering and rescaling, converge uniformly on compact sets towards the desired limit. So

$$\lim_{n \rightarrow \infty} \frac{\pi}{n^2} K_n \left( 1 - \frac{\alpha}{n}, 1 - \frac{\alpha}{n} \right) = \frac{1}{(\alpha + \bar{\alpha})^2} \left( 1 - e^{-(\alpha + \bar{\alpha})} \right) - \frac{1}{\alpha + \bar{\alpha}} e^{-(\alpha + \bar{\alpha})}$$

uniformly on compact sets of  $\mathbb{C}$ . Since  $|K_n(z, w)|^2 \leq K_n(z, z)K_n(w, w)$ , we obtain, by Montel's theorem, that  $\{K_n(z, \bar{w})\}_{n \in \mathbb{N}}$  is a normal family of holomorphic functions on  $\mathbb{C}^2$ . By noticing that their limit points are already determined on the set  $\{(z, \bar{z}) \in \mathbb{C}^2 : z \in \mathbb{C}\}$  we conclude that they are the same everywhere. This completes the proof.  $\square$

## 7 Proof about the inner and outer independence

We begin by proving Theorem 4.1 about the Coulomb gases and immediately after we proceed to the proof of Theorem 4.2 about Kac polynomials.

*Proof of Theorem 4.1.* By Lemma 9.2 it is enough to verify a convergence of the kernels. We shall find a simpler kernel for the same process. The usual kernel of  $\{x_k^{(n)} : k \in \{1, \dots, n\}\}$  is

$$\tilde{K}_n(z, w) = \sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k e^{-(n+1)V^\nu(|z|)} e^{-(n+1)V^\nu(|w|)}$$

where

$$(b_k^{(n)})^{-1} = 2\pi \int_0^\infty r^{2k+1} e^{-2(n+1)V^\nu(r)} dr.$$

Let  $R > 0$  and  $\bar{R} > 0$  be as in the hypotheses. If  $|z|, |w| < R$  the potential on the kernel is gone and we may write

$$\tilde{K}_n(z, w) = \sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k.$$

If  $|z|, |w| > \bar{R}$  the potential on the kernel is essentially, by (6), a logarithm

$$\begin{aligned} \tilde{K}_n(z, w) &= \sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k e^{-(n+1)(V^\nu(\bar{R}) - \log \bar{R} + \log |z|)} e^{-(n+1)(V^\nu(\bar{R}) - \log \bar{R} + \log |w|)} \\ &= \sum_{k=0}^{n-1} b_k^{(n)} e^{-2(n+1)(V^\nu(\bar{R}) - \log \bar{R})} \frac{z^k \bar{w}^k}{|z|^{n+1} |w|^{n+1}}. \end{aligned}$$

If  $|z| < R$  and  $|w| > \bar{R}$  we have a mixture of both

$$\begin{aligned} \tilde{K}_n(z, w) &= \sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k e^{-(n+1)(V^\nu(\bar{R}) - \log \bar{R} + \log |w|)} \\ &= \sum_{k=0}^{n-1} b_k^{(n)} e^{-(n+1)(V^\nu(\bar{R}) - \log \bar{R})} z^k \frac{\bar{w}^k}{|w|^{n+1}} \end{aligned}$$

and similarly for  $|z| > \bar{R}$  and  $|w| < R$ . By inverting the part in  $D_{\bar{R}}$  we find the kernel of the point process

$$\{x_k^{(n)} : k \in \{1, \dots, n\} \text{ and } |x_k^{(n)}| < R\} \sqcup \{1/x_k^{(n)} : k \in \{1, \dots, n\} \text{ and } |x_k^{(n)}| > \bar{R}\} \quad (13)$$

in the disjoint union  $D_R \sqcup D_{\bar{R}-1}$ . We obtain that this process is a determinantal point process associated to the sum of Lebesgue measures and to the kernel

$$K_n^I(z, w) = \begin{cases} \sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k & z \in D_R, w \in D_R \\ \sum_{k=0}^{n-1} b_k^{(n)} e^{-(n+1)(V^\nu(\bar{R})-\log \bar{R})} z^k \frac{\bar{w}^{-k}}{|w|^{1-n}} & z \in D_R, w \in D_{\bar{R}-1} \\ \sum_{k=0}^{n-1} b_k^{(n)} e^{-(n+1)(V^\nu(\bar{R})-\log \bar{R})} \frac{z^{-k}}{|z|^{1-n}} \bar{w}^k & z \in D_{\bar{R}-1}, w \in D_R \\ \sum_{k=0}^{n-1} b_k^{(n)} e^{-2(n+1)(V^\nu(\bar{R})-\log \bar{R})} \frac{z^{-k}}{|z|^{1-n}} \frac{\bar{w}^{-k}}{|w|^{1-n}} & z \in D_{\bar{R}-1}, w \in D_{\bar{R}-1} \end{cases}.$$

The terms  $|z|^{n-1}$  and  $|w|^{n-1}$  become  $z^{n-1}$  and  $\bar{w}^{n-1}$  if we consider a conjugation  $c(z)K_n(z, w)c(w)^{-1}$  where  $c(z) = (z/|z|)^{n-1}$  so the point process (13) is a determinantal point process associated to the sum of Lebesgue measures and to the kernel

$$K_n(z, w) = \begin{cases} \sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k & z \in D_R, w \in D_R \\ \sum_{k=0}^{n-1} b_k^{(n)} e^{-(n+1)(V^\nu(\bar{R})-\log \bar{R})} z^k \bar{w}^{n-1-k} & z \in D_R, w \in D_{\bar{R}-1} \\ \sum_{k=0}^{n-1} b_k^{(n)} e^{-(n+1)(V^\nu(\bar{R})-\log \bar{R})} z^{n-1-k} \bar{w}^k & z \in D_{\bar{R}-1}, w \in D_R \\ \sum_{k=0}^{n-1} b_k^{(n)} e^{-2(n+1)(V^\nu(\bar{R})-\log \bar{R})} z^{n-1-k} \bar{w}^{n-1-k} & z \in D_{\bar{R}-1}, w \in D_{\bar{R}-1} \end{cases}.$$

We already know, by the proof of Theorem 5.3 or by [32], that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(k+1)}{R^{2k}} z^k \bar{w}^k$$

uniformly on compact sets of  $D_R \times D_R$  and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} b_k^{(n)} e^{-2(n+1)(V^\nu(\bar{R})-\log \bar{R})} z^{n-1-k} \bar{w}^{n-1-k} = \frac{1}{\pi} \sum_{k=0}^{\infty} \bar{R}^{2k} (k+1) z^k \bar{w}^k$$

uniformly on compact sets of  $D_{\bar{R}-1} \times D_{\bar{R}-1}$ . We are going to prove now that

$$\sum_{k=0}^{n-1} b_k^{(n)} e^{-(n+1)(V^\nu(\bar{R})-\log \bar{R})} z^k \bar{w}^{n-1-k} \rightarrow 0$$

uniformly on compact sets of  $D_R \times D_{\bar{R}-1}$ . By Cauchy-Schwarz inequality we have

$$|K_n(z, w)| \leq \sqrt{K_n(z, z) K_n(w, w)}$$

where we have taken  $z \in D_R$  and  $w \in D_{\bar{R}-1}$ . This implies, in particular, that the sequence  $\{K_n(z, \bar{w})\}_{n \in \mathbb{N}}$  is locally uniformly bounded and thus it is a normal family of holomorphic functions in  $D_R \times D_{\bar{R}-1} \subset \mathbb{C}^2$ . It is enough to prove that  $K_n(z, w)$  converges pointwise to zero for  $(z, w)$  on an open set of  $D_R \times D_{\bar{R}-1}$ . But we can obtain this pointwise limit, for instance, as soon as  $|z/w| < R$  and  $|w| < e^{V^\nu(\bar{R})-\log \bar{R}}$  which completes the proof.  $\square$

*Proof of Theorem 4.2.* We have considered

$$p_n(z) = \sum_{k=0}^n a_k z^k.$$

Take

$$p_n^*(z) = \sum_{k=0}^n a_{n-k} z^k = z^n p_n(1/z).$$

We defined

$$I_n = \{z \in \mathbb{C} : p_n(z) = 0 \text{ and } |z| < 1\}$$

and

$$O_n = \{1/z \in \mathbb{C} : p_n(z) = 0 \text{ and } |z| > 1\} = \{z \in \mathbb{C} : p_n^*(z) = 0 \text{ and } |z| < 1\}.$$

The theorem will be a consequence of the following convergence in law (in the compact-open topology)

$$\left( \sum_{k=0}^n a_k z^k, \sum_{k=0}^n a_{n-k} z^k \right) \rightarrow \left( \sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} \tilde{a}_k z^k \right)$$

where  $\tilde{a}_0, \dots, \tilde{a}_k, \dots$  is an independent copy of  $a_0, \dots, a_k, \dots$ . Let  $N_n = \lfloor n/2 \rfloor$  and  $\tilde{N}_n = \lceil n/2 \rceil$  such that  $N_n + \tilde{N}_n = n$ . We have

$$\sum_{k=0}^{N_n} a_k z^k \rightarrow \sum_{k=0}^{\infty} a_k z^k$$

in law as  $n \rightarrow \infty$  and

$$\sum_{k=0}^{\tilde{N}_n-1} a_{n-k} z^k \rightarrow \sum_{k=0}^{\infty} a_k z^k$$

in law as  $n \rightarrow \infty$ . Since  $\sum_{k=0}^{N_n} a_k z^k$  is independent of  $\sum_{k=0}^{\tilde{N}_n-1} a_{n-k} z^k$  we get that

$$\left( \sum_{k=0}^{N_n} a_k z^k, \sum_{k=0}^{\tilde{N}_n-1} a_{n-k} z^k \right) \rightarrow \left( \sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} \tilde{a}_k z^k \right)$$

in law as  $n \rightarrow \infty$ . We can also notice that

$$\sum_{k=N_n+1}^n a_k z^k = z^{N_n+1} \left( \sum_{k=0}^{n-N_n-1} a_{N_n+1+k} z^k \right) = z^{N_n+1} \left( \sum_{k=0}^{\tilde{N}_n-1} a_{N_n+1+k} z^k \right)$$

and, since  $z^{N_n+1}$  goes to zero uniformly on compact sets of  $D_1$  and  $\sum_{k=0}^{\tilde{N}_n-1} a_{N_n+1+k} z^k$  converges in law to  $\sum_{k=0}^{\infty} a_k z^k$  then the product converges in law (in the compact-open topology) to zero so that

$$\sum_{k=N_n+1}^n a_k z^k \rightarrow 0$$

in law as  $n \rightarrow \infty$ . The same can be said for  $\sum_{k=\tilde{N}_n}^n a_{n-k} z^k$  and then

$$\left( \sum_{k=N_n+1}^n a_k z^k, \sum_{k=\tilde{N}_n}^n a_{n-k} z^k \right) \rightarrow (0, 0)$$

in law as  $n \rightarrow \infty$ . By Slutsky's theorem we have that

$$\left( \sum_{k=N_n+1}^n a_k z^k, \sum_{k=\tilde{N}_n}^n a_{n-k} z^k \right) + \left( \sum_{k=0}^{N_n} a_k z^k, \sum_{k=0}^{\tilde{N}_n-1} a_{n-k} z^k \right) \rightarrow (0, 0) + \left( \sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} \tilde{a}_k z^k \right)$$

and we conclude that

$$\left( \sum_{k=0}^n a_k z^k, \sum_{k=0}^n a_{n-k} z^k \right) \rightarrow \left( \sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} \tilde{a}_k z^k \right)$$

in law as  $n \rightarrow \infty$ . □

## 8 Proof about the behavior near the circle

We begin by proving the limiting behavior of the Coulomb gas at the unit circle and then we proceed to prove the limiting behavior of the zeros of Kac polynomials at the unit circle.

*Proof of Theorem 4.3.* This is a consequence of Theorem 6.8 together with the fact that

$$\lim_{n \rightarrow \infty} n \max \left\{ 0, \log \left| 1 - \frac{z}{n} \right| \right\} = -R(z) \quad (14)$$

uniformly on compact sets of  $\mathbb{C}$ . To prove this we first notice that

$$\lim_{n \rightarrow \infty} \frac{n}{2} \log \left| 1 - \frac{z}{n} \right|^2 = -\Re z$$

uniformly on compact sets of  $\mathbb{C}$ . This is a consequence of the differentiability of the logarithm and the differentiability of the square of the norm. Since  $\max\{x, y\} = (|x - y| + x + y)/2$  for every  $x, y \in \mathbb{R}$ , we obtain that if  $f_n$  and  $g_n$  converge uniformly on compact sets to  $f$  and  $g$  respectively then  $\max\{f_n, g_n\}$  converges uniformly on compact sets to  $\max\{f, g\}$ . Then (14) holds and we have completed the proof of the theorem.  $\square$

*Proof of Theorem 4.4.* Consider

$$q_n(z) = \frac{1}{\sqrt{n}} p_n \left( 1 - \frac{z}{n} \right).$$

Then, define

$$K_n(z, w) = \mathbb{E}[q_n(z) \bar{q}_n(w)] = \frac{1}{n} \sum_{k=0}^n \left( 1 - \frac{z}{n} \right)^k \left( 1 - \frac{\bar{w}}{n} \right)^k$$

that by a straightforward calculation converges uniformly on compact sets to

$$K(z, w) = \frac{1 - e^{-(z+\bar{w})}}{z + \bar{w}}.$$

With this in hand we may notice that

$$(q_n(z_1), \dots, q_n(z_l)) = \left( \frac{1}{\sqrt{n}} \sum_{k=0}^n a_k \left( 1 - \frac{z_1}{n} \right)^k, \dots, \frac{1}{\sqrt{n}} \sum_{k=0}^n a_k \left( 1 - \frac{z_l}{n} \right)^k \right)$$

converges to a Gaussian vector. This is simpler than Lindeberg central limit because all the variables are multiples of each other. Actually it can be obtained by calculating the asymptotic of the characteristic function.

Finally, the tightness of the sequence  $\{q_n\}_{n \in \mathbb{N}}$  can be obtained by Lemma 9.3 because we already know that  $K_n(z, z)$  is uniformly bounded on compact sets of  $\mathbb{C}$  and so  $\int_K K_n(z, z) d\ell_{\mathbb{C}}(z)$  is a bounded sequence for any compact set  $K \subset \mathbb{C}$ .  $\square$

## 9 Appendices

### 9.1 The correlation functions of the union of point processes

Consider  $(A_1, \mu_1)$  and  $(A_2, \mu_2)$  two measure spaces. If  $P_1$  is a point process on  $A_1$  and  $P_2$  is a point process on  $A_2$  independent of  $P_1$  we consider the union  $P_1 \cup P_2$  as a point process on the disjoint union  $(A_1 \amalg A_2, \mu_1 \oplus \mu_2)$ .

**Lemma 9.1** (Correlation function of an independent union). *Suppose  $\rho_k^{(1)}$  and  $\rho_k^{(2)}$  are the  $k$ -th correlation function of  $P_1$  and  $P_2$  respectively (with respect to the measures  $\mu_1$  on  $A_1$  and  $\mu_2$  on  $A_2$ ). Then the  $n$ -th correlation function of  $P_1 \cup P_2$  (with respect to the measure  $\mu_1 \oplus \mu_2$  on  $A_1 \amalg A_2$ ) is*

$$\rho_n = \sum_{k=0}^n \rho_k^{(1)} \odot \rho_{n-k}^{(2)}$$

where  $\rho_k^{(1)} \odot \rho_{n-k}^{(2)}$  is defined by

$$\rho_k^{(1)} \odot \rho_{n-k}^{(2)}(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = \rho_k^{(1)}(x_1, \dots, x_k) \rho_{n-k}^{(2)}(x_{k+1}, \dots, x_n)$$

if  $x_1, \dots, x_k \in A_1$  and  $x_{k+1}, \dots, x_n \in A_2$ . It is defined by the symmetric property if the argument contains  $k$  points in  $A_1$  and  $n - k$  points in  $A_2$  and it is defined as zero in the other cases.

*Proof.* Suppose  $C_1, \dots, C_n$  are  $n$  measurable sets in  $A_1 \amalg A_2$ . Write  $C_k = C_k^{(1)} \cup C_k^{(2)}$  where  $C_k^{(1)} \subset A_1$  and  $C_k^{(2)} \subset A_2$ . By the distribution property of multiplication over addition and by the distribution property of multiplication of sets over union of sets it is enough to suppose that  $C_1, \dots, C_k \subset A_1$  and  $C_{k+1}, \dots, C_n \subset A_2$  for some  $k$ . So we want to prove that

$$\begin{aligned} & \mathbb{E}[\#(C_1 \cap P_1) \dots \#(C_k \cap P_1) \#(C_{k+1} \cap P_2) \dots \#(C_n \cap P_2)] \\ &= \int_{C_1 \times \dots \times C_k \times C_{k+1} \times \dots \times C_n} \rho_k^{(1)}(x_1, \dots, x_k) \rho_{n-k}^{(2)}(x_{k+1}, \dots, x_n) d\mu_1^{\otimes k}(x_1, \dots, x_k) d\mu_2^{\otimes n-k}(x_{k+1}, \dots, x_n) \end{aligned}$$

which is a consequence of the independence and Fubini's theorem. □

This translates into a statement about independent union of determinantal point processes.

**Lemma 9.2** (Kernel of an independent union). *Suppose  $P_1$  and  $P_2$  are independent determinantal point processes with kernels  $K_1$  and  $K_2$ . Then their disjoint union  $P_1 \cup P_2$  is a determinantal point process with kernel  $K$  defined by*

$$K(x, y) = \begin{cases} K_1(x, y) & \text{if } x, y \in A_1 \\ K_2(x, y) & \text{if } x, y \in A_2 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* It is a consequence of Theorem 9.1 and the calculation of the determinant of a diagonal block matrix. □

## 9.2 Tightness for random analytic functions

We consider an open set  $U \subset \mathbb{C}$  and denote by  $\mathcal{O}(U)$  the space of holomorphic functions on  $U$  endowed with the topology of uniform convergence on compact sets also known as the compact-open topology. By Montel's theorem we are able to characterize the relatively compact sets of  $\mathcal{O}(U)$ . In the following lemma we consider a random version of it.

**Lemma 9.3** (Tightness characterization). *Let  $\{P_\lambda\}_{\lambda \in \Lambda}$  be a family of random analytic functions in a domain  $U$ . Then  $\{P_\lambda\}_{\lambda \in \Lambda}$  is tight if and only if for any compact set  $K \subset U$  and  $\varepsilon > 0$  we can find  $M > 0$  such that*

$$\mathbb{P} \left[ \sup_{x \in K} |P_\lambda(x)| > M \right] < \varepsilon \tag{15}$$

for every  $\lambda \in \Lambda$ . In particular, if the family  $\{I_\lambda^K\}_{\lambda \in \Lambda}$  defined by

$$I_\lambda^K = \int_K \mathbb{E}[|P_\lambda(z)|^2] d\ell_{\mathbb{C}}(z)$$

is bounded for any compact  $K$ , then  $\{P_\lambda\}_{\lambda \in \Lambda}$  is tight.

*Proof.* The characterization of tightness is a consequence of Montel's theorem. For the second assertion we notice that for any compact subset  $K \subset U$  there exists a compact set  $\tilde{K}$  that contains  $K$  and a constant  $C > 0$  such that

$$\sup_{x \in K} |f(x)|^2 \leq C \int_{\tilde{K}} |f(z)|^2 d\ell_{\mathbb{C}}(z)$$

for every  $f \in \mathcal{O}(U)$ . This is essentially a consequence of the subharmonicity of  $|f|^2$ . Then we write

$$\mathbb{P} \left[ \sup_{x \in K} |P_{\lambda}(x)|^2 > M^2 \right] \leq M^{-2} \mathbb{E} \left[ \sup_{x \in K} |P_{\lambda}(x)|^2 \right] \leq M^{-2} C I_{\lambda}^{\tilde{K}}.$$

Since  $I_{\lambda}^{\tilde{K}}$  is uniformly bounded on  $\lambda \in \Lambda$  we may choose  $M$  large enough such that (15) is satisfied. □

### 9.3 Gumbel distribution and weakly confining fluctuations

Here we establish a connection between the limit of the maxima in the very weakly confining case, Theorem 2.1, and the Gumbel distribution.

**Proposition 9.4** (Weakly confining and Gumbel distribution). *For each  $\chi > 0$  let  $X_{\chi}$  be a random variable with cumulative distribution function*

$$\mathbb{P}(X_{\chi} \leq t) = \prod_{k=0}^{\infty} (1 - t^{-2k-2\chi})$$

and let  $\varepsilon_{\chi} > 0$  denote the unique solution to

$$e^{\chi \varepsilon_{\chi}} \varepsilon_{\chi} = 1.$$

Then, as  $\chi \rightarrow \infty$ , we have

$$2\chi(X_{\chi} - 1 - \varepsilon_{\chi}/2) \rightarrow G$$

where  $G$  has a standard Gumbel distribution, i.e.

$$\mathbb{P}(G \leq a) = e^{-e^{-a}}$$

for every  $a \in \mathbb{R}$ .

*Proof.* Define  $b_{\chi}$  by

$$2b_{\chi} = \varepsilon_{\chi} + a/\chi.$$

We have to prove that

$$\lim_{\chi \rightarrow \infty} \prod_{k=0}^{\infty} (1 - (1 + b_{\chi})^{-2k-2\chi}) = e^{-e^{-a}}.$$

We proceed as in the proof of Theorem 2.5. We notice that, as  $\chi \rightarrow \infty$ ,

$$\log \prod_{k=0}^{\infty} (1 - (1 + b_{\chi})^{-2k-2\chi}) \sim - \sum_{k=0}^{\infty} (1 + b_{\chi})^{-2k-2\chi}$$

which is due to the fact that  $(1 + b_{\chi})^x \rightarrow \infty$  and  $\log(1 - x) \sim -x + o(x)$ . Then

$$\sum_{k=0}^{\infty} (1 + b_{\chi})^{-2k-2\chi} = \frac{(1 + b_{\chi})^{-2\chi}}{1 - (1 + b_{\chi})^{-2}} \sim \frac{e^{-2\chi b_{\chi} + \chi O(b_{\chi})^2}}{2b_{\chi}} \sim \frac{e^{-\chi \varepsilon_{\chi}}}{\varepsilon_{\chi}} e^{-a} = e^{-a}$$

which concludes the proof. □





## Chapter 6

# A large deviation principle for empirical measures

*This chapter corresponds to [51].*

**Abstract.** We prove a large deviation principle for a sequence of point processes defined by Gibbs probability measures on a Polish space. This is obtained as a consequence of a more general Laplace principle for the non-normalized Gibbs measures. We consider four main applications: Conditional Gibbs measures on compact spaces, Coulomb gases on compact Riemannian manifolds, the usual Gibbs measures in the Euclidean space and the zeros of Gaussian random polynomials. Finally, we study the generalization of Fekete points and prove a deterministic version of the Laplace principle known as  $\Gamma$ -convergence. The approach is partly inspired by the works of Dupuis and co-authors. It is remarkably natural and general compared to the usual strategies for singular Gibbs measures.

**Résumé.** Nous montrons un principe de grandes déviations pour une suite de processus ponctuels définis par des mesures de probabilité de Gibbs dans un espace polonais. Ce principe est obtenu comme conséquence d'un principe de Laplace pour des mesures de Gibbs non normalisées. Nous considérons quatre applications: des mesures de Gibbs conditionnées dans des espaces compacts, des gaz de Coulomb sur des variétés riemanniennes compactes, les mesures de Gibbs habituelles sur l'espace euclidien et les zéros des polynômes aléatoires gaussiens. Finalement, nous étudions la généralisation des points Fekete et nous démontrons une version déterministe du principe de Laplace appelée  $\Gamma$ -convergence. Notre approche est partiellement inspirée par les travaux de Dupuis et ses coauteurs. Cette approche est remarquablement naturelle et générale par rapport aux stratégies habituelles pour des mesures de Gibbs singulières.

*2010 MSC:* 60F10; 60K35; 82C22; 30C15

*Keywords:* Gibbs measure; Coulomb gas; empirical measure; large deviation principle; interacting particle system; singular potential; constant curvature; relative entropy; random polynomial; Fekete points.

*Mots-clés:* Mesure de Gibbs; gaz de Coulomb; mesure empirique; principe des grandes déviations; système de particules en interaction; potentiel singulière; courbure constante; entropie relative; polynôme aléatoire; points de Fekete.

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## 1 Introduction

The present article is inspired by part of the work of Dupuis, Laschos and Ramanan on large deviations for a sequence of point processes given by Gibbs measures associated to very general singular two-body interactions [48] but it differs from it in that we take a general sequence of interactions that includes, for instance, the interaction followed by the zeros of random polynomials as in [98]. We follow the philosophy of Dupuis and Ellis [47] about the use of variational formulas to make plausible and sometimes easier to find a Laplace principle. This

philosophy has already been used by Georgii in [54] to treat a system of random fields on  $\mathbb{Z}^d$  with interacting energies that converges uniformly to some limit functional.

We are interested in proving the Laplace principle and the large deviation principle for a very general sequence of energies in a not necessarily compact space. Part of our work has an overlap with the article of Berman [21] and it was developed independently. As in [21] the interest of this result is the generality of the sequence of energies: they do not need to be made of a two-body interaction potential but they may still be very singular. The key argument of the proof is a well-understood application of Jensen's inequality together with a general Laplace principle that has as its main ingredient a subadditivity property of the entropy. It is very simple compared to the ad hoc methods used in the usual proofs of the large deviation principles for Coulomb gases such as in [61], [60] [35] and [58]. In these methods, to prove a large deviation lower bound, the authors usually decompose the space in small regions and this decomposition may not be easy to achieve on a manifold and not so natural to look for. We give a more precise explanation of these methods in Remark 3.6.

Among the applications we can give we are particularly interested in explaining a simple case inspired by [24]. This is the case of a Coulomb gas on a two-dimensional Riemannian manifold. As a second application we study a large deviation principle for a conditional Gibbs measure, i.e. we fix the position of some of the particles and leave the rest of them random. The last applications we discuss are different proofs of already known results such as the special one-dimensional log-gas of [16] related to the Gaussian ensembles, the more general one-dimensional log-gas of [6, Section 2.6], the special two-dimensional log-gas [60] related to the Ginibre ensemble of random matrices and its generalization to an  $n$ -dimensional Coulomb gas in [35] and [48], the note in [58] about two-dimensional log-gases with a weakly confining potential and the Gaussian random polynomials of [98] and [30].

We now explain the contents of each section. The rest of Section 1 will be dedicated to the main definitions and assumptions we will need to state our results. Section 2 is about the usual mean-field case, the  $k$ -body interaction. We give sufficient conditions to be able to apply our result which will become important when we treat the Euclidean space case. In Section 3 we begin by giving an idea of the proofs which includes mainly a key variational formula. Then we give the proofs of the main theorem and of its corollary and we finish the section by giving some remarks about the usual proofs we may find in the literature. We discuss four particular examples in Section 4. More precisely, the conditional Gibbs measure, the Coulomb gas on a Riemannian manifold, a new way to obtain already known results in the Euclidean space about Coulomb gases and the assertion that the zeros of a Gaussian random polynomials may be treated by our main theorem. We conclude our article with Section 5 discussing a deterministic case which falls under the topic of Fekete points and which we consider as the natural deterministic analogue of the Laplace principle.

## 1.1 Model

Let  $M$  be a Polish space, i.e. a separable topological space metrizable by a complete metric. Endow it with the Borel  $\sigma$ -algebra associated to this topology, i.e. the least  $\sigma$ -algebra that contains the topology. Denote by  $\mathcal{P}(M)$  the space of probability measures in  $M$  and endow it with the smallest topology such that  $\mu \mapsto \int_M f d\mu$  is continuous for every bounded continuous function  $f : M \rightarrow \mathbb{R}$ . With this topology,  $\mathcal{P}(M)$  is also a Polish space (see [29, Section 2.4]). This is called the weak topology. Suppose we have a sequence  $\{W_n\}_{n \in \mathbb{N}}$  of symmetric measurable functions

$$W_n : M^n \rightarrow (-\infty, \infty]$$

and a sequence of non-negative numbers  $\{\beta_n\}_{n \in \mathbb{N}}$  that converges to some  $\beta \in (0, \infty]$ . Fix a probability measure  $\pi \in \mathcal{P}(M)$ . We shall be interested in the asymptotic behavior of the

**Gibbs measures**  $\gamma_n$  defined by

$$d\gamma_n = e^{-n\beta_n W_n} d\pi^{\otimes n}. \quad (1)$$

Define  $\tilde{W}_n : \mathcal{P}(M) \rightarrow (-\infty, \infty]$  by

$$\tilde{W}_n(\mu) = \begin{cases} W_n(x_1, \dots, x_n) & \text{if } \mu \text{ is atomic with } \mu = \frac{1}{n} \sum_{i=1}^n \delta_{x_i} \\ \infty & \text{otherwise.} \end{cases} \quad (2)$$

**Stable sequence (S).** We shall say that the sequence  $\{W_n\}_{n \in \mathbb{N}}$  is a **stable sequence** if it is uniformly bounded from below, i.e. if there exists  $C \in \mathbb{R}$  such that

$$W_n \geq C \text{ for all } n \in \mathbb{N}.$$

**Confining sequence (C).** We shall say that  $\{W_n\}_{n \in \mathbb{N}}$  is a **confining sequence** if the following is true. Let  $\{n_j\}_{j \in \mathbb{N}}$  be any increasing sequence of natural numbers and let  $\{\mu_j\}_{j \in \mathbb{N}}$  be any sequence of probability measures on  $M$ . If there exists a real constant  $A$  such that

$$\tilde{W}_{n_j}(\mu_j) \leq A$$

for every  $j \in \mathbb{N}$ , where  $\tilde{W}_n$  is defined in (2), then  $\{\mu_j\}_{j \in \mathbb{N}}$  is relatively compact in  $\mathcal{P}(M)$ .

In order to study the behavior as  $n \rightarrow \infty$  of  $\gamma_n$  we shall need a measurable function

$$W : \mathcal{P}(M) \rightarrow (-\infty, \infty].$$

**Definition 1.1** (Macroscopic limit). Suppose that  $\{W_n\}_{n \in \mathbb{N}}$  is a stable sequence (S). We say that a measurable function  $W : \mathcal{P}(M) \rightarrow (-\infty, \infty]$  is the **positive temperature macroscopic limit** of the sequence  $\{W_n\}_{n \in \mathbb{N}}$  if the following two conditions are satisfied.

- **Lower limit assumption (A1).** For every sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  of probability measures on  $M$  that converges to some probability measure  $\mu$  we have

$$\liminf_{n \rightarrow \infty} \tilde{W}_n(\mu_n) \geq W(\mu)$$

where  $\tilde{W}_n$  is defined in (2).

- **Upper limit assumption (A2).** For each  $\mu \in \mathcal{P}(M)$  we have that

$$\limsup_{n \rightarrow \infty} \mathbb{E}_{\mu^{\otimes n}}[W_n] \leq W(\mu).$$

We say that  $W$  is the **zero temperature macroscopic limit** of the sequence  $\{W_n\}_{n \in \mathbb{N}}$  if instead the lower limit assumption (A1) and the following condition are satisfied.

- **Regularity assumption (A2').** Define the set of ‘nice’ probability measures

$$\mathcal{N} = \left\{ \mu \in \mathcal{P}(M) : D(\mu \parallel \pi) < \infty \text{ and } \limsup_{n \rightarrow \infty} \mathbb{E}_{\mu^{\otimes n}}[W_n] \leq W(\mu) \right\}. \quad (3)$$

For every  $\mu \in \mathcal{P}(M)$  such that  $W(\mu) < \infty$  we can find a sequence of probability measures  $\{\mu_n\}_{n \in \mathbb{N}}$  in  $\mathcal{N}$  such that  $\mu_n \rightarrow \mu$  and  $\limsup_{n \rightarrow \infty} W(\mu_n) \leq W(\mu)$ .

Now we are ready to state the Laplace principles and the large deviation principles.

## 1.2 Main results

Let  $i_n : M^n \rightarrow \mathcal{P}(M)$  be the application defined by

$$i_n(x_1, \dots, x_n) \mapsto \frac{1}{n} \sum_{i=1}^n \delta_{x_i}, \quad (4)$$

the usual continuous ‘inclusion’ of  $M^n$  in  $\mathcal{P}(M)$ . Define the **free energy** with parameter  $\beta$  as

$$F = W + \frac{1}{\beta} D(\cdot \| \pi), \quad (5)$$

(we suppose  $0 \times \infty = 0$ ) where  $D(\mu \| \nu)$  denotes the relative entropy of  $\mu$  with respect to  $\nu$ , also known as the Kullback–Leibler divergence i.e.

$$D(\mu \| \nu) = \int_M \frac{d\mu}{d\nu} \log \left( \frac{d\mu}{d\nu} \right) d\nu \quad (6)$$

if  $\mu$  is absolutely continuous with respect to  $\nu$  and  $D(\mu \| \nu) = \infty$  otherwise.

**Theorem 1.2** (Laplace principle). *Let  $\{W_n\}_{n \in \mathbb{N}}$  be a stable sequence (S) and take a measurable function  $W : \mathcal{P}(M) \rightarrow (-\infty, \infty]$ . Take a sequence of positive numbers  $\{\beta_n\}_{n \in \mathbb{N}}$  that converges to some  $\beta \in (0, \infty]$ .*

*If  $\beta < \infty$  suppose that  $W$  is the positive temperature macroscopic limit of  $\{W_n\}_{n \in \mathbb{N}}$ .*

*If  $\beta = \infty$  suppose that  $W$  is the zero temperature macroscopic limit of  $\{W_n\}_{n \in \mathbb{N}}$  and suppose that  $\{W_n\}_{n \in \mathbb{N}}$  is a confining sequence (C).*

*Define the Gibbs measures  $\gamma_n$  by (1) and the free energy  $F$  by (5). Then, the following Laplace’s principle is satisfied.*

*For every bounded continuous function  $f : \mathcal{P}(M) \rightarrow \mathbb{R}$*

$$\frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f \circ i_n} d\gamma_n \xrightarrow{n \rightarrow \infty} - \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + F(\mu)\}.$$

This Laplace principle implies the following large deviation principle.

**Corollary 1.3** (Large deviation principle). *Suppose the same conditions as in Theorem 1.2. Define  $Z_n = \gamma_n(M^n)$ . Suppose  $Z_n > 0$  for every  $n$  and notice that, since  $W_n$  is bounded from below,  $Z_n < \infty$ . Take the sequence of probability measures  $\{\mathbb{P}_n\}_{n \in \mathbb{N}}$  defined by*

$$d\mathbb{P}_n = \frac{1}{Z_n} d\gamma_n. \quad (7)$$

*For each  $n \in \mathbb{N}$ , let  $i_n(\mathbb{P}_n)$  be the pushforward measure of  $\mathbb{P}_n$  by  $i_n$ . Then the sequence  $\{i_n(\mathbb{P}_n)\}_{n \in \mathbb{N}}$  satisfies a large deviation principle with speed  $n\beta_n$  and with rate function*

$$I = F - \inf F,$$

*i.e. for every open set  $A \subset \mathcal{P}(M)$  we have*

$$\liminf_{n \rightarrow \infty} \frac{1}{n\beta_n} \log \mathbb{P}_n(i_n^{-1}(A)) \geq - \inf_{\mu \in A} I(\mu)$$

*and for every closed set  $C \subset \mathcal{P}(M)$  we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n\beta_n} \log \mathbb{P}_n(i_n^{-1}(C)) \leq - \inf_{\mu \in C} I(\mu).$$

In the next section, Section 2, we shall study the usual case of  $k$ -body interaction. Section 4 will be about some more specific examples, such as the conditional Gibbs measure, the Coulomb gas on a compact Riemannian manifold, the usual Gibbs measures on a noncompact space such as the Euclidean space and the Gaussian random polynomials.

## 2 Example of a stable sequence: k-body interaction

We will give the most basic non-trivial example of a *stable sequence* (S). Take an integer  $k > 0$  and a symmetric lower semicontinuous function bounded from below  $G : M^k \rightarrow (-\infty, \infty]$ . Define the symmetric measurable functions  $W_n : M^n \rightarrow (-\infty, \infty]$  by

$$W_n(x_1, \dots, x_n) = \frac{1}{n^k} \sum_{\substack{\{i_1, \dots, i_k\} \subset \{1, \dots, n\} \\ \#\{i_1, \dots, i_k\} = k}} G(x_{i_1}, \dots, x_{i_k}).$$

and  $W : \mathcal{P}(M) \rightarrow (-\infty, \infty]$  by

$$W(\mu) = \frac{1}{k!} \int_{M^k} G(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k).$$

**Proposition 2.1** (Stability, lower and upper limit assumption, (A1) and (A2)).  $\{W_n\}_{n \in \mathbb{N}}$  is a *stable sequence* (S),  $W$  is lower semicontinuous and the pair  $(\{W_n\}_{n \in \mathbb{N}}, W)$  satisfies the *lower and upper limit assumption*, (A1) and (A2).

*Proof.* To see that  $\{W_n\}_{n \in \mathbb{N}}$  is a *stable sequence* (S) we notice that if  $C \leq G$  then  $\binom{n}{k} \frac{1}{n^k} C \leq W_n$ . The lower semicontinuity of  $W$  is a consequence of the lower semicontinuity of  $G$  and the fact that it is bounded from below. Now, let us prove that  $(\{W_n\}_{n \in \mathbb{N}}, W)$  satisfies the *lower and upper limit assumption*, (A1) and (A2).

• **Lower limit assumption (A1).** Let  $\mu \in \mathcal{P}(M)$ . Take  $N > 0$  and define  $G_N = G \wedge N$ . We will prove that

$$\tilde{W}_n(\mu) + \frac{N}{k! n^k} \left( n^k - \frac{n!}{(n-k)!} \right) \geq \frac{1}{k!} \int_{M^k} G_N(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k) \quad (8)$$

where  $\tilde{W}_n$  is the extension defined in (2). If  $\tilde{W}_n(\mu) = \infty$  there is nothing to prove. If  $\tilde{W}_n(\mu) < \infty$  then  $\mu = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$  for some  $(x_1, \dots, x_n) \in M^n$ . We have

$$\frac{1}{n^k} \sum_{\substack{\{i_1, \dots, i_k\} \subset \{1, \dots, n\} \\ \#\{i_1, \dots, i_k\} = k}} G_N(x_{i_1}, \dots, x_{i_k}) + \frac{N}{k! n^k} \left( n^k - \frac{n!}{(n-k)!} \right) \geq \frac{1}{k!} \int_{M^k} G_N(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k),$$

which due to the fact that  $G \geq G_N$  implies the inequality (8).

Let  $\mu_n \rightarrow \mu \in \mathcal{P}(M)$ . Then, using the inequality (8) and taking the lower limit we get

$$\liminf_{n \rightarrow \infty} \tilde{W}_n(\mu_n) \geq \frac{1}{k!} \int_{M^k} G_N(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k),$$

where we have used that  $G_N$  is lower semicontinuous and bounded from below. Finally, since  $G$  is bounded from below we can take  $N$  to infinity and use the monotone convergence theorem to get

$$\liminf_{n \rightarrow \infty} \tilde{W}_n(\mu_n) \geq \frac{1}{k!} \int_{M^k} G(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k).$$

• **Upper limit assumption (A2).** For this it is enough to take  $\mu \in \mathcal{P}(M)$  and notice that

$$\mathbb{E}_{\mu^{\otimes n}} [W_n] = \frac{1}{n^k} \binom{n}{k} \int_{M^k} G(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k).$$

□

Now we give a sufficient condition for a  $k$ -body interaction to be a *confining sequence* (C).

**Proposition 2.2** (*k*-body interaction and confining assumption). *Suppose  $G(x_1, \dots, x_k)$  tends to infinity when  $x_i \rightarrow \infty$  for all  $i \in \{1, \dots, k\}$ , i.e. suppose that for every  $C \in \mathbb{R}$  there exists a compact set  $K$  such that  $G|_{K^c \times \dots \times K^c} \geq C$ . Then  $\{W_n\}_{n \in \mathbb{N}}$  is a confining sequence (C).*

*Proof.* Without loss of generality we can suppose  $G$  positive. Remember the definition of  $\tilde{W}_n$  in (2). All we need is the following result.

**Lemma 2.3** (Bound on the number of particles outside a compact set). *Suppose that  $G$  is positive. Take  $n \in \mathbb{N}$ ,  $A \in \mathbb{R}$  and  $\mu \in \mathcal{P}(M)$  that satisfies*

$$\tilde{W}_n(\mu) \leq A.$$

*If  $K$  is a compact set such that  $G|_{K^c \times \dots \times K^c} \geq C$  with  $C > 0$ , then*

$$\mu(K^c) \leq \left(\frac{A}{C} k!\right)^{1/k} + \frac{k}{n}.$$

*Proof.* We first notice that  $\mu = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$  for some  $(x_1, \dots, x_n) \in M^n$ . By the hypotheses we can see that

$$\frac{1}{n^k} [\text{number of } k\text{-combinations outside } K] C \leq \frac{1}{n^k} \sum_{\substack{\{i_1, \dots, i_k\} \subset \{1, \dots, n\} \\ \#\{i_1, \dots, i_k\} = k}} G(x_{i_1}, \dots, x_{i_k}) \leq A. \quad (9)$$

where, more precisely, [number of  $k$ -combinations outside  $K$ ] denotes the cardinal of the following set,  $\{S \subset \{1, \dots, n\} : \#S = k \text{ and } \forall i \in S, x_i \notin K\}$ . But, if  $m$  denotes the number of points among  $x_1, \dots, x_n$  outside  $K$  and if  $k \leq m$ , we have

$$\frac{(m-k)^k}{k!} \leq \frac{m!}{(m-k)! k!} = [\text{number of } k\text{-combinations outside } K]$$

which, along with the inequality (9), implies

$$\frac{m}{n} \leq \left(\frac{A}{C} k!\right)^{1/k} + \frac{k}{n}.$$

Since  $\mu = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$  then  $\mu(K^c) = \frac{m}{n}$  which concludes the proof. □

Then we can conclude using Prokhorov's theorem and the fact that every single probability measure is tight. □

Finally we notice that in the *regularity assumption* (A2') we can replace finite entropy by absolute continuity with respect to  $\pi$ .

**Proposition 2.4** (*k*-body interaction and regularity assumption). *Let*

$$\mathcal{N}_1 = \{\mu \in \mathcal{P}(M) : D(\mu||\pi) < \infty\} \quad \text{and}$$

$$\mathcal{N}_2 = \{\mu \in \mathcal{P}(M) : \mu \text{ is absolutely continuous with respect to } \pi\}.$$

*Suppose that for every  $\mu$  with  $W(\mu) < \infty$ , there exists a sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  in  $\mathcal{N}_2$  such that  $\mu_n \rightarrow \mu$  and  $W(\mu_n) \rightarrow W(\mu)$  then the same is true if we replace  $\mathcal{N}_2$  by  $\mathcal{N}_1$ .*

*Proof.* It is enough to prove that for every  $\mu \in \mathcal{N}_2$  there exists a sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  in  $\mathcal{N}_1$  such that  $\mu_n \rightarrow \mu$  and  $W(\mu_n) \rightarrow W(\mu)$ . Let  $\rho$  be the density of  $\mu$  with respect to  $\pi$ , i.e.  $d\mu = \rho d\pi$ . For each  $n > 0$  define  $\mu_n \in \mathcal{N}_1$  by  $d\mu_n = \frac{\rho \wedge n}{\int_M \rho \wedge n d\pi}$ . Then, by the monotone convergence theorem we can see that  $\mu_n \rightarrow \mu$ . And, again, by the monotone convergence theorem, by supposing  $G \geq 0$ , we can see that  $W(\mu_n) \rightarrow W(\mu)$ . □

### 3 Proof of the theorem

This section is dedicated to the proof of the main theorem, i.e. Theorem 1.2. We start giving a sketch of the proof.

#### 3.1 Idea of the proof

We shall use the following very known result that tells us the Legendre transform of  $D(\cdot\|\mu)$ , defined in (6). See [47, Proposition 4.5.1] for a proof.

**Lemma 3.1** (Legendre transform of the entropy). *Let  $E$  be a Polish probability space,  $\mu$  a probability measure on  $E$  and  $g : E \rightarrow (-\infty, \infty]$  a measurable function bounded from below. Then*

$$\log \mathbb{E}_\mu [e^{-g}] = - \inf_{\tau \in \mathcal{P}(E)} \{ \mathbb{E}_\tau [g] + D(\tau\|\mu) \}.$$

Remember the definition of  $\gamma_n$  in (1) and  $F$  in (5). With the help of Lemma 3.1 we can write

$$\begin{aligned} \frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f \circ i_n} d\gamma_n &= \frac{1}{n\beta_n} \log \mathbb{E}_{\pi^{\otimes n}} [e^{-n\beta_n (f \circ i_n + W_n)}] \\ &= - \inf_{\tau \in \mathcal{P}(M^n)} \left\{ \mathbb{E}_{i_n(\tau)} [f] + \mathbb{E}_\tau [W_n] + \frac{1}{n\beta_n} D(\tau\|\pi^{\otimes n}) \right\}. \end{aligned}$$

where  $i_n(\tau)$  denotes the pushforward measure of  $\tau$  by  $i_n$ . So, we need to prove that

$$\inf_{\tau \in \mathcal{P}(M^n)} \left\{ \mathbb{E}_{i_n(\tau)} [f] + \mathbb{E}_\tau [W_n] + \frac{1}{n\beta_n} D(\tau\|\pi^{\otimes n}) \right\} \xrightarrow{n \rightarrow \infty} \inf_{\mu \in \mathcal{P}(M)} \{ f(\mu) + F(\mu) \}. \quad (10)$$

#### 3.2 Proof of Theorem 1.2: Case of finite $\beta$

In this subsection we shall prove the Laplace principle Theorem 1.2 and the large deviation principle Corollary 1.3 for the case of finite  $\beta$ .

To prove this we need the following properties of the entropy. The first one is analogous to the *lower limit assumption* (A1).

**Lemma 3.2** (Lower limit property of the entropy). *Let  $\{n_j\}_{j \in \mathbb{N}}$  be an increasing sequence in  $\mathbb{N}$ . For each  $j \in \mathbb{N}$  take  $\tau_j \in \mathcal{P}(M^{n_j})$ . If  $i_{n_j}(\tau_j) \rightarrow \zeta \in \mathcal{P}(\mathcal{P}(M))$ , then*

$$\mathbb{E}_\zeta [D(\cdot\|\pi)] \leq \liminf_{j \rightarrow \infty} \frac{1}{n_j} D(\tau_j\|\pi^{\otimes n_j}).$$

*Proof.* The idea of the proof is presented in [47]. We can also see [50]. It can be seen as equivalent to the large deviation upper bound of Sanov's theorem thanks to [77, Theorem 3.5].  $\square$

And the second one is analogous to the notion of *confining sequence* (C).

**Lemma 3.3** (Confining property of the entropy). *Let  $\{n_j\}_{j \in \mathbb{N}}$  be an increasing sequence in  $\mathbb{N}$ . For each  $j \in \mathbb{N}$  take  $\tau_j \in \mathcal{P}(M^{n_j})$ . If there exists a real constant  $C$  such that*

$$\frac{1}{n_j} D(\tau_j\|\pi^{\otimes n_j}) \leq C$$

*for every  $j \in \mathbb{N}$ , then the sequence  $\{i_{n_j}(\tau_j)\}_{j \in \mathbb{N}}$  is tight.*

*Proof.* The idea of the proof is presented in [47]. We can also see [50]. It can be seen as equivalent to the exponential tightness in Sanov's theorem thanks to [77, Theorem 3.3].  $\square$



Without loss of generality, we can suppose  $\beta_n = 1$  for every  $n$  by redefinition of  $W_n$  and  $W$ . Then the *Gibbs measure* (1) and the *free energy* (5) are

$$d\gamma_n = e^{-nW_n} d\pi^{\otimes n} \quad \text{and} \quad F = W + D(\cdot|\pi).$$

As explained in Subsection 3.1 we need to prove (10) which in this case is

$$\inf_{\tau \in \mathcal{P}(M^n)} \left\{ \mathbb{E}_{i_n(\tau)} [f] + \mathbb{E}_\tau [W_n] + \frac{1}{n} D(\tau|\pi^{\otimes n}) \right\} \xrightarrow{n \rightarrow \infty} \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + F(\mu)\}.$$

*Proof of Theorem 1.2: Case of finite  $\beta$ .* First, we will prove the **lower limit bound**

$$\liminf_{n \rightarrow \infty} \inf_{\tau \in \mathcal{P}(M^n)} \left\{ \mathbb{E}_{i_n(\tau)} [f] + \mathbb{E}_\tau [W_n] + \frac{1}{n} D(\tau|\pi^{\otimes n}) \right\} \geq \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + F(\mu)\}. \quad (11)$$

This is equivalent to say that for every increasing sequence of natural numbers  $\{n_j\}_{j \in \mathbb{N}}$  if we choose, for each  $j \in \mathbb{N}$ , a probability measure  $\tau_j \in \mathcal{P}(M^{n_j})$  we have

$$\lim_{j \rightarrow \infty} \left\{ \mathbb{E}_{i_{n_j}(\tau_j)} [f] + \mathbb{E}_{\tau_j} [W_{n_j}] + \frac{1}{n_j} D(\tau_j|\pi^{\otimes n_j}) \right\} \geq \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + F(\mu)\}. \quad (12)$$

where we can suppose that the limit exists and that it is finite and, in particular, the sequence is bounded from above.

Using that  $\{W_n\}_{n \in \mathbb{N}}$  is a *stable sequence* (S), we get that  $\frac{1}{n_j} D(\tau_j|\pi^{\otimes n_j})$  is uniformly bounded from above. By the *confining property of the entropy*, Lemma 3.3, we get that  $i_{n_j}(\tau_j)$  is tight. By taking a subsequence using Prokhorov's theorem, we shall assume it converges to some  $\zeta \in \mathcal{P}(\mathcal{P}(M))$ . Then, by the *lower limit property of the entropy*, Lemma 3.2, we get

$$\mathbb{E}_\zeta [D(\cdot|\pi)] \leq \liminf_{j \rightarrow \infty} \frac{1}{n_j} D(\tau_j|\pi^{\otimes n_j}).$$

Since  $\tilde{W}_n$  is measurable for every  $n$  (see [50, Proposition 7.6] for a proof) and the sequence  $\{\tilde{W}_n\}_{n \in \mathbb{N}}$  is uniformly bounded from below we may use the *lower limit assumption* (A1) to get (see [77, Proposition 3.2])

$$\mathbb{E}_\zeta [W] \leq \liminf_{j \rightarrow \infty} \mathbb{E}_{\tau_j} [W_{n_j}].$$

Then, by taking the lower limit when  $j$  tends to infinity in (12), we obtain

$$\lim_{j \rightarrow \infty} \left\{ \mathbb{E}_{i_{n_j}(\tau_j)} [f] + \mathbb{E}_{\tau_j} [W_{n_j}] + \frac{1}{n_j} D(\tau_j|\pi^{\otimes n_j}) \right\} \geq \mathbb{E}_\zeta [f + W + D(\cdot|\pi)] \geq \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + F(\mu)\}.$$

Now let us prove the **upper limit bound**

$$\limsup_{n \rightarrow \infty} \inf_{\tau \in \mathcal{P}(M^n)} \left\{ \mathbb{E}_{i_n(\tau)} [f] + \mathbb{E}_\tau [W_n] + \frac{1}{n} D(\tau|\pi^{\otimes n}) \right\} \leq \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + F(\mu)\}. \quad (13)$$

We need to prove that for every probability measure  $\mu \in \mathcal{P}(M)$

$$\limsup_{n \rightarrow \infty} \inf_{\tau \in \mathcal{P}(M^n)} \left\{ \mathbb{E}_{i_n(\tau)} [f] + \mathbb{E}_\tau [W_n] + \frac{1}{n} D(\tau|\pi^{\otimes n}) \right\} \leq f(\mu) + F(\mu).$$

It is enough to find a sequence  $\tau_n \in \mathcal{P}(M^n)$  such that

$$\limsup_{n \rightarrow \infty} \left\{ \mathbb{E}_{i_n(\tau_n)} [f] + \mathbb{E}_{\tau_n} [W_n] + \frac{1}{n} D(\tau_n|\pi^{\otimes n}) \right\} \leq f(\mu) + F(\mu).$$

We shall choose  $\tau_n = \mu^{\otimes n}$ . Then we know that, by the law of large numbers, we have the weak convergence  $i_n(\tau_n) \rightarrow \delta_\mu$ , so

$$\lim_{n \rightarrow \infty} \mathbb{E}_{i_n(\tau_n)} [f] = f(\mu).$$

In addition, by using that  $D(\tau_n \| \pi^{\otimes n}) = n D(\mu \| \pi)$  and the *upper limit assumption* (A2) we get that

$$\limsup_{n \rightarrow \infty} \left\{ \mathbb{E}_{i_n(\tau_n)} [f] + \mathbb{E}_{\tau_n} [W_n] + \frac{1}{n} D(\tau_n \| \pi^{\otimes n}) \right\} \leq f(\mu) + W(\mu) + D(\mu \| \pi)$$

completing the proof. □

### 3.3 Proof of Theorem 1.2: Case of infinite $\beta$

In this subsection we provide a proof for Theorem 1.2 for the case of infinite  $\beta$  by modifying the proof used in the case of finite  $\beta$ . Recall that from the definition of *Gibbs measure* (1) and *free energy* (5) now we have

$$d\gamma_n = e^{-n\beta_n W_n} d\pi^{\otimes n}, \quad \text{and} \quad F = W.$$

where  $\beta_n \rightarrow \infty$ .

We first notice that a *confining sequence* (C) satisfies an a priori stronger property.

**Proposition 3.4** (Confining property of the expected value of the energy). *Assume that  $\{W_n\}_{n \in \mathbb{N}}$  is a stable (S) and confining (C) sequence and take a sequence of probability measures  $\{\chi_j\}_{j \in \mathbb{N}}$  on  $\mathcal{P}(M)$ , i.e.  $\chi_j \in \mathcal{P}(\mathcal{P}(M))$ . Suppose there exists an increasing sequence  $\{n_j\}_{j \in \mathbb{N}}$  of natural numbers and a constant  $C < \infty$  such that  $\mathbb{E}_{\chi_j} [\tilde{W}_{n_j}] \leq C$  for every  $j \in \mathbb{N}$ . Then  $\{\chi_j\}_{j \in \mathbb{N}}$  is relatively compact in  $\mathcal{P}(\mathcal{P}(M))$ .*

*Proof.* The proof is left to the reader. See for instance [86, Lemma 2.1] for an idea or [50, Proposition 3.4] and [50, Proposition 7.4] for a full proof. □

Now we proceed with the proof of the theorem.

*Proof of Theorem 1.2: Case of infinite  $\beta$ .* Take  $f : \mathcal{P}(M) \rightarrow \mathbb{R}$  bounded continuous. By Subsection 3.1 about the idea of the proof we need to obtain (10). We start proving the **lower limit bound**

$$\liminf_{n \rightarrow \infty} \inf_{\tau \in \mathcal{P}(M^n)} \left\{ \mathbb{E}_{i_n(\tau)} [f] + \mathbb{E}_\tau [W_n] + \frac{1}{n\beta_n} D(\tau \| \pi^{\otimes n}) \right\} \geq \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + W(\mu)\}.$$

As in the proof used in the case of finite  $\beta$  we want to see that for every increasing sequence of natural numbers  $\{n_j\}_{j \in \mathbb{N}}$  and choosing for each  $j \in \mathbb{N}$  a probability measure  $\tau_j \in \mathcal{P}(M^{n_j})$  we have

$$\lim_{j \rightarrow \infty} \left\{ \mathbb{E}_{i_{n_j}(\tau_j)} [f] + \mathbb{E}_{\tau_j} [W_{n_j}] + \frac{1}{n_j \beta_{n_j}} D(\tau_j \| \pi^{\otimes n_j}) \right\} \geq \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + W(\mu)\}. \quad (14)$$

where we can suppose that the limit exists and it is finite. Since the entropy is non-negative we see that  $\mathbb{E}_{\tau_j} [W_{n_j}] = \mathbb{E}_{i_{n_j}(\tau_j)} [\tilde{W}_{n_j}]$  is a bounded sequence and, since  $\{W_n\}_{n \in \mathbb{N}}$  is a *confining sequence* (C), Proposition 3.4 tells us that  $i_{n_j}(\tau_j)$  is relatively compact in  $\mathcal{P}(\mathcal{P}(M))$ . We continue as in the proof used in the case of finite  $\beta$  where now  $W$  is bounded from below by the *regularity assumption* (A2') and because  $\{W_n\}_{n \in \mathbb{N}}$  is a *stable sequence* (S).

The proof of the **upper limit bound** follows the same reasoning as in the case of finite  $\beta$ . Take  $\mu \in \mathcal{P}(M)$ . Following the arguments used in the case of finite  $\beta$  we can prove that

$$\limsup_{n \rightarrow \infty} \inf_{\tau \in \mathcal{P}(M^n)} \left\{ \mathbb{E}_{i_n(\tau)} [f] + \mathbb{E}_\tau [W_n] + \frac{1}{n\beta_n} D(\tau \| \pi^{\otimes n}) \right\} \leq \inf_{\mu \in \mathcal{N}} \{f(\mu) + W(\mu)\} \quad (15)$$

where  $\mathcal{N}$  was defined in (3). By the *regularity assumption* (A2') we get

$$\inf_{\mu \in \mathcal{N}} \{f(\mu) + W(\mu)\} = \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + W(\mu)\}$$

completing the proof. □

### 3.4 Proof of Corollary 1.3

*Proof of Corollary 1.3.* We know that the large deviation principle is equivalent to the Laplace principle for the sequence  $i_n(\mathbb{P}_n)$  if the rate function has compact level sets (see [47, Theorem 1.2.1] and [47, Theorem 1.2.3])

If  $\beta < \infty$  this is the case because the entropy has compact level sets (see [47, Lemma 1.4.3 (c)]) and  $W$  is a lower semicontinuous function bounded from below. The lower semicontinuity of  $W$  is a consequence of the *lower and upper limit assumption*, (A1) and (A2).

If  $\beta$  is infinite then there is no entropy term and we can use that  $\{W_n\}_{n \in \mathbb{N}}$  is a *confining sequence* (C), and that  $(\{W_n\}_{n \in \mathbb{N}}, W)$  satisfies the *lower limit assumption* (A1) and the *regularity assumption* (A2') to prove that  $W$  has compact level sets.

Then we have to prove that, for every bounded continuous function  $f : \mathcal{P}(M) \rightarrow \mathbb{R}$ ,

$$\frac{1}{n\beta_n} \log \mathbb{E}_{i_n(\mathbb{P}_n)} \left[ e^{-n\beta_n f} \right] \xrightarrow{n \rightarrow \infty} - \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + F(\mu) - \inf F\}$$

or, using the measures  $\gamma_n$ ,

$$\frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f \circ i_n} \frac{d\gamma_n}{Z_n} \xrightarrow{n \rightarrow \infty} - \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + F(\mu) - \inf F\}.$$

This we can achieve by using Theorem 1.2 twice, for  $f$  and for the zero function. □

*Remark 3.5* (Other proof in the case of finite  $\beta$ ). When treating the case  $\beta < \infty$ , the proof we are aware of is [27]. It uses a “quasi-continuity” of the energy and it seems somewhat specific to the logarithmic energy.

*Remark 3.6* (Other proofs in the case of infinite  $\beta$ ). The proofs that treat the case  $\beta = \infty$  usually follow closely the approach we used for the large deviation upper bound. For the large deviation lower bound they proceed as follows. If  $A$  is an open set of  $\mathcal{P}(M)$  and  $\mu \in A$ , they try to obtain

$$\liminf_{n \rightarrow \infty} \frac{1}{n\beta_n} \log \int_{i_n^{-1}(A)} e^{-n\beta_n W_n} d\pi^{\otimes n} \geq -W(\mu).$$

For this, they search pairwise disjoint sets  $B_1, \dots, B_n$  such that  $i_n(B_1 \times \dots \times B_n) \subset A$  and such that  $\max_{B_1 \times \dots \times B_n} W_n \simeq W(\mu)$ . Then we may write

$$\int_{i_n^{-1}(A)} e^{-n\beta_n W_n} d\pi^{\otimes n} \geq \sum_{\sigma \in \mathcal{S}_n} \int_{B_{\sigma(1)} \times \dots \times B_{\sigma(n)}} e^{-n\beta_n W_n} d\pi^{\otimes n} \geq n! \pi(B_1) \dots \pi(B_n) e^{-n\beta_n \max_{B_1 \times \dots \times B_n} W_n}.$$

If we are able to choose those sets such that  $\pi(B_i) \geq \frac{C}{n}$  for some  $C$  independent of  $n$  we can obtain, using Stirling's formula,

$$\liminf_{n \rightarrow \infty} \frac{1}{n\beta_n} \log(n! \pi(B_1) \dots \pi(B_n)) \geq \liminf_{n \rightarrow \infty} \frac{1}{n\beta_n} (\log n! - n \log n) \geq 0$$

and conclude by using that  $\lim_{n \rightarrow \infty} \max_{B_1 \times \dots \times B_n} W_n = W(\mu)$ .

## 4 Applications

In this section we shall give the main applications we are thinking of: Conditional Gibbs measure, a Coulomb gas on a Riemannian manifold, the known results of Coulomb gases in Euclidean space and the zeros of Gaussian random polynomials.

### 4.1 Conditional Gibbs measure

In this subsection we treat the case of the Gibbs measure associated to a two-body interaction but with some of the points conditioned to be deterministic. We proceed by considering the deterministic points as a background charge and treat the interaction with this background as some potential energy that depends on  $n$ . More precisely, we use the following more general setup.

Let  $\{\nu_n\}_{n \in \mathbb{N}}$  be a sequence of probability measures on a compact metric space  $M$  that converges to some probability measure  $\nu \in \mathcal{P}(M)$ . Suppose we have a lower semicontinuous function  $G^E : M \times M \rightarrow (-\infty, \infty]$  that shall be thought of as the interaction energy between the particles and the environment and a symmetric lower semicontinuous function  $G^I : M \times M \rightarrow (-\infty, \infty]$  that will be interpreted as the interaction energy between the particles. More precisely we define two kinds of energy.

*External potential energy.* The probability measure  $\nu_n$  will interact with the  $n$  particles via the external potential  $V_n : M \rightarrow \mathbb{R}$  defined by  $V_n(x) = \int_M G^E(x, y) d\nu_n(y)$ . This gives rise to the external energy  $W_n^E : M^n \rightarrow (-\infty, \infty]$

$$W_n^E(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n V_n(x_i)$$

with a macroscopic external energy  $W^E : \mathcal{P}(M) \rightarrow (-\infty, \infty]$

$$W^E(\mu) = \int_{M \times M} G^E(x, y) d\mu(x) d\nu(y).$$

*Internal potential energy.* For each  $n$  we shall think of  $n$  particles interacting with the two-particle potential  $G^I$ . This would give rise to an internal energy  $W_n^I : M^n \rightarrow (-\infty, \infty]$

$$W_n^I(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i < j}^n G^I(x_i, x_j)$$

and a macroscopic internal energy  $W^I : \mathcal{P}(M) \rightarrow (-\infty, \infty]$

$$W^I(\mu) = \frac{1}{2} \int_{M \times M} G^I(x, y) d\mu(x) d\mu(y).$$

*Total potential energy.* For each  $n$  we define

$$W_n = W_n^E + W_n^I \quad \text{and} \quad W = W^E + W^I.$$

Then, it is not hard to see that  $\{W_n\}_{n \in \mathbb{N}}$  is a *stable sequence* (S) and  $W$  is a lower semicontinuous function. The example of a conditional Gibbs measure can be obtained essentially by choosing as  $\nu_n$  the empirical measure of some points and  $G^I = G^E$ . So, a particular case of the next theorem is a Coulomb gas conditioned to all but an increasing number of points.

**Theorem 4.1** (Varying environment). *Suppose that  $x \mapsto \int_M G^E(x, y) d\nu(y)$  is continuous.*

Let

$$\tilde{\mathcal{N}} = \left\{ \mu \in \mathcal{P}(M) : D(\mu \| \pi) < \infty \text{ and } y \mapsto \int_M G^E(x, y) d\mu(x) \text{ is continuous} \right\}$$

and suppose that for every  $\mu \in \mathcal{P}(M)$  such that  $W^I(\mu) < \infty$  there exists a sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  of probability measures in  $\tilde{\mathcal{N}}$  such that  $\mu_n \rightarrow \mu$  and  $W^I(\mu_n) \rightarrow W^I(\mu)$ .

Then  $W$  is the zero temperature macroscopic limit of  $\{W_n\}_{n \in \mathbb{N}}$ . In particular, if we choose  $\beta_n \rightarrow \infty$ , Theorem 1.2 and Corollary 1.3 may be applied for  $(\{W_n\}_{n \in \mathbb{N}}, W)$ .

*Proof.* Let us prove the lower limit assumption (A1).

**Lower limit assumption (A1).** By Proposition 2.1, we already know that  $(\{W_n^I\}_{n \in \mathbb{N}}, W^I)$  satisfies the lower limit assumption (A1). We only need to check this for  $(\{W_n^E\}_{n \in \mathbb{N}}, W^E)$ .

If  $\mu_n = i_n(x_1, \dots, x_n)$  then

$$\tilde{W}_n^E(\mu_n) = \int_M V_n d\mu_n = \int_{M \times M} G^E(x, y) d\mu_n(x) d\nu_n(y),$$

where  $\tilde{W}_n^E$  is defined in (2). So, the lower limit assumption (A1) is a consequence of the lower semicontinuity of  $G^E$ .

**Regularity assumption (A2').** To prove the regularity assumption (A2') we take  $\mu \in \mathcal{P}(M)$  such that  $W(\mu) < \infty$ . Then  $W^I(\mu) < \infty$ . By hypothesis, we know that there exists a sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  of probability measures in  $\tilde{\mathcal{N}}$  such that  $\mu_n \rightarrow \mu$  and  $W^I(\mu_n) \rightarrow W^I(\mu)$ . Since  $x \mapsto \int_M G^E(x, y) d\nu(y)$  is continuous we also have that  $W^E(\mu_n) \rightarrow W^E(\mu)$ . So,  $W(\mu_n) \rightarrow W(\mu)$ .

We have to prove that the sequence we chose is in the set  $\mathcal{N}$  defined in (3) by

$$\mathcal{N} = \left\{ \mu \in \mathcal{P}(M) : D(\mu \parallel \pi) < \infty \text{ and } \limsup_{n \rightarrow \infty} \mathbb{E}_{\mu^{\otimes n}} [W_n] \leq W(\mu) \right\},$$

i.e. we need to see that  $\tilde{\mathcal{N}} \subset \mathcal{N}$ .

Let  $\mu \in \tilde{\mathcal{N}}$ . Then

$$\begin{aligned} \mathbb{E}_{\mu^{\otimes n}} [W_n^E] &= \mathbb{E}_\mu [V_n] \\ &= \int_M \left( \int_M G^E(x, y) d\nu_n(y) \right) d\mu(x) \\ &= \int_M \left( \int_M G^E(x, y) d\mu(x) \right) d\nu_n(y). \end{aligned}$$

So, since  $y \mapsto \int_M G^E(x, y) d\mu(x)$  is continuous, we get

$$\mathbb{E}_{\mu^{\otimes n}} [W_n^E] \xrightarrow{n \rightarrow \infty} \int_{M \times M} G^E(x, y) d\nu(y) d\mu(x) = W^E(\mu).$$

By Proposition 2.1 we already know that  $\lim_{n \rightarrow \infty} \mathbb{E}_{\mu^{\otimes n}} [W_n^I] = W^I(\mu)$  and then  $\mu \in \mathcal{N}$ . □

For the sake of completeness we treat the case of a Coulomb gas conditioned to all points but a finite fixed number of them. Again, by considering the deterministic points as a background charge we can use the following more general framework. Suppose we have two compact metric spaces  $M$  and  $N$ , a probability measure  $\Pi$  on  $N$  and two lower semicontinuous functions  $G^E : N \times M \rightarrow (-\infty, \infty]$  and  $G^I : N \rightarrow (-\infty, \infty]$ . Let  $\{\nu_n\}_{n \in \mathbb{N}}$  be a sequence of probability measures on  $M$  that converges to some probability measure  $\nu \in \mathcal{P}(M)$ . We will consider one particle in  $N$  interacting with the environment via  $G^E$ , i.e. via a potential energy  $V_n : N \rightarrow (-\infty, \infty]$  defined by  $V_n(x) = \int_M G^E(x, y) d\nu_n(y)$ . This particle will also have a self-interaction given by  $\lambda_n G^I$  where  $\{\lambda_n\}_{n \in \mathbb{N}}$  is a sequence that converges to zero. The case of a Coulomb gas conditioned to all but  $k$  particles may be obtained by essentially taking  $N = M^k$ ,  $\Pi = \pi^{\otimes k}$ ,  $G^I(x_1, \dots, x_k) = \sum_{i < j} G(x_i, x_j)$ ,  $G^E((x_1, \dots, x_k), y) = \sum_{i=1}^k G(x_i, y)$ ,  $\lambda_n = \frac{1}{n}$  and  $\nu_n$  as the empirical measure of the deterministic particles.

**Theorem 4.2** (A particle in a varying environment). *Suppose that  $V : N \rightarrow \mathbb{R}$  defined by  $V(x) = \int_M G^E(x, y) d\nu(y)$  is (bounded and) continuous. Let*

$$\tilde{\mathcal{N}} = \left\{ \mu \in \mathcal{P}(N) : D(\mu \parallel \Pi) < \infty, \int_N G^I d\mu < \infty \text{ and } y \mapsto \int_N G^E(x, y) d\mu(x) \text{ is continuous} \right\}$$

and suppose that for every  $z \in N$  there exists a sequence of probability measures  $\{\mu_n\}_{n \in \mathbb{N}}$  in  $\tilde{\mathcal{N}}$  such that  $\mu_n \rightarrow \delta_z$ . Take a sequence of non-negative numbers  $\{\beta_n\}_{n \in \mathbb{N}}$  such that  $\beta_n \rightarrow \infty$  and define the measures  $\gamma_n^c$  by

$$d\gamma_n^c = e^{-\beta_n(V_n + \lambda_n G^I)} d\Pi.$$

Then, we have the following Laplace principle. For every continuous function  $f : N \rightarrow \mathbb{R}$

$$\frac{1}{\beta_n} \log \int_N e^{-\beta_n f} d\gamma_n^c \xrightarrow{n \rightarrow \infty} - \inf_{x \in N} \{f(x) + V(x)\}.$$

*Proof.* We use Lemma 3.1 to write

$$\frac{1}{\beta_n} \log \int_N e^{-\beta_n f} d\gamma_n^c = - \inf_{\mu \in \mathcal{P}(N)} \left\{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V_n] + \lambda_n \mathbb{E}_\mu[G^I] + \frac{1}{\beta_n} D(\mu \parallel \Pi) \right\}.$$

Following the same ideas used in the proofs of Theorem 1.2 and Theorem 4.1 we get

$$\liminf_{n \rightarrow \infty} \inf_{\mu \in \mathcal{P}(N)} \left\{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V_n] + \lambda_n \mathbb{E}_\mu[G^I] + \frac{1}{\beta_n} D(\mu \parallel \Pi) \right\} \geq \inf_{\mu \in \mathcal{P}(N)} \{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V] \}$$

and

$$\limsup_{n \rightarrow \infty} \inf_{\mu \in \mathcal{P}(N)} \left\{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V_n] + \lambda_n \mathbb{E}_\mu[G^I] + \frac{1}{\beta_n} D(\mu \parallel \Pi) \right\} \leq \inf_{\mu \in \mathcal{N}} \{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V] \}.$$

We shall think of  $N$  as included in  $\mathcal{P}(N)$  by the application  $z \mapsto \delta_z$ . Then, by the continuity of  $V$  and  $f$  and since we are assuming that elements of  $N$  are approximated by elements of  $\tilde{\mathcal{N}}$  we know that

$$\inf_{\mu \in \tilde{\mathcal{N}}} \{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V] \} = \inf_{\mu \in \tilde{\mathcal{N}} \cup N} \{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V] \}.$$

Since the infimum is achieved in  $N$  we get

$$\inf_{\mu \in \tilde{\mathcal{N}}} \{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V] \} = \inf_{x \in N} \{ f(x) + V(x) \} = \inf_{\mu \in \mathcal{P}(N)} \{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V] \}$$

concluding the proof. □

### 4.2 A Coulomb gas on a Riemannian manifold

Let  $(M, g)$  be a compact oriented  $n$ -dimensional Riemannian manifold without boundary where  $g$  denotes the Riemannian metric. We shall define a continuous function  $G : M \times M \rightarrow (-\infty, \infty]$  naturally associated to the Riemannian structure of  $M$ . This function along with the normalized volume form  $\pi$  of  $(M, g)$  will allow us to define the Gibbs measures  $\gamma_n$  of (1) and will put us in the context of Theorem 1.2.

For this we establish some notation. A signed measure  $\Lambda$  will be called a differentiable signed measure if it is given by an  $n$ -form or equivalently if it has a differentiable density with respect to  $\pi$ . From now on we shall identify  $\Omega^n(M)$  with the space of differentiable signed measures. Denote by  $\Delta : C^\infty(M) \rightarrow \Omega^n(M)$  the Laplacian operator, i.e.  $\Delta = d * d$  where  $*$  is the Hodge star operator or, equivalently,  $\Delta f = \nabla^2 f d\pi$  where  $\nabla^2$  is the Laplace-Beltrami operator. The function  $G$  we will be interested in is given by the following result.

**Proposition 4.3** (Green function). *Take any differentiable signed measure  $\Lambda$ . Then, there exists a symmetric continuous function  $G : M \times M \rightarrow (-\infty, \infty]$  such that for every  $x \in M$  the function  $G_x : M \rightarrow (-\infty, \infty]$  defined by  $G_x(y) = G(x, y)$  is integrable with respect to  $\pi$  and*

$$\Delta G_x = -\delta_x + \Lambda.$$

More explicitly, the previous equality can be written as follows. For every  $f \in C^\infty(M)$  we have

$$\int_M G_x \Delta f = -f(x) + \int_M f d\Lambda.$$

Such a function will be called a Green function associated to  $\Lambda$ . Furthermore  $G$  is integrable with respect to  $\pi \otimes \pi$ . If  $\mu$  is a differentiable signed measure then  $\psi : M \rightarrow \mathbb{R}$  defined by  $\psi(x) = \int_M G(x, y) d\mu(y)$  belongs to  $C^\infty(M)$  and

$$\Delta \psi = -\mu + \mu(M)\Lambda.$$

In particular, we can get that  $G$  is bounded from below,  $\int_M G_x d\Lambda$  does not depend on  $x \in M$  and a Green function associated to  $\Lambda$  is unique up to an additive constant.

*Proof.* This result is well known if  $\Lambda = \pi$ . See for instance [10, Chapter 4]. Then if  $H$  is a Green function associated to  $\pi$  we define  $\phi \in C^\infty(M)$  by  $\phi(x) = \int_M H(x, y) d\Lambda(y)$  and the function  $G : M \times M \rightarrow (-\infty, \infty]$  given by  $G(x, y) = H(x, y) - \phi(x) - \phi(y)$  is a Green function associated to  $\Lambda$ . □

We fix a differentiable signed measure  $\Lambda$ . For simplicity we choose the Green function  $G$  associated to  $\Lambda$  that satisfies  $\int_M G_x d\Lambda = 0$  for every  $x \in M$ . Define the symmetric function  $W_n : M^n \rightarrow \mathbb{R} \cup \{\infty\}$  by  $W_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i < j}^n G(x_i, x_j)$  and define  $W : \mathcal{P}(M) \rightarrow (-\infty, \infty]$  by  $W(\mu) = \frac{1}{2} \int_{M \times M} G(x, y) d\mu(x) d\mu(y)$ . Because  $G$  is bounded from below and lower semicontinuous we may apply Proposition 2.1 about the k-body interaction. In particular, we obtain that  $\{W_n\}_{n \in \mathbb{N}}$  is a *stable sequence* (S),  $W$  is lower semicontinuous and  $(\{W_n\}_{n \in \mathbb{N}}, W)$  satisfies the *lower limit assumption* (A1) and the *upper limit assumption* (A2).

We can prove a strong form of the regularity assumption for  $W$ .

**Proposition 4.4** (Regularity property of the Green energy). *Let  $\mu \in \mathcal{P}(M)$ . There exists a sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  of differentiable probability measures such that  $\mu_n \rightarrow \mu$  and  $W(\mu_n) \rightarrow W(\mu)$ .*

*Proof.* We can assume  $W(\mu) < \infty$ , otherwise any sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  of differentiable probability measures such that  $\mu_n \rightarrow \mu$  will satisfy  $W(\mu_n) \rightarrow W(\mu)$  due to the lower semicontinuity of  $W$ .

Using the proof of [15, Lemma 3.13] for the case of probability measures we know that the result is true for the Green function  $H$  associated to  $\pi$ . For general  $\Lambda$ , take  $\phi \in C^\infty(M)$  defined by  $\phi(x) = \int_M H(x, y) d\Lambda(y)$  as in the proof of Proposition 4.3. Then  $G : M \times M \rightarrow (-\infty, \infty]$  given by  $G(x, y) = H(x, y) - \phi(x) - \phi(y)$  is a Green function for  $\Lambda$  and for every  $\mu \in \mathcal{P}(M)$  we have

$$\int_{M \times M} G(x, y) d\mu(x) d\mu(y) = \int_{M \times M} H(x, y) d\mu(x) d\mu(y) - 2 \int_M \phi d\mu.$$

From this relation and the result for  $H$  we get the result for  $G$ . □

Then,  $(\{W_n\}_{n \in \mathbb{N}}, W)$  is a nice model where Theorem 1.2 and the results of Subsection 4.1 can be used.

**Corollary 4.5** (Macroscopic limit).  *$W$  is the zero temperature macroscopic limit and the positive temperature macroscopic limit of  $\{W_n\}_{n \in \mathbb{N}}$ , i.e.  $(\{W_n\}_{n \in \mathbb{N}}, W)$  satisfies all the conditions of Theorem 1.2. Additionally the results of Subsection 4.1 about the Conditional Gibbs measure may be applied.*

Now we shall enunciate a theorem that is our main motivation for choosing this model. Remember the definitions of  $i_n$ , (4), and  $\mathbb{P}_n$ , (7). Let  $\{X_n\}_{n \in \mathbb{N}}$  be a sequence of random variables taking values in  $\mathcal{P}(M)$  such that, for every  $n \in \mathbb{N}$ ,  $X_n$  has law  $i_n(\mathbb{P}_n)$ . By studying the minimizers of the free energy  $F$  defined in (5) we can understand the possible limit points of  $\{X_n\}_{n \in \mathbb{N}}$ . In particular, if  $F$  attains its minimum at a unique probability measure  $\mu_{eq}$ , we get

$$X_n \xrightarrow[n \rightarrow \infty]{a.s.} \mu_{eq}.$$

This is a consequence of Borel-Cantelli lemma and the large deviation principle in Corollary 1.3.

We specialize to the case of dimension two and finite  $\beta$  because the minimizer of  $F$  has a nice geometric meaning in this case.

**Theorem 4.6** (Minimizer of the free energy). *Let  $\rho$  be a strictly positive differentiable function such that*

$$\Delta \log \rho = \beta \mu_{eq} - \beta \Lambda \tag{16}$$

where  $\mu_{eq}$  denotes the probability measure defined by  $d\mu_{eq} = \rho d\pi$  (see [34] for the existence). Then  $F(\mu_{eq}) < F(\mu)$  for every  $\mu \in \mathcal{P}(M)$  different from  $\mu_{eq}$ . In particular, there exists only one strictly positive differentiable function that satisfies (16).

*Remark 4.7* (Scalar curvature relation). The motivation for studying a 2-dimensional manifold is that  $\mu_{eq}$  has a nice geometrical interpretation if we choose adequate  $\Lambda$  and  $\beta$ .

We shall suppose that  $\chi(M)$ , the Euler characteristic of  $M$ , is different from zero. If  $\bar{g}$  is any metric, we denote by  $R_{\bar{g}}$  the scalar curvature of  $\bar{g}$ . Choose

$$d\Lambda = \frac{R_g d\pi}{4\pi\chi(M)}.$$

It can be seen that if  $\bar{g} = \rho g$ , where  $\int_M \rho d\pi = 1$ , then

$$\Delta \log \rho = R_g d\pi - R_{\bar{g}} \rho d\pi.$$

With this identity we can prove that  $\rho$  is a solution to

$$R_{\bar{g}} = \left(1 + \frac{\beta}{4\pi\chi(M)}\right) R_g \rho^{-1} - \beta$$

where  $\bar{g} = \rho g$  if and only if  $\rho$  is a solution to

$$\Delta \log \rho = \beta \mu_{eq} - \beta \Lambda$$

where  $d\mu_{eq} = \rho d\pi$ . In particular, if  $\chi(M) < 0$  and  $\beta = -4\pi\chi(M)$  then  $\bar{g}$  satisfies

$$R_{\bar{g}} = 4\pi\chi(M),$$

i.e.  $\bar{g}$  is a metric with constant curvature. In other words, if  $\beta = -4\pi\chi(M)$ , the empirical measure converges almost surely to the volume form of the constant curvature metric conformally equivalent to the chosen metric.



The proof of Theorem 4.6 will be based on the fact that  $F$  is strictly convex and that we can calculate its derivative. We begin by proving its convexity.

**Proposition 4.8** (Convexity of  $W$ ).  *$W$  is convex.*

*Proof.* To prove the convexity it is enough to show that for every  $\mu, \nu \in \mathcal{P}(M)$

$$\frac{1}{2}W(\mu) + \frac{1}{2}W(\nu) \geq W\left(\frac{1}{2}\mu + \frac{1}{2}\nu\right) \tag{17}$$

due to the lower semicontinuity of  $W$ . If  $\mu$  and  $\nu$  are differentiable probability measures this is equivalent to

$$\int_M \|\nabla(f - g)\|^2 d\pi \geq 0$$

where  $f(x) = \int_M G(x, y) d\mu(y)$  and  $g(x) = \int_M G(x, y) d\nu(y)$ . For general  $\mu$  and  $\nu$  we can conclude using Proposition 4.4, and taking lower limits in the inequality (17) for differentiable probability measures.  $\square$

Since  $D(\cdot|\pi)$  is strictly convex (see [47, Lemma 1.4.3]) we obtain that the *free energy*  $F$  of parameter  $\beta < \infty$  is strictly convex.

Now we calculate the derivative of  $W$  and the entropy at  $\mu_{eq}$ .

**Lemma 4.9** (Derivative of  $W$  and the entropy). *Let  $\mu$  be any probability measure different from  $\mu_{eq}$  such that  $F(\mu) < \infty$ . Define*

$$\mu_t = t\mu + (1 - t)\mu_{eq}, \quad t \in [0, 1].$$

*Then,  $W(\mu_t)$  and  $D(\mu_t|\pi)$  are differentiable at  $t = 0$ , and*

$$\frac{d}{dt}W(\mu_t)|_{t=0} = \int_{M \times M} G(x, y) d\mu_{eq}(x) (d\mu(y) - d\mu_{eq}(y)), \tag{18}$$

$$\frac{d}{dt}D(\mu_t|\pi)|_{t=0} = \int_M \log \rho(y) (d\mu(y) - d\mu_{eq}(y)). \tag{19}$$

*Proof.* To get (18) we just notice that  $W(\mu_t)$  is a polynomial of degree 2 and to obtain (19) we use the monotone convergence theorem as said for instance in [23, Proposition 2.11].  $\square$

And now we are ready to finish the proof of Theorem 4.6.

*Proof of Theorem 4.6.* As in Lemma 4.9, let  $\mu$  be any probability measure different from  $\mu_{eq}$  such that  $F(\mu) < \infty$  and define

$$\mu_t = t\mu + (1 - t)\mu_{eq}, \quad t \in [0, 1].$$

Multiply (16) by  $G(x, y)$  and integrate in one variable to get

$$-\log \rho(y) + \int_M \log \rho(x) d\Lambda(x) = \beta \int_M G(x, y)\rho(x) d\pi(x).$$

Then, we have that

$$\begin{aligned}
& \frac{d}{dt} F(\mu_t)|_{t=0} \\
&= \int_{M \times M} G(x, y) d\mu_{eq}(x) (d\mu(y) - d\mu_{eq}(y)) + \frac{1}{\beta} \int_{M \times M} \log \rho(y) (d\mu(y) - d\mu_{eq}(y)) \\
&= \int_M \left( \int_M G(x, y) \rho(x) d\pi(x) + \frac{1}{\beta} \log \rho(y) \right) (d\mu(y) - d\mu_{eq}(y)) \\
&= \frac{1}{\beta} \int_M \left( \int_M \log \rho(x) d\Lambda(x) \right) (d\mu(y) - d\mu_{eq}(y)) \\
&= \frac{1}{\beta} \left( \int_M \log \rho(x) d\Lambda(x) \right) \left( \int_M (d\mu(y) - d\mu_{eq}(y)) \right) = 0.
\end{aligned}$$

This implies, due to the strict convexity of  $F(\mu_t)$  in  $t$ , that

$$F(\mu_{eq}) < F(\mu).$$

□

### 4.3 Usual Coulomb gases

In this subsection we provide different proofs to the large deviation principles associated to Coulomb gases studied in [58] and [48]. These models are usually motivated as describing the laws of eigenvalues of some random matrices and has as particular cases the models studied in [16], [61], [60] and [35]. We may see [6] for an introduction to random matrices. We would like to remark that the model studied in [17] may be treated by the same methods but does not fall directly in the regime of application of Theorem 1.2.

Suppose that  $l$  is a not necessarily finite measure on the Polish space  $M$ . Let  $V : M \rightarrow (-\infty, \infty]$  and  $G : M \times M \rightarrow (-\infty, \infty]$  be lower semicontinuous functions with  $G$  symmetric and such that  $(x, y) \mapsto G(x, y) + V(x) + V(y)$  is bounded from below. Define  $H_n : M^n \rightarrow (-\infty, \infty]$  by

$$H_n(x_1, \dots, x_n) = \sum_{i < j}^n G(x_i, x_j) + n \sum_{i=1}^n V(x_i)$$

and  $W : \mathcal{P}(M) \rightarrow (-\infty, \infty]$  by

$$W(\mu) = \frac{1}{2} \int_{M \times M} (G(x, y) + V(x) + V(y)) d\mu(x) d\mu(y).$$

Take a sequence  $\{\beta_n\}_{n \in \mathbb{N}}$  such that  $\beta_n \rightarrow \infty$  and let  $\gamma_n$  be the Gibbs measure defined by

$$d\gamma_n = e^{-\frac{\beta_n}{n} H_n} dl^{\otimes n}.$$

We shall give some hypotheses that imply that  $\gamma_n$  satisfies a Laplace principle.

The first example is related to [58]. More precisely, if we choose  $G(x, y) = -\beta \log \|x - y\|$ , condition (1.7) of [58] implies the first three conditions of the following theorem (see the proof of Proposition 4.12 for an idea) and the last condition is a consequence the nature of the logarithmic interaction and the required continuity of  $V$  in [58]. We remark that there is a slight typo in [58]: we should require  $\beta' > 2$  in dimension two.

**Theorem 4.10** (Weakly confining case). *Take  $\beta_n = n$ . Suppose that*

- $\int_M e^{-V} dl < \infty$ ,
- *the function  $(x, y) \mapsto G(x, y) + V(x) + V(y)$  is bounded from below,*
- *$G(x, y) + V(x) + V(y) \rightarrow \infty$  when  $x, y \rightarrow \infty$  at the same time, and*

• for every  $\mu \in \mathcal{P}(M)$  such that  $W(\mu) < \infty$ , there exists a sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  of probability measures absolutely continuous with respect to  $l$  such that  $\mu_n \rightarrow \mu$  and  $W(\mu_n) \rightarrow W(\mu)$ .

Then, for every bounded continuous function  $f : \mathcal{P}(M) \rightarrow \mathbb{R}$  we have

$$\frac{1}{n^2} \log \int_{M^n} e^{-n^2 f \circ i_n} d\gamma_n \xrightarrow{n \rightarrow \infty} - \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + W(\mu)\}.$$

*Proof.* Assume  $\int_M e^{-V} dl = 1$  for simplicity. We notice that

$$d\gamma_n = e^{-\left(\sum_{i < j}^n G(x_i, x_j) + (n-1) \sum_{i=1}^n V(x_i)\right)} d(e^{-V} l)^{\otimes n}.$$

If we define

$$\tilde{G}(x, y) = G(x, y) + V(x) + V(y)$$

and

$$W_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i < j}^n \tilde{G}(x_i, x_j)$$

we have

$$d\gamma_n = e^{-n^2 W_n} d(e^{-V} l)^{\otimes n}.$$

We now prove that  $\{W_n\}_{n \in \mathbb{N}}$  satisfies the conditions necessary to apply Theorem 1.2.

**Lower and upper limit assumption, (A1) and (A2).** By hypotheses,  $\tilde{G}$  is lower semicontinuous and bounded from below. We can apply Proposition 2.1 to get that  $\{W_n\}_{n \in \mathbb{N}}$  is a *stable sequence* (S) and that  $(\{W_n\}_{n \in \mathbb{N}}, W)$  satisfies the *lower limit assumption* (A1) and the *upper limit assumption* (A2).

**Regularity assumption (A2').** Since  $(\{W_n\}_{n \in \mathbb{N}}, W)$  satisfies the *upper limit assumption* (A2), the *regularity assumption* (A2') does not depend on  $\{W_n\}_{n \in \mathbb{N}}$  and we can use Proposition 2.4. Take  $\mu \in \mathcal{P}(M)$  such that  $W(\mu) < \infty$ . Then, by hypothesis, there exists a sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  of probability measures absolutely continuous with respect to  $l$  such that  $\mu_n \rightarrow \mu$  and  $W(\mu_n) \rightarrow W(\mu)$ . Since  $W(\mu) < \infty$  we can assume  $W(\mu_n) < \infty$  for every  $n \in \mathbb{N}$ . Fix  $n \in \mathbb{N}$ . We want to prove that  $\mu_n$  is absolutely continuous with respect to the measure defined by  $e^{-V} dl$ . For this it is enough to notice that

$$\mu_n(\{x \in M : V(x) = \infty\}) = 0.$$

We can see that the set  $\{(x, y) \in M \times M : V(x) = \infty \text{ and } V(y) = \infty\}$  is included in the set  $\{(x, y) \in M \times M : G(x, y) + V(x) + V(y) = \infty\}$ . The latter has zero measure because  $W(\mu) < \infty$  and we conclude by the definition of product measure.

**Confining sequence (C).** Using that  $\tilde{G}(x, y) \rightarrow \infty$  when  $x, y \rightarrow \infty$  at the same time and Proposition 2.2 we get that  $\{W_n\}_{n \in \mathbb{N}}$  is a *confining sequence* (C).

We can finally apply Theorem 1.2. □

The second example is related to the article this work is inspired on, i.e. [48]. More precisely, Assumptions C1-C3 of [48, Theorem 1.6] imply the conditions of the following theorem. We remark that there is a slight typo in [48]: Assumption A should be changed by any weaker assumption that guarantees the finiteness of the Gibbs measures.

**Theorem 4.11** (Strongly confining case).

Suppose that

- There exists  $\xi > 0$  such that  $\int_M e^{-\xi V} dl < \infty$ ,
- $V$  is bounded from below,
- there exists  $\epsilon \in [0, 1)$  such that  $(x, y) \mapsto G(x, y) + \epsilon V(x) + \epsilon V(y)$  is bounded from below,
- the function  $G(x, y) + V(x) + V(y)$  tends to infinity when  $x, y \rightarrow \infty$  at the same time, and

• for every  $\mu \in \mathcal{P}(M)$  such that  $W(\mu) < \infty$ , there exists a sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  of probability measures absolutely continuous with respect to  $l$  such that  $\mu_n \rightarrow \mu$  and  $W(\mu_n) \rightarrow W(\mu)$ .

Then, for every bounded continuous function  $f : \mathcal{P}(M) \rightarrow \mathbb{R}$  we have

$$\frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f \circ i_n} d\gamma_n \xrightarrow{n \rightarrow \infty} - \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + W(\mu)\}.$$

*Proof.* We can assume  $\int_M e^{-\xi V} dl = 1$  for simplicity. Then we can write

$$d\gamma_n = e^{-\frac{\beta_n}{n} \left( \sum_{i < j} G(x_i, x_j) + \left(n - \frac{n}{\beta_n} \xi\right) \sum_{i=1}^n V(x_i) \right)} d(e^{-\xi V l})^{\otimes n}.$$

which may only make sense for  $n$  large enough due to some positive and negative infinities. If we define

$$G^n(x, y) = G(x, y) + \frac{1}{n-1} \left( n - \frac{n}{\beta_n} \xi \right) V(x) + \frac{1}{n-1} \left( n - \frac{n}{\beta_n} \xi \right) V(y)$$

and

$$W_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i < j}^n G^n(x_i, x_j)$$

we have

$$d\gamma_n = e^{-n\beta_n W_n} d(e^{-\xi V l})^{\otimes n}.$$

Now we can try to apply Theorem 1.2 to get the Laplace principle. Define

$$G_1(x, y) = G(x, y) + \epsilon V(x) + \epsilon V(y), \quad W_n^1(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i < j}^n G_1(x_i, x_j),$$

$$G_2(x, y) = (1 - \epsilon)V(x) + (1 - \epsilon)V(y), \quad W_n^2(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i < j}^n G_2(x_i, x_j)$$

and

$$a_n = \frac{1}{1 - \epsilon} \left( \frac{1}{n-1} \left( n - \frac{n}{\beta_n} \xi \right) - \epsilon \right) \rightarrow 1.$$

These definitions allow us to write

$$W_n = W_n^1 + a_n W_n^2.$$

We start by proving the *lower limit assumption* (A1) and the *upper limit assumption* (A2).

**Lower and upper limit assumption, (A1) and (A2).** By the hypotheses, we can see that  $G_1$  and  $G_2$  are lower semicontinuous functions bounded from below. Then, we can apply Proposition 2.1 about the  $k$ -body interaction to get that  $\{W_n^1\}_{n \in \mathbb{N}}$  and  $\{W_n^2\}_{n \in \mathbb{N}}$  are *stable sequences* (S) and if we define the lower semicontinuous functions

$$W^1(\mu) = \frac{1}{2} \int_{M \times M} G_1(x, y) d\mu(x) d\mu(y) \quad \text{and} \quad W^2(\mu) = \frac{1}{2} \int_{M \times M} G_2(x, y) d\mu(x) d\mu(y),$$

then  $(\{W_n^1\}_{n \in \mathbb{N}}, W^1)$  and  $(\{W_n^2\}_{n \in \mathbb{N}}, W^2)$  satisfy the *lower limit assumption* (A1) and the *upper limit assumption* (A2).

Then, since  $a_n > 0$  for  $n$  large enough, we get that  $\{W_n\}_{n \in \mathbb{N}}$  is a *stable sequence* (S) for  $n$  large enough. Noticing that

$$W^1(\mu) + W^2(\mu) = W(\mu) = \frac{1}{2} \int_{M \times M} (G(x, y) + V(x) + V(y)) d\mu(x) d\mu(y).$$

we obtain that  $(\{W_n\}_{n \in \mathbb{N}}, W)$  satisfies the *lower limit assumption* (A1) and the *upper limit assumption* (A2).

**Confining sequence (C).** Using Proposition 2.2 about the confining assumption in the  $k$ -body interaction and by the fact that  $G(x, y) + V(x) + V(y) \rightarrow \infty$  when  $x, y \rightarrow \infty$  at the same time, we get that  $\{W_n^1 + W_n^2\}_{n \in \mathbb{N}}$  is a *confining sequence* (C). Along with the fact that  $\{W_n^1\}_{n \in \mathbb{N}}$  and  $\{W_n^2\}_{n \in \mathbb{N}}$  are *stable sequences* (S) and that  $a_n \rightarrow 1$  this implies that  $\{W_n\}_{n \in \mathbb{N}}$  is also a *confining sequence* (C).

**Regularity assumption (A2').** By an argument similar to the one given in the proof of Theorem 4.10 we can prove the *regularity assumption* (A2') for  $W$ .

We have proved the conditions to apply Theorem 1.2. □

#### 4.4 Gaussian random polynomials

In this subsection we will see that [98, Theorem 1] is a consequence of Corollary 1.3. Consider a probability measure  $\nu \in \mathcal{P}(\mathbb{C})$  and a continuous function  $\phi : \mathbb{C} \rightarrow \mathbb{R}$  such that

$$\liminf_{z \rightarrow \infty} \{\phi(z) - 2 \log \|z\|\} > -\infty. \tag{20}$$

Denote by  $\mathbb{C}_n[z]$  the space of complex polynomials of degree less or equal than  $n$  and denote by  $j_n : \mathbb{C}_n[z] \setminus \mathbb{C}_{n-1}[z] \rightarrow \mathcal{P}(\mathbb{C})$  the application that gives the empirical measure of the zeros of a polynomial, i.e.  $j_n$  is defined by

$$j_n(p) = \frac{1}{n} \sum_{i=1}^n \delta_{z_i} \text{ if } p(z) = a \prod_{i=1}^n (z - z_i) \text{ for some } a \neq 0.$$

We shall consider the complex Gaussian measure  $\mathcal{G}_n$  with covariance  $\langle \cdot, \cdot \rangle_n$  on  $\mathbb{C}_n[z]$  given by

$$\langle p, q \rangle_n = \int_{\mathbb{C}} \bar{p}(z) q(z) e^{-n\phi(z)} d\nu(z)$$

where we have supposed that  $\langle \cdot, \cdot \rangle_n$  is non-degenerate. We will see that the zeros of a random polynomial chosen according to  $\mathcal{G}_n$  can be treated by Corollary 1.3. In other words, we are interested in the pushforward measure of the restriction of  $\mathcal{G}_n$  to  $\mathbb{C}_n[z] \setminus \mathbb{C}_{n-1}[z]$  by  $j_n$ , that we will denote by  $j_n(\mathcal{G}_n)$  and that is still a probability measure because  $\mathcal{G}_n(\mathbb{C}_{n-1}[z]) = 0$ , and we want to write it in the form (1).

**Proposition 4.12** (Gibbs measure form of the zeros of a random polynomial). *Define the function  $G : \mathbb{C} \times \mathbb{C} \rightarrow (-\infty, \infty]$  by*

$$G(z, w) = -2 \log \|z - w\| + \phi(z) + \phi(w).$$

*Then, by the condition (20),  $G$  is a lower semicontinuous function bounded from below. Also, by (20),  $\int_{\mathbb{C}} e^{-2\phi(z)} d\mathcal{L}eb(z) < \infty$ . Define  $\pi \in \mathcal{P}(\mathbb{C})$  by*

$$d\pi(z) = \frac{e^{-2\phi(z)}}{\int_{\mathbb{C}} e^{-2\phi(z)} d\mathcal{L}eb(z)} d\mathcal{L}eb(z),$$

*the symmetric measurable function  $w_n : \mathbb{C}^n \rightarrow (-\infty, \infty]$  by*

$$w_n(z_1, \dots, z_n) = \frac{1}{n^2} \sum_{i < j} G(z_i, z_j) + \frac{n+1}{n^2} \log \left( \int_{\mathbb{C}} e^{-\sum_{i=1}^n G(z, x_i)} d\nu(z) \right) \tag{21}$$

*and the Gibbs measure  $\gamma_n$  by*

$$d\gamma_n = e^{-n^2 w_n} d\pi^{\otimes n}.$$

Then the zeros of a random polynomial chosen according to  $\mathcal{G}_n$  follows the law  $\frac{\gamma_n}{\gamma_n(\mathbb{C}^n)}$ . More precisely,

$$j_n(\mathcal{G}_n) = i_n \left( \frac{\gamma_n}{\gamma_n(\mathbb{C}^n)} \right)$$

where  $i_n \left( \frac{\gamma_n}{\gamma_n(\mathbb{C}^n)} \right)$  denotes the pushforward measure of  $\frac{\gamma_n}{\gamma_n(\mathbb{C}^n)}$  by  $i_n$ .

*Proof.* The lower semicontinuity of  $G$  follows from the continuity of the logarithm and the continuity of  $\phi$ . Since  $-2 \log \|z - w\| \geq -2 \log 2 - 2 \log \|z\| - 2 \log \|w\|$  if  $\|z\|, \|w\| \geq 1$  and using (20) we know that  $G$  is bounded from below. By (20) there exists  $C > 0$  such that  $e^{-2\phi(z)} \leq C \|z\|^{-4}$  if  $\|z\|$  is large enough and we obtain that  $\int_{\mathbb{C}} e^{-2\phi(z)} d\mathcal{L}eb(z) < \infty$ .

The statement about  $j_n(\mathcal{G}_n)$  is a consequence of [30, Theorem 5.1] and the fact that

$$\begin{aligned} \prod_{i < j} \|z_i - z_j\|^2 & \left( \int_{\mathbb{C}} \prod_{i=1}^n \|z - z_i\|^2 e^{-n\phi(z)} d\nu(z) \right)^{-(n+1)} d\mathcal{L}eb^{\otimes n}(z_1, \dots, z_n) \\ & = e^{-\sum_{i < j} G(z_i, z_j)} \left( \int_{\mathbb{C}} e^{-\sum_{i=1}^n G(z_i, z)} d\nu(z) \right)^{-(n+1)} \prod_{i=1}^n d\pi^{\otimes n}(z_1, \dots, z_n). \end{aligned}$$

□

The energy in (21) is a sum of an energy of the 2-body interaction type and a different kind of energy that we will try to understand. Under appropriate conditions in  $\phi$ , the authors of [98] extend  $G$  to  $\bar{\mathbb{C}} \times \bar{\mathbb{C}}$  so we shall only consider compact spaces.

Consider  $G : M \times M \rightarrow (-\infty, \infty]$  a lower semicontinuous function on a compact metric space  $M$ . Consider  $\nu \in \mathcal{P}(M)$  a probability measure on  $M$  and denote its support by  $K \subset M$ . Define  $W_n : M^n \rightarrow [-\infty, \infty)$  by

$$W_n(x_1, \dots, x_n) = \frac{1}{n} \log \left( \int_M e^{-\sum_{i=1}^n G(z, x_i)} d\nu(z) \right)$$

and  $W : \mathcal{P}(M) \rightarrow [-\infty, \infty)$  by

$$W(\mu) = - \inf_{x \in K} \left\{ \int_M G(x, y) d\mu(y) \right\}.$$

Notice that  $\{W_n\}_{n \in \mathbb{N}}$  is uniformly bounded from above and that it is not immediate to say that  $\{W_n\}_{n \in \mathbb{N}}$  is a *stable sequence* (S).

**Lemma 4.13** (Upper limit properties). *W is upper semicontinuous and for each  $\mu \in \mathcal{P}(M)$  we have that*

$$\limsup_{n \rightarrow \infty} \mathbb{E}_{\mu^{\otimes n}} [W_n] \leq W(\mu). \quad (22)$$

*Proof.* **W is upper semicontinuous.** This can be seen as a consequence of the lower semicontinuity of the function  $T : M \times \mathcal{P}(M) \rightarrow (-\infty, \infty]$  defined by  $T(x, \mu) = \int_M G(x, y) d\mu(y)$  as follows. Suppose  $\mu_n \rightarrow \mu$  in  $\mathcal{P}(M)$  and take  $x_n \in K$  such that  $T(x_n, \mu_n) \leq \inf_{x \in K} T(x, \mu_n) + \frac{1}{n}$ . Then

$$\liminf_{n \rightarrow \infty} T(x_n, \mu_n) \leq \liminf_{n \rightarrow \infty} \left[ \inf_{x \in K} T(x, \mu_n) \right].$$

Take a subsequence such that  $\lim_{j \rightarrow \infty} T(x_{n_j}, \mu_{n_j}) = \liminf_{n \rightarrow \infty} T(x_n, \mu_n)$  where, by taking a further subsequence if necessary, we may assume that  $x_n$  converge to some  $x_\infty \in K$ . The lower semicontinuity of  $T$  implies that  $T(x_\infty, \mu) \leq \lim_{j \rightarrow \infty} T(x_{n_j}, \mu_{n_j})$  and so

$$\inf_{x \in K} T(x, \mu) \leq \liminf_{n \rightarrow \infty} \left[ \inf_{x \in K} T(x, \mu_n) \right].$$

**Proof of (22).** Notice that

$$\tilde{W}_n(\hat{\mu}) = \frac{1}{n} \log \left( \int_M e^{-n \int_M G(z,x) d\hat{\mu}(x)} d\nu(z) \right)$$

if  $\hat{\mu} \in i_n(M^n)$ , where  $\tilde{W}_n$  is defined by (2). Then, if  $\hat{\mu} \in i_n(M^n)$ , we have

$$\tilde{W}_n(\hat{\mu}) \leq - \inf_{z \in K} \int_M G(z,x) d\hat{\mu}(x) = W(\hat{\mu}).$$

Let  $\mu \in \mathcal{P}(M)$ , then

$$\mathbb{E}_{\mu^{\otimes n}}[W_n] = \mathbb{E}_{i_n(\mu^{\otimes n})}[\tilde{W}_n] \leq \mathbb{E}_{i_n(\mu^{\otimes n})}[W]$$

and so

$$\limsup_{n \rightarrow \infty} \mathbb{E}_{\mu^{\otimes n}}[W_n] \leq \limsup_{n \rightarrow \infty} \mathbb{E}_{i_n(\mu^{\otimes n})}[W] \leq W(\mu)$$

by the upper semicontinuity and upper boundedness of  $W$ . □

We see that the *upper limit assumption*, (A2), with the sequence  $\{W_n\}_{n \in \mathbb{N}}$  not necessarily a *stable sequence* (S), is satisfied in a very general context. This is not the case for the *lower limit assumption*, (A1) and we will state the two main conditions that allow us to obtain it.

**Definition 4.14** (Bernstein-Markov condition). *For any  $\vec{x} = (x_1, \dots, x_n) \in M^n$  consider the application  $s_{\vec{x}} : M \rightarrow \mathbb{R}$  defined by  $s_{\vec{x}}(y) = e^{-\sum_{i=1}^n G(x_i, y)}$  and denote the support of  $\nu$  by  $K$ . We say that  $(G, \nu)$  satisfies the Bernstein-Markov condition if the following is true. For every  $\epsilon > 0$  there exists  $C > 0$  such that*

$$\sup_{y \in K} s_{\vec{x}}(y) \leq C e^{\epsilon n} \|s_{\vec{x}}\|_{L^1(M, \nu)}$$

for every  $\vec{x} \in M^n$  and for every  $n > 0$ .

**Definition 4.15** (Regular pair). *We will say that the pair  $(G, K)$  is regular if the following is true. For every probability measure  $\mu \in \mathcal{P}(M)$  and every  $\epsilon > 0$  there exists a probability measure  $\nu \in \mathcal{P}(M)$  such that  $\nu(\{x \in K : \int_M G(x, y) d\mu(y) \leq \inf_{z \in K} \int_M G(z, y) d\mu(y) + \epsilon\}) = 1$  and  $x \mapsto \int_M G(x, y) d\nu(y)$  is finite and continuous.*

Our Bernstein-Markov condition is an easy consequence of the Bernstein-Markov condition in the case of random polynomials (see [98, Lemma 9]) and our regular pair condition is a consequence of the non-thinness of  $K$  (see the proof of the second part of [98, Lemma 26]).

**Proposition 4.16** (Lower semicontinuity and lower boundedness). *Suppose the pair  $(G, K)$  is regular. Then  $W$  is lower semicontinuous and bounded from below.*

*Proof. W is bounded from below.* The regular pair condition implies, in particular, that there exists a probability measure  $\nu \in \mathcal{P}(M)$  supported on  $K$  such that  $x \mapsto \int_M G(x, y) d\nu(y)$  is continuous. So,

$$\begin{aligned} W(\mu) &\geq - \int_K \left( \int_M G(x, y) d\mu(y) \right) d\nu(x) \\ &= - \int_M \left( \int_K G(x, y) d\nu(x) \right) d\mu(y) \\ &\geq - \sup_{y \in M} \int_K G(x, y) d\nu(x) \end{aligned}$$

where we have used Fubini's theorem. Since  $x \mapsto \int_K G(x, y) d\nu(y)$  is continuous, it is bounded from above and we have thus proved that  $W$  is bounded from below.

**W is lower semicontinuous.** Let  $\{\mu_n\}_{n \in \mathbb{N}}$  be a sequence of probability measures converging to some  $\mu \in \mathcal{P}(M)$ . We want to prove that  $\liminf_{n \rightarrow \infty} W(\mu_n) \geq W(\mu)$ . For  $\epsilon > 0$  the regular pair condition says that there exists  $\nu \in \mathcal{P}(M)$  supported in  $K$  such that

$$\int_M G(x, y) d\mu(y) \leq \inf_{z \in K} \int_M G(z, y) d\mu(y) + \epsilon \tag{23}$$

for  $\nu$ -almost every  $x$  and  $x \mapsto \int_M G(x, y) d\nu(y)$  is bounded continuous. Then integrating 23 with respect to  $\nu$  we get

$$\int_{M \times M} G(x, y) d\nu(x) d\mu(y) \leq \inf_{z \in K} \int_M G(z, y) d\mu(y) + \epsilon$$

but, since  $x \mapsto \int G(x, y) d\nu(y)$  is bounded continuous we have that

$$\int_M \left( \int_M G(x, y) d\nu(x) \right) d\mu_n(y) \rightarrow \int_M \left( \int_M G(x, y) d\nu(x) \right) d\mu(y).$$

Since  $\nu$  is supported in  $K$  we know that

$$\inf_{z \in K} \int_M G(z, y) d\mu_n(y) \leq \int_M \left( \int_M G(x, y) d\mu_n(y) \right) d\nu(x).$$

Taking the upper limit and using Fubini's theorem we get

$$\limsup_{n \rightarrow \infty} \left[ \inf_{z \in K} \int_M G(z, y) d\mu_n(y) \right] \leq \inf_{z \in K} \int_M G(z, y) d\mu(y) + \epsilon.$$

Since this is true for every  $\epsilon > 0$  we conclude the proof. □

**Proposition 4.17** (Stability and lower limit assumption). *Suppose  $(G, K)$  is regular and that  $(G, \nu)$  satisfies the Bernstein-Markov condition. Then  $\{W_n\}_{n \in \mathbb{N}}$  is a stable sequence (S) and the pair  $(\{W_n\}_{n \in \mathbb{N}}, W)$  satisfies the lower limit assumption, (A1).*

*Proof.* If we take the logarithm on both sides of the Bernstein-Markov condition, we get

$$- \inf_{y \in K} \frac{1}{n} \sum_{i=1}^n G(x_i, y) \leq \frac{1}{n} \log(C) + \epsilon + W_n(\vec{x}).$$

Equivalently, we have that

$$W(\mu) \leq \frac{1}{n} \log(C) + \epsilon + \tilde{W}_n(\mu)$$

and, in particular, since  $W$  is bounded from below, we obtain that  $\{W_n\}_{n \in \mathbb{N}}$  is a stable sequence (S). If  $\mu_n \rightarrow \mu$  then

$$W(\mu) \leq \liminf_{n \rightarrow \infty} W(\mu_n) \leq \epsilon + \liminf_{n \rightarrow \infty} \tilde{W}_n(\mu_n).$$

Since this is true for every  $\epsilon > 0$  we conclude the proof. □

The following corollary immediately implies [98, Theorem 1].

**Corollary 4.18** (Zero temperature macroscopic limit). *Suppose that  $(G, K)$  is regular and that  $(G, \nu)$  satisfies the Bernstein-Markov condition. Suppose also that for every probability measure  $\mu \in \mathcal{P}(M)$  such that  $\int_{M \times M} G(x, y) d\mu(x) d\mu(y) < \infty$  there exists a sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  of probability measures on  $M$  such that  $D(\mu_n \| \pi) < \infty$  for every  $n \in \mathbb{N}$  and such that*

$$\lim_{n \rightarrow \infty} \int_{M \times M} G(x, y) d\mu_n(x) d\mu_n(y) = \int_{M \times M} G(x, y) d\mu(x) d\mu(y). \tag{24}$$



Define  $w_n : M^n \rightarrow (-\infty, \infty]$  and  $w : \mathcal{P}(M) \rightarrow (-\infty, \infty]$  by

$$w_n(z_1, \dots, z_n) = \frac{1}{n^2} \sum_{i < j} G(z_i, z_j) + \frac{n+1}{n^2} \log \left( \int_{\mathbb{C}} e^{-\sum_{i=1}^n G(z, x_i)} d\nu(z) \right)$$

and

$$w(\mu) = \frac{1}{2} \int_{M \times M} G(x, y) d\mu(x) d\mu(y) - \inf_{x \in K} \left\{ \int_K G(x, y) d\mu(y) \right\}.$$

Then  $w$  is the zero temperature macroscopic limit of  $\{w_n\}_{n \in \mathbb{N}}$ .

*Proof.* Using Propositions 4.17 and 2.1 we obtain that  $w_n$  is well defined,  $\{w_n\}_{n \in \mathbb{N}}$  is a stable sequence (S) and that  $(\{w_n\}_{n \in \mathbb{N}}, w)$  satisfies the lower limit assumption, (A1). The regularity assumption, (A2'), is implied by Proposition 2.1, the continuity of  $w$  and (24). □

## 5 Fekete points and the zero temperature deterministic case

We begin by a fact whose standard proof can be found in [50].

**Proposition 5.1** (Convergence of the infima). *If  $W$  is the positive temperature macroscopic limit or the zero temperature macroscopic limit of a stable (S) and confining (C) sequence  $\{W_n\}_{n \in \mathbb{N}}$  then*

$$\inf W_n \rightarrow \inf W.$$

In particular we get the following consequence.

**Theorem 5.2** (Deterministic Laplace principle). *If  $W$  is the positive temperature macroscopic limit or the zero temperature macroscopic limit of a stable (S) and confining (C) sequence  $\{W_n\}_{n \in \mathbb{N}}$  then for every bounded continuous function  $f : M \rightarrow \mathbb{R}$*

$$\inf \{W_n + f \circ i_n\} \rightarrow \inf \{W + f\}.$$

*Proof.* It is enough to notice that if  $W$  is the positive temperature macroscopic limit (respectively, the zero temperature macroscopic limit) of the sequence  $\{W_n\}_{n \in \mathbb{N}}$  then  $W + f$  is the positive temperature macroscopic limit (respectively, the zero temperature macroscopic limit) of the sequence  $\{W_n + f\}_{n \in \mathbb{N}}$  and use Proposition 5.1. □

This may be seen as a natural analogue of the Laplace principle. It is just (10) without the entropy term (as if  $\beta_n$  were infinity). This analogue is related to the notion of  $\Gamma$ -convergence (see [41] for an introduction to this topic) as is said in the following remark.

*Remark 5.3* ( $\Gamma$ -convergence). Theorem 5.2 can be used to prove the  $\Gamma$ -convergence of the sequence  $\tilde{W}_n$  defined in (2) (see [41, Theorem 9.4]). In fact, the confining property of  $\{W_n\}_{n \in \mathbb{N}}$  is not needed since we can obtain the  $\Gamma$ -convergence from the following standard statement if we take  $A_n$  to be equal to the graph of  $\tilde{W}_n$ .

Let  $E$  be a measurable space. Take a sequence  $\{A_n\}_{n \in \mathbb{N}}$  of measurable sets in  $E$  and choose  $x \in E$ . The following statements are equivalent.

a) There exists a sequence  $\{X_n\}_{n \in \mathbb{N}}$  of random variables taking values in  $E$  such that

$$\forall n \in \mathbb{N}, \mathbb{P}(X_n \in A_n) = 1 \quad \text{and} \quad X_n \xrightarrow{\mathbb{P}} x.$$

b) There exists a sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $E$  such that

$$\forall n \in \mathbb{N}, x_n \in A_n \quad \text{and} \quad x_n \rightarrow x.$$

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## RÉSUMÉ

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Nous explorons des modèles probabilistes appelés gaz de Coulomb. Ils apparaissent dans différents contextes comme par exemple dans la théorie des matrices aléatoires, l'effet Hall quantique fractionnaire de Laughlin et les modèles de supraconductivité de Ginzburg-Landau. Dans le but de mieux comprendre le rôle de l'espace ambiant, nous étudions des versions géométriques de ces systèmes. Nous exploitons trois structures sur ces modèles. La première est définie par l'interaction électrostatique provenant de la loi de Gauss. La deuxième est la structure déterminantale disponible que pour des valeurs précises de la température. La troisième est le principe de minimisation de l'énergie libre en physique, qui permet d'étudier des modèles plus généraux. Ces travaux conduisent à des nombreuses questions ouvertes et à une famille de modèles d'intérêt.

## MOTS CLÉS

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Gaz de Coulomb, mesure de Gibbs, grandes déviations, concentration de la mesure, polynôme aléatoire, variété riemannienne, processus ponctuel déterminantal, champ moyen.

## ABSTRACT

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We explore probabilistic models usually called Coulomb gases. They arise naturally in mathematics and physics. We can mention random matrix theory, the Laughlin fractional quantum Hall effect and the Ginzburg-Landau systems of superconductivity. In order to better understand the role of the ambient space, we study geometric versions of such systems. We exploit three structures. The first one comes from the electrostatic nature of the interaction given by Gauss's law. The second one is the determinantal structure which appears only for a specific temperature. The third one is the minimization of the free energy principle, coming from physics which gives us a tool to understand more general models. This work leads to many open questions on a whole family of models which can be of independent interest.

## KEYWORDS

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Coulomb gas, Gibbs measure, large deviations, concentration of measure, random polynomial, Riemannian manifold, determinantal point process, mean field theory.